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AN EXAMINATION OF THE INTERFACES BETWEEN OPERATIONS AND ADVERTISING STRATEGIES

by

Abdullahel Bari, BS, MS, MBA, DE

A Dissertation Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Business Administration

COLLEGE OF BUSINESS LOUISIANA TECH UNIVERSITY

May, 2010

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March 29, 2010 Date We hereby recommend that the dissertation prepared under our supervision by Abdullahel Bari entitled "An Examination of the Interfaces Between Operations and Advertising Strategies." accepted in partial fulfillment of the requirements for the Degree of be **Doctor of Business Administration - Management** Dissertation Research Supe visor of Head of Department Management and Information Systems Department urred in: Advisory Committee Approveg Approved: Director of Graduate Studies Dean of the Gradua GS Form 13a (6/07)

ABSTRACT

This dissertation is composed of three journals examining the interfaces between the marketing variable of advertising and various aspects of the operations function of the enterprise, namely, (1) production cost [Chapter 2], (2) inventory control [Chapter 3], and (3) service cost learning [Chapter 4]. The first journal identified the optimum advertising allocation policy over time in the presence of a quadratic convex/concave production cost function when the advertising response function is concave using a modified Vidale-Wolfe model. Through analytical proofs and numerical simulations, the results indicated the potential superiority of a pulsation policy in the presence of concavity in the advertising response function cost function is convex; otherwise, the uniform policy would be optimal. The study is seen as applicable to frequently purchased products in the maturity stage of their life cycles of dominant firms in their industries practicing a zero-inventory policy in a just-in-time environment.

The research objective pertaining to the second journal was to study how a firm would adapt optimum ordered quantity/production lot size and optimum advertising expenditure in response to changes in its own parameters, rival's parameters, or parameters that are common to all firms in a symmetric duopoly/oligopoly market. This was accomplished by developing comparative statics (sensitivity analysis) of a symmetric competitive inventory model with advertising-dependent demand based on a market share attraction model. Both optimum advertising expenditure and ordered quantity were found to be sensitive to changes in marketing and operations parameters. The robustness of the symmetric comparative statics was assessed by using data from the brewing industry in the US that represents an asymmetric oligopoly. The empirical analysis indicated that the theoretical results obtained for a symmetric oligopoly remained valid for an oligopoly where each firm had a market share less than 50% and the market shares were further apart from one another. The study is thought to be applicable to low-priced frequently purchased consumer items in competitive mature markets.

In the third journal, the original Bass model for new products was modified to incorporate advertising and customers' disadoption to characterize the optimum advertising policy over time for subscriber service innovations where service cost follows a learning curve. After characterizing the optimal policy for a general diffusion model, the results pertaining to a specific diffusion model for which advertising affects the coefficient of innovation were reported. On the empirical side, four alternative diffusion models were estimated and their predictive powers, using a one-step-ahead forecasting procedure, were compared. Empirical research findings suggest that the specific diffusion model considered in this study is not only of theoretical appeal, but also of notable empirical relevance. Taken together, the analytical and empirical findings argue in favor of advertising more heavily during the early stage of the diffusion process of the new subscriber service innovation and including a related message that would predominantly target innovators. Furthermore, it might be inappropriate to model the diffusion of subscriber services as if they were durable goods. The study is thought to be applicable to service innovations that are made available to customers periodically at a subscription fee. Typical examples include, but are not limited to, cable TV, health clubs, pest control, and the internet.

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ACKNOWLEDGEMENTS

I am grateful to my professors, colleagues, friends, and family members who have contributed to the successful completion of this dissertation and of my doctoral degree. First, I would like to thank Dr. Hani I. Mesak, chairman of my dissertation committee, for his constructive guidance, tremendous support, and hearty encouragement from the inception to the successful completion of this dissertation without which this research endeavor would not be successful.

I would also like to express my appreciation to my other committee members, Dr. Laura Birou and Dr. Anthony Jurkus for their time, patience, and efforts in their reading, editing, and suggesting improvements of my work. Special thanks also go to the entire graduate faculty, staff, and my fellow doctoral students for their support, expertise, and invaluable assistance. In this regard, I would like to thank Dr. Anthony Inman in particular for his continued support since I joined the DBA program at Louisiana Tech.

Finally, and most importantly, I would like to thank my wife, Imrana and our daughters, Tahsin and Tasnia, for their patience and sacrifice that allowed me to complete this dissertation.

CHAPTER 1

INTRODUCTION

Introduction

This dissertation adopts the two definitions shown below provided by Stevenson (2008), and Kotler and Keller (2009). "Operations Management is an area of business concerned with the production of quality goods and services, and involves the responsibility of ensuring that business operations are efficient and effective". "Marketing Management is a business discipline which is focused on the practical applications of marketing techniques and the management of a firm's marketing resources and activities". Obviously, both sides recognize the importance of the interface from a managerial standpoint. The above definitions imply that while the objective of marketing is to create customer demand, operations management focuses on the supply and fulfillment of that demand. Therefore, one would expect to see operations and marketing to be intimately connected in many business organizations. When the two areas are in conflict, one often sees a mismatch in demand and supply, leading to production inefficiencies and unsatisfied customers (Ho and Tang, 2004). When they are working in harmony, one frequently sees an improved firm competitiveness and profits (Hausman, Montgomery and Roth, 2002).

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Traditionally, operations and marketing are considered as two distinct, basic functions of an organization (Stevenson, 2008). Shapiro (1977) and Karmarkar and Lele (2004) note the conflicting nature among the two functions because of their horizontal separation. While operations has the objective to achieve economies of scale with centralized manufacturing, marketing and sales, on the other hand, focus on satisfying the demand of geographically disbursed customers. In achieving the common objectives of the enterprise, a conflict between these two separate entities occurs because their missions and reward systems are often different. While manufacturing aims at cost minimization, marketing aims at satisfying the needs and wants of customers. Operations/ marketing interfaces, causes of potential conflicts, and strategies for the integration of the two functions are discussed next.

Operations/Marketing Interfaces, Conflicts and Integration

Shapiro (1977) lists eight problem areas of "necessary cooperation but potential conflict" between production and marketing. They are (1) capacity planning and long-range sales forecasting, (2) production scheduling and short-range sales forecasting, (3) delivery and physical distribution, (4) quality assurance, (5) breadth of production line, (6) cost control, (7) new-product introduction, and (8) adjunct services, e.g., spare parts inventory, support, installation, and repair. Montgomery and Hausman (1986) cite the following as being the most important interfaces between the two functions: (1) strategy, (2) forecasting, (3) the order delivery cycle, (4) product line, (5) quality, and (6) customer service. It is interesting to note that all these issues are also listed by Shapiro (1977), with the exception of business strategy. To highlight the strategy interface, Montgomery and

Hausman (1986) mention that some companies have coupled a low cost manufacturing strategy with a highly differentiated, high cost marketing strategy which creates entry barriers for competitors. In this case, the low cost manufacturing strategy enables the company to keep close to competitors in cost, while offering expensive marketing alternatives in order to have considerable value in the market place. In this context, Karmarkar (1996), Karmarkar and Lele (1989, 2004), and O'Leary-Kelly and Flores (2002) consider the interface between operations and marketing as a legitimate research domain.

Achieving integration between manufacturing and marketing to resolve conflict could be achieved through pursuing a hybrid of three pure strategies: (i) concession (the conflicting parties make a concession by giving up their goals), (ii) compromise (the conflicting parties make a compromise by lowering their goal level), and (iii) intervention (a third party makes an intervention by identifying the intervention driver and weakening the conflict driver) (Omurgonulsen and Surucu, 2008). Typical examples of courses of action pertaining to each of the strategies mentioned above are (1) Manufacturing and marketing managers should understand each other better to facilitate close coupling between these functions. Cross-functional training, work experience, and rotating production and marketing people within the factory can substantially enhance understanding and cooperation (Montgomery and Hausman, 1986); (2) In organizing for new product development, many companies are moving to a team approach where R&D, manufacturing and marketing are all presented through a product's development and market introduction (Montgomery and Hausman, 1986); and (3) Top management should decide about its priorities in policy making, whether it is going to be manufacturing or

marketing-oriented. That is to say, if on-time delivery is determined as a priority, then the manufacturing department has to emphasize high inventory and bear this burden (Shapiro, 1977).

This dissertation considers the interface and integration between operations and marketing, focusing specifically on decision-making from the view point of management science/operations research. In this regard, as in Eliashberg and Steinberg (1993) and Celikbas, Shanthikumar and Swaminathan (1999), the integration of the studied two functions focuses on their coordination using a centralized approach. According to such an approach, decisions such as production quantity/ordered quantity and advertising expenditure are jointly determined by operations and marketing with the objective of maximizing the total performance of the firm as a whole.

Significance and Importance of Study

One of the first attempts to address Operations-Marketing joint decision-making is presented by Whitin (1955), who provides extensions of two very basic inventory models which model staple merchandise and style goods, respectively. The models Whitin (1955) proposed incorporate both lot-size analysis and demand functions which are controlled via price by the decision maker. Since then, several authors have conducted research aiming at coordinating production/procurement and marketing decisions. Eliashberg and Steinberg (1993), Parente (1998), Yano and Gilbert (2004), and Chan et al. (2004) provide lucid reviews of the literature on the subject. While several models integrating the operations and the pricing functions have been published, there is a dearth of published models integrating the operations and the advertising functions. Notable examples that are relevant to the scope and purposes of this dissertation include the studies of Subramanyam and Kumaraswamy (1981), Lee and Rosenblatt (1986), Urban (1992), Goyal and Gunasekaran (1995) and Pal, Bhunia and Mukherjee (2005).

Subramanyam and Kumaraswamy (1981) consider an inventory model that incorporates price, advertising, economies of scale, and the receipt of defective items. A shortcoming of their model is the restrictive treatment of defective items. The models developed by Lee and Rosenblatt (1986) have subsequently addressed this problem and identified different scenarios for dealing with defective items. Goyal and Gunasekran (1995) develop an integrated production-inventory-price-advertising for deteriorating items to determine the production quantity and economic order quantity. Pal, Bhunia and Mukherjee (2005) introduce a deterministic inventory model for which the demand rate is dependent upon selling price, advertising and display stock level in a showroom/shop with the objective of maximizing average profits. The above four reviewed models consider the effect of the number of times advertised (advertising frequency) on demand, but do not model the relationship between advertising expenditure on demand explicitly. The above issue is taken up in the model articulated by Urban (1992) who investigates a finite replenishment inventory model in which the demand of an item is a deterministic function of price and advertising expenditure. The formulated models by Urban (1992) also incorporate learning effects and the possibility of defective items in the production process to determine the optimal lot size, price mark-up and advertising expenditure simultaneously.

According to the *Statistical Abstracts of the United States*, total advertising expenditure in the US has grown from 187 billion dollars in 1997 to 279 billion dollars in 2007. At the individual level of the firm, it was reported that in 2007, the largest

advertiser (Proctor & Gamble) spent more than five billion dollars on advertising (\$5,230,100,000 to be exact) while the 100th largest advertiser (Walgreen Co.) spent \$355,400,000 (*Advertising Age*). Since a firm's marketing effort considerably relies on advertising, often requiring commitment of significant amount of resources, investigating better ways of spending such huge amounts of funds would undoubtedly be fruitful.

Having established that advertising is an important element in a firm's marketing efforts, to which significant amounts of resources are usually committed, and a scarcity of models integrating the operations and advertising functions, how to determine the optimum advertising policy that works in harmony with various operations management decisions that will be studied in this research becomes an important issue to both academicians and practitioners alike.

Objectives, Organization and Contribution of Study

This dissertation responds to a call made by Eliashberg and Steinberg (1993, p. 876) seventeen years ago who mention "Looking over the diversity of marketingproduction models available in the literature, we observe that the most common decision variable from the marketing side has been price (static or dynamic), and the most common decision variable from the production side has been order quantity or production rate. Future research should consider other marketing decision variables such as sales promotion and advertising expenditure, as well as decision variables corresponding to the timing of introduction of new products and their level of quality." This dissertation aims at partially filling a gap existing in the literature by examining the interfaces between the operations and advertising functions. This research work is written in a three-journal format. Reviews of the literature pertaining to the three journals are included in the ensuing chapters. The first journal (Chapter 2) investigates the impact of production cost on advertising strategy in a monopolistic market. For a given advertising budget, the objective is to identify the optimum advertising allocation policy over time in the presence of a quadratic convex/ concave production cost function when the advertising response function is basically concave, using a modified Vidale-Wolfe model (1957) articulated by Little (1979). To the current author's knowledge, the study represents, for the first time in the literature, an attempt that aims at examining the impact of production cost on the advertising strategy of the firm. The study is thought to be applicable to low priced, frequently purchased consumer items.

The second journal (Chapter 3) focuses on the development of a symmetric competitive inventory model with advertising-dependent demand. As in the study of Mesak (2003), the research employs a market share attraction model for which the attraction function of a competitor depends upon its own advertising effort. The objective of this research is to study how a firm would adapt optimum ordered quantity/production lot size and optimum advertising expenditure in response to changes in its own parameters, rival parameters, or the parameters that are common to all firms in a symmetric duopoly/oligopoly market. In this regard, the parameters are classified into two categories: operations parameters and marketing parameters. Operations parameters include ordering/setup costs, holding cost per unit of the product per unit time, and product unit cost. Marketing parameters include the parameters associated with the advertising attraction function for each firm, and market potential that is common to all

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firms. To the best knowledge of the author, this is the first work in the literature that aims at deriving and assessing the robustness of comparative statics (sensitivity analysis) of a symmetric competitive inventory model with advertising-dependent demand. The study is thought to be applicable to low priced, frequently purchased consumer items.

In the third journal (Chapter 4), the original Bass model (1969) is modified to incorporate advertising and customers' disadoption and used to determine the optimum advertising policy over time for subscriber service innovations where service cost follows a learning curve. To the current author's knowledge, this study represents the first attempt in the literature made to characterize and validate the optimal advertising trajectory for new subscriber services over time in the presence of service cost learning and customers' disadoption. The study is thought to be applicable to service innovations that are made available to customers periodically at a subscription fee. Typical examples include, but are not limited to, cable TV, health clubs, pest control, and the internet.

Chapter 5 offers an overview of the findings of the three studies integrating the advertising function with different aspects of operations management followed by the limitations, and directions for future research pertaining to each journal. To improve the readability of different chapters, derivation of related key formulas, proofs of all results and propositions, and supplementary materials are relegated to separate appendices located at the end of the dissertation, followed by a list of references to all chapters.

CHAPTER 2

IMPACT OF PRODUCTION COST ON THE ALLOCATION OF THE ADVERTISING BUDGET OVER TIME IN A JUST-IN-TIME ENVIRONMENT OF ZERO-INVENTORY POLICY

Introduction

The objective of marketing is to create customer demand while operations management focuses on the supply and fulfillment of that demand. A conflict between these two areas may lead to production inefficiencies and unsatisfied customers (Ho and Tang, 2004). A firm's marketing effort that considerably relies on advertising often requires a commitment of a significant amount of resources. Therefore, the issue of whether it is best to adopt a pulsation policy of advertising or one of even-spending that costs the same is of significant interest to both academicians and practitioners. While the superiority of pulsation policies in a monopolistic market have been examined by a few authors (Feinberg, 1992; Hahn and Hyun, 1991; Mesak and Darrat, 1992; Mesak and Ellis, 2009), none of them considered the effect of production cost on the advertising strategy of the firm. To the best knowledge of the author, the study reported herein is the first attempt in the literature to investigate the impact of production cost on the optimal advertising policy.

It is demonstrated in this study that, in the presence of a convex quadratic production cost function, a pulsation policy of advertising could be superior to its evenspending counterpart for a concave advertising response function or when the advertising budget lies in the concave region of an S-shaped advertising response function. The above results are novel and different from previous research findings in the literature such as those by Sasieni (1971) for a concave advertising response function and others by Mahajan and Muller (1986) for an S-shaped advertising response function. These authors have not considered the production cost function in their models. Since this research lies at the interface between the marketing and operations functions, review of relevant marketing literature is introduced first, followed by what is related to the operations literature.

Review of Related Marketing Literature

Several research works have been published relevant to the optimal allocation of a given advertising budget over time. Sasieni (1971) in his pioneering article has shown that with decreasing marginal returns to advertising spending (concave advertising response function), a uniform advertising policy is superior to other policies of the same cost in the long run. However, empirical evidence suggests that a pulsing advertising policy could be superior to uniform spending over time (Wells and Chnisky, 1965; Ackoff and Emshoff, 1975; Rao and Miller, 1975; Eastlack and Rao, 1986). Due to the contradiction between theoretical and empirical findings, few models have been published with the purpose of substantiating advertising pulsation. Notable studies in this respect in monopolistic markets are briefly reviewed below.

Mahajan and Muller (1986) and Sasieni (1989) found that for an S-shaped advertising response function, some form of pulsing is superior to the uniform advertising policy provided that average advertising spending over the planning horizon is less than the advertising rate at which the tangent drawn from the origin meets the advertising response function. Mesak (1985) derived the conditions under which an advertising pulsing policy can be superior to a uniform advertising policy for both stationary and non-stationary markets. Using Haley's (1978) model, Simon (1982) and Mesak (1992) found that an advertising pulsing policy can be optimal under either a constrained or unconstrained advertising budget. Mesak and Darrat (1992) compared five alternative advertising policies analytically using a modified Vidale-Wolfe model and considered the impact of the shape of the advertising response function on optimality. They found that for a linear or concave advertising response function, a policy of uniform spending was optimal, while for a convex response function, pulsing advertising policy provided the best performance. Analyzing the effect of different costs on the optimal advertising policy, Hahn and Hyun (1991) found that a pulsation policy is optimal when the ratio of pulsation costs to fixed advertising costs is sufficiently small. Feinberg (1992) mathematically illustrated that a pulsation policy can be optimal in presence of an Sshaped response function through the use of an exponential filtering mechanism. In a later study, Feinberg (2001) used optimal control theory to analyze a flexible class of Sshaped response functions and showed that under certain conditions non-constant advertising paths may be optimal. Alternative theories have also been proposed. Bronnenberg (1998) derives pulsing in the context of a monopolist facing a Markovian sales-response function. Naik et al. (1998) consider copy wear out as a source of pulsing.

In a recent work, Mesak and Ellis (2009) analytically showed that if the product of a concave market potential function and the linear or concave advertising response function is convex in advertising, an advertising pulsing policy is superior to its uniform counterpart for the same advertising spending. These results indicate a general agreement among researchers that the shape of the advertising response function is an influential factor in deciding whether a uniform advertising policy is superior (or inferior) to its pulsation counterparts. In this regard, empirical studies related to the shape of the advertising response function (e.g. Simon and Arndt, 1980), but a limited support to an S-shaped response function (e.g. Rao and Miller, 1975).

Review of Related Operations Literature

In this subsection, a review related to the production cost function and zeroinventory production relevant to the scope and the purposes of the paper is presented.

Production researchers have observed that by assigning production to the source with the lowest unit cost until its capacity is fulfilled and then proceeding to the next cheapest source results in a convex production cost function which is also supported by empirical evidence (Eliashberg and Steinberg, 1993). In another research work, Eliashberg and Steinberg (1987) cited production and economics literature that employed a convex quadratic cost function. The authors showed that if the capital inputs are fixed, the production cost function can take the form of $(1/K) Q^2$, where K is a constant and Q is the output. Hax and Candea (1984), however, offer that concave costs become an issue in situations involving setup (change-over) changes and economies of scale in the production process. With the advent of JIT, the goal of zero-inventory production has stimulated great interest among researchers (Hall, 1983). Bielecki and Kumar (1988) developed an analytical model to show the optimality of zero-inventory policy, even for an unreliable manufacturing system when the demand rate was constant. Chan et al. (2002a; 2002b) examined, for piecewise linear ordering cost (representing many less-than truckload shipments) the effectiveness of a zero-inventory-ordering policy (ZIO), a policy in which an order is placed only when the retailer's inventory levels drop to zero while the supplier does not hold any stock. These research works indicate the importance of zero-inventory policy for both academic and practical purposes.

In this study a modification of the original Vidale-Wolfe model (1957) is introduced for which the advertising response function can be either linear, concave or Sshaped. However, unlike previous studies, this investigation considers the impact of the shape of the production cost function on the profit and the advertising policy of the firm in a just-in-time environment of zero-inventory policy.

The rest of the study is organized as follows. The second section outlines the theoretical model. Then a comparison of alternative pulsation policies is presented in the third section. The fourth section presents numerical results as obtained by simulation of the proposed model. The fifth section summarizes and concludes the study.

The Model

The model described in this section can be applied to a business that sells a single frequently purchased item of low level consumer involvement in a monopolistic market, where the firm's marketing efforts are mostly limited to advertising. A monopolistic environment can be approximated in a condition where a firm dominates the market while facing competition from many small firms who are too small to influence the market in a noticeable manner. Campbell's soup in North America, Heinz ketchup in the United Kingdom, and Wrigley chewing gum in Norway are examples of such a monopolistic market (Lazich, 2005). The assumptions made to develop this model are as follows:

- (a) The advertising rate (advertising amplitude) is constant over a given period of time, and may take on one of two distinct levels in a cyclic fashion.
- (b) The advertising budget for each cycle and the cycle length are determined exogenously and they are the same for each cycle.
- (c) The aggregate sales response to advertising is deterministic.
- (d) Sale of a product are strictly influenced by advertising.
- (e) Production is equal to sales as no inventory is maintained.
- (f) Production cost is a quadratic function of sales revenue.
- (g) Operating costs other than those related to production and advertising are proportional to sales revenue.

Threshold effects of advertising were not considered due to their negligible role for frequently purchased items in mature markets (Vakratsas et al. 2004).

Figure 2.1 provides a schematic illustration of three alternative forms of practical pulsation policies and the firm has to choose among three alternative policies: (a) Uniform Advertising Policy (UAP), in which the firm advertises at a constant level throughout $(x_1 = x_2)$, where x_1 and x_2 indicate two different levels of advertising intensity, (b) Advertising Pulsing and Maintenance Policy (APMP), in which the firm alternates between a high level of advertising x_1 that lasts for a time period t_1 , followed by a lower

level x_2 , usually a maintenance level lasting for a duration of $(T-t_1)$, where T is the length of each cycle, and (c) Advertising Pulsing Policy (APP), in which the firm alternates between high and zero levels of advertising. The relationship between x_1 and x_2 and between t_1 and T pertaining to each policy is shown at the top of each configuration.



Figure 2.1. Different Forms of Advertising Pulsation Policies: (a) Uniform Advertising Policy (UAP), (b) Advertising Pulsing and Maintenance Policy (APMP), and (c) Advertising Pulsing Policy (APP).

Through laboratory study (Wells and Chnisky, 1965) and field experiments (Ackoff and Emshoff, 1975; Rao and Miller, 1975), it has been shown that an advertising pulsing and advertising pulsing/maintenance policy can be superior to the uniform advertising policy. These studies, and others, justify that the relatively narrow pulsation policy set considered above is not only of theoretical interest but also of practical relevance. Although the three studied pulsation policies are not by any means exhaustive, the inferiority of the uniform strategy implies the superiority of pulsing in general. The study is concerned with the long-term, steady-state response. Hauser and Wernerfelt (1989) assert that such a focus is most appropriate for strategic advertising decisions. The proposed model is constructed using Vidale and Wolfe model, and therefore, it is introduced in the ensuing subsection.

Dynamic Advertising Response Model

The Vidale and Wolfe model (1957) is one of the earliest and most intensively analyzed mathematical models of dynamic advertising response (e.g. Sasieni, 1971; Mahajan and Muller, 1986). According to this model, the instantaneous change in the sales rate is given by the following first-order linear differential equation:

$$\frac{dS}{dt} = bx(m-S) - aS, \qquad (2.1)$$

where S is the sales rate (\$/unit time); x is the advertising rate (\$/unit time); b is the advertising effectiveness parameter; m is the market potential or saturation sales, and a is the decay constant. The steady state advertising response S(x) is derived through setting dS/dt = 0, and solving Equation (2.1) for S to obtain:

$$S(x) = \frac{mbx}{a+bx}.$$
(2.2)

The linear advertising response function for the Vidale-Wolfe model is given by f(x) =bx, and the market potential, m are constant. Little (1979) proposed a modified version of the Vidale-Wolfe model for which f(x) takes a power function of the form f(x) = bx^{δ} , where b and δ are positive constants. Thus, f(x) becomes linear in x (f''(x) = 0) for δ = 1, concave in x (f''(x) < 0 as a sufficient condition) for $0 < \delta < 1$, and convex in x (f''(x) > 0 as a sufficient condition) for $\delta > 1$ (see also Figure 2.2). Sasieni (1989) used an S-shaped advertising response function of the form $f(x) = be^{-\delta x}$. This function is convex over the range $0 \le x \le x_t$ and concave for $x > x_t$. $x_t = \delta$ corresponds to the point at which the tangent drawn from the origin meets f(x) (see also Figure 2.3). This paper employs the modified version of Equation (2.1) for which the advertising response function can be linear, concave, or S-shaped. Turner and Wiginton (1976) and Mesak and Darrat (1992), among others, provide empirical evidence showing that the modified version of the Vidale-Wolfe model predicts well the sales rate as a function of the advertising rate.



Figure 2.2. Different Shapes of Advertising Response Function

By using the more general form for advertising response function in Equation (2.1), it can be shown that after rearrangement of terms (2.1) takes the following form:

$$\frac{dS}{dt} = \alpha(x)[S(x) - S]$$
(2.3)

where

$$\alpha(x) = a + f(x), \qquad (2.4)$$

$$S(x) = \frac{mf(x)}{a+f(x)}.$$
(2.5)

and



Figure 2.3. Details of an S-shaped Advertising Response Function

The steady state sales rate (2.5) is concave if f(x) is linear or concave, while it is S-shaped if f(x) is convex. The decay constant, a, in (2.4) and (2.5) reflects the effect of advertising from the competition. Gopalakrishna and Chatterjee (1992) argued that for a firm with dominant market share and competing against a fringe of many small suppliers, each too small to influence the market, parameter a could be regarded as a constant quantity.

Selecting a Performance Measure

An appropriate performance measure is needed to compare the effectiveness of the alternative advertising policies. Mesak and Ellis (2009) quoted Feinberg (1988) about the advantages and disadvantages of several performance measures, two of which are relevant to the scope of this research. They are: (i) the discounted profit over the infinite planning horizon that is sensitive to the initial sales level, and (ii) the average undiscounted profit over the infinite planning horizon that treats the profit made sooner as equal to the profit made later. Park and Hahn (1991) argued in favor of the second measure as it is independent of arbitrary initial conditions and its weakness is substantially mitigated due to the periodic nature of the advertising policies over time. Mesak (1992) showed that the second measure serves as a plausible approximation of the first one when the discount rate is small. Consequently, in this research, the second performance measure is employed.

Sales Response to Advertising Pulsation

Considering the general advertising pulsation policy for which $0 \le x_2 \le x_1$ (see Figure 2.4), the time axis is divided into equal similar cycles of duration *T*, in which advertising is at a high level x_1 , over a duration of time t_1 , and at a low level x_2 over a duration of time $(T-t_1)$. The first cycle starts at t = 0 at which the initial sales is S_0 , the sales rate grows to *M* at t_1 following the sales growth curve $g_1(t)$ while advertising is kept at its higher level. Afterwards, as the advertising level is decreased, the sales rate decays along the curve $g_2(t)$ until *T* and the cycle repeats with new initial conditions. It is shown in Appendix A (following Mesak and Ellis, 2009) that the system eventually reaches a quasi-steady state for which the steady state cycle starts and ends with the same level of sales rate. Also, it is shown in Appendix A that for an infinite planning horizon, the long run average sales rate of a firm is equal to the average sales rate over the steady state cycle. Therefore, the average profit per unit time at the steady state, *PRO*, in response to a general pulsation policy is chosen here to represent the performance measure of the firm.



Figure 2.4. Sales Response to Advertising at the Steady-state Cycle

Referring to Figure 2.4, the following quadratic production cost function is considered in the analysis:

$$q_i(t) = vg_i(t) + \frac{1}{K}g_i^2(t); i = 1, 2.$$
 (2.6)

The above production cost function is composed of two terms. The first term is linear in the sales rate, measured in dollars per unit time, whereas the second term is quadratic in the same. The quantity v is a positive constant fraction seen to represent the ratio of the linear production cost component to sales revenue and K is another constant conceived to represent the firm's process efficiency (Eliashberg and Steinberg, 1987).

The quadratic production cost Function (2.6) is convex for K > 0 and concave for K < 0. The situation of a convex production cost function for which K > 0 is analyzed first.

Considering the steady state cycle, the expressions for average sales revenue per unit time (R), the average production cost per unit time (PC), and the related average profit per unit time (PRO) are given by Expressions (2.7), (2.8) and (2.9), respectively.

$$R = \frac{1}{T} \left(\int_{0}^{t_{1}} g_{1}(t) dt + \int_{0}^{T-t_{1}} g_{2}(t) dt \right).$$
 (2.7)

$$PC = \nu R + \frac{1}{KT} \left(\int_{0}^{t_{1}} g_{1}^{2}(t) dt + \int_{0}^{T-t_{1}} g_{2}^{2}(t) dt \right), \qquad (2.8)$$

The first term in (2.8) is designated by *PC*1, whereas its second term is designated by *PC*2. Combining Equations (2.7) and (2.8) and introducing γ , with a given average rate of advertising spending x assumed to have been determined exogenously, net profit per unit time is given by

$$PRO = \frac{1}{T} \left\{ \gamma \left(\int_{0}^{t_{1}} g_{1}(t) dt + \int_{0}^{T-t_{1}} g_{2}(t) dt \right) - \frac{1}{K} \left(\int_{0}^{t_{1}} g_{1}^{2}(t) dt + \int_{0}^{T-t_{1}} g_{2}^{2}(t) dt \right) \right\} - x \quad (2.9)$$

where γ is a fraction less than 1, given by $\gamma = 1 - \varepsilon$, and $\varepsilon > v > 0$ represents the ratio of costs (other than those related to the nonlinear production cost and the advertising expenditure, such as the linear production cost and the physical distribution cost) to sales revenue which is assumed to be constant (Nicholson, 1983).

It is also shown in Appendix A that Expression (2.7) takes the following form:

$$R = \frac{1}{T} \left[S(x_1)t_1 + S(x_2)(T - t_1) + \{S(x_1) - S(x_2)\} \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right) \frac{(1 - e^{-\alpha_1 t_1})(1 - e^{-\alpha_2 (T - t_1)})}{(1 - e^{-(\alpha_1 t_1 + \alpha_2 (T - t_1)})} \right]$$
(2.10)

while the second term in Expression (2.8), PC2, takes the following form:
$$PC2 = \frac{1}{KT} \begin{bmatrix} S^{2}(x_{1})t_{1} + S^{2}(x_{2})(T - t_{1}) + 2\{S(x_{1}) - S(x_{2})\} \left(\frac{S(x_{2})}{\alpha_{2}} - \frac{S(x_{1})}{\alpha_{1}}\right) \\ \frac{(1 - e^{-\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})}{(1 - e^{-(\alpha_{1}t_{1} + \alpha_{2}(T - t_{1})})} + \frac{\{S(x_{1}) - S(x_{2})\}^{2}}{2(1 - e^{-(\alpha_{1}t_{1} + \alpha_{2}(T - t_{1})})^{2}} \\ \left\{\frac{(1 - e^{-2\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})^{2}}{\alpha_{1}} + \frac{(1 - e^{-\alpha_{1}t_{1}})^{2}(1 - e^{-2\alpha_{2}(T - t_{1})})}{\alpha_{2}}\right\} \end{bmatrix}$$
(2.11)

where

$$S(x_1) = mf(x_1)/(a + f(x_1)); \qquad S(x_2) = mf(x_2)/(a + f(x_2));$$

$$\alpha_1 = a + f(x_1); \qquad \alpha_2 = a + f(x_2).$$

Therefore, the net profit per unit time is given by (from Equation 2.9)

$$PRO = \gamma R - PC2 - x. \tag{2.12}$$

Comparison of Alternative Pulsation Policies

In this section, the notion of dominance among different classes within the pulsation group of policies is introduced. As in Mahajan and Muller (1986) and Mesak and Darrat (1992), an advertising policy c_i dominates another c_j that costs the same if when the firm employs policy c_i , the average profit per unit of time is larger than that under policy c_j . A class of policies C_i dominates a class of policies C_j if for any policy in C_j there exists a dominating policy in C_i .

Consider a given average advertising budget rate x per unit time in a cycle of duration of T for all three advertising pulsation policies (namely, UAP, APMP, and APP), then we must have for each policy

$$x_1 t_1 + x_2 (T - t_1) = xT, (2.13)$$

where x_1 and x_2 are the high and low levels of advertising respectively, while t_1 and $(T-t_1)$ are their related durations. These policies can be compared by conceiving them as policy sets, each characterized by a certain value of a policy parameter λ .

The policy parameter λ ($0 \le \lambda \le 1$) is defined as $x_2 = \lambda x$, such that different advertising policies can be characterized in terms of λ in the following manner: UAP is characterized by $\lambda = 1$ (Figure 2.1a), APMP is characterized by $0 < \lambda < 1$ (Figure 2.1b), and APP is characterized by $\lambda = 0$ (Figure 2.1c). Thus, for any given value of λ , the high advertising level x_1 can be uniquely determined from (2.13) by replacing x_2 by λx . Hence, the UAP set contains only one policy element for which $x_1 = x_2 = x$. On the other hand, the APMP set contains a number of policy elements given by the Cartesian product ($t_1 \ge x$ $T \ge \lambda$), while the APP set contains a number of policy elements given by the Cartesian product ($t_1 \ge T$). Based on expressions (2.10) – (2.12), we are in a position to introduce one result and one main proposition of which the proofs are provided in Appendix B. *Result 1:*

For an Advertising Pulsing Policy (APP),

Sign of
$$\left. \frac{d(PRO)}{dt_1} \right|_{t_1 \to T}$$
 is given by : Sign of $\left(\gamma - \frac{2S(x)}{K} \right)$. Sign of $\left\{ f(x) - xf'(x) \right\}$. (2.14)

From the above result and Figures 2.2 and 2.3, the following observations are made:

- (i) The sign of [f(x) xf'(x)] is positive for a concave advertising response function (see Figure 2.2),
- (ii) The sign of [f(x) xf'(x)] is negative for advertising budgets lying within the convex portion of an S-shaped advertising response function, the range of which

is determined by x = 0 and the advertising budget, x_t , given by the tangent drawn from the origin to meet that function (see Figure 2.3),

- (iii) The sign of [f(x) xf'(x)] is positive for the concave portion of an S-shaped advertising function determined by advertising budgets in excess of x_t (see Figure 2.3),
- (iv) It is observed from (2.2) that the maximum value that S(x) could take for a finite value of x is less than m. Therefore, if a solution of the equation $\gamma = 2S(x)/K$ does not exist ($\gamma K/2 \ge m$), the sign of [$\gamma 2S(x)/K$] is always positive, and
- (v) If a solution of the equation $\gamma = 2S(x)/K$ does exist $(\gamma K/2 < m)$, the solution would be unique at a value $x_s > 0$. In this case, the sign of $[\gamma - 2S(x)/K]$ is positive for advertising budgets $x < x_s$ and the sign of $[\gamma - 2S(x)/K]$ is negative for $x > x_s$.

Result 1 implies that if the sign of Expression (2.14) is negative, then APP dominates UAP as there will be at least one APP policy for which t_1 is smaller but close enough to *T* for which *PRO* is larger than the only unique UAP counterpart that costs the same. Based on Result 1, Proposition 1 is introduced for which the proof is found in Appendix B.

Proposition 1:

For a firm of a convex quadratic production cost function, if a solution to the equation $\gamma = 2S(x)/K$ exists at $x_s > 0$

(i) In the presence of a concave advertising response function, APP and APMP dominate UAP for all advertising budgets $x > x_s$.

(ii) In the presence of an S-shaped advertising response function, APP and APMP dominate UAP for all advertising budgets $x > Max \{x_t, x_s\}$.

The results depicted in Proposition 1 are in contradiction to previous research findings reported in the literature. The result of Proposition 1(i) is inconsistent with the findings of Sasieni (1971) who did not consider the production cost function in the modeling effort and asserted the optimality of UAP for a concave advertising response function. Similarly, the result of Proposition 1(ii) is inconsistent with the findings of Mahajan and Muller (1986) that did not consider the production cost function in their modeling effort and asserted the optimality of UAP for advertising budgets that lie in the concave portion of an S-shaped advertising response function.

In order to demonstrate that a quadratic convex production cost function is central to the potential superiority of a pulsation policy (APMP or APP) under a concave advertising response function, it is shown below that such possibility cannot materialize for a linear production cost function ($K = \infty$) or a concave one (K < 0). For that purpose, two results are introduced first followed by an additional proposition.

Result 2:

Considering the performance measure R and for a concave advertising response function, advertising pulsing/maintenance policy (APMP) dominates advertising pulsing policy (APP) but is dominated by uniform advertising policy (UAP).

The proof of the above result is found in Appendix B (see also Mesak and Darrat, 1992).

Result 3:

For a concave advertising response function, the absolute value of the quadratic term of the production cost function (*PC2*) attains its maximum value at a value of the policy parameter $\lambda = 1$ (or UAP).

The author was not able to prove analytically the above result. The validity of such a result, however, was examined by simulating the sensitivity of Expression (2.11), for the various values of the relevant parameters. It is concluded, without a formal proof, that for all values tried for the parameters the statement of the above results remained unchallenged. Using simulation to prove propositions is not an uncommon practice in the relevant literature. Notable examples are found in the studies of Mahajan and Muller (1986, p. 96) and Mesak (1992, p. 319). Based on results 2 and 3, Proposition 2 is introduced for which the proof is found in Appendix B.

Proposition 2:

For a firm of a concave quadratic production cost function, APMP dominates APP but is dominated by UAP for a concave advertising response function.

Proposition 2 broadens the scope of applicability of known results in the literature (e.g. Sasieni, 1971; Mesak and Darrat, 1992) to the situation of a concave quadratic production cost function that has not been examined before in the literature. More importantly, Proposition 2 attributes the potential superiority of a pulsation policy in the presence of concavity in the advertising response function predominantly to the convexity of the production cost function (Proposition 1). It is also shown in the next numerical section of the study that the findings of Proposition 2 appear applicable for advertising budgets lying in the concave portion of an S-shaped function.

A Numerical Investigation

The main purpose of the numerical investigation of this section are (1) to provide illustrative examples in support of the statements of Propositions 1 and 2 for both concave and S-shaped response functions in the presence of a convex quadratic production cost functions, and (2) to report the results of a simulation study that is aimed at validating the statement depicted in Result 3. The employed values of model parameters have been influenced a great deal by the results of the empirical estimation of modified versions of the Vidale-Wolfe model found in the studies of Mesak and Darrat (1992) and Mesak and Ellis (2009). The above two articles used the well-known Lydia Pinkham data, originally examined by Palda (1964), in the estimation of such models.

Illustrative Examples

For T = 0.50, $t_1 = 0.25$, $\gamma = 0.4$, and K = 40,000, the modified Vidale-Wolfe model (2.3) of a concave advertising response function, $f(x) = 0.01\sqrt{x}$, takes the form

$$dS / dt = (0.20 + 0.01\sqrt{x})[S(x) - S]$$
, and
 $S(x) = (16,000) \frac{0.01\sqrt{x}}{0.2 + 0.01\sqrt{x}}$.

Referring to Proposition 1, it can be shown that the solution of the equation $\gamma = 2S(x)/K$ is given by $x_s = 400$. Table 2.1 reports the quantities *R*, *PC2*, *PRO1*, and *PRO2* as a function of the policy parameter λ for two values of the advertising budgets $x = 300 < x_s$ and $x = 500 > x_s$. Consistent with Proposition 1, optimal *PRO1* occurs at a policy parameter λ less than 1 for x = 500, whereas optimal *PRO1* occurs at $\lambda = 1$ for x = 300. Consistent with Results 2 and 3 together with Proposition 2, the optimal values of *R*, *PC2*, and *PRO2* occur at $\lambda = 1$. The above observations related to a concave advertising response function appear reproducible for advertising budgets lying in the concave portion of an S-shaped advertising response function as shown in Table 2.2.

		x = 300				x = 500		_
λ	R	PC2	PRO1	PRO2	R	PC2	PRO1	PRO2
0.0	6075.959	923.124	1207.260	3053.508	7063.132	1247.456	1077.797	3572.709
0.1	6771.295	1146.355	1262.163	3554.873	7783.012	1514.506	1098.699	4127.710
0.2	6983.585	1219.325	1274.109	3712.759	7999.583	1599.917	1099.917	4299.750
0.3	7119.548	1267.243	1280.576	3815.062	8137.513	1655.535	1099.470	4410.541
0.4	7216.298	1301.904	1284.615	3888.423	8235.296	1695.542	1098.577	4489.660
0.5	7287.762	1327.807	1287.298	3942.911	8307.328	1725.318	1097.613	4548.250
0.6	7340.799	1347.196	1289.124	3983.515	8360.682	1747.541	1096.732	4591.813
0.7	7379.255	1361.342	1290.360	4013.044	8399.310	1763.719	1096.005	4623.443
0.8	7405.414	1371.007	1291.159	4033.172	8425.559	1774.755	1095.469	4644.978
0.9	7420.629	1376.644	1291.608	4044.896	8440.816	1781.185	1095.141	4657.512
1.0	7425.625	1378.498	1291.752	4048.748	8445.824	1783.299	1095.031	4661.628

Table 2.1. Numerical Results in Support of Propositions 1 and 2 – Concave Response Function

^a $PRO1 = \gamma R - PC2 - x$ (profit under a convex quadratic production cost function) ^b $PRO2 = \gamma R + PC2 - x$ (profit under a concave quadratic production cost function)

For T = 0.50, $t_1 = 0.25$, $\gamma = 0.4$, and K = 40,000, the modified Vidale-Wolfe model

(2.3) of an S-shaped advertising response function, $f(x) = 0.48e^{-350/x}$, takes the form

 $dS/dt = (0.20 + 0.48e^{-350/x})[S(x) - S], \text{ and}$ $S(x) = (16,000) \frac{0.48e^{-350/x}}{0.20 + 0.48e^{-350/x}}.$

It can be shown that the solution of the equation $\gamma = 2S(x)/K$ is given by $x_s = 399.79$. Furthermore, the considered S-shaped advertising response function is $x_t = 350$ (see Figure 2.3). Table 2.2 reports the quantities *R*, *PC2*, *PRO1*, and *PRO2* as a function of λ for two values of the advertising budget *x* lying in the concave portion of the above S-shaped advertising response function. They are $x_t = 350 < x = 370 < x_s = 399.79$ and $x = 430 > Max \{x_t, x_s\} = 399.79$. Again, consistent with Proposition 1, optimal *PRO1* occurs x = 370. Furthermore, the optimal values of *R*, *PC*2 and *PRO*2 occur at $\lambda = 1$.

		x = 370				x = 430		
λ	R	PC2	PRO1	PRO2	R	PC2	PRO1	PRO2
0.0	6844.444	1171.404	1196.374	3539.182	7103.958	1261.918	1149.665	3673.501
0.1	6747.666	1138.512	1190.554	3467.578	7021.259	1232.708	1145.796	3611.212
0.2	6697.205	1121.534	1187.348	3430.416	7030.222	1235.832	1146.257	3617.920
0.3	6799.364	1155.962	1193.783	3505.708	7222.575	1304.316	1154.714	3763.346
0.4	6996.192	1223.796	1204.681	3652.273	7481.329	1399.377	1163.154	3961.909
0.5	7206.720	1298.505	1214.183	3811.193	7722.623	1491.049	1168.001	4150.098
0.6	7391.101	1365.760	1220.680	3952.200	7918.916	1567.775	1169.791	4305.342
0.7	7535.693	1419.694	1224.584	4063.971	8066.251	1626.634	1169.867	4423.134
0.8	7637.803	1458.412	1226.709	4143.534	8167.564	1667.738	1169.288	4504.763
0.9	7698.233	1481.573	1227.721	4190.866	8226.605	1691.928	1168.714	4552.570
1.0	7718.208	1489.268	1228.015	4206.551	8245.983	1699.906	1168.487	4568.299

Table 2.2. Numerical Results in Support of Proposition 1 – S-shaped Response Function

^a $PRO1 = \gamma R - PC2 - x$ (profit under a convex quadratic production cost function) ^b $PRO2 = \gamma R + PC2 - x$ (profit under a concave quadratic production cost function)

A Simulation Study

In conducting the simulation study related to a concave advertising response function, different parameters were initialized at the following base values: m = 16,000, b = 0.01, $\delta = 0.50$, x = 400, $t_1 = 0.25$, T = 0.50 and K = 40,000. Different pulsation policies were produced through the use of policy parameter λ , and $\lambda = x_2/x$ (for APP, $\lambda = 0$; for APMP, $0 < \lambda < 1$; and for UAP, $\lambda = 1$). Eleven different values of λ were assumed ranging between 0 and 1 in increment of 0.1. Since K is basically a scaling parameter, only one value for it has been employed. The quadratic term of the production cost at the steady state cycle *PC2* (Equation 2.11) was computed for each possible combination of λ using different sets of values for the other parameters. In constructing such sets, only one parameter at a time was allowed to vary over a certain range while the other parameters were kept at their base values shown above. The different ranges used for the parameters (including the base values) are as follows:

m: 10,000 to 20,000 in increments of 1,000 (eleven values)

a: 0.001, 0.010, 0.050, 0.100, 0.200, 0.250 (six values)

b: 0.001, 0.005, 0.010, 0.015, 0.020 (five values)

 δ : 0.1 to 1.0 in increments of 0.1 (ten values)

x: 50 to 500 in increments of 50 (ten values)

(*T*, *t*₁): (0.2500, 0.0625), (0.2500, 0.1250), (0.2500, 0.1875), (0.5000, 0.1250), (0.5000, 0.2500), (0.5000, 0.3750), (1.0000, 0.2500), (1.0000, 0.5000), (1.0000, 0.7500) (nine values)

There were a total number of 605 computations performed for calculating *PC*2 (55 different values for the parameters *m* through t_1 multiplied by 11 different values of λ). For all the 55 relationships investigated between λ and *PC*2, the largest absolute value of *PC*2 occurs at $\lambda = 1$. Table 2.3 illustrates a sample of the relationship between λ and *PC*2 for two advertising budgets and two combinations of (T, t_1) .

	$(T, t_l) = (0.5000,$	0.1250)	$(T, t_1) = (0.2500, 0)$.1250)
	x = 50	x = 500	x = 100	x = 450
λ	PC2	PC2	<i>PC</i> 2	РС2
0.0	144.469	822.819	436.662	1175.480
0.1	262.380	1269.016	566.762	1433.828
0.2	308.940	1419.731	611.146	1516.823
0.3	342.341	1521.222	640.787	1570.960
0.4	368.322	1596.795	662.470	1609.944
0.5	389.087	1655.258	678.808	1638.984
0.6	405.704	1700.882	691.112	1660.669
0.7	418.710	1735.909	700.130	1676.463
0.8	428.312	1761.398	706.310	1687.240
0.9	434.435	1777.492	709.922	1693.521
1.0	436.656	1783.299	711.111	1695.586

Table 2.3. Relationship between *PC*2 and λ

Table 2.3 also reveals that the relationship between the absolute value of *PC*2 and λ is monotonically increasing exactly like that relationship between *R* and λ depicted in Table 2.1.

Conclusion

The literature reveals a contradiction between the theoretical results and the empirical research findings on the issue whether a firm should advertise at a constant rate or in a cyclic fashion in order to maximize its performance. Sasieni (1971) in his theoretical analysis showed that under linear or concave advertising response function, a pulsation or cyclic advertising policy can never be superior to a uniform advertising policy in the long run when the total cost of both policies are the same. On the other hand, based on empirical findings, Ackoff and Emshoff (1975) and Eastlack and Rao (1986) concluded that an advertising pulsation policy is superior to a policy of uniform spending over time that costs the same. After reviewing more than one hundred empirical

studies, Simon and Arndt (1980) concluded that the shape of the advertising response function is mostly concave or linear. While a few research attempted to resolve the above said difference (e.g. Feinberg, 1992; Hahn and Hyun, 1991; Mesak, 1992; Mesak and Ellis, 2009), the proposed solutions have been always within the marketing domain. Unlike the above attempts, this study offers a resolution within the operations function of the enterprise.

This paper attributes the potential superiority of pulsation in the presence of concavity in the advertising response function to the convexity of the production cost function and the advertising budget (Proposition 1). To substantiate the superiority of an advertising policy of pulsation, the convexity of the shape of the production cost function is envisioned to be a central requirement; otherwise, the uniform advertising policy becomes optimal (Results 2, 3 and Proposition 2). To that end, the theoretical and numerical analysis employ a dynamic advertising response model (modified Vidale-Wolfe model) in a just-in-time environment of zero-inventory policy.

Proposition 1, guided by the findings of Result 1, provides management with simple rules to assess the superiority of an advertising pulsation policy in the presence of a convex quadratic cost function for advertising budgets lying in the concavity region of the advertising response function. The implementation of such rules and their consequences has been illustrated by the results of a numerical investigation associated with the analyzed model in the fourth section (Tables 2.1 through 2.3).

The results of the study are not confined to the employed model but are rather of general applicability. The results should be reproducible for other models that can be cast in the form of model (2.3). Little (1979) showed that several notable theoretical models

can be formulated in the context of (2.3), including the models of Gould (1970), Little (1975), and Nerlove and Arrow (1962).

CHAPTER 3

ON COMPARATIVE STATICS OF A SYMMETRIC COMPETITIVE INVENTORY MODEL WITH ADVERTISING – DEPENDENT DEMAND

Introduction

While inventory is considered one of the most important elements of the supply chain in contemporary operations management, advertising is considered one of the most important promotional tools of modern marketing management. This study focuses on the development of a symmetric competitive inventory model with advertising-dependent demand. The objective of this research is to study how a firm would adapt optimum ordered quantity/production lot size and optimum advertising expenditure in response to changes in any of its own parameters, rival parameters, or the parameters that are common to all firms in a symmetric duopoly/oligopoly market. By a symmetric oligopoly, it is meant that the firms are as similar as possible in all economic respects who compete non-cooperatively with each other for the same potential buyers in an industry producing a storable, homogeneous product (Mesak, 2003). With symmetric competition, all rivals are assumed to have the same production costs, to be able to acquire real promotion on the same terms, to charge the same fixed price and to face symmetric demand functions (Schemalensee, 1976). As in the study of Mesak (2003) who has not explicitly considered inventory costs, this research employs a Nash equilibrium solution concept in conjunction with a market share attraction model for which the attraction function of a competitor depends upon its own advertising effort. Similar to the study of Min and Chen (1995) who have not explicitly considered advertising, each firm aims at maximizing its profit and ordering/setup as well as inventory holding costs which are assumed identical across firms in a situation for which all the assumptions of a traditional EOQ model hold. To the best knowledge of the author, this is the first work in the literature that attempts to derive and assess the robustness of sensitivity analysis (comparative statics) of a symmetric competitive inventory model with advertising-dependent demand (Here the two terms "sensitivity analysis," mainly adopted by operations research practitioners, and "comparative statics," mainly used by economists, are employed interchangeably to mean the same thing). The study is thought to be applicable to low priced, frequently purchased consumer items in the mature stage of their product life cycles.

This paper demonstrates that previous comparative statics obtained by Mesak (2003) remain valid after the incorporation of inventory costs in the modeling effort. However, optimal advertising is found sensitive to changes in the operations parameters of the model associated with ordering and inventory holding costs that were not considered by Mesak (2003). On the other hand, the analysis reveals that comparative statics obtained under a cost minimization of an inventory model could be very different from their counterparts under a profit maximization framework that considers an advertising-dependent demand. Furthermore, the optimal ordered quantity appears to be sensitive to changes in the marketing parameters of the model associated with the market

share attraction function and the advertising cost function. Because the model developed herein is static in orientation, a literature review of dynamic oligopoly models that use differential games in their analyses is beyond the scope of this exploratory study. Interested readers are referred to Dockner et al. (2000) and Erickson (2009a, 2009b) for a review of related literature.

Review of Related Marketing Literature

Whitin (1955) was the first to integrate purchasing decisions and marketing by incorporating the effect of price on demand within the inventory model where the retailer has to decide both the price and order quantity optimally. The above article triggered numerous research with the objective of integrating price-dependent demand with inventory models or ordering policies (Abad, 1988, 1996; Lee, 1993; Arcelus, Shah, and Srinivasan, 2003; Ray, Gerchak, and Jewkes, 2005; Viswananthan and Wang, 2003). The price-dependent demand model was later extended by another stream of research to integrate the inventory model with advertising dependent demand (Subramanyam and Kumaraswamy, 1981; Urban, 1992; Khouja and Robbins, 2003; Sana and Chaudhuri, 2008).

The research works mentioned above strictly considered a single firm, and therefore, competition was beyond their scope. Game theory is considered an appropriate tool for analyzing events where multiple agents are involved in a decision process and their actions are interrelated (Fudenberg, and Tirole, 1991). There has been limited research within a static framework on advertising competition in a duopoly/oligopoly using the game-theory approach (Friedman, 1958; Mills, 1961; Gupta and Krishnan, 1967; Mesak and Calloway, 1995, 1999; Mesak, 2003). The above research works,

however, have not considered inventory related costs in the modeling effort. Therefore, the current research contributes by investigating an inventory model with advertising-dependent demand in a duopoly/oligopoly.

Review of Related Operations Literature

To meet customer demand, firms face conflicting pressure to keep inventories low enough to reduce the holding cost but high enough to reduce ordering and setup costs with the objective of minimizing the total cost of inventory (Stevenson, 2008). This is accomplished by determining the right quantity to be ordered. There are several models that are used for this purpose of which the economic order quantity model (EOQ), along with its different variations, have been studied intensively assuming a constant demand rate (Buzacott, 1975; Chandra and Bahner, 1985; Chung, 1989; Moon and Lee, 2000). Noting that demand for inventory items increases with time in the growth phase, and decreases in the decline phase, researchers have extended the EOQ model to include a demand pattern that follows a certain distribution or varies with time (Bose, Goswami, and Chaudhuri, 1995; Haneveld and Teunter, 1998; Chang and Dye, 1999; Moon and Giri, 2001; Moon, Giri, and Ko, 2005).

However, all the research works mentioned above assume that the demand rate is determined exogenously and involve a single firm. Min and Chen (1995) extended the EOQ model to the case of a symmetric oligopoly consisting of several manufacturers who compete against each other for the same potential buyers assuming a constant demand rate. The above authors assume the availability of the options of investing to reduce the setup and inventory holding costs. The sensitivity analysis reported in the above theoretical paper was limited to studying the impact of increasing the number of competitors on the ordering cost, the holding cost and the ordered quantity per order. The present research differs from previous works as it employs an EOQ model to a symmetric oligopoly for which the demand is dependent on advertising expenditures.

The rest of the study is organized as follows. The second section outlines the theoretical model. The third section derives the sensitivity analysis for a monopoly and a symmetric duopoly. The fourth section estimates the market share attraction model using brewing industry data while the fifth section provides an empirical analysis to assess the robustness of comparative statics of the symmetric oligopoly. The sixth section summarizes and concludes the paper. Derivation of key formulas and mathematical proofs are reported in two separate Appendices (C and D).

A Mathematical Model of Inventory and Advertising Competition

Monopoly Model

A profit function in a monopoly is introduced first in this section. Denoting the advertising expenditure of the firm by x, the demand D(x) as a function of advertising is expressed as follows:

$$D(x) = \frac{mf(x)}{e+f(x)} = \frac{m\beta x^{\delta}}{e+\beta x^{\delta}} , \qquad (3.1)$$

where *m* is the market potential, f(x) is a concave attraction function such that $f'(x) \ge 0$, $f''(x) \le 0$ for $x \ge 0$, and *e* is a constant parameter. For the considered attraction power function, the β and δ are strictly positive parameters, and for a concave function, $0 < \delta <$ 1. It is shown in Appendix A that Expression (3.1) represents the steady state sales response of a modified version of the well-known Vidale-Wolfe model (1957) articulated by Little (1979). The parameter β represents the relative effectiveness of the advertising effort of the firm while the shaping parameter δ represents the elasticity of the advertising effort of the firm. To justify the assumption of a concave advertising attraction function, researchers have argued that a firm should first advertise in places where it gets the most response for each dollar spent. As advertising expenditure increases, the firm would spend more in previous places and expand its effort to the next best place. This argument of diminishing return justifies the assumption of a concave advertising function (Kotler, 1971; Simon and Arndt, 1980). Designating the unit price, the marginal cost, the advertising cost function, the quantity ordered, the ordering cost per order, the inventory holding cost per unit held per unit time, and fixed costs by *P*, *MC*, *C*(*x*), *Q*, *C*_o, *C*_h, and *F*, respectively, and introducing γ as the profit margin per unit sold (*P* - *MC*), the profit function, π of the firm takes the following form:

$$\pi = \gamma D(x) - C(x) - \frac{Q}{2}C_h - \frac{D(x)}{Q}C_o - F.$$
(3.2)

It is also assumed that the advertising cost function is convex and of the form $C(x) = dx^{\epsilon}$, where d and ϵ are positive constants such that d > 0, and $\epsilon > 1$ (Piconni and Olson, 1978). For a convex function, $C'(x) \ge 0$ and $C''(x) \ge 0$, for $x \ge 0$. Defining $\Pi = \pi / \gamma m$, for convenience of analysis, and replacing D(x) by Expression (3.1), the profit Function (3.2) takes the following form:

$$\Pi = \frac{f(x)}{e + f(x)} \left[1 - \frac{C_o}{\gamma Q} \right] - \frac{C(x)}{\gamma m} - \frac{C_h}{2\gamma m} Q - \frac{F}{\gamma m}.$$
(3.3)

The first-order optimality conditions require that the optimal advertising $x^* > 0$ and optimal ordered quantity $Q^* > 0$ satisfy the two equations:

$$\frac{\partial \Pi}{\partial x} = 0$$
, and $\frac{\partial \Pi}{\partial Q} = 0$. (3.4)

Also, for a global maximum of profit, the second order sufficiency conditions of optimality require that both $\frac{\partial^2 \Pi}{\partial x^2}$ and $\frac{\partial^2 \Pi}{\partial Q^2}$ are negative and $\frac{\partial^2 \Pi}{\partial x^2} \frac{\partial^2 \Pi}{\partial Q^2} - \left(\frac{\partial^2 \Pi}{\partial x \partial Q}\right)^2$ is positive (see Taha, 1992, p.718 for details). The detailed expressions of the necessary and sufficient conditions of optimality in conjunction with Expression (3.3) are found in Appendix C. The sensitivity analysis for a monopoly model is presented in the third section.

Duopoly Model

The model described in this section is an extension of the monopoly model introduced earlier and considers a market share attraction model in a duopoly market, with one advertising variable x_j . The market share of firm 1, $D(x_l)$, in a market of two competitors is expressed as follows:

$$D(x_1) = \frac{mf(x_1)}{f(x_1) + f(x_2)} = \frac{m\beta_1 x_1^{\delta_1}}{\beta_1 x_1^{\delta_1} + \beta_2 x_2^{\delta_2}} \text{ and } D(x_2) = m - D(x_1), \quad (3.5)$$

where *m* is the market potential and $f(x_1)$ and $f(x_2)$ are the attraction functions of firms 1 and 2, respectively. All rivals are assumed to have concave advertising attraction function, such that $f'(x_j) \ge 0$ and $f''(x_j) \le 0$ for $x_j \ge 0$. For all the attraction functions, the β 's and δ 's are strictly positive parameters, and for a concave function, $0 < \delta < 1$, as in the monopoly model. It is shown in Appendix D that Expression (3.5) represent the steady state sales response of a modified version of Kimball's (1957) competitive advertising model articulated by Little (1979). Similar to the monopoly model, designating the unit price, the marginal cost, the advertising cost function, the ordered quantity, the ordering cost per order, the inventory holding cost per unit held per unit time, and the fixed costs by P_j , MC_j , $C(x_j)$, Q_j , C_{oj} , C_{hj} , and F_j respectively, and introducing γ_j as the profit margin per unit sold ($P_j - MC_j$), the profit function, π_j of firm jtakes the form

$$\pi_{j} = \gamma_{j} D(x_{j}) - C(x_{j}) - \frac{Q_{j}}{2} C_{hj} - \frac{D(x_{j})}{Q_{j}} C_{oj} - F_{j}, \quad j = 1, 2.$$
(3.6)

As in the monopoly model, it is also assumed that the advertising cost function is convex and of the form $C(x_j) = d_j x_j^{\varepsilon_j}$, where d_j and ε_j are positive constants such that $d_j > 0$, and $\varepsilon_j > 1$ (Piconni and Olson, 1978) with underlying properties, $C'(x_j) \ge 0$ and $C''(x_j) \ge 0$, for $x_j \ge 0$. Defining $\prod_j = \pi_j / \gamma_j m$, for convenience of analysis, and substituting for $D(x_j)$, j = 1, 2, Expression (3.3) for a monopoly is adapted to a duopoly as shown below:

$$\Pi_{1} = \frac{f(x_{1})}{f(x_{1}) + f(x_{2})} \left[1 - \frac{C_{o1}}{\gamma_{1}Q_{1}} \right] - \frac{C(x_{1})}{\gamma_{1}m} - \frac{C_{h1}}{2\gamma_{1}m}Q_{1} - \frac{F_{1}}{\gamma_{1}m}.$$
(3.7)

$$\Pi_{2} = \frac{f(x_{2})}{f(x_{1}) + f(x_{2})} \left[1 - \frac{C_{o2}}{\gamma_{2}Q_{2}} \right] - \frac{C(x_{2})}{\gamma_{2}m} - \frac{C_{h2}}{2\gamma_{2}m}Q_{2} - \frac{F_{2}}{\gamma_{2}m}.$$
(3.8)

The analysis of this situation is based on Nash equilibrium solution concept of game theory. Researchers have widely used Nash equilibrium solution concept in an oligopoly (Mills, 1961; Mesak, 2003). The Nash equilibrium solution requires that no single rival unilaterally changes its advertising policy given the other rivals' optimal advertising policies. Since it is practically impossible for firms to monitor competitors' advertisement spending without time lags, and it takes time to change advertising budgets, a firm can expect a substantial time lag before any competitive reaction occurs.

This justifies the appropriateness of the Nash equilibrium solution concept (Schemalensee, 1976) and therefore is used in analyzing the model.

The first-order optimality conditions for an interior solution $(x_j^* > 0 \text{ and } Q_j^* > 0)$ implies that

$$\frac{\partial \Pi_j}{\partial x_j} = 0 \text{ and } \frac{\partial \Pi_j}{\partial Q_j} = 0, \ j = 1, 2.$$
 (3.9)

Also, to ensure that Expressions (3.9) indicate profit maximization, both the second-order derivatives $\partial^2 \Pi_j / \partial x_j^2$ and $\partial^2 \Pi_j / \partial Q_j^2$ must be negative and $\frac{\partial^2 \Pi_j}{\partial x_j^2} \frac{\partial^2 \Pi_j}{\partial Q_j^2} - \left(\frac{\partial^2 \Pi_j}{\partial x_j \partial Q_j}\right)^2$ must be positive for all *j*. A complete derivation of the first and second order conditions of optimality for firms 1 and 2 are presented in Appendix D. The

following section provides a brief derivation of the sensitivity analysis for both the monopoly and duopoly cases. Details are found in Appendices C and D.

Sensitivity Analysis

Sensitivity Analysis for a Monopoly

In this section, the impact of changes in each of the shift parameters γ , β , δ , m, d, ε , C_h , C_o , and e for a given firm on its optimal advertising expenditures x^* and optimal ordered quantity Q^* (self-sensitivity analysis) is derived theoretically. To study the sensitivity of x^* and Q^* to a change in one of the shift parameters θ , Equations (3.4) are partially differentiated with respect to θ then equated to zero afterwards to obtain

$$\frac{\partial^2 \Pi}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 \Pi}{\partial x \partial Q} \frac{\partial Q}{\partial \theta} + \frac{\partial^2 \Pi}{\partial x \partial \theta} = 0$$

and

$$\frac{\partial^2 \Pi}{\partial Q \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 \Pi}{\partial Q^2} \frac{\partial Q}{\partial \theta} + \frac{\partial^2 \Pi}{\partial Q \partial \theta} = 0.$$
(3.10)

Here, it is assumed that all second partial derivatives of Π with respect to each of the *equilibrium* values x^* and Q^* together with each of the *shift* parameters θ do exist and are continuous. The system of Equations (3.10) can be put in the following matrix form:

$$\begin{bmatrix} \frac{\partial^2 \Pi}{\partial x^2} & \frac{\partial^2 \Pi}{\partial x \partial Q} \\ \frac{\partial^2 \Pi}{\partial Q \partial x} & \frac{\partial^2 \Pi}{\partial Q^2} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial Q}{\partial \theta} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \Pi}{\partial x \partial \theta} \\ \frac{\partial^2 \Pi}{\partial Q \partial \theta} \end{bmatrix}.$$
(3.11)

Designating the 2 x 2 square matrix of the second-order partial derivatives in (3.11) by matrix **H**, which is assumed to be non-singular, and therefore, its inverse exists, Expression (3.11) can be written as

$$\begin{bmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial Q}{\partial \theta} \end{bmatrix} = -H^{I} \begin{bmatrix} \frac{\partial^{2} \Pi}{\partial x \partial \theta} \\ \frac{\partial^{2} \Pi}{\partial Q \partial \theta} \end{bmatrix}, \qquad (3.12)$$

and the elements of matrix **H** are obtained by partially differentiating (3.3). Complete derivations of those elements are presented in Appendix C. Upon applying (3.12) for a monopoly of profit maximizing behavior, Proposition 1 is introduced with related terms defined as shown below (see Appendix C for a proof):

$$a = \frac{ef'}{(e+f)^2} \frac{C_o}{\gamma Q^2} \text{ (positive term)}, \qquad (3.13)$$

$$b = \frac{e}{(e+f)^3} \left\{ f''(e+f) - 2f'^2 \right\} \left[1 - \frac{C_o}{\gamma Q} \right] - \frac{C''}{\gamma m} \text{ (negative term), and (3.14)}$$

$$\Delta = -\frac{2bf}{(e+f)} \frac{C_a}{\gamma Q^3} - a^2 \text{ (positive term).}$$
(3.15)

Proposition 1:

For a monopoly

- (i) An increase in parameter γ should be responded to by an increase in both the equilibrium advertising and ordered quantity.
- (ii) An increase in parameter C_h should be responded to by a decrease in both the equilibrium advertising and ordered quantity.
- (iii) An increase in parameter C_o should be responded to by a decrease in the equilibrium advertising and an increase (decrease) in the equilibrium ordered quantity if $\Delta a^2 > (<)0$.
- (iva) An increase in parameter β should be responded to by an increase (decrease)

in the equilibrium advertising if $\frac{C_o}{\gamma Q} < (>) \frac{2(f-e)}{2f-e}$ and an increase (decrease)

in the equilibrium ordered quantity if $\frac{C_o}{\gamma Q} < (>) \frac{2a^2(f-e)}{e\Delta + a^2(2f-e)}$.

- (ivb) An increase in parameter δ should be responded to by an increase (decrease) in the equilibrium advertising if $\frac{C_o}{\gamma Q} < (>) \frac{2\{(e+f)/\delta + (e-f)\ln x\}}{2(e+f)/\delta + (e-2f)\ln x}$ and an increase (decrease) in the equilibrium ordered quantity if $\frac{C_o}{\gamma Q} < (>) \frac{2a^2\{(e+f)/\delta + (e-f)\ln x\}}{2a^2(e+f)/\delta + (e-2f-e\Delta)\ln x}$.
- (v) An increase in either of the parameters d or ε should be responded to by a decrease in both the equilibrium advertising and ordered quantity.

(vi) An increase in parameter *e* should be responded to by an increase (decrease) in the equilibrium advertising if $\frac{C_o}{\gamma Q} < (>) \frac{2(f-e)}{2f-e}$ and an increase (decrease)

in the equilibrium ordered quantity if
$$\frac{C_a}{\gamma Q} < (>) \frac{2a^2(f-e)}{e\Delta + a^2(2f-e)}$$
.

(vii) An increase in parameter *m* should be responded to by an increase in both the equilibrium advertising and ordered quantity.

Part (i) of the above proposition suggests that as the advertising effectiveness increases, the firm should increase their advertising spending and ordered quantity to increase the profit. Part (ii) implies that as the inventory holding cost increases, the firm should reduce both advertising spending and ordered quantity to ensure optimal response. However, if the ordering cost increases as indicated in part (iii), a firm should always reduce the advertising spending. Though the direction of change in the ordered quantity will be an empirical question, the answer depends upon a combination of marketing and operations parameters together with the levels of the equilibrium advertising and ordered quantity themselves. Part (iva) and (ivb) imply that if either of the parameters associated with the advertising attraction function increases, the optimal response would be basically an empirical question, the answer of which would be governed by the related inequalities. On the other hand, if the parameters associated with the advertising cost functions change in such a way to increase advertising cost, part (v) indicates that optimality is insured by decreasing both advertising spending and ordered quantity. Similar to parts (iva) and (ivb), part (vi) suggests that if parameter e increases, optimal response would be an empirical question, the answer of which would be governed by the related inequalities. The situation associated with part (vii) argues that if the market potential increases, it

would be optimal for the firm to increase its advertising spending and ordered quantity to enhance its profitability.

Sensitivity Analysis for a Duopoly

This section provides a theoretical derivation of the impact of changes in each of the shift parameters, namely γ_j , β_j , δ_j , m, d_j , ε_j , C_{hj} , C_{oj} , for a given rival j on its optimal advertising expenditures x_j^* and optimal ordered quantity Q_j^* (self-sensitivity analysis) as well as on similar quantities related to the competitors (cross sensitivity analysis). In a symmetric duopoly, firms are as similar as possible in all economic respects that compete against each other for the same potential buyers in an industry producing a storable, homogeneous product. A symmetric competition also stipulates that rivals will have the same production cost, may acquire real promotion on the same terms, charge the same fixed price and face symmetric demand functions (Schemalensee, 1976). Therefore, all comparable shift parameters are assumed to be equal and firms are assumed to have equal market shares.

To study the sensitivity of each x_j^* and Q_j^* , j = 1, 2; to a change in one of the model parameters θ , Equations (3.9) are partially differentiated with respect to θ and then equated to zero to obtain

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} \frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_1} \frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial x_2} \frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_2} \frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} = 0 ,$$

$$\frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_1} \frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1^2} \frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_2} \frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial Q_2} \frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} = 0 ,$$

$$\frac{\partial^2 \Pi_2}{\partial x_2 \partial x_1} \frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial x_2 \partial Q_1} \frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_1 \partial x_2} \frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_1 \partial Q_2} \frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_1 \partial \theta} = 0 ,$$

and

$$\frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_1} \frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial Q_1} \frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_2} \frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2^2} \frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta} = 0.$$
(3.16)

Here, it is also assumed that all second partial derivatives of Π_j with respect to each of the *equilibrium* values x_j^* and Q_j^* together with each of the shift parameters θ do exist and are continuous. The system of Equations (3.16) can be put in the following matrix form:

$$\begin{bmatrix} \frac{\partial^{2}\Pi_{1}}{\partial x_{1}^{2}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial Q_{1}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial x_{2}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{1}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}^{2}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{2}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial x_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}^{2}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial Q_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}^{2}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial Q_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{2}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}^{2}} \end{bmatrix} \begin{bmatrix} \frac{\partial x_{1}}{\partial \theta} \\ \frac{\partial Q_{1}}{\partial \theta} \\ \frac{\partial Q_{1}}{\partial \theta} \\ \frac{\partial x_{2}}{\partial \theta} \\ \frac{\partial Q_{2}}{\partial \theta} \\ \frac{\partial Q_{2}}{\partial \theta} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial \theta} \\ \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial \theta} \\ \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial \theta} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial \theta} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial \theta} \end{bmatrix}.$$
(3.17)

Designating the 4 x 4 square matrix of the second-order partial derivatives in (3.17) by matrix **H**, Equation (3.17) can be rewritten as

$$\begin{bmatrix} \frac{\partial x_1}{\partial \theta} \\ \frac{\partial Q_1}{\partial \theta} \\ \frac{\partial x_2}{\partial \theta} \\ \frac{\partial Q_2}{\partial \theta} \end{bmatrix} = -H^{I} \begin{bmatrix} \frac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} \\ \frac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} \\ \frac{\partial^2 \Pi_2}{\partial x_2 \partial \theta} \\ \frac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta} \end{bmatrix}.$$
(3.18)

The derivations of the elements of matrix \mathbf{H} and all other terms related to Expression (3.18) are shown in Appendix D. It is also assumed that matrix \mathbf{H} is nonsingular, so that its inverse does exist. The model also assumes that the market potential m is constant. Gruca, Kumar, and Sudharshan (1992) asserted that negative

semi-definiteness of the square matrix \mathbf{H} is essential in ensuring uniqueness of the Nash equilibrium solution of the system of Equation (3.9). A complete proof of the negative semi-definiteness of matrix \mathbf{H} is provided in Appendix D.

Based on the solution of Equation (3.18) for a symmetric competitive EOQ inventory model with advertising-dependent demand, Proposition 2 is introduced with related terms defined as shown below (see Appendix D for a proof):

$$a = \frac{f'}{4f} \frac{C_o}{\gamma Q^2} \text{ (positive term)}, \tag{3.19}$$

$$b = \frac{\left(f''f - f'^2\right)}{4f^2} \left[1 - \frac{C_o}{\gamma Q}\right] - \frac{C''}{\gamma m} \text{ (negative term), and}$$
(3.20)

$$\Delta_1 = -\frac{bC_o}{\gamma Q^3} - a^2 \text{ (positive term).}$$
(3.21)

Proposition 2:

For a symmetric duopoly

- (i) An increase in the parameters γ , β , or δ of a rival should be responded to by an increase in the equilibrium advertising and ordered quantity of its own but a decrease in the equilibrium advertising and ordered quantity of its competitor.
- (ii) An increase in either of the parameters d or ε of a rival should be responded to by a decrease in the equilibrium advertising and ordered quantity of its own but an increase in the equilibrium advertising and ordered quantity of its competitor.
- (iii) An increase in parameter C_h of a rival should be responded to by a decrease in the equilibrium advertising and ordered quantity of its own but an increase in the equilibrium advertising and ordered quantity of its competitor.

- (iv) An increase in parameter C_o of a rival should be responded to by a decrease in the equilibrium advertising and an increase (decrease) in the equilibrium ordered quantity if $\Delta_1 - 2a^2 > (<)0$ of its own but an increase in the equilibrium advertising and ordered quantity of its competitor.
- (v) An increase in parameter *m* should be responded to by an increase in the equilibrium advertising and ordered quantity of both rivals.

Part (i) of the above proposition rejects the idea that improvements in the stated parameters of a firm would favor reducing its advertising spending and ordered quantity to achieve the same sales level as before and a simultaneous reduction in total inventory costs, as it would be suboptimal. The firm instead would make more profit by spending more on advertising and increasing its ordered quantity. Part (ii) implies that if the parameters associated with the advertising cost function increase, an optimal response for the firm is to decrease its advertising spending and ordered quantity to mitigate the impact on its advertising cost and reduce its total inventory costs. However, the rival should take advantage of the situation by increasing its advertising expenditure and ordered quantity to make more profit. Part (iii) indicates that if the inventory holding cost of a firm increases, the firm should decrease its advertising expenditure and ordered quantity for an optimal response. But it would be suboptimal for the rival not to take advantage of the situation by increasing its advertising expenditure and ordered quantity. However, if the ordering cost of a firm increases, as dictated in part (iv), the firm should respond by decreasing the advertising expenditure; but the optimal response in the ordered quantity will be dictated by other parameters. On the other hand, the rival's optimal policy will be to take advantage of the situation by increasing both the

advertising expenditure and ordered quantity. The case associated with part (v) argues that if the market size increases, it would be optimal for all competitors to increase their advertising spending and ordered quantity to increase their profit as supported by the empirical findings (Metwally, 1992). The optimal response for this case is dissimilar to the situations associated with parts (i) through (iv) of the proposition.

It is advantageous at this point to mention that parts (i), (ii) and (v) of Proposition 2 for a duopoly are consistent with the theoretical findings of Mesak (2003) who considered a symmetric oligopoly for which the number of competitors $n \ge 2$. Mesak (2003), however, has not considered inventory related costs in his modeling effort. It is conjectured here that the results depicted in Proposition 2 are generalized to an oligopoly of *n* competitors. The theoretical findings reported in the above proposition are derived under ideal symmetric conditions that almost never materialize in practice. The following two sections attempt to assess the robustness of the theoretical sensitivity results to deviations from these ideal conditions through examining the applicability of these results to a practical oligopoly setting. The next section estimates a market share attraction model in an oligopoly using actual data related to the United States brewing industry. The estimated parameters are then used in a following section to determine the elements of Expression (3.18), upon obtaining the equilibrium advertising and ordered quantity levels using Equation (3.9). Robustness is assessed by comparing the empirical sensitivity results obtained from applying Expression (3.18) with their theoretical counterparts highlighted in Proposition 2.

Estimation of a Market Share Attraction Model

The market share of firm j, $MSHR_j$, in a market of N competitors is expressed in Equation (3.22).

$$MSHR_{j} = \frac{f_{j}(x_{j})}{\sum_{i=1}^{N} f_{i}(x_{i})} = \frac{b_{j} x_{j}^{\delta_{j}}}{\sum_{i=1}^{N} b_{i} x_{i}^{\delta_{i}}} , \qquad (3.22)$$

where all parameters are defined in the second section. The market share attraction model for an oligopoly as given in Equation (3.22) is empirically estimated using actual data related to the United States brewing industry. Nelson (2005) provides firm specific sales and advertising data for Anheuser-Busch (firm 1), SAB-Miller (firm 2), Coors (firm 3) and an aggregate of other small brewers (firm 4). The same data set has been recently analyzed, but in a different research context, by Erickson (2009a). The current study employed data for the years 1994 through 2003. The raw data indicated an asymmetric nature of all the firms and yet each of them had a market share less than 50% for each year and further apart from one another making the data set suitable for examining the robustness of the theoretical sensitivity analysis associated with the symmetric competition as derived in the third section. Prior to analysis, the advertising expense data were converted to 1989 dollars. Defining,

 S_{jt} – cases in millions sold by firm j in year t,

 $MSHR_{jt}$ – market share of firm j in year t and

 x_{jt} – advertising in 1989 million dollars in year t for firm j.

The quantity x_{jt} is determined assuming t_0 be the year (1989 in this study) at which consumer price index = 100 and then defining

Advertising deflator in year t =

 $\frac{US \text{ population in year } t (18 \text{ years and older})}{US \text{ population in year } t_0 (18 \text{ years and older})} \times \text{consumer price index in year } t.$

Accordingly, advertising expenditure in year t for firm j, x_{jt} , expressed in t_0 dollars is given by:

Advertising in million dollars in year t for firm j/Advertising deflator in year t. Descriptive statistics of the data are shown in Table 3.1. The table indicates that all variables $MSHR_i$ and x_i assume considerable variability during the studied period.

Firm	Variable	Minimum	Average	Maximum
Anheuser-Busch	MSHR ₁	0.4421	0.4760	0.4962
	x_l^a	165.9807	214.6893	248.4003
	Q_I^{b}	204.7740	225.0320	253.9713
SAB-Miller	MSHR ₂	0.1843	0.2119	0.2379
	x_2^a	106.8985	158.8781	207.9111
	Q_2^{b}	101.8039	135.1450	152.6316
Coors	MSHR ₃	0.1014	0.1073	0.1134
	x_3^{a}	82.1813	103.7317	124.3615
	Q_3^{b}	48.7520	60.1266	86.6166
Others	MSHR₄	0.1947	0.2049	0.2224
	x_{4}^{a}	28.9650	96.8883	159.2536
	$Q_4^{\rm b}$	111.7474	130.2769	147.2750
Industry Sales ^b	S	2588.6	2708.24	2824.7

Table 3.1. Descriptive Statistics of Brewing Industry Data (1994 – 2003)

 $(1989 \ M)$, (Cases in Millions)

Also defining, $MSHR_t = (MSHR_{1t}, MSHR_{2t}, MSHR_{3t}, MSHR_{4t})^{1/4}$,

$$MSHR_{jt}^{*} = \ln (MSHR_{jt}/MSHR_{t}),$$

$$\widetilde{x}_{t} = (x_{1t}, x_{2t}, x_{3t}, x_{4t})^{1/4}, \text{ and}$$

$$x_{jt}^{*} = \ln (x_{jt}/\widetilde{x}_{t})$$

the market share attraction model (3.22) is estimated by ordinary least-squares regression after log-centering the data for each period *t* as defined above (Cooper and Nakanishi, 1998). The regression equation used to estimate the parameters of the market share attraction model is given by:

$$MSHR_{jl}^* = \alpha_1 + \alpha_2'd_2 + \alpha_3'd_3 + \alpha_4'd_4 + \delta x_{jl}^* + \varepsilon_{jl}^*,$$

where d_j is a dummy variable that takes on the value 1 for an observation related to firm jand 0 otherwise; $\alpha'_j = \alpha_j - \alpha_1$ for j = 2, 3, 4 (noting that $b_j = e^{\alpha_j}$ for all j); $\delta(=\delta_1 = \delta_2$ $= \delta_3 = \delta_4$) are parameters to be estimated; and ε^*_{ji} is an error term over j in period t.

Table 3.2 lists the parameter estimates and their statistical significance level (*p*-values) as obtained from linear regression. All parameters are statistically different from zero at better than the 0.05 level. The estimated model has face validity as the value of δ lies between 0 and 1 suggesting a concave advertising attraction function which is supported by the literature (Simon and Arndt, 1980; Friedman, 1983). Finally, the large value of R^2 indicates that the model fits the data fairly well.

Coefficient estimate of δ	0.081752 (0.0139)
Coefficient estimate of α_1	$0.746559 (3.5 \times 10^{-28})$
Coefficient estimate of α'_2	$-0.786013 (2.79 \times 10^{-27})$
Coefficient estimate of α'_3	-1.430452 (2.88x10 ⁻³²)
Coefficient estimate of α'_4	$-0.769772(1.54 \times 10^{-21})$
Number of observations	40
R^2	0.991965
	4

 Table 3.2. Regression Estimates of the Parameters Related to the Market Share

 Attraction Model

Note: *p*-values are in parentheses.

Derivation of Empirical Comparative Statics

The empirical comparative statics are derived by following the steps of the procedure described below:

- (i) All parameters associated with the market share attraction model (3.22) are estimated using information from Table 3.2.
- (ii) The parameters m, γ_j , advertising cost function $C(x_j)$, ordering cost C_{oj} , holding cost C_{hj} , and the order quantity Q_j variable are estimated. The above quantities are associated with the profit Functions (3.7 and 3.8), extended to four firms as shown below:

$$\Pi_{j} = \frac{f(x_{j})}{\sum_{i=1}^{4} f(x_{i})} \left[1 - \frac{C_{oj}}{\gamma_{j}Q_{j}} \right] - \frac{C(x_{j})}{\gamma_{j}m} - \frac{C_{hj}}{2\gamma_{j}m}Q_{j} - \frac{F_{j}}{\gamma_{j}m}, \qquad j = 1, 2, 3, 4.$$

The methods used to compute the quantities mentioned in step (ii) are illustrated in details later in this section. (iii) Noting that at optimality,

$$\frac{\partial \Pi_{j}}{\partial x_{j}} = \frac{f'(x_{j}) \sum_{i \neq j} f(x_{i})}{\left(\sum_{i} f(x_{i})\right)^{2}} \left[1 - \frac{C_{oj}}{\gamma_{j} Q_{j}}\right] - \frac{C'(x_{j})}{\gamma_{j} m} = 0 \text{ and}$$
$$\frac{\partial \Pi_{j}}{\partial Q_{j}} = \frac{f(x_{j})}{\sum_{i} f(x_{i})} \frac{C_{oj}}{\gamma_{j} Q_{j}^{2}} - \frac{C_{bj}}{2\gamma_{j} m} = 0.$$

Nash equilibrium x_j^* and Q_j^* is determined through numerically solving the system of eight simultaneous nonlinear equations given above.

- (iv) The elements of the Heissian matrix \mathbf{H}_{8x8} are computed which are similar to Expression (3.17), but extended to four firms. It should be noted that elements H_{jk} and H_{kj} need not be equal as these two terms belong to different rivals.
- (v) The entries of the post-multiplier column vector of Expression (3.18) extended to four firms is computed as follows:

(a)
$$\frac{\partial^{2} \Pi_{j}}{\partial x_{j} \partial \gamma_{j}} = \frac{f_{j}' \sum_{i \neq j} f_{i}}{\left(\sum_{i} f_{i}\right)^{2}} \left[\frac{C_{oj}}{Q_{j}} + \frac{C_{j}'}{m} \right] \frac{1}{\gamma_{j}^{2}} \text{ and } \frac{\partial^{2} \Pi_{j}}{\partial x_{j} \partial \gamma_{i}} = 0 \text{ for all } i \neq j.$$

(b)
$$\frac{\partial^{2} \Pi_{j}}{\partial Q_{j} \partial \gamma_{j}} = -\frac{f(x_{j})}{\sum_{i} f(x_{i})} \frac{C_{oj}}{\gamma_{j}^{2} Q_{j}^{2}} + \frac{C_{hj}}{2\gamma_{j}^{2} m} \text{ and } \frac{\partial^{2} \Pi_{j}}{\partial Q_{j} \partial \gamma_{i}} = 0 \text{ for all } i \neq j.$$

(c)
$$\frac{\partial^{2} \Pi_{j}}{\partial x_{j} \partial \beta_{j}} = \frac{\left(\sum_{i \neq j} f_{i}\right) \left(\sum_{i} f_{i} - 2f_{j}\right) \left(\partial f_{j}' / \partial \beta_{j}\right)}{\left(\sum_{i} f_{i}\right)^{3}} \left[1 - \frac{C_{oj}}{\gamma_{j} Q_{j}}\right] \text{ and } \frac{\left(\sum_{i} f_{i}\right)^{3}}{\left(\sum_{i} f_{i}\right)^{3}} \left[1 - \frac{C_{oj}}{\gamma_{j} Q_{j}}\right]$$

۰,

$$\frac{\partial^{2}\Pi_{j}}{\partial x_{j}\partial\beta_{i}} = \frac{\left(f_{j} - \sum_{i \neq j} f_{i}\right) (\partial f_{i} / \partial\beta_{i}) f_{j}'}{\left(\sum_{i} f_{i}\right)^{3}} \left[1 - \frac{C_{oj}}{\gamma_{j} Q_{j}}\right] \text{ for all } i \neq j.$$

$$(d) \frac{\partial^{2}\Pi_{j}}{\partial Q_{j} \partial\beta_{j}} = \frac{\left(\sum_{i \neq j} f_{i}\right) (\partial f_{i} / \partial\beta_{j})}{\left(\sum_{i} f_{i}\right)^{2}} \left[\frac{C_{oj}}{\gamma_{j} Q_{j}^{2}}\right] \text{ and}$$

$$\frac{\partial^{2}\Pi_{j}}{\partial Q_{j} \partial\beta_{i}} = -\frac{(\partial f_{i} / \partial\beta_{i}) f_{j}}{\left(\sum_{i} f_{i}\right)^{2}} \left[\frac{C_{oj}}{\gamma_{j} Q_{j}^{2}}\right] \text{ for all } i \neq j.$$

$$(e) \frac{\partial^{2}\Pi_{j}}{\partial x_{j} \partial\delta_{j}} = \frac{\left(\sum_{i \neq j} f_{i}\right) \left[\left(\sum_{i} f_{i}\right) (\partial f_{i} / \partial\delta_{j}) - 2f_{j}' (\partial f_{j} / \partial\delta_{j})\right]}{\left(\sum_{i} f_{i}\right)^{3}} \left[1 - \frac{C_{oj}}{\gamma_{j} Q_{j}}\right] \text{ and}$$

$$\frac{\partial^{2}\Pi_{j}}{\left(\sum_{i} f_{i}\right)^{3}} \left[\frac{f_{j} - \sum_{i} f_{i}}{\left(\sum_{i} f_{i}\right) (\partial f_{i} / \partial\delta_{i}) f_{j}'} \left[-C_{oj}\right]$$

$$\frac{\partial^2 \Pi_j}{\partial x_j \partial \delta_i} = \frac{\left(f_j - \sum_{i \neq j} f_i\right) (\partial f_i / \partial \delta_i) f_j'}{\left(\sum_i f_i\right)^3} \left[1 - \frac{C_{oj}}{\gamma_j Q_j}\right] \text{ for all } i \neq j.$$

(f)
$$\frac{\partial^2 \Pi_j}{\partial Q_j \partial \delta_j} = \frac{\left(\sum_{i \neq j} f_i\right) (\partial f_j / \partial \delta_j)}{\left(\sum_i f_i\right)^2} \left[\frac{C_{oj}}{\gamma_j Q_j^2}\right]$$
 and

$$\frac{\partial^2 \Pi_j}{\partial Q_j \partial \delta_i} = -\frac{\left(\partial f_i / \partial \delta_i\right) f_j}{\left(\sum_i f_i\right)^2} \left[\frac{C_{oj}}{\gamma_j Q_j^2}\right] \text{ for all } i \neq j.$$

(g)
$$\frac{\partial^2 \Pi_j}{\partial x_j \partial C_{hj}} = 0$$
 and $\frac{\partial^2 \Pi_j}{\partial x_j \partial C_{hi}} = 0$ for all $i \neq j$.

(h)
$$\frac{\partial^2 \Pi_j}{\partial Q_j \partial C_{hj}} = -\frac{1}{2\gamma_j m}$$
 and $\frac{\partial^2 \Pi_j}{\partial Q_j \partial C_{hi}} = 0$ for all $i \neq j$.
(i) $\frac{\partial^2 \Pi_j}{\partial x_j \partial C_{oj}} = -\frac{f'_j \left(\sum_{i \neq j} f_i\right)}{\left(\sum_i f_i\right)^2 \gamma_j Q_j}$ and $\frac{\partial^2 \Pi_j}{\partial x_j \partial C_{oi}} = 0$ for all $i \neq j$.
(j) $\frac{\partial^2 \Pi_j}{\partial Q_j \partial C_{oj}} = \frac{f_j}{\sum_i f_i} \left(\frac{1}{\gamma_j Q_j^2}\right)$ and $\frac{\partial^2 \Pi_j}{\partial Q_j \partial C_{oi}} = 0$ for all $i \neq j$.
(k) $\frac{\partial^2 \Pi_j}{\partial x_j \partial m} = \frac{C'_j}{\gamma_j m^2}$ and $\frac{\partial^2 \Pi_j}{\partial Q_j \partial m} = \frac{C_{hj}}{2\gamma_j m^2}$.

(vi) Expression (3.18) extended to four firms is applied in conjunction with the results obtained from steps (iv) and (v) to assess the signs of comparative statics.

For the current study, it is assumed for simplicity that the advertising cost function is linear, i.e., $C(x_j) = x_j$, thus, $C'_j = 1$ and $C''_j = 0$ for all *j*. The linearity of the advertising cost function in advertising spending has some empirical support (see Schmalensee, 1970, p. 232-233). After applying the above procedure, the results shown below were obtained.

From Tables 3.1 and 3.2, the following estimates were obtained:

m = 2708.24,	$\delta = 0.081752,$
$b_1 = e^{\alpha_1} = 2.109728$	$b_2 = e^{\alpha_1 + \alpha_2'} = 0.961314$
$b_3 = e^{\alpha_1 + \alpha'_3} = 0.504649$	$b_4 = e^{\alpha_1 + \alpha'_4} = 0.977054$

Operationalizing γ_j as the gross profit per case sold for firm *j* in year *t* expressed in 1989 dollars, this quantity was calculated for firms 1, 2 and 3 during the period 1994 – 2003 upon reviewing the annual published financial statements of the three companies.
For j = 1, 2 and 3, γ_j was computed as the average value of that quantity of firm j over the studied ten years. For firm 4, γ_4 was estimated as the average value of γ_1 , γ_2 and γ_3 . The obtained results are shown below:

$$\gamma_1 = 2.384, \quad \gamma_2 = 2.621, \quad \gamma_3 = 1.939, \quad \gamma_4 = 2.315.$$

Operationalizing C_{hjt} as the holding cost per year expressed in 1989 dollars, this quantity for j = 1, 2 and 3 in a given year t was computed as the product of the cost of case sold and the interest rate of 3 – months Treasury bill computed on annual basis (Irvine, 1981; Ramyantsev and Netessine, 2007). Following a similar procedure as that employed in computing the γ 's, the obtained results are shown below:

$$C_{h1} = 0.24, \quad C_{h2} = 0.21, \quad C_{h3} = 0.19, \quad C_{h4} = 0.21.$$

Operationalizing the ordered quantities Q_{jt} (in million of cases) as twice the number of cases sold for firm *j* divided by the annual inventory turnover in year *t*, Q_{1t} and Q_{3t} were computed as defined. Because inventory turnover figures were not available for the second and fourth companies, industry turnovers reported in different annual editions of *Almanac of Business and Industrial Financial Ratios* were used instead to calculate Q_{2t} and Q_{4t} . The calculations are conducted assuming that the assumptions of the economic order quantity (EOQ) model are applicable. The descriptive statics of variables Q_{jt} are depicted in Table 3.1.

Finally, the inverse optimization process (Troutt, 1995) was applied to provide estimates for C_{oj} , the per order cost of firm *j*. In this regard, the expression of the EOQ model for each firm *j* and year *t* given by $Q_{jt} = \sqrt{2D_{jt}C_{ojt}/C_{hjt}}$, where $D_{jt} = S_t MSHR_{jt}$, was used to obtain C_{ojt} . Upon averaging these quantities over the period 1994 – 2003, the estimates (in millions of 1989 dollars) are given below:

$$C_{o1} = 4.85, \quad C_{o2} = 3.6, \quad C_{o3} = 1.27, \quad C_{o4} = 3.39$$

Based on the above set of parameter estimates, Nash equilibrium advertising spending and optimal order quantity is obtained for all competitor according to step (iii) and is given below upon using MATLAB Version 6.5, Release 13:

 $x_1^* = 130.4319$, $x_2^* = 95.5651$, $x_3^* = 39.1818$, $x_4^* = 84.7214$,

$$Q_1^* = 227.873$$
, $Q_2^* = 139.8786$, $Q_3^* = 61.0179$, $Q_4^* = 136.1713$

Following step (iv), the Heissian matrix \mathbf{H} and its inverse \mathbf{H}^{-1} are obtained using the same above software as shown below:

	- 0.1200	0.0006	- 0.0031	0	- 0.0050	0	- 0.0035	0	
	0.0006	- 0.0160	- 0.0003	0	- 0.0004	0	- 0.0004	0	
	- 0.0003	0	- 0.1400	0.0010	- 0.0030	0	- 0.0021	0	
и_	- 0.0004	0	0.0010	-0.0210	- 0.0003	0	- 0.0003	0	Y 10 ⁻⁵
- 11	- 0.0003	0	- 0.0022	0	- 0.4500	0.0034	- 0.0025	0	N 10
	- 0.0005	0	- 0.0003	0	0.0034 -	0.0590	- 0.0004	0	
	- 0.0003	0	- 0.0021	0 -	-0.0035	0	- 0.1800	0.0013	
	- 0.0005	0	- 0.0003	0	- 0.0004	0	0.0013 ·	- 0.0250	

and

$$\mathbf{H}^{-1} = \begin{bmatrix} -0.8336 & -0.0319 & 0.0182 & 0.0009 & 0.0090 & 0.0005 & 0.0159 & 0.0008 \\ -0.0320 & -6.2512 & 0.0156 & 0.0007 & 0.0057 & 0.0003 & 0.0136 & 0.0007 \\ 0.0017 & 0.0001 & -0.7148 & -0.0340 & 0.0047 & 0.0003 & 0.0083 & 0.0004 \\ 0.0175 & 0.0007 & -0.0346 & -4.7635 & 0.0034 & 0.0002 & 0.0080 & 0.0004 \\ 0.0006 & 0.0000 & 0.0035 & 0.0002 & -0.2224 & -0.0128 & 0.0031 & 0.0002 \\ 0.0076 & 0.0003 & 0.0040 & 0.0002 & -0.0129 & -1.6957 & 0.0035 & 0.0002 \\ 0.0015 & 0.0001 & 0.0083 & 0.0004 & 0.0043 & 0.0002 & -0.5559 & -0.0282 \\ 0.0167 & 0.0006 & 0.0086 & 0.0004 & 0.0032 & 0.0002 & -0.0287 & -4.0015 \end{bmatrix}$$

Finally, upon applying Expression (3.18) extended to four competitors, the obtained comparative statics for different parameters θ are reported in Table 3.3.

	$\partial x^* / \partial \gamma_1$	$\partial x^* / \partial \gamma_2$	$\partial x^* / \partial \gamma_3$	$\partial x^* / \partial \gamma_4$	$\partial x * / \partial \beta_1$	$\partial x * / \partial \beta_2$	$\partial x^* / \partial \beta_3$
<i>θ</i> ℓ/ ¹ X	54.6849	-0.986	-0.8985	-1.1096	912.6154	-449	062-
$\frac{1}{x_2}$ / $\partial\theta$	-0.1113	38.8118	-0.4675	-0.5776	-18.1397	17604	-425
) _{x3} /∂θ	-0.0416	-0.1882	22.0816	-0.2132	-9.1984	-94	19363
$x_4/\partial\theta$	-0.0998	-0.4507	-0.4246	38.6936	-15.9863	-219	-386
	$\partial x^* / \partial \beta_4$	$\partial x * / \partial \delta_1$	$\partial x * / \partial \delta_2$	$\partial x * / \partial \delta_3$	$\partial x * / \partial \delta_4$	$\partial x * / \partial C_{h1}$	$\partial x * / \partial C_{h_2}$
ж, /д <i>Ө</i>	-437	1923.6	-50.4	-30.3329	-49	-2.4692	0.0608
)x ₂ /∂θ	-242	-172.5	1597.4	-41.6199	-89.9	0.005	-2.3951
) _{x3} /∂ <i>θ</i>	-92	-87	-43.2	658.3695	-43.1	0.0019	0.0116
$x_4/\partial\theta$	15182	-151	-79.4	-36.9584	1398	0.0045	0.0278
	$\partial x^*/\partial C_{h3}$	$\partial x * / \partial C_{h4}$	$\partial x * / \partial C_{ol}$	$\partial x * / \partial C_{o2}$	$\partial x * / \partial C_{o3}$	$\partial x * / \partial C_{o4}$	$\partial x^* / \partial m$
)x ₁ /∂ <i>θ</i>	0.0495	0.0646	-0.1196	0.0035	0.0071	0.0041	0.0454
$\frac{1}{x_2}$	0.0258	0.0336	0.0002	-0.139	0.0037	0.0021	0.0365
$\partial x_3 / \partial \theta$	-1.2166	0.0124	0.0001	0.0007	-0.1744	0.0008	0.0153
<i>x,1∂θ</i>	7200	-7 7573		0.0016	0.0024	0 144	0.0201

Table 3.3 Continued

Empirical Comparative Statics of Brewing Industry Data in the U.S.

	<i>эQ*/д</i> _/	$\partial Q^*/\partial \gamma_2$	$\partial Q^*/\partial \gamma_3$	$\partial Q*/\partial \gamma_4$	9 <i>Q</i> */∂β¦	$\partial Q^* / \partial \beta_2$	$\partial Q^* / \partial \beta_3$
<i>90</i> 1/3 <i>θ</i>	2.0996	-0.8471	-0.5664	-0.9438	90.9651	-396	-502
$\partial Q_2 / \partial heta$	-1.149	1.8755	-0.3338	-0.5537	-54.3778	925	-301
д <u>о</u> ,/дө	-0.5016	-0.216	1.2822	-0.2415	-37.4931	-111	1185
$\partial Q_4 / \partial \theta$	-1.0968	-0.4662	-0.316	1.9978	-51.1143	-226	-285
	$\partial Q^* / \partial \beta_4$	∂ <i>Q</i> * /∂δ¦	$\partial Q * / \partial \delta_2$	∂Q*/∂δ ₃	$\partial Q * / \partial \delta_4$	$\partial Q * / \partial C_{h1}$	$\partial Q * / \partial C_{h_2}$
90,/3 <i>0</i>	-382	380.5	-145.4	-59.6073	-138.3	-484.095	0.0523
$\partial Q_2 / \partial heta$	-234	-206.9	332.2	-37.5154	-89.2	0.0519	-335.54
<i>дQ</i> 3/д <i>θ</i>	-108	-91.9	-39.3	139.418	-39.1	0.0227	0.0133
$\partial Q_4 / \partial \theta$	853	-198.8	-85.6	-35.4519	308.1	0.0495	0.0288
	$\partial Q^*/\partial C_{h3}$	$\partial Q * / \partial C_{h4}$	$\partial Q * / \partial C_{ol}$	$\partial Q^* / \partial C_{o_2}$	$\partial Q^* / \partial C_{o3}$	$\partial Q^* / \partial C_{o4}$	$\partial Q^* / \partial m$
<i>ө</i> 0' / Э <i>Ө</i>	0.0312	0.0549	23.9329	0.003	0.0045	0.0035	0.0427
$\partial Q_2 / \partial \theta$	0.0184	0.0322	0.0025	19.5647	0.0026	0.0021	0.0261
д <u>о</u> ,/д <i>ө</i>	-161.452	0.0141	0.0011	0.0008	24.0577	0.000	0.0114
∂Q₄/∂ <i>θ</i>	0.0174	-319.116	0.0024	0.0017	0.0025	19.7526	0.0248

The table shows, without exception, that all of the 168 empirical comparative statics obtained possess signs that are consistent with their theoretical symmetric counterparts depicted in Proposition 2. Such a central finding is alarming provided that, for each year, the raw data summarized in Table 3.1 indicate that the market shares of competing firms were dispersed apart from each other.

Conclusion

This study involved theoretical derivation of comparative statics of a symmetric competitive inventory model with advertising-dependent demand associated with a market share attraction model of advertising competition. The validity and robustness of the theoretical model is empirically examined using the brewing industry data in the US for which competition is asymmetric. The results demonstrated remarkable consistency between the theoretical and empirical comparative statics. The validation procedure required estimation of the parameters of the market share attraction model (fourth section), which are used afterwards to operationalize Expression (3.18) extended to an oligopoly of four competitors. Then the signs of the empirical sensitivity results (fifth section) are compared with their theoretical counterparts involving symmetric competition in a duopoly (Proposition 2). The main conclusion drawn from the findings of the present study is of significant managerial appeal to both marketing practitioners and production managers. For an oligopolistic market that is not dominated by a single rival (largest market share is less than 50%), a firm would adjust its advertising spending and ordered quantity in response to changes in its own or competitor parameters in a manner consistent with the comparative statics reported in the proposition. Although the theoretical model was developed for a symmetric competition for simplicity, the

validation of the results using the empirical data from an asymmetric oligopolistic competition suggests that relaxing the assumption of symmetry for theoretical model development will make the analysis more complex without yielding much new insight. The estimated value of δ being less than 1 indicated a concave attraction function that favors constant advertising spending over time for all rivals (Mesak, 1999). While the comparative statics suggest how firms should respond to changes in model parameters, marketing practitioners as well as production/purchasing managers need to be aware of circumstances that may affect the model parameters. Notable examples are provided below.

The profit margin of a competitor, γ , could change due to the adoption of new technology, improved efficiency in production/operations, purchasing policies, etc. The parameters associated with the attraction function, namely β and δ of a competitor, can be affected in two distinct manners as identified by Little (1979). In the event a firm changes its advertising copy with the objective to communicate the same information better or provide a better demonstration of their product, parameter β is likely to be affected. However, if the goal is to reposition the product and appeal to a different group of consumers, parameter δ is likely to change. As mentioned in Mesak (2003), the rival's cost of advertising includes, in addition to the expenses of advertising messages, the opportunity cost of not utilizing such expenses in the rival's enterprise as well as the cost of obtaining the necessary funds to finance these expenses (Monahan, 1987). A change in the opportunity cost (perhaps due to changes in the rival's performance) could affect the *d* parameter. A change in the cost of funds (perhaps due to changes in the perceived risk of the rival by lenders) could influence the parameter ε . Stevenson (2008) mentions that

holding or carrying costs include opportunity costs associated with having funds which could be used elsewhere tied up in inventory, insurance, taxes (in some states), depreciation, obsolescence, deterioration, spoilage, pilferage, breakage, and warehouse costs (heat, light, rent, security). A change in one or more of the above cost items would result in a change of the C_o parameter. Also, the above reference mentions that ordering costs include determining how much is needed, preparing invoices, shipping costs, inspecting goods upon arrival for quality and quantity, and moving the goods to temporary storage. A change in one or more of the above cost items would cause a change in the C_h parameter. Finally, the market potential *m* can be affected by factors like population, average income per capita, and other factors that may influence the demand for the product (Lilien et al., 1992).

CHAPTER 4

OPTIMUM ADVERTISING POLICY OVER TIME FOR SUBSCRIBER SERVICE INNOVATIONS IN THE PRESENCE OF SERVICE COST LEARNING AND CUSTOMERS' DISADOPTION

Introduction

In the early 1900's, less than one third of the labor force in the United States was employed by the service sector. By 1950, the service industry employed almost half of the workforce. With the introduction of new services over the past three decades, such as cable TV, cell phones, internet, online banking/retailing, satellite radio, health clubs, etc., the service sector in the US currently employs most of the workforce, responsible for more than 80% of the gross domestic product, and growing considerably faster than the goods sector (US Department of Commerce, Bureau of Economic Analysis 2003; Zeithaml and Bitner, 2003).

Despite the growing role of the service sector and the declining role of the manufacturing sector in most Western societies (Metters and Marucheck, 2007), published research on the diffusion of new subscriber consumer service innovations remains scanty relative to the literature on new consumer durable (product) innovations (see Mahajan et al., 2000 and Meade and Islam, 2006 for further details). The importance of studying subscriber consumer services becomes self-evident in light of the fact that, at present, almost every US household is involved in one way or another, in

these services. The current study is applicable to service innovations that are made available through service providers to prospective consumers periodically at a subscription fee.

The modeling of diffusion of services provides unique challenges as stated by Libai et al. (2009a, p. 163):

A considerable influence on the market growth of a new service is customer attrition. Beginning with the initial stages of penetration into a market, there are customers who leave the service: They switch to competitors or, alternatively, leave the category. In this sense, the growth of a new service is similar to a leaking bucket – there is an inward flow of adopters and a concurrent outward flow of customers who leave.

Several researchers have modeled the diffusion of services as if they were durable goods, such as cell phones (Krishnan, Bass, and Kumar, 2000), landline phones (Jain, Mahajan, and Muller, 1991), and online banking (Hogan, Lemon, and Libai, 2003). In the literature, researchers often focused on cases when an exiting customer is acquired by a competitor (churning). However, customers can also disadopt and leave the service category altogether (Hogan et al., 2003). This is evidenced from the empirical research by Reichheld and Schefter (2000) and Meuter et al. (2005). Thus, attrition consists of both churning and disadopting customers and the attrition rate is the sum of the churn and disadoption rate (Libai et al., 2009a).

As mentioned above, this research is motivated by the recognition of the inherent differences between goods and services. The basic differences between services and goods are attributed to the assertion that services are intangible (cannot be seen, handled, smelled, etc.), heterogeneous (customized making its mass production difficult), produced and consumed simultaneously (lack of transportability), and perishable (unsold service on time is lost) (Zeithmal and Bitner, 2003). The above mentioned differences

between goods and services led Rust and Chung (2006, p.575) in their review article on service and relationships to advocate the use of *optimal control theory* as a viable tool to optimally manage the dynamic relationship with customers.

While optimal control theory has been applied within the context of diffusion of service innovations in a monopoly by a few researchers (e.g. Fruchter and Rao, 2001; Mesak and Darrat, 2002), none of them considered the effect of disadoption (termination of service subscription) on the optimal advertising policy of the firm.

Recent studies show that customer attrition, for which disadoption is a significant component, can have a considerable effect on growing markets (Hogan et al., 2003; Gupta et al., 2004; Libai et al., 2009a). To the best knowledge of the author, the study reported herein is the first attempt in the literature to investigate the impact of customers' disadoption on the optimal advertising policy of new subscriber services.

It is demonstrated in this paper that the advertising policy of the firm, in the presence of customers' disadoption, could be very different from the same when disadoption is ignored. The modeling effort employed in this study considers the learning cost curve and the discount rate. Subscriber services for which advertising is a main source of revenue instead of being an instrument for generating subscriptions such as newspapers, magazines and contemporary electronic media (internet websites) are beyond the scope of the present study. Interested readers are referred to Kumar and Sethi (2009) for a recent review of related literature.

Review of Related Marketing Literature

Notable examples of diffusion models for subscriber service innovations that have not considered marketing mix variables include Dodds (1973) who studied the diffusion of cable TV, Kim et al. (1995) who studied the diffusion of cell phones, Rai et al. (1998) who studied the diffusion of the internet and more recently Libai et al. (2009a, b) who studied cell phones, cable TV and online banking.

Subscriber services' studies that considered nonlinear pricing in the presence of network demand externalities (the benefits to a consumer increases with the number of other subscribers) include the works of Rohlfs (1974), Rabenau and Stahl (1974), and Dhebar and Oren (1985, 1986), and those that did not deal with demand externalities include the works of Maskin and Riley (1984), Wilson (1993), Fruchter and Rao (2001), and Danaher (2002). A nonlinear pricing includes an ongoing fee just to join (adopt) the service and usage fee such that heavy users pay more than light users.

Jain et al. (1999) examined the question of how the pricing of a complementary product such as the handset influences the pricing of a metered service, phone calls. They concluded that under certain cost conditions and competition in a two-period universe, the price of the handset decreases over time, whereas that of the calls is non-decreasing. Danaher et al. (2001) developed a model of first time sales and subscription for two generations of cellular phones in a European country. The authors found that, although the time path of estimated price elasticity in a multiple-generation setting closely follows those for a single generation, the interaction in price response across generations was significant. Mesak and Darrat (2002) developed and optimized diffusion systems for new subscriber consumer services. Each proposed system is composed of two differential equations, one for the retailer adoption process and another for the consumer adoption process that incorporate price. In a later empirical study (Mesak and Darrat, 2003), the authors concluded that a diffusion system with a two-way interaction between the two adoption processes performed slightly better than a single diffusion model of consumers that incorporate price.

The above literature review reveals that there are only two published studies that consider the disadoption rate in the diffusion of subscriber services. The first study (Libai et al., 2009a) does not consider any marketing variable in the modeling effort. The second study (Fruchter and Rao, 2001) considers price, but ignores service cost learning. The present paper considers both the disadoption rate and advertising in its modeling effort that also considers experience curves for subscriber services.

Review of Related Operations Literature

In this subsection a review related to the learning cost curves relevant to the scope and the purposes of the paper is presented.

The organizational learning curve refers to the increases in productivity that are obtained as organizations gain production experience (Pati and Reis, 2007). Operations literature usually associates learning curve phenomena with continuous improvement and quality management in processes (Adler and Clark, 1991; Zangwill and Kantor, 1998). Wright (1936) observed that in the production of airplanes the hours required to manufacture a unit falls at a constant rate as the number of units produced rises. Organizational learning curves exhibiting this inverse log-linear relationship between productivity improvements and cumulative experience have since been observed in a wide variety of industries (Argote and Epple, 1990; Conway and Schultz, 1959; Hirschmann, 1964; Yelle, 1979). As shown below, experience curves were found applicable in service operations.

Darr et al. (1995) evaluated the learning curve effects in a chain of 36 pizza stores using data collected over a year and half. They found that experience gained through "learning by doing" led to significant reductions in unit cost.

Goodale and Tunc (1998) explored the labor scheduling problems in service operations utilizing dynamic service rates based on the learning curve theory. They found that learning curve methodologies predominantly applied in manufacturing environments performed well in a customer contact environment with both part-time and full time workers.

Chambers and Johnston (2000) applied the experience curve in two very different service organizations. The first case showed how an experience curve has been calculated at a macro (organization) level for British Airways over a 20 year period, including the time at which it was privatized. The second example showed an application over the first year of operation of a high-volume paperwork processing operation within a financial service organization. The authors concluded that their studies demonstrated that experience curves can be applied to a great effect in high volume service organizations.

Boone et al. (2008) conducted an empirical study based on seven years of project data collected from an architectural engineering firm. The analysis showed professional service display learning curves.

In this study the marketing variable of advertising is incorporated into the diffusion model of services articulated by Libai et al. (2009a) that explicitly considers the disadoption rate. The related measure of performance that takes into account the cost learning curve is optimized afterwards using optimal control theory.

The rest of the study is organized as follows. The second section outlines a general dynamic diffusion model for new subscriber service innovations, formulates the problem and presents the solution method. The third section characterizes the optimal advertising policy. The fourth section is empirical in orientation and aims at estimating and choosing among alternative proposed diffusion models. The last section summarizes and concludes the paper. To improve exposition, derivation of key formulas and proofs of all propositions are relegated to a separate Appendix (Appendix E).

General Model Formulation and Solution Method

Let us consider adoptions of a new subscriber service in a monopolistic market. A firm manipulates its advertising expenditure U_t (assumed to be bounded from above) at each time t over a fixed planning period T, $0 \le t \le T$. The monopoly assumption may seem reasonable in situations in which the firm enjoys a patent protection, a proprietary technology, or a dominant market share. A general diffusion model is given by:

$$dN_t/dt = N'_t = f(N_t, U_t), N_0 > 0 \text{ and fixed},$$
 (4.1)

where, N_t and N'_t represent the number of subscribers by time *t* and the subscription rate respectively. Equation (4.1) suggests that the concurrent subscription rate is related to the current number of subscribers and the current rate of the marketing variable. Function *f* is assumed to be twice differentiable with the following properties related to the marketing variable where a subscript on a variable denotes partial differentiation with respect to that variable:

$$f \ge 0; f_U > 0; \text{ and } f_{UU} < 0.$$
 (4.2)

The inequalities (4.2) imply that the subscription rate is non-negative, decreases with an increase in the subscription charge, increases with an increase in advertising and is concave in the marketing variable (diminishing returns' phenomenon).

A cost learning curve is introduced next by assuming that marginal costs, denoted by C, depend on the number of subscribers such that marginal costs decrease with increasing the number of subscribers (experience),

$$C_t = C(N_t), \, dC(N_t)/dN_t = C'(N_t) \le 0.$$
 (4.3)

Note that marginal cost could be constant (C' = 0). C_t is mainly a function of efforts related to service activation (e.g. installation) and account maintenance (e.g. billing, computer service space, and help provided by the firm).

For a firm charging a fixed service fee rate P that aims to find the optimum trajectory U_t^* to maximize the discounted profit stream over the planning period T, the problem is formulated as follows for a discount rate $r \ge 0$: Find trajectory U_t^* to

Maximize
$$\int_{0}^{T} \left[\left(P - C(N_t) \right) N_t - E \frac{dN}{dt} - Q(U_t) \right] e^{-tt} dt .$$
 (4.4)

Subject to $\frac{dN(t)}{dt} = f(N_t, U_t)$ and the initial number of subscribers $N(0) \ge 0$ is fixed and N_T is free.

In Expression (4.4), PN_t represents the total revenue generated from subscribers N_t and $C(N_t) N_t$ is the related total variable cost. $E(dN_t/dt)$, as in Kim et al. (1995), stands for the cost associated with recruiting new subscribers (dN_t/dt) , so that constant E

represents the average incremental cost per additional new subscriber. For example, cellular phone firms have been willing to pay high commissions to distributors and have often given away hardware (handsets) in exchange for contractual agreements with new subscribers. In Expression (4.4), $Q(U_t)$ is the advertising cost function assumed to be non-negative and convex with respect to its argument with the properties Q' > 0, $Q'' \ge 0$ (Piconni and Olson, 1978). As in earlier monopolistic models (Dockner and Jorgensen, 1988a; Thompson and Teng, 1984; Kalish, 1985), no salvage value was assumed for the final number of subscribers at time *T*. Dockner and Jorgensen (1988b) assert that this assumption is particularly plausible when the firm is more concerned with its profit stream over the planning period than profit to be made after time *T*. This assumption also makes results for new subscriber services readily comparable with their counterparts related to new products.

The optimal control problem (4.4) can be solved by applying Pontryagin's maximum principle optimization technique (Pontryagin, 1962). To apply the maximum principle, the current value Hamiltonian (Sethi and Thompson, 1981) takes the form shown below:

$$H_t(N_t, U_t) = (P - C(N_t)) N_t - Q(U_t) + (\lambda_t - E) f(N_t, U_t),$$
(4.5)

where λ_t is a costate variable that must satisfy the ensuing equation:

$$\frac{d\lambda}{dt} = r\lambda - \frac{\partial H}{\partial N}, \ \lambda(T) = 0.$$
(4.6)

An economic interpretation of λ_t is found in Sethi and Thompson (1981). Briefly, λ_t , has the interpretation of a *shadow price* of the stock of subscribers N_t . In this paper, as in Dockner and Jorgensen (1988a), the considered admissible controls are twice differentiable in *t* and satisfy $U_t \ge 0$ for all relevant *t*. (In what follows, the time argument is eliminated to minimize confusion and improve clarity). Confining the interest to admissible controls, the partial derivatives of the current value Hamiltonian with respect to *U* along the optimal trajectory, as in Feichtinger (1982), must satisfy the following condition for an interior solution for which $0 \le U \le U \le \overline{U}$:

$$H_U = \frac{\partial H}{\partial U} = 0.$$
(4.7)

where \underline{U} and \overline{U} are the lower and upper bounds of U,

and
$$H_{UU} = \frac{\partial^2 H}{\partial U^2} < 0$$
. (4.8)

The above second-order condition $(H_{UU} < 0)$ is one of several sufficiency conditions of optimality implying that the Hamiltonian *H* is jointly concave in the control variable *U* and the state variable *N* (Seierstad and Sydsaeter, 1977). If a sufficiency condition is violated (e.g. $H_{UU} > 0$), the optimal advertising policy would be either the constant \underline{U} or the constant \overline{U} . By substituting into the Hamiltonian *H*, the constant that maximizes it would be chosen as U_t^* (Teng and Thompson, 1985, p. 192)

Optimal Advertising Policy for New Subscriber Services

(4.1) followed by an analysis related to specific diffusion models for subscriber services.

Analysis of the General Diffusion Model

Using Condition (4.7) in conjunction with Expressions (4.5), (4.6) and (4.8), it is shown in the Appendix that the first derivative of the optimal trajectory U^* with respect to time *t* is uniquely determined by the following equation:

$$-\frac{\partial^2 H}{\partial U^2}\frac{dU}{dt} = -\frac{Q'}{f_U}(f_N f_U - f_{UN} f) - f_U \{P - C - NC'\} + r\lambda f_U. \quad (4.9)$$

The optimum advertising trajectory is difficult to characterize using the general Expression (4.9) because several terms are of opposite as well as ambiguous signs. Therefore, the expression will be analyzed in conjunction with specific plausible cases. On the basis of the general Expression (4.9), two propositions are introduced for which the proofs are found in Appendix E.

Proposition 1:

For a low interest rate such that advertising elasticity of demand is a nonincreasing function of penetration, optimal advertising is decreasing over time in the presence of a service cost learning curve.

Proposition 1 implies from Expression (4.9) that when advertising elasticity of demand $\frac{\partial f}{\partial U} \left(\frac{U}{f} \right)$ is non-increasing in penetration *N*, it would be sufficient for marginal revenue with respect to an additional subscriber, or $\partial (PN)/\partial N$, to be larger than marginal cost with respect to the same, or $\partial (CN)/\partial N$, for optimal advertising to be decreasing over time. Continuous improvement of the quality of subscriber services is the rule rather than the exception. For example, a provider for cable TV is inclined to modify the service (cable channels) to meet consumer needs better (Gatignon and Robertson, 1985). Such continuous improvement in quality could be very well responsible for the decline in advertising elasticity. Declining advertising elasticities over the product life cycle have been observed in the empirical studies of Parsons (1975), Arora (1979), and Mesak and Clark (1998).

Proposition 2:

For a large interest rate, optimal advertising increases over time in the presence of a service cost learning curve.

Proposition 2 implies that a myopic service provider interested in maximizing short-term profit because of high uncertainty about the market (Bayus, 1994) would be inclined to increase advertising over time. The proposition asserts that the relationship between advertising and penetration would be approximately linear. The slope of this linear relationship is the interest rate r. Since Expression (4.9) does not clearly characterize the trajectory of the marketing variable U_t^* , several diffusion models with specified functional forms are analyzed next.

Analysis of Specific Diffusion Models

The original Bass model (1969) as modified by Libai et al. (2009a) to represent the diffusion of new subscriber services in continuous time that explicitly considers customers' disadoption is given by Equation (4.10):

$$\frac{dN_{t}}{dt} = p[M - N_{t}] + \frac{q(1 - \delta)N_{t}}{M}[M - N_{t}] - \delta N_{t} , \qquad (4.10)$$

where N_t is the number of subscribers by time t, M is the market potential, p is the coefficient of innovation, q is the coefficient of imitation, and δ is the disadoption rate. The above authors assumed that only those who did not disadopt spread positive word-of-mouth communications about the service. Therefore, the level of word-of-mouth promotion by retained customers remained the same (q), but its effective impact was reduced due to disadoption from $[qN_t]/M$ to $[q(1-\delta)N_t]/M$. As disadopters return to the market potential, the remaining market potential is $M - N_t$ and is not affected by the disadoption process (Libai et al. 2009a). Using diffusion data related to cell phones, cable

television and online banking in the US, Libai et al. (2009a) show that their service diffusion model is empirically appealing for the above service categories. The authors also illustrate that the following relationship among the parameters of their model holds: $q(1 - \delta) - p - \delta > 0.$

To incorporate advertising into diffusion models of new products, Horsky and Simon (1983) consider the coefficient of innovation p to be a function of advertising whereas Simon and Sebastian (1987) consider the coefficient of imitation q to be a function of advertising. Dockner and Jorgensen (1988a), on the other hand, consider advertising to affect both of the coefficients p and q. Employing an advertising efficiency function h(U) of the properties h > 0, $h_U = h' > 0$ and $h_{UU} = h'' < 0$ and following the above literature, Table 4.1 analyzes three alternative diffusion model specifications of subscriber services.

Table 4.1. Analysis of Alternative Model Specifications

M1	$f = ph(M - N) + q(1 - \delta)(N/M)(M - N) - \delta N$	$p(h'/M) [q(1 - \delta)(M - N)^2 - \delta M^2]$	-ph'
M2 _	$f = pM - (p + \delta)N + qh(1 - \delta)(N/M)(M - N)$	$-q(1-\delta)(h'/M) [p(M-N)^2 + \delta N^2]$	$qh'(1-\delta)(M-2N)/M$
M3 <i>j</i>	$f = ph(M - N) + qh(1 - \delta)(N/M)(M - N) - \delta N$	$-\delta(h'/M)(pM^2+q\delta N^2)$	$\frac{h'[-p+q(1-\delta)(M-2N)/M}{2N}$

p, q, M and δ are constant parameters, and $q(1 - \delta) - p - \delta > 0$

In model M1, advertising affects the coefficient of innovation. In model M2, advertising affects the coefficient of imitation. In model M3, advertising affects both of the above coefficients. Major investment decisions are typically made during the earliest phases of product/service launch (Parker, 1994, p. 364) for which $f_N > 0$. Dockner and Jorgensen (1988a, p. 123) assert that for a diffusion model that includes advertising to be

theoretically plausible, whenever $f_N > 0$ then f_{UN} should be negative. The properties $f_N > 0$ and $f_{UN} < 0$ mean that the number of new subscribers increases with penetration N and such an increase is weaker for higher levels of advertising (diminishing returns' phenomenon). A review of Table 4.1 indicates that only model M1 is appealing on theoretical grounds ($f_{UN} < 0$ at N = 0). In addition, models M2 and M3 are analytically intractable. As for such models, the first and second terms in (4.9) are of opposite signs. Therefore, model M1 will be the only model that will be considered for further analytical examination in this section. Its performance, however, will be empirically compared in the next section to models M2 and M3 together with the Bass (1969) model. The original Bass model does not incorporate marketing-mix variables and does not consider customers' disadoption. It is a special case of model (4.10) for which the disadoption rate $\delta = 0$.

Upon analyzing Expression (4.9) in conjunction with model M1 depicted in Table 4.1, two propositions are introduced below for which the proofs are found in Appendix E. *Proposition 3:*

For a low interest rate and the service diffusion model

dN/dt = ph(M - N) + q(N/M)(M - N), optimal advertising decreases over time in the presence of a service cost learning curve.

Proposition 3 implies that the optimal advertising policy is to advertise heavily when the service is introduced and then to reduce the level of advertising as the number of subscribers increases and the service moves through its life cycle.

Proposition 4:

For a low interest rate, service diffusion model

 $dN/dt = ph(M - N) + q(1 - \delta)(N/M)(M - N) - \delta N$ and presence of a service cost learning, optimal advertising is decreasing (or decreasing first then increasing later) for a penetration level N smaller or equal to (larger than) $M[1 - \sqrt{\delta/\{q(1-\delta)\}}]$.

Proposition 4 implies that the optimal advertising policy of the service firm in the presence of customers' disadoption could be different from its counterpart in the absence of the same (Proposition 3). In the presence of disadoption, advertising could be increasing over time no sooner than the time corresponding to the level of penetration N specified in the above proposition. It is noted that the specified value for N in Proposition 4 is positive as $\delta < q(1 - \delta)$ by Table 4.1 footnote. This inequality is confirmed later empirically by the findings of Table 4.3.

Empirical Analysis

Description of Data

Since the empirical analysis involves the estimation of several diffusion parameters, this required a comprehensive set of historical subscriber data along with advertising expenses. We were also interested in an industry where disadoption plays a significant role while the market is monopolistic at the same time. A plausible example of an industry with these attributes is cable TV. A cable TV service provider assumes a monopoly in the area in which it operates. It assembles selected cable channels into certain packages and makes them available to potential subscribers for a subscription fee. The selection of channels offered by a provider generally starts with a "basic" package for a fixed monthly charge. This package typically includes local and several distant broadcast stations. The "premium" package such as HBO, Cinemax, etc. are offered as optional packages for additional monthly charges. However, since the subscriptions to these premium packages require subscription to the basic service first, the empirical analysis reported herein is confined to the basic service only. In addition to the availability of relevant data, basic cable TV service was chosen in the empirical analysis as it represents an example of a service innovation that comprises solely the line of business of each service provider and is acquired only once per household (subscriber).

Aggregate annual data are available for Canada at the industry level (National and provincial) in terms of the number of subscribers (N_t) and advertising expense (U_t) . Following a similar mathematical approach as that employed in Mesak and Darrat (2003), it can be shown that if the consumer adoption process in each geographic area is represented by a Libai et al. (2009a) model, the consumer adoption process at the aggregate sub-national and national levels for the cable TV industry can be also represented by a Libai et al. (2009a) model. The data related to the cable industry in Canada and its provincial regions (Ouebec, Ontario and Nova Scotia & New Brunswick) were available from *Statistics Canada*. The annual data related to Canada and its three provincial regions covered the period 1976-1994, and as the record indicated the data were available as early as 1976. In 1995, a change in the definition of number of subscribers affected the count of subscribers and therefore, data from 1995 and beyond were not comparable with previous periods. These data are provided in Appendix F. As the data indicate, new subscribers, $n_t = N_t - N_{t-1}$, showed considerable variability over time. Such variability is important to distinguish between alternative model specifications.

Estimation and Model Selection Procedure

The discrete analogues of the Bass (1969) model, obtained from (4.10) upon substituting $\delta = 0$, together with the modified Libai et al. (2009a) models M1, M2 and M3 depicted in Table 4.1 are estimated and compared in this section. The models to be estimated take on the following forms:

$$N_{t} - N_{t-1} = p(M - N_{t-1}) + q \frac{(M - N_{t-1})N_{t-1}}{M}$$
 [Bass model]. (4.11)

$$N_{t} - N_{t-1} = p(M - N_{t-1})\ln(U_{t}) + q(1 - \delta)\frac{(M - N_{t-1})N_{t-1}}{M} - \delta N_{t-1}$$
[Model M1]. (4.12)

$$N_{t} - N_{t-1} = pM - (p+\delta)N_{t-1} + q(1-\delta)\frac{(M-N_{t-1})N_{t-1}}{M}\ln(U_{t}) \text{ [Model M2].}$$
(4.13)

$$N_{t} - N_{t-1} = p(M - N_{t-1})\ln(U_{t}) + q(1 - \delta)\frac{(M - N_{t-1})N_{t-1}}{M}\ln(U_{t}) - \delta N_{t-1}$$
[Model M3]. (4.14)

In models (4.11) through (4.14), as in the study of Horsky and Simon (1983), an advertising efficiency function of the form $h(U_t) = ln(U_t)$ is employed. The models (4.11) through (4.14) are nonlinear in the parameters p, q, M and δ Although nonlinear estimation procedures could be employed, such procedures entail the identification of initial values for the parameters to be estimated that could be a problematic task given that such parameters are unknown to begin with (Venkatesan, Krishnan and Kumar, 2004). Putsis and Srinivasan (2000, p. 267) mention that in practice, multiple runs from different starting values would be desirable. However, in many applications and for certain model specifications, it may be difficult or impossible to obtain convergence. Further, Van den Bulte and Lilien (1997) warn that nonlinear estimation procedures usually arrive at an estimate of market potential M that is close enough to the actual cumulative number of adopters at the end of the time series data of the diffusion process.

In this regard, Heeler and Hustad (1980) and Tigert and Farivar (1981) reported significant improvement in the quality of forecasting when using their diffusion model upon constraining the value of M to an intuitive estimate. In addition, Van den Bulte and Lilien (1997) suggest using an exogenous ceiling M to linearize the nonlinear regression equation to arrive at higher quality estimates of the parameters.

Initially, SAS nonlinear estimation procedure MARQUARDT was used to estimate models (4.11) through (4.14) that are nonlinear in the parameters using the available data sets without much success. In several runs all the estimated parameters were statistically insignificant and in others the procedure failed to attain convergence. Therefore, models (4.11) through (4.14) were linearized through assigning plausible values for the market potential M to each of the regions in Canada. To estimate model (4.11) using multiple regression analysis, the following variables were defined: $Y_t = N_t N_{t-1}$, $X_{1t} = M - N_{t-1}$ and $X_{2t} = (N_{t-1}/M) (M - N_{t-1})$. The estimates of the coefficients X_1 and X_2 provide the estimates for parameters p and q, respectively.

To estimate model (4.12) using multiple regression analysis, the following variables were defined: $Y_t = N_t - N_{t-1}$, $X_{1t} = (M - N_{t-1}) \ln(U_t)$, $X_{2t} = (N_{t-1}/M) (M - N_{t-1})$ and $X_{3t} = N_{t-1}$. Accordingly, for model (4.12) parameter δ estimate was set as the negative of the estimated coefficient of X_3 , parameter q estimate was equated to the estimated coefficient of X_2 divided by the estimate of $(1 - \delta)$. Finally, parameter p estimate was set

To estimate model (4.13) using multiple regression analysis, the following variables were defined: $Y_t = N_t - N_{t-1}$, $X_{1t} = N_{t-1}$, $X_{2t} = (N_{t-1}/M) (M - N_{t-1}) ln(U_t)$. The estimate of δ was given by subtracting the estimate of p from negative the estimate of

coefficient X_1 . Finally, the estimate of q was obtained as the estimate of the coefficient of X_2 divided by the estimate of $(1 - \delta)$.

To estimate model (4.14) using multiple regression analysis, the following variables were defined: $Y_t = N_t - N_{t-1}$, $X_{1t} = (M - N_{t-1}) \ln(U_t)$, $X_{2t} = (N_{t-1}/M) (M - N_{t-1}) \ln(U_t)$ and $X_{3t} = N_{t-1}$. The estimate of p was equated to the estimated coefficient of X_1 . The estimate of q was obtained as the estimate of the coefficient of X_2 divided by the estimate of $(1 - \delta)$. The estimate of δ was given by the negative of the estimate of variable X_3 coefficient.

The empirical literature suggests that a reasonable metric for judging rival models is their out-of-sample forecastability (Zanias, 1994; Pindyck and Rubinfeld, 1998). Bottomley and Fildes (1998) contend that goodness of fit raises issues regarding the robustness of the findings when models are estimated and "fitted" using much shorter data series as typifies new product/service forecasting applications. In addition, evidence in the literature (Rao, 1985; Young, 1993) demonstrates the lack of correlation between a model's fit and its forecasting performance. Therefore, the selection among models (4.11) through (4.14) is carried out according to a two-stage screening process. Only models of reasonable explanatory power (significance of at least one coefficient with the correct sign and insignificance of any coefficients with incorrect signs) are considered eligible candidates for the second stage. In the second stage, eligible models are compared based on their predictive power (measured by Theil's coefficient *UII*).

In computing coefficient *UII* for a given model and a certain region, the related data set is partitioned into two parts. The first large part (13 observations) is used to estimate the model, whereas the second smaller part (5 observations) of the data set

(holdout sample) is used to compare actual with the out-of-sample predicted values. Using one-step ahead forecasting process, the model is first fitted for 13 observations using ordinary least squares (OLS), then a new subscriber forecast $Y_{14} = N_{14} - N_{13}$ is obtained. The model is reestimated afterwards using 14 observations, and a forecast $Y_{15} =$ $N_{15} - N_{14}$ is obtained and so on. Advertising during a certain year is used to obtain new subscribers' forecast for the same year, as appropriate. Coefficient *UII* (Theil, 1965, p. 28) is computed using the expression given below:

$$UII = \frac{\left[\sum_{i=1}^{T} (A_i - PR_i)^2\right]^{1/2}}{\left[\sum_{i=1}^{T} A_i^2\right]^{1/2}},$$
(4.15)

where (A_i, PR_i) stand for a pair of an actual and predicted Y values for year *i*, and T represents the number of periods in the holdout sample (five years in this case). Finally, the selected service diffusion model is that eligible model with the smallest *UII* coefficient.

Estimation Results

The variables N_t , U_t and the parameter M were operationalized as shown below.

 N_t = Number of cable subscribers to the basic service by year *t* (in thousands).

 $U_t = A_t/A_l$, where A_t is advertising expenditure of cable TV (in thousands of Canadian dollars) divided by consumer price index (CPI) in year t and A_l is the advertising expenditure of cable TV (in thousands of Canadian dollars) divided by CPI in the first year of a given time series, so that $U_l = 1$.

M = Number of households wired for cable access in 2007 (in thousands).

After adding error terms to expressions (4.11) through (4.14), Tables 4.2 to 4.5 summarize the estimation results upon using the ordinary least squares (OLS) method of the SPSS software package.

Region	Year	р	q	M (in thousands)	SSE (in millions)	UII
Canada	1976-1994	0.027 (0.0005)	0.0226 (0.16)	12592.8	40.544	0.2554
Quebec	As above	0.0216 (0.0577)	0.034 (0.282)	3243.4	14.401	0.4960
Ontario	As above	0.015 (0.32)	0.0468 (0.16)	4605.7	16.586	
Nova Scotia & New Brunswick	As above	0.035 (0.006)	0.042 (0.164)	628.69	0.642	0.3439

Table 4.2. Estimation of Parameters for Bass Model (4.11)

Note: *p*-values are in parentheses.

Region	Year	p	$q(1-\delta)$	δ	M (in thousands)	SSE (in millions)	UII
Canada	1976- 1994	0.039 (0.002)	0.242 (3.9x10 ⁻⁸)	0.094 (1.18 x 10 ⁻⁵)	12592.8	21.287	0.2451
Quebec	As above	0.043 (0.013)	0.33 (5.05 x 10 ⁻⁵)	0.179 (0.001)	3243.4	8.669	0.4872
Ontario	As above	0.064 (0.029)	0.19 (0.0003)	0.064 (0.013)	4605.7	11.486	0.2152
Nova Scotia & New Brunswick	As above	0.0798 (0.002)	0.326 (2.84 x 10 ⁻⁶)	0.089 (0.0005)	628.69	0.428	0.1768

Table 4.3. Estimation of Parameters of Expression (4.12) Model

Note: *p*-values are in parentheses.

Region	Year	рМ	$p + \delta$	$q(1 - \delta)$	SSE (in millions)	UII
Canada	1976-1994	555.039 (3.4 x 10 ⁻⁶)	0.064 (0.003)	0.069 (0.035)	30.86	0.3173
Quebec	As above	202.126 (0.0005)	0.177 (0.01)	0.158 (0.014)	9.737	0.4782
Ontario	As above	199.095 (0.002)	0.06 (0.046)	0.113 (0.078)	13.842	0.3561
Nova Scotia & New Brunswick	As above	30.057 (1 x 10 ⁻⁵)	0.038 (0.023)	0.092 (0.072)	0.506	0.4639

Table 4.4. Estimation of Parameters of Expression (4.13) Model

Note: *p*-values are in parentheses.

Region	Year	р	q(1 – b)	δ	M (in thousands)	SSE (in millions)	UII
Canada	1976- 1994	0.267 (< 10 ⁻³)	-0.588 (< 10 ⁻³)	-0.06 (0.001)	12592.8	55.671	
Quebec	As above	0.137 (0.001)	-0.317 (< 10 ⁻³)	-0.06 (0.001)	3243.4	10.98	
Ontario	As above	0.548 (0.003)	-1.04 (0.002)	-0.047 (< 10 ⁻³)	4605.7	14.87	
Nova Scotia & New Brunswick	As above	-0.011 (0.94)	-0.019 (0.958)	-0.055 (< 10 ⁻³)	628.69	1.924	*

Table 4.5. Estimation of Parameters of Expression (4.14) Model

Note: *p*-values are in parentheses.

A review of Tables 4.2 – 4.5 reveals that model (4.14) does not pass the first stage of the proposed screening process for all regions as well as model (4.11) for the Ontario province. For model (4.14) in which advertising affects both of the coefficients of innovation and imitation, the coefficients $q(1 - \delta)$ and δ are mostly significant with the incorrect signs. For the Bass model (4.11), the coefficients p and q, though having the correct signs, were not significant in the Ontario province. Based on the second stage of screening process, it is concluded that model (4.12) for which advertising affects the

coefficient of innovation p is best. The related UII coefficients are considerably smaller than their counterparts related to both models (4.11) and (4.13) with the exception of Quebec province where UII of model (4.13) was slightly smaller. This conclusion seems credible in view of the fact that the Durbin-Watson (DW) test could not reject the null hypothesis of no autocorrelation in the residuals for all regions (at the 0.01 level, assuming that for the analyzed annual data errors follow a first-order auto-regressive model). Of course, absence of significant autocorrelation implies that the resultant t and Fvalues are credible and there is no serious omission of variables from the model. Additional support for the reliability of model (4.12) estimates comes from Ramsey's RESET test of misspecification (see Maddala, 1988; Thursby, 1989) and White's (1980) test of Heteroskedasticity of the error terms. The above diagnostic test statistics are reported in Table 4.6. Other evidence of the adequacy of model (4.12) is that all regression coefficients appear with the theoretically expected signs, and each is statistically significant at better than the 0.05 level for all regions. Furthermore, SSE for model (4.12) is the smallest across all models and regions.

Region	Year	Number of observations (n)	Durbin-Watson $(\alpha = 0.01)$ $d_L = 0.71, d_U = 1.42$	Ramsey's F $F_{0.01}$ (1, 14) = 6.3	White's χ^2 $\chi^2_{0.01, 18} = 20.09$
Canada	1976-1994	18	2.016	0.875	7.308
Quebec	As above	18	1.621	0.3294	12.294
Ontario	As above	18	2.80	0.1867	9.53
Nova Scotia & New Brunswick	As above	18	2.50	1.782	14.634

Table 4.6. Diagnostic Test Statistics – Model (4.12)

Because revenues in a given year are generated from the total number of subscribers N, Figures 4.1 – 4.4 depict the number of actual subscribers and their

predicted (fitted) values \hat{N} derived from model (4.12) for the different regions in Canada. In these figures, the fitted value \hat{N}_t is given by actual N_{t-1} plus the estimated \hat{Y}_t obtained from Equation (4.12).



Figure 4.1. Actual and Fitted Number of Subscribers – Canada.



Figure 4.2. Actual and Fitted Number of Subscribers – Quebec.



Figure 4.3. Actual and Fitted Number of Subscribers – Ontario.



Figure 4.4. Actual and Fitted Number of Subscribers – Nova Scotia and New Brunswick

Conclusion

The models analytically and empirically investigated in this study characterize and validate the optimal advertising policy of a new subscriber service provider over time. The approach to modeling diffusion of a new subscriber service considers demand dynamics, learning curve and discounting that are managerially relevant. Demand dynamics are reflected in the differential equation of the diffusion model through incorporating innovation, word of mouth, market saturation and customers' disadoption effects. The learning curve, on the other hand, reflects dynamics on the cost side, whereas discounting illustrates management preference for money.

The analytical findings of this study are summarized in Propositions 1 through 4. For a low interest rate such that advertising elasticity of demand is a non-increasing function of penetration, Propositions 1 and 3 argue in favor of advertising heavily at the beginning to build a large base of consumer adopters who keep on repurchasing the service upon their satisfaction with their initial experiences. Cost savings incurred due to service cost learning in one hand and positive word of mouth between initial adopters and those yet to adopt on the other hand lead to a reduction in the advertising level over time to enhance profitability. For a low interest rate and in the presence of customers' disadoption Proposition 4, unlike Proposition 3 where disadoption is absent, implies that the service provider may increase advertising later on, perhaps to regain back previous disadopters that remain to be inactive users of the service. When the interest rate is sufficiently high, Proposition 2 argues in favor of spending a low level of advertising by a myopic service provider at the beginning and increasing it monotonically over time as market acceptance of the new service industry becomes less uncertain.

The empirical findings, based on data from a single service category (Cable TV in Canada), suggest that a modified version of a service diffusion model articulated by Libai et al. (2009a) that includes customers' disadoption in which advertising affects the coefficient of innovation (p) is best. In terms of both explanatory power as well as predictive power, the above model is found to be superior to the original Bass (1969) model for new consumer durables. The original Bass model does not consider customers' disadoption or marketing-mix variables. In addition, the above modified Libai et al. (2009a) model is found to be more appealing than two counterparts for which advertising affects the coefficient of imitation (q) only or both coefficients (p and q).

Taken together, the analytical and empirical contributions highlighted above argue from a managerial point of view in favor of advertising more heavily during the early stage of the diffusion process of the new subscriber service innovation and the use of a message that would predominantly target innovators. Furthermore, it might be inappropriate to model the diffusion of subscriber services as if they were durable goods.

CHAPTER 5

CONCLUSION

Recent years have witnessed an increase of interest in managing the interface between the operations and marketing functions of the enterprise. Conflicts arise by nature between these two functions since, for a notable example, marketing wants to increase product diversity while manufacturing wants to reduce it through longer and stable production runs of a narrower product line (Shapiro, 1977). Consistent with the importance of managing the above said interface, the Journal of Operations Management published in 1991 and 2002 two special issues on the subject (Berry et al., 1991; Malhotra and Sharma, 2002). More recently, the Management Science journal published a special issue on the same subject in 2004 (Ho and Tang, 2004). Interdisciplinary research involving operations and/or marketing continues to be in demand. Recent calls for papers highlight the increasing interest in the subject. Examples are: A call for papers by Management Science for a special issue entitled "Marketing Within the Enterprise and Beyond" and another call for papers by the International Journal of Production Economics for a special issue entitled "Inter-disciplinary research in Operations Management."

This dissertation is a collection of three research journals examining the interface between the marketing variable of advertising and some aspects of the operations
function of the enterprise, namely, (1) production cost [Chapter 2], (2) inventory control [Chapter 3], and (3) service cost learning [Chapter 4].

The objective of the first journal (Chapter 2) was to identify the optimum advertising allocation policy over time in the presence of a quadratic convex/concave production cost function. Recognizing a contradiction between the theoretical results (superiority of uniform policy under a concave advertising response function) and the empirical research findings (superiority of a pulsation policy) on the issue of whether a firm should advertise at a constant rate (uniform) or in a cyclic fashion (pulsation) in order to maximize its performance, the first journal attempted to reconcile the difference by incorporating a production cost function in the modeling effort. Through analytical proofs and numerical simulations, the results indicated the potential superiority of a pulsation policy in the presence of concavity in the advertising response function only if the production cost was in the form of a quadratic convex function; otherwise, the uniform policy would be optimal. The theoretical and numerical analysis employed a dynamic advertising response model (modified Vidale-Wolfe model of proven empirical validity) in a just-in-time environment of zero-inventory policy.

The modeling effort developed in the first journal is exploratory, revealing many possibilities for future research. First, although most convex production cost functions analyzed in the literature assume a quadratic form (Eliashberg and Steinberg, 1993), it would be interesting and rather challenging to find out whether the results of the study remain applicable for any convex production cost function. Second, the study assumed price to be exogenous. Second, an engaging task would be to derive the optimal pricing and advertising policies in a unified framework. The review article by Gatignon (1993)

and the work of Lodish (1980) can provide guidance in developing research along the above direction. Third, the modeling effort in this study considered sales-advertising relationship to be deterministic and dealt with stationary markets for which the parameters of sales response are assumed to remain the same over different cycles. As mentioned in Mesak and Ellis (2009), relaxing these assumptions by introducing appropriate stochastic and non-stationary mechanisms would offer additional topics for future research. The papers by Horsky (1977) and Mesak (1985) offer guidance for research in the said direction. Fourth, the current study is confined to frequently purchased non-seasonal products for which advertising is the major element of the firm's marketing efforts. It would be interesting to examine seasonal products in future research. As indicated in Mesak and Ellis (2009), the problem may be analyzed for both infinite/finite planning horizons where the seasonality period is repeated infinitely (as in the case of toys)/finitely many times (as in the case of fashion clothing). The discussion in Jones (1999, p. 296) related to the advertising of seasonal products and the study of Favaretto et al. (1996) should be instructive in this regard. Finally, another plausible direction of future research would be to extend the modeling effort to incorporate competition for which advertising of each rival would affect market potential and/or its share of the market. The studies of Monahan (1987), Park and Hahn (1991), Villas-Boas (1993), Chintagunta and Vilcassim (1995), Dube et al. (2005) and Bass et al. (2005) would be instructive in developing this effort.

The research objective pertaining to the second journal (Chapter 3) was to study how a firm would adapt optimum ordered quantity/production lot size and optimum advertising expenditure in response to changes in its own parameters, rivals' parameters, or parameters that are common to all firms in a symmetric duopoly/oligopoly market. This was accomplished by developing comparative statics (sensitivity analysis) of a symmetric competitive inventory model with advertising-dependent demand based on a market share attraction model. Both optimum advertising expenditure and ordered quantity were found to be sensitive to changes in both marketing and operations parameters. The robustness of the symmetric comparative statics was validated by using data from the brewing industry in the US that represents an asymmetric oligopoly. The undertaken numerical and empirical analyses indicated that the theoretical results obtained for a symmetric oligopoly remained valid for an oligopoly where each firm had a market share less than 50% and the market shares were further apart from one another. Management of the enterprise would find the conclusions drawn from the research of significant appeal. Relaxing the assumption of symmetry would only complicate the analysis of an oligopolistic market without yielding much more insights.

Beyond its theoretical contribution and empirical validity, the model offers a framework for further analytical and empirical research in the future. First, an immediate direction for future research would be to replicate the analysis for other industries (e.g., the ready-to-eat cereal market and the carbonated soft drink market) to assess the generality of the conclusions arrived at in this study. Second, the empirical data used in this study suggested a concave shape of the advertising attraction function. However, studies by Rao and Miller (1975) and Rao (1978) indicated that the response function may have a convex region which favors a pulsing advertising policy. Since the shape of the attraction function is important in the analysis, it is recommended that data of shorter duration be used to uncover the true shape of the advertising attraction function (see

Mesak, 2003, p. 1801 for a related discussion of this issue). Another possible extension when using data of shorter duration could be to include the influence of additional promotional activities that have a short-term effect, along with advertising that is envisioned to have a long-term impact. Third, the current study employed the basic EOQ model which can be extended to incorporate inflation into account. Studies by Aggarwal (1981) and Chandra and Bahner (1985) could be instructional in this regard. Another possible direction for future research could be to incorporate price-quantity discount in the modeling effort. Elmaghraby and Keskinocak (2003) provide a lucid review on the subject. Finally, the study uses a static model. Future research may attempt deriving and validating comparative statics within a dynamic setting. However, considering the complexity of the analysis in a truly oligopolistic setting, simplification in the form of a symmetric competition, as done in this study, could make the analysis less difficult. Only recently, a progress has been made in symmetric dynamic advertising competition for oligopolies (see Erickson, 2009a for a notable example). Dynamic advertising competition in the past was mainly confined to a duopolistic structure (e.g. Erickson, 1985; Chintagunta and Vilcassim, 1994). These recent developments, however, do not consider decision-making within the operations function of the enterprise.

In the third journal (Chapter 4), the original Bass model was modified to incorporate advertising and customers' disadoption to characterize the optimum advertising policy over time for new subscriber service innovations where service cost follows a learning curve. After characterizing the optimal policy for a general diffusion model, the results pertaining to a specific diffusion model, for which advertising affects the coefficient of innovation, were reported. On the empirical side, four alternative diffusion models were estimated, and their predictive powers, using a one-step-ahead forecasting procedure were compared. Empirical research findings based on data related to the diffusion of basic cable TV in Canada suggest that the specific diffusion model considered in this study is not only of theoretical appeal, but also of notable empirical relevance. Taken together, the analytical and empirical findings argue in favor of advertising more heavily during the early stage of the diffusion process of the new subscriber service innovation and a related message that would predominantly target innovators. Furthermore, it might be inappropriate to model the diffusion of subscriber services as if they were durable goods.

The modeling effort developed in the third journal is amenable to further development in the future. First, the present study considers only the marketing variable of advertising. Other marketing-mix variables such as subscription fee/price (Mesak and Darrat, 2002) and quality (Rust, Zahorik and Keiningham, 1995) can also be considered. Optimal trajectories of such variables may be characterized through optimizing a proper measure of performance over the planning horizon, subject to the considered differential equations governing the service diffusion process. Second, this study assumed that each geographic area had only one service provider operating (as in the case of cable TV). An interesting direction for future research is to consider a situation in which two or more competing service providers operate (as in the case of cellular phones). Typically, competition is modeled in a (non-cooperative) game-theoretic framework. Service markets can take advantage from notions related to published new-product competitive diffusion models to modeling competition among new-service providers. Notable diffusion studies that consider advertising as the sole decision variable include Teng and

Thompson (1983), Horsky and Mate (1988), Dockner and Jorgensen (1992) and Nguyen and Shi (2006). Third, the empirical findings reported in the study were based on data from a single service category (Cable TV) in one country (Canada) for which data was readily available. An immediate direction for future empirical research is to investigate the role of advertising in the diffusion of other subscriber services in different countries to examine the robustness of the research findings of this study and to assess empirically alternative diffusion models of subscriber services that incorporate more than one marketing-mix variable. Fourth, the empirical work reported in this research is susceptible to further refinement and extension. For example, it was assumed in this work that the market potential M was stationary over time. This assumption could be relaxed by considering it as being dynamic and equals over time to the number of homes with at least one TV set. Examining the nature of the interaction between these two adoption processes would be of interest to academicians and marketing professionals alike. The empirical papers by Jones and Ritz (1991) and Mesak and Darrat (2003) should be informative in this respect. Fifth, the diffusion service models examined in this paper considered a single generation diffusion process, assumed to end by the saturation of market potential. The present modeling effort could be developed further to consider substitution with more advanced services technological generations. Examining the role of advertising in enhancing market potential by including new prospective adopters outside the old market potential and the substitution of old technology with a new one by current adopters (upgrading) is an interesting area for future research. The papers of Norton and Bass (1992), Mahajan and Muller (1996), Danaher et al. (2001) and Stremersch et al. (2010) should be enlightening in this respect. Finally, given that most of

the recent consumer innovations are either services (such as digital cable TV or instant messaging), or combined goods and services (such as cellular phones), it is disappointing to find out that empirical research on the impact of advertising on the diffusion of subscriber services to be lacking at best. The unavailability of advertising relevant data that goes along with related data on the diffusion of new subscriber services at a reasonable cost and/or collection effort is perhaps the main reason behind the current state of the art in this research domain. Therefore, the author shares Peres et al. (2010) assertion that more efforts should be made to collect necessary data to estimate alternative diffusion models of subscriber service innovations. The above article also provides additional plausible directions for future research that are closely related to the scope and purposes of this study.

While research on managing the interface between marketing and operations is limited, the most common decision variable from the marketing side has been price, and the most common decision variable from the operations side has been the order quantity or production rate. By considering advertising as the main marketing variable, this dissertation aimed at partially filling the gap that exists in the literature by examining interfaces between the operations and advertising functions.

APPENDIX A

DERIVATION OF KEY FORMULAS IN CHAPTER 2

Appendix A derives expressions (2.10) and (2.11) in Chapter 2. The derivation of Expression (2.10) relies heavily on ideas introduced earlier in the works of Mesak and Darrat (1992) and Mesak and Ellis (2009).

A.1. Derivation of expressions (2.10) and (2.11)

A.1.1. Sales revenue function

Considering the general advertising pulsation policy for which $0 \le x_2 \le x_1$, and $0 \le t_1 \le T$, the sales rate can be obtained by solving the differential Equation (2.3) in the following manner. Starting at t = 0, at which initial sales is S_0 while advertising is kept at the level x_1 in the interval $[0, t_1]$, sales rise exponentially with a growth rate $(a + f(x_1))$ towards $S(x_1)$ and the related growth rate sales curve, $g_1(t)$, is given by:

$$g_1(t) = S_0 + \{S(x_1) - S_0\} \left(1 - e^{-(a + f(x_1))t} \right), 0 \le t \le t_1, S_0 < S(x_1),$$
(A.1)

reaching a peak M given by

$$M = g_1(t_1) = S_0 e^{-(a+f(x_1))t_1} + S(x_1) \left(1 - e^{-(a+f(x_1))t_1}\right).$$
(A.2)

During interval $[t_1, T]$, advertising is at level $0 \le x_2 \le x_1$ and sales fall exponentially with a decay rate $(a + f(x_2))$ from *M* towards $S(x_2)$. The related decay sales rate curve, $g_2(t)$, would be given by

$$g_2(t) = M - (M - S(x_2))(1 - e^{-(a + f(x_2))(t - t_1)}), t_1 \le t \le T, M > S(x_2).$$
(A.3)

During the cycle *n*, the functions $g_{1n}(t)$ and $g_{2n}(t)$ represent the sales growth and sales decay curve, respectively, while initial sales are represented by $S_{0,n}$ during cycle *n* and M_n represents the sales peak in the cycle (see also Figure 2.4). Assuming the market is stationary, the quantities *a* and *f*(.) would maintain the same values over different cycles. Replacing $S_{0,n}$ for S_0 in (A.1) and $S_{0,n+1}$ for $g_2(T)$ in (A.3), then through the use of Equation (A.2), it can be shown that

$$S_{0,n+1} = S_{0,n} e^{-[\alpha_1 t_1 + \alpha_2 (T - t_1)]} + S(x_1) (1 - e^{-\alpha_1 t_1}) e^{-\alpha_2 (T - t_1)} + S(x_2) (1 - e^{-\alpha_2 (T - t_1)})$$
(A.4)

where

$$\alpha_1 = a + f(x_1), \ \alpha_2 = a + f(x_2), \ S(x_1) = mf(x_1)/\alpha_1, \ \text{and} \ S(x_2) = mf(x_2)/\alpha_2.$$

Putting $k = e^{-[\alpha_1 t_1 + \alpha_2 (T - t_1)]}$, 0 < k < 1, and $c = S(x_1) (1 - e^{-\alpha_1 t_1}) e^{-\alpha_2 (T - t_1)} + S(x_2) (1 - e^{-\alpha_2 (T - t_1)})$,

Equation (A.4) can be rewritten as

$$S_{0,n+1} = kS_{0,n} + c. (A.5)$$

It can be shown that a recursive expansion of (A.5) results in

$$S_{0,n+1} = k^n S_{0,1} + c \sum_{j=0}^{n-1} k^j = k^n S_{0,1} + c(1-k^n)/(1-k).$$
(A.6)

Taking the limit of (A.6) as $n \rightarrow \infty$, noticing that 0 < k < 1, implies

$$S_0 = \lim_{n \to \infty} S_{0,n+1} = c/(1-k),$$
 (A.7)

so that

$$S_{0} = \frac{S(x_{1})(1 - e^{-\alpha_{1}t_{1}})e^{-\alpha_{2}(T-t_{1})} + S(x_{2})(1 - e^{-\alpha_{2}(T-t_{1})})}{1 - e^{-[\alpha_{1}t_{1} + \alpha_{2}(T-t_{1})]}}.$$
 (A.8)

Substituting for the value of S₀ from (A.8) into (A.2), the limit *M* of the sequence $\{M_n\}$ is obtained as

$$M = \frac{S(x_1)(1 - e^{-\alpha_1 t_1})e^{-\alpha_2 (T - t_1)} + S(x_2)(1 - e^{-\alpha_2 (T - t_1)})e^{-\alpha_1 t_1}}{1 - e^{-[\alpha_1 t_1 + \alpha_2 (T - t_1)]}}.$$
 (A.9)

Substituting for $g_1(t)$ and $g_2(t)$ from (A.1) and (A.3) in (2.7) in the text produces, after carrying out the integration and rearrangements of terms

$$R = \frac{1}{T} \left[S(x_1)t_1 + S(x_2)(T - t_1) - \frac{\left(1 - e^{-\alpha_1 t_1}\right)\left(S(x_1) - S_0\right)}{\alpha_1} + \frac{\left(1 - e^{-\alpha_2(T - t_1)}\right)\left(M - S(x_2)\right)}{\alpha_2} \right].$$
 (A.10)

Substituting for the values of S_0 and M from (A.8) and (A.9) produces the following expressions for $S(x_1) - S_0$ and $M - S(x_2)$:

$$S(x_1) - S_0 = \frac{\left[S(x_1) - S(x_2)\right] \left(1 - e^{-\alpha_2(T - t_1)}\right)}{1 - e^{-\left[\alpha_1 t_1 + \alpha_2(T - t_1)\right]}}.$$
 (A.11)

$$M - S(x_2) = \frac{\left[S(x_1) - S(x_2)\right] \left(1 - e^{-\alpha_1 t_1}\right)}{1 - e^{-\left[\alpha_1 t_1 + \alpha_2 (T - t_1)\right]}}.$$
 (A.12)

Substituting for $S(x_1) - S_0$ and $M - S(x_2)$ from (A.11) and (A.12) in (A.10), the following Expression (A.13) is obtained which is the same as Expression (2.10) in the text.

$$R = \frac{1}{T} \left[S(x_1)t_1 + S(x_2)(T - t_1) + \{S(x_1) - S(x_2)\} \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right) \frac{\left(1 - e^{-\alpha_1 t_1}\right) \left(1 - e^{-\alpha_2 (T - t_1)}\right)}{\left(1 - e^{-(\alpha_1 t_1 + \alpha_2 (T - t_1))}\right)} \right].$$
 (A.13)

A.1.2. Production cost function

Similarly substituting for $g_1(t)$ and $g_2(t)$ from (A.1) and (A.3) in the second term of Expression (2.8) in the text produces, after carrying out the integration and rearrangements of terms

$$PC2 = \frac{1}{KT} \begin{bmatrix} S^{2}(x_{1})t_{1} + S^{2}(x_{2})(T - t_{1}) + \frac{\{S(x_{1}) - S_{0}\}(1 - e^{-\alpha_{1}t_{1}})}{\alpha_{1}} \left[\frac{\{S(x_{1}) - S_{0}\}(1 + e^{-\alpha_{1}t_{1}})}{2} - 2S(x_{1}) \right] \\ + 2S(x_{2})\{M - S(x_{2})\}\frac{(1 - e^{-\alpha_{2}(T - t_{1})})}{\alpha_{2}} + \{M - S(x_{2})\}^{2}\frac{(1 - e^{-2\alpha_{2}(T - t_{1})})}{2\alpha_{2}} \end{bmatrix}$$

.(A.14)

Substituting for $S(x_1) - S_0$ and $M - S(x_2)$ from (A.11) and (A.12) in (A.14), the following Expression (A.15) is obtained which is the same as Expression (2.11) in the text.

$$PC2 = \frac{1}{KT} \begin{bmatrix} S^{2}(x_{1})t_{1} + S^{2}(x_{2})(T - t_{1}) + 2\{S(x_{1}) - S(x_{2})\} \left(\frac{S(x_{2})}{\alpha_{2}} - \frac{S(x_{1})}{\alpha_{1}}\right) \\ \frac{(1 - e^{-\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})}{(1 - e^{-(\alpha_{1}t_{1} + \alpha_{2}(T - t_{1})})} + \frac{\{S(x_{1}) - S(x_{2})\}^{2}}{2(1 - e^{-(\alpha_{1}t_{1} + \alpha_{2}(T - t_{1})})^{2}} \\ \frac{\left(\frac{(1 - e^{-2\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})}{\alpha_{1}} + \frac{(1 - e^{-\alpha_{1}t_{1}})^{2}(1 - e^{-2\alpha_{2}(T - t_{1})})}{\alpha_{2}} \right\} \end{bmatrix}$$
(A.15)

A.2. Equality of Expression (2.10) to the long-run mean sales rate

The mean sales revenue related to cycle n, R_n , takes the same format as that of Expression (A.10) except that S_0 is replaced by $S_{0,n}$ and M is replaced by M_n , so that

$$R_{n} = \frac{1}{T} \left[S(x_{1})t_{1} + S(x_{2})(T - t_{1}) - \frac{\left(1 - e^{-\alpha_{1}t_{1}}\right)\left(S(x_{1}) - S_{0,n}\right)}{\alpha_{1}} + \frac{\left(1 - e^{-\alpha_{2}(T - t_{1})}\right)\left(M_{n} - S(x_{2})\right)}{\alpha_{2}} \right].$$
(A.16)

From (A.10) and (A.16), we obtain

$$R_n - R = \frac{1}{T} \left[\left(S_{0,n} - S_0 \right) \left(1 - e^{-\alpha_1 t_1} \right) / \alpha_1 + \left(M_n - M_1 \right) \left(1 - e^{-\alpha_2 (T - t_1)} \right) / \alpha_2 \right].$$
(A.17)

Referring to Expression (A.2), we can write

$$M_n - M = (S_{0,n} - S_0)e^{-\alpha_1 t_1}.$$
 (A.18)

Substituting for $(M_n - M)$ from (A.18) in (A.17) produces

$$R_n - R = \frac{1}{T} \left(S_{0,n} - S_0 \right) \left[\left(1 - e^{-\alpha_1 t_1} \right) / \alpha_1 + \left(1 - e^{-\alpha_2 (T - t_1)} \right) e^{-\alpha_1 t_1} / \alpha_2 \right].$$
(A.19)

But from (A.6) and (A.8), we can write

$$\sum_{n=1}^{\infty} \left(S_{0,n} - S_0 \right) = \sum_{n=1}^{\infty} \left[k^{n-1} S_{0,1} + c \frac{1 - k^{n-1}}{1 - k} - \frac{c}{1 - k} \right] = S_{0,1} \frac{1}{1 - k} - \frac{c}{(1 - k)^2} = \frac{1}{1 - k} \left(S_{0,1} - S_0 \right),$$
(A.20)

and k is a quantity less than 1, given by $e^{-[\alpha_1 t_1 + \alpha_2 (T - t_1)]}$ and $S_0 = c/(1-k)$ from (A.7).

From (A.19) and (A.20), it turns out that $\lim_{n\to\infty}(1/n)\sum_{n=1}^{\infty}(R_n - R) = 0$ as both the multipliers of $(S_{0,n} - S_0)$ in (A.19) and the right-hand side of (A.20) are independent of *n*. Accordingly, $\lim_{n\to\infty}(1/n)\left(\sum_{n=1}^{\infty}R_n\right)/n = R$, and hence proving our assertion.

APPENDIX B

PROOFS OF RESULTS AND PROPOSITIONS IN CHAPTER 2

Appendix B provides the proofs to three results and two propositions in Chapter 2. The proof of Result 2 is also found in Mesak and Darrat (1992) on pages 561 and 562. Proof of Result 1.

Since for APP

$$\frac{d(PRO)}{dt_1} = \frac{\partial(PRO)}{\partial t_1} + \frac{\partial(PRO)}{\partial x_1} \cdot \frac{dx_1}{dt_1}$$
(B.1)

and $PRO = \gamma R - PC2 - x$ (Equation 2.12), we have

$$\frac{\partial(PRO)}{\partial t_1} = \gamma \frac{\partial R}{\partial t_1} - \frac{\partial(PC)}{\partial t_1} - \frac{\partial x}{\partial t_1}, \qquad (B.2)$$

$$\frac{\partial(PRO)}{\partial x_1} = \gamma \frac{\partial R}{\partial x_1} - \frac{\partial(PC)}{\partial x_1} - \frac{\partial x}{\partial x_1}, \text{ and}$$
(B.3)

$$\frac{dx_1}{dt_1} = \frac{d}{dt_1} \left\{ \frac{xT}{t_1} \right\} = -\frac{xT}{t_1^2}.$$
(B.4)

For APP, as $t_1 \to T$, we have $x_1 = x$, $x_2 = 0$, $S(x_1) = S(x)$, $S(x_2) = 0$, $f(x_2) = 0$, $\alpha_2 = a$, and $\alpha_l = \alpha = a + f(x)$. Also defining $p = 1 - e^{-(\alpha_1 t_1 + \alpha_2 (T - t_1))}$, Equation (2.10) can be simplified S

$$R = \frac{1}{T} \left[S(x_1)t_1 + \{S(x_1)\} \left(\frac{1}{\alpha_2} - \frac{1}{\alpha_1} \right) \frac{\left(1 - e^{-\alpha_1 t_1}\right) \left(1 - e^{-\alpha_2 (T - t_1)}\right)}{p} \right], \quad (B.5)$$

and

$$\frac{\partial R}{\partial t_{1}} = \frac{1}{T} \left[S(x_{1}) + \{S(x_{1})\} \left(\frac{1}{\alpha_{2}} - \frac{1}{\alpha_{1}} \right) \left\{ \frac{p\{(1 - e^{-\alpha_{1}t_{1}})(-\alpha_{2}e^{-\alpha_{2}(T-t_{1})}) + (\alpha_{1}e^{-\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T-t_{1})})\}}{p^{2}} \right\} \right]$$

Noting that,
$$p\Big|_{t_1 \to T} = 1 - e^{-\alpha T}$$
, (B.7)

we obtain the following result upon minor arrangement of terms in (B.6)

$$\frac{\partial R}{\partial t_1}\Big|_{t_1 \to T} = \frac{1}{T} \left[S(x) + S(x) \left(\frac{1}{a} - \frac{1}{\alpha}\right) (-a) \right] = \frac{S(x)a}{T\alpha}.$$
 (B.8)

Similarly Equation (2.11) can be simplified as

$$PC2 = \frac{1}{KT} \begin{bmatrix} S^{2}(x_{1})t_{1} - \frac{2S^{2}(x_{1})(1 - e^{-\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})}{\alpha_{1}} + \frac{S^{2}(x_{1})}{2p^{2}} \\ \left\{ \frac{(1 - e^{-2\alpha_{1}t_{1}})(1 - e^{-\alpha_{2}(T - t_{1})})^{2}}{\alpha_{1}} + \frac{(1 - e^{-\alpha_{1}t_{1}})^{2}(1 - e^{-2\alpha_{2}(T - t_{1})})}{\alpha_{2}} \right\} \end{bmatrix}.$$
 (B.9)

After rearrangement of terms and carrying out the differentiation, it can be shown that Expression (B.9) takes the following form:

$$\frac{\partial(PC2)}{\partial t_1}\Big|_{t_1 \to T} = \frac{1}{KT} \left[S^2(x) - \frac{2S^2(x)(-a)}{\alpha} + \frac{S^2(x)(-2a)}{2a} \right] = \frac{2S^2(x)a}{KT\alpha} .$$
(B.10)

Similarly, carrying out the differentiation with respect to x_i , it can be shown that expressions (B.5) and (B.9) take the following forms respectively:

$$\frac{\partial R}{\partial x_1}\Big|_{t_1 \to T} = S'(x), \qquad (B.11)$$

and

$$\frac{\partial(PC2)}{\partial x_1}\Big|_{t_1 \to T} = \frac{1}{K} (2S(x)S'(x)).$$
(B.12)

After combining the results in expressions (B.2), (B.3), (B.4), (B.8), (B.10), (B.11), and

(B.12), noting that
$$\partial x / \partial t_1 = \partial x / \partial x_1 = 0$$
 and $\frac{dx_1}{dt_1}\Big|_{t_1 \to T} = -x/T$ from (B.4), Expression (B.1)

takes the following form:

$$\frac{d(PRO)}{dt_1}\Big|_{t_1 \to T} = \frac{\gamma S(x)a}{T \alpha} - \frac{1}{KT} \frac{2S^2(x)a}{\alpha} + \left(\gamma S'(x) - \frac{2S(x)S'(x)}{K}\right)\left(-\frac{x}{T}\right), \quad (B.13)$$

which can be rearranged into

$$\frac{d(PRO)}{dt_1}\Big|_{t_1 \to T} = \frac{1}{T} \left(\gamma - \frac{2S(x)}{K}\right) \left(\frac{S(x)a}{\alpha} - xS'(x)\right).$$
(B.14)

Substituting for S(x) = mf(x)/[a + f(x)], $\alpha = a + f(x)$ and after carrying out the differentiation of S(x) with respect to x, the following expression is obtained

$$\frac{S(x)a}{\alpha} - xS'(x) = \frac{amf(x)}{(a+f(x))^2} - mx \left\{ \frac{(a+f(x))f'(x) - f(x)f'(x)}{(a+f(x))^2} \right\}.$$
 (B.15)

After rearranging the terms in Expression (B.15) and incorporating it into Expression (B.14), we have

$$\frac{d(PRO)}{dt_1}\Big|_{t_1 \to T} = \frac{1}{T} \left(\gamma - \frac{2S(x)}{K}\right) \left\{\frac{am}{(a+f(x))^2}\right\} \left\{f(x) - xf'(x)\right\}.$$
 (B.16)

Since the second term of the right hand side of Expression (B.16) is always positive, we conclude that:

$$Sign of \left. \frac{d(PRO)}{dt_1} \right|_{t_1 \to T} \text{ is given by : } Sign of \left(\gamma - \frac{2S(x)}{K} \right) Sign of \left\{ f(x) - xf'(x) \right\}$$
(B.17)

which completes the proof.

Proof of Proposition 1

Considering Expression (B.17) since the first term of the right hand side is negative for $x > x_s$ and the second term of right hand side is positive for a concave advertising response function as well as for the concave portion of an S-shaped advertising response function in excess of Max { x_t , x_s }, we arrive at the following conclusion about the left hand side:

Sign of
$$\frac{d(PRO)}{dt_1}\Big|_{t_1 \to T}$$
 is negative, indicating that APP dominates UAP. Consider next an

APP for which the policy parameter $\lambda = 0$ and of $t_1 < T$ that dominates an UAP that costs the same for which the policy parameter $\lambda = 1$. Assuming that the *PRO* path over the interval $0 \le \lambda \le 1$ of the same t_1 and advertising budget x to be continuous and twice differentiable with respect to λ , there must exist at least one APMP for which $0 < \lambda < 1$ and close enough to $\lambda = 0$ for which *PRO* is larger than that corresponding to the considered UAP. Hence APMP does also dominate UAP. The proof becomes complete by realizing that the above relationships are true for all feasible values of x and T.

Proof of Result 2

Adding and subtracting the term $(\alpha_l t_l) \{ \alpha_2 (T - t_l) \} / \{ \alpha_l t_l + \alpha_2 (T - t_l) \}$ to and from the term $(1 - e^{-\alpha_1 t_1})(1 - e^{-\alpha_2 (T - t_1)}) / (1 - e^{-\{\alpha_l t_1 + \alpha_2 (T - t_1)\}})$ and substituting $S(x_l) = m f(x_l) / \alpha_l$ and $S(x_2) = m f(x_2) / \alpha_2$ in (2.10), it can be shown that R takes the following form:

$$R = R_1 + R_2, \tag{B.18}$$

where
$$R_{1} = \frac{m[f(x_{1})t_{1} + f(x_{2})(T - t_{1})]}{f(x_{1})t_{1} + f(x_{2})(T - t_{1}) + aT},$$
 (B.19)

$$R_{2} = \frac{am}{T} \left(\frac{1}{\alpha_{2}} - \frac{1}{\alpha_{1}}\right)^{2} \left[\frac{\left(1 - e^{-\alpha_{1}t_{1}}\right)\left(1 - e^{-\alpha_{2}(T-t_{1})}\right)}{1 - e^{-\left\{\alpha_{1}t_{1} + \alpha_{2}(T-t_{1})\right\}}} - \frac{(\alpha_{1}t_{1})\left\{\alpha_{2}(T-t_{1})\right\}}{\alpha_{1}t_{1} + \alpha_{2}(T-t_{1})}\right].$$
 (B.20)

It is shown next that R_1 is a non-decreasing function of $\lambda = x_2/x$ for $f'(.) \le 0$. By expressing x_2 in terms of λ and substituting in (2.13) for x_2 , x_1 can be expressed in terms of λ also resulting in

$$x_2 = \lambda x, x_1 = (x/t_1) \{ T - \lambda (T - t_1) \}.$$
 (B.21)

Since
$$\frac{dR_1}{d\lambda} = \frac{\partial R_1}{\partial x_1} \frac{dx_1}{d\lambda} + \frac{\partial R_2}{\partial x_2} \frac{dx_2}{d\lambda},$$
 (B.22)

employing (B.22) in conjunction with (B.19) and (B.21) produces

$$\frac{dR_1}{d\lambda} = \frac{amx(T-t_1)T[f'(x_2) - f'(x_1)]}{[f(x_1)t_1 + f(x_2)(T-t_1) + at]^2}.$$
(B.23)

It is evident from (B.23) that $dR_1/d\lambda$ should have the same sign as $[f'(x_2) - f'(x_1)]$ which is ≥ 0 for $f''(.) \leq 0$. Since R_2 is non-positive from (B.20), then R_2 attains its maximum at $\lambda =$ 1 as for that value $x_1 = x_1 = x$ so that $\alpha_1 = \alpha_2$ resulting in $R_2 = 0$. These results related to R_1 and R_2 imply from (B.18) that R attains its maximum at $\lambda = 1$ so that $R_{UAP} > R_{APP}$. Since R is continuous in λ , then there exists at least one $0 < \lambda < 1$ near $\lambda = 1$ for which

$$R_{APP} < R_{APMP} < R_{UAP}$$
.

The proof becomes complete by realizing that the relationship (B.24) is true for all feasible values of x, t_1 and T.

Proof of Proposition 2

For a concave advertising response function, $dR_1/d\lambda > 0$ by Result 2 and $d(PC2)/d\lambda > 0$ by Result 3. PRO = R + PC2 - x, so that $d(PRO)/d\lambda = dR/d\lambda + d(PC2)/d\lambda > 0$. As for UAP, $\lambda = 1$; for APMP, $0 < \lambda < 1$ and for APP, $\lambda = 0$, it turns out that for all feasible values x, t_1 and T, $PRO_{UAP} > PRO_{APMP} > PRO_{APP}$ implying that APMP dominates APP but is dominated by UAP.

APPENDIX C

COMPARATIVE STATICS IN A MONOPOLY DEVELOPED IN CHAPTER 3

Appendix C derives the fundamental equation of comparative statics in a monopoly in section C1. The proof to Proposition 1 in Chapter 3 is demonstrated in sections C2 and C3. Section C4 derives the steady state response of a modified Vidale-Wolfe model.

C1. Fundamental equation of comparative statics in a monopoly

Define the profit function
$$\pi = \gamma D(x) - C(x) - \frac{Q}{2}C_h - \frac{D(x)}{Q}C_o - F$$
, (C.1)

where,

x – advertising expenditure,

Q – ordered quantity,

 C_o – ordering cost per order,

 C_h – inventory holding cost per unit held per unit time,

C(x) – convex shaped advertising cost function such that C'(x) > 0 and C''(x) > 0,

D(x) – demand as a function of advertising given by,

$$D(x) = \frac{mf(x)}{e+f(x)}$$
, *m* is the market potential, $f(x)$ is a concave attraction function such

that f'(x) > 0 and f''(x) < 0, e is a constant parameter, and

F – fixed cost of the firm.

Defining $\Pi = \pi / \gamma m$, so that at optimal advertising x^* and optimal ordered quantity Q^*

$$\frac{\partial \Pi}{\partial x} = 0 \text{ and } \frac{\partial \Pi}{\partial Q} = 0.$$
 (C.2)

After substituting for D(x), Equation (C.1) can be rearranged to take on the following form:

$$\Pi = \frac{f(x)}{e+f(x)} \left[1 - \frac{C_o}{\gamma Q} \right] - \frac{C(x)}{\gamma m} - \frac{C_h}{2\gamma m} Q - \frac{F}{\gamma m}.$$
(C.3)

To study the sensitivity of x^* and Q^* to a change in one of the model parameters θ , Equation (C.2) are partially differentiated with respect to θ then equated to zero afterwards to obtain

$$\frac{\partial^2 \Pi}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 \Pi}{\partial x \partial Q} \frac{\partial Q}{\partial \theta} + \frac{\partial^2 \Pi}{\partial x \partial \theta} = 0,$$

$$\frac{\partial^2 \Pi}{\partial Q \partial x} \frac{\partial x}{\partial \theta} + \frac{\partial^2 \Pi}{\partial Q^2} \frac{\partial Q}{\partial \theta} + \frac{\partial^2 \Pi}{\partial Q \partial \theta} = 0.$$
 (C.4)

Here, it is assumed that all second partial derivatives of Π with respect to each of the *equilibrium* values x^* and Q^* together with each of the *shift* parameters θ do exist and are continuous. The system of Equations (C.4) can be put in the following matrix form:

$$\begin{bmatrix} \frac{\partial^2 \Pi}{\partial x^2} & \frac{\partial^2 \Pi}{\partial x \partial Q} \\ \frac{\partial^2 \Pi}{\partial Q \partial x} & \frac{\partial^2 \Pi}{\partial Q^2} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial Q}{\partial \theta} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^2 \Pi}{\partial x \partial \theta} \\ \frac{\partial^2 \Pi}{\partial Q \partial \theta} \end{bmatrix}.$$
 (C.5)

Designating the 2 x 2 square matrix of the second-order partial derivatives in (C.5) by matrix **H**, which is assumed to be non-singular, (C.5) can be written as

$$\begin{bmatrix} \frac{\partial x}{\partial \theta} \\ \frac{\partial Q}{\partial \theta} \end{bmatrix} = -H^{I} \begin{bmatrix} \frac{\partial^{2} \Pi}{\partial x \partial \theta} \\ \frac{\partial^{2} \Pi}{\partial Q \partial \theta} \end{bmatrix},$$
 (C.6)

and the elements of matrix \mathbf{H} are obtained by partially differentiating (C.3) in the following manner:

$$\frac{\partial \Pi}{\partial x} = \frac{ef'}{(e+f)^2} \left[1 - \frac{C_o}{\gamma Q} \right] - \frac{C'}{\gamma m} = 0, \qquad (C.7)$$

$$\frac{\partial^2 \Pi}{\partial x^2} = \frac{e}{\left(e+f\right)^3} \left\{ f''(e+f) - 2f'^2 \right\} \left[1 - \frac{C_o}{\gamma Q} \right] - \frac{C''}{\gamma m}, \qquad (C.8)$$

$$\frac{\partial^2 \Pi}{\partial x \partial Q} = \frac{\partial^2 \Pi}{\partial Q \partial x} = \frac{ef'}{(e+f)^2} \frac{C_o}{\gamma Q^2}, \qquad (C.9)$$

$$\frac{\partial \Pi}{\partial Q} = \frac{f}{(e+f)} \frac{C_o}{\gamma Q^2} - \frac{C_h}{2\gamma m} = 0, \text{ and} \qquad (C.10)$$

$$\frac{\partial^2 \Pi}{\partial Q^2} = -\frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3}.$$
 (C.11)

The following definitions are made for the derivations below:

$$a = \frac{ef'}{(e+f)^2} \frac{C_o}{\gamma Q^2} \text{ (positive term) and}$$
(C.12)

$$b = \frac{e}{\left(e+f\right)^{3}} \left\{ f''(e+f) - 2f'^{2} \right\} \left[1 - \frac{C_{o}}{\gamma Q} \right] - \frac{C''}{\gamma m} \text{ (negative term).} \quad (C.13)$$

Matrix **H**, therefore, takes the following form:

$$\boldsymbol{H} = \begin{bmatrix} b & a \\ a & -\frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} \end{bmatrix}.$$
 (C.14)

So that

$$\boldsymbol{H}^{I} = \frac{1}{\Delta} \begin{bmatrix} -\frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & -a \\ -a & b \end{bmatrix}, \text{ where all four terms are negative} \qquad (C.15)$$

and determinant
$$\Delta = -\frac{2bf}{(e+f)}\frac{C_o}{\gamma Q^3} - a^2$$
 is positive. (C.16)

(Note that the second order sufficiency conditions of optimality imply that $\frac{\partial^2 \Pi}{\partial x^2} < 0$, so

that from (C.8)
$$\left[1 - \frac{C_o}{\gamma Q}\right] > 0$$
 and $\Delta > 0$)

C2. Proof of proposition 1

C2.1 Comparative statics of γ

Replacing θ by γ , Equation (C.6) can be rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial \gamma} \\ \frac{\partial Q}{\partial \gamma} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{ef'}{(e+f)^2} \begin{bmatrix} \frac{C_o}{\gamma^2 Q} \end{bmatrix} + \frac{C'}{\gamma^2 m} \end{bmatrix}.$$
 (C.17)

Since every term is positive in the right-hand-side of Equation (C.17), it is concluded that

- both $\frac{\partial x}{\partial \gamma}$ and $\frac{\partial Q}{\partial \gamma}$ are *positive*.
- C2.2 Comparative statics of C_h

Replacing θ by C_h , Equation (C.6) is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial C_h} \\ \frac{\partial Q}{\partial C_h} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{2\gamma m} \end{bmatrix}.$$
 (C.18)

Carrying out the matrix multiplication, it can be shown that

$$\frac{\partial x}{\partial C_h} = -\frac{a}{2\Delta\gamma m} \implies negative.$$
$$\frac{\partial Q}{\partial C_h} = \frac{b}{2\Delta\gamma m} \implies negative.$$

C2.3 Comparative statics of m

Replacing θ by *m*, Equation (C.6) is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial m} \\ \frac{\partial Q}{\partial m} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{C'}{\gamma m^2} \\ \frac{C''}{\gamma m^2} \end{bmatrix}.$$
 (C.19)

Since every term is positive in the right-hand-side of Equation (C.19), it is concluded that

both $\frac{\partial x}{\partial m}$ and $\frac{\partial Q}{\partial m}$ are *positive*.

C2.4 Comparative statics of Co

Replacing θ by C_o , Equation (C.6) is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial C_o} \\ \frac{\partial Q}{\partial C_o} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} -\frac{ef'}{(e+f)^2} \frac{1}{\gamma Q} \\ \frac{f}{(e+f)\gamma Q^2} \end{bmatrix}.$$
 (C.20)

Carrying out the matrix multiplication, it can be shown that upon using (C.12) and (C.13)

$$\frac{\partial x}{\partial C_o} = -\frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} \left\{ \frac{ef'}{(e+f)^2} \frac{1}{\gamma Q} \right\} \frac{1}{\Delta} + \frac{af}{\Delta (e+f)} \frac{1}{\gamma Q^2} = -\frac{af}{\Delta (e+f)} \frac{1}{\gamma Q^2} \implies negative.$$

$$\frac{\partial Q}{\partial C_o} = -\frac{1}{\Delta} \left(\frac{aef'}{(e+f)^2} \frac{1}{\gamma Q} + \frac{bf}{(e+f)} \frac{1}{\gamma Q^2} \right).$$

Substituting for f' from (C.12) into the above expression, it can be shown after minor arrangement of terms that

$$\frac{\partial Q}{\partial C_o} = \frac{Q}{2\Delta C_o} \left(\Delta - a^2 \right), \text{ so that } \frac{\partial Q}{\partial C_o} \text{ becomes positive (negative) for } \Delta - a^2 > (<)0.$$

C2.5 Comparative statics of e

Replacing θ by *e*, Equation (C.6) is written as

$$\begin{bmatrix} \frac{\partial x}{\partial e} \\ \frac{\partial Q}{\partial e} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} -\frac{f'(f-e)}{(e+f)^3} \left(1 - \frac{C_o}{\gamma Q}\right) \\ -\frac{f}{(e+f)^2} \frac{C_o}{\gamma Q^2} \end{bmatrix}.$$
 (C.21)

Carrying out the matrix multiplication produces

$$\frac{\partial x}{\partial e} = \frac{2f(f-e)f'}{\Delta(e+f)^4} \left(1 - \frac{C_o}{\gamma Q}\right) \frac{C_o}{\gamma Q^3} - \frac{af}{\Delta(e+f)^2} \frac{C_o}{\gamma Q^2}$$
$$\frac{\partial Q}{\partial e} = \frac{a(f-e)f'}{\Delta(e+f)^3} \left(1 - \frac{C_o}{\gamma Q}\right) + \frac{bf}{\Delta(e+f)^2} \frac{C_o}{\gamma Q^2}.$$

Substituting for f' from (C.12) and b from (C.16) into the above expressions, it can be shown after minor arrangements of terms that

$$\frac{\partial x}{\partial e} = \frac{af}{\Delta e(e+f)^2 Q} \left[2(f-e) - \frac{C_o}{\gamma Q} (2f-e) \right].$$
$$\frac{\partial Q}{\partial e} = \frac{1}{2e\Delta(e+f)} \frac{\gamma Q^2}{C_o} \left[2a^2(f-e) - \frac{C_o}{\gamma Q} \left\{ e\Delta + a^2(2f-e) \right\} \right].$$

Therefore,

$$\frac{\partial x}{\partial e}$$
 is positive (negative) for $\frac{C_o}{\gamma Q} < (>) \frac{2(f-e)}{2f-e}$ and

$$\frac{\partial Q}{\partial e} \text{ is positive (negative) for } \frac{C_o}{\gamma Q} < (>) \frac{2a^2(f-e)}{e\Delta + a^2(2f-e)}.$$

(Note from (C.8) that since $1 - \frac{C_o}{\gamma Q} > 0$, then $0 < \frac{C_o}{\gamma Q} < 1$)

C3.1 Comparative statics of β and δ

Defining $f(x) = \beta x^{\delta}$, $0 < \delta < 1$, and replacing θ by β , Equation (C.6) takes the form

$$\begin{bmatrix} \frac{\partial x}{\partial \beta} \\ \frac{\partial Q}{\partial \beta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{e \delta x^{\delta-1} \left(e - \beta x^{\delta}\right)}{\left(e + \beta x^{\delta}\right)^3} \left(1 - \frac{C_o}{\gamma Q}\right) \\ \frac{e x^{\delta}}{\left(e + \beta x^{\delta}\right)^2} \frac{C_o}{\gamma Q^2} \end{bmatrix}.$$

The above expression is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial \beta} \\ \frac{\partial Q}{\partial \beta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{ef'(e-f)}{\beta(e+f)^3} \left(1 - \frac{C_o}{\gamma Q}\right) \\ \frac{ef}{\beta(e+f)^2 \gamma Q^2} \end{bmatrix}.$$
 (C.22)

Carrying out the matrix multiplication produces

$$\frac{\partial x}{\partial \beta} = \frac{2fe(e-f)f'}{\Delta\beta(e+f)^4} \frac{C_o}{\gamma Q^3} \left(1 - \frac{C_o}{\gamma Q}\right) + \frac{aef}{\Delta\beta(e+f)^2} \frac{C_o}{\gamma Q^2} \cdot \frac{\partial Q}{\partial\beta} = \frac{ae(e-f)f'}{\Delta\beta(e+f)^3} \left(1 - \frac{C_o}{\gamma Q}\right) - \frac{bef}{\Delta\beta(e+f)^2} \frac{C_o}{\gamma Q^2} \cdot \frac{\partial Q}{\partial\beta(e+f)^2} \cdot \frac{$$

Substituting for f' from (C.12) and b from (C.16) into the above expressions, it can be shown after minor arrangements of terms that

$$\frac{\partial x}{\partial \beta} = \frac{af}{\beta \Delta (e+f)^2 Q} \left[2(e-f) - \frac{C_o}{\gamma Q} (e-2f) \right].$$
$$\frac{\partial Q}{\partial \beta} = \frac{1}{2\beta \Delta (e+f)} \frac{\gamma Q^2}{C_o} \left[2a^2(e-f) - \frac{C_o}{\gamma Q} \left\{ a^2(e-2f) - e\Delta \right\} \right].$$

Therefore,

$$\frac{\partial x}{\partial \beta}$$
 is positive (negative) for $\frac{C_o}{\gamma Q} < (>) \frac{2(f-e)}{2f-e}$ and

$$\frac{\partial Q}{\partial \beta}$$
 is positive (negative) for $\frac{C_o}{\gamma Q} < (>) \frac{2a^2(f-e)}{e\Delta + a^2(2f-e)}$.

Similarly replacing θ by δ , Equation (C.6) takes the form

$$\begin{bmatrix} \frac{\partial x}{\partial \delta} \\ \frac{\partial Q}{\partial \delta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{e^2 \beta x^{\delta-1} (1+\delta \ln x) + e\beta^2 x^{2\delta-1} (1-\delta \ln x)}{(e+\beta x^{\delta})^3} (1-\frac{C_o}{\gamma Q}) \\ \frac{e\beta x^{\delta} \ln x}{(e+\beta x^{\delta})^2} \frac{C_o}{\gamma Q^2} \end{bmatrix}.$$

The above expression is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial \delta} \\ \frac{\partial Q}{\partial \delta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} \frac{ef'}{(e+f)^3} \left\{ \frac{e+f}{\delta} + (e-f) \ln x \right\} \left(1 - \frac{C_o}{\gamma Q} \right) \\ \frac{ef \ln x}{(e+f)^2} \frac{C_o}{\gamma Q^2} \end{bmatrix}.$$
 (C.23)

Carrying out the matrix multiplication produces

$$\frac{\partial x}{\partial \delta} = \frac{2fef'}{\Delta(e+f)^4} \left\{ \frac{e+f}{\delta} + (e-f)\ln x \right\} \frac{C_o}{\gamma Q^3} \left(1 - \frac{C_o}{\gamma Q} \right) + \frac{aef\ln x}{\Delta(e+f)^2} \frac{C_o}{\gamma Q^2} .$$
$$\frac{\partial Q}{\partial \delta} = \frac{aef'}{\Delta(e+f)^3} \left\{ \frac{e+f}{\delta} + (e-f)\ln x \right\} \left(1 - \frac{C_o}{\gamma Q} \right) - \frac{bef\ln x}{\Delta(e+f)^2} \frac{C_o}{\gamma Q^2} .$$

Substituting for f' from (C.12) and b from (C.16) into the above expressions, it can be shown after minor arrangements of terms that

$$\frac{\partial x}{\partial \delta} = \frac{af}{\Delta(e+f)^2 Q} \left[2 \left\{ \frac{e+f}{\delta} + (e-f) \ln x \right\} - \frac{C_o}{\gamma Q} \left\{ \frac{2(e+f)}{\delta} + (e-2f) \ln x \right\} \right].$$
$$\frac{\partial Q}{\partial \delta} = \frac{1}{2\Delta(e+f)} \frac{\gamma Q^2}{C_o} \left[\frac{2a^2 \left\{ \frac{e+f}{\delta} + (e-f) \ln x \right\} - \frac{C_o}{\gamma Q} \left\{ \frac{2a^2 (e+f)^2}{\delta} + (e-2f-e\Delta) \ln x \right\} \right].$$

Therefore,

$$\frac{\partial x}{\partial \delta} \text{ is positive (negative) for } \frac{C_o}{\gamma Q} < (>) \quad \frac{2\{(e+f)/\delta + (e-f)\ln x\}}{2(e+f)/\delta + (e-2f)\ln x} \text{ and}$$
$$\frac{\partial Q}{\partial \delta} \text{ is positive (negative) for } \frac{C_o}{\gamma Q} < (>) \quad \frac{2a^2\{(e+f)/\delta + (e-f)\ln x\}}{2a^2(e+f)/\delta + (e-2f-e\Delta)\ln x}.$$

C3.2 Comparative statics of d and ε

Defining $C(x) = dx^{\varepsilon}$, $\varepsilon > 1$, and replacing θ by d, Equation (C.6) is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial d} \\ \frac{\partial Q}{\partial d} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} -\frac{\varepsilon x^{\varepsilon-1}}{\gamma m} \\ 0 \end{bmatrix}.$$
 (C.24)

Carrying out the matrix multiplication, it can be shown that both

$$\frac{\partial x}{\partial d} = \frac{2f}{\Delta(e+f)} \frac{C_o}{\gamma Q^3} \left(-\frac{\varepsilon x^{\varepsilon-1}}{\gamma m} \right) \Rightarrow negative, \text{ and}$$
$$\frac{\partial Q}{\partial d} = \frac{a}{\Delta} \left(-\frac{\varepsilon x^{\varepsilon-1}}{\gamma m} \right) \Rightarrow negative.$$

Similarly replacing θ by ε , Equation (C.6) is rewritten as

$$\begin{bmatrix} \frac{\partial x}{\partial \varepsilon} \\ \frac{\partial Q}{\partial \varepsilon} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \frac{2f}{(e+f)} \frac{C_o}{\gamma Q^3} & a \\ a & -b \end{bmatrix} \begin{bmatrix} -\frac{d x^{\varepsilon-1}}{\gamma m} (1+\varepsilon \ln x) \\ 0 \end{bmatrix}.$$
 (C.25)

Carrying out the matrix multiplication, it can be shown that both

$$\frac{\partial x}{\partial \varepsilon} = \frac{2f}{\Delta(e+f)} \frac{C_o}{\gamma Q^3} \left(-\frac{d x^{\varepsilon-1}}{\gamma m} \right) (1+\varepsilon \ln x) \Rightarrow negative.$$
$$\frac{\partial Q}{\partial \varepsilon} = \frac{a}{\Delta} \left(-\frac{d x^{\varepsilon-1}}{\gamma m} \right) (1+\varepsilon \ln x) \Rightarrow negative.$$

C4. Steady state response of a modified Vidale-Wolfe model

The Vidale and Wolfe model (1957) takes the following form:

$$\frac{dS}{dt} = \beta x(m-S) - eS , \qquad (C.26)$$

where

dS/dt – the instantaneous change in the sales rate,

S – the sales rate,

x – the advertising rate,

 β – the advertising effectiveness parameter,

m – the market potential or saturation sales, and

e – the decay constant.

For the Vidale-Wolfe model, the advertising attraction function is linear and given

by $f(x) = \beta x$ and the market potential *m* is a constant quantity. Little (1979) proposed a modified version of the Vidale-Wolfe model for which f(x) takes a power function of the form $f(x) = \beta x^{\delta}$, where β and δ are positive constants.

The steady state sales response of the modified Vidale-Wolfe model is derived through setting dS/dt = 0, and solving Equation (C.26) for S to obtain

$$S(x) = \frac{m\beta x^{\delta}}{e + \beta x^{\delta}}.$$
 (C.27)

Expression (C.27) is the same as Equation (3.1) in the text.

APPENDIX D

COMPARATIVE STATICS IN A DUOPOLY DERIVED IN CHAPTER 3

This appendix derives the fundamental equation of comparative statics in a duopoly in section D1. The proof to Proposition 2 in Chapter 3 is demonstrated in section D2 and D3. Section D4 sheds light on the semi-definiteness of matrix **H**. Section D5 derives the steady-state response of a modified Kimball model.

D1. Fundamental equation of comparative statics in a duopoly

Define the profit function of firm *j* as

$$\pi_{j} = \gamma_{j} D(x_{j}) - C(x_{j}) - \frac{Q_{j}}{2} C_{hj} - \frac{D(x_{j})}{Q_{j}} C_{oj} - F_{j}, j = 1, 2.$$
(D.1)

Designating $\Pi_j = \pi_j / \gamma_j m$, so that at optimality

$$\frac{\partial \Pi_j}{\partial x_j} = 0 \text{ and } \frac{\partial \Pi_j}{\partial Q_j} = 0, \text{ for } j = 1, 2.$$
 (D.2)

For a duopoly, demands $D(x_1) = \frac{mf(x_1)}{f(x_1) + f(x_2)}$ and $D(x_2) = m - D(x_1)$, where *m* is the

market potential and $f(x_1)$ and $f(x_2)$ are the attraction functions of firms 1 and 2, respectively. After substituting for $D(x_j)$, j = 1, 2, Expression (C.3) for a monopoly is adapted to a duopoly as shown below

$$\Pi_{1} = \frac{f(x_{1})}{f(x_{1}) + f(x_{2})} \left[1 - \frac{C_{o1}}{\gamma_{1}Q_{1}} \right] - \frac{C(x_{1})}{\gamma_{1}m} - \frac{C_{h1}}{2\gamma_{1}m}Q_{1} - \frac{F_{1}}{\gamma_{1}m}.$$
 (D.3)

$$\Pi_{2} = \frac{f(x_{2})}{f(x_{1}) + f(x_{2})} \left[1 - \frac{C_{o2}}{\gamma_{2}Q_{2}} \right] - \frac{C(x_{2})}{\gamma_{2}m} - \frac{C_{h2}}{2\gamma_{2}m}Q_{2} - \frac{F_{2}}{\gamma_{2}m}.$$
 (D.4)

To study the sensitivity of each x_j^* and Q_j^* , j = 1, 2; to a change in one of the model parameters θ , Equations (D.2) are partially differentiated with respect to θ to obtain

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} \frac{\partial x_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_1} \frac{\partial Q_1}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial x_2} \frac{\partial x_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_2} \frac{\partial Q_2}{\partial \theta} + \frac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} = 0 ,$$

$$\frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{1}}\frac{\partial x_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}^{2}}\frac{\partial Q_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{2}}\frac{\partial x_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial Q_{2}}\frac{\partial Q_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial \theta} = 0 ,$$

$$\frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial x_{1}}\frac{\partial x_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{1}}\frac{\partial Q_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial x_{2}^{2}}\frac{\partial x_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{2}}\frac{\partial Q_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial \theta} = 0 , \text{ and}$$

$$\frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{1}}\frac{\partial x_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial Q_{1}}\frac{\partial Q_{1}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{2}}\frac{\partial x_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}^{2}}\frac{\partial Q_{2}}{\partial \theta} + \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial \theta} = 0 . \quad (D.5)$$

Here, it is also assumed that all second partial derivatives of Π_j with respect to each of the *equilibrium* values x_j^* and Q_j^* together with each of the *shift* parameters θ do exist and are continuous. The system of Equations (D.5) can be put in the following matrix form:

$$\begin{bmatrix} \frac{\partial^{2}\Pi_{1}}{\partial x_{1}^{2}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial Q_{1}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial x_{2}} & \frac{\partial^{2}\Pi_{1}}{\partial x_{1}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{1}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}^{2}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial x_{2}} & \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial x_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}^{2}} & \frac{\partial^{2}\Pi_{2}}{\partial x_{2}\partial Q_{2}} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial Q_{1}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial x_{2}} & \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}^{2}} \end{bmatrix} \begin{bmatrix} \frac{\partial x_{1}}{\partial \theta} \\ \frac{\partial Q_{1}}{\partial \theta} \\ \frac{\partial Q_{1}}{\partial \theta} \\ \frac{\partial x_{2}}{\partial \theta} \\ \frac{\partial Q_{1}}{\partial \theta} \\ \frac{\partial Q_{2}}{\partial \theta} \\ \frac{\partial Q_{2}}{\partial \theta} \end{bmatrix} = -\begin{bmatrix} \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial \theta} \\ \frac{\partial^{2}\Pi_{1}}{\partial Q_{1}\partial \theta} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial \theta} \\ \frac{\partial^{2}\Pi_{2}}{\partial Q_{2}\partial \theta} \end{bmatrix}. \quad (D.6)$$

Designating the 4 x 4 square matrix of the second-order partial derivatives in (D.6) by matrix \mathbf{H} , which is assumed to be non-singular, (D.6) can be written as

$$\begin{bmatrix} \frac{\partial x_1}{\partial \theta} \\ \frac{\partial Q_1}{\partial \theta} \\ \frac{\partial x_2}{\partial \theta} \\ \frac{\partial Q_2}{\partial \theta} \end{bmatrix} = -H^{-I} \begin{bmatrix} \frac{\partial^2 \Pi_1}{\partial x_1 \partial \theta} \\ \frac{\partial^2 \Pi_1}{\partial Q_1 \partial \theta} \\ \frac{\partial^2 \Pi_2}{\partial x_2 \partial \theta} \\ \frac{\partial^2 \Pi_2}{\partial Q_2 \partial \theta} \end{bmatrix}, \quad (D.7)$$

and the elements of matrix **H** are obtained by partially differentiating (D.3) and (D.4) in the following manner:

$$\frac{\partial \Pi_{1}}{\partial x_{1}} = \frac{f_{2}f_{1}'}{(f_{1} + f_{2})^{2}} \left[1 - \frac{C_{o1}}{\gamma_{1}Q_{1}} \right] - \frac{C_{1}'}{\gamma_{1}m} = 0, \qquad (D.8)$$

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} = \frac{f_2}{(f_1 + f_2)^3} \left\{ f_1''(f_1 + f_2) - 2f_1'^2 \right\} \left[1 - \frac{C_{o1}}{\gamma_1 Q_1} \right] - \frac{C_1''}{\gamma_1 m}, \quad (D.9)$$

$$\frac{\partial \Pi_1}{\partial Q_1} = \frac{f_1}{(f_1 + f_2)} \frac{C_{o1}}{\gamma_1 Q_1^2} - \frac{C_{h1}}{2\gamma_1 m} = 0, \qquad (D.10)$$

$$\frac{\partial^2 \Pi_1}{\partial Q_1^2} = -\frac{2f_1}{(f_1 + f_2)} \frac{C_{o1}}{\gamma_1 Q_1^3},$$
 (D.11)

$$\frac{\partial^2 \Pi_1}{\partial x_1 \partial x_2} = -\frac{f_1' f_2' (f_1 - f_2)}{(f_1 + f_2)^3} \left[1 - \frac{C_{o1}}{\gamma_1 Q_1} \right],$$
 D.12)

$$\frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_1} = \frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_1} = \frac{f_2 f_1'}{\left(f_1 + f_2\right)^2} \frac{C_{o1}}{\gamma_1 Q_1^2}, \qquad (D.13)$$

$$\frac{\partial^2 \Pi_1}{\partial x_1 \partial Q_2} = 0, \qquad (D.14)$$

$$\frac{\partial^2 \Pi_1}{\partial Q_1 \partial x_2} = -\frac{f_1 f_2'}{(f_1 + f_2)^2} \frac{C_{o1}}{\gamma_1 Q_1^2}, \qquad (D.15)$$

$$\frac{\partial \Pi_2}{\partial x_2} = \frac{f_1 f_2'}{(f_1 + f_2)^2} \left[1 - \frac{C_{o2}}{\gamma_2 Q_2} \right] - \frac{C_2'}{\gamma_2 m} = 0, \qquad (D.16)$$

$$\frac{\partial^2 \Pi_2}{\partial x_2^2} = \frac{f_1}{\left(f_1 + f_2\right)^3} \left\{ f_2''(f_1 + f_2) - 2f_2'^2 \right\} \left[1 - \frac{C_{o2}}{\gamma_2 Q_2} \right] - \frac{C_2''}{\gamma_2 m}, \qquad (D.17)$$

$$\frac{\partial \Pi_2}{\partial Q_2} = \frac{f_2}{f_1 + f_2} \frac{C_{o2}}{\gamma_2 Q_2^2} - \frac{C_{h2}}{2\gamma_2 m} = 0, \qquad (D.18)$$

$$\frac{\partial^2 \Pi_2}{\partial Q_2^2} = -\frac{2f_2}{f_1 + f_2} \frac{C_{o2}}{\gamma_2 Q_2^3},$$
 (D.19)

$$\frac{\partial^2 \Pi_2}{\partial x_2 \partial x_1} = -\frac{f_1' f_2' (f_2 - f_1)}{(f_1 + f_2)^3} \left[1 - \frac{C_{o2}}{\gamma_2 Q_2} \right], \qquad (D.20)$$

$$\frac{\partial^2 \Pi_2}{\partial x_2 \partial Q_2} = \frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_2} = \frac{f_1 f_2'}{\left(f_1 + f_2\right)^2} \frac{C_{o2}}{\gamma_2 Q_2^2},$$
 (D.21)

$$\frac{\partial^2 \Pi_2}{\partial x_2 \partial Q_1} = 0, \text{ and}$$
(D.22)

$$\frac{\partial^2 \Pi_2}{\partial Q_2 \partial x_1} = -\frac{f_2 f_1'}{(f_1 + f_2)^2} \frac{C_{o2}}{\gamma_2 Q_2^2}.$$
 (D.23)

Assuming a symmetric competition, the elements of matrix **H** can be simplified by putting $f = f_1 = f_2$, $Q = Q_1 = Q_2$, and so forth. The following definitions are also made for the derivations below:

$$a = \frac{f'}{4f} \frac{C_o}{\gamma Q^2}$$
 (positive term), and (D.24)

$$b = \frac{\left(f''f - f'^2\right)}{4f^2} \left[1 - \frac{C_o}{\gamma Q}\right] - \frac{C''}{\gamma m} \text{ (negative term).}$$
(D.25)

Matrix **H**, therefore, takes the following form:

$$\boldsymbol{H} = \begin{bmatrix} b & a & 0 & 0 \\ a & -\frac{C_o}{\gamma Q^3} & -a & 0 \\ 0 & 0 & b & a \\ -a & 0 & a & -\frac{C_o}{\gamma Q^3} \end{bmatrix}.$$
 (D.26)

Matrix **H** can be partitioned into four 2 x 2 matrices in the following manner.

$$\boldsymbol{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}.$$
 (D.27)

The inverse of matrix **H** (see Taha, 1992, p. 789 for details) is represented by the partitioned matrix \mathbf{H}^{-1} given by

$$\boldsymbol{H}^{I} = \begin{bmatrix} H^{11} & H^{12} \\ H^{21} & H^{22} \end{bmatrix}$$
(D.28)

where $H^{11} = (H_{11} - H_{12} \cdot H_{22}^{-1} \cdot H_{21})^{-1}$, and

$$H_{12} \cdot H_{22}^{-1} \cdot H_{21} = \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} \begin{bmatrix} b & a \\ a & -\frac{C_o}{\gamma Q^3} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}.$$
 (D.29)

Designating the determinant of H_{22} as $\Delta_1 = -\frac{bC_o}{\gamma Q^3} - a^2 > 0$ (D.30)

(Note that Δ_1 is positive per the second order sufficiency condition of optimality for firm 2).

Expression (D.29) takes on the following form:

$$H_{12} \cdot H_{22}^{-1} \cdot H_{21} = \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} \frac{1}{\Delta_1} \begin{bmatrix} -\frac{C_a}{\gamma Q^3} & -a \\ -a & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} 0 & 0 \\ -a^3 & 0 \end{bmatrix}.$$
 (D.31)

Matrix H^{11} , therefore, takes the following form:

$$H^{11} = \left(H_{11} - H_{12} \cdot H_{22}^{-1} \cdot H_{21}\right)^{-1} = \begin{bmatrix} b & a \\ a + \frac{a^3}{\Delta_1} & -\frac{C_o}{\gamma Q^3} \end{bmatrix}^{-1} = \frac{1}{\Delta_2} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} & -a \\ -a - \frac{a^3}{\Delta_1} & b \end{bmatrix}, \quad (D.32)$$

where determinant
$$\Delta_2 = -\frac{bC_o}{\gamma Q^3} - a^2 - \frac{a^4}{\Delta_1} = \Delta_1 - \frac{a^4}{\Delta_1} = \frac{(\Delta_1 - a^2)(\Delta_1 + a^2)}{\Delta_1} > 0.$$
 (D.33)

(Note that Δ_2 is positive since $\Delta_1 - a^2 > 0$ per the requirements of the positive semidefiniteness of matrix **H**. See subsection *D.4* of Appendix D for details). It should be noted that all the terms in H^{11} are negative. Turning now to matrix $H^{12} = -H^{11}.H_{12}.H_{22}^{-1}$, then

$$H^{12} = -\frac{1}{\Delta_2} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} & -a \\ -a - \frac{a^3}{\Delta_1} & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} \begin{bmatrix} b & a \\ a & -\frac{C_o}{\gamma Q^3} \end{bmatrix}^{-1}.$$
 (D.34)

After multiplication of the last two matrices, (D.34) takes the following form:

$$H^{12} = -\frac{1}{\Delta_1 \Delta_2} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} & -a \\ -a - \frac{a^3}{\Delta_1} & b \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{aC_o}{\gamma Q^3} & a^2 \end{bmatrix} = \frac{1}{\Delta_1 \Delta_2} \begin{bmatrix} \frac{a^2 C_o}{\gamma Q^3} & a^3 \\ -\frac{abC_o}{\gamma Q^3} & -a^2 b \end{bmatrix}.$$
 (D.35)

It should be noted that all the terms in H^{12} are positive.

Turning now to matrix $H^{21} = -H_{22}^{-1} \cdot H_{21} \cdot H^{11}$, and using the results from (D.32) and (D.33), H^{21} takes the following form:

$$H^{21} = -\frac{1}{\Delta_{1}} \begin{bmatrix} a^{2} & 0 \\ -ab & 0 \end{bmatrix} \frac{1}{\Delta_{2}} \begin{bmatrix} -\frac{C_{o}}{\gamma Q^{3}} & -a \\ -a - \frac{a^{3}}{\Delta_{1}} & b \end{bmatrix} = \frac{1}{\Delta_{1} \Delta_{2}} \begin{bmatrix} \frac{a^{2}C_{o}}{\gamma Q^{3}} & a^{3} \\ -\frac{abC_{o}}{\gamma Q^{3}} & -a^{2}b \end{bmatrix} = H^{12}.$$

Finally for matrix $H^{22} = H_{22}^{-1} - H^{21}H_{12}H_{22}^{-1}$, upon using the results obtained previously in (D.31) through (D.35), it can be shown that

$$H^{22} = \frac{1}{\Delta_1} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} & -a \\ -a & b \end{bmatrix} - \frac{1}{\Delta_1 \Delta_2} \begin{bmatrix} \frac{a^2 C_o}{\gamma Q^3} & a^3 \\ -\frac{abC_o}{\gamma Q^3} & -a^2b \end{bmatrix} \frac{1}{\Delta_1} \begin{bmatrix} 0 & 0 \\ \frac{aC_o}{\gamma Q^3} & a^2 \end{bmatrix}.$$

Multiplying the two matrices pertaining to the above expression produces

$$H^{22} = \frac{1}{\Delta_1} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} & -a \\ -a & b \end{bmatrix} - \frac{1}{\Delta_1^2 \Delta_2} \begin{bmatrix} \frac{a^4 C_o}{\gamma Q^3} & a^5 \\ -\frac{a^3 b C_o}{\gamma Q^3} & -a^4 b \end{bmatrix}, \text{ so that}$$
$$H^{22} = \frac{1}{\Delta_1} \begin{bmatrix} -\frac{C_o}{\gamma Q^3} \left(1 + \frac{a^4}{\Delta_1 \Delta_2} \right) & -a - \frac{a^5}{\Delta_1 \Delta_2} \\ -a + \frac{a^3 b C_o}{\Delta_1 \Delta_2 \gamma Q^3} & b + \frac{a^4 b}{\Delta_1 \Delta_2} \end{bmatrix}.$$
 (D.36)

It should be noted that all terms of Expression (D.36) are negative.

D2. Proof of proposition 2

D2.1 Comparative statics of γ

Omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculations and replacing θ by γ , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_1}{\partial \gamma_1} \\ \frac{\partial Q_1}{\partial \gamma_1} \\ \frac{\partial x_2}{\partial \gamma_1} \\ \frac{\partial Q_2}{\partial \gamma_1} \\ \frac{\partial Q_2}{\partial \gamma_1} \end{bmatrix} = \begin{bmatrix} \frac{C_o}{\Delta_2 \gamma Q^3} & - & - & - \\ \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) & - & - & - \\ -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & - \\ \frac{ab C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & - \\ 0 \\ \frac{ab C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & - \end{bmatrix} \begin{bmatrix} \left(\frac{f' C_o}{4fQ} + \frac{C'}{m} \right) \frac{1}{\gamma^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
(D.37)

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial \gamma_1} = \frac{C_o}{\Delta_2 \gamma Q^3} \left(\frac{f'C_o}{4fQ} + \frac{C'}{m} \right) \frac{1}{\gamma^2} \implies positive .$$

$$\frac{\partial Q_1}{\partial \gamma_1} = \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) \left(\frac{f'C_o}{4fQ} + \frac{C'}{m} \right) \frac{1}{\gamma^2} \implies positive .$$

$$\frac{\partial x_2}{\partial \gamma_1} = -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} \left(\frac{f'C_o}{4fQ} + \frac{C'}{m} \right) \frac{1}{\gamma^2} \implies negative .$$

$$\frac{\partial Q_2}{\partial \gamma_1} = \frac{abC_o}{\Delta_1 \Delta_2 \gamma Q^3} \left(\frac{f'C_o}{4fQ} + \frac{C'}{m} \right) \frac{1}{\gamma^2} \implies negative .$$

D2.2 Comparative statics of C_{h1}

Omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculations and replacing θ by C_{hl} , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_1}{\partial C_{h_1}} \\ \frac{\partial Q_1}{\partial C_{h_1}} \\ \frac{\partial x_2}{\partial C_{h_1}} \\ \frac{\partial Q_2}{\partial C_{h_1}} \\ \frac{\partial Q_2}{\partial C_{h_1}} \end{bmatrix} = \begin{bmatrix} - & \frac{a}{\Delta_2} & - & - \\ - & -\frac{b}{\Delta_2} & - & - \\ - & -\frac{a^3}{\Delta_1 \Delta_2} & - & - \\ - & \frac{a^2 b}{\Delta_1 \Delta_2} & - & - \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{2\gamma m} \\ 0 \\ 0 \end{bmatrix}.$$
(D.38)

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial C_{h1}} = \left(\frac{a}{\Delta_2}\right) \left(-\frac{1}{2\gamma m}\right) \implies negative.$$

$$\frac{\partial Q_1}{\partial C_{h1}} = \left(-\frac{b}{\Delta_2}\right) \left(-\frac{1}{2\gamma m}\right) \implies negative.$$

$$\frac{\partial x_2}{\partial C_{h1}} = \left(-\frac{a^3}{\Delta_1 \Delta_2}\right) \left(-\frac{1}{2\gamma m}\right) \implies positive.$$

$$\frac{\partial Q_2}{\partial C_{h1}} = \left(\frac{a^2 b}{\Delta_1 \Delta_2}\right) \left(-\frac{1}{2\gamma m}\right) \implies positive.$$

D2.3 Comparative statics of m

Replacing θ by *m*, Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_{1}}{\partial m} \\ \frac{\partial Q_{1}}{\partial m} \\ \frac{\partial Q_{2}}{\partial m} \end{bmatrix} = \begin{bmatrix} \frac{C_{o}}{\Delta_{2}\gamma Q^{3}} & \frac{a}{\Delta_{2}} & -\frac{a^{2}C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} \\ -\frac{a}{\Delta_{2}}\left(1+\frac{a^{2}}{\Delta_{1}}\right) & -\frac{b}{\Delta_{2}} & \frac{abC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} \\ -\frac{a^{2}C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} & \frac{C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}}\left(1+\frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) & \frac{a}{\Delta_{1}}\left(1+\frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \\ \frac{abC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} & \frac{a}{\Delta_{1}}\left(1-\frac{a^{2}bC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}}\right) & -\frac{b}{\Delta_{1}}\left(1+\frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \end{bmatrix} \begin{bmatrix} \frac{C'}{\gamma m^{2}} \\ \frac{C'}{\gamma m^{2}} \\ \frac{C'}{\gamma m^{2}} \\ \frac{C'}{\gamma m^{2}} \end{bmatrix}. \quad (D.39)$$

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial m} = \frac{C_o}{\gamma Q^3 \Delta_1 \Delta_2} \left(\Delta_1 - a^2 \right) \frac{C'}{\gamma m^2} + \frac{a}{\Delta_1 \Delta_2} \left(\Delta_1 - a^2 \right) \frac{C_h}{2\gamma m^2} \implies positive \ as \left(\Delta_1 - a^2 \right) > 0 \ .$$

 $\frac{\partial Q_1}{\partial m} = \frac{a}{\Delta_1 \Delta_2} \left(\Delta_1 + a^2 + \frac{bC_o}{\gamma Q^3} \right) \frac{C'}{\gamma m^2} - \frac{b}{\Delta_1 \Delta_2} \left(\Delta_1 - a^2 \right) \frac{C_h}{2\gamma m^2} \implies positive \ as \left(\Delta_1 - a^2 \right) > 0 \ and$

noting that $\Delta_1 + a^2 + \frac{bC_o}{\gamma Q^3} = 0$ from (D.30).

$$\frac{\partial x_2}{\partial m} = \frac{C_o}{\gamma Q^3 \Delta_1 \Delta_2} \left(\Delta_2 - a^2 + \frac{a^4}{\Delta_1} \right) \frac{C'}{\gamma m^2} + \frac{a}{\Delta_1 \Delta_2} \left(\Delta_2 - a^2 + \frac{a^4}{\Delta_1} \right) \frac{C_h}{2\gamma m^2} \implies positive \text{ and}$$

noting that $\Delta_2 - a^2 + \frac{a^4}{\Delta_1} > 0$ by (D.33).

$$\frac{\partial Q_2}{\partial m} = \left[\frac{a}{\Delta_1} + \frac{abC_o}{\Delta_1 \Delta_2 \gamma Q^3} \left(1 - \frac{a^2}{\Delta_1}\right)\right] \frac{C'}{\gamma m^2} - \frac{b}{\Delta_1 \Delta_2} \left(\Delta_2 - a^2 + \frac{a^4}{\Delta_1}\right) \frac{C_h}{2\gamma m^2} \quad \text{can be rewritten}$$

using (D.33) as

$$\frac{\partial Q_2}{\partial m} = \left[\frac{a\left(\Delta_1 + a^2 + bC_o/\gamma Q^3\right)}{\Delta_1(\Delta_1 + a^2)}\right] \frac{C'}{\gamma m^2} - \frac{b}{\Delta_1 \Delta_2} \left(\Delta_2 - a^2 + \frac{a^4}{\Delta_1}\right) \frac{C_h}{2\gamma m^2} \implies positive \text{ and}$$

noting from (D.33) that $\Delta_1 + a^2 + \frac{bC_o}{\gamma Q^3} = 0$ and $\Delta_2 - a^2 + \frac{a^4}{\Delta_1} > 0$.

D2.4 Comparative statics of C_{o1}

Omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculation and replacing θ by C_{ol} , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_{1}}{\partial C_{o1}} \\ \frac{\partial Q_{1}}{\partial C_{o1}} \\ \frac{\partial x_{2}}{\partial C_{o1}} \\ \frac{\partial Q_{2}}{\partial C_{o1}} \\ \frac{\partial Q_{2}}{\partial C_{o1}} \end{bmatrix} = \begin{bmatrix} \frac{C_{o}}{\Delta_{2}\gamma Q^{3}} & \frac{a}{\Delta_{2}} & - & -\\ \frac{a}{\Delta_{2}} \left(1 + \frac{a^{2}}{\Delta_{1}}\right) & -\frac{b}{\Delta_{2}} & - & -\\ -\frac{a^{2}C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} & - & -\\ \frac{abC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} & - & -\\ \end{bmatrix} \begin{bmatrix} -\frac{f'}{4f\gamma Q} \\ \frac{1}{2\gamma Q^{2}} \\ 0 \\ 0 \end{bmatrix}.$$
(D.40)

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_{1}}{\partial C_{o1}} = -\frac{f'}{4f} \frac{1}{\gamma Q} \left(\frac{C_{o}}{\Delta_{2} \gamma Q^{3}} \right) + \frac{1}{2} \frac{a}{\Delta_{2} \gamma Q^{2}} = -\frac{1}{2} \frac{a}{\Delta_{2} \gamma Q^{2}} \implies negative \text{ using (D.24).}$$

$$\frac{\partial Q_{1}}{\partial C_{o1}} = -\frac{f'}{4f} \frac{1}{\gamma Q} \frac{a}{\Delta_{2}} \left(1 + \frac{a^{2}}{\Delta_{1}} \right) - \frac{1}{2} \frac{b}{\Delta_{2} \gamma Q^{2}} = -\frac{b(\Delta_{1} - 2a^{2})}{2\Delta_{1} \Delta_{2} \gamma Q^{2}} \implies positive (negative) \text{ for }$$

$$(\Delta_{1} - 2a^{2}) > (<)0 \text{ by using expressions (D.24) and (D.30).}$$

$$\frac{\partial x_{2}}{\partial C_{o1}} = \frac{f'}{4f} \frac{1}{\gamma Q} \left(\frac{a^{2}C_{o}}{\Delta_{1} \Delta_{2} \gamma Q^{3}} \right) - \frac{1}{2} \frac{a^{3}}{\Delta_{1} \Delta_{2} \gamma Q^{2}} = \frac{1}{2} \frac{a^{3}}{\Delta_{1} \Delta_{2} \gamma Q^{2}} \implies positive \text{ by (D.24).}$$

$$\frac{\partial Q_{2}}{\partial C_{o1}} = -\frac{f'}{4f} \frac{1}{\gamma Q} \left(\frac{abC_{o}}{\Delta_{1} \Delta_{2} \gamma Q^{3}} \right) + \frac{1}{2} \frac{a^{2}b}{\Delta_{1} \Delta_{2} \gamma Q^{2}} = -\frac{1}{2} \frac{a^{2}b}{\Delta_{1} \Delta_{2} \gamma Q^{2}} \implies positive \text{ by (D.24).}$$

D3.1 Comparative statics of β_1 and δ_1

Define $f(x_j) = \beta_j x_j^{\delta_j}$, $0 < \delta_j < 1$, where, j = 1, 2. Omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculations and replacing θ by β_j , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_{1}}{\partial \beta_{1}} \\ \frac{\partial Q_{1}}{\partial \beta_{1}} \\ \frac{\partial A_{2}}{\partial \beta_{1}} \\ \frac{\partial A_{2}}{\partial \beta_{1}} \\ \frac{\partial Q_{2}}{\partial \beta_{1}} \\ \frac{\partial Q_{2}}{\partial \beta_{1}} \end{bmatrix} = \begin{bmatrix} \frac{C_{o}}{\Delta_{2}\gamma Q^{3}} & \frac{a}{\Delta_{2}} & - & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} \\ \frac{a}{\Delta_{2}} \left(1 + \frac{a^{2}}{\Delta_{1}}\right) & -\frac{b}{\Delta_{2}} & - & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} \\ -\frac{a^{2}C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} & - & \frac{a}{\Delta_{1}} \left(1 + \frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \\ \frac{abC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} & - & -\frac{b}{\Delta_{1}} \left(1 + \frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{C_{o}}{4\gamma Q^{2}\beta} \\ 0 \\ -\frac{C_{o}}{4\gamma Q^{2}\beta} \end{bmatrix}. \quad (D.41)$$

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial \beta_1} = \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) \frac{C_o}{4\gamma Q^2 \beta} \implies positive .$$

$$\frac{\partial Q_1}{\partial \beta_1} = -\frac{b}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) \frac{C_o}{4\gamma Q^2 \beta} \implies positive .$$

$$\frac{\partial x_2}{\partial \beta_1} = -\frac{a}{\Delta_1} \left(1 + \frac{a^4}{\Delta_1 \Delta_2} + \frac{a^2}{\Delta_2} \right) \frac{C_o}{4\gamma Q^2 \beta} \implies negative .$$

$$\frac{\partial Q_1}{\partial \beta_1} = \frac{b}{\Delta_1} \left(1 + \frac{a^4}{\Delta_1 \Delta_2} + \frac{a^2}{\Delta_2} \right) \frac{C_o}{4\gamma Q^2 \beta} \implies negative .$$

Similarly, omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculations and replacing θ by δ_l , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_{1}}{\partial \delta_{1}} \\ \frac{\partial Q_{1}}{\partial \delta_{1}} \\ \frac{\partial Q_{2}}{\partial \delta_{1}} \\ \frac{\partial Q_{2}}{\partial \delta_{1}} \\ \frac{\partial Q_{2}}{\partial \delta_{1}} \end{bmatrix} = \begin{bmatrix} \frac{C_{o}}{\Delta_{2}\gamma Q^{3}} & \frac{a}{\Delta_{2}} & - & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} \\ \frac{a}{\Delta_{2}} \left(1 + \frac{a^{2}}{\Delta_{1}}\right) & -\frac{b}{\Delta_{2}} & - & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} \\ -\frac{a^{2}C_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & -\frac{a^{3}}{\Delta_{1}\Delta_{2}} & - & \frac{a}{\Delta_{1}} \left(1 + \frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \\ \frac{abC_{o}}{\Delta_{1}\Delta_{2}\gamma Q^{3}} & \frac{a^{2}b}{\Delta_{1}\Delta_{2}} & - & -\frac{b}{\Delta_{1}} \left(1 + \frac{a^{4}}{\Delta_{1}\Delta_{2}}\right) \end{bmatrix} \begin{bmatrix} \frac{f'}{4\delta f} \left(1 - \frac{C_{o}}{\gamma Q}\right) \\ \frac{C_{o}}{4\gamma Q^{2}} \ln x \\ 0 \\ -\frac{C_{o}}{4\gamma Q^{2}} \ln x \end{bmatrix}. \quad (D.42)$$

Following similar steps as those employed to derive the comparative statics pertaining to

 β_l , it can be shown that

$$\frac{\partial x_1}{\partial \delta_1} > 0, \ \frac{\partial Q_1}{\partial \delta_1} > 0, \ \frac{\partial x_2}{\partial \delta_1} < 0, \ \frac{\partial Q_2}{\partial \delta_1} < 0.$$

D3.2 Comparative statics of d_1 and ε_1

Define $C(x_j) = d_j x_j^{\varepsilon_j}$, $\varepsilon_j > 1$, where, j = 1, 2. Omitting the terms of \mathbf{H}^{-1} which are not

pertinent to the calculations and replacing θ by d_I , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_1}{\partial d_1} \\ \frac{\partial Q_1}{\partial d_1} \\ \frac{\partial x_2}{\partial d_1} \\ \frac{\partial Q_2}{\partial d_1} \\ \frac{\partial Q_2}{\partial d_1} \end{bmatrix} = \begin{bmatrix} \frac{C_o}{\Delta_2 \gamma Q^3} & - & - & - \\ \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) & - & - & - \\ -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & - \\ \frac{a b C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & - \end{bmatrix} \begin{bmatrix} - & \frac{C \ln \varepsilon}{\gamma m d} \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$
(D.43)

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial d_1} = \frac{C_o}{\Delta_2 \gamma Q^3} \left(-\frac{C \ln \varepsilon}{\gamma m d} \right) \Rightarrow negative .$$
$$\frac{\partial Q_1}{\partial d_1} = \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) \left(-\frac{C \ln \varepsilon}{\gamma m d} \right) \Rightarrow negative .$$
$$\frac{\partial x_2}{\partial d_1} = -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} \left(-\frac{C \ln \varepsilon}{\gamma m d} \right) \Rightarrow positive .$$
$$\frac{\partial Q_2}{\partial d_1} = \frac{abC_o}{\Delta_1 \Delta_2 \gamma Q^3} \left(-\frac{C \ln \varepsilon}{\gamma m d} \right) \Rightarrow positive .$$

Similarly, omitting the terms of \mathbf{H}^{-1} which are not pertinent to the calculations and replacing θ by ε_l , Equations (D.7) take the form

$$\begin{bmatrix} \frac{\partial x_1}{\partial \varepsilon_1} \\ \frac{\partial Q_1}{\partial \varepsilon_1} \\ \frac{\partial x_2}{\partial \varepsilon_1} \\ \frac{\partial Q_2}{\partial \varepsilon_1} \\ \frac{\partial Q_2}{\partial \varepsilon_1} \end{bmatrix} = \begin{bmatrix} \frac{C_o}{\Delta_2 \gamma Q^3} & - & - & -\\ \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) & - & - & -\\ -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & -\\ \frac{abC_o}{\Delta_1 \Delta_2 \gamma Q^3} & - & - & -\\ \end{bmatrix} \begin{bmatrix} -\frac{C}{\gamma m} \left(\ln^2 \varepsilon + \frac{1}{\varepsilon} \right) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
(D.44)

After carrying out matrix multiplications, the following results are obtained:

$$\frac{\partial x_1}{\partial \varepsilon_1} = \frac{C_o}{\Delta_2 \gamma Q^3} \left\{ -\frac{C}{\gamma m} \left(\ln^2 \varepsilon + \frac{1}{\varepsilon} \right) \right\} \Rightarrow negative .$$
$$\frac{\partial Q_1}{\partial \varepsilon_1} = \frac{a}{\Delta_2} \left(1 + \frac{a^2}{\Delta_1} \right) \left\{ -\frac{C}{\gamma m} \left(\ln^2 \varepsilon + \frac{1}{\varepsilon} \right) \right\} \Rightarrow negative .$$
$$\frac{\partial x_2}{\partial \varepsilon_1} = -\frac{a^2 C_o}{\Delta_1 \Delta_2 \gamma Q^3} \left\{ -\frac{C}{\gamma m} \left(\ln^2 \varepsilon + \frac{1}{\varepsilon} \right) \right\} \Rightarrow positive .$$
$$\frac{\partial Q_2}{\partial \varepsilon_1} = \frac{abC_o}{\Delta_1 \Delta_2 \gamma Q^3} \left\{ -\frac{C}{\gamma m} \left(\ln^2 \varepsilon + \frac{1}{\varepsilon} \right) \right\} \Rightarrow positive (noting that b is negative)$$

D4. Negative semi-definiteness of matrix H

This property of matrix H is instrumental in ensuring uniqueness of the solution

related to the system of equations
$$\frac{\partial \Pi_1}{\partial x_1} = 0$$
, $\frac{\partial \Pi_1}{\partial Q_1} = 0$, $\frac{\partial \Pi_2}{\partial x_2} = 0$, and $\frac{\partial \Pi_2}{\partial Q_2} = 0$ (Gruca et

al., 1992). From the previous subsection matrix **H** for a symmetric duopoly takes on the form shown below

$$H = \begin{bmatrix} b & a & 0 & 0 \\ a & -\frac{C_o}{\gamma Q^3} & -a & 0 \\ 0 & 0 & b & a \\ -a & 0 & a & -\frac{C_o}{\gamma Q^3} \end{bmatrix}.$$

For matrix **H** to be negative semi-definite, the *k*th principal minor determinant Δ_{jj} of **H** would be either zero or has the sign of $(-1)^k$, k = 1, 2, 3, 4 (Taha, 1992, p. 789). For the above square matrix

$$\Delta_{11}=b<0,$$

 $\Delta_{22} = -\frac{bC_o}{\gamma Q^3} - a^2 = \Delta_1 > 0$, by the second order sufficiency condition of optimality of

firm 1 (or firm 2),

$$\Delta_{33} = b \left(-\frac{bC_o}{\gamma Q^3} - a^2 \right) = b \Delta_{22} < 0, \text{ which is negative as } b < 0 \text{ and } \Delta_{22} > 0, \text{ and}$$

$$\Delta_{44} = -\frac{C_o}{\gamma Q^3} (\Delta_1 - a^2).$$

For Δ_{44} to be ≥ 0 , $\Delta_1 - a^2$ should be ≥ 0 .

D5. Steady state response of a modified Kimball model

The Kimball model (1957) in a duopoly setting takes the following form:

$$\frac{dS_1}{dt} = -\frac{dS_2}{dt} = \beta_1 x_1 S_2 - \beta_2 x_2 S_1, \qquad (D.45)$$

and

$$S_1 + S_2 = m$$
, (D.46)

where

 dS_j/dt – the instantaneous change in the sales rate of firm j, j = 1, 2,

 S_j – the sales rate of firm j, j = 1, 2,

 x_j – the advertising effectiveness parameter of firm j, j = 1, 2, and

m – the market potential or saturation sales.

For the Kimball model, the advertising attraction function are linear and given by $f_j(x_j) = \beta_j x_j$, j = 1, 2, and the market potential *m* is a constant quantity. Little (1979) proposed a modified version of the Kimball model for which $f_j(x_j)$ takes a power function of the form $f_j(x_j) = \beta_j x_j^{\delta_j}$, where β_j and δ_j are positive constants, j = 1, 2.

The steady state sales response of the modified Kimball model are derived through setting $\frac{dS_1}{dt} = \frac{dS_2}{dt} = 0$, and solving Equations (D.45) and (D.46) for S_1 and S_2 to

produce

$$S_{1}(x_{1}) = \frac{m\beta_{1}x_{1}^{\delta_{1}}}{\beta_{1}x_{1}^{\delta_{1}} + \beta_{2}x_{2}^{\delta_{2}}} \text{ and } S_{2}(x_{2}) = m - S_{1}(x_{1}).$$
(D.47)

Expressions (D.47) are the same as Equation (3.5) in the text.

APPENDIX E

DERIVATION OF EXPRESSION (4.9) AND PROOFS OF PROPOSITIONS IN CHAPTER 4

This appendix derives Expression (4.9) in Chapter 4. Sections E2 through E5

provide the proofs of four propositions.

E1. Derivation of Expression (4.9)

The current value of Hamiltonian *H* is given by

$$H = (P - C) N - Ef - Q + \lambda f = (P - C) N - Q + (\lambda - E)f$$
(E.1)

The costate variable λ satisfies the following condition:

$$\frac{d\lambda}{dt} = r\lambda - \frac{\partial H}{\partial N} = r\lambda - [P - C - NC_N + (\lambda - E)f_N], \ \lambda(T) = 0.$$
(E.2)

The necessary condition of optimality implies that,

$$\frac{\partial H}{\partial U} = (\lambda - E)f_U - Q' = 0.$$
(E.3)

From (E.3),

$$\lambda - E = \frac{Q'}{f_U} > 0. \tag{E.4}$$

Given the properties of f and Q depicted in the text, then $\lambda > 0$.

Differentiating Expression (E.3) with respect to time t produces

$$\frac{\partial^2 H}{\partial U^2} \frac{dU}{dt} + \frac{\partial^2 H}{\partial U \partial N} \frac{dN}{dt} + \frac{\partial^2 H}{\partial U \partial \lambda} \frac{d\lambda}{dt} = 0.$$
(E.5)

Putting dN/dt = f, and substituting for $d\lambda/dt$ from (E.2) into (E.5) results, after minor arrangements of terms, in the following differential equation:

$$-\frac{\partial^2 H}{\partial U^2}\frac{dU}{dt} = \frac{\partial^2 H}{\partial U \partial N}f + \frac{\partial^2 H}{\partial U \partial \lambda}\left\{r\lambda - \frac{\partial H}{\partial N}\right\}.$$
 (E.6)

Furthermore, upon differentiating (E.3) using (E.4), the following quantities are obtained:

$$\frac{\partial^2 H}{\partial U^2} = (\lambda - E) f_{UU} - Q'' = \left(\frac{Q'}{f_U}\right) f_{UU} - Q'', \qquad (E.7)$$

which is negative as $f_{UU} < 0$ and $Q'' \ge 0$.

$$\frac{\partial^2 H}{\partial U \partial N} = (\lambda - E) f_{UN} = \left(\frac{Q'}{f_U}\right) f_{UN}, \text{ and}$$
(E.8)

$$\frac{\partial^2 H}{\partial U \partial \lambda} = f_U > 0. \tag{E.9}$$

From (E.1) and (E.4), the following expression is obtained:

$$\frac{\partial H}{\partial N} = P - C - NC_N + (\lambda - E)f_N = P - C - NC_N + \left(\frac{Q'}{f_U}\right)f_N.$$
(E.10)

Based on (E.7) through (E.10), (E.6) takes on the final form

$$-\frac{\partial^2 H}{\partial U^2}\frac{dU}{dt} = -\frac{Q'}{f_U}(f_N f_U - f_{UN} f) - f_U \{P - C - NC_N\} + r\lambda f_U.$$
(E.11)

Expression (E.11) is the same as Equation (4.9) depicted in the text.

E2. Proof of proposition 1

Defining advertising elasticity of demand, $\Lambda = \frac{\partial f}{\partial U} \left(\frac{U}{f} \right) = \frac{U f_U}{f}$,

$$\frac{\partial \Lambda}{\partial N} = -\frac{U(f_U f_N - f f_{UN})}{f^2}.$$
(E.12)

For
$$\frac{\partial \Lambda}{\partial N} \leq 0$$
, $f_U f_N - f f_{UN} \geq 0$. Putting $r = 0$ in (4.9), the sign of $\frac{dU}{dt}$ becomes negative

as
$$\frac{\partial^2 H}{\partial U^2}$$
 is negative by (E.7), $Q' > 0$, $Q'' > 0$, $f_U > 0$ and $P - C - NC_N > 0$ by assumptions.

E3. Proof of proposition 2

When r approaches ∞ , $\frac{dU}{dt}$ in (4.9) would have the same sign as the positive quantity

 $r\lambda f_U$ since $\lambda > 0$ from (E.4) and $f_U > 0$.

Noting the properties of f, Q, and C it can be deducted that in Expression (4.9), $\frac{\partial^2 H}{\partial U^2} < 0$

and $f_U \{ P - C - NC' \} > 0$. Therefore, in Expression (4.9), $\frac{dU}{dt} > 0$, which completes the

proof.

E4. Proof of proposition 3

Considering the specific diffusion model given by model M1 in main text

$$\frac{dN}{dt} = f(N,U) = ph(U)[M-N] + \frac{q(1-\delta)N}{M}[M-N] - \delta N$$
(E.13)

Therefore,

$$f_U = p(M - N)h'. \tag{E.14}$$

$$f_N = -ph + q(1 - \delta) \left\{ 1 - \frac{2N}{M} \right\} - \delta$$
 (E.15)

$$f_{UN} = -ph'. \tag{E.16}$$

Combining Expressions (E.13) through (E.16), it can be shown that

$$f_{N}f_{U} - f_{UN}f = q(1-\delta)pMh' - q(1-\delta)\frac{2N}{M}pMh' - \delta pMh' + q(1-\delta)\frac{ph'N^{2}}{M}$$
$$= f_{N}f_{U} - f_{UN}f = \frac{ph'}{M} \left\{ q(1-\delta)(M-N)^{2} - \delta M^{2} \right\}.$$
(E.17)

When $\delta = 0$, Expression (E.17) takes the following form:

 $f_N f_U - f_{UN} f = \frac{qph'}{M} (M - N)^2$, which is positive. Also noting that in Expression (4.9),

$$\frac{\partial^2 H}{\partial U^2} < 0$$
 and $f_U \{ P - C - NC' \} > 0$, so that for $r = 0$, $\frac{dU}{dt} < 0$, which completes the proof.

E5. Proof of proposition 4

In the presence of a disadoption rate δ , from Expression (E.17):

$$f_N f_U - f_{UN} f = \frac{ph'}{M} \left\{ q(1-\delta)(M-N)^2 - \delta M^2 \right\} \text{ which is positive for}$$
$$N \le M \left[1 - \sqrt{\frac{\delta}{q(1-\delta)}} \right], \text{ resulting in } \frac{dU}{dt} < 0, \text{ in Expression (4.9) when } r = 0 \text{ as}$$
$$\{ P - C - NC_N \} > 0. \text{ For } N > M \left[1 - \sqrt{\frac{\delta}{q(1-\delta)}} \right], \text{ however, } f_N f_U - f_{UN} f \text{ becomes}$$

negative and $\frac{dU}{dt}$ could be positive for a sufficiently long planning horizon.

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APPENDIX F

CABLE TV DIFFUSION DATA IN CANADA USED IN CHAPTER 4

The appendix depicts the annual diffusion data of basic cable TV analyzed in Chapter 4. The data cover the period 1976 – 1994 in Canada together with four of its provinces (Quebec, Ontario, Nova Scotia, and New Brunswick). The data for the last two provinces are available in a combined format.

Nova Scotia & New Brunswick	Number of subscribers	107 129	156	183	204	216	224	247	270	281	303	322	343	372	389	399	406	421	438
	Advertising expense	238 330	391	398	347	378	388	376	480	461	534	516	912	749	838	703	730	1024	1207
Ontario	Number of subscribers	1459 1518	1652	1761	1817	1923	1999	2024	2171	2274	2368	2479	2580	2692	2794	2846	2895	2969	3023
	Advertising expense	4405 4165	4650	5006	5539	5982	6074	6576	7710	8815	10523	10389	13501	14733	15912	14273	13968	14840	16720
Quebec	Number of subscribers	584 684	737	776	826	952	866	1028	1088	1194	1317	1408	1500	1573	1625	1677	1726	1768	1814
	Advertising expense	1971 2316	2716	3197	2647	4461	4507	4687	5194	7087	10058	12502	13784	15265	17343	18550	19772	20131	19310
Canada	Number of subscribers	3144 3417	3775	4084	4340	4700	4934	5141	5389	5673	6005	6310	6559	6886	7121	7286	7463	7657	7833
	Advertising expense	9345 9738	11420	12449	12022	14485	14866	15992	19055	22783	28155	29774	36165	40051	45286	42512	43990	44929	49650
	Year	1976 1977	1978	6261	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1661	1992	1993	1994

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Advertising expenses are in thousands of Canadian dollars and numbers of subscribers are in thousands of adopters.

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