

POWER SERIES STRESS FUNCTION FOR ANISOTROPIC AND ORTHOTROPIC BEAMS¹

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ABSTRACT

The characteristic fourth-order partial differential equation for two-dimensional elastic anisotropic and orthotropic materials is solved, using a doubly infinite power series. Two specific problems are presented to illustrate the use of power series; the simply supported anisotropic beam under a uniformly distributed load, and an orthotropic cantilever under triangular and concentrated end load. Results are compared with those of elementary bending theory.

INTRODUCTION

Polynomial or doubly infinite power series solutions of the Airy stress function (1863) have been applied to beam problems of isotropic bodies (Wang 1953; Timoshenko and Goodier 1951; Lekhnitskii 1947; Sechler 1961). Timoshenko and Goodier (1951) take polynomials of various degrees, suitably adjust their coefficients, and apply the stress function to a number of practically important two-dimensional elastic problems for isotropic materials. For a simply supported isotropic beam, they consider a polynomial of the fifth degree, and for an isotropic cantilever a polynomial of degree six is used. With appropriate boundary conditions, solutions for the stresses can be obtained. Wang (1953) integrates all the stress components in a given isotropic cantilever and substitutes the resulting unknown function into the stress equation. A solution is obtained, subjected to the prescribed boundary conditions. Sechler (1961) uses a technique similar to that of Timoshenko. Lekhnitskii (1947) has solved the stress function for an anisotropic beam by using polynomials. His results provide the only information relative to the use of power series for the solution of anisotropic

problems. Neau (1956) recently developed a scheme for applying doubly infinite power series to the stress function for the isotropic bodies. His system focuses on a systematic method for determining the constants. Problems in which boundary stresses can be described by means of power series are solvable by this method.

In this paper the method developed by Neau is extended to encompass orthotropic and anisotropic media. Two problems are examined as a means of portraying the general method of analysis: a simply supported anisotropic beam under uniform load and an orthotropic cantilever beam under triangular and concentrated end load. Solutions are compared with those obtained from elementary bending theory.

METHOD

The solution of the boundary value problem in plane elasticity hinges on the determination of Airy's stress function $F(x_1, x_2)$, satisfying the differential equation

$$\begin{aligned} K_1 \frac{\partial^4 F}{\partial x_1^4} + K_2 \frac{\partial^4 F}{\partial x_1^3 \partial x_2} + K_3 \frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} \\ + K_4 \frac{\partial^4 F}{\partial x_1 \partial x_2^3} + K_5 \frac{\partial^4 F}{\partial x_2^4} = 0 \end{aligned} \quad (1)$$

where the constants K_i are real functions of the known elastic moduli of the materials

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and defined as $K_1 = S_{2222}$; $K_2 = -4S_{2212}$; $K_3 = 2(S_{1122} + 2S_{1212})$; $K_4 = -4S_{1112}$; $K_5 = S_{1111}$.

For the orthotropic case, $K_2 = K_4 = 0$. Then, equation (1) reduces to the form

$$K_1 \frac{\partial^4 F}{\partial x_1^4} + K_3 \frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} + K_5 \frac{\partial^4 F}{\partial x_2^4} = 0 \quad (2)$$

The components σ_{11} , σ_{22} , σ_{12} of the stress tensor σ are related to F by the formulas

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2}; \quad \sigma_{22} = \frac{\partial^2 F}{\partial x_1^2}; \quad \sigma_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2} \quad (3)$$

It is assumed that the stress function F can be expressed in the form of a doubly infinite power series

$$F(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} x_1^m x_2^n \quad (4)$$

where m and n are positive integers and the C_{mn} are unknown constants to be determined for particular boundary conditions. It is helpful in the construction of solutions to record the array of C_{mn} of equation 4 as a matrix

$$\begin{bmatrix} C_{00} & C_{01} & C_{02} & C_{03} & C_{04} & C_{05} & C_{06} & \dots \\ C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} & \dots \\ C_{20} & C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & \dots \\ C_{30} & C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} & \dots \\ C_{40} & C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} & \dots \\ C_{50} & C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix} \quad (5)$$

Differentiating equation 4, we obtain the following expressions for fourth derivatives

$$\frac{\partial^4 F}{\partial x_1^4} = \sum_{m=4}^{\infty} \sum_{n=0}^{\infty} m(m-1)(m-2)(m-3) C_{mn} x_1^{m-4} x_2^n$$

$$\frac{\partial^4 F}{\partial x_1^3 \partial x_2} = \sum_{m=3}^{\infty} \sum_{n=1}^{\infty} mn(m-1)(m-2) C_{mn} x_1^{m-3} x_2^{n-1}$$

$$\frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} = \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} mn(n-1)(m-1) C_{mn} x_1^{m-2} x_2^{n-2}$$

and

$$\frac{\partial^4 F}{\partial x_1 \partial x_2^3} = \sum_{m=1}^{\infty} \sum_{n=3}^{\infty} mn(n-1)(n-2) C_{mn} x_1^{m-1} x_2^{n-3}$$

$$\frac{\partial^4 F}{\partial x_2^4} = \sum_{m=0}^{\infty} \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) C_{mn} x_1^m x_2^{n-4} \quad (6)$$

Substituting equations 6 into 1 and changing the range of summation gives

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left[K_1 m(m-1)(m+1)(m+2) C_{m+2, n-2} + K_2 m(m+1)(m-1)(n-1) C_{m+1, n-1} + K_3 mn(m-1)(n-1) C_{mn} + K_4 n(n+1)(n-1)(m-1) C_{m-1, n+1} + K_5 n(n-1)(n+1)(n+2) C_{m-2, n+2} \right] x_1^{m-2} x_2^{n-2} = 0 \quad (7)$$

Since x_1 and x_2 are in general non-zero, it follows that the expression in brackets in equation (7) must equal zero for different values of m and n . Three such equations

are tabulated below for particular values of m and n

for $m=2$ & $n=2$

$$24K_1C_{140} + 6K_2C_{231} + 4K_3C_{322} + 6K_4C_{413} + 24K_5C_{504} = 0$$

for $m=2$ & $n=3$

$$24K_1C_{41} + 12K_2C_{32} + 12K_3C_{23} + 24K_4C_{414} + 120K_5C_{505} = 0$$

for $m=3$ & $n=2$

$$120K_1C_{150} + 24K_2C_{241} + 12K_3C_{332} + 12K_4C_{423} + 24K_5C_{514} = 0$$

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(8)

It is readily apparent that these equations establish restrictive relations between the unknown constants. These constants occur along the diagonals of matrix (5) extending from lower left to upper right.

Since $K_2 = K_4 = 0$ for an orthotropic material, equation (7) reduces to

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \left[K_1 m(m-1)(m+1)(m+2) C_{m+2, n-2} + K_3 mn(m-1)(n-1) C_{mn} + K_5 n(n-1)(n+1)(n+2) C_{m-2, n-2} \right] x_1^{m-2} x_2^{n-2} = 0$$

(9)

AN ORTHOTROPIC CANTILEVER BEAM SUBJECTED TO A DISTRIBUTED LOAD AND CONCENTRATED END LOAD

As a first example, we examine a cantilevered orthotropic beam under a concen-

trated end load and a distributed load that varies linearly with distance along the beam. An orthotropic beam is located in the 1-2 plane, as shown in Fig. 1. Orthotropic and geometric axes are coincident. The reaction at the right end of the beam is considered to be distributed as shear over a vertical surface at $x_1 = \lambda$. Stress boundary conditions for the beam are as follows:

$$\begin{aligned} \sigma_{11} &= 0, \quad x_1 = 0 & |x_2| &\leq h \\ \sigma_{12} &= 0, \quad x_2 = \pm h & 0 \leq x_1 &\leq \lambda \\ \sigma_{22} &= 0, \quad x_2 = h & 0 \leq x_1 &\leq \lambda \\ \sigma_{22} &= qx_1, \quad x_2 = -h & 0 \leq x_1 &\leq \lambda \end{aligned}$$

$$\int_{-h}^h \sigma_{12} dx_2 = \begin{cases} -P - \frac{1}{2}q\lambda^2 & x_1 = \lambda \\ -P & x_1 = 0 \end{cases}$$

$$\int_{-h}^h \sigma_{11} dx_2 = 0 \quad x_1 = 0, \lambda$$

$$\int_{-h}^h \sigma_{11} x_2 dx_2 = \begin{cases} -P\lambda - \frac{1}{6}q\lambda^3 & x_1 = \lambda \\ 0 & x_1 = 0 \end{cases}$$

(10)

It remains now to determine the unknown constants C_{mn} in the stress function. Using equations (3), we obtain for the shear stress

$$\sigma_{12} = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn C_{mn} x_1^{m-1} x_2^{n-1}$$

(11)

Boundary condition (10b) gives

$$\sum_{n=1}^{\infty} n C_{mn} (h)^{n-1} = 0 \quad \left| \begin{array}{l} \text{for} \\ m \geq 1 \end{array} \right.$$

$$\sum_{n=1}^{\infty} n C_{mn} (-h)^{n-1} = 0 \quad \left| \begin{array}{l} \text{for} \\ m \geq 1 \end{array} \right.$$

(12)

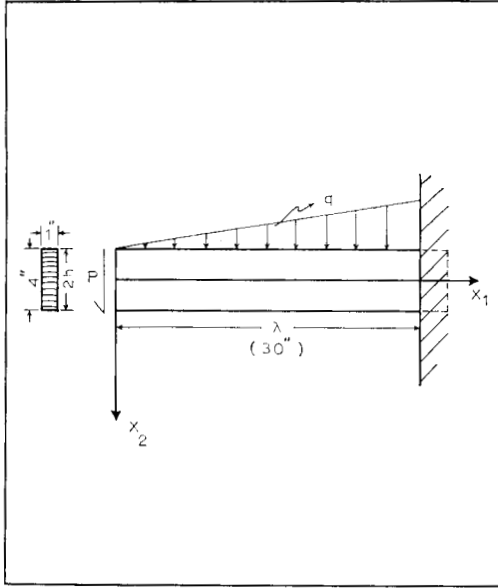


FIG. 1. An orthotropic cantilever subjected to an end load of P lb and triangular load of intensity q lb/in.

Multiplying both these equations by h , then adding and subtracting, two new equations are obtained

$$\left. \begin{aligned} \sum_{n=2,4,6}^{\infty} n C_{mn} h^n &= 0 \\ \sum_{n=1,3,5}^{\infty} n C_{mn} h^n &= 0 \end{aligned} \right\} \begin{array}{l} \text{for} \\ m \geq 1 \end{array} \quad (13)$$

Consider the boundary condition specified by (10c).

$$\sigma_{22} = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} m(m-1) C_{mn} x_1^{m-2} x_2^n$$

It follows that

$$\sum_{n=0}^{\infty} C_{mn} h^n = 0 \quad \text{for } m \geq 2 \quad (14)$$

Boundary condition (10d) gives

$$\sum_{m=2}^{\infty} \left[\sum_{n=0}^{\infty} C_{mn} (-h)^n \right] m(m-1) x_1^{m-2} = -q x_1$$

Hence

$$\left. \begin{aligned} \sum_{n=0}^{\infty} C_{2n} (-h)^n &= 0 \\ \sum_{n=0}^{\infty} C_{3n} (-h)^n &= -\frac{q}{6} \\ \sum_{n=0}^{\infty} C_{mn} (-h)^n &= 0 \end{aligned} \right\} m \geq 4 \quad (15)$$

Adding and subtracting equation (14) and the third of equation (15) gives

$$\left. \begin{aligned} \sum_{n=2,4,6}^{\infty} C_{mn} h^n &= 0 \\ \sum_{n=1,3,5}^{\infty} C_{mn} h^n &= 0 \end{aligned} \right\} m \geq 4 \quad (16)$$

Equations (13) and (16) require that

$$C_{4n} = 0 \quad \text{for } n = 0, 1, 2, \dots$$

With these results, it follows that for the orthotropic cantilever loaded as shown in Fig. 1, all unknown constants below the fourth row in (5) are zero. Then, from the restrictive relations (9) we conclude that

$$C_{06} = C_{24} = C_{25} = C_{26} = C_{34} = C_{35} = C_{36} = 0.$$

In addition, the constants C_{00} , C_{01} and C_{10} are disregarded because they do not appear in the equations for stress. Considering re-

sults to this point, the matrix of C_{mn} has been reduced to

$$\begin{bmatrix} \text{---} & \text{---} & C_{02} & C_{03} & C_{04} & C_{05} & \cdots \\ \text{---} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & \cdots \\ C_{20} & C_{21} & C_{22} & C_{23} & \cdot & \cdot & \cdots \\ C_{30} & C_{31} & C_{32} & C_{33} & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix} \quad (17)$$

Using the first of equations (13) with $m = 3$ gives

$$C_{32} = 0 .$$

Using this result with the restrictive relation given by (9) leads to

$$C_{14} = 0 .$$

Solving equations (13), (14), and the second of (15) simultaneously with $m = 3$, we obtain

$$C_{03} = -\frac{q}{12} , \quad C_{31} = \frac{q}{8h} , \quad C_{33} = -\frac{q}{24h^3} \quad (18)$$

Setting $m = 2$ in the first of equations (13) gives

$$C_{22} = 0 .$$

It follows from restrictive relation (9) that

$$C_{04} = 0 .$$

Simultaneous solution of the first of equations (15) with the second of (13) and also equation (14) for $m = 2$ results in

$$C_{20} = C_{21} = C_{23} = 0 .$$

Equation (9) with results from above gives

$$C_{05} = 0 .$$

Furthermore, by setting $m = 1$ in the first of equations (13), we find

$$C_{12} = 0 .$$

The second of equations (13) with $m = 1$ gives

$$C_{11}h + 3C_{13}h^3 + 5C_{15}h^5 = 0$$

Finally, using equations (9) and (18), we arrive at

$$C_{15} = -\frac{K_3 q}{80K_5 h^3} , \quad C_{11} = \frac{K_3 q h}{16K_5} - 3C_{13}h^2$$

Incorporating these preliminary results into the stress function $F(x_1, x_2)$ of equation (2) leads to

$$\begin{aligned} F(x_1, x_2) = & C_{02}x_2^2 + C_{03}x_2^3 + \left[\frac{K_3 q h}{16K_5} \right. \\ & \left. - 3C_{13}h^2 \right] x_1 x_2 + C_{13}x_1 x_2^3 - \frac{K_3 q}{80K_5 h^3} x_1 x_2^5 \\ & - \frac{q}{12} x_1^3 + \frac{q}{8h} x_1^3 x_2 - \frac{q}{24h^3} x_1^3 x_2^3 \end{aligned} \quad (19)$$

The remaining three unknown constants C_{02} , C_{03} , and C_{13} can be determined by using three of the remaining boundary conditions. Equation (10f) gives

$$C_{02} = 0 .$$

Finally, the boundary conditions (10e) and (10g) require that

$$C_{13} = -\frac{P}{4h^3} + \frac{K_3 q}{40K_5 h} \quad \text{and } C_{03} = 0 .$$

The complete stress function takes the form

$$\begin{aligned} F(x_1, x_2) = & \left[\frac{3P}{4h} - \frac{K_3 q h}{80K_5} \right] x_1 x_2 + \left[\frac{K_3 q}{40K_5 h} \right. \\ & \left. - \frac{P}{4h^3} \right] x_1 x_2^3 - \frac{K_3 q}{80K_5 h^3} x_1 x_2^5 - \frac{q}{12} x_1^3 \\ & + \frac{q}{8h} x_1^3 x_2 - \frac{q}{24h^3} x_1^3 x_2^3 , \end{aligned} \quad (20)$$

and the stress components are given by

$$\sigma_{11} = 6 \left[\frac{K_3 q}{40K_5 h} - \frac{P}{4h^3} \right] x_1 x_2 - \frac{K_3 q}{4K_5 h^3} x_1^3 x_2^3 - \frac{q}{4h^3} x_1^3 x_2^2 \quad (21)$$

$$\sigma_{22} = -\frac{q}{2} x_1 + \frac{3q}{4h} x_1 x_2 - \frac{q}{4h^3} x_1^3 x_2^3 \quad (22)$$

$$\sigma_{12} = \left[\frac{K_3 q h}{80K_5} - \frac{3P}{4h} \right] + 3 \left[\frac{P}{4h^3} - \frac{K_3 q}{40K_5 h} \right] x_2^2 + \frac{K_3 q}{16K_5 h^3} x_2^4 - \frac{3q}{8h} x_1^2 + \frac{3q}{8h^3} x_1^2 x_2^2 \quad (23)$$

It should be noted that equation (10f) does not require the σ_{11} to disappear at the end surfaces but rather requires the integral of σ_{11} with respect to x_2 to be zero. Nevertheless, σ_{11} proves to be zero at the free end ($x_1 = 0$) of this particular beam as can be verified from equation (21). For other beams, normal stress, σ_{11} , may be present on a free as well as an embedded end surface. An analysis in which this occurs will be discussed in the next example.

The variation of σ_{12} as well as σ_{11} , in a wood beam (yellow poplar) is pictured in Figs. 2 and 3. In Fig. 2 the distribution of σ_{11} with x_2 at $x_1 = \lambda/2$ is contrasted with the results of elementary bending theory. The figure demonstrates that power series analysis, based on a more fundamental approach to the elastic problem, indicates a moderate departure from a linear distribution of normal stress in an orthotropic beam. It is interesting to note the commonly accepted parabolic shape of the shear distribution as pictured in Fig. 3.

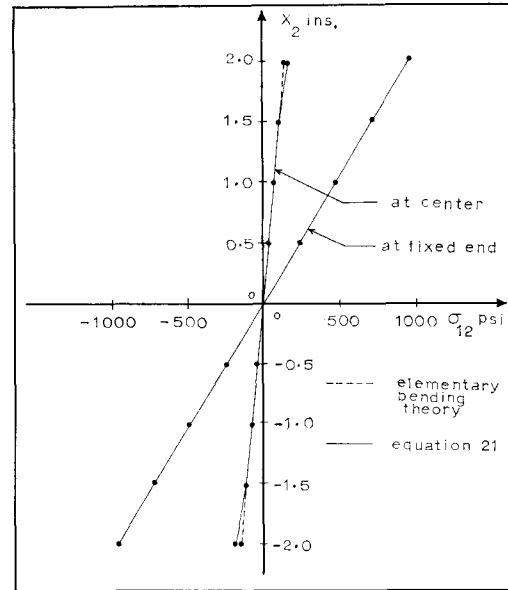


FIG. 2. Variation of σ_{11} with x_2 of an orthotropic cantilever of yellow poplar ($P = 10$ lb $q = 0.5$ lb/in.)

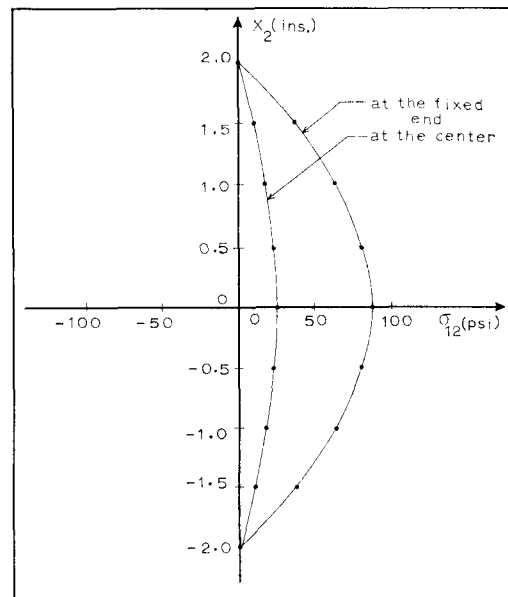


FIG. 3. Variation of σ_{12} with x_2 of an orthotropic cantilever of yellow poplar ($P = 10$ lb $q = 0.5$ lb/in.)

AN ANISOTROPIC BEAM SUBJECTED TO A UNIFORMLY DISTRIBUTED LOAD

An anisotropic beam under a uniformly distributed load is shown in Fig. 4. Com-

plete anisotropy is introduced in a wood beam by the non-coincidence of orthotropic symmetry and geometric axes as shown in Fig. 4. The beam is supported by means of shear distributed in an arbitrary manner on the two end surfaces. The stress boundary conditions are as follows:

$$\sigma_{22} = -q \quad \text{for } x_2 = h \quad |x_1| \leq \lambda$$

$$\sigma_{22} = 0 \quad x_2 = -h \quad |x_1| \leq \lambda$$

$$\sigma_{12} = 0 \quad x_2 = \pm h \quad |x_1| \leq \lambda$$

$$\int_{-h}^h \sigma_{12} dx_2 = \begin{cases} q\lambda & x_1 = \lambda \\ -q\lambda & x_1 = -\lambda \end{cases}$$

$$\int_{-h}^h \sigma_{11} x_2 dx_2 = 0 \quad x_1 = \pm \lambda$$

$$\int_{-h}^h \sigma_{11} dx_2 = 0 \quad x_1 = \pm \lambda$$

(24)

Using the first three boundary conditions of equations (24), and following the same procedure as followed for the orthotropic beam, we obtain a partially complete stress function.

$$\begin{aligned} F(x_1, x_2) = & C_{02} x_2^2 + C_{03} x_2^3 + C_{04} x_2^4 \\ & + \frac{K_4 - K_3 K_5}{80K_5^2 h^3} x_2^5 + 12 \frac{K_5}{K_4} C_{04} x_1 x_2 + \\ & \frac{K_4 q}{8K_5 h} x_1 x_2^2 - \frac{4K_5}{K_4} C_{04} x_1 x_2^3 - \frac{K_4 q}{16K_5 h^3} x_1 x_2^4 \\ & - \frac{q}{4} x_1^2 - \frac{3q}{8h} x_1^2 x_2 + \frac{q}{8h^3} x_1^2 x_2^3 \end{aligned} \quad (25)$$

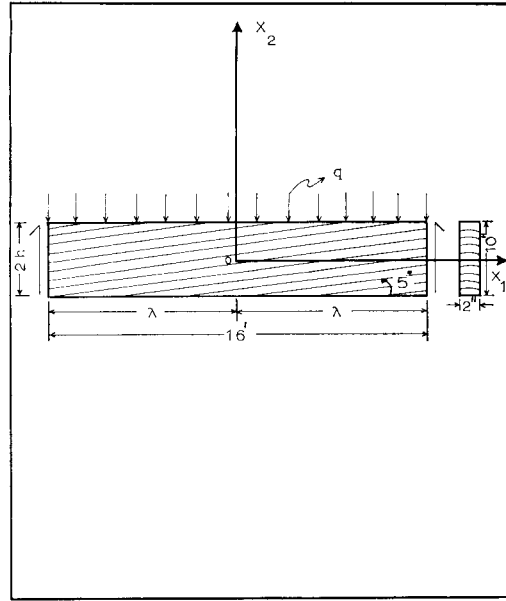


FIG. 4. A simply supported anisotropic beam subjected to a uniformly distributed load.

The three remaining boundary conditions (24d, e, f) require that

$$C_{04} = 0, \quad C_{03} = -\frac{q\lambda^2}{8h^3} - \frac{(K_4^2 - K_3 K_5)q}{40K_5^2 h},$$

and

$$C_{02} = 0$$

Hence, the stress function for the uniformly loaded anisotropic beam takes final form

$$\begin{aligned} F(x_1, x_2) = & q \left\{ \left[-\frac{\lambda^2}{8h^3} + \frac{K_4^2 - K_3 K_5}{40K_5^2 h} \right] x_2^3 \right. \\ & + \frac{K_4^2 - K_3 K_5}{80K_5^2 h^3} x_2^5 + \frac{K_4}{8K_5 h} x_1 x_2^2 - \\ & \frac{K_4}{16K_5 h^3} x_1 x_2^4 - \frac{1}{4} x_1^2 - \frac{3}{8h} x_1^2 x_2 \\ & \left. + \frac{1}{8h^3} x_1^2 x_2^3 \right\} \end{aligned} \quad (26)$$

and the stress components are

$$\sigma_{11} = q \left[-6 \left(\frac{\lambda^2}{8h^3} + \frac{K_4^2 - K_3 K_5}{40K_5^2 h} \right) x_2^2 + \frac{K_4^2 - K_3 K_5}{4K_5^2 h} x_2^3 + \frac{K_4}{4K_5 h} x_1 - \frac{3K_4}{4K_5 h^3} x_1 x_2^2 + \frac{3}{4h^3} x_1^2 x_2 \right] \tag{27}$$

$$\sigma_{22} = q \left[-\frac{1}{2} - \frac{3}{4h} x_2 + \frac{1}{4h^3} x_2^3 \right], \tag{28}$$

$$\sigma_{12} = q \left[-\frac{K_4}{4K_5 h} x_2 + \frac{K_4}{4K_5 h^3} x_2^3 + \frac{3}{4h} x_1 - \frac{3}{4h^3} x_1 x_2^2 \right] \tag{29}$$

Equation (24f) of the boundary conditions imposes the restriction that the net force at the end of the beam be zero whereas equation (24e) eliminates moment at the end. As a result the end surfaces of the beam are not entirely free of normal stress, σ_{11} . According to St. Venant's principle, the effect of this end stress is minimal a short distance from the ends. Consequently, the solution can be considered exact in the interior portions of the beam. Boundary condition (24d) requires that the reactions of the beam be distributed as shear over the end surfaces. This mode of support will modify the stress over the end surfaces. This mode of support will modify the stress field near the ends of the beam as contrasted with a beam supported for a short distance along the lower edge near the ends. Nevertheless, stress calculations in the interior of the beam will not be modified appreciably by this mode of support.

It is interesting to observe graphically the variation with position of the three components of stress in an anisotropic beam. As an example of an anisotropic beam we consider a wood beam (Douglas-fir) with a slope of grain of 5° as shown in Fig. 4. Because of the slope of grain, this beam is anisotropic in x_1-x_2 frame of reference. Using the transformation equation of a fourth-order Cartesian tensor, the elastic constants of the beam can be calculated.

$$S'_{ijkl} = a_{im} a_{jn} a_{ko} a_{lp} S_{mnop} .$$

From this equation and values for the orthotropic compliances of Douglas-fir as given by Hearmon (1951), we obtain

$$\begin{aligned} S'_{1111} &= 0.066 ; & S'_{1112} &= -0.031 ; \\ S'_{1212} &= 0.401 ; & S'_{2222} &= 0.082 ; \\ S'_{2212} &= -0.044 ; \\ S'_{1122} &= S'_{2211} = -0.015 , \end{aligned}$$

(in units of $0.145 \times 10^6 \text{ in}^2/\text{lb.}$)

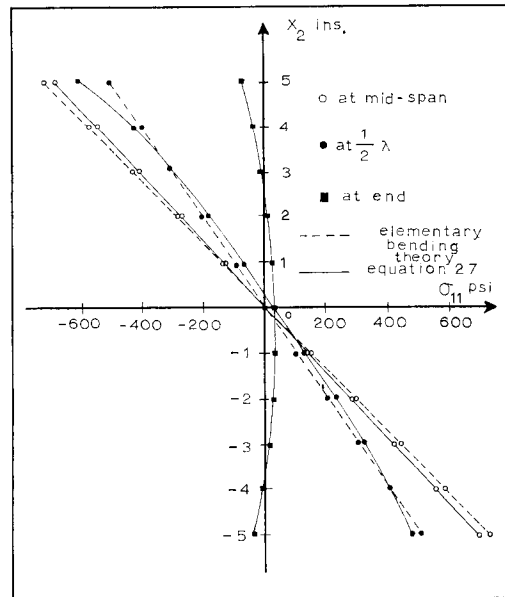


FIG. 5. Variation of σ_{11} with x_2 in an anisotropic beam of Douglas-fir (load, 5 lb/in.)

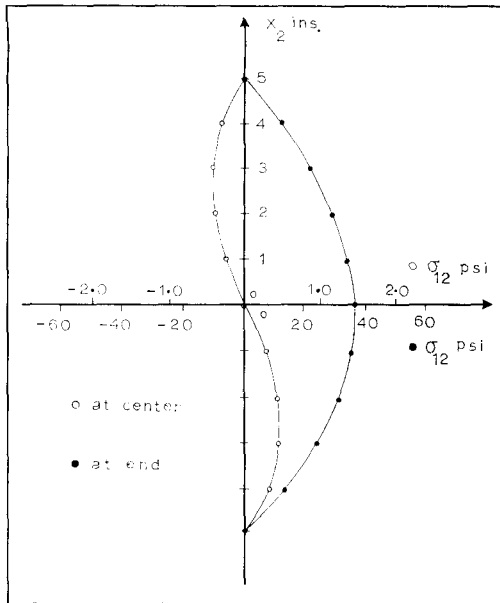


FIG. 6. Variation of σ_{12} with x_2 in uniformly loaded anisotropic beam of Douglas-fir (load, 5 lb/in.)

These values for the compliances are used to obtain the K_i of equation (1). With these constants the stresses can be calculated from equations (27), (28), and (29). The distribution of σ_{11} as well as σ_{12} , with position is pictured in Figs. 5 and 6. The variation of σ_{11} with x_2 at $x_1 = 0, \lambda/2$, and λ as calculated from equation (27) are contrasted with the values obtained from elementary bending theory in Fig. 5.

CONCLUSION

The doubly infinite power series leads to an approximate mathematical solution of two-dimensional elastic problems of ortho-

tropic and anisotropic bodies subjected to relatively simple boundary conditions. As shown in the body of the paper, all the components of stress in the plane can be evaluated. In contrast, elementary theory yields values for the stresses σ_{11} and σ_{12} only. The distribution of σ_{11} through the cross section of either an anisotropic simply supported beam or an orthotropic cantilever proved to be different than predicted from elementary bending theory.

If the load distribution is dependent on x_1 , σ_{11} at the end surface is zero. In contrast, if the applied load is constant, *i.e.* independent of x_1 , normal stress σ_{11} appears on the end surfaces but the integral of σ_{11} with respect to the x_2 is zero. The normal stress σ_{22} is independent of the properties of materials irrespective of whether the medium is orthotropic or anisotropic. It is also interesting to note that the shear stress σ_{12} is dependent on the elastic constants (S_{1112} and S_{1111}) other than the shear compliance (S_{1212}), and is numerically small over the cross section at the center of the anisotropic beam.

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