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Obstruction sets for classes of cubic graphs

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OBSTRUCTION SETS FOR CLASSES OF CUBIC GRAPHS

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ABSTRACT

This dissertation establishes two theorems which characterize the set of minimal obstructions for two classes of graphs. A minimal obstruction for a class of graphs is a graph that is not in the class but every graph that it properly contains, under some containment relation, is in the class. In Chapter 2, we provide a characterization of the class of cubic outer-planar graphs in terms of its minimal obstructions which are also called cubic obstructions in this setting. To do this, we first show that all the obstructions containing loops can be obtained from the complete set of loopless obstructions via an easily specified operation. We subsequently prove that there are only two loopless obstructions and then generate the complete list of 5 obstructions.

In Chapters 3 and 4, we provide a characterization for the more general class of outer-cylindrical graphs – those graphs that can be embedded in the plane so that there are two faces whose boundaries together contain all the vertices of the graph. In particular, in Chapter 3, we build upon the ideas of Chapter 2 by considering the operation used to generate all obstructions containing loops from those that are loopless and extend this operation to the class of outer-cylindrical graphs. We also provide a list of 26 loopless graphs and prove that each of these is a cubic obstruction for outer-cylindrical graphs. In Chapter 4, we prove that these 26 graphs are the only loopless cubic obstructions for outer-cylindrical graphs. Combining the results of Chapters 3 and 4, we then generate the complete list of 124 obstructions which is provided in an appendix.

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CHAPTER 1

INTRODUCTION

In this chapter, we provide a brief introduction to the basic ideas of graph theory that are used throughout this dissertation. The reader who is already familiar with graphs is encouraged to turn to Section 1.8. Unless stated otherwise, the terminology used here will follow West [15] and Diestel [5]. In addition, some concepts related to the fundamental structure of graphs are taken from Bondy and Murty [3] and Oxley [12].

1.1 Introduction to Graphs

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *endvertices* or *ends.* A *loop* is an edge whose endvertices are identical. *Parallel* or *multiple edges* are non-loop edges having the same pair of endvertices. Two distinct vertices u and v are *neighbors* if they are the ends of an edge, in which case u and v are also called *adjacent* vertices.

A *simple graph* is a graph having no loops or multiple edges. In the case of simple graphs an edge with ends u and v is usually written uv or vu, that is uv and vu are the same edge. If a vertex v is an endvertex of an edge e, then v and e are said to be *incident.* The *degree* of a vertex v in a graph G is the number of non-loop edges incident with v plus twice the number of loops incident with v. We denote the degree of v as $d(v)$ or $d_G(v)$. The minimum vertex degree of a graph G is denoted $\delta(G)$, and if all the vertices of G have degree k, then G is called *k-regular.* A graph that is 3-regular is also called a *cubic graph.*

A *subgraph* of a graph G is a graph H obtained by deleting edges or vertices of G. We write $H \subseteq G$ and say that H is a subgraph of G or *G contains H as subgraph.*

If V' is a set of vertices in a graph G, then $G - V'$ will denote the subgraph of G obtained by deleting all the vertices in V' and the edges incident with them. If V' consists of a single vertex v, then we also denote $G - \{v\}$ as $G - v$. If X is a set of edges in G, then $G \backslash X$ will denote the subgraph of G obtained by deleting all the edges in X. If X consists of a single edge e, then we also denote $G \backslash \{e\}$ as $G \backslash e$. If $S \subseteq G$, the *induced subgraph* on S can be obtained by deleting the set of vertices $V(G) - S$ from G. The induced subgraph is denoted by $G[S]$.

1.2 Standard Graphs and Graph Classes

A *path* is a non-empty graph P where $V(P) = \{x_0, x_1, \ldots, x_{k-1}\}$ and $E(P) = \{x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}\}$. The path P is often denoted by the string $x_0x_1 \ldots x_{k-1}$, and called a path *from* x_0 *to* x_{k-1}. A path from a vertex u to a vertex v is called a *(u,v)-path.* For a path $P = x_0x_1 \ldots x_{k-1}$ in G we denote the path $x_ix_{i+1} \ldots x_j$ as $P[x_i, x_j]$ as defined in [14].

The *end vertices* or *ends* of a (u,v)-path P are the vertices u and v; the other vertices are called the *internal vertices* of P and are denoted $I(P)$. Similarly, the *end edges* of a (u,v)-path are the edges of the path that are incident to u or v where the remaining edges are called the *internal edges* of the path. A path with n edges is denoted P_n. Two paths are called *independent* if $I(P) \cap I(Q) = \emptyset$. Similarly, P and Q are said to be *edge-disjoint* provided that $E(P) \cap E(Q) = \emptyset$. A *cycle* is a path $P = x_0 \ldots x_{k-1}$ together with an edge e not in P where the ends of e are x_0 and x_{k-1}. A cycle with n vertices is called an *n-cycle* and will be denoted C_n.

Figure 1.1: Graphs K_4 and $K_{2,3}$.

If any two vertices of a simple graph G are neighbors then G is called a *complete graph.* The complete graph with n vertices is denoted K_n. So, K_1 is a single vertex,

K_2 is an edge, and K_3 is a 3-cycle. A depiction of K_4 can be seen in Figure 1.1. A *complete bipartite graph* is a simple graph having a partition of its vertices into two disjoint sets S and T such that two vertices s and t are adjacent if and only if s is in S and t is in T. When the sets have sizes $|S| = m$ and $|T| = n$, the graph is denoted $K_{m,n}$. If G_1 and G_2 are subgraphs of a graph G, then the *union* of G_1 and G_2, denoted $G_1 \cup G_2$, is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint, we denote their union by $G_1 \mathring{\cup} G_2$.

1.3 Connectivity

A graph is *connected* if any two vertices are joined by a path. Otherwise it is *disconnected*. The *components* of a graph are its maximal connected subgraphs. Let S be a set of vertices in a graph G. The set S is a *vertex cut* of G if $G - S$ has more components than G. A vertex cut of size n is called an *n-vertex-cut*, and a graph is called *n-connected* if every vertex cut contains at least n vertices. If v is a vertex whose deletion increases the number of components, then v is called a *cut-vertex*. The *connectivity* or *vertex-connectivity* of G, denoted $\kappa(G)$, is the minimum size of a vertex cut of G. A connected graph that has no cut vertices is called a *block*. A *block of a graph* is a subgraph that is a block and is maximal with this property.

Now, if X is a subset of $E(G)$, then X is an *edge cut* of G if $G \backslash X$ has more components than G. The set X is said to be a *minimal edge cut* if X is an edge cut but, for any edge x in X, the set $X - \{x\}$ is not. An edge cut of size n is called an *n-edge-cut*, and a graph is called *k-edge-connected* if every edge cut contains at least k edges. A *cut-edge* is an edge whose deletion increases the number of components. The *edge-connectivity* of G, denoted $\lambda(G)$, is the minimum size of an edge cut of G.

Menger's Theorem [11] is very useful in proving structural results related to graph connectivity. We state the edge-form of Menger's Theorem followed by a lemma that shows that if G is a cubic graph, then k-edge-connectivity implies there are k independent paths between any two vertices.

Theorem 1.3.1 *(Menger[11]) A graph is k-edge-connected if and only if it contains k edge-disjoint paths between any two distinct vertices.*

Lemma 1.3.2 *A k-edge-connected cubic graph contains k independent paths between any two distinct vertices.*

Proof: Let G be a k-edge-connected cubic graph and let v and w be two vertices in G. From Theorem 1.3.1 there are k edge-disjoint paths from v and w. Trivially if $k = 0$ then there are at least $k = 0$ independent paths between any two vertices. If $k = 1$ then there is a path from v to w. Thus, we may assume that $k \geq 2$. Let $P_1, \ldots, P_k$ be a set of k edge-disjoint paths from v to w such that two of these paths share a common internal vertex.

Assume that P_1 and P_2 have a common internal vertex u. Since u is an internal vertex of P_1, there are 2 edges e_1 and e_1' incident to u that are in P_1. Likewise, u is an internal vertex of P_2, so there are 2 edges e_2 and e_2' incident to u that are in P_2. Since P_1 and P_2 are independent paths it is clear that $\{e_1, e_1'\} \cap \{e_2, e_2'\} = \emptyset$. Thus u is incident to at least four edges; a contradiction of G being cubic. Therefore, it must be that no two paths share a common internal vertex, and we conclude that there are k independent paths between any two vertices. ■

The following description for 2-connected graphs is taken directly from [12]. Suppose $k \geq 2$. A connected graph G is called a *generalized cycle* with *parts* $G_1, \ldots, G_k$ if the following conditions hold:

(i) Each G_i is a connected subgraph of G having a non-empty edge set, and, if $k = 2$, both G_1 and G_2 have at least three vertices.

(ii) The edge sets of $G_1, \ldots, G_k$ partition the edge set of G, and each G_i shares exactly two vertices, its *contact vertices*, with $\bigcup_{j \neq i} G_j$

(iii) If each G_i is replaced by an edge joining its contact vertices, the resulting graph is a cycle.

The following lemma is taken from [12, (5.3.4)] and guarantees that a graph of connectivity two has a representation as a generalized cycle where each part is a block.

Lemma 1.3.3 *(Oxley [12]) Let G be a block having at least four vertices and suppose that G is not 3-connected. Then G has a representation as a generalized cycle, each part of which is a block.*

1.4 Minors

A minor of a graph G is a graph obtained from G by performing a sequence of two easily specified operations. The first is the deletion of an edge. The second is called the *contraction* of an edge. The contraction of a non-loop edge e in G with endvertices u and v can be obtained by replacing u and v with a single vertex whose incident edges are the edges other that e that were incident to u or v in G where if the edge was incident to both u and v, the edge is now a loop. The contraction of a loop is the same as the deletion of the loop from G. In either case the contraction of an edge uv from G is denoted G/uv.

We say that a graph H is a *minor* of a graph G, and write $H \preceq G$, if H can be obtained from G through a sequence of single-edge deletions or contractions. The *minor order* is a partial ordering on the set of all graphs where the order is defined by $\preceq$. A class of graphs $\mathcal{G}$ is *minor-closed* if for any G in $\mathcal{G}$, every minor of G is also in $\mathcal{G}$.

The graph G' obtained from G by replacing edges of G with independent paths between their ends is called a *subdivision* of G, and we call these independent paths of G' *subdivisional paths.* If the ends of a subdivisional path P are u and v, then P is called a *(u,v)-subdivisional path.* In this context, we view $V(G)$ as a subset of $V(G')$ and call these vertices the *branch vertices* of G'. The other vertices of G' are called the *subdividing vertices* of G'. Now, if G contains a subgraph that is a subdivision of H, then H is also called a *topological minor* of G.

1.5 Embeddings

A *curve* is the image of a continuous map from $[0,1]$ to $\mathbb{R}^2$. It is called an *(x,y)-curve* when it starts at a point x and ends at a point y. A *drawing* or *embedding* in the plane of a graph G is a function ϕ defined on $V(G) \cup E(G)$ that assigns each vertex v a point $\phi(v)$ in the plane and assigns each edge from a vertex u to a vertex v a $(\phi(u), \phi(v))$-curve, where $\phi(u) = \phi(v)$ if and only if $u = v$. For distinct edges e and e' in $E(G)$, a point in $\phi(e) \cap \phi(e')$ that is not a common endpoint is called a *crossing.*

A graph is *planar* if it has a drawing in the plane without crossings. Such a drawing is a *planar embedding* of G, and a *plane graph* is a graph that has been embedded in the plane without crossings. Kuratowski characterized planar graphs by proving necessary and sufficient conditions for a graph to be planar in terms of a set of minimal non-planar graphs [9]. We now state this result and the analogue for simple cubic graphs as reported by Glover and Huneke [7].

Theorem 1.5.1 *(Kuratowski [9]) A graph G is planar if and only if G does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Theorem 1.5.2 *(Kuratowski [9]) A simple cubic graph G is planar if and only if G does not contain a subgraph that is a subdivision of $K_{3,3}$.*

An *open set* in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every point p in the set U, all points different from p within some small euclidean distance from p belong to U. A *region* is an open set U that contains a (p,q)-curve for every pair of points p, q in U. The *faces* of a plane graph G are the maximal regions of the plane that contain no point used in the embedding of G. A vertex v is said to *lie on the boundary* of a face f if for every open set U containing v, the set $U \cap f$ is not empty. Let e be an edge of a plane graph and r be a point of the (p,q)-curve corresponding to e. Suppose that r does not correspond to an end of e, then e is said to *lie on the boundary* of a face f if for every open set U containing r, the set $U \cap f$ is not empty. If a vertex or edge lies on the boundary of a face f, then it is also said to *border* f.

The following is taken from Oxley [12, page 284]. "Two planar embeddings G_1 and G_2 of G are said to be *equivalent* if the set of edges on the boundary of a face in G_1 always corresponds to the set of edges on the boundary of a face in G_2. We say that G is *uniquely embeddable in the plane* if any two planar embeddings of G are equivalent."

Theorem 1.5.3 *(Whitney[16]) Let G be a simple 3-connected planar graph. Then G is uniquely embeddable in the plane.*

A planar graph G is said to be *outer-planar* if it has an embedding such that all the vertices of G lie on the boundary of one face. A graph embedded in such a way is called an *outer-plane graph*. The following lemma is straightforward and is stated here for reference (see Oxley [12, Section 11.2]).

Lemma 1.5.4 *(Oxley [12]) Let G be a graph. Then*

(i) G is outer-planar if and only if its components are outer-planar.

(ii) G is outer-planar if and only if its blocks are outer-planar.

Just as Theorem 1.5.1 characterizes planar graphs, the following theorem is an analogous characterization for the class of outer-planar graphs, first proved by Chartrand and Harary [4].

Theorem 1.5.5 *(Chartrand, Harary 1967 [4]) A graph G is outer-planar if and only if G does not contain a subgraph that is a subdivision of K_4 or $K_{2,3}$.*

To complete this section, we now formalize some concepts related to the embedding of graphs that are generalized cycles. Let G be a plane generalized cycle with parts $G_1, G_2, \ldots, G_k$, and let C be the cycle corresponding to G as described in part iii of the definition of generalized cycle. Clearly there are two faces f and g incident to every edge of the embedding of C induced by the embedding of G. Likewise if we replace each edge of C with its corresponding part in G, we obtain an embedding of G where there are two faces $f' \subseteq f$ and $g' \subseteq g$ incident to every part of G. We call these faces the *central faces* of G and if the boundary of f or the boundary of g contains all the vertices of G_i, then G_i is said to be *centralized*.

1.6 Cubic Order

Cubic graphs have been of interest in a number of settings. For example, Glover and Huneke found the cubic obstructions for the class of projective planar graphs

[7]. Archdeacon and Bonnington found the cubic obstructions for the class of spindle graphs where they also describe the cubic order by saying, "The *cubic order* makes smaller [simple] graphs by edge deletions followed by suppressing the resulting degree-two vertices." [1]. In this context, a cubic graph G is a *cubic obstruction* for a class of graphs $\mathcal{G}$ if G is not in $\mathcal{G}$, but every graph that G properly contains in the cubic order is in $\mathcal{G}$. Here we formally define an operation over the class of all cubic graphs that produces the cubic order.

Let G be a cubic graph. The graph resulting from the *cubic edge-deletion* of an edge e of G, denoted $G\backslash_3 e$ is constructed from G by deleting e followed by the contraction of two edges. Specifically if e is not a loop, then we contract two non-adjacent edges incident with vertices of degree two and delete all resulting vertices of degree zero. The graph resulting from a *cubic edge-deletion* of a loop e' can be obtained by deleting e', contracting the edge that was adjacent to the loop, and then contracting an edge incident to the resulting vertex of degree two. This operation ensures that the cubic edge-deletion of any edge in a cubic graph results in a cubic graph or an empty graph. The *cubic order* is a partial ordering on the set of cubic graphs together with the operation of cubic edge-deletion.

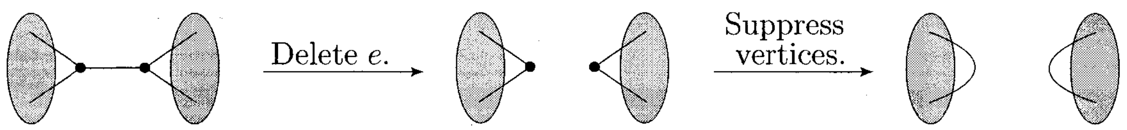


Figure 1.2: The cubic edge-deletion of a non-loop edge e

Recall that a graph H is a topological minor of a graph G provided that G contains a subgraph that is a subdivision of H. Archdeacon and Bonnington observe that the topological-minor order is equivalent to the cubic order for the class of cubic graphs [1]. Upon combining these two ideas, it is easy to derive the following proposition.

Proposition 1.6.1 *Let G and H be cubic graphs. Then G contains H in the cubic order if and only if G contains a subgraph that is a subdivision of H.*

We conclude this section by proving two lemmas related to a plane cubic graph G having a 2-cycle C. In the first lemma, we show that if a vertex of C lies on the boundary of a face f, then there is an edge of C that also lies on the boundary of f. In the second lemma, we show that the vertices of C border the same set of faces of G.

Lemma 1.6.2 *Let G be a plane cubic graph having a 2-cycle C. If f is a face bordered by a vertex of C then there is an edge of C that also borders f.*

Proof: Assume $V(C) = \{x, y\}$, $E(C) = \{e_1, e_2\}$, and f is a face bordered by x. Let e be the edge incident to x that is not in $E(C)$. Since x is not an isolated vertex, one of the edges, e_1, e_2, or e, incident to x borders f. If e borders f then, since G is cubic, one of e_1 or e_2 borders f. Thus one of the edges of C must border f. ■

Lemma 1.6.3 *Let G be a plane cubic graph having a 2-cycle C. Then, the vertices of C border the same set of faces.*

Proof: Assume $V(C) = \{x, y\}$, $E(C) = \{e_1, e_2\}$, and f is a face bordered by x. From Lemma 1.6.2 either e_1 or e_2 must border f. Since both e_1 and e_2 are incident to y, the vertex y must border f. ■

1.7 Structural Properties

In the previous section, we concluded with two straightforward lemmas related to a plane cubic graph having a 2-cycle. In this section, we show three lemmas that reveal some of the structure of graphs having a 2-edge-cut. Specifically, in the first lemma, we show that the edges of a 2-edge-cut of a plane 2-edge-connected graph border the same set of faces. In the second lemma, we identify some conditions for a graph having a 2-edge-cut that allows the graph to contain a K_4-subdivision. Finally, in the last lemma we identify some conditions for a cubic graph to have a subgraph of a certain type which we now describe.

A *storm* of Figure 1.3 is a 3-path with an additional edge in parallel to the internal edge of the 3-path. The *ends* of the 3-path are the ends of the storm. The graph of the storm is depicted in Figure 1.3.

Figure 1.3: A storm

Lemma 1.7.1 *Let G be a plane 2-edge-connected graph. The edges of any 2-edge-cut of G border the same set of faces.*

Proof: Let e_1 and e_2 be the edges of a 2-edge-cut of a plane 2-edge-connected graph G. Let G_1 and G_2 be the components of $G\backslash\{e_1, e_2\}$ and let $v_{i,j}$ be the end of e_j in G_i. Now there is no path from G_1 to G_2 in $G\backslash\{e_1, e_2\}$, so there must be a single face f of $G\backslash\{e_1, e_2\}$ having each of $v_{1,1}$, $v_{1,2}$, $v_{2,1}$, and $v_{2,2}$ on its boundary. Embed e_1 in f and we see that e_1 now lies in f. Upon the embedding of e_2, we see that there are two faces, f' and f'', whose boundaries contain e_1 and e_2. Thus e_1 and e_2 border the same faces in G. ■

Lemma 1.7.2 *Let G be a connected graph having a maximum degree of three and assume that there are three independent paths P_1, P_2, and P_3 in G having internal vertices and sharing ends w_1 and w_2. If the end edges of one of P_1, P_2, or P_3, do not form a 2-edge-cut in G, then G contains a subgraph that is a subdivision of K_4.*

Proof: Without loss of generality, assume that the end edges of P_1 do not form a 2-edge-cut. Let E_1 be the set of end edges of P_1 and recall that $I(P_i) = V(P_i) - \{w_i, w_2\}$. Now, since $G\backslash E_1$ is connected, there is a path from $I(P_1)$ to $I(P_2) \cup I(P_3)$ in $G\backslash E_1$. Let P be a shortest such path from $I(P_1)$ to $I(P_2) \cup I(P_3)$. Let v be the end of P in $I(P_1)$ and w be the end of P in $I(P_2) \cup I(P_3)$. Assume Q is the path in $\{P_2, P_3\}$

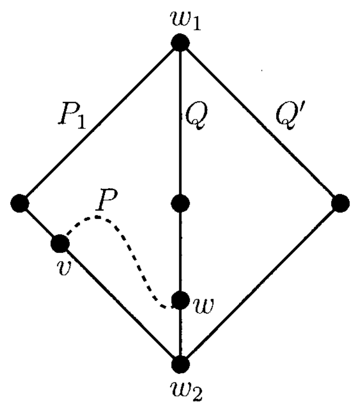


Figure 1.4: The end edges of each P_i form a 2-edge-cut.

that contains w and Q' is the other path in $\{P_2, P_3\}$ different from Q. We now show that P does not contain the end edges of Q or Q'.

If P contains an end edge of Q', then P contains an internal vertex of Q' so P is not a shortest path from $I(P_1)$ to $I(P_2) \cup I(P_3)$ – a contradiction to the choice of P. Thus we see that P avoids the end edges of Q'. Now, suppose that P contains an end edge of Q. Then P must also contain one of w_1 or w_2. Without loss of generality assume that P contains vertex w_1. Since the maximum degree of G is three and P is a path, it is easy to see that P must also contain an end edge of P_1 or Q' – a contradiction since P avoids the end edges of P_1 and Q'. So P does not contain the end edges of Q or Q' and it must be that G has a subgraph that is a subdivision of K_4 where v, w, w_1, and w_2 are the branch vertices of the K_4-subdivision and the subdivisional paths are P, Q', $P_1[w_1, v]$, $P_1[v, w_2]$, $Q[w_1, w]$, and $Q[w, w_2]$. ∎

Lemma 1.7.3 *Let G be a cubic 2-edge-connected graph and let P be a path in G. If the end edges of P form a 2-edge-cut, then the vertices of P are in a subgraph that is a subdivision of a storm S where the ends of P are the ends of S.*

Proof: Since the end edges of P form a 2-edge-cut, $I(P)$ is not empty. So assume v and w are the ends of P, vv' and ww' are the end edges of P, and G' is the component of $G \backslash \{vv', ww'\}$ containing the internal vertices of P. First, assume $v' = w'$ (see

Figure 1.5). Since $d(v') = 3$, let e be the edge that is not vv' or ww' and let x' denote the end of e different from v'. Since the edges vv' and ww' disconnect G, it is clear that x' is neither v nor w. If there is a path from x' to $V(G) - V(G')$ not using edge e, then the removal of vv' and ww' would not disconnect G. So every path from x' to $V(G) - V(G')$ contains e and so $G \backslash e$ is disconnected – a contradiction since G is 2-edge-connected. Thus, it must be that v' and w' are distinct.

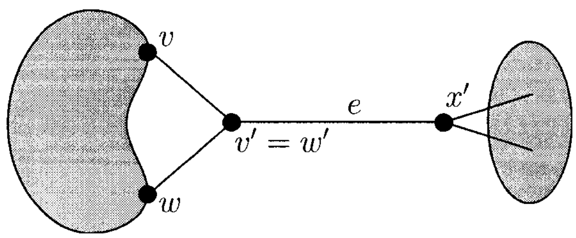


Figure 1.5: The case where $v' = w'$

Assume that v' has only two distinct neighbors v and y. Since $d(v') = 3$, the induced subgraph on v' and y must be a 2-cycle. Moreover, as v' has only two neighbors, then every path containing v' as an internal vertex must also contain its neighbors. Since P contains v' as an internal vertex then P must also contain y as an internal vertex, and it is easy to see that the vertices of P are in a subgraph that is a subdivision of a storm where v and w are the ends of the storm.

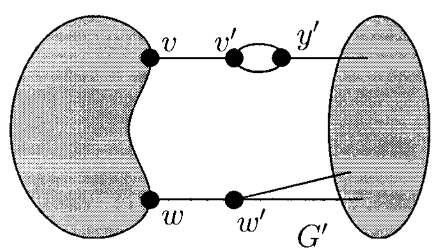


Figure 1.6: The case where v' is w'

Now assume that v' has three distinct neighbors v, y, and z. Since $d(v') = 3$ and v and v' are in P then at least one of y or z must be in P. Assume without loss of generality that y is in P. If z is also in P, it is clear that the vertices of P are in a subgraph that is a subdivision of a storm S where the ends of P are the ends of S and v' and z are the branch vertices of S of degree three. Now we consider the case where z is not in P.

Graph G is 2-edge-connected, so from Lemma 1.3.2 there are at least two internally disjoint paths from z to $I(P)$ one of which is not the edge $v'z$. Let P' denote a shortest such path from z to $I(P)$ and let z' denote the end of P' that is in $I(P)$. Since P' is in G', it is clear that $V(G) - V(G')$ and $V(P')$ are disjoint. So we see that the vertices of P lie in a subgraph that is a subdivision of a storm where the branch vertices of the 2-cycle are v' and z' and the ends of the storm are the ends of P. ■

1.8 Dissertation Overview

Here we provide an overview of the dissertation by discussing the types of results found. The collection of all cubic graphs is an important collection in many areas of graph theory. One of these areas is related to finding necessary and sufficient conditions for a graph to be in a minor-closed class of graphs. For example, Glover and Huneke [7] found the obstructions for the class of cubic projective planar graphs. This result pre-dates Glover, Huneke, and Wang's characterization of the class of all projective planar graphs in terms of those minor-minimal graphs not embeddable on the projective plane [8]. These graphs, which are called the *forbidden* or *excluded minors* for the projective plane, are graphs not embeddable on the projective plane but every proper minor is embeddable. Characterizations of this type began with the classic result by Kuratowski [9] which says that a graph G is planar if and only if G has neither K_5 nor $K_{3,3}$ as a minor. Thus, K_5 and $K_{3,3}$ are the excluded minors for the class of planar graphs.

In general, if one has a complete list of all excluded minors for a specific minor closed class of graphs, then this list yields structural information about the class. For example, the fact that K_5 and $K_{3,3}$ are the only two excluded minors for the class of planar graphs reveals valuable information about graph embeddability in the plane. Now, if S is a surface for which we would like a complete list of excluded minors, then in attempting to prove that a particular list of excluded minors is the complete list of excluded minors for the class of graphs embeddable on $\mathcal{S}$, a fundamental understanding of graph embeddability on the surface $\mathcal{S}$ is needed. Thus, a first approach to these types of problems is to consider a restriction on the class of graphs embeddable on $\mathcal{S}$. A class that is typically chosen in this setting is the class of cubic graphs. Having the complete list of cubic obstructions for a class, then reveals valuable structural information about both embeddability on the surface $\mathcal{S}$ and the set of all excluded minors for the class of graphs embeddable on $\mathcal{S}$. This is precisely what is reflected in the characterization of the class of projective planar graphs.

Presently, a characterization for every surface, other than the plane and projective plane, remains intractable. Specifically, the complete list of excluded minors for the torus is unknown as well as that of any surface of higher genus. A class related to that of graphs embeddable on the torus is the class of graphs embeddable on the pseudo-surface called the spindle surface, such graphs are called *spindle graphs*. The spindle surface is obtained by contracting a meridian of the torus; and thus graphs that are spindle are also embeddable on the torus. Archdeacon and Bonnington [1] provide a characterization for the class of cubic spindle graphs in an effort to gain insight into the collection of excluded minors for the class.

The interaction between results on cubic obstructions and excluded minors for various classes has generated a number of interesting research questions. While insight into embeddability on a surface is gained through finding the cubic obstructions for a class, this task is non-trivial. Moreover, recent results have shown that if one has the complete list of excluded minors for some surface, it is not simply a corollary to obtain the complete of cubic obstructions for the same surface. It is in this context that the research presented here was conducted. In [2] the complete list of excluded minors for the class of outer-cylindrical graphs is presented - outer-cylindrical graphs are those that can be embedded in the plane in such a way that the boundaries of two faces together contain all of the vertices of the graph. In this dissertation, we present the complete list of 124 cubic obstructions for the same class. In Chapter 3, we establish a number of results that help reduce the analysis of the list of obstructions by providing a construction that allows the generation of all cubic obstructions if one knows the

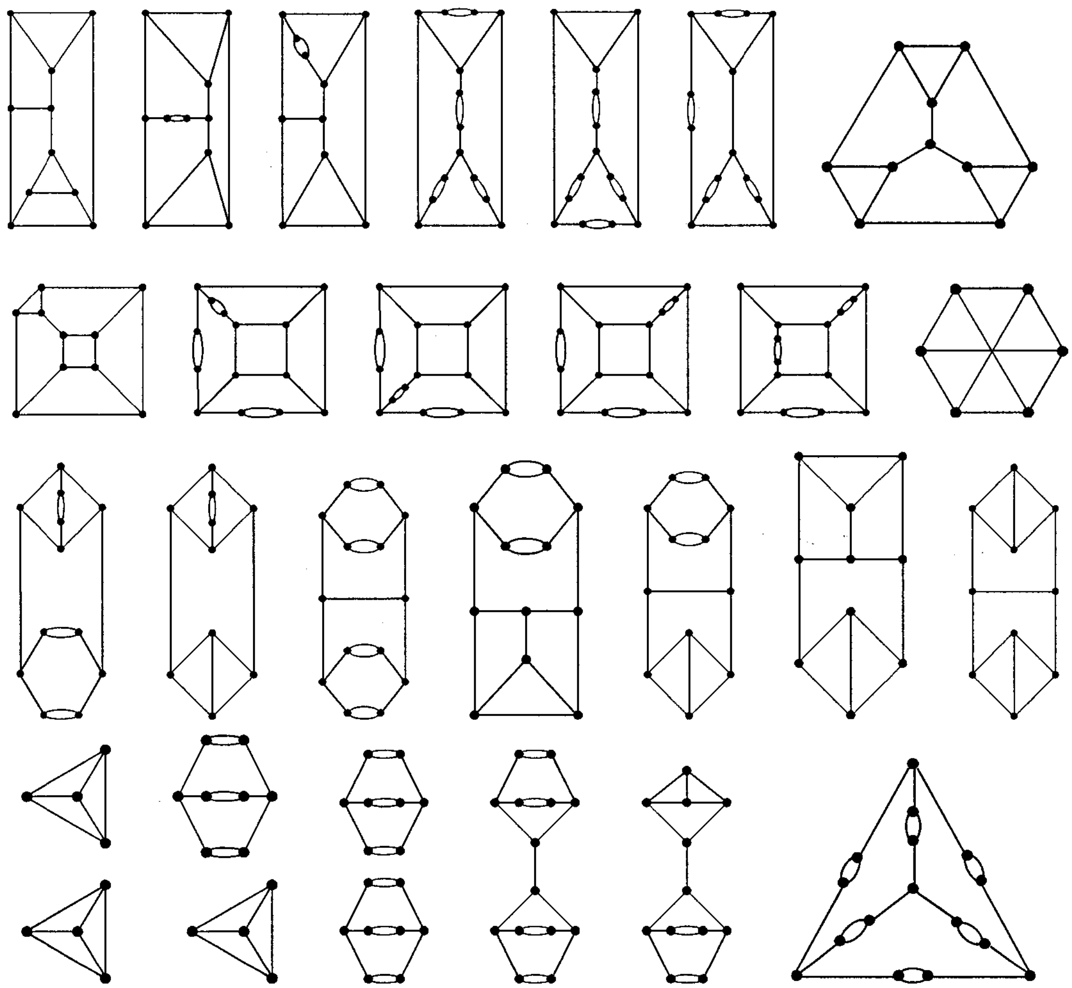


Figure 1.7: The loopless cubic obstructions for outer-cylindrical graphs.

list of all cubic obstructions having no loops. We then present a list of loopless cubic obstructions, which we have also included in Figure 1.7 in this overview, and then prove that each of these graphs is a cubic obstruction at the end of the chapter. In Chapter 4, we prove that the list of 26 loopless cubic obstructions for the class of outer-cylindrical graphs is the complete list of loopless cubic obstructions. Upon combining this with the results of Chapter 3, this allows us to generate the complete list of all cubic obstructions for the class which can be found on page 100.

In order to establish the characterization for outer-cylindrical graphs provided in Chapters 3 and 4, it was necessary to know the complete list of cubic obstructions for outer-planar graphs – graphs that can be embedded in the plane so that all the vertices lie on the boundary of one face. This result is presented in Chapter 2, and it provides the insight for the construction that allows the generation of all cubic obstructions from the list of all loopless cubic obstructions. In particular, the loopless cubic obstructions for outer-planar graphs are K_4 and $K_{2,3C_2}$ depicted in Figure 1.8. Taken together, the graphs depicted in Figure 1.8 is the complete list of cubic obstructions for outer-planar graphs.

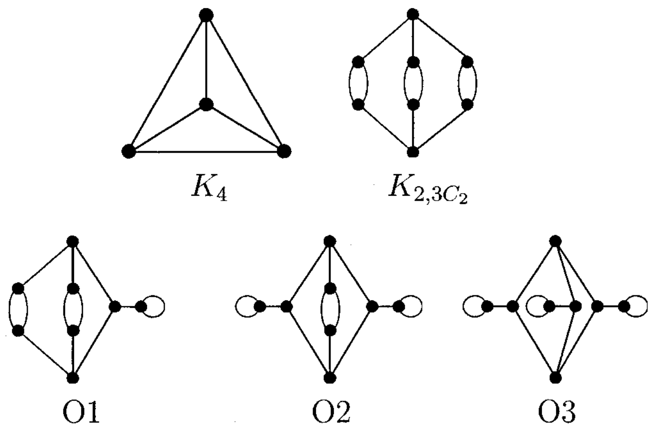


Figure 1.8: The cubic obstructions for outer-planar graphs.

The dissertation presented here provides a theoretical proof that the list of 124 cubic obstructions mentioned above form the complete list of cubic obstructions for outer-cylindrical graphs. While nothing related to computer analysis or graph generation via a computer appears in this dissertation, it is important to mention its utilization here. The obstructions were originally generated via a computer program developed through the support of the Louisiana Educational Quality Support Fund under grant LEQSF(2003-06)-RD-A-19. This program builds upon Brendan McKay's [10] work in his software program Nauty. Building on top of Nauty, we developed several algorithms to test for the excluded minors and cubic obstructions of a number of classes of graphs. The outer-cylindrical graphs were one such class, and the list was originally generated via this program. In order to determine that the computer generated the complete list, we needed to either theoretically prove that there is an upper bound on the number of vertices for any cubic obstructions – and then run the computer program through an exhaustive search to that point – or we needed an entirely theoretical proof. In the attempt to do the latter and develop additional algorithms to streamline the search processing time, the insight for a completely theoretical proof was obtained. Thus, the results of this dissertation are proved independent of any computer program.

CHAPTER 2

OBSTRUCTIONS FOR CUBIC OUTER-PLANAR GRAPHS

2.1 Outer-planar Graphs

Outer-planar graphs were introduced in Section 1.5, and Theorem 1.5.5 gives a characterization of outer-planar graphs first proved by Chartrand and Harary [4]. Specifically, they proved that any graph that is not outer-planar must contain a subgraph that is a subdivision of K_4 or $K_{2,3}$. The next theorem, which is the main result of this chapter, provides a characterization of the class of cubic outer-planar graphs.

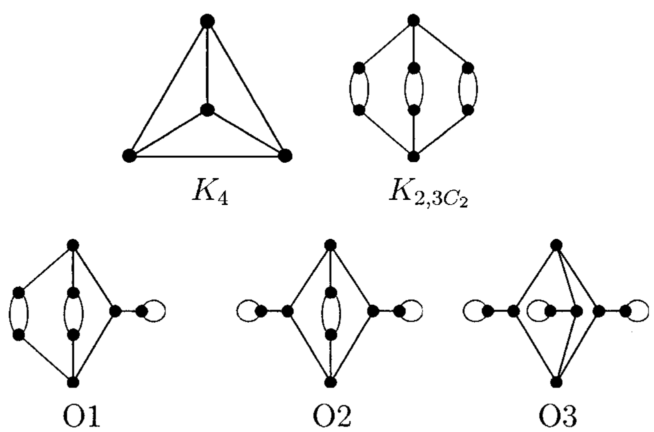


Figure 2.1: The cubic obstructions for outer-planar graphs.

Theorem 2.1.1 *A graph G is a cubic outer-planar graph if and only if G does not contain K_4, $K_{2,3C_2}$,* O1, O2, *or* O3 *(of Figure 2.1) in the cubic order.*

Upon combining Theorem 1.5.5 and Proposition 1.6.1, it is easy to show the following.

Lemma 2.1.2 *K_4 is a cubic obstruction for outer-planar graphs.*

Proof: First from Theorem 1.5.5, K_4 is not outer-planar. Since the cubic edge-deletion of any edge of K_4 results in a graph on two vertices joined by three edges, it is clear that this graph is outer-planar. Thus, K_4 is a cubic obstruction for outer-planar graphs. ■

Dirac [6] showed the following theorem which guarantees that a simple cubic graph contains a K_4-subdivision. Recall that $\delta(G)$ denotes the smallest degree of a vertex in G.

Theorem 2.1.3 *(Dirac[6]) Let G be a simple graph. If $\delta(G) \geq 3$, then G has a subgraph that is a subdivision of K_4*

Figure 2.2: A cubic graph with a noose (a) and a 2-cycle (b).

From this theorem, every simple cubic graph contains a subgraph that is a subdivision of K_4, and so simple cubic graphs are not outer-planar by Theorem 1.5.5. We conclude that every cubic outer-planar graph must not be simple. Now, consider a connected cubic graph G that is not simple. This means that G contains either a loop or a parallel edge. If G contains a loop l incident to a vertex k, then, since G is cubic, k must also be incident to a non-loop edge kt which is a bridge. We call loop l together with the edge kt a *noose* where t is the *top* and k is the *knot.* So if G contains a loop, then G contains a noose as depicted in Figure 2.2 (a). Thus, non-simple cubic graphs contain either a 2-cycle or a noose (see Figure 2.2).

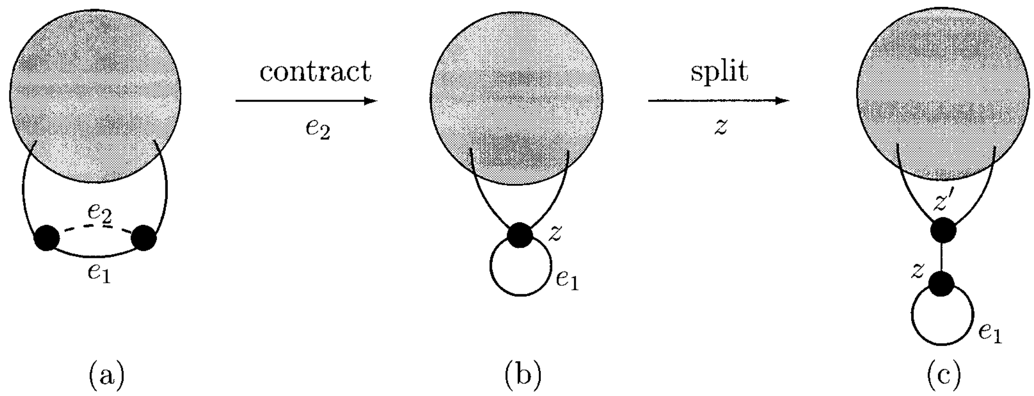


Figure 2.3: The 2-cycle-noose operation depicted.

Now, let G be a cubic graph with a 2-cycle C having edges e_1 and e_2 as depicted in Figure 2.2 (b). The operation of contracting e_2 of C results in a vertex z of degree four incident to a loop consisting of the edge e_1 (see Figure 2.3 (b)). Now splitting z into vertices z and z' so that z is incident to the loop e_1 and zz' is a bridge results in a cubic graph where the induced subgraph on e_1 and zz' is a noose. This operation will be called a *2-cycle-noose operation on C*. Figure 2.3 depicts the 2-cycle-noose operation just described.

Let G be a cubic graph with a noose N as depicted in Figure 2.2 (a). The operation of contracting edge kt to a vertex z and then splitting z so that G has parallel edges e_1 and e_2 will be called a *noose-2-cycle operation.* Figure 2.4 depicts a noose-2-cycle operation.

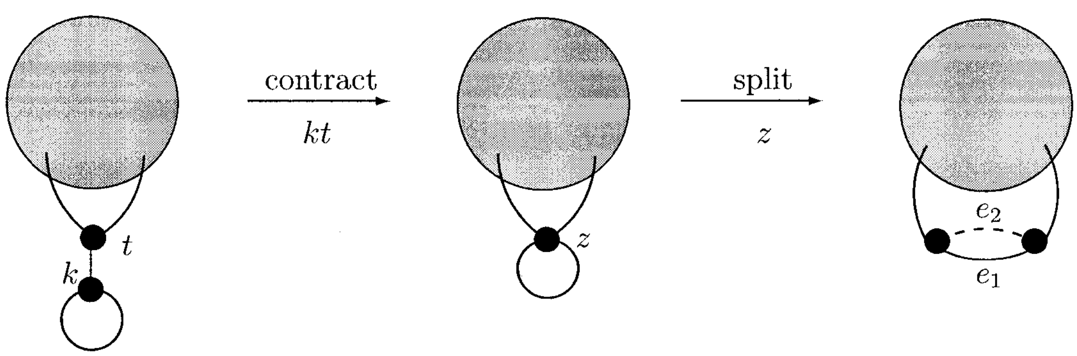


Figure 2.4: The noose-2-cycle operation depicted.

The next lemma provides a relationship between graphs obtained by 2-cycle-noose operations. Recall that if N is a noose, the edge joining the knot k with the top t is a bridge and we denote the loop of N by l. Also, recall that cubic edge-deletion (page 8) is denoted $\backslash_3$.

Lemma 2.1.4 *Let G be a cubic graph with a 2-cycle C whose edges are e_1 and e_2 and let G' be the graph with a noose N resulting from a 2-cycle-noose operation on C. Then $G\backslash_3 e_1 \cong G\backslash_3 e_2 \cong G'\backslash_3 tk \cong G'\backslash_3 l$.*

Proof: Clearly we see that $G\backslash_3 e_1 \cong G\backslash_3 e_2$. From the definition of the cubic edge-deletion of a noose, we also see that $G'\backslash_3 tk \cong G'\backslash_3 l$. Assume x and y are the vertices of the 2-cycle C, vertex x' is the neighbor of x in $V(G) - V(C)$, and y' is the neighbor of y in $V(G) - V(C)$. From the definition of the 2-cycle-noose operation, we see that x' and y' are the neighbors of the top, t of the noose N in G'. Likewise, it is easy to see that $G - V(C) \cong G' - \{t, k\}$. Now the graph $G\backslash_3 e_1$ is the graph $G - V(C)$ with an additional edge e from x' to y'. The graph $G'\backslash_3 tk$ is the graph $G' - \{t, k\}$ with an additional edge e' from x' to y'. Thus it is clear that $G\backslash_3 e_1 \cong G'\backslash_3 tk$. So we have $G\backslash_3 e_2 \cong G\backslash_3 e_1 \cong G'\backslash_3 tk \cong G'\backslash_3 l$. ■

The following two propositions will be used to determine the cubic obstructions for the class of outer-planar graphs. The first proposition establishes a condition for outer-planar graphs obtained by a 2-cycle-noose operation. The second proposition allows cubic obstructions for outer-planar graphs to be constructed by replacing 2-cycles with nooses in cubic obstructions that are already known.

Proposition 2.1.5 *Let G be a cubic graph with a 2-cycle C and G' the graph resulting from a 2-cycle-noose operation on C. Then G is outer-planar if and only if G' is outer-planar.*

Proof: Assume G is an outer-planar cubic graph with a 2-cycle C and let G' be the graph resulting from a 2-cycle-noose operation on C. Embed G in the plane so that every vertex of G lies on a single face f. Let $V(C) = \{x, y\}$ and $E(C) = \{e_1, e_2\}$.

Now we claim that at least one of the parallel edges of C borders f. Since G is cubic, there is an edge e' incident to x that is not e_1 or e_2. Let f_1 denote the face bordered by e' and e_1, f_2 denote the face bordered by e' and e_2, and f_3 denote the face whose boundary is C. Now f must be one of f_1, f_2, or f_3 since x borders f. From Lemma 1.6.2 one of e_1 or e_2, say e_1, borders f (see Figure 2.5 (a)).

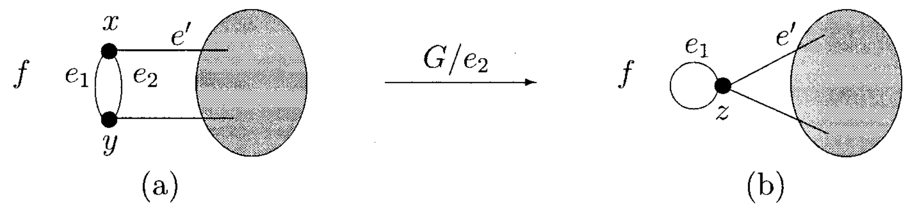


Figure 2.5: The contraction of e_2

Since e_1 borders f, edge e_2 can be contracted, resulting in a vertex z yielding an outer-plane graph where e_1 is now a loop and e_1 borders f as depicted in Figure 2.5. Now z can be split, resulting in an edge zz' where z is incident to the loop e_1 (see Figure 2.6). Thus the resulting graph is isomorphic to G' and is embedded with every vertex on face f. So G' is an outer-planar graph.

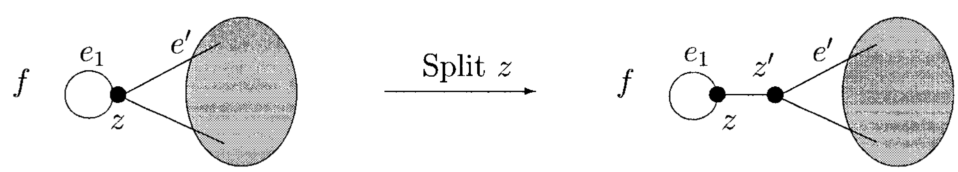


Figure 2.6: The splitting of the vertex of degree 4.

Assume G' is an outer-planar graph having a noose N arising from the 2-cycle-noose operation on C of G. Embed G' so that every vertex of G' lies on face f. Since k lies on the boundary of f then both l and kt are on the boundary of f as depicted in Figure 2.7. Now kt can be contracted yielding an outer-plane graph where l is still

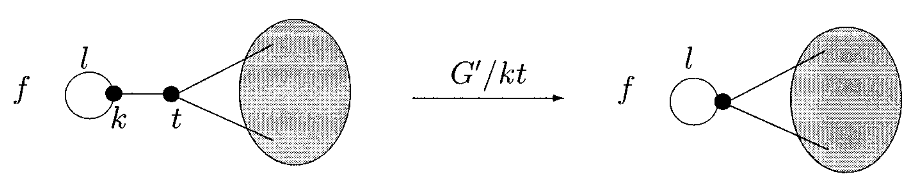


Figure 2.7: The graph G'.

on the boundary of f (as shown in Figure 2.7). Now, the resulting vertex of degree 4 can be split so that l is parallel to the new edge, the resulting graph being a cubic graph isomorphic to G. Notice that this graph is embedded so that every vertex lies on face f. Thus G is an outer-planar graph. We conclude that G is outer-planar if and only if G' is outer-planar. ■

Corollary 2.1.6 *Let G be a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Then G is outer-planar if and only if G' is outer-planar.*

Proof: Assume G is a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Let C be the 2-cycle of G' resulting from the noose-2-cycle operation on N of G. Then G is the graph resulting from a 2-cycle-noose operation on C of G'. Thus, by Proposition 2.1.5, G is outer-planar if and only if G' is outer-planar. ■

Proposition 2.1.7 *Let G be a cubic graph with a 2-cycle C, and G' the graph resulting from a 2-cycle-noose operation on C. Then G is a cubic obstruction for outer-planar graphs if and only if G' is a cubic obstruction for outer-planar graphs.*

Proof: Let G be a cubic obstruction for outer-planar graphs having a 2-cycle C and let N be the noose of G' obtained from G by a 2-cycle-noose operation on C. By Proposition 2.1.5, since G is not outer-planar, then G' is not outer-planar. Let e' be an edge in G'. Now if e' is not in N, then e' is in $E(G) - E(C)$. So it is clear that the 2-cycle-noose operation on C in $G\backslash_3 e'$ results in a graph that is isomorphic to $G'\backslash_3 e'$. Since G is a cubic obstruction for outer-planar graphs then $G\backslash_3 e'$ is outer-planar. So from Proposition 2.1.5 $G'\backslash_3 e'$ is outer-planar. Now, if e' is in N then by Lemma 2.1.4 $G'\backslash_3 e'$ is isomorphic to $G\backslash_3 e$ where e is in C. Thus $G'\backslash_3 e'$ is outer-planar.

We now suppose that G' is a cubic obstruction for outer-planar graphs and e is in $E(G)$. Again, Proposition 2.1.5 shows that G is not outer-planar. If e is not in C, then e is in $E(G') - E(N)$. So it is clear that the noose-2-cycle operation on N in

$G'\backslash_3 e$ results in a graph that is isomorphic to $G\backslash_3 e$. Since G' is a cubic obstruction for outer-planar graphs then $G'\backslash_3 e$ is outer-planar. Thus by Corollary 2.1.6, we have that $G\backslash_3 e$ is outer-planar. Now, if e is in C, then by Lemma 2.1.4, $G\backslash_3 e$ is isomorphic to $G'\backslash_3 e'$ where e' is in N. Thus, $G\backslash_3 e$ is outer-planar. With this the proposition is established. ■

Corollary 2.1.8 *Let G be a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Then G is a cubic obstruction for outer-planar graphs if and only if G' is a cubic obstruction for outer-planar graphs.*

Proof: Assume G is a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Let C be the 2-cycle of G' resulting from the noose-2-cycle operation of G on N. Then G is the graph resulting from a 2-cycle-noose operation of G' on C. Thus, by Proposition 2.1.7, G is a cubic obstruction for outer-planar graphs if and only if G' is a cubic obstruction for outer-planar graphs. ■

2.2 Loopless Outer-planar Cubic Obstructions

Proposition 2.1.7 and Corollary 2.1.8 allow us to replace 2-cycles with nooses and nooses with 2-cycles within a cubic obstruction for outer-planar graphs with the result being another cubic obstruction. Since a cubic graph has a noose if and only if it has a loop, the problem of finding all cubic obstructions for outer-planar graphs is now reduced to finding all loopless cubic obstructions. Specifically, in order to obtain the complete list of cubic obstructions for outer-planar graphs, we need only perform all possible combinations of 2-cycle-noose operations on the loopless cubic obstructions. The remainder of this section will be devoted to proving that the loopless cubic obstructions for outer-planar graphs are K_4 and $K_{2,3C_2}$ (see Figure 2.8) as we now state in the following proposition.

Proposition 2.2.1 *Let G be a loopless cubic obstruction for outer-planar graphs. Then G is isomorphic to K_4 or $K_{2,3C_2}$.*

Figure 2.8: The two loopless cubic obstructions for outer-planar graphs.

Before proving Proposition 2.2.1, we first show that all loopless cubic obstructions for outer-planar graphs are 2-edge-connected.

Proposition 2.2.2 *If G is a loopless cubic obstruction for outer-planar graphs, then G is 2-edge-connected.*

Proof: First, we will show that any cubic obstruction for outer-planar graphs is connected. So, suppose that G has more than one component, say $G_1, G_2, \ldots, G_n$. By Lemma 1.5.4 (i), a graph is outer-planar precisely when its components are outer-planar. Since G is not outer-planar, it must be the case that at least one of G_1,G_2,..., G_n is not. Without loss of generality, assume G_1 is not outer-planar and let e_2 be an edge of G_2. Since G_1 is not outer-planar and is a subgraph of $G\backslash_3 e_2$, it is clear that $G\backslash_3 e_2$ is not outer-planar – a contradiction of the assumption that G is a cubic obstruction. Thus G must be connected.

Now suppose G has an edge e for which $G\backslash e$ has two components G_1 and G_2. By Lemma 1.5.4 (ii) a graph is outer-planar precisely when its blocks are, and since G is not outer-planar, it must be the case that at least one of G_1, e, or G_2 is not outer-planar. Clearly, e is outer-planar, so assume one of G_1 and G_2 is not outer-planar, say G_1, and let e_2 be an edge of G_2. Since G is loopless, e_2 is not a loop and so e is an edge of $G\backslash_3 e_2$ which implies that G_1 is a subgraph of $G\backslash_3 e_2$. However, G_1 is not outer-planar, so it must be that $G\backslash_3 e_2$ in not outer-planar – a contradiction to G being a cubic obstruction for outer-planar graphs. Therefore G has no cut-edge and is thus 2-edge-connected. ■

Proof of Proposition 2.2.1: By Lemma 2.1.2, K_4 is a cubic obstruction for outer-planar graphs. Therefore we need to show that $K_{2,3C_2}$ is the only other loopless cubic obstruction for outer-planar graphs. Clearly, $K_{2,3C_2}$ contains a subgraph that is a subdivision of $K_{2,3}$, so by Theorem 1.5.5, $K_{2,3C_2}$ is certainly not outer-planar. However, the graph resulting from the cubic edge-deletion of any non-parallel edge is isomorphic to an outer-planar graph – an embedding of which is depicted in Figure 2.9 (graph A). The graph resulting from the cubic edge-deletion of any parallel edge of $K_{2,3C_2}$ is also isomorphic to an outer-planar graph – as depicted in Figure 2.9 (graph B). Thus we can conclude that $K_{2,3C_2}$ is a cubic obstruction for outer-planar graphs. Now we show that K_4 and $K_{2,3C_2}$ are the only loopless cubic obstructions.

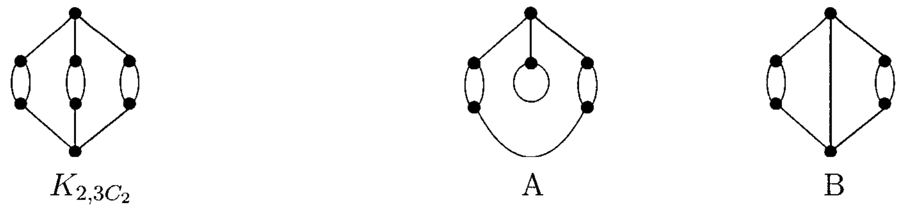


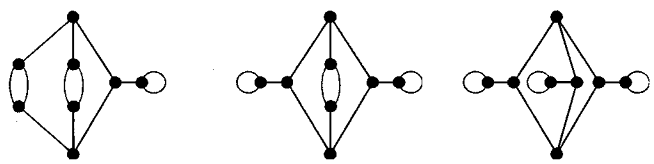
Figure 2.9: The graph $K_{2,3C_2}$ and those graphs resulting from a single cubic edge-deletion of edges of $K_{2,3C_2}$.

Assume that G is a loopless cubic obstruction for outer-planar graphs and is not isomorphic to either K_4 or $K_{2,3C_2}$. Clearly, G does not contain K_4 or $K_{2,3C_2}$ in the cubic order, and so by Proposition 1.6.1, G is a loopless non-outer-planar cubic graph with no subgraph that is a subdivision of K_4 and no subgraph that is a subdivision of $K_{2,3C_2}$. By Theorem 1.5.5, G must have a subgraph that is a subdivision of $K_{2,3}$ which we will denote G'. Let w_1 and w_2 be the vertices of G' that have degree three of the $K_{2,3}$-subdivision. Let P_1, P_2, and P_3 be the independent paths with internal vertices from w_1 to w_2 that form the $K_{2,3}$-subdivision and recall that $I(P_i) = V(P_i) - \{w_1, w_2\}$.

From Lemma 1.7.2 if any one of P_1, P_2, or P_3 does not form a 2-edge-cut then G contains a subgraph that is a subdivision of K_4. Since G is not isomorphic to K_4 then G would properly contain K_4 in the cubic order – a contradiction to G being a cubic obstruction. So it must be that the end edges of each of P_1, P_2, and P_3 form a 2-edge-cut.

Hence each P_i satisfies the hypothesis of Lemma 1.7.3 and so the vertices of each P_i is in a subgraph that is a subdivision of a storm in G where the ends of P_i are the ends of the storm. Thus G contains a subgraph that is a subdivision of $K_{2,3C_2}$ – a contradiction to the choice of G. So it must be that any loopless cubic obstruction for outer-planar graphs is isomorphic to either K_4 or $K_{2,3C_2}$. ■

Having established Proposition 2.2.1, we now prove Theorem 2.1.1, namely that G is a cubic obstruction for outer-planar graphs if and only if G does not contain K_4, $K_{2,3C_2}$, O1, O2, and O3 in the cubic order.



O1 O2 O3

Figure 2.10: The cubic obstructions for outer-planarity that contain loops.

Proof of Theorem 2.1.1: By Proposition 2.2.1 the only loopless cubic obstructions are K_4 and $K_{2,3C_2}$. Using Proposition 2.1.7, we can obtain all cubic obstructions by performing a 2-cycle-noose operation on every cubic obstruction having a 2-cycle. Graph O1 is obtained from $K_{2,3C_2}$ by a 2-cycle-noose operation, O2 is obtained from O1 by a 2-cycle-noose operation, and finally O3 is obtained from O2 by a 2-cycle-noose operation. By the symmetry of $K_{2,3C_2}$, every other graph obtained by 2-cycle-noose operations from $K_{2,3C_2}$, O1, and O2, results in a graph isomorphic to either O1, O2, or O3, and the theorem is established. ■

CHAPTER 3

SOME OBSTRUCTION SETS FOR CUBIC OUTER-CYLINDRICAL GRAPHS

3.1 Introduction

In [2], a class of graphs that extends the class of outer-planar graphs is characterized in terms of excluded minors. In that paper it says, "a graph is *outer-cylindrical* if it embeds in the plane so that there are at most two distinct faces whose boundaries together contain all the vertices." The class of outer-cylindrical graphs is minor-closed, and in [2] the complete set of 38 minor-minimal non-outer-cylindrical graphs is determined (see Appendix B). We now formally state this result.

Theorem 3.1.1 *(Archdeacon, Bonnington, Dean, Hartsfield, and Scott [2]) A graph G is outer-cylindrical if and only if G does not contain any of the graphs depicted in Appendix B in the minor order.*

In this chapter we establish some properties of cubic obstructions for outer-cylindrical graphs having loops. As was the case for the cubic obstructions of outer-planar graphs, one of the results in this chapter is that any cubic obstruction for outer-cylindrical graphs having a loop can be obtained from a loopless cubic obstruction. Moreover, we present a collection of loopless cubic obstructions for outer-cylindrical graphs (see Figure 3.1) and, in Section 3.3, we prove that each of these graphs is a cubic obstruction. The main result of the following chapter is to show that this same set of graphs, together with the properties established in this chapter, provide the complete list of cubic obstructions for the class of outer-cylindrical graphs. For this reason, the complete list is depicted in Appendix A, and only the loopless obstructions appear in Figure 3.1. So, the main result of Chapters 3 and 4 is the following theorem.

Theorem 3.1.2 *A cubic graph G is outer-cylindrical if and only if G does not contain any of the graphs depicted in Appendix A in the cubic order.*

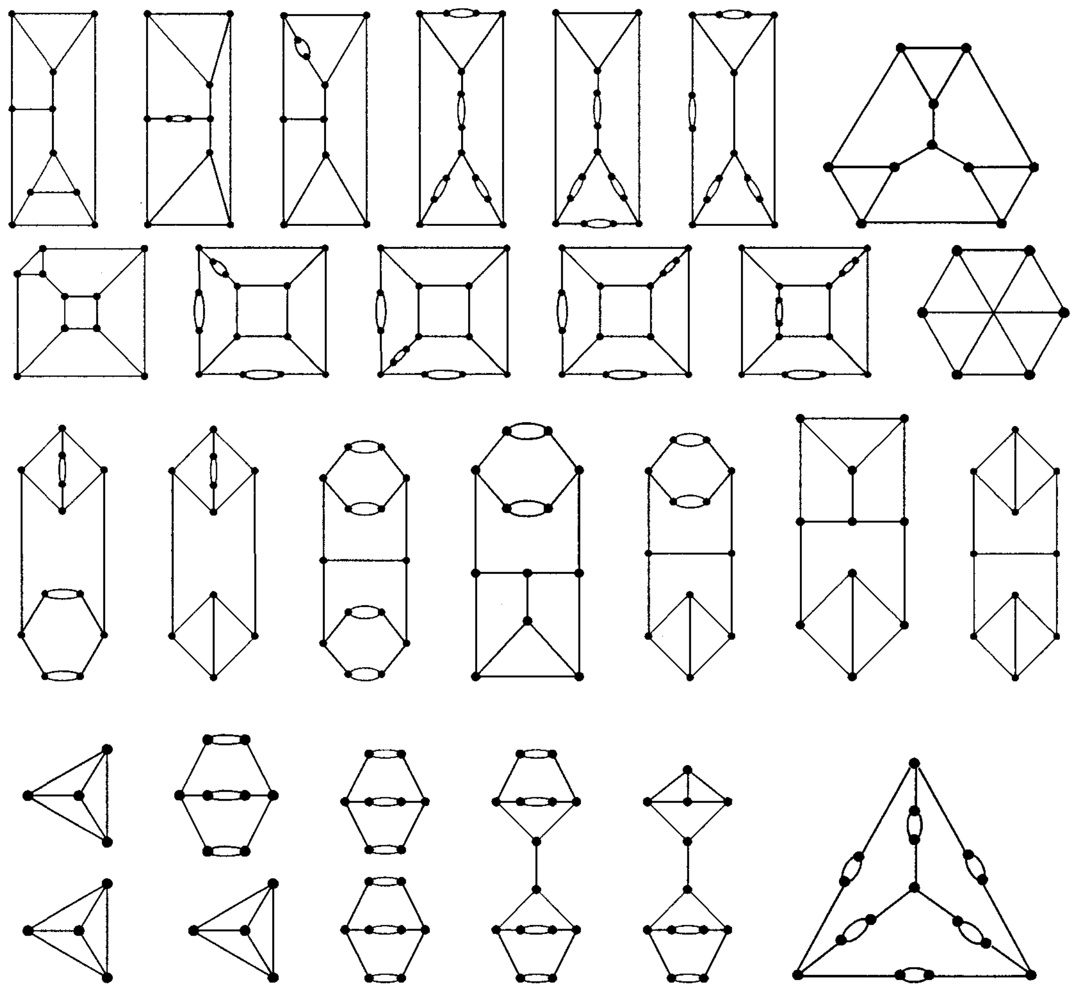


Figure 3.1: The loopless cubic obstructions for outer-cylindrical graphs.

3.2 Properties of Cubic Outer-cylindrical Graphs.

The following two propositions will be used to determine the cubic obstructions for the class of outer-cylindrical graphs. Just as in the case of outer-planar graphs, the first proposition establishes a condition for outer-cylindrical graphs obtained by a 2-cycle-noose operation. The second proposition allows cubic obstructions to be constructed by replacing 2-cycles with nooses in cubic obstructions that are already known.

Proposition 3.2.1 *Assume G is a cubic graph with a 2-cycle C and G' is the graph resulting from a 2-cycle-noose operation on C. Then G is outer-cylindrical if and only if G' is outer-cylindrical.*

Proof: Assume that N is the noose of G' resulting from the 2-cycle-noose operation on C. Label the vertices of C as x and y and label the edges of C as e_1 and e_2. Denote the top of N as t, the knot of N as k, and the loop of N as l.

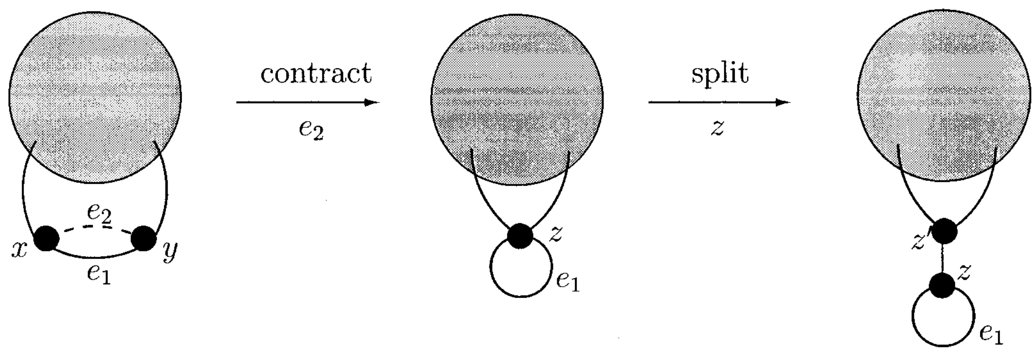


Figure 3.2: The 2-cycle-noose operation of G.

Assume G is outer-cylindrical and embed G in the plane so that there are two faces f and f' whose boundaries together contain all the vertices of G. Now x must lie on the boundary of one of f or f', so assume that x lies on the boundary of f. From Lemma 1.6.2 one of the edges of C, say e_1, must lie on the boundary of f. Now if f is the face whose boundary is C, let f'' denote the other face incident to e_1. In this case it is clear that f' and f'' are two faces whose boundaries together contain all the vertices on G. So without loss, assume f is f''.

Notice that e_2 can be contracted resulting in a vertex z yielding an outer-cylindrical graph where e_1 is now a loop and e_1 borders f. Now z can be split, resulting in an edge zz' where z is incident to a loop. Thus the resulting graph is isomorphic to G' and is embedded with every vertex lying on the boundary of face f or f', and so G' is outer-cylindrical.

Now, assume G' is outer-cylindrical and embed G' in the plane so that there are two faces f and f' whose boundaries together contain all the vertices of G. Assume

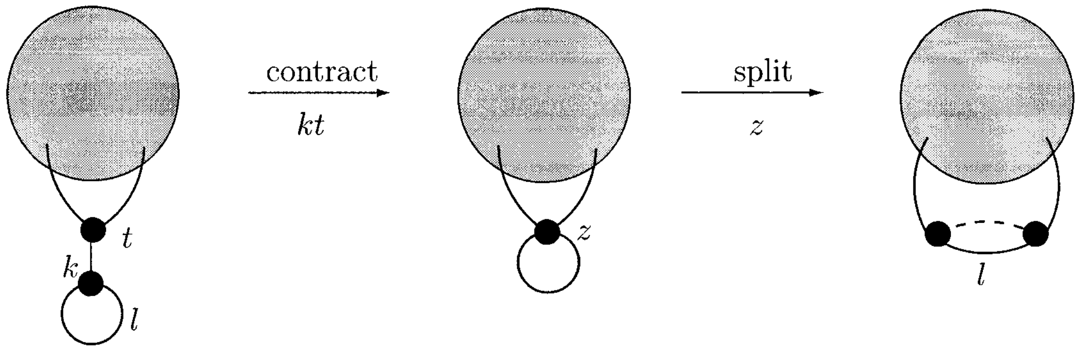


Figure 3.3: The 2-cycle-noose operation of G.

f is the face whose boundary contains the knot k of the noose N. Now if f is the face whose boundary is l, let f'' denote the other face incident to l. In this case it is clear that f' and f'' are two faces whose boundaries together contain all the vertices of G. So without loss, assume f is f''.

Notice that kt can be contracted into a vertex z yielding an outer-cylindrical graph where l is still on the boundary of f (see Figure 3.3). Now, the resulting vertex of degree 4, z, can be split so that l is an edge in parallel with the new edge. The result is a cubic graph isomorphic to G. Notice that this graph is embedded so that the boundaries of faces f and f' together contain all the vertices of G. Thus G is an outer-cylindrical graph. We conclude that G is outer-cylindrical if and only if G' is outer-cylindrical. ∎

Corollary 3.2.2 *Let G be a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Then G is outer-cylindrical if and only if G' is outer-cylindrical.*

Proof: Assume G is a cubic graph with a noose N and G' is the graph resulting from a noose-2-cycle operation on N. Let C be the 2-cycle of G' resulting from the noose-2-cycle operation of G on N. Then G is the graph resulting from a 2-cycle-noose operation of G' on C, and by Proposition 3.2.1, G' is outer-cylindrical if and only if G is outer-cylindrical. ∎

Proposition 3.2.3 *Let G be a cubic graph with a 2-cycle C, and G' the graph resulting from a 2-cycle-noose operation on C. Then G is a cubic obstruction for outer-cylindrical graphs if and only if G' is a cubic obstruction for outer-cylindrical graphs.*

Proof: Let G be a cubic graph with a 2-cycle C and let G' be the graph resulting from a 2-cycle-noose operation on C. Furthermore, assume that N is the noose of G' resulting from the 2-cycle-noose operation on C. Label the vertices of C as x and y and label the edges of C as e_1 and e_2. Label the non-parallel edge incident to x as e_x and label the non-parallel edge incident to y as e_y. Since G' is obtained from G by a 2-cycle-noose operation on C, then it is easy to see that e_x and e_y are incident to t of N (see Figure 3.4).

Figure 3.4: The graphs G and G'.

We first suppose that G is a cubic obstruction for outer-cylindrical graphs and let e' be an edge of G'. If e' is not in N then e' is in $E(G) - E(C)$. So it is clear that the 2-cycle-noose-operation on C in $G\backslash_3 e'$ results in a graph that is isomorphic to $G'\backslash_3 e'$. Since G is a cubic obstruction for outer-cylindrical graphs, then $G'\backslash_3 e'$ is outer-cylindrical. So from Proposition 3.2.1 $G'\backslash_3 e'$ is outer-cylindrical. Now if e' is in N, by Lemma 2.1.4, $G'\backslash_3 e'$ is isomorphic to $G\backslash_3 e_1$. Thus $G'\backslash_3 e'$ is outer-cylindrical.

We now suppose that G' is a cubic obstruction for outer-cylindrical graphs and e is in $E(G)$. Again, Proposition 3.2.1 shows that G is not outer-cylindrical. Now if e is not in C, then e is in $E(G') - E(N)$. So it is clear that the 2-cycle-noose-operation on N in $G'\backslash_3 e$ results in a graph that is isomorphic to $G\backslash_3 e$. Since G' is a cubic obstruction for outer-cylindrical graphs, $G'\backslash_3 e$ is outer-cylindrical. Thus, by Corollary 3.2.2, we have that $G\backslash_3 e$ is outer-cylindrical. Now, if e is in C, by Lemma

2.1.4, $G\backslash_3 e$ is isomorphic to $G'\backslash_3 tk$. Thus $G\backslash_3 e$ is outer-cylindrical. With this the proposition is established ■

Corollary 3.2.4 *Let G be a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Then G is a cubic obstruction for outer-cylindrical graphs if and only if G' is a cubic obstruction for outer-cylindrical graphs.*

Proof: Assume G is a cubic graph with a noose N and G' the graph resulting from a noose-2-cycle operation on N. Let C be the 2-cycle of G' resulting from the noose-2-cycle operation of G on N. Then G is the graph resulting from a 2-cycle-noose operation of G' on C. Thus, by Proposition 3.2.3, G is a cubic obstruction for outer-cylindrical graphs if and only if G' is a cubic obstruction for outer-cylindrical graphs. ■

We conclude this section with two technical lemmas which establish conditions on outer-cylindrical graphs having a 2-edge-cut and which will be used in the proof of Theorem 3.1.2. For any 2-edge-cut in an outer-cylindrical graph G, the first lemma finds two faces whose boundaries together would contain all the vertices of G along with the edges of the 2-edge-cut. The second sets a limit on the size of a generalized cycle of a cubic obstruction for outer-cylindrical graphs.

Lemma 3.2.5 *Let G be a cubic outer-cylindrical graph. If G is 2-edge-connected but not 3-edge-connected , then for every 2-edge-cut E' of G, there is an embedding of G with faces f' and f'' such that each vertex of G lies on the boundary of f' or f'' and both edges of E' border one of f' and f''.*

Proof: Let x_1x_2 and y_1y_2 be the edges of a 2-edge-cut E' of an outer-cylindrical graph G. Let G_1 and G_2 be the components of $G\backslash\{x_1x_2, y_1y_2\}$ where x_i and y_i are in $V(G_i)$. Embed G in the plane so that all the vertices of G border two faces which we label f' and f''.

We first show that if one of f' or f'' is bordered by vertices of both G_1 and G_2 then x_1x_2 and y_1y_2 borders the face. So, without loss of generality, assume f' is bordered by vertices from both G_1 and G_2. Now G is a plane cubic graph with edge-connectivity two. Thus the boundary of every face of G is a cycle. So there is a cycle C in G that forms the boundary of f'. Let v_1 and v_2 be vertices in C where v_1 is in G_1 and v_2 is in G_2. Now, there are two edge disjoint paths P_1 and P_2 from v_1 to v_2 in C. Since E' is a 2-edge-cut then the edge x_1x_2 must be in one of P_1 and P_2 and the edge y_1y_2 must be in the other. Suppose x_1x_2 is in $E(P_1)$ and y_1y_2 is in $E(P_2)$. Since C forms the boundary of f' and edges x_1x_2 and y_1y_2 lie in C, it is clear that x_1x_2 and y_1y_2 border f'. So we conclude that whenever f' is bordered by vertices of both G_1 and G_2, the edges x_1x_2 and y_1y_2 border f'. Now we consider the case where all the vertices of G_1 border one of f' and f'', and all the vertices of G_2 border the other.

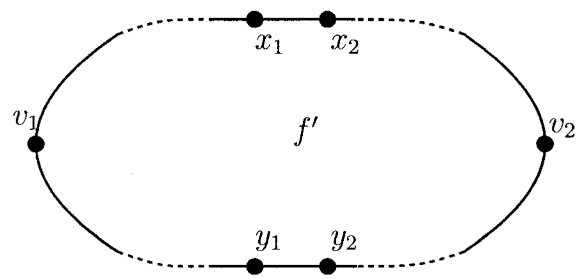


Figure 3.5: Cycle C forms the boundary of f'.

Without loss of generality, suppose all the vertices of G_1 border f' and all the vertices of G_2 border f''. Thus G_1 and G_2 are outer-planar and we can clearly embed G_1 and G_2 in the plane so that all the vertices of G_1 and G_2 lie on a face f. Now we can embed x_1x_2 in the face f so that each vertex of $G_1 \mathring{\cup} G_2 \cup \{x_1x_2\}$ lies on the boundary of one face, namely f. Adding y_1y_2 in the face f divides f into two faces f_1 and f_2 say. The resulting graph is a plane graph isomorphic to G in which each vertex of G lies on the boundary of one of f_1 or f_2, and x_1x_2 and y_1y_2 lie on the boundary of at least one of f_1 or f_2. ■

Lemma 3.2.6 *Let G be a 2-connected, but not 3-connected, cubic obstruction for outer-cylindrical graphs. Then G has a representation as a generalized cycle, each part of which is a block and the number of parts of this representation is no more than four.*

Proof: Let G be a cubic obstruction for outer-cylindrical graphs of connectivity two. From Lemma 1.3.3 we see that there is a representation of G as a generalized cycle where each part is a block. Assume that the number of parts of this generalized cycle is greater than four where $B_1, B_2, \ldots, B_n$ are the parts of G and $n > 4$. Let v_i denote the contact vertex of B_i and B_{i+1}. Since G is cubic it must be that B_i is an edge precisely when B_{i-1} and B_{i+1} are not edges, and thus n is even. For the rest of the proof, assume that $B_2, B_4, \ldots, B_n$ are edges of G.

Now, suppose that $B_1, B_3, \ldots, B_{n-1}$ are outer-planar. Observe that the boundaries of the central faces of the generalized cycle taken together contain all the vertices of G. Thus G is outer-cylindrical – a contradiction to G being a cubic obstruction. Thus it must be that one of $B_1, B_3, \ldots, B_{n-1}$ is not outer-planar. Without loss of generality assume that B_1 is not outer-planar.

We now claim that B_i is outer-planar for $i \neq 1$. This is clearly the case for $i \in \{2, 4, \ldots, n\}$. Now, suppose that one of B_3, B_5, $\ldots$, B_{n-1}, say B_3, is not outer-planar. For any edge e_5 in B_5, the blocks B_3 and B_1 are disjoint subgraphs in $G\backslash_3 e_5$. So we see that $G\backslash_3 e_5$ contains disjoint copies of either K_4 or $K_{2,3}$-subdivisions and as a result $G\backslash_3 e_5$ contains disjoint copies of either K_4 or $K_{2,3}$ as a minor. By Theorem 3.1.1, $K_4 \mathring{\cup} K_4$, $K_4 \mathring{\cup} K_{2,3}$, and $K_{2,3} \mathring{\cup} K_{2,3}$ are excluded minors for outer-cylindrical graphs, so we see that $G\backslash_3 e_5$ is not outer-cylindrical – a contradiction to G being a cubic obstruction. So it must be that B_i is outer-planar where $i \neq 1$.

Let e_3 be an edge in B_3 and embed the graph $G\backslash_3 e_3$ so that all the vertices of $G\backslash_3 e_3$ lie on the boundary of one of two faces labeled f and g. Assume that a vertex v of B_5 lies on the boundary of f. We now show that $B_3\backslash_3 e_3, B_5, \ldots, B_{n-1}$ are centralized parts. So assume that one of these parts is not and let w be the vertex in the non-centralized part of $B_3\backslash_3 e_3, B_5, \ldots, B_{n-1}$ that does not lie on the boundary of f. Since $G\backslash_3 e_3$ is outer-cylindrical, w must lie on the boundary of g. And since G is a generalized cycle, there is a (v, w)-path, P, in $G\backslash_3 e_3$ whose internal vertices do not contain vertices from B_1. Consider the deletion of B_2, $B_3\backslash_3 e_3$, B_4, B_5, $\ldots$, B_n from the embedded graph of $G\backslash_3 e_3$, the result being B_1. Let f' be the face of B_1 containing f and let g' be the face of B_1 containing g. Since B_1 is not outer-planar,

$f' \neq g'$. Thus re-embedding P so that v is in $f \subseteq f'$ and w is in $g \subseteq g'$ induces a crossing of an edge of P with an edge of B_1. This implies that the embedded graph of $G\backslash_3 e_3$ has a crossing – a contradiction to the outer-cylindrical embedding of $G\backslash_3 e_3$. So it must be the case that all the vertices of $B_3\backslash_3 e_3, B_5, \ldots, B_{n-1}$ lie on f. Thus B_5 is a centralized part.

Likewise for an edge e_5 in B_5 we can embed the graph $G\backslash_3 e_5$ so that all the vertices of $G\backslash_3 e_5$ lie on one of two faces labeled h and k. If vertices of B_3 lie on the boundary of h, then all the vertices of $B_3, B_5\backslash_3 e_5, \ldots, B_{n-1}$ lie on the boundary of h. Now since B_5 was a centralized part in the embedding of $G\backslash_3 e_3$, we see that the replacement of $B_5\backslash_3 e_5$ with B_5 is a graph embedded in the plane where there are two faces, namely h and k, whose boundaries together contain all the vertices of G. Thus, G is an outer-cylindrical graph – a contradiction to G being a cubic obstruction. With this we conclude that the number of parts of G is no more than four. ■

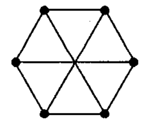
3.3 Obstructions for Outer-cylindrical Graphs

This section is devoted to showing that the graphs depicted in Figure 3.1 are cubic obstructions for outer-cylindrical graphs. Before describing the approach we take, we will prove the following lemma.

Lemma 3.3.1 *The only non-planar cubic obstruction for outer-cylindrical graphs is* $K_{3,3}$.

Proof: Here we show that $K_{3,3}$ is a cubic obstruction. Since $K_{3,3}$ is not planar it cannot be embedded in the plane. Thus by the definition, $K_{3,3}$ is not outer-cylindrical. For any edge e, the graph $K_{3,3}\backslash_3 e$ is isomorphic to K_4 which is outer-cylindrical. So we conclude that $K_{3,3}$ is a cubic obstruction.

Now suppose, that G is a non-planar cubic obstruction for outer-cylindrical graphs. By Theorem 1.5.2, the graph G contains a subgraph that is a subdivision of $K_{3,3}$, and by Proposition 1.6.1 G contains $K_{3,3}$ in the cubic order. Since $K_{3,3}$ is a cubic obstruction, G must be isomorphic to $K_{3,3}$. Therefore, $K_{3,3}$ is the only non-planar cubic obstruction for outer-cylindrical graphs. ■



$K_{3,3}$

Figure 3.6: The non-planar cubic obstruction for outer-cylindrical graphs.

The graphs depicted in Figure 3.1 were shown, via computer check, to be cubic obstructions for outer-cylindrical graphs. Table 3.1 occurs from pages 38 – 50, and it is here that the graphs depicted in Figure 3.1 are shown to be cubic obstructions for outer-cylindrical graphs. Specifically, each row of the table contains a graph depicted in Figure 3.1 and a proof that this graph is a cubic obstruction. The labeled graph G in the right column of each row of Table 3.1 is a loopless plane graph from Figure 3.1. The chart in the left column of each row of Table 3.1 has two columns, describing an outer-cylindrical embedding of G upon the cubic edge-deletion of some edge of G.

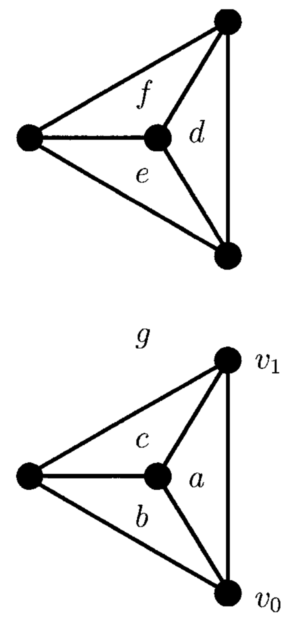
Vertices of each graph in the right column of Table 3.1 will be labeled v_i, for some $i \in \{0, 1, \dots\}$. The faces of the graph are labeled $a, b, \dots, h$. The left column of the chart in each row of Table 3.1 contains all the edges of G up to symmetry. Each entry of the right column of the chart contains a list of the faces of the graph resulting from the cubic edge-deletion of the corresponding edge in the left column of the chart. To explain how this information proves that the graph G is an obstruction, we need to describe an operation. Now upon the cubic edge-deletion of an edge of G, a new face is created by joining two faces of G. Formally, if a new face is created by the cubic edge-deletion of an edge xy of G, then the new face contains two faces, say a and b, and this new face is defined to be $a \cup b \cup \phi\{xy\} - \{\phi(x), \phi(y)\}$ which will be denoted $a \overset{*}{\vee} b$.

Now for each edge in the left column of the chart in each row of Table 3.1, the list of faces in the right column of the chart has two faces that are enclosed in a $\boxed{\text{box}}$. These are the two faces whose boundaries together contain all the vertices of the graph obtained by the cubic edge-deletion of the edge in the left column. So for the cubic edge-deletion of any edge e of G, we demonstrate an outer-cylindrical embedding of $G\backslash_3 e$.

Table 3.1: The loopless outer-cylindrical cubic obstructions.

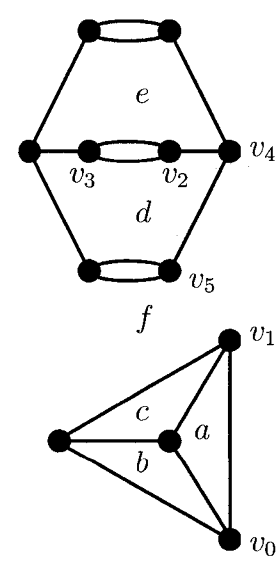
$K_4 \mathring{\cup} K_4$

edge xy	faces of $K_4 \mathring{\cup} K_4 \backslash_3 xy$
$v_0 v_1$	$\boxed{a \overset{*}{\vee} g}$, b, c, $\boxed{d}$, e, f



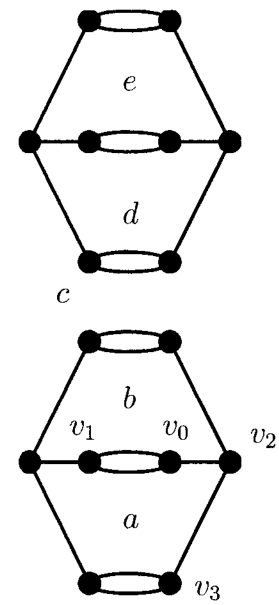
$K_4 \mathring{\cup} K_{2,3C_2}$

edge xy	faces of $K_4 \mathring{\cup} K_{2,3C_2} \backslash_3 xy$
$v_0 v_1$	$\boxed{a \overset{*}{\vee} f}$, b, c, $\boxed{d}$, e
$v_2 v_3$	$\boxed{a}$, b, c, d, e, $\boxed{f}$
$v_4 v_5$	$\boxed{a}$, b, c, $\boxed{d \overset{*}{\vee} f}$, e



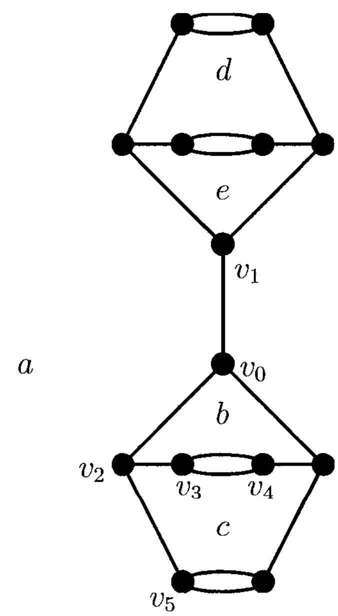
$K_{2,3C_2}\mathring{\cup}K_{2,3C_2}$

edge xy	faces of $K_{2,3C_2}\mathring{\cup}K_{2,3C_2}\setminus_3 xy$
v_0v_1	a, b, $\boxed{c}$, $\boxed{d}$, e
v_2v_3	$\boxed{a \overset{*}{\vee} c}$, b, c, $\boxed{d}$, e



D1

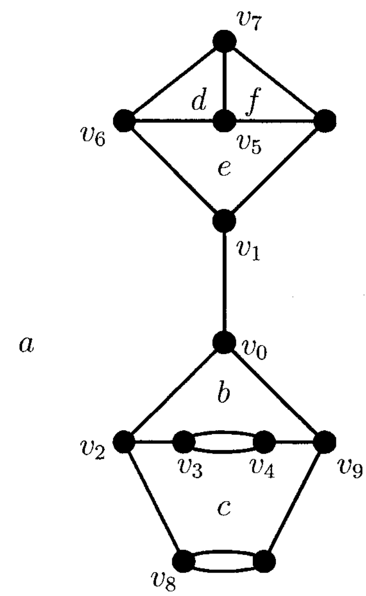
edge xy	faces of D1$\setminus_3 xy$
v_0v_1	a, b, $\boxed{c}$, $\boxed{d}$, e
v_0v_2	$\boxed{a \overset{*}{\vee} b}$, c, $\boxed{d}$, e
v_2v_5	$\boxed{a \overset{*}{\vee} c}$,$b$, $\boxed{d}$, e
v_3v_4	$\boxed{a}$, b, c, $\boxed{d}$, e



D2

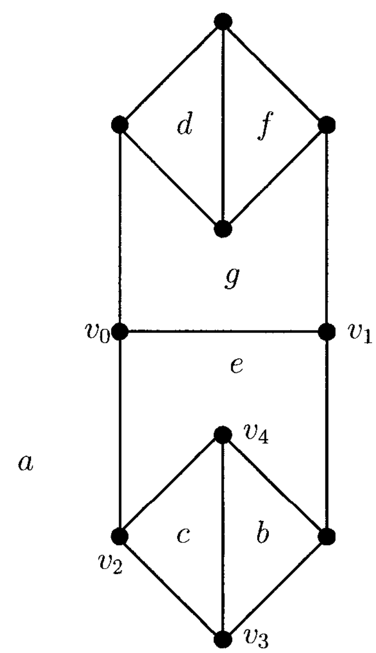
edge xy	faces of $\mathrm{D2}\backslash_3 xy$
v_0v_1	see below *
v_0v_2	$\boxed{a \overset{*}{\vee} b}$, c, $\boxed{d}$, e, f
v_2v_8	$\boxed{a \overset{*}{\vee} c}$, b, $\boxed{d}$, e, f
v_3v_4	$\boxed{a}$, b, c, $\boxed{d}$, e, f
v_1v_6	$\boxed{a \overset{*}{\vee} e}$, $\boxed{b}$, c, d, f
v_5v_6	$\boxed{a}$, $\boxed{b}$, c, $d \overset{*}{\vee} e$, f
v_5v_7	$\boxed{a}$, $\boxed{b}$, c, $d \overset{*}{\vee} f$, e
v_6v_7	$\boxed{a \overset{*}{\vee} d}$, $\boxed{b}$, c, e, f

* Embed v_2v_9 of $G\backslash_3 v_0v_1$ in face c then faces $a \overset{*}{\vee} b$ and d contain all vertices of $G\backslash_3 v_0v_1$.



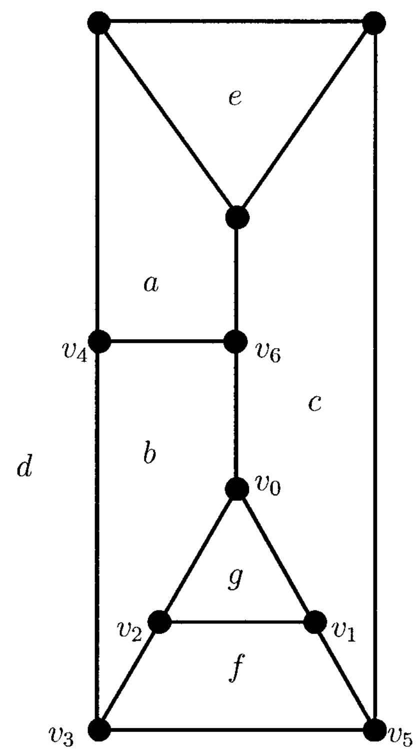
C21

edge xy	faces of $\mathrm{C21}\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, c, d, $\boxed{e \overset{*}{\vee} g}$, f
v_0v_2	$\boxed{a \overset{*}{\vee} e}$, b, c, d, f, $\boxed{g}$
v_2v_3	$\boxed{a \overset{*}{\vee} c}$, b, d, e, f, $\boxed{g}$
v_3v_4	$\boxed{a}$, $b \overset{*}{\vee} c$, d, e, f, $\boxed{g}$



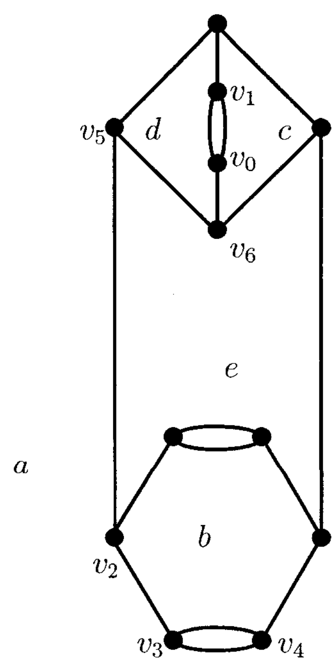
C9

edge xy	faces of $C9\backslash_3 xy$
v_0v_1	a, b, $\boxed{c \overset{*}{\vee} g}$, $\boxed{d}$, e, f
v_0v_2	a, $b \overset{*}{\vee} g$, $\boxed{c}$, $\boxed{d}$, e, f
v_0v_6	$\boxed{a}$, $\boxed{b \overset{*}{\vee} c}$, d, e, f, g
v_1v_2	a, b, $\boxed{c}$, $\boxed{d}$, e, $f \overset{*}{\vee} g$
v_1v_5	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} f}$, d, e, g
v_2v_3	$\boxed{a}$, $b \overset{*}{\vee} f$, $\boxed{c}$, d, e, g
v_3v_4	a, $\boxed{b \overset{*}{\vee} d}$, $\boxed{c}$, e, f, g
v_3v_5	a, b, $\boxed{c}$, $\boxed{d \overset{*}{\vee} f}$, e, g



T1

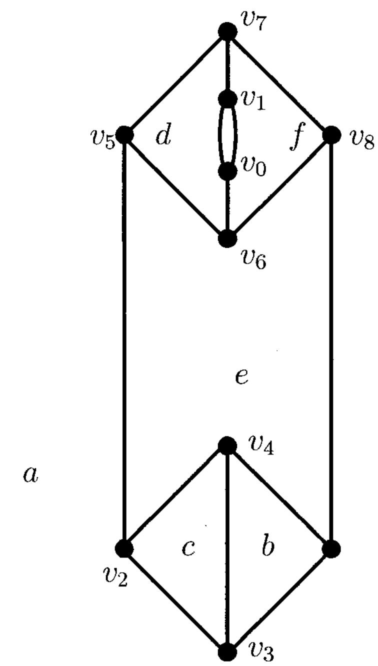
edge xy	faces of $T1\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, c, d, $\boxed{e}$
v_0v_6	a, $\boxed{b}$, $\boxed{c \overset{*}{\vee} d}$, e
v_2v_5	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$
v_2v_3	$\boxed{a \overset{*}{\vee} b}$, c, $\boxed{d}$, e
v_3v_4	a, b, c, $\boxed{d}$, $\boxed{e}$
v_5v_6	$\boxed{a}$, b, c, $\boxed{d \overset{*}{\vee} e}$



T2

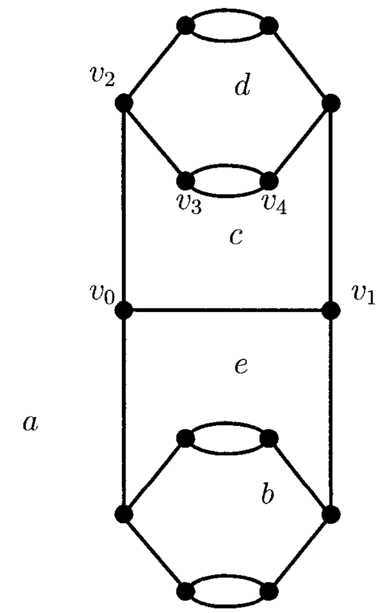
edge xy	faces of $\mathrm{T2}\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, c, d, $\boxed{e}$, f
v_0v_6	see below *
v_2v_5	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$, f
v_2v_3	$\boxed{a \overset{*}{\vee} c}$, b, $\boxed{d}$, e, f
v_3v_4	$\boxed{a}$, $b \overset{*}{\vee} c$, $\boxed{d}$, e, f
v_5v_6	$\boxed{a}$, b, c, $\boxed{d \overset{*}{\vee} e}$, f

* Embed edge v_5v_8 of $G\backslash_3 v_0v_6$ in face a. Then faces a and e contain all vertices of $G\backslash_3 v_0v_6$.



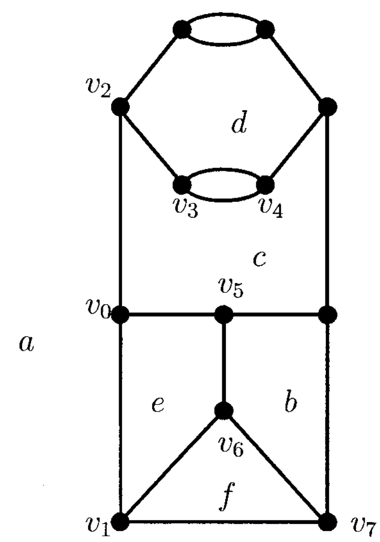
T3

edge xy	faces of $\mathrm{T3}\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} e}$, d
v_0v_2	$\boxed{a \overset{*}{\vee} c}$, b, d, $\boxed{e}$
v_2v_3	a, b, $\boxed{c \overset{*}{\vee} d}$, $\boxed{e}$
v_3v_4	$\boxed{a}$, b, c, d, $\boxed{e}$



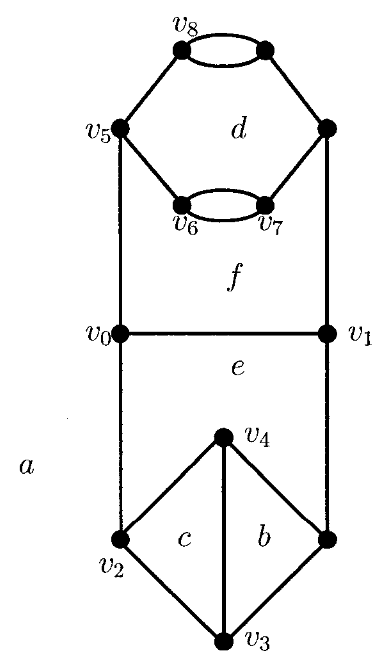
T4

edge xy	faces of $\mathrm{T4}\backslash_3 xy$
v_0v_1	$\boxed{a \overset{*}{\vee} e}$, b, $\boxed{c}$, d, f
v_0v_2	$\boxed{a \overset{*}{\vee} c}$, b, d, $\boxed{e}$, f
v_0v_5	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} e}$, d, f
v_1v_6	$\boxed{a}$, b, $\boxed{c}$, d, $e \overset{*}{\vee} f$
v_1v_7	$\boxed{a \overset{*}{\vee} f}$, b, $\boxed{c}$, d, e
v_2v_3	a, b, $\boxed{c \overset{*}{\vee} d}$, e, $\boxed{f}$
v_3v_4	$\boxed{a}$, b, c, d, $\boxed{e}$, f
v_5v_6	$\boxed{a}$, $b \overset{*}{\vee} e$, $\boxed{c}$, d, f



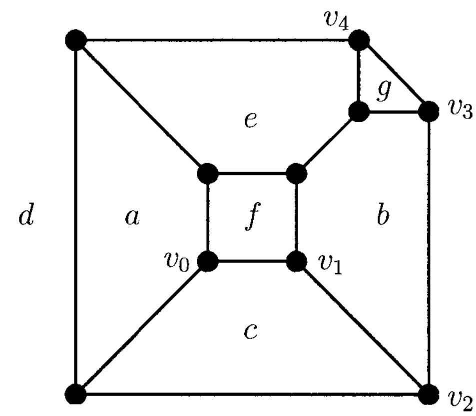
T5

edge xy	faces of $\mathrm{T5}\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, c, d, $\boxed{e \overset{*}{\vee} f}$
v_0v_2	$\boxed{a \overset{*}{\vee} e}$, b, c, d, $\boxed{f}$
v_0v_5	$\boxed{a \overset{*}{\vee} f}$, b, c, d, $\boxed{e}$
v_2v_3	$\boxed{a \overset{*}{\vee} c}$, b, d, e, $\boxed{f}$
v_3v_4	$\boxed{a}$, $b \overset{*}{\vee} c$, d, e, $\boxed{f}$
v_5v_8	$\boxed{a \overset{*}{\vee} d}$, b, c, $\boxed{e}$, f
v_6v_7	$\boxed{a}$, b, c, d, $\boxed{e}$, f



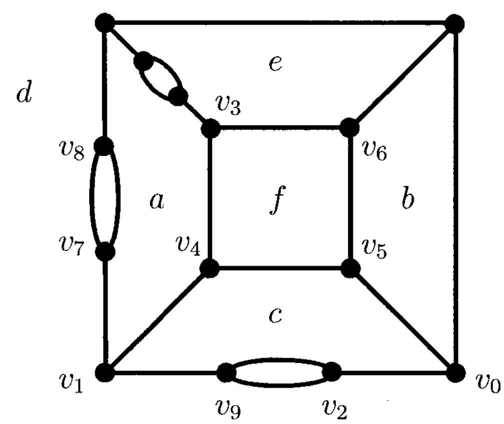
C16

edge xy	faces of C16$\backslash_3 xy$
v_0v_1	a, b, $c \overset{*}{\vee} f$, $\boxed{d}$, $\boxed{e}$, g
v_1v_2	a, $\boxed{b \overset{*}{\vee} c}$, d, $\boxed{e}$, f, g
v_2v_3	$\boxed{a}$, $\boxed{b \overset{*}{\vee} d}$, c, e, f, g
v_3v_4	a, b, $\boxed{c}$, $d \overset{*}{\vee} g$, $\boxed{e}$, f



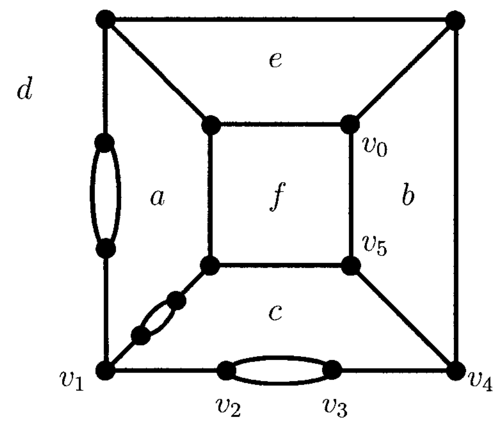
Q1

edge xy	faces of Q1$\backslash_3 xy$
v_0v_2	a, b, $\boxed{c \overset{*}{\vee} d}$, $\boxed{e}$, f, g
v_0v_5	$\boxed{a}$, $\boxed{b \overset{*}{\vee} c}$, d, e, f, g
v_1v_4	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e, f, g
v_1v_7	$\boxed{a \overset{*}{\vee} d}$, $\boxed{b}$, c, e, f, g
v_1v_9	a, b, $\boxed{c \overset{*}{\vee} d}$, $\boxed{e}$, f, g
v_2v_9	$\boxed{a}$, $\boxed{b}$, c, d, e, f, g
v_3v_4	$\boxed{a \overset{*}{\vee} f}$, b, c, $\boxed{d}$, e, g
v_4v_5	a, b, $c \overset{*}{\vee} f$, $\boxed{d}$, $\boxed{e}$, g
v_5v_6	a, $\boxed{b \overset{*}{\vee} f}$, c, $\boxed{d}$, e, g
v_7v_8	a, b, $\boxed{c}$, d, $\boxed{e}$, f, g



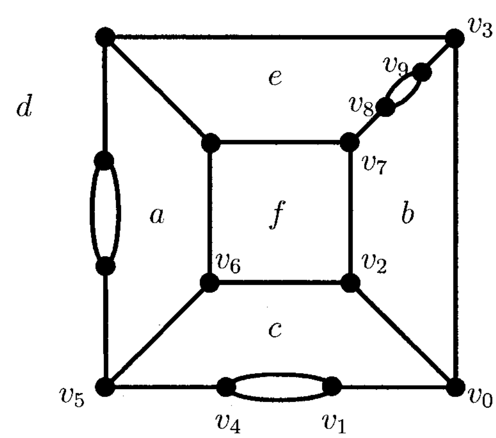
Q2

edge xy	faces of $Q2\backslash_3 xy$
v_0v_5	$\boxed{a}$, $b \overset{*}{\vee} f$, c, $\boxed{d}$, e, g
v_1v_2	a, b, $\boxed{c \overset{*}{\vee} d}$, e, $\boxed{f}$, g
v_2v_3	$\boxed{a}$, $\boxed{b}$, c, d, e, f, g
v_3v_4	a, b, $\boxed{c \overset{*}{\vee} d}$, e, $\boxed{f}$, g
v_4v_5	$\boxed{a}$, $\boxed{b \overset{*}{\vee} c}$, d, e, f, g



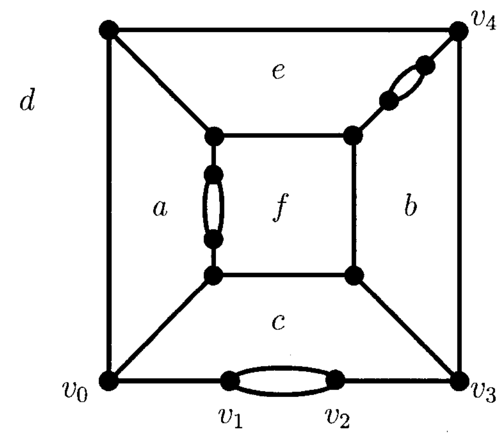
Q3

edge xy	faces of $Q3\backslash_3 xy$
v_0v_1	a, b, $\boxed{c \overset{*}{\vee} d}$, $\boxed{e}$, f, g
v_0v_2	$\boxed{a}$, $\boxed{b \overset{*}{\vee} c}$, d, e, f, g
v_0v_3	a, $\boxed{b \overset{*}{\vee} d}$, c, e, $\boxed{f}$, g
v_1v_4	$\boxed{a}$, $\boxed{b}$, c, d, e, f, g
v_2v_6	a, b, $c \overset{*}{\vee} f$, $\boxed{d}$, $\boxed{e}$, g
v_2v_7	a, $\boxed{b \overset{*}{\vee} f}$, c, $\boxed{d}$, e, g
v_3v_9	a, $b \overset{*}{\vee} e$, c, $\boxed{d}$, $\boxed{f}$, g
v_4v_5	a, b, $\boxed{c \overset{*}{\vee} d}$, $\boxed{e}$, f, g
v_5v_6	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e, f, g
v_7v_8	a, $b \overset{*}{\vee} e$, c, $\boxed{d}$, $\boxed{f}$, g
v_8v_9	a, b, c, $\boxed{d}$, e, $\boxed{f}$, g



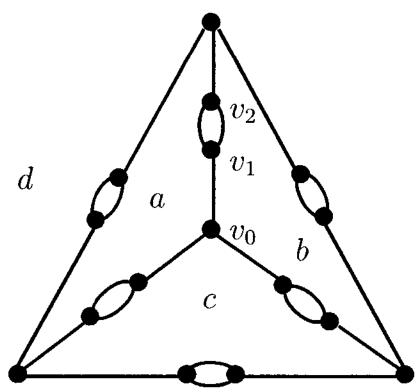
Q4

edge xy	faces of $Q4\backslash_3 xy$
v_0v_1	$\boxed{a}$, $\boxed{b}$, $c \overset{*}{\vee} d$, e, f, g
v_1v_2	$\boxed{a}$, $\boxed{b}$, c, d, e, f, g
v_2v_3	$\boxed{a}$, $\boxed{b}$, $c \overset{*}{\vee} d$, e, f, g
v_3v_4	$\boxed{a}$, $\boxed{b \overset{*}{\vee} d}$, c, e, f, g



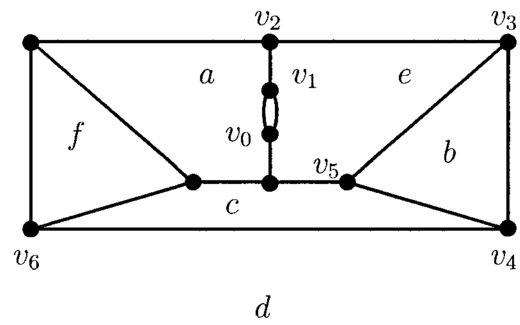
F1

edge xy	faces of $F1\backslash_3 xy$
v_0v_1	$\boxed{a \overset{*}{\vee} b}$, $\boxed{c}$, d
v_1v_2	a, b, $\boxed{c}$, $\boxed{d}$



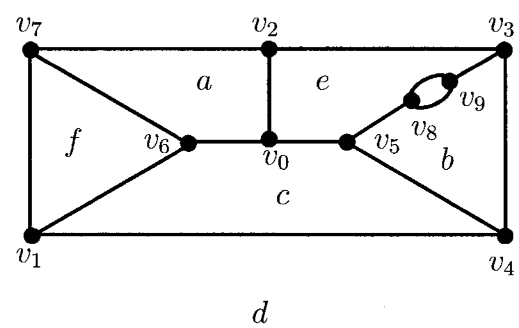
P1

edge xy	faces of $P1\backslash_3 xy$
v_0v_1	a, b, $\boxed{c}$, $\boxed{d}$, e, f
v_1v_2	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$, f
v_2v_3	a, b, $\boxed{c}$, $\boxed{d \overset{*}{\vee} e}$, f
v_3v_4	$\boxed{a}$, $\boxed{b \overset{*}{\vee} d}$, c, e, f
v_3v_5	$\boxed{a}$, $b \overset{*}{\vee} e$, c, $\boxed{d}$, f
v_4v_6	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} d}$, e, f



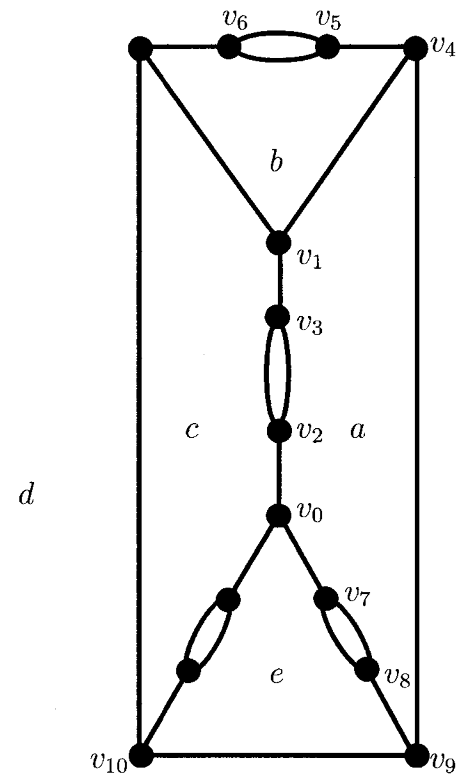
P2

edge xy	faces of $P2\backslash_3 xy$
v_4v_5	a, $\boxed{b \overset{*}{\vee} c}$, $\boxed{d}$, e, f
v_5v_8	a, $\boxed{b \overset{*}{\vee} e}$, c, d, $\boxed{f}$
v_8v_9	a, b, $\boxed{c}$, $\boxed{d}$, e, f
v_0v_5	a, b, $\boxed{c \overset{*}{\vee} e}$, $\boxed{d}$, f
v_0v_2	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$, f
v_0v_6	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e, f
v_6v_7	$\boxed{a \overset{*}{\vee} f}$, $\boxed{b}$, c, d, e
v_1v_6	a, b, $\boxed{c \overset{*}{\vee} f}$, d, $\boxed{e}$
v_1v_4	a, $\boxed{b}$, $\boxed{c \overset{*}{\vee} d}$, e, f



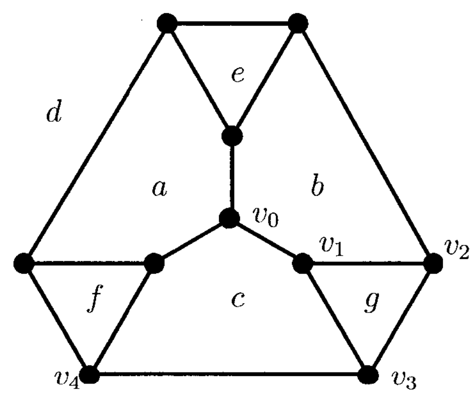
P3

edge xy	faces of P3$\backslash_3 xy$
v_0v_2	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e
v_0v_7	$\boxed{a \overset{*}{\vee} e}$, $\boxed{b}$, c, d
v_1v_3	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e
v_1v_4	$\boxed{a \overset{*}{\vee} b}$, $\boxed{c}$, d, e
v_2v_3	a, $\boxed{b}$, c, d, $\boxed{e}$
v_4v_5	$\boxed{a}$, $b \overset{*}{\vee} d$, $\boxed{c}$, e
v_4v_9	$\boxed{a \overset{*}{\vee} d}$, b, c, $\boxed{e}$
v_5v_6	$\boxed{a}$, b, $\boxed{c}$, d, e
v_7v_8	a, b, $\boxed{c}$, $\boxed{d}$, e
v_8v_9	$\boxed{a \overset{*}{\vee} e}$, $\boxed{b}$, c, d
v_9v_{10}	$\boxed{a}$, b, c, $\boxed{d \overset{*}{\vee} e}$



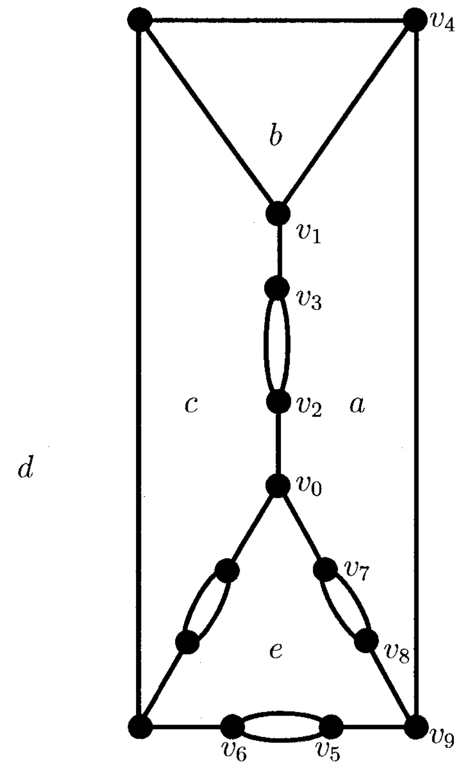
C15

edge xy	faces of C15$\backslash_3 xy$
v_0v_1	$\boxed{a}$, $b \overset{*}{\vee} c$, $\boxed{d}$, e, f, g
v_1v_3	$\boxed{a}$, b, $c \overset{*}{\vee} g$, $\boxed{d}$, e, f
v_2v_3	$\boxed{a}$, b, c, $\boxed{d \overset{*}{\vee} g}$, e, f
v_3v_4	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} d}$, e, f, g



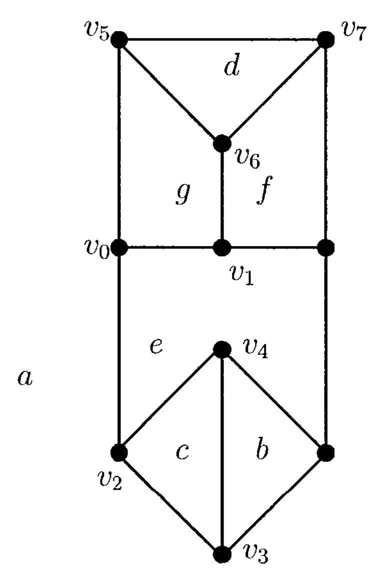
P4

edge xy	faces of $\mathrm{P4}\backslash_3 xy$
v_0v_2	$\boxed{a \overset{*}{\vee} c}$, b, d, $\boxed{e}$
v_0v_7	$\boxed{a \overset{*}{\vee} e}$, $\boxed{b}$, c, d
v_1v_3	$\boxed{a \overset{*}{\vee} c}$, b, d, $\boxed{e}$
v_1v_4	$\boxed{a \overset{*}{\vee} b}$, c, d, $\boxed{e}$
v_2v_3	a, $\boxed{b}$, c, d, $\boxed{e}$
v_4v_9	$\boxed{a \overset{*}{\vee} d}$, b, c, $\boxed{e}$
v_5v_9	a, b, $\boxed{c}$, $\boxed{d \overset{*}{\vee} e}$
v_5v_6	$\boxed{a}$, b, $\boxed{c}$, d, e
v_7v_8	a, b, $\boxed{c}$, $\boxed{d}$, e
v_8v_9	$\boxed{a \overset{*}{\vee} e}$, $\boxed{b}$, c, d



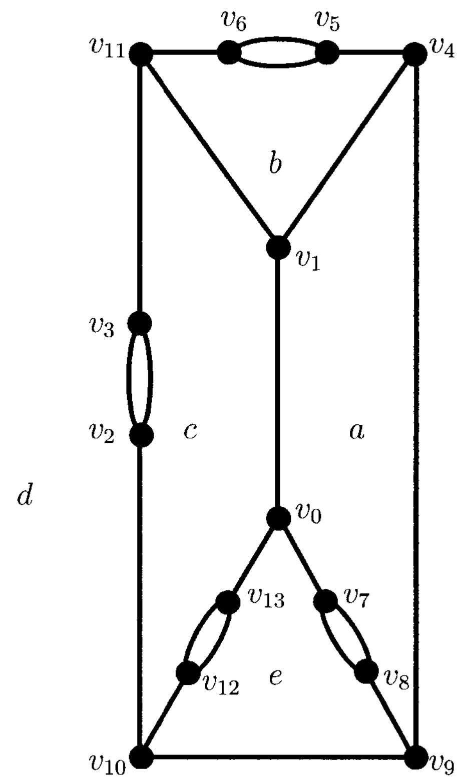
C14

edge xy	faces of $\mathrm{C14}\backslash_3 xy$
v_0v_1	$\boxed{a}$, b, c, d, $\boxed{e \overset{*}{\vee} g}$, f
v_0v_2	$\boxed{a \overset{*}{\vee} e}$, b, c, d, f, $\boxed{g}$
v_2v_3	$\boxed{a \overset{*}{\vee} c}$, b, d, e, f, $\boxed{g}$
v_3v_4	$\boxed{a}$, $b \overset{*}{\vee} c$, d, e, f, $\boxed{g}$
v_0v_5	$\boxed{a \overset{*}{\vee} g}$, b, c, d, $\boxed{e}$, f
v_1v_6	$\boxed{a}$, b, c, d, $\boxed{e}$, $f \overset{*}{\vee} g$
v_5v_6	$\boxed{a}$, b, c, $d \overset{*}{\vee} g$, $\boxed{e}$, f
v_5v_7	$\boxed{a \overset{*}{\vee} d}$, b, c, $\boxed{e}$, f, g



P5

edge xy	faces of $\mathrm{P5}\backslash_3 xy$
v_0v_1	$\boxed{a \overset{*}{\vee} c}$, $\boxed{b}$, d, e
v_0v_7	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$
v_0v_{13}	a, $\boxed{b}$, $\boxed{c \overset{*}{\vee} e}$, d
v_1v_4	$\boxed{a \overset{*}{\vee} b}$, $\boxed{c}$, d, e
v_1v_{11}	$\boxed{a}$, $\boxed{b \overset{*}{\vee} c}$, d, e
v_2v_3	a, $\boxed{b}$, c, d, $\boxed{e}$
v_2v_{10}	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} d}$, e
v_3v_{11}	$\boxed{a}$, b, $\boxed{c \overset{*}{\vee} d}$, e
v_4v_9	$\boxed{a \overset{*}{\vee} d}$, b, c, $\boxed{e}$
v_4v_5	a, $\boxed{b \overset{*}{\vee} d}$, c, $\boxed{e}$
v_5v_6	$\boxed{a}$, b, $\boxed{c}$, d, e
v_6v_{11}	a, $\boxed{b \overset{*}{\vee} d}$, c, $\boxed{e}$
v_7v_8	a, b, $\boxed{c}$, $\boxed{d}$, e
v_8v_9	$\boxed{a \overset{*}{\vee} e}$, b, c, $\boxed{d}$
v_9v_{10}	a, $\boxed{b}$, c, $\boxed{d \overset{*}{\vee} e}$
$v_{10}v_{12}$	a, $\boxed{b}$, $\boxed{c \overset{*}{\vee} e}$, d
$v_{12}v_{13}$	$\boxed{a}$, b, c, $\boxed{d}$, e



CHAPTER 4

CUBIC OBSTRUCTIONS FOR OUTER-CYLINDRICAL GRAPHS

4.1 Introduction

The main result of this chapter is to show that the set of all graphs depicted in Appendix A is the complete set of cubic obstructions for outer-cylindrical graphs. In Section 4.2, we characterize cubic obstructions that are disconnected, 1-edge-connected, and those having a 2-edge-cut whose deletion results in a graph that has no components which are 2-cycles. In Section 4.3, we characterize those cubic obstructions containing a subgraph that is a subdivision of a cube. We complete this chapter in Section 4.4 by characterizing the remaining cubic obstructions not already characterized in the previous sections.

If G is a cubic obstruction for outer-cylindrical graphs, Proposition 3.2.3 and Corollary 3.2.4 allow us to replace 2-cycles with nooses and nooses with 2-cycles within G and obtain another cubic obstruction. Since a cubic graph has a noose if and only if it has a loop, the problem of finding all cubic obstructions for outer-cylindrical graphs is reduced to finding all loopless cubic obstructions. In order to obtain the complete list of cubic obstructions for outer-cylindrical graphs, we need only perform all possible combinations of 2-cycle-noose operations on the loopless cubic obstructions. Thus, the remainder of this chapter will be devoted to finding all loopless cubic obstructions for outer-cylindrical graphs.

4.2 Obstructions from Outer-planar Obstructions

In this section we show that the disconnected cubic obstructions for outer-cylindrical graphs are the disjoint unions of two cubic obstructions for outer-planar graphs. Furthermore, we show that loopless cubic obstructions of edge-connectivity one are D1 and D2 which were presented in Table 3.1 on pages 39 and 40. Finally, we show that if none of the components resulting from a 2-edge-cut of a cubic obstruction is a 2-cycle then the obstruction is either C9, C14, T1, T2, T3, T4, or T5. From Lemma 3.3.1, the problem of finding all the cubic obstructions for outer-cylindrical graphs is reduced to finding only the loopless cubic obstructions that are planar. We now prove two propositions, the first of which characterizes the cubic obstructions that are disconnected. The second proposition gives the loopless cubic obstructions having edge-connectivity one.

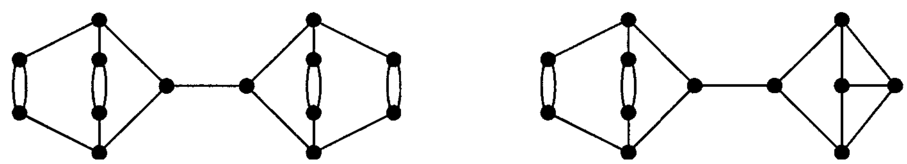
Proposition 4.2.1 *The planar disconnected cubic obstructions for outer-cylindrical graphs are the disjoint unions of two cubic obstructions for outer-planar graphs.*

Proof: From Table 3.1 (pages 38 – 39), Theorem 2.1.1, and Proposition 3.2.3, it is clear that disjoint union of two cubic obstructions for outer-planar graphs is a cubic obstruction for outer-cylindrical graphs. We claim that there no others. Assume that G is a disconnected cubic obstruction for outer-cylindrical graphs, and G is not isomorphic to the disjoint union of two cubic obstructions for outer-planar graphs. Since G is disconnected then G has at least two components G_1 and G_2. We now show that neither G_1 nor G_2 is outer-planar.

Without loss of generality assume that G_1 is outer-planar. Then, since G is a cubic obstruction, $G - V(G_1)$ can be embedded so that every vertex lies on at most two faces. Consider such an embedding of $G - V(G_1)$ and let f and g be the two faces of the embedding whose boundaries together contain all the vertices of $G - V(G_1)$. Since G_1 is outer-planar embed G_1 in f so that every vertex of G_1 lies on f. So there are two faces of an embedding of G whose boundaries together contain all the vertices of G, and G is an outer-cylindrical graph – a contradiction to the choice of

G. So it must be that G_1 is not outer-planar. We conclude that neither G_1 nor G_2 is outer-planar.

Since G_1 is cubic and not outer-planar, from Theorem 2.1.1, G_1 contains a cubic obstruction for outer-planar graphs, likewiseG_2 contains a cubic obstruction for outer-planar graphs. Because G_1 and G_2 are disjoint then G contains a disjoint union of two cubic obstructions for outer-planar graphs. Now G is a cubic obstruction for outer-cylindrical graphs and G contains a disjoint union of two cubic obstructions for outer-planar graphs in the cubic order. Thus G is isomorphic to a disjoint union of cubic obstructions for outer-planar graphs – a contradiction to the choice of$G$. With this contradiction the proposition is established. ■



D1 D2

Figure 4.1: The cubic obstructions for outer-cylindrical graphs of edge-connectivity one.

Proposition 4.2.2 *Let G be a loopless planar cubic obstruction for outer-cylindrical graphs. If G is of edge-connectivity one, then G is isomorphic to* D1 *or* D2.

Proof: In Theorem 3.1.1, the excluded minors for outer-cylindrical graphs are presented, the disconnected ones being $K_4 \mathring{\cup} K_4$, $K_4 \mathring{\cup} K_{2,3}$, and $K_{2,3} \mathring{\cup} K_{2,3}$. So if G contains disjoint copies of K_4 or $K_{2,3}$ subdivisions, then clearly G contains disjoint copies of K_4 or $K_{2,3}$ as a minor. So, by Theorem 3.1.1 we can state the following.

4.2.3 *If G contains a subdivision of $K_4 \mathring{\cup} K_4$, $K_4 \mathring{\cup} K_{2,3}$, or $K_{2,3} \mathring{\cup} K_{2,3}$, then G is not outer-cylindrical.*

Let G be a loopless cubic obstruction for outer-cylindrical graphs that is not isomorphic to one of D1 and D2, and suppose $G \backslash v_1 v_2$ is a disconnected subgraph that has components G_1 and G_2 where v_1 is in G_1 and v_2 is in G_2.

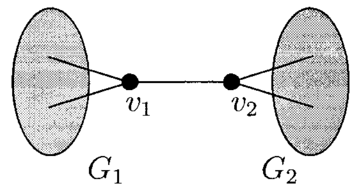


Figure 4.2: The cut edge of G.

We now show that G_1 and G_2 are not outer-planar. Without loss of generality, assume that G_1 is outer-planar and let e' be an edge of G_1. We will now construct an outer-cylindrical embedding of G. Since G is loopless, G_2 is a subgraph of $G\backslash_3 e'$ and so G_2 is outer-cylindrical. Embed G_2 so that the boundaries of two faces, say f and g, together contain all the vertices of G_2. Now suppose that f is the face having v_2 on its boundary. Since G_1 is outer-planar, embed v_1v_2 and G_1 in f such that all the vertices of G_1 lie in f. Thus G is embedded in the plane so that there are two faces whose boundaries together contain every vertex of G. So G is outer-cylindrical – a contradiction to G being a cubic obstruction. Therefore, it must be that G_1 and G_2 are not outer-planar, and by Theorem 1.5.5 both G_1 and G_2 contain a subgraph that is a subdivision of either K_4 or $K_{2,3}$.

Assume that either G_1 or G_2 is not 2-edge-connected. So without loss of generality assume there is an edge $v'v''$ that disconnects G_2 into components G' and G'' where v' is in G' and v'' is in G''. If G' and G'' are outer-planar then G_2 would be outer-planar by Lemma 1.5.4. So assume G' is not outer-planar and consider an edge e'' in G''. Since G'' is not a loop and e'' is not either v_1v_2 or $v'v''$ then G' and G_1 are disjoint and subgraphs of $G\backslash_3 e''$. Thus $G\backslash_3 e''$ contains disjoint subgraphs which are subdivisions of either K_4 or $K_{2,3}$. So from 4.2.3, it is clear that $G\backslash_3 e''$ is not outer-cylindrical – a contradiction to G being a cubic obstruction. Thus G_1 and G_2 are 2-edge-connected.

Assume that both components contain a subgraph that is a subdivision of K_4. The ends of v_1v_2 are not of degree three in either component thus they are not the vertices of degree three of either subgraph that is a subdivision of K_4. So, the graph $G\backslash_3 v_1v_2$ contains a subgraph that is a subdivision of $K_4 \mathring{\cup} K_4$ which means that $G\backslash_3 v_1v_2$ is not outer-cylindrical by Proposition 4.2.1 – a contradiction to G being a cubic obstruction. So it must be one component does not contain a subgraph that is a subdivision of K_4.

Now, for the remainder of this proof, without loss of generality, assume that G_2 does not contain a subgraph that is a subdivision of K_4. By Theorem 1.5.2, since G_2 is not outer-planar, G_2 must contain a subgraph that is a subdivision of $K_{2,3}$ which we will denote H_2. Now again, by Theorem 1.5.5, we have the following.

4.2.4 *Since G_1 is not outer-planar, it contains a subgraph that is a subdivision of K_4 or $K_{2,3}$ which we denote H_1.*

Since G is 1-edge-connected there is at least one path from H_1 to H_2. Assume R is a shortest such path and let h_1 be the end of R in H_1 and let h_2 be the end of R in H_2. Since H_2 is a subgraph that is a subdivision of $K_{2,3}$, let w_1 and w_2 denote the vertices of degree three, and P_1, P_2, and P_3 be the internally disjoint paths of H_2 whose ends are w_1 and w_2. Since h_2 is in H_2, then h_2 is an internal vertex of either P_1, P_2, or P_3. Without loss of generality, assume h_2 is in P_2.

Assume that the end edges of one of P_1, P_2, or P_3 do not form a 2-edge-cut in G_2. From Lemma 1.7.2, G_2 contains a subgraph that is a subdivision of K_4 – a contradiction to G_2 not containing a subgraph that is a subdivision of K_4. Thus it must be that the end edges of each of P_1, P_2, and P_3 form a 2-edge-cut in G_2.

Since G is cubic and R is a shortest path from H_1 to H_2, then R avoids P_1 and P_3. Clearly, R utilizes the bridge v_1v_2. So it is clear that vertex v_2 is not in either P_1 or P_3. Thus we see that the end edges of each of P_1 and P_3 form a 2-edge-cut in G.

Now by Lemma 1.7.3 we see that each of P_1 and P_3 are paths whose vertices are in a subgraph that is a subdivision of a storm and the ends of P_1 and P_3 are the ends of the storm. Let x_1 and y_1 denote the internal vertices of P_1 that are in a storm and let x_3 and y_3 denote the internal vertices of P_3 that are in a storm. We now consider the two cases implicit in 4.2.4 above.

Suppose that H_1 is a subgraph that is a subdivision of K_4. Let k_1, k_2, k_3, and k_4 be the branch vertices of H_1 having degree three and let h_1 be in the (k_1,k_4)-subdivisional path. Thus we see that G contains a subgraph that is a subdivision of D2 denoted D

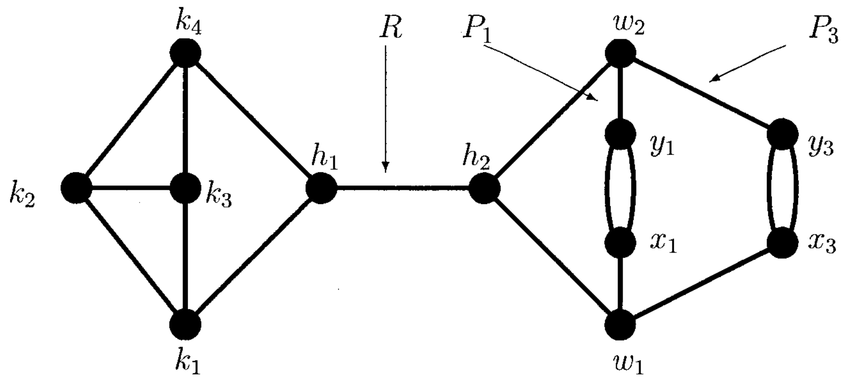


Figure 4.3: A subgraph that is a subdivision of D2

where the branch vertices of degree three that are not in any 2-cycle are k_1, k_2, k_3, k_4, h_1, h_2, w_1, and w_2 and the branch vertices of D that are in a 2-cycle are x_1, y_1, x_3, and y_3. So it must be that if G_1 contains a subgraph that is a subdivision of K_4 then G contains a subgraph that is a subdivision of D2. Now G is a cubic obstruction and G contains D2 in the cubic order, thus G is isomorphic to D2 – a contradiction to the choice of G. Thus, for the remainder of this proof assume that G_1 does not contain a subgraph that is a subdivision of K_4.

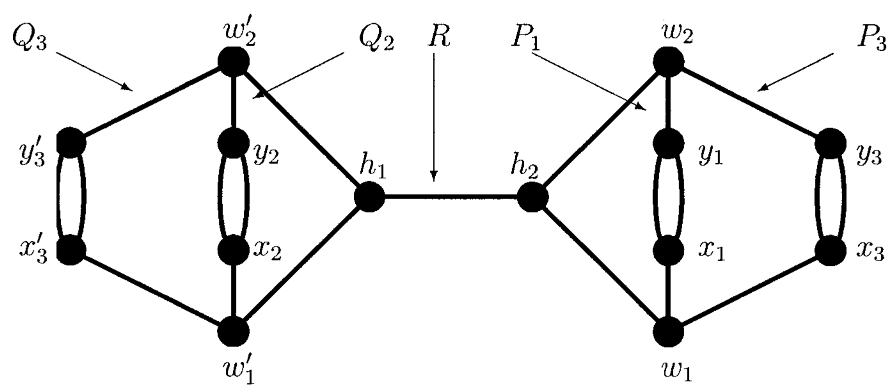


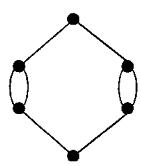
Figure 4.4: A subgraph that is a subdivision of D1.

In this case, H_1 is a subgraph that is a subdivision of $K_{2,3}$. Let w_1' and w_2' denote the vertices of degree three of H_1, and let Q_1, Q_2, and Q_3 be the internally disjoint paths of H_1 joining w_1' and w_2'. Without loss of generality, assume h_1 is in Q_1. Since the degree of w_1' and w_2' is three, notice that $h_1 \neq w_1'$ and $h_1 \neq w_2'$.

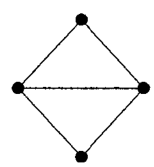
Assume that the end edges of one of Q_1, Q_2, or Q_3 do not form a 2-edge-cut in G_1. Then from Lemma 1.7.2, G_1 contains a subgraph that is a subdivision of K_4 – a contradiction to G_1 not containing a subgraph that is a subdivision of K_4. Thus it must be that the end edges of each of Q_1, Q_2, and Q_3 form a 2-edge-cut in G_1.

Since R is a shortest path from H_1 to H_2, path R avoids Q_2 and Q_3. Now clearly R utilizes the bridge v_1v_2. So it is clear that vertex v_1 is not in either Q_2 or Q_3. Thus we see that the end edges of each of Q_2 and Q_3 form a 2-edge-cut in G.

Now by Lemma 1.7.3 we see that each of Q_2 and Q_3 are paths whose vertices are in a subgraph that is a subdivision of a storm and the ends of Q_2 and Q_3 are the ends of the storm. Let x_2 and y_2 denote the internal vertices Q_2 that are in a storm and let x_3' and y_3' denote the internal vertices of Q_3 that are in a storm. Thus G contains a subgraph that is a subdivision of D1 where the branch vertices of degree three that are not in a 2-cycle are w_1', w_2', h_1, h_2, w_1, and w_2 and the branch vertices of D that are in a 2-cycle are x_1, y_1, x_2, y_2, x_3, y_3, x_3', and y_3',. So it must be that G_1 contains a subgraph that is a subdivision of D1. Now G is a cubic obstruction and G contains D1 in the cubic order, thus G is isomorphic to D1 – a contradiction to the choice of G. ■



a



b

Figure 4.5: The halo and the kite

With the establishment of Propositions 4.2.1 and 4.2.2, the problem of finding all the cubic obstructions for outer-cylindrical graphs is now reduced to finding only the planar cubic obstructions that are 2-edge-connected. In the next proposition, we characterize the planar cubic obstructions for which there is a 2-edge-cut, E', where neither component of the graph obtained from deleting E' is a 2-cycle. Before proving the next proposition, we need to formally develop some of the terminology used in the proof. Let G be a 6-cycle and e_1 and e_2 be distinct edges of G where the endvertices of e_1 and e_2 are not neighbors. The *halo* is obtained by adding one edge in parallel to e_1 and one edge in parallel to e_2. The halo is depicted in Figure 4.5 (a). The *kite* is obtained by deleting one edge of K_4 (see Figure 4.5 (b)).

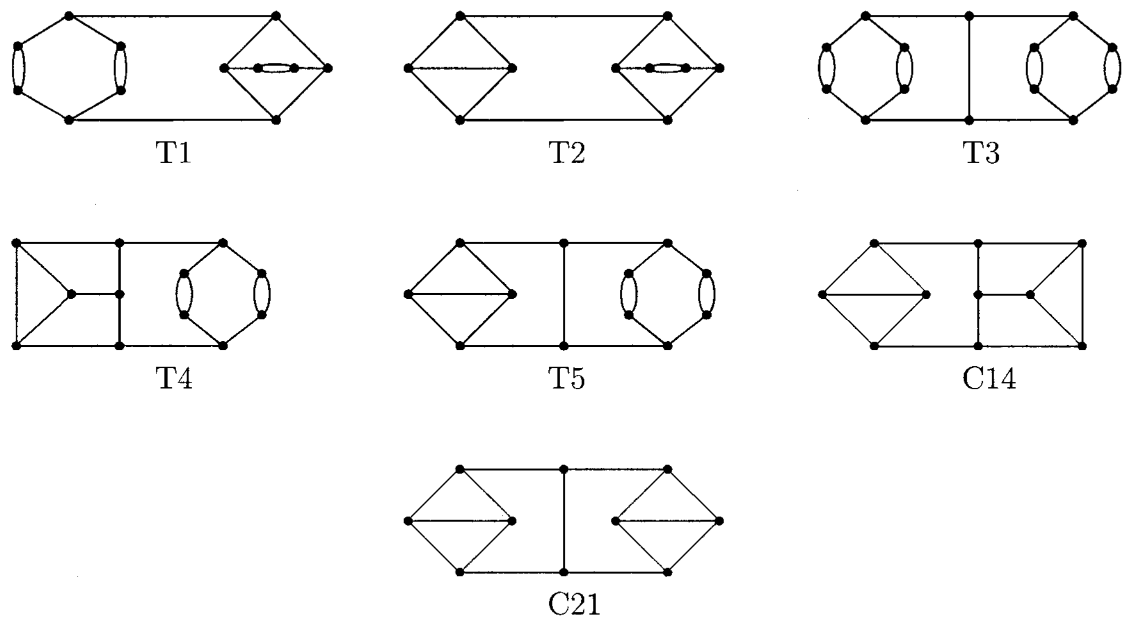


Figure 4.6: The set $\mathcal{C}$ of graphs which is a subset of the set of loopless cubic obstructions for outer-cylindrical graphs.

Proposition 4.2.5 *Let G be a planar cubic obstruction for outer-cylindrical graphs that has edge-connectivity two. If neither component obtained from some 2-edge-cut of G is a 2-cycle, then G is isomorphic to one of* C14, C21, T1, T2, T3, T4, *or* T5.

Proof: Assume that G is a loopless cubic obstruction that is not isomorphic to C14, C21, T1, T2, T3, T4, or T5. Let X and Y denote the components resulting from the deletion of a 2-edge-cut $\{x'y', x''y''\}$ of G, where x', x'' are vertices in X and y', y'' are vertices in Y, and suppose X and Y are not 2-cycles.

Assume that X and Y are outer-planar and embed X and Y in the plane so that all the vertices of X and Y lie on the boundary of one face denoted f. Embed $x'y'$ in f and we see that $x'y'$ lies in f. Upon the embedding of $x''y''$, we see that there are

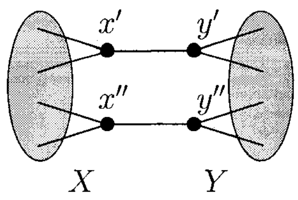


Figure 4.7: A depiction of a 2-edge-cut of G.

two faces f' and f'' whose boundaries together contain all the vertices of X and Y and the edges $x'y'$ and $x''y''$. Thus G is outer-cylindrical – a contradiction since G is a cubic obstruction. So it must be that at least one of X and Y is not outer-planar. For the remainder of this proof we will assume that X is not outer-planar. We now show the following.

Lemma 4.2.6 *y' and y'' are not neighbors.*

Proof: Assume that y' and y'' are neighbors (see Figure 4.8). Because the components resulting from $G\backslash\{x'y', x''y''\}$ are not 2-cycles, y' has a neighbor $z' \neq y''$ in Y and y'' has a neighbor $z'' \neq y'$ in Y. Since G is cubic and 2-edge-connected, $z'' \neq z'$. So the 2-edge-connected graph $G\backslash_3 y'y''$ has a 2-edge-cut $\{x'z', x''z''\}$ having X as a component. Let S be the component of $(G\backslash_3 y'y'')\backslash\{x'z', x''z''\}$ that is not X. It is easy to see that $y'y''$ is not part of a 2-edge-cut of G and as a result, $G\backslash_3 y'y''$ is 2-edge-connected.

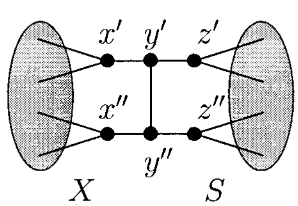


Figure 4.8: A 2-edge-cut of G.

By Proposition 3.2.5 there is an outer-cylindrical embedding of $G\backslash_3 y'y''$ with faces f_1 and f_2 such that each vertex of $G\backslash_3 y'y''$ borders either f_1 or f_2 and $x'z'$, $x''z''$ border one of f_1 and f_2, say f_1. From Lemma 1.7.1, $x'z'$ and $x''z''$ border the same faces so let f' and f'' be the faces bordered by $x'z'$ and $x''z''$. Clearly f_1 is one of f' or f'', so, without loss of generality, assume that $f_1 = f'$. If f_2 is f'', then $G - V(S) = X$ would be outer-planar – a contradiction to X not being outer-planar. So it must be that f'' is not f_2. Since X is not outer-planar, vertices of X lie on the boundary of both f_1 and f_2

Assume that there is a vertex s of S that borders f_2. Since $G\backslash_3 y'y''$ is 2-edge-connected, the boundary of f_2 is a cycle C. So we see that s is a vertex of C. Since vertices of X are in f_2, let x be a vertex of X that is also in C. Now there are two disjoint paths from x to s that are in C. Since $x'z'$ and $x''z''$ is a 2-edge-cut, then they must also be in C. And since C is the boundary of f_2, edges $x'z'$ and $x''z''$ lie on the boundary of f_2 which implies $f'' = f_2$. Since this is a contradiction to $f'' \neq f_2$, it must be that vertices of S do not border f_2. So we see that all the vertices of S border f_1.

Let G' be the graph obtained by subdividing $x'z'$ and $x''z''$ once, obtaining new vertices w_1 and w_2 and embedding the edge w_1w_2 in face f''. Since w_1 and w_2 lie on $f_1(= f')$, each vertex of G' lies on the boundary of either f_1 or f_2. Thus G' is an outer-cylindrical graph. But G' is isomorphic to G – a contradiction. We conclude that y' and y'' are not neighbors and Lemma 4.2.6 is established. ■

Lemma 4.2.7 *We now show that Y has one of two subgraphs of a certain type. Y has a subgraph that is a subdivision of a kite or a halo, denoted Y', where y' and y'' are the branch vertices of degree two of Y'.*

Proof: From Lemma 3.2.6 the number of parts of a generalized cycle of a cubic obstruction is not larger than four and every part is a block; so Y must be a block. By Theorem 1.3.1, in Y there are two disjoint paths Y_1 and Y_2 from y' to y''. Since $y'y''$ is not an edge of Y, the paths Y_1 and Y_2 must each have at least one internal vertex. If there is a path from $I(Y_1)$ to $I(Y_2)$ not containing the end edges of Y_1 and Y_2, then by taking a shortest such path in Y together with Y_1 and Y_2, it is easy to see that Y contains a subgraph that is a subdivision of a kite where the branch vertices of degree two are y' and y''. If there is no path from $I(Y_1)$ to $I(Y_2)$, then the end edges of Y_1 form a 2-edge-cut and the end edges of Y_2 form a 2-edge-cut. Thus, from Lemma 1.7.3, the vertices of Y_i are in a subgraph that is a subdivision of a storm where the ends of Y_i are the ends of the storm subdivision. Hence, Y must contain a subgraph that is a subdivision of a halo where the vertices of degree two are y' and y'', and Lemma 4.2.7 is established. ■

From Lemma 4.2.7, Y contains a subgraph Y' that is a subdivision of either a halo or a kite where y' and y'' are degree two branch vertices of Y'. Since X is not outer-planar, by Theorem 1.5.5, X contains a subgraph that is a subdivision of K_4 or $K_{2,3}$ which we denote X'. Since G is 2-edge-connected, there are 2 disjoint paths from X' to Y'. Let P_1 and P_2 be the two of the shortest such paths. Let x_1' be the end of P_1 in X' and x_2' be the end of P_2 in X'. Since y' and y'' are ends of the edges of the edge cut E', then y' and y'' must be the ends of P_1 and P_2 that are in Y'. Without loss of generality assume that y' is the end of P_1 in Y' and y'' is the end of P_2 in Y'.

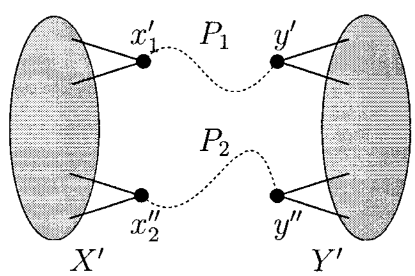


Figure 4.9: The paths P_1 and P_2.

We conclude that one of the following four cases occurs.

Case 1: X' is a subgraph that is a subdivision of K_4 and Y' is a subgraph that is a subdivision of a kite.

Case 2: X' is a subgraph that is a subdivision of K_4 and Y' is a subgraph that is a subdivision of a halo.

Case 3: X' is a subgraph that is a subdivision of $K_{2,3}$ and Y' is a subgraph that is a subdivision of a kite.

Case 4: X' is a subgraph that is a subdivision of $K_{2,3}$ and Y' is a subgraph that is a subdivision of a halo.

The remainder of this proof will be devoted to showing that in each case G must contain a subgraph that is a subdivision of one of C14, C21, T1, T2, T3, T4, or T5. In Cases 1 and 2 we assume that X' is a subgraph that is a subdivision of K_4.

Case 1: Assume Y' is a subgraph that is a subdivision of a kite where the branch vertices of degree two are y' and y''. Let k_1, k_2, k_3, and k_4, be the branch vertices of degree three of X' and let y_1 and y_2 be the branch vertices of degree three of Y'.

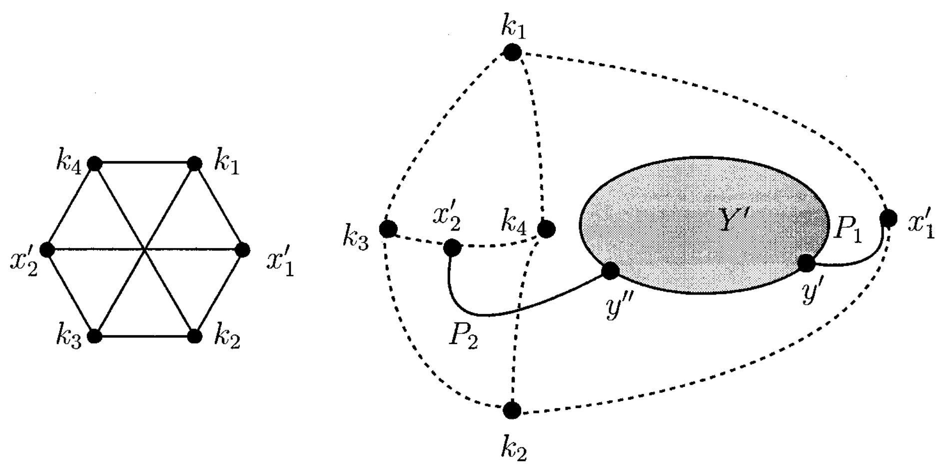


Figure 4.10: A subdivision of $K_{3,3}$

First we show that it is not possible for the ends of P_1 and P_2 to be on different subdivisional paths of X' that do not share the same ends in X'. Without loss of generality, assume x'_1 lies on the (k_1,k_2)-subdivisional path and x'_2 lies on the (k_3,k_4)-subdivisional path (see Figure 4.10). Then G contains a subgraph that is a subdivision of $K_{3,3}$ where k_1, k_2, k_3, k_4, x'_1, and x'_2 are the branch vertices of degree three of a $K_{3,3}$-subdivision. Thus, G is non-planar and from Lemma 3.3.1, G must be isomorphic to $K_{3,3}$ – a contradiction since $\lambda(G) = 2$. So it must be that either the ends of P_1 and P_2 are on the same subdivisional path of X' or the ends of P_1 and P_2 are on different subdivisional paths that share an end in X'.

Now consider the case when x'_1 and x'_2 lie on the same subdivisional path of X', say the (k_1,k_2)-subdivisional path. Then G contains a subgraph that is a subdivision of C21 where k_1, k_2, k_3, k_4, x'_1, x'_2, y_1, y_2, y', and y'' are the branch vertices of degree three as shown in Figure 4.11.

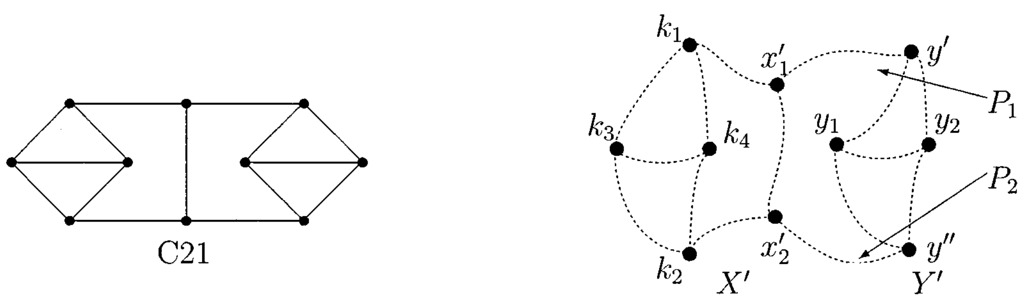


Figure 4.11: A subdivision of C21

Last, consider the case when x'_1 and x'_2 lie on different subdivisional paths of X' that share an end. Without loss of generality, assume that x'_1 lies on the (k_1,k_2)-subdivisional path and x'_2 lies on the (k_2,k_4)-subdivisional path. Then G contains a subgraph that is a subdivision of C14 where k_1, k_2, k_3, k_4, x'_1, x'_2, y_1, y_2, y', and y'' are the branch vertices of degree three as depicted in Figure 4.12.

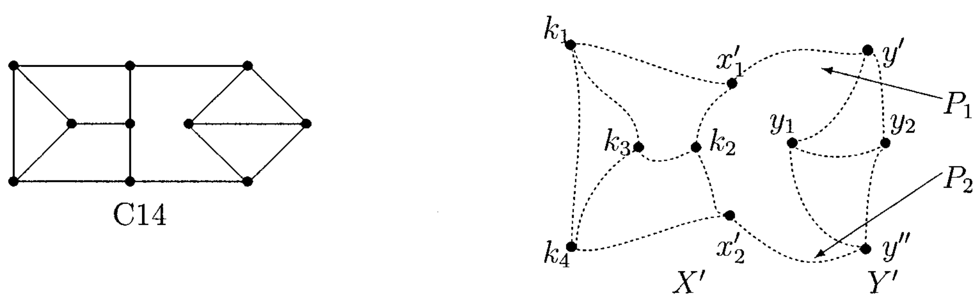


Figure 4.12: A subdivision of C14

Case 2: Assume that Y' is a subgraph that is a subdivision of a halo where the branch vertices of degree two are y' and y''. Let k_1, k_2, k_3, and k_4 be the branch vertices of degree three of X' and C_1 and C_2 be the two cycles in Y' that are the subdivisions of 2-cycles.

First we show that it is not possible for the ends of P_1 and P_2 to be on different subdivisional paths of X' that do not share the same ends in X'. Without loss of

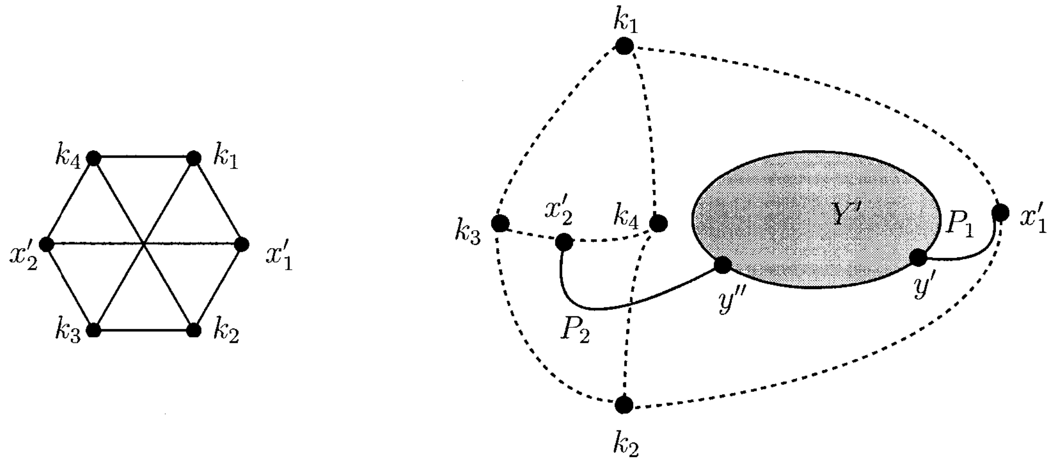


Figure 4.13: A subdivision of $K_{3,3}$

generality, assume x_1' lies on the (k_1,k_2)-subdivisional path and x_2' lies on the (k_3,k_4)-subdivisional path (see Figure 4.13). Then G contains a subgraph that is a subdivision of $K_{3,3}$ where k_1, k_2, k_3, k_4, x_1', and x_2' are the branch vertices of degree three of a $K_{3,3}$-subdivision. Thus, G is non-planar and from Lemma 3.3.1, G must be isomorphic to $K_{3,3}$ – a contradiction since $\lambda(G) = 2$. So it must be that either the ends of P_1 and P_2 are on the same subdivisional path or the ends of P_1 and P_2 are on different subdivisional paths that share an end in X'.

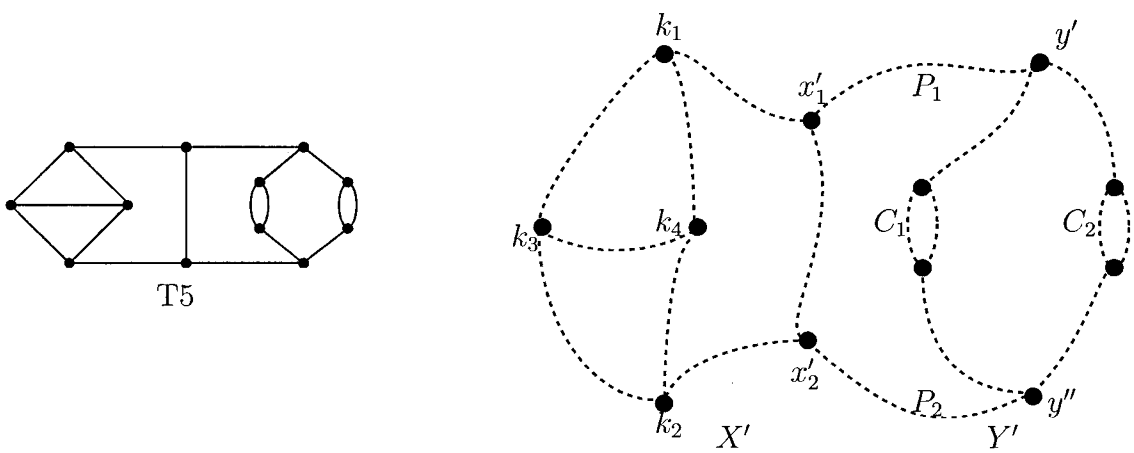


Figure 4.14: A subdivision of T5

Now, consider the case when x_1' and x_2' lie on the same subdivisional path of X', say the (k_1,k_2)-subdivisional path. Then G contains a subgraph that is a subdivision of T5 where k_1, k_2, k_3, k_4, x'_1, x'_2, y', and y'' are the branch vertices of degree three and C_1 and C_2 are the cycles of T5 as shown in Figure 4.14.

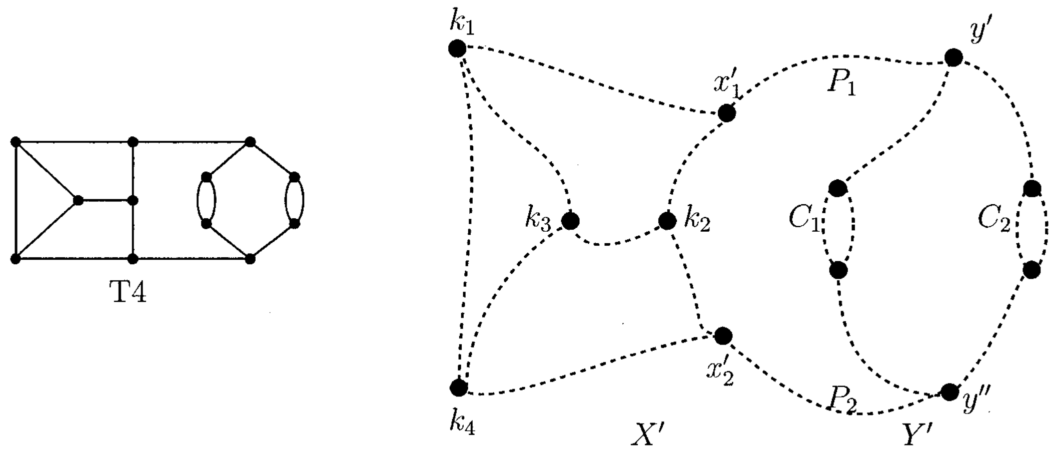


Figure 4.15: A subdivision of T4

Finally, consider the case when x'_1 and x'_2 lie on different subdivisional paths of X' that share an end. Without loss of generality, assume that x'_1 lies on the (k_1,k_2)-subdivisional path and x'_2 lies on the (k_2,k_4)-subdivisional path. Then G contains a subgraph that is a subdivision of T4 where k_1, k_2, k_3, k_4, x'_1, x'_2, y', and y'' are the branch vertices of degree three and C_1 and C_2 are the cycles of T4 (see Figure 4.15).

Having settled Cases 1 and 2, for the next two cases, we assume that X' is a subgraph that is a subdivision of a $K_{2,3}$ where Q_1, Q_2, or Q_3 are independent paths of the $K_{2,3}$-subdivision and vertices w_1 and w_2 are the ends of each of Q_1, Q_2, and Q_3. Additionally we may assume, from Cases 1 and 2, that X does not have a subgraph that is a subdivision of K_4. If the end edges of one of Q_1, Q_2, or Q_3 do not form a 2-edge-cut in X, then, from Lemma 1.7.2, X contains a subgraph that is a subdivision of K_4 – a contradiction to X not containing a subgraph that is a subdivision of K_4. Thus it must be that the end edges of each of Q_1, Q_2, and Q_3 form a 2-edge-cut in X.

Case 3: Assume Y' is a subgraph that is a subdivision of a kite where the branch vertices of degree two are y' and y'' and vertices y_1 and y_2 are the branch vertices of degree three of Y'.

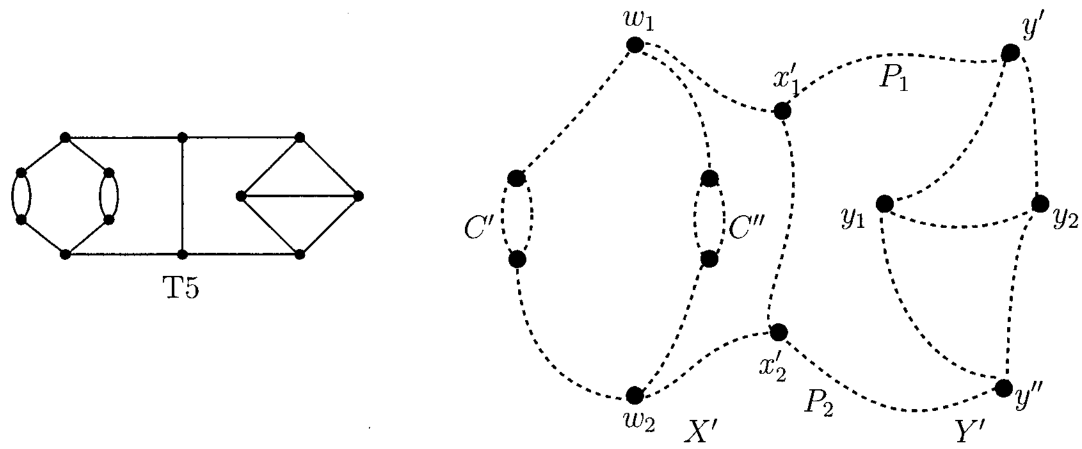


Figure 4.16: A subdivision of T5

First, consider the case where x'_1 and x'_2 lie on the same Q_i, say Q_1, of X'. Since P_1 and P_2 are two shortest, disjoint paths from X' to Y', then both avoid Q_2 and Q_3. Since edge $x'y'$ is in P_1 and edge $x''y''$ is in P_2, the vertex x' is not in either Q_2 or Q_3, and x'' is not in either Q_2 or Q_3. Thus we see that the end edges of each of Q_2 and Q_3 form a 2-edge-cut in G. So, from Lemma 1.7.3, the vertices of Q_2 lie in a storm and the vertices of Q_3 lie in a storm and w_1 and w_2 are the ends of both storms. Let C' be the 2-cycle of the storm having vertices in Q_2 and let C'' be the 2-cycle of Q_3 having vertices in Q_3. Thus it is easy to see that G contains a subgraph that is a subdivision of T5 where the branch vertices of degree three that are not in a 2-cycle are x'_1, x'_2, w_1, w_2, y', y'', y_1, and y_2 and the 2-cycles of the storms are C' and C'' as shown in Figure 4.16.

Second, consider the case where x'_1 and x'_2 lie on distinct subdivisional paths, say Q_1 and Q_2, of X'. Since P_1 and P_2 are two shortest disjoint paths from X' to Y', both avoid Q_3. Since edge $x'y'$ is in P_1 and edge $x''y''$ is in P_2, the vertices x' and x'' are not in Q_3. Thus, we see that the end edges of each of Q_3 form a 2-edge-cut in G. So, from Lemma 1.7.3, we see that the internal vertices of Q_3 lie in a storm of G and w_1 and w_2 are the ends of the storm. Let C denote the 2-cycle of the storm. Thus, G contains a subgraph that is a subdivision of T2 where the branch vertices of degree three that are not in a 2-cycle are x'_1, x'_2, w_1, w_2, y_1, y_2, y', and y'' and the 2-cycle of the storm is C as shown in Figure 4.17.

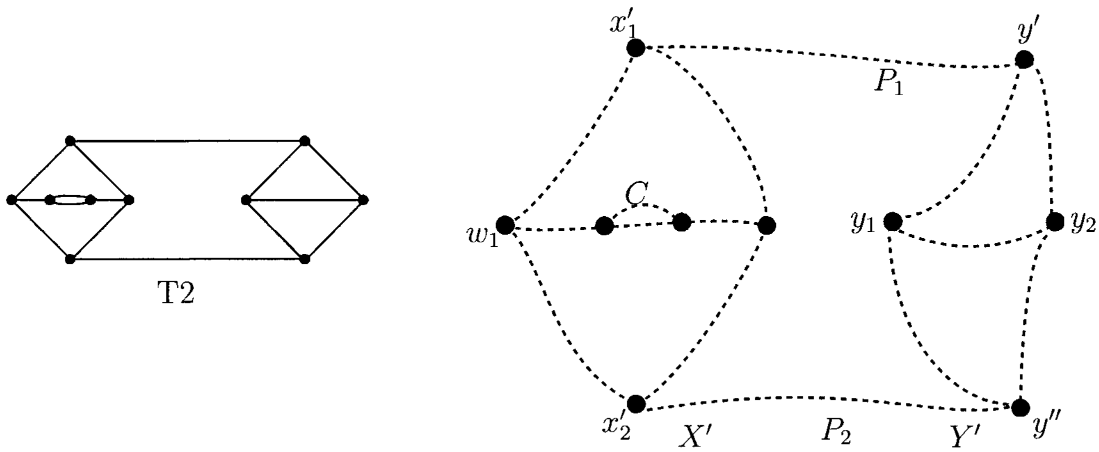


Figure 4.17: A subdivision of T2

Case 4: Assume Y' is a subgraph that is a subdivision of a halo where the branch vertices of degree two are y' and y'' and the 2-cycles of Y' are C_1 and C_2.

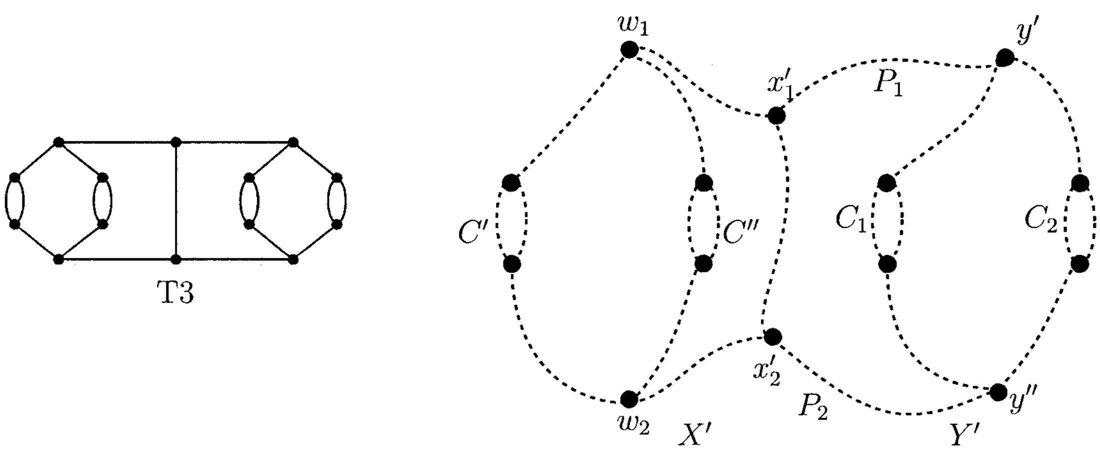


Figure 4.18: A subdivision of T3

First, consider the case where x_1' and x_2' lie on the same subdivisional path, say Q_1, of X'. Since P_1 and P_2 are two shortest, disjoint paths from X' to Y', both paths avoid Q_2 and Q_3. Since edge $x'y'$ is in P_1 and edge $x''y''$ is in P_2, the vertex x' is not in either Q_2 or Q_3, and x'' is not in either Q_2 or Q_3. Thus, we see that the end edges of each of Q_2 and Q_3 form a 2-edge-cut in G. So, from Lemma 1.7.3, it is clear that the vertices of Q_2 lie in a storm and the vertices of Q_3 lie in a storm where w_1 and w_2 are the ends of both storms. Let C' denote the 2-cycle containing vertices of Q_2

and let C'' denote the 2-cycle containing vertices of Q_3. Thus, it is easy to see that G contains a subgraph that is a subdivision of T3 where the branch vertices of degree three are x_1', x_2', w_1, w_2, y', and y'' as shown in Figure 4.18 and the 2-cycles are C' and C''.

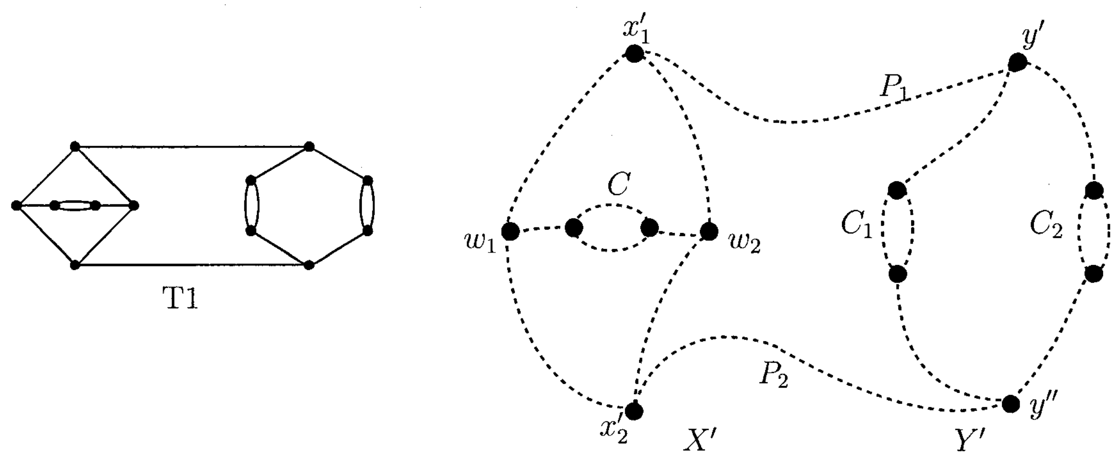


Figure 4.19: A subdivision of T1

Now, consider the case where x_1' and x_2' lie on different subdivisional paths, say Q_1 and Q_2, of X'. Since P_1 and P_2 are two shortest disjoint paths from X' to Y', both paths avoid Q_3. Since edge $x'y'$ is in P_1 and edge $x''y''$ is in P_2, the vertices x' and x'' are not in Q_3. Thus, we see that the end edges of Q_3 form a 2-edge-cut in G. So, from Lemma 1.7.3, it is clear that the vertices of Q_3 lie in a storm where w_1 and w_2 are the ends of the storm. Let C denote the 2-cycle of the storm. Thus, it is easy to see that G contains a subgraph that is a subdivision of T1 where the branch vertices of degree three that are not in a 2-cycle are x_1', x_2', w_1, w_2, y', and y'' as shown in Figure 4.19.

Now G is a cubic obstruction and G contains either C14, C21, T1, T2, T3, T4, or T5 in the cubic order, thus G is isomorphic to either C14, C21, T1, T2, T3, T4, or T5 – a contradiction to the choice of G. So it must be that if both of the components obtained from a 2-edge-cut of a loopless 2-edge-connected cubic obstruction is not a 2-cycle then the obstruction is isomorphic to one of C14, C21, T1, T2, T3, T4, or T5, and Proposition 4.2.5 holds. ■

For the remainder of this chapter, $\mathcal{C}$ will represent the set of obstructions {C14, C21, T1, T2, T3, T4, T5} as just described in Proposition 4.6 and depicted in Figure 4.6.

Corollary 4.2.8 *Let G be a loopless planar cubic obstruction for outer-cylindrical graphs that is not isomorphic to any graph in $\mathcal{C}$. Then if E' is a 2-edge-cut of G, one of the components of $G \backslash E'$ is a 2-cycle.*

Proof: Assume the components G_1 and G_2 of $G \backslash E'$ are not 2-cycles. By Proposition 4.2.5 G is in $\mathcal{C}$ – a contradiction. Thus if E' is a 2-edge-cut of G, one of the components of $G \backslash E'$ is a 2-cycle. ■

Now the problem of finding all the cubic obstructions for outer-cylindrical graphs has been reduced to finding all the planar 2-edge-connected obstructions not in $\mathcal{C}$. From Corollary 4.2.8, if E' is a 2-edge-cut of a loopless 2-edge-connected obstruction not in $\mathcal{C}$, then one of the components of $G \backslash E'$ is a 2-cycle. In the next section we will characterize those obstructions that contain the cube, Q (see Figure 4.20) in the cubic order.

4.3 Obstructions Containing the Cube

In Proposition 4.3.1 we identify the loopless cubic obstructions that contain a cube in the cubic order. Specifically, we show that the cubic obstructions containing the cube are C16, Q1, Q2, Q3, and Q4 (see Figure 4.22). Before stating this proposition, we need some additional terminology.

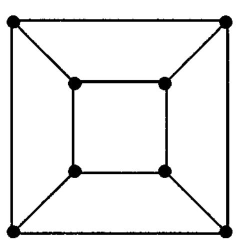


Figure 4.20: The graph of the cube, denoted Q.

The graph of Figure 4.20 is a *cube* and will be denoted Q. Since the cube is 3-connected, it is easy to see that no member of $\mathcal{C}$ contains the cube in the cubic order.

A *circular ladder* (see Figure 4.21) is a graph that consists of two disjoint cycles of the same size $C = l_1 l_3 \ldots l_{2n-1}$ and $D = l_2 l_4 \ldots l_{2n}$ where l_{2i-1} and l_{2i} where $1 \leq i \leq n$ are joined by an edge called a *rung*. Here, the cycles C and D are called the *rails*. A circular ladder with n rungs will be called an *n-circular-ladder* and is denoted CL_n. Notice that the cube is a 4-circular-ladder.

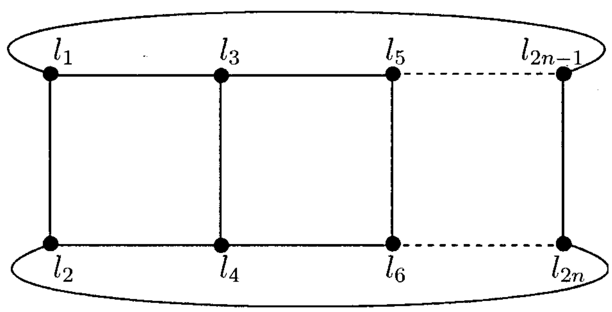


Figure 4.21: A labeled circular ladder.

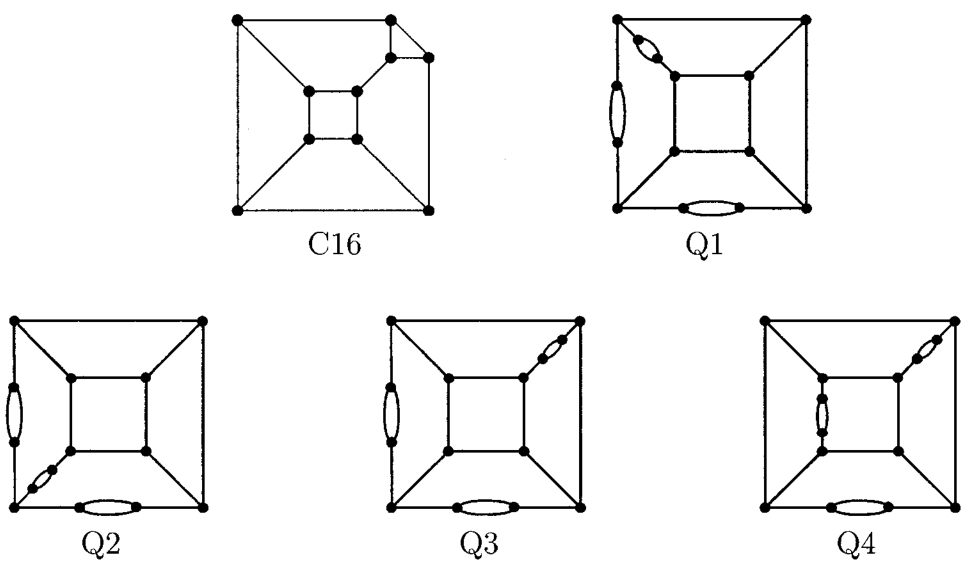


Figure 4.22: The loopless cubic obstructions for outer-cylindrical graphs having a Q-subdivision.

Proposition 4.3.1 *Let G be a planar loopless cubic obstruction for outer-cylindrical graphs. If G contains the cube in the cubic order then G is isomorphic to one of* C16, Q1, Q2, Q3, *and* Q4.

Proof: Suppose G is a loopless planar cubic obstruction for outer-cylindrical graphs containing the cube in the cubic order. In this proof we show that G contains a subgraph that is a subdivision of C16, Q1, Q2, Q3, or Q4. By Proposition 1.6.1, G contains a subgraph that is a subdivision of the cube. Since the cube is a 4-circular-ladder, the graph G contains an n-circular-ladder where $n \geq 4$. Let n be an integer such that G contains a subgraph that is a subdivision of the n-circular-ladder CL_n, which we label L, and G does not contain a k-circular-ladder where $k > n$. Assume the rails of this copy of CL_n are $C = l_1 l_3 \ldots l_{2n-1}$ and $D = l_2 l_4 \ldots l_{2n}$ and the rungs are $l_1 l_2, l_3 l_4, \ldots l_{2n-1} l_{2n}$. Let $L_{i,j}$ denote the subdivisional path of L from l_i to l_j of CL_n. Notice that any n-circular-ladder is outer-cylindrical by embedding CL_n in the plane and observing that the rails of the n-circular-ladder bound two faces. Since G is cubic and not outer-cylindrical, G is not isomorphic to an n-circular-ladder and thus G is not isomorphic to L. Moreover, as all the vertices of L have degree three and G is connected, at least one of the subdivisional paths of L, say L', has an internal vertex (see Figure 4.21).

We claim that every path from the internal vertices of L' to $V(L) - V(L')$ contains an end edge of L'. Let P be a shortest such path in G from $I(L')$ to $V(L) - V(L')$ not containing an end edge of L', vertex v be the end of P in $I(L')$ and w be the end of P in $V(L) - V(L')$.

First suppose L' is one of $L_{i,i+2}$, $L_{2n-1,1}$, or $L_{2n,2}$. By the symmetry of the graph L, assume L' is $L_{1,3}$ as depicted in Figure 4.23. Since the cube is uniquely embeddable, w must be a vertex in $L_{1,2}$, $L_{3,4}$, $L_{2,4}$, $L_{2i-1,2i+1}$ where $2 \leq i \leq n-1$, or $L_{2n-1,1}$ or G would not be planar. If w is in $L_{1,2}$, $L_{3,4}$, $L_{3,5}$, or $L_{2n-1,1}$, then G contains a subgraph that is a subdivision of C16.

If w is in $L_{2,4}$, then G contains a subgraph that is a subdivision of an $(n+1)$-circular-ladder – a contradiction of the choice of L. Suppose w is in $L_{2i-1,2i+1}$ where $2 \leq i \leq n-1$, then G contains a subgraph that is a subdivision of C16 if $n \geq 5$. If

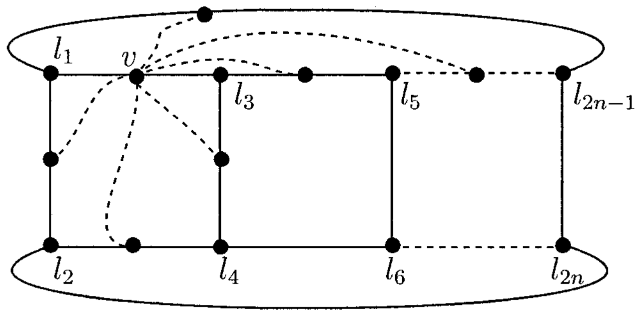


Figure 4.23: Possible paths starting from the internal vertices of $L_{1,3}$

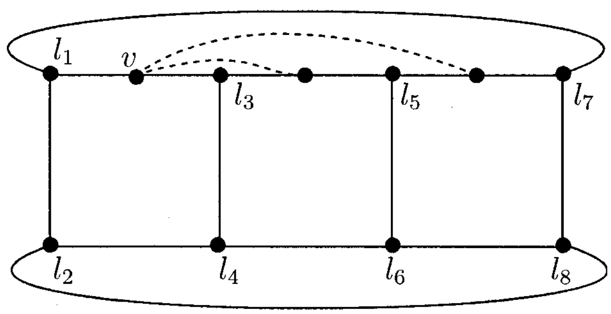


Figure 4.24: Subdivisional paths $L_{3,5}$ and $L_{5,7}$ have internal vertices

$n = 4$ then i is 2 or 3 and w is in either $L_{3,5}$ or $L_{5,7}$ as depicted in Figure 4.24. If w is in $L_{3,5}$ then G contains a subgraph that is a subdivision of C16 where v, w, and l_3 are vertices of a cycle of L that correspond to the triangle of C16. If w is in $L_{5,7}$, then G contains a 5-circular-ladder where cycles $l_1 v l_3 l_4 l_2$ and $l_7 w l_5 l_6 l_8$ are the rails of the 5-circular-ladder. Thus it must be that L' is not one of $L_{i,i+2}$, $L_{2n-1,1}$, or $L_{2n,2}$.

Second, suppose L' is one of the subdivisional paths of L corresponding to a rung of $L_{i,i+1}$. By symmetry of L, assume L' is $L_{3,4}$ as depicted in Figure 4.25. Now w must be a vertex in $L_{1,2}$, $L_{1,3}$, $L_{2,4}$, $L_{3,5}$, $L_{4,6}$, or $L_{5,6}$, or G would not be planar. If w is in $L_{1,3}$, $L_{2,4}$, $L_{3,5}$, or $L_{4,6}$, then G contains a subgraph that is a subdivision of C16. So it must be that w is in either $L_{1,2}$ or $L_{5,6}$. By the symmetry of L, assume w is in $L_{5,6}$.

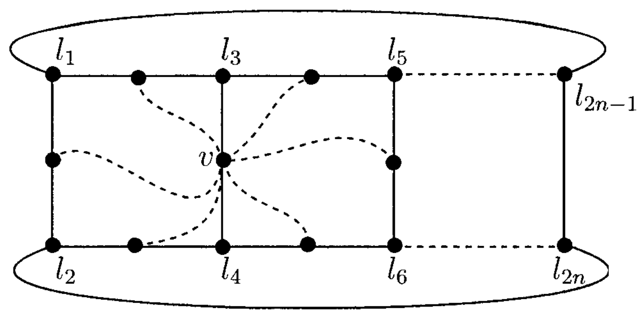


Figure 4.25: Possible paths starting from the internal vertices of $L_{3,4}$

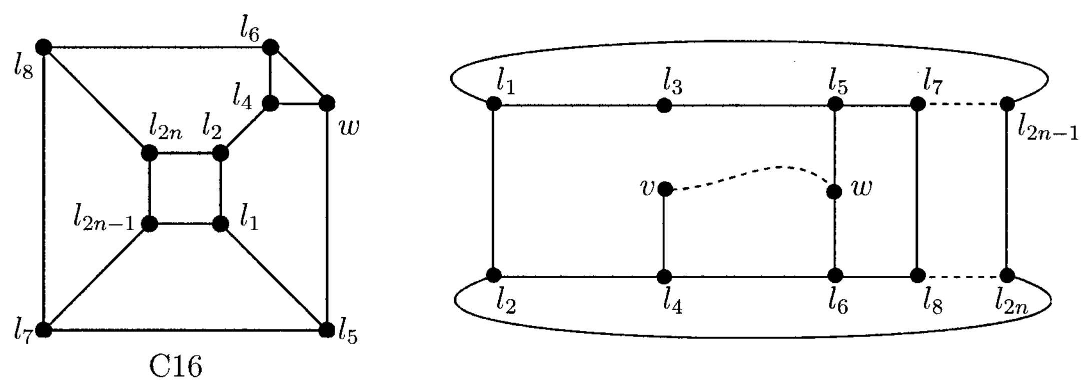


Figure 4.26: The subdivisional path $L_{5,6}$ has internal vertices.

If $n > 4$, G contains a subgraph that is a subdivision of C16 by deleting edges in the path $L_{3,4}[l_3, v]$ where w, l_4, and l_6 are the vertices of the cycle of G that corresponds to the triangle of C16. This is depicted in Figure 4.26. If n is 4 (see Figure 4.27) then G contains a 5-circular-ladder where $l_1l_3vl_4l_2$ and $l_7l_5wl_6l_8$ are the rails and l_1l_7, l_3l_5, vw, l_4l_6, and l_2l_8 the are the rungs of the 5-circular-ladder. Since this is a contradiction to the choice of L, every path from the internal vertices of L' to $V(L) - V(L')$ contains an end edge of L' and by Corollary 4.2.8, the induced graph on the internal vertices of any subdivisional path of L is a 2-cycle.

Notice that if none of the rungs of L have internal vertices then G would be outer-cylindrical. We see this by embedding G in the plane and observing that the boundaries of the faces bounded by the cycle of L corresponding to the rails of an

n-circular-ladder together contain all the vertices of G. This means that one of the rungs of L must contain internal vertices.

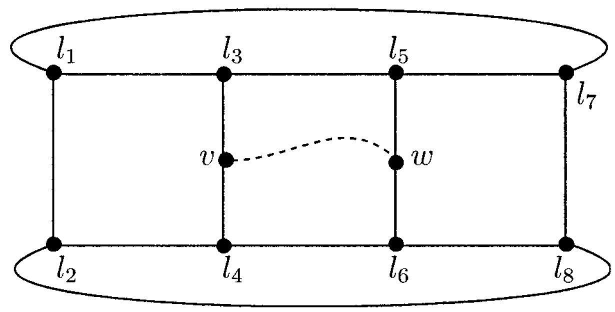


Figure 4.27: The subdivisional path $L_{5,6}$ has internal vertices.

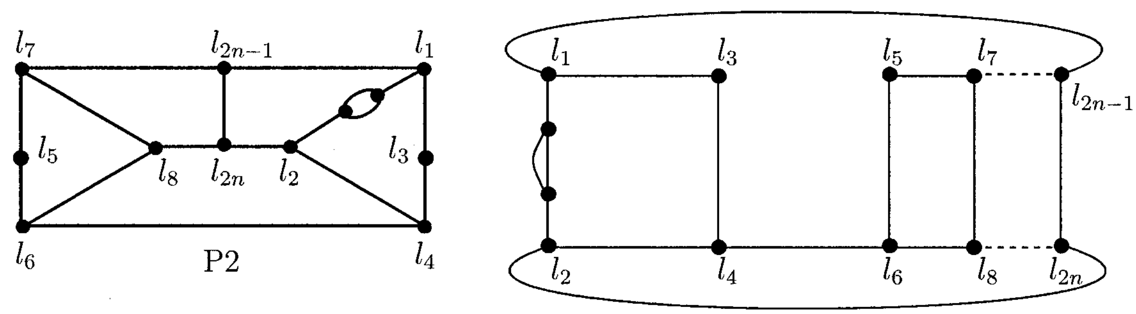


Figure 4.28: P2 with the induced labeling from n-circular-ladder. When $I(L_{1,2})$ form a 2-cycle, G contains P2 in the cubic order.

Without loss of generality, assume that $L_{1,2}$ has internal vertices. Since the end edges of $L_{1,2}$ form a 2-edge-cut, by Corollary 4.2.8, we see that the vertices of $L_{1,2}$ lie in a storm in G. If $n \geq 5$, then by deleting the edges of path $L_{3,5}$ we see that G contains a subgraph that is a subdivision of P2 as depicted in Figure 4.28. Since G is a cubic obstruction and G contains P2 in the cubic order, then G must be isomorphic to P2 since P2 is a cubic obstruction. But P2 does not contain the cube in the cubic order – a contradiction to the choice of G. So it must be that $n = 4$, and L is a subdivision of the cube Q.

Since $n = 4$, at least two subdivisional paths have internal vertices, or G is outer-cylindrical. Now suppose that exactly two subdivisional paths of L, say L' and L'', have internal vertices. Clearly if there is a face f whose boundary contains $I(L')$ and

$I(L'')$, then G would be an outer-cylindrical graph. So suppose $I(L')$ and $I(L'')$ do not lie on the boundary of a common face. Without loss of generality assume that L' is the $L_{1,2}$ of the graph depicted in Figure 4.29.

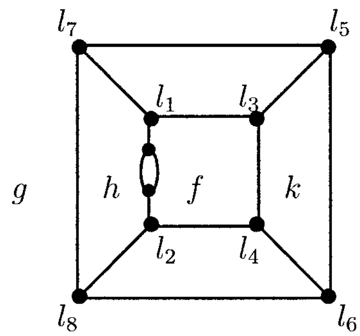


Figure 4.29: Faces of the cube.

Now L'' must be one of $L_{5,7}$, $L_{5,6}$, $L_{5,3}$, $L_{4,6}$, or $L_{6,8}$. If L'' is one of $L_{5,7}$, $L_{5,6}$, or $L_{6,8}$ then we see that the boundaries of the faces f and g together contain all the vertices of G. Thus G is an outer-cylindrical graph – a contradiction to G being a cubic obstruction. If L'' is one $L_{5,3}$ or $L_{4,6}$ then the boundaries faces of h and k together contain all the vertices of G. Thus G is an outer-cylindrical graph – a contradiction to G being a cubic obstruction. Since both cases lead to a contradiction, it must be that at least three subdivisional paths of L say L_1, L_2, and L_3 have internal vertices whose induced subgraph in G is a 2-cycle.

Lemma 4.3.2 *Let L' and L'' be distinct subdivisional paths of L having a common endvertex, then G contains a subgraph that is a subdivision of either* Q1, Q2, *or* Q3 *depicted in Figure 4.22.*

Proof: By the symmetry of the cube, assume $L' = L_{1,2}$ and $L'' = L_{1,3}$. Now at least one of L_1, L_2, and L_3, say L_1, is not $L_{1,2}$ or $L_{1,3}$. Consider the cubic edge-deletion of the parallel edge e of L_1 and embed $G\backslash_3 e$. Since the cube is 3-connected, from Theorem 1.5.3, we see that $G\backslash_3 e$ has the unique embedding depicted in Figure 4.30. Now the only pair of faces whose boundaries together would contain all the branch vertices of the cube along with the internal vertices of $L_{1,2}$ and $L_{1,3}$ is the face f incident to both the internal vertices of $L_{1,2}$ and the face g.

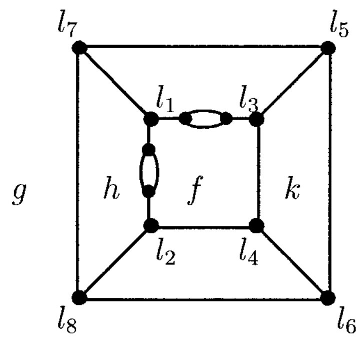


Figure 4.30: Faces of the cube.

If L_1 is one of $L_{2,4}$, $L_{3,4}$, $L_{5,7}$, $L_{5,6}$, $L_{6,8}$, or $L_{7,8}$, then G is outer-cylindrical since the vertices of all these paths lie on the boundaries of f and g. So we may assume that L_1 is one of $L_{1,7}$, $L_{2,8}$, $L_{3,5}$, or $L_{4,6}$. If L_1 is $L_{1,7}$ then G contains a subgraph that is a subdivision of Q2 (see Figure 4.30). If L_1 is $L_{2,8}$ or $L_{3,5}$ then G contains a subgraph that is a subdivision of Q1. If L_1 is $L_{4,6}$ then G contains a subgraph that is a subdivision of Q3.

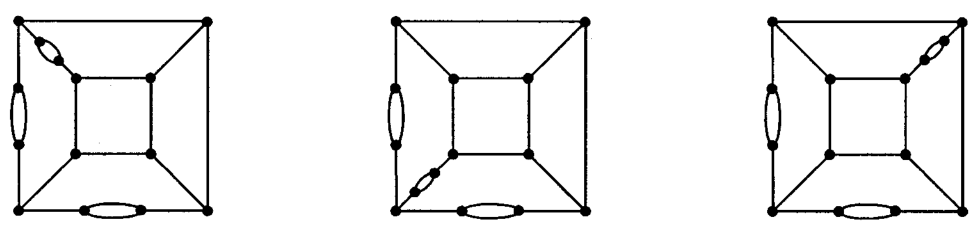


Figure 4.31: The graphs Q1, Q2, and Q3

Thus if L' and L'' are distinct subdivisional paths of G having internal vertices such that $V(L') \cap V(L'') \neq \emptyset$, then G contains a subgraph that is a subdivision of Q1, Q2, or Q3 and Lemma 4.3.2 holds. ■

From Lemma 4.3.2 we may assume that paths L_1, L_2, and L_3 do not share an end.

Lemma 4.3.3 *Let L' and L'' be distinct subdivisional paths of L having internal vertices then L' and L'' do not lie on the boundary of a common face for the embedding of G.*

Proof: Assume that for the embedding of G, two subdivisional paths of L, say L' and L'', having internal vertices lie on the boundary of a common face. Now at least one of L_1, L_2, and L_3 is not L' or L'' so assume that L_1 is neither L' nor L''. Consider the cubic edge-deletion of a parallel edge e of L_1 and embed $G\backslash_3 e$. Since the cube is uniquely embeddable, the embedding of $G\backslash_3 e$ results in a plane graph where L' and L'' lie on the boundary of a common face f. By Lemma 4.3.2 we may assume that L' and L'' do not share an end, so without loss of generality assume that L' is $L_{1,2}$ and L'' is $L_{3,4}$.

Again by Lemma 4.3.2, subdivisional paths of L having internal vertices must not share an end, and so L_1 is one of $L_{5,6}$, $L_{5,7}$, $L_{6,8}$, or $L_{7,8}$. But in all of these cases, G would is an outer-cylindrical graph by considering faces f and g – a contradiction to G being an obstruction. Thus Lemma 4.3.3 holds. ■

Since L has at least three subdivisional paths with internal vertices, three of which we have labeled L_1, L_2, and L_3, assume by symmetry of the cube that $L_1 = L_{1,2}$. By Lemmas 4.3.2 and 4.3.3, we need only consider the cases where L_2 is one of $L_{5,7}$, $L_{5,6}$, $L_{5,3}$, $L_{4,6}$ or $L_{6,8}$. By symmetry of the cube, we only consider the case where L_2 is $L_{5,7}$ or $L_{5,6}$ (see Figure 4.32).

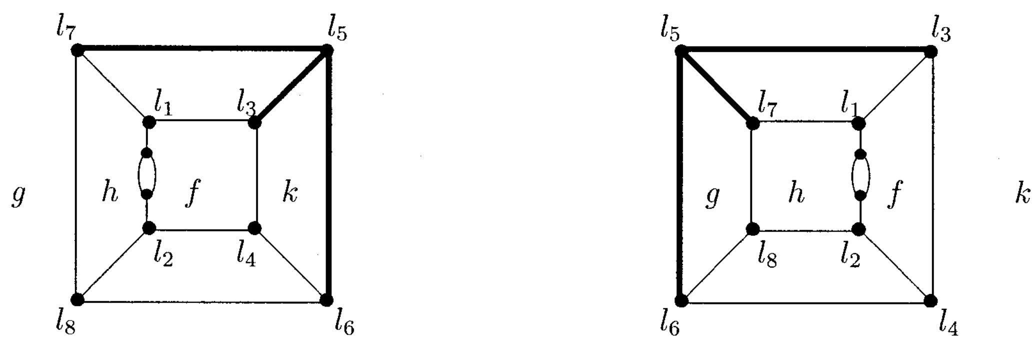


Figure 4.32: The symmetry of the cube.

Now if $L_2 = L_{5,7}$ then the only subdivisional path that does not share an end with either L_1 or L_2 and does not lie on the boundary of a face incident to either L_1 or L_2 is $L_{4,6}$; now if $L_3 = L_{4,6}$ then G contains a subgraph that is a subdivision of Q4. Now if it were the case that $L_2 = L_{5,6}$ then the only subdivisional path that does not share an end with either L_1 or L_2 and does not lie on the boundary of a face incident to either L_1 or L_2 is $L_{6,8}$; now if $L_3 = L_{6,8}$ then G contains a subgraph that is a subdivision of Q4. So we have shown that G must contain a subgraph that is a subdivision of C16, Q1, Q2, Q3, or Q4. Since G is a cubic obstruction and G contains either C16, Q1, Q2, Q3, or Q4 in the cubic order, then G must be isomorphic to C16, Q1, Q2, Q3, or Q4. Thus, we see that if G contains a cube and is a cubic obstruction for outer-cylindrical graphs, then G either C16, Q1, Q2, Q3, or Q4 and Proposition 4.3.1 holds. ■

To conclude this section, we introduce some additional terminology that will be used to complete the proof of Theorem 3.1.2. Let $\mathcal{H}$ be a set of graphs. We say that graph G is *$\mathcal{H}_3$-less* if for any H in $\mathcal{H}$, the graph G does not contain H in the cubic order. If $\mathcal{H}$ consists of a single graph H, we denote $\{H\}_3$-less also as H_3-less.

Now from Proposition 4.3.1 and Proposition 4.2.5 the problem of finding all the cubic obstructions for outer-cylindrical graphs has been reduced to finding the loopless, planar, 2-edge-connected, Q_3-less, cubic obstructions for outer-cylindrical graphs that are not in $\mathcal{C}$. For ease of notation, in the remainder of this chapter we will call a planar 2-edge-connected Q_3-less cubic obstruction for outer-cylindrical graphs that is not in $\mathcal{C}$ a *restricted obstruction.*

4.4 Restricted Obstructions

In this section we prove that the set of restricted obstructions is {F1, P1, P2, P3, P4, P5, C9, C15} (see Figure 4.34). Before we prove this, we introduce a family of graphs and a canonical labeling of the members of this family.

For an integer $k \geq 1$ a *k-house* is a graph H_k obtained from a path which is labeled $h_{2k}h_{2k-2}\dots h_2h_0h_1h_3\dots h_{2k-1}$ along with edges $h_{2i-1}h_{2i}$ for $1 \leq i \leq k$. A 0-*house* is the graph H_0 that is comprised of a single vertex h_0.

Example 4.4.1 *We depict a labeled n-house in Figure 4.33.*

Assume $x \leq y \leq z$ are non-negative integers. Let $\mathcal{A}$ be an x-house obtained from the path $a_{2x}a_{2x-2}\dots a_2a_0a_1a_3\dots a_{2x-1}$, let $\mathcal{B}$ be a y-house obtained from the

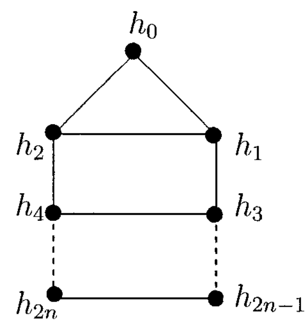


Figure 4.33: A labeled n-house.

path $b_{2y}b_{2y-2}\dots b_2b_0b_1b_3\dots b_{2y-1}$, and let $\mathcal{C}$ be a z-house obtained from the path $c_{2z}c_{2z-2}\dots c_2c_0c_1c_3\dots c_{2z-1}$. An *$(x, y, z)$-die* denoted $D_{x,y,z}$ is the graph obtained from $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ by adding a vertex d together with edges da_0, db_0, dc_0, $a_{2x}b_{2y-1}$, $b_{2y}c_{2z-1}$, and $c_{2z}a_{2x-1}$ (see Figure 4.35). This labeling of vertices and edges of $D_{x,y,z}$ is called a canonical labeling of $D_{x,y,z}$. In order to allow for effective labeling of dies, we also label a_0 as a_{-1}, b_0 as b_{-1}, and c_0 as c_{-1}.

Recall that the subdivisional paths of a subdivision of a graph G are the paths of the subdivision that correspond to the edges of G. We now discuss a canonical labeling of the subdivisional paths of a subgraph that is a subdivision of $D_{x,y,z}$. Let D be a subgraph that is a subdivision of $D_{x,y,z}$ and assume that the branch vertices of D receive the canonical labeling of $D_{x,y,z}$. The (d,a_0)-subdivisional path of D is labeled A_0, the (d,b_0)-subdivisional path of D is labeled B_0, and the (d,c_0)-subdivisional path of D is labeled C_0. The (a_i,a_j)-subdivisional path of D is labeled $A_{i,j}$, the (b_i,b_j)-subdivisional path of D is labeled $B_{i,j}$, and the (c_i,c_j)-subdivisional path of D is labeled $C_{i,j}$. The (a_{2x},b_{2y-1})-subdivisional path of D is labeled A, the (b_{2y},c_{2z-1})-subdivisional path of D is labeled B, the (c_{2z},a_{2x-1})-subdivisional path of D is labeled C. Now we define an order $\leq_D$ on dies.

Definition 4.4.2 *We say that $D_{x,y,z} \leq_D D_{x',y',z'}$ whenever:*

$x < x'$ or

$x = x'$ and $y < y'$ or

$x = x'$, $y = y'$, and $z \leq z'$.

For dies D' and D'' we say that D'' is larger than D' if $D' \leq_D D''$.

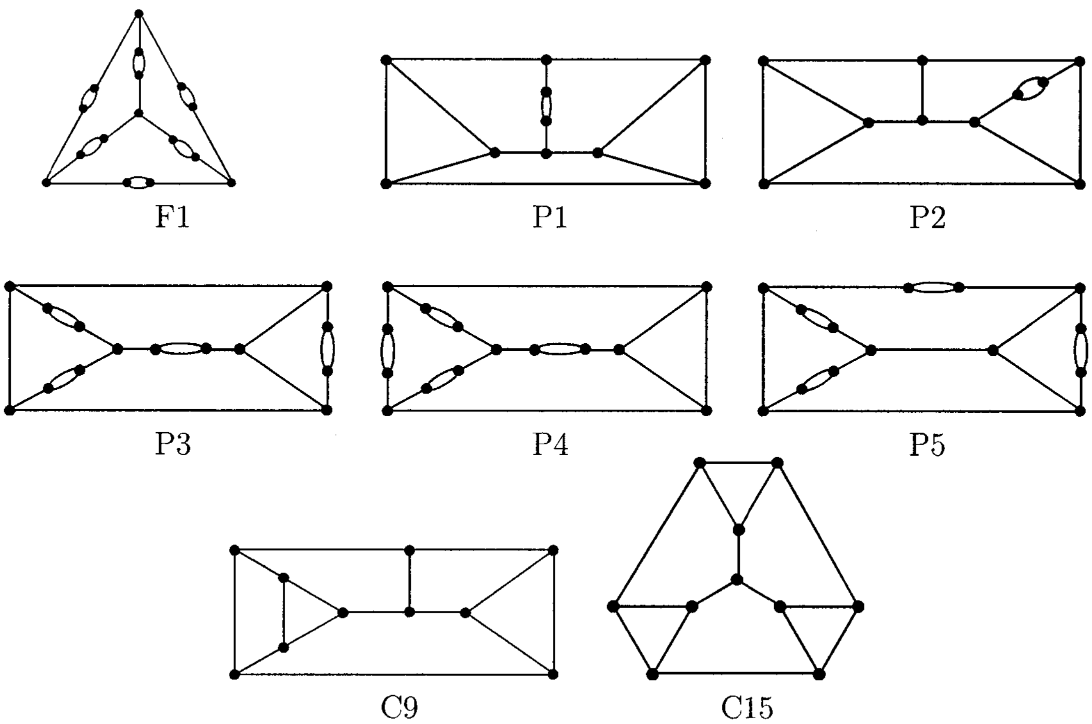


Figure 4.34: Restricted obstructions

In order to prove that the set of restricted obstructions is {F1, P1, P2, P3, P4, P5, C9, C15}, we will use the next proposition which shows us that every restricted obstruction contains K_4 in the cubic order and, since $K_4 \cong D_{0,0,0}$, every such obstruction contains a subgraph that is a subdivision of a die. From this we will be able to build all of the restricted obstructions. Here we recall that the following is the list of the properties of a restricted obstruction.

- loopless
- planar
- 2-edge-connected
- Q_3-less
- One of the components resulting from any 2-edge-cut is a 2-cycle.

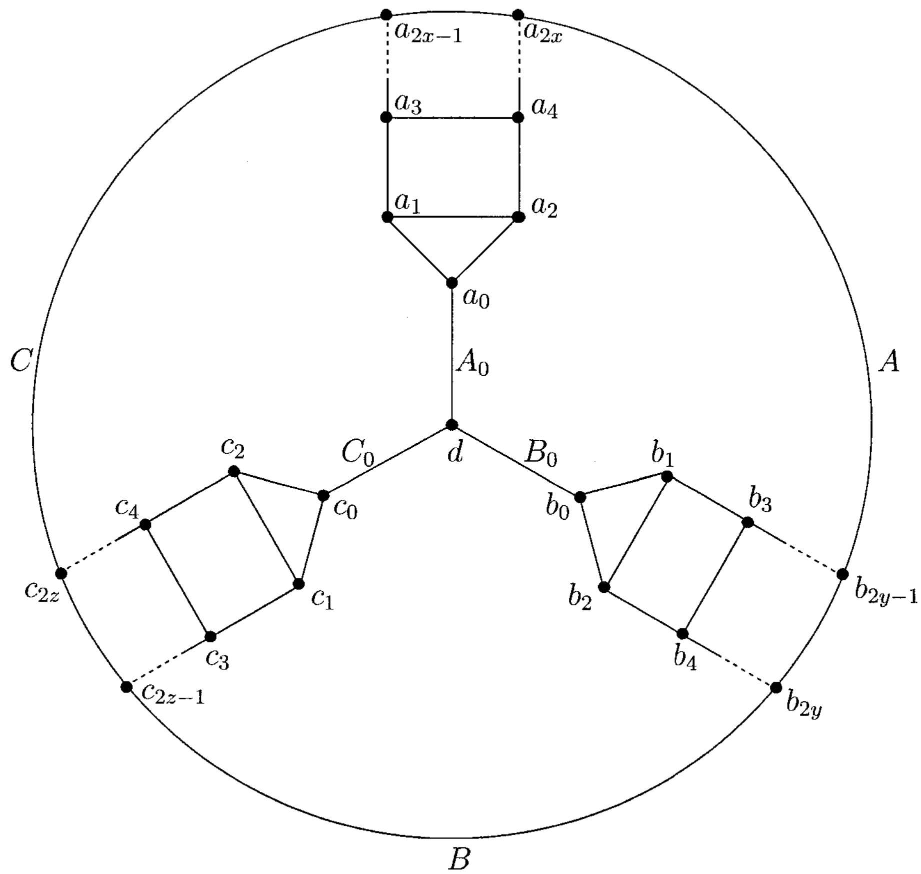


Figure 4.35: The canonical labeling of a die

Proposition 4.4.3 *Every restricted obstruction contains a K_4 in the cubic order.*

Proof: Let G be a restricted obstruction and let $E' = \{e_1, e_2, e_3, \ldots, e_m\}$ be a set of edges of G consisting of one edge from every pair of parallel edges of G. Let G' denote the graph obtained by the cubic edge-deletion of every edge in E' and let G'' denote the graph $G \backslash E'$.

Since G is cubic and 2-edge-connected and G'' is obtained from G by deleting only parallel edges of G, G'' must also be 2-edge-connected. By definition of cubic edge-deletion, G'' is a subdivision of G', thus G' must also be 2-edge-connected. So G' is loopless because loops in cubic graphs occur only in nooses and nooses contain a bridge. Let P_i denote the subdivisional path of G'' corresponding to the edge e_i in $E(G')$. Now G'' was obtained by the deletion of only parallel edges of G. So, the

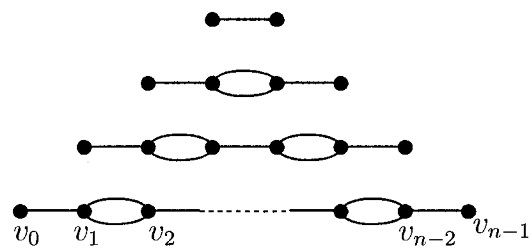


Figure 4.36: Possible graphs for $G[P_i]$.

induced subgraph $G[P_i]$ of G is either an edge or is a path $v_0v_1 \ldots v_{n-1}$, where n is even, together with edges parallel to v_iv_{i+1} for each $i \in \{1, 3, \ldots, n-3\}$ (see Figure 4.36). Such a subgraph will be called a *2-cycle-path.* Note that a 2-cycle-path could be an edge.

We will now show that G' is simple. So, suppose G' has a pair of parallel edges. Without loss of generality, assume this pair is e_1 and e_2. Let x and y denote the ends of e_1 and e_2 and label the edge different from e_1 and e_2 incident to x as e_3 and label the edge different from e_1 and e_2 incident to y as e_4. (see Figure 4.37).

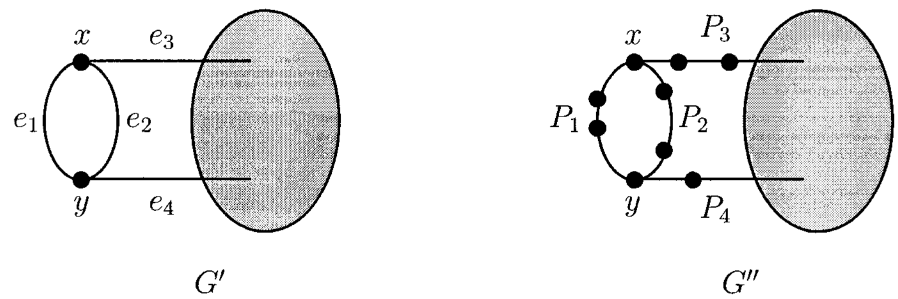


Figure 4.37: The graphs G' and G''.

Since the induced subgraphs in G, $G[V(P_1)]$, $G[V(P_2)]$, $G[V(P_3)]$, and $G[V(P_4)]$ are isomorphic to 2-cycle-paths, we have that the end edge of P_3 incident to x and the end edge of P_4 incident to y form a 2-edge-cut which we denote E''. Let M and N be the components of the graph of $G \backslash E''$. Without loss of generality, assume that P_1 and P_2 are in component M.

Since G' was obtained by the cubic edge-deletion of one of every pair of parallel edges of G, then e_1 and e_2 are not parallel edges of G. Thus, at least one of P_1 or P_2

has internal vertices. Without loss of generality, assume that P_1 has internal vertices. So it is clear that M is not a 2-cycle.

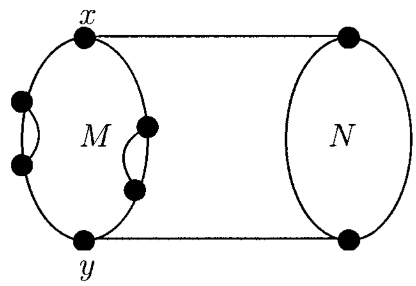


Figure 4.38: Component N is a 2-cycle.

Now, if N is not a 2-cycle, then from Proposition 4.2.5 we see that G is in the collection $\mathcal{C}$ – a contradiction to G being a restricted obstruction. So N is a 2-cycle as depicted in Figure 4.38. It then follows that G'' is a subdivision of $K_{2,3}$. Thus there are three faces associated with any embedding of G''. Furthermore, the boundaries together of any two of the three faces contain all the vertices of G''. Now upon the embedding of the edges of E' that are in parallel to edges of G'', we see that G has been embedded so that all the vertices of G lie on the boundary of at most two faces. Thus G is an outer-cylindrical graph – a contradiction to G being a restricted obstruction. So it must be that G' does not have parallel edges.

Now, since G' does not have loops or parallel edges, G' is a simple graph. So, by Theorem 2.1.3, we see that G' contains a subgraph that is a subdivision of K_4. Therefore, from Theorem 1.6.1, it follows that G contains K_4 in the cubic order. So it must be that any restricted obstruction contains a K_4 in the cubic order. ∎

Proposition 4.4.3 ensures that each member of the set of restricted obstructions contains a subgraph that is a subdivision of a die. Now, to finish the proof of Theorem 3.1.2 we take a largest die D in a restricted obstruction and find all the cubic obstructions that have a subgraph that is a subdivision of D. We prove four lemmas which will complete the proof of Theorem 3.1.2. We will determine all the restricted obstructions by considering the cubic obstructions containing a largest die, with respect to $\leq_D$, of $D_{0,0,0}$, $D_{0,0,1}$, $D_{0,1,1}$, and finally $D_{a,b,c}$ where $a \geq 0$, $b \geq 1$, and $c \geq 2$.

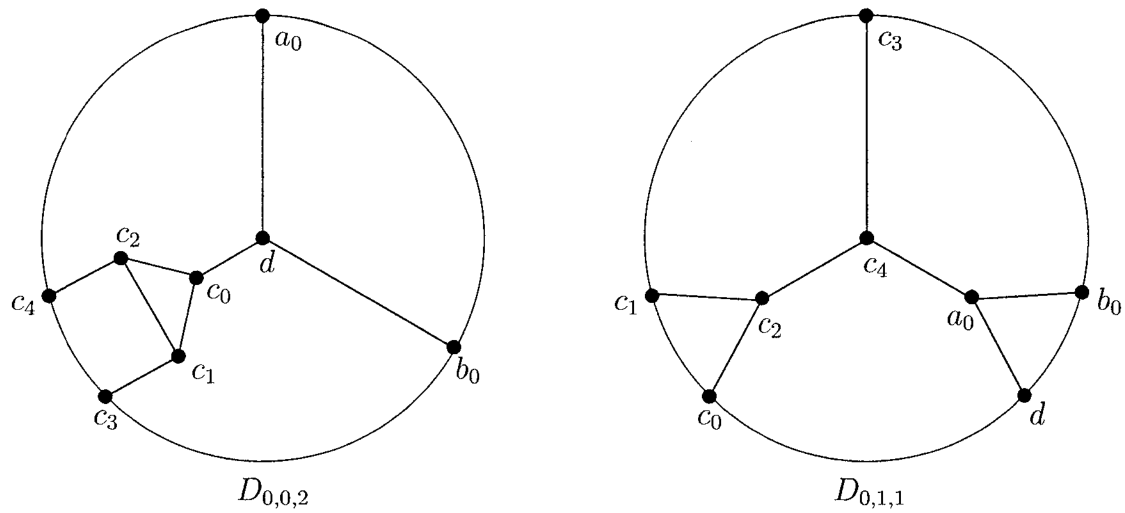


Figure 4.39: $D_{0,0,2} \cong D_{0,1,1}$. $D_{0,1,1}$ has the appropriate labels of $D_{0,0,2}$ showing the isomorphism.

Since $D_{0,0,2}$ is isomorphic to $D_{0,1,1}$ (see Figure 4.39), if G contains a subgraph that is a subdivision of $D_{0,0,c}$ where $c \geq 2$ then G contains a subgraph that is a subdivision of $D_{0,1,1}$.

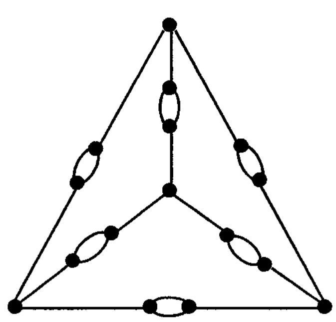


Figure 4.40: F1

Lemma 4.4.4 *Any restricted obstruction containing a largest die in the cubic order isomorphic to* $D_{0,0,0}$ *is* F1.

Proof: Let G be a restricted obstruction containing a largest die D that is isomorphic to $D_{0,0,0}$ where the vertices of D are canonically labeled as described on page 79. Since D is outer-cylindrical, D is not isomorphic to G and so one of the subdivisional paths of D has an inner vertex. By the symmetry of D, assume that path A has an internal vertex. Now if there were a path P from $I(A)$ to D - $I(A)$ not containing an end edge of P then G would contain a subgraph that is a subdivision of $D_{0,0,1}$ or $K_{3,3}$ – a contradiction to the choice of G. This is depicted in Figure 4.41.

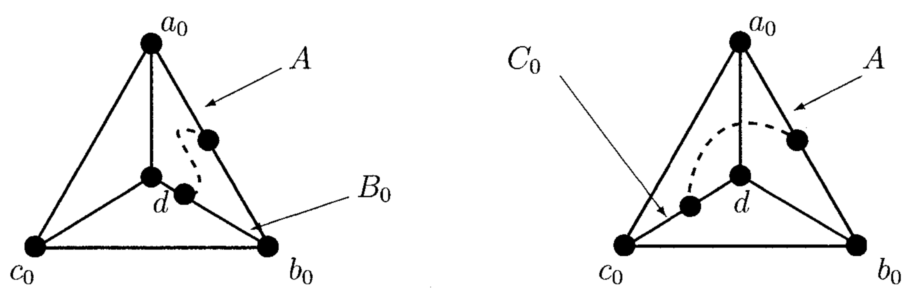


Figure 4.41: Subdivisional paths of $D_{0,0,0}$

Now it must be that the end edges of A form a 2-edge-cut. By Proposition 4.2.5, since G is not in $\mathcal{C}$, the induced subgraph on $I(A)$ is a 2-cycle. Moreover, since A was arbitrarily chosen, due to the symmetry of D, this is true for every subdivisional path P of D that has internal vertices.

We claim that for every subdivisional path P of D, the set $I(P)$ is nonempty. Assume that this is not the case and by symmetry of the graph assume $I(B)$ is empty. Since $I(B)$ is empty, F1 is not isomorphic to G and F1 properly contains G in the cubic order. Since F1 is a cubic obstruction, it must be that G is outer-cylindrical – a contradiction since G is a cubic obstruction. So it must be that any subdivisional path of a restricted obstruction containing a $D_{0,0,0}$ as a largest die in the cubic order has internal vertices and therefore G is isomorphic to F1. ■

In the next lemma, we will show that any restricted obstruction containing a $D_{0,0,1}$ and no larger die contains one of P3, P4, or P5 in the cubic order. Figure 4.42 depicts P3, P4, and P5 and this, together with the depictions of $D_{0,0,1}$ in Figure 4.43, reveals that each of P3, P4, and P5 contain $D_{0,0,1}$ in the cubic order.

Lemma 4.4.5 *Any restricted obstruction containing a largest die in the cubic order isomorphic to* $D_{0,0,1}$ *is one of* P3, P4, *or* P5.

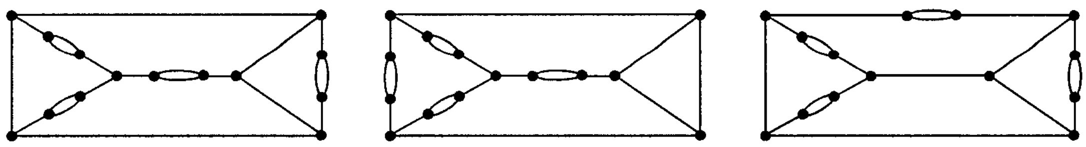


Figure 4.42: The graphs P3, P4, and P5

Proof: Let G be a restricted obstruction containing a subgraph that is a subdivision of a largest die isomorphic to $D_{0,0,1}$ we will denote D. Assume the vertices of D are canonically labeled (see Figure 4.43) and G is not isomorphic to P3, P4, or P5. Since D is outer-cylindrical, D is not isomorphic to G; so one of the subdivisional paths of D has an internal vertex.

The graph of the *prism* is isomorphic to the die $D_{0,0,1}$ and is depicted in Figure 4.43 with the die-labeling.

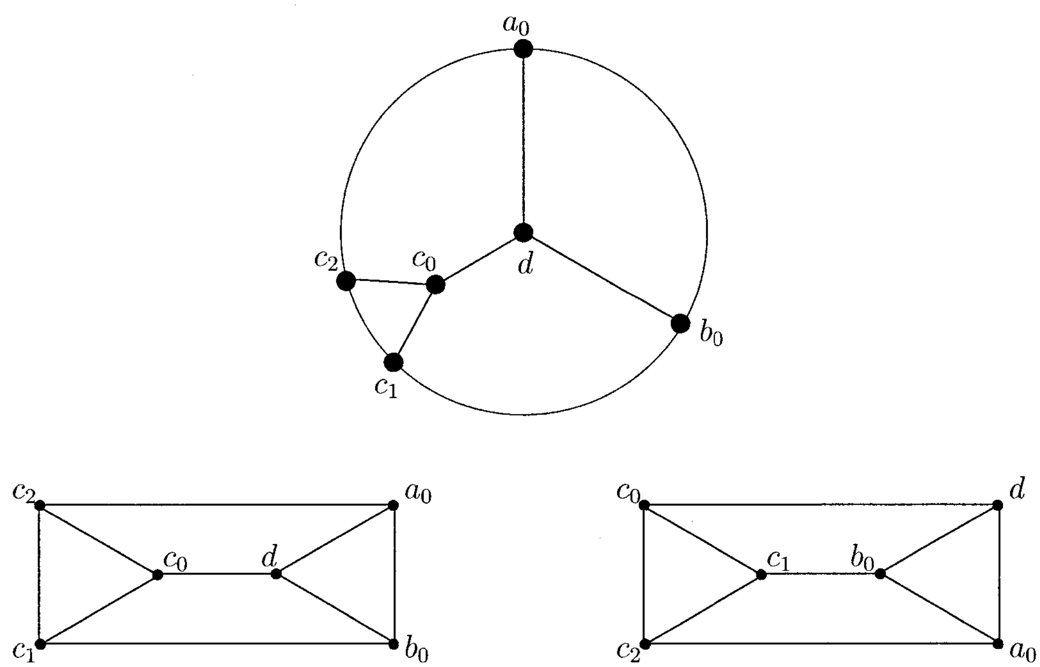


Figure 4.43: The die $D_{0,0,1}$ and two depictions of the prism with the die-labeling.

We claim that the end edges of every subdivisional path D' of D having internal vertices form a 2-edge-cut. So assume there is a path from $I(D')$ to D not containing the end edges of D'. Let P be a shortest such path where v is the end of P in $I(D')$ and w is the end of P in D.

Assume that v is a vertex in a cycle of D corresponding to a triangle in $D_{0,0,1}$. Then v is an internal vertex of one of A_0, B_0, A, $C_{0,1}$, $C_{0,2}$, or $C_{1,2}$. By the symmetry of D (see Figure 4.43) we can assume v is in the subdivisional path A_0 (see Figure 4.44). From Theorem 1.5.3 since D is 3-connected, D is uniquely embeddable in the plane. Since G is cubic, w must be an internal vertex of a subdivisional path of D. Thus w must be a vertex in A, B_0, C, C_0, or $C_{0,2}$. If w is a vertex in C or A, then G contains a subgraph that is a subdivision of $D_{0,1,1}$ where d and c_0 receive the same labels, v is labeled b_0, and b_0 is labeled a_0 – a contradiction to the choice of D.

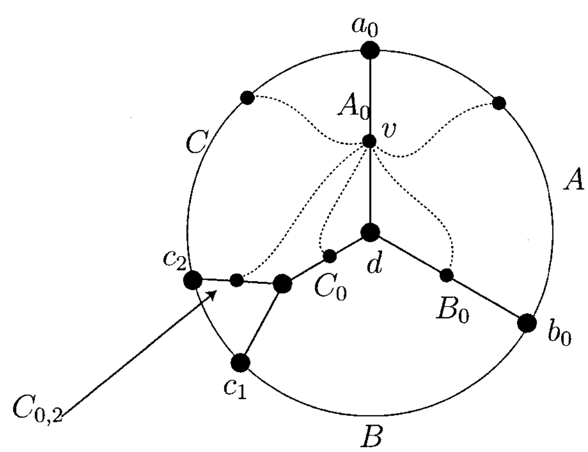


Figure 4.44: Labeled vertices of $D_{0,0,1}$

If w is a vertex in B_0 or C_0, then G contains a subgraph that is a subdivision of $D_{0,1,1}$ where a_0 is labeled d, b_0 is labeled a_0, v is labeled b_0, and c_2 is labeled c_0 – a contradiction to the choice of D. And if w is a vertex in $C_{0,2}$, then G contains a subgraph that is a subdivision of the cube where a_0vwc_2 and $b_0dc_0c_1$ are the rails of the cube – a contradiction to G being a restricted obstruction.

Assume that v is a vertex in a cycle of D that does not correspond to a triangle in $D_{0,0,1}$. Then v is in one of C, C_0, or B (see Figure 4.45). By the symmetry of D assume that v is in C_0. Since the end edges of subdivisional paths of D corresponding to a triangle in $D_{0,0,1}$ form 2-edge-cuts, w must be in either B or C. By the symmetry of D assume that w is in B. Then G contains a subgraph that is a subdivision of $D_{0,1,1}$ where w is labeled d, v is labeled a_0, b_0 receives the same label, and c_1 is labeled c_0, – a contradiction to the choice of D. So it must be that the end edges of every

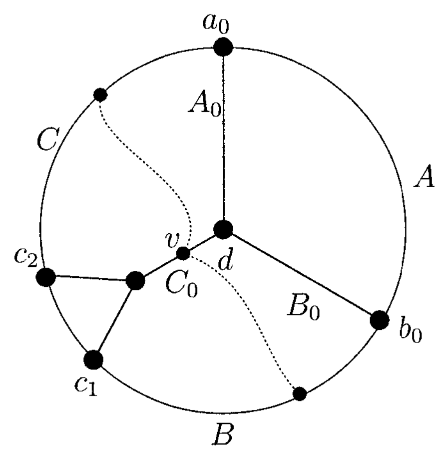


Figure 4.45: Labeled vertices of $D_{0,0,1}$

subdivisional path of D having internal vertices form a 2-edge-cut. We now show that D has specific subdivisional paths containing internal vertices. Specifically, we prove the following lemma.

Lemma 4.4.6 *At least one cycle of D corresponding to a triangle of $D_{0,0,1}$ has at least two subdivisional paths with internal vertices.*

Proof: Assume that neither cycle of D corresponding to a triangle of $D_{0,0,1}$ has at least two internal subdivisional paths with internal vertices. If neither cycle of D corresponding to a triangle of $D_{0,0,1}$ has subdivisional paths with has internal vertices, then G has an outer-cylindrical embedding where the boundaries of f_1 and f_2 together contain all the vertices of G (see Figure 4.46). This is a contradiction to G being a cubic obstruction.

Now, assume that exactly one of the cycles of D corresponding to a triangle of $D_{0,0,1}$ has exactly one subdivisional path with internal vertices. Without loss of generality, assume that $C_{0,1}$ has internal vertices. Then it is clear that the boundaries of faces f_1 and f_2 together contain all the vertices of G.

Hence we can assume that both cycles of D corresponding to a triangle of $D_{0,0,1}$ each have exactly one subdivisional path with internal vertices. We have two cases. By the symmetry of D, if $C_{0,1}$ and B_0 both have internal vertices then G has an

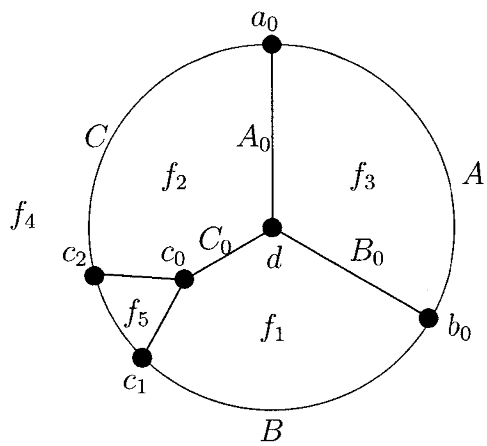


Figure 4.46: Labeled vertices of $D_{0,0,1}$

outer-cylindrical embedding where the boundaries of f_1 and f_2 together contain all the vertices of G – a contradiction to G being a cubic obstruction. By the symmetry of D, if $C_{0,1}$ and A_0 both have internal vertices then G has an outer-cylindrical embedding where the boundaries of f_1 and f_2 together contain all the vertices of G – a contradiction to G being a cubic obstruction. So there is a cycle of D corresponding to a triangle of $D_{0,0,1}$ having at least two subdivisional paths with internal vertices and Lemma 4.4.6 holds. ∎

From Lemma 4.4.6, we assume for the remainder of this proof that two subdivisional paths of a cycle of D corresponding to a triangle of $D_{0,0,1}$ contain internal vertices. By the symmetry of D assume that $C_{0,1}$ and $C_{0,2}$ contain internal vertices. If $C_{0,1}$ and $C_{0,2}$ were the only subdivisional paths of D with internal vertices then G would be outer-cylindrical where the boundaries of f_1 and f_2 would contain all the vertices of G. So we know there is another subdivisional path of D that has internal vertices and we have one of the following cases.

(i) $C_{1,2}$ is a subdivisional path with internal vertices.

(ii) C or B is a subdivisional path with internal vertices.

(iii) C_0 is a subdivisional path with internal vertices.

(iv) A_0, B_0, or A is a subdivisional path with internal vertices.

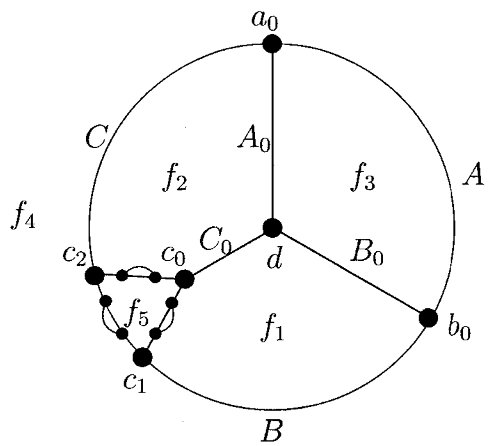


Figure 4.47: The paths $C_{0,1}$, $C_{0,2}$, and C have internal vertices

Suppose (i) holds, that is, $C_{1,2}$ is a subdivisional path with internal vertices (see Figure 4.47). Now if $C_{0,1}$, $C_{0,2}$, and $C_{1,2}$ were the only subdivisional paths of D with internal vertices then G would be outer-cylindrical since the boundaries of f_3 and f_5 contain all the vertices of G. So assume there is another subdivisional path P'' of D that has internal vertices. The induced subgraph on P'' is a 2-cycle, so let e be a parallel edge of $I(P'')$. Embed $G\backslash_3 e$ so that all the vertices lie on two faces. Now the only two faces whose boundaries together contain all the vertices of $G\backslash_3 e$ are f_5 and f_3. If P'' is one of the subdivisional paths A_0, B_0, or A then we see that G is an outer-cylindrical graph. So it must be that P'' is one of C, B, or C_0. But in all three of these cases it is easy to see that G contains a subgraph that is a subdivision of P4. Now G is a cubic obstruction and G contains P4 in the cubic order, thus G is isomorphic to P4 – a contradiction to the choice of G.

Suppose (ii) holds, that is C or B is a subdivisional path with internal vertices. By the symmetry of D we may assume that C is such a path (see Figure 4.48). Now if $C_{0,1}$, $C_{0,2}$, and C were the only subdivisional paths of D with internal vertices then G would be outer-cylindrical where the boundaries of faces f_1 and f_2 together would contain all the vertices of G. So we know there is another subdivisional path P'' of D that has internal vertices. The induced subgraph on $I(P'')$ is a 2-cycle, so let e be a parallel edge of P''. Embed $G\backslash_3 e$ so that all the vertices lie on the boundary of at most two faces.

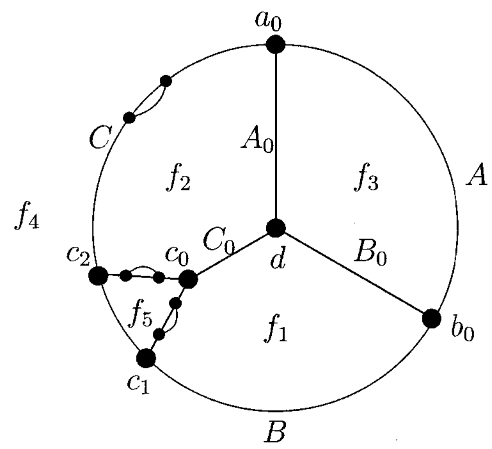


Figure 4.48: The paths $C_{0,1}$, $C_{0,2}$, and C have internal vertices

Notice that the boundary of f_5 along with the boundary of any other face does not contain all the vertices of $G\backslash_3 e$. Likewise the boundary of f_4 along with the boundary of any other face does not contain all the vertices of G. In addition, the boundary of f_3 along with the boundary of any other face does not contain all the vertices of $G\backslash_3 e$. Thus the only two faces whose boundaries together contain all the vertices of $G\backslash_3 e$ are f_1 and f_2.

Now if P'' is one of the subdivisional paths A_0, B_0, C_0, or B, then G is an outer-cylindrical graph with all the vertices lying on the boundaries of f_1 or f_2. So it must be that P'' is A. But then it is easy to see that G contains a subgraph that is a subdivision of P5. Now G is a cubic obstruction and G contains P5 in the cubic order, thus G is isomorphic to P5 – a contradiction to the choice of G.

Suppose (iii) holds, that is C_0 is a subdivisional path with internal vertices (see Figure 4.49). If $C_{0,1}$, $C_{0,2}$, and C_0 were the only subdivisional paths of D with internal vertices then G would be outer-cylindrical where the boundaries of faces f_1 and f_2 together would contain all the vertices of G. So we know there is another subdivisional path P'' of D that has internal vertices. The induced subgraph on $I(P'')$ is a 2-cycle, so let e be a parallel edge of P''. Embed $G\backslash_3 e$ so that all the vertices lie on the boundary of at most two faces. Now the only pair of faces whose boundaries together contain all the vertices of $G\backslash_3 e$ is f_1 and f_2. If P'' is one of the subdivisional paths A_0, B_0, C, or B then we see that G is an outer-cylindrical graph – a contradiction to G

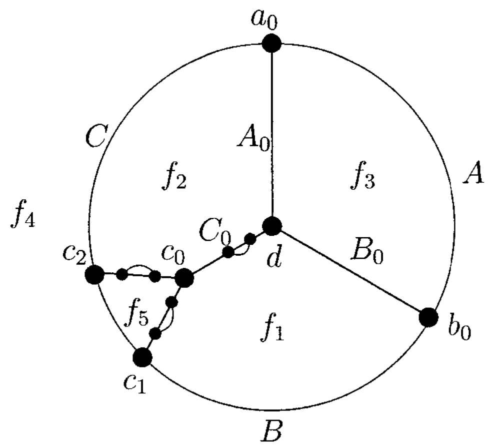


Figure 4.49: The paths $C_{0,1}$, $C_{0,2}$, and C_0 have internal vertices

being a restricted obstruction. So it must be that P'' is A. But then it is easy to see that G contains a subgraph that is a subdivision of P3. Now G is a cubic obstruction and G contains P3 in the cubic order, thus G is isomorphic to P3 – a contradiction to the choice of G.

Having settled cases (i), (ii), and (iii), the only subdivisional paths containing internal vertices are those described in case (iv). But in this case, we see that G is an outer-cylindrical graph by observing that the boundary of faces f_5 and f_3 together contain all the vertices of G – a contradiction to G being a restricted obstruction. With this last contradiction Lemma 4.4.5 is established. ∎

The next proposition shows that any restricted obstruction containing a largest die in the cubic order isomorphic to $D_{0,1,1}$ is P1 or P2. Before proving this, we require another definition.

A *boat* is a graph consisting of two vertices b and s called the *bow* and *stern* respectively and three disjoint (b,s)-paths. One of the paths is an edge called the *tether* and the other two paths are of equal length and are denoted $bw_1w_3 \ldots w_{2n-1}s$ and $bw_2w_4 \ldots w_{2n}s$. An edge, called a *rib*, joins w_{2i-1} to w_{2i}. A boat with n ribs is called an n-boat. If G is a subgraph that is a subdivision of an n-boat then the subdivisional path whose ends are b and w_1 is W_1, the subdivisional path whose ends are the b and w_2 is W_2. Similarly, the subdivisional path whose ends are s

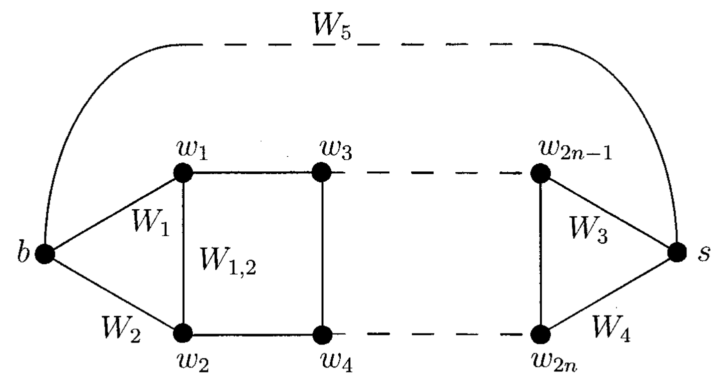


Figure 4.50: An n-boat

and w_{2n-1} is W_3, and the subdivisional path whose ends are s and w_{2n} is W_4. The subdivisional path corresponding to the tether (whose ends are the b and s) is W_5, and the subdivisional path whose ends are the w_i and w_j is $W_{i,j}$. This is the canonical labeling of an n-boat.

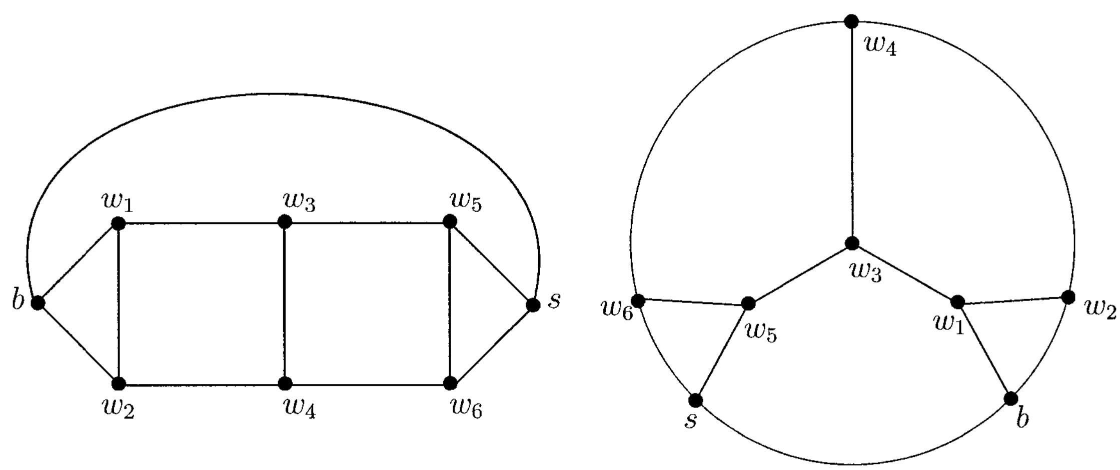


Figure 4.51: The 3-boat is isomorphic to $D_{0,1,1}$.

Notice that a 3-boat is the die $D_{0,1,1}$ by mapping w_3 to d, w_4 to a_0, w_1 to b_0, w_5 to c_0, w_2 to b_1, b to b_2, s to c_1, and w_6 to c_2.

Lemma 4.4.7 *Any restricted obstruction having a largest die in the cubic order isomorphic to $D_{0,1,1}$ is isomorphic to* P1 *or* P2.

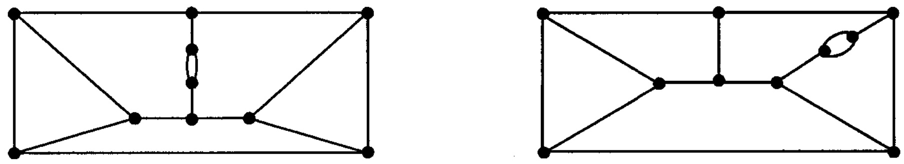


Figure 4.52: The graphs P1 and P2

Proof: Let G be a restricted obstruction containing a subgraph that is a subdivision of a largest die isomorphic to $D_{0,1,1}$ we will denote D. Assume that G is not isomorphic to P1 or P2 and that the vertices of D have a canonical labeling. Let n be an integer such that G contains a subgraph that is a subdivision of an n-boat we denote B and G does not contain a k-boat for $k > n$. Then n is at least three since $D_{0,1,1}$ is isomorphic to a 3-boat. Notice that an n-boat is outer-cylindrical by considering the faces bounded by the cycles $bw_1w_3 \ldots w_{2n-1}s$ and $bw_2w_4 \ldots w_{2n}s$. Thus there are subdivisional paths of B that contain internal vertices.

Lemma 4.4.8 *The end edges of every subdivisional path B' of B having internal vertices form a 2-edge-cut*

Proof: Assume there is a path from $I(B')$ to $V(B)- I(B')$ not containing the end edges of B'. Let P be a shortest such path where v is the end of P in B' and w is the end of P in $V(B)- I(B')$. By symmetry of the boat, we have four cases for P.

(1) v an internal vertex of W_1, W_2, W_3, or W_4

(2) v an internal vertex of $W_{i,i+1}$

(3) v an internal vertex of $W_{i,i+2}$ where i is odd

(4) v an internal vertex of W_5

First suppose that (1) v is in one of W_1, W_2, W_3, or W_4. By the symmetry of B assume that v is in W_1. Since G is planar then w must be in W_2, W_5, $W_{1,2}$, $W_{1,3}$, W_3, or $W_{2i-1,2i+1}$ where $2 \leq i \leq n-1$. If w is in W_2 or W_5 then G contains a

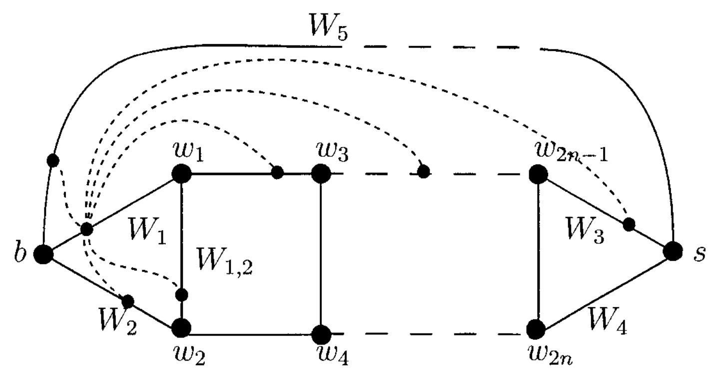


Figure 4.53: The paths of W_1

$(n+1)$-boat – a contradiction to G not containing a k-boat where $k > n$. If w is in $W_{1,2}$ then G contains a die isomorphic to $D_{0,1,2}$ where d is w_3, a_0 is w_4, b_0 is w_{2n-1}, and c_0 is w_1 – a contradiction to G not containing a die larger than $D_{0,1,1}$. If w is in $W_{1,3}$ then G contains a die isomorphic to $D_{0,1,2}$ where d is w_3, a_0 is w_4, b_0 is w_{2n-1}, and c_0 is w – a contradiction to G not containing a die larger than $D_{0,1,1}$. If w is in W_3 then G contains a subgraph that is a subdivision of a cube having rails $bvws$ and $w_2w_1w_{2n-1}w_{2n}$ – a contradiction to G being a restricted obstruction. If w is in $W_{2i-1,2i+1}$, where $2 \leq i \leq n-1$, then G contains a subgraph that is a subdivision of a cube having rails $bvws$ and $w_2w_1w_{2i-1}w_{2i}$– a contradiction to G being a restricted obstruction. It then follows that if W_1, W_2, W_3, or W_4 are subdivisional paths having internal vertices, then the end edges of W_1, W_2, W_3, or W_4 form a 2-edge-cut.

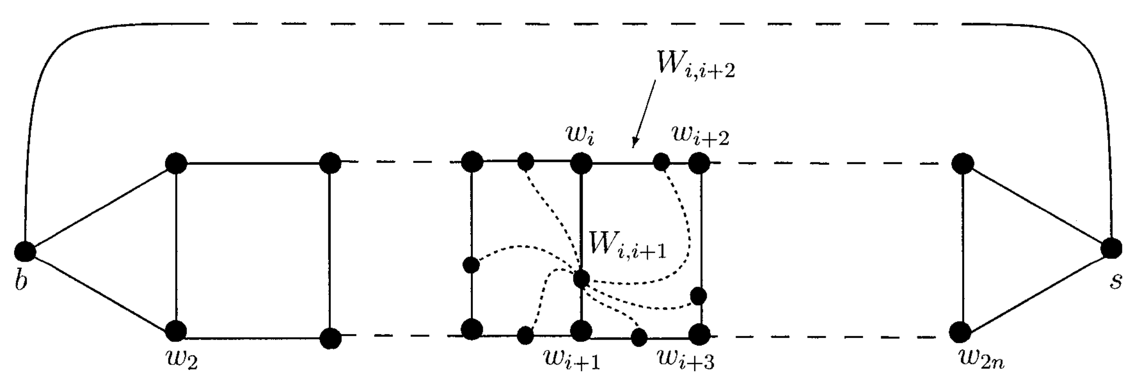


Figure 4.54: The paths of $W_{i,i+1}$

Now assume that (2) v is in $W_{i,i+1}$. Now we can assume by case (1) that vertex w is not in one of W_1, W_2, W_3, or W_4. So we have two cases, the first being that w is in one of $W_{i,i+2}$, $W_{i+1,i+3}$, $W_{i-2,i}$, or $W_{i-1,i+1}$ and the second is that w is in $W_{i+2,i+3}$ or $W_{i-2,i-1}$ if the respective paths exists. For the first case, by the symmetry of D, assume that w is in $W_{i,i+2}$, then G contains a die isomorphic to $D_{1,1,1}$ where d is w_{i+1}, a_0 is v and b_0 is w_2 and c_0 is w_{2n} – a contradiction to the choice of D. Considering the second case, from the symmetry of D, assume that w is in $W_{i+2,i+3}$. Then we see that G contains a subgraph that is a subdivision of a cube where the rails are $vww_{i+2}w_i$ and $w_{i+1}w_{i+3}sb$ – a contradiction to G being a restricted obstruction. Thus we see that the end edges of $W_{i,i+1}$ form 2-edge-cut.

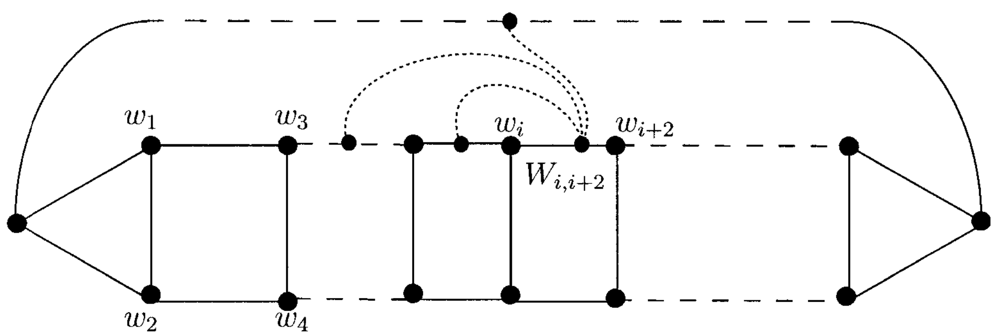


Figure 4.55: The paths of $W_{i,i+2}$

Next we assume that (3) v is in $W_{i,i+2}$ where i is odd. Now by cases (1) ane (2) vertex w is not in one of W_1, W_2, W_3, W_4, $W_{i+1,i+3}$, $W_{i,i+1}$, or $W_{i+2,i+3}$. So we have two cases, the first being that w is in $W_{j,j+2}$ for $j \neq i$ and the second is that w is in W_5. For the first case, assume by the symmetry of the n-boat, that w is in $W_{j,j+2}$ for $j < i$. Then B contains a $D_{1,1,1}$ subdivision where d is w_{i+1}, a_0 is w_i, b_0 is w_2, and c_0 is w_{2n} – a contradiction to the choice of D.

For the last case, where w is in W_5, we have two cases. In the first case assume that $i = 1$. Then G contains a subdivision of a die isomorphic to $D_{0,1,2}$ where d is w_4, a_0 is w_3, b_0 is w_{2n} and c_0 is w_2 – a contradiction to the choice of D. In the case that $i \neq 1$, G contains a subdivision of a die isomorphic to $D_{0,1,2}$ where d is w_{i+1}, a_0 is

w_i, b_0 is w_2 and c_0 is w_{2n} – a contradiction to the choice of D. Thus we see that the end edges of $W_{i,i+2}$ form a 2-edge-cut where i is odd. Likewise by the symmetry of D we see that $W_{i,i+2}$ form a 2-edge-cut where i is even. Now case (4) cannot occur, because if v is in W_5, there is no place for w. We conclude that the end edges of every subdivisional path of B form a 2-edge-cut. ■

Consider an outer-cylindrical embedding of an n-boat where the boundaries of faces f and g together contain all the vertices of the n-boat. Now we know that n is at least three and since the n-boat is an outer-cylindrical graph, at least one of the subdivisional paths of B must have internal vertices.

Assume that one of f or g, say g, is not incident to W_5, the subdivisional path corresponding to the tether. If g is the face bounded by the cycle of B corresponding to a triangle of the n-boat, then the boundary of any other face, together with this face would not contain all the vertices of G. If g is the face bounded by the cycle of B corresponding to a 4-cycle of the n-boat, then the boundary of g, together with any other face would not contain all the vertices of B. So it must be that f and g both must be incident to the path W_5. Now the only subdivisional paths of B not lying on the boundary of f or g are the ones denoted $W_{i,i+1}$. Since G is not an outer-cylindrical graph and G contains B, it then follows that one of the $W_{i,i+1}$ subdivisional paths of B has internal vertices.

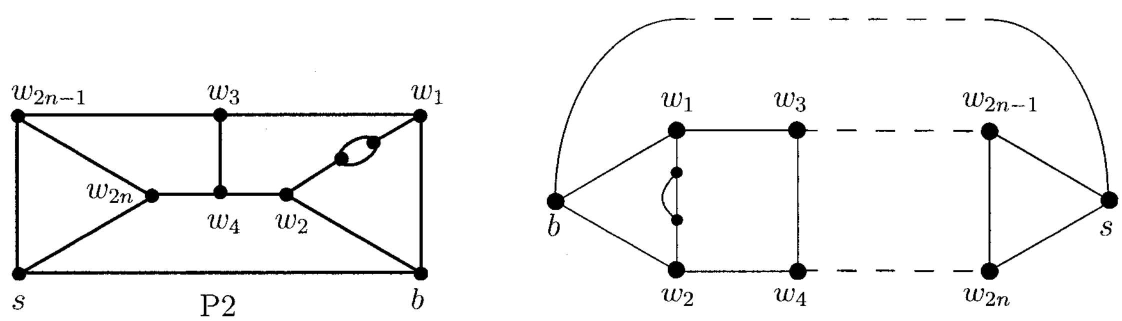


Figure 4.56: When $I(W_{1,2})$ forms a 2-cycle, G contains P2.

Suppose $W_{1,2}$ or $W_{2n-1,2n}$ has internal vertices. By the symmetry of B, assume that $W_{1,2}$ has internal vertices. Then B has a P2-subdivision as depicted in Figure

4.56. Now G is a cubic obstruction and G contains P2 in the cubic order, thus G is isomorphic to P2 – a contradiction to the choice of G.

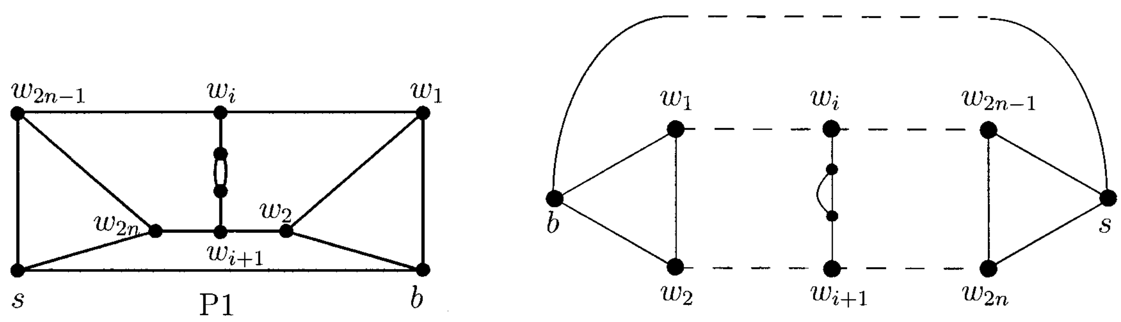


Figure 4.57: When $I(W_{i,i+1})$ forms a 2-cycle, G contains P1.

If $W_{i,i+1}$ where $i \neq 1$ and $i \neq 2n-1$ has internal vertices then G contains a subgraph that is a subdivision of P1 as depicted in Figure 4.57. Now G is a cubic obstruction and G contains P1 in the cubic order, thus G is isomorphic to P1 – a contradiction to the choice of G. So, it must be that any restricted obstruction containing a largest die D isomorphic to $D_{0,1,1}$ is isomorphic to P1 or P2. ■

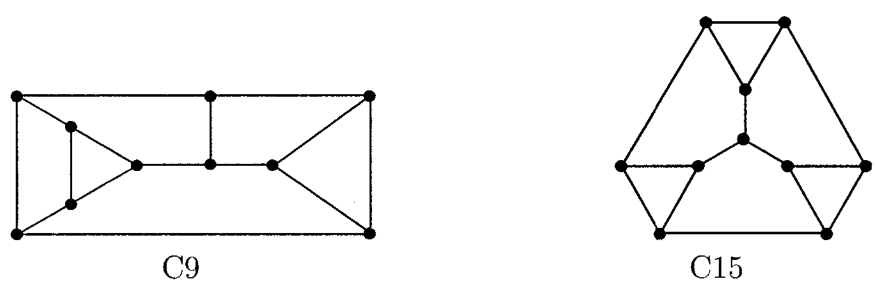


Figure 4.58: The graphs C9 and C15

Lemma 4.4.9 *Any restricted obstruction having a largest die in the cubic order isomorphic to $D_{a,b,c}$ where $a \geq 0$, $b \geq 1$, and $c \geq 2$ is* C9 *or* C15.

Proof: Let G be a restricted obstruction containing a largest die D isomorphic to $D_{a,b,c}$ where $a \geq 0$, $b \geq 1$, and $c \geq 2$ and assume that G is not isomorphic to C9 or C15. But C9 is a cubic obstruction for outer-cylindrical graphs and is isomorphic to

$D_{0,1,2}$. Thus if G contains a die $D_{0,b,c}$ where $b \geq 1$ and $c \geq 2$ then G would properly contain C9 in the cubic order – a contradiction to G being a cubic obstruction. So it must be that G contains a largest die isomorphic to $D_{1,1,1}$. But C15 is a cubic obstruction for outer-cylindrical graphs and is isomorphic to $D_{1,1,1}$. Thus G properly contains C15 in the cubic order – a contradiction to G being a cubic obstruction. Thus it must be that any restricted obstruction having a largest die in the cubic order isomorphic to $D_{a,b,c}$ where $a \geq 0$, $b \geq 1$, and $c \geq 2$ is C9 or C15. This finishes the proof of Theorem 3.1.2. ∎

APPENDIX A

CUBIC OBSTRUCTIONS FOR OUTER-CYLINDRICAL GRAPHS

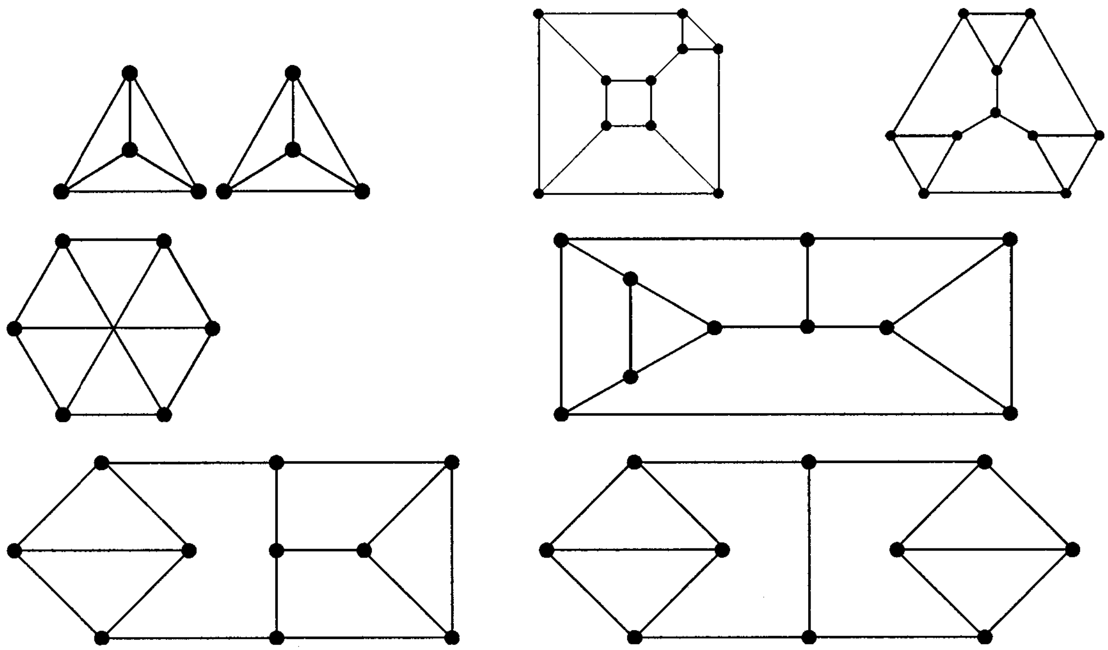


Figure A.1: The simple cubic obstructions for outer-cylindrical graphs.

In Table A.1 (on page 102) we give all cubic obstructions for outer-cylindrical graphs that contain loops. The graphs in the left column of the page are the non-simple loopless cubic obstructions which are also listed in Figure A.2. The family of graphs in the right column of the page is every graph, up to isomorphism, created from a set of 2-cycle-noose operations on 2-cycles of the graph in the left column of the page.

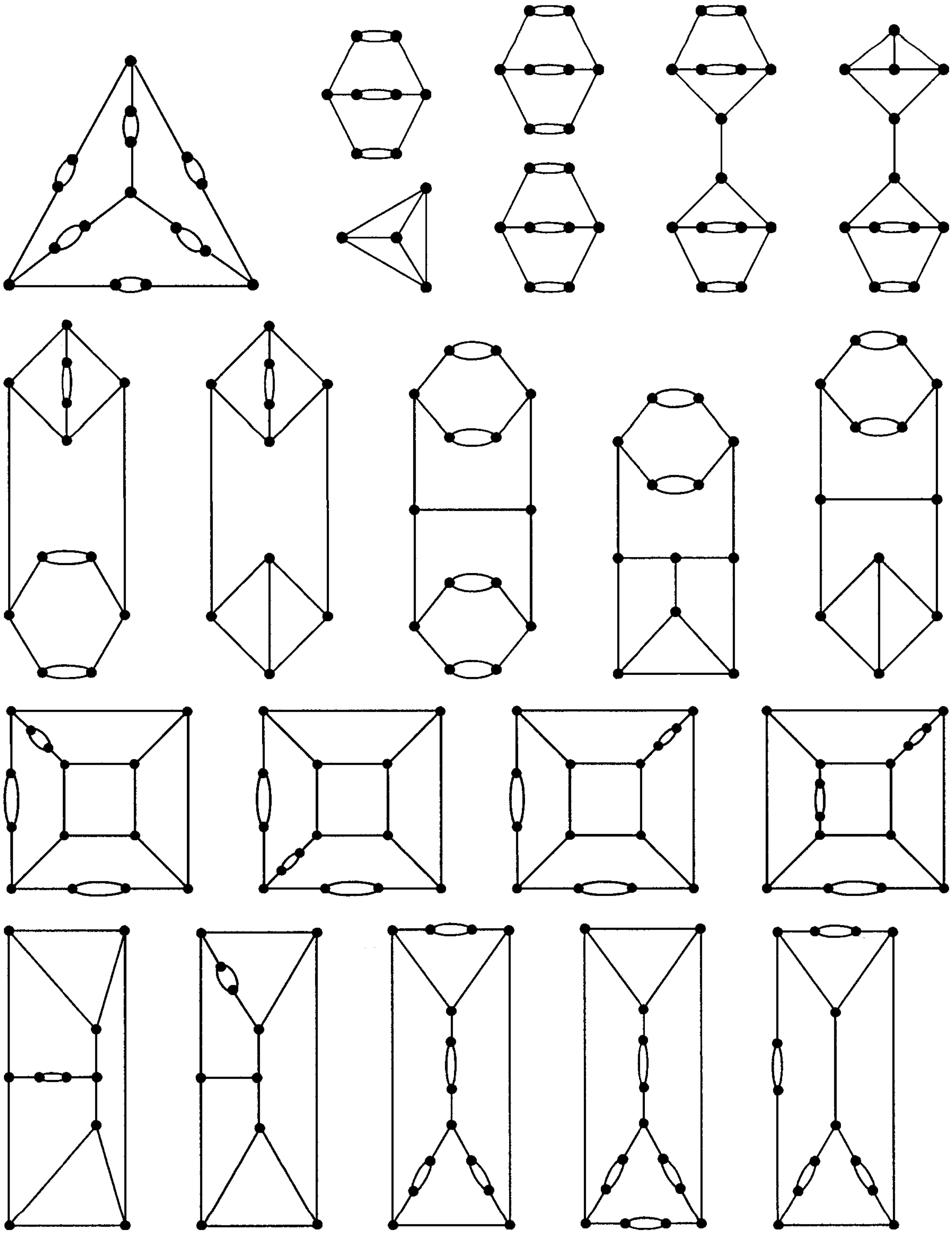


Figure A.2: The non-simple cubic obstructions for outer-cylindrical graphs.

Table A.1: The families of cubic obstructions containing loops.

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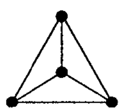
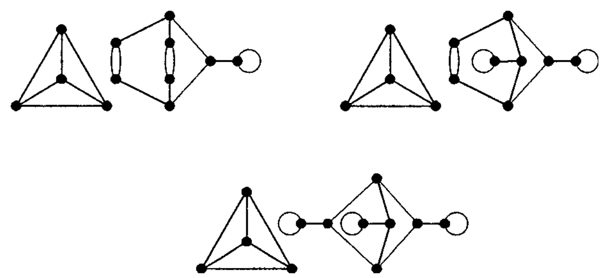
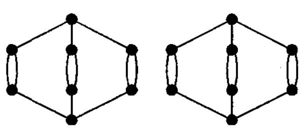
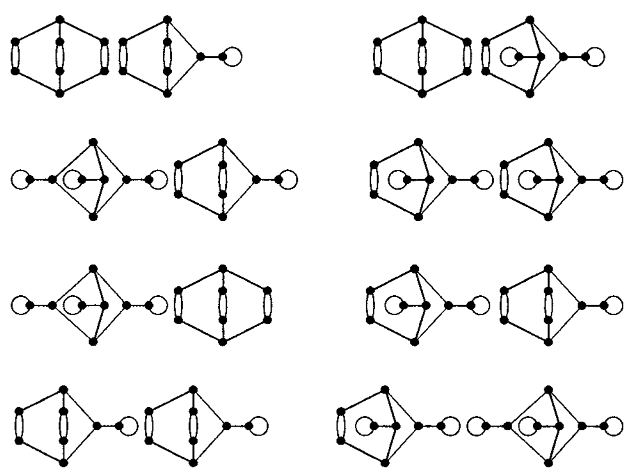
	
$K_{2,3C_2} \mathring{\cup} K_{2,3C_2}$	
	
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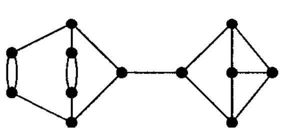	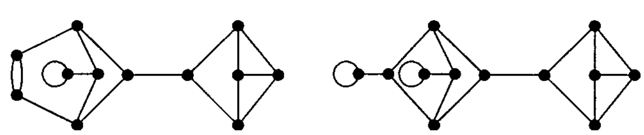

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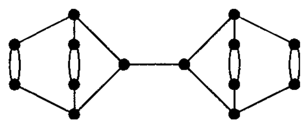	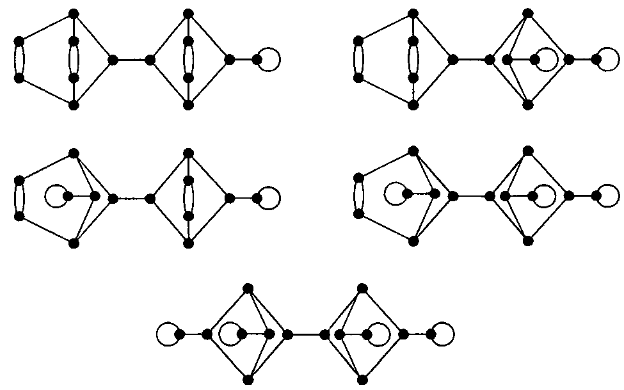

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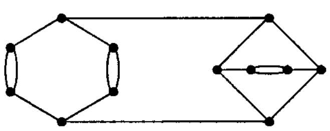	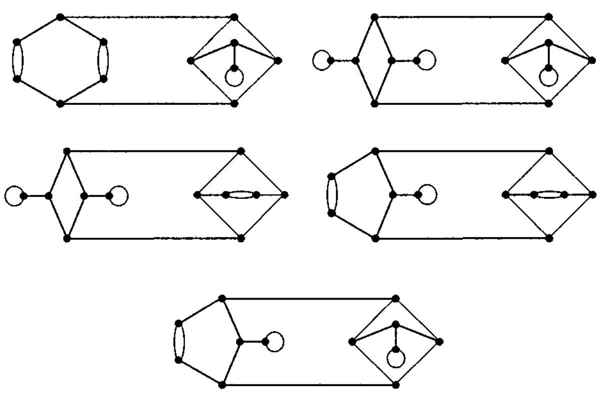

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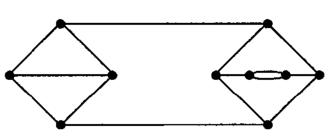	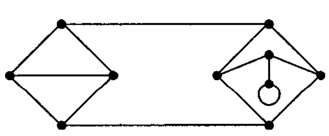

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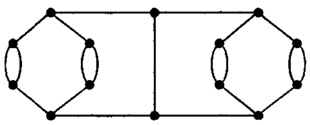	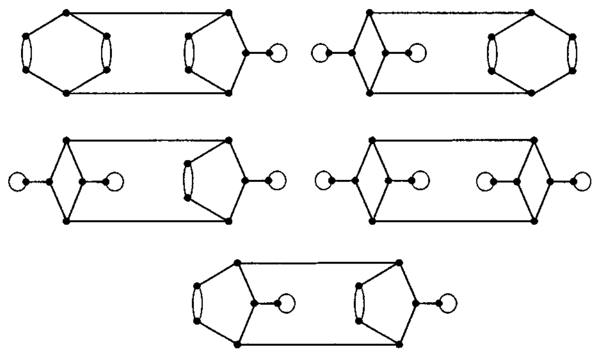
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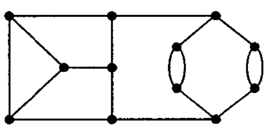	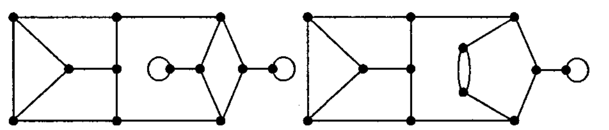
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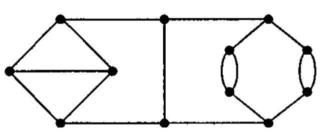	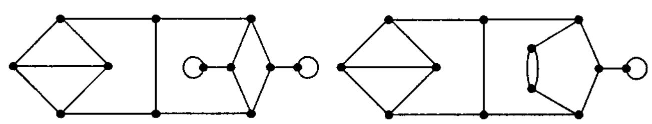

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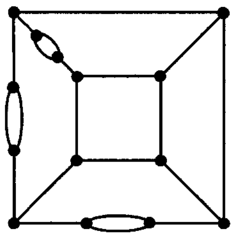
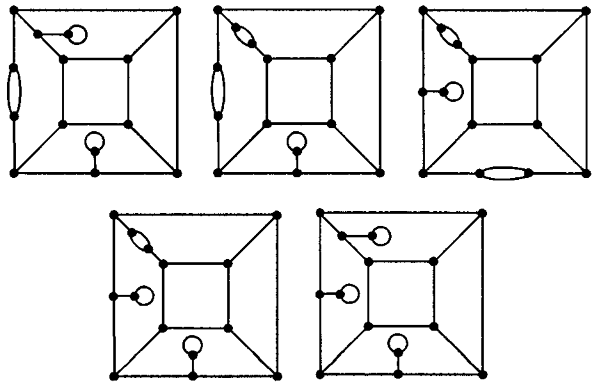
<tr><td></td><td></td></tr>
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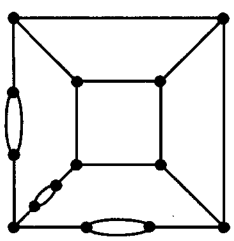
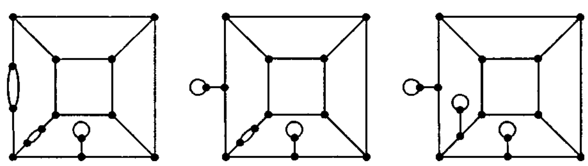
<tr><td></td><td></td></tr>
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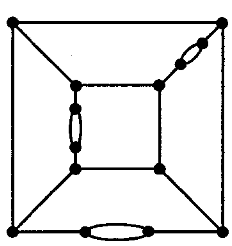
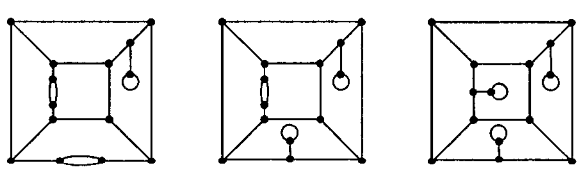
<tr><td></td><td></td></tr>
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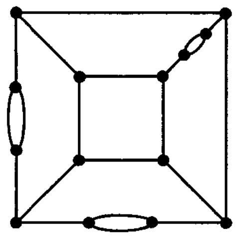	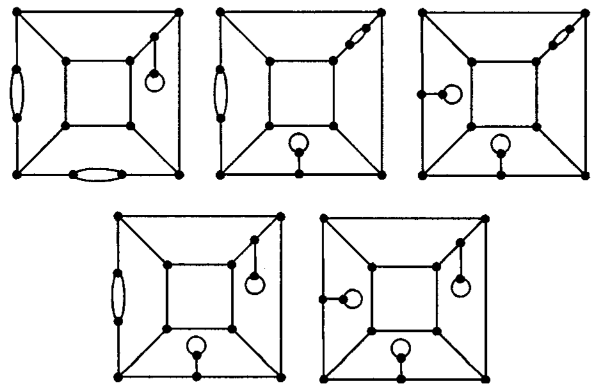
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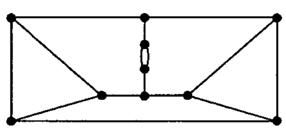	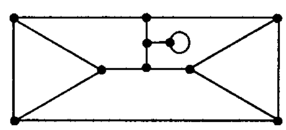
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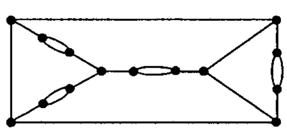	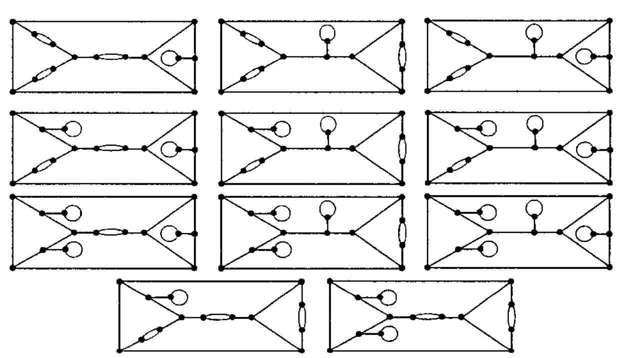

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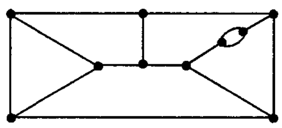	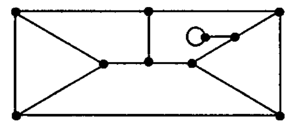
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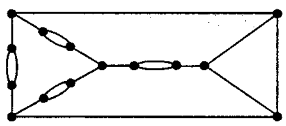	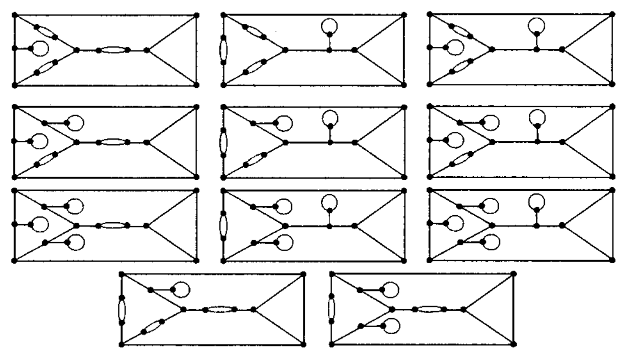
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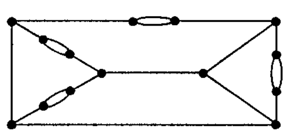	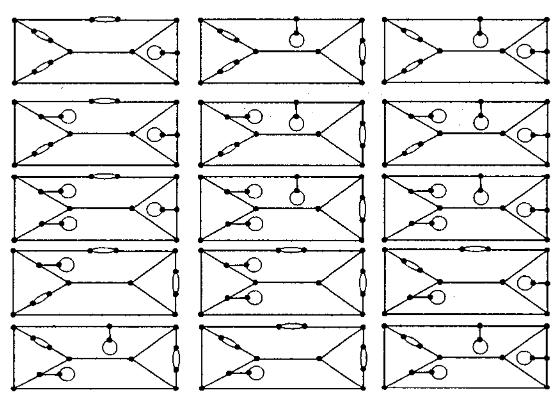

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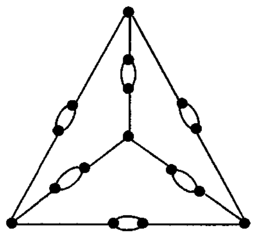
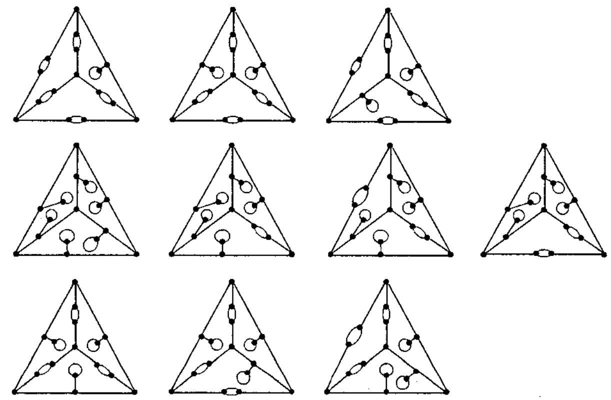
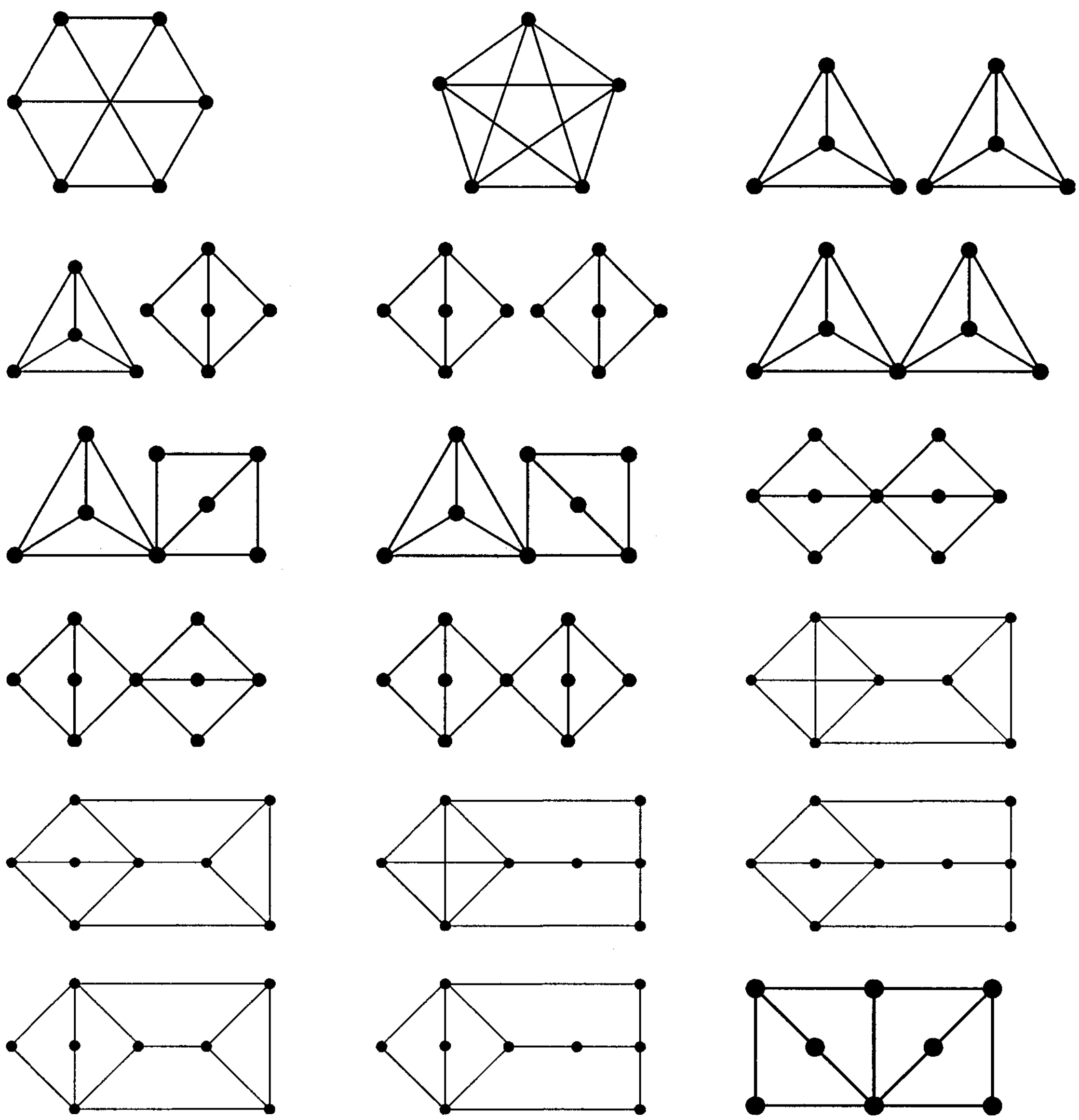
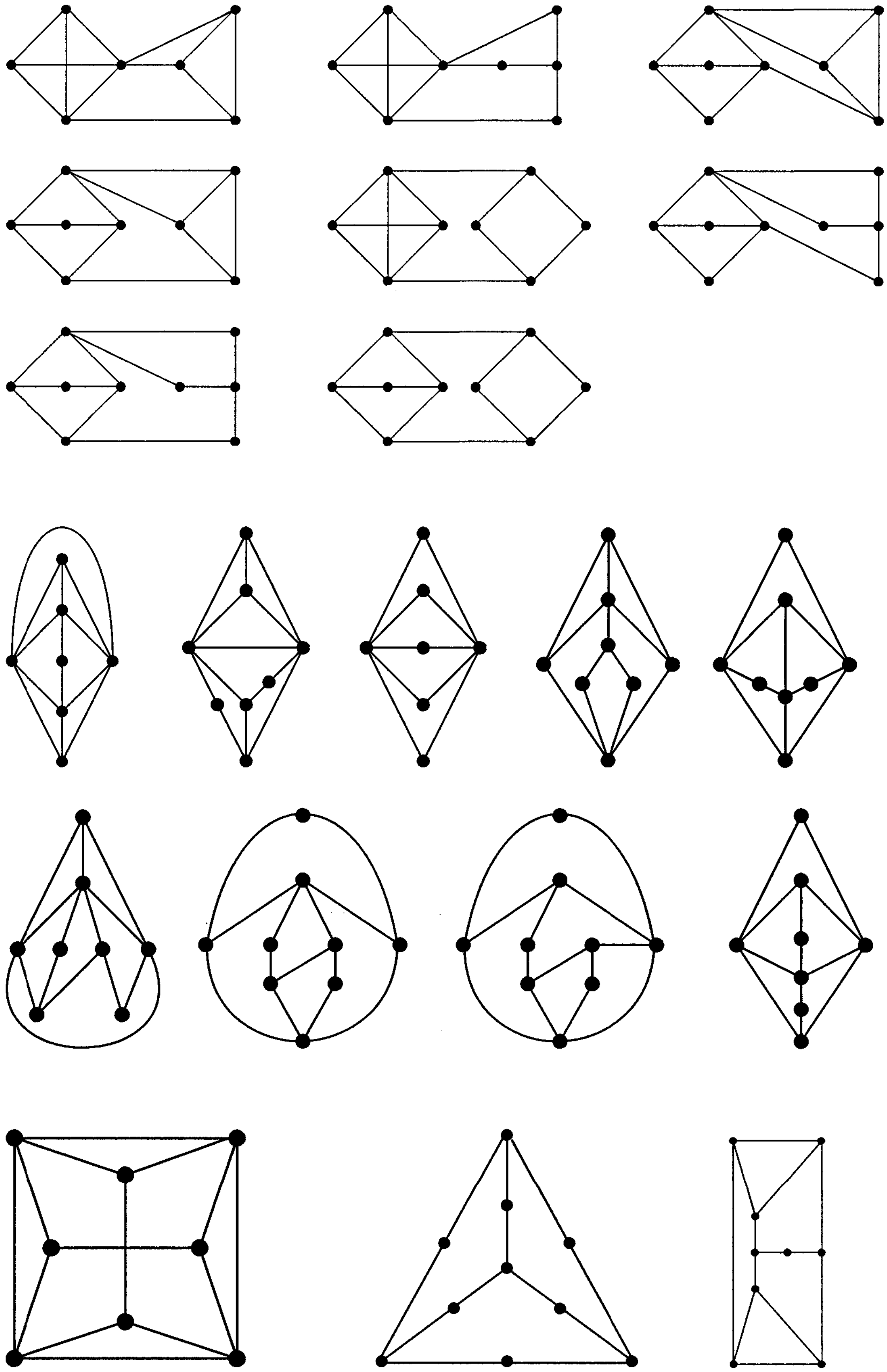
	

APPENDIX B

EXCLUDED MINORS FOR OUTER-CYLINDRICAL GRAPHS





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