Four Hierarchies of ω–Regular Languages

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Abstract

We argue several decompositions of ω-regular sets into rational $\mathbf{G}_\delta$ sets. We measure the complexity of ω-regular sets by the number of rational $\mathbf{G}_\delta$ sets obtained by the decompositions. Barua (1992) studied a hierarchy $\mathcal{R}_n(n = 1, 2, 3, \cdots)$, where $\mathcal{R}_n$ is a class of ω-regular sets which are decomposed into n rational $\mathbf{G}_\delta$ sets forming a decreasing sequence. On the other hand, Kaminski (1985) defined a hierarchy $\mathbf{B}_m(m = 1, 2, 3, \cdots)$, where $\mathbf{B}_m$ is a class of ω-regular sets which are decomposed into $2m$ rational $\mathbf{G}_\delta$ sets not necessarily forming a decreasing sequence. Already it is reported that $\mathbf{B}_n = \mathcal{R}_{2n}$ by Takahashi(1995). And besides we show $\mathcal{B}_n = \mathcal{R}_{2n}$, where $\mathcal{B}_n$ is a class of ω-regular sets whose defining condition is more lenient than that of $\mathcal{R}_{2n}$. In conclusion, we state that various hierarchies are reduced to four types of hierarchies.

1. Introduction

Barua[1] obtained some important results about a hierarchy of ω-regular sets $\mathcal{R}_n(n = 1, 2, 3, \cdots)$, where $\mathcal{R}_n$ is a class of ω-regular sets L satisfying (1) below:

There exist rational $\mathbf{G}_\delta$ sets $G_0, G_1, \cdots, G_{n-1}$ such that

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{n-1}$$

and

$$L = \bigcup_{i:even}^{n-1} (G_i - G_{i+1}). \qquad (1)$$

In the meanwhile, Kaminski[2] had researched some hierarchies of ω-regular sets $\mathbf{B}_n$, $\mathbf{RB}_n$, $\mathbf{LB}_n$, and $\mathbf{LRB}_{n+1}$ $(n = 1, 2, 3, \cdots)$, where $\mathbf{B}_n$ is a class of ω-regular sets L satisfying (2):

There exists rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_n, B_n$ such that

$$L = \bigcup_{i=1}^{n} (A_i - B_i). \quad (2)$$

There are differences in appearance between (1) and (2). In (1) the component sets constructing L are linearly ordered w.r.t. set inclusion, whereas in (2) the component sets don't have any order w.r.t. set inclusion.

Recently, Takahashi[5] has shown that $\mathbf{B}_n = \mathcal{R}_{2n}$ $(n = 1, 2, 3, \cdots)$. The truth is that $\mathbf{B}_n$ and $\mathbf{LRB}_{n+1}$ have mutually dual relation, and besides $\mathbf{RB}_n$ and $\mathbf{LB}_n$ are dual. Therefore we define a class of ω-regular sets $\mathcal{L}_n$ which is the dual class of $\mathcal{R}_n$. Moreover, we define two classes $\mathbf{R}_n$ and $\mathbf{L}_n$ of ω-regular sets whose component sets are not rational $\mathbf{G}_\delta$, but rational $\mathbf{F}_\sigma$. And then we consider mutual relation among four classes $\mathcal{R}_n, \mathcal{L}_n, \mathbf{R}_n$, and $\mathbf{L}_n$.

2. Preliminary and background

Let Σ be an alphabet containing at least two elements. We denote the set of all words over Σ including the empty word ε by Σ^*. Σ^* without ε is denoted by Σ^+. Let ω be the set of all natural numbers. A mapping from ω to Σ is called an ω-word over Σ. By Σ^ω we denote the set of all ω-words over Σ. An ω-word $\alpha \in \Sigma^\omega$ is written as $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ where $\alpha_i = \alpha(i)$ $(i = 0, 1, 2, \cdots)$. We call a subset of Σ^* (Σ^ω, resp) a language (ω-language) over Σ. For $A \subseteq \Sigma^*$ and $B \subseteq \Sigma^* \cup \Sigma^\omega$, we define the catenation of A and B as

$$AB = \{xy \in \Sigma^* \cup \Sigma^\omega | \ x \in A, \ y \in B\}.$$

The ω-power of $L \subseteq \Sigma^*$ is an ω-language defined as

$$L^\omega = \{x_0x_1x_2\cdots \in \Sigma^\omega \mid x_i \in L - \{\varepsilon\} \ \ for \ \ all \ \ i \in \omega\}.$$

For $x \in \Sigma^*$ and $z \in \Sigma^* \cup \Sigma^\omega$, if $z = xy$ for some $y \in \Sigma^* \cup \Sigma^\omega$, x is called an *initial segment* of z, and we denote the relation by $x < z$.

Definition 2.1. For each $x \in \Sigma^*$, we define an *open base* for x as follows:

$$N_x = \{\alpha \in \Sigma^\omega \mid x < \alpha\}.$$

An ω-language $A \subseteq \Sigma^\omega$ is an *open* set of the product topology on Σ^ω if $A = \cup_{x \in B} N_x$ for some $B \subseteq \Sigma^*$. An ω-language is *closed* if its complement is open. Let $\mathbf{G}$ ($\mathbf{F}$) denote the set of all open (closed) sets. $\mathbf{F}_\sigma$ ($\mathbf{G}_\delta$) is the set of all denumerable unions (intersections) of closed (open) sets. $\mathbf{G}_{\delta\sigma}$ ($\mathbf{F}_{\sigma\delta}$) is the set of all denumerable unions (intersections) of $\mathbf{G}_\delta$ ($\mathbf{F}_\sigma$) sets, respectively. The rest of the *Borel hierarchy* is defined in the same manner.

Definition 2.2. For a given Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$, we define the following sets:

For $q \in Q$ and $u \in \Sigma^+$, let $R(q,u) = \{\delta(q,v) \mid v < u\}$. We also define $\mathcal{M}_q = \{R(q,u) \mid \delta(q,u) = q \;\; for \;\; some \;\; u \in \Sigma^+\}$, and set $\mathcal{R}(M) = \cup_{q \in Q} \mathcal{M}_q$.

Definition 2.3. Given a Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$ and an ω-word $\alpha \in \Sigma^\omega$, the *run* r of M on α is a mapping from ω to Q such that $r(0) = q_0, \quad r(n+1) = \delta(r(n), \alpha(n)) \qquad \text{for } n \geq 0$.

Then we formulate a set of states occurring infinitely many times while M runs on $\alpha \in \Sigma^\omega$, as follows:

$$In(\alpha, M) = \{q \in Q \mid \mathbf{card}(r^{-1}(q)) = \aleph_0\}.$$

Given a finite automaton $\langle M, F \rangle$, we call the ω-language $L(\langle M, F \rangle) = \{\alpha \in \Sigma^\omega \mid In(\alpha, M) \cap F \neq \phi\}$ (which is Büchi-accepted by $\langle M, F \rangle$) a *rational* $\mathbf{G}_\delta$ set. A *rational* $\mathbf{F}_\sigma$ set is a set whose complement is a rational $\mathbf{G}_\delta$ set. We denote the set of all rational $\mathbf{G}_\delta$ sets ($\mathbf{F}_\sigma$ sets, resp) by $\mathcal{O}_3$ ($\mathcal{O}_4$) (cf. Kobayashi et al.[3]).

For a given Σ-table $M = \langle Q, \Sigma, \delta, q_0 \rangle$ and a family of state sets $\mathcal{F} \subseteq \mathcal{R}(M)$, we call $\langle M, \mathcal{F} \rangle$ a Muller automaton. Given a Muller automaton $\langle M, \mathcal{F} \rangle$, we define the ω-language Muller-accepted by $\langle M, \mathcal{F} \rangle$ as follows:

$$L(\langle M, \mathcal{F} \rangle) = \{\alpha \in \Sigma^\omega \mid In(\alpha, M) \in \mathcal{F}\}.$$

Kaminski[2] studied the following four classes.

Definition 2.4. We define four classes $\mathbf{RB}_n$, $\mathbf{B}_n$, $\mathbf{LB}_n$ and $\mathbf{LRB}_n$ of ω-regular sets as follows.

(a) $L \in \mathbf{RB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$L = \bigcup_{i=1}^{n-1} (A_i - B_i) \cup A_n.$$

(b) $L \in \mathbf{B}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_n, B_n$ such that

$$L = \bigcup_{i=1}^{n} (A_i - B_i) .$$

(c) $L \in \mathbf{LB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_n, B_n$ such that

$$L = -B_1 \cup \bigcup_{i=2}^{n} (A_i - B_i) ,$$

where $-B_1 = \overline{B}_1$.

(d) $L \in \mathbf{LRB}_n$ $(n \geq 2)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$L = -B_1 \cup \bigcup_{i=2}^{n-1} (A_i - B_i) \cup A_n.$$

On the basis of the Büchi-McNaughton theorem, we can conclude that any ω-regular set is in $\mathbf{F}_{\sigma\delta} \cap \mathbf{G}_{\delta\sigma}$. Accordingly, by restricting the number of quantifiers to 2 in the theorem of Kuratowski[4, §37. III], we obtain the following corollary.

Corollary 2.5. *A set $A \subseteq \Sigma^\omega$ is in both $\mathbf{F}_{\sigma\delta}$ and $\mathbf{G}_{\delta\sigma}$ if and only if there exists a countable transfinite ordinal μ such that*

$$A = \bigcup_{\lambda:even}^{\mu} (G_\lambda - G_{\lambda+1})$$

with decreasing sequence $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_\mu$, where each G_λ is a $\mathbf{G}_\delta$ set in Σ^ω. Here if μ is even, let $G_{\mu+1} = \phi$.

Corresponding to the ordinal number μ, Barua[1] defined the class $\mathcal{D}_{\mu+1}$ which consists of such set A's as mentioned in Corollary 2.5. In particular, $\mathcal{D}_1 = \mathbf{G}_\delta$. He constructed a class $\mathcal{R}_n (n \geq 1)$ of ω-regular sets taking the finite ordinal $n \in \omega$ as μ, as Definition 2.6.(a).

Definition 2.6.

(a) For each $n \geq 0$ we define a class $\mathcal{R}_{n+1}$ of ω-regular sets as follows.

L is in $\mathcal{R}_{n+1}$ *iff* there exist rational $\mathbf{G}_\delta$ sets $G_0, G_1, \cdots, G_n$ such that

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \quad \text{and}$$

$$L = \bigcup_{i:even}^{n} (G_i - G_{i+1}) .$$

$G_0, G_1, \cdots, G_n$ are called the *component* sets of L. In particular, $\mathcal{R}_1 = \mathcal{O}_3$.

We define a class $\mathcal{L}_n$ as the dual class of $\mathcal{R}_n$.

(b) For each $n \geq 0$ we define a class $\mathcal{L}_{n+1}$ of ω-regular sets as follows.

L is in $\mathcal{L}_{n+1}$ *iff* there exist rational $\mathbf{G}_\delta$ sets $G_0, G_1, \cdots, G_n$ such that

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \quad \text{and}$$

$$L = -G_0 \cup \bigcup_{i:odd}^{n} (G_i - G_{i+1}) .$$

The Barua hierarchy $\mathcal{R}_n$ $(n \geq 1)$ is composed of ω-regular sets whose component sets are all multiplicative ($\mathbf{G}_\delta$). Hence, by replacing the multiplicative sets by the additive ($\mathbf{F}_\sigma$) sets, we define two classes of ω-regular sets whose component sets are all rational $\mathbf{F}_\sigma$.

(c) For each $n \geq 0$ we define a class $\mathbf{R}_{n+1}$ of ω-regular sets as follows.

L is in $\mathbf{R}_{n+1}$ *iff* there exist rational $\mathbf{F}_\sigma$ sets $F_0, F_1, \cdots, F_n$ such that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \quad \text{and}$$

$$L = \bigcup_{i:even}^{n} (F_i - F_{i+1}) .$$

(d) For each $n \geq 0$ we define a class $\mathbf{L}_{n+1}$ of ω-regular sets as follows.

L is in $\mathbf{L}_{n+1}$ *iff* there exist rational $\mathbf{F}_\sigma$ sets $F_0, F_1, \cdots, F_n$ such that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \quad \text{and}$$

$$L = -F_0 \cup \bigcup_{i:odd}^{n} (F_i - F_{i+1}) .$$

From the definitions, we immediately obtain the following two lemmas.

Lemma 2.7. *For* $n \geq 1$
(a) $\mathcal{L}_n \subseteq \mathcal{L}_{n+1}$,
(b) $\mathbf{R}_n \subseteq \mathbf{R}_{n+1}$,
(c) $\mathbf{L}_n \subseteq \mathbf{L}_{n+1}$.
These inclusions turn out to be proper in Theorem 3.6.

Lemma 2.8.
(a) $L \in \mathcal{L}_n$ *iff* $\overline{L} \in \mathcal{R}_n$.
(b) $L \in \mathbf{L}_n$ *iff* $\overline{L} \in \mathbf{R}_n$.

As for $\mathbf{RB}_n, \mathbf{B}_n, \mathbf{LB}_n, \mathbf{LRB}_{n+1}$ and $\mathcal{R}_n, \mathcal{L}_n$ $(n = 1, 2, 3, \cdots)$, we have already known the following theorem.

Theorem 2.9. (Takahashi[5, Theorem 3.11]) *For* $n \geq 0$
(a) $\mathbf{RB}_{n+1} = \mathcal{R}_{2n+1}$,
(b) $\mathbf{B}_{n+1} = \mathcal{R}_{2n+2}$,
(c) $\mathbf{LB}_{n+1} = \mathcal{L}_{2n+1}$,
(d) $\mathbf{LRB}_{n+2} = \mathcal{L}_{2n+2}$.

The following result is obtained using $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ for any $n \geq 1$.

Theorem 2.10. (Barua[1, Theorem 6.3])
$\mathcal{O}_3 = \mathcal{R}_1 \subset \mathcal{R}_2 \subset \mathcal{R}_3 \subset \mathcal{R}_4 \subset \cdots$.

3. Various kinds of hierarchies

Recently, the close relations between the Barua hierarchy and the Kaminski hierarchy were shown by Takahashi[5]. The component sets constructing an ω-regular set in the Barua hierarchy always make a decreasing sequence w.r.t. set inclusion. But the component sets constructing an ω-regular set in the Kaminski hierarchy don't necessarily make a decreasing sequence. Therefore we investigate an intermediate decreasing condition of these two modes. That is, we consider the following four kinds of classes of ω-regular sets which are composed of component sets A_i, B_i $(i = 1, 2, \cdots, n)$ with the alternately decreasing sequence

(*) $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$,
instead of the successively decreasing sequence
(**) $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots \supseteq A_n \supseteq B_n$,
required in the definition of $\mathcal{R}_n, \mathcal{L}_n$. We can conclude that there is no distinction between the classes constructed with decrements (*) and (**). In other words, even if we consider Kaminski's four kinds of classes of ω-regular sets, we can assume decrement (*) or (**) according to the situation. Furthermore we can determine some relations among $\mathcal{R}-, \mathcal{L}-, \mathbf{R}-$, and $\mathbf{L}-$classes. Consequently we can conclude the existence of four hierarchies $\{\mathcal{R}_n\}, \{\mathcal{L}_n\}, \{\mathbf{R}_n\}$, and $\{\mathbf{L}_n\}$.

Definition 3.1. We define four classes of ω-regular sets as follows.

(a) $L \in \mathcal{RB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_{n-1} \quad \text{and} \quad L = \bigcup_{i=1}^{n-1}(A_i - B_i) \cup A_n.$$

(b) $L \in \mathcal{B}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_n, B_n$ such that

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \quad \text{and} \quad L = \bigcup_{i=1}^{n}(A_i - B_i).$$

(c) $L \in \mathcal{LB}_n$ $(n \geq 1)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_n, B_n$ such that

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \quad \text{and} \quad L = -B_1 \cup \bigcup_{i=2}^{n}(A_i - B_i).$$

(d) $L \in \mathcal{LRB}_n$ $(n \geq 2)$

$\overset{\text{def}}{\Longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $B_1, A_2, B_2, \cdots, A_{n-1}, B_{n-1}, A_n$ such that

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_{n-1} \quad \text{and} \quad L = -B_1 \cup \bigcup_{i=2}^{n-1}(A_i - B_i) \cup A_n.$$

We note the difference between Definition 2.4 and Definition 3.1. In Definition 2.4, the component sets do not necessarily possess decreasing property. On the other hand, in Definition 3.1 the component sets always have an alternate decreasing property. First, we prove the following theorem.

Theorem 3.2. *For* $n \geq 1$

(a) $\mathcal{RB}_n = \mathcal{R}_{2n-1}$,

(b) $\mathcal{B}_n = \mathcal{R}_{2n}$,

(c) $\mathcal{LB}_n = \mathcal{L}_{2n-1}$,

(d) $\mathcal{LRB}_{n+1} = \mathcal{L}_{2n}$.

Proof. First we demonstrate point (b). Let L be in $\mathcal{B}_n$, i.e.,

$$L = (A_1 - B_1) \cup \cdots \cup (A_n - B_n)$$

for some rational $\mathbf{G}_\delta$ sets $A_1, B_1, \cdots, A_n, B_n$ with $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n$. Construct rational $\mathbf{G}_\delta$ sets G_i $(i = 0, 1, \cdots, 2n-1)$ as follows.

For $j = 0, 1, \cdots, n-1$, set $G_{2j} = \bigcup_{i=j+1}^{n}(A_i \cup B_i)$ and $G_{2j+1} = B_{j+1} \cup \bigcup_{i=j+2}^{n}(A_i \cup B_i)$. Then we immediately see that $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{2n-1}$. Furthermore, L can be transformed as follows[‡].

$$\begin{aligned}
L &= A_1 \cdot \overline{B}_1 \cdot \overline{B}_2 \cdots \overline{B}_n + A_2 \cdot \overline{B}_2 \cdot \overline{B}_3 \cdots \overline{B}_n + \cdots + A_{n-1} \cdot \overline{B}_{n-1} \cdot \overline{B}_n \\
&\quad + A_n \cdot \overline{B}_n \\
&= ((\cdots((A_1 \cdot \overline{B}_1 + A_2)\overline{B}_2 + A_3)\overline{B}_3 + \cdots + A_{n-1})\overline{B}_{n-1} + A_n)\overline{B}_n \\
&= ((\cdots((A_1 \cdot \overline{B}_1 \cdot \overline{A}_2 + A_2)\overline{B}_2 \cdot \overline{A}_3 + A_3)\overline{B}_3 \cdot \overline{A}_4 + \cdots + A_{n-1}) \\
&\quad \overline{B}_{n-1} \cdot \overline{A}_n + A_n)\overline{B}_n \\
&= A_1 \cdot \overline{B}_1 \cdot \overline{A}_2 \cdot \overline{B}_2 \cdot \overline{A}_3 \cdots \overline{A}_n \cdot \overline{B}_n \\
&\quad + A_2 \cdot \overline{B}_2 \cdot \overline{A}_3 \cdots \overline{A}_n \cdot \overline{B}_n \\
&\quad \cdots \\
&\quad + A_{n-1} \cdot \overline{B}_{n-1} \cdot \overline{A}_n \cdot \overline{B}_n \\
&\quad + A_n \cdot \overline{B}_n \\
&= \bigcup_{i:even}^{2n-1} (G_i - G_{i+1})
\end{aligned}$$

Hence $\mathcal{B}_n \subseteq \mathcal{R}_{2n}$. Since the reverse inclusion is trivial, we obtain point (b). Point (a) is a corollary of point (b). Points (d) and (c) are also obtained similarly to points (b) and (a). ∎

From Theorem 2.9 and Theorem 3.2 we obtain Theorem 3.3. Theorem 3.3 indicates that the alternate decreasing property of component sets is not essential.

[‡] $+, \cdot, -$ indicate union, intersection, and complement, respectively.

Theorem 3.3. *For* $n \geq 1$

(a) $\mathcal{RB}_n = \mathbf{RB}_n$,

(b) $\mathcal{B}_n = \mathbf{B}_n$,

(c) $\mathcal{LB}_n = \mathbf{LB}_n$,

(d) $\mathcal{LRB}_{n+1} = \mathbf{LRB}_{n+1}$.

Note: In Theorem 3.3, it is obvious from the definitions that the left-hand sides are included in the right-hand sides.

In the following, we study the relations among four classes $\mathcal{R}_n, \mathcal{L}_n, \mathbf{R}_n$, and $\mathbf{L}_n$.

Lemma 3.4. *For* $n \geq 1$

(a) $\mathbf{R}_{2n} = \mathcal{R}_{2n}$,

(b) $\mathbf{R}_{2n-1} = \mathcal{L}_{2n-1}$,

(c) $\mathbf{L}_{2n} = \mathcal{L}_{2n}$,

(d) $\mathbf{L}_{2n-1} = \mathcal{R}_{2n-1}$.

Proof. We prove only point (a), because points (b), (c) and (d) are also proved in the same manner.

$L \in \mathbf{R}_{2n}$

$\longleftrightarrow$ There exist rational $\mathbf{F}_\sigma$ sets $F_0, F_1, \cdots, F_{2n-1}$ such that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{2n-1} \quad \text{and} \quad L = \bigcup_{i:even}^{2n-1} (F_i - F_{i+1}).$$

$\overset{\P}{\longleftrightarrow}$ There exist rational $\mathbf{G}_\delta$ sets $G_0, G_1, \cdots, G_{2n-1}$ such that

$$G_0 \supseteq G_1 \supseteq \cdots \supseteq G_{2n-1} \quad \text{and} \quad L = \bigcup_{i:even}^{2n-1} (G_i - G_{i+1}).$$

$$\P \left(\begin{array}{l} \text{For } i = 0, 1, \cdots, 2n-1, \text{ let} \\ G_i = \Sigma^\omega - F_{(2n-1)-i} \text{ for only if part, and let} \\ F_i = \Sigma^\omega - G_{(2n-1)-i} \text{ for if part.} \end{array} \right)$$

$\longleftrightarrow$ $L \in \mathcal{R}_{2n}$ ∎

Lemma 3.5. *For any* $n \geq 1$

(a) $\mathbf{R}_n = \mathbf{R}_{n+1}$ *iff* $\mathbf{L}_n = \mathbf{L}_{n+1}$,

(b) $\mathcal{R}_n = \mathcal{R}_{n+1}$ *iff* $\mathcal{L}_n = \mathcal{L}_{n+1}$.

Proof. We prove only point (a), because point (b) is proved similarly to point (a).

(*only if part*): We assume $\mathbf{R}_n = \mathbf{R}_{n+1}$. Then the following equivalence holds.

$$\begin{array}{lll} & L \in \mathbf{L}_n & \\ \longleftrightarrow & \overline{L} \in \mathbf{R}_n & \text{by Lemma 2.8} \\ \longleftrightarrow & \overline{L} \in \mathbf{R}_{n+1} & \text{by the assumption} \\ \longleftrightarrow & L \in \mathbf{L}_{n+1} & \text{by Lemma 2.8} \end{array}$$

(*if part*): This is proved in a similar manner to the only if part. ∎

Theorem 3.6. *For* $n \geq 1$

(a) $\mathcal{R}_n \subset \mathcal{R}_{n+1}$,

(b) $\mathcal{L}_n \subset \mathcal{L}_{n+1}$,

(c) $\mathbf{R}_n \subset \mathbf{R}_{n+1}$,

(d) $\mathbf{L}_n \subset \mathbf{L}_{n+1}$.

Proof. By Lemma 2.7, it suffices to show that the inclusions are strict.

(a) follows from Theorem 2.10 (Barua[1, Theorem 6.3]).

(b) Suppose $\mathcal{L}_n = \mathcal{L}_{n+1}$ for some $n \geq 1$. Then we would have $\mathcal{R}_n = \mathcal{R}_{n+1}$ from Lemma 3.5 (b). This contradicts point (a).

($c^\dagger$) Suppose $\mathbf{R}_n = \mathbf{R}_{n+1}$ for some even $n \geq 1$. Then we have the following.

$$\begin{array}{rcll} \mathcal{R}_{n+1} & = & \mathbf{L}_{n+1} & \text{by Lemma 3.4 (d)} \\ & = & \mathbf{L}_n & \text{by Lemma 3.5 (a)} \\ & = & \mathcal{L}_n & \text{by Lemma 3.4 (c)} \\ & \subset & \mathcal{L}_{n+1} & \text{by point (b)} \\ & = & \mathbf{R}_{n+1} & \text{by Lemma 3.4 (b)} \\ & = & \mathbf{R}_n & \text{by assumption} \\ & = & \mathcal{R}_n & \text{by Lemma 3.4 (a)} \end{array}$$

This contradicts point (a). Hence $\mathbf{R}_n \subset \mathbf{R}_{n+1}$ for any even $n \geq 1$. The proof can be given similarly for odd $n \geq 1$.

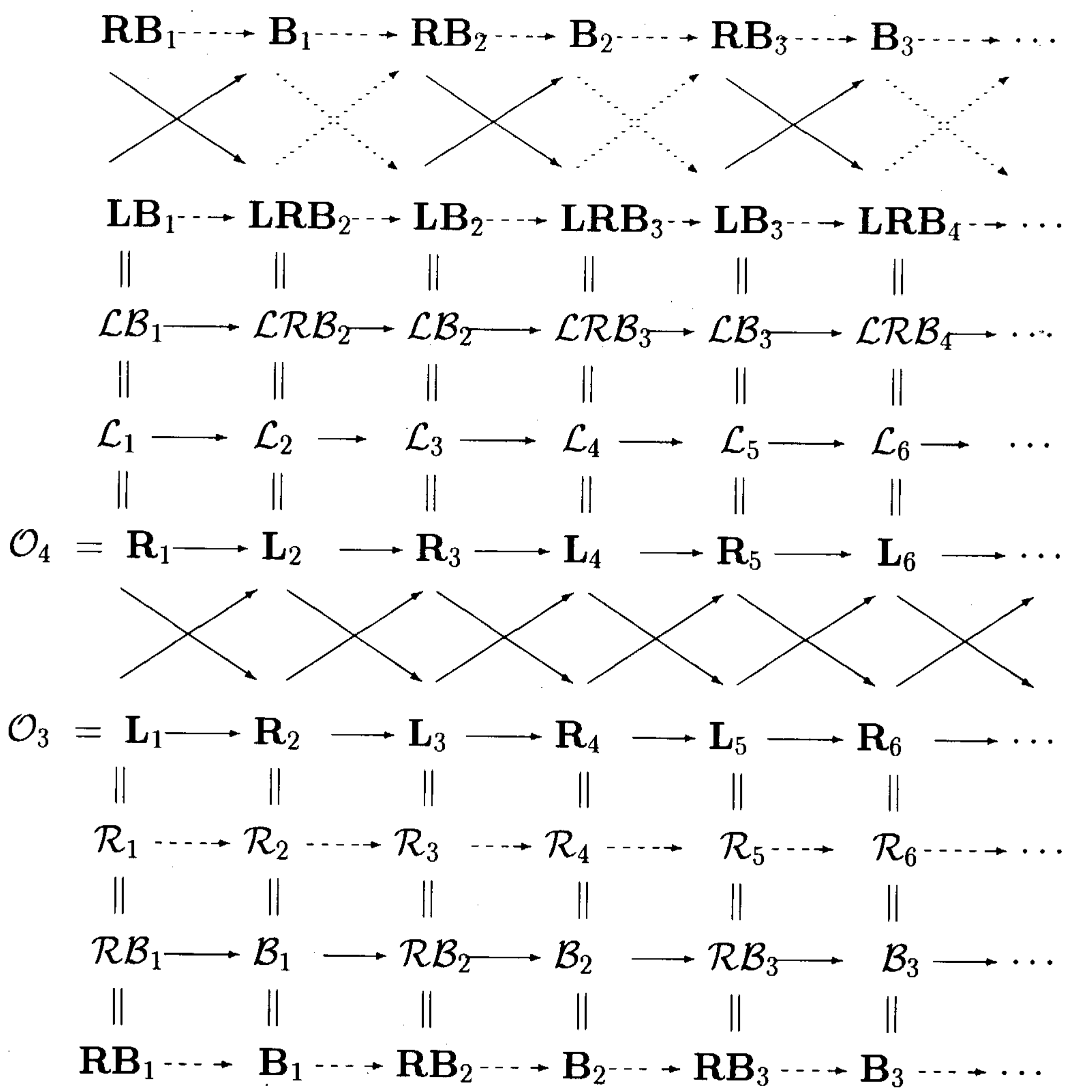
(d) If $\mathbf{L}_n = \mathbf{L}_{n+1}$ for some $n \geq 1$, then we have $\mathbf{R}_n = \mathbf{R}_{n+1}$ by Lemma 3.5 (a). This contradicts point (c). ∎

4. Concluding remarks

We present our results as the diagram in Fig.1, which shows that the hierarchy studied by Wagner[6] and Kaminski[2] coincides with

†The existence of class $\mathbf{L}_n$ and the proof of (c) were suggested by Professor H. Yamasaki (private communication).

that studied by Barua[1]. Kaminski's hierarchy is based on the density of designated state sets of Muller automata. On the other hand, Barua's hierarchy is based on classical descriptive set theory. Since these two hierarchies coincide with each other in spite of the different backgrounds, we can conclude that the classes form a stable hierarchy of ω-regular sets.



The arrows used here express strict inclusion.

$\dashrightarrow$ shows already known results. $\longrightarrow$ shows new results.

Fig.1.

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