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On calculating residuated approximations and the structure of finite lattices of small width

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ON CALCULATING RESIDUATED APPROXIMATIONS AND THE STRUCTURE OF FINITE LATTICES OF SMALL WIDTH

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ABSTRACT

The concept of a residuated mapping relates to the concept of Galois connections; both arise in the theory of partially ordered sets. They have been applied in mathematical theories (e.g., category theory and formal concept analysis) and in theoretical computer science. The computation of residuated approximations between two lattices is influenced by lattice properties, e.g. distributivity.

In previous work, it has been proven that, for any mapping $f : L \to Q$ between two complete lattices L and Q, there exists a largest residuated mapping ρ_f dominated by f, and the notion of "the shadow σ_f of f" is introduced. A complete lattice Q is completely distributive if, and only if, the shadow of any mapping $f : L \to Q$ from any complete lattice L to Q is residuated.

Our objective herein is to study the characterization of the skeleton of a poset and to initiate the creation of a structure theory for finite lattices of small widths. We introduce the notion of the skeleton $\tilde{L}$ of a lattice L and apply it to find a more efficient algorithm to calculate the umbral number for any mapping from a $\sim$ finite lattice to a complete lattice.

We take a maximal autonomous chain containing x as an equivalent class $[x]$ of x. The lattice $\tilde{L}$ is based on the sets $\{ [x] \mid x \in L \}$. The umbral number

for any mapping $f : L \to Q$ between two complete lattices is related to the property of $\tilde{L}$. Let L be a lattice satisfying the condition that $[x]$ is finite for all $x \in L$; such an L is called $\sim$ finite. We define $L_o = \{\bigwedge[x] \mid x \in L\}$ and $f_o = f|_{L_o}$. The umbral number for any isotone mapping f is equal to the umbral number for f_o, and $\sigma_{f_o}^{(\alpha)} = \sigma_f^{(\alpha)}|_{f_o}$ for any ordinal number α. Let $u_{L,Q}$ be the maximal umbral number for all isotone mappings $f : L \to Q$ between two complete lattices. If L is a $\sim$ finite lattice, then $u_{L,Q} = u_{L_o,Q}$. The computation of $u_{L_o,Q}$ is less than or equal to that of $u_{L,Q}$, we have a more efficient method to calculate the umbral number $u_{L,Q}$.

The previous results indicate that the umbral number $u_{L,Q}$ determined by two lattices is determined by their structure, so we want to find out the structure of finite lattices of small widths. We completely determine the structure of lattices of width 2 and initiate a method to illuminate the structure of lattices of larger width.

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NOMENCLATURE

(a, b)	open interval, page 13
$(a, b]$	left-open, right-closed interval, page 13
$:=$	equals by definiton, page 4
$[a, b)$	left-closed, right-open interval, page 13
$[a, b]$	closed interval, page 13
$[x]$	maximal autonomous chain containing an element x in a poset, page 15
$\bigcap S$	intersection of a family of sets, page 8
$\bigcup S$	union of a family of sets, page 8
$\circ$	function composition: $g \circ f$, page 28
$\downarrow x$	principal ideal generated by x, page 6
$\emptyset$	empty set, page 4
$\in$	membership: $a \in A$, page 4
$\leq$	partial order relation, page 5
$\leq^*$	converse of $\leq$, page 5
π_P^k	the set of antichains of P which contain k elements, page 60
π_P	the set of antichains of P, page 60
$\mathscr{P}(M)$	power set of a set M, page 8
$\mid P \mid$	number of elements in a set P, page 4
$\notin$	nonmembership: $a \notin A$, page 13
$\pi(x)$	set of elements parallel with x, page 13
$\prod_{i \in I} S_i$	product of a family of sets, page 20

ρ_f	the residuated approximation of f, page 35
$\sigma_f^{(\alpha)}$	umbral mappings corresponding f, page 35
σ_f	the shadow of f, page 33
$\sqsubseteq$	partial order relation in a set of antichains, page 62
$\tilde{P}$	the order skeleton of a poset P, page 15
$\uparrow x$	principal filter generated by x, page 6
$\{x \mid P(x)\}$	set of all elements x which satisfy $P(x)$, page 4
$A \subseteq B$	set inclusion, page 5
$A \cap B$	intersection of two sets A and B, page 19
$A \cong B$	isomorphism structures, page 5
$A \cup B$	union of two sets A and B, page 13
$A \sqcap B$	meet of two antichains A and B, page 62
$A \sqcup B$	join of two antichains A and B, page 62
$A \overset{\varphi}{\cong} B$	isomorphism φ, page 5
$A \times B$	product of sets, page 4
$a \vee b$	join of a and b, page 7
$a \wedge b$	meet of a and b, page 7
A^*	dual structure of A, page 5
$f(A)$	image of set A under f, page 4
$f(a)$	image of element a under f, page 4
$f : A \to B$	mapping of A into B, page 4
$f^{-1}(a)$	inverse image of a under f, page 5
gcd	greatest common divisor, page 58
$HS(L_i : i \in I)$	horizontal sum, page 57
$J(L)$	the set of join-irreducible elements, page 24
$J^{\sim}(L)$	the set of join-reducible elements, page 24

$J_c^{\sim}(L)$	the set of completely join-reducible elements, page 24
$J_c(L)$	the set of completely join-irreducible elements, page 24
L_o	β_o-skeleton of a lattice L, page 49
lcm	least common multiple, page 58
$Lex\{P_t \mid t \in T\}$	lexicographic sum, page 19
$M(L)$	the set of meet-irreducible elements, page 24
$M^{\sim}(L)$	the set of meet-reducible elements, page 24
$M_c^{\sim}(L)$	the set of completely meet-reducible elements, page 24
$M_c(L)$	the set of completely meet-irreducible elements, page 24
$Max(S)$	the set of maximal elements of S, page 6
$Min(S)$	the set of minimal elements of S, page 7
$QVS(P, Q, [x, 1]_P)$	quasi vertical sum, page 40
S_P^{β}	β-skeleton of a poset P, page 20
u_f	umbral number of f, page 36
$VS(P, Q)$	vertical sum, page 41
$x < y$	$x \leq y$ and $x \neq y$, page 6
$x \prec y$	x is covered by y, page 6

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CHAPTER 1

INTRODUCTION

1.1 General Overview

As an algebraic system, a partially ordered set consists of a set and a binary operation (a partially order relation between two elements). A residuated mapping is a special morphism between two partially ordered sets and has better property than another mapping, Galois connections. Recently, M. F. Janowitz has suggested that residuated mappings may play a role in the theory of cluster analysis [11].

For any mapping $f : L \to Q$ from a complete lattice L to a complete lattice Q, Andréka, Greechie and Strecker developed the "shadow" of f to obtain the maximal residuated mapping "dominated by" f (the residuated approximation ρ_f of f). If a mapping f is not residuated, iterations of umbral mapping converge to ρ_f for f, the least ordinal number α such that $\sigma_f^{(\alpha)} = \rho_f$ is called the umbral number of f, u_f. It is proven that (1) Q is completely distributive iff[1] σ_f is residuated for any complete lattice L and any mapping $f : L \to Q$, (2) L is completely distributive iff $f^{(+)}$ is residual for any complete lattice Q and any mapping $f : L \to Q$, (3) L is infinitely

[1]We use "iff" to represent "if and only if".

distributive iff $f^{(+)}$ is residual for every finite lattice Q and any mapping $f : L \to Q$. Hence, the maximal number $u_{L,Q}$ for any $f : L \to Q$ between two complete lattices L and Q is determined by the properties of such two lattices.

1.2 Research Objectives

Our object is to study (1) the relation between the structure of L and the umbral number for an isotone mapping $f : L \to Q$ between two lattices L and Q, and (2) the structure of finite lattices.

First of all, we present the concept of "the order skeleton of a poset" and use it to solve the previous two problems. The order skeleton $\tilde{P}$ of a poset P is based on an equivalent relation $\sim$, and $[x]$ is the equivalent class under $\sim$. When we apply $\sim$ to a lattice L, the relation $\sim$ becomes a congruence relation on L. Let $\beta[x]$ be a fixed element in $[x]$, for any element x in L; we obtain the β-skeleton S_L^{β} of L which is a copy of $\tilde{L}$, and for any β, it is always true that $S_L^{\beta} \cong \tilde{L}$.

In a $\sim$ finite lattice L, the maximal autonomous chain $[x]$ is finite for any x in L, and there exists a join subcomplete sub-semilattice L_o of L which is the union of $\bigwedge[x]$ for x. L_o is a special β-skeleton of L and $L_o \cong \tilde{L}$. If f is isotone, then $f_o = f|_{L_o}$ is isotone and $u_f = u_{f_o}$. By using of f_o we can speed up the computation of the umbral number u_f, and we prove that $u_{L,Q}$ from a $\sim$ finite lattice L to a complete lattice Q is less than or equal to 1 if L_o is distributive.

Further, we study the structure of a finite lattice with its skeleton and obtain some useful results about finite lattices of width 2 and 3.

1.3 Organization

Chapter 1 gives a brief introduction about the research and the objectives. Chapter 2 provides the background and the previous work. In Chapter 3, we introduce order skeletons of posets and lattices, and in Chapter 4, we focus on the computation of umbral mappings.

Chapter 5 introduces and studies concepts with the potential of increasing our understanding of the structure of finite lattices. Chapter 6 gives a complete description of the structure of finite lattices of width 2 and it discusses salient results in lattices of width 3. The summary of our work and some suggestions about future work are given in Chapter 7.

CHAPTER 2

REVIEW OF POSETS AND LATTICES

2.1 Posets

The **cardinality** of a set P, denoted by $|P|$, is the number of elements of P. For a non-negative integer n, the **product set** $A_1 \times \ldots \times A_n$ of n sets, $A_1, \ldots, A_n$, is defined by $A_1 \times \ldots \times A_n := \{(x_1, \ldots, x_n) \mid x_1 \in A_1, \ldots, x_n \in A_n\}$. [1] If $A_1 = \ldots = A_n = A$, then we write $A^n := A_1 \times \ldots \times A_n$, and for $n = 0$, we define $A^0 := \{\emptyset\}$. An **n-ary operation** (or **function**) on A is any function f from A^n to A, n is the **arity** (or **type**) of f; an operation f on A is **unary**, **binary**, **ternary** or **finitary** if its arity is 1, 2, 3, or a finite non-negative integer.

An **algebraic structure** consists of a pair $(A, \{f_\alpha \mid \alpha \in I\})$ where A is a nonempty set and, for each α in the indexing set I, there is an n such that $f_\alpha : A^n \to A$ is a finitary operation on A; if $I = \{\alpha_1, \ldots, \alpha_k\}$ is finite and f_{α_i} is of type n_i where, by convention, $n_i \geq n_{i+1}$ $(i = 1, \ldots, k-1)$, then the algebraic structure is of **type** $n_1, \ldots, n_k$. If a subset B of an algebra A is itself an algebra under the operation of A restricted to B, (i.e., $f_\alpha(x_1, \ldots, x_{n_i}) \in B$ for $x_1, \ldots, x_{n_i} \in B$), then B is called a **subalgebra** of A. Let (A, f_α) and (B, g_α) for $\alpha \in I$ be algebraic structures of type $n_1, \ldots, n_k$; a **homomorphism**

[1] We use ":=" to indicate that the object to the left of the equality is defined by the object to the right of the equality.

$\varphi : A \to B$ is a mapping between algebraic structures A and B such that $\varphi(f_\alpha(x_1, ..., x_{\alpha_i})) = g_\alpha(\varphi(x_1), ..., \varphi(x_{\alpha_i}))$ for each $\alpha \in I$, $x_i \in A$, φ maps A onto B; if the mapping φ is also one-to-one, i.e., for any $y_i \in B$, there exists an unique $\varphi^{-1}(y_i) \in A$ such that $f_\alpha(\varphi^{-1}(y_1), \dots, \varphi^{-1}(y_{\alpha_i})) = \varphi^{-1}(g_\alpha(y_1, \dots, y_{\alpha_i}))$, then φ is called an **isomorphism**, expressed by $A \overset{\varphi}{\cong} B$. If $A = B$, then we usually substitute **endomorphism** for homomorphism and **automorphism** for isomorphism.

Let P be a nonempty set, a **relation** on P is a subset $R \subseteq P \times P$. R may satisfy some of these properties:

(a) **reflexive**. If $x \in P$, then xRx holds.

(b) **symmetric**. If, for $x, y \in P$, xRy holds, then yRx holds.

(c) **antisymmetric**. If, for $x, y \in P$, xRy and yRx hold, then $x = y$.

(d) **transitive**. If, for $x, y, z \in P$, xRy and yRz hold, then xRz holds.

A relation R on a set P is an **equivalence relation** iff it is reflexive, symmetric and transitive. A relation R on a set P is a **partial order relation** iff it is reflexive, antisymmetric and transitive. A partially relation is usually written $\leq$. The pair $(P, \leq)$ is called a **partially ordered set** (or **poset**); we sometimes refer to it simply as P. The **converse** of a relation $\leq$ is the relation $\leq^*$ such that $x \leq^* y$ iff $y \leq x$. The **dual poset** of a poset $(P, \leq)$ is $(P^*, \leq^*)$ where $P^* = P$ and $\leq^*$ is the converse of $\leq$. If the dual poset of a poset P is isomorphic with the poset P, then the poset P is a **self-dual poset**. A nonempty subset S of a poset P is a **subposet** of P if there is a one-to-one function $f : S \to P$ such that $x \leq y \Rightarrow f(x) \leq f(y)$ for $x, y \in S$. Let P and Q be two posets. If there is a bijection function $f : P \to Q$ such that, for all $a, b \in P$, $a \leq b \Rightarrow f(a) \leq f(b)$ and, for all $c, d \in Q$, $c \leq d \Rightarrow f^{-1}(c) \leq f^{-1}(d)$, then f is a **poset isomorphism** and $P \cong Q$.

The **strict inequality** $x < y$ (or, equivalently, $y > x$) means that $x \leq y$ and $x \neq y$. We say that x **is covered by** y (or y **covers** x), in symbol $x \prec y$ (or $y \succ x$), if $x < y$, and there is no element z such that $x < z < y$.

A poset P can be represented by a **Hasse diagram**. In such a diagram, a point represents an element in P and a line that goes from x up to y means that $x \prec y$.

A nonempty subset I of a poset P is called an **order ideal** (or simply an ideal) of P if $x \in I$ and $y \leq x$ imply $y \in I$. A nonempty subset F of a poset P is called an **order filter** (or simply a filter) of P if $x \in F$ and $y \geq x$ imply $y \in F$.

Let P be a poset, $x \in P$ and $A \subseteq P$, then

$$\downarrow x := \{y \in P \mid y \leq x\}, \uparrow x := \{y \in P \mid x \leq y\}, \downarrow A := \bigcup_{a \in A} (\downarrow a).$$

The subset $\downarrow x$ (resp., $\uparrow x$) is called the **principal ideal** (resp., **principal filter**) generated by x. Let $S \subseteq P$ and let $s \in S$, then the element s is called a **maximal** (resp.[2], **minimal**) element of S if, for $x \in S$, $s \leq x$ (resp., $x \leq s$) implies $x = s$. For $S \subseteq P$, **Max(S)** is the set of all maximal elements of S, i.e, $Max(S) := \{x \in S \mid x \text{ is a maximal element in } S\}$; **Min(S)** is the set of all

[2] We use "resp." to represent "respectively".

minimal elements of S, i.e, $Min(S) := \{x \in S \mid x \text{ is a minimal element in } S\}$. Every finite poset has at least one minimal and one maximal element, but an infinite poset may have neither. For example, the set of integers, with natural ordering, is an infinte poset having neither minimal nor maximal element. In some posets there are more than one minimal or maximal elements.

Let $S \subseteq P$ and $s \in P$, s is an **upper** (resp., **lower**) **bound** of S if $x \leq s$ (resp., $s \leq x$) holds for all $x \in S$. S may have no bound or many different bounds. If, for $S \subseteq P$, there exists an upper (resp., lower) bound $a \in P$ such that $a \leq x$ (resp., $x \leq a$) holds for all upper (resp., lower) bounds x of S, then a is called the **least upper** (resp., **greatest lower**) **bound** of S. The least upper (resp., greatest lower) bound of a subset S of P, if it exists, is unique. The least upper bound of S is sometimes called the **supremum** or **join** of S denoted by $\bigvee S$, and the greatest lower bound of S is called the **infimum** or **meet** of S denoted by $\bigwedge S$. For any a and b of a poset P, the join and meet of $\{a, b\}$, if they exist, are denoted by $a \vee b$ and $a \wedge b$, respectively. Let $S \subseteq P$. If an element x in P is the least upper bound (resp., greatest lower bound) of S and also $x \in S$, then x is called the **greatest** (resp., **least**) element of S and $Max(S) = \{x\}$ (resp., $Min(S) = \{x\}$). The least element of a poset, if it exists, is called 0, and the greatest element of a poset, if it exists, is called 1. A poset P is a **bounded poset** if it has both 0 and 1. If P has a least element 0, then an element x of P is an **atom** if $0 \prec x$; if P has a largest element 1, then an element x of P is a **coatom** if $x \prec 1$.

2.2 Lattices

If, for any x and y in a poset L, the join (resp., meet) of the set $\{x,y\}$ exists, then the poset L is a **join-semilattice** (resp. **meet-semilattice**). If L is both join-semilattice and meet-semilattice, it is a **lattice**. Often the lattice L is expressed as $(L, \vee, \wedge)$. A **bounded lattice** is a lattice which has both 0 and 1. If there is a join (resp., a meet) for every subset of a lattice L, then L is **join complete** (resp., **meet complete**). Any join complete lattice is also a meet complete lattice, and vice versa. If, for any subset S of a poset L, both $\bigvee S$ and $\bigwedge S$ exist, then L is a **complete lattice**. A complete lattice is a bounded lattice. The **power set** of a set M, written $\mathscr{P}(M)$, is the set of all subsets of M. $(\mathscr{P}(M), \subseteq)$ is an example of a complete lattice in which, for $S \subseteq \mathscr{P}(M)$, $\bigvee S = \bigcup S$ and $\bigwedge S = \bigcap S$. A finite lattice is a complete lattice, but a complete lattice may not be a finite lattice. For example, the set of real numbers in [0, 1] under natural ordering is a complete lattice, not a finite lattice.

A lattice $(L, \vee, \wedge)$ is an algebraic structure where $\vee$ and $\wedge$ are binary operations, this algebra satisfies the following axioms for all $x, y, z \in L$:

Commutative laws: $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$.

Associative laws: $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

Absorption laws: $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.

Idempotent laws: $x \vee x = x$ and $x \wedge x = x$.

Any algebra $(L, \vee, \wedge)$ of type 2, 2 that is a lattice is said to correspond to the poset $(L, \leq)$ in case, for $a, b \in L$, $a \leq b$ is equivalent to $a \vee b = b$ and $a \wedge b = a$.

Let $(L, \vee, \wedge)$ and $(M, \sqcup, \sqcap)$ be two lattices, and there exists a function $\phi : L \rightarrow M$. The function ϕ is a **lattice isomorphic** iff ϕ is a bijection function such that, for all $x, y \in L$, $\phi(x \vee y) = \phi(x) \sqcup \phi(y)$ and $\phi(x \wedge y) = \phi(x) \sqcap \phi(y)$.

If $(L, \vee, \wedge)$ is a lattice corresponding to the poset $(L, \leq)$, then $(L^*, \vee^*, \wedge^*)$ is the lattice corresponding to the poset $(L^*, \leq^*)$ such that $a \vee^* b = a \wedge b$ and $a \wedge^* b = a \vee b$ for $a, b \in L = L^*$; the lattice $(L^*, \vee^*, \wedge^*)$ is called the **dual lattice** of $(L, \vee, \wedge)$. If the dual lattice of a lattice L is isomorphic with the lattice L, then the lattice L is a **self-dual lattice**.

We call a statement A a **lattice theoretical proposition** if it includes only $\vee$, $\wedge$ and variables. If we interchange $\vee$ and $\wedge$ in A, we obtain the **dual** of proposition A.

Lattice Theoretical Duality Principle: [17]

The dual of any true lattice theoretical proposition is itself a true lattice theoretical proposition.

Theorem 2.2.1. *In a lattice L, the following are equivalent:*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \textit{for all } x, y, z \in L \tag{2.2.1}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \textit{for all } x, y, z \in L \tag{2.2.2}$$

Proof. Let $x, y, z \in L$ and let $a := (x \wedge y) \vee (x \wedge z)$. If the Formula 2.2.1 holds, then we have $a = ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z)$ by the Formula 2.2.1; thus $a = x \wedge ((x \vee z) \wedge (y \vee z))$, since $(x \wedge y) \vee x = x$ by commutative law and absorption law, and $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ by commutative law and the Formula 2.2.1; so $a = (x \wedge (x \vee z)) \wedge (y \vee z) = x \wedge (y \vee z)$ by associative law and absorption law, hence the Formula 2.2.2 holds. Dually, if the Formula 2.2.2 holds, then the Formula 2.2.1 holds. □

A lattice L is a **distributive lattice** if it satisfies the Formula 2.2.1 or the Formula 2.2.2.

A nonempty subset S of a lattice L is a **sublattice** of L if S is a subalgebra of L. Thus, if $S \subseteq L$ and $S \neq \emptyset$, then, for $a, b \in S$, $a \vee b \in S$ and $a \wedge b \in S$. A nonempty subset S of a lattice L is a **join sub-semilattice** (resp. **meet sub-semilattice**) if $S \subseteq L$ and $S \neq \emptyset$, then, for $a, b \in S$, $a \vee b \in S$ (resp., $a \wedge b \in S$).

A lattice is distributive iff no sublattice of it is isomorphic with either the lattice M_3 of Figure 2.1 or the lattice N_5 of Figure 2.2 [1].

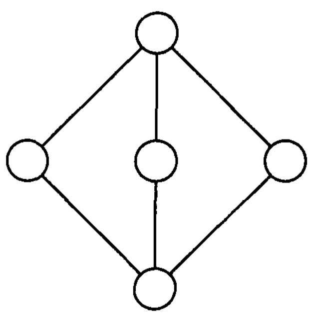


Figure 2.1 M_3.

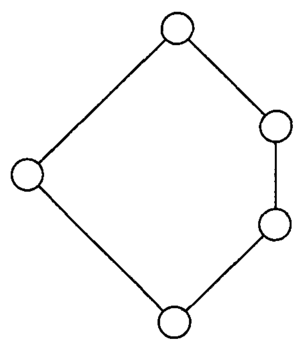


Figure 2.2 N_5.

Lemma 2.2.2. *A lattice L is a distributive lattice iff the dual lattice L^* is a distributive lattice.*

Proof. A lattice L is a distributive lattice iff L has no sublattice isomorphic with either M_3 or N_5; L has no sublattice isomorphic with either M_3 or N_5 iff L^* has no sublattice isomorphic with either M_3 or N_5; L^* has no sublattice isomorphic with either M_3 or N_5 iff L^* is a distributive lattice. Hence, L is a distributive lattice iff L^* is a distributive lattice. □

By the notation $\bigvee\limits_{i\in I} y_i$ (resp., $\bigwedge\limits_{i\in I} y_i$) we mean the least upper bound (resp., greatest lower bound) of the family $\{y_i\}_{i\in I}$. Let L be a complete lattice with $x \in L$, I be an index set and $\{y_i \mid i \in I\}$ be a subset of L. We consider the following **infinite distributive laws**:

$$x \wedge (\bigvee_{i\in I} y_i) = \bigvee_{i\in I} (x \wedge y_i), \tag{2.2.3}$$

$$x \vee (\bigwedge_{i\in I} y_i) = \bigwedge_{i\in I} (x \vee y_i). \tag{2.2.4}$$

If L is join complete and L satisfies the Formula 2.2.3, then L is an **infinitely distributive lattice**. The following example shows that the property of being infinitely distributive is not a self-dual. Let $\mathbb{N}$ be the set of non-negative integers and $a, b \in \mathbb{N}$, $a \wedge b$ is their greatest common divisor, and the $a \vee b$ is their least common multiple. $(\mathbb{N}, \vee, \wedge)$ is a complete lattice where the least element is 1 and the greatest element is 0, which follows from the representation of the greatest common divisor and the least common multiple in terms of their prime factors. Also, from this representation, one can prove that the lattice $\mathbb{N}$ satisfies the Formula 2.2.4; but it does not satisfy the Formula 2.2.3. For example, $\{a_i\}(i = 1, \ldots, \infty)$ is the set of all odd positive integers, then $2\wedge(\bigvee\limits_{i=1}^{\infty} a_i) = 2\wedge 0 = 2$, but $\bigvee\limits_{i=1}^{\infty}(2\wedge a_i) = \bigvee\limits_{i=1}^{\infty} 1 = 1 \neq 2 = 2\wedge(\bigvee\limits_{i=1}^{\infty} a_i)$.

Let I be an index set and $K(i)$ be a set for each $i \in I$. Define $F := \{\, f \mid f : I \to \bigcup_{i \in I} K(i)$ and $f(i) \in K(i) \,\}$. Then F is the set of all functions choosing, for each index i of I, some element $f(i)$ in $K(i)$. A complete lattice L is **completely meet distributive** if $\bigwedge_{i \in I} \bigvee_{k \in K(i)} x_{i,k} = \bigvee_{f(i) \in F} \bigwedge_{i \in I} x_{i,f(i)}$ holds; a complete lattice L is **completely join distributive** if $\bigvee_{i \in I} \bigwedge_{k \in K(i)} x_{i,k} = \bigwedge_{f(i) \in F} \bigvee_{i \in I} x_{i,f(i)}$ holds. In fact, a completely join distributive lattice is a completely meet distributive lattice. In the light of this comment, we have that a lattice L is **completely distributive** iff it is completely join distributive.

In the light of this comment, we have that a completely meet distributive (resp., join distributive) lattice is an infinitely meet distributive (resp., join distributive) lattice, an infinitely meet distributive (resp., join distributive) lattice is a distributive lattice.

CHAPTER 3

ORDER SKELETONS

3.1 Order Skeleton of a Poset

If A and B are sets, then $A - B := \{x \in A \mid x \notin B\}$. A poset P is called a **linearly ordered set** or **chain** if $x \leq y$ or $y \leq x$ holds for all $x, y \in P$. Let x and y be two elements in P. If $x \nleq y$ and $y \nleq x$, then x is **parallel with** y, we express it by $x \parallel y$.

Let $a, b \in P$. Define four types interval as follows,

closed interval: $[a, b] := \{x \in P \mid a \leq x \leq b\}$,

open interval: $(a, b) := \{x \in P \mid a < x < b\}$,

left-open, right-closed interval: $(a, b] := \{x \in P \mid a < x \leq b\}$,

left-closed, right-open interval: $[a, b) := \{x \in P \mid a \leq x < b\}$.

Note that we define $[a, b]$ even if $a \nleq b$, in which case $[a, b] = \emptyset$. A similar comment also applies to the other intervals just defined.

Let x be an element in a poset P. Define

$$\pi(x) := \{y \in P \mid y \parallel x\}.$$

Note that $\pi(x) = P - ((\uparrow x) \cup (\downarrow x))$. The set $\pi(x)$ is the set of elements which are parallel with x.

Let $\sim$ be the binary relation defined by $x \sim y$ for $x, y \in P$ in case

$x \nparallel y$ and $\pi(x) = \pi(z)$ for all $z \in [x, y] \cup [y, x]$.

A nonempty subset S of a poset P is called an **autonomous set** iff, for any $x \in P - S$, (1) $x \leq y$ for some $y \in S$ implies that $x \leq s$ for all $s \in S$, and (2) $y \leq x$ for some $y \in S$ implies that $s \leq x$ for all $s \in S$. An **autonomous chain** is a chain that is also an autonomous set.

Lemma 3.1.1. *Let P be a poset and $x, y \in P$. Then the following statements are equivalent.*

(1) $x \sim y$,

(2) *we have* $m \nparallel n$, $(\uparrow m - \uparrow n) \cup (\uparrow n - \uparrow m) = [m, n] \cup [n, m]$ *and*

$(\downarrow m - \downarrow n) \cup (\downarrow n - \downarrow m) = [m, n] \cup [n, m]$ *for* $m, n \in [x, y] \cup [y, x]$,

(3) $[x, y] \cup [y, x]$ *is an autonomous chain.*

Proof. (1) $\Rightarrow$ (2) Let $x \sim y$. Suppose $m, n \in [x, y] \cup [y, x]$. Then $m \nparallel n$ and $\pi(m) = \pi(x) = \pi(n)$. We may assume that $m \leq n$. Then $[m, n] \subseteq \uparrow m - \uparrow n$ and $(\uparrow m - \uparrow n) \cup (\uparrow n - \uparrow m) = \uparrow m - \uparrow n$. Let $a \in \uparrow m - \uparrow n$, then $a \notin \pi(m) = \pi(n)$ and $n \nleq a$, so $a \leq n$ and $a \in [m, n]$. Thus $\uparrow m - \uparrow n \subseteq [m, n] = [m, n] \cup [n, m]$. Hence $(\uparrow m - \uparrow n) \cup (\uparrow n - \uparrow m) = [m, n] = [m, n] \cup [n, m]$. Similarly $(\downarrow m - \downarrow n) \cup (\downarrow n - \downarrow m) = [m, n] \cup [n, m]$.

(2) $\Rightarrow$ (3) Suppose that condition (2) holds. Then $[x, y] \cup [y, x]$ is a chain. We claim that $[x, y] \cup [y, x]$ is an autonomous set. Let $p \in P - ([x, y] \cup [y, x])$ and $a \in [x, y] \cup [y, x]$. If $p \leq a$, then $p \leq x$ and $p \leq y$, thus $p \leq b$ for all $b \in [x, y] \cup [y, x]$. If $a \leq p$, then $x \leq p$ and $y \leq p$, thus $b \leq p$ for all $b \in [x, y] \cup [y, x]$.

(3) $\Rightarrow$ (1) Suppose that $[x, y] \cup [y, x]$ is an autonomous chain. We may

assume that $x \leq y$. Let $z \in [x,y]$, then $\uparrow x = [x,z] \cup \uparrow z$ and $\downarrow z = [x,z] \cup \downarrow x$, thus $(\uparrow x) \cup (\downarrow x) = [x,z] \cup (\uparrow z) \cup (\downarrow x) = ([x,z] \cup \downarrow x) \cup (\uparrow z) = (\downarrow z) \cup (\uparrow z)$, so $\pi(x) = \pi(z)$. Hence $x \sim y$. $\square$

When $x \leq y$, we have the following restatement of this Lemma.

Corollary 3.1.2. *Let P be a poset, $x, y \in P$ and $x \leq y$. Then the following statements are equivalent.*

(1) *$x \sim y$,*

(2) *for any $m, n \in [x,y]$, we have $m \not\parallel n$; if $m \leq n$, then $\uparrow m - \uparrow n = [m,n]$ and $\downarrow n - \downarrow m = [m,n]$,*

(3) *$[x,y]$ is an autonomous chain.*

Lemma 3.1.3. *The relation $\sim$ defined on P is an equivalent relation.*

Proof. The relation $\sim$ is reflexive since $x \not\parallel x$ and $[x,x] = \{x\}$.

We claim that $\sim$ is symmetric. If $x \sim y$, then $y \sim x$ by the definition of $\sim$.

We claim that $\sim$ is transitive. Let $x, y, z \in P$ with $x \sim y$ and $y \sim z$. Since $\pi(x) = \pi(y) = \pi(z)$, the three elements x, y and z are in a chain. We may assume that $x \leq z$. If $y \leq x$, then $[x,z]$ is a subset of an autonomous chain $[y,z]$, thus $[x,z]$ is an autonomous chain. If $z \leq y$, then $[x,z]$ is a subset of an autonomous chain $[x,y]$, so $[x,z]$ is an autonomous chain. If $y \in [x,z]$, then $[x,z]$ is the union of two autonomous chains $[x,y]$ and $[y,z]$, it follows that $[x,z]$ is an autonomous chain. Therefore $x \sim z$. $\square$

Let P be a poset with $x \in P$, define $[x] := \{ y \mid x \sim y \}$.

The **order skeleton** (or simply the **skeleton**) **of** P, $(\tilde{P}, \leq)$, is defined as follows:

$$\tilde{P} := \{\, [x] \mid x \in P \,\},$$

$$[x] \leq [y] \text{ iff there exist } x_1 \in [x] \text{ and } y_1 \in [y] \text{ such that } x_1 \leq y_1.$$

Lemma 3.1.4. *Let P be a poset. Then*

(1) $(\tilde{P}, \leq)$ *is a poset;*

(2) *for $x \in P$, $[x]$ is a maximal autonomous chain containing x.*

Proof. (1) The relation $\leq$ is reflexive since $x \in [x]$ and $x \leq x$.

We claim that $\leq$ is antisymmetric. If $[x] \leq [y]$ and $[y] \leq [x]$, then there exist $x_1, x_2 \in [x]$ and $y_1, y_2 \in [y]$ such that $x_1 \leq y_1$ and $y_2 \leq x_2$. Since $y_1 \sim y_2$ and $x_1 \leq y_1$, we have $x_1 \not\parallel y_2$ by the definition of $\sim$, i.e., $x_1 \leq y_2$ or $y_2 \leq x_1$. If $x_1 \leq y_2$, then $y_2 \in [x_1, x_2]$, so $y \sim y_2 \sim x$; if $y_2 \leq x_1$, then $x_1 \in [y_2, y_1]$, so $x \sim x_1 \sim y$. Hence $x \sim y$ and $[x] = [y]$.

We claim that $\leq$ is transitive. Let $[x], [y], [z] \in \tilde{P}$ with $[x] \leq [y]$ and $[y] \leq [z]$, then there exist $x_1 \in [x]$, $y_1, y_2 \in [y]$ and $z_1 \in [z]$ such that $x_1 \leq y_1$ and $y_2 \leq z_1$. Since $y_1 \sim y_2$ and $x_1 \leq y_1$, we have $x_1 \not\parallel y_2$. If $x_1 \leq y_2$, then $x_1 \leq y_2 \leq z_1$, so $[x] \leq [z]$; if $y_2 \leq x_1$, then $x_1 \in [y_2, y_1]$ and $x \sim x_1 \sim y$, so $[x] = [y] \leq [z]$.

(2) Since $x \sim x$, we have $x \in [x]$. Suppose that $[a, b]$ is an autonomous chain such that $[x] \subseteq [a, b]$. For any $c \in [a, b]$, $[x, c] \cup [c, x]$ is a subset of the autonomous chain $[a, b]$, so $[x, c] \cup [c, x]$ is an autonomous chain, thus $x \sim c$ by Lemma 3.1.1. Hence, $c \in [x]$ and $[x] = [a, b]$. Therefore, $[x]$ is a maximal autonomous chain containing x. □

Lemma 3.1.5. *Let P be a poset with $x, y \in P$, then*

(1) $[x] = [y] \Leftrightarrow x \sim y$,

(2) $x \parallel y$ *in* $P \Leftrightarrow [x] \parallel [y]$ *in* $\tilde{P}$,

(3) *if there exist* $x_1 \in [x]$ *and* $y_1 \in [y]$ *such that* $x_1 < y_1$ *and* $x_1 \nsim y_1$ *hold, then* $[x] < [y]$,

(4) *if* $[x] < [y]$, *then* $x_1 < y_1$ *and* $x_1 \nsim y_1$ *hold for all* $x_1 \in [x]$ *and* $y_1 \in [y]$,

(5) $[x] \leq [y] \Leftrightarrow x \sim y$ *or* $x < y$ *with* $x \nsim y$,

(6) $[x] \sim [y]$ *in* $\tilde{P} \Leftrightarrow [x] = [y]$,

(7) $x \leq y$ *and* $\pi(x) = \pi(y)$ *iff* $\uparrow x = [x, y] \cup \uparrow y$ *and* $\downarrow y = [x, y] \cup \downarrow x$,

(8) *if* $y \in [x]$ *and* $\bigvee[x]$ *exists, then* $\uparrow y = [y, \bigvee[x]] \cup \uparrow (\bigvee[x])$,

(9) *if* $y \in [x]$ *and* $\bigwedge[x]$ *exists, then* $\downarrow y = [\bigwedge[x], y] \cup \downarrow (\bigwedge[x])$,

(10) *if* $\bigwedge[x]$ *exists, then* $(\bigwedge[x], x] \subseteq [x]$,

(11) *if* $\bigvee[x]$ *exists, then* $[x, \bigvee[x]) \subseteq [x]$,

(12) *if* $\bigwedge[x]$ *exists and* $\bigwedge[x] < y \leq x$, *then* $(\bigwedge[x], x] = (\bigwedge[x], y] \cup [y, x]$,

(13) *if* $\bigvee[x]$ *exists and* $x \leq y < \bigvee[x]$, *then* $[x, \bigvee[x]) = [x, y] \cup [y, \bigvee[x]]$,

(14) *if* $\bigwedge[x]$ *and* $\bigvee[x]$ *exist, then* $[x] \cup \{\bigwedge[x], \bigvee[x]\} = [\bigwedge[x], x] \cup [x, \bigvee[x]]$.

Proof. (1) We only need prove that $[x] = [y]$ implies $x \sim y$. Since $x \in [x]$ and $[x] = [y]$, we have $x \in [y]$ and $x \sim y$.

(2) Let $x \parallel y$. Then, for $a \in [x]$ and $b \in [y]$, we have $y \in \pi(x) = \pi(a)$, thus, $a \in \pi(y) = \pi(b)$. Hence $[x] \nleq [y]$ and $[y] \nleq [x]$, i.e., $[x] \parallel [y]$. To prove the converse, let $[x] \parallel [y]$, then $x \nleq y$ and $y \nleq x$ from the definition of the ordering in $\tilde{P}$. Hence $x \parallel y$.

(3) This proof follows from the definition of $\tilde{P}$ and part (1).

(4) Let $[x] < [y]$ with $x_1 \in [x]$ and $y_1 \in [y]$. Since $[x] < [y]$, we have $y_1 \nleq x_1$; also since $x_1 \nparallel y_1$ by part (2), it follows that $x_1 < y_1$. Since

$[x_1] = [x] \neq [y] = [y_1]$, we have $x_1 \nsim y_1$ by part (1).

(5) The proof of $\Leftarrow$ follows from part (1) and (3), the proof of $\Rightarrow$ follows from part (1) and (4).

(6) We only need to prove $[x] \sim [y]$ in $\tilde{P}$ implies $[x] = [y]$. Let $[x] \sim [y]$ in $\tilde{P}$. Then $[x] \not\parallel [y]$ in $\tilde{P}$ and $x \not\parallel y$ in P. We claim that $x \sim y$. But suppose not. Then, there exists $z \in [x, y] \cup [y, x]$ such that $\pi(x) \neq \pi(z)$, thus $[z] \in [[x], [y]] \cup [[y], [x]]$ and $\pi([x]) \neq \pi([z])$, so $[x] \nsim [y]$ in $\tilde{P}$ contradicting the hypothesis $[x] \sim [y]$ in $\tilde{P}$. Hence, $x \sim y$ and $[x] = [y]$.

(7) Let $x \leq y$ and $\pi(x) = \pi(y)$. Since $x \leq y$, we have $[x, y] \cup (\uparrow y) \subseteq \uparrow x$. Let $p \in \uparrow x$, then $p \notin \pi(x) = \pi(y)$, i.e $p \leq y$ or $y \leq p$, thus $p \in [x, y]$ or $p \in \uparrow y$. So $\uparrow x \subseteq [x, y] \cup \uparrow y$. Hence $\uparrow x = [x, y] \cup \uparrow y$. Similarly $\downarrow y = [x, y] \cup \downarrow x$.

Suppose that $\uparrow x = [x, y] \cup \uparrow y$ and $\downarrow y = [x, y] \cup \downarrow x$. Then $x \leq y$. Also, we have $(\uparrow x) \cup (\downarrow x) = [x, y] \cup (\uparrow y) \cup \downarrow x = (\uparrow y) \cup ([x, y] \cup \downarrow x) = (\uparrow y) \cup (\downarrow y)$. Hence, $\pi(x) = \pi(y)$.

(8) Assume that $y \in [x]$ and $\bigvee[x]$ exists in P. Then $y \leq \bigvee[x]$ and $[y, \bigvee[x]] \cup \uparrow (\bigvee[x]) \subseteq \uparrow y$. Let $z \in \uparrow y$, then $[y] \leq [z]$. If $[y] = [z]$, then $z \in [y] = [x]$, thus $z \in [y, \bigvee[x]]$; if $[y] < [z]$, then $w < z$ for all $w \in [x]$, so $\bigvee[x] \leq z$. Thus $\uparrow y \subseteq [y, \bigvee[x]] \cup \uparrow (\bigvee[x])$. Hence $\uparrow y = [y, \bigvee[x]] \cup \uparrow (\bigvee[x])$.

(9) This proof is dual to the proof of (8).

(10) Let $z \in (\bigwedge[x], x]$. Then $z \not\parallel x_1$ for $x_1 \in [x]$ since $z \notin \pi(x) = \pi(x_1)$, and z is not less than all element in $[x]$ since $\bigwedge[x] < z$. Hence, there exists $p \in [x]$ such that $p \leq z$. Thus, $z \in [p, x] \subseteq [x]$. Hence $(\bigwedge[x], x] \subseteq [x]$.

(11) This proof is similar to that of (10).

(12) Let $y \in [\bigwedge[x], x]$, then $(\bigwedge[x], y]$ and $[y, x]$ are the subsets of the autonomous chain $[x]$ by (10); also since the intersection of $(\bigwedge[x], y]$ and $[y, x]$ is not empty, we have $(\bigwedge[x], x] = (\bigwedge[x], y] \cup [y, x]$.

(13) This proof is dual to that of (12).

(14) We have $(\bigwedge[x], x] \subseteq [x]$ and $[x, \bigvee[x]) \subseteq [x]$ by part (10) and part (11), so that $[\bigwedge[x], x] \cup [x, \bigvee[x]] \subseteq [x] \cup \{\bigwedge[x], \bigvee[x]\}$. If $y \in [x]$, then $\bigwedge[x] \leq y \leq \bigvee[x]$ and $y \nparallel x$, thus $y \in [\bigwedge[x], x] \cup [x, \bigvee[x]]$. It follows that $[x] \cup [\bigwedge[x], \bigvee[x]] \subseteq [\bigwedge[x], x] \cup [x, \bigvee[x]]$. Hence the formula holds. $\square$

Let T be a nonempty poset as an index set and P_t $(t \in T)$ be a family of pairwise disjoint nonempty posets that are disjoint from T. In [14] the **lexicographic sum** $Lex\{P_t \mid t \in T\}$ is defined to be $\bigcup_{t \in T} P_t$, and, for any $p_1, p_2 \in Lex\{P_t\}$, $p_1 \leq p_2$ iff

(1) for some $t_1, t_2 \in T$, $t_1 < t_2$, $p_1 \in P_{t_1}$ and $p_2 \in P_{t_2}$, or

(2) for some $t \in T$, $p_1, p_2 \in P_t$ and $p_1 \leq_{P_t} p_2$.

Theorem 3.1.6. *Any poset P is isomorphic to the lexicographic sum of a family of maximal autonomous chains $[x]$ for $x \in P$, i.e., $P \cong Lex\{P_t \mid t \in T\}$ when $T = \tilde{P}$ and for $t \in T$, $P_t = [x]$, when $t = [x]$; in brief $P \cong Lex\{[x] \mid [x] \in \tilde{P}\}$.*

Proof. We take $T = \tilde{P}$. $P = \bigcup_{t \in T} P_t$ becomes $P = \bigcup_{[x] \in \tilde{P}} [x]$, (i.e., $P = \bigcup \tilde{P}$) which is true, since $\sim$ is reflexive. Since $\tilde{P} \subseteq \mathscr{P}(P)$ and $[x] \in \mathscr{P}(P)$ for all $x \in P$, we have $\tilde{P} \in \mathscr{P}(\mathscr{P}(P))$ and $\tilde{P} \cap [x] = \emptyset$ for all $[x] \in \tilde{P}$. For all $p \in P$, there exists $t \in T$ such that $p \in P_t$ and $t = [x]$. Let $p_1 \in P_{t_1}$ with $t_1 = [x_1]$

and $p_2 \in P_{t_2}$ with $t_2 = [x_2]$. Then $p_1 \leq p_2$ iff (1) $p_1 \leq p_2$ with $p_1 \nsim p_2$ or, (2) $p_1 \leq p_2$ with $p_1 \sim p_2$, i.e., (1) $t_1 < t_2$, $p_1 \in P_{t_1}$ and $p_2 \in P_{t_2}$ or, (2) $t_1 = t_2$, $p_1, p_2 \in P_{t_1} = P_{t_2}$ and $p_1 \leq p_2$. Hence, P is the lexicographic sum of $[x]$ for $[x] \in \tilde{P}$. □

A poset P is a **skeleton** if $P \cong \tilde{P}$ via the bijection mapping $\sim$ sending x to $[x]$.

Lemma 3.1.7. *A poset P is a skeleton iff $[x] = \{x\}$ for any $x \in P$.*

Proof. Assume that P is a skeleton, so that $P \cong \tilde{P}$. Let $x \in P$ and $y \in [x]$, then $[x] = [y]$; also since $\sim$ is one-to-one, we have $y = x$. Hence, $[x] = \{x\}$.

Assume that $[x] = \{x\}$ for all $x \in P$. Then $\sim : P \to \tilde{P}$ is one-to-one. Since $\sim : P \to \tilde{P}$ is a onto by the definition of $\sim$, the mapping $\sim$ is a poset isomorphism. Hence $P \cong \tilde{P}$. □

Let P be a poset. We define

$$\prod_{[x]\in\tilde{P}} [x] := \{\, \beta : \tilde{P} \to P \mid \beta([x]) \in [x] \,\},$$

and, for $\beta \in \prod_{[x]\in\tilde{P}} [x]$,

$$S_P^\beta := \beta(\tilde{P}) = \{\, \beta([x]) \mid [x] \in \tilde{P} \,\} \subseteq P,$$

We call $(S_P^\beta, \leq|_{S_P^\beta})$ the β-**skeleton** of P. We write $\leq_{S_P^\beta}$ for $\leq|_{S_P^\beta}$.

Lemma 3.1.8. *Let P be a poset and let $\beta \in \prod_{[x]\in\tilde{P}} [x]$. Then*

(1) *$x \sim \beta([x])$ for all $x \in P$;*

(2) *$x \in S_P^\beta \Leftrightarrow x = \beta([x])$;*

(3) *let $x, y \in S_P^\beta$ with $x \sim y$, then $x = y$;*

(4) *let $x, y \in S_P^\beta$ with $x \neq y$, then $x \nsim y$;*

(5) S_P^β *is a subposet of* P, *i.e., for* $x, y \in S_P^\beta$, $x \leq_{S_P^\beta} y \Leftrightarrow x \leq y$;

(6) $\tilde{P} \cong S_P^\beta$ *for any* $\beta \in \prod_{[x] \in \tilde{P}} [x]$;

(7) $\tilde{P} \cong \tilde{\tilde{P}}$.

Proof. (1) and (2) These proofs follow directly from the definition of the function β.

(3) Let $x, y \in S_P^\beta$ with $x \sim y$. Then $[x] = [y]$. So $x = \beta([x]) = \beta([y]) = y$.

(4) This proof follows from (3).

(5) The proof follows from the definition of $\leq_{S_P^\beta}$,

(6) The function $\beta : \tilde{P} \rightarrow S_P^\beta$ is onto by the definition of β-skeleton. Let $\beta([x]) = \beta([y])$, then $[x] = [\beta(x)] = [\beta(y)] = [y]$ since $\beta([x]) \in [x]$ and $\beta([y]) \in [y]$, so β is one-to-one.

Let $[x] \leq [y]$. If $[x] < [y]$, then $\beta([x]) < \beta([y])$ by Lemma 3.1.5 part (4); if $[x] = [y]$, then $\beta([x]) = \beta([y])$. Hence $\beta([x]) \leq \beta([y])$. Suppose that $\beta([x]) \leq \beta([y])$. There are two cases: in the first case, $\beta([x]) \sim \beta([y])$, thus $x \sim \beta([x]) \sim \beta([y]) \sim y$; in the second case, $\beta([x]) \nsim \beta([y])$, thus $[x] = [\beta([x])] < [\beta([y])] = [y]$ since $\beta([x]) < \beta([y])$ with $\beta([x]) \in [x]$ and $\beta([y]) \in [y]$. Hence, $[x] \leq [y]$.

Therefore, $\tilde{P} \cong S_P^\beta$.

(7) By abuse of notation, we define $\sim : \tilde{P} \rightarrow \tilde{\tilde{P}}$ by $\sim([x]) = [[x]]$. Then $\sim$ is clearly onto. Let $[[x]], [[y]] \in \tilde{\tilde{P}}$ with $[[x]] = [[y]]$, then $[x] \sim [y]$ in $\tilde{P}$ by Lemma 3.1.5 part (1), and $[x] = [y]$ by Lemma 3.1.5 part (6). Hence $\sim$ is one-to-one.

If $[x] \leq [y]$ in $\tilde{P}$, then $[[x]] \leq [[y]]$ in $\tilde{\tilde{P}}$. If $[[x]] \leq [[y]]$, then $[x] \sim [y]$ or $[x] < [y]$ with $[x] \nsim [y]$ by Lemma 3.1.5 part (5); $[x] \sim [y]$ implies $[x] = [y]$, since $[[x]] = [[y]]$ and $\sim$ is one-to-one. Hence, $[x] \leq [y]$.

Therefore, $\tilde{P} \cong \tilde{\tilde{P}}$. □

Lemma 3.1.9. *Let P be a poset and $\beta \in \prod_{[x] \in \tilde{P}} [x]$.*

(1) *If $x, y \in P$ with $x \leq y$, then $\beta([x]) \leq \beta([y])$.*

(2) *If $T \subseteq S_P^\beta$ and u is a minimal upper bound of T in P, then $\beta([u])$ is a minimal upper bound of T in S_P^β.*

(3) *If $T \subseteq S_P^\beta$ and v is a maximal lower bound of T in P, then $\beta([v])$ is a maximal lower bound of T in S_P^β.*

(4) *If $x \wedge y$ exists for $x, y \in P$, then $[x] \wedge [y]$ exists in $\tilde{P}$ and $[x] \wedge [y] = [x \wedge y]$.*

(5) *If $x \vee y$ exists for $x, y \in P$, then $[x] \vee [y]$ exists in $\tilde{P}$ and $[x] \vee [y] = [x \vee y]$.*

Proof. (1) Let $x \leq y$, then $[\beta([x])] = [x] \leq [y] = [\beta([y])]$, by Lemma 3.1.5 (5) either $\beta([x]) \sim \beta([y])$ or $\beta([x]) < \beta([y])$ with $\beta[x] \nsim \beta[y]$. Thus either $\beta([x]) = \beta([y])$ by Lemma 3.1.8 part (3) or $\beta([x]) \leq \beta([y])$. Hence, in either case, $\beta([x]) \leq \beta([y])$.

(2) Let $T \subseteq S_P^\beta$ and u be a minimal upper bound of T in P, then $t \leq u$ for all $t \in T$, thus $[t] \leq [u]$ and $t = \beta([t]) \leq \beta([u])$ for all $t \in T$ by (1). Hence, $\beta([u])$ is an upper bound of T in S_P^β, and therefore, also in P.

Let z be any upper bound of T in S_P^β. Then z is a upper bound of T in P and $z \nless u$ since u is a minimal upper bound of T in P. If $z \sim u$, then $z \sim \beta([u])$ since $\beta([u]) \sim u$, thus $z = \beta([u])$ by Lemma 3.1.8 part (3) since

$z, \beta([u]) \in S_P^\beta$; if $z \nsim u$, then $z \notin [u]$; since $\beta([u]) \sim u$ and $z \nless u$, we have $z \nless \beta([u])$. So $z \nless_{S_P^\beta} \beta([u])$. Hence $\beta([u])$ is a minimal upper bound of T in S_P^β.

(3) This proof is dual to that of (2).

(4) Let $x, y \in P$, then either $x \nparallel y$ or $x \parallel y$ holds. If $x \nparallel y$, we may assume that $x \leq y$. So $[x] \leq [y]$ and $[x] \wedge [y] = [x] = [x \wedge y]$. Hence, we may assume $x \parallel y$, then $[x] \parallel [y]$ by Lemma 3.1.5 part (2). Since $x \wedge y \leq x$, we have $[x \wedge y] \leq [x]$. Similarly, $[x \wedge y] \leq [y]$. Hence, $[x \wedge y]$ is a lower bound of $[x]$ and $[y]$. Let $[s]$ be any lower bound of $[x]$ and $[y]$. Then $[s] < [x]$ and $[s] < [y]$ since $[x] \parallel [y]$. Thus, $s < x$ and $s < y$ by Lemma 3.1.5 part (4), so $s \leq x \wedge y$ and $[s] \leq [x \wedge y]$. Therefore, $[x] \wedge [y]$ exists in $\tilde{P}$ and $[x] \wedge [y] = [x \wedge y]$.

(5) This proof is dual to that of (4). □

3.2 Order Skeleton of a Lattice

For completeness, we provide a direct proof that the relation $\sim$ on a lattice L is a congruence relation.

Lemma 3.2.1. *Let L be a lattice. Then $\tilde{L}$ is also a lattice, in fact, $\tilde{L} = L/\sim$.*

Proof. Let $a \sim a_1$ and $b \sim b_1$. We claim that $a \vee b \sim a_1 \vee b_1$. Lemma 3.1.9 part (5) indicates that $[a] \vee [b]$ and $[a_1] \vee [b_1]$ exist, moreover $[a \vee b] = [a] \vee [b]$ and $[a_1 \vee b_1] = [a_1] \vee [b_1]$; also since $[a] = [a_1]$ and $[b] = [b_1]$, it follows that $[a \vee b] = [a_1 \vee b_1]$. By Lemma 3.1.5 part (1) we have $a \vee b \sim a_1 \vee b_1$. Dually $a \wedge b \sim a_1 \wedge b_1$. □

Corollary 3.2.2. *Let L be a lattice, then $\sim : L \to \tilde{L}$ is a lattice homomorphism. Hence, for $x, y \in L$, $[x \vee y] = [x] \vee [y]$ and $[x \wedge y] = [x] \wedge [y]$.*

An element x of a lattice L is called **join-irreducible** (resp., **meet-irreducible**) if $x = \bigvee A$ (resp., $x = \bigwedge A$) for any finite subset $A \subseteq L$ implies that $x \in A$. An element x is called **join-reducible** (resp., **meet-reducible**) if there exists a finite subset $A \subseteq L$ such that $x = \bigvee A$ (resp., $x = \bigwedge A$) and $x \notin A$. Thus, an element $x \in L$ is join-irreducible (resp., meet-irreducible) iff it is not join-reducible (resp., meet-reducible). Let $J(L)$ and $M(L)$ be the set of all join-irreducible elements of L and the set of all meet-irreducible elements of L, respectively. Let $J^{\sim}(L)$ and $M^{\sim}(L)$ be the set of all join-reducible elements of L and the set of all meet-reducible elements of L, respectively. Thus, $M^{\sim}(L) = L - M(L)$ and $J^{\sim}(L) = L - J(L)$. Note that $1 \in M^{\sim}(L)$ and $0 \in J^{\sim}(L)$, since $\emptyset$ is a finite set, $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. An element x is called **completely join-irreducible** (resp., **completely meet-irreducible**) if, for any $A \subseteq L$, $x = \bigvee A$ (resp., $x = \bigwedge A$) implies that $x \in A$. An element x is called **completely join-reducible** (resp., **completely meet-reducible**) if there exists some $A \subseteq L$ such that $x = \bigvee A$ (resp., $x = \bigwedge A$) and $x \notin A$. The set of all completely join-reducible elements and the set of all completely meet-reducible elements of L are denoted by $J_c^{\sim}(L)$ and $M_c^{\sim}(L)$, respectively; the set of all completely join-irreducible elements and the set of all completely meet-irreducible elements of L are denoted by $J_c(L)$ and $M_c(L)$, respectively. Thus $M_c^{\sim}(L) = L - M_c(L)$ and $J_c^{\sim}(L) = L - J_c(L)$. Obviously, $J^{\sim}(L) \subseteq J_c^{\sim}(L)$ and $M^{\sim}(L) \subseteq M_c^{\sim}(L)$. Note that (1) $1 \in M_c^{\sim}(L)$, since $1 = \bigwedge \emptyset$ with $1 \notin \emptyset$; (2) $0 \in J_c^{\sim}(L)$, since $0 = \bigvee \emptyset$ with $0 \notin \emptyset$.

Lemma 3.2.3. *Let L be a complete lattice with $x \in L$.*

(1) *If $[x] \cap M^{\sim}(L) \neq \emptyset$, then $[x] \cap M^{\sim}(L) = \{\bigvee[x]\}$ and $[x] \in M^{\sim}(\tilde{L})$.*

(2) *If $[x] \cap J^{\sim}(L) \neq \emptyset$, then $[x] \cap J^{\sim}(L) = \{\bigwedge[x]\}$ and $[x] \in J^{\sim}(\tilde{L})$.*

(3) *If $[x] \cap M^{\sim}(L) \cap J^{\sim}(L) \neq \emptyset$, then $[x] = \{x\}$ and $[x] \in M^{\sim}(\tilde{L}) \cap J^{\sim}(\tilde{L})$.*

(4) *$|M^{\sim}(\tilde{L})| = |M^{\sim}(L)|$ and $|J^{\sim}(\tilde{L})| = |J^{\sim}(L)|$.*

(5) *Let $\bigvee[x] \in [x]$. If $\bigvee[x] \in M_c(L)$ and $\bigvee[x] < 1$, then $|C_{\bigvee[x]}| = 1$.*

(6) *Let $\bigwedge[x] \in [x]$. If $\bigwedge[x] \in J_c(L))$ and $0 < \bigwedge[x]$, then $|C^{\bigwedge[x]}| = 1$.*

(7) *Let $a \in L$ and let $y \in [x]$. If $a \parallel x$, then $x \vee a = y \vee a$ and $x \wedge a = y \wedge a$.*

Proof. (1) Suppose that $[x] \cap M^{\sim}(L) \neq \emptyset$. Let $y \in [x] \cap M^{\sim}(L)$, then there exist $a, b \in L$ such that $y = a \wedge b$ and $y \notin \{a, b\}$, so $a \parallel b$ and $\pi(y) \neq \pi(a)$; thus, we have $y < a$ and $y \nsim a$, so $[y] < [a]$ by Lemma 3.1.5 part (3); therefore, $z < a$ for all $z \in [y] = [x]$ by Lemma 3.1.5 part (4). Hence, $\bigvee[x] \leq a$; similarly $\bigvee[x] \leq b$; it follows that $\bigvee[x] \leq a \wedge b = y$. Since $y \in [x]$, we have $y \leq \bigvee[x]$. Therefore, $y = \bigvee[x]$, i.e., $[x] \cap M^{\sim}(L) = \{\bigvee[x]\}$. Since $[x] = [y] = [a \wedge b] = [a] \wedge [b]$ and $[a] \parallel [b]$, we have $[x] \notin \{[a], [b]\}$. Hence, $[x] \in M^{\sim}(\tilde{L})$.

(2) This proof is dual to that of (1).

(3) Let $[x] \cap M^{\sim}(L) \cap J^{\sim}(L)$, then $\bigvee[x] = \bigwedge[x]$ with $[x] \in M^{\sim}(\tilde{L}) \cap J^{\sim}(\tilde{L})$ by (1) and (2). Since $\bigwedge[x] \leq x \leq \bigvee[x]$, it follows that $[x] = \{x\}$.

(4) Define $\sim : M^{\sim}(L) \rightarrow M^{\sim}(\tilde{L})$ by $\sim (x) = [x]$. We claim that $\sim$ is onto. Since $[0] \in M^{\sim}(\tilde{L})$ and $0 \in M^{\sim}(L)$, we may assume that $[x] \in M^{\sim}(\tilde{L})$ and $[x] \neq [0]$. Then there exists $[a], [b] \in \tilde{L}$ such that $[x] = [a] \wedge [b]$

and $x \notin \{[a],[b]\}$. Thus, $[a] \parallel [b]$ and $a \parallel b$ by Lemma 3.1.5 part (2); so $a \wedge b \in M^{\sim}(L)$; since $[x] = [a] \wedge [b] = [a \wedge b]$ by Corollary 3.2.2, we have $\sim(x) = \sim(a \wedge b) = [x]$. We claim that $\sim$ is one-to-one. Let $[y], [z] \in M^{\sim}(\tilde{L})$ with $[y] = [z]$. By (1) $[y] \cap M^{\sim}(L) = \bigvee[y]$ and $[z] \cap M^{\sim}(L) = \bigvee[z]$, so that $\sim^{-1}([y]) = \bigvee[y] = \bigvee[z] = \sim^{-1}([z])$. Hence $|M^{\sim}(\tilde{L})| = |M^{\sim}(L)|$. Dually $|J^{\sim}(\tilde{L})| = |J^{\sim}(L)|$.

(5) Let $y = \bigvee[x] \in M_c(L) \cap [x]$ and $y < 1$. We claim that $C_y \neq \emptyset$. Suppose not. Then $y = \bigwedge(\uparrow y - \{y\})$ contradicting $y \in M_c(L)$. Let $a \in C_y$. For any $z > y$, we have $z \not\parallel a$, otherwise, $y = a \wedge z \notin \{a, z\}$ contradicts $y \notin M_c^{\sim}(L)$; also $z \nprec a$ since $y \prec a$, thus $a \leq z$. Hence, $|C_{\bigvee[x]}| = |C_y| = |\{a\}| = 1$.

(6) This proof is dual to that of (5).

(7) Let $y \in [x]$. By definition of $\sim$, $a \parallel x$ iff $a \parallel y$. Since $a \in \pi(x) - \pi(x \vee a)$, we have $x \nsim x \vee a$. Since $x \leq x \vee a$ and $y \sim x$, we have $y \leq x \vee a$, so $y \vee a \leq x \vee a$. Similarly $x \vee a \leq y \vee a$. Hence, $x \vee a = y \vee a$. Dually $x \wedge a = y \wedge a$. □

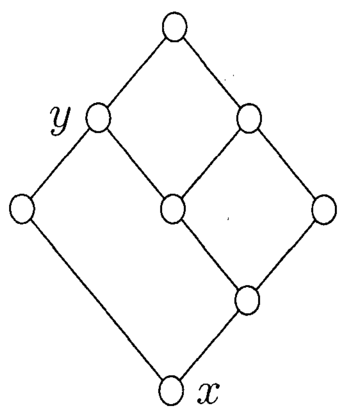


Figure 3.1 An example of a skeleton containing an interval which is not a skeleton.

A lattice L is a skeleton if $L \cong \tilde{L}$ via a lattice isomorphism $\sim$. Figure 3.1 presents an example of a skeleton having an interval $[x, y]$, which is not a skeleton.

CHAPTER 4

RESIDUATED APPROXIMATIONS AND UMBRAL MAPPINGS

4.1 Residuated Mappings

Let X, Y, Z be sets, we denote the **composition of functions** $f : X \to Y$ and $g : Y \to Z$ by $g \circ f$; so $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Let P, Q be posets, a mapping $f : P \to Q$ is **isotone** if it satisfies $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in P$; the mapping f is **antitone** if $x \leq y$ implies $f(y) \leq f(x)$ for all $x, y \in P$.

An isotone mapping $f : P \to Q$ is a **residuated mapping** iff there is an isotone mapping $g : Q \to P$ such that $(g \circ f)(p) \geq p$ for all $p \in P$ and $(f \circ g)(q) \leq q$ for all $q \in Q$. Such a mapping g is called a **residual mapping** corresponding to f. The following Lemma proves that such a g is uniquely determined by f.

Lemma 4.1.1. *Let $f : P \to Q$ be a residuated mapping between two posets, then the residual of f is unique.*

Proof. Let g and h be residual mappings corresponding to f. Then for all $q \in Q$, $g(q) \leq (h \circ f)(g(q)) = (h \circ f \circ g)(q) = h((f \circ g)(q)) \leq h(q)$ since h and g are residual mappings corresponding to f; so $g(q) \leq h(q)$ and, by symmetry, we have $h(q) \leq g(q)$. Hence $g(q) = h(q)$ for all $q \in Q$, i.e., $g = h$. □

The following lemma proves that the concept of being a residuated mapping is dual to the concept of being a residual mapping.

Lemma 4.1.2. *Let P, Q be two posets. If $f : P \to Q$ is a residuated mapping, then $f : P^* \to Q^*$ is a residual mapping, and vice verse.*

Proof. Suppose $f : P \to Q$ is a residuated mapping, then there is a residual mapping $g : Q \to P$ such that $(g \circ f)(p) \geq p$ holds for all $p \in P$ and $(f \circ g)(q) \leq q$ holds for all $q \in Q$, so $(g \circ f)(p) \leq^* p$ holds for all $p \in P^*$ and $(f \circ g)(q) \geq^* q$ holds for all $q \in Q^*$. The mapping g is isotone from Q^* to P^* since, for $q_1, q_2 \in Q^*$ with $q_1 \leq q_2$, $q_1 \geq q_2$ holds in Q and $g(q_1) \geq g(q_2)$ holds in P, thus $g(q_1) \leq g(q_2)$ holds in P^*. Hence f is a residual mapping from P^* to Q^*. Similarly if f is a residual mapping from P to Q, then f is a residuated mapping from P^* to Q^*. □

A residuated mapping f is also uniquely determined by the residual mapping g corresponding to f. Often we write $g := f^+$ and $f := g^-$, so $f = g^- = (f^+)^- := f^{+\,-}$ and $g = f^+ = (g^-)^+ := g^{-\,+}$. We write $Res(P, Q)$ for the set of all residuated mappings from P to Q and $Res^+(Q, P)$ for the set of all residual mappings from Q to P.

Let $f : P \to Q$ and $g : Q \to P$ be two antitone mappings between two posets P and Q. The pair (f, g) is a **Galois connection** iff $(g \circ f)(p) \geq p$ for all $p \in P$ and $(f \circ g)(q) \geq q$ for all $q \in Q$. We define $Gal(P, Q) :=$

$\{f \mid f : P \to Q$ and $g : Q \to P$ such that (f, g) is a Galois connection between P and $Q\}$. The following lemma shows the connection between Galois connections and residuated mappings.

Lemma 4.1.3. *Let P and Q be posets. Every isotone mapping $f : P \to Q$ is residuated iff the pair of mappings (f, f^+) is a Galois connections between P and Q^*, i.e. $Res(P, Q) = Gal(P, Q^*)$.*

Proof. Let $f \in Res(P, Q)$, then there exists an isotone $f^+ : Q \to P$ such that $(f^+ \circ f)(p) \geq p$ holds for all $p \in P$ and $(f \circ f^+)(q) \leq q$ holds for all $q \in Q$, thus $(f^+ \circ f)(p) \geq p$ holds in P and $(f \circ f^+)(q) \geq^* q$ holds in Q^*. The mapping $f : P \to Q^*$ is antitone since $f : P \to Q$ is isotone, and the mapping $f^+ : Q^* \to P$ is antitone since $f^+ : Q \to P$ is isotone. So $f \in Gal(P, Q^*)$. Hence $Res(P, Q) \subseteq Gal(P, Q^*)$. It is similar to prove that $Gal(P, Q^*) \subseteq Res(P, Q)$. Therefore $Res(P, Q) = Gal(P, Q^*)$. □

Information obtained regarding the computation of residuated mappings may be thought of as information obtained regarding the computation of Galois connection. In the sequel we consider only residuated mappings.

By Lemma 4.1.2 and Lemma 4.1.3 we have the following corollary.

Corollary 4.1.4. *Let P and Q be posets. Then $Res(P, Q) = Res^+(P^*, Q^*)$ and $Res^+(P^*, Q^*) = Gal(P, Q^*)$.*

Let $f : P \to Q$ be a residuated mapping between two posets with $q \in Q$. We define $f^{-1}(\downarrow q) := \{p \in P \mid f(p) \leq q\}$. The following lemma gives a necessary and sufficient condition that an isotone mapping f is residuated.

Lemma 4.1.5. *Given two posets P and Q, an isotone mapping $f : P \to Q$ is residuated iff, for any $q \in Q$, there exists a greatest element in $f^{-1}(\downarrow q)$.*

Proof. If f is residuated, then there is an isotone mapping $g : Q \to P$ such that (1) $g(f(p)) \geq p$ for all $p \in P$, (2) $f(g(q)) \leq q$ for all $q \in Q$. We have $g(q) \in f^{-1}(\downarrow q)$ for any $q \in Q$ by (2). Now let $z \in f^{-1}(\downarrow q)$, so that $f(z) \leq q$ and $z \leq g(f(z))$ by (1). Since g is isotone and $f(z) \leq q$, we have $g(f(z)) \leq g(q)$, so $z \leq g(f(z)) \leq g(q)$; therefore, $g(q)$ is the greatest element in $f^{-1}(\downarrow q)$.

Conversely assume that, for any $q \in Q$, there exists a greatest element in $f^{-1}(\downarrow q)$; call it $g(q)$ so that $g : Q \to P$. Then g is isotone since, for $q_1 \leq q_2$, it follows that $f^{-1}(\downarrow q_1) \subseteq f^{-1}(\downarrow q_2)$, so $g(q_1) \leq g(q_2)$. Since $g(q) \in f^{-1}(\downarrow q)$, $f(g(q)) \leq q$ holds. Now, let $p \in P$, then $f(p) \in Q$ and $g(f(p))$ is the greatest element in $f^{-1}(\downarrow f(p))$ by the definition of g; since p is in $f^{-1}(\downarrow f(p))$, we have $p \leq g(f(p))$. Hence f is residuated. □

For $f : L \to Q$ between two complete lattices, define

$$A_f(x) := \{q \in Q \mid x \leq \bigvee f^{-1}(\downarrow q)\}.$$

Lemma 4.1.6. *Let $f : P \to Q$ be a residuated mapping between two posets and I be an index set. If $\bigvee_{\alpha\in I} x_\alpha$ exists in P, then $\bigvee_{\alpha\in I} f(x_\alpha)$ exists in Q and $f(\bigvee_{\alpha\in I} x_\alpha) = \bigvee_{\alpha\in I} f(x_\alpha)$.*

Proof. Let $f : P \to Q$ be residuated. If $I = \emptyset$, then there exists 0 in P and $\bigvee_{\alpha\in I} x_\alpha = 0$; since f is isotone, we have $f(\bigvee_{\alpha\in I} x_\alpha) \geq \bigvee_{\alpha\in I} f(x_\alpha) = 0$ and there exists 0 in Q; moreover, $\{x_\alpha \mid \alpha \in I\} \subseteq f^{-1}(\downarrow 0)$ and $0 \in A_f(\bigvee_{\alpha\in I} x_\alpha) = A_f(0)$, so that $f(\bigvee_{\alpha\in I} x_\alpha) = f(0) = 0 = \bigvee_{\alpha\in I} f(x_\alpha)$. If $I \neq \emptyset$. We have $f(\bigvee_{\alpha\in I} x_\alpha) \geq f(x_\alpha)$ since f is isotone. Let $q \geq f(x_\alpha)$ for all $\alpha \in I$, then $f^+(q) \geq f^+(f(x_\alpha)) = (f^+ \circ f)(x_\alpha) \geq x_\alpha$ and therefore $f^+(q) \geq \bigvee_{\alpha\in I} x_\alpha$; hence $q \geq (f \circ f^+)(q) = f(f^+(q)) \geq f(\bigvee_{\alpha\in I} x_\alpha)$. Therefore $\bigvee_{\alpha\in I} f(x_\alpha)$ exists in Q and $f(\bigvee_{\alpha\in I} x_\alpha) = \bigvee_{\alpha\in I} f(x_\alpha)$. □

Let P be a poset and $S \subseteq P$, we denote $f(S) = \{f(s) \mid s \in S\}$.

The following corollary shows that an isotone mapping between two complete lattices is residuated iff it preserves all existing joins.

Corollary 4.1.7. *Let $f : P \to Q$ be an isotone mapping between two complete lattices P and Q. Then the mapping f is a residuated mapping iff $f(\bigvee_{\alpha\in I} x_\alpha) = \bigvee_{\alpha\in I} f(x_\alpha)$ for all index sets I with $x_\alpha \in P$.*

Proof. By the Lemma 4.1.6, if f is residuated, then $f(\bigvee_{\alpha\in I} x_\alpha) = \bigvee_{\alpha\in I} f(x_\alpha)$ for all subsets $\{x_\alpha \mid \alpha \in I\} \subseteq P$. Assume that $f(\bigvee_{\alpha\in I} x_\alpha) = \bigvee_{\alpha\in I} f(x_\alpha)$. Let $g(q) = \bigvee f^{-1}(\downarrow q)$ for $q \in Q$. If $f^{-1}(\downarrow q) = \emptyset$, then $f^{-1}(\downarrow q) \subseteq f^{-1}(\downarrow 0)$ and $g(q) = 0$, thus $f(0) = 0$. We may assume that $f^{-1}(\downarrow q) \neq \emptyset$. Then g is isotone since, for $q_1 \leq q_2$, $f^{-1}(\downarrow q_1) \subseteq f^{-1}(\downarrow q_2)$ holds, so $g(q_1) \leq g(q_2)$. For $p \in P$, we have $g(f(p)) = \bigvee f^{-1}(\downarrow f(p))$; since $p \in f^{-1}(\downarrow f(p))$, it follows $g(f(p)) = \bigvee f^{-1}(\downarrow f(p)) \geq p$. Now let $q \in Q$; then $f(g(q)) = f(\bigvee f^{-1}(\downarrow q)) = \bigvee f(f^{-1}(\downarrow q))$, and $f(z) \leq q$ holds for all $z \in f^{-1}(\downarrow q)$, so $f(g(q)) = \bigvee f(f^{-1}(\downarrow q)) \leq q$. Hence, f is residuated. □

4.2 Umbral Mappings

4.2.1 Umbral Number

In this section, let $f : L \to Q$ be a mapping between two complete lattices L and Q.

Define the mapping $f^{(+)} : Q \to L$ as follows; for $q \in Q$,

$$f^{(+)}(q) := \bigvee\{x \in L \mid f(x) \leq q\} = \bigvee f^{-1}(\downarrow q).$$

Define the mapping $f^{(-)} : Q \to L$ as follows; for $q \in Q$,

$$f^{(-)}(q) := \bigwedge\{x \in L \mid q \leq f(x)\} = \bigwedge f^{-1}(\uparrow q).$$

Define the mapping $\sigma_f : L \to Q$ as follows; for $x \in L$,

$$\sigma_f(x) := \bigwedge\{q \in Q \mid x \leq \bigvee f^{-1}(\downarrow q)\}.$$

The mapping σ_f is called the **shadow** of f in [2]. Note that

$$\begin{aligned}\sigma_f(x) &= \bigwedge A_f(x) = \bigwedge\{q \in Q \mid x \leq \bigvee f^{-1}(\downarrow q)\}\\ &= \bigwedge\{q \in Q \mid x \leq f^{(+)}(q)\}\\ &= (f^{(+)})^{(-)}(x) = f^{(+)(-)}(x).\end{aligned}$$

A mapping f may be not isotone, but $f^{(+)}$ and $f^{(-)}$ are isotone. Since, for $q_1, q_2 \in Q$ with $q_1 \leq q_2$, we have $\{x \in L \mid f(x) \leq q_1\} \subseteq \{x \in L \mid f(x) \leq q_2\}$, so that $f^{(+)}(q_1) = \bigvee\{x \in L \mid f(x) \leq q_1\} \leq \bigvee\{x \in L \mid f(x) \leq q_2\} = f^{(+)}(q_2)$. Hence, $f^{(+)}$ is isotone. Similarly, we can prove that $f^{(-)}$ is isotone.

The proofs of the following theorems in this section are given in [2] or [8].

Theorem 4.2.1. *Let $f : L \to Q$ be a mapping between two complete lattices.*

(1) *If f is residuated, then $f^{(+)}$ is the residual mapping f^+, i.e., $f^{(+)} = f^+$.*

(2) *If f is residual, then $f^{(-)}$ is the residuated mapping f^-, i.e., $f^{(-)} = f^-$.*

(3) *If $f^{(+)} = f^+$, then σ_f is residuated.*

(4) *If $f^{(-)} = f^-$, then σ_f is residuated.*

Proof. (1) The mapping $f^{(+)}$ for any mapping f is always isotone. Let $y \in L$, then $y \in \{x \in L \mid f(x) \leq f(y)\}$, so $y \leq \bigvee\{x \in L \mid f(x) \leq f(y)\} = f^{(+)}(f(y))$. Let $q \in Q$, then $f^{(+)}(q) = \bigvee\{x \in L \mid f(x) \leq q\}$. Let $q \in Q$. By Corollary 4.1.7 we have $f(f^{(+)}(q)) = f(\bigvee\{x \in L \mid f(x) \leq q\}) = \bigvee_{z\in\{x\in L|f(x)\leq q\}} f(z) = \bigvee_{f(z)\leq q} f(z) \leq q$. Hence $f^{(+)}$ is a residual mapping corresponding to f. Since f^+ is the unique residual of f by Lemma 4.1.1, we have $f^{(+)} = f^+$.

(2) This proof is dual to (1).

(3) Note that $\sigma_f(x) = \bigwedge A_f(x) = \bigwedge\{q \in Q \mid x \leq f^{(+)}(q)\}$. If $f^{(+)}$ is residual, then $A_f(x)$ has a least element q_x. Also $x \leq f^{(+)}(q_x) \leq f^{(+)}(q)$ holds for any $q \in A_f(x)$ since $f^{(+)}$ is isotone. Hence $q \in A_f(x) \Leftrightarrow q_x \leq q \Leftrightarrow x \leq f^{(+)}(q)$.

(4) This proof is similar to (3). □

Let $f : L \to Q$ and $g : L \to Q$ be two mappings between two posets. Define $f \leq g$ to mean that $f(x) \leq g(x)$ for each $x \in L$, sometimes we write $g \geq f$ for $f \leq g$. We say f is "dominated by" g if $f \leq g$.

Theorem 4.2.2. *For any mapping $f : L \to Q$ between two complete lattices L and Q, there is a largest residuated mapping ρ_f dominated by f.*

Proof. Let $H := \{h : L \to Q \mid h \le f$ and h is residuated$\}$. Then $H \neq \emptyset$ since there exists 0 in Q and the mapping g satisfying $g(x) = 0$ for all $x \in L$ is contained in H. Define $\rho_f(x) := \bigvee_{h \in H} h(x)$ for all $x \in L$. Since $h \le f$, we have $\rho_f = \bigvee_{h \in H} h \le f$.

We claim that ρ_f is residuated. Let $S \subseteq L$. Since $h(\bigvee S) = \bigvee_{s \in S} h(s)$ for any $h \in H$, we have $\rho_f(\bigvee S) = \bigvee_{h \in H} h(\bigvee S) = \bigvee_{h \in H} (\bigvee_{s \in S} h(s)) = \bigvee_{s \in S} (\bigvee_{h \in H} h(s)) = \bigvee_{s \in S} \rho_f(s)$. By Corollary 4.1.7 ρ_f is residuated. Since any residuated mapping dominated by f is less than or equal to ρ_f, the residuated mapping ρ_f is the largest residuated mapping dominated by f. □

Following the definition in [2], we call the ρ_f in the previous theorem the **residuated approximation** of f.

For any ordinal number α, we define the **umbral mapping** $\sigma_f^{(\alpha)}$ as follows:

$$\sigma_f^{(\alpha)}(x) := \begin{cases} f, & \alpha = 0, \\ \sigma_f^{(\alpha-1)}, & \text{for a successor ordinal } \alpha, \\ \bigwedge_{\beta<\alpha} \sigma_f^{(\beta)}, & \text{for a limit ordinal.} \end{cases}$$

In Theorem 4.2.3 we prove that, for any mapping $f : L \to Q$ between two complete lattices L and Q, $f \ge \sigma_f \ge \sigma_f^{(2)} \ge ... \ge \rho_f$ is true. Moreover, we prove, in Corollary 4.2.10, that there always exists a least ordinal α such that $\sigma_f^{(\alpha)} = \rho_f$. We call this α the **umbral number** u_f of f.

Theorem 4.2.3. *Let $f : L \to Q$ be a mapping between two complete lattices. Then*

(1) *σ_f is isotone;*

(2) *$\rho_f \le \sigma_f \le f$;*

(3) *if σ_f is residuated, then $\sigma_f = \rho_f$;*

(4) *if f is residuated, then $f = \sigma_f = \rho_f$.*

Proof. (1) Since $\sigma_f = (f^{(+)})^{(-)}$ is the composition of two isotone functions $f^{(+)}$ and $f^{(-)}$, σ_f is isotone.

(2) For $x \in L$, we have $\sigma_f(x) \le f(x)$ since $f(x) \in A_f(x)$. Let $q \in A_f(x)$, then $x \le \bigvee f^{-1}(\downarrow q)$; since ρ_f is a residuated mapping under f, we have $\rho_f(x) \le \rho_f(\bigvee f^{-1}(\downarrow q)) = \bigvee\limits_{w \in f^{-1}(\downarrow q)} \rho_f(w) = \bigvee\limits_{f(w) \le q} \rho_f(w) \le \bigvee\limits_{f(w) \le q} f(w) \le q$. Hence $\rho_f(x) \le \bigwedge A_f(x) = \sigma_f(x)$, so $\rho_f \le \sigma_f$.

(3) If σ_f is residuated, then $\sigma_f \le \rho_f$ by Theorem 4.2.2; and since $\rho_f \le \sigma_f$ by (2), we have $\sigma_f = \rho_f$.

(4) If f is residuated, then $f \le \rho_f$; and since $\rho_f \le f$ by (2), we have $f = \rho_f$. Therefore $f = \sigma_f = \rho_f$. □

Let Q^L be the set of all mappings from a complete lattice L to a complete lattice Q, i.e., $Q^L := \{f \mid f : L \to Q\}$. We define $u_{L,Q} := \bigvee\limits_{f \in Q^L} u_f$ and $u_L := \bigvee\{u_{L,Q} \mid Q \text{ is any complete lattice}\}$ if u_L exists.

Theorem 4.2.4. *Let L be a complete lattice. If L is completely distributive, then, for any complete lattice Q and any $f : L \to Q$, $f^{(+)}$ is residual and σ_f*

is residuated, i.e., $u_L = 1$.

Proof. Let $K \subseteq Q$. For any $q \in K$, $\{x \mid f(x) \leq \bigwedge K\} \subseteq \{x \mid f(x) \leq q\}$; i.e., $A_{\bigwedge K} \subseteq A_q$. Thus $\bigvee A_{\bigwedge K} \leq \bigvee A_q$. Since L is completely distributive,

$$\bigvee A_{\bigwedge K} \leq \bigwedge_{q \in K} (\bigvee A_q) = \bigvee_{m \in \prod_{q \in K} A_q} (\bigwedge_{q \in K} m(q)).$$

Let $m \in \prod_{q \in K} A_q$ and $q_0 \in K$. Then $m(q_0) \in A_{q_0}$ and $f(m(q_0)) \leq q_0$. Since $\bigwedge_{q \in K} m(q) \leq m(q_0)$, we have $f(\bigwedge_{q \in K} m(q)) \leq f(m(q_0)) \leq q_0$, thus $\bigwedge_{q \in K} m(q) \in A_{q_0}$ and $\bigwedge_{q \in K} m(q) \in \bigcap_{q \in K} A_q = A_{\bigwedge K}$, hence $\bigwedge_{q \in K} m(q) \leq \bigvee A_{\bigwedge K}$ and

$$\bigvee_{m \in \prod_{q \in K} A_q} (\bigwedge_{q \in K} m(q)) \leq \bigvee A_{\bigwedge K}.$$

Therefore, $\bigvee A_{\bigwedge K} = \bigwedge_{q \in K} (\bigvee A_q)$, i.e., $f^+(\bigwedge K) = \bigwedge_{q \in K} f^+(q)$. By Theorem 4.2.1 part (3) σ_f is residuated. $\square$

The proof of the following corollary is similar to the above proof.

Corollary 4.2.5. *Let* $f : L \to Q$ *be a mapping from a complete lattice* L *to a finite lattice* Q*. If* L *is infinitely distributive, then the mapping* $f^{(+)}$ *is residual.*

4.2.2 Some Insight into Umbral Mappings

Lemma 4.2.6. *Let* L *and* Q *be complete lattices. If* $f : L \to Q$ *is an isotone mapping with* $x \in L$*, then*

(1) $\uparrow f(x) \subseteq A_f(x)$ *and* $\sigma_f(x) = f(x) \wedge (\bigwedge(A_f(x) - \uparrow f(x)))$,

(2) $\sigma_f(x) < f(x)$ *implies* $A_f(x) - \uparrow f(x) \neq \emptyset$.

Proof. (1) Let $q \in \uparrow f(x)$, then $x \leq f^{(+)}(f(x)) \leq f^{(+)}(q)$ and $q \in A_f(x)$. Thus $\uparrow f(x) \subseteq A_f(x)$. Hence

$$\begin{aligned}\sigma_f(x) &= \bigwedge A_f(x)\\ &= \bigwedge((\uparrow f(x)) \cup (A_f(x) - \uparrow f(x)))\\ &= (\bigwedge(\uparrow f(x))) \wedge (\bigwedge(A_f(x) - \uparrow f(x)))\\ &= f(x) \wedge (\bigwedge(A_f(x) - \uparrow f(x))).\end{aligned}$$

(2) This can be deduced directly from (1). □

Lemma 4.2.7. *Let $f : L \to Q$ be an isotone mapping between two complete lattices L and Q.*

(1) *If $q \in A_f(x)$, then $\uparrow q \subseteq A_f(x)$.*

(2) *If $q \notin A_f(x)$, then $\downarrow q \cap A_f(x) = \emptyset$.*

(3) *If $L = (\downarrow a) \cup (\uparrow a)$ and $f(a) \nleq q$, then $f^{(+)}(q) \leq a$.*

Proof. (1) Let $q \in A_f(x)$. For $q_0 \in \uparrow q$, we have $x \leq f^{(+)}(q) \leq f^{(+)}(q_0)$ and $q_0 \in A_f(x)$. Hence $\uparrow q \subseteq A_f(x)$.

(2) Let $q \notin A_f(x)$, then $x \nleq f^{(+)}(q)$. If $q_0 \in \downarrow q$, then $f^{(+)}(q_0) \leq f^{(+)}(q)$ and $x \nleq f^{(+)}(q_0)$, so $q_0 \notin A_f(x)$. Hence $\downarrow q \cap A_f(x) = \emptyset$.

(3) Let $L = (\downarrow a) \cup (\uparrow a)$ and $f(a) \nleq q$. Suppose $f^{(+)}(q) \nleq a$. Then $a < f^{(+)}(q)$ holds since $L = (\downarrow a) \cup (\uparrow a)$. There exists $x_0 \in \{x \in L \mid f(x) \leq q\}$ such that $a < x_0$; otherwise, any element in $\{x \in L \mid f(x) \leq q\}$ is less than or equal to a, so $f^{(+)}(q) = \bigvee\{x \in L \mid f(x) \leq q\} \leq a$ contradicting $f^{(+)}(q) \nleq a$. Thus, $f(a) \leq f(x_0) \leq q$ since f is isotone, which contradicts $f(a) \nleq q$. □

Theorem 4.2.8. *Let $f : L \to Q$ be an isotone mapping between two complete lattices. Then f is residuated iff $\sigma_f(x) = f(x)$ for all $x \in J_c^{\sim}(L)$.*

Proof. By Corollary 4.1.7 we need only to show the sufficiency. We claim that $f(\bigvee A) = \bigvee_{a \in A} f(a)$ for any $A \subseteq L$. We have $f(\bigvee A) \geq \bigvee_{a \in A} f(a)$ since f is isotone. If $\bigvee A \in A$, we have $f(\bigvee A) = \bigvee_{a \in A} f(a)$ because $\bigvee A \in A$. If $\bigvee A \notin A$, then $\bigvee A \in J_c^{\sim}(L)$ and $f(\bigvee A) = \sigma_f(\bigvee A)$ by the hypothesis. Since $a \in f^{(-1)}(\bigvee_{a \in A} f(a))$ for all $a \in A$, it follows that $A \subseteq f^{(-1)}(\bigvee_{a \in A} f(a))$ and $\bigvee A \leq \bigvee f^{(-1)}(\bigvee_{a \in A} f(a))$, thus $\bigvee_{a \in A} f(a) \in A_f(\bigvee A)$, so $\sigma_f(\bigvee A) \leq \bigvee_{a \in A} f(a)$; hence, $f(\bigvee A) = \sigma_f(\bigvee A) \leq \bigvee_{a \in A} f(a)$. Theorefore, $f(\bigvee A) = \bigvee_{a \in A} f(a)$. □

The **length of a chain** consisting of $r+1$ elements, say $x_0 < x_1 < \ldots < x_r$, is r. The **height** of a poset P is the least cardinality which is greater than or equal to the length of any chain in P; we denote it by *height*(P).

Example 4.2.1. For the isotone mapping f in Figure 4.1, $\sigma_f^{(3)}$ is residuated, but $\sigma_f^{(2)}$ is not, so $u_f = 3 > 2 = \mathit{height}(Q)$. Thus, u_f is not necessarily bounded by the height of Q.

If $f : L \to Q$ and $a, b \in P$ with $a < b$, then the *restriction of f to the interval* $[a, b]$ is denoted by $f|_{[a,b]}$. Thus $\sigma_{f|_{[a,b]}} : [a, b] \to Q$ is the shadow of the function $f|_{[a,b]}$, and $\sigma_f|_{[a,b]}$ is the restriction of σ_f to the interval $[a, b]$. We now show that these functions may be different.

Example 4.2.2. For the isotone mapping f in Figure 4.2, $\sigma_{f|_{[a,b]}}$ is a one-to-one mapping while $\sigma_f|_{[a,b]}$ is the zero mapping, which is not one-to-one, so $\sigma_{f|_{[a,b]}} \neq \sigma_f|_{[a,b]}$.

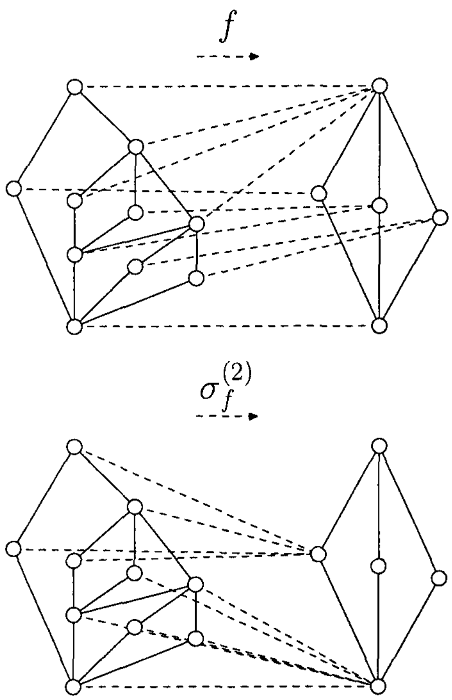


Figure 4.1 An isotone mapping f with $u_f > height(Q)$.

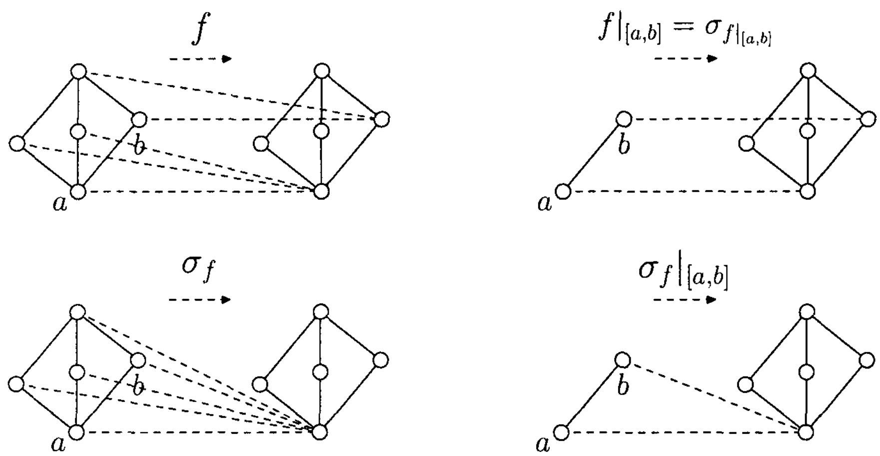


Figure 4.2 An example of a mapping f with $\sigma_{f|_{[a,b]}} \neq \sigma_f|_{[a,b]}$.

Let P and Q be disjoint lattices with $x, 1 \in P$ and $0, y \in Q$ such that $[x, 1]_P \cong [0, y]_Q$, and let $\varphi : [x, 1]_P \to [0, y]_Q$ be a lattice isomorphism. Then the **quasi vertical sum** of P and Q (over $[x, 1]$, via φ), denoted by the symbol $QVS(P, Q, [x, 1]_P, \varphi)$, or $QVS(P, Q)$ if $[x, 1]_P$ and φ are understood, is the disjoint union of $P - [x, 1]$ with Q and is partially ordered by the rule $u \leq v$ in $QVS(P, Q, [x, 1]_P)$ if

(1) $u, v \in P - [x, 1]_P$ with $u \leq_P v$, or

(2) $u \in P - [x, 1]_P$ and $v \in [0, y]_Q$ with $u \leq_P \varphi^{-1}(v)$, or

(3) $u, v \in Q$ with $u \leq_Q v$, or

(4) $u \in P - [x, 1]_P$, $v \in Q - [0, y]_Q$ and there exists $w \in [0, y]_Q$ such that $w \leq_Q v$ and $u \leq_P \varphi^{-1}(w)$.

The Hasse diagram for $QVS(P, Q, [x, 1]_P)$ is obtained by placing the diagram for Q over the one for P, overlapping $[x, 1]_P$ with $[0, y]_Q$.

If $x_P = 1_P$ (and, therefore, $y_Q = 0_Q$), then $QVS(P, Q, [x, 1]_P)$ is the quasi vertical sum of P and Q over $\{1_P\}$; we call it the **vertical sum** of P and Q, and denote it by $VS(P, Q)$. Thus, every vertical sum of lattices is a (trivial) quasi vertical sum of lattices. The Hasse diagram for $VS(P, Q)$ is obtained by placing the diagram for Q above the one for P, then removing 1_P from the resulting diagram. Loosely speaking, one may think of identifying 0_Q with 1_P and taking the transitive closure of $\leq_Q \cup \leq_P$ for the ordering on $VS(P, Q)$. The most important elementary fact about $VS(P, Q)$ is this: if $x \in P$ and $y \in Q$, then $x \leq y$ in $VS(P, Q)$.

Example 4.2.3. Let n be an integer greater than 0. Figure 4.3 presents a lattice L_n and an isotone mapping $f : L_n \to L_n$ with $u_f = n$, showing that the umbral number of an isotone mapping f might be any positive integer. We now define the lattice L_n inductively. The M_3 is the nondistributive lattice having five elements consisting of the bounds and three atoms a, b, c given in Figure 2.1, thus M_3 is a quintuple $\langle a, b, c, 0, 1\rangle$ such that a, b and c are both atoms and coatoms. Let P_i be a family of disjoint copies of M_3, let a_i be the element of P_i corresponding to a in M_3. Define L_1 to be P_1 and L_2 to be the quasi vertical sum of P_2 and L_1 over $[a_2, 1]_{P_2}$; there exists a lattice isomorphism φ such that $\varphi(a_2) = 0_{L_1}$ and $\varphi(1_{P_2}) = a_1$, i.e., $L_2 = QVS(P_2, L_1, [a_2, 1]_{P_2})$. Assuming that the lattice L_n has been defined, the lattice L_n is the quasi vertical sum of P_n and L_{n-1} over $[a_n, 1]_{P_n}$, i.e, $L_n = QVS(P_n, L_{n-1}, [a_n, 1]_{P_n})$. The bounds of L_n are a_{n+1} and a_0. For each i, $L_{i-1} \cong [a_i, a_0]$. For convenience, we regard L_{i-1} as the sublattice of L_i for $1 \leq i \leq n$, in the natural way $L_{i-1} = L_i|_{[a_i, a_0]}$.

The isotone mapping $f : L_n \to L_n$ is defined to be:

$$f(x) := \begin{cases} a_1, & \text{if } x \in \{a_0, a_1\}, \\ a_2, & \text{if } x \in (a_2, a_0) - \{a_1\}, \\ x, & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$\sigma_f(x) = \begin{cases} a_2, & \text{if } x \in \{a_0, a_1\}, \\ a_3, & \text{if } x \in (a_3, a_1) - \{a_2\}, \\ f(x), & \text{otherwise.} \end{cases}$$

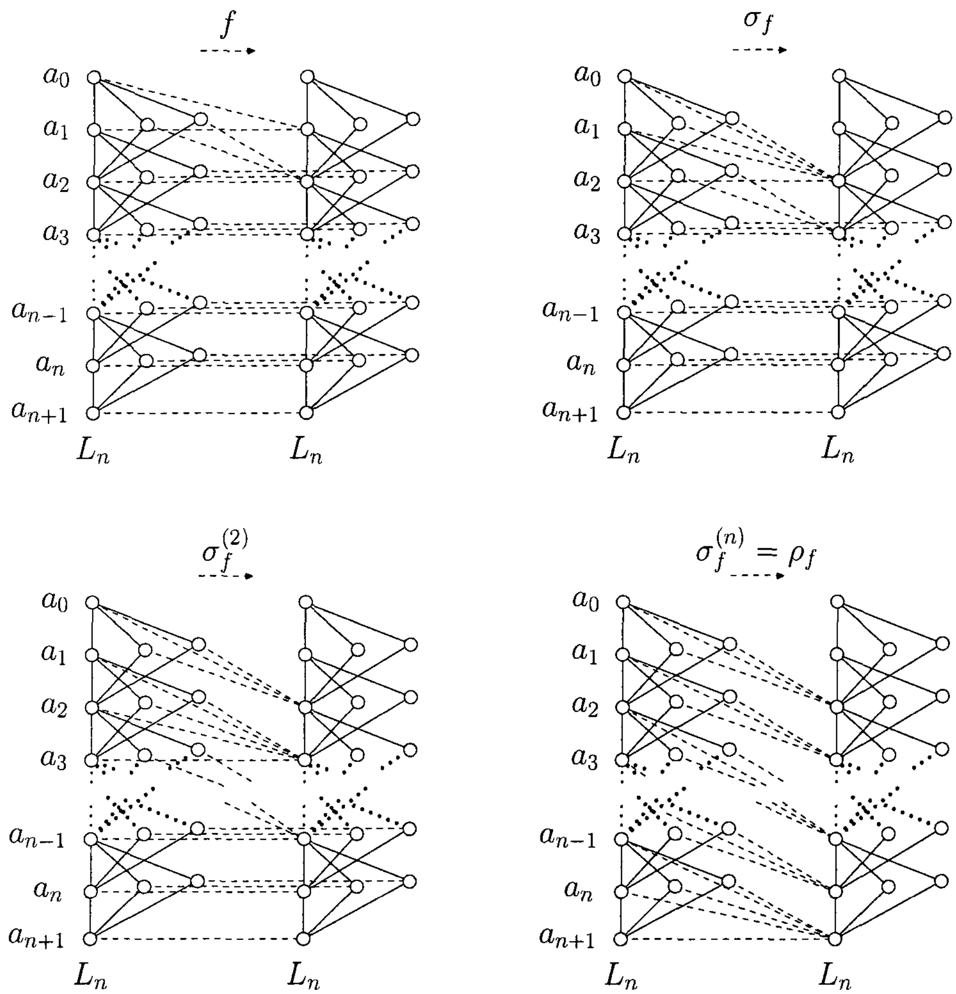


Figure 4.3 An isotone mapping $f : L_n \to L_n$ with $u_f = n$.

Iterating to obtain the umbral mapping $\sigma_f^{(i)}$, we have

$$\sigma_f^{(i)}(x) = \begin{cases} a_{i+1}, & \text{if } x \in \{a_{i-1}, a_i\}, \\ a_{i+2}, & \text{if } x \in (a_{i+2}, a_i) - \{a_{i+1}\}, \\ \sigma_f^{(i-1)}(x), & \text{otherwise.} \end{cases}$$

Finally,

$$\sigma_f^{(n)}(x) = \begin{cases} a_{n+1}, & \text{if } x \in [a_{n+1}, a_{n-1}], \\ \sigma_f^{(n-1)}(x), & \text{otherwise.} \end{cases}$$

It is easy to verify that $\sigma_f^{(n)}$ preserves joins, so that the mapping $\sigma_f^{(n)}$ is residuated, i.e., $\sigma_f^{(n)} = \rho_f$ and $u_f = n$.

Let $0 \notin \bigcup_{i=1}^{\infty} L_i$ and define the lattice $L_\omega := \{0\} \cup \bigcup_{i=1}^{\infty} L_i$, with the induced ordering (making 0 the bottom element). Since L_n is a sublattice of L_m if $n \leq m$, the function $f_n : L_n \to L_n$ induces a function $f : L_\omega \to L_\omega$ as follows: for $x \in L_\omega$, $f(x) = f_n(x)$ if $x \in L_n$; if $x \in L_n \cap L_m$ with $n \leq m$, then L_n is a sublattice of L_m, so $f_m(x) = f_n(x)$. The umbral number u_f might be ω.

It is an open question as to whether or not, for any ordinal number α, there exists a lattice M_α and a function $f : M_\alpha \to M_\alpha$ such that $u_f = \alpha$.

Example 4.2.4. Figure 4.4 gives another example to show that, for an isotone mapping $f : L \to Q$ from a lattice L of height 2 to a lattice Q, the umbral number u_f of f might be any positive integer. Let L be the horizontal sum of $2n$ ($n \geq 2$) chains of height 2 and $Q := \{0, 1\} \cup \{b_{i,1} \mid 1 \leq i \leq n-1\} \cup \{b_{i,j} \mid 1 \leq i \leq n-1, 2i \leq j < 4i-1\}$. Define $\leq$ on L to the following transitive closure of the relation

$$\leq_{Q_n} = \begin{cases} (b_{i,1}, b_{i+1,1}), & 0 \leq i \leq n-1 \\ (b_{i,1}, b_{i+1,j}), & 0 \leq i \leq n-1, 2i \leq j \leq 4i-1 \\ (b_{i,1}, b_{i-1,j}, & 1 \leq i \leq n, 2(i-1) \leq j \leq 4i-1) \\ (b_{i,j}, b_{i+1,j}, & 1 \leq i \leq n-2, 2i \leq j \leq 4i-1) \end{cases}$$

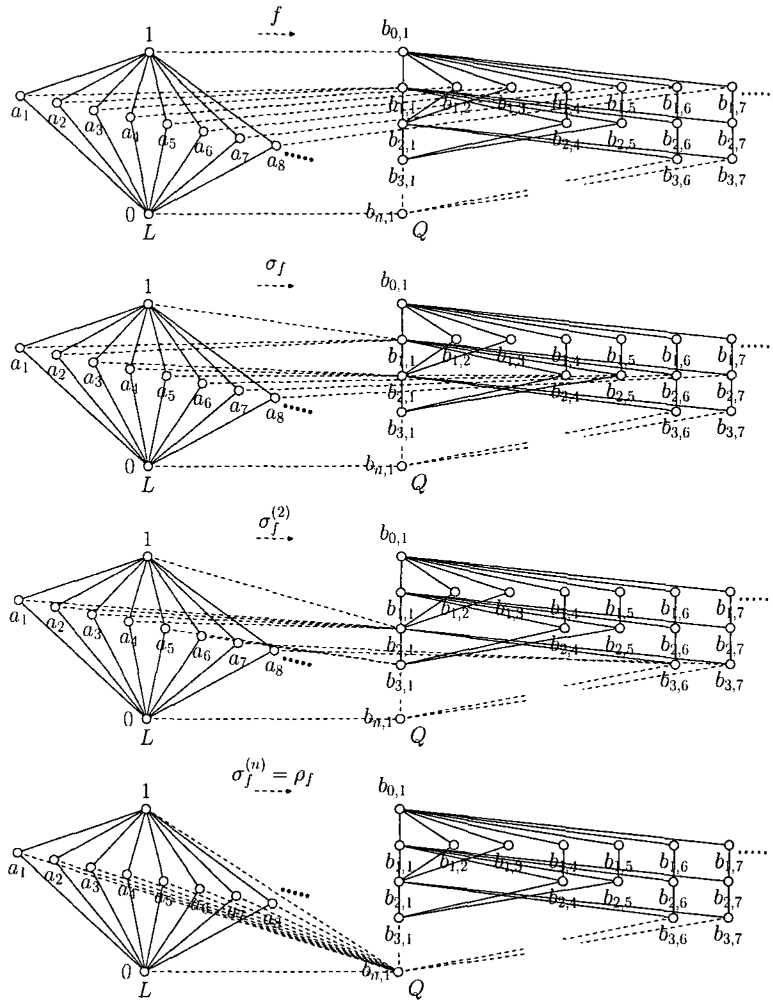


Figure 4.4 An isotone mapping f from a lattice L of height 2 to a lattice Q such that $u_f = n$.

The isotone mapping $f : L \to Q$ is defined to be:

$$f(x) := \begin{cases} b_{0,1}, & \text{if } x = 1, \\ b_{n,1}, & \text{if } x = 0, \\ b_{1,1}, & \text{if } x \in \{a_1, a_2\}, \\ b_{1,j}, & \text{if } x = a_{j+1},\ j \geq 2. \end{cases}$$

It is easy to verify that

$$\sigma_f(x) = \begin{cases} b_{1,1}, & \text{if } x \in \{1, a_1, a_2\}, \\ b_{0,1}, & \text{if } x = 0, \\ b_{2,1}, & \text{if } x \in \{a_3, a_4\}, \\ b_{2,j}, & \text{if } x = a_{j+1},\ j \geq 4. \end{cases}$$

Iterating to obtain the umbral mapping $\sigma_f^{(i)}$, we have

$$\sigma_f^{(i)}(x) = \begin{cases} b_{i,1}, & \text{if } x \in \{1, a_1, a_2, ..., a_{2i-1}, a_{2i}\}, \\ b_{0,1}, & \text{if } x = 0, \\ b_{i+1,1}, & \text{if } x \in \{a_{2i+1}, a_{2i+2}\}, \\ b_{i+1,j}, & \text{if } x = a_{j+1},\ j \geq 2i. \end{cases}$$

Finally, $\sigma_f^{(n)}(x) = 0$ for all $x \in L$, hence $u_f = n$.

Let $f : L \to Q$ be a mapping between two finite lattices. If $u_f > 2$, then the decreasing sequence $f > \sigma_f > \sigma_f^{(2)} > ... > \rho_f$ is finite. The image of any element decreases at most *height*(Q) times before it reaches 0; thus all elements are mapped to 0 in less than $|L| \times$ *height*(Q) iterations of umbral mapping. Hence $u_f \leq |L| \times$ *height*(Q).

Given two complete L and Q, the following lemma in [7] gives the upper bound of u_f for all $f : L \to Q$.

Lemma 4.2.9. *Let $f : L \to Q$ be any mapping between two complete lattices. Then $u_{L,Q} \leq |L||Q|$.*

Proof. Let $\mathcal{S} := \{\sigma_f^{(\alpha)} \subseteq L \times Q \mid \alpha$ is an ordinal number and $\sigma_f^{(\alpha)} \neq \sigma_f^{(\alpha+1)}\}$. Since $\sigma_f^{(\alpha)} \in Q^L$ for all ordinals number α, it follows that $|\mathcal{S}| \leq |Q|^{|L|}$ and $\mathcal{S}$ is a set. For any ordinal number α, there exists $x_\alpha \in L$ and $q_\alpha \in Q$ such that $\sigma_f^{(\alpha)}(x_\alpha) = q_\alpha > \sigma_f^{\alpha+1}(x_\alpha)$. Define $g : \mathcal{S} \to L \times Q$ by $g(\sigma_f^{(\alpha)}) := (x_\alpha, q_\alpha)$. We claim that g is one-to-one. Suppose that $\sigma_f^{(\alpha)}, \sigma_f^{(\beta)} \in \mathcal{S}$ and $g(\sigma_f^{(\alpha)}) = g(\sigma_f^{(\beta)})$, then $(x_\alpha, q_\alpha) = g(\sigma_f^{(\alpha)}) = g(\sigma_f^{(\beta)}) = (x_\beta, q_\beta)$, so $x_\alpha = x_\beta$ and $q_\alpha = q_\beta$. We claim that $\alpha = \beta$. Suppose not. We may assume that $\alpha < \beta$; since $\alpha + 1 \leq \beta$ and $x_\alpha = x_\beta$, we have $q_\beta = \sigma_f^{(\beta)}(x_\beta) = \sigma_f^{(\beta)}(x_\alpha) \leq \sigma_f^{(\alpha+1)}(x_\alpha) < \sigma_f^{(\alpha)}(x_\alpha) = q_\alpha$ contradicting $q_\alpha = q_\beta$. So $\alpha = \beta$ and $\sigma_f^{(\alpha)} = \sigma_f^{(\beta)}$. Therefore, g is one-to-one and $|\mathcal{S}| \leq |L||Q|$. □

Corollary 4.2.10. *Let $f : L \to Q$ be a mapping between two complete lattices. Then there exists an ordinal number α such that $\sigma_f^{(\alpha)} = \rho_f$. Hence the residuated approximation can always be calculated using the umbral mappings.*

4.3 Umbral Mappings Based on ∼ Finite Lattices

A lattice L is called a ∼ **finite lattice** if, for all $x \in L$, $[x]$ is a finite chain.

Lemma 4.3.1. *Let L be a ∼ finite lattice with $x \in L$.*

(1) *The subset $[x]$ is a finite chain.*

(2) *Both $\bigwedge[x]$ and $\bigvee[x]$ exist, and $\bigwedge[x], \bigvee[x] \in [x]$.*

(3) *If $x \in J_c^{\sim}(L)$, then $x = \bigwedge[x]$.*

(4) *If $x \in M_c^{\sim}(L)$, then $x = \bigvee[x]$.*

(5) *Let $A \subseteq L$. If $\bigvee A \notin A$, then, for any $u \in A$, there exists $v \in A$ such that $[u] \parallel [v]$ or $[u] < [v]$, i.e., there is no $w \in A$ such that $[a] \leq [w]$ for all $a \in A$.*

(6) *Let $A \subseteq L$. If $\bigvee A \notin A$, then $\bigvee_{a \in A}(\bigwedge[a]) = \bigvee_{a \in A}(\bigvee[a])$.*

Proof. (1) and (2) come directly from the definition of $\sim$ finite lattice.

(3) Let $x \in J_c^{\sim}(L)$. Then there exists $A \subseteq L$ such that $x = \bigvee A$ and $x \notin A$. By Lemma 3.1.5 part (9) we have $\downarrow x = [\bigwedge[x], x] \cup \downarrow(\bigwedge[x])$, so that $x = \bigvee A = \bigvee(A \cap \downarrow x) = \bigvee(A \cap([\bigwedge[x], x] \cup \downarrow(\bigwedge[x]))) = (\bigvee(A \cap [\bigwedge[x], x])) \vee (\bigvee(A \cap \downarrow(\bigwedge[x])))$. We claim that $A \cap [\bigwedge[x], x] = \emptyset$, otherwise it is a subset of a finite chain $[x]$, namely $A \cap [\bigwedge[x], x]$, and $x = \bigvee(A \cap [\bigwedge[x], x])$, thus $x \in A$ contradicting $x \notin A$. Hence $x = \bigvee(A \cap \downarrow(\bigwedge[x]))$ and $x \leq \bigwedge[x]$; also since $\bigwedge[x] \leq x$ is always true, it follows that $x = \bigwedge[x]$.

(4) This proof is dual to (3).

(5) Let $x = \bigvee A$. Suppose that there exists $w \in A$ such that $[a] \leq [w]$ for all $a \in A$. Then $a \leq \bigvee[w]$ for all $a \in A$, so $x = \bigvee A \leq \bigvee[w]$. Since $x = \bigvee A \notin A$, the element x is a completely join-reducible element; thus, $x = \bigwedge[x]$ by part (3). Since $w < x = \bigwedge[x]$ and $[w]$ is finite, we have $[w] < [x]$ and $\bigvee[w] < x$. Hence $x \leq \bigvee[w] < x$, a contradiction.

(6) We have $\bigvee_{a \in A}(\bigwedge[a]) \leq \bigvee A \leq \bigvee_{a \in A}(\bigvee[a])$ since $\bigwedge[a] \leq a \leq \bigvee[a]$ for all $a \in A$. Let $z = \bigvee_{a \in A}(\bigwedge[a])$. Since L is $\sim$ finite and $\bigwedge[a] \leq z$ for all $a \in A$, we have $[a] = [\bigwedge[a]] \leq [z]$ for all $a \in A$. Since there is no element w in A such that $[a] \leq [w]$ for all $a \in A$, but $[a] \leq [z]$ for all $a \in A$ by part (5), it follows

that $z \notin A$, so that $[a] < [z]$ for all $a \in A$, thus, $\bigvee[a] \leq z$ for all $a \in A$, so $\bigvee_{a\in A}(\bigvee[a]) \leq z = \bigvee_{a\in A}(\bigwedge[a])$. Hence $\bigvee A = \bigvee_{a\in A}(\bigwedge[a]) = \bigvee_{a\in A}(\bigvee[a])$. □

Let L be $\sim$ finite with $x \in L$. Then $\bigwedge[x]$ exists for all $x \in L$ since $[x]$ is finite. We define

$$x_o := \bigwedge[x].$$

The mapping β_o, defined by $\beta_o(x) := x_o$, is in $\prod_{[x]\in\tilde{L}}[x]$. The β_o-skeleton, $(L_o, \leq|_{L_o})$, of a $\sim$ finite lattice L is defined to be

$$L_o := \beta_o(\tilde{L}).$$

A nonempty subset M of a lattice L is **a join subcomplete sub-semilattice** (resp., **meet subcomplete sub-semilattice**) if it is closed under arbitrary existing joins (resp., meets) of nonempty subsets of M, as calculated in L, i.e., if $\emptyset \neq S \subseteq M$ and $\bigvee_L S$ (resp., $\bigwedge_L S$) exists, then $\bigvee_L S \in M$ (resp., $\bigwedge_L S \in M$). A nonempty subset of a lattice is a **subcomplete sublattice** if it is both a join subcomplete sub-semilattice and meet subcomplete sub-semilattice.

Theorem 4.3.2. *Let L be a $\sim$ finite lattice.*

(1) $J_c^\sim(L) = J_c^\sim(L_o)$.

(2) L_o *is a join subcomplete sub-semilattice of L.*

Proof. (1) Let $x \in J_c^\sim(L)$. Then there exists $A \subseteq L$ such that $x = \bigvee A \notin A$. By Lemma 4.3.1 part (6) $x = \bigvee_{a\in A} a_o$ and by Lemma 4.3.1 part (3) $x = x_o$, thus $x \in L_o$ and $a_o \in L_o$ for all $a \in A$. So x is the least upper bound of $\{ a_o \mid a \in A \}$ in L_o and $x \notin \{ a_o \mid a \in A \}$ since $a_o \leq a < x$ for all $a \in A$. Hence, $x \in J_c^\sim(L_o)$ and $J_c^\sim(L) \subseteq J_c^\sim(L_o)$.

Now let $y \in J_c^{\sim}(L_o)$. Then there exists $T \subseteq L_o$ such that $y = \bigvee_{L_o} T$ and $y \notin T$. Since $T \subseteq L$ and $y \in L$, y is a upper bound of T in L and $y \notin T$. Let z be any upper bound of T in L, then $[t] \leq [z]$ for all $t \in T$, so by Lemma 4.3.1 part (5) $[t] < [z]$ for all $t \in T$. Thus, $z_o \in L_o$ and $t < z_o$ for all $t \in T$. Since z_o is a upper bound of T in L_o and $y = \bigvee_{L_o} T$, we have $y \leq z_o \leq z$, so that y is the least upper bound of T in L. Hence, $y \in J_c^{\sim}(L)$ and $J_c^{\sim}(L_o) \subseteq J_c^{\sim}(L)$. Therefore, $J_c^{\sim}(L) = J_c^{\sim}(L_o)$.

(2) This follows from part (1). □

Let $f : L \to Q$ be a mapping from a $\sim$ finite lattice L to a complete lattice Q, define $f_o : L_o \to Q$ by

$$f_o := f|_{L_o}.$$

Lemma 4.3.3. *Let $f : L \to Q$ be an isotone mapping from a $\sim$ finite lattice L to a complete lattice Q, then $\sigma_{f_o}^{(\alpha)} = \sigma_f^{(\alpha)}|_{L_o}$ for any ordinal number α.*

Proof. If $\alpha = 0$, then $\sigma_{f_o}^{(0)} = f_o = f|_{L_o} = \sigma_f^{(0)}|_{L_o}$ by the definition of f_o.

Suppose that $\sigma_{f_o}^{(\beta)} = \sigma_f^{(\beta)}|_{L_o}$ for all $\beta < \alpha$. If α is a limit ordinal number, then $\sigma_{f_o}^{(\alpha)} = \bigwedge_{\beta<\alpha} \sigma_{f_o}^{(\beta)} = \bigwedge_{\beta<\alpha} (\sigma_f^{(\beta)}|_{L_o}) = \sigma_f^{(\alpha)}|_{L_o}$. If α is not a limit ordinal, then $\sigma_{f_o}^{(\alpha-1)} = \sigma_f^{(\alpha-1)}|_{L_o}$ holds for an ordinal α. We claim $\sigma_{f_o}^{(\alpha)} = \sigma_f^{(\alpha)}|_{L_o}$. Recall that, for all $x \in L$, $A_{\sigma_f^{(\alpha-1)}}(x) = \{q \in A \mid x \leq \bigvee(\sigma_f^{(\alpha-1)})^{-1}(\downarrow q)\}$ and, for all $x \in L_o$, $A_{\sigma_{f_o}^{(\alpha-1)}}(x) = \{q \in Q \mid x \leq \bigvee(\sigma_{f_o}^{(\alpha-1)})^{-1}(\downarrow q)\}$. Now fix $x \in L$ and let $y \in A_{\sigma_{f_o}^{(\alpha-1)}}(x)$. Then $x \leq \bigvee(\sigma_{f_o}^{(\alpha-1)})^{-1}(\downarrow y)$. Moreover, we have $x \leq \bigvee(\sigma_{f_o}^{(\alpha-1)})^{-1}(\downarrow y) \leq \bigvee(\sigma_f^{(\alpha-1)})^{-1}(\downarrow y)$ since $(\sigma_{f_o}^{(\alpha-1)})^{-1}(\downarrow y) \subseteq (\sigma_f^{(\alpha-1)})^{-1}(\downarrow y)$ holds, thus $y \in A_{\sigma_f^{(\alpha-1)}}(x)$ and $A_{\sigma_{f_o}^{(\alpha-1)}}(x) \subseteq A_{\sigma_f^{(\alpha-1)}}(x)$. Now let $z \in A_{\sigma_f^{(\alpha-1)}}(x)$

and $S := (\sigma_f^{(\alpha-1)})^{-1}(\downarrow z)$, then $x \le \bigvee(\sigma_f^{(\alpha-1)})^{-1}(\downarrow z) = \bigvee S$. If $\bigvee S \in S$, then $\sigma_{f_o}^{(\alpha-1)}(x) = \sigma_f^{(\alpha-1)}(x) \le \sigma_f^{(\alpha-1)}(\bigvee S) \le z$, thus $z \in A_{\sigma_{f_o}^{(\alpha-1)}}(x)$ by Lemma 4.2.6 part (1), so $A_{\sigma_f^{(\alpha-1)}}(x) \subseteq A_{\sigma_{f_o}^{(\alpha-1)}}(x)$. If $\bigvee S \notin S$, then $\bigvee S = \bigvee_{s \in S} s_o$ by Lemma 4.3.1 part (6); thus $\bigvee_{s \in S} \sigma_{f_o}^{(\alpha-1)}(s_o) \in A_{\sigma_{f_o}^{\alpha-1}}(x)$ and $\sigma_{f_o}^{(\alpha-1)}(s_o) = \sigma_f^{(\alpha-1)}(s_o) \le \sigma_f^{(\alpha-1)}(s) \le z$ for all $s \in S$. Hence $\bigvee_{s \in S} \sigma_{f_o}^{(\alpha-1)}(s_o) \le \bigvee_{s \in S} \sigma_f^{(\alpha-1)}(s) \le z$ and $z \in A_{\sigma_{f_o}^{(\alpha-1)}}(x)$ by Lemma 4.2.7 part (1). So $A_{\sigma_f^{(\alpha-1)}}(x) \subseteq A_{\sigma_{f_o}^{(\alpha-1)}}(x)$. Therefore, $A_{\sigma_f^{(\alpha-1)}}(x) = A_{\sigma_{f_o}^{(\alpha-1)}}(x)$ and $\sigma_{f_o}^{(\alpha)} = \sigma_f^{(\alpha)}|_{L_o}$. □

By the Lemma 4.3.3 we have the following conclusion.

Theorem 4.3.4. *Let $f : L \to Q$ be an isotone mapping from a $\sim$ finite lattice L to a complete lattice Q. Then*

(1) *$\sigma_{f_o}^{(\alpha)}$ is residuated iff $\sigma_f^{(\alpha)}$ is residuated for any ordinal number α,*

(2) $u_f = u_{f_o}$.

Proof. (1) By Theorem 4.2.8 $\sigma_{f_o}^{(\alpha)}$ is residuated iff $\sigma_{f_o}^{(\alpha+1)}(x) = \sigma_{f_o}^{(\alpha)}(x)$ for all $x \in J_c^{\sim}(L_o)$; we have $\sigma_f^{(\alpha+1)}(x) = \sigma_{f_o}^{(\alpha+1)}(x)$ and $\sigma_f^{(\alpha)}(x) = \sigma_{f_o}^{(\alpha)}(x)$ for all $x \in J_c^{\sim}(L)$ by Theorem 4.3.2 part (1) and Lemma 4.3.3, $\sigma_f^{(\alpha+1)}(x) = \sigma_f^{(\alpha)}(x)$ for all $x \in J_c^{\sim}(L)$ iff $\sigma_f^{(\alpha)}$ is residuated by Theorem 4.2.8. Hence, $\sigma_{f_o}^{(\alpha)}$ is residuated iff $\sigma_f^{(\alpha)}$ is residuated.

(2) This follows directly from part (1). □

Corollary 4.3.5. *Let $f : L \to Q$ be an isotone mapping from a $\sim$ finite lattice L to a complete lattice Q. If L_o is completely distributive, then σ_f is residuated.*

Proof. Since L is $\sim$ finite, L_o exists. Let $f : L \to Q$ be an isotone mapping and $f_o = f|_{L_o}$. Since L_o is completely distributive, the shadow σ_{f_o} is residuated by Theorem 4.2.4; thus, σ_f is residuated by Theorem 4.3.4. $\square$

Let L and Q be finite lattices. Then there are $|Q|^{|L|}$ different mappings $f : L \to Q$ from L to Q. In order to calculate $u_{L,Q}$, we need to check every mapping f; if f is isotone, then we iterate its umbral mappings until we obtain ρ_f, the iteration number is u_f. We use the constant c to stand for the average time for the calculation of one shadow, such a constant c depends on the program that calculates the shadow of f and the computer running the program. The total time $Total(u_{L,Q})$ to calculate $u_{L,Q}$ is approximately $Total(u_{L,Q}) = c * \sum_{f \in Q^L}^{|Q|^{|L|}} u_f$; let $f_o = f|_{L_o}$, then the total time $Total(u_{L_o,Q})$ to calculate $u_{L_o,Q}$ is $Total(u_{L_o,Q}) = c * \sum_{f_o \in Q^{L_o}}^{|Q|^{|L_o|}} u_{f_o}$. If $|L_o| < |L|$, then $Total(u_{L_o,Q}) \ll Total(u_{L,Q})$ yet $u_{L,Q} = u_{L_o,Q}$.

Theorem 4.3.6. *Let S be a subcomplete lattice of a complete lattice L and the mapping $f : S \to Q$ be an isotone mapping from S into a complete lattice Q with $f(0_S) = 0$. Then there exists an isotone mapping $\hat{f} : L \to Q$ such that $\sigma_{\hat{f}}^{(\alpha)}|_S = \sigma_f^{(\alpha)}$ for any ordinal number α.*

Proof. Define $x_S := \bigwedge((\uparrow x) \cap S)$ for $x \in L$. Then $x_S \in S$. For $x \in S$, $x_S = x$, we define $\hat{f}(x) := f(x_S)$. We claim that $\hat{f}$ is isotone. Let $x, y \in L$ with $x \le y$, then $(\uparrow y) \cap S \subseteq (\uparrow x) \cap S$, thus $x_S \le y_S$. Hence $\hat{f}(x) = f(x_S) \le f(y_S) = \hat{f}(y)$. We will prove that $\sigma_f^{(\alpha)} = \sigma_{\hat{f}}^{(\alpha)}|_S$ and $\sigma_f^{(\alpha)}(0_S) = 0$ for any ordinal number α.

Let $\alpha = 0$, then $\sigma_{\hat{f}}^{(0)}|_S = \hat{f}|_S = f = \sigma_f^{(0)}$ and $\hat{f}(0) = f(0) = 0$ by the hypothesis.

Suppose that $\sigma_{\hat{f}}^{(\beta)}|_S = \sigma_f^{(\beta)}$ and $\sigma_{\hat{f}}^{(\beta)}|_S(0) = \sigma_f^{(\beta)}(0) = 0$ for all $\beta < \alpha$. If α is a limit ordinal number, then $\sigma_{\hat{f}}^{(\alpha)}|_s = \bigwedge_{\beta<\alpha} (\sigma_{\hat{g}}^{(\beta)}|_S) = \bigwedge_{\beta<\alpha} \sigma_g^{(\beta)} = \sigma_f^{(\alpha)}$ and $\sigma_{\hat{f}}^{(\alpha)}|_s(0) = \sigma_f^{(\alpha)}(0) = 0$. If α is not a limit ordinal number, then $\sigma_{\hat{f}}^{(\alpha-1)}|_S = \sigma_f^{(\alpha-1)}$ and $\sigma_f^{(\alpha-1)}(0_S) = 0$. Obviously $\sigma_f^{(\alpha)}(0_S) = 0$. Let $g := \sigma_f^{(\alpha-1)}$ and $\hat{g} := \sigma_{\hat{f}}^{(\alpha)-1}$. Then $g^{-1}(\downarrow q)$ and $\hat{g}^{-1}(\downarrow q)$ for any $q \in Q$ are nonempty since $g(0_S) = 0$. If $\hat{g}(x) < g(1_S)$, then $(\uparrow x) \cap S \neq \emptyset$, thus $x \leq \bigwedge((\uparrow x) \cap S) = x_S$. If $q < g(1_S)$, then

$$\begin{aligned}
\bigvee \hat{g}^{-1}(\downarrow q) &= \bigvee \{x \in L \mid \hat{g}(x) = \sigma_{\hat{f}}^{(\alpha)-1}(x) = \sigma_f^{\alpha-1}(x_S) = g(x_S) \leq q\} \\
&= \bigvee \{x_S \in S \mid g(x_S) \leq q\} \\
&= \bigvee g^{-1}(\downarrow q).
\end{aligned}$$

we have that, for $x \in S$,

$$\begin{aligned}
\sigma_{\hat{f}}^{(\alpha)}(x) &= \sigma_{\hat{g}}(x) \\
&= \bigwedge \{q \in Q \mid x \leq \bigvee \hat{g}^{-1}(\downarrow q)\} \\
&= \hat{g}(x) \wedge \bigwedge \{q < \hat{g}(1) \mid \hat{g} \not\leq q \text{ and } x \leq \bigvee \hat{g}^{-1}(\downarrow q)\} \\
&= g(x_S) \wedge \bigwedge \{q < g(1_S) \mid g(x_S) \not\leq q \text{ and } x_S \leq \bigvee g^{-1}(\downarrow q)\} \\
&= \bigwedge \{q \in Q \mid x_S \leq \bigvee g^{-1}(\downarrow q)\} \\
&= \sigma_g(x_S) = \sigma_f^{(\alpha)}(x_S).
\end{aligned}$$

Therefore, $\sigma_{\hat{f}}^{(\alpha)}|_S = \sigma_f^{(\alpha)}$ and $\sigma_{\hat{f}}^{(\alpha)}|_S(0) = \sigma_f^{(\alpha)}(0)$ for any ordinal α. $\square$

A **pentagon** is a quintuple $\langle a, b, c, u, v\rangle$ such that $a, b, c, u, v \in L$, $c \wedge a = v$, $c \vee b = u$ and $v < b < a < u$. Let $\mathscr{F} := \{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}$ be the lattices presented in Figure 4.5.

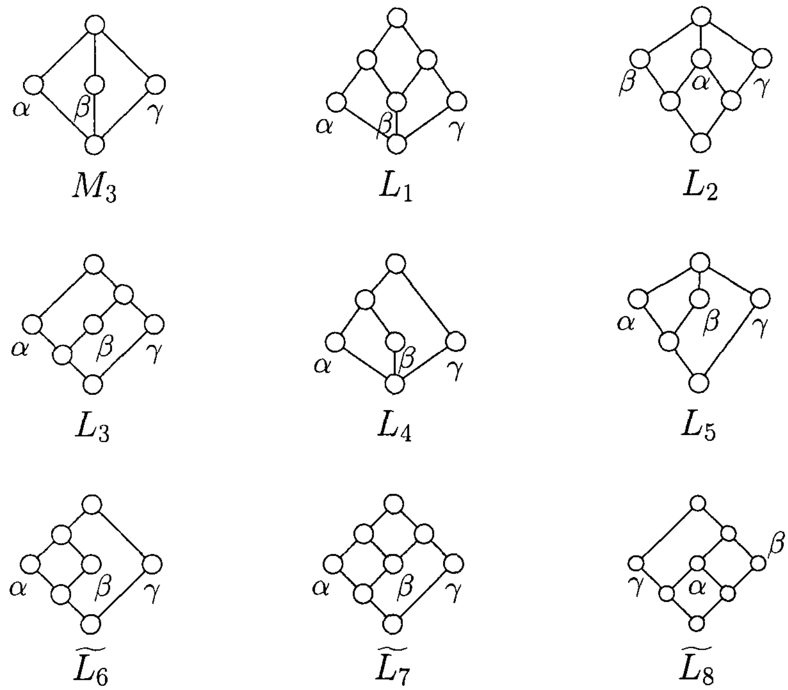


Figure 4.5 Non-distributive lattices.

Lemma 4.3.7. *Let L be a lattice. If σ_f is residuated for any isotone mapping $f : L \to N_5$, then L does not contain a sublattice isomorphic to a lattice in $\mathscr{F}$.*

Proof. Suppose that L contains a sublattice S isomorphic to a lattice in $\mathscr{F}$. Use the labeling in $\mathscr{F}$, note that $S = \{0, 1, \alpha, \beta, \gamma, \alpha \vee \beta, \beta \vee \gamma, \alpha \wedge \beta, \beta \wedge \gamma\}$ where $0, 1, \alpha, \beta, \gamma$ are distinct and fixed while others might not be distinct. Let $Q := \{u, v, w, 1_Q, 0_Q\}$ be a pentagon.

Let $f_S : S \to Q$ be the following mapping,

$$f_S(x) = \begin{cases} 0_Q, & \text{if } x \in \{0, \alpha \wedge \beta\}, \\ 1_Q, & \text{if } x \in \{1, \alpha \vee \beta, \beta \vee \gamma\}, \\ v, & \text{if } x = \alpha, \\ w, & \text{if } x = \beta, \\ u, & \text{if } x = \gamma, \\ v, & \text{if } x = \alpha \wedge \gamma > 0. \end{cases}$$

It is easy to verify that f_S is isotone and

$$\bigvee f_S^{-1}(\downarrow q) = \begin{cases} \alpha \wedge \beta, & \text{if } q = 0_Q, \\ 1, & \text{if } q \in \{u, 1_Q\}, \\ \alpha, & \text{if } q = v, \\ \beta, & \text{if } q = w. \end{cases}$$

Thus, $\sigma_{f_S}(\alpha) = \bigwedge\{v, u, 1_Q\} = v$ and $\sigma_{f_S}(\beta) = \bigwedge\{u, w, 1_Q\} = 0$, but $\sigma_{f_S}(\alpha \vee \beta) = u$. It follows that $\sigma_{f_S}(\alpha \vee \beta) = u \neq v \vee 0 = \sigma_{f_S}(\alpha) \vee \sigma_{f_S}(\beta)$. By Theorem 4.3.6, there is an isotone mapping $f : L \to Q$ such that $\sigma_f(\alpha \vee \beta) = \sigma_{f_S}(\alpha \vee \beta) \neq \sigma_{f_S}(\alpha) \vee \sigma_{f_S}(\beta) = \sigma_f(\alpha) \vee \sigma_f(\beta)$. Hence σ_f is not residuated. □

Lemma 4.3.8. *Let $f : L \to N_5$ be a mapping from a $\sim$finite lattice L into N_5. If σ_f is residuated for any isotone mapping f, then L has no sublattice isomorphic to a lattice in $\mathscr{F}$.*

Proof. Suppose that L has sublattice S isomorphic to a lattice in $\mathscr{F}$. By Lemma 4.3.7 there exists $g : S \to N_5$ such that σ_g is not residuated, by theorem 4.3.6 there exists $f : L \to N_5$ with $f|_S = g$ such that σ_f is not residuated. □

The following lemma is given in [16].

Lemma 4.3.9. *Let L be a lattice. The following statements are equivalent.*

(1) *no sublattice of L is isomorphic to a lattice in* $\mathscr{F}$.

(2) $\tilde{L}$ *is distributive.*

Specializing to the case in which L is a lattice with no infinite chains, we have the following theorem.

Theorem 4.3.10. *Let L be a lattice with no infinite chains. The following statements are equivalent.*

(1) σ_f *is residuated for any complete lattice Q and isotone mapping* $f : L \to Q$.

(2) *L has no sublattice isomorphic to a lattice in* $\mathscr{F}$.

(3) $\tilde{L}$ *is distributive.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 4.3.8. (2) $\Rightarrow$ (3) follows from Lemma 4.3.9. We only need to prove (3) $\Rightarrow$ (1). Let $f : L \to Q$ be an isotone mapping from L to a complete lattice Q. Let $\tilde{L}$ be distributive. Since L has no infinite chains, $\tilde{L}$ has no infinite chains and, for any $S \subseteq \tilde{L}$, there exists a finite subset A of $\tilde{L}$ such that $\bigvee S = \bigvee A$. Thus, $\tilde{L}$ is distributive implies $\tilde{L}$ is completely distributive. Since $L_o \cong \tilde{L}$, it follows that L_o is completely distributive. Thus, σ_{f_o} is residual by Theorem 4.2.4. Hence σ_f is residuated by Theorem 4.3.4. □

Let $\{L_i\}_{i\in I}$ be a collection of pairwise disjoint bounded lattice such that $|L_i| > 3$ and $|I| > 1$. Let $P = \bigcup_{i\in I}(L_i - \{0_i, 1_i\})$ be partially ordered by the rule $x \leq y$ iff there exists $i \in I$ such that $\{x, y\} \in L_i - \{0_i, 1_i\}$ with $x \leq y$ in L_i. In [1], the **horizontal sum** of $\{L_i\}_{i\in I}$ is the lattice formed by adjoining a largest element 1 and a least element 0 to P and is denoted $L = HS(L_i : i \in I)$.

Some elementary facts about horizontal sum of $\{L_i\}_{i\in I}$ are:

(1) $|C_0| > 1$ and $|C^1| > 1$.

(2) If $x \in L_i - \{0_i, 1_i\}$ and $y \in L_j - \{0_j, 1_j\}$ with $i \neq j$, then $x \parallel y$ in $HS(L_i : i \in I)$.

(3) $0 \not\prec 1$.

The lattice L in Figure 4.6 is not $\sim$ finite. The skeleton $\tilde{L}$ is distributive does not imply that σ_f for an isotone mapping f is residuted. In this example, $[a, b]$ is an interval by $\mathbb{N} \oplus 1$ and Q is the horizontal sum of a chain with 3 elements and $\mathbb{N} \oplus 1$. In this figure, $\mathbb{N}$ is denoted by the dotted lines and the mappings are depicted by the dashed lines; any integer i_L in $[a, b]$ is mapped to the corresponding integer i_Q of Q by the isotone mapping f. Note that $\tilde{L}$ is isomorphic to $\mathbf{3}^2$ which is a distributive lattice of width 3 and 9 elements, but σ_f is not residuated because $\bigvee_{i_L\in\mathbb{N}} \sigma_f(i_L) = 0 < \sigma_f(\bigvee \mathbb{N})$.

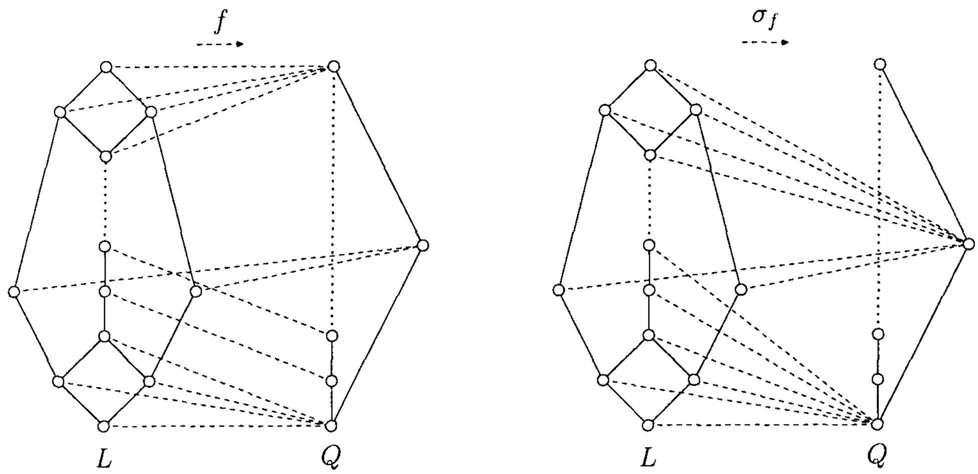


Figure 4.6 An example such that $\tilde{L}$ is distributive, but σ_f is not residuated.

In [8] it is proven that if $f : L \to Q$ is an isotone mapping from an infinitely distributive lattice L to a finite lattice Q, then σ_f is residuated. But σ_f might not be residuated if L is not infinitely distributive. Let L_1 be the subcomplete lattice of $\langle \mathbb{N}_0; \text{lcm}, \text{gcd} \rangle$ generated by $\{2^i \mid i \in \mathbb{N}\} \cup \{3^j \mid j \in \mathbb{N}\}$, let Q be the horizontal sum of three-element chain and $\mathbb{N} \oplus 1$ (the non-negative integers starting from 0 with a new top element 1). The lattice Q is drawn in the codomain of the function f in Figure 4.6. Let z be the unique atom in the three-element chain. Define the mapping f as follows:

$$f(x) := \begin{cases} 0_Q, & \text{if } x = 0_L, \\ z, & \text{if } x = 2^i, i \in \mathbb{N}, \\ j, & \text{if } x = 3^j, j \in \mathbb{N}, \\ 1_Q, & \text{otherwise.} \end{cases}$$

Note that, $\sigma_f(1_{L_1}) = z, \sigma_f(3^j) = 0$ and $\bigvee_{j \in \mathbb{N}} 3^j = 1_{L_1}$. Thus, $\sigma_f^{-1}(0)$ is not a principle ideal in L_1 and σ_f is not residuated.

The L_1 in the previous example provides an example showing that the converse of Lemma 4.3.8 does not hold. The complete lattice L_1 is a $\sim$ finite lattice and no sublattice of L_1 is isomorphic to a lattice in $\mathscr{F}$. But σ_f is not residuated.

Discussion

A complete lattice L is a **bi-skeletonizable** lattice if $[x] = [\bigwedge[x], \bigvee[x]]$ for all $x \in L$. Note that, for a bi-skeletonizable lattice L, L_o is a join sub-complete sub-semilattice of L, but $J_c^{\sim}(L_o) \subset J_c^{\sim}(L)$ might happen.

Let $f : L \rightarrow Q$ be an isotone mapping from a bi-skeletonizable lattice L to a complete lattice Q, $u_{L,Q} \neq u_{L_o,Q}$ might happen. We know that if $\tilde{L}$ is completely distributive and L is $\sim$ finite, then $u_{L,Q} \leq 1$.

(Question 1) Is $u_{L,Q} \leq 2$ true if $\tilde{L}$ is completely distributive and bi-skeletonizable?

Let $f : L \rightarrow Q$ be an isotone mapping between any two complete lattices.

(Question 2) Does $u_{L,Q} \leq Min(|\tilde{L}|, |\tilde{Q}|)$ hold?

CHAPTER 5

FINITE LATTICES

5.1 Introduction

Our initial investigation focused on the calculation of the residuated approximation of a mapping between two complete lattices. In order to develop the theory, we concentrated on lattices of small widths, in particular lattices of width 2 and, to a lesser extent, width 3. After obtaining our results on residuated approximations, we returned to develop a theory of lattices of small widths. We give a complete description of the structure of lattices of width 2 in the next chapter. This description, while it has not yet yielded a complete description of lattices of width 3, has provided insight on the structure of lattices of larger width.

An **antichain** S is a subset of a poset P such that $a \parallel b$ holds for any two distinct elements $a, b \in S$. The set of antichains of a poset P is denoted by π_P. We define

$$\pi_P^k := \{\{x_1, \ldots, x_k\} \mid x_1, \ldots, x_k \in P \text{ and } x_i \parallel x_j \text{ for } 1 \le i < j \le k\},$$

$$\pi_P := \bigcup_{k \ge 1} \pi_P^k.$$

The **width**, denoted $w(P)$, of a poset P is the least cardinality which is greater than or equal to the cardinality of any antichain of P. Obviously any poset of width 1 is a chain.

Theorem 5.1.1. ***(Dilworth's Chain Decomposition Theorem*** [6])

Let P be a finite poset of width w. Then P is a union of w chains.

This theorem is known to be true for any poset of width w [14], but we shall apply it only to posets of finite width.

Corollary 5.1.2. *A poset P of width w is a union of w disjoint chains.*

Proof. Let P be a poset of width 1. Then P is a chain.

Assume that any poset of width k is a union of k disjoint chains. Let P be a poset of width $k + 1$. We take C_0 as a maximal chain in P, then $P - C_0$ is a poset of width k. By assumption $P - C_0$ is a union of k disjoint chains C_i $(1 \leq i \leq k)$, i.e., $P - C_0 = \bigcup_{1 \leq i \leq k} C_i$. Since C_0 is disjoint with other k chains C_i, P is a union of $k + 1$ disjoint chains C_i $(0 \leq i \leq k)$, i.e., $P = \bigcup_{0 \leq i \leq k} C_i$. □

The following definitions and lemmas are given in [9]. The proofs of the first two lemmas are straightforward and we omit them.

Lemma 5.1.3. *If $\mathfrak{S}$ is a nonempty set of ideals of a finite poset P, then $\bigcup \mathfrak{S}$ and $\bigcap \mathfrak{S}$ are ideals of P.*

Lemma 5.1.4. *Let $\mathfrak{F}$ be a family of sets such that, for $A, B \in \mathfrak{F}$, $A \cap B, A \cup B \in \mathfrak{F}$. Then $(\mathfrak{F}, \subseteq)$ is a distributive lattice in which the operations $\vee$ and $\wedge$ coincide with $\cup$ and $\cap$, respectively.*

Lemma 5.1.5. *Let $\mathfrak{I}$ be the set of all ideals of a finite poset $(P, \leq)$. Then $(\mathfrak{I}, \subseteq)$ is a distributive lattice.*

Proof. This follows immediately from Lemma 5.1.3 and Lemma 5.1.4. □

Recall that π_P is the set of all antichains of the poset P. Note that for any $\emptyset \neq S \subseteq P$, if $Max(S) \neq \emptyset$, then $Max(S) \in \pi_P$.

Definition 5.1.1. *Let P be a finite poset. For $A, B \in \pi_P$, $A \sqsubseteq B$ iff $\downarrow A \subseteq \downarrow B$, i.e., for every element a in A, there exists $b \in B$ such that $a \leq b$. For $A, B \in \pi_P$, we define $A \sqcap B := Max((\downarrow A) \cap (\downarrow B))$ and $A \sqcup B := Max(A \cup B)$. Thus, $A \sqcap B$ and $A \sqcup B$ are again antichains in P, since P is finite.*

We now prove that π_P is a lattice and the operations $\sqcup$ and $\sqcap$ are the join and meet operations of π_P.

Lemma 5.1.6. *Let P be a finite poset. Then*

(1) *$\sqsubseteq$ is a partial order relation on the set π_P;*

(2) *$(\pi_P, \sqcup, \sqcap)$ is a lattice.*

Proof. (1) The reflexivity and transitivity follow directly from the definition of $\sqsubseteq$. We only need to prove it is antisymmetric.

Let $A, B \in \pi_P$, $A \sqsubseteq B$ and $B \sqsubseteq A$. For any $a_1 \in A$, there is a $b \in B$ such that $a_1 \leq b$; also for such a b, there is $a_2 \in A$ such that $b \leq a_2$. Thus, $a_1 \leq b \leq a_2$. Since $a_1, a_2 \in A$ and $A \in \pi_P$, we have $a_1 = a_2 = b$, so $A \subseteq B$. By symmetry, $B \subseteq A$. Hence, $A = B$.

(2) Let $A, B \in \pi_P$, then $Max(A \cup B) \in \pi_P$ and $\downarrow A, \downarrow B \subseteq \downarrow Max(A \cup B)$, thus $A \sqsubseteq Max(A \cup B)$ and $B \sqsubseteq Max(A \cup B)$. Hence, $Max(A \cup B)$ is a upper bound of A and B in π_P. Suppose that Y is a upper bound of A and B in π_P, so that $A \sqsubseteq Y$ and $B \sqsubseteq Y$. Then $\downarrow A \subseteq \downarrow Y$ and $\downarrow B \subseteq \downarrow Y$, thus

$(\downarrow A) \cup (\downarrow B) \subseteq \downarrow Y$. Since P is finite, we have $(\downarrow A) \cup (\downarrow B) = \downarrow Max(A \cup B)$. So $Max(A \cup B) \sqsubseteq Y$. Therefore, $Max(A \cup B)$ is the least upper bound of A and B, i.e., $A \sqcup B$ is the least upper bound of A and B in π_P.

Let $A, B \in \pi_P$, then $Max((\downarrow A) \cap (\downarrow B)) \in \pi_P$, $\downarrow (Max((\downarrow A) \cap (\downarrow B))) \subseteq \downarrow A$ and $\downarrow (Max((\downarrow A) \cap (\downarrow B))) \subseteq \downarrow B$, thus $Max((\downarrow A) \cap (\downarrow B)) \sqsubseteq A$ and $Max((\downarrow A) \cap (\downarrow B)) \sqsubseteq B$. Hence $Max((\downarrow A) \cap (\downarrow B))$ is a lower bound of A and B in π_P. Suppose that X is any lower bound of A and B in π_P. Then $\downarrow X \subseteq \downarrow A$ and $\downarrow X \subseteq \downarrow B$, thus, $\downarrow X \subseteq (\downarrow A) \cap (\downarrow B)$. Since P is finite, we have $(\downarrow A) \cap (\downarrow B) = \downarrow (Max((\downarrow A) \cap (\downarrow B)))$. So $X \sqsubseteq Max((\downarrow A) \cap (\downarrow B))$. Therefore, $Max((\downarrow A) \cap (\downarrow B))$ is the greatest lower bound of A and B, i.e., $A \sqcap B$ is the greatest lower bound of A and B. □

The following lemma shows the relation between ideals and antichains.

Lemma 5.1.7. *Let $(P, \leq)$ be a poset, $\Im$ its set of ideals and π_P its set of antichains. Define $f : \pi_P \to \Im$ to be $f(A) := \downarrow A$ for $A \in \pi_P$. Then f is an injection of π_P into $\Im$.*

Proof. Note that $\downarrow A \in \Im$ for any nonempty subset A of P. Let $B, C \in \pi_P$ and $f(B) = f(C)$, then $\downarrow B = \downarrow C$ so $B \sqsubseteq C$ and $C \sqsubseteq B$. Since $\sqsubseteq$ is antisymmetric, we have $B = C$. Therefore, f is an injection of π_P into $\Im$. □

We now prove that, if P is finite, then the function f in Lemma 5.1.7 is a bijection.

Lemma 5.1.8. *If P is a finite poset, then the f in Lemma 5.1.7 is an order isomorphism from $(\pi_P, \sqsubseteq)$ onto $(\Im, \subseteq)$.*

Proof. We proved that f is an injection in Lemma 5.1.7. To see that f preserves order, note that, for $X, Y \in \pi_P$, $X \sqsubseteq Y$ implies that $\downarrow X \subseteq \downarrow Y$, i.e., $f(X) \subseteq f(Y)$. We need to prove f is surjective.

Let $I \in \Im$ and $A = Max(I)$, then A is an antichain. We claim that $I = \downarrow A$. Since P is finite, any element of I is less than or equal to a maximal element of I, so $I \subseteq \downarrow A$; since $A \subseteq I$ and every element of $\downarrow A$ is less than or equal to an element of A, we have $\downarrow A \subseteq I$ by the definition of ideal. Hence $I = \downarrow A = f(A)$. It follows that f is surjective. □

If P is not finite, then the function f in Lemma 5.1.7 might not be surjection. For example, P is the set of rational numbers with usual ordering and π_P is the set of one-element subsets of P; thus, π_P is countable, but the set $\Im$ is equipotent with the set of real numbers, hence, $|\Im| > |\pi_P|$.

Lemma 5.1.9. *If P is a finite poset, then $(\pi_P, \sqsubseteq)$ is a distributive lattice with smallest element $\emptyset$ and largest element $Max(P)$.*

Proof. This follows from Lemma 5.1.4 and Lemma 5.1.8. □

A poset $(P, \leq)$ of width w is a union of w disjoint chains $C_1, \ldots, C_w$ by the Corollary to Dilworth's Theorem. Recall that π_P^w is the set of antichains of the poset P which have w elements. If $A \in \pi_P^w$, then A has exactly one element a_i in common with each chain C_i for $i = 1, \ldots, w$. We now prove that π_P^w is a sublattice of the lattice π_P.

Lemma 5.1.10. *Let P be a poset of width w and P be a union of chains $C_1, \ldots, C_w$. If $A = \{a_1, \ldots, a_w\} \in \pi_P^w$ and $B = \{b_1, \ldots, b_w\} \in \pi_P^w$, where $a_i, b_i \in C_i$*

for $i = 1, \ldots, w$, then

(1) $A \sqsubseteq B \Leftrightarrow a_i \leq b_i$ *for* $i = 1, \ldots, w$;

(2) *let* $s_i = Max(\{a_i, b_i\})$ *and* $l_i = Min(\{a_i, b_i\})$ *for* $1 \leq i \leq w$, *and let* $S = \{s_i \mid 1 \leq i \leq w\}$ *and* $L = \{l_i \mid 1 \leq i \leq w\}$, *then* $S, L \in \pi_P^w$;

(3) *furthermore,* $A \sqcup B = S$ *and* $A \sqcap B = L$ *so that* π_P^w *is a sublattice of* π_P.

Proof. (1) Assume that $A \sqsubseteq B$. Let i be a fixed element of $\{1, \ldots, w\}$. Then there exists b_j such that $a_i \leq b_j$. If $j = i$, then we are done. If $j \neq i$, then $a_i \leq b_j$. We have $a_i \not\parallel b_i$ since $a_i, b_i \in C_i$. We claim that $a_i \leq b_i$, otherwise $b_i < a_i \leq b_j$, a contradiction. Hence $a_i \leq b_i$ for all $i \in \{1, \ldots, w\}$. The proof of $\Leftarrow$ is immediate from the definition of $\sqsubseteq$.

(2) We now prove that $S \in \pi_P$. Suppose not. There exist $i, j \in \{1, \ldots, w\}$ with $i \neq j$ and $s_i \leq s_j$. The following four cases are the possible cases, since $s_k = Max(\{a_k, b_k\})$ for $k \in \{1, 2, \ldots, w\}$.

Case 1. $s_i = a_i, s_j = a_j$. Thus, $a_i = s_i \leq s_j = a_j$ contradicting $A \in \pi_P^w$.

Case 2. $s_i = b_i, s_j = b_j$. Thus, $b_i = s_i \leq s_j = b_j$ contradicting $B \in \pi_P^w$.

Case 3. $s_i = a_i, s_j = b_j$. Thus, $b_i \leq a_i$ and $a_i = s_i \leq s_j = b_j$, so that $b_i \leq b_j$, contradicting $B \in \pi_P^w$.

Case 4. $s_i = b_i, s_j = a_j$. Thus, $a_i \leq b_i$ and $b_i = s_i \leq s_j = a_j$, so that $a_i \leq a_j$, contradicting $A \in \pi_P^w$.

The assumption that $s_i \leq s_j$ leads to a contradiction, since we arrive at a contradiction in each case. Therefore, $S \in \pi_P$ and also $|S| = w$, i.e., $S \in \pi_P^w$.

The proof of $L \in \pi_P^w$ is dual to that of $S \in \pi_P^w$.

(3) Since $a_i, b_i \leq s_i$ for all $i \in \{1, ..., w\}$, we have $A \sqsubseteq S$ and $B \sqsubseteq S$, thus S is a upper bound of A and B. Let $U := \{u_i \mid 1 \leq i \leq w\} \in \pi_P^w$ with $A, B \sqsubseteq U$. Since P is a union of the w chains $C_1, ..., C_w$ and, for each i, $a_i, b_i, u_i \in C_i$, it follows that $a_i \not\parallel b_i$, $a_i \leq u_i$ and $b_i \leq u_i$, thus $s_i = Max(\{a_i, b_i\}) \leq u_i$. Hence, $S \sqsubseteq U$. Therefore, $A \sqcup B = S$ and $A \sqcup B \in \pi_P^w$. Dually $A \sqcap B \in \pi_P^w$ and $A \sqcap B = L$. So π_P^w is a sublattice of π_P. □

If P is finite, then π_P^w is a finite lattice with bounds $\bigsqcup(\pi_P^w)$ and $\bigsqcap(\pi_P^w)$; moreover, π_P^w is a distributive lattice since π_P is a distributive lattice.

5.2 Significant Intervals and Components in Finite Lattices

For $X \in \pi_L$, let $i_X := [\bigwedge X, \bigvee X]$. An interval $[a, b]$ is ***k*-determined** if there is $X \in \pi_L^k$ such that $[a, b] = i_X$; X is called a ***k*-determinant** of $[a, b]$. Given $[a, b] \in L$ with $a < b$, it is possible that $[a, b] = i_X = i_Y$ where $X \in \pi_L^k$, $Y \in \pi_L^j$ with $j \neq k$, so an interval can have different determinants of different sizes. For example, the boolean lattice with 3 atoms, which we denote as $\mathbf{2}^3$, has the set of the three atoms as a 3-determinant, the set of three coatoms as a 3-determinant and $\{x, x'\}$ as a 2-determinant for any x such that $0 < x < 1$, where x' is the complement of x. Not all intervals have a k-determinant for some k, e.g. a chain with more than 2 elements has no k-determinant for any k.

For $2 \leq k \leq w$, $\mathscr{D}_k(L) := \{i_X \mid X \in \pi_L^k\}$. The set $\mathscr{D}_k$ under $\subseteq$ is a finite poset. A k-determined interval $[a, b]$ is in $Min(\mathscr{D}_k(L))$ (resp., $Max(\mathscr{D}_k(L))$) if

$[c, d] \subseteq [a, b]$ (resp., $[a, b] \subseteq [c, d]$) for $[c, d] \in \mathscr{D}_k(L)$ implies $[c, d] = [a, b]$.

Define

$$\mathscr{D} := \bigcup_{2 \leq k \leq w} \mathscr{D}_k(L).$$

An element x in L is a **nodal** element iff $L = (\uparrow x) \cup (\downarrow x)$. It is easy to see that x is nodal iff $\pi(x) = \emptyset$. An interval $[a, b]$ of L is a **nodal interval** if $a < b$ and $[a, b]$ consists only of nodal elements of L.

Note that

(1) every nodal interval has more than 1 element;

(2) any nodal interval is an autonomous chain;

(3) there is no determinant for a nodal interval, i.e., for a nodal interval $[a, b]$, there is no k such that $X \in \pi_L^k$ and $[a, b] = i_X$;

(4) for a lattice L, no nodal interval exists in $\tilde{L}$.

Lemma 5.2.1. *Let L be a lattice.*

(1) *If* $[x, y], [y, z]$ *are two nodal intervals in L, then* $[x, z] = [x, y] \cup [y, z]$ *is a nodal interval in L.*

(2) *If* $[x, y], [x, z]$ *are two nodal intervals in L, then* $[x, y \vee z] = [x, y] \cup [x, z]$ *is a nodal interval in L and* $y \vee z \in \{y, z\}$.

(3) *If* $[y, x], [z, x]$ *are two nodal intervals in L, then* $[y \wedge z, x] = [y, x] \cup [z, x]$ *is a nodal interval in L and* $y \wedge z \in \{y, z\}$.

(4) *If* $[x, y]$ *is a nodal interval in L and* $z \in [x, y]$, *then* $[x, z]$ *and* $[z, y]$ *are nodal intervals.*

Proof. (1) Obviously $[x, y] \cup [y, z] \subseteq [x, z]$. Let $w \in [x, z]$. Since y is a nodal element, we have $w \leq y$ or $y \leq w$; thus, $w \in [x,y] \cup [y,z]$, so $[x,z] \subseteq [x,y] \cup [y,z]$ and w is a nodal element. Hence $[x,z] = [x,y] \cup [y,z]$ and $[x,z]$ is a nodal interval.

(2) Since y is a nodal element, we have $y \leq z$ or $z \leq y$. We may assume that $y \leq z$. Then $[x,y] \subseteq [x,z]$. Hence, $[x,y] \cup [x,z] = [x,z] = [x, y \vee z]$.

(3) This is dual to (2).

(4) This statement follows directly from the definition of nodal interval.

□

Definition 5.2.1. *Let L be a finite lattice of width w. We now define the notion of a significant interval. We first define a w-significant interval, then a j-significant interval with* $1 < j < w$, *after that a 1-significant interval.*

An interval i_X *is a w-**significant interval** if* $i_X \in Min(\mathscr{D}_w(L))$.

Assume that the k-significant intervals have been defined for all k with $j < k \leq w$. *An interval* i_X *is a j-**significant interval** with* $1 < j < w$ *if* $i_X \in Min(\mathscr{D}_j(L))$ *and there is no k-significant interval* i_Y *with* $j < k \leq w$ *such that* $i_X \subseteq i_Y$.

*An interval is a **1-significant interval** if it is a maximal nodal interval.*

*An interval is a **significant interval** if it is a k-significant interval for some integer k in* $[1, w]$.

For $1 \leq k \leq w$, *let* $\mathscr{S}\mathscr{I}_k(L)$ *be the set of k-significant intervals of L, and let* $\mathscr{S}\mathscr{I}(L)$ *be the set of significant intervals of L, i.e,*

$$\mathscr{S}\mathscr{I}_k(L) := \{i_X \mid i_X \text{ is a k-significant interval }\},$$

$$\mathcal{SI}(L) := \bigcup_{1 \le k \le w} \mathcal{SI}_k(L) = \text{the set of all significant intervals of } L.$$

Note that

(1) for $X, Y \in \pi_L$ with $i_X \in \mathcal{SI}_{|X|}(L)$ and $i_Y \in \mathcal{SI}_{|Y|}(L)$, if $|X| = |Y|$, then $i_X = i_Y$; if $|X| < |Y|$, then $i_X \not\subseteq i_Y$;

(2) for $i_X \in \mathcal{SI}_{|X|}(L)$ with $X \in \pi_L$ and $|X| > 1$, if there is an interval i_Y such that $i_Y \subset i_X$, then $w(i_Y) \le w(i_X) - 1$, since the i_X is a minimal interval in $\mathcal{D}_{|X|}(L)$;

(3) for any $i_X = [\bigwedge X, \bigvee X]$ with $X \in \pi_L$ and $|X| > 1$, $\bigwedge X \in M^{\sim}(L)$ and $\bigvee X \in J^{\sim}(L)$ hold.

The next lemma follows from the minimality of a j-significant interval.

Lemma 5.2.2. *Let L be a finite lattice and i_X be a j-significant interval with $X \in \pi_L^j$ and $j > 1$.*

(1) *If $i_Y \subseteq i_X$ with $Y \in \pi_L^j$, then $i_Y = i_X$.*

(2) *If $z \in (\bigwedge X, \bigvee X)$, then z is not a nodal element.*

Proof. (1) Since a j-significant interval i_X does not contain any j-determined interval as a proper sublattice by the definition of significant interval, $i_Y \subseteq i_X$ with $Y \in \pi_L^j$ implies $i_Y = i_X$.

(2) Suppose that there is a nodal element z in $(\bigwedge X, \bigvee X)$. Then either $X \subseteq \uparrow z$ or $X \subseteq \downarrow z$, since $X \in \pi_L$. We may assume that $X \subseteq \uparrow z$. Thus, $\bigvee X \le z < \bigvee X$. Since we arrive at a contradiction, there is no such z. □

We now prove that the width of a significant interval $i_X \in \mathcal{SI}_{|X|}(L)$ is $|X|$ and that the width of the intersection of distinct significant intervals is less than the width of either of them.

Theorem 5.2.3. *Let L be a finite lattice of width w with $w > 1$.*

(1) *Let $X \in \pi_L$ with $2 \leq |X| \leq w$ and $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$, then $w(i_X) = |X|$.*

(2) *Let $X, Y \in \pi_L$ with $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$ and $i_Y \in \mathscr{S}\mathscr{I}_{|Y|}(L)$, and let $2 \leq |X|, |Y|$. If $w(i_X \cap i_Y) = w(i_X)$ or $w(i_Y)$, then $i_X = i_Y$. Alternatively, if $i_X \neq i_Y$, then $w(i_X \cap i_Y) \leq Min(|X| - 1, |Y| - 1)$, i.e., $w(i_X \cap i_Y) \neq w(i_X)$ and $w(i_X \cap i_Y) \neq w(i_Y)$.*

Proof. (1) Let $X \in \pi_L$ with $2 \leq |X| \leq w$ and $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$, so $w(i_X) \geq |X|$. We claim that $w(i_X) \not> |X|$. Suppose not, then $w(i_X) > |X|$ and there exists $S \in \pi_L$ such that $S \subseteq i_X$ and $|S| > |X|$, thus $i_S \subseteq i_X$ and $i_S \in \mathscr{D}_{|S|}(L)$. There exists $i_T \subseteq i_S$ such that $i_T \in Min(\mathscr{D}_{|S|}(L))$ since L is finite. Let $Y \subseteq T$ and $Y \in \pi_L^{|X|}$. Then $|Y| = |X|$ and $i_Y \subseteq i_T \subseteq i_S \subseteq i_X$; and since $i_Y \in \mathscr{D}_{|X|}(L)$ and $i_X \in Min(\mathscr{D}_{|X|}(L))$, we have $i_Y = i_T = i_X$. Since $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$, there is no k-significant interval i_Z with $|X| < k$ such that $i_X \subseteq i_Z$; and since $i_X = i_T$, there is no k-significant interval i_Z with $|X| < k$ such that $i_T \subseteq i_X$; since $|T| = |S| > |X|$, there is no k-significant interval i_Z with $|T| < k$ such that $i_T \subseteq i_Z$. Hence, $i_T \in \mathscr{S}\mathscr{I}_{|S|}(L)$ with $|S| > |X|$ and $i_T = i_X$, contradicting $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$. Therefore, $w(i_X) = |X|$.

(2) Let $X, Y \in \pi_L$ with $i_X \in \mathscr{S}\mathscr{I}_{|X|}(L)$, $i_Y \in \mathscr{S}\mathscr{I}_{|Y|}(L)$ and $w(i_X \cap i_Y) = w(i_X)$. We claim that $i_X = i_Y$. Suppose $i_X \neq i_Y$. Since $i_X, i_Y \in \mathscr{S}\mathscr{I}(L)$, we have $i_X \not\subseteq i_Y$ and $i_X \cap i_Y \subset i_X$. Since $w(i_X \cap i_Y) = w(i_X)$, there exists $Z \subseteq i_X \cap i_Y$ and $Z \in \pi_L^{|X|}$, so $Z \subseteq i_X$ and $i_Z \subseteq i_X$; similarly $Z \subseteq i_Y$ and $i_Z \subseteq i_Y$.

Hence $i_Z \subseteq i_X \cap i_Y \subset i_X$ and $i_Z \in \mathscr{D}_{|X|}(L)$, contradicting $i_X \in Min(\mathscr{D}_{|X|}(L))$. Hence, $i_X = i_Y$. Similarly, if $w(i_X \cap i_Y) = w(i_Y)$, then $i_X = i_Y$. □

The following lemma and corollary indicate that there exist a least j-determinant and a largest j-determinant in a j-significant interval $(2 \leq j)$.

Lemma 5.2.4. *Let $j > 1$ and let i_X be a j-significant interval in a finite lattice L. If Y, Z are j-determinants of i_X, then $Y \sqcap Z$ and $Y \sqcup Z$ are j-determinants of i_X.*

Proof. By applying Lemma 5.1.10 part (2) and (3) with $P = i_X$, we have $Y \sqcap Z \subseteq i_X$ and $Y \sqcap Z \in \pi_L^j$; since i_X is a j-significant interval, we have $i_{Y \sqcap Z} = i_X$. Hence $Y \sqcap Z$ is a j-determinant of i_X. Similarly, $Y \sqcup Z$ is also a j-determinant of i_X. □

Corollary 5.2.5. *Let $j > 1$ and let i_X be a j-significant interval in a finite lattice L, then there exist a least j-determinant and a largest j-determinant of i_X.*

Proof. Since L is finite, the interval i_X is finite and there are only finitely many j-determinants of i_X. By Lemma 5.2.4, if $\{X_1, \ldots, X_n\}$ is the set of all j-determinants of the j-significant interval i_X, then $X_1 \sqcap X_2 \sqcap \ldots \sqcap X_n$ is the least j-determinant of i_X and $X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n$ is the largest j-determinant of i_X in the ordering $\sqsubseteq$. □

The following lemmas show the relation between two w-significant intervals.

Lemma 5.2.6. *Let L be a lattice of width w with $w > 1$ and let $X, Y \in \pi_L^w$ with $i_X, i_Y \in \mathscr{S}\mathscr{I}_w(L)$.*

(1) $(\bigwedge X) \parallel (\bigwedge Y)$ *iff* $(\bigvee X) \parallel (\bigvee Y)$,

(2) $\bigwedge X < \bigwedge Y$ *iff* $\bigvee X < \bigvee Y$,

(3) $\bigwedge X = \bigwedge Y$ *iff* $\bigvee X = \bigvee Y$.

Proof. (1) Let $(\bigwedge X) \parallel (\bigwedge Y)$. We claim that $(\bigvee X) \parallel (\bigvee Y)$. Suppose not, we may assume that $(\bigvee X) \leq (\bigvee Y)$. Since $X, Y \sqsubseteq X \sqcup Y$, by Lemma 5.1.10 $X \sqcup Y \in \pi_L^w$ and we have $\bigwedge Y < (\bigwedge X) \vee (\bigwedge Y) \leq \bigwedge(X \sqcup Y)$ and $\bigvee Y = (\bigvee X) \vee (\bigvee Y) \leq \bigvee(X \sqcup Y)$. Let $z \in X \sqcup Y$, then $z \in X$ or $z \in Y$, thus $z \leq \bigvee X$ or $z \leq \bigvee Y$; hence $z \leq \bigvee Y$ since $\bigvee X \leq \bigvee Y$. It follows $\bigvee(X \sqcup Y) \leq \bigvee Y$. Hence $\bigvee(X \sqcup Y) = \bigvee Y$. Since $\bigwedge Y < \bigwedge X \sqcup Y \leq \bigvee X \sqcup Y = \bigvee Y$, we have $\bigwedge Y \in i_Y - i_{X \sqcup Y}$ so that $i_{X \sqcup Y} \subset i_Y$, contradicting the fact that i_Y is a w-significant interval. The proof of the converse is dual to the above proof.

(2) Suppose $\bigwedge X < \bigwedge Y$. Since $(\bigwedge X) \nparallel (\bigwedge Y)$, we have $(\bigvee X) \nparallel (\bigvee Y)$ by part (1). If $\bigvee X \not< \bigvee Y$, then $\bigvee Y \leq \bigvee X$; since $\bigwedge X \notin i_Y$ and $\bigwedge X \in i_X$, we have $i_Y = [\bigwedge Y, \bigvee Y] \subset [\bigwedge X, \bigvee X] = i_X$, contradicting the fact that i_X is a w-significant interval. Hence $\bigvee X < \bigvee Y$. The proof of the converse is dual to the above proof.

(3) Suppose $\bigwedge X = \bigwedge Y$. By (1) $(\bigvee X) \nparallel (\bigvee Y)$ since $(\bigwedge X) \nparallel (\bigwedge Y)$; and by (2) $\bigvee X \not< \bigvee Y$ since $\bigwedge X \not< \bigwedge Y$; similarly $\bigvee Y \not< \bigvee X$. Hence $\bigvee X = \bigvee Y$.

The proof of the converse is dual to the above proof. $\square$

For two significant intervals i_X and i_Y in a finite lattice of width $w > 1$, we define

$$i_X \leq i_Y \text{ iff } \bigwedge X \leq \bigwedge Y;$$

$$i_X \parallel i_Y \text{ iff } \bigwedge X \parallel \bigwedge Y.$$

Lemma 5.2.7. *Let L be a finite lattice of width w with $w > 1$ and let i_X, i_Y be two w-significant intervals in L with $X, Y \in \pi_L^w$. If $i_X \parallel i_Y$, then*

(1) $\bigwedge(X \sqcup Y) < \bigvee(X \sqcap Y)$,

(2) $(\bigwedge X) \vee (\bigwedge Y) \in i_{X \sqcap Y}$ *and* $(\bigvee X) \vee (\bigvee Y) \in i_{X \sqcup Y}$.

Proof. (1) By Lemma 5.1.10 part (2) we have $X \sqcap Y, X \sqcup Y \in \pi_L^w$, so $\bigvee(X \sqcap Y)$ is not parallel with all elements in $X \sqcup Y$. Since $i_X \parallel i_Y$, it follows that $X \not\sqsubseteq Y$ and $Y \not\sqsubseteq X$, otherwise, $X \sqsubseteq Y$ or $Y \sqsubseteq X$, thus $\bigwedge X \leq \bigwedge Y$ or $\bigwedge Y \leq \bigwedge X$ by Lemma 5.1.10 part (1), i.e., $i_X \nparallel i_Y$, contradicting $i_X \parallel i_Y$. Since $w = |X| = |X \sqcap Y|$ also holds, we have $X \sqcap Y \not\subseteq X$ and $X \sqcap Y \not\subseteq Y$, thus there exist $u \in X \cap (X \sqcap Y)$ and $v \in Y \cap (X \sqcap Y)$ such that $u, v < \bigvee(X \sqcap Y)$ since $X \sqcap Y \in \pi_L$. Also, since $X \sqcup Y \subseteq X \cup Y$, $u \in X$, $v \in Y$ and $X, Y \in \pi_L$, we have $\bigvee(X \sqcap Y) \nless z$ for any $z \in X \sqcup Y$. Also, since $X \sqcup Y \in \pi_L^w$ and $w(L) = w$, $\bigvee(X \sqcap Y)$ is not parallel with some $s \in X \sqcup Y$, so $s \leq \bigvee(X \sqcap Y)$. Hence, $\bigwedge(X \sqcup Y) < s \leq \bigvee(X \sqcap Y)$.

(2) Since $X \sqcap Y \sqsubseteq X, Y \sqsubseteq X \sqcup Y$, we have $\bigwedge X, \bigwedge Y \in [\bigwedge(X \sqcap Y), \bigwedge(X \sqcup Y)]$, so $\bigwedge(X \sqcap Y) \leq (\bigwedge X) \vee (\bigwedge Y) \leq \bigwedge(X \sqcup Y)$ and $\bigwedge(X \sqcup Y) < \bigvee(X \sqcap Y)$ by part (1). Hence, $(\bigwedge X) \vee (\bigwedge Y) \in i_{X \sqcap Y}$. The proof of $(\bigvee X) \wedge (\bigvee Y) \in i_{X \sqcup Y}$ is a dual argument. $\square$

The next lemma states that in a finite lattice L of width $w > 1$ whenever two sets X, Y in π_L^w determine distinct significant intervals $i_X \neq i_Y$, the intersection $X \cap Y$ has cardinality at most $w - 2$.

Lemma 5.2.8. *Let L be a finite lattice of width $w > 1$ with $X, Y \in \pi_L^w$ and $|X \cap Y| = w - 1$. If $i_X, i_Y \in \mathscr{S}\mathscr{I}_w(L)$, then $i_X = i_Y$.*

Proof. Let $X - (X \cap Y) = \{a\}$ and $Y - (X \cap Y) = \{b\}$. Since $a \in X - Y$ and $b \in Y - X$, we have $a \neq b$. If $a \parallel b$, then, by the hypothesis that $|X \cap Y| = w - 1$, $(X \cap Y) \cup \{a, b\} \in \pi_L^{w+1}$ contradicting $w(L) = w$. Thus, $a \nparallel b$. We may assume that $a < b$. Note that

$$\bigwedge X = (\bigwedge(X \cap Y)) \wedge a \leq (\bigwedge(X \cap Y)) \wedge b = \bigwedge Y$$
$$\bigvee X = (\bigvee(X \cap Y)) \vee a \leq (\bigvee(X \cap Y)) \vee b = \bigvee Y.$$

Let $c = ((\bigwedge(X \cap Y)) \wedge b) \vee a$, then $a = ((\bigwedge(X \cap Y)) \wedge a) \vee a \leq c \leq ((\bigwedge(X \cap Y)) \wedge b) \vee b = b$. We claim that $\bigwedge X \nless \bigwedge Y$. Suppose $\bigwedge X < \bigwedge Y$. First note that for $z \in X \cap Y$, $z \neq a, b$. We claim that for $z \in X \cap Y$, $c \parallel z$ holds; for if $z \leq c$, then $z \leq c \leq b$, contradicting $z, b \in Y$ with $z \neq b$; if $c \leq z$, then $a \leq c \leq z$, contradicting $a, z \in X$ with $z \neq a$. Thus, $N := (X \cap Y) \cup \{c\} \in \pi_L^w$. Since $\bigwedge Y = (\bigwedge(X \cap Y)) \wedge b \leq c \leq (\bigwedge(X \cap Y)) \vee a \leq (\bigvee(X \cap Y)) \vee a = \bigvee X$, it follows that $N \subseteq [(\bigwedge(X \cap Y)) \wedge b, (\bigvee(X \cap Y) \vee a)] = [\bigwedge Y, \bigvee X]$. Also since $\bigwedge X < \bigwedge Y$, we have $[\bigwedge N, \bigvee N] \subseteq [\bigwedge Y, \bigvee X] \subset [\bigwedge X, \bigvee X]$, so $i_N \in \mathscr{D}_w(L)$ and $i_N \subset i_X$, contradicting $i_X \in Min(\mathscr{D}_w(L))$. Hence, $\bigwedge X \nless \bigwedge Y$. Since $\bigwedge X \leq \bigwedge Y$, we have $\bigwedge X = \bigwedge Y$, so that $\bigvee X = \bigvee Y$ by Lemma 5.2.6 part (3). Therefore, $i_X = i_Y$. $\square$

Definition 5.2.2. *Let L be a finite lattice of width w. Now we define the notion of a component. We first define a w-component, then a j-component with $1 < j < w$, after that a 1-component.*

*An interval i_X is a w-**component** if $i_X \in Max(\mathscr{D}_w(L))$.*

*Assume that the k-components have been defined for all k with $j < k \leq w$. An interval i_X is a j-**component** with $1 < j < w$ if $i_X \in Max(\mathscr{D}_j(L))$ and there is no k-component i_Y with $j < k \leq w$ such that $i_X \subseteq i_Y$.*

*An interval interval is a **1-component** if it is a maximal nodal interval.*

*An interval is a **component** if it is a k-component for some $1 \leq k \leq w$.*

For $1 \leq k \leq w$, let $Comp_k(L)$ be the set of k-components of L, and let $Comp(L)$ be the set of components of L, i.e.,

$$Comp_k(L) := \{i_X \mid i_X \text{ is a } k\text{-component of } L\},$$

$$Comp(L) := \bigcup_{1 \leq k \leq w} Comp_k(L).$$

Note that

(1) $Comp(L)$ is the set of all components of L;

(2) the 1-components of L are precisely the 1-significant intervals of L;

(3) for any j-component i_X with $X \in \pi_L^j$ and $j \geq 2$, both $\bigwedge X \in M^{\sim}(L)$ and $\bigvee X \in J^{\sim}(L)$ hold.

Let P be a finite poset, we define

$$C^x := \{p \in P \mid p \prec x\}, C_x := \{p \in P \mid x \prec p\}.$$

So C^x is the set of all elements that x covers, and C_x is the set of all elements that cover x. Note that if $C^x = \{y\}$, then $\downarrow y = \downarrow x - \{x\}$.

Let $X \in \pi_P$,

$$C_X := Min(\{y \in P \mid x \prec y \text{ for some } x \in X\}).$$

Note that

(1) $C_X \in \pi_P$, $C_X = Min(\uparrow X - X)$ and $X \cap C_X = \emptyset$;

(2) if $X = \{x\}$, then $C_x = C_X$;

(3) $C_X = \emptyset$ iff $X \subseteq Max(P)$.

Theorem 5.2.9. *Every finite lattice L is the union of its components, i.e.,* $L = \bigcup Comp(L)$.

Proof. Let L be a finite lattice with $|L| > 2$. Obviously $\bigcup Comp(L) \subseteq L$. In order to prove the reverse inclusion, we prove that, for $x \in L$, there exists a component which contains x. The following three cases are an exhaustive set of possible cases.

Case 1: such x is a nodal element with $|C_x| = 1$ or $|C^x| = 1$. We may assume $|C_x| = 1$. Let $C_x = \{y\}$, then y is a nodal element and $[x, y]$ is a nodal interval. Thus, x is in a nodal interval, so x is in some 1-component.

Case 2: such x is a nodal element with $|C_x| > 1$ or $|C^x| > 1$. We may assume $|C_x| = n > 1$. Then $x \in [x, \bigvee C_x]$ and $[x, \bigvee C_x] \in \mathscr{D}_n(L)$. Thus there exists $i_Z \in Max(\mathscr{D}_n(L))$ such that $[x, \bigvee C_x] \subseteq i_Z$. Suppose that x is not in any k-component with $n < k$. Thus, i_Z is not contained in any k-component with $n < k$. Hence $i_Z \in Comp_n(L)$ with $x \in i_Z$.

Case 3: such x is not a nodal element. Thus, $\pi(x)$ is a nonempty poset. Let $j = w(\pi(x))$. Suppose that x is not in any k-component with $j + 1 < k$.

Then there exists $Y \in \pi_L^j$ such that $Y \subseteq \pi(x)$. Thus, $i_{\{x\} \cup Y} \in \mathscr{D}_{j+1}(L)$ with $\{x\} \cup Y \in \pi_L^{j+1}$. Since L is finite, there exists $i_Z \in Max(\mathscr{D}_{j+1}(L))$ such that $i_{\{x\} \cup Y} \subseteq i_Z$. Since x is not in any $i_X \in Comp_k(L)$ with $j+1 < k$, it follows that i_Z is not contained in any $i_X \in Comp_k(L)$ with $j+1 < k$. So, $i_Z \in Comp_{j+1}(L)$ with $x \in i_Z$.

Therefore L is the union of its components. □

Since $\sim : L \to \tilde{L}$ is a lattice homomorphism, we have the following results:

(1) $X \in \pi_L$ iff $\tilde{X} \in \pi_{\tilde{L}}$;

(2) $i_X \in \mathscr{D}_{|X|}(L)$ iff $i_{\tilde{X}} \in \mathscr{D}_{|\tilde{X}|}(\tilde{L})$;

(3) $i_X \in \mathscr{S}\mathscr{I}_k(L)$ iff $i_{\tilde{X}} \in \mathscr{S}\mathscr{I}_k(\tilde{L})$, and $|\mathscr{S}\mathscr{I}(L)| = |\mathscr{S}\mathscr{I}(\tilde{L})|$;

(4) $i_X \in Comp_k(L)$ iff $i_{\tilde{X}} \in Comp_k(\tilde{L})$, and $|Comp(L)| = |Comp(\tilde{L})|$.

Recall that a lattice L is a skeleton if $L \cong \tilde{L}$ via the mapping $\sim : L \to \tilde{L}$.

Lemma 5.2.10. *Let L be a finite skeleton.*

(1) *If $x \in M(L)$, then $C_x \subseteq J^{\sim}(L)$.*

(2) *If $x \in J(L)$, then $C^x \subseteq M^{\sim}(L)$.*

(3) *If $x \in M(L) \cap J(L)$, then $C_x \subseteq J^{\sim}(L)$ and $C^x \subseteq M^{\sim}(L)$.*

(4) *There is no 1-significant interval in L.*

(5) *If $Y \in \pi_L$, then $\uparrow Y - Y = \uparrow C_Y$.*

Proof. (1) Let $x \in M(L)$. Then $|C_x| = 1$ holds since L is finite. Let $C_x = \{y\}$. Then $\uparrow x = [x,y] \cup \uparrow y$. We claim that $y \in J^{\sim}(L)$; otherwise, $C^y = \{x\}$ and $[x,y] \cup \downarrow x = \downarrow y$, so $(\uparrow x) \cup (\downarrow x) = ([x,y] \cup \uparrow y) \cup (\downarrow x) = (\uparrow y) \cup ([x,y] \cup \downarrow x) =$

$(\uparrow y) \cup (\downarrow y)$ and, since $[x, y] = \{x, y\}$, we have $y \sim x$; thus by Lemma 3.1.7 $y = x$, a contradiction. Hence, $C_x \subseteq J^{\sim}(L)$.

(2) This is dual to (1).

(3) This follows directly from (1) and (2).

(4) Suppose that $[x, y]$ is a nodal interval. Then $x < y$ and $x \sim y$, by Lemma 3.1.7 $y = x$ contradicting $x < y$. Hence no nodal interval exists in L.

(5) If $Y = \{1\}$, then $\uparrow Y - Y = \emptyset = \uparrow C_Y$. Here we may assume that $Y \neq \{1\}$. Then $\uparrow Y - Y \neq \emptyset$. Clearly $\uparrow C_Y \subseteq \uparrow Y - Y$. To show the reverse inclusion, let $p \in \uparrow Y - Y$. Since L is finite, there exists $a \in Min(\uparrow Y - Y)$ such that $a \leq p$, thus $a \in C_Y$ and $p \in \uparrow C_Y$, so $\uparrow Y - Y \subseteq \uparrow C_Y$. Therefore, $\uparrow Y - Y = \uparrow C_Y$. $\square$

CHAPTER 6

FINITE LATTICES OF WIDTH 2 AND 3

6.1 Finite Lattices of Width 2

In this section we apply the concepts of significant interval and component to lattices of width 2. We create a complete description, accounting to a structure theory, of finite lattices of width 2.

6.1.1 Properties of Finite Lattices of Width 2

For any x in a lattice L of width 2, $\pi(x)$ has the following properties.

Lemma 6.1.1. *Let L be a lattice of width 2 and let $x \in L$ with $\pi(x) \neq \emptyset$. Then*

(1) *$\pi(x)$ is a chain;*

(2) *if $y < \bigwedge \pi(x)$, then $y < x$;*

(3) *if $y > \bigvee \pi(x)$, then $y > x$.*

Proof. (1) Let $a, b \in \pi(x)$. Then $a \parallel x$ and $b \parallel x$. Since $w(L) = 2$, we have $a \nparallel b$. Hence $\pi(x)$ is a chain.

(2) Let $y < \bigwedge \pi(x)$. Then $y \notin \pi(x)$ and $y \nparallel x$. We claim that $x \nleq y$, otherwise, $x \leq y < \bigwedge \pi(x) \leq a$ for any $a \in \pi(x)$, so $x < a$ contradicting $a \in \pi(x)$. Hence $y < x$.

(3) This proof is dual to (2). □

The following lemma indicates that, in a lattices of width 2, all meet-reducible elements form a chain and all join-reducible elements form a chain.

Lemma 6.1.2. *Let L be a lattice of width 2, then*

(1) *$M^{\sim}(L)$ is a chain.*

(2) *$J^{\sim}(L)$ is a chain.*

Proof. (1) Since $w(L) = 2$, there exists $\{a, b\} \in \pi_L^2$, so $a \wedge b \in M^{\sim}(L)$ and therefore $M^{\sim}(L) \neq \emptyset$. Let $x, y \in M^{\sim}(L)$ with $x \neq y$, then there exist $\{a_1, a_2\}, \{b_1, b_2\} \in \pi_L^2$ such that $x = a_1 \wedge a_2$ and $y = b_1 \wedge b_2$. By Dilworth's theorem L is a union of chains C_1 and C_2. Let $a_1, b_1 \in C_1$ and $a_2, b_2 \in C_2$, then $a_1 \not\parallel b_1$ and $a_2 \not\parallel b_2$. We may assume that $a_1 \leq b_1$. If $a_2 \leq b_2$, then $x = a_1 \wedge a_2 \leq b_1 \wedge b_2 = y$. Thus we may assume that $b_2 \leq a_2$, so that $x \leq b_1 \wedge a_2$ and $y \leq b_1 \wedge a_2$, and hence $x \vee y \leq b_1 \wedge a_2$. Since $a_1 \leq b_1$ and $b_1 \parallel b_2$, it follows that $b_2 \nleq a_1$. Since $b_2 \leq a_2$ and $a_1 \parallel a_2$, it follows that $a_1 \nleq b_2$. Hence $a_1 \parallel b_2$. Since $x \vee y \leq a_2$ and $a_1 \parallel a_2$, we have $a_1 \nleq x \vee y$, similarly $b_2 \nleq x \vee y$. Also since $a_1 \parallel b_2$ and $w(L) = 2$, either $x \vee y \leq a_1$ or $x \vee y \leq b_2$. Thus, since $x \vee y \leq b_1 \wedge a_2$, $x \leq x \vee y \leq a_1 \wedge a_2 = x$ or $y \leq x \vee y \leq b_1 \wedge b_2 = y$. Hence, $x \vee y = x$ or $x \vee y = y$, i.e., $y \leq x$ or $x \leq y$.

(2) This proof is dual to (1). □

The following lemma indicates the relation between a meet-reducible element (or a join-reducible element) x and the element y parallel with x.

Lemma 6.1.3. *Let L be a lattice of width 2 with $x, y \in L$.*

(1) *If $x \in M^{\sim}(L)$ and $y \parallel x$, then $x \prec x \vee y$.*

(2) *If $x \in J^{\sim}(L)$ and $y \parallel x$, then $x \wedge y \prec x$.*

Proof. (1) Suppose that $x \nprec x \vee y$. Then there exists $z \in (x, x \vee y)$. Since $x \in M^{\sim}(L)$, there exists $\{a, b\} \in \pi_L^2$ such that $x = a \wedge b$. Since $a \parallel b$ and $w(L) = 2$, we have $y \nparallel a$ or $y \nparallel b$; since $y \parallel x$, it follows $a \nleq y, b \nleq y$ and $y \nleq a \wedge b$, i.e., $y < a$ and $y \parallel b$, or $y \parallel a$ and $y < b$. We may assume that $y < a$ and $y \parallel b$. We claim that $z \parallel y$ and $z \parallel b$. Since $x < z$ and $x \parallel y$, it follows that $z \nleq y$; also since $z < x \vee y$, we have $y \nleq z$, so $z \parallel y$. Since $z < a$ and $a \parallel b$, it follows that $b \nleq z$; also since $a \wedge b = x < z$, we have $z \nleq b$, so $z \parallel b$. Thus, $\{y, b, z\} \in \pi_L^3$ contradicting $w(L) = 2$. Hence, $x \prec x \vee y$.

(2) This proof is dual to that of (1). □

In a finite lattice L of width 2, $\mathscr{S}\mathscr{I}_k(L) = \emptyset$ and $Comp_k(L) = \emptyset$ for any $k > 2$ since $\pi_L^k = \emptyset$ for any $k > 2$.

The following lemma shows the property of a 2-significant interval.

Theorem 6.1.4. *Let L be a finite lattice of width 2 and $x, y \in L$ with $x < y$. The following statements are equivalent:*

(1) *$[x, y]$ is a 2-significant interval;*

(2) *$|C_x| = 2$ and $y = \bigvee C_x$;*

(3) *$|C^y| = 2$ and $x = \bigwedge C^y$;*

(4) *$[x, y]$ is the horizontal sum of 2 chains;*

(5) *there are two distinct chains* C_1, C_2 *in* L *such that both are maximal autonomous chains in* L *and* $[x,y] = \{x,y\} \cup C_1 \cup C_2$*;*

(6) *there are* $a, b \in [x,y]$ *such that* $[x,y] \cong HS(\{x,y\} \cup [a], \{x,y\} \cup [b])$.

Proof. (1) $\Rightarrow$ (2) : Since $[x,y]$ is a 2-significant interval in L, there is $X \in \pi_L^2$ such that $[x,y] = [\bigwedge X, \bigvee X]$, so $x \in M^{\sim}(L)$ and $|C_x| = 2$. Let $C_x = \{a,b\}$. Then $i_{\{a,b\}} \subseteq i_X$. Since $i_X \in \mathscr{S}\mathscr{I}_2(L)$, we have $i_{\{a,b\}} = i_X$ and $y = \bigvee C_x$.

(2) $\Rightarrow$ (3) : Let $C_x = \{a,b\}$ and $y = a \vee b$. Then $y \in J^{\sim}(L)$. Also since $w(L) = 2$, we have $|C^y| = 2$. Since $\{a,b\}, C^y \in \pi_L^2$ and $\{a,b\}, C^y \subseteq \downarrow y$, it follows that $\{a,b\} \sqsubseteq C^y$ and $x = a \wedge b \leq \bigwedge C^y$. We claim that $x = \bigwedge C^y$. Suppose not. Then $x < \bigwedge C^y$; thus, $a \leq \bigwedge C^y$ or $b \leq \bigwedge C^y$ since $C_x = \{a,b\}$; since $\bigwedge C^y < y = a \vee b$, it is not possible that both $a \leq \bigwedge C^y$ and $b \leq \bigwedge C^y$. We may assume $a \leq \bigwedge C^y$ and $b \parallel \bigwedge C^y$. Then $y = a \vee b \leq (\bigwedge C^y) \vee b \leq y$ so that $y = (\bigwedge C^y) \vee b$. Since $\bigwedge C^y \prec (\bigwedge C^y) \vee b$ by Lemma 6.1.3 part (1), we have $|C^y| = 2$ and $\bigwedge C^y \prec y$, a contradiction.

(3) $\Rightarrow$ (4) : Let $C^y = \{d,e\}$ and $x = \bigwedge C^y$. Then $(x,d] \cap (x,e] = \emptyset$; otherwise, let $z \in (x,d] \cap (x,e]$, then $x < z \leq d \wedge e = \bigwedge C^y = x$, a contradiction, so $[x,y]$ is the horizontal sum of two chains $\{y\} \cup [x,d]$ and $\{y\} \cup [x,e]$.

(4) $\Rightarrow$ (5) : Let $C_x = \{a,b\}$, $C_1 = [a,y)$ and $C_2 = [b,y)$. Then C_1 and C_2 are finite disjoint chains. We only need to prove C_1 and C_2 are autonomous chains. Let $t \in L - C_1$ with $t \leq u$ for some $u \in C_1$. If $t \parallel b$, then $t \nparallel a$ since $a \parallel b$ and $w(L) = 2$; thus, $t \leq a$ since $t \notin C_1$. Moreover, if $t \nparallel b$, then $b \nleq t$.

Otherwise, $b \leq t \leq u$ and $a \leq u < y$, so $y = a \vee b \leq u < y$, a contradiction; so $t \leq b$ and $t \leq u \wedge b = x$ since $[x, y]$ is the horizontal sum of $C_1 \cup \{x, y\}$ and $C_2 \cup \{x, y\}$; hence, $t \leq a$. In either case, t is less than or equal to all elements in C_1. Dually, if $u \leq t$ for some $u \in C_1$, then t is larger than or equal to the largest element in C_1. Hence, C_1 is an autonomous chain. Since $x \prec a$ and $\bigvee[a, y) \prec y$ with $x, y \notin [a]$, the chain C_1 is a maximal autonomous chain. Similarly, C_2 is an maximal autonomous chain.

(5) $\Rightarrow$ (6) : Suppose that L contains two distinct maximal autonomous chains C_1 and C_2 where $[x, y] = \{x, y\} \cup C_1 \cup C_2$. Let $a \in C_1$ and $b \in C_2$, then $C_1 = [a]$ and $C_2 = [b]$ since C_1 and C_2 are autonomous chains, thus $[x, y]$ is the horizontal sum of two chains $\{x, y\} \cup [a]$ and $\{x, y\} \cup [b]$, i.e., $[x, y] \cong HS(\{x, y\} \cup [a], \{x, y\} \cup [b])$.

(6) $\Rightarrow$ (1) : Let $[x, y] \cong HS(\{x, y\} \cup [a], \{x, y\} \cup [b])$. We have $a \parallel b$ and $[x, y] \in \mathscr{D}_2(L)$. Since $[x, y]$ is the horizontal sum of two chains $\{x, y\} \cup [a]$ and $\{x, y\} \cup [b]$, it follows that $[x, y] \in Min(\mathscr{D}_2(L))$. Hence, $[x, y]$ is a 2-significant interval. □

Theorem 6.1.4 shows that any 2-significant interval is the horizontal sum of two chains, as depicted in Figure 6.1, the number of elements in each chain may be any number larger than 3; in the Hasse diagram of $[x, y]$ in order to embrace this universality, we use a dotted-line to connect the least element and the largest element in an autonomous chain.

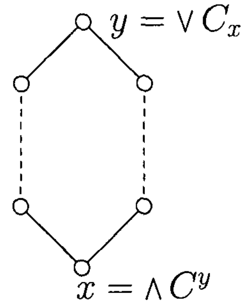


Figure 6.1 A generic 2-significant interval $[x, y]$.

Corollary 6.1.5. *Let a finite lattice L be a skeleton of width 2 and $x, y \in L$ with $x < y$. The following statements are equivalent:*

(1) $[x, y]$ *is a 2-significant interval;*

(2) $|C_x| = 2$ *and* $y = \bigvee C_x$*;*

(3) $|C^y| = 2$ *and* $x = \bigwedge C^y$*;*

(4) $[x, y]$ *is the horizontal sum of 2 chains;*

(5) *there are two distinct chains* C_1, C_2 *in L such that both are autonomous chains in L and* $[x, y] = \{x, y\} \cup C_1 \cup C_2$*;*

(6) *there are* $a, b \in [x, y]$ *such that* $[x, y] \cong HS(\{x, y\} \cup [a], \{x, y\} \cup [b])$*;*

(7) $x \in M^{\sim}([x, y])$, $y \in J^{\sim}([x, y])$ *and* $|[x, y]| = 4$*;*

(8) $C_x = C^y$ *and* $|C_x| = 2$.

Proof. The statements of (1) – (6) are the same as those in Theorem 6.1.4.

(6) $\Rightarrow$ (7): Since $[x, y] \cong HS(\{x, y\} \cup [a], \{x, y\} \cup [b])$, both $x \in M^{\sim}([x, y])$ and $y \in J^{\sim}([x, y])$ hold. Since L is a finite skeleton, we have $|[a]| = 1$ and $|[b]| = 1$, so that $[x, y] = \{x, y\} \cup [a] \cup [b] = \{x, y, a, b\}$ and $|[x, y]| = 4$.

(7) $\Rightarrow$ (8): Since $|[x, y]| = 4$, we have $|(x, y)| = 2$. Let $(x, y) = \{a, b\}$, then $a \parallel b$ since $x \in M^{\sim}([x, y])$; hence $C_x = \{a, b\} = C^y$ and $|C_x| = 2$.

(8) $\Rightarrow$ (6): Suppose $|C_x| = 2$ and $C_x = C^y$. Let $C_x = \{a, b\}$, then $[x, y] = \{x, y, a, b\} = HS(\{x, y\} \cup [a], \{x, y\} \cup [b])$. □

The following lemma shows that if $[x, y]$ is a 2-significant interval; then the open interval (x, y) contains at most one join-reducible element of L and that join-reducible element, if it exists, covers x.

Lemma 6.1.6. *Let L be a finite lattice of width 2. If $[x, y]$ is a 2-significant interval with $C_x = \{a, b\}$, then*

(1) $(a, y) \subseteq \pi(b)$ and $(b, y) \subseteq \pi(a)$;

(2) $(a, y) \subseteq J(L)$ and $(b, y) \subseteq J(L)$;

(3) $J^{\sim}(L) \cap (x, y) \subseteq \{a, b\}$ and $|J^{\sim}(L) \cap (x, y)| \leq 1$.

Proof. (1) Let $z \in (a, y)$. Since $a \parallel b$ and $a < z$, we have $z \nleq b$; since $a < z$ and $z < y = a \vee b$, it follows that $b \nleq z$; so $z \in \pi(b)$. Hence, $(a, y) \subseteq \pi(b)$. Similarly, $(b, y) \subseteq \pi(a)$.

(2) Let $z \in (a, y)$. Suppose $z \in J^{\sim}(L)$. Then $z \parallel b$ by (1). Thus, we have $z \wedge b \prec z$, by Lemma 6.1.3 part (2), and $b \in J(L)$, by Lemma 6.1.2 part (2); also since $x \prec b$, we have $z \wedge b = x < a < z$ contradicting $z \wedge b \prec z$. Hence $z \in J(L)$. Similarly, $(b, y) \subseteq J(L)$.

(3) From part (1) and (2) we have $(x, y) \cap J^{\sim}(L) \subseteq \{a, b\}$. Since $a \parallel b$ and $J^{\sim}(L)$ is a chain by Lemma 6.1.3 part (2), it follows that $J^{\sim}(L) \cap (x, y) \subset \{a, b\}$ and $|J^{\sim}(L) \cap (x, y)| \leq 1$. □

The following lemma shows that if $[x, y]$ is a 2-significant interval, then the open interval (x, y) contains at most one meet-reducible element of L

and that meet-reducible element, if it exists, is covered by y.

Lemma 6.1.7. *Let L be a finite lattice of width 2. If $[x,y]$ is a 2-significant interval with $C^y = \{c,d\}$, then*

(1) $(x,c) \subseteq \pi(d)$ *and* $(x,d) \subseteq \pi(c)$;

(2) $(x,c) \subseteq M(L)$ *and* $(x,d) \subseteq M(L)$;

(3) $M^{\sim}(L) \cap (x,y) \subseteq \{c,d\}$ *and* $|M^{\sim}(L) \cap (x,y)| \leq 1$.

Proof. This proof is dual to that of Lemma 6.1.6. □

The following lemma shows the relation between any two distinct 2-significant intervals.

Lemma 6.1.8. *Let L be a finite lattice of width 2, let $[x_1,y_1]$ and $[x_2,y_2]$ be distinct 2-significant intervals. Then*

(1) $x_1 < x_2$ *or* $x_2 < x_1$,

(2) $x_1 < x_2$ *iff* $y_1 < y_2$,

(3) $x_2 < x_1$ *iff* $y_2 < y_1$.

Proof. (1) By Theorem 6.1.4 $x_1, x_2 \in M^{\sim}(L)$, and $x_1 \not\parallel x_2$ by Lemma 6.1.2 part (1); since $[x_1,y_1]$ and $[x_2,y_2]$ are distinct 2-significant intervals, we have $x_1 \neq x_2$ by Lemma 5.2.6 part (3). Hence, $x_1 < x_2$ or $x_2 < x_1$.

(2) and (3) follow from Lemma 5.2.6 part (2). □

In a finite lattice L of width 2, Theorem 6.1.4 shows that, for $x \in M^{\sim}(L)$, there is a 2-significant interval $[x, \bigvee C_x]$ with $\bigvee C_x \in J^{\sim}(L)$; dually, for $y \in J^{\sim}(L)$, there is a 2-significant interval $[\bigwedge C^y, y]$ with $\bigwedge C^y \in M^{\sim}(L)$. Lemma 6.1.2 indicates that the set consisting of the least elements of the

2-significant intervals is a chain and, dually, the set consisting of the largest elements of the 2-significant intervals is a chain. Lemma 6.1.8 shows that for two distinct elements x_1 and x_2 in $M^{\sim}(L)$, $[x_1, \bigvee C_{x_1}]$ and $[x_2, \bigvee C_{x_2}]$ are two distinct 2-significant; dually for two distinct elements y_1 and y_2 in $J^{\sim}(L)$, $[\bigwedge C_{y_1}, y_1]$ and $[\bigwedge C_{y_2}, y_2]$ are two distinct 2-significant intervals. Hence, there is a bijective correspondence between $\mathscr{S}\mathscr{I}_2(L)$ and $M^{\sim}(L)$ and between $\mathscr{S}\mathscr{I}_2(L)$ and $J^{\sim}(L)$. Thus $|M^{\sim}(L)| = |\mathscr{S}\mathscr{I}_2(L)| = |J^{\sim}(L)|$. We may write $M^{\sim}(L) = \{m_i \mid 1 \leq i \leq |M^{\sim}(L)| \text{ and } m_1, m_2 < \ldots < m_{|M^{\sim}(L)|}\}$, $J^{\sim}(L) = \{j_i \mid 1 \leq i \leq |J^{\sim}(L)| \text{ and } j_1 < j_2 < \ldots < j_{|J^{\sim}(L)|}\}$. These are called the standard numberings of the $M^{\sim}(L)$ and $J^{\sim}(L)$, respectively. Let $\mathscr{S}\mathscr{I}_2(L) = \{[x_i, y_i] \mid 1 \leq i \leq |M^{\sim}(L)|, x_i \in M^{\sim}(L) \text{ and } y_i \in J^{\sim}(L)\}$. It gives the standard numbering of $\mathscr{S}\mathscr{I}_2(L)$.

Lemma 6.1.9. *Let L be a finite lattice of width 2, let $m_i \in M^{\sim}(L)$ and $j_i \in J^{\sim}(L)$ for $1 \leq i \leq |M^{\sim}(L)|$. Then $[m_i, j_i]$ is a 2-significant interval and exactly one of the following obtains:*

(1) $m_{i+1}, j_{i-1} \notin (m_i, j_i)$,

(2) $m_{i+1} \in (m_i, j_i)$ *and* $j_{i-1} \notin (m_i, j_i)$,

(3) $j_{i-1} \in (m_i, j_i)$ *and* $m_{i+1} \notin (m_i, j_i)$,

(4) $m_{i+1}, j_{i-1} \in (m_i, j_i)$ *and* $m_{i+1} \parallel j_{i-1}$,

(5) $m_{i+1}, j_{i-1} \in (m_i, j_i)$ *and* $j_{i-1} \leq m_{i+1}$.

Proof. The conditions are evidently mutually exclusive. If $i = 1$, then $[m_1, j_1]$ is a 2-significant interval and, either (1) or (2) holds, since m_1 is the least meet-reducible element and j_1 is the least join-reducible element in L.

Suppose that, for $i = k$, $[m_k, j_k]$ is a 2-significant interval and it is one of the above 5 intervals. We claim that $[m_{k+1}, j_{k+1}]$ is a 2-significant interval and it is one of the above 5 intervals. Since $[m_k, j_k] \in \mathscr{S}\mathscr{I}_2(L)$, we have $j_k = \bigvee m_k$ by Theorem 6.1.4 part (2); also since $m_k < m_{k+1}$ and j_{k+1} is the least join-reducible element in $\uparrow j_k - \{j_k\}$, it follows that $j_{k+1} = \bigvee m_{k+1}$ by Lemma 6.1.8 part (2), so $[m_{k+1}, j_{k+1}]$ is a 2-significant interval by Theorem 6.1.4 part (2). If $m_{k+1} \notin (m_k, j_k)$, then $j_k \notin (m_{k+1}, j_{k+1})$, so that either (1) or (2) holds. If $m_{k+1} \in (m_k, j_k)$, then $j_k \in (m_{k+1}, j_{k+1})$ by Lemma 6.1.6 part (3). There exists at most one meet-reducible element in (m_{k+1}, j_{k+1}), namely (if it is in the interval) m_{k+2}, by Lemma 6.1.7 part (3). $m_{k+2} \notin (m_{k+1}, j_{k+1})$ implies that (3) holds. If $m_{k+2} \in (m_{k+1}, j_{k+1})$ and $m_{k+2} \parallel j_k$, then (4) holds; if $m_{k+2} \in (m_{k+1}, j_{k+1})$ and $m_{k+2} \nparallel j_k$, then $m_{k+1} \prec j_k$ by Lemma 6.1.6 part (3) and $m_{k+2} \prec j_{k+1}$ by Lemma 6.1.7 part (3); since $[m_{k+1}, j_{k+1}] \in \mathscr{S}\mathscr{I}_2(L)$, we have $j_k \leq m_{k+2}$, i.e., (5) holds. Therefore, $[m_i, j_i]$ is a 2-significant interval for $1 \leq i \leq |M^{\sim}(L)|$ and one of (1)-(5) holds. □

Lemma 6.1.9 shows that any 2-significant interval is isomorphic to one of the six types in Figure 6.2. In the diagrams we use the symbol Δ to represent the least element in another 2-significant interval and the symbol ∇ to represent the largest element in another 2-significant interval; if one point is the least element in a 2-significant interval and the largest element element in another 2-significant interval, then we use the symbol ✡ to represent it. For $1 \leq i \leq 5$, a 2-significant interval $[x, y]$ is of type $i+1$ iff $[x, y]$ satisfies (i) of Lemma 6.1.9.

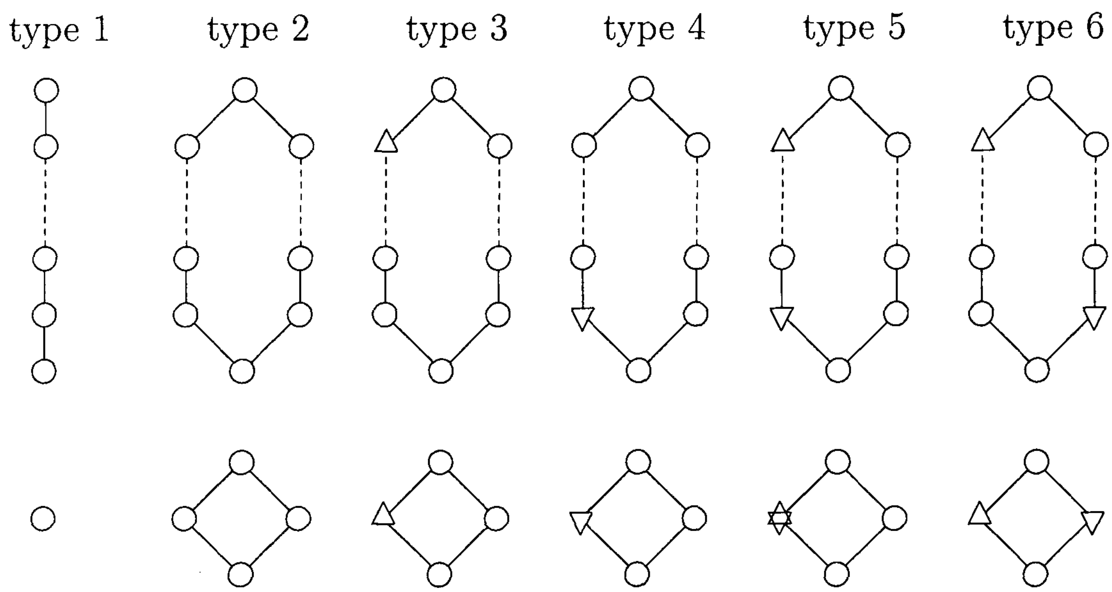


Figure 6.2 Six types of significant intervals in a finite lattice of width 2 and their skeletons.

6.1.2 The Structure of a Finite Lattice of Width 2

Now, we are close to being able to characterize the structure of finite lattices of width 2.

Theorem 6.1.10. *Every finite lattice L of width 2 is the union of its significant intervals.*

Proof. Let $x \in L$. If $|C_x| = 2$, then $[x, \bigvee C_x]$ is a 2-significant interval by Theorem 6.1.4.

If $|C^x| = 2$, then $[\bigwedge C^x x, x]$ is a 2-significant interval by Theorem 6.1.4.

If $|C_x| = |C^x| = 1$ and $\pi(x) = \emptyset$, then x is a nodal element, and there is a maximal nodal interval containing x, such interval is a 1-significant interval.

If $|C_x| = |C^x| = 1$ and $\pi(x) \neq \emptyset$. Then $x \wedge (\bigwedge \pi(x))$ is a meet-reducible element of L in $\downarrow x$ and $x \vee (\bigvee \pi(x))$ is a join-reducible element of L in $\uparrow x$. Thus, $M^{\sim}(L) \cap \downarrow x \neq \emptyset$ and $J^{\sim}(L) \cap \uparrow x \neq \emptyset$. Since the set of all meet-reducible elements in $\downarrow x$ is a finite chain and the set of all join-reducible elements is another finite chain in $\uparrow x$, there exist the largest meet-reducible element m in $\downarrow x$ and the least join-reducible element j in $\uparrow x$, so x is in the interval $[m, j]$. We claim that $[m, j]$ is a 2-significant interval. Let $C_m = \{a, b\}$, then $a \leq x$ or $b \leq x$ since $m < x$ and $C_m = \{a, b\}$. We may assume that $a \leq x$, then $x \nleq b$ since $a \parallel b$, and $b \nleq x$ since $|C_x| = 1$, so $x \parallel b$. We have $a \vee b \nleq x$. We claim that $a \vee b \not\parallel x$, otherwise, $m < a \leq (a \vee b) \wedge x < x$ and $(a \vee b) \wedge x \in M^{\sim}(L)$ contradicting that m is the largest element in $M^{\sim}(L) \cap \downarrow x$. Hence, $x < a \vee b$ and $j \leq a \vee b$ since $j \in J^{\sim}(L) \cap (\uparrow x)$. We have $j \nless b$ since $x \leq j$ and $x \parallel b$. Also $j \not\parallel b$, otherwise, we have $b \wedge j \prec j$ by Lemma 6.1.3 part (2) and $m \leq b \wedge j < b$ since $m < a \leq x < j$ and $m < b$; thus $m = b \wedge j \prec j$ contradicting $m < x < j$. So $b < j$ and $a \leq x \leq j$. Hence $a \vee b \leq j$ and $j = a \vee b$. Therefore, $[m, j]$ is a 2-significant interval by Theorem 6.1.4. □

Let $\mathbf{C}_k = \{0, 1, 2, \ldots, k\}$ be a chain with the $k + 1$ elements. We use bold font in order to distinguish the set C_x, of elements in L covering x, from the chain $\mathbf{C}_k$.

In a finite lattice L of width 2, a significant interval $[x,y]$ is a 1-significant or 2-significant interval. It is one of six types intervals shown in Figure 6.2.

We denote the types of significant intervals $[x,y]$ as follows:

Type($[x,y]$) = $(1, \mathbf{C}_s)$, if $[x,y]$ is a 1-significant interval with $[x,y] \cong \mathbf{C}_s$;

Type($[x,y]$) = $(2, \mathbf{C}_s, \mathbf{C}_t)$, if $[x,y]$ is a 2-significant interval with $[x,y] \cong HS(\mathbf{C}_s, \mathbf{C}_t)$, and all elements in (x,y) are meet-irreducible and join-irreducible;

Type($[x,y]$) = $(3, \mathbf{C}_s, \mathbf{C}_t)$, if $[x,y]$ is a 2-significant interval with $[x,y] \cong HS(\mathbf{C}_s, \mathbf{C}_t)$, there is one meet-reducible element in $\mathbf{C}_s \cap C^y$ and all elements in (x,y) are join-irreducible;

Type($[x,y]$) = $(4, \mathbf{C}_s, \mathbf{C}_t)$, if $[x,y]$ is a 2-significant interval with $[x,y] \cong HS(\mathbf{C}_s, \mathbf{C}_t)$, there is one join-reducible element in $\mathbf{C}_s \cap C_x$ and all elements in (x,y) are meet-irreducible;

Type($[x,y]$) = $(5, \mathbf{C}_s, \mathbf{C}_t)$, if $[x,y]$ is a 2-significant interval with $[x,y] \cong HS(\mathbf{C}_s, \mathbf{C}_t)$, there are one meet-reducible element in $\mathbf{C}_s \cap C^y$ and one join-reducible element in $\mathbf{C}_t \cap C_x$;

Type($[x,y]$) = $(6, \mathbf{C}_s, \mathbf{C}_t)$, if $[x,y]$ is a 2-significant interval with $[x,y] \cong HS(\mathbf{C}_s, \mathbf{C}_t)$, there are one meet-reducible element in $\mathbf{C}_s \cap C^y$ and one join-reducible element in $\mathbf{C}_s \cap C_x$.

We abbreviate the denotations of the types of significant intervals as follows:

$(1, \mathbf{C}_s)$ is abbreviated $(1, s)$,

$(2, \mathbf{C}_s, \mathbf{C}_t)$ is abbreviated $(2, s, t)$,

$(3, \mathbf{C}_s, \mathbf{C}_t)$ is abbreviated $(3, s, t)$,

$(4, \mathbf{C}_s, \mathbf{C}_t)$ is abbreviated $(4, s, t)$,

$(5, \mathbf{C}_s, \mathbf{C}_t)$ is abbreviated $(5, s, t)$,

$(6, \mathbf{C}_s, \mathbf{C}_t)$ is abbreviated $(6, s, t)$.

Note that in a finite lattice L of width 2, a 1-significant interval is self-dual; since a 2-significant interval is the horizontal sum of two chains, a 2-significant interval is self-dual; so any significant interval $[x, y]$ is self-dual, i.e., $[x, y] \cong [x, y]^*$. Note that $[x, y] \in \mathscr{S}\mathscr{I}(L)$ iff $[x, y]^* \in \mathscr{S}\mathscr{I}(L^*)$, and $[x, y] \in \mathscr{S}\mathscr{I}_k(L)$ iff $[x, y]^* \in \mathscr{S}\mathscr{I}_k(L^*)$ for $k = 1, 2$.

Lemma 6.1.11. *Let $[x, y]$ and $[z, w]$ be significant intervals in a finite lattice L of width 2, then $x \not\parallel z$ and $y \not\parallel w$.*

Proof. By duality, we need only prove that $x \not\parallel z$. If x or z is nodal, the result holds. Hence we may assume that $[x, y]$ and $[z, w]$ are 2-significant intervals, then $x \not\parallel z$ by Lemma 6.1.8. □

Since the set of the least elements of significant intervals forms a chain, we may list all significant intervals in increasing order of least element of these intervals, (i.e., $\mathscr{S}\mathscr{I}(L) = \{[x_i, y_i] \mid i = 1, \dots, k, \text{ and } x_1 < \dots < x_k\}$). We say that the intervals $[x_i, y_i]$ and $[x_{i+1}, y_{i+1}]$ are **adjacent** for $1 \le i \le k - 1$. Note that

(1) $x_1 = 0$ and $y_k = 1$;

(2) two 1-significant intervals are not adjacent.

Lemma 6.1.12. *Let L be a finite lattice of width 2 and $[x_i, y_i] \in \mathcal{SI}(L)$ for $i > 1$.*

(1) *If x_i is a nodal element, then $[x_1, y_i] \cong VS([x_1, y_{i-1}], [x_i, y_i])$.*

(2) *If x_i is not nodal, then $[x_1, y_i] \cong QVS([x_1, y_{i-1}], [x_i, y_i]; [x_i, y_{i-1}])$.*

Proof. (1) Since $[x_i, y_i] \in \mathcal{SI}(L)$ and x_i is a nodal element, x_i is the largest element in the significant interval $[x_{i-1}, y_{i-1}]$, so $x_i = y_{i-1}$; also since $[x_1, y_i] = [x_1, y_{i-1}] \cup [x_i, y_i]$, we have $[x_1, y_i] = [x_1, y_{i-1}] \cup [x_i, y_i] = VS([x_1, y_{i-1}], [x_i, y_i])$.

(2) Since x_i is not nodal and $[x_i, y_i]$ is a significant interval, $[x_i, y_i]$ must be a 2-significant interval and $\pi(x_i) \neq \emptyset$; by Lemma 6.1.6 part (3) x_i is covered by some join-reducible element y; by corollary 6.1.5 part (3) $[\bigwedge C^y, y]$ is a 2-significant interval which is adjacent to the 2-significant interval $[x_i, y_i]$, so $x_i \prec y = y_{i-1}$ and $[x_1, y_{i-1}] \cap [x_i, y_i] = \{x_i, y_{i-1}\}$; it follows that $[x_1, y_i]$ is the quasi vertical sum of $[x_1, y_{i-1}]$ and $[x_i, y_i]$ over $[x_i, y_{i-1}]$, (i.e., $[x_1, y_i] \cong QVS([x_1, y_{i-1}], [x_i, y_i], [x_i, y_{i-1}])$). □

Theorem 6.1.13. *Let L be a finite lattice of width 2 and let $[x_i, y_i]$ be any significant interval in L with $1 \leq i \leq n$ and $x_1 < x_2 < \ldots < x_n$. Let $L_1 = [x_1, y_1]$, $L_2 = [x_1, y_2] = QVS(L_1, [x_2, y_2])$, ... , and $L_n = [x_1, y_n] = QVS(L_{n-1}, [x_n, y_n])$, then $L \cong L_n$.*

Proof. We claim that $L_n = QVS(L_{n-1}, [x_n, y_n])$.

Let $n = 2$. Then $[x_1, y_2] = QVS([x_1, y_1], [x_2, y_2]) =$ by Lemma 6.1.12.

Suppose that, for $n = k$, $L_k = QVS(L_{k-1}, [x_k, y_k])$ holds. Let $n = k + 1$. Then $L_{k+1} = [x_1, y_{k+1}] = QVS(L_k, [x_{k+1}, y_{k+1}])$ by Lemma 6.1.12.

Since $x_1 = 0$, $y_n = 1$ and L is the union of all significant intervals by Theorem 6.1.10, we have $L \cong L_n$. □

The previous theorem states that a finite lattice of width 2 is "the quasi vertical sum of a sequence of its significant intervals".

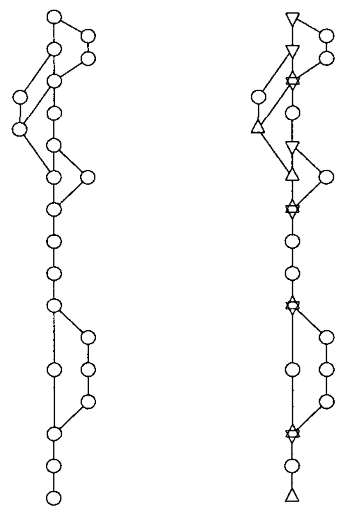
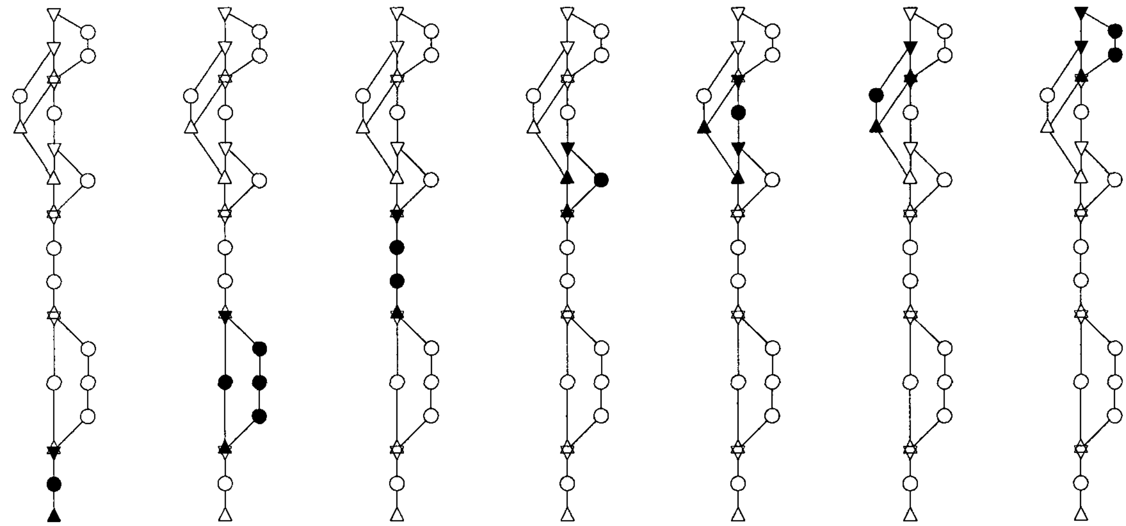


Figure 6.3 The Hasse diagram and the enhanced Hasse diagram of a prototypical finite lattice, L_{19}, of width 2.

A finite lattice L of width 2 is the union of a collection of significant intervals which are 1-significant or 2-significant intervals. The **significant interval identification notation (siin)** for a 1-significant or 2-significant interval $[x, y]$ is defined as follows: the first number is 0, the second number is an ordinal number identifying where this interval lies in the sequence of intervals, the remaining 2 or 3 numbers are the numbers (1, s) or (i, s, t) as described above. Thus we can represent L as a sequence of siins.



0,1,1,2   0,2,2,2,4   0,3,1,3   0,4,3,2,2   0,5,5,2,3   0,6,6,2,2   0,7,4,2,3

Figure 6.4 The *i*th significant interval of L_{19} is highlighted, notated and the siin presented below it.

Figure 6.3 shows the Hasse diagram of a finite lattice of width 2 and all its siins. When we use Δ , ∇ and/or ✡ to indicate significant intervals in a Hasse diagram, we refer to the diagram as an **enhanced Hasse diagram**. In the enhanced Hasse diagram, for each element labelled Δ , the interval $[\Delta, \nabla]$ from Δ up to the least element labelled ∇ is a significant interval. In Figure 6.3 we show the Hasse diagram of a finite lattice L_{19} of width 2; furthermore we redraw the Hasse diagram with the symbols Δ, ∇ and Δ ∇ ; in Figure 6.4 we highlight every significant interval on L_{19} and give the siin notation for an interval. Thus we can describe the lattice L_{19} by the sequence of numbers of siins $(0, 1, 1, 2)$, $(0, 2, 2, 2, 4)$, $(0, 3, 1, 3)$, $(0, 4, 3, 2, 2)$, $(0, 5, 5, 2, 3)$, $(0, 6, 6, 2, 2)$, and $(0, 7, 4, 2, 3)$. Dropping the parentheses and concatenating we arrive at a complete determination of the lattice L_{19} by the sequence of numbers "0, 1, 1, 2, 0, 2, 2, 2, 4, 0, 3, 1, 3, 0, 4, 3, 2, 2, 0, 5, 5, 2, 3, 0, 6, 6, 2, 2, 0, 7, 4, 2, 3".

Figure 6.5 shows the lattice L_{19} and its skeleton, and there is no 1-significant interval in its skeleton.

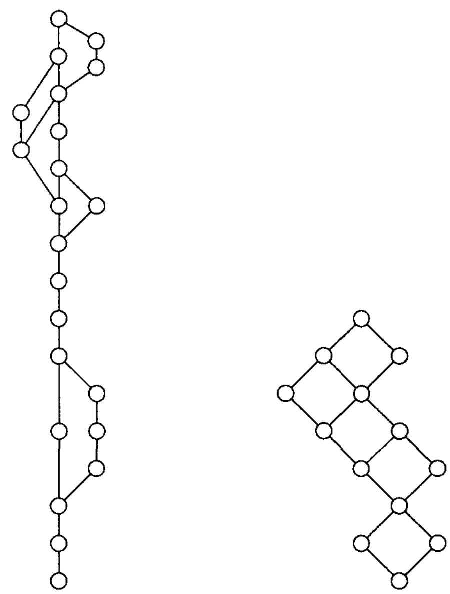


Figure 6.5 The lattice L_{19} and its skeleton.

In this section we will study the structure of a finite skeleton of width 2.

Lemma 6.1.14. *Let L be a finite skeleton of width 2. Then*

(1) *L is the union of 2-significant intervals;*

(2) *L is the union of 2-components;*

(3) *every 2-significant interval has 4 elements and a unique 2-determinant;*

(4) *L is a distributive lattice.*

Proof. (1) This follows from Theorem 6.1.10.

(2) This follows from Theorem 5.2.9.

(3) Let $[x,y]$ be a 2-significant interval. There are $a, b \in [x,y]$ such that $[x,y] \cong HS(\{x,y\} \cup [a], \{x,y\} \cup [b])$ by Theorem 6.1.4 part (6). Since L is a skeleton, $|[a]| = |[b]| = 1$ by Lemma 3.1.7, thus $[x,y] \cong HS(\{x,y,a\}, \{x,y,b\})$. Hence, $|[x,y]| = |\{x,y,a,b\}| = 4$ and $[x,y]$ has unique 2-determinant $\{a,b\}$.

(4) Since L is of width 2, L has no sublattice isomorphic to M_3. We claim that L has no sublattice isomorphic to N_5. Suppose not. Let $\langle a,b,c,u,v\rangle$ be a pentagon in L. Since L is a skeleton, we have $a \not\sim b$, so there is some $d \in [b,a)$ such that $d \in M^{\sim}(L)$ or some $d \in (b,a]$ such that $d \in J^{\sim}(L)$. We may assume that there is $d \in [b,a)$ such that $d \in M^{\sim}(L)$, it follows that $c \vee d = c \vee b$ and $d \prec c \vee d$ by Lemma 6.1.3 part (1); since $c \parallel a$, we have $c \vee d \nleq a$; since $d < a$ and $d < c \vee d$, we have $a \nless c \vee d$, thus $a \parallel c \vee d$, so $a \parallel c \vee b$ contradicting the fact that $\langle a,b,c,u,v\rangle$ is a pentagon. Hence, L is distributive. $\square$

In fact any significant interval i_X in a skeleton of width 2 is a 2-significant interval since there is no 1-significant interval in a skeleton, and $i_X \cong \widetilde{i_X}$ by the function $\sim : i_X \to \widetilde{i_X}$.

Lemma 6.1.15. *Let L be a finite skeleton of width 2 and $[x,y]$ be a 2-component. If $\{a,b\}$ is a 2-determinant of $[x,y]$, then*

(1) *$\{a,b\} \subseteq M(L) \cap J(L)$,*

(2) *$\{a,b\} \cap C_x \neq \emptyset$ and $\{a,b\} \cap C^y \neq \emptyset$.*

Proof. (1) Let $[x,y]$ be a 2-component and $\{a,b\}$ be a 2-determinant of $[x,y]$. Then $x = a \wedge b$ and $y = a \vee b$. We claim that $a \in M(L)$. Suppose that

$a \in M^{\sim}(L)$. Then $a \prec a \vee b = y$ by Lemma 6.1.3 part (1). Let $w = C_a - \{y\}$. Then $w \nleq b$ since $a < w$ and $a \parallel b$. We have $b \nleq w$, otherwise, $y = a \vee b \leq w$ contradicting $C_a = \{y, w\}$. So $w \parallel b$. Since $w \parallel y$ and $y \in J^{\sim}(L)$, we have $w \in J(L)$. Also since $w \wedge b < w$ and $C^w = \{a\}$, it follows that $w \wedge b \leq a$ and $x = a \wedge b \leq w \wedge b \leq a \wedge b = x$. Hence, $x = w \wedge b$. Since $y = a \vee b \leq w \vee b$ and $w \nleq y$, we have $y < w \vee b$. Thus $[x, y] \subset [w \wedge b, w \vee b] \in \mathscr{D}_2(L)$ contradicting the maximality of a 2-component $[x, y]$. Dually, $a \in J(L)$. Hence $a \in M(L) \cap J(L)$. By symmetry $b \in M(L) \cap J(L)$.

(2) Suppose that $\{a, b\} \cap C_x = \emptyset$. Since $a \parallel b$ and $w(L) = 2$, we have $a \nparallel \bigvee C_x$ or $b \nparallel \bigvee C_x$; also since $[x, \bigvee C_x] = \{x, \bigvee C_x\} \cup C_x$ by Lemma 6.1.14 part (3), it follows that $\bigvee C_x \leq a$ or $\bigvee C_x \leq b$. We may assume $\bigvee C_x \leq a$, then $\bigvee C_x \leq b$ or $\bigvee C_x \parallel b$ since $a \parallel b$. If $\bigvee C_x \leq b$, then $x < \bigvee C_x \leq a \wedge b$ contradicting $a \wedge b = x$. If $\bigvee C_x \parallel b$, then $(\bigvee C_x) \wedge b \prec \bigvee C_x$ by Lemma 6.1.3 part (2), thus $x < (\bigvee C_x) \wedge b \leq a \wedge b$ contradicting $a \wedge b = x$. Hence, $\{a, b\} \cap C_x \neq \emptyset$. Dually, $\{a, b\} \cap C^y \neq \emptyset$. □

6.2 Finite Lattices of Width 3

6.2.1 Properties of Finite Lattices of Width 3

Let i_X and i_Y be two 3-significant intervals in L with $X, Y \in \pi_L^3$. The following Theorem shows the relation between i_X and i_Y.

Theorem 6.2.1. *Let L be a finite lattice of width 3 and let i_X and i_Y be 3-significant intervals in L with $X, Y \in \pi_L^3$. Then $\bigwedge X \leq \bigwedge Y$ and $\bigvee X \leq \bigvee Y$, or $\bigwedge Y \leq \bigwedge X$ and $\bigvee Y \leq \bigvee X$.*

Proof. The lattice L is a union of three chains C_1, C_2 and C_3 by Dilworth's Theorem. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ with $x_i, y_i \in C_i$ for each $i \in \{1, 2, 3\}$. We claim that $(\bigwedge X) \nparallel (\bigwedge Y)$. Suppose that $(\bigwedge X) \parallel (\bigwedge Y)$. Then $(\bigvee X) \parallel (\bigvee Y)$ by Lemma 5.2.6 part (1), thus $X \not\sqsubseteq Y$ and $Y \not\sqsubseteq X$. We may assume that $y_1 \leq x_1$, $y_2 \leq x_2$ and $x_3 \leq y_3$. Then $X \bigsqcup Y = \{x_1, x_2, y_3\}$, $X \bigsqcap Y = \{y_1, y_2, x_3\}$, $\bigwedge Y < y_1 \leq x_1 < \bigvee X$, $\bigwedge Y < y_2 \leq x_2 < \bigvee X$ and $\bigwedge X < x_3 \leq y_3 < \bigvee Y$. Let $a = (\bigwedge X) \vee (\bigwedge Y)$ and $b = (\bigvee X) \wedge (\bigvee Y)$, then $\bigwedge X, \bigwedge Y < a \leq b < \bigvee X, \bigvee Y$. Since $[a, \bigvee X] \subset i_X$, we have $i_X = ([a, \bigvee X]) \cup (i_X - [a, \bigvee X])$. Since $\bigwedge Y \leq x_1$ and $\bigwedge X \leq x_1$, we have $a \leq x_1$, similarly $a \leq x_2$, $a \leq y_3$ and $x_3 \leq b$ hold. Hence, $2 \leq |X \cap [a, \bigvee X]| \leq 3$. But $|X \cap [a, \bigvee X]| \neq 3$, since $\bigwedge X < a$ follows from the supposition that $\bigwedge X \parallel \bigwedge Y$. Therefore, $|X \cap [a, \bigvee X]| = 2$ and $a \nleq x_3$.

We claim that $\{x_1, x_2, a \vee x_3\} \in \pi_L^3$ and $\{x_1, x_2, a \vee x_3\} \subseteq [a, \bigvee X]$. Since $a \leq y_3$ and $x_3 \leq y_3$, we have $a \vee x_3 \leq y_3$; also since $X \sqcup Y = \{x_1, x_2, y_3\} \in \pi_L^3$ by Lemma 5.1.10 part (2) and (3), we have $x_1 \nleq a \vee x_3$; similarly $x_2 \nleq a \vee x_3$. Since $\{x_1, x_2, x_3\} \in \pi_L^3$, it follows that $a \vee x_3 \nleq x_1$ and $a \vee x_3 \nleq x_2$. Hence $\{x_1, x_2, a \vee x_3\} \in \pi_L^3$. Since $a \leq x_1 \leq \bigvee X$ and $a \leq x_2 \leq \bigvee X$, it follows that $x_1, x_2 \in [a, \bigvee X]$. We have $a \leq a \vee x_3 \leq b \leq \bigvee X$ since $x_3 \leq b$ and $a \leq b$. Thus $x_1, x_2, a \vee x_3 \in [a, \bigvee X]$. Hence, $i_{\{x_1, x_2, a \vee x_3\}} \subseteq [a, \bigvee X] \subset i_X$ contradicting i_X is a 3-significant interval. Therefore, $(\bigwedge X) \nparallel (\bigwedge Y)$. By Lemma 5.2.6 part (2), if $\bigwedge X \leq \bigwedge Y$, then $\bigvee X \leq \bigvee Y$; by Lemma 5.2.6 part (3), if $\bigwedge Y \leq \bigwedge X$, then $\bigvee Y \leq \bigvee X$. $\square$

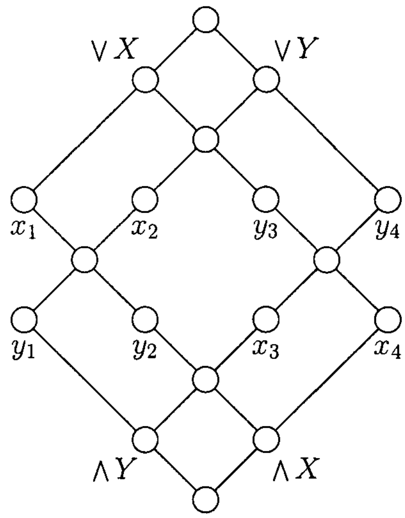


Figure 6.6 Two significant intervals i_X and i_Y in a lattice with $\bigwedge X \parallel \bigwedge Y$ and $\bigvee X \parallel \bigvee Y$.

For $i_X, i_Y \in \mathscr{S}\mathscr{I}_4(L)$, $\bigwedge X \parallel \bigwedge Y$ might happen. Figure 6.6 gives an example. In this example, $X = \{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$.

The following corollary to Theorem 6.2.1 shows that, in a finite lattice L of width 3, the union of two 3-significant intervals is a sublattice of L.

Corollary 6.2.2. *Let i_X, i_Y be two 3-significant intervals in a finite lattice L of width 3. Then $i_X \cup i_Y$ is a sublattice of L.*

Proof. We have $\bigwedge X \leq \bigwedge Y$ and $\bigvee X \leq \bigvee Y$, or $\bigwedge Y \leq \bigwedge X$ and $\bigvee Y \leq \bigvee X$ by Theorem 6.2.1. We may assume that $\bigwedge X \leq \bigwedge Y$ and $\bigvee X \leq \bigvee Y$. Let $a, b \in i_X \cup i_Y$. If $a, b \in i_X$, then $a \wedge b, a \vee b \in i_X$, so $a \wedge b, a \vee b \in i_X \cup i_Y$. Similarly if $a, b \in i_Y$, then $a \wedge b, a \vee b \in i_X \cup i_Y$. We may assume that $a \in i_X$ and $b \in i_Y$, so that $\bigwedge X \leq a \leq \bigvee X$ and $\bigwedge Y \leq b \leq \bigvee Y$; thus, $\bigwedge X = (\bigwedge X) \wedge (\bigwedge Y) \leq a \wedge b \leq a \leq \bigvee X$ and $\bigwedge Y \leq b \leq a \vee b \leq (\bigvee X) \vee (\bigvee Y) = \bigvee Y$. Hence $a \wedge b, a \vee b \in i_X \cup i_Y$. Therefore, $i_X \cup i_Y$ is a sublattice of L. □

6.2.2 The Structure of a Finite Lattice of Width 3

Lemma 6.2.3. *Let L be a finite lattice of width 3, let* $[x_1, y_1]$ *and* $[x_2, y_2]$ *be distinct 3-significant intervals. Then*

(1) $x_1 < x_2$ *or* $x_2 < x_1$,

(2) $x_1 < x_2$ *iff* $y_1 < y_2$,

(3) $x_2 < x_1$ *iff* $y_2 < y_1$.

Proof. (1) Since $[x_1, y_1]$ and $[x_2, y_2]$ are 3-significant intervals, there exist Z_1 and Z_2 in π_L^3 such that $[x_1, y_1] = i_{Z_1}$ and $[x_2, y_2] = i_{Z_2}$. By Theorem 6.2.1 we have $\bigwedge i_{Z_1} \nparallel \bigwedge i_{Z_2}$, i.e., $x_1 \nparallel x_2$; since $[x_1, y_1]$ and $[x_2, y_2]$ are distinct 3-significant intervals, we have $x_1 \neq x_2$ by Lemma 5.2.6 part (3). Hence $x_1 < x_2$ or $x_2 < x_1$.

(2) and (3) follow from Lemma 5.2.6 part (2). □

Since the set of the least elements of the 3-significant intervals forms a chain, we may list all 3-significant intervals in increasing order of least element of these intervals, i.e., $\mathscr{S}\mathscr{I}_3(L) = \{[x_i, y_i] \mid x_1 < x_2 < \ldots < x_k\}$.

Lemma 6.2.4. *Let L be a lattice of width 3 and let* $[x_1, y_1], [x_2, y_2] \in \mathscr{D}_3(L)$ *with* $x_1 < x_2$. *If* $w([x_2, y_1]) = 2$, *then* $[x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2])$ *is an empty set or a chain.*

Proof. Since $w([x_2, y_1]) = 2$, there exists $\{a, b\} \in \pi^2_{[x_2, y_1]}$. Suppose that $[x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2]) \neq \emptyset$. Let $z \in [x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2])$.

We claim that $z \parallel a$ and $z \parallel b$. Since $x_2 \leq a$, $z \leq y_2$ and $z \notin [x_2, y_2]$, we have $a \nleq z$; since $a \leq y_1$, $x_1 \leq z$ and $z \notin [x_1, y_1]$, we have $z \nleq a$; so $a \parallel z$; by symmetry $b \parallel z$. Since $\{a, b, z\} \in \pi_L^3$ for $z \in [x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2])$, any two elements in $[x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2])$ are comparable. Hence, $[x_1, y_2] - ([x_1, y_1] \cup [x_2, y_2])$ is a chain. $\square$

CHAPTER 7

SUMMARY AND DISCUSSION

In this dissertation, we develop the equivalent relation $\sim$ on a poset P, the blocks of which are the maximal autonomous chains of the poset P. The order skeleton $\tilde{P}$ of P is the poset formed by the blocks of $\sim$. The mapping $\sim : P \twoheadrightarrow \tilde{P}$ from a poset P to its order skeleton $\tilde{P}$ is an order-epimorphism, P may be recaptured from $\tilde{P}$ as a lexicographic sum of $\tilde{P}$ and the chains $\{[x] \mid x \in P\}$. Specifying to a lattice L, $\sim$ is a congruence relation on L and the mapping $\sim: L \twoheadrightarrow \tilde{L}$ is the canonical epimorphism, so that $\tilde{L}$ is realized as $L/\sim$.

In order to solve some problems about a lattice, firstly, we can create a similar problem based on its order skeleton, then work on the similar problem; finally, we seek the solution for the initial problem. We handle the following two problems by applying the order skeleton. The first one concerns the calculation of the residuated approximation, the second one concerns the structure of finite lattices of small widths.

For any isotone mapping $f : L \to Q$ between two complete lattices, it has been known that the umbral number u_f of f is small given sufficient distributivity of L or Q. We proved that in fact, u_f is small when there is sufficient distributivity of $\tilde{L}$ or $\tilde{Q}$. For a $\sim$ finite lattice L, we proved that $u_f = u_{f_o}$, where f_o is essentially a function from $\tilde{L}$ to Q, induced by f.

We find that the skeleton of any finite lattices of width 2 is distributive. Moreover, we have some results about the structure of finite lattices of width 3. Combining order skeleton and antichain helps to reveal the structure of finite lattices of larger width. Future research will focus on developing a structure theory of finite lattices.

APPENDIX A

SOURCE CODE

To support this research, we developed a software named ISRMap which provided the following functions:

(1) Draw and edit the Hasse diagrams of posets.

(2) Judge whether a poset is a lattice.

(3) Calculate and list all isotone mappings between two finite posets.

(4) Calculate and list all umbral mappings for any isotone mapping between two finite lattices.

(5) Calculate the umbral number u_f for any mapping $f : L \to Q$ between two finite lattices and list the $u_{L,Q}$.

We used this program to examine many examples which generated intuition for the creation of the relevant theorems. For some examples, we uploaded the computing task on LONI (Louisiana Optical Network Initiative) to finish the computation. Having so generated intuition, we proceeded to write the dissertation without any reference to the program. Our results depend on the program in no way – except for the initial intuition. However, for completeness, we present the source code.

We use a one-dimension array called "points" to represent elements in a lattice and a two-dimension array called "partial" to represent the partial ordering relation between two elements in a lattice.

```
struct points {
  char label[6];
  char imagelabel[6];
  int index;
  int imageindex;
};


struct partial{
 int index;  //to save the index of upper point.
 CString label;  //to save the label of upper point.
 int distance;//to save the distance between two points.
};
```

Now we describe the purposes of the following functions:

1. the function CISRMapDoc::fplus is to compute the $f^{(+)}$;
2. the function CISRMapDoc::fminus is to compute the $f^{(-)}$;
3. the function CISRMapDoc::OnCalSha is to compute σ_f.

```
void CISRMapDoc::fplus(struct partial p[][MAXNODE],
                       struct partial q[][MAXNODE]){
  int i, j;
```

```
bool nullmap;
for(i = 0; i<nq; i++){
  backplus[i]=qpoint[i].imageindex;
  qpoint[i].imageindex = -1;
  qpoint[i].imagelabel = "";
}
for(i = 0; i<np; i++){
 if(qpoint[ppoint[i].imageindex].imageindex == -1){
    qpoint[ppoint[i].imageindex].imageindex = i;
    qpoint[ppoint[i].imageindex].imagelabel =
                                        ppoint[i].label;
 }
 else{
  qpoint[ppoint[i].imageindex].imageindex=pjoin[i]
   [qpoint[ppoint[i].imageindex].imageindex][1];
  qpoint[ppoint[i].imageindex].imagelabel=ppoint
   [qpoint[ppoint[i].imageindex].imageindex].label;
 }
}
nullmap = true;
if(qpoint[qlowest[1]].imageindex == -1){
 qpoint[qlowest[1]].imageindex=plowest[1];
 qpoint[qlowest[1]].imagelabel=ppoint[plowest[1]].label;
```

```
}
while(nullmap){
  nullmap = false;
  for(i = 0; i<nq; i++){
   if(qpoint[i].imageindex == -1)
    nullmap = true;
   //to get fplus for point i.
   for(j = 0; j<nq; j++){
    if((i!=j)&&(qpartial[i][j].label==
      qpoint[i].label)){
     //point j under point i.
     if(qpoint[j].imageindex == -1){
      //point j not mapped any point in P.
      break;
     }
     if(qpoint[i].imageindex == -1){
      qpoint[i].imageindex = qpoint[j].imageindex;
      qpoint[i].imagelabel = qpoint[j].imagelabel;
     }
     else{
      qpoint[i].imageindex=pjoin[qpoint[i].imageindex]
       [qpoint[j].imageindex][1];
      qpoint[i].imagelabel =
```

```
          ppoint[qpoint[i].imageindex].label;
        }
       }
      }
     }
   }
}

void CISRMapDoc::fminus(struct partial p[][MAXNODE],
                        struct partial q[][MAXNODE]){
  int i, j, k;
  bool nullmap;
  for(i = 0; i<np; i++)
    backpmap[i] = -1;
  for(i = 0; i<nq; i++){
    if(backpmap[qpoint[i].imageindex] == -1)
      backpmap[qpoint[i].imageindex] = i;
    else
      backpmap[qpoint[i].imageindex] =
          qmeet[i][backpmap[qpoint[i].imageindex]][1];
  }
  nullmap = true;
  if(backpmap[plowest[1]] == -1)
```

```
    backpmap[plowest[1]] = qlowest[1];
  while(nullmap){
    nullmap = false;
    for(i = 0; i<np; i++){
      if(backpmap[i] == -1)
        nullmap = true;
      //to get fminus for point i.
      for(j = 0; j<np; j++){
       if((i!=j)&&(ppartial[i][j].label==
        ppoint[j].label)){
        //point j above point i.
         if(backpmap[j] == -1)//point j not mapped.
          continue;
         if(backpmap[i] == -1)
          backpmap[i] = backpmap[j];
         else
          backpmap[i]=qmeet[backpmap[i]][backpmap[j]][1];
        }
      }
    }
  }
}
```

```
void CISRMapDoc::OnCalSha(){
  char num[12];
  if(!checkres()){
    numsh++;
    preprocess();
    fplus(qpartial, ppartial);
    fminus(ppartial, qpartial);
    UpdateAllViews(NULL);
    showhint("", "");
  }
  memset(num, '\0', 12);
  memcpy(num, "SHADOW: ",8);
  _itoa(numsh, num+8, 10);
  showhint("", num);
}
```

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