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Defined Benefit Pension Annuity and Defined Contribution Pension Annuity

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1. Introduction

We build option model for pricing individual defined contribution pension and examined whether individual defined contribution pensions can reduce inefficiency in defined benefit pensions in this paper.

The nature of the pension contract depends on whether the plan is a defined benefit (DB) or defined contribution (DC) pension. In the case of DC pension, contributions are invested, often at the participant's direction, in stocks, bonds, or other financial instruments. In the United States, the fastest–growing DC plan is the 401(k). The 401(k) participants must have a choice of at least three funds to choose from a stock index fund, a bond fund, and usually a money market. In the case of a funded DB plan, the plan sponsor is expected to contribute to the plan in an olderly fashion according to actuarial standards, so the needed funds are available when the worker retires.

As the period of annuity is long, the value of DB pension fluctuates. The DC pension sustains the value by giving consumers part of the investment income.

In the case of DC pension, the insured is given a part of investment income to sustain the value of annuity. If the investment income is higher than minimum guaranteed insurance, an excess is added to insurance. By contrast, if it is lower than minimum insurance, minimum insurance is paid. The DC pension, therefore, is thought of as derivative financial assets with payments that depend upon changes in the investment income. Then the insurers give put option and receive option premium in return.

When we approximate the price of annuity to a Brownian motion process, the Black–Scholes approach leads to the price of annuity, which is the sum of the future price of annuity and option premium. But we can not know the exact future price. Hence, the premium is high in expecting that the future price rises, and it is low in expecting that the future price falls.

We use the option pricing model for examining the relation between DC pension and the price fluctuation. The remainder of this paper is organized as follows. In Section 2, we explain defined benefit (DB) pension and defined contribution (DC) pension. In Section 3, we prove the inefficiency of the defined benefit pension. Section 4 provides the optimal demand for the defined benefit pension and the effect of price fluctuation. Section 5 provides option model for pricing individual defined contribution (DC) pension and the effect of price fluctuation. The last section provides a summary and conclusions.

2. Defined benefit and defined contribution

Workers forecast the future and save enough to smooth consumption over time. But the lifetime budget constraint includes many uncertain components – real earnings, public and private pension benefits, investment returns, healthcare needs, and job loss or disability. Pensions play a key role in protecting

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against these risks.

Pensions offer a mechanism for people to share longevity risk with a group of fellow-employees, but the insurance market for individual annuities is skewed by adverse selection because people who buy individual annuities are likely to live much longer than the population average. For this reason, a firm-based or occupational pension plan can afford workers a valuable precommitment device.

While pensions can help mitigate these potent individual risks, they in turn expose employees covered by a firm-based pension to risk associated with the plan sponsor – misdirection of investments or spending on too much administrative costs. Protection against such behavior often motivates government regulations.

The plan sponsor risk, which arises only in DB pension, is a risk associated with under-funded plan terminations. Protection against the risk of plan sponsor bankruptcy is to have a pension solvency guarantee system. In Germany, DB pension insurance takes the form of a pay-as-you-go solvency fund.

Also county-specific risks occurs in saving via pensions. These include macroeconomic risks associated with inflation and recession, and also country-specific fluctuations in capital markets.

People must seek international portfolio diversification to protect against these risks, since a country's economic and political environment fluctuates in the long run. What is worse, little can be done privately to protect against the negative financial consequences of the potentially catastrophic crises – depression, war. For this reason, the huge unfounded government sponsored social security system started. A government-sponsored defined benefit system transfers income from workers to the retired generation and redistributed these risks from the then old to future taxpayers.

In developing countries, DC pension has grown over the past several decades. In the United States, the fastest-growing plan type is the popular 401(k).

3. Inefficiency in DB pension

At first, we explain the inefficiency of DB pension. The market economy is composed of individuals and firms. The model proposed here is comprised of three periods. The difference of between the first and the second period of life is only the difference of the probability of individuals' death. An individual is born at the beginning of the first period. The probability that he dies at the end of the first period is p_1 and the probability that he dies at the end of the second period is p_2.

Individuals work only in the first period of life, earn a wage of w and do not leave their property. Individuals live for one or two or three periods. They consume part of their first-period income and, save the rest or purchase annuities to finance their retirement consumption. Let c_t denote consumption in period t. If they live for three periods, they derive utility $U(c_1, c_2, c_3)$. If they live for two periods, they derive utility $V(c_1, c_2)$. If they live for only one period, they derive utility $H(c_1)$. Then individuals' expected utility u are described as follows.

$$u(c_1, c_2, c_3)=(1-p_1)(1-p_2)U(c_1, c_2, c_3)+(1-p_1)p_2V(c_1, c_2)+p_1H(c_1) \quad (3-1)$$

Using the rate of time preference, we can reformalize as follows

$$U(c_1, c_2, c_3)=\nu(c_1)+\frac{\nu(c_2)}{1+\rho}+\frac{\nu(c_3)}{(1+\rho)^2} \quad (3-2)$$

$$V(c_1, c_2)=\nu(c_1)+\frac{\nu(c_2)}{1+\rho} \quad (3-3)$$

$$H(c_1)=\nu(c_1) \quad (3-4)$$

The parameter ρ is the rate of time preference and the function $\nu(\bullet)$ is the utility function of consumption; $\nu(\bullet)$ is nonnegative and a concave increasing function of the consumption.

Suppose that individuals purchase annuities to maximize utility and they pay fair premium that equals the expected value of annuities.

$$z=\frac{1-p_1}{1+r}a_2+\frac{(1-p_1)(1-p_2)}{(1+r)^2}a_3, \quad (3-5)$$

where z is the premium of annuities, a_t is the annuity in period t and r is the rate of riskless interest.

When individuals can purchase such an annuity and have no incentive to leave their property, they don't save. Because annuities, which hold (3−5), have higher profitability than saving. If they save to finance their retirement consumption, saving s they need is as follows.

$$s=\frac{1}{1+r}c_2+\frac{1}{(1+r)^2}c_3 \quad (3-6)$$

$$s>z$$

Then the budget constraint in the first, second, and third period are as follows respectively.

$$w=c_1+z \quad (3-7)$$

$$a_2=c_2 \quad (3-8)$$

$$a_3=c_3 \quad (3-9)$$

We can derive the lifetime budget constraint from (3−5, 7, 8, 9).

$$w=c_1\frac{1-p_1}{1+r}c_2+\frac{(1-p_1)(1-p_2)}{(1+r)^2}c_3 \quad (3-10)$$

Consider a representative individual. His maximization problem is

$$\max \quad u(c_1, c_2, c_3)=(1-p_1)(1-p_2)U(c_1, c_2, c_3)+(1-p_1)p_2V(c_1, c_2)+p_1H(c_1)$$

$$\text{subject to} \quad w=c_1\frac{1-p_1}{1+r}c_2+\frac{(1-p_1)(1-p_2)}{(1+r)^2}c_3.$$

The first–order conditions for maximum are

$$\nu'(c_1)-\mu=0 \quad (3-12)$$

$$\frac{1-p_1}{1+\rho}\nu'(c_2)-\frac{1-p_1}{1+r}\mu=0 \quad (3-13)$$

$$\frac{(1-p_1)(1-p_2)}{(1+\rho)^2}\nu'(c_3)-\frac{(1-p_1)(1-p_2)}{(1+r)^2}\mu=0. \quad (3-14)$$

where μ is a Lagrange multiplier. We can derive the following from (3−12, 13, 14).

$$\nu'(c_2)=\frac{1-\rho}{1+r}\nu'(c_1) \quad (3-15)$$

$$\nu'(c_3)=\frac{1-\rho}{1+r}\nu'(c_2) \quad (3-16)$$

We can derive the following results from (3−15, 16).

If $\rho \geq r$, then $c_1 \geq c_2 \geq c_3$.
If $\rho < r$, then $c_1 < c_2 < c_3$.

Proposition 1

When the rate of time preference is higher (lower) than the interest rate, optimal consumption increases (decreases) with age. In the case of the DB pension, annuity in the second period equals annuity in the third period. Therefore the DB pension is inefficient.

4. Demand for DB pension annuity and interest rate

How does a rise of price have an effect on demand for DB pension annuity and inefficiency of DB pension? In Section 4–1, we derive the optimal demand for DB pension when individuals save part of income to make up for the shortage of DB pension. In Section 4–2, comparative statics are used to prove how the fluctuation in the interest rate has effect on the demand for DB pension.

4–1 Basic Model

In the case of DB pension, annuity in the second period equals annuity in the third period. Therefore if we denote fixed annuity as a, then $a = a_2 = a_3$ and $(3-5)$ is rewritten as follows.

$$z = \frac{1-p_1}{1+r}a + \frac{(1-p_1)(1-p_2)}{(1+r)^2}a \quad (4-1)$$

Since consumption changes with age, the DB pension annuity is not enough for individuals to consume efficiently. Then individuals save part of income to make up for the shortage. Introducing saving, the budget constraints in each period are rewritten as follows.

$$w = c_1 + z + s_1 \quad (4-2)$$
$$(1+r)s_1 + a = c_2 + s_2 \quad (4-3)$$
$$(1+r)s_2 + a = c_3, \quad (4-4)$$

where $s_1 \equiv$ saving in the first period
$s_2 \equiv$ saving in the second period

Individuals choose consumptions in each period, saving and annuity to maximize their expected utility under budget constraints $(4-2, 3, 4)$. When we solve $(4-1)$ for a, we derive the following equation.

$$a = \frac{(1+r)^2}{(1-p_1)(1+r)(1-p_1)(1-p_2)} \quad (4-5)$$

When we assume $c_i\,(i = 1, 2, 3)$, $s_i\,(i = 1, 2)$, z are all nonnegative, individuals' Lagrange function is as follows.

$$L = u(c_1, c_2, c_3) + \mu_1(w - c_1 - z - s_1) + \mu_2[(1+r)s_1 + a - c_2 - s_2] + \mu_3[(1+r)s_2 + a - c_3]$$
$$+ \mu_4(a - \xi z) + q_1c_1 + q_2c_2 + q_3c_3 + q_4s_1 + q_5s_2 + q_6z, \quad (4-6)$$

where are $\mu_i\,(i = 1, 2, 3)$ the Lagrange multipliers on budget constraints and is a Lagrange multiplier on $(4-5)$ when ξ is defined as follows.

$$\xi = \frac{(1+r)^2}{(1-p_1)(1+r)(1-p_1)(1-p_2)} \quad (4-7)$$

$q_i\,(i = 1, 2, ..., 6)$ are Lagrange multipliers on nonnegative constrains. Then the first–order conditions are as follows.

$$\frac{\partial L}{\partial c_1} = \nu'(c_1) - \mu_1 + q_1 = 0 \quad (4-8)$$

$$\frac{\partial L}{\partial c_2} = \frac{1-p_1}{1+\rho}\nu'(c_2) - \mu_2 + q_2 = 0 \quad (4-9)$$

$$\frac{\partial L}{\partial c_3} = \frac{(1-p_1)(1-p_2)}{(1+\rho)^2}\nu'(c_3) - \mu_3 + q_3 = 0 \quad (4-10)$$

$$\frac{\partial L}{\partial s_1} = \mu_1 + (1+r)\mu_2 + q_4 = 0 \qquad (4-11)$$

$$\frac{\partial L}{\partial s_2} = \mu_2 + (1+r)\mu_3 + q_5 = 0 \qquad (4-12)$$

$$\frac{\partial L}{\partial z} = -\mu_1 - \xi\mu_4 + q_6 = 0 \qquad (4-13)$$

$$\frac{\partial L}{\partial z} = \mu_2 + \mu_3 + \mu_4 = 0 \qquad (4-14)$$

$$\frac{\partial L}{\partial \mu_1} = w - c_1 - z - s_1 = 0 \qquad (4-15)$$

$$\frac{\partial L}{\partial \mu_2} = (1+r)s_1 + a - c_2 - s_2 = 0 \qquad (4-16)$$

$$\frac{\partial L}{\partial \mu_3} = (1+r)s_2 + a - c_3 = 0 \qquad (4-17)$$

$$\frac{\partial L}{\partial \mu_4} = a - \xi z = 0 \qquad (4-18)$$

$$q_i c_i = 0 \ (i = 1, 2, 3) \qquad (4-19)$$

$$q_i 3c_i = 0 \ (i = 1, 2) \qquad (4-20)$$

$$q_6 z = 0 \qquad (4-21)$$

$q_i \ (i = 1, 2, ..., 6) = 0$ from $(4-19, 20, 21)$

$\left[\frac{(1+r)^2}{(2+r)\xi} - 1\right]\mu_1 = 0$ from $(4-11, 12, 13, 14)$

From $(4-7)$,

$$\left[\frac{(1-p_1)(1+r)(1-p_1)(1-p_2)}{2+r} - 1\right]\mu_1 = 0 \qquad (4-22)$$

Since p_1, p_2 are probability, $0 \le p_i \le 1$ $(i = 1, 2)$. Therefore $(4-22)$ holds for any μ_1 when $p_1 = p_2 = 0$. That is, $s_i > 0$ $(i = 1, 2)$, $z > 0$ hold at once when $p_1 = p_2 = 0$.

4–2 Comparative Statics

As shown in the previous section, when the probability of death and demand for annuity are positive, both savings in the first period and the second period are not positive. Then we analyze the following two cases : (1) $s_1 > 0, s_2 = 0$, (2) $s_1 = 0, s_2 > 0$.

(1) $s_1 > 0, s_2 = 0$ $(r < \rho)$

As shown in the previous section, when the rate of time preference is higher than the interest rate, optimal consumption decreases with age. In this case, individuals can consume enough within DB pension annuity, so that saving in the second period is zero.

At the beginning of the first period, individuals choose consumption, saving and annuity to maximize expected lifetime utility $(3-1, 2, 3, 4)$ under budget constraints $(4-1, 2, 3, 4, 5, 6)$. Since $s_2 = 0$ by assumption, the budget constraints in the second and third period is rewritten as follows.

$$(1+r)s_1 + a = c_2 \qquad (4-23)$$

$$a = c_3 \qquad (4-24)$$

From $s_1 > 0$ and (4−23, 24),

$$c_2 > c_3 \qquad (4-25)$$

From (4−23, 24), the lifetime budget constraint is as follows.

$$w = c_1 + \frac{c_2}{1+r} + \frac{(1-p_1)(1+r)+(1-p_1)(1-p_2)-(1+r)}{(1+r)^2}c_3 \quad (4-26)$$

Also, if $c_i\ (i = 1, 2, 3)$ is assumed to be nonnegative, then the Lagrange function is as follows.

$$L = u(c_1, c_2, c_3) + \mu\left[w - c_1 + \frac{c_2}{1+r} + \frac{(1-p_1)(1+r)+(1-p_1)(1-p_2)-(1+r)}{(1+r)^2}c_3\right]$$
$$+ q_1(c_2 - c_3) + q_2c_1 + q_3c_2 + q_4c_3, \qquad (4-27)$$

where μ, $q_i\ (i = 1, 2, 3, 4)$ are the Lagrange multipliers.

Now examine how does the increase of the real interest rate have effects on the demand for annuity.

From the Lagrange function (4−27),

$$\frac{da}{dr} = \frac{dc_3}{dr} = \frac{1}{|H|}[L_{11}L_{42}(L_{3r}L_{24} - L_{2r}L_{34}) + L_{22}L_{3r}], \quad (4-28)$$

where

$$L_{ij} = \frac{\partial^2 L}{\partial c_i \partial c_j}(i, j = 1, 2, 3);\ L_{i4} = \frac{\partial^2 L}{\partial c_i \partial \mu}(i = 1, 2, 3);\ L_{ir} = \frac{\partial^2 L}{\partial c_i \partial r}(i = 1, 2, 3).$$

The Hessian matrix of the second derivatives $|H|$ is proved to be negative, where

$$L_{11} = v''(c_1) < 0,\ L_{22} = \frac{1-p_1}{1+\rho}v''(c_2) < 0,\ L_{24} = L_{42} = -\frac{1}{1+r} < 0$$

$$L_{2r} = -\frac{\mu}{(1+r)^2} > 0,\ L_{3r} = -\frac{p_1}{(1+r)^2}\mu < 0.$$

As we define K as follows, $K/(1+r)$ is the discounted rate of consumption in the third period.

$$K \equiv \frac{(1-p_1)(1+r)+(1-p_1)(1-p_2)-(1+r)}{1+r} > 0 \quad (4-29)$$

From (4−29),

$$L_{34} = -K/(1+r) < 0$$
$$da/dr < 0 \quad (4-30)$$

Proposition 2

The rise of the interest rate increases the consumption in the second period but decreases the consumption in the third period. Individuals increase saving in the first period to make up for an excess of consumption over DB pension annuity. However profitability of saving is lower than annuity. Therefore when the rise of the interest rate increases saving in the first period, the inefficiency of DB pension annuity increases.

(2) $s_1 = 0, s_2 > 0\ (r > p)$

When the rate of time preference is smaller than the interest rate, the consumption in the second period is smaller than the consumption in the third period and saving in the second period is positive. Since $s_1 = 0$ by assumption, the budget constraints in the second and third period are as follows.

$$a = c_2 + s_2 \qquad (4-31)$$
$$(1+r)s_2 + a = c_3 \qquad (4-32)$$

From $s_2 > 0$ and $(4-31, 32)$,

$$c_3 > c_2 \qquad (4-33)$$

From $(4-1, 2, 5, 31, 32)$, we can derive the following lifetime budget constraint.

$$w = c_1 + \delta c_2 + \frac{\delta}{1+r} c_3, \quad (4-34)$$

where

$$\delta \equiv \frac{(1-p_1)(1+r)+(1-p_1)(1-p_2)}{(1+r)(2+r)}$$

Also, $c_i\,(i=1, 2, 3)$ is assumed to be nonnegative. At the beginning of the first period, individuals choose consumption, saving and annuity to maximize their lifetime utility $(3-1, 2, 3, 4)$ under budget constraints $(4-33, 34)$. Then the Lagrange function is as follows.

$$L = u(c_1, c_2, c_3) + \mu\left[w - c_1 + \delta c_2 - \frac{\delta}{1+r} c_3\right] + q_1(c_3 - c_2) + q_2 c_1 + q_3 c_2 + q_4 c_3, \quad (4-36)$$

where μ, $q_i\,(i=1, 2, 3, 4)$ are the Lagrange multipliers.

Now we examine how the rise of price has effects on the demand for annuity. From the Lagrange function $(4-36)$, we can derive the following result.

$$\frac{dc_1}{dr} = \frac{1}{|H|}(L_{3r}L_{22}L_{34} - L_{2r}\delta L_{33}) < 0 \quad (4-37)$$

From $L_{22} = \dfrac{1-p_1}{1+\rho}v''(c_2) < 0$

$$L_{33} = \frac{(1-p_1)(1-p_2)}{(1+\rho)^2}v''(c_3) < 0$$

$$L_{34} = \frac{\delta}{1+r} < 0$$

$$L_{23} = \frac{(1-p_1)(1-r)^2 + (1-p_1)(1-p_2)(2r+3)}{(1+r)^2(2+r)^2}\mu > 0$$

$$L_{3r} = \frac{(1-p_1)(2r+3)^2 + (1-p_1)(1-p_2)(3r+5)}{(1+r)^2(2+r)^2}\mu > 0$$

Proposition 3

When the rate of time preference is smaller than the interest rate, the rise of interest rate decreases consumption in the first period but increases the demand for DB pension annuity. In this case, the rise of interest rate decreases inefficiency of annuity. However, the optimal consumption increases with age when the rate of time preference is smaller than the interest rate. Note that inefficiency of DB pension annuity never vanishes.

5. DC Pension Annuity

In the previous section, we prove that the rise of interest rate increases inefficiency of DB pension annuity when the rate of time preference is larger than the interest rate. The reason is that the rise of interest rate decreases the real value of DB pension annuity. In this section, we examine DC pension annuity. In Section 5–1, the price of DC pension annuity is derived. In Section 5–2, we examine how inflation has effects on DC pension annuity.

5–1 Price of DC Pension Annuity

DC pension annuities, whose part of premium are invested on special account, change with investment income. If investment income is negative, the minimum coverage is paid. On the contrary, if investment income is positive, it is added to the minimum coverage.

Let the minimum coverage of DC pension annuity denote F dollars and the investment assets, which is investment income added to minimum coverage, denote E_t dollars in the period T. If E_t is smaller than F, insurers have to pay F dollars. On the contrary, if $E_t \geq F$, then E_t is paid. Then a claim cost in the period t is $\max(F, E_t) = E_t + \max(0, F - E_t)$. In this case, $\max(0, F - E_t)$ is called a (European) put option spread, and the minimum coverage F is called an exercise price.

Now, we assume the evolution of DC pension annuity E can be approximated by a Brownian motion process. Then we can formally write the dynamics of this annuity's value in stochastic differential equation form as

$$dE = gEdt + \sigma Edw, \quad (5-1)$$

where g is the instantaneous expected rate of growth on this DC pension annuity ; σ^2 is the instantaneous variance (constant) of the return ; dw is a standard Gauss–Wiener process. Then the Black–Scholes approach leads to the following formula for the return on this DC pension annuity :

$$I(F, E_0, t) = E_t + E\left[\max(0, F - E_t)\right] = E_t - E_0 e^{-\delta t} N(-x^*) + Fe^{-rt} N(-x^* + \sigma\sqrt{t}) \quad (5-2)$$

$$x^* \equiv \frac{\ln(E_0/F) + t(\alpha - \delta + \sigma^2/2)}{\sigma\sqrt{t}},$$

where E_0 is the evaluation of DC pension annuity before invested and it equals the premium ; α is the nominal rate of return on this annuity ; δ is the real rate of return on this annuity, $\delta \equiv \alpha - g$; $N(\bullet)$ is the standard normal distribution function. $-E_0 e^{-\delta t} N(-x^*) + Fe^{-rt} N(-x^* + \sigma\sqrt{t})$ is the put option premium. Since g is included in $(5-2)$, the price of DC pension annuity is independent of individuals' risk types. Consequently, the option price of DC pension annuity for risk–neutral consumers is the same as the one for risk–averse consumers. In the risk–neutral case, future price of DC pension annuity is discounted by its expectation.

5–2 The fluctuation of the investment income

If the minimum coverage of a DC pension annuity is smaller than the DB pension annuity, risk–averse consumers purchase DB pension annuities. In other words, if risk–averse consumers purchase DC pension annuities, its minimum coverage is larger than DB pension annuity.

The claim of a DC pension annuity is composed of investment income added to minimum coverage and option premium. However, since we can not know future investment income exactly, the claim depends on the expected investment income. Therefore, if the expected investment income is high, the claim is high, and vice versa.

Let claims of DC pension annuities in the second and third period denote α_2, α_3 respectively. When the rise of interest rate increases the expected rate of return on DC pension annuities, the expected rate of growth on DC pension annuity, the minimum coverage and the claim increase respectively. Let the price of this annuity at the beginning of the first period denote E_1, and the real rate of return on this annuity in the first and second period denote δ_1, δ_2 respectively. Then the prices of this annuity in the second and the third period are $E_1(1+\delta_1), E_1(1+\delta_1)(1+\delta_2)$ respectively. Also, we assume the minimum coverage equals the claim of a DB pension annuity and let it denote a. When the change of the interest rate is expected, (5−2) is rewritten as follows.

$$\alpha_2 = E_1(1+\delta_1) - E_1 N(-x_2{}^*) + aN(-x_2{}^* + \sigma\sqrt{2}) \quad (5-4)$$
$$\alpha_3 = E_1(1+\delta_1)(1+\delta_2) - E_1 N(-x_3{}^*) + aN(-x_3{}^* + \sigma) \quad (5-5)$$
$$x_j{}^* = \frac{\ln(E_1/a) + j(\alpha - \delta_j + \sigma^2/2)}{\sigma\sqrt{j}}, \ (j = 1, 2, 3) \quad (5-6)$$

When the interest rate increases, we assume the real rate of return on this annuity in the second and third period increase at the same rate. Then the claims on the annuities in the second changes as follows.

$$\frac{d\alpha_2}{d\delta_1} = E_1 > 0 \quad (5-7)$$

$$\frac{d\alpha_2}{d\delta_2} = -\left[x_2{}^* E_1 N(-x_2{}^*) - a(-x_2{}^* + \sigma\sqrt{2})N(-x_2{}^* + \sigma\sqrt{2})\right]\frac{dx_2{}^*}{d\delta_2} > 0 \quad (5-8)$$

From $x_2{}^* - \sigma\sqrt{2} < x_2{}^*$

$$N(-x_2{}^*) > N(x_2{}^* + \sigma\sqrt{2})$$

$$\frac{dx_2{}^*}{d\delta_2} = \frac{-2}{\sigma\sqrt{2}} < 0$$

Consequently, in the case of DC pension annuities, the rise of return rate increases the claim of DC pension annuities in the second period. When the rate of time preference is very high, the effects that the rise of the return rate increases inefficiency in DC pension annuities are larger than DB pension annuities.

6. Conclusion

In many countries, many elderly people depend on government and employment–based pensions. The modal pension in the developed countries was of the defined benefit several years ago. A governed–based defined benefit system transfers income from workers to the retired generation, at least in part. But the world's population is rapidly growing aging, so the risks of underfunded plan terminations is increasing and defined contribution plans are becoming more the norm in the developed countries. We build option model for pricing individual defined contribution pension and examined weather individual defined contribution pensions can reduce inefficiency in defined benefit pensions in this paper.

When the rate of time preference is very large, a rise of interest rate can increase inefficiency in defined benefit (DB) annuity but reduce the increase of inefficiency. By contrast, when the rate of time preference is smaller than the interest rate, a rise of interest rate reduces the increase of inefficiency in DB pension annuity.

When DC pension annuity has positive option premium, its price is larger than DB pension annuity. Consequently, when the rate of time preference is large, we had better choose DC pension annuity. When the rate of time preference is small, we had better choose DB pension annuity.

In the case of DB pension, current contributions may be used to support current retirees, at least in part. Our model is not indexed to inflation, but the DB pension annuity is indexed to inflation only for workers vesting after a fixed period of years. In this case, inefficiency in defined benefit pensions reduces. When the aging population grows, we can not sustain DB pension institutions. We must examine the relation between the growth rate of population and pension institutions.

In the case of DC pension, contributions are invested, often at the participant's direction, in stocks, bonds, or other financial instruments. The DC pension sustains the value by giving consumers part of the investment income. If the investment income is higher than minimum coverage, an excess is added to insurance. By contrast, if it is lower than minimum coverage, the minimum coverage is paid.

Of course, the investment income risk is still there and unexpected economic fluctuation may let many insurers bankrupt. Our model is not included the probability of insurers' bankrupt. The factor of bankrupt has effect of minimum coverage down and risk–averse consumers may shift DB pension annuity. We must examine the relation between option price DC pension annuity and the probability of insurers' bankrupt.

REFERENCES

Ahrend, Peter (1996). "Pension Financial Security in Germany," In Securing Employer–Based Pensions : An International Perspective, Eds. Z. Bodie and O. S. Mitchell. Philadelphia, PA : Pension Research Council and University of Pennsylvania Press, 73－120

Black, Fisher and Myron Scholes (1973), "The Pricing of Options and Cooperate Liabilities," *Journal of Political Economy*, 81, 637－654.

Bodie, Zvi and Robert C. Merton (1993). "Pension Benefit Guarantees in the United States : A Functional Analysis." In *The Future of Pensions in the United States*, Ed. R. Schmitt. Philadelphia, PA : Pension Research Council and University of Pennsylvania Press, 194－234.

Binsbergen, Jules H. van and Michael W. Brandt (2007), "Solving Dynamic Portfolio Choice Problems by Recursing on Optimized Portfolio Weights or on the Value Function?", *Computational Economics*, Vol.29, Issue 3－4, 355－367.

Brandt, Michael W., Amit Goyal, Pedro Santa–Clara and Jonathan R. Stroud (2005), "A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning about Return Predictability", *Review of Financial Studies*, 18, 831－873.

Brandt, Michael W. and Pedro Santa–Clara (2006), "Dynamic Portfolio Selection by Augmenting the Asset Space", *The Journal of Finance*, Vol.LXI, No.5, 2187－2218.

Dahl, Mikkel and Thomas Moller (2006), "Valuation and Hedging of Life Insurance Liabilities with Systematic Morality Risk", *Insurance : Mathematics and Economics*, 39, 193－217.

Davis, E. Philip (1996). "An International Comparison of the Financing of Occupational Pensions," in Securing Employed–Based Pensions : An International Perspective, Eds. Z. Bodie and O. S. Mitchell. Philadelphia, PA : Pension Research Council and University of Pennsylvania Press, 244－285

Diamond, Peter A. and James A Mirrlees (1985). "Insurance Aspects of Pensions," in *Pensions, Labor, and Individual Choice*. David A. Wise, ed. Chicago : The University of Chicago Press, 317－356.

Diamond, Peter A. and E. Sheshinsky (1995). "Economic Aspects of Optimal Disability Benefits," *Journal of Public Economics*, 1－23.

Feldstein, Martin (1983). "Should Private Pensions be Indexed?" in *Financial Aspects of the United States Pension System*, ed. Zvi Bodie and John Shoven, Chicago, University of Chicago Press, 211－230

Geanakoplos, John, Olivia S. Mitchell and Stephen Zeldes (1999). "Social Security Money's Worth," In *Prospects for Social Security Reform*. O.S. Mitchell, R. Myers and H. Young, eds. Pension Research Council. Philadelphia, PA : University of Pennsylvania Press, 79－151.

International Association of Insurance Supervisors (2006). "Standard on Asset–Liability Management", *Standard*, No.3.

Ippolito, Richard (1985). "The Labor Contract and True Economic Pension Liabilities." *American Economic Review*. 75(5), 1031－1043.

Ippolito, Richard (1989). *The Economic of Pension Insurance*. Pension Research Council. Homewood, Illinois : Dow Jones–Irwin.

James, Estelle and Dmitri Vittas (1996). "Mandatory Savings Schemes : Are They an Answer to the Old Age Security Problem?" in *Securing Employer–Based Pensions : An International Perspective*, Z. Bodie and O. Mitchell, eds. Philadelphia, PA : Pension Research Council and University of Pennsylvania Press, 151－192.

Longstaff, Francis A. and Eduardo Schwartz (2001). "Valuing American Option by Simulation : A Simple

Least–Squares Approach", *The Review of Financial Studies*, Vol.14. No.1, 113－147.

Merton, Robert ; Zvi Bodie and Alan J. Marcus (1987). "Pension Plan Integration as Insurance Against Social Security Risk." In *Issues In Pension Economics*. J. Shoven and D. Wise, eds. Chicago : University of Chicago Press, 147－169.

Mitchell, Olivia S., Jim Poterba, Mark Warshawsky and Jeff Brown (1999). "New Evidence on the Money's Worth of Individual Annuities." *American Economic Review*, Vol.89. No.5, 1299－1318.

Monden, Sinichi (2009). "A Dynamic Optimal Portfolio Strategy for the Insurance Liability Exposed to Uncertainty in the Capital Market", *Journal of the Japanese Association of Risk, Insurance and Pension*, Vol.4. No.1, 67－99.(In Japanese)

Pesando, James (1996). "The Government's Role in Insuring Pensions," in *Securing Employer–Based Pensions : An International Perspective*, Z. Bodie and O.S. Mitchell, eds. Philadelphia, PA : Pension Research Council and University of Pennsylvania Press, 286－316.

Pelsser, Antoon (2003). "Pricing and Hedging Annuity Options via Static Option Replication", *Insurance : Mathematics and Economics*, 33, 283－296.

Sass, Steven (1997). *The Promise of Private Pensions : The First Hundred Years*, Cambridge, MA : Havard University Press.

Yaari, M. E. (1965). "Unception Lifetime, Life Insurance and the Theory of the Consumer," *Review of Economic Studies* 32, 137－150.

Zimbidis, Alexandros A. (2011). "Optimal Management of a Variable Annuity Invested in a Black–Scholes Market Driven by a Multidimensional Fractional Bronwnian Motion", *Stochastic Analysis and Applications*, 29, 61－77.

Defined Benefit Pension Annuity
and
Defined Contribution Pension Annuity

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Abstract : In many countries, many elderly people depend on the government and employment–based pensions. The modal pension in the developed countries was of the defined benefit several years ago. A governed–based defined benefit system transfers income from workers to the retired generation, at least in part. But the world's population is rapidly growing aging, so the risks of underfunded plan terminations is increasing and defined contribution plans are becoming more the norm in the developed countries. We build an option model for pricing individual defined contribution pension and examined whether individual defined contribution pensions can reduce inefficiency in defined benefit pensions in this paper.

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