A CHARACTERIZATION OF RAMSEY GRAPHS FOR R(3,4)

by

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ABSTRACT

The Ramsey number $R(\omega,\alpha)$ is the minimum number n such that every graph G with $|V(G)| \geq n$ has an induced subgraph that is isomorphic to a complete graph on ω vertices, K_ω, or has an independent set of size α, N_α. Graphs having fewer than n vertices that have no induced subgraph isomorphic to K_ω or N_α form a class of Ramsey graphs, denoted $\mathscr{R}(\omega,\alpha)$. This dissertation establishes common structure among several classes of Ramsey graphs and establishes the complete list of $\mathscr{R}(3,4)$.

The process used to find the complete list for $\mathscr{R}(3,4)$ can be extended to find other Ramsey numbers and Ramsey graphs. The technique for finding a complete list for $\mathscr{R}(\omega,\alpha)$ is inductive on n vertices in that a complete list of all graphs in $\mathscr{R}(\omega,\alpha)$ having exactly n vertices can be used to find the complete list $n+1$ vertices. This process can be repeated until any extension is not in $\mathscr{R}(\omega,\alpha)$, and thus $R(\omega,\alpha)$ has been determined. We conclude by showing how to extend methods presented in proving $R(3,4)$ in finding $R(5,5)$.

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CHAPTER 1

INTRODUCTION

This chapter will provide the basics of graph theory used in this dissertation. The reader familiar with graphs can skip to Section 1.5. The terminology used will follow West[17] and Diestel [3].

1.1 Introduction to Graphs

A *graph* $G = (V, E)$ is an ordered pair of sets consisting of a *vertex set* V, or $V(G)$, and an *edge set* denoted E, or $E(G)$. The elements of $V(G)$ are called *vertices*, and the elements of $E(G)$, called *edges*, are unordered pairs of vertices in $V(G)$. Every edge e in $E(G)$ has two vertices called its *endvertices* or *endpoints*. A *loop* is an edge whose endvertices are identical. Two non-loop edges e and f are said to be *parallel* if they have the same pair of endvertices. A graph with no loops or parallel edges is said to be *simple*. Unless stated otherwise, all graphs in this dissertation will be finite and simple.

In a simple graph G, an edge $e \in E(G)$ that has the endpoints $u, v \in V(G)$ is denoted uv or vu. A vertex $v \in V(G)$ is *incident* with an edge $e \in E(G)$ if v is an endpoint of e. Two vertices $u, v \in V(G)$ are *adjacent*, or *neighbors*, if and only if uv is an edge in $E(G)$. The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident to v. A vertex with degree equal to zero is an *isolated* vertex. The number $\delta(G) := min\{d(v)|v \in V(G)\}$ is the minimum degree of G, and the number $\Delta(G) := max\{d(v)|v \in V(G)\}$ is the maximum degree of G.

Two graphs $G = (V,E)$ and $G' = (V',E')$ are *isomorphic*, written $G \cong G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'$ for all $x, y \in V$. The map φ is called an *isomorphism*.

1.2 Trees and Subgraphs

A graph G has a *path* $P_n = v_1e_1v_2e_2\cdots v_{n-1}e_{n-1}v_n$ where the endpoints of each edge e_i are the vertices v_{i-1} and v_i, and the vertices of P are distinct. A graph that is a path with n vertices and $n-1$ edges is called P_nA *cycle* $C_n = v_0e_1v_1e_2v_2\cdots e_nv_0$ in a graph is similar to a path except that the first and last vertices coincide. A graph that is a cycle with n vertices and n edges is called C_n. A graph G that contains no cycle is *acyclic.*

A graph G is *connected* if a path exists between any two vertices in $V(G)$; otherwise, G is *disconnected.* The *components* G_1, G_2, $\cdots$, G_n of a graph G are the maximally connected subgraphs of G and $G \cong G_1 \mathring{\bigcup} G_2 \mathring{\bigcup} \cdots G_n$. A graph G is *minimally connected* if G is connected and for any edge $e \in E(G)$, the graph $G - e$ is a disconnected graph. G is called k-connected if $|V(G)| > k$ and the deletion of any set of k vertices in $V(G)$ results in a graph that is still connected.

A *tree* is a connected acyclic graph. Clearly, for any $u, v \in V(T)$ such that $uv \notin E(T)$, the graph $T + uv$ will contain a cycle. A *leaf* in tree T is a vertex v such that $d(v) = 1$. If each component of a graph is a tree, then the graph is a *forest.* The followig theorems are straightforward and standard in any textbook.

Theorem 1.2.1. *The following assertions are equivalent for a graph T.*

1. *T is a tree;*
2. *any two vertices of T have a unique path in T;*
3. *T is minimally connected;*
4. *T is maximally acyclic.*

Theorem 1.2.2. *If T is a tree with $\Delta(T) = k$, then T has at least k leaves.*

Corollary 1.2.3. *If T is a tree with $\Delta(T) \leq 2$, then T is a path.*

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, denoted $G' \subseteq G$. A subgraph H of G is *induced* when uv in $E(H)$ if and only if uv in $E(G)$.

A *spanning subgraph* H of a graph G is a subgraph that contains every vertex in $V(G)$. A *spanning tree* is a spanning subgraph that is a tree. A *spanning cycle* is a spanning subgraph that is isomorphic to a cycle. Spanning cycles are also called *Hamilton cycles.* Similarly, a *Hamilton path* is a path containing all the vertices in a graph.

1.3 Cliques and Independent Sets

The *complete graph* G on n vertices, denoted K_n, is such that every pair of distinct vertices are adjacent. A *clique* of G is a subgraph H that is isomorphic to a complete graph. The size of the largest clique of G is denoted $\omega(G)$. The clique on three vertices, K_3, is also called a *triangle*. Thus, graphs having no clique of size three are called *triangle free*.

The *complement* of a graph G is denoted $\overline{G}$ where $V(G) = V(\overline{G})$ and $e \in E(\overline{G})$ if and only if $e \notin E(G)$. The *totally disconnected graph* is the graph that has no edges. The totally disconnected graph on α vertices is denoted $N_\alpha \cong \overline{K_n}$. If a graph G has an induced subgraph isomorphic to N_α, then the set of vertices is said to be independent; moreover, the size of a maximum independent set of G is denoted $\alpha(G)$.

1.4 Ramsey Graphs

The class of Ramsey graphs, $\mathscr{R}(\omega . \alpha)$, consists of all graphs that contain no subgraph isomorphic to K_ω, and no independent set of size α, N_α. The minimum number of vertices, n, for which every graph with at least n vertices contains an induced subgraph isomorphic to K_ω or N_α is called the Ramsey number, denoted $R(\omega, \alpha) = n$.

A common way to introduce Ramsey numbers, using $R(3,3)$ as an example, is to ask the question "What is the minimum number of people that must be invited to a party such that there are three people who are all mutual acquaintances (each one knows the other two) or mutual strangers (each one does not know either of the other two)?" This question can be represented as a graph G. Let the vertices of the graph represent the people at the party. If two people know each other, then their corresponding vertices are joined by an edge; otherwise, there is no edge present. If there are three people who mutually know each other, then this will form a triangle in the graph, or K_3. If there are three people who are mutual strangers, then this will form a set of three independent vertices in the graph, or N_3. Now the question becomes what is the minimum number of vertices, n, such that all graphs on n vertices either have a K_3 or N_3. Harary [7] presented Theorem 1.4.1 to establish $R(3,3) = 6$.

Theorem 1.4.1. *(Harary [7]) For any graph G with six vertices, G or $\overline{G}$ contains a triangle as a subgraph.*

Proof. Consider the cycle on five vertices, depicted in Figure 1.1, which is easily seen to be a Ramsey graph in $\mathscr{R}(3,3)$. No triangles exist in the cycle C_5, so there is no subgraph isomorphic to K_3. Also, there is no set of three vertices that do not have an edge in C_5. Thus we say the cycle on five vertices is a Ramsey graph, or $C_5 \in \mathscr{R}(3,3)$. This means that $R(3,3) > 5$, so we must consider graphs with six vertices. The only connected graph on six vertices that has $\Delta(G) = 2$ is the path or cycle on six vertices, but it is clear that three independent vertices exist, so $\Delta(G) \geq 3$. Considering the neighbors of a vertex v, of degree three, if no edge exists between them then there are three independent vertices. Also if an edge exists between any two, a triangle exists. Thus there are no Ramsey graphs on six vertices and $R(3,3) = 6$. □

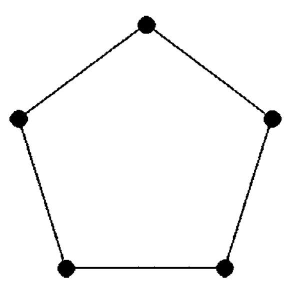


Figure 1.1: The cycle on five vertices has no K_3 or N_3.

Theorem 1.4.2. *(Ramsey 1930 [14]) For every $r \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that every graph of order at least n contains either K_r or N_r as an induced subgraph.*

Ramsey [14] proved that for any natural number r, the Ramsey number $R(r,r)$ is finite. Few Ramsey numbers are known. From a survey of Ramsey numbers[13], the known numbers and best bounds are updated periodically. The known Ramsey numbers are listed in Table 1.1.

Table 1.1: Table of known Ramsey numbers[13].

	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10
3		6	9	14	18	23	28	36	[40,43]
4			18	25	[35,41]				
5				[43,49]					
6									

Theorem 1.4.3. *(Greenwood and Gleason [6])* $R(\omega,\alpha) \leq R(\omega-1,\alpha)+R(\omega,\alpha-1)$.

The Ramsey numbers for $R(5,5)$, $R(4,6)$ and $R(3,10)$ are not known, but we do know upper and lower bounds for these. Greenwood and Gleason [6] proved Theorem 1.4.3 which gives an upper bound on a Ramsey number. The range for the remaining Ramsey numbers grows very large as α and ω increase.

To determine a particular Ramsey number, the Ramsey graphs with $R(\omega,\alpha)-1$ vertices become a focus. Such graphs are called Ramsey *critical.* The complete set of Ramsey critical graphs are known for the Ramsey numbers $R(3,3)$, $R(3,4)$, $R(3,5)$, $R(3,6)$, $R(3,7)$, and $R(4,4)$. Some critical graphs for other Ramsey numbers are known, but we do not know if these collections are complete [13]. Many lower bounds for finding Ramsey numbers are created by finding a Ramsey graph where there is no K_ω or N_α on n vertices forcing $R(\omega,\alpha) > n$.

Theorem 1.4.4 proves that if a graph H satisfies the clique or independent set for a $R(\omega,\alpha)$, then any graph G that contains H as an subgraph will also satisfy the clique or independent set. This is useful since searching a subgraph can be considerably faster than searching the entire graph.

Theorem 1.4.4. *Let H be an induced subgraph of G. If* $H \notin \mathscr{R}(\omega, \alpha)$*, the* $G \notin \mathscr{R}(\omega, \alpha)$.

Proof. Let $H \notin \mathscr{R}(\omega, \alpha)$. Let G be a graph such that H is an induced subgraph of G. Since $H \notin \mathscr{R}(\omega, \alpha)$, there exists a K_ω or N_α subgraph of H. One of the following is true: $K_\omega \subseteq H \subseteq G$ or the independent set N_α in H is also an indpendent set of G. In both cases, $G \notin \mathscr{R}(\omega, \alpha)$. □

Corollary 1.4.5. *If there are no* $\mathscr{R}(\omega, \alpha)$ *graphs on n vertices, then there are no* $\mathscr{R}(\omega, \alpha)$ *graphs on* $n+1$ *vertices*

The proof of Corollary 1.4.5 is easily seen since there is a K_ω or N_α for every graph on n vertices, and any graph on $n+1$ vertices must also have a K_ω or N_α.

Lemma 1.4.6. *If* $G \in \mathscr{R}(3, \alpha)$ *then G is triangle free and* $\Delta(G) \leq \alpha - 1$.

Proof. Suppose $G \in \mathscr{R}(3, \alpha)$. Since G has no subgraph isomorphic to K_3, G is triangle free. Suppose $\Delta(G) > \alpha - 1$. There is a vertex $v \in V(G)$ such that $d(v) \geq \alpha$. Since G is triangle free, no two neighbors of v can be adjacent to one another without forming a triangle. This means that $N(v)$ forms an independent set of size at least α, but $\alpha(G) < \alpha$, a contradiction. Thus G is triangle free and $\Delta(G) \leq \alpha - 1$. □

1.5 Dissertation Overview

An overview of the dissertation is provided here discussing the results found. Ramsey theory can be applied to many areas in mathematics and computer science. A survey of the applications of Ramsey theory include topics from algebra to theoretical computer science [16].

There are very few Ramsey numbers for which the exact value has been calculated. The most likely unkown Ramsey number to be determined next is $R(5,5)$. The best known bounds are $43 \leq R(5,5) \leq 49$. Since there are approximately 10^{208} nonisomorphic graphs on 42 vertices [8], a brute force approach of searching all graphs on 42 vertices is not feasible with current computational resources. The research presented in this dissertation reduces the number of graphs that must be tested and provides a framework that shows a common structure among graphs in $\mathscr{R}(5,5)$.

Ramsey [14] started the work in this area in 1930 by proving that for any $\alpha, \omega \in \mathbb{N}$ there is a minimum number n such that every graph G on n vertices has either a K_ω or N_α. While we know these numbers exist, and can find a range where they will be, finding the exact value of n has proved to be intractable.

Chapter 2 discusses patterns that are found in various Ramsey graphs. These patterns were discovered by creating algorithms to find all the Ramsey graphs for certain values of α and ω. Due to the computation complexity of the algorithms developed, the resources of the Louisiana Optical Network Initiative (LONI) were utilized. Then the graphs found were analyzed for a common structure between the set of Ramsey numbrers. The analyzed classes were $R(3,3)$, $R(3,4)$, $R(3,5)$, $R(3,6)$, and $R(4,4)$. The analysis is summarized in the tables showing the relation of the number of Ramsey graphs based on the vertices and edges.

Greenwood and Gleason [6] proved $R(3,4) = 9$ in 1955. Their proof found a maximal Ramsey graph and used bounds on $R(3,4)$ to show that no larger Ramsey graphs exist. An independent proof of $R(3,4)$ is shown in this dissertations by using the property that every graph $G \in \mathscr{R}(3,4)$ has a spanning tree, and extensions from all possible spanning trees are done to find the complete list for $\mathscr{R}(3,4)$. Instead of using a brute force approach on all spanning trees, Chapter 3 shows that Ramsey graphs with at least five vertices must have a Hamilton path or be a tree, so the paths P_5, P_6, P_7, P_8, and P_9 are the only initial trees needed. These paths are then extended until all graphs in $\mathscr{R}(3,4)$ are found.

By showing that the list of all graphs in $\mathscr{R}(3.4)$ has been found, in Chapter 4, we show that no Ramsey graphs on nine vertices are possible. This being the smallest such number leads to an independent proof of $R(3,4) = 9$. In Chapter 5, we show how to extend the method created here to aid in finding $R(5,5)$. In particular, we show that all graphs in $\mathscr{R}(5,5)$ having 42 vertices are Hamiltonian that are also 14-connected.

CHAPTER 2

OBSERVATIONS OF RAMSEY GRAPHS

2.1 Introduction

A graph G is in the family of Ramsey graphs, denoted $\mathscr{R}(\omega,\alpha)$, if G contains no induced subgraph isomorphic to K_ω or N_α. The Ramsey number $R(\omega,\alpha)$ is the minimum number of vertices such that every graph with at least $R(\omega,\alpha)$ vertices has an induced subgraph isomorphic to K_ω or N_α. Ramsey [14] showed that the number $R(\omega,\alpha)$ exists for any $\omega,\alpha \in \mathbb{N}$. Thus, there are a finite number of Ramsey graphs since there are a finite number of graphs up to a set number of vertices. Hence, all Ramsey graphs in $\mathscr{R}(\omega,\alpha)$ will have at most $R(\omega,\alpha)-1$ vertices.

Using the algorithms developed in this chapter, each Ramsey number $R(3,3)$, $R(3,4)$, $R(3,5)$, $R(3,6)$, and $R(4,4)$ was tested by determining all necessary graphs up to $R(\omega,\alpha)$ vertices. This evaluation determines if a graph contains a K_ω or N_α and is summarized in the tables at the end of Section 2.3.

2.2 Parallel Algorithms

A computer algorithm was developed to find the Ramsey graphs to analyze. Due to the number of graphs that needed to be searched, a regular computer was insufficeint to run the program. By parallelization, the algorithm was able to search through many more graphs than on a single computer. The main computer used in this dissertation was part of the Louisiana Optical Network Initiative (LONI) system. LONI consists of five Power5 575 AIX clusters, six five teraflop Dell Linux clusters, and one Dell 50 teraflop Linux cluster. By using the P5 supercomputers, the computation for testing graphs with a higher number of vertices became feasible. For example, to test $\mathscr{R}(4,4)$ on 18 vertices, it took eight nodes, 64 processors, on a Power5 495684 seconds = 139.69 hours = 5.73 days. If this test was performed on a single processor, an estimated time of approximately one year of continuous operation would be needed.

Brendan McKay developed a graph generation program called *nauty*[10]. *Nauty* can generate all graphs up to isomorphism with a maximum of 32 vertices. This limit is based on how a graph is stored in the program. A graph is represented in the bit-level of an integer, which is four bytes or 32 bits. *Nauty* can be passed many parameters in the generation of graphs. The parameters are passed to *nauty* when the program is called for a given vertex class. These parameters include connectivity, the edge range, minimum degree, and maximum degree. For example, we know from Lemma 1.4.6 that for every $G \in \mathscr{R}(3,\alpha)$, G has a maximum degree of $\alpha - 1$, so we can instruct *nauty* to only generate graphs with $\Delta(G) \leq \alpha - 1$. Any other restrictions based on the class $\mathscr{R}(\omega,\alpha)$ can be passed as a parameter to *nauty* to reduce the number of graphs that will be tested.

Since the algorithms used to test a graph are independent, graphs can be tested in parallel to speed up computation time. Many methods were developed to optimize the program, but Algorithm 2.1 is what lies at the core of the program. The computational complexity of the algorithm allows the generation of graphs on a small number of vertices, but is not feasible for larger numbers of vertices due to the large number of graphs that must be tested and given current computational resources on LONI. A parallelized version of the algorithm was used to create the tables in Section 2.3. Even with parallezation, the number of graphs that must be tested grows too fast to be feasible on current computer hardware. LONI is very useful extending the feasibility, but the exponential growth in the number of graphs as vertices increase is too much for the current hardware available.

Nauty is able to generate all graphs up to isomorphism on up to 32 vertices. A graph with a small number of vertices, about 10 to 15 depending on conditions of the graphs, works well on a regular laptop or desktop computer, but the growth of the number of graphs as vertices increase make continuing computation this way infeasible. In order to test all graphs on a higher number of vertices in a reasonable amount of time, parallelization was developed to increase the computational speed.

Algorithm 2.1 Test all graphs on $|V(G)|$ vertices for $\mathscr{R}(\omega,\alpha)$.

```
FUNCTION CreateListRwa( ω, α )
  int i = 1;
  repeat
    |V(G)| = i;
    for every G on |V(G)| vertices do
      if NOT TestRwa(G,ω,α) then
        ListRwa[|V(G)|].Add(G);                    // Add G to list
      end if
    end for
    ++i;
  until List[i-1].Size() == 0
  ListRwa.Print();                                 // All graphs in R(ω, α)
ENDFUNCTION
```

The first step of parallelizing the program finds independent sections that run simultaneously with minimal communication. By partitioning the graphs of a vertex class, we can test several partitions in parallel. The partitioning is done through *nauty* in such a way that no two partitions have an isomorphic graph, but the partitions may vary in size. Once the graphs are partitioned into separate classes, each processor needs to be assigned a set of partitions to test. These assignments can be done statically or dynamically.

A static allocation assigns each processor approximately the same number of partitions and requires no communication time other than the collection of the results at the programs completion. However, this does not work well for speeding up the program since each graph can vary on the number of computation cycles required to determine if the graph is in $\mathscr{R}(\omega,\alpha)$ or not. This leads to some processors working on its assigned partitions while the other processors sit idle.

By creating a *Master Thread*, shown in Algorithm 2.2, to dynamically allocate partitions to processors, the idle time can be reduced. The *Master Thread* coordinate with all the computation nodes to test for Ramsey graphs. One modification to *Master Thread* is to sort the size of the partitions to increase efficiency. While the size of the partition is not directly related to the time it takes to test it, it usually is a good indicator. Since the partitions are not necessarily similar in size, initially the graphs are counted in each partition without being tested. This produces a slight amount of extra work, but the time used counting graphs is negligible compared to testing each graph. The partitions are sorted by size and the *Master Thread* assigns each processor a partition starting with the largest partition size. This is done since the smaller partitions will be assigned to processors as they become idle and thus will reduce processor idle time. After all partitions have been tested, the results are collected and summarized by the *Master Thread*.

Algorithm 2.2 *Master Thread* controls the program.

```
FUNCTION Master Thread( ω, α, 𝒢 )
  np := number of processors;
  nc := 100 * np;
  int class = 0;
  // Send initial class to each processor to test.
  for each processor do
    Send( pid, class );
    ++class;
  end for
  // Wait for a processor to finish and then send new class to test.
  repeat
    Receive(p);
    Send(p, 𝒢[class]);
    ++class;
  until class == nc;
  // Wait for all processors to finish.
  for each remaing processor do
    Recieve(p);
    Send(p, "STOP");
  end for
  // Collect all graphs found from processors.
  CollectGraphs();
ENDFUNCTION
```

The *Compute Node*, shown in Algorithm 2.3, does most of the computation. First, it gets the initial partition of graphs to test from the *Master Thread*. Each graph is tested to determine if the graph is the Ramsey class of graphs set by the initial parameters. When a *Compute Node* is finished with its partition, the *Master Thread* is contacted for a new class to test. This process is repeated until the *Master Thread* finishes assigning all classes and orders the *Compute Node* to stop. When all *Compute Nodes* are finished, the data collected is sent to the *Master Thread* for summarizing.

Algorithm 2.3 *Compute Node* takes instruction from *Master Thread*.

```
FUNCTION Compute Node( ω, α )
  Receive(ℋ);
  repeat
    for every graph G in ℋ vertices do
      if NOT TestRwa(G,ω,α) then
        ListRwa.Add(G);                         // Add G to list
      end if
    end for
    Send(Master Thread)
  until Receive(ℋ) == "STOP"
  ReturnInfo(ListRwa);
ENDFUNCTION
```

These methods work well for computationally intensive programs that do not need much communication. The only communications required are integers representing a class and its size between the *Master Thread* thread and the processors. With enough graphs in each partition, the communication time is minimal compared to the computation time. Where np is the number of processors, usually around $100 * np$ partitions would result in a good parallel speedup.

The P5 computers for LONI contain 14 nodes with eight processors on each node. The wall clock time, maximum time to run a program, on LONI is limited to one week. If we tested all graphs on n vertices and the program took five days, it can easily break the wall clock time on $n+1$ vertices and the program will be shut down automatically. So another layer of parallelization was developed to test vertex classes that would take longer than one week. At this level, the partitions created as before remain the same, but more partitions are created initially since many more graphs need to be tested. When running the program, a subset of the total partitions will be tested to fit in the wall clock time. Suppose we create 5000 partitions to test. If we would run the program as before, the wall clock time would ellapse and kill the program. A *job* is created that would take the subset of partitions from 0 to 1000 to be tested in one program. Four other jobs would also be created to test all the partitions in the set. In this way, a program that would have taken five weeks can still be done without violating the wall clock time by spreading the jobs out over the network. The number of partitions and jobs that must be created are highly variable based on the vertex class and any restrictions that can be placed on the graphs needed.

2.3 Tables for Ramsey Graphs

Tables 2.1 through 2.5 show the number of Ramsey graphs for each vertex and edge class for the Ramsey numbers R(3,3), R(3,4), R(3,5), R(3,6), and R(4,4). The tables are organized in a matrix where the columns are vertex classes, rows are the edge classes, and each cell has the number of Ramsey graphs for that particular vertex and edge class. The bottom row is the number of Ramsey graph for each vertex class. The right column is the number of Ramsey graphs for each edge class. Summing the total, the total number of Ramsey graphs for each Ramsey number is found in the bottom right corner.

Table 2.1: Distribution of $\mathscr{R}(3,3)$ graphs. R(3,3) = 6.

R(3,3)								
	Vertices							
		1	2	3	4	5	6	Total
	0	1	1					2
E	1		1	1				2
d	2			1	1			2
g	3				1			1
e	4				1			1
s	5					1		1
	6							0
Total		1	2	2	3	1	0	9

Table 2.2: Distribution of $\mathscr{R}(3,4)$ graphs. R(3,4) = 9.

R(3,4)											
	Vertices										
		1	2	3	4	5	6	7	8	9	Total
	0	1	1	1							3
E	1		1	1	1						3
d	2			1	2	1					4
g	3				2	2	1				5
e	4				1	3	1				5
s	5					2	4				6
	6					1	4	1			6
	7						3	2			5
	8						1	3			4
	9						1	2			3
	10							1	1		2
	11								1		1
	12								1		1
Total		1	2	3	6	9	15	9	3	0	48

Table 2.3: Distribution of $\mathscr{R}(3,5)$ graphs. R(3,5) = 14.

R(3,5)																
	Vertices															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
	0	1	1	1	1											4
E	1		1	1	1	1										4
d	2			1	2	2	1									6
g	3				2	3	3	1								9
e	4				1	4	6	2	1							14
s	5					2	8	7	1							18
	6					1	7	13	5							26
	7						4	17	13	1						35
	8						2	15	27	3						47
	9						1	10	39	11						61
	10							4	41	28	1					74
	11							1	27	59	2					89
	12							1	15	73	10					99
	13								6	62	32					100
	14								2	33	69					104
	15								1	14	86	1				102
	16								1	4	65	6				76
	17									2	32	19				53
	18										12	31				43
	19										3	30				33
	20										1	13	1			15
	21											4	2			6
	22											1	5			6
	23												2			2
	24												2			2
	25															0
	26													1		1
Total		1	2	3	7	13	32	71	179	290	313	105	12	1	0	1029

Table 2.4: Distribution of $\mathscr{R}(3,6)$ graphs. R(3,6) = 18.

R(3,6)														
	Vertices													
		···	7	8	9	10	11	12	13	14	15	16	17	Total
	0	··												5
E	1	···												5
d	2	···	1											8
g	3	···	3	1										13
e	4	···	7	3	1									23
s	5	···	13	10	2	1								37
	6	·	20	23	8	1								60
	7	···	20	44	26	5								99
	8	···	18	63	70	16	1							170
	9	···	11	73	142	60	3							290
	10	···	5	63	234	175	16							493
	11	···	1	40	284	451	64	1						841
	12	··	1	21	267	864	265	4						1422
	13	···		9	185	1255	900	20						2369
	14	·		3	106	1344	2353	119						3925
	15	···		2	47	1114	4444	644	1					6252
	16			1	22	707	6134	2693	4					9561
	17	···			8	377	6239	7968	45					14637
	18	···			3	167	4823	16445	375					21813
	19	···			1	71	2885	23986	2402					29345
	20	··			1	28	1405	25267	10176	1				36878
	21	···				13	565	19704	27975	16				48273
	22					4	206	11672	51188	177				63247
	23	···				2	64	5404	64221	1588				71279
	24					1	20	2016	56809	8494				67340
	25	···				1	6	630	36312	27013	1			63963
	26	···					2	169	17208	53157	7			70543
	27	···						41	6189	67224	101			73555
	28	··						8	1729	56478	822			59037
	29	···						1	377	32235	3998			36611
	30	··							66	12784	10910			23760
	31	···							8	3550	17552			21110
	32	···							1	699	16896	5		17601
	33	···								94	9957	39		10090
	34	···								9	3587	200		3796
	35	···								1	794	547		1342
	36	·									100	803		903
	37	···									7	634		641
	38	··										275		275
	39	···										62		62
	40	···										11	2	13
	41	···											3	3
	42	··											2	2
Total		···	100	356	1407	6657	30395	116792	275086	263520	64732	2576	7	761692

Table 2.5: Distribution of $\mathscr{R}(4,4)$ graphs. R(4,4) = 18.

R(4,4)													
	Vertices												
		...	8	9	10	11	12	13	14	15	16	17	
	0	...											3
E	1	...											3
d	2	...											4
g	3	...											8
e	4	...											9
s	5	...											15
	6	...											23
	7	...	1										34
	8	...	3										51
	9	...	11	1									83
	10	...	38	1									124
	11	...	111	5									196
	12	...	244	18									320
	13	...	398	73									500
	14	...	467	257	1								737
	15	...	398	768	5								1175
	16	...	244	1719	34								1998
	17	...	111	2831	177								3119
	18	...	38	3355	814								4207
	19	...	11	2831	2963	1							5806
	20	...	3	1719	8193	7							9922
	21	...	1	768	16396	72							17237
	22	...		257	23270	546							24073
	23	...		73	23270	3201							26544
	24	...		18	16396	13695							30109
	25	...		5	8193	41553							49751
	26	...		1	2963	87361	8						90333
	27	...		1	814	126742	177						127734
	28	...			177	126742	1906						128825
	29	...			34	87361	13332						100727
	30	...			5	41553	58131						99689
	31	...			1	13695	163757						177453
	32	...				3201	302088						305289
	33	...				546	370368	20					370934
	34	...				72	302088	535					302695
	35	...				7	163757	6339					170103
	36	...				1	58131	37825					95957
	37	...					13332	127138					140470
	38	...					1906	257711					259617
	39	...					177	325095					325272
	40	...					8	257711					257719
	41	...						127138	40				127178
	42	...						37825	872				38697
	43	...						6339	6247				12586
	44	..						535	20901				21436
	45	...						20	37348				37368
	46	...							37348				37348
	47	...							20901				20901
	48	...							6247				6247
	49	...							872				872
	50	...							40	13			53
	51	...								96			96
	52	...								211			211
	53	...								211			211
	54	...								96			96
	55	...								13			13
	56	...	...	...	...	...	...	...	...	...	...	...	...
Total		...	2079	14701	103706	546356	1449166	1184231	130816	640	2	1	3432184

2.4 Analyzing Ramsey Graphs

An analysis of each table reveals several patterns which are present in each class of Ramsey numbers. This section will look at $R(3,5)$ in detail, but the patterns hold for the other Ramsey numbers.

Figure 2.1 shows a perspective with regard to the number of Ramsey graphs. The distribution is formed by plotting the number of graphs on each vertex class of $\mathscr{R}(3,5)$. Notice how on 11 and 12 vertices the number of Ramsey graphs falls dramatically. We can use Theorem 1.4.4 and a complete list of Ramsey graphs on a vertex class past the maximum to find the ramsey number with less computation than testing all graphs up to $R(\omega,\alpha)$ vertices. By having the list, each graph can have a vertex added in all possible ways to find the list of Ramsey graphs on the next vertex class. When a vertex class is reached where adding any vertex will form a K_ω or N_α, we will have the Ramsey number.

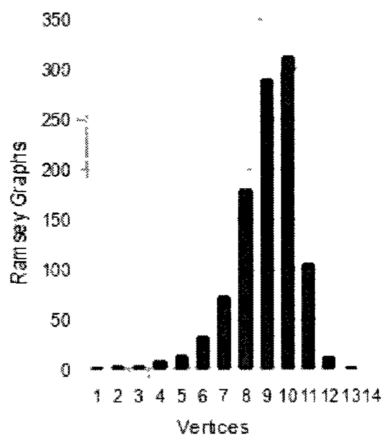


Figure 2.1: The hill over vertices for $\mathscr{R}(3,5)$.

Observing that the maximum occurs at $|V| = 10$, we isolate the Ramsey graphs in $\mathscr{R}(3,5)$ having exactly 10 vertices, considering how the graphs are distributed by the number of edges in each Ramsey graph. The plot is depicted in Figure 2.2. When plotting with vertices for $\omega = \alpha$, the maximum is on the edge class $\lfloor \frac{|E|}{2} \rfloor = \lfloor \frac{n(n-1)}{4} \rfloor$ where n is the vertex class. The edges classes surrounding $\lfloor \frac{n(n-1)}{4} \rfloor \pm x$ are symmetric since if G is in $\mathscr{R}(3,4)$, then $\overline{G}$ is also in $\mathscr{R}(3,4)$ when $\omega = \alpha$. For the Ramsey numbers $\mathscr{R}(3,3)$, $\mathscr{R}(3,4)$, $\mathscr{R}(3,5)$, and $\mathscr{R}(3,6)$ in the tables shown, the maximum is less than or equal to $\frac{|E|}{3}$.

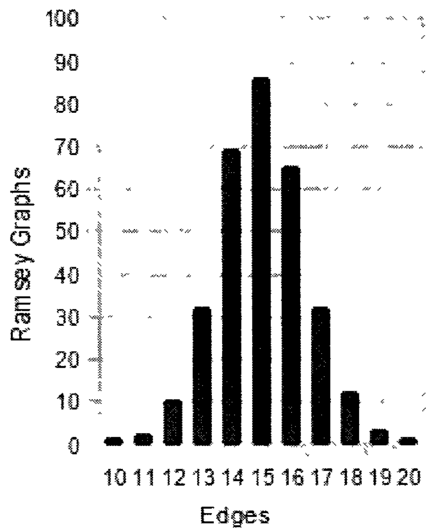


Figure 2.2: The hill over edges for vertex class 10 of $\mathscr{R}(3,5)$.

In a plot where the vertex classes and edge classes form a plane with the height being the number of Ramsey graphs for that vertex and edge class, this 3-dimensional plot looks similar to a mountain (Figure 2.3). This pattern appears in all the tables listed. When $\omega = \alpha$, such as $\mathscr{R}(4,4)$ the mountain is symmetric since if $G \in \mathscr{R}(4,4)$ then $\overline{G} \in \mathscr{R}(4,4)$ also. As with the vertex class, after the peak of the mountain the number of Ramsey graphs in each edge class drops fast.

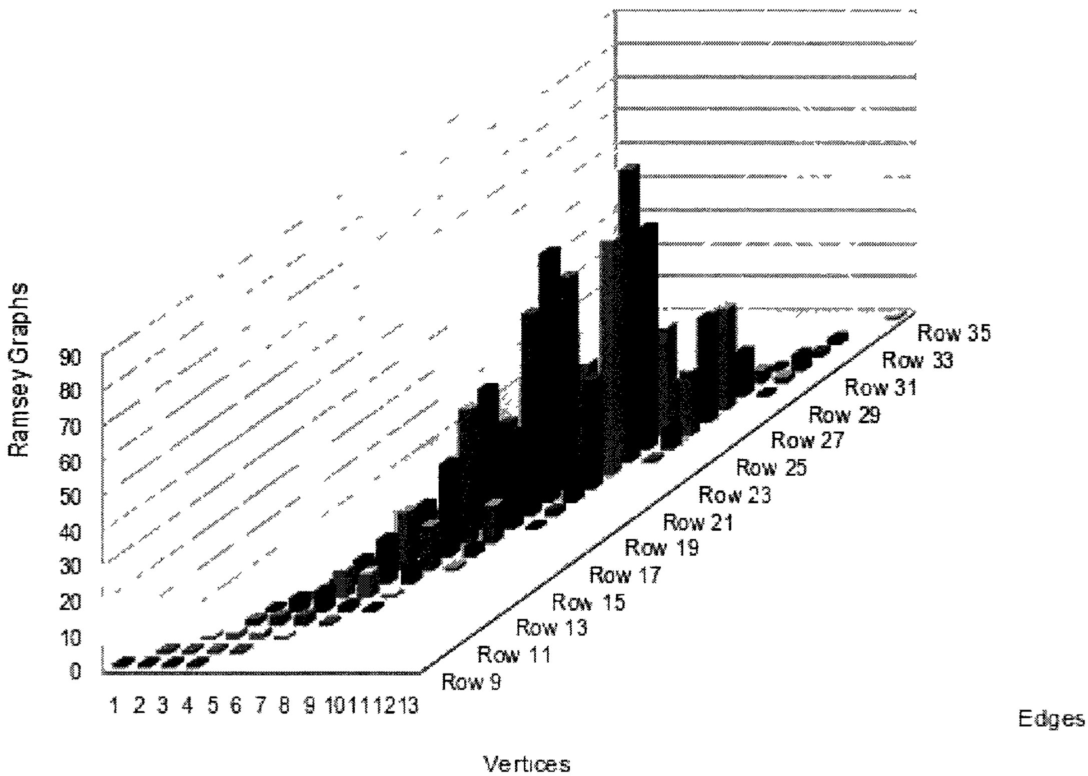


Figure 2.3: The mountain for $\mathscr{R}(3,5)$.

CHAPTER 3

HAMILTON PATHS IN GRAPHS IN $\mathscr{R}(3,4)$

3.1 Overview

The Ramsey number, $R(3,4)$, is defined to be the minimum number n such that any graph with at least n vertices has an induced subgraph isomorphic to K_3 or N_4. The class $\mathscr{R}(3,4)$ contains all graphs having no induced subgraphs isomorphic to K_3 or N_4. From the class $\mathscr{R}(3,4)$, the Ramsey number $R(3,4)$ can be easily obtained by finding a graph G in $\mathscr{R}(3,4)$ with the maximum number of vertices among all graphs of $\mathscr{R}(3,4)$. In this case, $R(3,4) = |V(G)| + 1 = 8 + 1 = 9$. The main result of this dissertation summarized in Theorem 3.1.1 by showing the complete family of $\mathscr{R}(3,4)$.

To prove Theorem 3.1.1, we will first determine the disconnected graphs. A consequence of this process will also allow us to show the list of graphs G in $\mathscr{R}(3,4)$ where the independence number of G is one or two. In Section 3.4, we will prove that any connected graph in $\mathscr{R}(3,4)$ that is not a tree will have a Hamilton path. This is used to prove Theorem 4.2.1 which establishes the complete list of graphs G where $\alpha(G) = 3$. Combining these results will show the list of graphs represented in Figure 3.1 to be $\mathscr{R}(3,4)$. Also, since the list is complete and the maximum number of vertices of any graph in $\mathscr{R}(3,4)$ is eight, we get an independent proof for $R(3,4) = 9$.

Theorem 3.1.1. *If $G \in \mathscr{R}(3,4)$, then G is isomorphic to a graph depicted in Figure 3.1.*

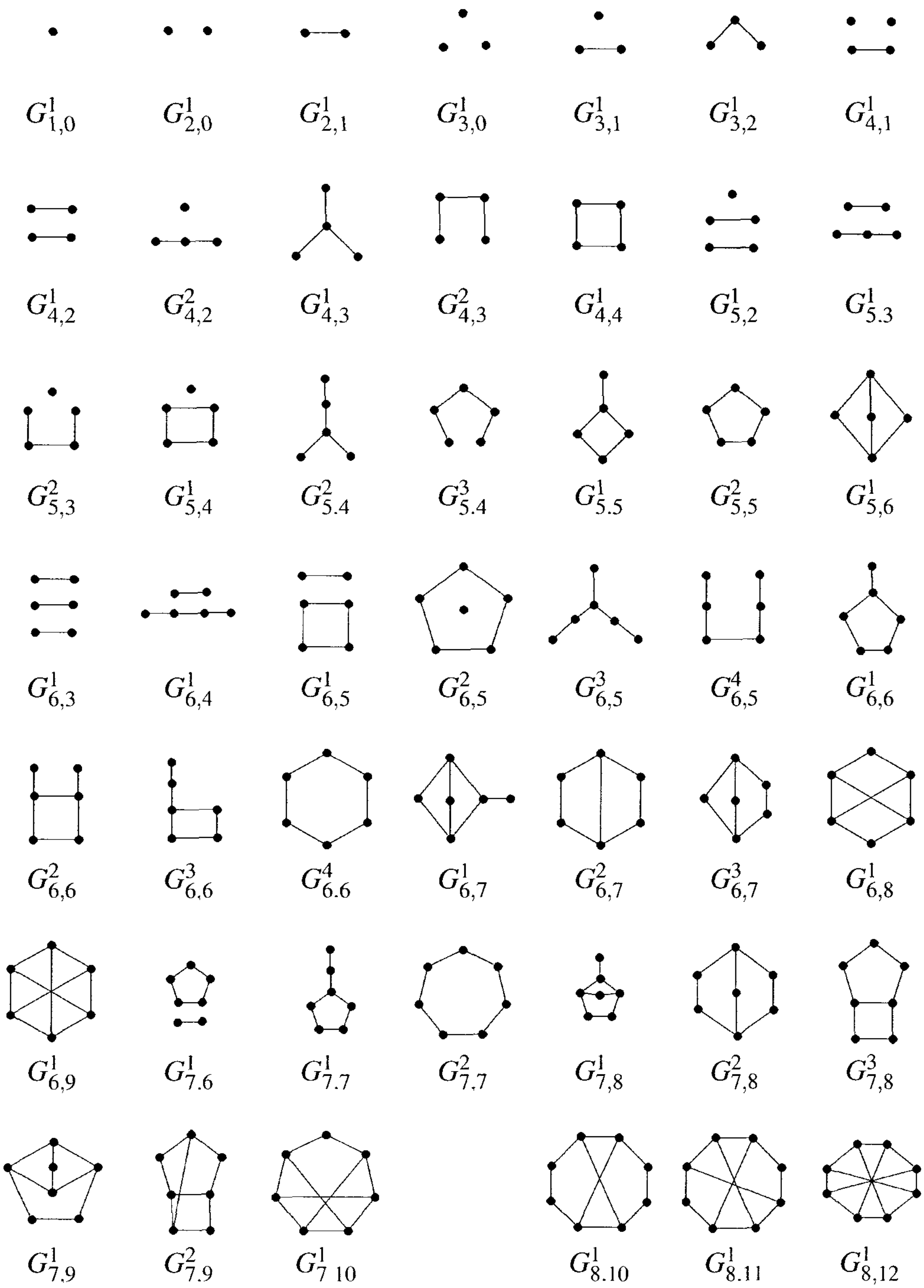


Figure 3.1: Complete list of graphs in $\mathscr{R}(3,4)$.

The following two straightfoward lemmas reveal some of the structure for graphs in $\mathscr{R}(3,4)$ and will be used in the proof of Theorem 3.1.1. Lemma 3.1.2 says that all graphs in $\mathscr{R}(3,4)$ are triangle free and have maximum degree of three and is easily derived from Lemma 1.4.6.

Lemma 3.1.2. *If $G \in \mathscr{R}(3,4)$ then $\Delta(G) \leq 3$ and G triangle free.*

Lemma 3.1.3. *Let H be a spanning subgraph of G such that $H \notin \mathscr{R}(3,4)$ and $G \in \mathscr{R}(3,4)$. If S is an independent set of size four in H, then there is a pair $\{x,y\} \subseteq S$ such that $xy \in E(G)$.*

Proof. Let H be a spanning subgraph of G such that $H \notin \mathscr{R}(3,4)$ and $G \in \mathscr{R}(3,4)$. H is triangle free since $H \subseteq G$ and G is triangle free. Thus $\alpha(H) \geq 4$. Since $\alpha(H) \geq 4$, there is an independent set S of size four in H. Suppose that for every $\{x,y\} \subseteq S$ there is no edge $xy \in E(G)$. This means that the independent set S in H is also an independent set in G, but $\alpha(G) < 4$ and $|S| \geq 4$, a contradiction. Thus there is a pair $\{x,y\} \subseteq S$ such that $xy \in E(G)$. □

Lemma 3.1.3 focuses on subgraphs H not in $\mathscr{R}(3,4)$ of a graph G in $\mathscr{R}(3,4)$. For any independent set S of size four in H, an edge must exist in that set in $E(G)$. This is used in the proofs of Theorem 3.4.2 and Theorem 4.2.1. It does not matter if larger independent sets exist since a larger independent set will consist of multiple independent sets of size four.

The graphs depicted in Figure 3.1 are organized notationally by the number of vertices and edges. Specifically, we write $G^i_{|V|.|E|}$ where $|V|$ is the number of vertices, $|E|$ is the number of edges, and i is an index number for classes with more than one $\mathscr{R}(3,4)$-graph with $|V|$ vertices and $|E|$ edges.

3.2 Disconnected Graphs for $\mathscr{R}(3,4)$

Let G_1 and G_2 be components of G, where $u \in V(G_1)$ and $v \in V(G_2)$. Since u and v are in different components of G, there is no uv-path in G. This also means that u and v are independent vertices in G. Since a vertex in one component will always be independent from all vertices not in that component, the graphs in $\mathscr{R}(3,4)$ have a maximum of three components; otherwise, such a graph will have four independent vertices. By finding all connected graphs G in $\mathscr{R}(3,4)$ with $\alpha(G) \leq 2$, we can find all the disconnected graphs in $\mathscr{R}(3,4)$. Lemma 3.2.1 characterizes all connected graphs with $\alpha(G) = 1$, and Lemma 3.2.2 characterizes all connected graphs with $\alpha(G) = 2$. Using these two lemmas, we prove Theorem 3.2.3 which provides the list of all $G \in \mathscr{R}(3,4)$ where G is disconnected.

Lemma 3.2.1. *Let G be a connected graph with $\alpha(G) = 1$. Then $G \in \mathscr{R}(3,4)$ if and only if G is isomorphic to K_1 or K_2 depicted in Figure 3.2.*

$K_1 \cong G^1_{1,0}$ $K_2 \cong G^1_{2,1}$

Figure 3.2: List of connected $\mathscr{R}(3,4)$ graphs with $\alpha(G) = 1$.

Proof. It is easy to see that K_1 and K_2 are in $\mathscr{R}(3,4)$ and $\alpha(K_1) = \alpha(K_2) = 1$. Let $G \in \mathscr{R}(3,4)$ be connected and $\alpha(G) = 1$. Since $\alpha(G) = 1$, G must be isomorphic to a complete graph; otherwise, if $xy \notin E(G)$, then x and y are independent and $\alpha(G) > 1$. If $|V(G)| \geq 3$ and G is a complete graphs, then $G \notin \mathscr{R}(3,4)$ since G would contain a subgraph isomorphic to K_3. Thus, $|V(G)| = 1$ or $|V(G)| = 2$, which are the graphs $K_1 \cong G^1_{1,0}$ and $K_2 \cong G^1_{2.1}$. □

Lemma 3.2.2. *Let G be a connected graph with $\alpha(G) = 2$. Then $G \in \mathscr{R}(3,4)$ if and only if G is isomorphic to P_3, P_4, C_4, or C_5 depicted in Figure 3.3.*

$P_3 \cong G^1_{3,2}$ $P_4 \cong G^2_{4,3}$ $C_4 \cong G^1_{4,4}$ $C_5 \cong G^2_{5,5}$

Figure 3.3: List of connected R(3,4) graphs with $\alpha(G) = 2$.

Proof. It is easy to see that P_3, P_4, C_4, and C_5 are in $\mathscr{R}(3,4)$ and each has exactly two independent vertices. So, let G be a connected graph with $\alpha(G) = 2$. Let us consider the maximum degree of G. If $\Delta(G) = 0$ or $\Delta(G) = 1$ then either $G \cong K_1$ or $G \cong K_2$, both of which have $\alpha(G) = 1$, a contradiction. Thus suppose $\Delta(G) = 2$. From Corollary 1.2.3, if G is a tree then G must be a path. The only paths that have an independence number two are $P_3 \cong G^1_{3,2}$ and $P_4 \cong G^2_{4,3}$. Thus suppose G must not be a tree. Since $\Delta(G) = 2$ and G is not a tree then G is a cycle. Similarly, there are two cycles with an independence number two, namely $C_4 \cong G^1_{4,4}$ and $C_5 \cong G^2_{5,5}$. Finally, for $\Delta(G) \geq 3$, there is a vertex $v \in V(G)$ such that $d(v) \geq 3$ and the neighborhood of v has at least three independent vertices since G is triangle free. Hence G must be isomorphic to P_3, P_4, C_4, or C_5. □

Theorem 3.2.3. *Let G be a disconnected graph in $\mathscr{R}(3,4)$. Then G is isomorphic to one of the graphs depicted in Figure 3.4.*

Proof. Let G be a disconnected graph in $\mathscr{R}(3,4)$ such that $G \cong G_1 \mathring{\bigcup} G_2 \mathring{\bigcup} \cdots \mathring{\bigcup} G_n$ where G_i are components of G. Since $\alpha(G) < 4$ and the components of G are disjoint, $n \leq 3$; otherwise, G would not be in the class $\mathscr{R}(3,4)$ since vertices in different components of G are independent. This means $\alpha(G_1) + \alpha(G_2) = \alpha(G)$ or $\alpha(G_1) + \alpha(G_2) + \alpha(G_3) = \alpha(G)$. Suppose $\alpha(G_1) + \alpha(G_2) + \alpha(G_3) = \alpha(G) < 4$, it is clear that $\alpha(G_1) = \alpha(G_2) = \alpha(G_3) = 1$. Thus the components of G must be disjoint unions of graphs from Lemma 3.2.1. The disjoint unions are $K_1 \mathring{\bigcup} K_1 \mathring{\bigcup} K_1 \cong G^1_{3,0}$, $K_1 \mathring{\bigcup} K_1 \mathring{\bigcup} K_2 \cong G^1_{4,1}$, $K_1 \mathring{\bigcup} K_2 \mathring{\bigcup} K_2 \cong G^1_{5,2}$, and

$K_2 \mathring{\bigcup} K_2 \mathring{\bigcup} K_2 \cong G^1_{6,3}$. This covers the cases where G has exactly three components, so now suppose G has exactly two components which are the following two cases. (1) Let $G \cong G_1 \mathring{\bigcup} G_2$ have $\alpha(G_1) = \alpha(G_2) = 1$. Again we use the graphs found in Lemma 3.2.1 to find G to be $K_1 \mathring{\bigcup} K_1 \cong G^1_{2,0}$, $K_1 \mathring{\bigcup} K_2 \cong G^1_{3,1}$, or $K_2 \mathring{\bigcup} K_2 \cong G^1_{4,2}$. (2) Let $\alpha(G_1) = 1$ and $\alpha(G_2) = 2$. Similarly, we can use disjoint unions of the graphs such that G_1 is from Lemma 3.2.1 and G_2 is from Lemma 3.2.2. If $G_1 \cong K_1$ we have G isomorphic to $K_1 \mathring{\bigcup} P_3 \cong G^2_{4,2}$, $K_1 \mathring{\bigcup} P_4 \cong G^2_{5,3}$, $K_1 \mathring{\bigcup} C_4 \cong G^1_{5,4}$, or $K_1 \mathring{\bigcup} C_5 \cong G^2_{6,5}$. If $G_1 \cong K_2$ we have G isomorphic to $K_2 \mathring{\bigcup} P_3 \cong G^1_{5,3}$, $K_2 \mathring{\bigcup} P_4 \cong G^1_{6,4}$, $K_2 \mathring{\bigcup} C_4 \cong G^1_{6,5}$, or $K_2 \mathring{\bigcup} C_5 \cong G^1_{7,6}$. Thus the case for G having two components is complete. No analysis for one component is needed since a component is connected. Hence the graphs depicted in Figure 3.4 are all the disconnected graphs in $\mathscr{R}(3,4)$. $\square$

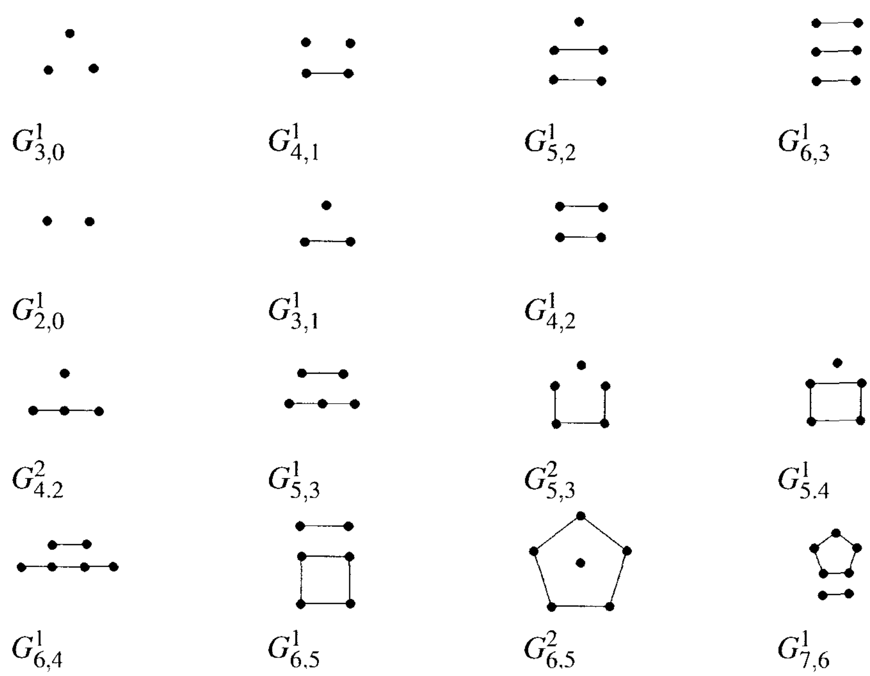


Figure 3.4: List of disconnected $\mathscr{R}(3,4)$ graphs.

Thus we have characterized all disconnected graphs in $\mathscr{R}(3,4)$. Also we have shown all connected graphs G in $\mathscr{R}(3,4)$ where $\alpha(G) = 1$ and $\alpha(G) = 2$. In the remainder of the chapter we assume that $G \in \mathscr{R}(3,4)$ is connected and $\alpha(G) = 3$.

3.3 Trees with $\Delta(T) \leq 3$

Every connected graph G has a spanning subgraph T that is a tree that contains all the vertices of G. These trees are called spanning trees. The proof of $\mathscr{R}(3,4)$ uses this property to find the complete list of graphs G in $\mathscr{R}(3,4)$ where $\alpha(G) = 3$. By extending all possible trees in all possible ways on a given number of vertices, the complete list of $\mathscr{R}(3,4)$ can be found on that number of vertices since any graph G in $\mathscr{R}(3,4)$ would have one of these spanning trees. The main result of this section is Theorem 3.3.1 which shows the list of trees up to nine vertices with a maximum degree of three.

Theorem 3.3.1. *If T is a tree such that $|V(T)| \leq 9$ and $\Delta(T) \leq 3$, then T is isomorphic to a graph depicted in Figure 3.5.*

Proof. First, Lemma 3.3.2 shows all trees with a maximum degree of three up to six vertices. After this, Lemma 3.3.3, Lemma 3.3.4, and Lemma 3.3.5 show the list of trees with a maximum degree of three for seven, eight and nine vertices respectively. By combining these four lemmas, we get the trees depicted in Figure 3.5. □

To help prove the lemmas, we define M_P to be a maximum path in the tree T. The proofs for the following lemmas are similar. The main case is when $\Delta(T) = 3$, which will be broken down into cases based on a longest path in the tree, M_P. The remaing vertices must be attached to the maximum path M_P. The leaf vertices of M_P cannot be part of the attachment since this will have a longer maximum path in T. Based on how many vertices must be attached and how many vertices in M_P have degree three, the different possible attachments will find all the possible trees. By looking at each case up to isomorphism, the graphs in Figure 3.5 will be shown to be complete.

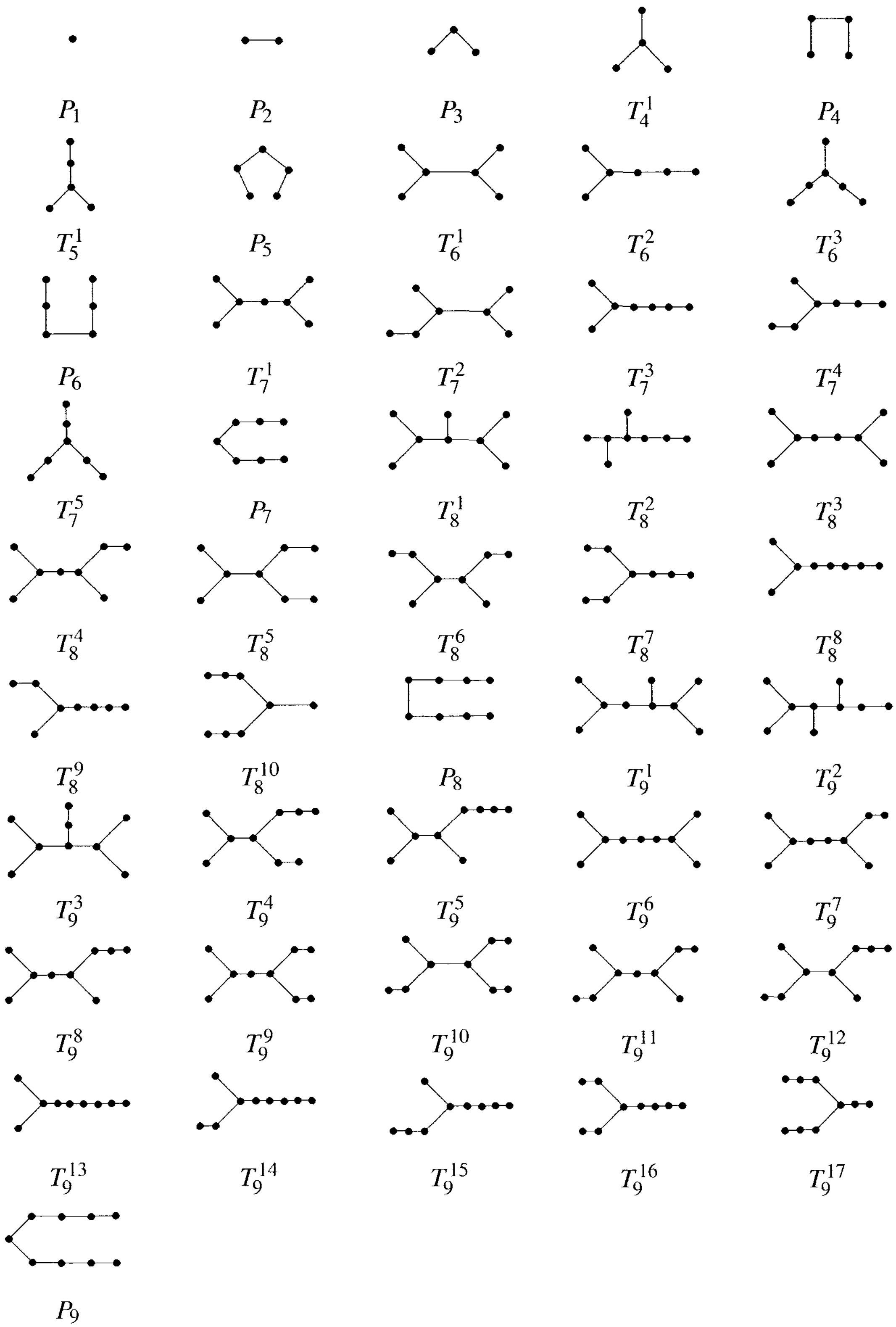


Figure 3.5: All trees up to nine vertices with $\Delta(T) \leq 3$.

Lemma 3.3.2. *If T is a tree such that $1 \leq |V(T)| \leq 6$ and $\Delta(T) \leq 3$, then T is in Figure 3.6.*

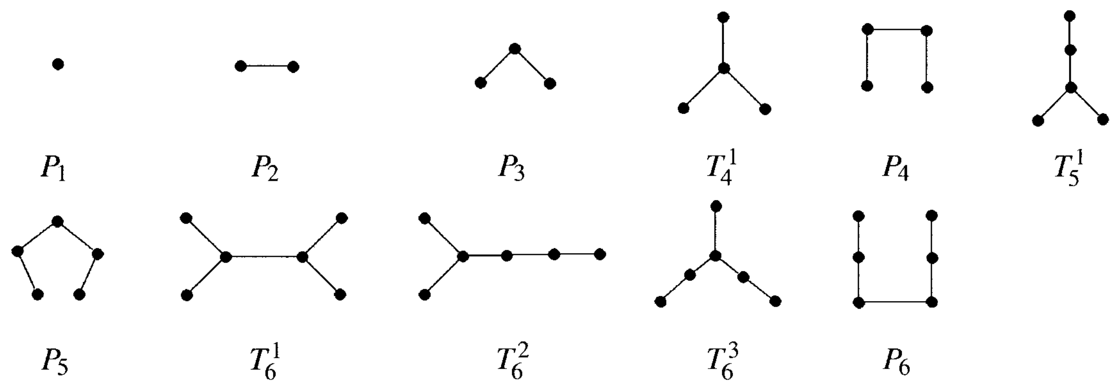


Figure 3.6: Trees up to six vertices with $\Delta(T) \leq 3$.

Proof. Let T be a tree with at most six vertices. It may be worth noting that this tree is a connected graph. If $\Delta(T) \leq 2$ we can use Corollary 1.2.3 to show that T must be a path. The trees up to six vertices with $\Delta(T) \leq 2$ are P_1, P_2, P_3, P_4, P_5, and P_6. Now we may assume that $\Delta(T) = 3$. We may also assume that $|V(T)| \geq 4$ since the only graph that is not a path with fewer than four vertices is K_3.

Suppose $\Delta(T) = 3$ and $|V(T)| = 4$. The length of a maximum path M_P of T must be exactly three. Let $M_P = v_1v_2v_3$. The vertex v_4 can only be attached to v_2 without creating a longer maximum path. Thus, the resulting graph is isomorphic to T_4^1. Now we may assume that $|V(T)| \geq 5$.

Suppose $|V(T)| = 5$. The length of the maximum path M_P of T must be exactly four. Let $M_P = v_1v_2v_3v_4$. The vertex v_5 can be attached to v_2 or v_3 without creating a longer maximum path. Both of the resulting graphs are isomorphic to T_5^1. Now we may assume that $|V(T)| = 6$.

Suppose $|V(T)| = 6$. The length of the maximum path M_P of T can be either four or five. Suppose $M_P = v_1v_2v_3v_4$. The vertices v_5 and v_6 must be attached to M_P without creating a longer maximum path. If $v_5v_6 \in E(T)$, then any attachment to M_P would result in a longer maximum path. Thus $v_5v_6 \notin E(T)$. Since $\Delta(T) = 3$, the only graph up to isomorphism is formed by adding the edges v_2v_5 and v_3v_6. The resulting graph is isomorphic to T_6^1. Now we Suppose $M_P = v_1v_2v_3v_4v_5$. The vertex v_6 must be attached. The attachment by adding the edge v_2v_6 or v_4v_6 forms the tree T_6^2. The other attachment by adding the edge v_3v_6 forms the tree T_6^3. Thus if T is a tree such that $1 \leq |V(T)| \leq 6$ and $\Delta(T) \leq 3$, then T is in Figure 3.6. □

Lemma 3.3.3. *If T is a tree such that $|V(T)| = 7$ and $\Delta(T) \leq 3$, then T is in Figure 3.7.*

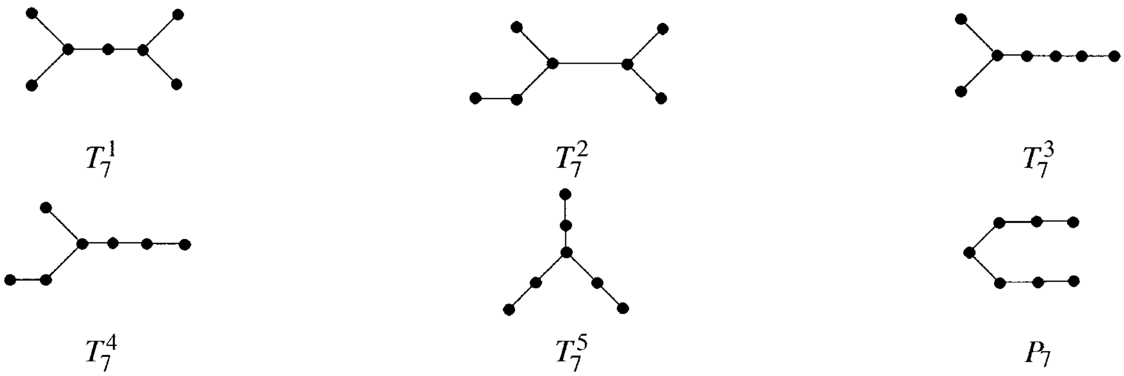


Figure 3.7: Trees on seven vertices with $\Delta(T) \leq 3$.

Proof. Suppose T is a tree such that $|V(T)| = 7$ and $\Delta(T) \leq 3$. From Corollary 1.2.3 we know if $\Delta(T) \leq 2$ then T would be isomorphic to P_7. Thus we may assume $\Delta(T) = 3$. Let M_P be a maximum path in T.

The length of the maximum path M_P of T can be at most six. Now, suppose $M_P = v_1v_2v_3v_4v_5v_6$. Exactly one vertex, v_7, in $V(T)$ that does not lie on M_P. Adding an edge to v_1 or v_6 will result in a longer maximum path. Thus exactly one vertex in $\{v_2, v_3, v_4, v_5\}$ can have degree three to attach v_7. The graph formed by adding v_2v_7 or v_5v_7 is isomorphic to T_7^3. The graph formed by adding v_3v_7 or v_4v_7 is isomorphic to T_7^4. Now M_P can be at most five vertices.

Suppose $M_P = v_1v_2v_3v_4v_5$. Exactly two vertices $v_6, v_7 \in V(T)$ do not lie on M_P. Adding an edge to v_1 or v_5 will result in a longer maximum path. Thus at most two vertices in $\{v_2, v_3, v_4\}$ can have degree three to attach v_6 and v_7. Suppose $v_6v_7 \in E(T)$. The graph formed by adding v_2v_6, v_2v_7, v_4v_6 or v_4v_7 will result in a longer maximum path. The graph formed by adding v_3v_6 or v_3v_7 is isomorphic to T_7^5. Now we may suppose $v_6v_7 \notin E(T)$. Since $\Delta(T) = 3$, exactly two vertices in $\{v_2, v_3, v_4\}$ have degree three. The graphs formed by adding the attachment v_6 and v_7 up to isomorphism are T_7^1 and T_7^2.

Now the maximum path can be at most four. However, since $\Delta(T) \leq 3$, a longer maximum path will be created by attaching the three vertices that do not lie on M_P. Hence the graphs depicted in Figure 3.7 are the trees on seven vertices with a maximum degree of three. □

Lemma 3.3.4. *If T is a tree such that $|V(T)| = 8$ and $\Delta(T) \leq 3$, then T is in Figure 3.8.*

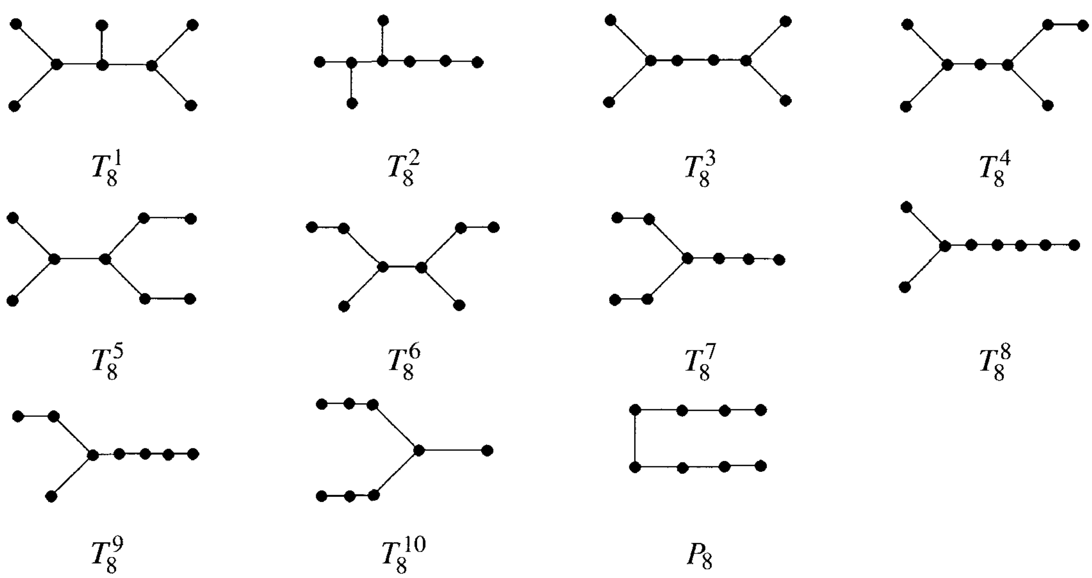


Figure 3.8: Trees on eight vertices with $\Delta(T) \leq 3$.

Proof. Suppose T is a tree such that $|V(T)| = 8$ and $\Delta(T) \leq 3$. From Corollary 1.2.3 we know if $\Delta(T) \leq 2$ then T would be isomorphic to P_8. Thus we may assume $\Delta(T) = 3$. Let M_P be a maximum path in T.

The length of the maximum path M_P of T can be at most seven. Suppose $M_P = v_1v_2v_3v_4v_5v_6v_7$. Exactly one vertex, v_8, in $V(T)$ that does not lie on M_P. Adding an edge to v_1 or v_7 will result in a longer maximum path. Thus exactly one vertex in $\{v_2, v_3, v_4, v_5, v_6\}$ can have degree three to attach v_8. The graph formed by adding v_2v_8 or v_6v_8 is isomorphic to T_8^8. The graph formed by adding v_3v_8 or v_5v_8 is isomorphic to T_8^9. The graph formed by adding v_4v_8 is isomorphic to T_8^{10}. Now M_P can be at most six vertices.

Suppose $M_P = v_1v_2v_3v_4v_5v_6$. Exactly two vertices, $v_7, v_8 \in V(T)$ do not lie on M_P. Adding an edge to v_1 or v_6 will result in a longer maximum path. Thus at most two vertices in $\{v_2, v_3, v_4, v_5\}$ can have degree three to attach v_7 and v_8. Suppose $v_7v_8 \in E(G)$. The graph formed by adding v_2v_7, v_2v_8, v_5v_7 or v_5v_8 will result in a longer maximum path. The graph formed by adding v_3v_7, v_3v_8, v_4v_7 or v_4v_8 is isomorphic to T_8^7. Now we may suppose $v_7v_8 \notin E(T)$. Since $\Delta(T) = 3$, exactly two vertices in $\{v_2, v_3, v_4, v_5\}$ have degree three. Suppose $v_2v_7 \in E(T)$. The graph formed by adding v_3v_8 is isomorphic to T_8^2. The graph formed by adding v_4v_8 is isomorphic to T_8^4. The graph formed by adding v_5v_8 is isomorphic to T_8^3. Now we suppose $v_3v_7 \in E(T)$. The graph formed by adding v_4v_8 is isomorphic to T_8^6. The graph formed by adding v_5v_8 is isomorphic to a graph in the previous case, T_8^4. Now suppose $v_4v_7 \in E(T)$. The graph formed by adding v_5v_8 is isomorphic to a previous case, T_8^2. Now M_P can be at most five vertices.

Suppose $M_P = v_1v_2v_3v_4v_5$. Exactly three vertices, $v_6, v_7, v_8 \in V(T)$ do not lie on M_P. Adding an edge to v_1 or v_6 will result in a longer maximum path. Thus at most three vertices in $\{v_2, v_3, v_4\}$ can have degree three to attach v_6, v_7, and v_8. Without loss of generality, let $v_3v_6 \in E(T)$. The vertices v_7 and v_8 can be added to $\{v_2, v_4, v_6\}$ since $d(v_3) = 3$. Suppose $v_7v_8 \in E(G)$. The graph formed by adding v_2v_7, v_2v_8, v_4v_7, v_4v_8, v_6v_7, or v_6v_8 will result in a longer maximum path. Now we may assume that $v_7v_8 \notin E(T)$. Suppose $v_2v_7 \in E(T)$. The graph formed by adding v_6v_8 is isomorphic to T_8^5. The graph formed by adding v_4v_8 is isomorphic to T_8^1. Now if we suppose $v_4v_7 \in E(T)$, we would have isomorphic graphs. This leaves us to assume $v_6v_7 \in E(T)$. The graph formed by adding v_4v_8 is isomorphic to a previous case, T_8^5. The graph formed by adding v_6v_8 is isomorphic to a previous case, T_8^5. Now we suppose $v_3v_7 \in E(T)$. Now M_P can be at most four vertices.

Now the maximum path can be at most four. However, since $\Delta(T) \leq 3$, a longer maximum path will be created by attaching the four vertices that do not lie on M_P. Hence the graphs depicted in Figure 3.8 are the trees on eight vertices with a maximum degree of three. □

Lemma 3.3.5. *If T is a tree such that $|V(T)| = 9$ and $\Delta(T) \leq 3$, then T is in Figure 3.9.*

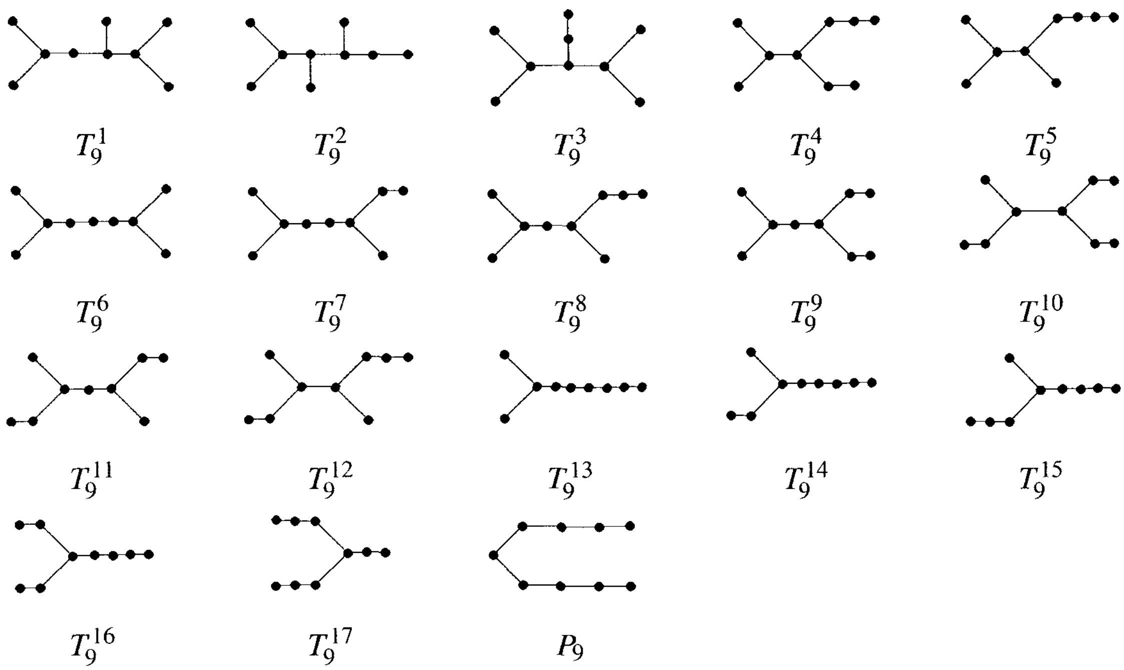


Figure 3.9: Trees on nine vertices with $\Delta(T) \leq 3$.

Proof. Suppose T is a tree such that $|V(T)| = 9$ and $\Delta(T) \leq 3$. From Corollary 1.2.3 we know if $\Delta(T) \leq 2$ then T would be isomorphic to P_9. Thus we may assume $\Delta(T) = 3$. Let M_P be the maximum path in T.

The length of the maximum path M_P of T can be at most eight. Suppose M_P is isomorphic to P_8 and $M_P = v_1v_2v_3v_4v_5v_6v_7v_8$. Exactly one vertex $v_9 \in T$ does not lie on M_P. Since $d(v_1) = d(v_8) = 1$, there is exactly one vertex on M_P has degree three in T. If v_2(or v_7) has degree three in T, then $T \cong T_9^{13}$. If v_3(or v_6) has degree three in T, then $T \cong T_9^{14}$. If v_4(or v_5) has degree three in T, then $T \cong T_9^{15}$. Thus the case for a maximum path of eight is concluded.

Suppose M_P is isomorphic to P_7 and $M_P = v_1v_2v_3v_4v_5v_6v_7$. Exactly two vertices $v_8, v_9 \in T$ do not lie on M_P. Since $d(v_1) = d(v_7) = 1$, either one or two vertices on M_P have degree three in T. Suppose one vertex of M_P has degree three in T. Both vertices v_8 and v_9 must be attached through the same vertex of M_P. Since $\Delta(T) = 3$, we know that $v_8v_9 \in E(T)$. Without loss of generality, we can assume that $d(v_8) = 2$ in T. If $d(v_2) = 3$(or v_6), then a longer maximum path would exist in T. If $d(v_3) = 3$(or v_5), then $T \cong T_9^{16}$. If $d(v_4) = 3$, then $T \cong T_9^{17}$. This concludes one vertex of degree three in T. Suppose two vertices in T have degree three. If $d(v_2) = d(v_3) = 3$ (or v_5 and v_6), then $T \cong T_9^5$. If $d(v_2) = d(v_4) = 3$ (or v_4 and v_6), then $T \cong T_9^8$. If $d(v_2) = d(v_5) = 3$ (or v_3 and v_6), then $T \cong T_9^7$. If $d(v_2) = d(v_6) = 3$, then $T \cong T_9^6$. If $d(v_3) = d(v_4) = 3$ (or v_4 and v_5), then $T \cong T_9^{12}$. If $d(v_3) = d(v_5) = 3$, then $T \cong T_9^{11}$. This concludes the case for a maximum path of seven.

Suppose M_P is isomorphic to P_6 and $M_P = v_1v_2v_3v_4v_5v_6$. There are exactly three vertices $v_7, v_8, v_9 \in T$ do not lie on M_P. Since $d(v_1) = d(v_6) = 1$, either one, two, or three vertices on M_P have degree three in T. Suppose one vertex of M_P has degree three in T. Since v_7, v_8, and v_9 must attach to M_P through one vertex, we can assume $v_7v_8 \in E(T)$ and $v_8v_9 \in E(T)$ without loss of generality. If $d(v_2) = 3$ (or v_5), then a longer maximum path would exist in T regardless of how it is attached. Suppose $d(v_3) = 3$ (or v_4). If $v_3v_7 \in E(T)$(or v_3v_9), then the path $v_9v_8v_7v_3v_4v_5v_6$ is longer than M_P. If $v_3v_8 \in E(T)$, then $T \cong T_9^4$. Suppose exactly two vertices of M_P has degree three in T. Clearly two attachments to M_P must be made, one of size two, and the other of size one. Without loss of generatlity, we assume that $v_7v_8 \in E(T)$ and $d(v_7) = 2$ as one partition and v_9 as the second partition. If $v_2v_7 \in E(T)$(or v_5), then a longer maximum path would exist. Suppose $v_3v_7 \in E(T)$ (or v_4v_7). Now v_9 must be attached to a vertex in M_P other than v_1, v_3, or v_6. If $v_2v_9 \in E(T)$, then $T \cong T_9^4$, a graph which was found in the previous case. If $v_4v_9 \in E(T)$, then $T \cong T_9^10$. If $v_5v_9 \in E(T)$, then $T \cong T_9^9$. This concludes two vertices of M_P have degree three. Suppose exactly three vertices of M_P has degree three in T. Clearly three attachments to M_P each of size one must be made. If $d(v_2) = d(v_3) = d(v_4) = 3$ (or $d(v_3) = d(v_4) = d(v_5) = 3$), then $T \cong T_9^2$. If $d(v_2) = d(v_3) = d(v_5) = 3$ (or $d(v_2) = d(v_4) = d(v_5) = 3$), then $T \cong T_9^1$. Thus the final case for M_P of size six is concluded.

Suppose M_P is isomorphic to P_5 and $M_P = v_1v_2v_3v_4v_5$. There are exactly four vertices $v_6, v_7, v_8, v_9 \in T$ do not lie on M_P. Since $d(v_1) = d(v_5) = 1$, either one, two, three, or four vertices on M_P have degree three in T. Suppose exactly one vertex of M_P has degree three in T. The vertices v_6, v_7, v_8, and v_9 must all be attached through one vertex of M_P. Since $\Delta(T) = 3$, this is impossible to do without creating a longer maximum path. A similar argument exists for the case with two vertices of M_P having degree three in T. Suppose exactly three vertices of M_P have degree three. The three vertices of M_P with degree three must be v_2, v_3, and v_4. The size of the attachments must be one, one, and two since there are four vertices and three partitions. Without loss of generality, let $v_6v_7 \in E(T)$ and $d(v_6) = 2$. The edge $v_3v_6 \in E(T)$ otherwise T would have a longer maximum path. The remaining vertices can be attached either way to show $T \cong T_9^3$. If four vertices of M_P had degree three, a longer maximum path would exist in T. This concludes the trees with maximum path of five.

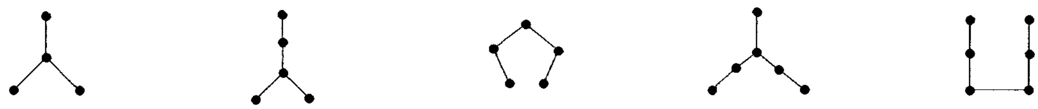
Now the maximum path can be at most four. However, since $\Delta(T) \leq 3$, a longer maximum path will be created by attaching the four vertices that do not lie on M_P. Hence the graphs depicted in Figure 3.9 are the trees on nine vertices with a maximum degree of three. □

Using the list proved by Theorem 3.3.1, we will extend these trees in the next section to prove Theorem 3.4.2 to reduce the work needed by the main theorem of this dissertation.

3.4 Trees in $\mathscr{R}(3,4)$

In this section we consider the case when $G \in \mathscr{R}(3,4)$ is a tree. Let T be a tree in $\mathscr{R}(3,4)$. From Lemma 3.2.1 and Lemma 3.2.2, the trees for $\alpha = 1$ and $\alpha = 2$ are easily found. These trees are K_1, K_2, P_3, and P_4. The remaining trees in $\mathscr{R}(3,4)$ must have $\alpha = 3$.

Lemma 3.4.1. *If $T \in \mathscr{R}(3,4)$ is a tree and $\alpha(T) = 3$, then T is isomorphic to $G^1_{4,3}$, $G^2_{5,4}$, $G^3_{6,5}$, P_5, or P_6 depicted in Figure 3.10.*



$G^1_{4,3}$ $\quad$ $G^2_{5,4}$ $\quad$ $P_5 \cong G^3_{5,4}$ $\quad$ $G^3_{6,5}$ $\quad$ $P_6 \cong G^4_{6,5}$

Figure 3.10: List of $\mathscr{R}(3,4)$ trees with $\alpha(G) = 3$.

Proof. By Corollary 1.2.3, if $\Delta(T) \leq 2$, then T is a path. The paths P_5 and P_6 are the only paths with an independence number of exactly three, which are both in the list. Thus we may assume $\Delta(T) = 3$. Let $v \in V(T)$ such that $d(v) = 3$ and let $r,s,t \in N(v)$. Removing the vertex v from T, three components will be formed, so let $T - v \cong T_r \mathring{\bigcup} T_s \mathring{\bigcup} T_t$ where T_r, T_s, T_t are the components that contain the vertices r, s, and t respectively. Since there are no edges between components and $T \in \mathscr{R}(3,4)$, we know $\alpha(T_r) + \alpha(T_s) + \alpha(T_t) \leq 3$. This means each component can have a maximum independent set of 1. Using Lemma 3.2.1, the components can be K_1 or K_2. We now add v back to get four possible graphs. The case where $T - v = K_2 \mathring{\bigcup} K_2 \mathring{\bigcup} K_2$ will have $\alpha(T) = 4$ as shown in Figure 3.11, so it is not in $\mathscr{R}(3,4)$. Thus $T - v$ must be isomorphic to one of $G^1_{4,3}$, $G^2_{5,4}$, and $G^3_{6.5}$. Lemma 3.1.2 states no graph T exists such that $\Delta(T) \geq 4$. This completes the list of connected trees. ☐

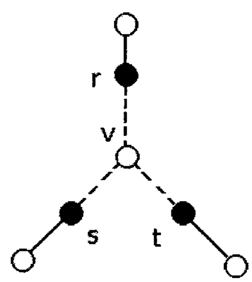


Figure 3.11: The tree $T - v = K_2 \mathring{\bigcup} K_2 \mathring{\bigcup} K_2$.

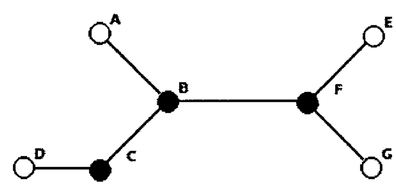
Lemma 3.4.1 gives all the trees in $\mathscr{R}(3,4)$ with $\alpha(T) = 3$. The next theorem will help for construction of the remaining graphs in $\mathscr{R}(3,4)$. Any connected graph has a spanning tree. If $G \in \mathscr{R}(3,4)$, then $\alpha(G) < 4$, but a spanning tree $T \subseteq G$ does not need to have $\alpha(T) < 4$ since $E(T) \subseteq E(G)$. This means $\alpha(T) \geq \alpha(G)$.

Theorem 3.4.2 will use the tables at the end of the section in its proof to show the graph has a Hamilton path. First, we will describe how to interpret the tables and navigate the proof.

The table starts with a list of all trees that have $\Delta(T) = 3$ from Theorem 3.3.1 and have $\alpha(T) \geq 4$. Because $\alpha(T) \geq 4$, an edge must be added to every independent set of size four to decrease the independence number of the graph to three. In order to understand how to read the tables, we will refer to Table 3.1 as an example. The vertices of the tree are labeled in the figure on the right side. An independent set S of size four is on the left(and are represented by hollow vertices in the graph). From Lemma 3.1.3, an edge must be added between two vertices of S. From the four vertices in S, there are six possible edge additions that cover all cases, and the table on the left explains the result of each edge addition. For example, in Table 3.1, the goal is to show that an extension of T, T^*, will have a Hamilton path, a triangle, or a vertex of degree four. If T^* has a triangle or vertex of degree four, T^* will not be a subgraph of any graph in $\mathscr{R}(3,4)$, and generation may stop on the condition listed in Lemma 3.1.2. Generation may also stop when a Hamilton path is found since T^* will have the same Hamilton path with any further edge additions. Temporary graphs $H^{\iota}_{|V|,|E|+1}$ may be formed where $\alpha(H^{\iota}_{|V|,|E|+1}) \geq 4$, $\Delta(G) \leq 3$, and H^{ι} is triangle free. For example, by extending all trees to a T^* or one of the stopping conditions, we will prove Theorem 3.4.2.

Lets take a look at how the tree T_7^2 in Table 3.1 eventually forms a Hamilton path. In Table 3.1, the set $S = \{A, D, E. G\}$ is independent, and from Lemma 3.1.3, if T_7^1 is a subgraph of some $G \in \mathscr{R}(3,4)$, then there is a pair $\{x, y\} \subseteq S$ such that $xy \in E(G)$. We now look at the six possible edges formed by S, namely AC, AE, AG, CE, CG, and EG. These edges are listed in the left panel of the table, with the result of $T_7^2 + e$ also listed where e is a possible edge addition. The graphs $T_7^2 + EG$ creates a triangle EFG, and thus will not be in the class of $\mathscr{R}(3,4)$. The generation along this path stops here. When one of the four edges, AE, AG, DE, and DG is added to T_7^2, a Hamilton path is formed and is labeled in the result. Since this is the result we are looking for, generation stops since any graph that contains this graph will also have a Hamilton path. The remaining edge, AD form a graph isomorphic to $H_{7,7}^2$. The generation continues on a graph further in the table.

Table 3.1: Example tree T_7^2.

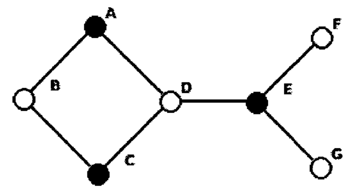
<table>
<tr><td colspan="2">T_7^2</td></tr>
<tr><td>$S=\{A,D,E,G\}$</td><td>$\alpha \geq 4$</td></tr>
<tr><td>
<table>
<tr><th>Add Edge</th><th>Result</th></tr>
<tr><td>AD</td><td>$H_{7,7}^2$</td></tr>
<tr><td>AE</td><td>Hamilton Path $GF\mathbf{EA}BCD$</td></tr>
<tr><td>AG</td><td>Hamilton Path $EF\mathbf{GA}BCD$</td></tr>
<tr><td>DE</td><td>Hamilton Path $ABC\mathbf{DE}FG$</td></tr>
<tr><td>DG</td><td>Hamilton Path $ABC\mathbf{DG}FE$</td></tr>
<tr><td>EG</td><td>Triangle EFG</td></tr>
</table>
</td><td>
</td></tr>
</table>

Continuing to the graph $H_{7,7}^2$ in Table 3.2, we have a set $S=\{B,D,F,G\}$ of four mutaully independent vertices. Since $\alpha(H_{7,7}^2) \geq 4$, Lemma 3.1.3 states there must exist an edge in the set of a graph in $\mathscr{R}(3,4)$. The possible edge additions are BD, BF, BG, DF, DG, and FG. The vertex D has a degree of three, so by adding any edge incident with D will make $d(D) \geq 4$ contradicting Lemma 3.1.2, so BD, DF, and DG stop generation. Adding either edge BF or BG a Hamilton path is formed and is labeled. The final edge, FG, results in a triangle formed, so the resulting graph will not be a subgraph of a graph in $\mathscr{R}(3,4)$. This concludes the part of the proof for $\alpha(T_7^2) \geq 4$. This will be used in the next section to say that if for $G \in \mathscr{R}(3,4)$, T is a spanning tree of G such that $\alpha(T) \geq 4$ then G has a spanning path. The following tables can be used for the proof of Theorem 3.4.2 and can be read in the same way as the above example.

Table 3.2: Example of temporary graph H_7^2.

$H^2_{7,7}$	
$S=\{B,D,F,G\}$	$\alpha \geq 4$

Add Edge	Result
BD	$d(D) \geq 4$
BF	Hamilton Path GE**F**$\mathbf{B}ADC$
BG	Hamilton Path FE**G**$\mathbf{B}ADC$
DF	$d(D) \geq 4$
DG	$d(D) \geq 4$
FG	Triangle EFG



This leads to the main theorem of this chapter, Theorem 3.4.2, which shows that a graph in $\mathscr{R}(3,4)$ is a tree or has a Hamilton path.

Theorem 3.4.2. *Let $G \in \mathscr{R}(3,4)$ be connected with at most nine vertices. If $\alpha(G)=3$ then G is a tree or has a Hamilton path.*

Proof. Let G in $\mathscr{R}(3,4)$ be connected with at most nine vertices. If G is acyclic, then G is a tree. Thus we may assume G is not acyclic. There exists some spanning tree $T \subseteq G$. Clearly, there is a set of edges such that $G \cong T+e_1+e_2+\cdots+e_n$. Since we want to find all graphs in $\mathscr{R}(3,4)$, we can extend all trees on n vertices to find all graphs in $\mathscr{R}(3,4)$ with n vertices. The remainder of this proof will focus on the construction of G from the tree T.

Let T be a tree. If $\alpha(T) \leq 3$ then T must be isomporphic to one of the graphs shown in Lemma 3.2.1, Lemma 3.2.2, or Lemma 3.4.1, which all of these graphs have a Hamilton path or are a tree. Thus, we may assume $\alpha(T) \geq 4$.

From Lemma 3.1.3, each independent set of size four in T must have an edge added to decrease the independence number. Since every set must have an edge added, the order in which the edges are added does not matter, so we may choose any independent set S of size four initially. Up to six graphs can be formed from the six possible edge additions. By exending all trees found in in this way, we get all graphs G in $\mathscr{R}(3,4)$ with $|V(T)|$

vertices. We start this process on six vertices since trees with fewer vertices will not have an independence number of four.

We are only concerned about graphs in $\mathscr{R}(3,4)$, so we can limit new graphs based on necessary conditions to be a subgraph of a graph in $\mathscr{R}(3,4)$. There is no need to continue generation if addin an edge forms a triangle, a vertex of degree four, or a Hamilton path. Also, we only need to search the trees found in Theorem 3.3.1 since any tree not in the list cannot be a subgraph of a graph in $\mathscr{R}(3,4)$.

The arrangment of the Tables 3.3 through Table 3.6 prevents a circular argument. The tables are arranged in such a way that $|V(G)|$ is constant throughout the table. The graphs are ordered in such a way that $|E(G_i)| \leq |E(G_{i+1})|$ where i is the position of the graph on the table. For each graph G, two possible results can happen. From an edge addition, the generation may terminate if no edge exists that can be added to G to satisfy the necessary conditions, for $G \in \mathscr{R}(3,4)$. The other case is to add a new graph to the table, namely $G+e$. The new graph will always have exactly one more edge than G. By only advancing foward to a graph with exactly one more edge, we will not encounter a circular logic since all generations terminate in the end. □

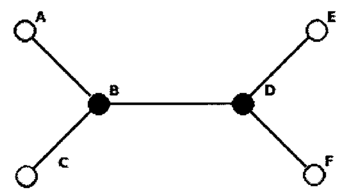
Table 3.3: G in $\mathscr{R}(3,4)$ contains a Hamilton path on six vertices.

|E| = 5

T_6^1

$S = \{A,C,E,F\}$ — $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AE	Hamilton Path $CB\mathbf{AE}DF$
AF	Hamilton Path $CB\mathbf{AF}DE$
CE	Hamilton Path $AB\mathbf{CE}DF$
CF	Hamilton Path $AB\mathbf{CF}DE$
EF	Triangle DEF



T_6^2

$S = \{A,C,D,F\}$ — $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AF	Hamilton Path $CB\mathbf{AF}ED$
CD	Triangle BCD
CF	Hamilton Path $AB\mathbf{CF}ED$
DF	Triangle DEF

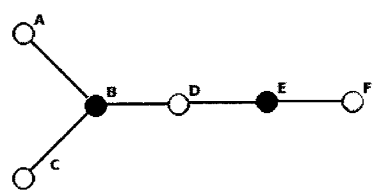


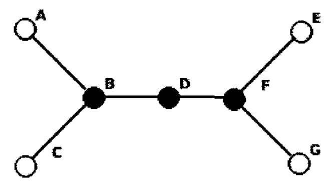
Table 3.4: G in $\mathscr{R}(3,4)$ contains a Hamilton path on seven vertices.

|E| = 6

T_7^1

$S = \{A, C, E, G\}$	$\alpha \geq 4$

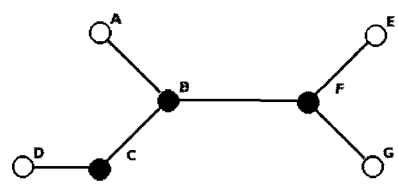
Add Edge	Result
AC	Triangle ABC
AE	$H_{7.7}^1$
AG	$H_{7.7}^1$
CE	$H_{7,7}^1$
CG	$H_{7,7}^1$
EG	Triangle EFG



T_7^2

$S = \{A, D, E, G\}$	$\alpha \geq 4$

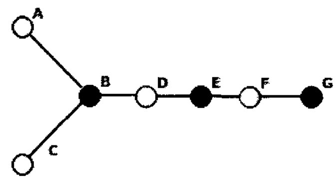
Add Edge	Result
AD	$H_{7.7}^2$
AE	Hamilton Path $GF\mathbf{EA}BCD$
AG	Hamilton Path $EF\mathbf{GA}BCD$
DE	Hamilton Path $ABC\mathbf{DE}FG$
DG	Hamilton Path $ABC\mathbf{DG}FE$
EG	Triangle EFG



T_7^3

$S = \{A, C, D, F\}$	$\alpha \geq 4$

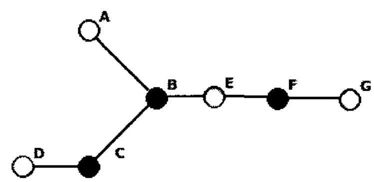
Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AF	$H_{7,7}^1$
CD	Triangle BCD
CF	$H_{7,7}^1$
DF	Triangle DEF



T_7^4

$S = \{A, D, E, G\}$	$\alpha \geq 4$

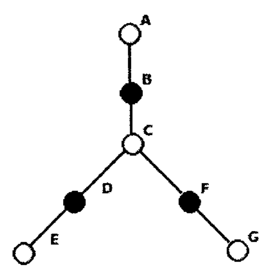
Add Edge	Result
AD	Hamilton Path **AD**$CBEFG$
AE	Triangle ABE
AG	Hamilton Path $DCBEF$**GA**
DE	Hamilton Path ABC**DE**FG
DG	Hamilton Path ABC**DG**FE
EG	Triangle EFG



T_7^5

$S = \{A, C, E, G\}$	$\alpha \geq 4$

Add Edge	Result
AC	$d(C) \geq 4$
AE	Hamilton Path $GFCD$**EA**B
AG	Hamilton Path $EDCF$**GA**B
CE	$d(C) \geq 4$
CG	$d(C) \geq 4$
EG	Hamilton Path $ABCD$**EG**F

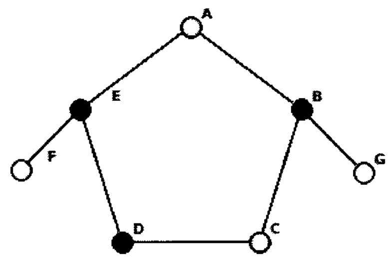


$|E| = 7$

$H^1_{7,7}$

$S = \{A, C, F, G\}$ | $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AF	Triangle AEF
AG	Triangle ABG
CF	Hamilton Path $GBAED\mathbf{CF}$
CG	Triangle BCG
FG	Hamilton Path $CDE\mathbf{FG}BA$



$H^2_{7,7}$

$S = \{B, D, F, G\}$ | $\alpha \geq 4$

Add Edge	Result
BD	$d(D) \geq 4$
BF	Hamilton Path $GE\mathbf{FB}ADC$
BG	Hamilton Path $FE\mathbf{GB}ADC$
DF	$d(D) \geq 4$
DG	$d(D) \geq 4$
FG	Triangle EFG

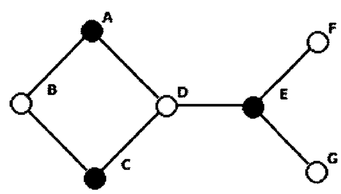


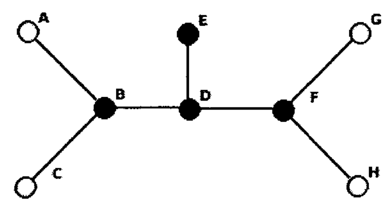
Table 3.5: G in $\mathscr{R}(3,4)$ contains a Hamilton path on eight vertices.

|E| = 7

T_8^1

$S=\{A,C,G,H\}$	$\alpha \geq 4$

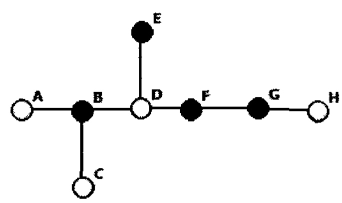
Add Edge	Result
AC	Triangle ABC
AG	$H_{8,8}^4$
AH	$H_{8,8}^4$
CG	$H_{8,8}^4$
CH	$H_{8,8}^4$
GH	Triangle FGH



T_8^2

$S=\{A,C,D,H\}$	$\alpha \geq 4$

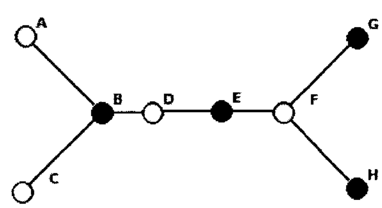
Add Edge	Result
AC	Triangle ABC
AD	$d(D) \geq 4$
AH	Hamilton Path $CB\mathbf{AH}GFDE$
CD	$d(D) \geq 4$
CH	Hamilton Path $AB\mathbf{CH}GFDE$
DH	$d(D) \geq 4$



T_8^3

$S=\{A,C,D,F\}$	$\alpha \geq 4$

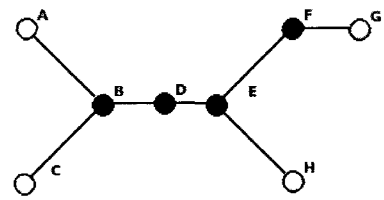
Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AF	$d(F) \geq 4$
CD	Triangle BCD
CF	$d(F) \geq 4$
DF	$d(F) \geq 4$



T_8^4

$S = \{A,C,G,H\}$	$\alpha \geq 4$

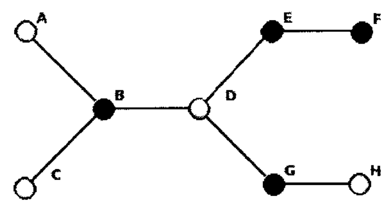
Add Edge	Result
AC	Triangle ABC
AG	$H_{8,8}^{8}$
AH	$H_{8,8}^{7}$
CG	$H_{8,8}^{8}$
CH	$H_{8,8}^{7}$
GH	$H_{8,8}^{5}$



T_8^5

$S = \{A,C,D,H\}$	$\alpha \geq 4$

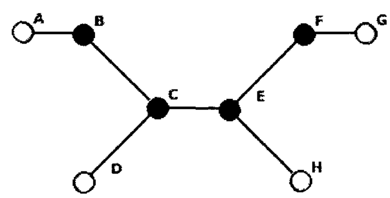
Add Edge	Result
AC	Triangle ABC
AD	$d(D) \geq 4$
AH	Hamilton Path $FEDG\mathbf{HA}BC$
CD	$d(D) \geq 4$
CH	Hamilton Path $FEDG\mathbf{HC}BA$
DH	$d(D) \geq 4$



T_8^6

$S = \{A,D,H,G\}$	$\alpha \geq 4$

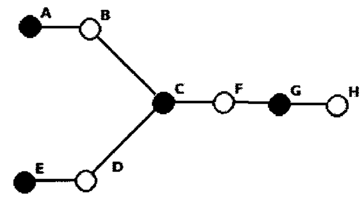
Add Edge	Result
AD	$H_{8,8}^{6}$
AH	Hamilton Path $DCB\mathbf{AH}EFG$
AG	Hamilton Path $DCB\mathbf{AG}FEH$
DH	Hamilton Path $ABC\mathbf{DH}EFG$
DG	Hamilton Path $ABC\mathbf{DG}FEH$
HG	$H_{8,8}^{6}$



T_8^7

$S = \{B, D, F, H\}$	$\alpha \geq 4$

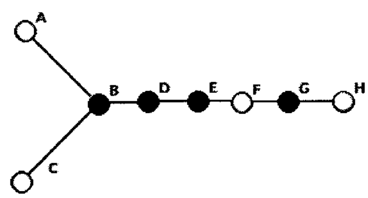
Add Edge	Result
BD	Triangle BCD
BF	Triangle BCF
BH	Hamilton Path $A\mathbf{BH}GFCDE$
DF	Triangle CDF
DH	Hamilton Path $E\mathbf{DH}GFCBA$
FH	Triangle FGH



T_8^8

$S = \{A, C, F, H\}$	$\alpha \geq 4$

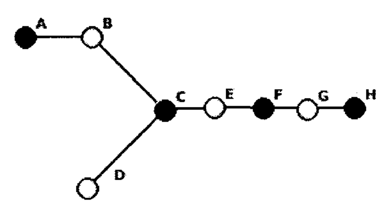
Add Edge	Result
AC	Triangle ABC
AF	$H_{8,8}^7$
AH	Hamilton Path $CB\mathbf{AH}GFED$
CF	$H_{8,8}^7$
CH	Hamilton Path $AB\mathbf{CH}GFED$
FH	Triangle FGH



T_8^9

$S = \{B, D, E, G\}$	$\alpha \geq 4$

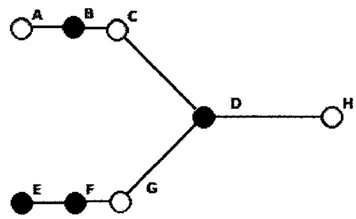
Add Edge	Result
BD	Triangle BCD
BE	Triangle BCE
BG	$H_{8,8}^4$
DE	Triangle CDE
DG	$H_{8,8}^7$
EG	Triangle EFG



T_8^{10}

$S = \{A, C, G, H\}$	$\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AG	Hamilton Path EF**GA**$BCDH$
AH	Hamilton Path **HA**$BCDGFE$
CG	Triangle CDG
CH	Triangle CDH
GH	Triangle DGH

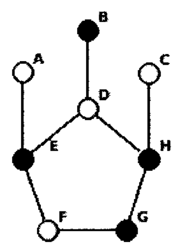


|E| = 8

$H_{8,8}^4$

$S = \{A, C, D, F\}$	$\alpha \geq 4$

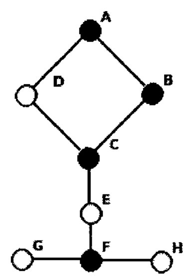
Add Edge	Result
AC	Hamilton Path BDE**AC**HGF
AD	$d(D) \geq 4$
AF	Triangle AEF
CD	$d(D) \geq 4$
CF	$H_{8,9}^1$
DF	$d(D) \geq 4$



$H_{8,8}^5$

$S = \{D, E, G, H\}$	$\alpha \geq 4$

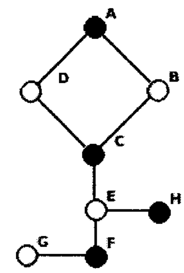
Add Edge	Result
DE	Triangle CDE
DG	Hamilton Path $HFECBA$**DG**
DH	Hamilton Path $GFECBA$**DH**
EG	Triangle EFG
EH	Triangle EFH
GH	Triangle FGH



$H^6_{8,8}$

$S = \{B, D, E, G\}$ — $\alpha \geq 4$

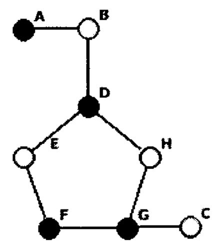
Add Edge	Result
BD	Triangle ABD
BE	$d(E) \geq 4$
BG	Hamilton Path F**GB**$ADCEH$
DE	$d(E) \geq 4$
DG	Hamilton Path HEF**GD**ABC
EG	$d(E) \geq 4$



$H^7_{8,8}$

$S = \{B, C, E, H\}$ — $\alpha \geq 4$

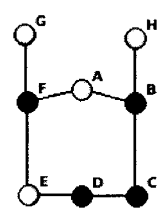
Add Edge	Result
BC	Hamilton Path A**BC**$GHDEF$
BE	Triangle BDE
BH	Triangle BDH
CE	Hamilton Path $ABDHG$**CE**F
CH	Triangle CGH
EH	Triangle DEH



$H^8_{8,8}$

$S = \{A, E, G, H\}$ — $\alpha \geq 4$

Add Edge	Result
AE	Triangle AEF
AG	Triangle AFG
AH	Triangle ABH
EG	Triangle EFG
EH	Hamilton Path $GFAB$**HE**DC
GH	Hamilton Path **GH**$BAFEDC$



Edges = 9	
$H^1_{8,9}$	
$S = \{B, C, G, H\}$	$\alpha \geq 4$

Add Edge	Result
BC	Triangle *ABC*
BG	Hamilton Path *HED***GB***ACF*
BH	Hamilton Path *GDE***HB***ACF*
CG	Hamilton Path *HED***GC***ABF*
CH	Hamilton Path *GDE***HC***ABF*
GH	Hamilton Path *CABFE***HG***D*

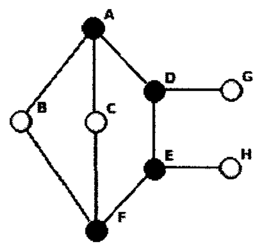


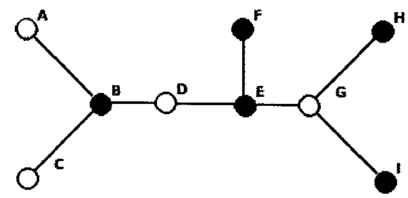
Table 3.6: G in $\mathscr{R}(3,4)$ contains a Hamilton path on nine vertices.

|E| = 8

T_9^1

$S = \{A,C,D,G\}$ — $\alpha \geq 4$

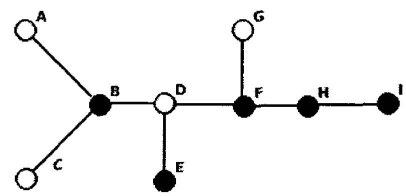
Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AG	$d(G) \geq 4$
CD	Triangle BCD
CG	$d(G) \geq 4$
DG	$d(G) \geq 4$



T_9^2

$S = \{A,C,D,G\}$ — $\alpha \geq 4$

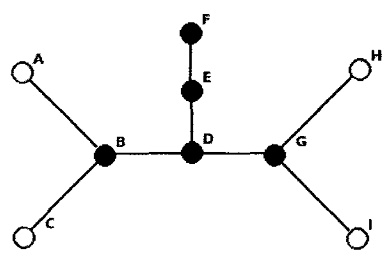
Add Edge	Result
AC	Triangle ABC
AD	$d(D) \geq 4$
AG	$H_{9.9}^4$
CD	$d(D) \geq 4$
CG	$H_{9.9}^4$
DG	$d(D) \geq 4$



T_9^3

$S = \{A,C,H,I\}$ — $\alpha \geq 4$

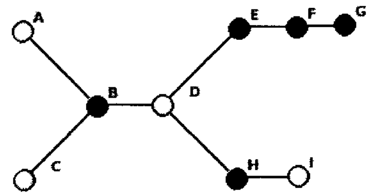
Add Edge	Result
AC	Triangle ABC
AH	$H_{9.9}^{11}$
AI	$H_{9.9}^{11}$
CH	$H_{9.9}^{11}$
CI	$H_{9.9}^{11}$
HI	Triangle GHI



T_9^4

$S = \{A,C,D,I\}$	$\alpha \geq 4$

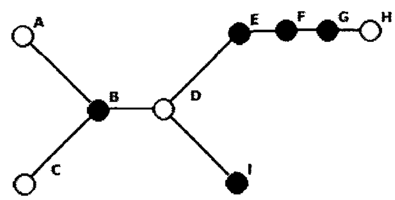
Add Edge	Result
AC	Triangle ABC
AD	$d(D) \geq 4$
AI	Hamilton Path $CBA\mathbf{I}HDEFG$
CD	$d(D) \geq 4$
CI	Hamilton Path $ABC\mathbf{I}HDEFG$
DI	$d(D) \geq 4$



T_9^5

$S = \{A,C,D,H\}$	$\alpha \geq 4$

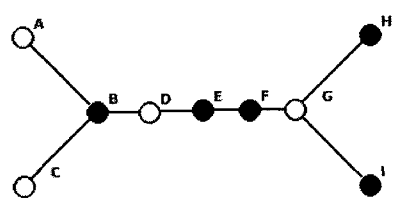
Add Edge	Result
AC	Triangle ABC
AD	$d(D) \geq 4$
AH	Hamilton Path $CBA\mathbf{H}GFEDI$
CD	$d(D) \geq 4$
CH	Hamilton Path $ABC\mathbf{H}GFEDI$
DH	$d(D) \geq 4$



T_9^6

$S = \{A,C,D,G\}$	$\alpha \geq 4$

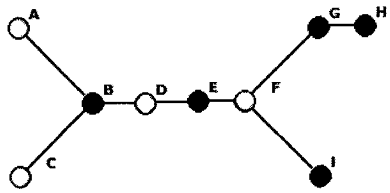
Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AG	$d(G) \geq 4$
CD	Triangle BCD
CG	$d(G) \geq 4$
DG	$d(G) \geq 4$



T_9^7

$S = \{A, C, D, F\}$ — $\alpha \geq 4$

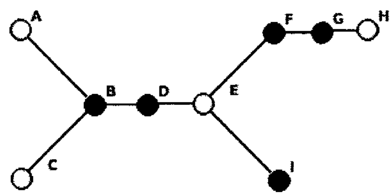
Add Edge	Result
AC	Triangle ABC
AD	Triangle ABD
AF	$d(F) \geq 4$
CD	Triangle BCD
CF	$d(F) \geq 4$
DF	$d(F) \geq 4$



T_9^8

$S = \{A, C, E, H\}$ — $\alpha \geq 4$

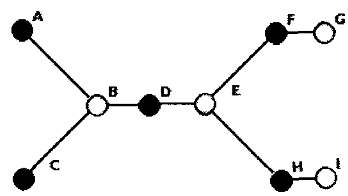
Add Edge	Result
AC	Triangle ABC
AE	$d(E) \geq 4$
AH	$H_{9,9}^5$
CE	$d(E) \geq 4$
CH	$H_{9,9}^5$
EH	$d(E) \geq 4$



T_9^9

$S = \{B, E, G, I\}$ — $\alpha \geq 4$

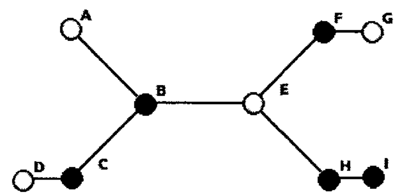
Add Edge	Result
BE	$d(B), d(E) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
EG	$d(E) \geq 4$
EI	$d(E) \geq 4$
GI	$H_{9,9}^7$



T_9^{10}

$S = \{A, D, E, G\}$ | $\alpha \geq 4$

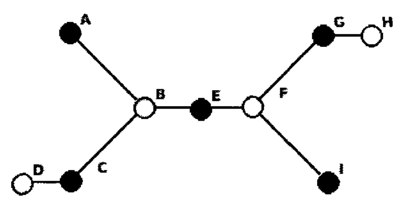
Add Edge	Result
AD	$H_{9,9}^6$
AE	$d(E) \geq 4$
AG	Hamilton Path $DCB\mathbf{AG}FEHI$
DE	$d(E) \geq 4$
DG	Hamilton Path $ABC\mathbf{DG}FEHI$
EG	$d(E) \geq 4$



T_9^{11}

$S = \{B, D, F, H\}$ | $\alpha \geq 4$

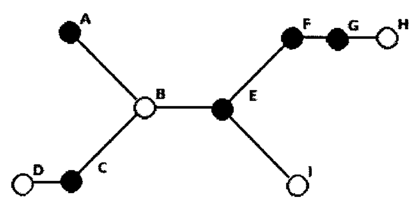
Add Edge	Result
BD	$d(B) \geq 4$
BF	$d(B), d(F) \geq 4$
BH	$d(B) \geq 4$
DF	$d(F) \geq 4$
DH	$H_{9,9}^5$
FH	$d(F) \geq 4$



T_9^{12}

$S = \{B, D, H, I\}$ | $\alpha \geq 4$

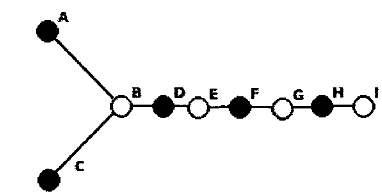
Add Edge	Result
BD	$d(B) \geq 4$
BH	$d(B) \geq 4$
BI	$d(B) \geq 4$
DH	Hamilton Path $ABC\mathbf{DH}GFEI$
DI	Hamilton Path $ABC\mathbf{DI}EFGH$
HI	$H_{9,9}^9$



T_9^{13}

$S = \{B, E, G, I\}$ | $\alpha \geq 4$

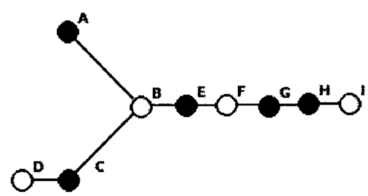
Add Edge	Result
BE	$d(B) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
EG	Triangle EFG
EI	$H_{9,9}^{7}$
GI	Triangle GHI



T_9^{14}

$S = \{B, D, F, I\}$ | $\alpha \geq 4$

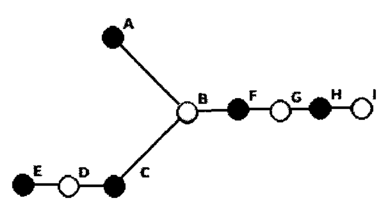
Add Edge	Result
BD	$d(B) \geq 4$
BF	$d(B) \geq 4$
BI	$d(B) \geq 4$
DF	$H_{9,9}^{10}$
DI	Hamilton Path $ABCDIHGFE$
FI	$H_{9,9}^{3}$



T_9^{15}

$S = \{B, D, G, I\}$ | $\alpha \geq 4$

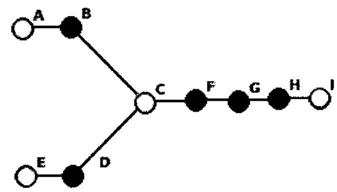
Add Edge	Result
BD	$d(B) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
DG	$H_{9,9}^{8}$
DI	$H_{9,9}^{5}$
GI	Triangle GHI



T_9^{16}

$S = \{A, C, E, I\}$ — $\alpha \geq 4$

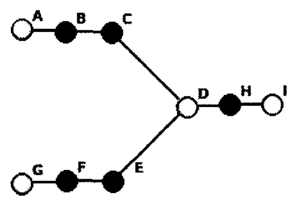
Add Edge	Result
AC	$d(C) \geq 4$
AE	Hamilton Path $B\mathbf{AE}DCFGHI$
AI	Hamilton Path $EDCB\mathbf{AI}HGF$
CE	$d(C) \geq 4$
CI	$d(C) \geq 4$
EI	Hamilton Path $ABCD\mathbf{EI}HGF$



T_9^{17}

$S = \{A, D, G, I\}$ — $\alpha \geq 4$

Add Edge	Result
AD	$d(D) \geq 4$
AG	Hamilton Path $IHDCB\mathbf{AG}FE$
AI	Hamilton Path $GFEDH\mathbf{IA}BC$
DG	$d(D) \geq 4$
DI	$d(D) \geq 4$
GI	Hamilton Path $ABCDEF\mathbf{GI}H$

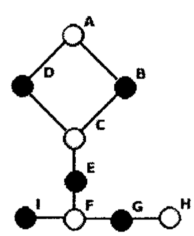


$|E| = 9$

$H^3_{9,9}$

$S = \{A, C, F, H\}$ — $\alpha \geq 4$

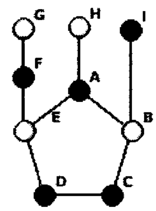
Add Edge	Result
AC	$d(C) \geq 4$
AF	$d(F) \geq 4$
AH	$H^3_{9.10}$
CF	$d(C), d(F) \geq 4$
CH	$d(C) \geq 4$
FH	$d(F) \geq 4$



$H^4_{9,9}$

$S = \{B, E, G, H\}$ | $\alpha \geq 4$

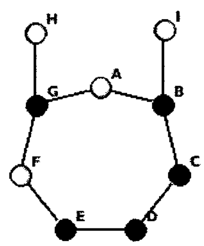
Add Edge	Result
BE	$d(B), d(E) \geq 4$
BG	$d(B) \geq 4$
BH	$d(B) \geq 4$
EG	$d(E) \geq 4$
EH	$d(E) \geq 4$
GH	Hamilton Path $IBCDEF\mathbf{GH}A$



$H^5_{9,9}$

$S = \{A, F, H, I\}$ | $\alpha \geq 4$

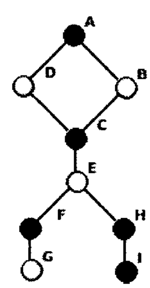
Add Edge	Result
AF	Triangle AFG
AH	Triangle AGH
AI	Triangle ABI
FH	Triangle FGH
FI	Hamilton Path $HGAB\mathbf{IF}EDC$
HI	Hamilton Path $ABCDEFG\mathbf{HI}$



$H^6_{9,9}$

$S = \{B, D, E, G\}$ | $\alpha \geq 4$

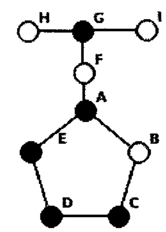
Add Edge	Result
BD	Triangle ABD
BE	$d(E) \geq 4$
BG	Hamilton Path $IHECDA\mathbf{BG}F$
DE	$d(E) \geq 4$
DG	Hamilton Path $IHECBA\mathbf{DG}F$
EG	$d(E) \geq 4$



$H_{9,9}^{7}$

$S = \{B, F, H, I\}$	$\alpha \geq 4$

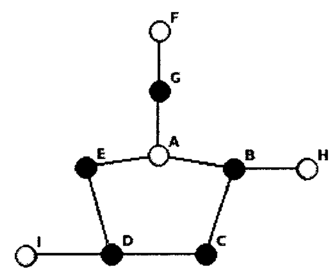
Add Edge	Result
BF	Triangle ABF
BH	Hamilton Path $IGFAEDC\mathbf{BH}$
BI	Hamilton Path $HGFAEDC\mathbf{BI}$
FH	Triangle FGH
FI	Triangle FGI
HI	Triangle GHI



$H_{9,9}^{8}$

$S = \{A, F, H, I\}$	$\alpha \geq 4$

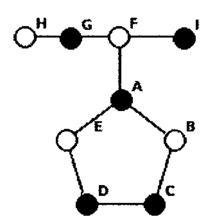
Add Edge	Result
AF	$d(A) \geq 4$
AH	$d(A) \geq 4$
AI	$d(A) \geq 4$
FH	Hamilton Path $IDEAG\mathbf{FH}BC$
FI	Hamilton Path $HBCD\mathbf{IF}GAE$
HI	Hamilton Path $FGAED\mathbf{IH}BC$



$H_{9,9}^{9}$

$S = \{B, E, F, H\}$	$\alpha \geq 4$

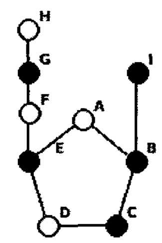
Add Edge	Result
BE	Triangle ABE
BF	$d(F) \geq 4$
BH	Hamilton Path $IFG\mathbf{HB}CDEA$
EF	$d(F) \geq 4$
EH	Hamilton Path $IFG\mathbf{HE}ABCD$
FH	$d(F) \geq 4$



$H_{9,9}^{10}$

$S = \{A, D, F, H\}$ | $\alpha \geq 4$

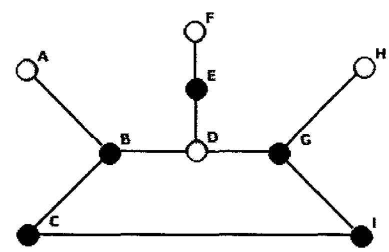
Add Edge	Result
AD	Triangle ADE
AF	Triangle AEF
AH	Hamilton Path $IB\mathbf{AH}GFEDC$
DF	Triangle DEF
DH	Hamilton Path $IBC\mathbf{DH}GFEA$
FH	Triangle FGH



$H_{9,9}^{11}$

$S = \{A, D, F, H\}$ | $\alpha \geq 4$

Add Edge	Result
AD	$d(D) \geq 4$
AF	Hamilton Path $HGICB\mathbf{AF}ED$
AH	Hamilton Path $FEDGIB\mathbf{AH}$
DF	$d(D) \geq 4$
DH	$d(D) \geq 4$
FH	Hamilton Path $ABCIGDE\mathbf{FH}$

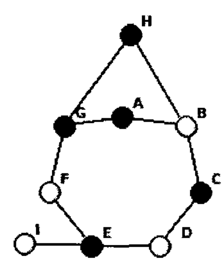


|E| = 10

$H_{9,10}^{3}$

$S = \{B, D, F, I\}$ | $\alpha \geq 4$

Add Edge	Result
BD	$d(B) \geq 4$
BF	$d(B) \geq 4$
BI	$d(B) \geq 4$
DF	Triangle DEF
DI	Triangle DEI
FI	Triangle EFI



CHAPTER 4

COMPLETING $\mathscr{R}(3,4)$

4.1 Introduction

The main result of this chapter is to show that the set of all graphs depicted in Figure 3.1 is the complete set of all graphs in $\mathscr{R}(3,4)$. In Chapter 3, all graphs G in $\mathscr{R}(3,4)$ that were disconnected, $\alpha(G)=1$, $\alpha(G)=2$, and trees were proved. We must prove the remaining case of G in $\mathscr{R}(3,4)$ such that G has an independence number of three and is not a tree which is Theorem 4.2.1. Once this is done, Theorem 3.1.1 can then be used to find an independent proof for $R(3,4)=9$.

4.2 Completeness Of $\mathscr{R}(3,4)$

For any graph G in $\mathscr{R}(3,4)$ that we have not found, we know G must be triangle free and have an independence number of three. Lemma 3.2.1 and Lemma 3.2.2 proved all graphs G in $\mathscr{R}(3,4)$ with an independence number less than or equal to two, so we may assume that $\alpha(G)=3$. Also, Theorem 3.2.3 proved all the disconnected graphs in $\mathscr{R}(3,4)$, so we may also assume G is connected. Since Lemma 3.4.1 proved all the trees with independence number three, we may assume that G is not a tree. Thus in this section, we may assume G is in $\mathscr{R}(3,4)$ such that G is connected, G is not a tree, and $\alpha(G)=3$.

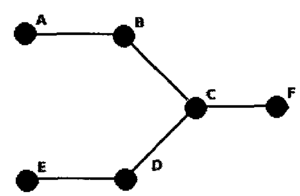
Since G is a connected graph, G must contain a spanning tree T. The spanning tree T will be triangle free like G, but the independence number of T can be different than G. The independence number must be $\alpha(T) \leq \alpha(G)$ since T is a subgraph of G. From Theorem 3.4.2, we know that any connected graph $G \in \mathscr{R}(3,4)$ up to nine vertices with $\alpha(G)=3$ is a tree or has a Hamilton path. Since G is not a tree, we know G must contain a Hamilton path. We will let the spanning tree T of G be a Hamilton path. By adding edges to T in all possible ways, we will find all graphs in $\mathscr{R}(3,4)$ since they will all contain this Hamilton path as a spanning tree.

To start construction of all graphs G in $\mathscr{R}(3,4)$, we start by labeling the vertices of a Hamilton path T. We want to extend this graph by adding one edge in all possible ways to form a new set of graphs up to isomorphism. This process produces the graphs $T+e_1$, $T+e_2, \cdots, T+e_n$. We know that for any edge e that is added to T, $\alpha(T+e) \leq \alpha(T)$ and $\omega(T+e) \geq \omega(T)$. Since we are looking for graphs in $\mathscr{R}(3,4)$, we have restrictions on the independence number and clique number of the graphs. Only graphs that are triangle free and have an independence number of at least three need to be extended in the next round. This process repeats until no more extensions are possible due to the restrictions placed on it by $\mathscr{R}(3,4)$.

We will use Table 4.1 to show how to read the tables used in the main proof of this chapter. Two types of graphs are listed on the table. If the name is of the form $H^i_{|V|,|E|}$ then this is a graph with $|V|$ vertices, $|E|$ edges, and i as an index number. All graphs denoted $H^i_{|V|,|E|}$ have $\alpha(H^i_{|V|,|E|}) \geq 4$ which are not graphs in $\mathscr{R}(3,4)$. These graphs are only temporary placeholders since $H^i_{|V|,|E|}$ can be a subgraph of a graph that is in $\mathscr{R}(3,4)$. This graph is extended the same way as the graphs in Theorem 3.1.1 were extended by adding an edge to an independent set of size four to reduce the independence number. Adding edges in this way will produce a triangle, a temporary graph, or a graph in $\mathscr{R}(3,4)$. A graph with a triangle will not be a subgraph of a graph in $\mathscr{R}(3,4)$, so generation will stop. Both temporary graphs and graphs in $\mathscr{R}(3,4)$ can be subgraphs of a graph in $\mathscr{R}(3,4)$; thus, generation is continued until all extended graphs cannot be a subgraph of a graph in $\mathscr{R}(3,4)$. When no graphs are left to extend, generation is stopped.

If the name is of the form $G^i_{|V|\,|E|}$, then this is a graph in $\mathscr{R}(3,4)$ with $|V|$ vertices, $|E|$ edges, and i as the index. The left side of the next row contains the set of vertices whose degree is equal to three, denoted by $d_3(v)$. The right side displays the independence number of the graph. The left panel is a list of all possible edge additions to the graph and the results from adding that particular edge, while the right panel shows the labeled graph that is being used.

Table 4.1: Initial graph $G^3_{6,5}$.

$G^3_{6,5}$		
$d_3(v) = \{C\}$		$\alpha = 3$
Add edge	Result	
AD	$G^2_{6,6}$	
AE	$G^1_{6,6}$	
AF	$G^3_{6,6}$	
BD	Triangle BCD	
BE	$G^2_{6,6}$	
BF	Triangle BCF	
DF	Triangle CDF	
EF	$G^3_{6,6}$	

The list of degree three vertices is determined by finding all vertices in $V(G)$ that have degree equal to three. From Lemma 3.1.2, we know that $\Delta(G) \leq 3$ since $G \in \mathscr{R}(3,4)$; thus, no edges can be added to a vertex of degree three since the maximum degree would be increased to four, contradicting Lemma 3.1.2.

The list of possible edges given is reduced from the list of all possible edges which have $\frac{|V|\,|V-1|}{2}$ edges. Since parallel edges are not recognized, the edges already in $E(G)$ are removed. Any edge that has an end vertex from one of the vertices of degree three is also removed since the addition of this edge creates a vertex of degree four. The edges remaining in the list are the possible edge additions to the graph. When a graph is a cycle, all the vertices are symmetric, so only one vertex is used to create the list of possible edge additions. The edges listed are the possible edge additions to the graph. Adding any of these edges from the list can result in creating a triangle or creating a graph in $\mathscr{R}(3,4)$. As mentioned earlier, if a graph has a triangle, then this graph will not be in $\mathscr{R}(3,4)$ and is not a subgraph of any graph in $\mathscr{R}(3,4)$, so the generation stops along this route. Since a graph in $\mathscr{R}(3,4)$ can be a subgraph of another graph in $\mathscr{R}(3,4)$, generation of the new graph is continued further in the table. This process is repeated until no graphs are left to generate, at which point all graphs found form the complete list of connected graphs in $\mathscr{R}(3,4)$ on $|V|$ vertices with an independence number of three.

Now we use Table 4.1 to walk through an example of generating all graphs in $\mathscr{R}(3,4)$ that have six vertices. The first graph is $G^3_{6,5}$ which is a tree from Lemma 3.4.1 with an independence number of three. The set $d_3(v)$ contains the vertex C which is the only vertex with degree three. Initially, there are $\frac{6 \cdot 5}{2} = 15$ possible edges, but by removing the five edges already in the graph, the number of possible edge additions is reduced to 10. Removing any edge addition with an endvertex of C further reduces the number of edges to eight. These eight edges are AD, AE, AF, BD, BE, BF, DF, and EF. Now, for each edge, a new graph is formed $G^3_{6.5}+e$ where e is one of the edge additions. The result of adding any one of the edges BD, BF, or DF forms a triangle in the resultant graph; thus, generation along this route will stop. Adding the edge AD or BE forms a graph isomorphic to $G^2_{6,6}$. Adding either edge AF or EF will form a graph isomorphic to $G^3_{6.6}$. The remaining edge AE will result in a graph isomorphic to $G^1_{6.6}$. The new set of graphs will continue generation further in the table.

By repeating this process on $P_6 \cong G^4_{6,5}$ we can find the complete list of graphs in $\mathscr{R}(3,4)$ with six vertices. Just like before, the possible edge additions and results will form graphs isomorphic to $G^1_{6.6}$, $G^2_{6,6}$, $G^3_{6,6}$, or $G^4_{6.6}$. Since we can assume that G in $\mathscr{R}(3,4)$ has a Hamilton path, we do not need to extend $G^3_{6,5}$ since each graph produced is isomorphic to a graph produced from $G^4_{6.5}$. It is only included for clarity in the example.

By using this process for five, six, seven, and eight vertices, we get the list of connected graphs G in $\mathscr{R}(3,4)$ for $\alpha(G)=3$. On nine vertices, no graphs are created from P_9, so this is the smallest number for which every graph has a K_3 or N_4; thus, no graphs with at least nine vertices will be in $\mathscr{R}(3,4)$. This leads to Theorem 4.2.1.

Theorem 4.2.1. *If $G \in \mathscr{R}(3,4)$, G is connected, $\alpha(G) = 3$, and G is not a tree; then G is one of the graphs in Figure 4.1*

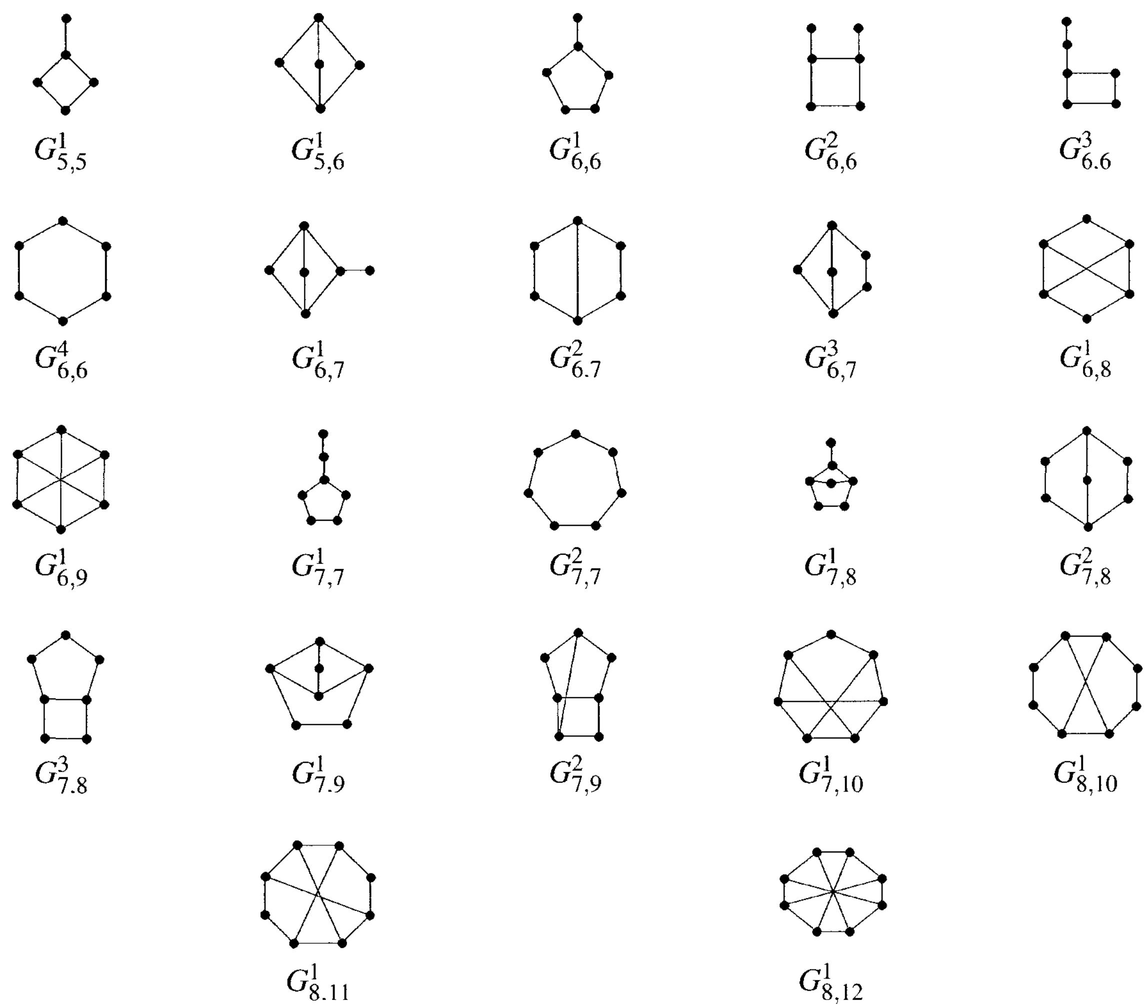


Figure 4.1: List of connected R(3,4) graphs that are not trees and $\alpha(G) = 3$.

Proof. Let G be in $\mathscr{R}(3,4)$, connected, $\alpha(G) = 3$, and G is not a tree. We start with a Hamilton path on $|V|$ vertices. Adding edges in all possible ways as described previously will give us the complete list of graphs meeting our conditions on $|V|$ vertices. Since no graphs with fewer than five vertices have $\alpha(G) = 3$ and are not trees, we start with Table 4.2 for five vertices. The process is repeated for six, seven, eight, and nine vertices in the other tables. Since no graphs were found on nine vertices, we also do not need any tables that are larger than nine vertices to complete the proof. By analyzing the results from Table 4.2 through Table 4.6 we get the complete list in of graphs depicted in Figure 4.1. □

As in Chapter 3, the arrangment of the tables prevents a circular argument from being made. The tables are arranged in such a way that $|V(G)|$ is constant throughout the table. In each table, the graphs are ordered in such a way that $|E(G_i)| \leq |E(G_{i+1})|$ where i is the position of the graph on the table, so no infinite loop can exist as we progress through the tables. For each graph G, two possibilities can happen by adding an edge. First, the generation may terminate if no edge exists that can be added to G to satisfy the necessary conditions. The other way will be to consider a new graph that is further in the table, $G+e$. The new graph will always have exactly one more edge than G. By only going foward to a graph with exactly one more edge, we will not encounter a circular logic proof since the generation will terminate in the end.

Now that Theorem 4.2.1 has been established, it can be used with what we also proved in Chapter 3 to prove Theorem 3.1.1.

Theorem 3.1.1 If $G \in \mathscr{R}(3,4)$, then G is isomorphic to one of the graphs depicted in Figure 3.1.

Proof. Let $G \in \mathscr{R}(3,4)$. If G is disconnected then G is a graph from Theorem 3.2.3. Thus we may assume G is connected. If $\alpha(G) = 1$, then G is proved in Lemma 3.2.1. If $\alpha(G) = 2$, then G is proved in Lemma 3.2.2. Now $\alpha(G) = 3$. If G is a tree, then G is proved in Lemma 3.4.1. The remaining case is G is connected and not a tree, which is proved in Theorem 4.2.1. Since no G is in $\mathscr{R}(3,4)$ when $\alpha(G) \geq 4$, the union of these lists shown in Figure 3.1 is the complete list of $\mathscr{R}(3,4)$. □

Theorem 4.2.2. $R(3,4) = 9$.

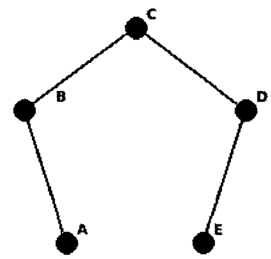
Proof. Theorem 3.1.1 establishes the complete list of Ramsey graphs. By analyzing the list of graphs, we know for every graph $G \in \mathscr{R}(3.4)$ the number of vertices must be less than or equal to eight. Thus, $R(3,4) \geq 9$. Since is was also established that no Ramsey graph can exists on nine vertices, from Corollary 1.4.5, we know no Ramsey graph can exist on more than nine vertices; thus, $R(3,4) \leq 9$. Hence, we get an independent proof that $R(3,4) = 9$. □

Table 4.2: $\mathscr{R}(3,4)$ generation on five vertices.

$G^3_{5,4}$

$d_3(v) = \emptyset$ | $\alpha = 3$

Add Edge	Result
AC	Triangle ABC
AD	$G^1_{5,5}$
AE	$G^2_{5,5}$
BD	Triangle BCD
BE	$G^1_{5,5}$
CE	Triangle CDE

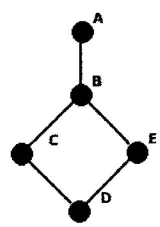


|E| = 5

$G^1_{5,5}$

$d_3(v) = \{B\}$ | $\alpha = 3$

Add Edge	Result
AC	Triangle ABC
AD	$G^1_{5,6}$
AE	Triangle ABE
CE	Triangle BCE

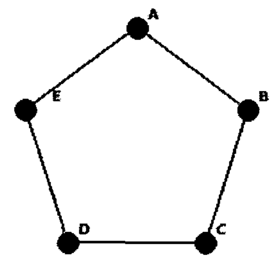


$G^2_{5,5}$

$d_3(v) = \emptyset$

$\alpha = 2$

Add Edge	Result
AC	Triangle *ABC*
AD	Triangle *ADE*
BD	Triangle *BCD*
BE	Triangle *ABE*
CE	Triangle *CDE*



|E| = 6

$G^1_{5,6}$

$d_3(v) = \{A, B\}$

$\alpha = 3$

Add Edge	Result
CD	Triangle *ACD*
CE	Triangle *ACE*
DE	Triangle *ADE*

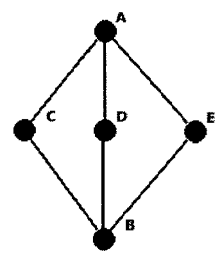
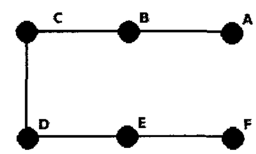


Table 4.3: $\mathscr{R}(3,4)$ generation on six vertices.

$G^4_{6,5}$

$d_3(v) = \emptyset$ | $\alpha = 3$

Add Edge	Result
AC	Triangle ABC
AD	$G^3_{6,6}$
AE	$G^1_{6,6}$
AF	$G^4_{6,6}$
BD	Triangle BCD
BE	$G^2_{6,6}$
BF	$G^1_{6,6}$
CE	Triangle CDE
CF	$G^2_{6,6}$
DF	Triangle DEF

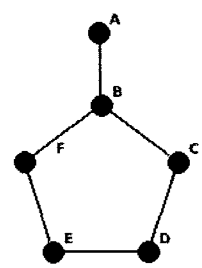


|E| = 6

$G^1_{6,6}$

$d_3(v) = \{B\}$ | $\alpha = 3$

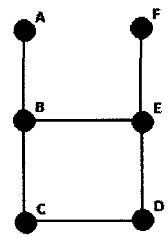
Add Edge	Result
AC	Triangle ABC
AD	$G^3_{6,7}$
AE	$G^3_{6,7}$
AF	Triangle ABF
CE	Triangle CDE
CF	Triangle BCF
DF	Triangle DEF



$G^2_{6,6}$

$d_3(v) = \{B, E\}$ | $\alpha = 3$

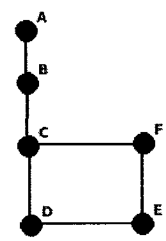
Add Edge	Result
AC	Triangle ABC
AD	$G^1_{6,7}$
AF	$G^2_{6,7}$
CF	$G^1_{6,7}$
DF	Triangle DEF



$G^3_{6,6}$

$d_3(v) = \{C\}$ | $\alpha = 3$

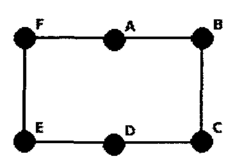
Add Edge	Result
AD	$G^2_{6,7}$
AE	$G^3_{6,7}$
AF	$G^2_{6.7}$
BD	Triangle BCD
BE	$G^1_{6,7}$
BF	Triangle BCF
DF	Triangle DEF



$G^4_{6,6}$

$d_3(v) = \emptyset$
All vertices in graph are symmetric. | $\alpha = 3$

Add Edge	Result
AC	Triangle ABC
AD	$G^2_{6.7}$
AE	Triangle AEF

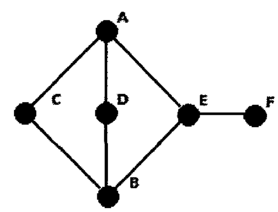


|E| = 7

$G_{6,7}^{1}$

$d_3(v) = \{A, B, E\}$	$\alpha = 3$

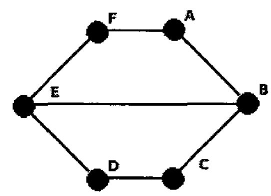
Add Edge	Result
CD	Triangle ACD
CF	$G_{6,8}^{1}$
DF	$G_{6,8}^{1}$



$G_{6,7}^{2}$

$d_3(v) = \{B, E\}$	$\alpha = 3$

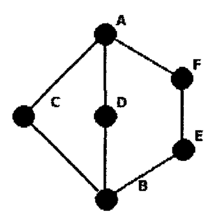
Add Edge	Result
AC	Triangle ABC
AD	$G_{6,8}^{1}$
CF	$G_{6,8}^{1}$
DF	Triangle DEF

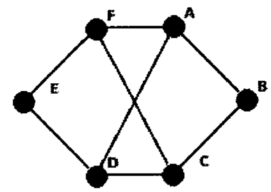


$G_{6,7}^{3}$

$d_3(v) = \{A, B\}$	$\alpha = 3$

Add Edge	Result
CD	Triangle ACD
CE	Triangle BCE
CF	Triangle ACF
DE	Triangle BDE
DF	Triangle ADF



|E| = 8	
$G^1_{6,8}$	
$d_3(v) = \{A,C,D,F\}$	$\alpha = 3$
Add Edge | Result BE | $G^1_{6,9}$	

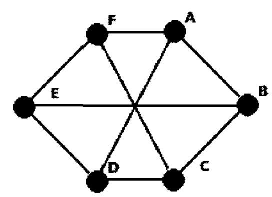
|E| = 9	
$G^1_{6,9}$	
$d_3(v) = \{A,B,C,D,E,F\}$ Every vertex has degree three.	$\alpha = 3$
Add Edge | Result $\emptyset$ | No possible edge additions.	

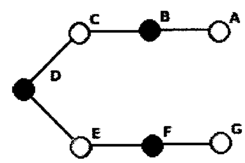
Table 4.4: $\mathscr{R}(3,4)$ generation on seven vertices.

|E| = 6

$H^1_{7,6}$

$S = \{A,C,E,G\}$ | $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AE	$G^1_{7,7}$
AG	$G^2_{7,7}$
CE	Triangle CDE
CG	$G^1_{7,7}$
EG	Triangle EFG

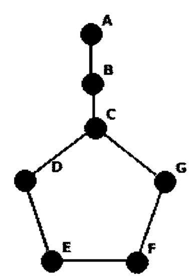


|E| = 7

$G^1_{7,7}$

$d_3(v) = \{C\}$ | $\alpha = 3$

Add Edge	Result
AD	$G^3_{7,8}$
AE	$G^2_{7,8}$
AF	$G^2_{7,8}$
AG	$G^3_{7,8}$
BD	Triangle BCD
BE	$G^1_{7.8}$
BF	$G^1_{7.8}$
BG	Triangle BCG
DF	Triangle DEF
DG	Triangle CDG
EG	Triangle EFG

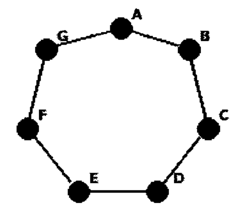


$G^2_{7,7}$

$d_3(v) = \emptyset$
All vertices in graph are symmetric. (A)

$\alpha = 3$

Add Edge	Result
AC	Triangle ABC
AD	$G^3_{7,8}$
AE	$G^3_{7,8}$
AF	Triangle AFG



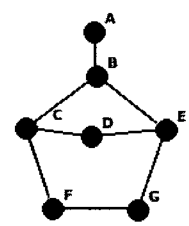
|E| = 8

$G^1_{7,8}$

$d_3(v) = \{B, C, E\}$

$\alpha = 3$

Add Edge	Result
AD	$G^1_{7,9}$
AF	$G^2_{7,9}$
AG	$G^2_{7,9}$
DF	Triangle CDF
DG	Triangle DEG

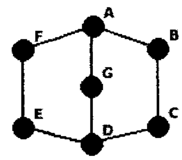


$G^2_{7,8}$

$d_3(v) = \{A, D\}$

$\alpha = 3$

Add Edge	Result
BE	$G^2_{7,9}$
BF	Triangle ABF
BG	Triangle ABG
CE	Triangle CDE
CF	$G^2_{7,9}$
CG	Triangle CDG
EG	Triangle DEG
FG	Triangle AFG

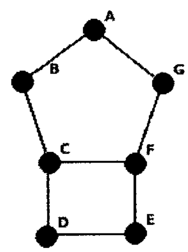


$G^3_{7,8}$

$d_3(v) = \{C, F\}$

$\alpha = 3$

Add Edge	Result
AD	$G^2_{7,9}$
AE	$G^2_{7,9}$
BD	Triangle BCD
BE	$G^1_{7,9}$
BG	Triangle ABG
DG	$G^1_{7,9}$
EG	Triangle EFG

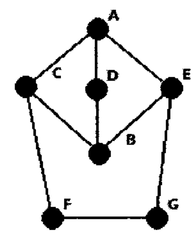


|E| = 9

$G^1_{7,9}$

$d_3(v) = \{A, B, C, E\}$	$\alpha = 3$

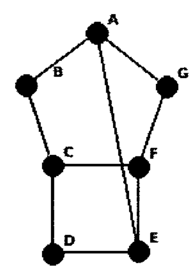
Add Edge	Result
DF	$G^1_{7,10}$
DG	$G^1_{7,10}$



$G^2_{7,9}$

$d_3(v) = \{A, C, E, F\}$	$\alpha = 3$

Add Edge	Result
BD	Triangle BCD
BG	Triangle ABG
DG	$G^1_{7,10}$



<table>
<tr><td colspan="2">|E| = 10</td></tr>
<tr><td colspan="2">$G^{1}_{7,10}$</td></tr>
<tr><td>$d_3(v) = \{B, C, D, E, F, G\}$
Only one vertex with $d(v) \leq 3$</td><td>$\alpha = 3$</td></tr>
<tr><td>Add Edge | Result
$\emptyset$ |

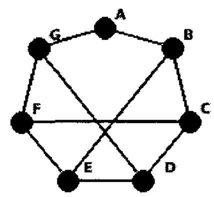
No possible edge additions.</td><td>
</td></tr>
</table>

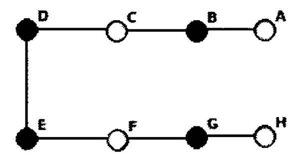
Table 4.5: $\mathscr{R}(3,4)$ generation on eight vertices.

|E| = 7

$H^1_{8,7}$

$S = \{A,C,F,H\}$ | $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AF	$H^1_{8,8}$
AH	$H^3_{8,8}$
CF	$H^2_{8,8}$
CH	$H^1_{8,8}$
FH	Triangle FGH

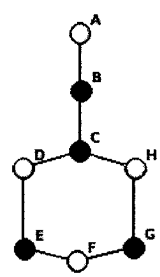


|E| = 8

$H^1_{8,8}$

$S = \{A,D,F,H\}$ | $\alpha \geq 4$

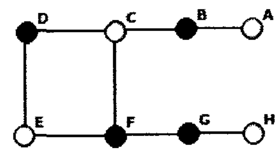
Add Edge	Result
AD	$H^4_{8,9}$
AF	$H^6_{8,9}$
AH	$H^4_{8,9}$
DF	Triangle DEF
DH	Triangle CDH
FH	Triangle FGH



$H^2_{8,8}$

$S = \{A,C,E,H\}$ | $\alpha \geq 4$

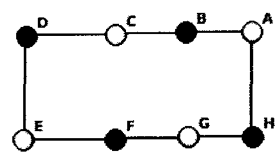
Add Edge	Result
AC	$d(C) \geq 4$
AE	$H^2_{8.9}$
AH	$H^4_{8,9}$
CE	$d(C) \geq 4$
CH	$d(C) \geq 4$
EH	$H^5_{8,9}$



$H^3_{8,8}$

$S = \{A,C,E,G\}$ | $\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AE	$H^3_{8.9}$
AG	Triangle AGH
CE	Triangle CDE
CG	$H^3_{8,9}$
EG	Triangle EFG

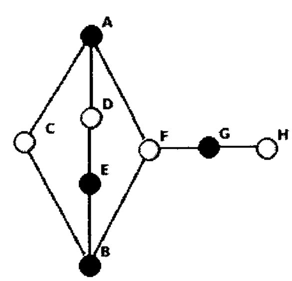


|E| = 9

$H^2_{8,9}$

$S = \{C,D,F,H\}$ | $\alpha \geq 4$

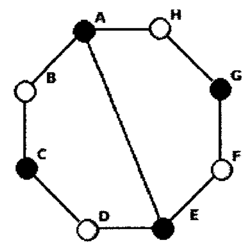
Add Edge	Result
CD	Triangle ACD
CF	$d(F) \geq 4$
CH	$G^1_{8.10}$
DF	$d(F) \geq 4$
DH	$H^1_{8.10}$
FH	$d(F) \geq 4$



$H^3_{8,9}$

$S = \{B,D,F,H\}$ | $\alpha \geq 4$

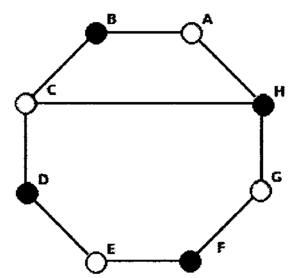
Add Edge	Result
BD	Triangle BCD
BF	$G^1_{8,10}$
BH	Triangle ABH
DF	Triangle DEF
DH	$G^1_{8,10}$
FH	Triangle FGH



$H^4_{8,9}$

$S = \{A,C,E,G\}$ | $\alpha \geq 4$

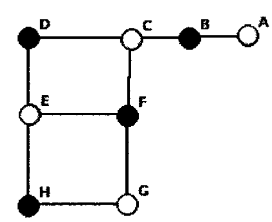
Add Edge	Result
AC	$d(C) \geq 4$
AE	$H^2_{8,10}$
AG	Triangle AGH
CE	$d(C) \geq 4$
CG	$d(C) \geq 4$
EG	Triangle EFG



$H^5_{8,9}$

$S = \{A,C,E,G\}$ | $\alpha \geq 4$

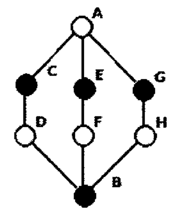
Add Edge	Result
AC	$d(C) \geq 4$
AE	$d(E) \geq 4$
AG	$H^3_{8.10}$
CE	$d(C), d(E) \geq 4$
CG	$d(C) \geq 4$
EG	$d(E) \geq 4$



$H^6_{8,9}$

$S = \{A, D, F, H\}$ | $\alpha \geq 4$

Add Edge	Result
AD	Triangle ACD
AF	Triangle AEF
AH	Triangle AGH
DF	Triangle BDF
DH	Triangle BDH
FH	Triangle BFH

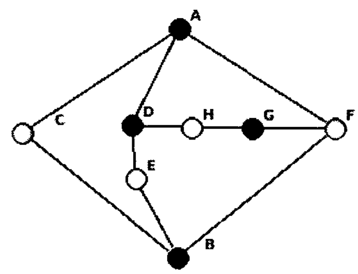


|E| = 10

$H^1_{8,10}$

$S = \{C, E, F, H\}$ | $\alpha \geq 4$

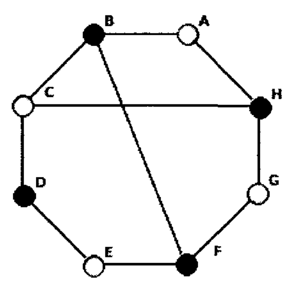
Add Edge	Result
CE	Triangle BCE
CF	$d(F) \geq 4$
CH	$G^1_{8.11}$
EF	$d(F) \geq 4$
EH	Triangle DEH
FH	$d(F) \geq 4$



$H^2_{8,10}$

$S = \{A, C, E, G\}$ | $\alpha \geq 4$

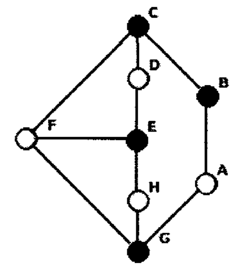
Add Edge	Result
AC	$d(C) \geq 4$
AE	$G^1_{8,11}$
AG	Triangle AGH
CE	$d(C) \geq 4$
CG	$d(C) \geq 4$
EG	Triangle EFG



$H^3_{8,10}$

$S = \{A,D,F,H\}$	$\alpha \geq 4$

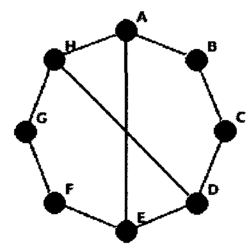
Add Edge	Result
AD	$H^1_{8,11}$
AF	$d(F) \geq 4$
AH	Triangle AGH
DF	$d(F) \geq 4$
DH	Triangle DEH
FH	$d(F) \geq 4$



$G^1_{8,10}$

$d_3(v) = \{A,D,E,H\}$	$\alpha = 3$

Add Edge	Result
BF	$G^1_{8,11}$
BG	$G^1_{8,11}$
CF	$G^1_{8,11}$
CG	$G^1_{8,11}$

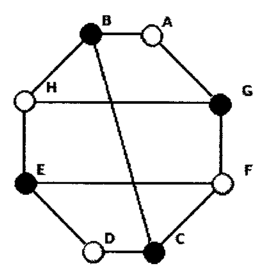


|E| = 11

$H^1_{8,11}$

$S = \{A,D,F,H\}$	$\alpha \geq 4$

Add Edge	Result
AD	$G^1_{8,12}$
AF	$d(F) \geq 4$
AH	$d(H) \geq 4$
DF	$d(F) \geq 4$
DH	$d(H) \geq 4$
FH	$d(F), d(H) \geq 4$

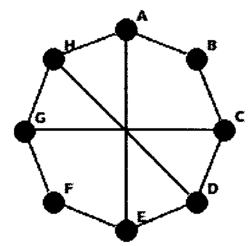


$G^1_{8,11}$

$d_3(v) = \{A,C,D,E,G,H\}$

$\alpha = 3$

Add Edge	Result
BF	$G^1_{8,12}$



|E| = 12

$G^1_{8,12}$

$d_3(v) = \{A,B,C,D,E,F,G,H\}$
Every vertex has degree three.

$\alpha = 3$

Add Edge	Result
$\emptyset$	All $v \in V(G), d(v) \geq 3$

No possible edge additions.

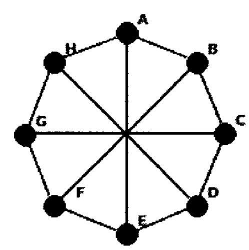


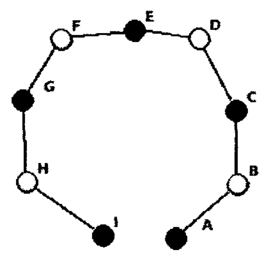
Table 4.6: $\mathscr{R}(3,4)$ generation on nine vertices.

|E| = 8

$H^1_{9,8}$

$S = \{B,D,F,H\}$	$\alpha \geq 4$

Add Edge	Result
BD	Triangle BCD
BF	$H^1_{9,9}$
BH	$H^2_{9,9}$
DF	Triangle DEF
DH	$H^1_{9,9}$
FH	Triangle FGH

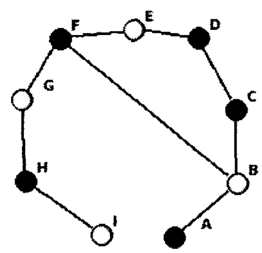


|E| = 9

$H^1_{9,9}$

$S = \{B,E,G,I\}$	$\alpha \geq 4$

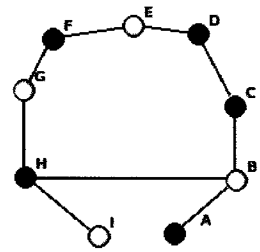
Add Edge	Result
BE	$d(B) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
EG	Triangle EFG
EI	$H^1_{9,10}$
GI	Triangle GHI



$H^2_{9,9}$

$S = \{B, E, G, I\}$	$\alpha \geq 4$

Add Edge	Result
BE	$d(B) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
EG	Triangle EFG
EI	$H^2_{9.10}$
GI	Triangle GHI

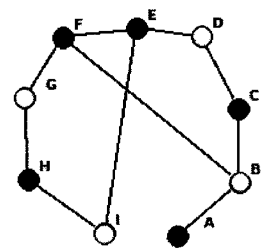


|E| = 10

$H^1_{9,10}$

$S = \{B, D, G, I\}$	$\alpha \geq 4$

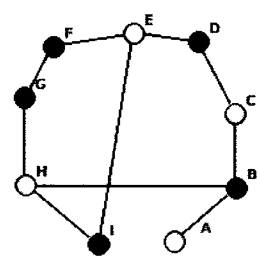
Add Edge	Result
BD	$d(B) \geq 4$
BG	$d(B) \geq 4$
BI	$d(B) \geq 4$
DG	$H^1_{9,11}$
DI	Triangle DEI
GI	Triangle GHI



$H^2_{9,10}$

$S = \{A, C, E, H\}$	$\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AE	$d(E) \geq 4$
AH	$d(H) \geq 4$
CE	$d(E) \geq 4$
CH	$d(H) \geq 4$
EH	$d(E), d(H) \geq 4$

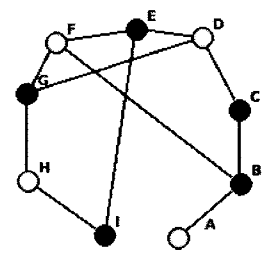


|E| = 11

$H^1_{9,11}$

$S = \{A,D,F,H\}$	$\alpha \geq 4$

Add Edge	Result
AD	$d(D) \geq 4$
AF	$d(F) \geq 4$
AH	$H^1_{9,12}$
DF	$d(D), d(F) \geq 4$
DH	$d(D) \geq 4$
FH	$d(F) \geq 4$

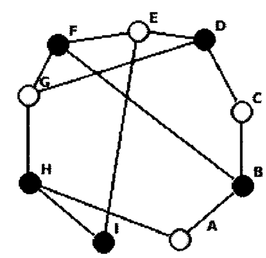


|E| = 12

$H^1_{9,12}$

$S = \{A,C,E,G\}$	$\alpha \geq 4$

Add Edge	Result
AC	Triangle ABC
AE	$d(E) \geq 4$
AG	$d(G) \geq 4$
CE	$d(E) \geq 4$
CG	$d(G) \geq 4$
EG	$d(E), d(G) \geq 4$



CHAPTER 5

OTHER RAMSEY NUMBERS

5.1 Some Known Results

The Ramsey number, $\mathscr{R}(\omega,\alpha)$, is defined to be the minimum number n such that any graph with at least n vertices has a K_ω or N_α as a subgraph This dissertation has focused on the class of Ramsey graphs $\mathscr{R}(3,4)$ by finding the underlying structure of Hamilton paths in these graphs A brute force approach, testing all graphs up to $R(3,4) = 9$ vertices, could have been done with a computer, but this method does not scale well for testing graphs on a larger number of vertices

The Ramsey number $R(3,4) = 9$, $R(3,5) = 14$, and $R(4,4) = 18$ were found by Greenwood and Gleason [6] from a critical Ramsey graph to create a lower bound Then, by using Theorem 1 4 3, the lower and upper bounds were shown to be tight Mckay and Radziszowski proved $R(4,5) = 25$ [11]

The next Ramsey number for which the exact value is unkown is $R(5,5)$ The best known bounds for $R(5,5)$ are 43 and 49 The lower bound of 43 was found by Exoo [4] by showing a graph on 42 vertices with no K_5 or N_5 subgraph This means that $R(5\ 5) > 42$ Mckay and Radziszowski [12] used the graph Exoo described and were able to manipulate it to find 656 $\mathscr{R}(5,5)$-graphs on 42 vertices, that is, they found 328 graphs and their complements None of these graphs were able to be extended to 43 vertices It is unknown whether more Ramsey graphs exist on 42 vertices or if any exist on 43 vertices

5.2 Forced Subgraphs for $R(5,5)$

A brute force approach to find $R(5,5)$ is not computationally feasible since the number of possible graphs grows exponentially compared to the number of vertices. Some overall structure must be found to significantly narrow the search for the graphs. This section finds that the critical graphs in $R(5,5)$ are highly connected and must contain Hamilton cycles. We start this proof with Theorem 5.2.1, which uses the symmetry present in Ramsey numbers $R(\omega,\alpha) = R(\alpha,\omega)$.

Theorem 5.2.1. *If $G \in \mathscr{R}(5,5)$ and $|V(G)| > 28$, then G is connected.*

Proof. Let $G \in \mathscr{R}(5,5)$ and $|V(G)| > 28$. We know that G has no K_5 subgraph and no independent set of five vertices. Since each independent set of each component of G can be added together for the independent set of G, G has at most four components; otherwise, $\alpha(G) \geq 5$.

Suppose G has four components and $G \cong G_1 \overset{\circ}{\bigcup} G_2 \overset{\circ}{\bigcup} G_3 \overset{\circ}{\bigcup} G_4$. Each component must have a maximal independent set of size one; otherwise, $\alpha(G) \geq 5$. Knowing this, each component G_i must be in the class of $\mathscr{R}(5,2)$ graphs. Since $R(5,2) = 5$, we get a bound on the number of vertices in G_i to be $|V(G_i)| \leq R(5,2) - 1 = 5 - 1 = 4$ vertices. Thus, G can have at most four components with each component having at most four vetrices. Hence, $|V(G)| \leq 4 \cdot 4 = 16$. This contradicts our assumption that $|V(G)| > 28$, so G does not have four components.

Suppose G has three components and $G \cong G_1 \overset{\circ}{\bigcup} G_2 \overset{\circ}{\bigcup} G_3$. Without loss of generality, we let G_1 and G_2 be in $\mathscr{R}(5,2)$ and let G_3 be in $\mathscr{R}(5,3)$; otherwise, $\alpha(G_1) + \alpha(G_2) + \alpha(G_3) = \alpha(G) \geq 5$. The components G_1 and G_2 can have at most four vertices. Since $R(5,3) = 14$, G_3 must have at most 13 vertices. Thus, our bound on the number of vertices is $|V(G)| = 4 + 4 + 13 = 21$. However, $|V(G)| > 28 > 21$ so G does not have three components.

Suppose G has two components and $G \cong G_1 \overset{\circ}{\bigcup} G_2$, then the components can be either 1) $G_1 \in \mathscr{R}(5,2)$ and $G_2 \in \mathscr{R}(5.4)$ or 2) $G_1 \in R(5.3)$ and $G_2 \in R(5,3)$. We know the Ramsey number $R(4.5) = R(5,4) = 25$. In the first case, the bound on the number of vertices of G is $(R(5,2)-1)+(R(5,4)-1)=4+24=28$. In the second case, our bound is $|V(G)| \leq (R(5,3)-1)+(R(5,3)-1)=13+13=26$. In both cases, we get a contradiction since we assume $|V(G)|>28$, thus G cannot have two components.

Hence, if $|V(G)|>28$, then G is connected. □

Now we know that any graph G in $\mathscr{R}(5,5)$ with at least 29 vertices is connected. Every connected graph has a minimum cut set of vertices, S, whose removal from G will disconnect the resultant graph. Theorem 5.2.2 proves the size of S in G based on the number of vertices that G contains.

Theorem 5.2.2. *If $G \in \mathscr{R}(5,5)$ and $|V(G)|>28$, then G is $(|V(G)|-28)$-connected.*

Proof. Let $G \in \mathscr{R}(5,5)$ with $|V(G)|>28$. From Theorem 5.2.1, we know that G is connected. Some set $S \subseteq V(G)$ exists that is a minimum cutset of G, thus we let $G-S \cong G_1 \overset{\circ}{\bigcup} G_2$.

Suppose G is 1-connected and not 2-connected. This means S is a cut-vertex and $|S|=1$. Since $G \in \mathscr{R}(5,5)$ and vertex deletion does not increase the maximum clique, we know that G_1 and G_2 do not contain a K_5. Thus, we concentrate on the independence numbers of G_1 and G_2. We can add the independent sets of G_1 and G_2 for a lower bound on the independent set of G. Two cases are presented for the bound of the components G_1 and G_2. First, let $G_1, G_2 \in \mathscr{R}(5,3)$. Since $R(5,3)=14$ and $|S|=1$, we have $|V(G)|=|V(G_1)|+|V(G_2)|+|V(S)| \leq 13+13+1=27$, which contradicts $|V(G)|>28$, thus is impossible. Second, without loss of generallity, we let $G_1 \in \mathscr{R}(5,2)$ and $G_2 \in \mathscr{R}(5,4)$. We get $|V(G)|=|V(G_1)|+|V(G_2)|+|V(S)| \leq 4+24+1=29$. This shows for G to be 1-connected and not 2-connected $|V(G)| \leq 29$. If $|V(G)|>29$, then G must be at least 2-connected.

Note that the size of G_1 and G_2 cannot increase as the connectivity of the graph increases. Thus the increase in connectivity only increases the size of S. Similar arguments can be used for higher connectivity. Hence we get G is at least $(|V(G)|-28)$-connected. □

Since $43 \leq R(5,5) \leq 49$ are the best known bounds, Corollary 5.2.3 considers the connectivity of Ramsey graphs on 42 vertices.

Corollary 5.2.3. *If $G \in \mathscr{R}(5,5)$ and $|V(G)| = 42$, then G is 14-connected, $\delta(G) \geq 14$, and $\Delta(G) \leq 27$*

Proof. Let G be in $\mathscr{R}(5,5)$ and $|V(G)| = 42$. From Theorem 5.2.2, we get G is 14-connected.

Since G is 14-connected it must have a minimum degree of at least 14; thus, $\delta(G) \geq 14$. Also, if $G \in \mathscr{R}(5,5)$, then $\overline{G} \in \mathscr{R}(5,5)$. The complement of G must also be 14-connected. This yields the maximum degree $\Delta(G) \leq (42-1) - 14 = 27$. □

Theorem 5.2.4. *(Chvatal, Erdös [2]) Let G be an s-connected graph containing no independent set of s vertices. Then G is Hamiltonian-connected (i.e. every pair of vertices is joined by a Hamiltonian path).*

We can use Corollary 5.2.3 and the properties of $\mathscr{R}(5,5)$ to show a Ramsey graph on at least 33 vertices satisfies the conditions of Theorem 5.2.4. The start of the underlying structure of $\mathscr{R}(5,5)$ graphs is that they are Hamiltonian-connected. We use this fact to reduce the test set of approximately 10^{208} graphs on 42 vertices. The test set of graphs have been reduced to only Hamiltonian graphs that are also 14-connected with $14 \leq \delta(G) \leq \Delta(G) \leq 27$.

Since looking for a Hamiltonian cycle in a graph is equivalent to the Traveling Salesman Problem, an NP problem, a better way is to generate the test set starting with a Hamilton cycle on 42 vertices. From this initial cycle, just like the Hamilton path in the proof of $\mathscr{R}(3,4)$, we can add edges between vertices. The edge additions will follow lemmas equivelant to the lemmas used in $\mathscr{R}(3,4)$ but for $\mathscr{R}(5,5)$ and the additional information we know. An edge added should not form a K_5, and this is easily checked by comparing the neighborhoods of the endpoints of the edge to be added. Similar to Lemma 3.1.3, $\alpha(G) < 4$, if an independent set of size five is found, an edge must be added in that set for a graph to be in $\mathscr{R}(5,5)$. This produces only 10 edges that need to be traced, similar to the six edges in $\mathscr{R}(3,4)$. One additional property would be the minimum degree of 14 and the maximum degree of 27 on graphs with 42 vertices.

The goal in the end is to reduce the set of graphs that need testing to become feasible for computation. By using LONI and reducing the number of graphs needing to be tested, eventually the two should meet when $R(5,5)$ will be found.

APPENDIX A

ALGORITHMS FOR RAMSEY GRAPHS

Our first algorithm tests if G is in $\mathscr{R}(\omega,\alpha)$, determining if G contains a subgraph that is isomorphic to a clique of size ω. Algorithm A.1 determines if graph G has a K_ω for some ω vertices in $V(G)$. The general form of this algorithm searches every subset $S \subseteq V(G)$ such that $|S| = \omega$ to determine if the induced subgraph S of G is isomorphic to K_ω or N_α in G.

Algorithm A.1 Algorithm to test K_ω.

```
FUNCTION TestKw( graph G, int ω )
  int S[ω];
  bool flag;
  // for every set of ω vertices in V(G) assign to v
  for every S ⊆ V(G) of size ω do
    flag = true;
    for i = 0 to ω-2 do
      for j = i+1 to ω-1 do
        if S[i]S[j] ∉ E(G) then
          flag = FALSE;                          // v is not K_ω
        end if
      end for
    end for
    if flag then
      return TRUE;                               // v is K_ω
    end if
  end for
  return FALSE;                                  // No K_ω in G
ENDFUNCTION
```

We can increase optimization of this program by reducing the search time for a small clique. For example, consider a graph G such that $|V(G)| = 15$ to determine if G contains a triangle. If we used the general algorithm to test for K_3, then we would need to look through $\binom{15}{3} = 455$ sets of vertices to confirm that no triangles exist. Since every triangle in G must contain an edge e in $E(G)$, we can limit our search to edges in $E(G)$ because any two nonadjacent vertices cannot be part of the same triangle. For each edge $e \in E(G)$, $e = v_1v_2$, we need to compare the intersection of the neighborhoods, $N(v_1) \cap N(v_2)$. If this intersection is not empty, then there is a triangle in G. Conversely, if the intersection is empty, then the edge e is not part of a triangle in G. By limiting the test to only edges in $E(G)$, a maximum of $\frac{15 \cdot 14}{2} = 105$ comparisons can save unnecessary computation time while also creating a list of K_3-subgraphs. The algorithm described here is depicted in Algorithm A.2. A similar algorithm can be developed to test if a graph is K_4-free by looking for a triangle in the induced subgraphs of the neighborhoods of each vertex since $K_4 - v \cong K_3$.

Algorithm A.2 Algorithm to test K_3-free.

```
FUNCTION TestK3-free( graph G )
  // for every edge in E(G), test if part of triangle
  for every e = v1v2 ∈ E(G) do
    N(v1 ∩ v2) = N(v1) ∩ N(v2);
    if N(v1 ∩ v2) ≠ φ then
      return TRUE;                                  // e is part ofK3
    end if
  end for
  return FALSE;                                     // No K3 in G
ENDFUNCTION
```

The test for the independence number in Algorithm A.3 is equivalent to test for a clique in the complement of a graph. An easy improvement is to modify the code for testing K_ω to test for N_α and not wasting computation time on creating a temporary complement of the graph.

Algorithm A.3 Algorithm to test N_α.

```
FUNCTION TestNa( graph G, int α )
  return TestKw( G̅, α );                            // α(G) = ω(G̅)
ENDFUNCTION
```

To determine if $G \in \mathscr{R}(\omega, \alpha)$, neither a K_ω nor N_α can be present in G. Algorithm A.4 tests both of these conditions to determine if $G \in \mathscr{R}(\omega, \alpha)$. If either K_ω or N_α is found in graph G, then *FALSE* is returned since $G \notin \mathscr{R}(\omega, \alpha)$; otherwise, when neither is found *TRUE* is returned. This function is used to build the complete list of graphs in $\mathscr{R}(\omega, \alpha)$.

Algorithm A.4 Algorithm to test if graph is in $\mathscr{R}(\omega,\alpha)$.

FUNCTION TestRwa(graph G, int ω, int α)
  if TestKw(G, ω) OR TestNa(G, α) **then**
    return TRUE; // K_ω or N_α in G
  else
    return FALSE; // $G \in \mathscr{R}(\omega,\alpha)$
  end if
ENDFUNCTION

The complete list for $\mathscr{R}(\omega,\alpha)$ can be found by testing all graphs up to $R(\omega,\alpha)$ vertices. When $R(\omega,\alpha)$ is unknown, finding the complete list can be done from starting at one vertex. After finding all Ramsey graphs in a vertex class, the list for the next vertex class can be generated. When a vertex class n is reached that does not have a graph in $\mathscr{R}(\omega,\alpha)$ then the list is complete. Corollary 1.4.5 states that the first vertex class n where $\mathscr{R}(\omega,\alpha)$ is empty implies that the Ramsey number $R(\omega,\alpha) = n$.

The algorithms so far test an individual graph. Since many graphs must be tested to build $\mathscr{R}(\omega,\alpha)$, several graphs can be tested simultaneously. This is the basis of the parallel algorithm used to find the complete list of $\mathscr{R}(\omega,\alpha)$ graphs.

APPENDIX B

ALL TREES WITH $\Delta(T) \leq 3$ AND $|V(G)| \leq 9$

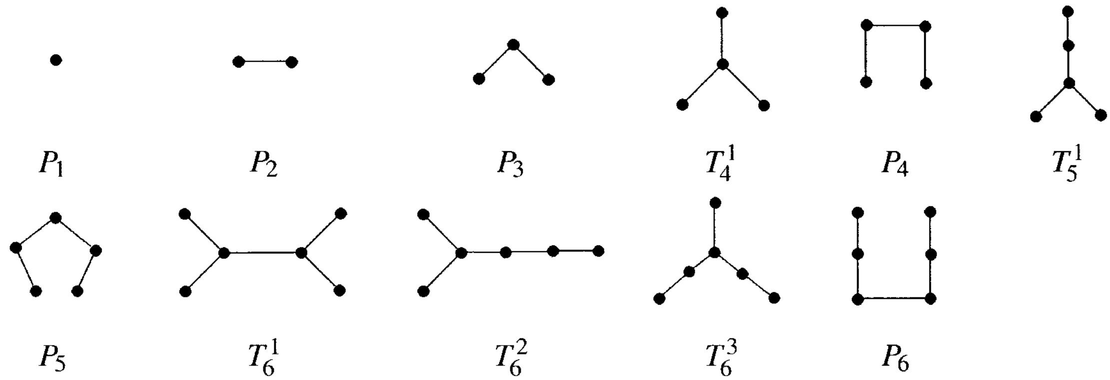


Figure B.1: All trees up to six vertices with $\Delta(T) \leq 3$.

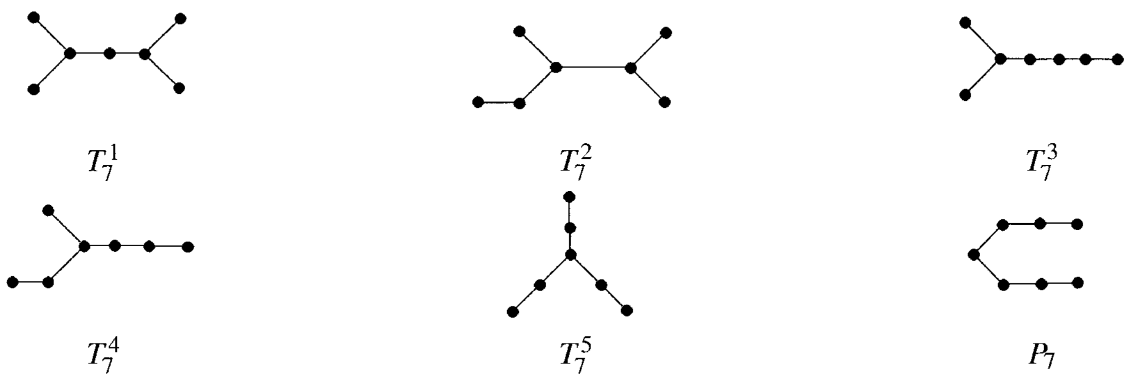


Figure B.2: Trees on seven vertices with $\Delta(T) \leq 3$.

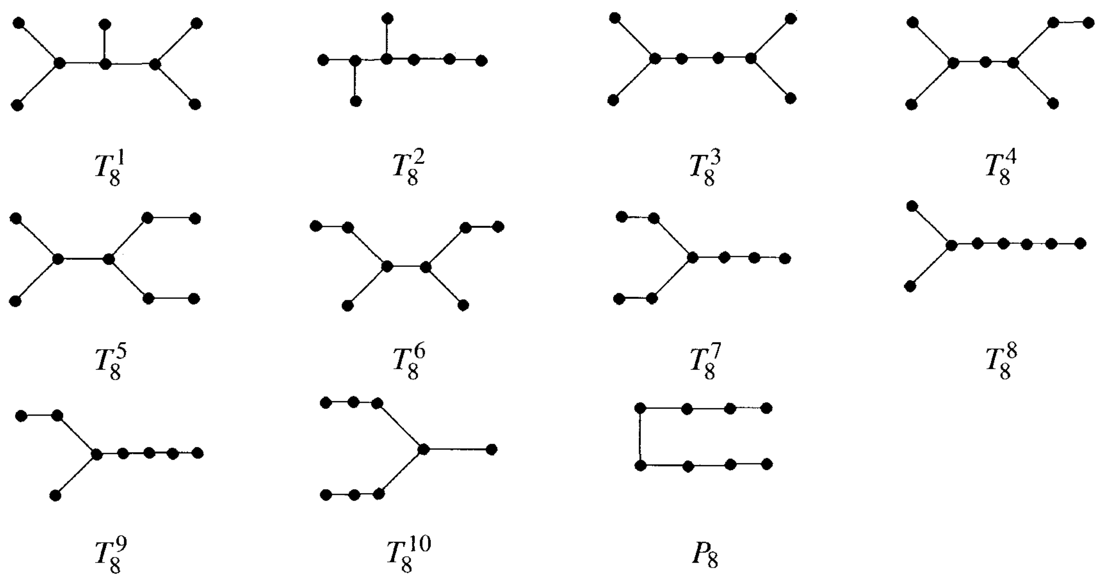


Figure B.3: Trees on eight vertices with $\Delta(T) \leq 3$.

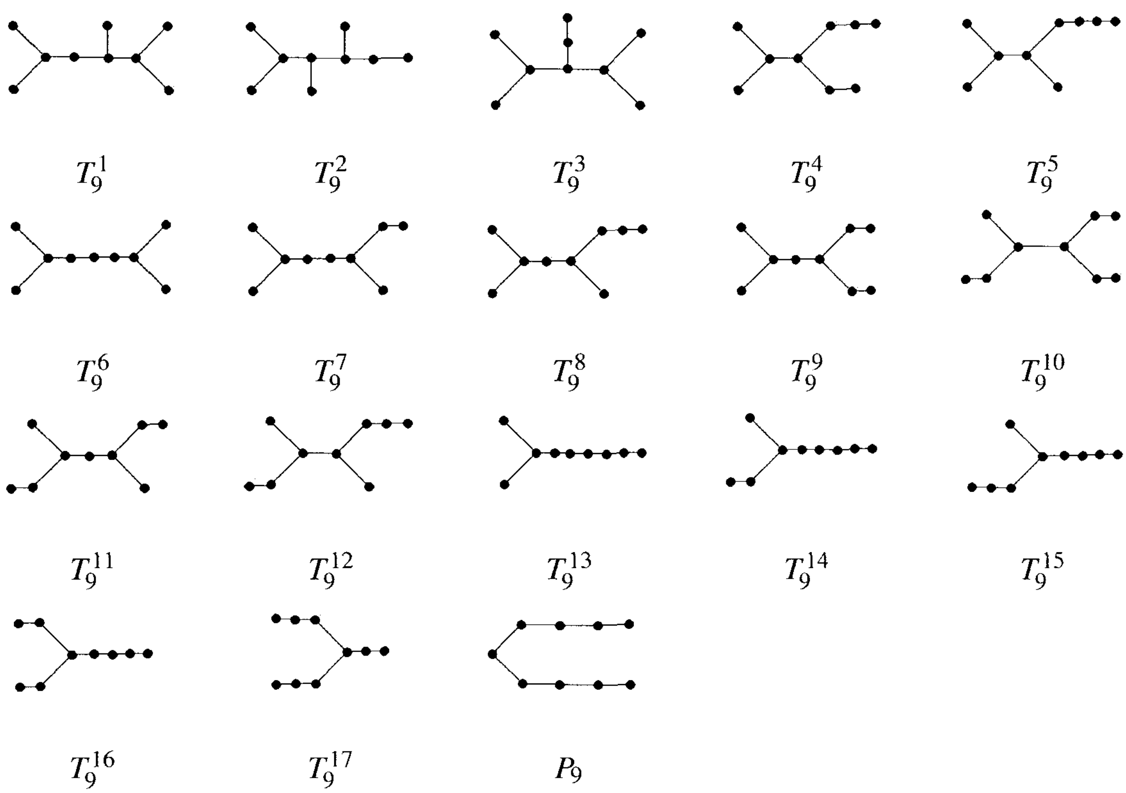


Figure B.4: Trees on nine vertices with $\Delta(T) \leq 3$.

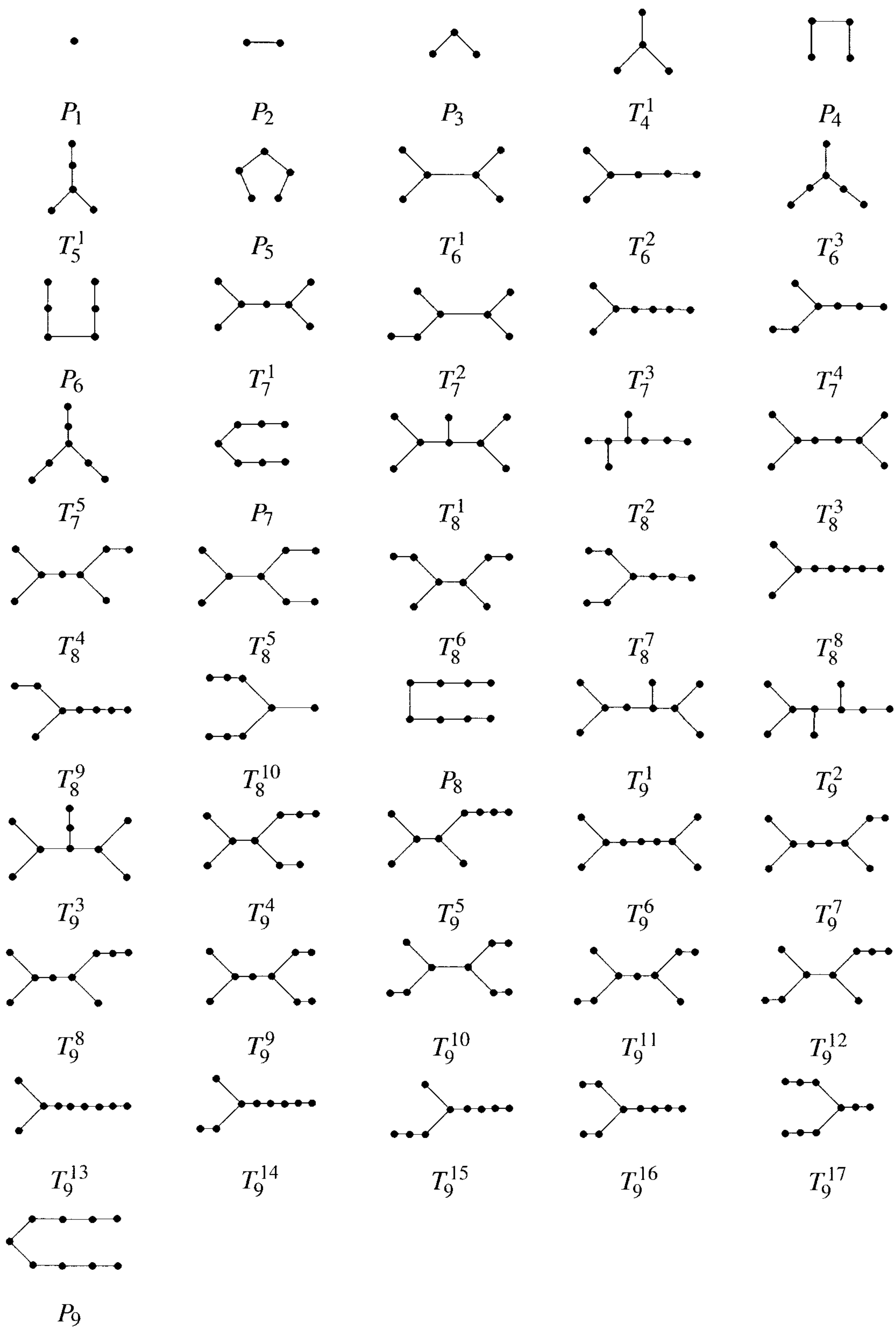


Figure B.5: All trees up to 9 vertices with $\Delta(T) \leq 3$.

APPENDIX C

TABLES OF RAMSEY GRAPHS

Table C.1: Distribution of $\mathscr{R}(3,4)$ graphs. R(3,4) = 9.

R(3,4)											
	Vertices										
		1	2	3	4	5	6	7	8	9	Total
	0	1	1	1							3
E	1		1	1	1						3
d	2			1	2	1					4
g	3				2	2	1				5
e	4				1	3	1				5
s	5					2	4				6
	6					1	4	1			6
	7						3	2			5
	8						1	3			4
	9						1	2			3
	10							1	1		2
	11								1		1
	12								1		1
Total		1	2	3	6	9	15	9	3	0	48

Table C.2: Distribution of $\mathscr{R}(3,5)$ graphs. R(3,5) = 14.

R(3,5)																
	Vertices															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
	0	1	1	1	1											4
E	1		1	1	1	1										4
d	2			1	2	2	1									6
g	3				2	3	3	1								9
e	4				1	4	6	2	1							14
s	5					2	8	7	1							18
	6					1	7	13	5							26
	7						4	17	13	1						35
	8						2	15	27	3						47
	9						1	10	39	11						61
	10							4	41	28	1					74
	11							1	27	59	2					89
	12							1	15	73	10					99
	13								6	62	32					100
	14								2	33	69					104
	15								1	14	86	1				102
	16								1	4	65	6				76
	17									2	32	19				53
	18										12	31				43
	19										3	30				33
	20										1	13	1			15
	21											4	2			6
	22											1	5			6
	23												2			2
	24												2			2
	25															0
	26													1		1
Total		1	2	3	7	13	32	71	179	290	313	105	12	1	0	1029

Table C.3: Distribution of $\mathscr{R}(3,6)$ graphs. R(3,6) = 18.

R(3,6)														
	Vertices													
		···	7	8	9	10	11	12	13	14	15	16	17	Total
	0	···												5
E	1	···												5
d	2	· ·	1											8
g	3	···	3	1										13
e	4	···	7	3	1									23
s	5	···	13	10	2	1								37
	6		20	23	8	1								60
	7	···	20	44	26	5								99
	8	· ·	18	63	70	16	1							170
	9	···	11	73	142	60	3							290
	10	··	5	63	234	175	16							493
	11	···	1	40	284	451	64	1						841
	12	· ·	1	21	267	864	265	4						1422
	13	···		9	185	1255	900	20						2369
	14	·		3	106	1344	2353	119						3925
	15	···		2	47	1114	4444	644	1					6252
	16	···		1	22	707	6134	2693	4					9561
	17	···			8	377	6239	7968	45					14637
	18	··			3	167	4823	16445	375					21813
	19	···			1	71	2885	23986	2402					29345
	20	·			1	28	1405	25267	10176	1				36878
	21	···				13	565	19704	27975	16				48273
	22					4	206	11672	51188	177				63247
	23	···				2	64	5404	64221	1588				71279
	24	··				1	20	2016	56809	8494				67340
	25	···				1	6	630	36312	27013	1			63963
	26	···					2	169	17208	53157	7			70543
	27	···						41	6189	67224	101			73555
	28	···						8	1729	56478	822			59037
	29	···						1	377	32235	3998			36611
	30	···							66	12784	10910			23760
	31	···							8	3550	17552			21110
	32	···							1	699	16896	5		17601
	33	···								94	9957	39		10090
	34									9	3587	200		3796
	35	···								1	794	547		1342
	36	··									100	803		903
	37	···									7	634		641
	38	· ·										275		275
	39	···										62		62
	40											11	2	13
	41	···											3	3
	42	· ·											2	2
Total		···	100	356	1407	6657	30395	116792	275086	263520	64732	2576	7	761692

Table C.4: Distribution of $\mathscr{R}(4,4)$ graphs. R(4,4) = 18.

	R(4,4)												
	Vertices												
		...	8	9	10	11	12	13	14	15	16	17	
	0	...											3
E	1	...											3
d	2	...											4
g	3	...											8
e	4	...											9
s	5	...											15
	6	...											23
	7	...	1										34
	8	...	3										51
	9	...	11	1									83
	10	...	38	1									124
	11	...	111	5									196
	12	...	244	18									320
	13	...	398	73									500
	14	...	467	257	1								737
	15	...	398	768	5								1175
	16	...	244	1719	34								1998
	17	...	111	2831	177								3119
	18	...	38	3355	814								4207
	19	...	11	2831	2963	1							5806
	20	...	3	1719	8193	7							9922
	21	...	1	768	16396	72							17237
	22	...		257	23270	546							24073
	23	...		73	23270	3201							26544
	24	...		18	16396	13695							30109
	25	...		5	8193	41553							49751
	26	...		1	2963	87361	8						90333
	27	...		1	814	126742	177						127734
	28	...			177	126742	1906						128825
	29	...			34	87361	13332						100727
	30	...			5	41553	58131						99689
	31	...			1	13695	163757						177453
	32	...				3201	302088						305289
	33	...				546	370368	20					370934
	34	...				72	302088	535					302695
	35	...				7	163757	6339					170103
	36	...				1	58131	37825					95957
	37	...					13332	127138					140470
	38	...					1906	257711					259617
	39	...					177	325095					325272
	40	...					8	257711					257719
	41	...						127138	40				127178
	42	...						37825	872				38697
	43	...						6339	6247				12586
	44	...						535	20901				21436
	45	...						20	37348				37368
	46	...							37348				37348
	47	...							20901				20901
	48	...							6247				6247
	49	...							872				872
	50	...							40	13			53
	51	...								96			96
	52	...								211			211
	53	...								211			211
	54	...								96			96
	55	...								13			13
	56	..	...	...	...	...	...	...	...	...	...	...	...
Total		...	2079	14701	103706	546356	1449166	1184231	130816	640	2	1	3432184

APPENDIX D

RAMSEY GRAPHS FOR $\mathscr{R}(3,4)$

$K_1 \cong G^1_{1,0}$ $K_2 \cong G^1_{2,1}$

Figure D.1: List of connected $\mathscr{R}(3,4)$ graphs with $\alpha(G) = 1$.

$P_3 \cong G^1_{3,2}$ $P_4 \cong G^2_{4,3}$ $C_4 \cong G^1_{4,4}$ $C_5 \cong G^2_{5,5}$

Figure D.2: List of connected $\mathscr{R}(3,4)$ graphs with $\alpha(G) = 2$.

$G^1_{3,0}$ $G^1_{4.1}$ $G^1_{5.2}$ $G^1_{6,3}$

$G^1_{2.0}$ $G^1_{3.1}$ $G^1_{4,2}$

$G^2_{4,2}$ $G^1_{5,3}$ $G^2_{5,3}$ $G^1_{5.4}$

$G^1_{6,4}$ $G^1_{6.5}$ $G^2_{6,5}$ $G^1_{7.6}$

Figure D.3: List of disconnected $\mathscr{R}(3,4)$ graphs.

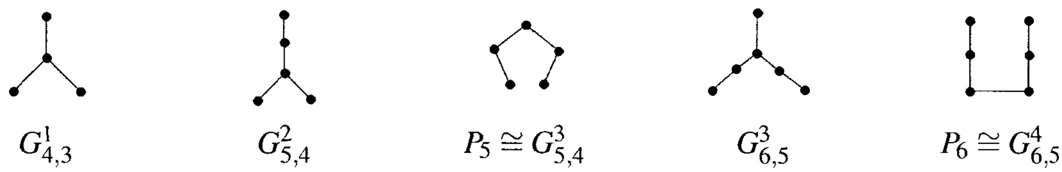


Figure D.4: List of $\mathscr{R}(3,4)$ trees with $\alpha(G) = 3$.

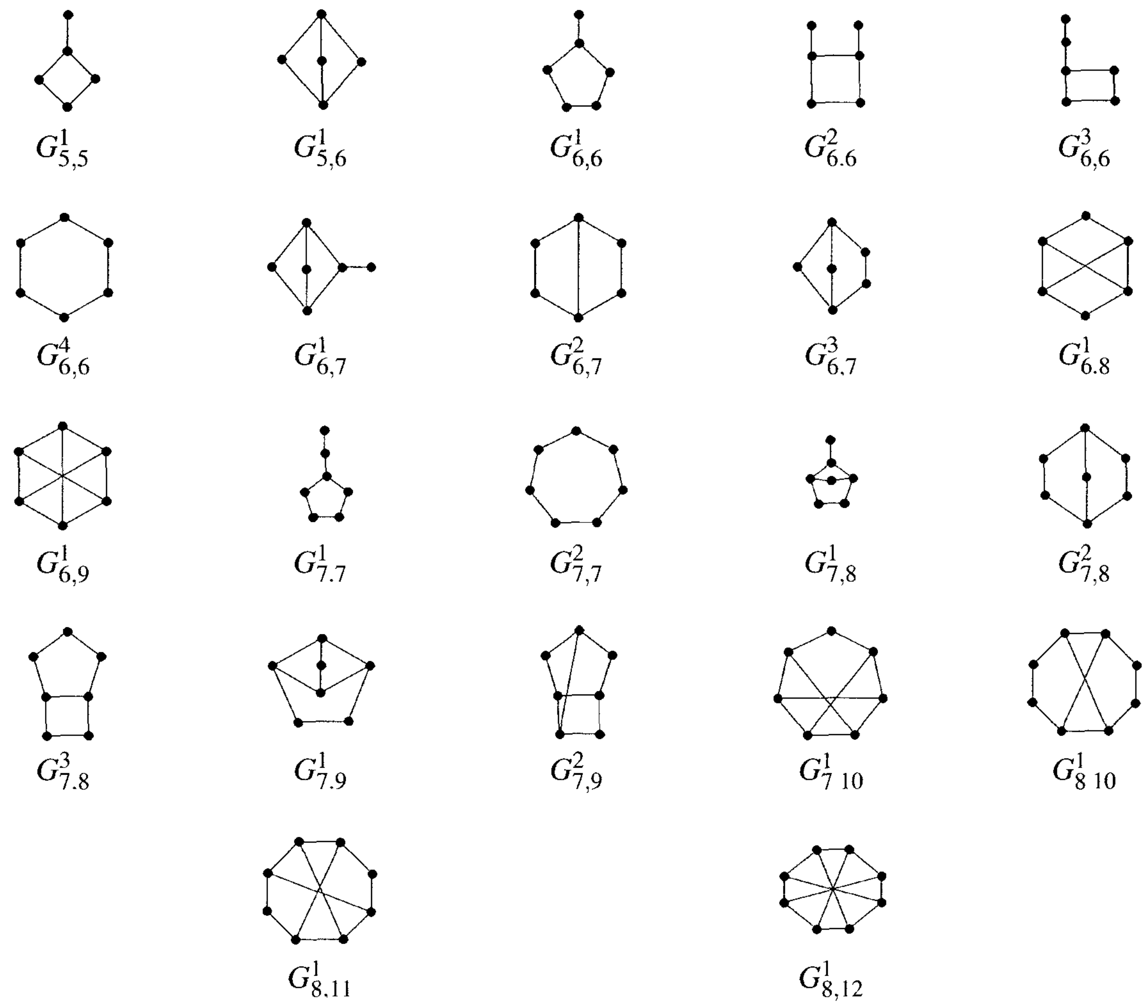


Figure D.5: List of connected$\mathscr{R}(3,4)$ graphs that are not trees and $\alpha(G) = 3$.

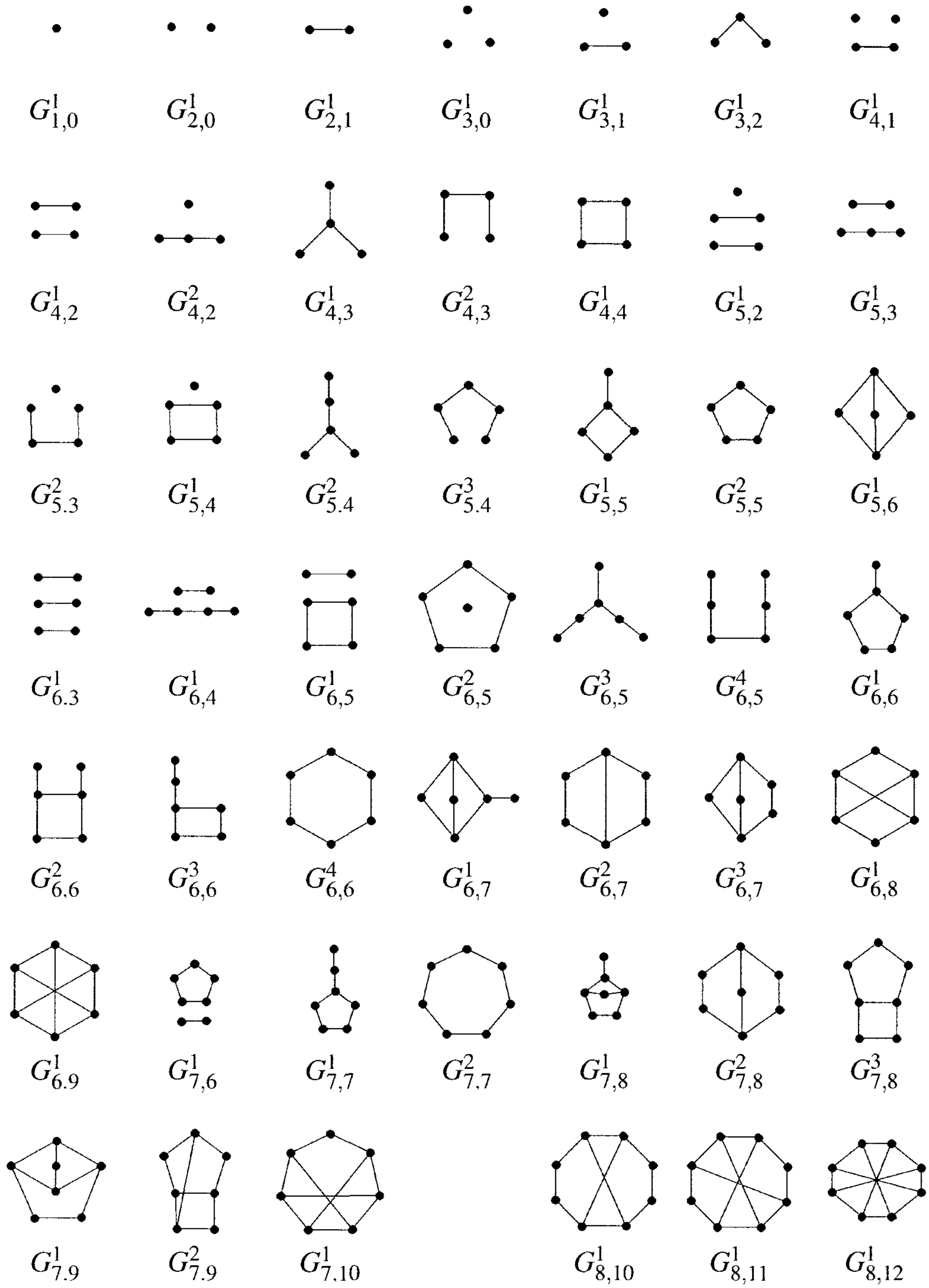


Figure D.6: Complete list of graphs in $\mathscr{R}(3,4)$.

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