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Results in lattices, ortholattices, and graphs

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RESULTS IN LATTICES, ORTHOLATTICES, AND GRAPHS

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ABSTRACT

This dissertation contains two parts: lattice theory and graph theory.

In the lattice theory part, we have two main subjects. First, the class of all distributive lattices is one of the most familiar classes of lattices. We introduce "π-versions" of five familiar equivalent conditions for distributivity by applying the various conditions to 3-element antichains only. We prove that they are inequivalent concepts, and characterize them via exclusion systems. A lattice L satisfies $D_{0\pi}$ if $a \wedge (b \vee c) \leq (a \wedge b) \vee c$ for all 3-element antichains $\{a, b, c\}$. We consider a congruence relation $\sim$ whose blocks are the maximal autonomous chains and define the order-skeleton of a lattice L to be $\widetilde{L} := L/\sim$. We prove that the following are equivalent for a lattice L: (i) L satisfies $D_{0\pi}$, (ii) $\widetilde{L}$ satisfies any of the five π-versions of distributivity, (iii) the order-skeleton $\widetilde{L}$ is distributive.

Second, the symmetric difference notion for Boolean algebra is well-known. Matoušek introduced the orthocomplemented difference lattices (ODLs), which are ortholattices associated with a symmetric difference. He proved that the class of ODLs forms a variety. We focus on the class of all ODLs that are set-representable and prove that this class is not locally finite by constructing an infinite set-representable ODL that is generated by three elements.

In the graph theory part, we prove generating theorems and splitter theorems for 5-regular graphs. A generating theorem for a certain class of graphs tells us how to generate all graphs in this class from a few graphs by using some graph operations. A splitter theorem tells us how to build up any graph G from any graph H if G "contains" H. In this dissertation, we find generating theorems for 5-regular graphs and 5-regular loopless graphs for various edge-connectivities. We also find splitter theorems for 5-regular graphs for various edge-connectivities.

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PART I

LATTICE THEORY

CHAPTER 1

REVIEW OF LATTICES

1.1 Posets and Lattices

In this chapter, we provide some basic concepts and results on lattices and distributive lattices.

Definition 1.1.1. Let P be a set. A *partial order* on P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$,

(*i*) $x \leq x$ (*reflexivity*),

(*ii*) $x \leq y$ and $y \leq x$ imply $x = y$ (*antisymmetry*),

(*iii*) $x \leq y$ and $y \leq z$ imply $x \leq z$ (*transitivity*).

The pair $\langle P; \leq \rangle$ is called a *partially ordered set* (or *poset*); when there is no ambiguity, we sometimes refer to it as P.

Let P be a poset with $x, y \in P$. If $x \leq y$ or $y \leq x$, then we say that x and y are *comparable*, denoted by $x \nparallel y$; otherwise, x and y are *parallel* (or *incomparable*), denoted by $x \parallel y$. A poset P is a *chain* if every two elements of P are comparable. A poset P is an *antichain* if every two distinct elements of P are parallel.

Given two posets P and Q, a mapping ϕ from P onto Q is an (*order-*)*isomorphism* if $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q. Two posets P and Q are (*order-*) *isomorphic*, denoted by $P \cong Q$, if there exists an isomorphism from P onto Q.

We use the symbol ":=" to mean "equals by definition." Given any poset $\langle P; \leq \rangle$ we can form a new poset $\langle P^*; \leq^* \rangle$, called the *dual* of $\langle P; \leq \rangle$, by defining $P^* := P$ and $x \leq^* y$ holds in P^* if and only if $y \leq x$ holds in P. A poset $\langle P; \leq \rangle$ is *self-dual* if P is isomorphic to its dual P^*. To each statement T about the poset P there corresponds a *dual statement* T^* about the poset P^* obtained by reversing the ordering in S. Given a statement about posets that is true in all posets, the dual statement is also true in all posets. This is the *Duality Principle for Posets.*

In this dissertation, we write $A \subseteq B$ to mean that A is a subset of B; we write $A \subset B$ to mean $A \subseteq B$ and $A \neq B$. We use $\mathbb{N}$ for the positive integers $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0$ for the non-negative integers $\{0, 1, 2, \dots\}$. Given a set S, we use ${}^{\#}S$ for the *cardinality* of S.

Let P be a poset and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. An element $x \in P$ is the *least upper bound* of S, or *supremum* of S ($\sup S$) if x is an upper bound of S, and $s \leq y$ for all $s \in S$ implies $x \leq y$. Dually, we can define what it means for x to be a *lower bound* of S, and for x to be the *greatest lower bound* of S, also called the *infimum* of S ($\inf S$). If $\sup S$ (resp., $\inf S$) exists, then we denote it by $\bigvee S$ (resp., $\bigwedge S$) and call it the *join* (resp., *meet*) of S. We sometimes write $\bigvee_P S$ (resp., $\bigwedge_P S$) to emphasize that $\bigvee S$ (resp., $\bigwedge S$) is calculated in P. For $x, y \in P$, if $\sup\{x, y\}$ (resp., $\inf\{x, y\}$) exists, then we denote it by $x \vee y$ (resp., $x \wedge y$) and call it the *join* (resp., *meet*) of x and y.

Definition 1.1.2. Let L be a non-empty poset. If both $x \vee y$ and $x \wedge y$ exist for all $x, y \in L$, then L is called a *lattice*. If both $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq L$, then L is called a *complete lattice*. A lattice L is usually expressed as $\langle L; \vee, \wedge \rangle$. A lattice L

has a *top element* (resp., *bottom element*), usually denoted by 1 (resp., 0), if $x \leq 1$ (resp., $0 \leq x$) for all $x \in L$. A lattice L with top and bottom elements is called a *bounded lattice*. A nonempty subset S of a lattice L is called a *sublattice* if $x \vee y \in S$ and $x \wedge y \in S$ for all $x, y \in S$.

An *n-ary operation* on a set A is a function that takes n elements of A and returns a single element of A. A *nullary* operation is a 0-ary operation. A *unary* operation is a 1-ary operation. A *binary* operation is a 2-ary operation. An *operation* is an n-ary operation for some $n \in \mathbb{N}_0$. An *algebra* (or *algebraic structure*) is a tuple $\langle A; \mathcal{F} \rangle$ where A is a set and $\mathcal{F}$ is a collection of operations on A. If $\mathcal{F}$ is finite, say $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ for some $k \in \mathbb{N}$, then we write $\langle A; f_1, f_2, \ldots, f_k\}$ for $\langle A; \mathcal{F} \rangle$. We say that $\langle A; f_1, f_2, \ldots, f_k \rangle$ is an *algebra of type* $(n_1, n_2, \ldots, n_k)$ if, for each $i \in \{1, 2, \ldots, k\}$, f_i is an n_i-ary operation.

Let $\langle A; \mathcal{F}_A \rangle$ and $\langle B; \mathcal{F}_B \rangle$ be two algebras of the same type. Then $\langle B; \mathcal{F}_B \rangle$ is a *subalgebra* of $\langle A; \mathcal{F}_A \rangle$ if $B \subseteq A$ and every operation in $\mathcal{F}_B$ is the corresponding operation in $\mathcal{F}_A$ restricted to B. Note that the intersection of a family of subalgebras is a subalgebra. Hence, for each non-empty subset S of an algebra A, there exists a smallest subalgebra of A containing S; we call this the subalgebra of A *generated* by S, denoted by $\Gamma(S)$. If $S = \{a, b, c\}$, then we write $\Gamma\{a, b, c\}$ for $\Gamma(S)$. An algebra A is 3-*generated* if A is generated by a 3-element subset of A.

A lattice L can be defined alternatively as an algebra $\langle L; \vee, \wedge \rangle$ of type $(2, 2)$ satisfying

(*i*) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$ (commutative laws),

(*ii*) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = z \wedge (y \wedge z)$ (associative laws),

(iii) $x \vee x = x$, $x \wedge x = x$ (idempotent laws), and

(iv) $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$ (absorption laws).

Given a lattice $\langle L; \vee, \wedge \rangle$, one can define a partial order on L by setting $x \leq y$ if and only if $x = x \wedge y$ for all $x, y \in L$.

Note that the dual of a statement about lattices phrased in terms of $\vee$ and $\wedge$ is obtained simply by interchanging the symbols $\vee$ and $\wedge$. This is called the *Duality Principle for Lattices.*

Let L and M be two lattices. A mapping $\phi : L \to M$ is said to be a *(lattice-) homomorphism* if, for all $x, y \in L$, $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$. A bijective homomorphism is called a *(lattice-)isomorphism.* Two lattices L and M are *isomorphic*, denoted by $L \cong M$, if there exists an isomorphism from L onto M. An injective homomorphism is called a *(lattice-)embedding.* A lattice M is *(lattice-) embeddable* in a lattice L if there exists an embedding from L into M.

Let L and M be two lattices. The *cartesian product* of L and M is $\langle L \times M; \vee, \wedge \rangle$ where $\vee$ and $\wedge$ are defined by $(a, b) \vee (c, d) := (a \vee c, b \vee d)$ and $(a, b) \wedge (c, d) := (a \wedge c, b \wedge d)$ for all $a, c \in L$ and $b, c \in M$. Note that the cartesian product is a lattice. We denote this lattice by $L \times M$ when there is no ambiguity. In this dissertation, we write $\mathbf{n}$ for an n-element chain. For example, $\mathbf{3}$ is a 3-element chain. Given a lattice L and an integer $n \geq 1$, we denote by L^n the cartesian product of n copies of L. For example, $\mathbf{2}^3$ is an 8-element lattices.

1.2 Distributive Lattices

Distributive lattices are perhaps the most familiar class of lattices. They are ubiquitous but rather specific structures.

Definition 1.2.1. Let L be a lattice with $a, b, c \in L$. We define

$M(a, b, c)$ to mean $a \wedge (b \vee (a \wedge c)) = (a \wedge b) \vee (a \wedge c)$, and

$M^*(a, b, c)$ to mean $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.

A lattice L is *modular* if $M(a, b, c)$ holds for all $a, b, c \in L$.

The prototypical non-modular example is the lattice N_5 which is presented in Figure 1.1. Dedekind characterized modular lattices by the following theorem.

Theorem 1.2.2 (Dedekind). *A lattice L is non-modular if and only if N_5 can be embedded into L.*

Theorem 1.2.3. *Let L be a lattice. The following statements are equivalent.*

(i) L is modular, i.e., $M(a, b, c)$ holds for all $a, b, c \in L$.

(ii) $M^*(a, b, c)$ holds for all $a, b, c \in L$.

Proof. Note that N_5 is self-dual, so that the theorem follows from Theorem 1.2.2 and its dual statement. □

Definition 1.2.4. Let L be a lattice with $a, b, c \in L$. We define

$D(a, b, c)$ to mean $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

$D^*(a, b, c)$ to mean $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

$D_m(a, b, c)$ to mean $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, and

$D_0(a, b, c)$ to mean $a \wedge (b \vee c) \leq (a \wedge b) \vee c$.

A lattice L is *distributive* if $D(a, b, c)$ holds for all $a, b, c \in L$.

The prototypical non-distributive examples are M_3 and N_5 which are presented in Figure 1.1. Birkhoff characterized distributive lattices by the following theorem.

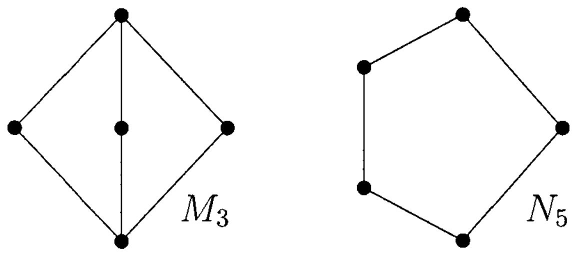


Figure 1.1: Two standard non-distributive lattices

Theorem 1.2.5 (Birkhoff). *A lattice L is non-distributive if and only if M_3 or N_5 can be embedded into L.*

Theorem 1.2.6. *Let L be a lattice. The following statements are equivalent.*

(*i*) *L is distributive, i.e., $D(a, b, c)$ holds for all $a, b, c \in L$.*

(*ii*) *$D^*(a, b, c)$ holds for all $a, b, c \in L$.*

(*iii*) *$D_m(a, b, c)$ holds for all $a, b, c \in L$.*

(*iv*) *$D_0(a, b, c)$ holds for all $a, b, c \in L$.*

Proof. (*i*) $\Leftrightarrow$ (*ii*) Since M_3 and N_5 are self-dual, this follows immediately from Theorem 1.2.5 and its dual statement.

(*i*) $\Rightarrow$ (*iii*) Suppose that $D(a, b, c)$ holds for all $a, b, c \in L$. Hence $D^*(a, b, c)$ holds for all $a, b, c \in L$. Let $d, e, f \in L$. We have that

$$(d \wedge e) \vee (e \wedge f) \vee (f \wedge d) = (e \wedge (d \vee f)) \vee (d \wedge f)$$

$$=(e \vee (d \wedge f)) \wedge (d \vee f) = (d \vee e) \wedge (e \vee f) \wedge (f \vee d).$$

(*iii*) $\Rightarrow$ (*iv*) Suppose that $D_m(a, b, c)$ holds for all $a, b, c \in L$. Let $d, e, f \in L$. Then, $d \wedge (e \vee f) \leq (d \vee e) \wedge (d \vee f) \wedge (e \vee f) = (d \wedge e) \vee (d \wedge f) \vee (e \wedge f) \leq (d \wedge e) \vee f$.

$(iv) \Rightarrow (\imath)$ Suppose that $D_0(a,b,c)$ holds for all $a,b,c \in L$. Let $d,e,f \in L$. Since $d \wedge e \leq d \wedge (e \vee f)$ and $d \wedge f \leq d \wedge (e \vee f)$, we have $(d \wedge e) \vee (d \wedge f) \leq d \wedge (e \vee f)$. We also have that

$$\begin{aligned} d \wedge (e \vee f) &= d \wedge (d \wedge (f \vee e)) \leq d \wedge ((d \wedge f) \vee e) \\ &= d \wedge (e \vee (d \wedge f)) \leq (d \wedge e) \vee (d \wedge f). \end{aligned}$$

Thus, $d \wedge (e \vee f) = (d \wedge e) \vee (d \wedge f)$. $\square$

CHAPTER 2

ORDER-SKELETONS

2.1 Order-skeleton on a Lattice

Let L be a lattice and let $a, b \in L$. As usual, we write $[a, b] := \{x \in L \mid a \leq x \leq b\}$ and $[a, b) := \{x \in L \mid a \leq x < b\}$. We allow for the possibility that $a \not\leq b$, in which case, of course, both sets are empty. Define $\pi(a) := \{b \in L \mid b \parallel a\}$. We denote the set of antichains in L by π_L and the set of n-element antichains in L by π_L^n, where $n \geq 1$. The following definition plays an important role.

$$a \sim b \text{ means } a \nparallel b \text{ and } \pi(a) = \pi(c) \text{ for all } c \in [a, b] \cup [b, a].$$

Following [32], we define a non-empty subset S of L to be (*order-*)*autonomous* in case, for all $p \notin S$, (1) if there is an $s \in S$ with $s \leq p$, then $x \leq p$ for all $x \in S$, and (2) if there is an $s \in S$ with $p \leq s$, then $p \leq x$ for all $x \in S$.

All the results in this section can be found in [12, 34]. We present the proofs for completeness.

Lemma 2.1.1. *Let L be a lattice with $a, b \in L$. The following are equivalent.*

(*i*) *$a \sim b$.*

(*ii*) *$[a, b]$ or $[b, a]$ is a chain and $\pi(a) = \pi(b)$.*

(*iii*) *$[a, b]$ or $[b, a]$ is an autonomous chain.*

Proof $(i) \Rightarrow (ii)$ We may assume that $a \sim b$ and $a \leq b$ We need only to show that $[a, b]$ is a chain For any $c, d \in [a, b]$, we have $\pi(c) = \pi(a) = \pi(d)$, so that $c \nparallel d$ It follows that $[a, b]$ is a chain

$(ii) \Rightarrow (iii)$ Since $\pi(a) = \pi(b)$, we have $a \nparallel b$ We may assume $a \leq b$, so that $[a, b]$ is a chain Let $c \in [a, b]$ and $p \notin [a, b]$ If $p \leq c \leq b$, then since $\pi(a) = \pi(b)$, we have $p \nparallel a$, so that $p \leq a$, thus, $p \leq x$ for all $x \in [a, b]$ Dually, if $c \leq p$, then $x \leq p$ for all $x \in [a, b]$ Therefore, $[a, b]$ is autonomous

$(iii) \Rightarrow (i)$ We may assume that $[a, b]$ is an autonomous chain Let $c \in [a, b]$ and $x \notin [a, c]$ Note that, $x \leq a$ if and only if $x \leq c$, and $a \leq x$ if and only if $c \leq x$ Thus, $x \parallel a$ if and only if $x \parallel c$ It follows that $\pi(a) = \pi(c)$, so that $a \sim b$ $\square$

Given a lattice L and a binary relation R on L, then R is *reflexive* if xRx for all $x \in L$, R is *symmetric* if xRy implies yRx for all $x, y \in L$, R is *transitive* if xRy and yRz imply xRz for all $x, y, z \in L$ A binary relation on a lattice L is an *equivalence relation* if it is reflexive, symmetric, and transitive An equivalence relation on a lattice L is a *congruence relation* if, for any $a, b, c, d \in L$, $a \; \theta \; b$ and $c \; \theta \; d$ imply that $(a \vee c) \; \theta \; (b \vee d)$ and $(a \wedge c) \; \theta \; (b \wedge d)$

Lemma 2.1.2. *Let L be a lattice and θ be an equivalence relation on L Then θ is a congruence relation on L if and only if, for any $a, b, c \in L$, $a \; \theta \; b$ implies that $(a \vee c) \; \theta \; (b \vee c)$ and $(a \wedge c) \; \theta \; (b \wedge c)$*

Proof $(\Rightarrow)$ This follows immediately from the definition

$(\Leftarrow)$ Let $a, b, c, d \in L$ with $a \; \theta \; b$ and $c \; \theta \; d$ We have that $(a \vee c) \; \theta \; (b \vee c) \; \theta \; (b \vee d)$ and $(a \wedge c) \; \theta \; (b \wedge c) \; \theta \; (b \wedge d)$ $\square$

Lemma 2.1.3. *Let L be a lattice and let A, B be two autonomous chains in L such that $A \cap B \neq \emptyset$ Then $A \cup B$ is an autonomous chain*

Proof Let $c \in A \cap B$ We first prove that $A \cup B$ is a chain Let $a \in A$ and $b \in B$ We may assume that $b \notin A$ Since A is autonomous and $b \nparallel c$, we have $b \nparallel a$ Hence, $A \cup B$ is a chain We now prove that $A \cup B$ is autonomous Let $p \notin A \cup B$ If there exists an $s \in A \cup B$ such that $s \leq p$, then $c \leq p$, so that $x \leq p$ for all $x \in A \cup B$ Similarly, if there exists an $x \in A \cup B$ such that $p \leq s$, then $p \leq x$ for all $x \in A \cup B$ Thus, $A \cup B$ is autonomous □

Lemma 2.1.4. *The relation $\sim$ defined on a lattice L is an equivalence relation on L*

Proof The reflexivity and symmetry follow directly from the definition The transitivity follows from the fact that the subsets of autonomous chains are autonomous chains and Lemma 2 1 3 □

Lemma 2.1.5. *The relation $\sim$ defined on a lattice L is a congruence relation on L*

Proof By Lemma 2 1 4, $\sim$ defines an equivalence relation on L Let $a, b, c \in L$ with $a \sim b$ Since $a \nparallel b$, we may assume that $a \leq b$ We shall argue that $a \vee c \sim b \vee c$ by the following two cases

Case 1 Suppose that $a \nparallel c$ Since $\pi(a) = \pi(b)$, we have $b \nparallel c$ Thus, $\{a, b, c\}$ is a chain If $c \leq a \leq b$, then $a \vee c = a \sim b = b \vee c$ If $a \leq c \leq b$, then $a \vee c = c \sim b = b \vee c$ If $a \leq b \leq c$, then $a \vee c = c \sim c = b \vee c$ Therefore, in all cases, $a \vee c \sim b \vee c$

Case 2 Suppose that $a \parallel c$ Since $a \leq b$, we have $a \vee c \leq b \vee c$ By Lemma 2 1 1 part (iii), $[a, b]$ is an autonomous chain Since $a \leq a \vee c$, we have $b \leq a \vee c$, so that $b \vee c \leq a \vee c$ Thus, $a \vee c = b \vee c$, and therefore, $a \vee c \sim b \vee c$

By a dual argument, we have $a \wedge c \sim b \wedge c$. Therefore, by Lemma 2.1.2, $\sim$ defines a congruence relation. □

Define $[a] := \{b \mid a \sim b\}$ and $\widetilde{L} := L/\sim = \{[a] \mid a \in L\}$. We call the quotient lattice $\langle \widetilde{L}; \vee_{\widetilde{L}}, \wedge_{\widetilde{L}} \rangle$ the *order-skeleton* of L.

Lemma 2.1.6. *Let L be a lattice with $a, b \in L$. Then*

(i) *$[a] \leq_{\widetilde{L}} [b]$ if and only if there exist $a_1 \in [a]$ and $b_1 \in [b]$ such that $a_1 \leq_L b_1$,*

(ii) *$[a] <_{\widetilde{L}} [b]$ if and only if $a <_L b$ and $[a] \neq [b]$;*

(iii) *$[a] \vee_{\widetilde{L}} [b]$ exists and equals $[a \vee_L b]$;*

(iv) *$[a] \wedge_{\widetilde{L}} [b]$ exists and equals $[a \wedge_L b]$;*

(v) *$a \parallel_L b$ if and only if $[a] \parallel_{\widetilde{L}} [b]$; and*

(vi) *$\pi_L(a) = \pi_L(b)$ if and only if $\pi_{\widetilde{L}}([a]) = \pi_{\widetilde{L}}([b])$*

Proof. (i), (ii), (iii), and (iv) follow directly from the fact that $\sim$ is a congruence relation (cf. [9]). Also, (v) follows from (i) and the definition of $\sim$, and (vi) follows from (v). □

For a lattice L with $a \in L$, the element a is *join-reducible* if there exist $b, c < a$ such that $a = b \vee c$ A *meet-reducible* element is defined dually. An element a is *doubly-reducible* if it is both join-reducible and meet-reducible. Note that under this definition, 0 is not join-reducible and 1 is not meet-reducible.

For convenience of notation, we use a, b, c for elements in L and x, y, z for elements in $\widetilde{L}$ The following lemma ensures that there is at most one join-reducible (resp , meet-reducible) element of L in each $[a] \in \widetilde{L}$.

Lemma 2.1.7. *Let L be a lattice with $a \in L$ and $x \in \widetilde{L}$.*

(i) *If a is join-reducible in L, then $\bigwedge_L [a]$ exists and $\bigwedge_L [a] = a$.*

(*ii*) *If x is join-reducible in $\widetilde{L}$, then $\bigwedge_L x$ exists and $\bigwedge_L x \in x$.*

(*iii*) *If a is meet-reducible in L, then $\bigvee_L [a]$ exists and $\bigvee_L [a] = a$.*

(*iv*) *If x is meet-reducible in $\widetilde{L}$, then $\bigvee_L x$ exists and $\bigvee_L x \in x$.*

Proof By duality, we need only prove (*i*) and (*ii*).

(*i*) Let $a \in L$ be join-reducible. We need only show that a is the lower bound of $[a]$. There exist $b, c \in L$ such that $b \parallel_L c$ and $a = b \vee_L c$. Since $b \parallel_L c$ and $a \not\parallel_L c$, we have $[a] \neq [b]$. By Lemma 2.1.6 (*i*), we have $[b] \leq_{\widetilde{L}} [a]$, so that $[b] <_{\widetilde{L}} [a]$. Let $u \in [a]$. By Lemma 2.1.6 (*ii*), $b <_L u$. Similarly, $c <_L u$. Hence, $a = b \vee_L c \leq_L u$.

(*ii*) Since x is join-reducible in $\widetilde{L}$, there exist $b, c \in L$ such that $[b], [c] <_{\widetilde{L}} x$ and $x = [b] \vee_{\widetilde{L}} [c]$. By Lemma 2.1.6 (*ii*), for all $a \in x$, we have $b, c < a$, so that $b \vee_L c \leq a$. By Lemma 2.1.6 (*iii*), $x = [b \vee_L c]$, so $b \vee_L c \in x$. Therefore $b \vee_L c$ is the smallest element in x, i.e., $\bigwedge_L x = b \vee_L c \in x$. □

Lemma 2.1.8. *Let L be a lattice with $x \in \widetilde{L}$. Then*

(*i*) *x is a maximal autonomous chain in L;*

(*ii*) *$\widetilde{\widetilde{L}} := (\widetilde{\widetilde{L}}) = \{\{x\} \mid x \in \widetilde{L}\}$, i.e., $\sim_{\widetilde{L}}$ is equality on $\widetilde{L}$.*

(*iii*) *$L \cong \widetilde{L}$ if and only if $\widetilde{L} = \{\{a\} \mid a \in L\}$*

Proof. (*i*) Let $x \in \widetilde{L}$. For any $b, c \in x$, we have $b \sim c$, so that $b \not\parallel_L c$; hence x is a chain. Let $p \notin x$ and $b, c \in x$. If $b \leq_L p$, then by Lemma 2.1.1, we have that $[b, c]$ or $[c, b]$ is autonomous, so that $c \leq_L p$. Similarly, $p \leq_L b$ implies $p \leq_L c$. Hence, x is autonomous. We now show that x is maximal Let $S \subseteq L$ be an autonomous chain containing x For $a \in x$ and $s \in S$, we have that $[a, s]$ or $[s, a]$ is autonomous, so that by Lemma 2 1.1, $a \sim s$. Thus, $S \subseteq [a] = x$, i.e., x is a maximal autonomous chain.

(*ii*) Let $x, y \in \widetilde{L}$ and $x \sim_{\widetilde{L}} y$. Since $x \not\parallel_{\widetilde{L}} y$, we may assume that $x \leq_{\widetilde{L}} y$, and hence, there exist $a \in x$ and $b \in y$ such that $a \leq_L b$. For any $c \in L$ with $a \leq_L c \leq_L b$, we have $x \leq_{\widetilde{L}} [c] \leq_{\widetilde{L}} y$. Thus, $\pi_{\widetilde{L}}(x) = \pi_{\widetilde{L}}([c])$, so that by Lemma 2 1.6 (*vi*), $\pi(a) = \pi(c)$. It follows that $a \sim b$, so that $x = y$. Therefore, $\widetilde{x} = \{x\}$.

(*iii*) If $\widetilde{L} = \{\{a\} \mid a \in L\}$, then the function $a \mapsto [a]$ is easily seen to be an isomorphism. For the converse, assume that $L \cong \widetilde{L}$ via the isomorphism $f : L \to \widetilde{L}$. Let $a, b \in L$ with $a \sim b$. We may assume that $a \leq b$. Since f is an isomorphism, we have $f(a) \leq_{\widetilde{L}} f(b)$. Let x be an arbitrary element in $[f(a), f(b)]$ and let $c := f^{-1}(x)$. We have $c \in [a, b]$, and thus, $\pi(c) = \pi(a)$ since $a \sim b$. Since f is an isomorphism, we have $\pi_{\widetilde{L}}(x) = \pi_{\widetilde{L}}(f(a))$. Therefore, by definition, $f(a) \sim_{\widetilde{L}} f(b)$. By (*ii*), we have $f(a) = f(b)$, so that $a = b$. Therefore, $\sim$ is equality on L, and $\widetilde{L} = \{\{a\} \mid a \in L\}$. $\square$

The following lemma utilizes the Axiom of Choice.

Lemma 2.1.9. *Let L be a lattice. Consider the following conditions.*

(*i*) *Every doubly-reducible element in $\widetilde{L}$ is a singleton subset of L*

(*ii*) *There exists an* embedding *$\beta : \widetilde{L} \hookrightarrow L$ such that $\beta(x) \in x$ for every $x \in \widetilde{L}$.*

(*iii*) *$\widetilde{L}$ is* embeddable *in L.*

(*iv*) *The cardinality of the set of doubly-reducible elements in L equals the cardinality of the set of doubly-reducible elements in $\widetilde{L}$*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) Moreover, if L contains finitely many doubly-reducible elements, then the four conditions are equivalent to each other

Proof (*i*) $\Rightarrow$ (*ii*) Assume that every doubly-reducible element in $\widetilde{L}$ is a singleton subset of L Note that, by Lemma 2.1.7, if x is join-reducible (resp., meet-reducible) in $\widetilde{L}$, then $\bigwedge_L x$ (resp., $\bigvee_L x$) exists and $\bigwedge_L x \in x$ (resp., $\bigvee_L x \in x$). By assumption, if

x is doubly-reducible in $\widetilde{L}$, then $\bigwedge_L x = \bigvee_L x$. Thus, there exists a selection function $\beta : \widetilde{L} \to L$ such that

(1) $\beta(x) \in x$,

(2) if x is join-reducible, then $\beta(x) = \bigwedge_L x$, and

(3) if x is meet-reducible, then $\beta(x) = \bigvee_L x$.

For all $x, y \in \widetilde{L}$, if $\beta(x) = \beta(y)$, then $x = [\beta(x)] = [\beta(y)] = y$. Hence, the mapping $\beta : \widetilde{L} \to L$ is one-to-one. We now show that β is a homomorphism. Let $[a], [b] \in \widetilde{L}$. If $[a] \nparallel_{\widetilde{L}} [b]$, then $\beta([a] \vee_{\widetilde{L}} [b]) = \beta([a]) \vee_L \beta([b])$. If $[a] \parallel_{\widetilde{L}} [b]$, then both $[a] \vee_{\widetilde{L}} [b]$ and $a \vee_L b$ are join-reducible, so that by Lemma 2.1.6 (iii) and Lemma 2.1.7, $\beta([a] \vee_{\widetilde{L}} [b]) = \beta([a \vee_L b]) = \bigwedge_L [a \vee_L b] = \beta([a]) \vee_L \beta([b])$. Dually, we have $\beta([a] \wedge_{\widetilde{L}} [b]) = \beta([a]) \wedge_L \beta([b])$. Therefore, β is an embedding.

$(ii) \Rightarrow (i)$ Let $\beta : \widetilde{L} \hookrightarrow L$ be an embedding such that $\beta(x) \in x$ for every $x \in \widetilde{L}$. Let $x \in \widetilde{L}$ be a doubly-reducible element. Since x is join-reducible in $\widetilde{L}$, there exist $y, z \in \widetilde{L}$ such that $y \parallel_{\widetilde{L}} z$ and $x = y \vee_{\widetilde{L}} z$. By Lemma 2.1.6 (v), $\beta(y) \parallel_L \beta(z)$, so that $\beta(x) = \beta(y) \vee_L \beta(z)$ is join-reducible in L. By Lemma 2.1.7, $\beta(x) = \bigwedge_L x$. Dually, $\beta(x) = \bigvee_L x$, so that $\bigwedge_L x = \bigvee_L x$, i.e., x is a singleton subset of L.

$(ii) \Rightarrow (iii)$ This follows immediately from the definition.

$(iii) \Rightarrow (iv)$ Let $f : \widetilde{L} \hookrightarrow L$ be an embedding. Let c_L and $c_{\widetilde{L}}$ be the cardinality of the doubly-reducible elements in L and $\widetilde{L}$ respectively. Note that f maps the doubly-reducible elements in $\widetilde{L}$ to the doubly-reducible elements in L, so that $c_{\widetilde{L}} \leq c_L$. Also note that every doubly-reducible element $a \in L$ corresponds to a doubly-reducible element $[a] \in \widetilde{L}$, so that $c_L \leq c_{\widetilde{L}}$. Therefore, $c_L = c_{\widetilde{L}}$.

Now assume that L contains finitely many doubly-reducible elements

$(iv) \Rightarrow (i)$ Suppose that x is doubly-reducible in $\widetilde{L}$ but not a singleton subset of L Since every doubly-reducible element $a \in L$ corresponds to a doubly-reducible element $[a] \in \widetilde{L}$, we have $c_L \leq c_{\widetilde{L}}$ Also note that there is no doubly-reducible element in L that corresponds to x It follows that $c_L < c_{\widetilde{L}}$ which contradicts (iv) □

Let $\{L_\alpha\}_{\alpha \in I}$ be a collection of pairwise disjoint bounded lattices such that $^{\#}L_\alpha \geq 2$ and $^{\#}I > 1$ Let $L = \{0, 1\} \cup \left(\bigcup_{\alpha \in I} (L_\alpha - \{0_{L_\alpha}, 1_{L_\alpha}\}) \right)$ with the partial ordering defined by $x \leq y$ in L if and only if $x = 0$, $y = 1$, or $x \leq_{L_\alpha} y$ for some $\alpha \in I$ Then L is called the *horizontal sum* of $\{L_\alpha\}_{\alpha \in I}$, denoted by $L = \mathrm{HS}(\{L_\imath \;\; \imath \in I\})$ Given two bounded lattices L_1 and L_2 with $^{\#}L_1 \geq 3$ and $^{\#}L_2 \geq 3$, we denote the horizontal sum of L_1 and L_2 by $\mathrm{HS}(L_1, L_2)$ Note that the horizontal sum of $\mathbf{2}$ and a lattice L with $^{\#}L \geq 2$ is isomorphic to L

Note that, in Lemma 2 1 9, parts (i), (iii), and (iv) are not in general equivalent to each other For example, let L be the horizontal sum of L_{15} (see Figure 3 1) with countably many copies of the $\mathbf{3}^2$ Then $\widetilde{L}$ is the horizontal sum of countably many copies of the $\mathbf{3}^2$ Then $\widetilde{L}$ is embeddable in L, but L contains a horizontal summand isomorphic to L_{15} so that $\widetilde{L}$ contains a doubly-reducible element which is not a singleton subset of L, i e , $(iii) \not\Rightarrow (i)$ Now let $M = \langle \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}, \cup, \cap \rangle$ which contains a doubly-reducible element $\{a, b\}$ Let Q be the horizontal sum of L_{15} with countably many copies of M In this example, both Q and $\widetilde{Q}$ have countably many doubly-reducible elements, but $\widetilde{Q}$ is not embeddable in Q, i e , $(iv) \not\Rightarrow (iii)$

Define a *pentagon* in a lattice L to be a quintuple $\langle a, b, c, u, v \rangle$ (see Figure 2.1) such that $a, b, c, u, v \in L$ and $v < b < a < u,\ c \wedge a = v,\ c \vee b = u$.

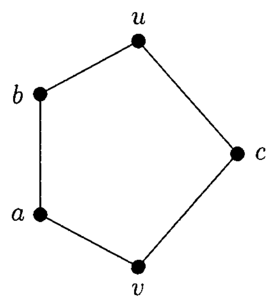


Figure 2.1: A pentagon

Lemma 2.1.10. *Let L be a lattice If $\widetilde{L}$ is non-modular, then $\widetilde{L}$ contains a pentagon $\langle x, y, z, u, v \rangle$ and there exists an element $w \in \widetilde{L}$ such that either*

(i) $x = y \vee_{\widetilde{L}} w$ *and* $\{w, y, z\} \in \pi^3_{\widetilde{L}}$; *or*

(ii) $y = x \wedge_{\widetilde{L}} w$ *and* $\{w, x, z\} \in \pi^3_{\widetilde{L}}$

Proof Let $\langle x_1, y, z, u, v \rangle$ be a pentagon in $\widetilde{L}$ (see Theorem 1.2 2). Since $x_1 \not\sim_{\widetilde{L}} y$, there exists $x_2 \in \widetilde{L}$ such that $y <_{\widetilde{L}} x_2 \le_{\widetilde{L}} x_1$ and $\pi(y) \neq \pi(x_2)$. Note that, $\langle x_2, y, z, u, v \rangle$ is a pentagon. Since $\pi(y) \neq \pi(x_2)$, we have either (i*) there exists $w \in \widetilde{L}$ with $w \parallel_{\widetilde{L}} y$ and $w \nparallel_{\widetilde{L}} x_2$, or (ii*) there exists $w \in \widetilde{L}$ with $w \parallel_{\widetilde{L}} x_2$ and $w \nparallel_{\widetilde{L}} y$. By duality, we may assume that (i*) holds Since $y \le_{\widetilde{L}} x_2$ and $y \not\le_{\widetilde{L}} w$, we have $x_2 \not\le_{\widetilde{L}} w$. Since $x_2 \not\le_{\widetilde{L}} w$ and $w \nparallel_{\widetilde{L}} x_2$, we have $w \le_{\widetilde{L}} x_2$ Let $x = y \vee_{\widetilde{L}} w$. We have $y <_{\widetilde{L}} x \le_{\widetilde{L}} x_2$, and thus, $\langle x, y, z, u, v \rangle$ is a pentagon Since $w \not\le_{\widetilde{L}} y$ and $x \wedge_{\widetilde{L}} z = v \le_{\widetilde{L}} y$, we have $w \not\le_{\widetilde{L}} x \wedge_{\widetilde{L}} z$ Since $w \not\le_{\widetilde{L}} x \wedge_{\widetilde{L}} z$ and $w \le_{\widetilde{L}} x$, we have $w \not\le_{\widetilde{L}} z$ Since $w \le_{\widetilde{L}} x$ and $z \not\le_{\widetilde{L}} x$, we have $z \not\le_{\widetilde{L}} w$ Thus, $w \parallel_{\widetilde{L}} z$, and $\{w, y, z\} \in \pi^3_{\widetilde{L}}$. Therefore, (i) holds In the dual case (ii*), (ii) holds □

CHAPTER 3

π-VERSIONS OF DISTRIBUTIVITY

In this chapter, we define five π-versions of distributivity and characterize them via exclusion system.

3.1 π-Versions of Distributivity

Recall that a lattice L is distributive if any one, and hence by Theorem 1.2.6 all, of the following equivalent conditions hold:

(*i*) $D(a, b, c)$ for all $a, b, c \in L$,

(*ii*) $D^*(a, b, c)$ for all $a, b, c \in L$,

(*iii*) $D_m(a, b, c)$ for all $a, b, c \in L$,

(*iv*) $D_0(a, b, c)$ for all $a, b, c \in L$.

By a *π-version* of distributivity we mean that version of distributivity assumed to hold only for antichains. More specifically, we make the following definitions. A lattice L is *π-meet-distributive* (resp., *π-join-distributive*) if $D(a, b, c)$ (resp., $D^*(a, b, c)$) holds for all $\{a, b, c\} \in \pi^3_L$. A lattice L is *π-distributive* if it is both π-meet-distributive and π-join-distributive. A lattice L satisfies the *π-median law* if $D_m(a, b, c)$ holds for all $\{a, b, c\} \in \pi^3_L$. A lattice L satisfies $D_{0\pi}$ if $D_0(a, b, c)$ holds for all $\{a, b, c\} \in \pi^3_L$. We have resisted considering π-semi-distributivity because it is equivalent to semi-distributivity as defined in [8].

The following theorem tells the importance of the condition $D_{0\pi}$.

Theorem 3.1.1. *Let L be a lattice The following statements are equivalent*

(i) *L satisfies $D_{0\pi}$.*

(ii) *$\widetilde{L}$ satisfies $D_{0\pi}$*

(iii) *$\widetilde{L}$ is distributive.*

Proof. First, we prove that (i) and (ii) are equivalent.

(i) $\Leftrightarrow$ (ii) Assume that L satisfies $D_{0\pi}$. For $\{[a],[b],[c]\} \in \pi^3_{\widetilde{L}}$, we have $\{a,b,c\} \in \pi^3_L$ and $[a]\wedge_{\widetilde{L}}([b]\vee_{\widetilde{L}}[c]) = [a\wedge(b\vee c)] \leq_{\widetilde{L}} [(a\wedge b)\vee c] = ([a]\wedge_{\widetilde{L}}[b])\vee_{\widetilde{L}}[c]$. Thus, $\widetilde{L}$ satisfies $D_{0\pi}$. We now assume that L does not satisfy $D_{0\pi}$. Then there exists $\{a,b,c\} \in \pi^3_L$ such that $a\wedge(b\vee c) \nleq (a\wedge b)\vee c$. Let $d = a\wedge(b\vee c)$ and $e = (a\wedge b)\vee c$, so that $d \nleq e$. Since $\{a,b,c\} \in \pi^3_L$, we have $\{[a],[b],[c]\} \in \pi^3_{\widetilde{L}}$. Since $d \leq a$, $c \leq e$, and $c \nleq a$, we have $e \nleq d$. Thus, $d \parallel e$. It follows, $[d] \parallel_{\widetilde{L}} [e]$. Since $[a]\wedge_{\widetilde{L}}([b]\vee_{\widetilde{L}}[c]) = [d] \nleq_{\widetilde{L}} [e] = ([a]\wedge_{\widetilde{L}}[b])\vee_{\widetilde{L}}[c]$, we have $D_0([a],[b],[c])$ does not hold. Thus, $\widetilde{L}$ does not satisfy $D_{0\pi}$.

We now prove that (ii) and (iii) are equivalent.

(ii) $\Leftrightarrow$ (iii) Assume that $\widetilde{L}$ is distributive. For any $\{x,y,z\} \in \pi^3_{\widetilde{L}}$, $x\wedge_{\widetilde{L}}(y\vee_{\widetilde{L}} z) = (x\wedge_{\widetilde{L}} y)\vee_{\widetilde{L}}(x\wedge_{\widetilde{L}} z) \leq_{\widetilde{L}} (x\wedge_{\widetilde{L}} y)\vee_{\widetilde{L}} z$. Thus, $\widetilde{L}$ satisfies $D_{0\pi}$. We now assume that $\widetilde{L}$ is not distributive Then $\widetilde{L}$ contains a sublattice isomorphic to M_3 or N_5. It is easy to verify that M_3 does not satisfy $D_{0\pi}$. We may assume that $\widetilde{L}$ contains a pentagon $\langle x,y,z,u,v\rangle$. By Lemma 2 1.10, we may assume that there exists $w \in \widetilde{L}$ such that $x = y\vee_{\widetilde{L}} w$ and $\{w,y,z\} \in \pi^3_{\widetilde{L}}$. Since $w\wedge_{\widetilde{L}} z \leq_{\widetilde{L}} x\wedge_{\widetilde{L}} z = v \leq_{\widetilde{L}} y$, we have $w\wedge_{\widetilde{L}}(z\vee_{\widetilde{L}} y) = w \nleq_{\widetilde{L}} y = (w\wedge_{\widetilde{L}} z)\vee_{\widetilde{L}} y$, i.e., $\widetilde{L}$ does not satisfy $D_{0\pi}$. $\square$

Lemma 3.1.2. *Let L be a modular lattice. The following statements are equivalent.*

(i) *L is distributive.*

(ii) *L is π-distributive.*

(iii) *L is π-meet-distributive.*

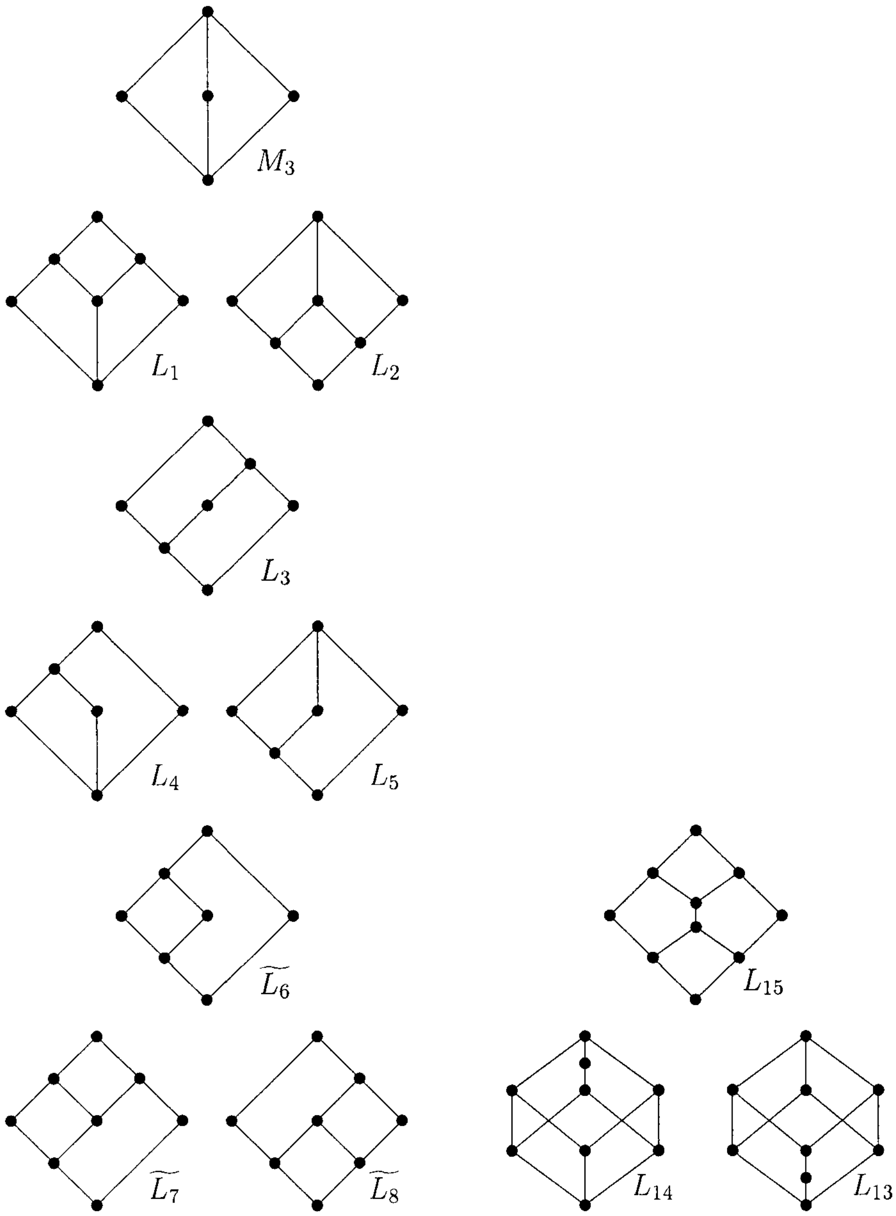
(iv) *L is π-join-distributive.*

(v) *L satisfies the π-median law.*

(vi) *L satisfies $D_{0\pi}$.*

Proof By definition, distributivity implies each of the five π-versions of distributivity. We now suppose that L is not distributive. Since L is modular, by Theorems 1.2.5 and 1.2.2, L contains a sublattice isomorphic to M_3, which does not satisfies any of the five π-versions of distributivity. Therefore, the lemma is proved. □

Lemma 3.1.2 tells us that, in a modular lattice, each of the five π-versions of distributivity is equivalent to distributivity. However, in general, they are not equivalent to each other. Note that in Figure 3.1, L_{13} is π-meet-distributive but not π-join-distributive, while L_{14} is π-join-distributive but not π-meet-distributive. L_{15} satisfies $D_{0\pi}$ but does not satisfy the π-median law. Also, both L_{13} and L_{14} satisfy $D_{0\pi}$ and the π-median law, but do not satisfy π-distributivity.



(a) An exclusion system for $\mathcal{D}_{0\pi} \subset \mathcal{L}$

(b) Additional excluded lattices for π-distributivity

Figure 3.1: An exclusion system for $\mathcal{D}_\pi \subset \mathcal{L}$

3.2 Exclusion Systems for π-Versions of Distributivity

In this section, we characterize the five π-versions of distributivity via exclusion systems.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of algebras such that $\mathcal{C}_1 \subset \mathcal{C}_2$; an *exclusion system* for $\mathcal{C}_1 \subset \mathcal{C}_2$ is a class $\mathcal{S} \subset \mathcal{C}_2 - \mathcal{C}_1$ such that, for $L \in \mathcal{C}_2$, $L \notin \mathcal{C}_1$ if and only if there exists $S \in \mathcal{S}$ isomorphic to a subalgebra of L. We denote by $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{M}$ the classes of lattices, distributive lattices, and modular lattices, respectively. Let $\mathcal{D}_{0\pi}$, $\mathcal{D}_{m\pi}$, $\mathcal{D}_{\wedge\pi}$, $\mathcal{D}_{\vee\pi}$, and $\mathcal{D}_{\pi}$ be the classes of lattices satisfying $D_{0\pi}$, the π-median law, π-meet-distributivity, π-join-distributivity, and π-distributivity, respectively. Recall that N_5 is the 5-element non-modular lattice and M_3 is the 5-element modular non-distributive lattice (see Figure 1.1). Theorem 1.2.5 states that $\{M_3, N_5\}$ is an exclusion system for $\mathcal{D} \subset \mathcal{L}$. We write 1_K (resp., 0_K) for the top (resp., bottom) element of a sublattice K of a lattice L.

The following lemma follows immediately from Theorems 1.2.5 and 1.2.2.

Lemma 3.2.1. *The singleton set $\{M_3\}$ is an exclusion system for $\mathcal{D} \subset \mathcal{M}$.*

Recall that $D_0(a, b, c)$ means $a \wedge (b \vee c) \leq (a \wedge b) \vee c$. Dually, we can define $D_0^*(a, b, c)$ to mean $(a \vee b) \wedge c \leq a \vee (b \wedge c)$. Note that $D_0^*(a, b, c) = D_0(c, b, a)$.

We now present three lemmas about the condition $D_0(a, b, c)$ and the property $D_{0\pi}$.

Lemma 3.2.2. *Let L be a lattice with $\{a, b, c\} \in \pi_L^3$.*

(i) *If $a \wedge (b \vee c) \notin \{a \wedge b, a \wedge c\}$, then $\{a \wedge (b \vee c), b, c\} \in \pi_L^3$.*

(ii) *If $a \vee (b \wedge c) \notin \{a \vee b, a \vee c\}$, then $\{a \vee (b \wedge c), b, c\} \in \pi_L^3$*

(*iii*) *If* $a \vee b < 1_{\Gamma\{a,b,c\}}$ *and* $a \vee c < 1_{\Gamma\{a,b,c\}}$, *then* $\{(a \vee b) \wedge (a \vee c), b, c\} \in \pi_L^3$.

(*iv*) *If* $0_{\Gamma\{a,b,c\}} < a \wedge b$ *and* $0_{\Gamma\{a,b,c\}} < a \wedge c$, *then* $\{(a \wedge b) \vee (a \wedge c), b, c\} \in \pi_L^3$.

Proof. By duality, we need only prove (*i*) and (*iii*).

(*i*) Let $a_1 := a \wedge (b \vee c)$ and assume that $a_1 \notin \{a \wedge b, a \wedge c\}$. Since $a_1 \neq a \wedge b$ and $a \wedge b \leq a_1$, we have $a \wedge b < a_1$. Since $a_1 \wedge b = a \wedge (b \vee c) \wedge b = a \wedge b < a_1$ and $a_1 \wedge b = a \wedge b < b$, we have $a_1 \parallel b$. By symmetry, $a_1 \parallel c$. It follows that $\{a_1, b, c\} \in \pi_L^3$.

(*iii*) Let $a_u := (a \vee b) \wedge (a \vee c)$. Since $a \leq a_u$ and $a \not\leq b$, we also have $a_u \not\leq b$. We have $b \not\leq a_u$, otherwise, $b \leq a_u = (a \vee b) \wedge (a \vee c) \leq a \vee c$, which implies $1_{\Gamma\{a,b,c\}} = a \vee b \vee c = a \vee c$, contradicting $a \vee c < 1_{\Gamma\{a,b,c\}}$. Hence, $a_u \parallel b$. By symmetry, $a_u \parallel c$. Therefore, $\{a_u, b, c\} \in \pi_L^3$. □

Lemma 3.2.3. *Let L be a lattice with $\{a, b, c\} \in \pi_L^3$. The following statements are equivalent.*

(*i*) *$D_0(a, b, c)$ holds.*

(*ii*) $c \vee (a \wedge (b \vee c)) = (a \wedge b) \vee c$

(*iii*) $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b))$

(*iv*) $\{a \wedge (b \vee c), b, (a \wedge b) \vee c\} \notin \pi_L^3$

Proof (*i*) $\Rightarrow$ (*ii*) Assume that $a \wedge (b \vee c) \leq (a \wedge b) \vee c$. Then

$$(a \wedge b) \vee c = (a \wedge (b \vee c)) \vee ((a \wedge b) \vee c) = a \wedge (b \vee c) \vee c = c \vee (a \wedge (b \vee c)).$$

(*ii*) $\Rightarrow$ (*iii*) Assume that $c \vee (a \wedge (b \vee c)) = (a \wedge b) \vee c$. Since $a \wedge (b \vee c) \leq c \vee (a \wedge (b \vee c)) = (a \wedge b) \vee c$, we have that

$$a \wedge (b \vee c) = (a \wedge (b \vee c)) \wedge ((a \wedge b) \vee c) = a \wedge ((a \wedge b) \vee c) = a \wedge (c \vee (a \wedge b)).$$

(*iii*) $\Rightarrow$ (*iv*) Assume that $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b))$. Since $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b)) \leq c \vee (a \wedge b) = (a \wedge b) \vee c$, we have that $\{a \wedge (b \vee c), b, (a \wedge b) \vee c\} \notin \pi_L^3$

$(iv) \Rightarrow (i)$ Let $a_1 := a \wedge (b \vee c)$ and $c_1 := (a \wedge b) \vee c$. Note that $a_1 \leq a$ and $c \leq c_1$. Assume that $\{a_1, b, c_1\} \notin \pi_L^3$. We have that $a_1 \not\parallel b$, $b \not\parallel c_1$, or $a_1 \not\parallel c_1$. Note that, since $b \not\leq a$ and $a_1 \leq a$, we have $b \not\leq a_1$. Therefore, if $a_1 \not\parallel b$, then since $b \not\leq a_1$, we have $a_1 < b$, so that $a_1 \leq a \wedge b \leq (a \wedge b) \vee c = c_1$. Similarly, since $c \not\leq b$ and $c \leq c_1$, we have $c_1 \not\leq b$. Therefore, if $b \not\parallel c_1$, then since $c_1 \not\leq b$, we have $b < c_1$, so that $a_1 = a \wedge (b \vee c) \leq b \vee c \leq c_1$. Thus, we may assume that $a_1 \not\parallel c_1$. Since $c \leq c_1$, $a_1 \leq a$, and $c \not\leq a$, we have $c_1 \not\leq a_1$. Thus, $a_1 \leq c_1$. $\square$

Lemma 3.2.4. *Let L be a lattice. The following statements are equivalent.*

(i) *The lattice L does not satisfy $D_{0\pi}$.*

(ii) *There exists $\{a, b, c\} \in \pi_L^3$ such that $a \leq b \vee c$ and $a \wedge b \leq c$.*

(iii) *There exists $\{a, b, c\} \in \pi_L^3$ such that $a \wedge b = 0_{\Gamma\{a,b,c\}}$ and $b \vee c = 1_{\Gamma\{a,b,c\}}$.*

Proof. Let $a_1 := a \wedge (b \vee c)$ and $c_1 := (a \wedge b) \vee c$. Note that $a_1 \leq a$ and $c \leq c_1$.

$(i) \Rightarrow (ii)$ Since L does not satisfy $D_{0\pi}$, there exists $\{a, b, c\} \in \pi_L^3$ such that $a_1 \not\leq c_1$. By Lemma 3.2.3 (iv), $\{a_1, b, c_1\} \in \pi_L^3$. Thus, $a_1 \wedge b = a \wedge b \leq (a \wedge b) \vee c = c_1$. Similarly, $a_1 \leq b \vee c_1$.

$(ii) \Rightarrow (iii)$ Since $a \wedge b \leq c$, $a \wedge b = a \wedge b \wedge c = 0_{\Gamma\{a,b,c\}}$. Similarly, $b \vee c = 1_{\Gamma\{a,b,c\}}$.

$(iii) \Rightarrow (i)$ We have that $a \wedge (b \vee c) = a \not\leq c = (a \wedge b) \vee c$, i.e., $D_0(a, b, c)$ does not hold. $\square$

Notice that distributivity implies $D_{0\pi}$, but $D_{0\pi}$ does not imply distributivity. For example, L_{15} is a non-distributive lattice satisfying $D_{0\pi}$. Moreover, for $\{a, b, c\} \in \pi_L^3$, if either $D(a, b, c)$ or $D^*(c, b, a)$ holds, then $D_0(a, b, c)$ holds. But the converse is not true. Figure 3.2 is an example of a lattice satisfying $D_{0\pi}$, but not $D(a, b, c)$ or $D^*(c, b, a)$.

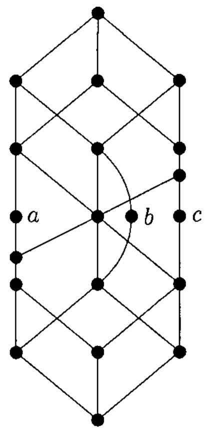


Figure 3.2: A lattice L satisfying $D_{0\pi}$ but satisfying neither $D(a, b, c)$ nor $D^*(c, b, a)$.

Recall that a lattice L is a *subdirect product* of a family $(L_\imath)_{\imath \in I}$ of lattices if

($\imath$) L is a sublattice of $\prod_{\imath \in I} L_\imath$, and

($\imath\imath$) the projection mapping $\pi_\imath$ satisfies $\pi_\imath(L) = L_\imath$ for each $\imath \in I$.

An embedding $\alpha\colon L \to \prod_{\imath \in I} L_\imath$ is *subdirect* if $\alpha(L)$ is a subdirect product of $L_\imath$. A lattice L is *subdirectly irreducible* if, for every subdirect embedding $\alpha\colon L \to \prod_{\imath \in I} L_\imath$, there is an $\imath \in I$ such that $\pi_\imath \circ \alpha\colon L \to L_\imath$ is an isomorphism.

We follow the notation from [22, 16, 17]. Note that M_3, L_1, L_2, L_3, L_4, L_5, L_{13}, L_{14} and L_{15} are subdirectly irreducible lattices and each $\widetilde{L_\imath}$ is the order-skeleton of the corresponding lattice $L_\imath$ for $\imath = 6, 7, 8$ as found in these references.

In [10], Davey and Rival proved the following lemma.

Lemma 3.2.5. *Let L be a lattice containing a pentagon $\langle a, b, c, u, v\rangle$ and an element d such that $a = b \vee d$ and $\{b, c, d\} \in \pi_L^3$. Then L contains a sublattice isomorphic to L_1, L_3, L_4, $\widetilde{L_6}$, $\widetilde{L_7}$, or $\widetilde{L_8}$.*

Observe that L_1 and L_2 are dual, L_4 and L_5 are dual By Lemma 2 1 10, Lemma 3 2 5 and its dual, and the fact that the eight lattices $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, and $\widetilde{L_8}$ are not modular, we have the following corollary

Corollary 3.2.6. *Let L be a lattice The order-skeleton $\widetilde{L}$ is modular if and only if $\widetilde{L}$ contains no sublattice isomorphic to $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, or $\widetilde{L_8}$*

Note that the lattices $M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}$, and $\widetilde{L_8}$ satisfy the condition (ι) in Lemma 2 1 9 Therefore, we have the following corollary

Corollary 3.2.7. *For $F \in \{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}$, if $\widetilde{L}$ contains a sublattice isomorphic to F, then L contains a sublattice isomorphic to F*

Let $\langle \mathcal{L}_F, \leq \rangle$ be the poset of all finite lattices with the ordering defined by *order-embedding*, i e , $L_1 \leq L_2$ if and only if there exists a one-to-one mapping $f \quad L_1 \hookrightarrow L_2$ such that $x \leq y$ if and only if $f(x) \leq f(y)$ for all $x, y \in L_1$ One can verify that the half open interval $[M_3, \mathbf{3}^2)$ of $\mathcal{L}_F$ is precisely $\{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}$

We now characterize the condition $D_{0\pi}$

Theorem 3.2.8. *The interval $[M_3, \mathbf{3}^2) = \{M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}\}$ is an exclusion system for $\mathcal{D}_{0\pi} \subset \mathcal{L}$*

Proof Observe that the nine lattices in $[M_3, \mathbf{3}^2)$ do not satisfy $D_{0\pi}$ We now argue that, if L does not satisfy $D_{0\pi}$, then L contains a sublattice isomorphic to a lattice in $[M_3, \mathbf{3}^2)$

Suppose that L does not satisfy $D_{0\pi}$ By Theorem 3 1 1, $\widetilde{L}$ is not distributive If $\widetilde{L}$ is modular, then by Lemma 3 2 1, it contains a sublattice isomorphic to M_3 If $\widetilde{L}$ is non-modular, then by Corollary 3 2 6, it contains a sublattice isomorphic to one of $L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}$ Hence, by Corollary 3 2 7, L contains a sublattice isomorphic to one of $M_3, L_1, L_2, L_3, L_4, L_5, \widetilde{L_6}, \widetilde{L_7}, \widetilde{L_8}$ □

Recall that a lattice L satisfies the *π-median law* if and only if $D_m(a,b,c)$ holds for all $\{a,b,c\} \in \pi_L^3$ Two elements a, b of a lattice L are *complements* if and only if $a \vee b = 1$ and $a \wedge b = 0$ For convenience of notation, for $a, b, c \in L$, define $a_u = (a \vee b) \wedge (a \vee c)$, $b_u = (a \vee b) \wedge (b \vee c)$, and $c_u = (a \vee c) \wedge (b \vee c)$ Dually, define $a_l = (a \wedge b) \vee (a \wedge c)$, $b_l = (a \wedge b) \vee (b \wedge c)$, and $c_l = (a \wedge c) \vee (b \wedge c)$ Also, define $m_u = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ and $m_l = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

It is easy to verify that the nine lattices in $[M_3, \mathbf{3}^2)$ do not satisfy the π-median law, so that $\mathcal{D}_{m\pi} \subset \mathcal{D}_{0\pi}$

Lemma 3.2.9. *Let L be a lattice such that $\widetilde{L} \cong \mathbf{3}^2$ If L does not satisfy the π-median law, then L_{15} is embeddable in L*

Proof Note that $\widetilde{L} \cong \mathbf{3}^2$ is generated by its 3-element antichain Since L does not satisfy the π-median law, there exists $\{a,b,c\} \in \pi_L^3$ such that $D_m(a,b,c)$ does not hold Since $\{[a],[b],[c]\} \in \pi_{\widetilde{L}}^3$, we have $\widetilde{L} = \Gamma_{\widetilde{L}}\{[a],[b],[c]\}$ Without loss of generality, we may assume that $[a]$ and $[c]$ are complements in $\widetilde{L}$ Since $D_m(a,b,c)$ does not hold, we have $m_l = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) < (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = m_u$ Since $m_l = b_l \leq b \leq b_u = m_u$ and $m_l \sim m_u$, we have $[m_l] = [b] = [m_u]$ Consider the inverse image of $\sim$ $L \to \widetilde{L}$ in L It is straightforward to verify that $\{0, 1, a, b_u, b_l, c, a \wedge b, b \wedge c, a \vee b, b \vee c\} \cong L_{15}$ □

Lemma 3.2.10. *The singleton set $\{L_{15}\}$ is an exclusion system for $\mathcal{D}_{m\pi} \subset \mathcal{D}_{0\pi}$*

Proof It is easy to verify that L_{15} does not satisfy the π-median law Now assume that L satisfies $D_{0\pi}$, but does not satisfy the π-median law There exists $\{a,b,c\} \in \pi_L^3$ such that $D_m(a,b,c)$ does not hold, so that $m_l < m_u$ Since L satisfies $D_{0\pi}$, by Theorem 3 1 1, $\widetilde{L}$ is distributive, so that $m_l \sim m_u$

We claim that at least two of a, b, and c are parallel to m_u Otherwise, we may assume that $m_u \nparallel a$ and $m_u \nparallel b$ If $m_u \leq a$ and $m_u \leq b$, then $m_u \leq a \wedge b \leq m_l$, contradicting $m_l < m_u$ If $a < m_u$ and $b < m_u$, then $a \leq m_l$ and $b \leq m_l$, so that $m_u \leq a \vee b \leq m_l$, contradicting $m_l < m_u$ If $a \leq m_u$ and $m_u \leq b$, then $a \leq b$, contradicting $a \parallel b$ If $b \leq m_u$ and $m_u \leq a$, then $b \leq a$, contradicting $a \parallel b$ Therefore, two of a, b, and c are parallel to m_u

We may assume that $m_u \parallel a$ and $m_u \parallel b$, so that $\{a, b, m_u\} \in \pi_L^3$ Observe that $a \wedge b \leq m_u \leq a \vee b$, i e , a and b are complement elements in $\Gamma\{a, b, m_u\}$ One can verify that $\Gamma_{\widetilde{L}}\{[a], [b], [m_u]\} \cong \mathbf{3}^2$, so that by Lemma 3 2 9, L_{15} is a sublattice of L □

By Theorem 3 2 8 and Lemma 3 2 10, we have the following theorem

Theorem 3.2.11. *The set* $[M_3, \mathbf{3}^2) \cup \{L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{m\pi} \subset \mathcal{L}$

It is easy to verify that the ten lattices in $[M_3, \mathbf{3}^2) \cup \{L_{15}\}$ are not π-meet-distributive, so that $\mathcal{D}_{\wedge\pi} \subset \mathcal{D}_{m\pi}$

Lemma 3.2.12. *The singleton set* $\{L_{14}\}$ *is an exclusion system for* $\mathcal{D}_{\wedge\pi} \subset \mathcal{D}_{m\pi}$

Proof It is easy to verify that L_{14} is not π-meet-distributive Now assume that L satisfies the π-median law, but does not satisfy the π-meet-distributivity There exists $\{a_1, b, c\} \in \pi_L^3$ such that $D(a_1, b, c)$ does not hold Let $a = a_1 \wedge (b \vee c)$ Since $a_1 \wedge b \leq (a_1 \wedge b) \vee (a_1 \wedge c) < a_1 \wedge (b \vee c) = a$, we have $a \neq a_1 \wedge b$ By symmetry, $a \neq a_1 \wedge c$ By Lemma 3 2 2 (i), $\{a, b, c\} \in \pi_L^3$ We have $(a \wedge b) \vee (a \wedge c) = (a_1 \wedge b) \vee (a_1 \wedge c) <$

$a_1 \wedge (b \vee c) = a \wedge (b \vee c)$, i.e., $D(a, b, c)$ does not hold. Let $F = \Gamma\{a, b, c\}$. We have $b \vee c = a \vee b \vee c = 1_F$.

Since L satisfies the π-median law, we have $m_l = m_u$. Since $b \vee c = 1_F$, $m_u = (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \vee b) \wedge (a \vee c) = a_u$. Since $a_l = (a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c) = a \wedge 1_F = a \leq a_u$ and $a_l \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = m_l = m_u = a_u$, we have $0_F < b \wedge c$. Since $a \leq a_u$ and $a \not\leq b$, we have $a_u \not\leq b$. Since $a_u \not\leq b$ and $b_l \leq b$, we have $a_u \neq b_l$. Since $a_u \neq b_l$ and $b_l \vee (a \wedge c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = m_l = m_u = a_u$, we have $0_F < a \wedge c$. By symmetry, $0_F < a \wedge b$. By Lemma 3.2.2 (iv), $\{a_l, b, c\} \in \pi_L^3$, $\{a, b_l, c\} \in \pi_L^3$, and $\{a, b, c_l\} \in \pi_L^3$. Since $b_l \not\leq c$ and $c_l \leq c$, we have $b_l \not\leq c_l$. By symmetry, $c_l \not\leq b_l$. Thus, $b_l \parallel c_l$. It follows, $\{a, b_l, c_l\} \in \pi_L^3$. Since $a \leq a_u = m_u = m_l = b_l \vee c_l$, $(a \wedge b_l) \vee (a \wedge c_l) \leq (a \wedge b) \vee (a \wedge c) < a = a \wedge (b_l \vee c_l)$, i.e., $D(a, b_l, c_l)$ does not hold. Since $a \wedge b \leq a_l \leq a$ and $a \wedge b \leq b_l \leq b$, we have $a \wedge b \leq a_l \wedge b_l \leq a \wedge b$, so that $a_l \wedge b_l = a \wedge b$. Similarly, $b_l \wedge c_l = b \wedge c$ and $a_l \wedge c_l = a \wedge c$. Observe that $a_l \vee b_l = b_l \vee c_l = a_l \vee c_l = m_l$, so that $\Gamma\{a_l, b_l, c_l\} \cong \mathbf{2}^3$. Since $a_l \leq a \leq m_l$, we have $\Gamma\{a, b_l, c_l\} \cong L_{14}$. □

By Theorem 3 2 11, Lemma 3.2 12, and their dual statements, we have the following three theorems.

Theorem 3.2.13. *The set* $[M_3, \mathbf{3}^2) \cup \{L_{14}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\wedge\pi} \subset \mathcal{L}$

Theorem 3.2.14. *The set* $[M_3, \mathbf{3}^2) \cup \{L_{13}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\vee\pi} \subset \mathcal{L}$.

Theorem 3.2.15. *The set* $[M_3, \mathbf{3}^2) \cup \{L_{13}, L_{14}, L_{15}\}$ *is an exclusion system for* $\mathcal{D}_{\pi} \subset \mathcal{L}$

Comment 3.2.16. By definition, π-distributivity implies both π-meet-distributivity and π-join-distributivity. By comparing the exclusion systems of the π-versions of distributivity, one can also see that either one of π-meet-distributivity and π-join-distributivity implies the π-median law, and the π-median law implies $D_{0\pi}$. In particular, π-distributivity implies $D_{0\pi}$.

By using the previous theorems concerning exclusion systems with Theorem 3.1.1, we have the following corollary.

Corollary 3.2.17. *Let L be a lattice. The following statements are equivalent*

(*i*) *L satisfies $D_{0\pi}$.*

(*ii*) *L contains no sublattice isomorphic to a lattice in $[M_3, \mathbf{3}^2)$.*

(*iii*) *$\widetilde{L}$ is distributive.*

(*iv*) *$\widetilde{L}$ is π-distributive*

(*v*) *$\widetilde{L}$ is π-meet-distributive.*

(*vi*) *$\widetilde{L}$ is π-join-distributive.*

(*vii*) *$\widetilde{L}$ satisfies the π-median law.*

(*viii*) *$\widetilde{L}$ satisfies $D_{0\pi}$.*

This corollary tells us that, if a lattice L is isomorphic to its own order-skeleton, then all these π-properties are equivalent to distributivity. We conclude this section by observing that no two of the properties (*iii*) - (*viii*) are equivalent for general lattices.

3.3 Other Versions of Weak Distributivity

We now discuss the relation between the π-versions of distributivity to some weakened conditions found in the references [11] and [20].

Note that both $\mathbf{2}$ and N_5 are π-distributive lattices, but $\mathbf{2} \times N_5$ does not satisfy $D_{0\pi}$ Thus, by Comment 3 2 16, both $\mathbf{2}$ and N_5 satisfy the five π-versions of distributivity, but $\mathbf{2} \times N_5$ does not satisfy any of these five conditions Therefore, the five classes of lattices satisfying the various π-versions of distributivity are not lattice varieties since they are not closed under products

A lattice L is *semi distributive* whenever, for every $a, b, c \in L$,

(SD1) $a \wedge b = a \wedge c$ implies $a \wedge b = a \wedge (b \vee c)$, and

(SD2) $a \vee b = a \vee c$ implies $a \vee b = a \vee (b \wedge c)$

It is easy to show that the π-version of semi-distributivity defined as before by applying the conditions only to $\{a, b, c\} \in \pi_L^3$ is equivalent to semi-distributivity Davey, Poguntke, and Rival proved that $\{M_3, L_1, L_2, L_3, L_4, L_5\}$ is an exclusion system for $\mathcal{SD} \subset \mathcal{L}$ where $\mathcal{SD}$ is the class of all semi-distributive lattices [8]

A lattice L is *near distributive* whenever, for every $a, b, c \in L$,

(ND1) $a \wedge (b \vee c) = a \wedge (b \vee (a \wedge (c \vee (a \wedge b))))$ and

(ND2) $a \vee (b \wedge c) = a \vee (b \wedge (a \vee (c \wedge (a \vee b))))$

As with semi-distributivity, it is not difficult to show that the π-version of near-distributivity is equivalent to near-distributivity It is also easy to show that near-distributivity implies semi-distributivity The following lemma shows that $D_{0\pi}$ implies near-distributivity

Lemma 3.3.1. *Let L be a lattice If L satisfies $D_{0\pi}$, then L is near-distributive*

Proof Since the π-version of near-distributivity is equivalent to near-distributivity, we need only to show that the π-version holds Let $\{a, b, c\} \in \pi_L^3$ By Lemma 3 2 3 (*iii*), we know that $a \wedge (b \vee c) = a \wedge (c \vee (a \wedge b))$ Let $r = a \wedge (b \vee (a \wedge (c \vee (a \wedge b))))$ be the right hand side of (ND1). Since $a\wedge(c\vee(a\wedge b))\leq a$ and $a\wedge(c\vee(a\wedge b))\leq b\vee(a\wedge(c\vee(a\wedge b)))$, we have $a\wedge(c\vee(a\wedge b)\leq r$. Also, we have $r\leq a\wedge(b\vee(a\wedge(c\vee b)))\leq a\wedge(b\vee(c\vee b))=a\wedge(b\vee c)$. Therefore, $a\wedge(b\vee c)=r=a\wedge(b\vee(a\wedge(c\vee(a\wedge b))))$, i.e., (ND1) holds. By duality, (ND2) holds. Therefore, L is near-distributive. □

A lattice L is *almost-distributive* if it is near-distributive and for every $x,y,z,u,v\in L$,

(AD1) $v\wedge(u\vee c)\leq u\vee(c\wedge(v\vee a))$; and

(AD2) $v\vee(u\wedge c')\geq u\wedge(c'\vee(v\wedge a'))$,

where $a=(x\wedge y)\vee(x\wedge z)$, $c=x\wedge(y\vee(x\wedge z))$, $a'=(x\vee y)\wedge(x\vee z)$, and $c'=x\vee(y\wedge(x\vee z))$. Note that a' is the dual of a, c' is the dual of c, and (AD2) is the dual of (AD1).

Lemma 3.3.2. *Let L be a lattice. If L satisfies $D_{0\pi}$, then L is almost distributive.*

Proof. Let L be a lattice satisfying $D_{0\pi}$. By Lemma 3.3.1, L is near-distributive. Thus, by duality, we need only to show that (AD1) holds.

Since L satisfies $D_{0\pi}$, by Theorem 3.1.1, $\widetilde{L}$ is distributive. Recall that $\sim$ is a congruence relation on L. In $\widetilde{L}$, $[a]=[(x\wedge y)\vee(x\wedge z)]=([x]\wedge_{\widetilde{L}}[y])\vee_{\widetilde{L}}([x]\wedge_{\widetilde{L}}[z])=[x]\wedge_{\widetilde{L}}([y]\vee_{\widetilde{L}}[z])=[x\wedge(y\vee z)]=[c]$. Thus, $a\sim c$ and clearly, $a\leq c$. If $a\parallel v$, then $c\parallel v$ and by Lemma 2.1.7, $v\vee a=\bigwedge[v\vee a]=\bigwedge[v\vee c]=v\vee c$, so that $v\wedge(u\vee c)\leq u\vee c=u\vee(c\wedge(v\vee c))=u\vee(c\wedge(v\vee a))$. If $v\leq a$, then $v\wedge(u\vee c)\leq v\leq a\leq u\vee a=u\vee(c\wedge(v\vee a))$. If $c\leq v$, then $v\wedge(u\vee c)\leq u\vee c=u\vee(c\wedge(v\vee a))$. If $a\leq v\leq c$, then $v\wedge(u\vee c)=v\leq u\vee v=u\vee(c\wedge(v\vee a))$. Thus, (AD1) holds. □

Note that the converse of this lemma is not true. For example, $\widetilde{L_6}$ is almost distributive but does not satisfy $D_{0\pi}$.

Day introduced in [3] the "doubling" construction which can "double" an interval in a lattice. Here we consider the special case when the interval is a singleton subset. Let L be a lattice with $d \in L$. We define $L[d] := (L \backslash \{d\}) \cup \{d_1, d_2\}$ with the partial order such that $x \leq y$ in $L[d]$ if and only if one of the following conditions hold:

(i) $x, y \in L \backslash \{d\}$ and $x \leq y$ in L;

(ii) $x \in \{d_1, d_2\}$, $y \in L \backslash \{d\}$, and $d \leq y$;

(iii) $x \in L \backslash \{d\}$, $y \in \{d_1, d_2\}$, and $x \leq d$;

(iv) $(x, y) \in \{(d_1, d_1), (d_1, d_2), (d_2, d_2)\}$.

Note that $\widetilde{D[d]} \cong \widetilde{D}$. If $\widetilde{D} \cong D$, then the order-skeleton $\widetilde{D[d]}$ is isomorphic to D, and every block of the order-skeleton is a singleton subset except one block which is a doubleton subset.

In [31], Rose proved that for any subdirectly irreducible lattice L, L is almost distributive if and only if $L \cong D[d]$ for some distributive lattice D and $d \in D$ (see also [16, 20]).

Lemma 3.3.3. *Let $L \cong D[d]$ for some distributive lattice D with $d \in D$. If L contains a pentagon $\langle a, b, c, u, v \rangle$ and θ is the smallest congruence relation that identifies a and b, then $L/\theta \cong D$ and θ is the congruence relation that identifies only a and b*

Lemma 3.3.4. *Let L be a π-distributive lattice with $x, y, z \in L$ and $L \cong D[d]$ for some distributive lattice D with $d \in D$. If L contains a pentagon $\langle a, b, c, u, v \rangle$ and $x \vee v = y \vee v$, then $(x \wedge z) \vee v = (y \wedge z) \vee v$.*

Proof Assume that $(x \wedge z) \vee v \neq (y \wedge z) \vee v$. Let θ be the smallest congruence relation that identifies a and b. By Lemma 3.3.3, L/θ is distributive, so that $[(x \wedge z) \vee v]_\theta = ([x]_\theta \wedge_\theta [z]_\theta) \vee_\theta [v]_\theta = ([x]_\theta \vee_\theta [v]_\theta) \wedge_\theta ([z]_\theta \vee_\theta [v]_\theta) = [(x \vee v) \wedge (z \vee v)]_\theta = [(y \vee v) \wedge (z \vee v)]_\theta =$

$[(y \wedge z) \vee v]_\theta$ Since $(x \wedge z) \vee v \; \theta \; (y \wedge z) \vee v$, we have $\{(x \wedge z) \vee v, (y \wedge z) \vee v\} = \{a, b\}$ and we may assume that $(x \wedge z) \vee v = a$ and $(y \wedge z) \vee v = b$ Since $a \not\leq b$, we have $x \not\leq y$ Note that $x \vee a = x \vee v = y \vee v = y \vee b$ Since $a \leq x \vee a = y \vee b$ and $a \not\leq b$, we have $y \not\leq b$ Since $x \leq x \vee a = y \vee v$ and $x \not\leq y$, we have $b \not\leq y$ Thus, $y \parallel b$ Since $b \leq y \vee b = y \vee v \leq y \vee c$ and $b \not\leq c$, we have $y \not\leq c$ Since $y < y \vee b = y \vee v \leq y \vee c$, we have $c \not\leq y$ Hence, $y \parallel c$, so that $\{y, b, c\} \in \pi_L^3$ Since $b \wedge y < b$ and $a \wedge y \; \theta \; b \wedge y$, we have $a \wedge y = b \wedge y$ Since $a \leq x \vee a = y \vee v \leq y \vee c$ and L is π-distributive, we have $a = a \wedge (y \vee c) = (a \wedge y) \vee (a \wedge c) = (b \wedge y) \vee (b \wedge c) \leq b$, contradicting $b < a$ Therefore, $(x \wedge z) \vee v = (y \wedge z) \vee v$ □

Lemma 3.3.5. *Let L be a subdirectly irreducible π-distributive lattice If L contains a pentagon $\langle a, b, c, u, v \rangle$, then $u = 1$ and $v = 0$*

Proof Let L be a subdirectly irreducible π-distributive lattice and let $\langle a, b, c, u, v \rangle$ be a pentagon in L Assume that $u \neq 1$ or $v \neq 0$ By duality, we may assume $v \neq 0$ Define a relation α on L by $x \; \alpha \; y$ if and only if $x \vee v = y \vee v$ It is easy to see that α is an equivalence relation such that for any $x, y, z \in L$ with $x \; \alpha \; y$, we have $x \vee z \; \alpha \; y \vee z$ and, by Lemma 3 3 4, $x \wedge z \; \alpha \; y \wedge z$, so that α is a congruence relation Let θ be the smallest congruence relation that identifies a and b By Lemma 3 3 3, θ identifies only a and b Note that, since $a \vee v = a \neq b = b \vee v$, we have $a \not\alpha \; b$, so that $\theta \neq \alpha$ Let β be a congruence relation with $\beta \subseteq \alpha$ and $\beta \subseteq \theta$, and suppose that $c, d \in L$ with $c \; \beta \; d$ Since $c \; \theta \; d$, we have $c = d$ or $\{c, d\} = \{a, b\}$ Since $c \; \alpha \; d$, we have $c = d$ Therefore, β is equality, which implies that L is not subdirectly irreducible, contradicting the assumption □

Theorem 3.3.6. *Let L be a subdirectly irreducible lattice with $L \neq N_5$ Then L is distributive if and only if L is π-distributive*

Proof Let L be a subdirectly irreducible π-distributive lattice with $L \neq N_5$ By Comment 3 2 16, L satisfies $D_{0\pi}$, and by Lemma 3 3 2, L is almost distributive, so that $L \cong D[d]$ for some distributive lattice D and $d \in D$ Assume that L is not distributive, so that L contains a pentagon $\langle a, b, c, u, v\rangle$ By Lemma 3 3 5, $u = 1$ and $v = 0$ Since $L \neq N_5$, there exists $e \in L$ such that $e \notin \{a, b, c, 0, 1\}$ We have $a \not< e$, for if $a < e$, then $\langle a, b, c \wedge e, b \vee (c \wedge e), 0\rangle$ is a pentagon, so that, by Lemma 3 3 5 again, $1 = b \vee (c \wedge e) \leq e$, contradicting $e \neq 1$ Similarly, $e \not< b$ Since L is subdirectly irreducible, $e \notin [b, a]$, so that $e \parallel a$ and $e \parallel b$ Since $\{a, b, e, a \vee e, a \wedge e\}$ is a pentagon, by Lemma 3 3 5, $a \vee e = b \vee e = 1$ and $a \wedge e = b \wedge e = 0$ Let θ be the smallest congruence relation that identifies a and b By Lemma 3 3 3, θ identifies only a and b Since $[c]_\theta = [c \wedge (a \vee e)]_\theta = [(c \wedge a) \vee (c \wedge e)]_\theta = [c \wedge e]_\theta$, we have $c \,\theta\, c \wedge e$, so that $c = c \wedge e$ Similarly, $c = c \vee e$, so that $c = e$, contradicting the assumption □

Theorem 3.3.7. *Let L be a subdirectly irreducible lattice Then L is almost distributive if and only if L satisfies $D_{0\pi}$*

Proof By Lemma 3 3 2, $D_{0\pi}$ implies almost distributivity We now prove the sufficiency Let L be a subdirectly irreducible almost distributive lattice There exists a distributive lattice D and an element $d \in D$ such that $L \cong D[d]$ Notice that the order-skeleton $\widetilde{L} \cong \widetilde{D[d]} \cong \widetilde{D}$ is distributive, by Corollary 3 2 17, L satisfies $D_{0\pi}$ □

In [11], Erné introduce n-zipper-distributivity and the conditions of H_n where $n \geq 3$ It turns out that the π-version of n-zipper-distributivity is also equivalent to

n-zipper-distributivity. In Figure 3.3, we present a diagram indicating the implications between the various of weakened distributive conditions discussed above. We observe that nothing collapses except the five π-versions of distributivity discussed in Corollary 3.2.17 even if a lattice L is isomorphic to its own order-skeleton.

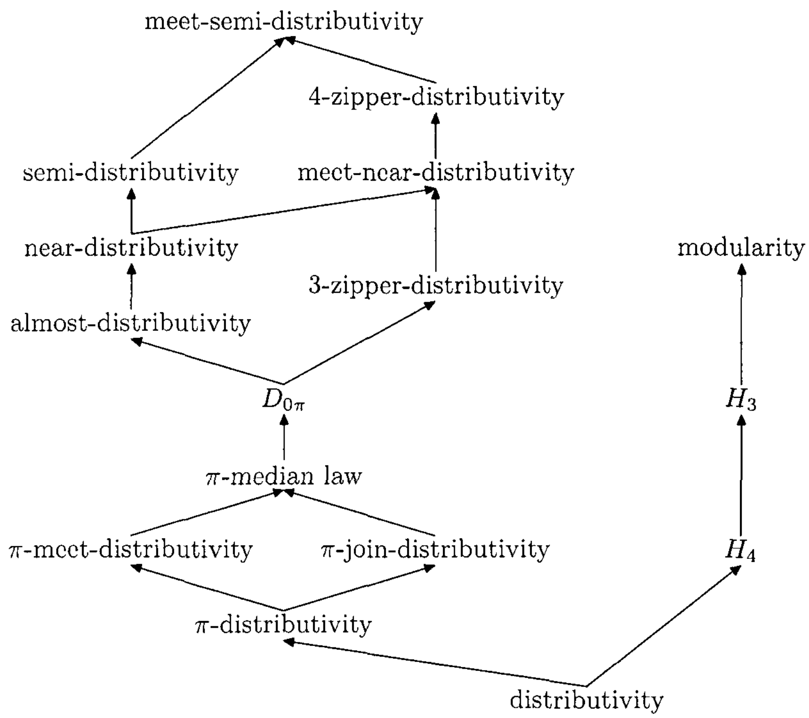


Figure 3.3: The relations between various weakened distributive conditions.

CHAPTER 4

ORTHOCOMPLEMENTED DIFFERENCE LATTICES

4.1 Introduction

Recall that a lattice L is bounded if and only if there exist $0, 1 \in L$ such that $0 \wedge x = 0$ and $1 = 1 \vee x$ for all $x \in L$.

Definition 4.1.1. An algebra $\langle L; \vee, \wedge, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ is called an *ortholattice* (OL) or *orthocomplemented lattice*, if $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice satisfying

(*i*) $x \vee x' = 1$ and $x \wedge x' = 0$,

(*ii*) $x'' = x$, and

(*iii*) $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

An OL L is said to be an *orthomodular lattice* (OML) if, for all $x, y \in L$ with $x \leq y$, $x \vee (x' \wedge y) = y$. A subset S of an OML L is a *sub-OML* if, for all $x, y \in S$, $x \vee y \in S$, $x \wedge y \in S$, and $x' \in S$.

Matoušek introduced the notion of symmetric difference for OLs in [23]. He and Pták developed the notion in [24, 25, 26, 27]. This extends the standard set-theoretic symmetric difference for power sets.

Definition 4.1.2. A (*symmetric*) *difference algebra* is an algebra $\langle D; \Delta, 0, 1\rangle$ of type $(2, 0, 0)$ in which the following identities hold:

$(DA1)$ $x\Delta(y\Delta z) = (x\Delta y)\Delta z$,

$(DA2)$ $x\Delta 0 = x$,

$(DA3)$ $x\Delta x = 0$,

$(DA4)$ If there exists a non-zero element, then $1 \neq 0$

Note that, for all $x, y \in D$, we have that $x\Delta y = (x\Delta y)\Delta 0 = (x\Delta y)\Delta(x\Delta y\Delta y\Delta x) = (x\Delta y)\Delta(x\Delta y)\Delta(y\Delta x) = 0\Delta(y\Delta x) = y\Delta x$ Hence, difference algebra is symmetric

The most familiar example of a difference algebra is $\langle \mathcal{P}(X), \Delta, \emptyset, X\rangle$ where X is a set and Δ is the standard set-theoretic symmetric difference on the power set $\mathcal{P}(X)$ of X

A class $\mathcal{V}$ of algebras of type $\mathcal{F}$ is called a *variety* if $\mathcal{V}$ is the class of all algebras of type $\mathcal{F}$ satisfying a given set of identities Recall that, a lattice can be defined so that the operations $\vee$ and $\wedge$ satisfy the commutative, associative, idempotent, and absorption law, which are identities Hence, the class $\mathcal{L}$ of all lattices forms a variety The class of all OLs is a variety since OL is defined by identities

Definition 4.1.3. An algebra $\langle L, \vee, \wedge, \Delta, ', 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ is called an *orthocomplemented difference lattice* (ODL) if $\langle L, \vee, \wedge, ', 0, 1\rangle$ is an OL and the following identities hold

$(D1)$ $x\Delta(y\Delta z) = (x\Delta y)\Delta z$,

$(D2)$ $x\Delta 1 = 1\Delta x = x'$,

$(D3)$ $x\Delta y \leq x \vee y$

Matoušek proved in [23] that for an ODL $\langle L, \vee, \wedge, \Delta, ', 0, 1\rangle$, the OL $\langle L, \vee, \wedge, ', 0, 1\rangle$ is an OML, denoted by L_{supp}, the algebra $\langle L, \Delta, 0, 1\rangle$ is a difference algebra, denoted by $\delta(L)$ For simplicity of notation, we also use $\langle L_{\text{supp}}, \Delta\rangle$ to represent the ODL L

We denote the class of all ODLs by $\mathcal{ODL}$ and note that this class forms a variety since ODL is defined by equations. For definitions and elementary results related to OMLs not explicitly given here, we refer to [18].

Recall the definition of subalgebra in Section 1.1. To indicate which structure we are working on, we use sub-OML (resp., sub-ODL) to mean the subalgebra of an OML (resp., ODL). Given a subset S, we also use $\Gamma^{\Delta}(S)$, $\Gamma^{\mathrm{OML}}(S)$, and $\Gamma^{\mathrm{ODL}}(S)$ to mean the subalgebra generated by S in a difference algebra, OML, or ODL, respectively.

In this chapter, we focus on the class of all ODLs that are set-representable. We construct an example showing that there exists a 3-generated infinite set-representable ODL. This answers the final open question posed by Matoušek in [27].

4.2 ODLs and Set-Representable ODLs

In this section, we introduce the notions of ODLs and set-representable ODLs.

Let L be a lattice. Two elements $a, b \in L$ are *orthogonal*, denoted by $a \perp b$, if $a \leq b'$. An element $a \neq 0$ of L is an *atom* if there is no element $b \in L$ such that $0 < b < a$. A lattice L is *atomic* if, for each $x \neq 0$ in L, there exists an atom $a \in L$ such that $a \leq x$. A lattice L is *atomistic* if every $x \neq 0$ in L is the join of a set of atoms in L. It is well-known ([18], page 140) that an OML is atomic if and only if it is atomistic.

Given an OML L, a mapping $s : L \to \{0, 1\} \subseteq \mathbb{R}$ is called a *dispersion free state* on L if $s(1) = 1$ and $x \perp y$ in L implies $s(x \vee y) = s(x) + s(y)$. A set $\mathcal{S}$ of dispersion free states is *full* in case $x \leq y$ in L if and only if $s(x) \leq s(y)$ for all $s \in \mathcal{S}$. An OML L is *concrete* if it has a full set of dispersion free states.

Lemma 4.2.1. *An atomic* OML L has a full set of dispersion free states if and only if, for all non-orthogonal atoms $a, b \in L$, there exists a dispersion free state s on L such that $s(a) = s(b) = 1$.

Proof. ($\Rightarrow$) Suppose that L has a full set of dispersion free states. Let $a, b \in L$ be two non-orthogonal atoms in L. Then $a \not\leq b'$, so that there exists a dispersion free state s on L such that $s(a) = 1$ and $s(b') = 0$, i.e., $s(a) = s(b) = 1$.

($\Leftarrow$) Let $x, y \in L$ with $x \not\leq y$. Since $x \not\leq y$, there is an atom $a \in L$ such that $a \leq x$ and $a \not\leq y$. Since $y' \not\leq a'$, there is an atom $b \in L$ such that $b \leq y'$ and $b \not\leq a'$. Hence $a \perp b$, so that there exists a dispersion free state on L such that $s(a) = s(b) = 1$. Since $a \leq x$ and $s(a) = 1$, we have $s(x) = 1$. Since $b \leq y'$ and $s(b) = 1$, we have $s(y') = 1$, i.e., $s(y) = 0$. $\square$

Definition 4.2.2. Let X be a set and let $\Omega \subseteq \mathcal{P}(X)$. Then the pair (X, Ω) is said to be a *D-ring* if $X \in \Omega$ and $A \Delta B \in \Omega$ for all $A, B \in \Omega$, where Δ is the set-theoretic symmetric difference.

Definition 4.2.3. An ODL $\langle L; \vee, \wedge, \Delta_L, 0, 1\rangle$ is said to be a *set-representable ODL* (*SRODL*) if there exists a D-ring (X, Ω) such that $\langle L; \leq, \Delta_L, 0, 1\rangle$ is isomorphic to $\langle \Omega; \subseteq, \Delta, \emptyset, X\rangle$. We denote the class of SRODLs by $\mathcal{SRODL}$.

Note that the 2-element addition abelian group $\langle \mathbb{Z}_2; \oplus\rangle$ forms a difference algebra $\langle \mathbb{Z}_2; \oplus, 0, 1\rangle$. We write $\mathbb{Z}_2$ for this difference algebra when there is no ambiguity and write $0 \leq 1$ in $\mathbb{Z}_2$

Definition 4.2.4. Let L be an ODL and $s : L \to \mathbb{Z}_2$ be a mapping. Then s is said to be an *ODL-evaluation* on L if the following properties hold for all $x, y \in L$:

(i) $s(1) = 1$,

(ii) $x \leq y \Rightarrow s(x) \leq s(y)$,

(iii) $s(x\Delta y) = s(x) \oplus s(y)$

A set $\mathcal{S}$ of ODL-evaluations on L is *full* in case $x \leq y$ in L if and only if $s(x) \leq s(y)$ for all $s \in \mathcal{S}$

For a mapping $s\colon L \to \mathbb{Z}_2$ from an ODL L to $\mathbb{Z}_2$, we define $\bar{s}\colon L_{\text{supp}} \to \{0,1\} \subseteq \mathbb{R}$ to be the mapping from the OML L_{supp} to $\{0,1\} \subseteq \mathbb{R}$ such that, for all $x \in L$, $s(x) = 1$ in $\mathbb{Z}_2$ if and only if $\bar{s}(x) = 1$ in $\mathbb{R}$

Let L_1 and L_2 be two ODLs It is easy to see that the Cartesian product $L_1 \times L_2$ is also an ODL by defining all the operations coordinatewise For fixed $i \in \{1,2\}$, if s_i is an ODL-evaluation on L_i, then the mapping $s\colon L_1 \times L_2 \to \mathbb{Z}_2$ defined by $s((x_1, x_2)) = s_i(x_i)$ is an ODL-evaluation on $L_1 \times L_2$ This can also be generalized to the Cartesian product of any family of ODLs

Lemma 4.2.5. *Let L be an* ODL with $x, y \in L$ Then $(x' \wedge y) \vee (x \wedge y') \leq x\Delta y \leq (x \vee y) \wedge (x' \vee y')$

Proof By definition $x\Delta y \leq x \vee y$ Since $x\Delta y = x\Delta 1\Delta 1\Delta y = x'\Delta y' \leq x' \vee y'$, we have $x\Delta y \leq (x \vee y) \wedge (x' \vee y')$ Since $(x\Delta y)' = x\Delta y\Delta 1 = x\Delta y' \leq x \vee y'$, $x' \wedge y = (x \vee y')' \leq x\Delta y$ Similarly, $x \wedge y' \leq x\Delta y$ Hence, $(x' \wedge y) \vee (x \wedge y') \leq x\Delta y$ □

Lemma 4.2.6. *Let L be an* ODL with $x, y \in L$ Then $x \perp y$ if and only if $x\Delta y = x \vee y$

Proof (⇒) Suppose $x \perp y$ Since $x \perp y$, we have $x \wedge y = 0$, so that $x' \vee y' = 1$ We also have $x \leq y'$ and $y \leq x'$ Hence, by Lemma 4 2 5, we have that $x\Delta y \leq (x \vee y) \wedge (x' \vee y') = (x \vee y) \wedge 1 = x \vee y = (x \wedge y') \vee (x' \wedge y) \leq x\Delta y$ Thus, $x\Delta y = x \vee y$

($\Leftarrow$) Suppose $x \Delta y = x \vee y$ Since $x \Delta y' = x \Delta y \Delta 1 = (x \Delta y)' = (x \vee y)' = x' \wedge y'$, we have that $x = x \Delta 0 = x \Delta (y' \Delta y') = (x \Delta y') \Delta y' \leq (x \Delta y') \vee y' = (x' \wedge y') \vee y' = y'$ Hence, $x \perp y$ $\square$

Lemma 4.2.7. *Let L be an* ODL and s be an ODL-evaluation Then $\bar{s}$ is a dispersion free state on the OML L_{supp}

Proof Let $x, y \in L$ with $x \perp y$ By Lemma 4 2 6, $x \vee y = x \Delta y$ We need only to show that $\bar{s}(x \vee y) = \bar{s}(x) + \bar{s}(y)$ Since $s(x) \oplus s(y) = s(x \Delta y) = s(x \vee y) \not< s(x), s(y)$, we know that $s(x)$ and $s(y)$ cannot evaluate to one simultaneously Thus, $\bar{s}(x \vee y) = \bar{s}(x) + \bar{s}(y)$ $\square$

Matoušek proved the following lemma in [23]

Lemma 4.2.8. *An* ODL L is an SRODL if and only if L has a full set of ODL-evaluations

4.3 $\mathcal{SRODL}$ is not Locally Finite

A lattice L is *locally finite* if the sublattice generated by a finite subset of L is finite In this section, we construct a 3-generated set-representable ODL that has infinitely many elements We conclude that the class $\mathcal{SRODL}$ of all set-representable ODLs is not locally finite and therefore the class $\mathcal{ODL}$ of all ODLs is not locally finite

In [27], Matoušek and Pták introduce a "labeling of atoms" and give a constructive proof showing that finite cubic OMLs with certain properties are ODL-embeddable Here we distill the idea of labeling atoms and generalize it to the infinite case

Let L be an OML For $x, y \in L$, x and y are *compatible*, denoted by $x \, C \, y$, if $x = (x \wedge y) \vee (x \wedge y')$ For $x \in L$, we define $C(x) = \{y \in L \quad x \, C \, y\}$ A *block* of an

OML L is a maximal compatible sublattice. An OML L is *cubic* if each of its blocks is isomorphic to $\mathbf{2}^3$. We denote the set of atoms in L by $At(L)$ and the set of blocks in L by $Bl(L)$.

Lemma 4.3.1. *Let L be a Boolean algebra and Δ be a symmetric difference defined on L such that $x\Delta 1 = x'$ for all $x \in L$. Then the following two statements are equivalent.*

(*i*) $x\Delta y \leq x \vee y$ *for all* $x, y \in L$, *i.e.*, $\langle L; \Delta\rangle$ *is an* ODL.

(*ii*) $x\Delta y = x \vee y$ for all $x, y \in L$ with $x \perp y$.

Furthermore, if $L \cong \mathbf{2}^3$ with $At(L) = \{a, b, c\}$, then each condition is equivalent to the following condition.

(*iii*) $a\Delta b\Delta c = 1$.

Proof. (*i*) $\Rightarrow$ (*ii*) If (*i*) holds then $\langle L; \Delta\rangle$ is an ODL since $x\Delta 1 = x'$ for all $x \in L$, so that (*ii*) follows from Lemma 4.2.6.

(*ii*) $\Rightarrow$ (*i*) Let L be a Boolean algebra with a symmetric difference Δ that satisfies (*ii*) such that $x\Delta 1 = x'$ for all $x \in L$. Let $x, y \in L$. We need only to show that $x\Delta y \leq x \vee y$. Since $(x \wedge y) \perp (x \wedge y')$, we have $x = (x\wedge y) \vee (x \wedge y') = (x \wedge y)\Delta(x \wedge y')$. Similarly, $y = (y \wedge x)\Delta(y \wedge x')$. Since $(x \wedge y') \perp (y \wedge x')$, we have,

$$x\Delta y = (x \wedge y)\Delta(x \wedge y')\Delta(y \wedge x)\Delta(y \wedge x')$$
$$= (x \wedge y')\Delta(y \wedge x') = (x \wedge y') \vee (y \wedge x') \leq x \vee y.$$

Now suppose that $L \cong \mathbf{2}^3$ with atoms a, b, c.

(*ii*) $\Rightarrow$ (*iii*) We have $a\Delta b\Delta c = (a \vee b)\Delta c = c'\Delta c = c' \vee c = 1$.

$(iii) \Rightarrow (ii)$ Let $x, y \in L$ with $x \perp y$. We will argue that $x\Delta y = x \vee y$. We may assume that $x, y \notin \{0, 1\}$. Since $x \perp y$, one of x, y has to be an atom, say $x = a$. If y is a coatom, then $y = a'$, so that $x\Delta y = a\Delta a' = 1 = x \vee y$. Otherwise, we may assume $y = b$, so that $x\Delta y = a\Delta b = c\Delta 1 = c' = a \vee b$. $\square$

In [23], Matoušek proved the following lemma.

Lemma 4.3.2. *Let L be an* ODL. Let $x, y, z \in L$ with $x\,C\,y$ and $x\,C\,z$. Then $x\,C\,(y\Delta z)$ and $x \wedge (y\Delta z) = (x \wedge y)\Delta(x \wedge z)$.

By a straightforward calculation, one sees that for an ODL L and for any $a \in L$, $\langle [0, a]; \Delta|_{[0,a]}\rangle$ is an ODL. We write $[0, a]$ instead of $\langle [0, a]; \Delta|_{[0,a]}\rangle$ if there is no ambiguity.

Lemma 4.3.3. *Let L be an* ODL with $c \in L$. Then xCc for all $x \in L$ if and only if $L \cong [0, c] \times [0, c']$ as ODLs.

Proof. $(\Rightarrow)$ Define $\phi : L \to [0, c] \times [0, c']$ by $\phi(x) := (x \wedge c, x \wedge c')$ for all $x \in L$. We know that ϕ is an OML-isomorphism (see [18]). Let $x, y \in L$. By Lemma 4.3.2, we have $(x\Delta y) \wedge c = (x \wedge c)\Delta(y \wedge c)$ and $(x\Delta y) \wedge c' = (x \wedge c')\Delta(y \wedge c')$. Hence, $\phi(x\Delta y) = ((x\Delta y) \wedge c, (x\Delta y) \wedge c') = (x \wedge c, x \wedge c')\Delta(y \wedge c, y \wedge c') = \phi(x)\Delta\phi(y)$, i.e., ϕ is an ODL-isomorphism. Thus, $L \cong [0, c] \times [0, c']$ as an ODL.

$(\Leftarrow)$ If $L \cong [0, c] \times [0, c']$ as an ODL, then $L_{\text{supp}} \cong [0, c]_{\text{supp}} \times [0, c']_{\text{supp}}$ as an OML. By Theorem 1 in Section 3 in [18], xCc for all $x \in L$. $\square$

Definition 4.3.4. Let $\lambda \colon K \hookrightarrow D$ be an injective mapping from an OML K into a difference algebra D. We say that λ is a *D-labeling* of K if

(i) $\lambda(1_K) = 1_D$,

(ii) for all $a, b \in K$ with $a \perp_K b$, we have $\lambda(a \vee_K b) = \lambda(a)\, \Delta_D\, \lambda(b)$, (*iii*) for all $a, b \in K$ with $a \vee_K b < 1_K$, there exists $c \in K$ such that $c \leq a \vee_K b$ and $\lambda(c) = \lambda(a) \, \Delta_D \, \lambda(b)$.

We say that the OML K is *labeled* by the difference algebra D via the mapping λ.

Lemma 4.3.5. *Let K be an OML and let D be a difference algebra. Let $\lambda\colon K \hookrightarrow D$ be a D-labeling. Then $\lambda(0_K) = 0_D$ and $\lambda(x') = \lambda(x)\Delta_D 1_D$ for all $x \in K$.*

Proof. Since $0_K \perp_K 0_K$, we have that $\lambda(0_K) = \lambda(0_K \vee_K 0_K) = \lambda(0_K)\Delta_D\lambda(0_K)$, so that $\lambda(0_K) = 0_D$. For $x \in K$, we have that $1_D = \lambda(1_K) = \lambda(x \vee_K x') = \lambda(x)\Delta_D\lambda(x')$, so that $\lambda(x') = \lambda(x)\Delta_D 1_D$. $\square$

Lemma 4.3.6. *Let $(K_\alpha)_{\alpha\in\kappa}$ be a disjoint family of* OMLs and let D be a difference algebra. Each K_α is labeled by D via a mapping $\lambda_\alpha\colon K_\alpha \hookrightarrow D$. If $\lambda_\alpha(x) \neq \lambda_\beta(y)$ for all $\alpha, \beta \in \kappa$ with $\alpha \neq \beta$ and $x \in K_\alpha \backslash \{0_{K_\alpha}, 1_{K_\alpha}\}$, $y \in K_\beta \backslash \{0_{K_\beta}, 1_{K_\beta}\}$, then $K := \mathrm{HS}\left((K_\alpha)_{\alpha\in\kappa}\right)$ is labeled by D via the mapping $\lambda := \bigcup\limits_{\alpha\in\kappa} \lambda_\alpha$.

Proof. Let $a, b \in K$ and assume that $a, b \notin \{0_K, 1_K\}$. If $a \perp_K b$, then $a \perp_{K_\alpha} b$ for some $\alpha \in \kappa$, so that $\lambda(a \vee_K b) = \lambda_\alpha(a \vee_{K_\alpha} b) = \lambda_\alpha(a)\Delta_D\lambda_\alpha(b) = \lambda(a)\Delta_D\lambda(b)$. If $a \vee_K b < 1_K$, then $a \vee_{K_\beta} b < 1_{K_\beta}$ for some $\beta \in \kappa$, so that there exists $c \in K_\beta$ such that $c \leq a \vee_{K_\beta} b$ and $\lambda(c) = \lambda_\beta(c) = \lambda_\beta(a)\Delta_D\lambda_\beta(b) = \lambda(a)\Delta_D\lambda(b)$. Hence, λ is a D-labeling of K. $\square$

For a set X, we say that a subset S is *co-finite* if its complement S^c is finite. In this section, we choose $\mathrm{FC}(\mathbb{Z}) := \{S \subseteq \mathbb{Z} \mid \text{either } S \text{ is finite or } S \text{ is co-finite}\}$, the family of finite or co-finite subsets of $\mathbb{Z}$, with the set-theoretical symmetric difference Δ to do the labeling. We denote the OML obtained from the horizontal sum of κ copies of $\mathbf{2}^2$ by MO_κ.

Definition 4.3.7. Given an OML K. We say that K is a *base* of an ODL L in case

(i) K is a sub-OML of L_{supp},

(ii) $a\Delta_L b \in K$ for all $a, b \in K$ with $a \vee_K b < 1_K$, and

(iii) L is generated by K as an ODL.

If K is a base of an ODL L and $L_{\text{supp}} \cong \text{HS}(K, T)$ for some OML T, then we call T a *tail* of L. If $T \cong MO_\kappa$ for some cardinal number T, then we say the tail T a *standard tail* of L.

Definition 4.3.8. Let K be an OML and let D be a difference algebra. We say that D is an *envelope* for K if

(i) $K \subseteq D$,

(ii) $0_D = 0_K$ and $1_D = 1_K$, and

(iii) $\langle C_K(x); \Delta_D|_{C_K(x)}\rangle$ is an ODL for all $x \in K$ with $x \neq 0_K, 1_K$.

Lemma 4.3.9. *Let K be an* OML and let D be a difference algebra. Let $\lambda\colon K \to D$ be a D-labeling.

(i) Let $T^\lambda := \Gamma^\Delta(\lambda(K))\backslash\lambda(K) \cup \{0_D, 1_D\}$. Then T^λ can be organized into an OML such that $T^\lambda \cong MO_\kappa$ for some cardinal number κ and $x' = x\Delta_D 1_D$ for all $x \in T$.

(ii) Let $L := K^\lambda := \text{HS}(K, T^\lambda)$. For $x, y \in L$, define $x\Delta y := \lambda^{-1}(\lambda(x)\Delta_D\lambda(y))$ if $\lambda(x)\Delta_D\lambda(y) \in \lambda(K)$ and $x\Delta y := \lambda(x)\Delta_D\lambda(y)$ otherwise. Then L is an ODL.

(iii) The OML K is a base of the ODL L with T^λ a standard tail.

Proof. (i) For each $x \in \Gamma^\Delta(\lambda(K))\backslash\lambda(K)$, we organize $\{0_D, 1_D, x, x\Delta_D 1_D\}$ into $\mathbf{2}^2$ with $x' := x\Delta_D 1_D$, denoted by B_x. Then T^λ can be organized into the OML

$\mathrm{HS}\left(\{B_x\colon x \in \Gamma^{\Delta}(\lambda(K))\backslash\lambda(K)\}\right) \cong MO_\kappa$ for some cardinal number κ such that $x' = x\Delta_D 1_D$ for all $x \in T$.

($\imath\imath$) Note that Δ on L is associative and $x' = x\Delta 1_D$ for all $x \in L$. Let $y, z \in L$ and assume that $y \vee_L z < 1_L$. Then $y, z \in K$, so that there exists a $w \in K$ such that $w \leq y \vee_K z = y \vee_L z$ and $\lambda(w) = \lambda(y)\Delta_D\lambda(z)$. Hence $y\Delta z = w \leq y \vee_L z$. Therefore, L is an ODL.

($\imath\imath\imath$) Note that K is a sub-OML of $L_{\text{supp}} = \mathrm{HS}(K, T^\lambda)$ and L is generated by K as an ODL. Let $a, b \in K$ with $a \vee_K b < 1_K$. Then there exists an element $c \in K$ such that $c \leq a \vee_K b$ and $\lambda(c) = \lambda(a)\Delta_D\lambda(b)$. Hence $a\Delta b = c \in K$. Therefore, K is a base of L with T^λ a standard base. □

Definition 4.3.10. Let K be an OML and let D be a difference algebra. Let $\lambda\colon K \to D$ be a D-labeling. We define K^λ the ODL in Lemma 4.3.9 and T^λ the standard tail in Lemma 4.3.9.

Lemma 4.3.11. *If an* OML K is a base of an ODL L, then

($\imath$) the difference algebra $\delta(L)$ is an envelop for K, and

($\imath\imath$) the OML K is labeled by the difference algebra $\delta(L)$ via the identity mapping $\imath\colon K \hookrightarrow \delta(L)$.

Proof. ($\imath$) Since K is a sub-OML of L_{supp}, we have that $K \subseteq \delta(L)$ and $0_{\delta(L)} = 0_K$, $1_{\delta(L)} = 1_K$. For $x \in K$ with $x \neq 0_K, 1_K$, we have that both $\langle [0,x]_K; \Delta_D|_{[0,x]_K}\rangle$ and $\langle [0,x']_K; \Delta_D|_{[0,x']_K}\rangle$ are ODLs and $C_K(x) \cong [0,x]_K \times [0,x']_K$, so that $\langle C_K(x); \Delta_D|_{C_K(x)}\rangle$ is an ODL.

(ii) We have $\imath(1_K) = 1_L$. For $a, b \in K$ with $a \perp_K b$, we have $\imath(a \vee_K b) = a \vee_K b = a\Delta_D b = \imath(a)\Delta_D \imath(b)$. For $c, d \in K$ with $c \vee_K d < 1_K$, let $c := a\Delta_D b \leq a \vee_K b$, so that we have $\imath(c) = c = a\Delta_D b = \imath(a)\Delta_D \imath(b)$. □

The following lemma generalize Theorem 3.10 of [27] and give the relations between labeling, base, and envelop.

Lemma 4.3.12. *Let* K *be an* OML and D be a difference algebra such that $K \subset D$ and $D = \Gamma^{\Delta}(K)$. Let $\imath\colon K \hookrightarrow D$ be the identity mapping. Then the following statements are equivalent.

(i) $K^{\imath}$ is an ODL and K is a base of $K^{\imath}$ with $T^{\imath}$ a standard tail.

(ii) The difference algebra D is an envelop for K.

(iii) The OML K is labeled by the difference algebra D via the identity mapping $\imath\colon K \hookrightarrow D$.

Proof (i) $\Rightarrow$ (ii) This follows from Lemma 4.3.11.

(ii) $\Rightarrow$ (iii) We have $\imath(1_K) = 1_D$. Let $a, b \in K$ with $a \perp_K b$ and assume $a \neq 0_K, 1_K$. Since $\langle C_K(a); \Delta|_{C_K(a)}\rangle$ is an ODL and $a, b \in C_K(a)$, we have that $\imath(a \vee_K b) = a \vee_K b = a\Delta_D b = \imath(a)\Delta_D \imath(b)$. Let $c, d \in K$ with $c \vee_K d < 1_K$ and assume $c, d \neq 0_K$. Since $\langle C_K(c \vee_K d); \Delta|_{C_K(c\vee_K d)}\rangle$ is an ODL and $c, d \in C_K(c \vee_K d)$, we have that $c\Delta_D d \leq c \vee_K d$ and $\imath(c\Delta_D d) = c\Delta_D d = \imath(c)\Delta_D \imath(d)$. Therefore, $\imath$ is a D-labeling.

(iii) $\Rightarrow$ (i) This follows from 4.3.9. □

Example. Let FC($\mathbb{Z}$) be the family of finite or co-finite subsets of $\mathbb{Z}$. Let Δ be the standard set-theoretic symmetric difference and let ^{c} be the set complementation. Then $\langle \mathrm{FC}(\mathbb{Z}); \Delta, \emptyset, \mathbb{Z}\rangle$ is a difference algebra. As usual, we represent a block B of a cubic OML by the set of atoms in B. We use certain subsets of FC($\mathbb{Z}$) to present the

Greechie diagram of a cubic OML K_0 whose atoms $At(K_0)$ and blocks $Bl(K_0)$ are the following:

$$\begin{aligned}
At(K_0) = \Big\{ & \{n+1\}, \{n, n+1\}, \{n, n+4\}, \{n, n+1, n+2, n+3\}, \\
& \{n, n+2\}^c, \{n, n+3\}^c, \{n, n+1, n+3\}^c, \\
& \{n, n+2, n+3\}^c, \{n, n+2, n+4\}^c \,:\, n \in \mathbb{Z} \Big\}, \\
Bl(K_0) = \Big\{ & \{\{n, n+1\}, \{n+1, n+2\}, \{n, n+2\}^c\}, \\
& \{\{n+1, n+2\}, \{n, n+1, n+2, n+3\}, \{n, n+3\}^c\}, \\
& \{\{n+1\}, \{n, n+1, n+2, n+3\}, \{n, n+2, n+3\}^c\}, \\
& \{\{n+2\}, \{n, n+1, n+2, n+3\}, \{n, n+1, n+3\}^c\}, \\
& \{\{n+2\}, \{n, n+4\}, \{n, n+2, n+4\}^c\} \,:\, n \in \mathbb{Z} \Big\}.
\end{aligned}$$

Note that the Greechie diagram given in Figure 4.1 spirals infinitely in both directions. The ordering in K_0 induced by the block structure is not set inclusion. In fact, no structure, from FC($\mathbb{Z}$) is used in K_0 other than the symmetric difference Δ. Also note that the elements of K_0 are precisely the empty set, the singleton subset of $\mathbb{Z}$, the doubleton subsets of $\mathbb{Z}$ consisting of two integers differing by at most 4, the 3-element subsets of $\mathbb{Z}$ consisting of three non-consecutive integers such that one of them differs from the other two by at most 2, the 4-element subsets of $\mathbb{Z}$ consisting of consecutive integers, and complements of such subsets. (One can replace $\mathbb{Z}$ by $\mathbb{N}$ and remove the three dots below the elements 12 and 2 in Figure 4.1 to get another example which spirals infinitely in one direction only; we prefer the example given because the proofs are shorter due to the symmetry.)

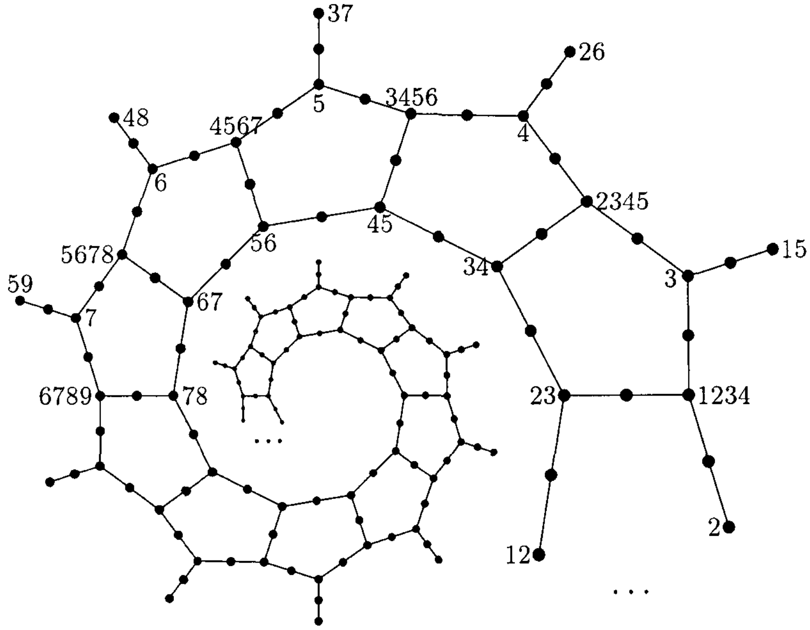


Figure 4.1: The partially labeled Greechie diagram of the infinite cubic OML K_0. Additional labels are obtained from the Δ inherited from FC($\mathbb{Z}$). We represent a set such as $\{i, j, k, l\}$ by the string $ijkl$ where $i < j < k < l$.

Remark. We choose the elements of K_0 from FC($\mathbb{Z}$) and choose the identity mapping $i\colon K_0 \to \mathrm{FC}(\mathbb{Z})$ so that each element of K_0 coincides with its labeling. In Figure 4.1, we give the partial labeling for K_0 such that, for each block in the diagram, two atoms are labeled. To obtain the complete labeling for K_0, we first label the top element by $\mathbb{Z}$ and bottom element by $\emptyset$; for each block, we label the third atom by the symmetric difference of the other two atoms with $\mathbb{Z}$; and then we label each coatom by symmetric difference of the corresponding atom with $\mathbb{Z}$. Note that, for each block, the symmetric difference of the three atoms is the top element $\mathbb{Z}$.

Lemma 4.3.13. (*i*) *The difference algebra* FC($\mathbb{Z}$) *is an envelop for* K_0.

(*ii*) *The* OML K_0 is labeled by the difference algebra FC($\mathbb{Z}$) via the identity mapping $i\colon K_0 \hookrightarrow \mathrm{FC}(\mathbb{Z})$.

(iii) $L_0 := K_0^\imath$ is an ODL with $\delta(L_0) = \mathrm{FC}(\mathbb{Z})$.

(iv) The OML K_0 is a base of the ODL L_0 with $T^\imath$ a standard tail of L_0.

Proof. (i) It is sufficient to show that $\langle C(x); \Delta|_{C(x)}\rangle$ is an ODL for all $x \in K_0$ with $x \neq 0_{K_0}, 1_{K_0}$. Note that $C(x)$ is either a block or three blocks whose intersection is $\{0_{K_0}, 1_{K_0}, x, x'\}$. If $C(x)$ is a block, then by Lemma 4.3.1, $\langle C(x); \Delta|_{C(x)}\rangle$ is an ODL. Otherwise, we may assume $x \in \{\{1\}, \{1,2\}, \{1,2,3,4\}\}$. One can verify that in each case, $\langle C(x); \Delta|_{C(x)}\rangle$ is an ODL.

Parts (ii), (iii), and (iv) follow from Lemma 4.3.12. □

Convention. Henceforth we use K_0 exclusively for the cubic OML presented in Example 4.3 and L_0 for the ODL generated by K_0. We point out that the ODL L_0 contains all the elements of $\mathrm{FC}(\mathbb{Z})$. The OML $(L_0)_{\mathrm{supp}}$ is the horizontal sum of the OML K_0 and the standard tail $T^\imath$.

The following lemma shows that the ODL L_0 is not locally finite.

Lemma 4.3.14. *The* ODL L_0 *is a 3-generated infinite* ODL.

Proof. It is clear that L_0 is infinite. For $n \in \mathbb{Z}$, let $G_n := \{\{n-1\}, \{n\}, \{n+1\}\}$. For simplicity, we write Γ for Γ^{ODL} in this proof. We claim that $G_n \subseteq \Gamma(G_0)$ for all $n \in \mathbb{Z}$. By symmetry, it is enough to show that $G_n \subseteq \Gamma(G_0)$ for all $n \geq 0$. We will prove this by induction.

We have $G_0 \subseteq \Gamma(G_0)$. Suppose that $G_k \subseteq \Gamma(G_0)$ for some $k \geq 0$. Since $\{k-1, k, k+1, k+2\} = (\{k\} \vee \{k+1\})' \in \Gamma(G_0)$ and

$$\{k+2\} = \{k-1\}\Delta\{k\}\Delta\{k+1\}\Delta\{k-1, k, k+1, k+2\} \in \Gamma(G_0),$$

we have $G_{k+1} \subseteq \Gamma(G_0)$. Thus, by induction, $G_n \subseteq \Gamma(G_0)$ for all $n \geq 0$; and hence, $G_n \subseteq \Gamma(G_0)$ for all $n \in \mathbb{Z}$, which implies that L_0 is generated by G_0. □

Definition 4.3.15. Let M be a subset of $\mathbb{Z}$. We define a function $s_M : L_0 \to \mathbb{Z}_2$ by

$$s_M(X) := \begin{cases} 0, & X \text{ is finite and } {}^{\#}(X \cap M) \text{ is even,} \\ 1, & X \text{ is finite and } {}^{\#}(X \cap M) \text{ is odd,} \\ 1 \oplus s_M(X^c), & X \text{ is co-finite.} \end{cases}$$

Note that M is not necessarily finite or co-finite. Also note that every ODL-evaluation s on L_0 is of the form s_M where $M = \{n \in \mathbb{Z} : s(\{n\}) = 1\}$.

Lemma 4.3.16. *Let $M, N \subseteq \mathbb{Z}$ and let $A, B \in L_0$. Then*

(i) $s_M(A \Delta B) = s_M(A) \oplus s_M(B)$, *and*

(ii) $s_{M\Delta N}(A) = s_M(A) \oplus s_N(A)$.

Proof (i) Since $(A\Delta B) \cap M = (A \cap M)\Delta(B \cap M)$, we have $s_M(A\Delta B) = s_M(A) \oplus s_M(B)$.

(ii) Since $A\cap(M\Delta N) = (A\cap M)\Delta(A\cap N)$, we have $s_{M\Delta N} A = s_M(A) \oplus s_N(A)$. ☐

Lemma 4.3.17. *Let M be a subset of $\mathbb{Z}$. The following statements are equivalent.*

(i) *The mapping s_M is an* ODL-evaluation on L_0.

(ii) The mapping $\bar{s}_M$ is a dispersion free state on the OML $(L_0)_{\text{supp}}$.

(iii) For all $n \in M$, (1) if $n + 1 \notin M$, then $[n - 3, n + 2] \cap M = \{n - 1, n\}$; (2) if $n - 1 \notin M$, then $[n - 2, n + 3] \cap M = \{n, n + 1\}$.

(iv) For each block B with atoms a, b, c, we have $\bar{s}_M(a) + \bar{s}_M(b) + \bar{s}_M(c) = 1$.

Proof (i) $\Rightarrow$ (ii) This follows from Lemma 4.2.7.

(ii) $\Rightarrow$ (iii) Suppose that $\bar{s}_M$ is a dispersion free state on $(L_0)_{\text{supp}}$. By Lemma 4.2 1, orthogonal atoms cannot evaluate one simultaneously. We may assume $M \notin \{\emptyset, \mathbb{Z}\}$ and let $n \in M$. We may assume $n = 0$. By symmetry, it is enough to show that $1 \notin M$ implies $[-3, 2] \cap M = \{-1, 0\}$. Assume that $1 \notin M$. Since $\{-1, 0\}$ and

$\{0,1\}$ are orthogonal atoms of K_0 and $\bar{s}_M(\{0,1\}) = 1$, we have $\bar{s}_M(\{-1,0\}) = 0$, so that $-1 \in M$. Since $\{0\}$ and $\{-2,-1,0,1\}$ are orthogonal atoms and $\bar{s}_M(\{0\}) = 1$, we have $\bar{s}_M(\{-2,-1,0,1\}) = 0$, so that $-2 \notin M$. By a similar argument, we have that $-3 \notin M$ and $2 \notin M$. Therefore, $[-3,2] \cap M = \{-1,0\}$.

$(iii) \Rightarrow (iv)$ Let B be a block of L_0 with atoms a, b, c. Since $s_M(a) \oplus s_M(b) \oplus s_M(c) = s_M(\mathbb{Z}) = 1$, we have that either $\bar{s}_M(a) + \bar{s}_M(b) + \bar{s}_M(c) = 1$ or $\bar{s}_M(a) = \bar{s}_M(b) = \bar{s}_M(c) = 1$. Assume on the contrary that $\bar{s}_M(a) = \bar{s}_M(b) = \bar{s}_M(c) = 1$. We inspect the five types of blocks. By symmetry, we select a specific block for each type; we pick the list below:

$$B_1 := \{\{1,2\}, \{2,3,\}, \{1,3\}^c\},$$

$$B_2 := \{\{1,2\}, \{0,1,2,3\}, \{0,3\}^c\},$$

$$B_3 := \{\{2\}, \{1,2,3,4\}, \{1,3,4\}^c\},$$

$$B_4 := \{\{2\}, \{0,1,2,3\}, \{0,1,3\}^c\},$$

$$B_5 := \{\{2\}, \{0,4\}, \{0,2,4\}^c\}.$$

If $B \in \{B_3, B_4, B_5\}$, then since $s_M(\{2\}) = 1$, we have $2 \in M$. By part (iii), we have $s_M(\{1,2,3,4\}) = s_M(\{0,1,2,3\}) = s_M(\{0,4\}) = 0$, a contradiction. We now assume that $B \in \{B_1, B_2\}$, so that $s_M(\{1,2\}) = 1$. By symmetry, we may assume that $1 \in M$ and $2 \notin M$, so that $0 \in M$ and $3 \notin M$. Hence, $s_M(\{2,3\}) = s_M(\{0,1,2,3\}) = 0$, a contradiction.

$(iv) \Rightarrow (i)$ Note that the mapping s_M satisfies parts (i) and (iii) of Definition 4.2.4. We need only to verify part (ii). Let $x, y \in L_0$ with $x \leq y$. We may assume that $x \neq y$ and $x, y \notin \{\emptyset, \mathbb{Z}\}$. Then x is an atom and y is a coatom. Note that $x, y \in B$

for some block B of L_0 with atoms x, a, b and $y = x \vee a = x \Delta a$. We may assume $s_M(x) = 1$. Then $s_M(a) = s_M(b) = 0$, so that $s_M(y) = s_M(x) \oplus s_M(a) = 1 \oplus 0 = 1$, which implies that $s_M(x) \leq s_M(y)$. $\square$

Lemma 4.3.18. *Let $F, G \subset \mathbb{Z}$ be two non-empty finite subsets that are not orthogonal in L_0. If there is no* ODL-evaluation t on L_0 such that $t(F) = t(G) = 1$, then

(i) for all $f \in F$ and $g \in G$, $|f - g| \leq 2$,

(ii) if $n \in \mathbb{Z}$ and $^{\#}(\{n, n+1\} \cap F) = 1$, then either $\{n, n+1\} \subseteq G$ or $\{n, n+1\} \cap G = \emptyset$,

(iii) if $^{\#}F$ is odd, then $^{\#}G$ is even.

Proof. (i) Assume on the contrary that there exist $f \in F$ and $g \in G$ such that $|f - g| \geq 3$. We may assume $f - g \geq 3$. Let f_1 be the maximum in F and g_1 be the minimum in G. Let $t_1 := s_{\{f_1, f_1+1\}}$, $t_2 := s_{\{g_1, g_1-1\}}$, and $t_3 := s_{\{f_1, f_1+1, g_1, g_1-1\}}$. Since $f_1 + 1 \notin F$ and $g_1 - 1 \notin G$, we have $t_1(F) = t_2(G) = 1$, so that by the assumption, $t_1(G) = t_2(F) = 0$, which implies that $t_3(F) = t_1(F) \oplus t_2(F) = 1 \oplus 0 = 1$ and $t_3(G) = t_1(G) \oplus t_2(G) = 0 \oplus 1 = 1$, contradicting the assumption.

(ii) Since $s_{\{n,n+1\}}(F) = 1$, we have $s_{\{n,n+1\}}(G) = 0$, so that either $\{n, n+1\} \subseteq G$ or $\{n, n+1\} \cap G = \emptyset$.

(iii) Since $^{\#}F$ is odd, we have $s_{\mathbb{Z}}(F) = 1$, so that $s_{\mathbb{Z}}(G) = 0$, which implies that $^{\#}G$ is even. $\square$

Lemma 4.3.19. *If $F, G \subset \mathbb{Z}$ are two non-empty finite subsets that are not orthogonal in L_0, then there exists an* ODL-evaluation t on L_0 such that $t(F) = t(G) = 1$.

Proof. Assume that there is no ODL-evaluation t on L_0 such that $t(F) = t(G) = 1$; we prove that F and G are orthogonal in L_0. By Lemma 4.3.18 part (i), $|f - g| \leq 2$ for all $f \in F$ and $g \in G$. If F is a singleton, say $F = \{0\}$, then $\emptyset \neq G \subseteq \{-2,-1,0,1,2\}$, so that by Lemma 4.3.18 part (ii), either $\{-1,0,1\} \subseteq G$ or $\{-1,0,1\} \cap G = \emptyset$; by Lemma 4.3.18 part (iii), ${}^{\#}G$ is even, so that $G = \{-2,-1,0,1\}$, $G = \{-1,0,1,2\}$, or $G = \{0,4\}$, each of which is orthogonal to $F = \{0\}$ in L_0. Thus, we may assume that ${}^{\#}F \geq 2$ and ${}^{\#}G \geq 2$. By symmetry, we may assume that $0 \in F$ is the minimum of $F \cup G$, so that $G \subseteq \{0,1,2\}$ and $F \subseteq \{0,1,2,3,4\}$ since $|f-g| \leq 2$ for all $f \in F$ and $g \in G$. Since ${}^{\#}(\{-1,0\} \cap F) = 1$ and $-1 \notin G$, by Lemma 4.3.18 part (ii), we have $0 \notin G$, so that $G = \{1,2\}$ and therefore $F \subseteq \{0,1,2,3\}$. Since ${}^{\#}(\{0,1\} \cap G) = 1$, by Lemma 4.3.18 part (ii), we have $1 \in F$. Since ${}^{\#}(\{2,3\} \cap G) = 1$, by Lemma 4.3.18 part (ii), we have that either $\{2,3\} \subseteq F$ or $\{2,3\} \cap F = \emptyset$, so that either $F = \{0,1,2,3\}$ or $F = \{0,1\}$, each of which is orthogonal to G. $\square$

Lemma 4.3.20. *Let $F, G \in L_0$ with $F \not\leq G$ and $F, G \notin \{\emptyset, \mathbb{Z}\}$. Then $F \not\perp G^c$, $F \not\perp F \Delta G$, and $G^c \not\perp F^c \Delta G^c$.*

Proof. Since $F \not\leq G = G^{cc}$, we have $F \not\perp G^c$. We have $F \not\perp F\Delta G$; otherwise, $F \leq F \vee (F\Delta G) = F\Delta(F\Delta G) = G$. We have $G^c \not\perp F^c\Delta G^c$; otherwise, $G^c \leq G^c \vee (F^c\Delta G^c) = G^c\Delta(F^c\Delta G^c) = F^c$, so that $F \leq G$. $\square$

Lemma 4.3.21. *The set of all* ODL-evaluations on L_0 is full.

Proof. Let $F, G \in L_0$ with $F \not\leq G$. We will prove that there exists an ODL-evaluation t on L_0 such that $t(F) = 1$ and $t(G) = 0$. We may assume that $F, G \notin \{\emptyset, \mathbb{Z}\}$, so that by Lemma 4.3.20, $F \not\perp G^c$, $F \not\perp F\Delta G$, and $G^c \not\perp F^c\Delta G^c$. In case F is co-finite, if G is finite, then $s_\emptyset(F) = 1$ and $s_\emptyset(G) = 0$; if G is co-finite, then G^c is finite, so that $F^c\Delta G^c$ is finite; since $G^c \not\perp F^c\Delta G^c$, by Lemma 4.3.19, there exists an ODL-evaluation t_3 on L_0 such that $t_3(G^c) = t_3(F^c\Delta G^c) = 1$, so that $t_3(F) = t_3(F^c) \oplus 1 =$

$t_3((F^c \Delta G^c)\Delta G^c) \oplus 1 = t_3(F^c \Delta G^c) \oplus t_3(G^c) \oplus 1 = 1$ and $t_3(G) = 0$. Thus, we may assume that F is finite. If G is finite, then $F\Delta G$ is finite and $F \not\perp F\Delta G$, so that by Lemma 4.3.19, there exists an ODL-evaluation t_1 on L_0 such that $t_1(F) = t_1(F\Delta G) = 1$, which implies that $t_1(F) = 1$ and $t_1(G) = 0$. If G is co-finite, then G^c is finite and $F \not\perp G^c$, so that by Lemma 4.3.19, there exists an ODL-evaluation t_2 on L_0 such that $t_2(F) = t_2(G^c) = 1$, which implies that $t_2(F) = 1$ and $t_2(G) = 0$. $\square$

Theorem 4.3.22. *$\mathcal{SRODL}$ is not locally finite.*

Proof. By Lemma 4.3.21 and 4.2.8 , we have that L_0 is an SRODL. By Lemma 4.3.14, L_0 is a 3-generated ODL. Hence, L_0 is a 3-generated SRODL, so that $\mathcal{SRODL}$ is not locally finite. $\square$

PART II

GRAPH THEORY

CHAPTER 5

GRAPH THEORY

5.1 Introduction of Graph Theory

In this chapter, we provide some basic graph theory notation and results. Unless stated otherwise, we will follow the notation and terminology of West [35].

A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$ where each edge is *incident* to two vertices (not necessarily distinct) called its *endpoints*. A graph is a *null graph* if both of its vertex set and edge set are empty. If two distinct vertices u and v are the endpoints of an edge e (denoted by $e = uv$), then we say that u and v are *adjacent*, and u and e are *incident*. If two edges e and f share a common endpoint, then we say that e and f are *adjacent*. An edge with identical endpoints is called a *loop* Edges that have the same pair of endpoints are called *parallel edges* or *multiple edges*. If the two distinct endpoints of parallel edges have exactly $k \geq 2$ common incident edges, then we call each edge a *k-parallel-edge*. An edge is *simple* if it is neither a loop nor a parallel edge. A graph G is *loopless* if it does not contain a loop. A graph G is *simple* if it is loopless and does not contain any parallel edges. The *degree* of a vertex u, denoted by $d(u)$, is the number of non-loop edges plus twice the number of loops incident to u. A graph G is *r-regular* if every vertex of G is of degree r. An *isomorphism* from a graph G to a graph H is a bijection $f\colon V(G) \rightarrow V(H)$

such that f and f^{-1} preserve the adjacency of vertices. We say that G is *isomorphic* to H if there is an isomorphism from G to H. All graphs in this dissertation are finite.

5.2 Connectivity

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list of vertices. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. Note that we consider a vertex with a loop to be a cycle. The *girth* of G is the minimum size among all cycles in G. A graph G is *connected* if every pair of vertices can be joined by a path in G, and is *disconnected* otherwise. A graph H is a *subgraph* of a graph G, written as $H \subseteq G$, if both $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *component* of a graph G is a maximal connected subgraph. A component (or a graph) is *trivial* if it has no edges; otherwise it is *non-trivial.* To *delete* an edge e from G, denoted by $G\backslash e$, we remove e from $E(G)$. To *delete* a vertex v from G, denoted by $G - v$, we remove v from $V(G)$ and remove the edges incident to v from $E(G)$. To *contract* an edge e from G, we replace the two endpoints of e by a single vertex whose incident edges are the edges other than e that were incident to the two endpoints of e. To *contract* a subgraph H of G means to contract all the edges in H.

Let G be a graph and let k be an integer. A *cut-edge* of G is an edge whose deletion increases the number of components. Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T. An *edge-cut* is an edge set of the form $[S, \bar{S}]$, where S is a nonempty proper subset of $V(G)$ and $\bar{S}$ denotes $V(G)\backslash S$. We define $m_G(S,T) := |[S,T]|$ and $m_G(S) := |[S,\bar{S}]|$. When there is no ambiguity, we write $m(S,T)$ and $m(S)$ instead of $m_G(S,T)$ and $m_G(S)$. A connected graph G is *k-edge-connected* if G cannot be disconnected by deleting fewer than k edges. A graph G is *minimally k-edge-connected* if G is k-edge-connected and the result of any edge deletion is not k-edge-connected. A graph G is *essentially k-edge-connected* if, for any non-trivial partition $[S, \bar{S}]$, $m(S) \geq k$. We point out that contraction does not decrease edge-connectivity.

Mader proved the following theorem in [21].

Theorem 5.2.1. *Every minimally k-edge-connected graph has a vertex of degree k.*

5.3 Edge-Block Tree

A graph with no cycle is *acyclic*. A *tree* is a connected acyclic graph. A *leaf* is a vertex of degree one. In a connected graph, an *edge-block* is a maximal 2-edge-connected subgraph. For a connected graph G, we define G_B to be the graph obtained from G by contracting every edge-block in G. The following lemma shows that G_B is a tree and we call G_B the *edge-block tree*.

Lemma 5.3.1. *Let G be a connected graph. Then G_B is a tree.*

Proof. It is clear that G_B is connected. Also, G_B has no cycle since every edge in G_B is a cut-edge. Thus, G_B is a tree. □

5.4 Immersion

We define the containment relationship of immersion between two graphs The immersion relationship was first introduced by Nash-Williams [28, 29] and is weaker than the minor or topological-minor relations As a matter of fact, this relation has been studied for over forty years

A graph H is a *minor* of a graph G if a copy of H can be obtained from G by deleting or contracting the edges of G In a graph G, *subdivision* of an edge xy is the operation of replacing xy with a path xzy through a vertex z not in G An *H-subdivision* (or *subdivision* of H) is a graph obtained from a graph H by successive edge subdivisions A graph H' is a *pseudo-subdivision* of a graph H if $V(H) \subseteq V(H')$ and there exists a family $\{H'_e\}_{e \in E(H)}$ where each H'_e is a subgraph of H' such that

(1) if $e \in E(H)$ joins two distinct vertices x and y then H'_e is an xy-path and $V(H'_e) \cap V(H) = \{x, y\}$,

(2) if $e \in E(H)$ joins a vertex x to itself then H'_e is a cycle and $V(H'_e) \cap V(H) = \{x\}$,

(3) $E(H'_e) \cap E(H'_f) = \emptyset$ for every pair e, f of distinct edges of H,

(4) $V(H') = V(H) \cup \bigcup_{e \in E(H)} V(H'_e)$, and

(5) $E(H') = \bigcup_{e \in E(H)} E(H'_e)$

Note that if we add the condition $V(H'_e) \cap V(H'_f) \subseteq V(H)$ to (3), then H' is a subdivision of H and H is a topological minor of a minor of H' We say that a graph H is *immersed* in a graph G or G contains H as an immersion, denoted by $H \prec G$ if G has a subgraph H', which is isomorphic to a pseudo-subdivision of H We distinguish vertices x in H' or in G if x corresponds to a vertex in H In this case, we say that

x is *from* H. We also color all vertices and edges in G *white* if they do not belong to H'. A component C is called *white* if all the edges and vertices in C are white. After applying a graph operation to G, we say that the operation *keeps* H if the resulting graph maintains H as an immersion.

5.5 Graph Operations

By *splitting* a degree-four vertex x in a graph G, we mean the operation illustrated in Figure 5.1. Note that we allow the vertex x to have incident loops. If x has exactly one incident loop, then to split x is equivalent to deleting the incident loop and contracting an incident edge. If x has two incident loops, then to split x is equivalent to delete x with the two incident loops.

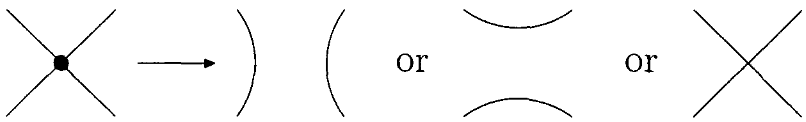


Figure 5.1: Splitting a vertex

We use $5K_2$, $3K_2^L$, and K_2^{2L} for the only 5-regular graphs on two vertices with zero, two, and four loops, respectively. Let G be a graph and let H be a component of G. We define the graph operation $\mathcal{O}_0(H)$ to mean deleting the component H in G. A *basic graph operation* on an edge $e = xy$ in a 5-regular graph G is to delete e and split x and y. We define $\mathcal{O}_k$, $k = 1, 2, 3, 4$, to mean the basic graph operation that can be applied to a k-parallel-edge. These four basic graph operations are illustrated by Figures 5.2 to 5.5. Note that $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}_0(5K_2)$ are all the possible basic graph operations. We want to point out that $\mathcal{O}_3$ is equivalent to contracting the three parallel edges to a vertex x and split x. The graph operation $\mathcal{O}_4$ is equivalent to contracting the four parallel edges with an edge incident to them, so that $\mathcal{O}_4$ does not reduce edge-connectivity.

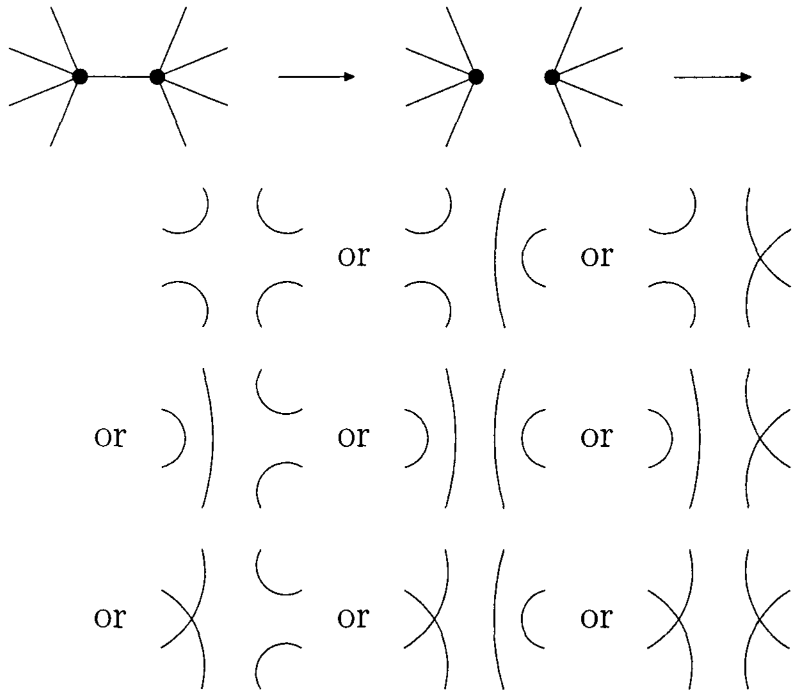


Figure 5.2: Operation $\mathcal{O}_1$

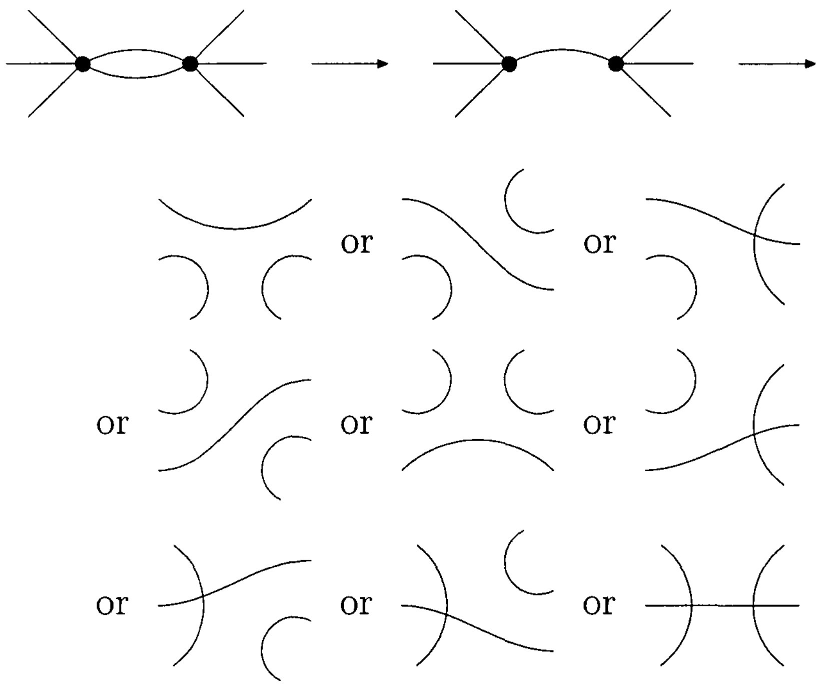


Figure 5.3: Operation $\mathcal{O}_2$

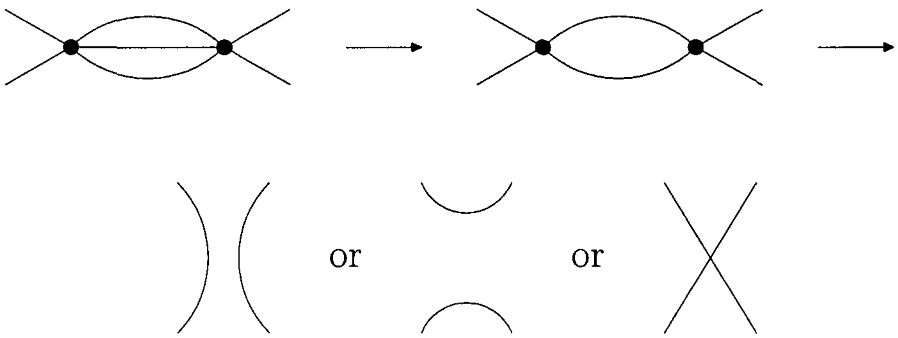


Figure 5.4: Operation $\mathcal{O}_3$

Figure 5.5: Operation $\mathcal{O}_4$

5.6 Generating and Splitter Theorems

A *generating theorem* for a class $\mathcal{G}$ of graphs tells us how to construct all the members of $\mathcal{G}$ from a set of graphs by using a set of graph operations. Ideally, the set of graphs and the set of graph operations are small.

Let k and g be two integers with $1 \leq k \leq 5$ and $g \geq 0$. Let $\Phi_{k,g}$ be the class of k-edge-connected 5-regular graphs of girth at least g. For example, $\Phi_{1,2}$ is the class of connected 5-regular loopless graphs. Let $G, H \in \Phi_{k,g}$. We say that G can be *reduced* to H within $\Phi_{k,g}$ by a set $\mathcal{O}$ of graph operations if there is a sequence G_0, G_1, $\ldots$, G_t of graphs in $\Phi_{k,g}$ such that $G_0 = G$, $G_t = H$, and each G_i is obtained from G_{i-1} by applying a single graph operation in $\mathcal{O}$. Moreover, $G_i \prec G_{i-1}$ holds for each $i \geq 1$. Under this terminology, a *splitter theorem* is a result claiming the existence of a set $\mathcal{O}$ of graph operations such that if $G, H \in \Phi_{k,g}$ and $H \prec G$ then G can be reduced within $\Phi_{k,g}$ to H.

Steinitz and Rademacher [33] proved that the class $\mathcal{G}$ of 3-connected 3-regular simple planar graphs can be generated from the tetrahedron by *adding handles* (see Figure 5.6). Kanno [19] proved generating and splitter theorems for 3- and 4-regular graphs with various connectivities and girth. Ding, Kanno, and Su [7] proved generating theorems for 5-regular planar graphs with certain restrictions for edge-connectivity. In Chapter 6, we find generating theorems for 5-regular graphs and 5-regular loopless graphs with different edge-connectivities. We also find splitter theorems for 5-regular graphs with various edge-connectivities.

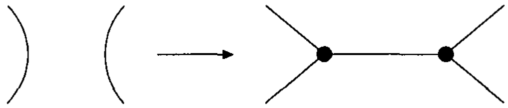


Figure 5.6: Adding a handle

CHAPTER 6

GENERATING AND SPLITTER THEOREMS

6.1 Generating Theorems for 5-Regular Graphs

In this section, we prove a sequence of theorems which state that if a 5-regular graph G is k-edge-connected with girth at least g, then it can be reduced to one of $5K_2$, $3K_2^L$, or K_2^{2L} within $\Phi_{k,g}$ by select graph operations from $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, $\mathcal{O}_5$, and $\mathcal{O}_0(5K_2)$. To prove these results, it is sufficient to prove that G can be reduced by one step.

In [7], we prove generating theorems for 5-regular planar graphs with certain restrictions for edge-connectivity. Note that generating theorems for 5-regular graphs are inferred by the paper since planarity is not essential in the paper. In this dissertation, we do not require planarity, so that we can use fewer graph operations and give simpler proofs for generating theorems.

A graph G is called an *alternating path* if it has vertices $x_1, x_2, ..., x_{2t}$ with $t \geq 1$ such that there are three parallel edges from x_{2i-1} to x_{2i} ($i = 1, 2, ..., t$) and two parallel edges from x_{2i} to x_{2i+1} ($i = 1, 2, ..., i - 1$). If we add two more edges from x_1 to x_{2t}, then the resulting graph is called an *alternating cycle*. Notice that an alternating cycle is 5-regular, and an alternating path is "almost" 5-regular - other than the two ends, all its vertices have degree five.

Lemma 6.1.1. *Let G be a 5-regular connected graph. If G has no simple edges, then G is either an alternating cycle or an alternating path with a loop at each of its ends*

Proof Let H be the graph obtained from G by deleting all loops and then deleting all but one edge from each parallel family. Clearly, H is connected. The maximum degree of H is at most two and thus H is either a cycle or a path. If H has only two vertices, then G is either the alternating path $3K_2^L$ or the alternating cycle $5K_2$. If H has more than two vertices, then G is an alternating cycle when H is a cycle and G is an alternating path with two loops when H is a path. □

Corollary 6.1.2. *Let G be a 5-regular connected graph If G is not isomorphic to $5K_2$ and has no simple edges, then G contains a 3-parallel-edge*

Lemma 6.1.3. *Let G be a 5-regular connected graph If $\mathcal{O}_1$ and $\mathcal{O}_3$ cannot be applied within $\Phi_{0,1}$, then G is isomorphic to $5K_2$.*

Proof Suppose that G is not isomorphic to $5K_2$. Since $\mathcal{O}_1$ cannot be applied in G, there is no simple edges. By Corollary 6.1.2, $\mathcal{O}_3$ can be applied. □

Theorem 6.1.4. *Every 5-regular graph can be reduced to $5K_2$, $3K_2^L$, or K_2^{2L} within $\Phi_{0,1}$ by $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_0(5K_2)$*

Proof It is clear that $5K_2$, $3K_2^L$, and K_2^{2L} are the only possible 5-regular graphs with no more than two vertices. Let G be a 5-regular graph with more than two vertices. If G contains $5K_2$, then $\mathcal{O}_0(5K_2)$ can be applied; otherwise, by Lemma 6.1.3, $\mathcal{O}_1$ or $\mathcal{O}_3$ can be applied in a component in G □

Lemma 6.1.5. *Let x be a degree-four vertex in a connected graph. Then at least one splitting at x results in a connected graph, unless every edge incident to x is a cut-edge*

Proof Since x is of degree four, it has four incident edges e_1, e_2, e_3, and e_4, where $e_i = e_j$ for $i \neq j$ only if the edge e_ie_j is a loop Suppose that some e_i, say e_1, is not a cut-edge Then there is a cycle containing both e_1 and e_j for some $j \neq 1$ By symmetry, we may assume $j = 2$ Then the splitting $\{e_1, e_4\}$-$\{e_2, e_3\}$ results in a connected graph □

Lemma 6.1.6. *Let G be a 5-regular 2-edge-connected graph If $\mathcal{O}_1$ and $\mathcal{O}_3$ cannot be applied within $\Phi_{1\,1}$, then G is isomorphic to $5K_2$*

Proof If G has no simple edge, then by Lemma 6 1 1, either $\mathcal{O}_3$ can be applied within $\Phi_{1\,1}$, or G is isomorphic to $5K_2$ Now assume that G has a simple edge $e = xy$ Note that at most one of the four edges incident to x in $G\backslash e$ is a cut-edge in $G\backslash e$ Similarly, at most one of the four edges incident to y in $G\backslash e$ is a cut-edge in $G\backslash e$ Note that applying $\mathcal{O}_1$ to e is equivalent to deleting the edge e and splitting the two degree-four vertices By Lemma 6 1 5, $\mathcal{O}_1$ can be applied within $\Phi_{1,1}$ □

Lemma 6.1.7. *Let G be a 5-regular connected graph and B be an edge-block in G with $m(B) = 1$ If $\mathcal{O}_1$ and $\mathcal{O}_3$ cannot be applied to B within $\Phi_{1\,1}$, then B is isomorphic to $2L$*

Proof The proof is essentially the same as the proof of Lemma 6 1 6 □

We introduce a new graph operation $\mathcal{O}_5$, illustrated by Figure 6 1, which is needed in the following Theorem

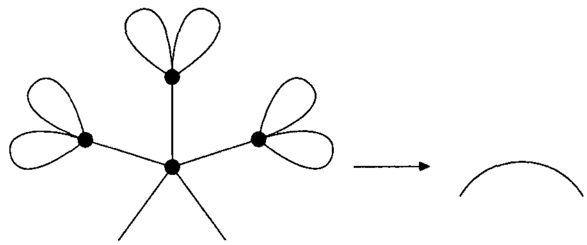


Figure 6 1 Operation $\mathcal{O}_5$

Theorem 6.1.8. *Every 5-regular connected graph can be reduced to $5K_2$, $3K_2^L$, or K_2^{2L} within $\Phi_{1\,1}$ by $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_5$*

Proof Let G be a 5-regular connected graph with more than two vertices and suppose that $\mathcal{O}_1$ and $\mathcal{O}_3$ cannot be applied within $\Phi_{1\,1}$ If G is 2-edge connected, then by Lemma 6 1 6, G is isomorphic to $5K_2$ Otherwise, by Lemma 6 1 7, every edge-block B with $m(B) = 1$ has to be $2L$, i e , every leaf in G_B is $2L$ in G In the tree G_B, pick a vertex as a root and build a tree structure Pick a leaf x with longest path to the root Let $e = xy$ be the edge incident to this leaf Note that the children of y are all leaves Since $\mathcal{O}_1$ cannot be applied to e within $\Phi_{1\,1}$, by Lemma 6 1 5, y is a single vertex in G and all the edges incident to y are cut-edges Thus, y has four children, i e , y is incident to four $2L$'s in G, so that $\mathcal{O}_5$ can be applied within $\Phi_{1\,1}$ □

Lemma 6.1.9. *Let G be a k-edge-connected graph with $k \in \{2, 3, 4\}$ Let x be a vertex of degree four in G Then there exists a k edge connected outcome by splitting x*

Proof Suppose that no outcome by splitting x is k-edge-connected If x is incident to a loop, then the outcome is clearly k-edge-connected So we may assume that x is not incident to a loop

Let xx_1, xx_2, xx_3, xx_4 be the four edges incident to x. Consider the splitting $\{xx_1, xx_2\} - \{xx_3, xx_4\}$. Since the resulting graph is not k-edge-connected, $V(G\backslash x)$ has a partition $[X_{12}, X_{34}]$ such that $x_1, x_2 \in X_{12}$, $x_3, x_4 \in X_{34}$, and $m(X_{12}, X_{34}) \leq k-1$. Similarly, the other two splitting give us the partitions $[X_{13}, X_{24}]$ and $[X_{14}, X_{23}]$. For simplicity, we define $X_{ji} := X_{ij}$ in case $i < j$. For $i \in \{1,2,3,4\}$, let $X_i := \bigcap_{j \neq i} X_{ij}$ and $Y_i := \bigcap_{j \neq i, k \neq i, j \neq k} X_{jk}$. Note that for $i \in \{1,2,3,4\}$, $x_i \in X_i$, so that X_i is non-empty.

For convenient of notation, we define $a_{ij} := m(X_i, X_j)$, $b_{ij} := m(X_i, Y_j)$, and $c_{ij} := m(Y_i, Y_j)$. Observe that $X_{12} = X_1 \cup X_2 \cup Y_3 \cup Y_4$ and $X_{34} = Y_1 \cup Y_2 \cup X_3 \cup X_4$. Since $m(X_{12}, X_{34}) \leq k-1$, we have

$$(a_{13}+a_{14}+a_{23}+a_{24})+(b_{12}+b_{21}+b_{34}+b_{43})+(b_{11}+b_{22}+b_{33}+b_{44}) \leq m(X_{12}, X_{34}) \leq k-1.$$

By symmetry, we will have two other similar inequalities about $m(X_{13}, X_{24})$ and $m(X_{14}, X_{23})$. Sum up the three inequalities, we have

$$\sum_{i=1}^{4} \sum_{j=1, j\neq i}^{4} (a_{ij} + b_{ij}) + 3\sum_{i=1}^{4} b_{ii} \leq 3(k-1). \tag{6.1}$$

Since $m_{G\backslash x}(X_1) \geq k-1$, we have

$$(a_{12} + a_{13} + a_{14}) + b_{11} + (b_{12} + b_{13} + b_{14}) = m_{G\backslash x}(X_1) \geq k-1.$$

By symmetry, we will have three other similar inequalities about $m_{G\backslash x}(X_2)$, $m_{G\backslash x}(X_3)$, and $m_{G\backslash x}(X_4)$. Sum up the four inequalities, we have

$$\sum_{i=1}^{4} \sum_{j=1, j\neq i}^{4} (a_{ij} + b_{ij}) + \sum_{i=1}^{4} b_{ii} \geq 4(k-1). \tag{6.2}$$

Compare the inequalities 6.1 and 6.2, we have $4(k-1) \leq 3(k-1)$, which implies $k \leq 1$, contradicting that $k \in \{2,3,4\}$. □

Theorem 6.1.10. *Every 5-regular 2-edge-connected graph can be reduced to* $5K_2$ *or* $3K_2^L$ *within* $\Phi_{2,1}$ *by* $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, *and* $\mathcal{O}_4$.

Proof. Let G be a 5-regular 2-edge-connected graph with more than two vertices. If there exists a non-loop edge e such that $G\backslash e$ is 2-edge-connected, then by Lemma 6.1.9, one of $\mathcal{O}_i$, $i = 1, 2, 3, 4$ can be applied within $\Phi_{2,1}$. Now assume that every non-loop edge in G is contained in an edge-cut of size two. Let G_1 be the graph obtained from G by deleting all the loops. Then G_1 is minimally 2-edge-connected. By Lemma 5.2.1, there is a vertex x of degree two in G_1. By the way we construct G_1 from the 5-regular graph G, x has an odd degree, contradicting that x has degree two. $\square$

Lemma 6.1.11. *Let G be a 5-regular essentially 4-edge-connected graph with more than two vertices. Let e be a non-loop edge in G. Then one of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied to e within $\Phi_{3,1}$.*

Proof. Note that $G\backslash e$ is essentially 3-edge-connected. In $G\backslash e$, if x is incident to a loop, then we split x by contracting the loop and an edge incident to x. Similarly, if y is incident to a loop, then we split y. Let H be the outcome. Note that H is essentially 3-edge-connected and every vertex in H is of degree at least 3. Thus, H is 3-edge-connected. By Lemma 6.1.9, we can split the degree-four vertices (if exist) in H within $\Phi_{3,1}$. Therefore, G can be reduced within $\Phi_{3,1}$. $\square$

Theorem 6.1.12. *Every 5-regular 3-edge-connected graph can be reduced to $5K_2$ or $3K_2^L$ within $\Phi_{3,1}$ by $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Proof. Let G be a 5-regular 3-edge-connected graph with more than two vertices. If G is essentially 4-edge-connected, then by Lemma 6.1.11, pick a non-loop edge that is not a 3-parallel-edge, then G can be reduced within $\Phi_{3,1}$ by $\mathcal{O}_1$ or $\mathcal{O}_2$. Otherwise, there exists a non-trivial edge-cut of size three. Choose a non-trivial edge-cut of size three with a smallest component A (in the sense of number of vertices). Contract the other component $\bar{A}$ to a vertex x and add a loop to this vertex. Let H be the resulting graph. Note that H is essentially 4-edge-connected. Pick a non-loop edge e in H that is not a 3-parallel-edge and not incident to x. By Lemma 6.1.11, H can be reduced within $\Phi_{3,1}$ by $\mathcal{O}_1$ or $\mathcal{O}_2$. We do the same graph operation on e in G. Then G is reduced within $\Phi_{3,1}$. $\square$

Theorem 6.1.13. *Every 5-regular 4-edge-connected graph can be reduced to $5K_2$ within $\Phi_{4,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$.*

Proof Let G be a 5-regular 4-edge-connected graph with more than two vertices. Note that G has no loop since G is 4-edge-connected. If there exists an edge e such that $G\backslash e$ is 4-edge-connected, then by Lemma 6.1.9, one of $\mathcal{O}_i$, $i = 1, 2, 3$ can be applied within $\Phi_{4,1}$. Now assume that every edge in G is contained in an edge-cut of size four. Then G is minimally 4-edge-connected. By Theorem 5.2.1, there is a vertex x in G of degree four, contradicting that G is 5-regular. $\square$

Lemma 6.1.14. *Let G be a 5-regular 5-edge-connected graph with more than two vertices. If there is an edge e such that $G\backslash e$ is essentially 5-edge-connected, then G can be reduced within $\Phi_{5,1}$ by $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Proof. Use exactly the same argument in 6.1.9 on the two degree-four vertices in $G\backslash e$. $\square$

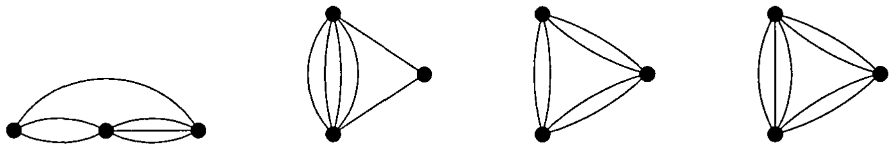
Theorem 6.1.15. *Every 5-regular 5-edge-connected graph can be reduced to $5K_2$ within $\Phi_{5,1}$ by $\mathcal{O}_1$ and $\mathcal{O}_2$*

Proof Let G be a 5-regular 5-edge-connected graph with more than two vertices. If there is an edge e such that $G\backslash e$ is essentially 5-edge-connected, then by Lemma

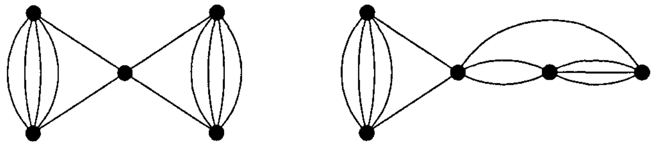
6.1.14, G can be reduced within $\Phi_{5,1}$. Otherwise, there exists a non-trivial edge-cut of size five, pick one with a smallest component (in the sense of number of vertices), contract the other component to one vertex. The resulting graph has to be $5K_2$, contradicting the way we choose the edge-cut. □

6.2 Generating Theorems for 5-Regular Loopless Graphs

Let G be a 5-regular loopless k-edge-connected graph where $k \in \{0, 1, 2, 3\}$. We replace each of the subgraphs in G that is isomorphic to one of L_1, L_2, and L_3, shown in Figure 6.2, by a vertex with an incident loop, replace each of the subgraphs in G that is isomorphic to L_4, shown in Figure 6.2, by a vertex with two incident loops, and let G_L be the outcome. Note that each replacement is equivalent to contracting the subgraph to a vertex and attach enough loops to this vertex so that it has degree five. Hence, the resulting graph G_L is 5-regular k-edge-connected. We denote a vertex with two incident loops by $2L$. Notice that, each $2L$ in G_L is a subgraph isomorphic to one of L_4, L_5, and L_6 in G. We say that an edge e in G is *incident* to a subgraph S of G if, S is isomorphic to L_i for some $i \in \{1, 2, \dots, 6\}$, and the corresponding edge e in G_L is incident to the loop in G_L corresponding to S in G. An edge e in G is said to be *special associate with* a subgraph S isomorphic to L_1 if it is incident to S and is incident to a 3-parallel-edge in S; the corresponding edge e in G_L is said to be *special associate with* the loop in G_L corresponding to S in G. Note that, if a non-loop edge $e = xy$ in G_L is not special and x is incident to a loop, then there is a splitting at x in $G \backslash e$ that does not create a loop and the resulting graph is k-edge-connected.



(a) L_1 (b) L_2 (c) L_3 (d) L_4



(e) L_5 (f) L_6

Figure 6 2 The "loop(s)" in G_L

We now present seven lemmas

Lemma 6.2.1. *Let G be a 5-regular loopless graph If G_L contains no $2L$'s, then G_L contains a non-loop edge that is not special*

Proof Assume on the contrary that all non-loop edges in G_L are special Suppose that we have s special edges and l loops Since every special edge is incident to a loop in G_L, we have $s \leq l$ Since every non-loop edge is incident to at most two loops and every loop is incident to exactly three edges, we have $3l \leq 2s$ Thus, $3s \leq 3l \leq 2s$, so that $s = 0$, a contradiction □

Lemma 6.2.2. *Let G be a 5-regular loopless graph If G_L is not isomorphic to $3K_2^L$ and contains no $2L$'s, then G_L contains at least two non-loop edges that are not special*

Proof Assume on the contrary that G_L contains at most one non-loop edge that is not special Suppose that we have e edges and l loops Since every special edge incident to a loop, we have $e - 1 \leq l$ Since every non-loop edge is incident to at most two loops and every loop is incident to exactly three edges, we have $3l \leq 2e$. Thus, $3(e-1) \leq 3l \leq 2e$, so that $e \leq 3$. Hence, G_L is isomorphic to $3K_2^L$, a contradiction. □

Lemma 6.2.3. *Let G be a loopless graph and x be a vertex of degree four in G. If every splitting at x produces a loop, then x is incident to a 3- or 4-parallel-edge.*

Proof Let xx_1, xx_2, xx_3, and xx_4 be the four incident edges. Consider the splitting $\{xx_1, xx_2\} - \{xx_3, xx_4\}$. By symmetry, we may assume that x_1x_2 is a loop in the resulting graph, i.e., $x_1 = x_2$. Now consider the splitting $\{xx_2, xx_3\} - \{xx_4, xx_1\}$. By symmetry, we may assume that x_2x_3 is a loop in the resulting graph, i.e., $x_2 = x_3$. Hence, x is incident to x_1 which is a 3- or 4-parallel-edge. □

Lemma 6.2.4. *Let G be a 5-regular loopless graph such that $\mathcal{O}_3$ and $\mathcal{O}_4$ cannot be applied within $\Phi_{0,2}$. If e is a 3- or 4-parallel-edge, then e is in a subgraph isomorphic to one of L_1, L_2, and L_4.*

Proof. If e is a 4-parallel-edge, then since $\mathcal{O}_4$ cannot be applied, e is in a subgraph isomorphic to L_2. We now assume that e is a 3-parallel-edge. Note that applying $\mathcal{O}_3$ to e is equivalent to contracting the three parallel edges to a vertex x and splitting x. Let H be the outcome by contracting the three parallel edges to x. By Lemma 6.2.3, x is incident to a 3- or 4-parallel-edge in H. Hence, e is in a subgraph isomorphic to L_1 or L_4. □

Lemma 6.2.5. *Let G be a 5-regular loopless graph such that $\mathcal{O}_3$ and $\mathcal{O}_4$ cannot be applied within $\Phi_{0,2}$. If $e = xy$ is a simple edge in G such that every splitting at x in $G \backslash e$ produces a loop, then e is either in or incident to a subgraph isomorphic one of L_1, L_2, and L_4.*

Proof. By Lemma 6.2.3, we may assume that x is incident to a 3- or 4-parallel-edge f. By Lemma 6.2.4, f is in a subgraph S isomorphic to one of L_1, L_2, and L_4. Thus, e is either in or incident to the subgraph S that is isomorphic to one of L_1, L_2, and L_4. □

Lemma 6.2.6. *Let G be a 5-regular loopless graph with more than two vertices. Let e be an edge of G. If no basic graph operations can be applied to e within $\Phi_{0,2}$, then e is either a special edge or in a subgraph isomorphic to one of L_1, L_2, L_3, and L_4.*

Proof. Let e be a k-parallel-edge with endpoints x and y in G where $k \in \{1, 2, 3, 4, 5\}$ and suppose that e is not in a subgraph isomorphic to one of L_1, L_2, and L_4. Since $\mathcal{O}_0(5K_2)$ cannot be applied within $\Phi_{0,2}$, $k \neq 5$. By Lemma 6.2.4, $k \neq 3, 4$. If $k = 1$, then by Lemma 6.2.5, e is incident to a subgraph isomorphic to one of L_1, L_2, and L_4; since $\mathcal{O}_1$ cannot be applied within $\Phi_{0,2}$, e is a special edge. We now assume $k = 2$. Note that applying $\mathcal{O}_2$ to e can be viewed as contracting the two 2-parallel-edges to a vertex z and splitting the degree-six vertex x, i.e., delete the vertex x and pairing the six incident edges. By a similar argument to Lemma 6.2.3, we know that x is incident to a 4-, 5-, or 6-parallel-edge. Hence, e is in L_1, L_3, or L_4. □

Lemma 6.2.7. *Let G be a 5-regular loopless connected graph with more than two vertices If no basic graph operations can be applied in G within $\Phi_{0,2}$, then every non-loop edge in G_L is special*

Proof. Let e be a non-loop edge in G_L. Then e in G is not in a subgraph isomorphic to one of L_1, L_2, L_3, and L_4. By Lemma 6.2.6, e is special. □

Here we introduce two graph operations illustrated in Figures 6.3 and 6.4, which are needed in the following theorem.

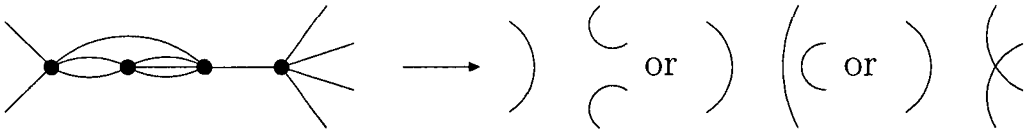


Figure 6.3: Operation $\mathcal{O}_6$

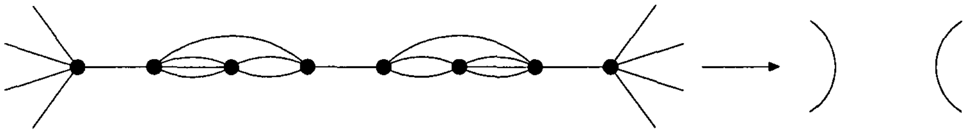


Figure 6.4: Operation $\mathcal{O}_7$

Theorem 6.2.8. *Every 5-regular loopless graph can be reduced to $5K_2$ within $\Phi_{0,2}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, $\mathcal{O}_0(5K_2)$, $\mathcal{O}_6$, and $\mathcal{O}_7$.*

Proof. Let G be a 5-regular loopless graph with more than two vertices. Suppose that no basic graph operation can be applied in G within $\Phi_{0,2}$. By Lemma 6.2.7, every non-loop edge in G_L is special. Let e be an edge in G_L. If e is special associate with one loop, then $\mathcal{O}_6$ can be applied within $\Phi_{0,2}$ in G. If e is special associate with two loops, then $\mathcal{O}_7$ can be applied within $\Phi_{0,2}$ in G. $\square$

Lemma 6.2.9. *Let G be a 5-regular loopless connected graph and B be an edge-block in G_L with $m(B) = 1$. If no basic graph operations can be applied to an edge in B within $\Phi_{1,2}$, then B is isomorphic to $2L$ in G_L.*

Proof. Let B' be a disjoint copy of B and connect B and B' by the cut-edge. Let H be the outcome. By Lemma 6.2.2, H is isomorphic to $3K_2^L$ or K_2^{2L}. Since H has a cut-edge, it has to be isomorphic to K_2^{2L}, so that B is isomorphic to $2L$ in G_L. $\square$

Here we introduce the graph operations $\mathcal{O}_8$, $\mathcal{O}_9$, and $\mathcal{O}_{10}$ illustrated in Figures 6.5, 6.6, and 6.7, respectively. For a graph operation $\mathcal{O}$, we use $\mathcal{O}(L)$ to mean the graph operation obtained from $\mathcal{O}$ by replacing each of the $2L$'s by one of L_4, L_5, and L_6, and each of the single loop by one of L_1, L_2, and L_3.

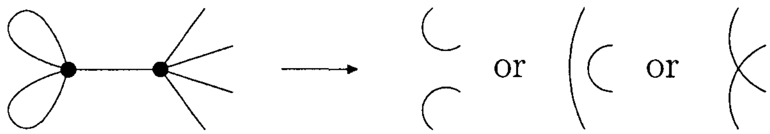


Figure 6.5: Operation $\mathcal{O}_8$

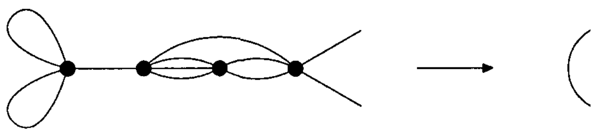


Figure 6.6: Operation $\mathcal{O}_9$

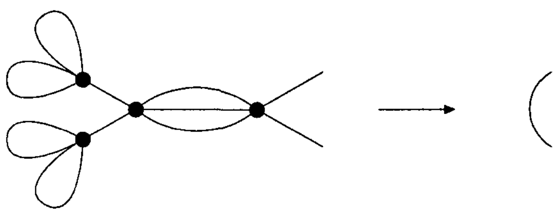


Figure 6.7: Operation $\mathcal{O}_{10}$

Theorem 6.2.10. *Every 5-regular loopless connected graph can be reduced to $5K_2$ within $\Phi_{1,2}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, $\mathcal{O}_5(L)$, $\mathcal{O}_8(L)$, $\mathcal{O}_9(L)$, and $\mathcal{O}_{10}(L)$.*

Proof. Let G be a 5-regular loopless connected graph with more than two vertices. Suppose that no basic graph operations can be applied within $\Phi_{1,2}$. In the edge-block tree $(G_L)_B$, we choose a longest path and let x be a leaf in this path. Let $e = xy$

be the edge incident to x If y is a single vertex in G_L, then since $\mathcal{O}_1$ cannot be applied within $\Phi_{1\,2}$, by Lemma 6 1 5, all the edges incident to y are cut-edges, thus, y is incident to at least four leaves in $(G_L)_B$, i e , y is incident to at least four $2L$'s in G_L, so that $\mathcal{O}_5(L)$ can be applied within $\Phi_{1,2}$ We may assume that y is not a single vertex If e is not incident to a 3-parallel-edge in G, then $\mathcal{O}_8$ can be applied within $\Phi_{1\,2}$ Now assume that e is incident to 3-parallel-edges in G If e is a special edge, then $\mathcal{O}_9$ can be applied within $\Phi_{1\,2}$ Otherwise, y in $(G_L)_B$ is the three 3-parallel-edges in G_L and at least three edges incident to y are cut-edges Hence, $O_{10}(L)$ can be applied within $\Phi_{1\,2}$ □

Lemma 6.2.11. *Let G be a k-edge-connected graph with $k \in \{0, 1, 2, 3, 4\}$ Let x be a vertex of degree four in G If there exists a splitting at x producing a new loop and the outcome is k-edge-connected, then every splitting at x is k-edge-connected*

Proof Since a splitting at x producing a new loop, there are two parallel edges e_1 and e_2 having common endpoints x and y Change the pairing of the split and notice that the new outcome can be obtained from contracting e_1 and e_2 Hence, the new outcome is also k-edge-connected □

Lemma 6.2.12. *Let G be a 5-regular loopless k-edge-connected graph with more than two vertices where $k \in \{2, 3\}$ If no basic graph operations can be applied in G within $\Phi_{k\,2}$, then every non-loop edge in G_L is either a special edge or in an edge-cut of size k in G_L*

Proof Suppose that $e = xy$ in G_L is a non-loop edge that is not in an edge-cut of size k Then $G_L \backslash e$ is k-edge-connected, so that by Lemma 6 1 5, there exists a basic graph operation that can be applied in G within $\Phi_{k,1}$ Since $\mathcal{O}$ cannot be applied in G

within $\Phi_{k,2}$, there exists a outcome containing a loop, so that by Lema 6.2.11, every outcome contains a loop, i.e., $\mathcal{O}$ cannot be applied within $\Phi_{0,2}$. By Lemma 6.2.6, e has to be a special edge. $\square$

Theorem 6.2.13. *Every 5-regular loopless 2-edge-connected graph can be reduced to $5K_2$ within $\Phi_{2,2}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, and $\mathcal{O}_6$.*

Proof. Let G be a 5-regular loopless 2-edge-connected graph with more than two vertices and suppose that no basic graph operations can be applied within $\Phi_{2,2}$. If G_L is 3-edge-connected, then by Lemma 6.2.12, every edge in G_L is special. This contradicts Lemma 6.2.1. Suppose that G_L is not 3-edge-connected, so that it contains an edge-cut of size two. Choose an edge-cut $\{f_1, f_2\}$ with a smallest component (in the sense of number of vertices). We replace the edge-cut $\{f_1, f_2\}$ in this component by an edge e and let H be the outcome. Then H is 3-edge-connected. By Lemma 6.2.12, every edge in H other than e is special. By Lemma 6.2.2, H is isomorphic to $3K_2^L$. Replace the non-loop edge that is not special by the 2-edge-cut set and one can verify that $\mathcal{O}_6$ can be applied within $\Phi_{2,2}$ in G in this case. $\square$

We introduce the graph operations $\mathcal{O}_{11}$ and $\mathcal{O}_{12}$ illustrated in Figures 6.8 and 6.9, respectively. These two operations are needed in the following theorem.

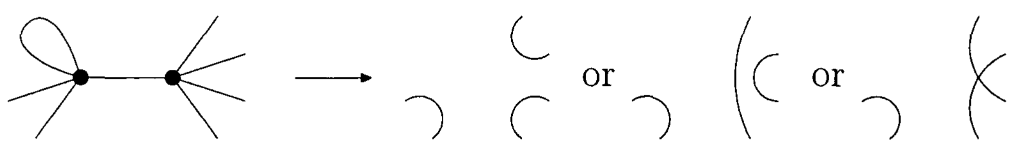


Figure 6.8: Operation $\mathcal{O}_{11}$

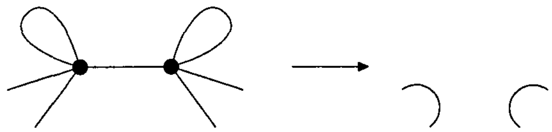


Figure 6.9: Operation $\mathcal{O}_{12}$

Theorem 6.2.14. *Every 5-regular loopless 3-edge-connected graph can be reduced to $5K_2$ within $\Phi_{3,2}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_{11}(L)$, and $\mathcal{O}_{12}(L)$.*

Proof. Let G be a 5-regular loopless 3-edge-connected graph with more than two vertices and assume on the contrary that none of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied within $\Phi_{3,2}$. If there is no non-trivial edge-cut of size three in G_L, then $\mathcal{O}_{11}$ or $\mathcal{O}_{12}$ can be applied within $\Phi_{3,2}$ in G. Otherwise, we choose a non-trivial edge-cut F of size three such that $G\backslash F$ has a smallest component H (in the sense of number of vertices). Note that each edge in H is in a trivial edge-cut of size three, so that is incident to loop(s). Hence, $\mathcal{O}_{11}$ or $\mathcal{O}_{12}$ can be applied within $\Phi_{3,2}$ in G. $\square$

Notice that, a 5-regular 4-edge-connected graph is loopless. Thus, the following two corollaries are implied by Theorem 6.1.13 and Theorem 6.1.15.

Corollary 6.2.15. *Every 5-regular 4-edge-connected graph can be reduced to $5K_2$ within $\Phi_{4,2}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$.*

Corollary 6.2.16. *Every 5-regular 5-edge-connected graph can be reduced to $5K_2$ within $\Phi_{5,2}$ by $\mathcal{O}_1$ and $\mathcal{O}_2$.*

6.3 Splitter Theorems for 5-Regular Graphs

In this section, we find splitter theorems of 5-regular graphs for different edge-connectivities. We denote by $\Phi_{k,g}(H)$ the class of k-edge-connected 5-regular graphs of girth at least g that contains H as an immersion.

Theorem 6.3.1. *If $G, H \in \Phi_{0,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{0,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, or $\mathcal{O}_0(5K_2)$.*

Proof Suppose $G \neq H$. We only need to show that G can be reduced one step in $\Phi_{0,1}(H)$. Choose an arbitrary white edge e in G. If e is in a component isomorphic to $5K_2$, then apply $\mathcal{O}_0(5K_2)$. Otherwise, apply one of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, and $\mathcal{O}_4$ by deleting e and splitting the two degree-four vertices. □

Lemma 6.3.2. *Let $G, H \in \Phi_{1,1}$ and $H \prec G$. Let e be a white cut-edge in G and C be a white component with more than one vertices in $G\backslash e$. Then one of $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_5$ can be applied within $\Phi_{1,1}(H)$.*

Proof Make a copy C' of C and connect C and C' by the edge e. Denote this graph by G'. By Theorem 6.1.8, one of $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_5$ can be applied to G' without disconnecting the graph. Note that, the operation cannot be applied to the edge e; otherwise, the resulting graph would be disconnected. Thus, the operation could be applied to either C or C'. By symmetry, we may assume that it is applied to C. Apply the same operation to the corresponding edge(s) in G. Since C is white, the resulting graph is in $\Phi_{1,1}(H)$. □

Theorem 6.3.3. *If $G, H \in \Phi_{1,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{1,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, or $\mathcal{O}_5$.*

Proof. We only need to show that G can be reduced in $\Phi_{1,1}(H)$ for one step.

Note that, if e is a white cut-edge in G and C is a white component in $G \backslash e$ is a white component with one vertex, then C is a vertex with two incident loops.

(i) Suppose that there exists a white cut-edge $e = xy$ in G and C is a white component in $G \backslash e$. If C has more than one vertices, then by Lemma 6.3.2, G can be reduced by $\mathcal{O}_1$, $\mathcal{O}_3$, or $\mathcal{O}_5$. We may assume that y is incident to $2L$ and splitting x disconnects the graph. Since we have a white component in the resulting graph, we can change the pairing for the splitting. By Lemma 6.1.5, we may assume that x has four incident cut-edges in $G \backslash e$. Then at least three of them are white cut-edges. By a similar argument, we may assume that each of the three white cut-edges incident to a $2L$, so that $\mathcal{O}_5$ can be applied within $\Phi_{1,1}(H)$.

(ii) Suppose that there is no white cut-edge. Let $e = xy$ be a white edge. Then $H \prec G \backslash e$. If a splitting at x disconnects the graph, then we can change the pairing for the splitting since we have a white component in the resulting graph, so that by Lemma 6.1.5, we can split x to obtain a connected outcome. Suppose that splitting x gives us a connected outcome G_1. Assume that splitting y at G_1 disconnects the graph. Similarly, we can change the pairing, so that by Lemma 6.1.5, we can split y to obtain a connected outcome since y does not have four incident cut-edges. $\square$

Lemma 6.3.4. *Let $G, H \in \Phi_{2,1}$ and $H \prec G$. Let $\{e_1, e_2\}$ be an edge-cut $\{e_1, e_2\}$ with e_1 as a white edge. Then one of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, and $\mathcal{O}_4$ can be applied within $\Phi_{2,1}(H)$.*

Proof. Note that both e_1 and e_2 are white, so that there exists a white component C in $G \backslash \{e_1, e_2\}$. Make a copy C' of C and connect with C through e_1, e_2. Let G' be the outcome. By Theorem 6.1.10, one of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, and $\mathcal{O}_4$ can be applied within

$\Phi_{2,1}$. Apply the same operation to the corresponding edge(s) in G. Then G is reduced within $\Phi_{2,1}(H)$. $\square$

Theorem 6.3.5. *If $G, H \in \Phi_{2,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{2,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, or $\mathcal{O}_4$.*

Proof. Let $e = xy$ be a white edge. By Lemma 6.3.4, we may assume that $G \backslash e$ is 2-edge-connected. If splitting x in $G \backslash e$ reduces the edge-connectivity, then the outcome has a white component, so that we can change the pairing and keep the edge-connectivity by Lemma 6.1.9. Similarly, if splitting y reduces the edge-connectivity, then the outcome has a white component, so that we can change the pairing and keep the edge-connectivity. Thus, G can be reduced within $\Phi_{2,1}(H)$. $\square$

Lemma 6.3.6. *Let $\{e_1, e_2, e_3\}$ be an edge-cut and C be a white component with more than one vertices in $G \backslash \{e_1, e_2, e_3\}$. Then one of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied within $\Phi_{3,1}(H)$.*

Proof. Make a copy C' of C and connect with C through e_1, e_2, e_3. Let G' be the outcome. By Theorem 6.1.12, one of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied within $\Phi_{3,1}$. Apply the same operation to the corresponding edge(s) in G. Then G is reduced within $\Phi_{3,1}(H)$. $\square$

Theorem 6.3.7. *If $G, H \in \Phi_{3,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{3,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, or $\mathcal{O}_3$.*

Proof. Let $e = xy$ be a white edge. If $G \backslash e$ is not 3-edge-connected, then there is a white component C in $G \backslash e$. If C has more than one vertices, then by Lemma 6.3.6, G can be reduced within $\Phi_{3,1}(H)$. Otherwise, we can apply graph operation to e within $\Phi_{3,1}(H)$.

Now suppose that $G\backslash e$ is 3-edge-connected. If splitting x reduces the edge-connectivity, then the outcome has a white component, so that we can change the pairing and keep the edge-connectivity by Lemma 6.1.9. Similarly, if splitting y reduces the edge-connectivity, then the outcome has a white component, so that we can change the pairing and keep the edge-connectivity. Thus, G can be reduced within $\Phi_{3,1}(H)$. □

Lemma 6.3.8. *Let $\{e_1, e_2, e_3, e_4\}$ be an edge-cut and C is a white component in $G\backslash\{e_1, e_2, e_3, e_4\}$. Then one of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied within $\Phi_{4,1}(H)$.*

Proof. Make a copy C' of C and connect with C through e_1, e_2, e_3, e_4. Let G' be the outcome. By Theorem 6.1.13, one of $\mathcal{O}_1$, $\mathcal{O}_2$, and $\mathcal{O}_3$ can be applied within $\Phi_{4,1}$. Apply the same operation to the corresponding edge(s) in G. Then G is reduced within $\Phi_{4,1}(H)$. □

Theorem 6.3.9. *If $G, H \in \Phi_{4,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{4,1}$ by $\mathcal{O}_1$, $\mathcal{O}_2$, or $\mathcal{O}_3$.*

Proof. Let $e = xy$ be a white edge. By Lemma 6.3.8, we may assume that $G\backslash e$ is 4-edge-connected. If splitting x reduces the edge-connectivity, then the outcome has a white component, so that we can change the pairing and keep the edge-connectivity by Lemma 6.1.9. Similarly, if splitting y reduces the edge-connectivity, then we have a white component, so that we can change the pairing and keep the edge-connectivity. Thus, G can be reduced within $\Phi_{4,1}(H)$. □

Theorem 6.3.10. *If $G, H \in \Phi_{5,1}$ and $H \prec G$, then G can be reduced to H within $\Phi_{5,1}$ by $\mathcal{O}_1$ or $\mathcal{O}_2$.*

Proof. Let $e = xy$ be a white edge. If splitting x results an edge-cut of size four that do not incident to y, then we have a white component, so that we can change the pairing and apply Lemma 6.1.14. Similarly, if splitting y results an edge-cut of size four, then we have a white component, so that we can change the pairing and obtain a 5-regular outcome. Thus, G can be reduced within $\Phi_{5,1}(H)$. $\square$

CHAPTER 7

CONCLUSIONS AND FUTURE WORK FOR PART I AND PART II

This dissertation consists of two parts: lattice theory and graph theory.

In the lattice theory part, we define and characterize five π-versions of distributivity via exclusion systems. A possible extension for future research is to find the π-versions of modularity and characterize them via exclusion systems. In this dissertation, we restrict the distributivity conditions on 3-elements antichains. A natural extension is to find similar results restricting to n-element antichains for $n \geq 4$ or for infinite antichains.

We also introduce a labeling method and use this method to construct an infinite 3-generated SRODL, a consequence of which is that $\mathcal{ODL}$ is not locally finite. We know that not every OML can be embedded into an ODL. It is natural to ask which OMLs can be embedded into ODLs. We know that the class of all SRODLs is a variety. But it remains open to us that whether or not the class of all OMLs that can be embedded into ODLs is a variety.

In the graph theory part, we find generating and splitter theorems for 5-regular graphs of various edge-connectivities. We also find generating theorems for 5-regular loopless graphs of various edge-connectivities. A natural direction for future work is to find generating theorems and splitter theorems with other restrictions of girth and edge-connectivity. Another possible direction is to find generating and splitter theorems for planar graphs or other surfaces, e.g., the projective plane and the klein bottle.

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