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Embedding oriented graphs in books

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EMBEDDING ORIENTED GRAPHS IN BOOKS

by

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ABSTRACT

A *book* consists of a line L in $\mathbb{R}^3$, called the *spine*, and a collection of half planes, called *pages*, whose common boundary is L. A k-*book* is book with k pages. A k-page book embedding is a continuous one-to-one mapping of a graph G into a book such that the vertices are mapped into L and the edges are each mapped to either the spine or a particular page, such that no two edges cross in any page. Each page contains a planar subgraph of G. The *book thickness*, denoted $bt(G)$, is the minimum number of pages for a graph to have a k-page book embedding. We focus on oriented graphs, and propose a new way to embed oriented graphs into books, called an *oriented book embedding*, and define *oriented book thickness*.

We investigate oriented graphs having oriented book thickness k using k-*page critical oriented graphs*, oriented graphs with oriented book thickness equal to k, but, for each arc, the deletion of that arc yields an oriented graph with oriented book thickness equal to $k-1$. We discuss several classes of two-page critical oriented graphs, and use them to characterize oriented graphs with oriented book thickness equal to one that are *strictly uni-dicyclic graphs*, oriented graphs having exactly one cycle, which is a directed cycle. We give a similar result for *strictly bi-dicyclic graphs*, oriented graphs having exactly two cycles, which are directed cycles.

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CHAPTER 1

INTRODUCTION

Graph theory originated in 1735 with the Königsberg Bridge Problem, a problem in which Leonard Euler attempted to determine if it were possible to find a path along seven bridges without recrossing a bridge. From its recreational beginnings, graph theory then became its own branch of study in mathematics with applications in computer science, operations research, and the social sciences. Graph theory also contains some famous problems including the Traveling Salesman Problem and the Four Color Theorem.

One branch of graph theory, network theory, studies relationships between discrete objects. Examples of study in network theory include computer networks, traffic networks, and social networks. To study these relationships, we focus on *oriented graphs* and follow the definition of West [17]. Oriented graphs can be more useful than un-oriented graphs in situations where order is important, such as job scheduling, in which one job must be completed before another.

In this dissertation, we characterize certain classes of graphs having *oriented book thickness* k, for an integer k. We do this using forbidden graphs, which we call k-page-critical graphs. The analogous problem for un-oriented graphs, simply called *book thickness*, has been studied extensively, and is equivalent to the *stack number* of a graph, which has many applications in computer science.

In this chapter, we provide the basic definitions of graphs and oriented graphs, some basic classes of graphs and oriented graphs, and give a general overview of the dissertation. Unless otherwise stated, terminology will follow that of West [17].

1.1 Basic Definitions

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of a nonempty, finite vertex set $V(G)$ and a finite edge set $E(G)$. The cardinality of $V(G)$ is called the *order* of G, and the cardinality of $E(G)$ is called the *size* of G. Each edge $e \in E(G)$ is associated with a nonempty set of at most two vertices in $V(G)$, called the *endpoints* of e; different edges can be associated with the same set of endpoints. For an edge $e \in E(G)$ with endpoints $x, y \in V(G)$, we say that e is *incident* to both x and y, and that x and y are *adjacent*. For a vertex $x \in V(G)$, the subset of $V(G)$ consisting of the vertices adjacent to x is called the *neighborhood* of x, denoted $N(x)$. For a graph G, a *loop* is an edge in $E(G)$ whose endpoints are not distinct. Edges which have exactly the same set of endpoints in G are called *multiple edges.* A graph is *simple* if it contains no loops or multiple edges. *All graphs in this dissertation will be simple graphs.* For a simple graph G, if $e \in E(G)$ is associated with vertices x and y, we write $e = \{x, y\}$ or $e = \{y, x\}$. The *degree* of a vertex $x \in V(G)$ is the cardinality of $N(x)$, denoted $|N(x)|$. A *subgraph* of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$ such that, for each edge $e \in E(G')$, it must be true that $e \in E(G)$, and both endpoints of e are in $V(G')$. A *proper subgraph* of G is a subgraph G'

such that $V(G') \subsetneq V(G)$ or $E(G') \subsetneq E(G)$. An *isomorphism* from a graph G to a graph H is bijection $\gamma : V(G) \to V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\gamma(u), \gamma(v)\} \in E(H)$. Then we say that G and H are *isomorphic.*

An *oriented graph* $\vec{D}$ is an ordered pair $(V(\vec{D}), A(\vec{D}))$ consisting of a nonempty, finite vertex set $V(\vec{D})$ and a finite arc set $A(\vec{D})$; each arc $\vec{a} \in A(\vec{D})$ is associated with exactly one ordered pair of distinct vertices in $V(\vec{D})$ such that no two arcs are associated with the same pair of vertices. For an arc $\vec{a} = (u, v)$, we call u the *tail* of $\vec{a}$ and v the *head* of $\vec{a}$, and we say that u *is directed* to v. We say that u and v are the *endpoints* of $\vec{a}$. The endpoints of an arc are *adjacent* and an arc is *incident* to both of its endpoints. For an oriented graph $\vec{D}$, if we remove the directions of the arcs, yielding edges, we obtain a simple graph, which we call the *underlying graph* of $\vec{D}$. If an oriented graph $\vec{D}$ has underlying graph G, we say that $\vec{D}$ is an *orientation* of G.

Since we restrict all undirected graphs in this dissertation to be simple, we focus on oriented graphs, rather than directed graphs. The definition of an oriented graph is slightly different from the definition of a directed graph (see [5]); the underlying graph of a oriented graph must be simple, which is not always the case for a directed graph. Figure 1.1 depicts an oriented graph and a directed graph.

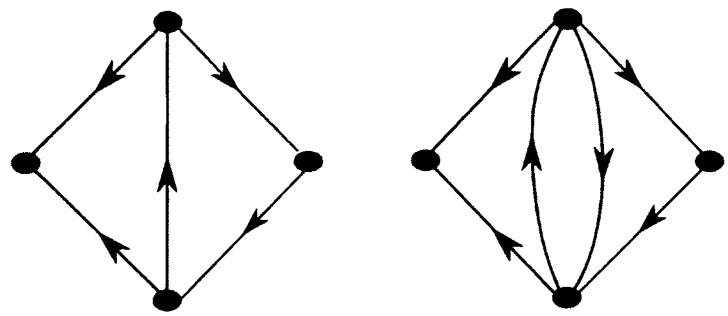

Figure 1.1: An oriented graph (left) and a directed graph (right).

An *oriented subgraph* of an oriented graph $\vec{D}$ is an oriented graph $\vec{D'}$ such that $V(\vec{D'}) \subseteq V(\vec{D})$, $A(\vec{D'}) \subseteq A(\vec{D})$, and for each arc $\vec{a} = (u, v) \in A(\vec{D'})$, it must be true that $\vec{a} \in A(\vec{D})$ and $\{u, v\} \subseteq V(\vec{D'})$. A *proper oriented subgraph* of an oriented graph $\vec{D}$ is an oriented subgraph $\vec{D'}$ of $\vec{D}$ such that either $V(\vec{D'}) \subsetneqq V(\vec{D})$ or $A(\vec{D'}) \subsetneqq A(\vec{D})$.

For an oriented graph $\vec{D}$ containing an arc (u, v), we say that u is an *in-neighbor* of v and that v is an *out-neighbor* of u. For a vertex $u \in V(\vec{D})$, we denote the set of all in-neighbors of u as $N^-(u)$ and the set of all out-neighbors of u as $N^+(u)$. We also say that the *in-degree* of u is the cardinality of $N^-(u)$, denoted $|N^-(u)|$, and the *out-degree* of u is the cardinality of $N^+(u)$, denoted $|N^+(u)|$. The *neighborhood* of u is defined to be $N(u) := N^+(u) \cup N^-(u)$, and the *degree* of u is $|N^+(u)| + |N^-(u)|$. If $N(u) = N^-(u)$, then u is said to be a *sink*; if $N(u) = N^+(u)$, then u is said to be a *source*.

For a graph G with edge e, the *deletion of* e is an operation that yields a graph with edge set $E(G) \setminus \{e\}$ and vertex set $V(G)$; for simplicity, we will simply denote the graph G with e deleted as $G \setminus e$. The deletion of an arc $\vec{a}$ in an oriented graph $\vec{D}$ is defined analogously and is written $\vec{D} \setminus \vec{a}$. The *converse* of an arc $\vec{a} = (u, v)$ is the arc (v, u), denoted $\vec{a}^*$. If we replace an arc $\vec{a}$ with its converse $\vec{a}^*$, we obtain an oriented graph with an arc set $A(\overrightarrow{D} \setminus \overrightarrow{a}) \cup \{\overrightarrow{a}^*\}$ and a vertex set $V(\overrightarrow{D})$; we denote the resulting oriented graph $\overrightarrow{D}(\overrightarrow{a}^*)$. We call this replacement *switching the direction of the arc* $\overrightarrow{a}$. If we switch the direction of every arc in an oriented graph $\overrightarrow{D}$, we call the resulting oriented graph the *converse* of $\overrightarrow{D}$ and denote it $\overrightarrow{D}^*$. For a graph G with vertex v, the *deletion of* v is an operation that yields a graph with vertex set $V(G) \setminus \{v\}$ and edge set $E(D) \setminus \{e_1, e_2, ..., e_k\}$, such that e_i, $1 \leq i \leq k$, is incident to v. We denote the graph G with vertex v deleted as $G - v$. The deletion of a vertex v in an oriented graph $\overrightarrow{D}$ is defined analogously and is denoted $\overrightarrow{D} - v$.

For $n \in \mathbb{N}$, let $[n]$ denote $\{1, 2, ..., n\}$. We use this notation to define the following types of graphs and oriented graphs. Later, for convenience, we may refer to a graph having only one vertex and no edges as a *trivial path*.

Definition 1.1. *For $n \geq 2$, the standard path, denoted $\boldsymbol{P}_n$, is the graph with $V(\boldsymbol{P}_n) = [n]$ and $E(\boldsymbol{P}_n) = \{\{i, i+1\} \mid i \in [n-1]\}$. An n-path is a graph isomorphic to $\boldsymbol{P}_n$, for some n; a path is an n-path, for some n. We denote a path P_n, or simply, P. For a path P_n, we call the vertices in P_n corresponding to the vertices* 1 *and n in $\boldsymbol{P}_n$ the endpoints of P_n. For a path P_n with endpoints x and y, we may also call P_n an x, y-path.*

A graph G is *connected* if, for each pair of vertices $u, v \in V(G)$, there exists a subgraph of G that is a path with endpoints u and v. If a graph is not connected, it is *disconnected*; a maximal connected subgraph of a graph G is called a *component* of G.

Definition 1.2. *For $n \geq 3$, the standard cycle, denoted $\boldsymbol{C_n}$, is the graph with $V(\boldsymbol{C_n}) =$ $[n]$ and $E(\boldsymbol{C_n}) = E(\boldsymbol{P_n}) \cup \{1, n\}$. An n-cycle is a graph isomorphic to $\boldsymbol{C_n}$, for some n; a cycle is an n-cycle, for some n. We denote a cycle C_n, or simply, C.*

For a graph G which contains a cycle C as a subgraph, a *chord* of C is an edge $e = \{x, y\}$ in G such that $x, y \in V(C)$, but $e \notin E(C)$.

We call an oriented graph whose underlying graph is a path or a cycle an *oriented path* or *oriented cycle*, respectively.

Definition 1.3. *The standard directed path, denoted $\overrightarrow{\boldsymbol{P_n}}$, is the oriented graph such that $V(\overrightarrow{\boldsymbol{P_n}}) = [n]$ and $A(\overrightarrow{\boldsymbol{P_n}}) = \{(i, i+1) \mid i \in [n-1]\}$. An n-directed path, or n-dipath, is an oriented graph isomorphic to $\overrightarrow{\boldsymbol{P_n}}$; a directed path, or dipath, is an n-dipath, for some n.*

Definition 1.4. *The standard directed cycle, denoted $\overrightarrow{\boldsymbol{C_n}}$, is the oriented graph with $V(\overrightarrow{\boldsymbol{C_n}}) = [n]$ and $A(\overrightarrow{\boldsymbol{C_n}}) = A(\overrightarrow{\boldsymbol{P_n}}) \cup (n, 1)$. An n-directed cycle, or n-dicycle, is an oriented graph isomorphic to $\overrightarrow{\boldsymbol{C_n}}$ (for some n); a directed cycle, or dicycle is an n-dicycle, for some n. We denote a cycle $\overrightarrow{C_n}$, or simply, $\overrightarrow{C}$.*

For $n \in \mathbb{N}$, and $n \geq 3$, the *standard complete graph*, denoted $\mathbf{K}_n$, is the graph with $V(\mathbf{K}_n) = [n]$ and $E(\mathbf{K}_n) = \{\{i, j\} \mid i, j \in V(G), i \neq j\}$. A *complete graph on n vertices* is a graph isomorphic to $\mathbf{K}_n$; we sometimes denote a complete graph by K_n. A *complete graph* is a complete graph on n vertices. A *tournament* is an orientation of a complete graph, for some n. The *standard complete bipartite graph*, denoted $\mathbf{K}_{m,n}$, is the graph where $V(\mathbf{K}_{m,n}) = [m+n]$ and edge set $E(\mathbf{K}_{m,n}) = \{\{i, j\} \subset [m+n] \,|\, i \leq m$ and $m < j\}$. A *complete bipartite graph* is a graph isomorphic to $\mathbf{K}_{m,n}$, for some

$m, n \in \mathbb{N}$; we sometimes denote a complete bipartite graph by $K_{m,n}$. A *bipartite graph* is a graph whose vertex set can be partitioned into two sets such that no two vertices in the same set are adjacent.

For a graph G, a *subdivision* of an edge $\{u, v\}$ in G is the replacement of $\{u, v\}$ with a u, v-path on three vertices, through a vertex w such that $w \notin V(G)$. A *subdivision of* G is a graph obtained from a graph G by successive subdivisions. A *contraction* of an edge $\{u, v\}$ is the replacement of u and v with a single vertex which is incident to the edges other than $\{u, v\}$ that were incident to u or v. Since we focus on simple graphs, if a contraction yields a set of multiple edges, we delete all but one edge of the set.

1.2 Embedding Graphs into $\mathbb{R}^2$

In this section we define an *embedding* of a graph in $\mathbb{R}^2$ and give a known result for graphs that have an embedding in $\mathbb{R}^2$. We assume the reader is familiar with $\mathbb{R}^1$ and $\mathbb{R}^2$, endowed with the usual topologies induced by the usual Euclidean metrics: for $x, y \in \mathbb{R}^1$, $d(x, y) = |x - y| = \sqrt{(x-y)^2}$ and for $x, y \in \mathbb{R}^2$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. We first define a *simple arc* on $\mathbb{R}^2$ as a subset $S \subseteq \mathbb{R}^2$ such that there exists a one-to-one continuous function, $f : [0, 1] \to S$, from $[0, 1]$ onto S; we call $f(0)$ and $f(1)$ the *boundary points* of the simple arc.

Definition 1.5. *A graph G has an embedding in $\mathbb{R}^2$ if there exists a graph G' on $\mathbb{R}^2$, with $G \cong G'$, such that $V(G') \subseteq \mathbb{R}^2$ and $E(G')$ is a set of simple arcs, in which the endpoints of each edge are the boundary points of the corresponding simple arc. If G*

has an embedding on $\mathbb{R}^2$ such that two edges intersect only at a common endpoint, then we say that G has a planar embedding in $\mathbb{R}^2$ and we call G a planar graph.

Not every graph is a planar graph, in [13], Kuratowski characterizes all planar graphs by the following result:

Theorem 1.1. (*Kuratowski, [13]*) *A graph G is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.*

We now give a few well known properties for planar graphs. For a planar graph G with order n, the size of G is bounded by $3n - 6$ [17]. Also, for a planar graph G, performing an arbitrary number of subdivisions to the edges of G will always result in a planar graph. Similarly, an arbitrary number of contractions of the edges of a planar graph G will result in a planar graph.

The complement of the image of a plane graph in $\mathbb{R}^2$ is a family of open regions, called *faces*, each homeomorphic to an open disk. For a plane graph G, one face is unbounded, and is referred to as the *outer face* of the graph. An *outerplanar graph* is a planar graph which has an embedding in $\mathbb{R}^2$ such that each vertex lies on the boundary of the outer face. For an outerplanar graph G of order n, the size of G is bounded above by $2n - 3$ [17]. As a corollary of Kuratowski's Theorem, Chartrand and Harary [3] proved a similar characterization for outerplanar graphs.

Corollary 1.1. (*Chartrand, Harary [3]*) *graph G is outerplanar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_4 or $K_{2,3}$.*

1.3 Book Embeddings and Oriented Book Embeddings

We now describe a type of embedding that will be focused on in this dissertation. We first define the surface itself, called a *book*. Book embeddings are formally defined by Bernhart and Kainen [1], and we follow their definition, shown below.

Definition 1.6. *For an integer $k \geq 0$, a k-book, or a book with k pages, consists of a line L in $\mathbb{R}^3$, called the spine, and k closed half planes, called pages, whose common boundary is L. If $k = 0$, a k-book is simply the line L. The interior of a page p is the open half plane, $p \setminus L$. A book is a k-book for some k.*

Definition 1.7. *A graph G has a k-page book embedding in a book $\mathbb{B}$ if there exists a graph G', with $G \cong G'$ such that:*

(i) each vertex $v \in V(G')$ is a distinct point in L;

(ii) each edge $e \in E(G')$ is either a simple arc in L, or is a simple arc in the interior of a single page;

(iii) two edges in L, or two edges in the interior of the same page, may intersect only at a common endpoint.

A book embedding is a k-book embedding, for some k. A particular book embedding of a graph G is denoted $e(G)$.

Convention 1.1. *For a graph G with $|V(G)| = n$ and book embedding $e(G)$, we identify the spine with the z-axis and place the vertices at the integers $[n]$ along the positive z-axis. We then label the vertices $v_1, v_2, \ldots, v_n$ such that for $i \in [n]$, the vertex v_i is placed at i on the z-axis. This ordering of the vertices is called the spine order of the book embedding.*

Using Convention 1.1, if G is a graph with book embedding $e(G)$, we say that a vertex v_i is *below* v_j *in* $e(G)$ and that v_j is *above* v_i *in* $e(G)$ if $i < j$; we say that v_i is *between vertices* v_j *and* v_k *in* $e(G)$ if $j \leq i \leq k$ or $k \leq i \leq j$.

For a graph G with a book embedding $e(G)$, each non-empty page, including the spine, must contain a planar subgraph of G. Thus, if two edges are mapped onto the spine, the images of the edges cannot intersect, except possibly at a common endpoint. For an edge in $e(G)$ that is embedded onto a page p, but is not embedded onto the spine, we say that edge is *properly embedded* in p. For a pair of edges that are both properly embedded onto the same page p, the images of the two edges do not intersect, except possibly at a common endpoint on L. For two edges both properly embedded into the same page p, we say the edges follow the *planarity rule*.

Definition 1.8. *The book thickness of a graph, denoted* $bt(G)$ *is the minimum* k *required for* G *to have a* k*-page book embedding such that every two edges properly embedded onto the same page follow the planarity rule.*

For a graph G, we call an edge in G that is mapped into the spine of $e(G)$ a *tight edge*, and we call an edge in G that is not a tight edge a *loose edge.* We call a vertex v of G a *tight vertex* if it is the common endpoint of exactly two tight edges in $e(G)$; we call v a *half-loose vertex* if it is the endpoint of exactly one tight edge in $e(G)$, and we call v a *loose vertex* if it is the endpoint of no tight edge in $e(G)$. For a graph G, we say that an edge $\{u, v\}$ in G *shields* a vertex $x \in V(G)$ in $e(G)$ if x is between u and v in the spine order of $e(G)$. Now consider two edges e_1, e_2 that are in the same page of a particular book embedding of G. We say that e_2 *shields* e_1 if e_1

shields at least one endpoint of e_2 in the embedding. If e_1 shields exactly one endpoint of e_2 in the embedding, then the two edges must share an endpoint; otherwise, they would cross in the page. We now give a straightforward result, which will be used in later proofs.

Lemma 1.1. *Let G be a graph such that $bt(G) = 1$ and let P_n be a proper subgraph which in and x_1, x_n-path with $n > 1$. In a one-page book embedding of G, if x_1 is shielded by an edge $\{u, v\}$ in the embedding, such that $u, v \notin V(P_n)$, then each vertex in P_n is shielded by $\{u, v\}$ in the embedding.*

Proof. Assume for a contradiction, there exists a one-page book embedding G such that x_1 is shielded by $\{u, v\}$, but there exists at least one vertex in P_n not shielded by $\{u, v\}$. Label the vertices of the path $x_1, x_2, ..., x_n$, $n > 1$, so that P_n has edges $\{x_i, x_{i+1}\}$ for $1 \leq i \leq n - 1$. We may assume that x_k, $1 < k \leq n$ is not shielded by $\{u, v\}$, and each x_j, $1 \leq j \leq k - 1$ is shielded by $\{u, v\}$. Then, one endpoint of $\{u, v\}$ is located between x_{k-1} and x_k in the spine; thus, the edge $\{x_{k-1}, x_k\}$ must be properly embedded in the page of the embedding. However, $\{u, v\}$ and $\{x_{k-1}, x_k\}$ are located in the same page, and since they do not follow the planarity rule, we have a contradiction. □

For small page number, characterizing graphs having book thickness k is relatively straightforward. In 1979, Bernhart and Kainen [1] characterized graphs having book thickness $k = 0, 1, 2$ (see Theorem 1.2). Kainen later wrote a second paper, [2], which gave the book thickness for more families of graphs and discussed applications of finding book thickness. Shortly after, Chung, Leighton, and Rosenburg [4] authored a paper giving more applications, mainly for VLSI design; moreover, they gave bounds for book thickness based on structural properties of certain graphs. In 1989, Mihalis Yanakakis [18] provided an algorithm that embeds all planar graphs into four pages. So far, no such results have been found for graphs having book thickness three.

Theorem 1.2. (*Bernhart and Kainen,* 1979 [*1*]) *Let G be connected, then the following hold:*

1. $bt(G) = 0$ *if and only if G is a path,*
2. $bt(G) \leq 1$ *if and only if G is outerplanar, and*
3. $bt(G) \leq 2$ *if and only if G is a subgraph of a hamiltonian planar graph.*

Definition 1.9. *A graph G is a k-page-critical graph if $bt(G) = k$, and for every edge $e \in E(G)$, we have that $bt(G \setminus e) = k - 1$.*

By a corollary of Kuratowski's Theorem [3] and Theorem 1.2, K_4 and $K_{2,3}$ are two-page-critical graphs. All k-page-critical graphs are connected; otherwise, suppose that G was k-page-critical with two distinct components H_1, H_2, therefore, $bt(G) = max\{bt(H_1), bt(H_2)\} = k$. If $max\{bt(H_1), bt(H_2)\} = bt(H_1)$, then $bt(G) = bt(H_1)$. Delete an edge $e' \in E(H_2)$, thus $bt(G \setminus e') = k$, and G is not a k-page-critical graph.

So far, study has been done on the stack number of an oriented graph, described in [6] and studied for certain families of oriented graphs in [9], [10], and [11]. Also, in [7], Frati, Fulek, and Ruiz-Vargas discuss a function that gives the page number based on connectedness. However, in these works, the authors restricted themselves to *directed acyclic graphs*, oriented graphs having no directed cycle. Our focus is on oriented graphs containing directed cycles, and thus we define an *oriented book embedding* below.

We are now prepared to define an oriented book embedding of an oriented graph $\overrightarrow{D}$. Similarly to a book embedding of a graph G, we will map the vertices to the spine of a book and the arcs to the spine or to at most one of the pages. It is natural to add restrictions to the definition that account for the directions of the arcs. To do so, let $\overrightarrow{D}$ be an oriented graph with n vertices; recall Convention 1.1. For an arc (v_i, v_j) mapped to a particular page of a book, if $i < j$, we say that (v_i, v_j) is an *upwards arc*, if $j < i$, we say that (v_i, v_j) is a *downwards arc*. If two arcs in an embedding are both upwards arcs, or are both downwards arcs, then we say that the two arcs *agree*. If every arc properly embedded in a particular page is a downwards arc, then we say that the page is a *downwards page*. If every arc properly embedded onto a particular page is an upwards arc, then we say that the page is an *upwards page*. We now give the formal definition of an oriented book embedding.

Definition 1.10. *An oriented graph $\overrightarrow{D}$ has a k-page oriented book embedding in a book $\mathbb{B}$ if there exists an oriented graph $\overrightarrow{D}'$, with $\overrightarrow{D} \cong \overrightarrow{D}'$ such that:*

(i) each vertex $v \in V(\overrightarrow{D}')$ is a distinct point in L;

(ii) each arc $\overrightarrow{a} \in A(\overrightarrow{D}')$ is either a simple arc in L, or is a simple arc in the interior of a single page;

(iii) two arcs in L, or two arcs in the interior of the same page, may intersect only at a common endpoint;

(iv) the direction of each pair of arcs in L agree;

(v) each page must either be a downwards page or an upwards page.

An oriented book embedding is a k-page oriented book embedding, for some k. A particular oriented book embedding of an oriented graph $\overrightarrow{D}$ *is denoted* $e(\overrightarrow{D})$.

For an oriented graph $\overrightarrow{D}$ with an oriented book embedding $e(\overrightarrow{D})$, each non-empty page, including the spine, must contain a planar oriented subgraph of $\overrightarrow{D}$. Thus, if two arcs are mapped onto the spine, the images of the arcs cannot intersect, except possibly at a common endpoint. For an arc in $e(\overrightarrow{D})$ that is embedded onto a page p, but is not embedded onto the spine, we say that arc is *properly embedded* in p. For a pair of arcs that are both properly embedded onto the same page p, the images of the two arcs do not intersect, except possibly at a common endpoint on L. For two arcs both properly embedded into the same page p, we say the arcs follow the *planarity rule*.

Definition 1.11. *The oriented book thickness, denoted* $obt(\overrightarrow{D})$, *is the minimum* k *required for* $\overrightarrow{D}$ *to have a* k*-page oriented book embedding such that every two arcs located in the same page follow the planarity rule.*

We do not restrict the direction of all arcs in an oriented book embedding to agree, just arcs embedded onto the spine or those properly embedded onto the same page; an oriented book embedding may have an upwards page and a downwards page. From the definition of oriented book thickness, for an oriented graph $\overrightarrow{D}$ with underlying graph D, we have that $bt(D) \leq obt(\overrightarrow{D})$.

For an oriented graph $\overrightarrow{D}$ with oriented book embedding $e(\overrightarrow{D})$, we call an arc in $\overrightarrow{D}$ that is located in the spine of $e(\overrightarrow{D})$ a *tight arc*, and we call an arc in $\overrightarrow{D}$ that is not a tight arc a *loose arc*. We call a vertex v of $\overrightarrow{D}$ a *tight vertex* if it is the common endpoint of exactly two tight arcs in $e(\overrightarrow{D})$; we call v a *half-loose vertex* if it is the endpoint of exactly one tight arc, and we call v a *loose vertex* if it is the endpoint of no tight arc. We say that an arc (u, v) in $\overrightarrow{D}$ *shields* a vertex $x \in V(\overrightarrow{D})$ if x is between u and v in the spine order of $e(\overrightarrow{D})$. For two arcs $\overrightarrow{a}_1, \overrightarrow{a}_2$ that are in the same page, we say that $\overrightarrow{a}_2$ *shields* $\overrightarrow{a}_1$ if $\overrightarrow{a}_1$ shields at least one endpoint of $\overrightarrow{a}_2$; if $\overrightarrow{a}_1$ shields exactly one endpoint of $\overrightarrow{a}_2$, then the two arcs must share an endpoint; otherwise, they would violate the planarity rule.

Lemma 1.2. *Let $\overrightarrow{D}$ be an oriented graph such that $obt(\overrightarrow{D}) = 1$. In a one-page oriented book embedding of $\overrightarrow{D}$ such that the direction of the page is upwards, if a vertex $x \in V(\overrightarrow{D})$ has out-degree (resp. in-degree) greater than one, then at least one out-neighbor (resp. in-neighbor) is located above (resp. below) x.*

Proof. Assume for a contradiction that in a one-page oriented book embedding of $\overrightarrow{D}$ such that the direction of the page is upwards, a vertex $x \in V(\overrightarrow{D})$ has two out-neighbors that are located below x in the spine. However, since at most one arc can be located in the spine below x, at least one arc with tail x must be located in the page with downwards direction, a contradiction. □

Definition 1.12. *An oriented graph $\overrightarrow{D}$ is a k-page-critical oriented graph if $obt(\overrightarrow{D}) = k$, and for every arc $\overrightarrow{a} \in A(\overrightarrow{D})$, we have that $obt(\overrightarrow{D} \setminus \overrightarrow{a}) = k - 1$. The class of all k-page-critical oriented graphs is denoted $\mathcal{M}^k$.*

Recall that $K_{2,3}$ and K_4 are two-page critical graphs. Let $\mathcal{D}_{2,3}$ be the class of oriented graphs whose underlying graph is $K_{2,3}$ and $\mathcal{D}_4$ be the class of oriented graphs whose underlying graph is K_4. It is true that $\mathcal{D}_{2,3} \subseteq \mathcal{M}^2$, but $\mathcal{D}_4 \nsubseteq \mathcal{M}^2$, which is discussed in Chapter 4.

1.4 Dissertation Overview

We begin Chapter 2 by characterizing oriented graphs having 0-page oriented book thickness using two oriented graphs and the class of directed cycles. Then we focus on the class of *strictly uni-dicyclic graphs*, the class of oriented graphs which contain exactly one oriented cycle, which is a directed cycle. The main result of this chapter, Theorem 2.4, characterizes strictly uni-dicyclic graphs having oriented book thickness equal to one using three classes of strictly uni-dicyclic graphs that are two-page-critical. The content of this chapter has been submitted for publication and is available in [14].

In Chapter 3, we focus on *strictly bi-dicyclic graphs*, the class of oriented graphs that contain exactly two oriented cycles, which are directed cycles. The main result of this chapter, Theorem 3.1, characterizes strictly bi-dicyclic graphs having oriented book thickness equal to one using Theorem 2.4 and five classes of strictly bi-dicyclic oriented graphs that are two-page-critical.

In Chapter 4, we describe a subclass of k-critical graphs, called *switching k-page-critical graphs*, the class of all oriented graphs $\overrightarrow{D}$ such that $obt(\overrightarrow{D}) = k$, but for each arc $a \in A(\overrightarrow{D})$, we have that $obt(\overrightarrow{D}(a^*)) = k - 1$. In this chapter, we discuss two classes of oriented graphs in $\mathcal{M}_s^2$ and one class of oriented graphs in $\mathcal{M}_s^3$. We introduce a class of oriented graphs $\mathcal{C}^d$, which contains all oriented graphs whose underlying graph is outerplanar and consists of a cycle with d chords, and characterize strictly two-page-critical graphs in $\mathcal{C}^1$.

In Chapter 5, we characterize strictly two-page critical graphs in $\mathcal{C}^2$, and for $d \geq 3$, we describe an obstruction, called a z-structure, for oriented graphs in $\mathcal{C}^d$ having oriented book thickness equal to one. At the end of this chapter, we discuss possible directions for future work to follow this dissertation.

CHAPTER 2

STRICTLY UNI-DICYCLIC GRAPHS

In this chapter, we define a class of graphs called *strictly uni-dicyclic graphs*, denoted $\mathcal{U}$, and characterize two-page-critical oriented graphs in $\mathcal{U}$ using three subclasses $\mathcal{I}$, $\mathcal{R}$, and $\mathcal{T}$. We first discuss one-page oriented book embeddings of oriented cycles and oriented trees; we also give results that will be required in later proofs.

2.1 Oriented Cycles and Oriented Trees

In this section, we discuss the oriented book embeddings of two fundamental types of oriented graphs: oriented cycles and oriented trees. We will use these oriented graphs to construct strictly uni-dicyclic graphs in Section 2.3. We first discuss book embeddings of (undirected) cycles and characterize the spine order of every one-page book embedding of a cycle (see Theorem 2.1). We then describe all possible one-page oriented book embeddings of directed cycles in Corollary 2.1.

Let C be a cycle on n vertices. By definition, there exists an isomorphism $\phi : [n] \to V(C)$ such that the edge set of C consists of $\{\phi(i), \phi(i+1)\}$ for $i \in [n-1]$ and edge $\{\phi(1), \phi(n)\}$. This allows us to define the following way to order the vertices on the spine, called a *translation cycle*.

Definition 2.1. *For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, a translation cycle of $V(C)$ is a mapping $\phi(i) \to \phi(i+k)$ (mod n), $i \in [n]$.*

Theorem 2.1. *Let C be a cycle having a k-page book embedding. Then k is minimal if and only if the spine order is a translation cycle of $V(C)$.*

Proof. Since C is not a path, by Theorem 1.2, $bt(C) \geq 1$ and since C is outerplanar, $bt(C) \leq 1$. Therefore, k is minimal if and only if $k = 1$. If the spine order is a translation cycle of $V(C)$, then $k = 1$. If $n = 3$, then every possible spine order of $V(C)$ is a translation cycle, thus we may assume that $n > 3$ and suppose that there exists a 1-page book embedding of C such that $(*)$ the spine order is not a translation cycle of $V(C)$; then, there exists at least one pair of vertices, $\phi(i)$, $\phi(i+1)$ in C, with all indices taken *mod n*, that are not consecutive in the spine, implying that the edge $\{\phi(i), \phi(i+1)\}$ is a loose edge. Considering all such pairs, $(**)$ choose i_0 such that no other such pair exists between $\phi(i_0)$ and $\phi(i_0+1)$ in the spine. By $(*)$, there is at least one vertex $\phi(j)$ embedded between $\phi(i_0)$ and $\phi(i_0+1)$ in the spine. Since $\phi(j)$ is shielded by the edge $\{\phi(i_0), \phi(i_0+1)\}$, by the planarity rule, vertex $\phi(j+1)$ or vertex $\phi(j-1)$ must also be shielded by the edge $\{\phi(i_0), \phi(i_0+1)\}$. Since $n > 3$, either $\phi(j+1)$ or $\phi(j-1)$ is not in $\{\phi(i_0), \phi(i_0+1)\}$. Thus, vertex except $\phi(i_0)$ and $\phi(i_0+1)$ is embedded between $\phi(i_0)$ and $\phi(i_0+1)$; otherwise, the planarity rule is violated. By $(**)$, the spine order must be a translation cycle of $V(C)$. □

Corollary 2.1 follows from Theorem 2.1 and provides the location of the arcs in a one-page oriented book embedding of a directed cycle.

Corollary 2.1. *Let $\vec{D}_n$ be an n-dicycle. Then $\vec{D}_n$ has a one-page oriented book embedding if and only if the spine ordering is a translation cycle of $V(\overrightarrow{D_n})$ and every arc is tight, except the arc between the top vertex and bottom vertex in the spine.*

We now show that every oriented cycle has a one-page oriented book embedding.

Theorem 2.2. *Let $\vec{C}$ be an oriented cycle. Then $obt(\vec{C}) = 1$.*

Proof. Let C be the underlying graph of $\vec{C}$. Since $bt(C) \geq 1$, we have that $obt(\vec{C}) \geq 1$. To show that $obt(\vec{C}) \leq 1$, embed the vertices into the spine so that the spine order is a translation cycle of $V(\vec{C})$, say $\phi(1), \phi(2), ..., \phi(n)$. Embed the arc $\vec{a}$ with endpoints $\phi(1)$,$\phi(n)$ into the page. For $1 < i \leq n-1$, embed every arc having the same direction as a into the page, and embed each arc having the opposite direction of $\vec{a}$ into the spine. Thus, we obtain a 1-page oriented book embedding, and $obt(\vec{C}) \leq 1$. Since we have shown that $1 \leq obt(\vec{C}) \leq 1$, we conclude that $obt(\vec{C}) = 1$. □

We now discuss oriented book embeddings of oriented trees and oriented forests, and introduce a type of oriented tree called a *fountain tree*, which will be utilized in Section 3. A *tree*, denoted T, is a connected graph having no cycle as a subgraph. An *oriented tree*, denoted $\vec{T}$, is an oriented graph whose underlying graph is a tree.

The authors in [10] focus on oriented graphs which contain no directed cycle; they use a restricted definition of oriented book embeddings which requires the direction of all arcs in every page to agree and do not allow arcs to be embedded into the spine, i.e., each vertex must be loose in the embedding. They prove that every oriented tree has a one-page oriented book embedding such that an arbitrarily chosen vertex called the *root* is unshielded. We translate their result below.

Theorem 2.3. (*Heath, Pemmaraju, and Trenk, 1999 [10]*) *For every oriented tree $\vec{T}$ with arbitrarily chosen root v, there exists a spine order of $V(\vec{T})$ that yields a one-page oriented book embedding of $\vec{T}$ in which the root v is unshielded, every vertex is loose, and the page is upwards.*

The above result only proves existence. We give an algorithm, called the Oriented Tree Spine Order Algorithm (OTSO Algorithm), such that the output is a spine order which yields a one-page oriented book embedding as described in Theorem 2.3. For convenience, we will always assume the direction of the page is upwards, unless otherwise noted. For an oriented tree $\vec{T}$, we can choose the root to be a sink in $\vec{T}$. The following lemma proves that there exists a one-page oriented book embedding of $\vec{T}$ such that the root is the top vertex in the spine.

Lemma 2.1. *If a vertex x is a sink in $\vec{T}$, then there exists a spine order of $V(\vec{T})$ which yields a one-page oriented book embedding of $\vec{T}$ such that x is the top vertex in the spine, and each vertex is loose in the oriented book embedding.*

Proof. Since x is a sink, let $N(x) = N^-(x) = \{v_1, v_2, ..., v_k\}$. Delete x to obtain k disconnected oriented trees, denoted $\overrightarrow{T_{v_i}}$, with root v_i, $1 \leq i \leq k$. Let β_i be the spine order of $\overrightarrow{T_{v_i}}$ in the one-page oriented book embedding guaranteed by Theorem 2.3. To obtain a one-page oriented book embedding of $\vec{T}$, embed the vertices into the spine with spine order $(x, \beta_1, \beta_2, ..., \beta_k)$ such that x is the top vertex in the spine. By Theorem 2.3, every arc of $\overrightarrow{T_{v_i}}$, $1 \leq i \leq k$, can be placed into the interior of the page such that v_i is unshielded, each arc is upwards, and each vertex is loose. Then place each arc (v_i, x) into the interior of the page, and since all arcs (v_i, x) share x as a common endpoint, they do not cross in the page. Therefore, the statement holds. $\square$

Let $\overrightarrow{T}$ be an oriented tree with sink x. For convenience, we call an oriented book embedding as described in Lemma 2.1 a *sink-top oriented book embedding*; we abbreviate the spine order of such an embedding $(x; \alpha_x)$, such that x is the top vertex in the spine, and α_x represents $(\beta_1, \beta_2, ..., \beta_k)$. Similarly, we can construct a *source-bottom oriented book embedding*.

We are now prepared to define a *hook fountain tree*, which has a specific one-page oriented book embedding that will be useful in Section 3. To do this, we use the standard dipath $\overrightarrow{P_n}$, with arcs $(i, i+1)$, $1 \leq i \leq n-1$, and n oriented trees $\overrightarrow{T_{x_i}}$, $1 \leq i \leq n$, such that each $\overrightarrow{T_{x_i}}$ contains a sink x_i. By Lemma 2.1, each $\overrightarrow{T_{x_i}}$ has a one-page sink-top oriented book embedding with spine order (x_i, α_i), such that x_i is the top vertex in the spine. Embed $V(\overrightarrow{P_n})$ and $V(\overrightarrow{T_{x_i}})$, $1 \leq i \leq n$, into the spine of a book with spine order $(1, 2, ..., n, (x_n; \alpha_n), (x_{n-1}; \alpha_{n-1}), ..., (x_1; \alpha_1))$ such that the vertex 1 is the top vertex in the spine. Place all arcs in $A(\overrightarrow{P_n})$ into the spine, and place all arcs in $A(\overrightarrow{T_{x_i}})$, $1 \leq i \leq n$, into the interior of the page. Therefore, we have a one-page oriented book embedding of $\overrightarrow{P_n} \cup \bigcup_{i=1}^{n} \overrightarrow{T_{x_i}}$. We now identify each $x_i \in V(\overrightarrow{T_{x_i}})$ with $i \in V(\overrightarrow{P_n})$, changing the spine order to $(x_1 = 1, x_2 = 2, ..., x_n = n, \alpha_n, \alpha_{n-1}, ..., \alpha_1)$, as shown in Figure 2.1 for $n = 2$. This does not increase the number of pages required since, for $1 \leq i \leq n$, each set of arcs with head x_i will be shielded by the set of arcs with head x_{i-1}. This yields an oriented tree containing an n-dipath. We call such an oriented tree a *hook fountain tree* and call the one-page oriented book embedding described above a *hook fountain oriented book embedding*. We denote the spine order of a such an embedding $(x_1; \gamma_{x_1})^f$, where x_1 is the top vertex in the spine and γ_{x_1} represents $(x_2, ..., x_n, \alpha_n, \alpha_{n-1}, ..., \alpha_1)$. Similarly, we can construct a *flower fountain tree* and *flower fountain oriented book embedding* such that each $x_i \in V(\overrightarrow{T_{x_1}})$ is a source.

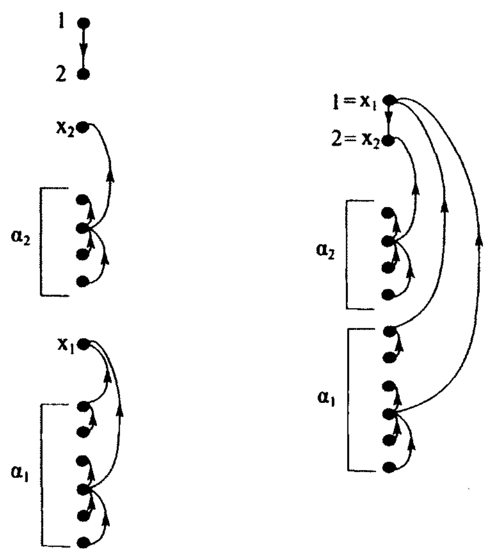

Figure 2.1: Constructing a hook fountain tree.

2.2 One-page Critical Oriented Graphs

We now characterize the class, denoted $\mathcal{M}^1$, of one-page critical oriented graphs. The only connected oriented graph on n vertices which can be embedded into a 0-page book, i.e., only the spine, is a directed path $\overrightarrow{P_n}$. Since $\overrightarrow{P_n}$ has exactly one source and exactly one sink, to find minimal obstructions we consider oriented graphs having a source or sink of degree at least two. Thus, we define S^+ to be the oriented 3-path containing a source of degree two, and we define its converse $(S^+)^*$ to be S^-, which contains a sink of degree two. If an oriented graph is an oriented tree that is not an oriented path, then it contains a vertex of degree greater than or equal to three and must also contain an oriented subgraph isomorphic to either S^+ or S^-; therefore, the only other oriented graphs to consider are oriented cycles. If an oriented cycle is not a directed cycle, then it must contain an oriented subgraph that is isomorphic to either S^+ or S^-. While an n-dicycle, denoted $\overrightarrow{D}_n$, contains no S^+ or S^-, by Theorem 2.2, $obt(\overrightarrow{D_n}) = 1$. However, for each arc $\overrightarrow{a} \in A(\overrightarrow{D_n})$ the deletion $\overrightarrow{a}$ results in a dipath; therefore, $\overrightarrow{D}_n$, $n \geq 3$, is in $\mathcal{M}^1$, giving the following result:

Lemma 2.2. $\mathcal{M}^1 = \{S^+, S^-, \overrightarrow{D}_n \mid n \geq 3\}$.

2.3 Strictly Uni-dicyclic Critical Graphs

In this section, we give the complete description for the class of *strictly uni-dicyclic graphs*, oriented graphs having exactly one oriented cycle, which is a directed cycle; we denote the class of strictly uni-dicyclic graphs $\mathcal{U}$. We will discuss three subclasses, $\mathcal{I}, \mathcal{T}, \mathcal{R}$, of $\mathcal{U}$ that are two-page-critical, which we use to characterize $\mathcal{M}^2 \cap \mathcal{U}$ in Theorem 2.4.

For $n \geq 3$, let $\overrightarrow{D}_n$ be an n-dicycle, with arcs $(\phi(i), \phi(i+1))$, for $1 \leq i \leq n-1$, and arc $(\phi(n), \phi(1))$; then, $\overrightarrow{D}_n$ is a member of $\mathcal{U}$ and we can construct every oriented graph in $\mathcal{U}$ as follows. Following the definition of 1-sum for undirected graphs, found in [16], we define the 1-*sum* of two oriented graphs $\overrightarrow{D}$ and $\overrightarrow{D}'$, via y and y', to be the oriented graph obtained by identifying a vertex $y \in V(\overrightarrow{D})$ with a vertex $y' \in V(\overrightarrow{D}')$; we denote the resulting oriented graph $\overrightarrow{D}(y) +_1 \overrightarrow{D}'(y')$. For convenience, we call a single vertex a *trivial oriented tree*. For n distinct, possibly trivial, oriented trees $\overrightarrow{T}_i$, $1 \leq i \leq n$, we describe a member of $\mathcal{U}$ to be $\bigcup_{i=1}^{n}(\overrightarrow{D}_n(\phi(i)) +_1 \overrightarrow{T}_i(y_i))$, where

$\phi(i) \in V(\overrightarrow{D}_n)$ and $y_i \in V(\overrightarrow{T_i})$. For an oriented graph $\overrightarrow{D}$ in $\mathcal{U}$, if $\overrightarrow{T_i}$ is not a trivial oriented tree, i.e., if $\phi(i)$ has degree at least three, we call the vertex $\phi(i) = y_i$ in $\overrightarrow{D}$ a *heavy vertex*. In fact, each member of $\mathcal{I}$, $\mathcal{T}$, and $\mathcal{R}$ have at most three heavy vertices.

Consider a strictly uni-dicyclic graph $\overrightarrow{D}$ containing a dicycle $\overrightarrow{D_m}$. For an arc $\overrightarrow{h} \in A(\overrightarrow{D_m})$, if each endpoint of $\overrightarrow{h}$ is a heavy vertex, we call $\overrightarrow{h}$ a *heavy arc*. The next two results consider an oriented book embedding of a strictly uni-dicyclic graph which has exactly one heavy vertex and an oriented book embedding of a strictly uni-dicyclic graph which has a heavy arc.

Lemma 2.3. *Let $\overrightarrow{D} \in \mathcal{U}$. If $\overrightarrow{D}$ has exactly one heavy vertex, then there exists a one-page oriented book embedding of $\overrightarrow{D}$ such that the page is upwards and the endpoints of the loose arc of $\overrightarrow{D_n}$ are half-loose in the embedding.*

Proof. Let $\overrightarrow{D}_n$ be an n-dicycle with arcs $(\phi(i), \phi(i+1))$, for $1 \leq i \leq n-1$, and arc $(\phi(n), \phi(1))$. We may assume the heavy vertex in $\overrightarrow{D}$ is $\phi(1)$. Then there exists a non-trivial oriented tree $\overrightarrow{T}$, containing a vertex y, such that $\overrightarrow{D} = \overrightarrow{D_n}(\phi(1)) +_1 \overrightarrow{T}(y)$. Applying Theorem 2.3, embed $\overrightarrow{T}$ into a one-page book such that each vertex is loose and the direction of the page is upwards. Since $y = \phi(1)$ is loose in the spine, we can insert the directed path $\phi(1), \phi(2), ..., \phi(n)$ into the spine below y, so that each arc of the dipath is downwards in the spine, and place the arc $(\phi(n), \phi(1))$ into the interior of the page. Thus, we have a one-page oriented book embedding of $\overrightarrow{D}$ with $\phi(1)$ and $\phi(n)$ half-loose in the embedding. $\square$

Corollary 2.2. *Let $\overrightarrow{D} \in \mathcal{U}$ such that $\overrightarrow{D}$ contains a heavy arc and $obt(\overrightarrow{D}) = 1$. In a one-page oriented book embedding of $\overrightarrow{D}$, the heavy arc is loose in the embedding.*

Proof. Assume for a contradiction that there exists a heavy arc in $\overrightarrow{D}$ that is tight in the embedding. By Corollary 2.1, the arc of the dicycle that is loose in the embedding, call it $\overrightarrow{l}$, shields at least one of the endpoints of the heavy arc, call it x, and each vertex of the dicycle is tight in the spine, except the endpoints of $\overrightarrow{l}$. Since x is heavy, there is a neighbor of x *not* shielded by $\overrightarrow{l}$. By Lemma 1.1, we have a contradiction. $\square$

We now define the first subclass of strictly uni-dicyclic graphs, $\mathcal{I}$, as follows.

Definition 2.2. *An oriented graph $\overrightarrow{D}$ is a member of $\mathcal{I}$ if $\overrightarrow{D}$ contains an n-dicycle $\overrightarrow{D_n}$, with $n \geq 4$, and for two vertices $\phi(i), \phi(j) \in V(\overrightarrow{D_n})$ with $|i - j| > 1$, there exist two arcs in $A(\overrightarrow{D})$, call them $\overrightarrow{a_i}$ and $\overrightarrow{a_j}$, that each have one endpoint of degree 1 and the other endpoint $\phi(i)$ or $\phi(j)$, respectively, in $\overrightarrow{D_n}$, such that $\overrightarrow{D} = \overrightarrow{D_n} \cup \{\overrightarrow{a_i}, \overrightarrow{a_j}\}$.*

Three members of $\mathcal{I}$, with $n = 4$, are shown in Figure 2.2.

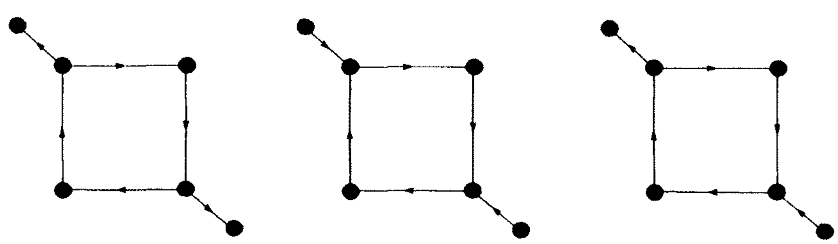

Figure 2.2: Members of $\mathcal{I}$ (based on a 4-dicycle)

Lemma 2.4. *If $\overrightarrow{D} \in \mathcal{I}$, then $obt(\overrightarrow{D}) = 2$.*

Proof. Suppose for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. Since $\overrightarrow{D}$ contains dicycle, by Lemma 2.1, each vertex of $\overrightarrow{D}_n$ is tight in the embedding, except the endpoints of the loose arc. However, $\phi(i)$ and $\phi(j)$ are not adjacent; thus, at least one of $\{\phi(i), \phi(j)\}$, say $\phi(i)$, is shielded by the loose arc of $\overrightarrow{D_n}$; let u_i be the other endpoint of the $\overrightarrow{a_i}$. Since u_i must be located in the spine above all vertices of the dicycle, or below all vertices of the dicycle, $\overrightarrow{a_i}$ crosses the loose arc of the dicycle, a contradiction to the planarity rule, thus $obt(\overrightarrow{D}) \geq 2$. In Figure 2.3, we show a two-page oriented book embedding of a member of $\mathcal{I}$; each member of $\mathcal{I}$ can be embedded this way, thus $obt(\overrightarrow{D}) \leq 2$. Therefore, $obt(\overrightarrow{D}) = 2$. $\square$

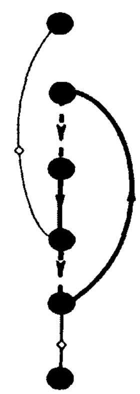

Figure 2.3: A two-page oriented book embedding of a member of $\mathcal{I}$.

For members of the class $\mathcal{I}$, the length of the directed cycle is required to be at least four. Therefore, to address cycles of length exactly three, we define the class $\mathcal{T}$ as follows.

Definition 2.3. *An oriented graph $\overrightarrow{D}$ is a member of $\mathcal{T}$ if $\overrightarrow{D}$ contains a 3-dicycle and there exist three arcs $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3} \in A(\overrightarrow{D})$ that each have one endpoint of degree one and other endpoint $\phi(1), \phi(2), \phi(3)$, respectively, in $\overrightarrow{D_n}$, with $\overrightarrow{D} = \overrightarrow{D_n} \cup \{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\}$.*

As with $\mathcal{I}$, the direction of each arc not contained in the directed cycle is not unique. Since the size of the dicycle in elements of $\mathcal{T}$ is restricted, we are able to list, in Figure 2.4, the four members of $\mathcal{T}$.

Lemma 2.5. *If $\overrightarrow{D} \in \mathcal{T}$, then $obt(\overrightarrow{D}) = 2$.*

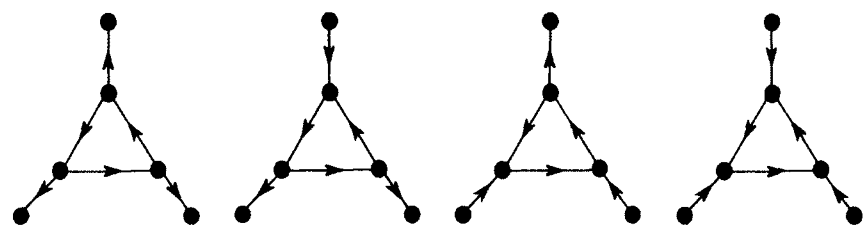

Figure 2.4: Members of $\mathcal{T}$

To introduce the last class, $\mathcal{R}$, of strictly uni-dicyclic graphs, we first define an oriented tree, called an *antler*, using the oriented paths S^+, S^-, and a standard j-dipath $\overrightarrow{P_j}$. Let s^+, s^- be the vertices of degree two in S^+, S^-, respectively. For an integer $j \geq 2$, we define a *positive j-antler*, $\overrightarrow{A}_j^+ := S^+(s^+) +_1 \overrightarrow{P_j}(j)$; we also define the positive 1-antler $A_1^+ := S^+$. We define a *negative j-antler* A_j^- to be the converse, $(A_j^+)^*$, of a positive j-antler. A_3^+ and A_4^- can be seen in Figure 2.5. It is important to note that if an oriented tree contains *no* positive antler, it is a hook fountain tree, and similarly, if an oriented tree contains *no* negative antler, it is a flower fountain tree.

Figure 2.5: A_3^+ and A_4^-

In later proofs, we use the following definition that describes the location of an antler, when it is a subgraph of a strictly uni-dicyclic graphs with dicycle $\overrightarrow{D_m}$.

Definition 2.4. *If $\overrightarrow{h}$ is a heavy arc of $\overrightarrow{D_m}$ having endpoints w and v, such that $\overrightarrow{T_v}$ is an antler, we say that $\overrightarrow{T_v}$ is properly positioned with respect to $\overrightarrow{h}$ if one of the following is true:*

- $\overrightarrow{h} = (v, w)$ *and $\overrightarrow{T_v}$ is a positive antler, or*
- $\overrightarrow{h} = (w, v)$ *and $\overrightarrow{T_v}$ is a negative antler.*

If there is no ambiguity to $\overrightarrow{h}$, we simply say $\overrightarrow{T_v}$ is properly positioned.

We now define $\mathcal{R}$, using a positive antler A_j^+, a negative antler A_k^-, and the standard n-dicycle, $\overrightarrow{D_n}$.

Definition 2.5. *In $\overrightarrow{D_n}$, choose an arbitrary arc $(\phi(i), \phi(i+1))$ and call it (x, y). Let $\overrightarrow{T_x}$ be a positive antler with source x, and $\overrightarrow{T_y}$ be a negative antler with sink y. Then $\mathcal{R}$ contains each oriented graph $\overrightarrow{D_m} \cup \overrightarrow{T_x} \cup \overrightarrow{T_y}$. Notice that both $\overrightarrow{T_x}$ and $\overrightarrow{T_y}$ are properly positioned with respect to (x, y).*

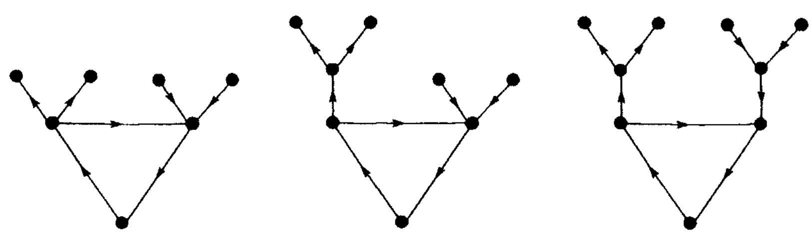

Figure 2.6: Members of $\mathcal{R}$

Lemma 2.6. *Let $\overrightarrow{D} \in \mathcal{R}$. Then $obt(\overrightarrow{D}) = 2$.*

Proof. Let $\overrightarrow{D} \in \mathcal{R}$. Suppose for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. We will first prove the statement for $\overrightarrow{D}$ such that $\overrightarrow{T_x} = S^+$ and

$\overrightarrow{T_y} = S^-$; we have that $s^+ = x$ and $s^- = y$. Let a_1, a_2 be the sinks of the positive antler, and let b_1, b_2 be the sources of the negative antler. By Corollary 2.1, the spine order of the embedding must be a translation cycle of $V(\overrightarrow{D_n})$ and each arc except one, call it $\overrightarrow{l}$, is tight in the embedding and each vertex in $V(\overrightarrow{D_n})$, except the endpoints of l, is shielded by $\overrightarrow{l}$ in the embedding. Therefore, no vertex that is not contained in $V(\overrightarrow{D_n})$ can appear between two vertices of $\overrightarrow{D_n}$ in the spine and by Corollary 2.2, $\overrightarrow{l} = (x, y)$. We may assume then that x is below y in the spine, and the direction of the page is upwards.

Since each arc between x and y is tight, by Lemma 1.2, at least of a_1, a_2, say a_2 is above y in the spine. Then y is shielded by (x, a_2). Then both b_1 and b_2 must appear between y and a_2 in the spine, thus either (b_1, y) or (b_2, y) is embedded into the page, as a downwards arc. However, since (x, y) is an upwards arc, we obtain a contradiction and $obt(\overrightarrow{D}) \geq 2$. In Figure 2.7, we show a two-page oriented book embedding of a member of $\mathcal{R}$; each member of $\mathcal{R}$ can be embedded this way, thus $obt(\overrightarrow{D}) \leq 2$. Therefore, $obt(\overrightarrow{D}) = 2$. The proofs for other members of $\mathcal{R}$ are similar. □

To prove that each oriented graph $\overrightarrow{D} \in \mathcal{T} \cup \mathcal{I} \cup \mathcal{R}$ is two-page-critical, we must not only show that $obt(\overrightarrow{D}) > 1$, but also that for each arc $\overrightarrow{a} \in A(\overrightarrow{D})$, we have that $obt(\overrightarrow{D} \setminus \overrightarrow{a}) \leq 1$.

Lemma 2.7. $\mathcal{I} \cup \mathcal{T} \cup \mathcal{R} \subseteq \mathcal{M}^2$

Proof. Let $\overrightarrow{D} \in \mathcal{I} \cup \mathcal{T} \cup \mathcal{R}$. By Lemmas 2.4, 2.5, and 2.6, we have that $obt(\overrightarrow{D}) > 1$. We now discuss the deletion of an arc $\overrightarrow{a} \in A(\overrightarrow{D})$. If $\overrightarrow{a}$ is contained in the dicycle of

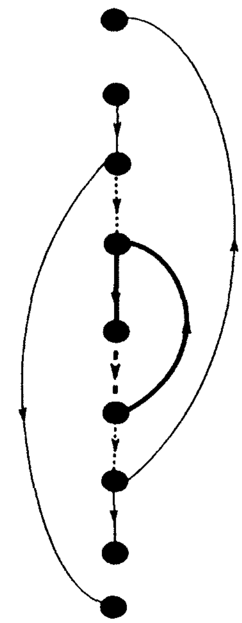

Figure 2.7: A two-page oriented book embedding of a member of $\mathcal{R}$.

$\overrightarrow{D}$, then $\overrightarrow{D} \setminus \overrightarrow{d}$ is an oriented tree, and by Theorem 2.3, $obt(\overrightarrow{D} \setminus \overrightarrow{d}) \leq 1$. Thus, we may focus on the arcs of $\overrightarrow{D}$ not contained in the dicycle. We have three cases.

i. Suppose $\overrightarrow{D} \in \mathcal{T}$. If we delete an arc $\overrightarrow{d}$ not contained in the dicycle, then exactly two, adjacent vertices, call them x and y, have a neighbor not contained in the dicycle. Embed the dicycle so that the arc between x and y is loose. Then the remaining two arcs can be placed into the spine or the page, depending on their direction, and we obtain a one-page oriented book embedding of $\overrightarrow{D} \setminus \overrightarrow{d}$.

ii. Suppose $\overrightarrow{D} \in \mathcal{I}$. If we delete an arc $\overrightarrow{d}$ not contained in the dicycle, then exactly one vertex, call it x, has a neighbor not contained in the dicycle. Embed the dicycle so that x is an endpoint of the loose arc of the dicycle. Then the remaining arc can be placed into the spine or the page, depending on its direction, and we obtain a one-page oriented book embedding of $\overrightarrow{D} \setminus \overrightarrow{d}$.

iii. Suppose $\vec{D} \in \mathcal{R}$. We first prove the statement for $\vec{a}$ in the positive antler of $\vec{D}$. Then there are two possibilities for $\vec{a}$ such that $\vec{a}$ is not contained in the dicycle, either a can be incident to one of the sinks of the positive antler as on the left of Figure 2.8, or not, as seen on the right of Figure 2.8. In either case, we can find a one-page oriented book embedding of $\vec{D} \setminus \vec{a}$ as seen in Figure 2.8 below. If $\vec{a}$ is in the negative antler of $\vec{D}$, the proof is similar.

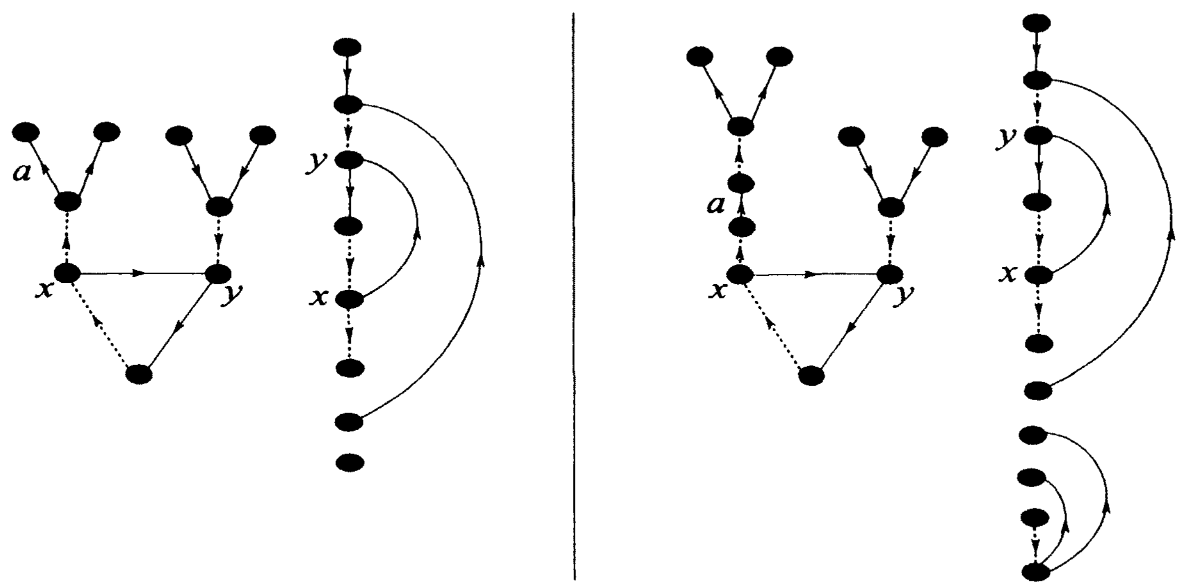

Figure 2.8: one-page oriented book embedding of $\vec{D} \setminus \vec{a}$.

□

The next theorem shows that $\mathcal{M}^2 \cap \mathcal{U}$, the class of two-page-critical, strictly uni-dicyclic graphs, is completely characterized by the three classes of two-page-critical oriented graphs.

Theorem 2.4. *Let $\overrightarrow{D} \in \mathcal{U}$. Then $obt(\overrightarrow{D}) \leq 1$ if and only if $\overrightarrow{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, or $\mathcal{R}$; in other words, $\mathcal{M}^2 \cap \mathcal{U} = \mathcal{I} \cup \mathcal{T} \cup \mathcal{R}$.*

Proof. To prove the necessary condition, consider the contrapositive, which states: If $\overrightarrow{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, or $\mathcal{R}$ then $obt(\overrightarrow{D}) > 1$. This is true by Lemma 2.4, Lemma 2.5, and Lemma 2.6. To prove the sufficient condition, let $\overrightarrow{D}$ be a strictly uni-dicylic graph containing no member of $\mathcal{I}$, $\mathcal{T}$, or $\mathcal{R}$, and we now construct a one-page oriented book embedding of $\overrightarrow{D}$. Let $\overrightarrow{D}$ contain an n-dicycle $\overrightarrow{D}_n$. Since $\overrightarrow{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$ as an oriented subgraph, $\overrightarrow{D_n}$ has at most two heavy vertices x and y, such that the arc $(x, y) \in A(\overrightarrow{D_n})$.

Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, either $\overrightarrow{T_x}$ does not contain a positive antler, or $\overrightarrow{T_y}$ does not contain a negative antler. We may assume that $\overrightarrow{T_x}$ does not contain a positive antler. We first consider a one-page oriented book embedding of $\overrightarrow{D_n} \cup \overrightarrow{T_y}$, guaranteed by Lemma 2.3. Since x has degree two in $\overrightarrow{D_n} \cup \overrightarrow{T_y}$, x is half-loose in the spine. We now put $\overrightarrow{T_x}$ back into $\overrightarrow{D}$. Since $\overrightarrow{T_x}$ contains no positive antler, $|N^+(x)| \leq 1$. If $|N^+(x)| = 0$, then x is a sink, and by Lemma 2.1, there is a sink-top oriented book embedding of $\overrightarrow{T_x}$ with spine order $(x; \alpha_x)$. Thus, place α_x below x in the spine and we are done. If $|N^+(x)| = 1$, then $\overrightarrow{T_x}$ must be a hook fountain tree, and there is a sink fountain oriented book embedding of $\overrightarrow{T_x}$ with spine order $(x; \gamma_x)^f$. Thus, place γ_x below x in the spine and we are done. In either case, we obtain an oriented book embedding of $\overrightarrow{D}$. The case in which $\overrightarrow{T_y}$ does not contain a negative antler is similar.

□

CHAPTER 3

STRICTLY BI-DICYCLIC GRAPHS

In the previous chapter, we discussed the class $\mathcal{U}$ of strictly uni-dicyclic graphs, and characterized two-page-critical oriented graphs in $\mathcal{U}$ by $\mathcal{I} \cup \mathcal{R} \cup \mathcal{T}$. Thus, if an oriented graph $\overrightarrow{D}$ contains a member of $\mathcal{I} \cup \mathcal{R} \cup \mathcal{T}$ as a proper oriented subgraph, then $\overrightarrow{D}$ has an oriented book thickness greater than one, and cannot be two-page-critical. In this chapter, we discuss the class of *strictly bi-dicyclic graphs*, oriented graphs which contain exactly two oriented cycles, which are directed cycles, for $m, n \geq 3$, $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$ and a possibly trivial oriented path $\overrightarrow{P}$ between them, such that $\overrightarrow{P}$ is arc disjoint from $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$, and has an endpoint x_1 in $\overrightarrow{D_m}$ and an endpoint x_k in $\overrightarrow{D_n}$. We denote this common minimal subgraph $\overrightarrow{D_m} \cup \overrightarrow{P} \cup \overrightarrow{D_n}$ as $\overrightarrow{H}$, which is an example of a bicircular matroid [15]. We will denote the class of strictly bi-dicylic graphs by $\mathcal{B}$, and characterize two-page-critical oriented graphs in $\mathcal{B}$.

For an oriented graph $\overrightarrow{G} \in \mathcal{B}$ containing $\overrightarrow{H}$ as an oriented subgraph and each vertex $v \in V(\overrightarrow{H})$, let $\overrightarrow{T_v}$ denote the possibly trivial, oriented tree in $\overrightarrow{G} \setminus A(\overrightarrow{H})$. Recall the following definition from Chapter 2: we call a vertex v in $\overrightarrow{D_m}$ or $\overrightarrow{D_n}$ a *heavy vertex* if v has degree at least three. It is possible for vertices in $\overrightarrow{P}$ to be heavy, as well as vertices in $\overrightarrow{D_m}$ or $\overrightarrow{D_n}$.

Also, for an oriented graph $\overrightarrow{G} \in \mathcal{B}$, if there exist at least two vertices in one of the dicycles, say $\overrightarrow{D_m}$, such that for $v \in V(\overrightarrow{D_m})$ each $\overrightarrow{T_v}$ is a non-trivial oriented tree, then $\overrightarrow{G}$ contains an oriented subgraph isomorphic to a member of $\mathcal{T}$ as an oriented subgraph, because x_1 is a heavy vertex. Moreover, if there exists a heavy vertex v in a dicycle *not* adjacent to one of the endpoints of $\overrightarrow{P}$, then $\overrightarrow{G}$ contains a member of $\mathcal{I}$ as an oriented subgraph.

From the observation above, we can focus on the class $\mathcal{Y}$, described below, in order to characterize two-page-critical oriented graphs in $\mathcal{B}$. Since we only consider one neighbor of each endpoint of $\overrightarrow{P}$ in $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$, respectively, let $y_1 \in V(\overrightarrow{D_m})$ be a neighbor of x_1, and let $y_2 \in V(\overrightarrow{D_n})$ be a neighbor of x_k. Then each oriented graph in $\mathcal{Y}$ is of the form $\overrightarrow{T_{y_1}} \cup \overrightarrow{H} \cup \overrightarrow{T_{y_2}}$.

If y_1 and y_2 are both in-neighbors of x_1 and x_k, respectively, or y_1 and y_2 are both out-neighbors of x_1 and x_k, respectively, then we say that y_1 and y_2 are *consistent neighbors* of x_1 and x_k, respectively; otherwise, we say that y_1 and y_2 are *inconsistent neighbors* of x_1 and x_k, respectively.

Definition 3.1. *For $i = 1, 2$, the class $\mathcal{Y}$ consists of the following five sub-classes, where each antler is properly positioned:*

$\mathcal{B}_1 : |V(\overrightarrow{P})| = 1$, *$y_i$ are consistent; $\overrightarrow{T_{y_i}}$ are single arcs;*

$\mathcal{B}_2 : |V(\overrightarrow{P})| = 1$, *$y_i$ are inconsistent; $\overrightarrow{T_{y_i}}$ are antlers;*

$\mathcal{B}_3 : |V(\overrightarrow{P})| > 1$, *$y_i$ are inconsistent; $\overrightarrow{T_{y_i}}$ are antlers;*

$\mathcal{B}_4 : |V(\overrightarrow{P})| > 1$, *$y_i$ are consistent; $\overrightarrow{T_{y_1}}$ is an antler and $\overrightarrow{T_{y_2}}$ is a single arc;*

$\mathcal{B}_5 : |V(\overrightarrow{P})| > 1$, *$\overrightarrow{P}$ is not a dipath, y_i are consistent; $\overrightarrow{T_{y_i}}$ are antlers.*

Notice that for a member $\overrightarrow{D}$ of $\mathcal{B}_5$, $\overrightarrow{P}$ cannot be a dipath, because otherwise, $\overrightarrow{D}$ would have a member of $\mathcal{B}_4$ as a proper oriented subgraph, and would not be minimal. The main result of this chapter, Theorem 3.1 proves that these are the only strictly bi-dicyclic oriented graphs that are two-page-critical. For convenience, we show Figure 3.1, which depicts a member of each $\mathcal{B}_i$. In Figure 3.1, a square on an arc denotes arbitrary direction, a square on a dashed line denotes an oriented path, and an arrow on a dashed line denotes a directed path.

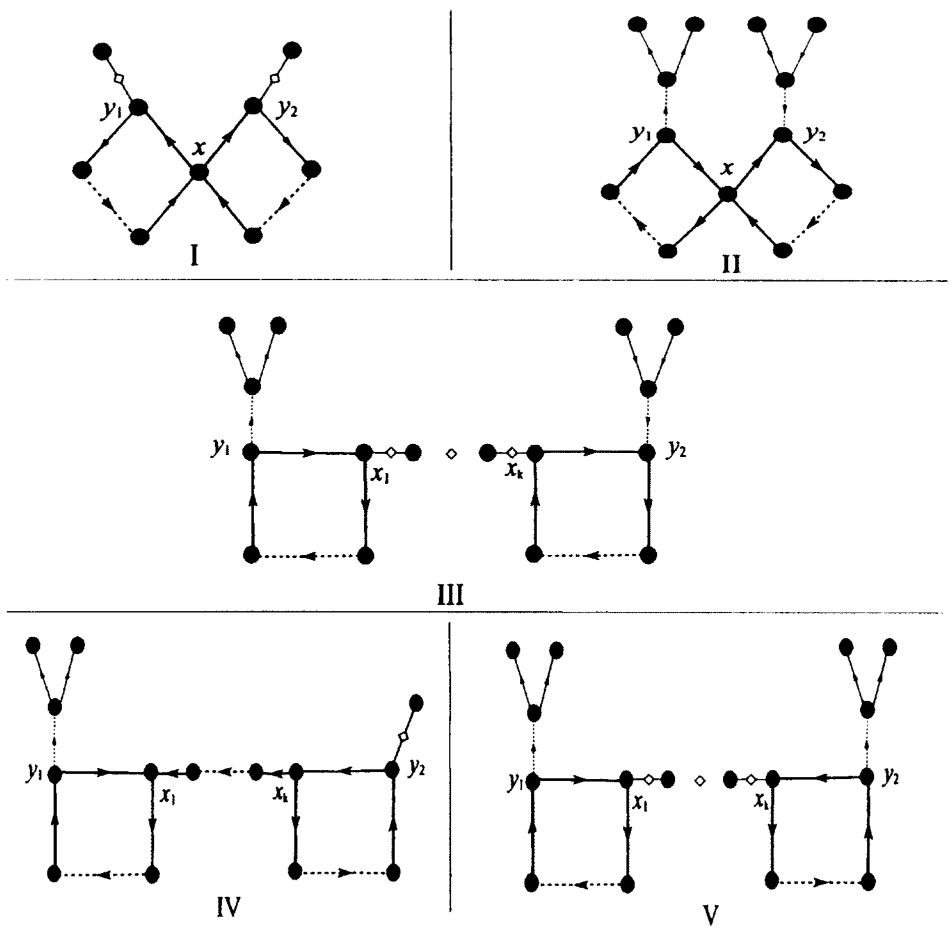

Figure 3.1: Members of $\mathcal{Y} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5$.

We will prove Theorem 3.1 using Propositions 3.1, 3.2, 3.3, and 3.4, and to prove the propositions, we require Lemmas 3.1 - 3.9.

Theorem 3.1. *Let $\overrightarrow{D}$ be a strictly bi-dicyclic graph. Then, $obt(\overrightarrow{D}) \leq 1$ if and only if $\overrightarrow{D}$ contains no member of $\mathcal{I} \cup \mathcal{T} \cup \mathcal{R} \cup \mathcal{Y}$.*

The next result shows that we can embed the oriented subgraph $\overrightarrow{H}$ into a 1-book. Note that this does not describe the exact embedding itself; it simply proves that one exists. Let $V(\overrightarrow{P}) = \{x_i : i \in [k]\}$ and $A(\overrightarrow{P})$ consist of arcs with endpoints x_i and x_{i+1}, $1 \leq i \leq k-1$.

Lemma 3.1. $obt(\overrightarrow{H}) \leq 1$.

Proof. By Corollary 2.1, to embed the directed cycles $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$ into a 1-book, the spine order must be a translation cycle of the vertices of $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$. As usual, we construct a one-page oriented book embedding such that the page is upwards. Embed the vertices of $\overrightarrow{D_m}$ into the spine, using a translation cycle, such that x_1 is the tail of the loose arc of $\overrightarrow{D_m}$. Then embed the vertices of $\overrightarrow{D_n}$ into the spine, using a translation cycle, such that each vertex in $\overrightarrow{D_n}$ is below each vertex of $\overrightarrow{D_m}$ and x_k is the head of the loose arc of $\overrightarrow{D_n}$. Then the vertices of $\overrightarrow{P}$ can be placed consecutively in the spine between x_1 and x_k. Then place the downwards arcs of $\overrightarrow{P}$, i.e., the arcs of the form (x_i, x_{i+1}), $1 \leq i \leq k-1$, into the spine, and place the upwards arcs of $\overrightarrow{P}$, i.e., the arcs of the form (x_{i+1}, x_i), $1 \leq i \leq k-1$, into the page. Thus, we obtain a one-page oriented book embedding. $\square$

The next result will be utilized in later proofs. Recall in Corollary 2.1 that the one-page oriented book embedding of a directed cycle has exactly one arc, the loose arc, embedded into the page, and all other arcs are embedded into the spine.

Lemma 3.2. *Let $\overrightarrow{D}$ be an oriented graph consisting of exactly two dicycles, which share at most one vertex. Then, in a one-page oriented book embedding of $\overrightarrow{D}$, no loose arc in the embedding shields another loose arc.*

Proof. Assume for a contradiction, that for a one-page oriented book embedding of $\overrightarrow{D}$ there are two arcs, call them $\overrightarrow{a}$ and $\overrightarrow{b}$, such that $\overrightarrow{b}$ shields $\overrightarrow{a}$. By Corollary 2.1, in a one-page oriented book embedding of a dicycle, the vertices are located in the spine, in a translation cycle, and each arc, except one, is tight in the spine. However, the dicycle containing $\overrightarrow{b}$ contains both endpoints of $\overrightarrow{a}$, unless $\overrightarrow{a}$ is a loop, which we do not allow; therefore, the two dicycles share more than one vertex, a contradiction. $\square$

We are ready to formally define each of the sub-classes of $\mathcal{Y}$.

Definition 3.2. *An oriented graph is in $\mathcal{B}_1$ if it consists of the oriented graph $\overrightarrow{T_{y_1}} \cup \overrightarrow{H} \cup \overrightarrow{T_{y_2}}$ such that $V(\overrightarrow{P}) = \{x\}$, for $i = 1, 2$, y_i are consistent neighbors of x, and $\overrightarrow{T_{y_i}}$ are single arcs. See the oriented graph of I in Figure 3.1.*

Lemma 3.3. *Let $\overrightarrow{D} \in \mathcal{B}_1$. Then $obt(\overrightarrow{D}) = 2$.*

Proof. Let $\overrightarrow{D} \in \mathcal{B}_1$. Assume for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. We may assume the page is upwards, and y_1 and y_2 are both out-neighbors of x. According to Corollary 2.1, the vertices of each dicycle must be located in the spine in a translation cycle of the dicycle, and each dicycle has exactly one loose arc. Since (x, y_1) and (x, y_2) are heavy arcs, by Corollary 2.2, (x, y_1) and (x, y_2) are loose in the embedding, and by Lemma 3.2, the arc (x, y_1) does not shield

(x, y_2), and (x, y_2) does not shield (x, y_1). Since y_1 and y_2 are both out-neighbors of x, either y_1 or y_2 are located below x is the spine, say y_2. However, since (x, y_2) is located in the page, and is a downwards arc, we have a contradiction to the direction rule, thus $obt(\vec{D}) \geq 2$. In Figure 3.2, we show a two-page oriented book embedding of a member of $\mathcal{B}_1$; each member of $\mathcal{B}_1$ can be embedded this way, thus $obt(\vec{D}) \leq 2$. Therefore, $obt(\vec{D}) = 2$. $\square$

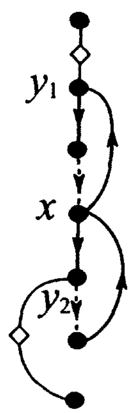

Figure 3.2: A two-page oriented book embedding of a member of $\mathcal{B}_1$.

For the remaining cases of $\mathcal{Y}$, each member contains a positive or negative antler as an oriented subgraph. To prove the oriented book thickness of these oriented graphs, we now give the following result:

Lemma 3.4. *Let $\overrightarrow{D}$ be a strictly uni-dicyclic graph with exactly one heavy vertex x (or y) such that $\overrightarrow{T_x}$ (or $\overrightarrow{T_y}$) is a positive antler (or a negative antler) that is properly positioned with respect to an arc $\overrightarrow{h} = (x, y)$. In a one-page oriented book embedding of $\overrightarrow{D}$ such that $\overrightarrow{h}$ is loose in the embedding, there exists some arc of the antler that shields $\overrightarrow{h}$.*

Proof. Suppose $\overrightarrow{T_x}$ is a positive antler, and assume for a contradiction that for a one-page oriented book embedding of $\overrightarrow{D}$ such that the direction of the page is upwards and $\overrightarrow{h}$ is loose in the embedding, then $\overrightarrow{h}$ is not shielded by an arc of $\overrightarrow{T_x}$. Since the page is upwards, x is below y in the spine, and thus each vertex of $\overrightarrow{T_x}$ must be below x in the spine. To avoid a downwards arc in page, the arcs of the directed path from x to the vertex of degree three, call it x', in $\overrightarrow{T_x}$ must be located in the spine, and then each vertex between y and x' is tight in the spine. However, since x' has out-degree two, by Lemma 1.2, one of its out-neighbors, call it a, must be located above x', and therefore above y, in the spine. Then (x', a) shields $\overrightarrow{h}$, a contradiction. If $\overrightarrow{T_x}$ is a negative antler, the proof is similar. □

Notice that the arc, call it (w, z), that shields $\overrightarrow{h} = (x, y)$ is not necessarily incident to one of the sinks of $\overrightarrow{T_x}$; there are three possibilities for (w, z), seen in Figure 3.3. We now define the second sub-class of $\mathcal{Y}$, denoted $\mathcal{B}_2$.

Definition 3.3. *An oriented graph is in $\mathcal{B}_2$ if it consists of the oriented graph $\overrightarrow{T_{y_1}} \cup \overrightarrow{H} \cup \overrightarrow{T_{y_2}}$ such that $V(\overrightarrow{P}) = \{x\}$, y_1 and y_2 are inconsistent neighbors of x, and each $\overrightarrow{T_{y_i}}$ is an antler that is properly positioned with respect to the heavy arc in each dicycle, $i = 1, 2$. See the oriented graph of II in Figure 3.1.*

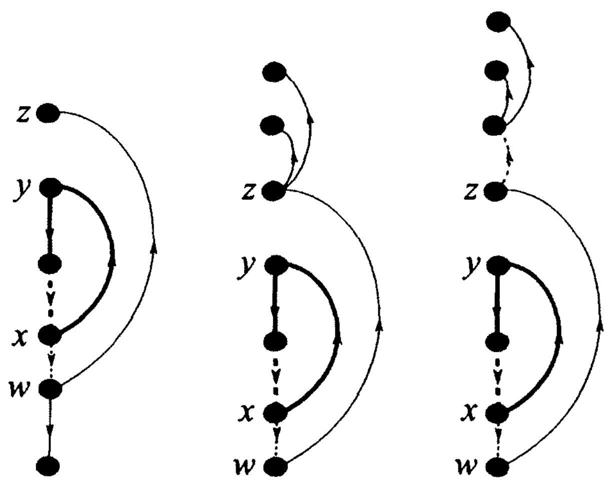

Figure 3.3: Three ways to shield $\overrightarrow{h} = (x, y)$

Lemma 3.5. *Let* $\overrightarrow{D} \in \mathcal{B}_2$. *Then* $obt(\overrightarrow{D}) = 2$.

Proof. Let $\overrightarrow{D} \in \mathcal{B}_2$. Suppose for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. We may assume the page is upwards. According to Corollary 2.1, the vertices of each dicycle must be located in the spine in a translation cycle of the dicycle, and each dicycle has exactly one arc which is loose in the embedding. By Corollary 2.2, the heavy arcs of $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$, (y_1, x) and (x, y_2), respectively, are loose in the embedding; by Lemma 3.2, neither (y_1, x) or (x, y_2) shields the other, and since we assume the page is upwards, y_1 is located below x in the spine and y_2 is located above x in the spine. Then, each vertex between y_1 and y_2 is tight in the spine. Since $\overrightarrow{T_{y_1}}$ is a positive antler, by Lemma 3.4, there is an arc (w, z) of $\overrightarrow{T_{y_1}}$ such that (w, z) shields (y_1, x); therefore, z is above y_1, and thus y_2, and either $w = y_1$, or w is below y_1 in the spine and each vertex between y_1 and w is tight. Since y_2 is shielded by (w, z), by Lemma 1.1, every vertex of $\overrightarrow{T_{y_2}}$ must be shielded by (w, z). Since $\overrightarrow{T_{y_2}}$ is a negative antler, by Lemma 3.4, there is an arc (u, v) of $\overrightarrow{T_{y_2}}$ such that (u, v) shields

(x, y_2). Since each vertex between y_1 and w is tight, u must be located below w in the spine, and then it is not shielded by (w, z), a contradiction to Lemma 1.1. Thus, $obt(\vec{D}) \geq 2$. In Figure 3.4, we show a two-page oriented book embedding of a member of $\mathcal{B}_2$; each member of $\mathcal{B}_2$ can be embedded this way, thus $obt(\vec{D}) \leq 2$. Therefore, $obt(\vec{D}) = 2$. ☐

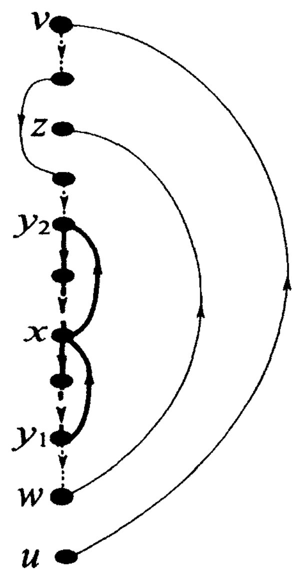

Figure 3.4: A two-page oriented book embedding of a member of $\mathcal{B}_2$.

Definition 3.4. *An oriented graph is in $\mathcal{B}_3$ if it consists of the oriented graph $\overrightarrow{T_{y_1}} \cup \vec{H} \cup \overrightarrow{T_{y_2}}$ such that $V(\vec{P}) = \{x_i : i \in [k],\ k > 1\}$, y_1 and y_2 are inconsistent neighbors of x_1 and x_k, respectively, and each $\overrightarrow{T_{y_i}}$ is an antler that is properly positioned with respect to the heavy arc in each dicycle, $i = 1, 2$. See the oriented graph of III in Figure 3.1.*

Lemma 3.6. *Let $\vec{D} \in \mathcal{B}_3$. Then $obt(\vec{D}) = 2$.*

Proof. Let $\vec{D} \in \mathcal{B}_3$. We may assume that $\vec{D}$ has heavy arcs (y_1, x_1) and (x_k, y_2). Suppose for a contradiction that $obt(\vec{D}) < 2$. Since $\vec{D}$ contains a dicycle, $obt(\vec{D}) \neq 0$.

Now suppose that there exists a one-page oriented book embedding of $\vec{D}$. We may assume the page is upwards. By Corollary 2.1, the vertices of each dicycle must be located in the spine in a translation cycle of the dicycle, and each dicycle has exactly one arc that is loose in the embedding. Consider $\overrightarrow{D_m}$ first. By Corollary 2.2, the heavy arc (y_1, x_1) in $\overrightarrow{D_m}$ is loose in the embedding, and since the page is upwards, y_1 is located below x_1 in the spine. Since $\overrightarrow{T_{y_1}}$ is a positive antler, it contains a vertex of out-degree two; by Lemma 3.4, there is an arc (w, z) of $\overrightarrow{T_{y_1}}$ such that (w, z) shields (y_1, x_1); thus, z is above x_1 and either $w = y_1$ or w is below y_1 and there is a directed path from y_1 to w such that each vertex, except possibly w, in the directed path is tight. Since x_1 is shielded by (w, z), and there is an oriented path between x_1 and each vertex in $\overrightarrow{D_n}$ and $\overrightarrow{T_{y_2}}$, by Lemma 1.1, every vertex of $\overrightarrow{D_n}$ and $\overrightarrow{T_{y_2}}$ must also be shielded by (w, z). By Lemma 2.2, the heavy arc (x_k, y_2) in $\overrightarrow{D_n}$ is loose in the embedding, and since the page is upwards, x_k is located below y_2 in the spine. By Lemma 3.2, neither (y_1, x_1) or (x_k, y_2) shields the other. Since $\overrightarrow{T_{y_2}}$ is a negative antler, it contains a vertex of in-degree 2; by Lemma 3.4, there is an arc (u, v) of $\overrightarrow{T_{y_2}}$ such that (u, v) shields (x_k, y_2); thus, u is below x_k and either $v = y_2$ or v is below y_2 and there is a directed path from v to y_2 such that each vertex, except possibly v, in the directed path is tight. There is an oriented path between x_k and x_1; however, x_k is shielded by (u, v) and x_1 is not. Thus, by Lemma 1.1, we have a contradiction to Lemma 1.1. Therefore, $obt(\vec{D}) \geq 2$. In Figure 3.5, we show a two-page oriented book embedding of a member of $\mathcal{B}_3$; each member of $\mathcal{B}_3$ can be embedded this way, thus $obt(\vec{D}) = 2$. □

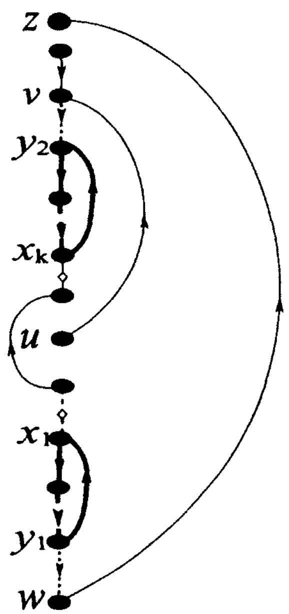

Figure 3.5: A two-page oriented book embedding of a member of $\mathcal{B}_3$.

Definition 3.5. *An oriented graph is in $\mathcal{B}_4$ if it consists of the oriented graph $\overrightarrow{T_{y_1}} \cup \overrightarrow{H} \cup \overrightarrow{T_{y_2}}$ such that $\overrightarrow{P}$ is a non-trivial directed path, y_1 and y_2 are consistent neighbors of x_1 and x_k, respectively, and one of the following holds:*

- *y_1 and y_2 are in-neighbors of x_1 and x_k, respectively, x_1 is the sink of $\overrightarrow{P}$, $\overrightarrow{T_{y_1}}$ is a positive antler and $\overrightarrow{T_{y_2}}$ is a single arc,*
- *y_1 and y_2 are out-neighbors of x_1 and x_k, respectively, x_1 is the source of $\overrightarrow{P}$, $\overrightarrow{T_{y_1}}$ is a negative antler and $\overrightarrow{T_{y_2}}$ is a single arc.*

The first case is the oriented graph of IV in Figure 3.1.

Lemma 3.7. *Let $\overrightarrow{D} \in \mathcal{B}_4$. Then $obt(\overrightarrow{D}) = 2$.*

Proof. Let $\overrightarrow{D} \in \mathcal{B}_4$ such that y_1 and y_2 are in-neighbors of x_1 and x_k, respectively, x_1 is the sink of $\overrightarrow{P}$, $\overrightarrow{T_{y_1}}$ is positive antler and $\overrightarrow{T_{y_2}}$ is a single arc. Assume for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. We may assume the page is upwards. By Corollary 2.1, the vertices of each dicycle must be located in the spine in a translation cycle of the dicycle, and each dicycle has exactly one loose arc. Consider $\overrightarrow{D_m}$ first. By Corollary 2.2, the heavy arc (y_1, x_1) in $\overrightarrow{D_m}$ is loose in the embedding, and since the page is upwards, y_1 is located below x_1 in the spine. By Lemma 3.4, there is an arc (w, z) of $\overrightarrow{T_{y_1}}$ such that (w, z) shields (y_1, x_1). Thus, z_1 is above x_1 and either $w = y_1$ or w is below y_1 and there is a directed path from y_1 to w such that each vertex, except possibly w, in the directed path is tight. Now consider $\overrightarrow{D_n}$. Since x_1 is shielded by (w, z), and there is a directed path between x_1 and x_k, by Lemma 1.1, the vertices of $\overrightarrow{D_n}$ must be located between x_1 and w in the spine. By Lemma 2.2, the heavy arc (y_2, x_k) in $\overrightarrow{D_n}$ is loose in the embedding, and since the page is upwards, y_2 is located below x_k in the spine. By Lemma 3.2, neither (y_1, x_1) or (y_2, x_k) shields the other. However, y_2 is between x_1 and x_k in the spine, and there is a directed path from x_k to x_1. Thus, some arc of the directed path must be a downwards arc in the page, a contradiction to the direction rule. Therefore, $obt(\overrightarrow{D}) \geq 2$. In Figure 3.6, we show a two-page oriented book embedding of a member of $\mathcal{B}_4$; each member of $\mathcal{B}_4$ can be embedded this way, thus $obt(\overrightarrow{D}) = 2$. The proofs other members of $\mathcal{B}_4$ are similar. $\square$

Definition 3.6. *An oriented graph is in $\mathcal{B}_5$ if it consists of the oriented graph $\overrightarrow{T_{y_1}} \cup \overrightarrow{H} \cup \overrightarrow{T_{y_2}}$ such that $\overrightarrow{P}$ is a non-trivial oriented path that not a directed path, y_1 and y_2 are consistent neighbors of x_1 and x_k, respectively, and each $\overrightarrow{T_{y_i}}$ is an antler that is properly positioned with respect to the heavy arc in each dicycle, $i = 1, 2$. See the oriented graph of V in Figure 3.1.*

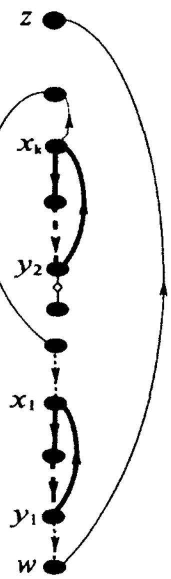

Figure 3.6: A two-page oriented book embedding of a member of $\mathcal{B}_4$.

Lemma 3.8. *Let $\overrightarrow{D} \in \mathcal{B}_5$. Then $obt(\overrightarrow{D}) = 2$.*

Proof. Let $\overrightarrow{D} \in \mathcal{B}_5$, such that y_1 and y_2 are in-neighbors of x_1 and x_k, respectively. Suppose for a contradiction that $obt(\overrightarrow{D}) < 2$. Since $\overrightarrow{D}$ contains a dicycle, $obt(\overrightarrow{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\overrightarrow{D}$. We may assume the page is upwards. By Corollary 2.1, the vertices of each dicycle must be located in the spine in a translation cycle of the dicycle, and each dicycle has exactly one loose arc. By Corollary 2.2, the heavy arc (y_1, x_1) in $\overrightarrow{D_m}$ is loose in the embedding, and since the page is upwards, y_1 is located below x_1 in the spine. By Lemma 3.4, there is an arc (w_1, z_1) of $\overrightarrow{T_{y_1}}$ such that (w_1, z_1) shields (y_1, x_1). Thus, z_1 is above x_1 and either $w_1 = y_1$ or w_1 is below y_1 and there is a directed path from y_1 to w_1 such that each vertex, except possibly w, in the directed path is tight. Now consider $\overrightarrow{D_n}$. Since x_1 is shielded by (w_1, z_1), and there is an oriented path between x_1 and x_k, by Lemma 1.1, the vertices of $\overrightarrow{D_n}$ must be located between x_1 and w_1 in the spine. By

Lemma 2.2, the heavy arc (y_2, x_k) in $\overrightarrow{D_n}$ is loose in the embedding, and since the page is upwards, y_2 is located below x_k in the spine. By Lemma 3.2, neither (y_1, x_1) or (y_2, x_k) shields the other. Consider $\overrightarrow{D_m}$ first. By Lemma 3.4, there is an arc (w_2, z_2) of $\overrightarrow{T_{y_2}}$ such that (w_2, z_2) shields (y_2, x_k). Thus, z_2 is above x_2 and either $w_2 = y_2$ or w_2 is below y_2 and there is a directed path from y_2 to w_2 such that each vertex, except possibly w_2, in the directed path is tight.

However, since x_k is shielded by (w_2, z_2) and x_1 is not, by Lemma 1.1, we have a contradiction. Therefore, $obt(\vec{D}) \geq 2$. In Figure 3.7, we show a two-page oriented book embedding of a member of $\mathcal{B}_5$; each member of $\mathcal{B}_5$ can be embedded this way, thus $obt(\vec{D}) = 2$. $\square$

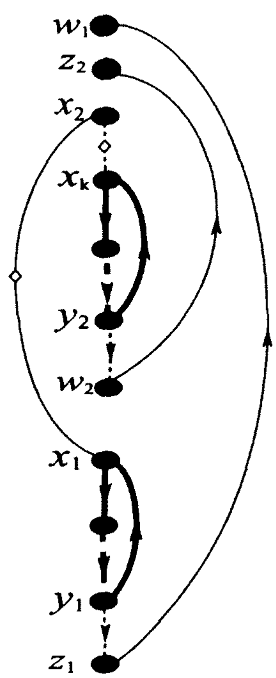

Figure 3.7: A two-page oriented book embedding of a member of $\mathcal{B}_5$.

For Lemmas 3.3-3.8, it was our goal to prove the oriented book thickness was equal to two. To prove that oriented graphs in $\mathcal{Y}$ are two-page-critical, and then to prove that these are the only strictly bi-dicyclic two-page-critical oriented graphs, we will construct one-page oriented book embeddings. In a one-page oriented book embedding of a strictly bi-dicyclic graph, the location of the directed cycles in the embedding is more strict than the embeddings of the oriented trees. Thus, we embed $\overrightarrow{H}$ first, such that the direction of the page is upwards, and then embed $\overrightarrow{T_u}$, such that $u \in V(\overrightarrow{H})$ is heavy. Given a one-page oriented book embedding of $\overrightarrow{H}$, we now discuss cases in which $\overrightarrow{T_u}$ can also be embedded into the 1-book. Recall the definitions of flower fountain tree and hook fountain tree from Chapter 2. We summarize these cases in Table 3.1.

Suppose we embed $\overrightarrow{H}$ into a 1-book, and u is a heavy vertex of $\overrightarrow{H}$. If u is loose, then $\overrightarrow{T_u}$ can be embedded into the 1-book. If u is loose-above, then $\overrightarrow{T_u}$ can be embedded into the 1-book if $\overrightarrow{T_u}$ is a flower fountain tree; similarly, if u is loose-below, then $\overrightarrow{T_u}$ can be embedded into the 1-book if $\overrightarrow{T_u}$ is a hook fountain tree.

For the case in which a vertex in $\overrightarrow{H}$ is heavy and tight, we require the following definition. For an oriented graph $\overrightarrow{D}$ that is embedded into a k-book, an open interval $\mathbb{I}$ is a subset of the spine that is disjoint from $\overrightarrow{D}$. A vertex u has *upper tree space* in the embedding if $\mathbb{I}$ is above u and for a given page there is no arc located in the page with one endpoint between u and $\mathbb{I}$ and other endpoint either below u or above $\mathbb{I}$. Similarly, a vertex u can have *lower tree space.* If a vertex has both upper tree space and lower tree space, then we say the vertex has *tree spaces.* If a vertex is loose-above (or loose-below), then it has upper tree space (or lower tree space). However, the converse is not necessarily true.

If a vertex u that is heavy in $\overrightarrow{H}$ has tree spaces in the embedding of $\overrightarrow{H}$, then $\overrightarrow{T_u}$ can be embedded into the one-page book. If u is a source in $\overrightarrow{T_u}$, then $\overrightarrow{T_u}$ can be embedded into the 1-book if u has upper tree space in the embedding of $\overrightarrow{H}$. If u is a sink in $\overrightarrow{T_u}$, then $\overrightarrow{T_u}$ can be embedded into the 1-book if u has lower tree space in the embedding of $\overrightarrow{H}$.

Table 3.1: Summary of embedding $\overrightarrow{T_u}$

Property of u:	Restriction on $\overrightarrow{T_u}$:
loose	$\overrightarrow{T_u}$ can contain positive or negative antler
loose-above	flower fountain tree
loose-below	hook fountain tree
tree spaces	$\overrightarrow{T_u}$ can contain positive or negative antler
upper tree space	u is a source in $\overrightarrow{T_u}$
lower tree space	u is a sink in $\overrightarrow{T_u}$

In Figure 3.8, y_2 is loose-above, x and y_1 have upper tree space with $\mathbb{I}$ between y_2 and z but are not loose-above, and the vertices v, z, w, and u have tree spaces.

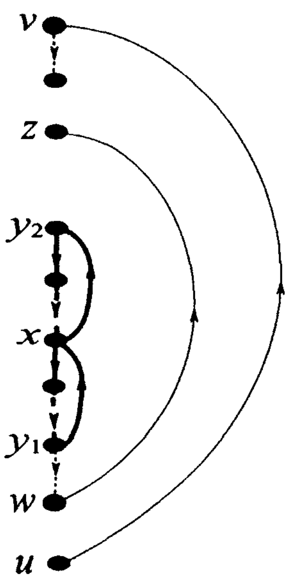

Figure 3.8: A one-page oriented book embedding.

Lemma 3.9. $\mathcal{Y} \subseteq \mathcal{M}^2$

Proof. By Lemmas 3.3-3.8, if $\vec{D} \in \mathcal{Y}$, then $obt(\vec{D}) = 2$. To show that for each arc $\vec{a} \in V(\vec{D})$, we have that $obt(\vec{D} \setminus \vec{a}) = 1$, we will consider the arcs of $\vec{H}$ first. If we delete an arc $\vec{a}$ of either dicycle, we obtain a strictly uni-dicyclic graph containing no member of $\mathcal{I} \cup \mathcal{T} \cup \mathcal{R}$, and thus $obt(\vec{D} \setminus \vec{a}) = 1$. For $\vec{D} \in \mathcal{B}_i$, $3 \leq i \leq 5$, if we delete an arc $\vec{a}$ of $\vec{P}$, we obtain two disconnected strictly uni-dicyclic oriented graphs, each containing no member of $\mathcal{I} \cup \mathcal{T} \cup \mathcal{R}$, and thus $obt(\vec{D} \setminus \vec{a}) = 1$. Therefore, we need only focus on deleting arcs *not* contained in $\vec{H}$. If $\vec{a} \in A(\overrightarrow{T_{y_1}})$, the deletion of $\vec{a}$ disconnects $\vec{D}$, and we obtain two components, an oriented tree (possibly trivial) and a member of $\mathcal{B}$ such that $\overrightarrow{T_{y_1}}$ in $\vec{D} \setminus \vec{a}$ is a (possibly trivial) directed path. By Theorem 2.3, the component of $\vec{D} \setminus \vec{a}$ that is an oriented tree can be embedded into a one-page book. Then we need only show that the component of $\vec{D} \setminus \vec{a}$ that is a strictly bi-dicyclic graph has a one-page oriented book embedding, which is seen in the following by showing the embedding of $\overrightarrow{H}$ in Figure 3.9. In each case, except case IV, we let $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$, and we embed the oriented subgraph $\overrightarrow{H}$ of $\overrightarrow{D} \setminus \overrightarrow{d}$ as shown in Figure 3.9, and y_2 has tree spaces with infinitely large open interval $\mathbb{I}$, and the case for $\overrightarrow{d} \in A(\overrightarrow{T_{y_2}})$ is similar. In case 4, we show the embedding of $\overrightarrow{H}$ when $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$ and when $\overrightarrow{d} \in A(\overrightarrow{T_{y_2}})$:

I) Let $\overrightarrow{D} \in \mathcal{B}_1$ and $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$.

II) Let $\overrightarrow{D} \in \mathcal{B}_2$ and $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$.

III) Let $\overrightarrow{D} \in \mathcal{B}_3$ and $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$.

IV) Let $\overrightarrow{D} \in \mathcal{B}_4$. Assume that 1) $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$, and 2) $\overrightarrow{d} \in A(\overrightarrow{T_{y_2}})$.

V) Let $\overrightarrow{D} \in \mathcal{B}_5$ and $\overrightarrow{d} \in A(\overrightarrow{T_{y_1}})$. Since $\overrightarrow{P}$ is not a directed path from x_k to x_1, there exists an arc, (x_{i+1}, x_i), $1 \leq i \leq k-1$, in $\overrightarrow{P}$ that prevents $\overrightarrow{P}$ from being a directed path from x_k to x_1.

□

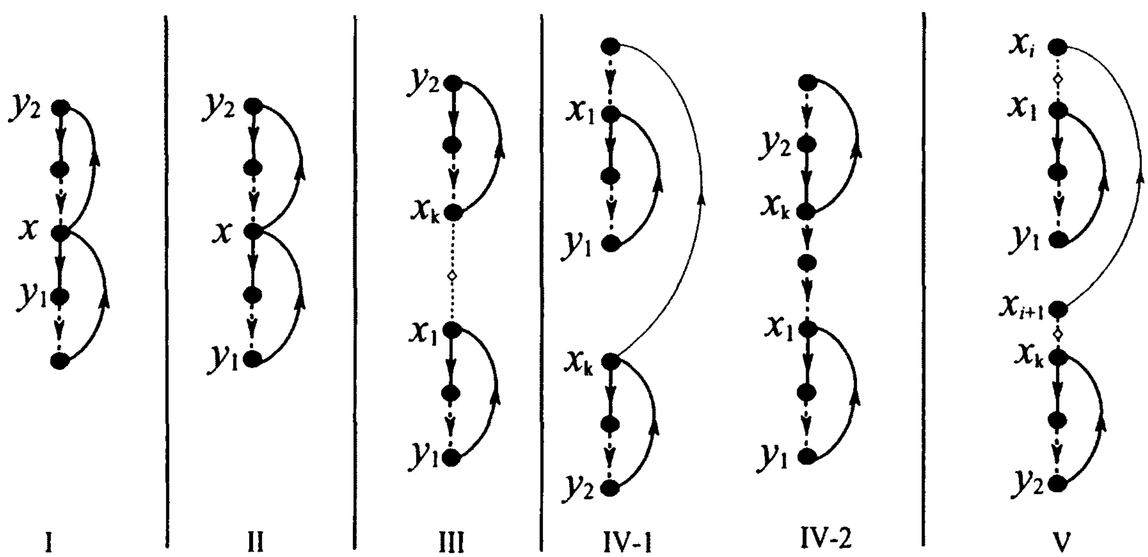

Figure 3.9: Lemma 3.9

We are now prepared to characterize strictly bi-dicyclic graphs having oriented book thickness equal to one using $\mathcal{I} \cup \mathcal{T} \cup \mathcal{R} \cup \mathcal{Y}$. To do this, we give the following four propositions, which will make up the main result, Theorem 3.1.

Proposition 3.1. *Let $\vec{D}$ be a strictly bi-dicyclic graph with oriented subgraph $\vec{H}$, such that $\vec{P}$ is a trivial oriented path. Then $obt(\vec{D}) \leq 1$ if and only if $\vec{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, $\mathcal{B}_1$, or $\mathcal{B}_2$.*

Proof. To prove the necessary condition, consider the contrapositive which states: If $\vec{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, $\mathcal{B}_1$, or $\mathcal{B}_2$, then $obt(\vec{D}) > 1$. This holds by Lemmas 2.4, 2.5, 2.6, 3.3, and 3.5, respectively. To prove the sufficient condition, assume that $\vec{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, $\mathcal{B}_1$, or $\mathcal{B}_2$, and we will construct a one-page oriented book embedding. $V(\vec{P}) = \{x\}$, let y_1, y_2 be the out-neighbors of x in $\vec{H}$, and z_1, z_2 be the in-neighbors of x in $\vec{H}$. Since $\vec{D}$ contains no $\mathcal{B}_1$, y_1 and y_2 cannot both be heavy and z_1 and z_2 cannot both be heavy; since $\vec{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$, y_1 and z_1 cannot both be heavy and y_2 and z_2 cannot both be heavy. We may then assume that $\overrightarrow{T_{y_1}}$, $\vec{T_x}$, and $\overrightarrow{T_{z_2}}$ are the only non-trivial oriented trees in $\vec{D}$. Since $\vec{D}$ contains no member of $\mathcal{B}_2$, either $\overrightarrow{T_{y_1}}$ contains no negative antler with sink y_1 (i.e., is flower fountain tree), or $\overrightarrow{T_{z_2}}$ contains no positive antler with source z_2 (i.e., is hook fountain tree). We have two cases:

i. Suppose $\overrightarrow{T_{y_1}}$ is a flower fountain tree, and $\overrightarrow{T_{z_2}}$ contains a positive antler with source z_2. Since $\vec{D}$ contains no member of $\mathcal{R}$, it must be true that x is a source in $\vec{T_x}$. Embed $\vec{H}$ as seen in Figure 3.10. Since y_1 is loose-above and $\overrightarrow{T_{y_1}}$ is a flower fountain tree, x has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{y_1}}$ and x is a source in $\overrightarrow{T_x}$, and z_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is above the highest vertex of $\overrightarrow{T_x}$ and $\mathbb{I}'$ is directly below z_2; thus, we obtain a one-page oriented book embedding.

ii. Suppose $\overrightarrow{T_{z_2}}$ is a hook fountain tree, and $\overrightarrow{T_{y_1}}$ contains a negative antler with sink y_1. Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, it must be true that x is a sink in $\overrightarrow{T_x}$. Embed $\overrightarrow{H}$ as seen in Figure 3.10. Since z_2 is loose-below and $\overrightarrow{T_{z_2}}$ is a hook fountain tree, x has lower tree space with open interval $\mathbb{I}$ below the lowest vertex of $\overrightarrow{T_{z_2}}$ and x is a sink in $\overrightarrow{T_x}$, and y_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above y_1 and $\mathbb{I}'$ below the lowest vertex of $\overrightarrow{T_x}$. Thus, we obtain a one-page oriented book embedding.

□

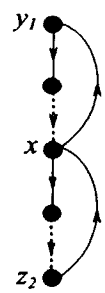

Figure 3.10: Embedding $\overrightarrow{H}$ in Proposition 3.1.

We now consider strictly bi-dicyclic graphs such that the oriented path $\overrightarrow{P}$ is non-trivial. Since an oriented path is also an oriented tree, we may have a sink-top oriented book embedding of $\overrightarrow{P}$ or a source-bottom oriented book embedding of $\overrightarrow{P}$, as defined in Chapter 2.

Proposition 3.2. *Let $\overrightarrow{D}$ be a strictly bi-dicyclic graph with oriented subgraph $\overrightarrow{H}$ such that $\overrightarrow{P}$ is non-trivial, y_1 and y_2 are heavy, inconsistent neighbors of x_1 and x_k,* *respectively. Then* $obt(\overrightarrow{D}) \leq 1$ *if and only if* $\overrightarrow{D}$ *contains no member of* $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, *or* $\mathcal{B}_3$.

Proof. To prove the necessary condition, consider the contrapositive which states: If $\overrightarrow{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_3$ then $obt(\overrightarrow{D}) > 1$. This holds by Lemmas 2.4, 2.5, 2.6, and 3.6, respectively. To prove the sufficient condition, assume that $\overrightarrow{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_3$ and we will construct a one-page oriented book embedding. Since $\overrightarrow{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$, y_1, x_1 and y_2, x_k are the *only* heavy vertices in each dicycle. Since $\overrightarrow{D}$ contains no member of $\mathcal{B}_3$, either $\overrightarrow{T_{y_1}}$ contains no positive antler with source y_1 (i.e., is a hook fountain tree) or $\overrightarrow{T_{y_2}}$ contains no negative antler with sink y_2 (i.e., is a flower fountain tree). Then we have the following two cases:

i. Suppose that $\overrightarrow{T_{y_1}}$ is a hook fountain tree, and $\overrightarrow{T_{y_2}}$ contains a negative antler with sink y_2. Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, $\overrightarrow{T_{x_k}}$ is a hook fountain tree. Embed the dicycles into a one-page book so that y_2 is the top vertex in the spine and y_1 is bottom vertex in the spine. Then x_k is above x_1 in the spine as well. We have two cases to embed $\overrightarrow{P}$.

   a. Suppose that $(x_{k-1}, x_k) \in A(\overrightarrow{P})$. Then, by Lemma 2.1, $\overrightarrow{P}$ has a sink-top oriented book embedding such that x_k is above every other vertex in $\overrightarrow{P}$, and for $1 < i < k$, each x_i is loose in the embedding. An example can be seen in Figure 3.12. Then y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is above y_2 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_u}$, y_1 is loose-below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$, x_k is loose-below, and for $2 \leq i \leq k-1$, each x_i is loose, and thus we obtain a one-page oriented book embedding.

b. Suppose that there is a directed path from x_k to some x_j, $2 \leq j \leq k-1$, such that $(x_{j-1}, x_j) \in A(\overrightarrow{P})$. Then, to avoid $\overrightarrow{D}$ containing a member of $\mathcal{R}$ as an oriented subgraph, for $j \leq i \leq k$, each x_i is a sink in $\overrightarrow{T_{x_i}}$. Place the vertices $x_j, x_{j+1}, \ldots, x_{k-1}$ into the spine consecutively below x_k. Then, by Lemma 2.1, the oriented path between x_1 and x_j has a sink-top oriented book embedding such that x_j is above every other vertex in the oriented path, and for $1 < i < j$, each vertex x_i is loose in the embedding. An example can be seen in Figure 3.12. Then y_1 is loose-below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x_1 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$. For $2 \leq h \leq j-1$, each x_h is loose, x_j has lower tree space with open interval $\mathbb{I}$ below the lowest vertex of $\overrightarrow{T_u}$. For $j+1 \leq i \leq k$, each x_i has lower tree space with open interval $\mathbb{I}$ below the lowest vertex of $\overrightarrow{T_{x_{i-1}}}$, and y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above y_2 and $\mathbb{I}'$ below the lowest vertex of $\overrightarrow{T_{x_k}}$. Thus, we obtain a one-page oriented book embedding.

c. If there is a directed path from x_k to x_1, then to avoid a member of $\mathcal{R}$, each x_i is a sink in $\overrightarrow{T_{x_i}}$, $1 \leq i \leq k$. An example can be seen in Figure 3.12. Then y_1 is loose-below, x_1 has lower tree space with open interval $\mathbb{I}$ below the lowest vertex of $\overrightarrow{T_{y_1}}$, and each x_i, $2 \leq i \leq k$, has lower tree space with open interval $\mathbb{I}$ below the lowest vertex of $\overrightarrow{T_{x_{i-1}}}$, and y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is above y_2 and $\mathbb{I}'$ below the lowest vertex of $\overrightarrow{T_{x_k}}$. Thus, we obtain a one-page oriented book embedding.

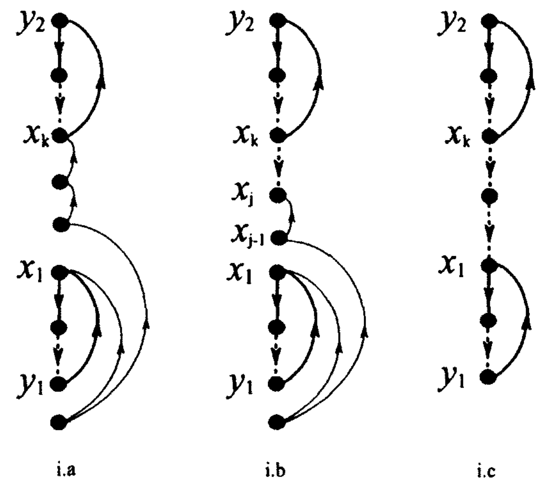

Figure 3.11: Embedding $\overrightarrow{H}$ in Proposition 3.2 for case i.

ii. Suppose that $\overrightarrow{T_{y_2}}$ is a flower fountain tree, and $\overrightarrow{T_{y_1}}$ contains a positive antler with source y_1. Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, $\overrightarrow{T_{x_1}}$ is a flower fountain tree. Embed the dicycles into a one-page book so that y_2 is the top vertex in the spine and y_1 is the bottom vertex in the spine. Then x_k is above x_1 in the spine as well. We have two cases to embed $\overrightarrow{P}$.

a. Suppose that $(x_1, x_2) \in A(\overrightarrow{P})$. Then, by Lemma 2.1, $\overrightarrow{P}$ has a source-bottom oriented book embedding such that x_1 is below every other vertex in $\overrightarrow{P}$, and for $1 < i < k$, each x_i is loose in the embedding. An example can be seen in Figure 3.12. Then x_1 is loose above, y_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below y_1 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_u}$, y_2 is loose-above, x_k has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below x_k and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{y_2}}$, x_1 is loose-above, each x_i, $2 \le i \le k-1$, is loose, and thus we obtain a one-page oriented book embedding.

b. Suppose there is a directed path to x_1 from some x_j, $2 < j \le k-1$, such that $(x_j, x_{j+1}) \in A(\overrightarrow{P})$. Then, to avoid a member of $\mathcal{R}$, for $1 \le i \le j$, each x_i is a source in $\overrightarrow{T_{x_i}}$. Place the vertices $x_j, x_{j-1}, ..., x_2$ into the spine consecutively above x_1. Then, by Lemma 2.1, the oriented path between x_j and x_k has a source-bottom oriented book embedding such that x_j is below every other vertex in the oriented path, and for $j < i < k$, each vertex x_i is loose in the embedding. An example can be seen in Figure 3.12. Then y_2 is loose-above, x_k has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below x_k and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{y_2}}$, for $j+1 \le h \le k-1$, each x_h, is loose, x_j has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_u}$, for $1 \le i \le j-1$, each x_i has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{x_{i+1}}}$, and y_1 has tree spaces with open interval $\mathbb{I}$ directly below y_1 and $\mathbb{I}'$ above the highest vertex of $\overrightarrow{T_{x_1}}$. Thus. we obtain a one-page oriented book embedding.

c. If there is a directed path from x_k to x_1, then for $1 \le i \le k$, each x_i is a source in $\overrightarrow{T_{x_i}}$. An example can be seen in Figure 3.12. Then y_2 is loose-above, x_k has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{y_2}}$, and for $2 \le i \le k-1$, each x_i has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{x_{i+1}}}$, and y_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below y_1 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{x_1}}$. Thus, we obtain a one-page oriented book embedding.

□

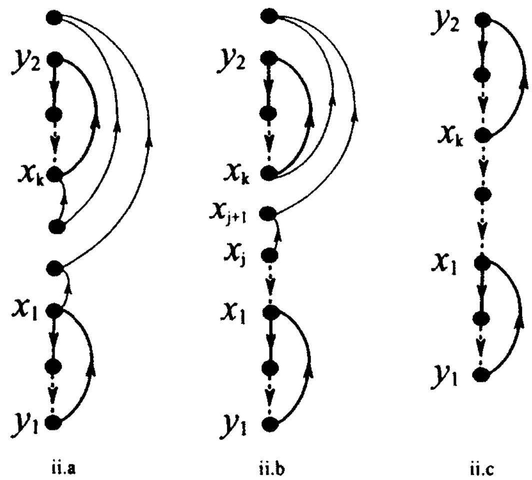

Figure 3.12: Embedding $\overrightarrow{H}$ in Proposition 3.2 for case ii.

Proposition 3.3. *Let $\overrightarrow{D}$ be a strictly bi-dicyclic graph with oriented subgraph $\overrightarrow{H}$ such that $\overrightarrow{P}$ is a non-trivial directed path, y_1 and y_2 are heavy, consistent neighbors of x_1 and x_k, respectively. Then $obt(\overrightarrow{D}) \leq 1$ if and only if $\overrightarrow{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_4$.*

Proof. To prove the necessary condition, consider the contrapositive which states: If $\overrightarrow{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_4$, then $obt(\overrightarrow{D}) > 1$. This holds by Lemmas 2.4, 2.5, 2.6, and 3.7. To prove the sufficient condition, assume that $\overrightarrow{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_4$, and we will construct a one-page oriented book embedding. We may assume that y_1 and y_2 are in-neighbors of x_1 and x_k, respectively, and that there is a directed path from x_k to x_1. Since $\overrightarrow{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$, y_1, x_1, y_2 and x_k are the only vertices that can be heavy in each dicycle. Since $\overrightarrow{D}$ contains no member of $\mathcal{B}_4$, either $\overrightarrow{T_{y_2}}$ is a trivial oriented tree or $\overrightarrow{T_{y_1}}$ does not contain a positive antler with source y_1 (i.e., is a hook fountain tree). We have two cases:

i. Suppose that $\overrightarrow{T_{y_2}}$ is a trivial oriented tree and $\overrightarrow{T_{y_1}}$ is contains a positive antler with source y_1. Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, for $1 \leq i \leq k$, each x_i must be a source in $\overrightarrow{T_{x_i}}$. Then we may embed $\overrightarrow{H}$ in $\overrightarrow{D}$ as seen in Figure 3.13, and since x_k has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{H}$. For $1 \leq i \leq k-1$, each x_i has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{x_{i+1}}}$, and y_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below y_1 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{x_1}}$. Thus, we obtain a one-page oriented book embedding.

ii. Suppose that $\overrightarrow{T_{y_1}}$ is a hook fountain tree and $\overrightarrow{T_{y_2}}$ is a non-trivial oriented tree. Since $\overrightarrow{D}$ contains no member of $\mathcal{R}$, either $\overrightarrow{T_{x_k}}$ does not contain a negative antler with sink x_k (i.e., is a flower fountain tree), or $\overrightarrow{T_{y_2}}$ does not contain a positive antler with source y_2 (i.e., is a hook fountain tree). In either case, embed $\overrightarrow{H}$ as seen in Figure 3.13.

a. Suppose that $\overrightarrow{T_{x_k}}$ is a flower fountain tree and $\overrightarrow{T_{y_2}}$ contains a positive antler with source y_2. Since y_1 is loose below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x_1 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$ and x_k is loose above. For $2 \leq i \leq k-1$, each x_i is loose, y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below y_2 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{x_1}}$. Thus, we obtain a one-page oriented book embedding.

b. Suppose that $\overrightarrow{T_{y_2}}$ is a hook fountain tree and $\overrightarrow{T_{x_k}}$ contains a negative antler with sink x_k. Since y_1 is loose below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x_1 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$, y_2 is loose above. For $2 \leq i \leq k-1$, each x_i is loose, x_k has tree spaces with $\mathbb{I}$ directly above x_k and $\mathbb{I}'$ below the lowest vertex of $\overrightarrow{T_{y_2}}$. Thus, we obtain a one-page oriented book embedding.

$\square$

Proposition 3.4. *Let $\overrightarrow{D}$ be a strictly bi-dicyclic graph with oriented subgraph $\overrightarrow{H}$ such that $\overrightarrow{P}$ is a non-trivial oriented (but not directed) path, such that y_1 and y_2 are heavy, consistent neighbors of x_1 and x_k, respectively. Then $obt(\overrightarrow{D}) \leq 1$ if and only if $\overrightarrow{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_5$.*

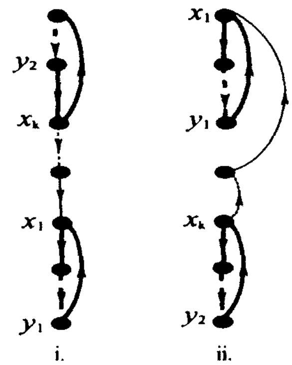

Figure 3.13: Embedding $\vec{H}$ in Proposition 3.3.

Proof. To prove the necessary condition, consider the contrapositive which states: If $\vec{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_5$, then $obt(\vec{D}) > 1$. This holds by Lemmas 2.4, 2.5, 2.6, and 3.8. To prove the sufficient condition, assume that $\vec{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{R}$, or $\mathcal{B}_5$, and we will construct a one-page oriented book embedding. We may assume that y_1 and y_2 are in-neighbors of x_1 and x_k, respectively, and that there is a non-trivial oriented (but not directed) path between x_k and x_1. Since $\vec{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$, y_1, x_1, y_2 and x_k are the only vertices that can be heavy in each dicycle. Since $\vec{D}$ contains no member of $\mathcal{B}_5$ either $\overrightarrow{T_{y_1}}$ does not contain a positive antler with source y_1, i.e., $\overrightarrow{T_{y_1}}$ is a hook fountain tree, or $\overrightarrow{T_{y_2}}$ does not contain a positive antler with source y_2, i.e., $\overrightarrow{T_{y_2}}$ is a hook fountain tree. In either case, the proof is symmetric. We may assume that $\overrightarrow{T_{y_1}}$ is a hook fountain tree and $\overrightarrow{T_{y_2}}$ contains a positive antler with source y_2. To avoid a member of $\mathcal{R}$, $\overrightarrow{T_{x_k}}$ must be a flower fountain tree. Embed the dicycles into a one-page book so that x_1 is the top vertex in the spine and y_2 is the bottom vertex in the spine. Then x_1 is above x_k in the spine as well. We have two cases to embed $\vec{P}$.

i. Suppose that $(x_k, x_{k-1}) \in A(\vec{P})$. Then, by Lemma 2.1, $\vec{P}$ has a source-bottom oriented book embedding such that x_k is below every other vertex in $\vec{P}$, and for $1 < i < k$, each x_i is loose in the embedding. An example can be seen in Figure 3.14. Then x_k is loose-above, For $2 \leq i \leq k-1$, each x_i is loose, y_1 is loose-below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x_1 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$; y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly below y_2 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_u}$, and thus we obtain a one-page oriented book embedding.

ii. Suppose that there is a directed path from some x_j to x_k, $2 \leq j \leq k-1$, such that $(x_j, x_{j-1}) \in A(\overrightarrow{P})$. This x_j must exist, as $\overrightarrow{P}$ is not a directed path. Then, to avoid a member of $\mathcal{R}$, for $j \leq i \leq k$, each x_i must be a source in $\overrightarrow{T_{x_i}}$. Place the vertices $x_j, x_{j+1}, ..., x_{k-1}$ into the spine consecutively above x_k. Then, by Lemma 2.1, the oriented path between x_1 and x_j has a source-bottom oriented book embedding such that x_j is below every other vertex in the oriented path, and each vertex x_i, $1 < i < j$ is loose in the embedding. An example can be seen in Figure 3.14. Then, y_1 is loose-below, x_1 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is directly above x_1 and $\mathbb{I}'$ is below the lowest vertex of $\overrightarrow{T_{y_1}}$; x_j has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{x_1}}$. For $j+1 \leq i \leq k$, each x_i has upper tree space with open interval $\mathbb{I}$ above the highest vertex of $\overrightarrow{T_{x_{i+1}}}$, and y_2 has tree spaces with open intervals $\mathbb{I}$ and $\mathbb{I}'$, such that $\mathbb{I}$ is below y_2 and $\mathbb{I}'$ is above the highest vertex of $\overrightarrow{T_{x_k}}$, and thus we obtain a one-page oriented book embedding.

□

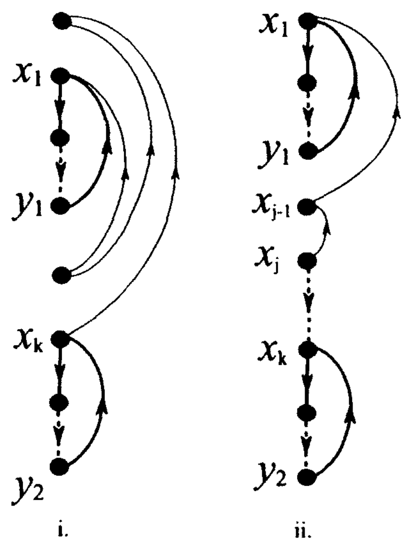

Figure 3.14: Embedding $\overrightarrow{H}$ in Proposition 3.4.

CHAPTER 4

SWITCHING CRITICAL GRAPHS

In the previous two chapters, we discussed k-page critical oriented graphs, oriented graphs such that for each arc, the deletion of that arc decreases the oriented book thickness. In this chapter, we focus on switching the direction of each arc. Recall that the converse of an arc $\vec{a}$ is denoted $\vec{a}^*$, and for an oriented graph $\vec{D}$ with arc $\vec{a}$, the oriented graph obtained by replacing $\vec{a} = (x, y)$ with $\vec{a}^* = (y, x)$ is denoted $\vec{D}(\vec{a}^*)$.

Definition 4.1. *An oriented graph $\vec{D}$ is a switching k-page-critical graph if $obt(\vec{D}) = k$ and for every arc $\vec{a} \in A(\vec{D})$, we have that $obt(\vec{D}(\vec{a}^*)) = k - 1$. The class of all switching k-critical graphs is denoted $\mathcal{M}_s^k$.*

$\mathcal{M}_s^k \subseteq \mathcal{M}^k$, since if switching the direction of an arc decreases the oriented book thickness, then deleting the arc will as well. We will discuss the class $\mathcal{R}$, defined in Chapter 2, and show that $\mathcal{R} \subseteq \mathcal{M}_s^2$; we will also define two new classes, $\mathcal{N}$ and $\mathcal{W}^4$, and show that $\mathcal{N} \subseteq \mathcal{M}_s^2$ and $\mathcal{W}^4 \subseteq \mathcal{M}_s^3$.

Let $\vec{D} \in \mathcal{R}$. In Lemma 2.6 in Chapter 2, we proved that $obt(\vec{D}) = 2$, so to prove that $\vec{D} \in \mathcal{M}_s^2$, we now must show for each arc $\vec{a} \in A(\vec{D})$, we have that $obt(\vec{D}(\vec{a}^*)) = 1$. Recall that $\mathcal{R} \subseteq \mathcal{U}$, where $\mathcal{U}$ is the class of strictly uni-dicyclic oriented graphs. If we switch the direction of an arc $\vec{a}$ in the directed cycle, we no longer have a dicycle, but an oriented cycle which contains a directed hamilton path. The result below, translated from [10], shows how to embed such an oriented graph so that each vertex in the oriented cycle of $\vec{D}(\vec{a}^*)$ is loose, i.e., has tree spaces in the embedding. This allows us to embed the antlers of $\vec{D}(\vec{a}^*)$ without requiring a second page.

Theorem 4.1. (*Heath, Pemmaraju, and Trenk,* 1999 [*10*]) *Let $\vec{D}$ be an oriented graph that contains exactly one oriented cycle, $\vec{C}$, which is not a directed cycle. If $\vec{C}$ contains a directed hamilton path, then $obt(\vec{D}) = 1$.*

We are now prepared to prove that the class $\mathcal{R}$ is switching two-page critical.

Lemma 4.1. $\mathcal{R} \subseteq \mathcal{M}_s^2$.

Proof. Let $\vec{D} \in \mathcal{R}$. By Lemma 2.6 in Chapter 2, $obt(\vec{D}) = 2$. To show, for all $\vec{a} \in A(\vec{D})$, we have that $obt(\vec{D}(\vec{a}^*)) = 1$, first let $\vec{a}$ be an arc in the directed cycle of $\vec{D}$. Then $\vec{D}(\vec{a}^*)$ contains an oriented cycle $\vec{C}$, that is not a dicycle, such that $\vec{C}$ contains a directed hamilton path. By Theorem 4.1, $obt(\vec{D}(\vec{a}^*)) = 1$. Now let $\vec{a}$ be an arc that is not part of the directed cycle; by Theorem 2.4 in Chapter 2, $obt(\vec{D}(\vec{a}^*)) = 1$. Thus $\vec{D} \in \mathcal{M}_s^2$. $\square$

By Theorem 2.4 in Chapter 2, $\mathcal{M}^2 \cap \mathcal{U} = \mathcal{I} \cup \mathcal{T} \cup \mathcal{R}$. Clearly, $\mathcal{I}$ and $\mathcal{T}$ are not in $\mathcal{M}_s^2$, as certain arcs of members of $\mathcal{I}$ and $\mathcal{T}$ have arbitrary direction. Since $\mathcal{M}_s^2 \subseteq \mathcal{M}^2$, we have the following result.

Lemma 4.2. $\mathcal{M}_s^2 \cap \mathcal{U} = \mathcal{R}$.

The next class, $\mathcal{N}$, of oriented graphs in $\mathcal{M}_s^2$ is a subset of the following class of oriented graphs, denoted $\mathcal{C}^d$. Recall that for a graph D containing a cycle C, a *chord* of C is an edge $\{x, y\} \in E(D)$ such that $x, y \in V(C)$, but $\{x, y\} \notin E(C)$. We define the class $\mathcal{C}^d$ to be the class of all oriented graphs $\overrightarrow{D}$ whose underlying graph is outerplanar, and consists of a cycle with d chords, called the *outer cycle*. For a member of $\mathcal{C}^d$, we call the oriented subgraph whose underlying graph is the outer cycle the *outer oriented cycle*. The next result helps us prove that, for an oriented graph $\overrightarrow{D} \in \mathcal{N}$, it is true that $obt(\overrightarrow{D}) > 1$.

Lemma 4.3. *Let $\overrightarrow{D} \in \mathcal{C}^d$. If there exists a one-page oriented book embedding of $\overrightarrow{D}$, then, in such an embedding, each chord is embedded into the interior of the page.*

Proof. Let $\overrightarrow{D} \in \mathcal{C}^d$ such that there exists a one-page oriented book embedding of $\overrightarrow{D}$. By Theorem 2.1 in Chapter 2, the spine order is a translation cycle of the oriented cycle; since each chord has endpoints which are non-consecutive in every translation cycle of the oriented cycle, each chord must be embedded into the interior of the page. □

We are now prepared to describe the next class of two-page switching-critical graphs. The class $\mathcal{N}$ consists of each oriented graph $\overrightarrow{D} \in \mathcal{C}^1$ such that $\overrightarrow{D}$ contains exactly two directed cycles which share exactly one arc, the chord of $\overrightarrow{D}$, and share no vertex other than the endpoints of the chord. For convenience, we denote the two dicycles $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$, for $m, n \geq 3$, and the chord (x, y). See Figure 4.1.

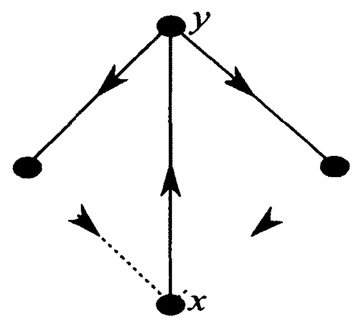

Figure 4.1: A general member of $\mathcal{N}$.

Lemma 4.4. *Let $\vec{D} \in \mathcal{N}$. Then $obt(\vec{D}) = 2$.*

Proof. Assume, for a contradiction, that $obt(\vec{D}) < 2$. Since $\vec{D}$ contains an oriented cycle, $obt(\vec{D}) \neq 0$. Now suppose that there exists a one-page oriented book embedding of $\vec{D}$. Let (x, y) be the chord of $\vec{D}$; by Lemma 4.3, (x, y) is embedded into the interior of the page. As $\vec{D}$ also contains exactly two directed cycles, by Corollary 2.1 in Chapter 2, each directed cycle in $\vec{D}$ has exactly one arc embedded into the interior of the page. Since (x, y) is contained in both directed cycles, and is already located in the interior of the page, no other arc of $\vec{D}$ is embedded into the page. However, each arc in $A(\vec{D}) \setminus (x, y)$ must be located in the spine, a contradiction, since the arcs in $A(\vec{D}) \setminus (x, y)$ form an oriented cycle. Thus, $obt(\vec{D}) \geq 2$. In Figure 4.2, we show a two-page oriented book embedding of a member of $\mathcal{N}$; each member of $\mathcal{N}$ can be embedded this way, thus $obt(\vec{D}) \leq 2$. Therefore, $obt(\vec{D}) = 2$. □

To show that $\mathcal{N} \subseteq \mathcal{M}_s^2$, we first prove the following, stronger result that allows us to switch the direction of more than one arc in $\vec{D}$.

Theorem 4.2. *Let $\vec{D} \in C^1$. Then $obt(\vec{D}) \leq 1$ if and only if $\vec{D} \notin \mathcal{N}$.*

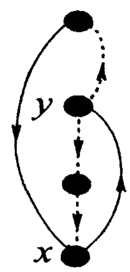

Figure 4.2: A two-page oriented book embedding of a member of $\mathcal{N}$.

Proof. To prove the necessary condition, consider the contrapositive which states: if $\vec{D} \in \mathcal{N}$, then $obt(\vec{D}) > 1$. This is true by Lemma 4.4. To prove the sufficient condition, suppose $\vec{D} \notin \mathcal{N}$. Let the chord be (x, y). Then there cannot be two arc-disjoint directed paths from y to x; thus, we have three possibilities for $\vec{D}$. In each case, $\vec{D}$ can be embedded into a one-page book as seen in Figure 4.3.

I. The tail of the chord x has out-degree at least 2.

II. The head of the chord y has in-degree at least 2.

III. There exists a source in $\vec{D}$.

□

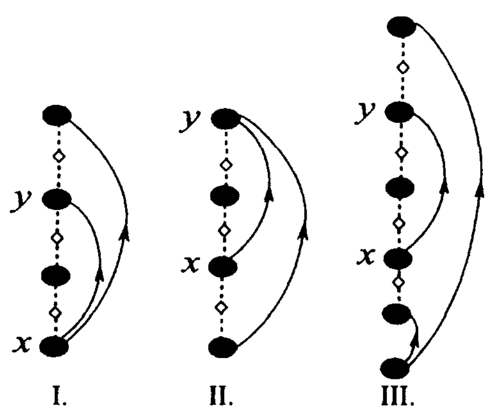

Figure 4.3: A one-page oriented book embedding of $\vec{D} \in \mathcal{C}^1 \setminus \mathcal{N}$.

For an arc a in an oriented graph $\vec{D} \in \mathcal{N}$, we have that $\vec{D}(a^*) \in \mathcal{C}^1$ and $\vec{D}(a^*) \notin \mathcal{N}$; thus, we have the following corollary of Theorem 4.2.

Corollary 4.1. $\mathcal{N} \subseteq \mathcal{M}_s^2$.

We now describe another class of oriented graphs, which we denote $\mathcal{W}^4$.

Definition 4.2. *For $n \geq 4$, let $\overrightarrow{C_n}$ be an oriented cycle on n vertices which contains exactly two sources, y_1 and y_2, and exactly two sinks, z_1 and z_2. Then an oriented graph in $\mathcal{W}^4$ is an oriented graph with vertex set $V(\overrightarrow{C_n}) \cup \{x\}$ and with arc set $A(\overrightarrow{C_n}) \cup \{(x, y_1), (x, y_2), (z_1, x), (z_2, x)\}$. We denote a member of $\mathcal{W}^4$ as $\overrightarrow{W_n^4}$. Figure 4.4 shows $\overrightarrow{W_4^4}$.*

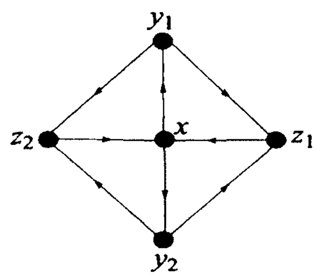

Figure 4.4: $\overrightarrow{W_4^4}$

Lemma 4.5. $obt(\overrightarrow{W_n^4}) > 2$

Proof. Suppose for a contradiction that there exists a two-page oriented book embedding of $\overrightarrow{W_n^4}$. Suppose $n = 4$. There are four possible cyclic orderings for y_1, y_2, z_1, z_2 to appear in the spine order: *i)* y_1, y_2, z_1, z_2 *ii)* y_1, z_1, y_2, z_2, *iii)* y_1, z_1, z_2, y_2, and *iv)* z_1, y_1, y_2, z_2. This is because of the symmetry of $\overrightarrow{W_4^4}$; for instance, the ordering y_1, y_2, z_1, z_2 is the same as y_1, y_2, z_2, z_1 and y_2, y_1, z_1, z_2, since y_1 and y_2 are both out-neighbors of x, z_1 and z_2 are both in-neighbors of x and out-neighbors of both y_1, y_2. In each case, wherever x may appear in the spine order in relation to y_1, y_2, z_1, z_2, three pages are required. Figure 4.5 shows the four cases. If $n > 4$, the proof is similar.

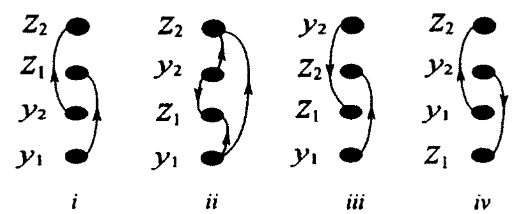

Figure 4.5: Possibilites for y_1, y_2, z_1, z_2.

□

Lemma 4.6. $\overrightarrow{W_n^4} \in \mathcal{M}_s^3$.

Proof. We will first prove the statement for $n = 4$. To show that $\overrightarrow{W_4^4} \in \mathcal{M}_s^3$, we first show the statement is true for the arcs $\overrightarrow{W_4^4}((y_1, z_1)^*)$ and $\overrightarrow{W_4^4}((x, y_1)^*)$. The two-page oriented book embedding of $\overrightarrow{W_4^4}(y_1, z_1)^*$ and $\overrightarrow{W_4^4}(x, y_1)^*$ are shown in Figure 4.6 below. If a is an arc other than (y_1, z_1) or (x, y_1), the case is symmetric. For $n > 4$, the proof is similar.

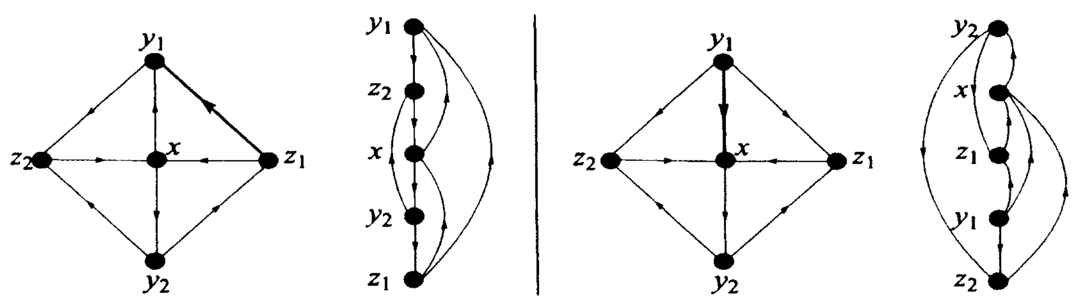

Figure 4.6: $\overrightarrow{W_4^4}(y_1, z_1)^*$ (left) and $\overrightarrow{W_4^4}(x, y_1)^*$ (right).

□

CHAPTER 5

MISCELLANEOUS RESULTS AND FUTURE WORK

In the last chapter, we introduced the class of all oriented graphs whose underlying graph is outerplanar and consists of a cycle with d chords, denoted $\mathcal{C}^d$, and characterized two-page switching critical graphs in $\mathcal{C}^1$ (see the first picture of Figure 5.1) by the class $\mathcal{N}$. In this chapter, we consider the class $\mathcal{C}^d$ for $d > 1$. In the first section, we focus on oriented cycles in $\mathcal{C}^2$ (see the second picture of Figure 5.1), and define a class of oriented graphs, which we use to characterize two-page critical graphs in $\mathcal{C}^2$ along with the classes $\mathcal{I}, \mathcal{T}$, and $\mathcal{N}$. In the second section, we discuss a structure that may appear in oriented graphs in $\mathcal{C}^d$, for $d \geq 3$, called a *z-structure* (see the third picture of Figure 5.1), and show that if an oriented graph in $\mathcal{C}^d$ contains a z-structure as an oriented subgraph, then it has oriented book thickness greater than one. In the last section, we discuss future work to follow this dissertation.

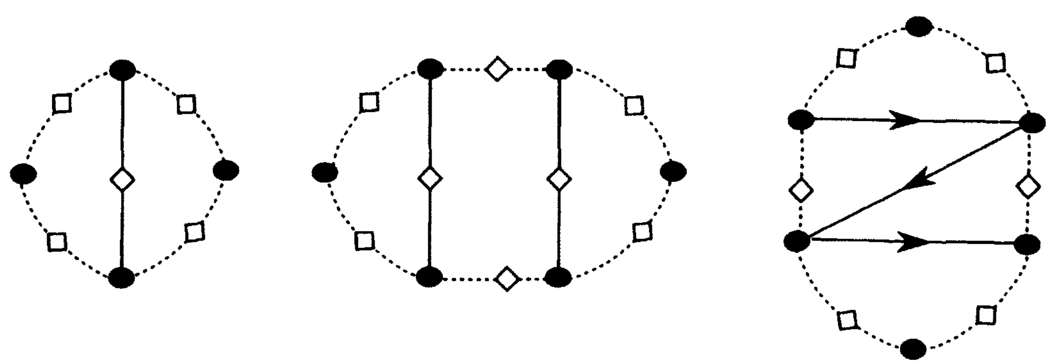

Figure 5.1: An oriented graph in $\mathcal{C}^1$,$\mathcal{C}^2$, and a z-structure.

5.1 Oriented Cycles with Two Chords

Each oriented graph in $\mathcal{C}^2$ contains an oriented subgraph such that the underlying graph is a strictly bi-dicyclic oriented graph. However, strictly bi-dicyclic graphs require oriented cycles to be directed cycles, while the direction of each arc in an oriented graph in $\mathcal{C}^2$ is arbitrary. To discuss two-page critical oriented graphs in $\mathcal{C}^2$, like strictly bi-dicyclic graphs, we will consider oriented graphs in $\mathcal{C}^2$ which contain at least two, arc-disjoint directed cycles. In the following definition of the class $\mathcal{O}$, the direction of the chords is extremely important; in a planar embedding of a member of $\mathcal{O}$, the direction of the chords create dicycles with "opposite direction", one dicycle having clockwise direction and the other counter-clockwise. If this is not the case, we can embed the oriented graph into a 1-book.

We state condition B. in the definition below to avoid a member of $\mathcal{O}$ either having multiple edges or containing a member of $\mathcal{B}_1$ as an oriented subgraph, see the left of Figure 5.2; we state condition C. to avoid an oriented graph in $\mathcal{O}$ containing a member of $\mathcal{N}$ as an oriented subgraph, see the right of Figure 5.2. Three members of $\mathcal{O}$ can be seen in Figure 5.3.

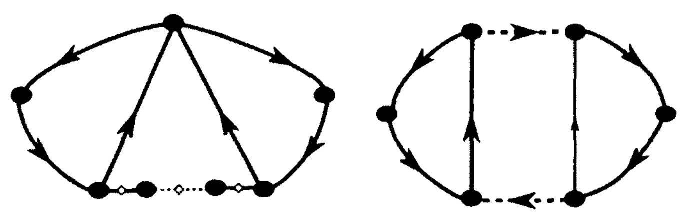

Figure 5.2: Members of $\mathcal{C}^2$ that contain a member of $\mathcal{B}_1$ (left) and $\mathcal{N}$ (right).

Definition 5.1. *The class $\mathcal{O}$ is the class of each oriented graph in $\mathcal{C}^2$ which contains exactly two, arc-disjoint directed cycles $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$ that share at most one vertex,* *such that each dicycle contains one of the chords, and two oriented paths $\overrightarrow{P}$ and $\overrightarrow{P}'$, that are arc-disjoint from $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$ such that the following three conditions hold:*

A. *The endpoints of $\overrightarrow{P}$ are the tails of the chords, and the endpoints of $\overrightarrow{P}'$ are the heads of the chords;*

B. *If $\overrightarrow{P}$ is a trivial oriented path, then $|V(\overrightarrow{P}')| = 2$ or $|V(\overrightarrow{P}')| = 3$; and*

C. *If $\overrightarrow{P}$ and $\overrightarrow{P}'$ are both non-trivial directed paths, then the source of each $\overrightarrow{P}$ and $\overrightarrow{P}'$ must be an endpoint of the same chord.*

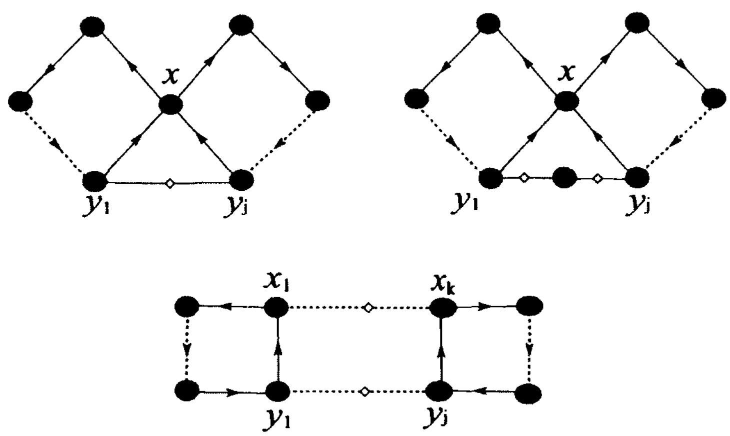

Figure 5.3: Members of $\mathcal{O}$

Then, each oriented graph $\overrightarrow{D}$ in $\mathcal{O}$ contains $\overrightarrow{H} = \overrightarrow{D_m} \cup \overrightarrow{P} \cup \overrightarrow{D_n}$, defined in Chapter 3, as a proper oriented subgraph, where $\overrightarrow{P}$ has endpoints x_1 and x_k in the dicycles $\overrightarrow{D_m}$ and $\overrightarrow{D_n}$, respectively. Let y_1 and y_j be the endpoints of $\overrightarrow{P}'$. Then y_1 and y_j are consistent neighbors of x_1 and x_k, respectively, and the chords are heavy arcs in $\overrightarrow{D}$. Notice, that in each translation cycle of the outer oriented cycle of a member of $\mathcal{O}$, the endpoints of the oriented path cannot alternate, i.e., x_1 (or y_1) cannot appear between y_1 and y_j (or x_1 and x_k) in the spine while x_k (or y_j) is not between y_1 and y_j (or x_1 and x_k) in the spine.

Lemma 5.1. *Let $\vec{D} \in \mathcal{O}$. Then $obt(\vec{D}) = 2$.*

Proof. Let $\vec{D} \in \mathcal{O}$ and suppose for a contradiction that $obt(\vec{D}) < 2$. Since $\vec{D}$ contains a dicycle, $obt(\vec{D}) \neq 0$. Now suppose there exists a one-page oriented book embedding of $\vec{D}$. We may assume the page is upwards, and that y_1 and y_j are in-neighbors of x_1 and x_k, respectively, i.e., the chords of $\vec{D}$ are (y_1, x_1) and (y_j, x_k). Consider the oriented subgraph $\vec{H}$ of $\vec{D}$. Since (y_1, x_1) and (y_j, x_k) are heavy arcs in $\vec{D}$, by Lemma 2.2, (y_1, x_1) and (y_j, x_k) must be loose in the embedding. By Lemma 3.2, neither (y_1, x_1) nor (y_j, x_k) shields the other. We may assume that y_1 and x_1 are below y_j and x_k. Since the page is upwards, y_1 is below x_1 and y_j is below x_k in the spine. However, this is not a translation cycle of the outer cycle of $\vec{D}$, as x_1 is between y_1 and y_j in the spine, a contradiction to Theorem 2.1. To obtain a two-page embedding of $\vec{D}$, we may assume that $\vec{P}$ is a non-trivial oriented path and contains an arc with endpoints x_i and x_{i+1}; if $(x_i, x_{i+1}) \in A(\vec{P})$, see the left side of Figure 5.4, and if $(x_{i+1}, x_i) \in A(\vec{P})$, see the right side of Figure 5.4, thus $obt(\vec{D}) \leq 2$. Therefore, $obt(\vec{D}) = 2$. □

We now show that each member of $\mathcal{O}$ is two-page critical.

Lemma 5.2. $\mathcal{O} \subseteq \mathcal{M}^2$

Proof. Let $\vec{D} \in \mathcal{O}$. By Lemma 5.1, $obt(\vec{D}) = 2$. We now must show that for each arc $\vec{a} \in A(\vec{D})$, we have that $obt(\vec{D} \setminus \vec{a}) = 1$. Label the chords (y_1, x_1) and (y_j, x_k). We have three cases: $\vec{a}$ can be in one of the oriented paths, $\vec{a}$ can be a chord, or neither.

i. Let $\vec{a}$ be an arc of one of the oriented paths. We may assume that $\vec{P}$ is a non-trivial oriented path, and let $\vec{a} \in A(\vec{P})$. Then $\vec{D} \setminus \vec{a}$ is a strictly

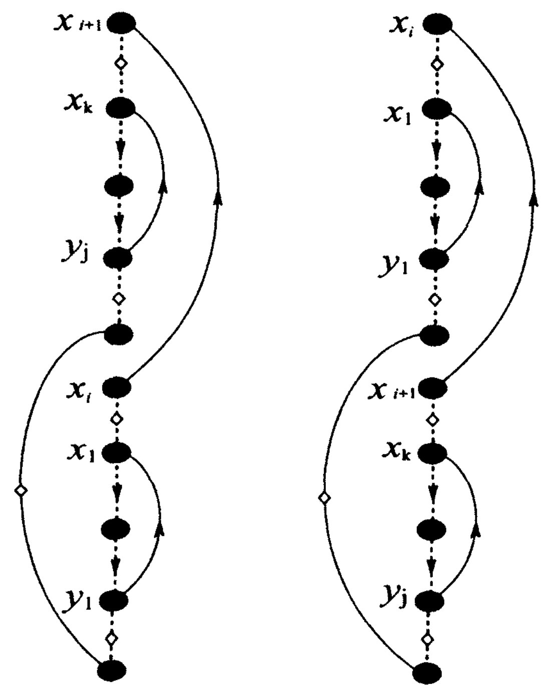

Figure 5.4: A two-page oriented book embedding of a member of $\mathcal{O}$.

bi-dicyclic graph. The vertices x_1, y_1, x_k, and y_j are the only heavy vertices of $\overrightarrow{D}$, thus $\overrightarrow{D} \setminus \overrightarrow{a}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$ as an oriented subgraph. $\overrightarrow{D} \setminus \overrightarrow{a}$ contains no antler; thus, it contains no member of $\mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$, or $\mathcal{B}_5$ as an oriented subgraph. If $\overrightarrow{P}'$ is trivial, by the definition of $\mathcal{O}$, we have that $\overrightarrow{D} \setminus \overrightarrow{a}$ contains no member of $\mathcal{B}_1$ as an oriented subgraph. Therefore, by Theorem 3.1 we have that $obt(\overrightarrow{D} \setminus \overrightarrow{a}) = 1$. If $\overrightarrow{P}'$ is non-trivial and we let $\overrightarrow{a} \in A(\overrightarrow{P}')$, the proof is symmetric.

ii. Let $\overrightarrow{a}$ be a chord of $\overrightarrow{D}$, say $\overrightarrow{a} = (y_1, x_1)$. Suppose there exists an arc $(y_i, y_{i-1}) \in A(\overrightarrow{P}')$, as seen in the first oriented graph of Figure 5.5. Embed $\overrightarrow{D} \setminus \overrightarrow{a}$ as seen in the first oriented book embedding of Figure 5.5, thus $obt(\overrightarrow{D} \setminus \overrightarrow{a}) = 1$. If no such arc exists, i.e., $\overrightarrow{P}'$ is a dipath from y_1 to y_j, then by condition C., $\overrightarrow{P}$ is not a dipath from x_k to x_1, i.e., there exists an arc $(x_i, x_{i+1}) \in A(\overrightarrow{P})$, as seen in the second oriented graph of Figure 5.5. Embed $\vec{D} \setminus \vec{a}$ as seen in the second oriented book embedding of Figure 5.5, and we obtain a one-page oriented book embedding. If $\vec{a} = (y_j, x_k)$, the proof is symmetric.

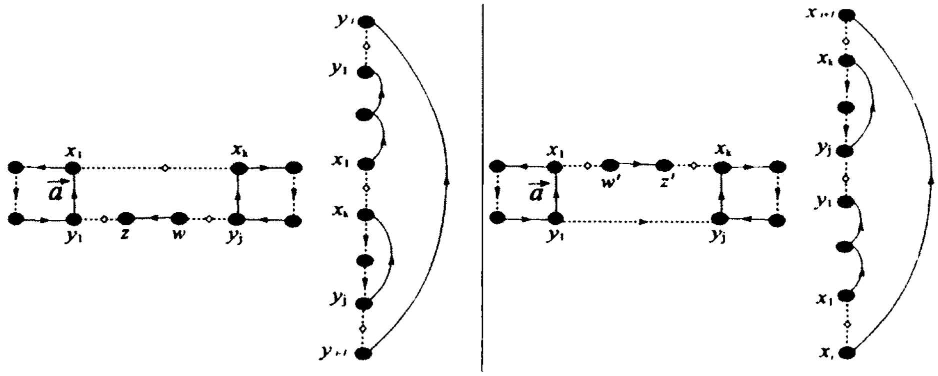

Figure 5.5: A one-page oriented book embedding of $\vec{D} \setminus \vec{a}$, where $\vec{a}$ is a chord.

iii. Let $\vec{a}$ be an arc that is neither in one of the oriented paths nor a chord. Embed $\vec{D} \setminus \vec{a}$ as seen on the right in Figure 5.6, and we obtain a one-page oriented book embedding. If $\vec{a}$ in an arc of $\overrightarrow{D_n}$ other than (y_j, x_k), the case is symmetric.

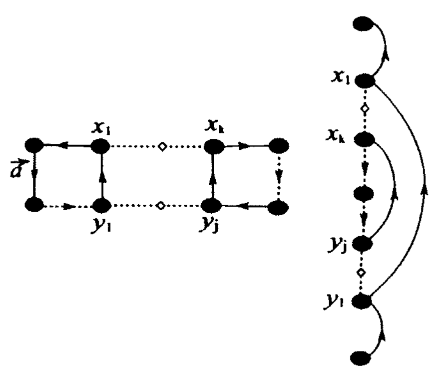

Figure 5.6: A one-page oriented book embedding of $\vec{D} \setminus \vec{a}$, where $\vec{a} \in \overrightarrow{D_m}$ and is not the chord.

□

Theorem 5.1. *Let $\vec{D} \in C^2$. Then $obt(\vec{D}) \leq 1$, if and only if $\vec{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{B}_1$, $\mathcal{N}$, or $\mathcal{O}$.*

Proof. To prove the necessary condition, consider the contrapositive, which states: if $\vec{D}$ contains a member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{N}$, or $\mathcal{O}$, then $obt(\vec{D}) > 1$. This is true by Lemmas 2.4, 2.5, 4.4, and 5.1. To prove the sufficient condition, suppose that $\vec{D}$ contains no member of $\mathcal{I}$, $\mathcal{T}$, $\mathcal{N}$, or $\mathcal{O}$, and we will construct a one-page oriented book embedding of $\vec{D}$. Label the endpoints of one chord with x_1 and y_1 and the other with x_k and y_j, so that there is an oriented path between x_1 and x_k that is arc-disjoint from an oriented path between y_1 and y_j. There are two cases for the direction of the chords, either (y_1, x_1) and (x_k, y_j) are the chords of $\vec{D}$, or (y_1, x_1) and (y_j, x_k) are the chords of $\vec{D}$.

i. Suppose (y_1, x_1) and (x_k, y_j) are the chords of $\vec{D}$. By the direction of the chords, $\vec{D}$ does not contain a member of $\mathcal{B}_1$ or $\mathcal{O}$. Since $\vec{D}$ contains no member of $\mathcal{I}$ or $\mathcal{T}$, either the oriented path between x_1 and x_k is not a directed path from x_1 to x_k, or the oriented path between y_1 and y_j is not a directed path from y_j to y_1. In either case, the proof is symmetric; thus, we may assume the oriented path between x_1 and x_k is not a directed path from x_1 to x_k. Then there is a arc (x_{i+1}, x_i) as seen on the left in Figure 5.7, and we can embed $\vec{D}$ into a one-page book as seen on the right side of Figure 5.7.

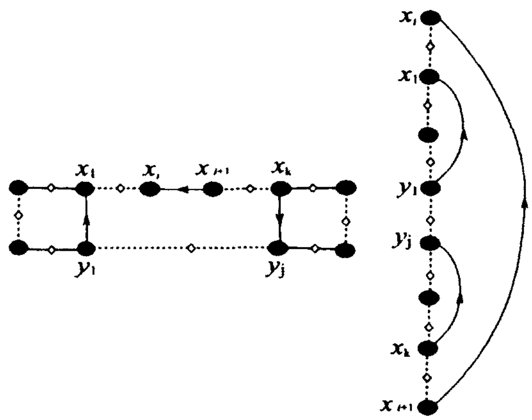

Figure 5.7: $\vec{D} \in \mathcal{C}^2$ and its one-page oriented book embedding.

ii. Suppose that (y_1, x_1) and (y_j, x_k) are the chords of $\vec{D}$. Then, since $\vec{D}$ contains no member of $\mathcal{B}_1$, $\mathcal{N}$ or $\mathcal{O}$, either there is no directed path from x_1 to y_1, which contains neither x_k nor y_j, or there is a no directed path from x_k to y_j, which contains neither x_1 nor y_1. In either case, the proof is symmetric; thus, we may assume there is no directed path from x_1 to y_1 which contains neither x_k nor y_j. Then there exists an arc (p, q), as seen on the left side in Figure 5.8, and we may embed $\vec{D}$ into a one-page book as seen on the right side in Figure 5.8.

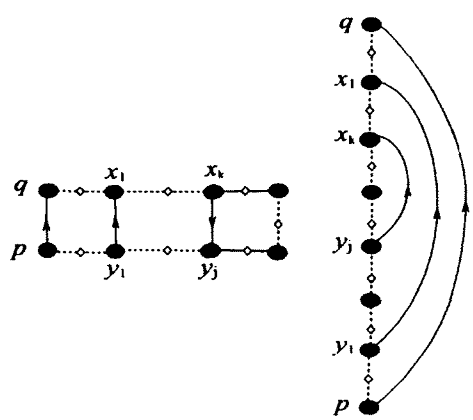

Figure 5.8: $\vec{D} \in \mathcal{C}^2$ and its one-page oriented book embedding.

□

5.2 Z-Structures

Now that we have characterized oriented graphs in C^2 having oriented book thickness one, we move on to oriented graphs in C^d for $d \geq 3$ by discussing a certain structure that may occur in oriented graphs in C^d having at least three chords, called a z-structure, formally defined below, such that the chords may or may not share an endpoint.

Definition 5.2. *An oriented graph $\vec{D} \in C^d$, for $d \geq 3$, contains a z-structure if $\vec{D}$ contains three chords, $\vec{a}$, $\vec{b}$, and $\vec{c}$, and an oriented path $\vec{P}$, containing no chord, such that the endpoints of $\vec{P}$ are the head of $\vec{a}$ and the head of $\vec{c}$, and $\vec{P}$ contains the tail of $\vec{b}$, but no other endpoint of $\vec{a}$, $\vec{b}$, or $\vec{c}$. Figure 5.9 depicts two oriented graphs in C^3 containing z-structure.*

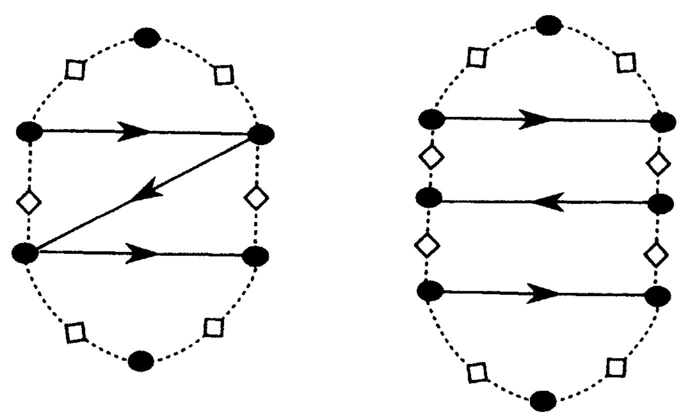

Figure 5.9: Members of C^3 containing a z-structure

To prove that if an oriented graph in C^d, $d \geq 3$, contains a z-structure, then it has oriented book thickness greater than one, recall Theorem 2.1, which states if a cycle has a one-page oriented book embedding, then the spine order of the embedding is a translation cycle of the vertices of the cycle. This holds for the outer oriented cycle of $\overrightarrow{D}$.

Theorem 5.2. *Let $\overrightarrow{D} \in C^d$, $d \geq 3$, such that $\overrightarrow{D}$ contains a z-structure. Then $obt(\overrightarrow{D}) > 1$.*

Proof. Suppose $\overrightarrow{D}$ contains a z-structure with chords $\overrightarrow{a} = (x_a, y_a)$, $\overrightarrow{b} = (x_b, y_b)$, and $\overrightarrow{c} = (x_c, y_c)$, and suppose for a contradiction that $obt(\overrightarrow{D}) \leq 1$. Since $\overrightarrow{D}$ contains an oriented cycle $obt(\overrightarrow{D}) \neq 0$, we assume that there exists a one-page oriented book embedding of $\overrightarrow{D}$, and we may assume the page is upwards. By Lemma 4.3, $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ must be located in the page, and the spine order must be a translation cycle of the vertices of the outer oriented cycle of $\overrightarrow{D}$. Consider the following two cases:

i. Suppose that x_b is below x_a in the spine. Since the page is upwards, y_a must be located above x_a in the spine, and by the translation cycle, x_c must be located between x_a and x_b, and y_c must be located between x_b and x_c; however, this results in a downwards arc in the page, a contradiction. See the left side in Figure 5.10.

ii. Suppose that x_b is above x_a in the spine. If y_a is above x_b in the spine, then by the translation cycle, y_b is between x_a and x_b in the spine, resulting in a downwards arc in the page, a contradiction. See the center of Figure 5.10. Thus, y_a is between x_a and x_b in the spine. Also, y_b must be located above x_b in the spine; by the translation cycle, y_c must be between x_b and y_b, and y_c must be between y_c and y_b, resulting in a downwards arc in the page, a contradiction. See the right side in Figure 5.10.

□

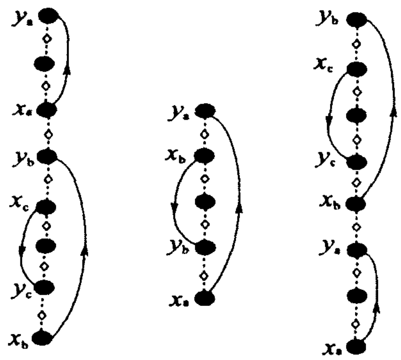

Figure 5.10: Theorem 5.2

5.3 Future Work

We now discuss possible directions for future work which may follow this dissertation. Some problems, but not all, mentioned below are related directly to the work in this dissertation.

For the unoriented case, there is no known characterization for graphs having oriented book thickness equal to three. In Chapter 4, we discussed a class of oriented graphs, $\mathcal{W}^4$, that was switching three-page-critical; we would like to further investigate the class of switching three-page-critical graphs, and three-page-critical oriented graphs in order to better understand three-page embeddings. We would also like to further investigate the class of switching k-critical graphs. We have the following conjecture. Recall from Chapter 3 that for a member of the class $\mathcal{B}_2$, each arc had a specified direction.

Conjecture 5.1. $\mathcal{B}_2 \subseteq \mathcal{M}_s^2$.

We would also like to investigate the oriented book thickness of oriented graphs and their *minors*, defined below. For a graph G and edge $e = \{u, v\}$ of G, a *subdivision* of e is the addition of a new vertex w to $V(G)$, the removal of e from $E(G)$, and the addition of the path uwv. A *contraction* of an edge $e = \{u, v\}$ is the removal of u and v from $V(G)$ and the addition of a new vertex w to $V(G)$ whose incident edges are those that were incident to u or v in the original graph. A graph H is a *minor* of a graph G if G can be obtained from H by certain deletions or contractions of the edges of G. In the future, we would like to study minors in relation to oriented book embeddings.

We must be careful, however, with definitions of subdivisions and contractions for oriented graphs. Recall the class $\mathcal{R}$, defined in Chapter 2. If we were to subdivide the arc between x and y, as seen below in Figure 5.11, we obtain an oriented graph $\vec{D}$ such that $obt(\vec{D}) = 1$. Therefore, since $\vec{D}$ has a member of $\mathcal{R}$ as a minor (according the definition in the un-directed case), we can immediately see that oriented book thickness is not minor closed.

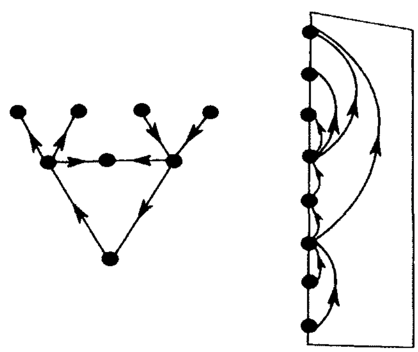

Figure 5.11: $obt(\vec{D}) = 1$

Therefore, we need to reconsider the definition of minor for oriented graphs. In [12], the author defines a type of minor, called a *butterfly minor*, that forbids a contraction that creates a new directed cycle. This definition would avoid the case shown in Figure 5.11. It is our goal to investigate further the relationship between oriented graphs having oriented book thickness k, and their butterfly minors.

At the beginning of this dissertation, we mentioned that book thickness for un-oriented graphs is analogous to stack-number of *directed acyclic graphs* (DAGS), oriented graphs containing no directed cycle. In [10], the authors discuss stack number, and a related parameter, *queue number*. A *queue layout* is a linear ordering of the vertices, where the arcs are then places into queues, such that no two arcs are nested in a queue. Figure 5.12 is an example of a DAG, and its 2-queue layout.

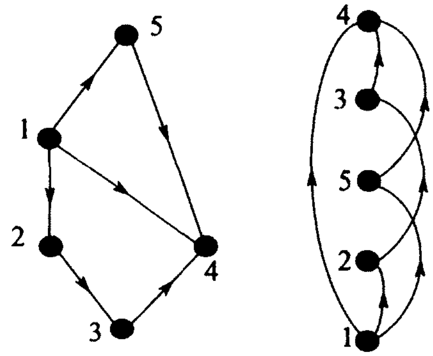

Figure 5.12: A 2-queue layout

Study of queue layouts, like stack layouts, has so far been restricted to DAGS. Since our definition of oriented book embeddings allows directed cycles, we would like to develop a similar definition such that no two arcs are nested in any page, while allowing for directed cycles. In [10], the authors described a type of embedding, called a *directed arched leveled-planar embedding* that is equivalent to a 1-queue layout. We would like to develop a similar embedding that allows for directed cycles.

Another class of oriented graphs whose oriented book thickness will be investigated is that of tournaments. In [1], Bernhart and Kainen proved the following for complete graphs.

Theorem 5.3. *(Bernhart and Kainen, [1]) For* $n \geq 4$, $bt(K_n) = \lceil \frac{n}{2} \rceil$.

A *transitive tournament*, denoted $\overrightarrow{TT_n}$ is a tournament on n vertices such that for $x, y, z \in V(\overrightarrow{TT_n})$, if $\{(x,y),(y,z)\} \in A(\overrightarrow{TT_n})$, then it must be true that $(x,z) \in A(\overrightarrow{TT_n})$. Therefore, every transitive tournament contains a directed hamilton path. If we embed the vertices into the spine using the directed hamilton path, then the direction of every arc will be upward, and therefore, the directions can be dropped, giving the following.

Observation 5.1. *Let* $\overrightarrow{TT_n}$ *be a transitive tournament on* n *vertices. Then* $obt(\overrightarrow{TT_n}) = bt(K_n)$.

Besides transitive tournaments, there are many other types of tournaments, such as regular tournaments and strong tournaments. We intend to investigate these classes and hope to obtain upper bounds for their oriented book thickness. In investigating tournaments, the use of computers will become imperative, since as n increases, the number of non-isopmorphic tournaments on n vertices increases extremely quickly. For example, while there are only four tournaments on four vertices, there are twelve on five vertices and fifty-six on six vertices. This is another reason we wish to consider the types of tournament separately.

APPENDIX

ALGORITHMS

Algorithm 1 OTSO Algorithm

Require: An oriented tree $\overrightarrow{T}$, an arbitrarily fixed vertex $x \in V(\overrightarrow{T})$, and two empty lists L and S.

Ensure: A list S that yields the spine order of a 1-page oriented book embedding of $\overrightarrow{T}$ such that each vertex is loose in the embedding, the direction of all arcs agree, and x is uncovered.

```
while L ≠ ∅ do
  if |N^+(x)| = n > 0, n ∈ ℕ then
    Let N^+(x) = {u_1, u_2, ..., u_n}
    Add u_1 to the end of L
    Add u_1 to the beginning of S
    if n ≥ 2 then
      for i = 2, i ≤ n, i + + do
        if u_i ∉ L then
          add u_i to L between u_{i-1} and x
          add u_i to S between u_{i-1} and x
        end if
      end for
    end if
  end if
  if |N^-(x)| = m > 0, m ∈ ℕ then
    Let N^-(x) = {v_1, v_2, ..., v_n}
    Add u_v to the end of L
    Add v_1 to the end of S
    if m ≥ 2 then
      for i = 2, i ≤ m, i + + do
        if v_i ∉ L then
          add v_i to L between v_{i-1} and x
          add v_i to S between v_{i-1} and x
        end if
      end for
    end if
  end if
end while
```

Algorithm 2 Uni-cyclic Checking Algorithm

Require: A uni-cyclic oriented graph $\overrightarrow{D}$, consisting of an oriented cycle $\overrightarrow{C_n}$ with $V(\overrightarrow{C_n}) = \{1, 2, ..., n\}$ and n (possibly trivial) oriented trees.

Ensure: Whether or not $\overrightarrow{D}$ has a 1-page oriented book embedding.

```
Let k = 1
for j = 1, j ≤ n, j++ do
  if d^+_{C_n}(j) > 0 then
    Label j with 1. Label an out-neighbor of j in C_n with n. Label the other neighbor
    of n in C_n with n − 1 and continue this until 2 is reached.
    Embed 1, 2, ..., n − 1, n into the spine such that 1 is the bottom vertex in the spine
    and n is the top.
    Embed A(C_n) into the spine or the page, such that the direction of the page is
    upwards.
    if 1 is tight above and T_1 contains a positive antler then
      k = 2
    else if n is tight below and T_n contains a negative antler then
      k = 2
    else

      for (i = 2, i ≤ n − 1, i++) do
        if i is tight and T_i is non-trivial then
          k = 2
          exit loop
        else if i is tight below and T_i contains a negative antler then
          k = 2
          exit loop
        else if i is tight above and T_i contains a positive antler then
          k = 2
          exit loop
        else
          break
        end if
      end for
    end if
    if k = 2 then
      break
    else
      Return True
      STOP
    end if
  else
    break
  end if
end for
```

Index

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