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On the Geometric Realisation of Equal Tempered Music

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Abstract

Since the time of Pythagoras (c.550BC), mathematicians interested in music have asked, "What governs the whole number ratios that emerge from derivations of the harmonic series?" Simon Stevin (1548-1620) devised a mathematical underlay (where a semitone equals $2^{\frac{1}{12}}$) that gave rise to the equal temperament tuning system we still use today. Beyond this, the structure of formalised musical orderings have eluded many of us. Music theorists use the tools and techniques of their trade to peer into the higher-order musical structures that underpin musical harmony. These methods of investigating music theory and harmony are difficult to learn (and teach), as complex abstract thought is required to imagine the components of a phenomenon that cannot be seen. This paper outlines a method to understanding the mathematical underpinnings of the equal tempered tuning system. Using this method, musical structure can be quantitatively modelled as a series of harmonic elements at each pulse of musical time.

Keywords: Western Music, Mathematics, Harmonic Structure, Geometry

Mathematics Subject Classification (2010): 00A65

1. Introduction

This paper outlines equal tempered musical structure using mixed approaches. As researchers, we identify as hermeneutic phenomenologists. Three languages we adopt in this paper use my understandings of words, mathematics and music theory as their vehicles,

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where each appeal to different aspects of the hermeneutic conversation with regard to equal tempered musical structure. Words inform a social tool for communication and understanding of musical structure. Mathematics inform ways of thinking that describe operations of musical structure. Music theory informs sets of tools that contain cyclic structures within the part /whole nature of harmony. We aim to triangulate these languages to approach learning equal tempered music structure.

As stated by Cooper & Barger (2009), “the many connections between music and mathematics are well known”[2]. From a physical standpoint, the creation of string instruments from since medieval times emerged from the simple observation that plucking a taught string produces a consistent tone. It then followed that plucking a taught string that is half the length of the original produces a higher version of the original tone. Likewise, plucking a string that is twice the length of the original tone creates a lower version of the same tone. The distances between these tonal similarities is known as an octave. It follows that an interval of one octave is either half or double the frequency of a given tone. The tuning systems throughout history have argued different approaches to which notes should be used between an octave. A monochord is the traditional tool used to divide an octave measurably to create new pitches and, “end results of such a monochord division are an array of pitches (which can be arranged in a scale) and a set of intervallic relationships between them specifically defined by numeric ratios (a tuning system)” [4]. The different tuning systems for each set of pitches and intervals used to divide an octave have been constantly redefined throughout musical periods [7]. The consistent problem has been a question of balancing the richest resonances between notes derived from an octave, and being able to replicate the patterns that underlie them in a modulatory manner. The equal-tempered tuning system is used today to tune pianos, and solves the previously mentioned modulatory problem with a slight cost to resonance. In this system, each *semitone*–the smallest interval that separates any two notes are equidistant. The remainder of this paper will use mathematics to describe how this system can be used to geometrically map musical structure.

2. The Chromatic Scale and Enharmonic Equivalence

We can assign letter names and a number to each pitch within a chromatic scale (the ordered set of all notes within an octave span) to tabulate the data so it looks like a one octave span of a piano keyboard:

Of course, a piano keyboard spans more than one octave, with Table 1 repeating for each octave span. The distance between any se-

n	0	1	2	3	4	5	6	7	8	9	10	11	0
Name	C	C#/ Db	D	D#/ Eb	E / (Fb)	F / (E#)	F#/ Gb	G	G#/ Ab	A	A#/ Bb	B / (Cb)	C / (B#)

Table 1: All pitches within a chromatic (piano keyboard) ordering.

quential pitch in Table 1 is one semitone, and an octave spans twelve semitones. To a music theorist, one octave is all that is required to make sense of the entire range of a keyboard due to the axiom of *octave equivalence* (Eq. 6, Section 4.). A span of zero semitones is called a unison (i. e., the distance from C to the same C), which is equivalent to an octave span (i. e., the distance from C to the next C). Since the names of the two pitches remain the same, they are considered equivalent. There are twelve pitches between any octave, which can be signified using letter names or numbers from zero to eleven. Hence, this system can be seen as counting in *modulo* 12 (*mod* 12). According to Mathieson (2006), "since pitches are related to the number of motions of a string, the pitches of notes are comprised of certain numbers of parts; thus, they can be described and compared in numerical terms and ratios"[6]. Hence, pitches (values for n in Table 1) can be described as either note names or interval spans.

Most pitches in Table 1 have two names due to the axiom of enharmonic equivalence. Technically, every note and interval has multiple names; those listed in Table 1 are simply the most common enharmonics. For instance, the names C# (C sharp) and Db (D flat) signify exactly the same sounds aurally, yet each are notated differently in a score. There is a system behind this note-naming convention. Scales are ordered patterns which define particular musical atmospheres (known as *tonal centres* or *keys*). Each scale begins on a note that contains the least amount of tension in relation to any other notes being played (called the *tonic*). A lone tone is a tonic, and hence the first note of any piece is that piece's tonic, which establishes its key. Composers can use methods to exchange any tonic to *modulate*: the action of transitioning a piece into a new key. An interval pattern then tells which notes belong to a certain *quality* (type) of scale that defines its key. For example, we can create an ascending major scale quality using the 'white' notes of Table 1 $\{0; 2; 4; 5; 7; 9; 11; 0\}$. Due to octave equivalence, the set of notes $\{0; 2; 4; 5; 7; 9; 11; 0\}$ is the same as $\{0; 2; 4; 5; 7; 9; 11\}$. Adding any scale quality as a set of intervals above any tonic note constructs that tonic's scale quality. For instance, adding the major scale quality to D (2) results in a D major scale; also referred to as, "the key of D major"(Section 6.): i. e.,

$$\{0; 2; 4; 5; 7; 9; 11\} + 2 = 2; 4; 6; 7; 9; 11; 1 \rightarrow \{D; E; F\#; G; A; B; C\#\}$$

In this way, all major scale qualities can be formed using any tonic as a base. To determine which note names are written, there cannot be two of the same letter name in any scale quality. For instance, D major contains the enharmonic notes $\{1, 6\}$. C# is written for 1 as D $\{2\}$ already exists in the set. Likewise, F# is written for 6 because a G $\{7\}$ already exists in the set. This system is used to consistently name pitches contained within different keys. Scale qualities sound similar to each other as they contain the same interval structure; yet sound different as each tonic changes (leading to specific note names in each scale set). Hence, keys are established using intervallic scale qualities that form a collection of notes related to a tonic.

The axioms of octave and enharmonic equivalence are fundamental to the part / whole nature of music. Throughout this paper we will be building upon these axioms to describe the fundamental interactions of pitch structures from a music theorists' perspective. In this way, general readers can explore musical structures using a common system of relativistic measurement. Semitones are used to define pitches as notes or intervals, which form the basis of musical structure.

3. The Problems with Pitches

To reiterate, an interval is the relative distance between notes, measured in semitones. The interval between any sequential note in Table 2. is one semitone, and an octave spans twelve semitones. Any interval can be measured sequentially through time (melodic interval) or simultaneously (harmonic interval). Harmonic and melodic intervals are equivalent measurements of semitone distances between notes. The concept of a semitone is easy enough to understand from a musical perspective intuitively, but to address just what a semitone represents in a mathematical sense is actually rather difficult, as there are more factors to consider.

To consider pitches purely from a musician's perspective neglects a common mathematical perspective. For instance, musical notes can be described to contain frequency f and wavelength λ. An octave interval contains a frequency ratio of 2, meaning an octave interval above or below a given note has its f doubled ($n\uparrow 8\text{ve} = 2f$) or halved ($n\downarrow 8\text{ve} = f/2$). Opposingly, each note's λ above or below a given note is halved ($n\uparrow 8\text{ve} = \lambda/2$) or doubled ($n\downarrow 8\text{ve} = 2\lambda$). Hence, the direction of an interval is determined by the multiplicative operator used to calculate a note's f or λ. As multiplication is the opposite of division, f and λ are inversionally equivalent. More specifically, any pair of intervals which total an octave but have opposing directions are inversionally equivalent. Stevin's identity can be used to calculate the f or λ of pitches above or below a given note. There are twelve pitches between each octave span, and to preserve an octave's frequency ratio of 2, Stevin defined an equal tempered semitone equal to $2^{\frac{1}{12}}$. Transpositions of this identity can be used to measure all interval sizes [3].

Musicians use a system of simple addition and subtraction of semitones to theorise with pitches, where sets of pitches are used to define larger harmonic structures. This means that semitones must also behave logarithmically. Fortunately, a logarithmic system is already established for the measurement of semitones in cents, where one semitone equals one hundred cents. This makes it possible to add or subtract semitones in the way musicians translate pitches. Musically, the measurement of cents makes it easy to determine the proximity of pitches, and guides intonation (tuning). As each semitone value is equal to one hundred cents, it measures a relative percentage for each semitone; and manipulating these percentages can lead to differing tuning styles. Moreover, melodic and harmonic intervals also contain a particular type of resonance, known as their interval quality. Interval qualities are (individually and culturally) perceived to contain consonance or dissonance.

As f and λ propagate through time, their velocity can be determined. Velocity is affected by temperature. For instance; in warmer areas, molecules contain more energy, so can vibrate faster than in colder areas, hence sound can propagate faster. Thus, temperature affects intonation; as a difference in temperature affects velocity, which in turn affects f and λ.

We can summarise these aspects using the following scaffold toward defining pitches:

1. Pitches allow for *octave equivalence*, and are notated in a score using *enharmonic equivalence*.

2. Pitches can be interpreted as note or interval, signified by an integer n.

3. Harmonic and melodic intervals are equivalent measurements of semitone distances between notes.

4. Intervals that add to an octave are *inversionally equivalent* via direction.

5. Intervals contain a particular degree of resonance, known as their *interval quality*.

6. Translations by semitone interval n must be the same as multiplicative changes of n for f and λ.

7. Stevin's identity defines a semitone interval as $I = 2^{\frac{1}{12}}$, which preserves an octave's frequency ratio of 2.

8. Transpositions of this identity lead to $I^n = 2^{\frac{n}{12}}$ [3].

9. Converting to cents: $[log(I^n)/log(2)] * 1200$[7]; assists with *intonation*.

10. Temperature affects intonation, as temperature affects velocity, which affects notes, since $v = f\lambda$.

Using this scaffold to bound my phenomenon of inquiry, we attempt to describe semitones (the building blocks of musical structures) from differing, yet intersecting mathematical approaches. We aim to demonstrate how these points intersect throughout the remainder of this paper. It is hoped that by doing so, the general reader will be able to understand the above scaffold from a perspective that links musical and mathematical approaches. The next section will unpack deliberations algebraically, focusing on the intersection between core items from this scaffold. Building from this juncture, trigonometry, set theory and matrices will be used to describe methods for interacting, categorising and exploring musical structures.

4. Pitch Identities and Octave Equivalence

In order to align mathematical and musical perspectives of pitches, the stretching (multiplication) and compressing (division) of frequencies and wavelengths must align with the translation (addition or subtraction) of semitones. One way to accommodate this issue considers the argument or angle to be the operator of pitch motion. The amount of rotation (radians θ or degrees ^{o}) of any angle remains consistent despite any translative or multiplicative motion along a radius to a point on a circle's circumference. The circumference of any circle can be found using the formula $C = 2\pi r$. Taking a unit circle (a circle that has a radius length $r = 1$), its circumference is 2π and $C = 2\pi$. In order to calculate the angle θ inherent to each of the twelve pitches, we simply take the circumference of a unit circle and divide it by twelve (Eq. (1)):

$$\theta = C/12 \quad \& \quad \theta = \pi/6 \tag{1}$$

This means we can plot all twelve pitches as points around a unit circle, each separated by $\pi/6$ radians. Hence, each semitone can be measured as a point (x, y) using standard Pythagorean trigonometric functions (cos(θ), sin(θ)). Since angles are mapped around a circle, we can also use polar form to approach pitches. Since any pitch n can be described as a note or interval, a duality exists that suggests also using exponential form (i. e., notes and intervals signified by cos(θn) + i sin(θn) and the term, 'pitches' refer to $e^{i\theta n}$ to measure angles on the complex plane. Using the complex plane also solves a problem graphically, as $\cos(\theta) + \sin(\theta)$ yields one superimposed waveform (i. e., not congruent with note and interval descriptions of pitch n); whereas the polar form $\cos(\theta) + \mathrm{i}\sin(\theta)$ yields the required dual waveforms, which can be used as references to map any pitch n. This may all be a lot to take in at once, so we will show this progressively.

In the complex plane, a point z can be defined by the translation $x+yi$, where $i = \sqrt{-1}$. Another way to represent a point z is to express its coordinate in polar form (r, θ). Here r is the distance from the origin to the point (radius of a unit circle), and θ is the angle from the positive X-axis to the radius (anticlockwise, in radians). There are conversion formulas for switching between polar and Cartesian coordinates (Eq. (2)), giving the polar form of a point z in terms of the trigonometric functions (Eq. (3)):

$$(x, y) \rightarrow (r, \theta) : x = rcos\theta, y = rsin\theta \tag{2}$$

Therefore,

$$z = r(cos\theta + isin\theta) \tag{3}$$

The complex plane allows the use of waveforms simultaneously for visual mapping (Section 5.). Since $r = 1$ on a unit circle, we can reduce the equation further to get something very similar to Euler's formula. Using Euler's formula, the polar form of z can be notated in *exponential form*.

$$z = r(cos\theta + isin\theta),$$

where $r = 1$.

$$e^{i\theta} = cos\theta + isin\theta$$

Hence,

$$z = e^{i\theta} \tag{4}$$

where, $\theta = \pi/6$.

Hence z represents a point of interest – a pitch that can be a note or interval. The angle θ is also called the argument, denoted by $\theta = \arg(z)$. Using this convention, we can define octave equivalence. For each complete rotation around the circle ($\pm 2\pi$), the resulting angle θ remains the same. As each note can be represented as an angle, the distance between each octave higher or lower is $\pm 2\pi k$ (Eq. (6)).

Hence, each argument has multiple solutions. The principal argument refers to the angle that falls within the range $\{-\pi < \arg(z) \leq \pi\}$, and is denoted using a capital letter Arg(z). It follows that arg(z) is equivalent to Arg(z)$\pm 2\pi k$ (Eq. (7)).

$$\theta_{note} - \theta_{8ve(k)} = 2\pi k \tag{5}$$

$$arg(z) = Arg(z) \pm 2\pi k \tag{6}$$

Hence (Eq. (6)) defines the axiom of octave equivalence, where the span of any numbers of octaves up or down from any pitch can be considered equal to its unison. Returning to (Eq. (4)), exponential form was used to denote pitches. Here, arg(z) refers to the angle θ of any pitch i. Pitches behave multiplicatively and in polar or exponential forms, de Moivre's formula (Eq. (8)) enables a path forward. With de Moivre's formula, we can define the exponential form of a pitch's argument using values of n (Eqs.(9)-(11)).

$$z^n = e^{(i\theta)n} \tag{7}$$

$$(cos\theta + isin\theta)^n = cos\theta n + isin\theta n \tag{8}$$

$$z^n = e^{i\theta n} = e^{i\pi n/6} \tag{9}$$

Therefore,

$$Arg(z^n) = \theta n = \pi n/6 \tag{10}$$

Therefore,

$$arg(z^n) = \theta n \pm 2\pi k = \pi n/6 \pm 2\pi k \tag{11}$$

This gives an interesting insight to what the values of z and n represent. Here, n acts as an intervallic scalar factor of each point z. Pitches allow for octave equivalence as: (a.) each $Arg(z^n)$ can be determined by multiplying θ by an integer from 0 to 11; and, (b.) any integer value outside this range yields some $arg(z^n)$ (i. e., $n = -12 \equiv -2\pi \pm 2\pi k, n = 13 \equiv \pi 6 \pm 2\pi k, \ldots$). This scalar multiplication is why in musical parlance, a formal collection of melodic pitches are called scale sets (simplified to scales); which outline tonal atmospheres (known as keys, or tonal centres). The multiplicative nature of pitches can also be shown when the frequency f and wavelength λ of a pitch are considered. The span of one octave is a twelve semitone span, which equates to a frequency twice that of a unison (Eq. (12)) so, $f * I^{12} = 2f$. Therefore, we have the Steven's identity,

$$I = 2^{\frac{1}{12}} \tag{12}$$

Thus arriving at Simon Stevin's identity of an equal-tempered semitone. To solve for all twelfth roots of two, simply multiply (Eq.(12)) by (Eq.(9)). i. e., $I * (z^n)$ for $\{0 \le \pi \le 11\}$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$I(z^n)$	$2^{1/12}$	$2^{1/12} e^{i\pi/6}$	$2^{1/12} e^{i\pi/3}$	$i\ 2^{1/12}$	$2^{1/12} e^{i2\pi/3}$	$2^{1/12} e^{i5\pi/6}$	$-2^{1/12}$	$2^{1/12} e^{i7\pi/6}$	$2^{1/12} e^{i4\pi/3}$	$-i\ 2^{1/12}$	$2^{1/12} e^{i5\pi/3}$	$2^{1/12} e^{i11\pi/6}$

Table 2: All 12^{th} roots of 2.

There are twelve solutions to the twelfth root of two due to the fundamental theorem of algebra (any polynomial of degree x has x roots). Coincidentally, in musical terminology the, "n^{th} scale degree" refers to, "the pitch at the n^{th} position of a scale set," and is denoted by an integer with a hat, known as a carat: Stevin's identity can also be used to measure all interval sizes[3]. We have notated (Eq. (13)) to show how this relationship can be formed in the necessary multiplicative sense.

$$I^n = (2^{\frac{1}{12}})^n = (2^{\frac{n}{12}}) \tag{13}$$

When we substitute $\{0 \le n \le 11\}$ into (Eq. (13)), we get the frequency ratios underpinning Stevin's identity:

n	0	1	2	3	4	5	6	7	8	9	10	11
$\sqrt[12]{2^n}$	1	1.059...	1.122...	1.189...	1.2599...	1.3348...	1.414...	1.498...	1.587...	1.682...	1.782...	1.88775...

Table 3: Frequency ratios for the 12th roots of 2 using Stevin's Identity (Eq.13).

The values in Table 4. have been approximated as they contain ongoing decimals, yet if these values are kept in exact form (Eq. (13)), they can be used to accurately calculate interval sizes in cents [7].

$$I^n_{cents} = [log(2^{\frac{n}{12}}/log(2))] * 1200 \tag{14}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
Cents	0	100	200	300	400	500	600	700	800	900	1000	1100

Table 4: Interval sizes in cents for all values of $\{0 \leq n \leq 11\}$; n: cents = 1:100.

Performing the calculation (Eq. (14)) yields the values shown in Table 4.,

Hence an octave contains 1200 cents, and each semitone contains 100 cents. Cents are a logarithmic measure, which means that interval sizes can be translated (added or subtracted) as needed. Table 4. also demonstrates that n is one percent of its value in cents. Hence, the measurements of n and its value in cents contains a part/whole ratio of 1:100. Instrumentalists often adjust their tunings whilst performing, and it is not uncommon to adjust the tuning of specific chordal members to adhere to just intonation. For instance, tuning in just intonation for a major chord quality demands its chordal third thirteen cents flat and fifth two cents sharp from its tonic. In other words, play the major third interval 13% flatter than usual, and the perfect fifth interval 2% sharper. Equal temperament and just intonation are still both used today: the former as an aligned scaffold that is completely modulatory, the latter as a performance methodology that contains richer consonances.

Using (Eq. (13)), we can calculate the f and λ of any pitch above or below any given note by any interval n. As Stevin's identity focuses solely on defining n as interval measurements, a starting note is required to calculate f and λ of any note separated by some interval n. To calculate the frequency of an interval n semitones above a note, simply multiply f by $2^{\frac{n}{12}}$. To get the frequency of a note below, divide f by $2^{\frac{n}{12}}$. Inversely, to calculate the wavelength of a note n semitones above, divide λ by $2^{\frac{n}{12}}$; for below, multiplyλ by $2^{\frac{n}{12}}$. Instead of dividing, one can also multiply by negative n. This works for any interval n between any two notes' f and λ. Any f or λ of a note above or below a given reference note's f or λ by interval n is given by (Eq. (15)),

$$f_2 \uparrow = f_1 * 2^{\frac{n}{12}} \wedge \lambda_2 \downarrow = \lambda_1 * 2^{\frac{n}{12}}; f_2 \downarrow = f_1 * 2^{\frac{-n}{12}} \wedge \lambda_2 \uparrow = \lambda_1 * 2^{\frac{-n}{12}} \quad (15)$$

Combining these aspects, we can show the f and λ of each n above or below any pitch; say, middle C (C_4).

The formulae in (Eq. (15)) show the multiplicative nature of f and λ by ratios of n. They state that when a note moves up by n, f is stretched by a factor of $2^{\frac{n}{12}}$, whilst λ is compressed by that same scalar factor. Conversely, when a note moves down by n, f is compressed and λ is stretched by $2^{\frac{n}{12}}$. Hence, when we hear musical pitches move up or down in a seemingly linear motion (translating in an additive or subtractive sense), the waveforms are actually scaling up or down in

n	$Arg(z^n)$	I^n_{cents}	$n(f_1)$	$f_1\,(Hz)$	$\lambda_1\,(m)$	$n(f_2)\uparrow$	$f_2\uparrow(Hz)$	$\lambda_2\uparrow(m)$	$n(f_2)\downarrow$	$f_2\downarrow(Hz)$	$\lambda_2\downarrow(m)$
0	0	0	C_4	261.63	1.3187	C_4	261.63	1.3187	C_4	261.63	1.3187
1	$\pi/6$	100	C_4	261.63	1.3187	$C^\#_4/D^b_4$	277.18	1.2447	B_3	246.94	1.3971
2	$\pi/3$	200	C_4	261.63	1.3187	D_4	293.66	1.1748	$A^\#_3/B^b_3$	233.08	1.4802
3	$\pi/2$	300	C_4	261.63	1.3187	$D^\#_4/E^b_4$	311.13	1.1089	A_3	220.00	1.5682
4	$2\pi/3$	400	C_4	261.63	1.3187	E_4	329.63	1.0466	$G^\#_3/A^b_3$	207.65	1.6614
5	$5\pi/6$	500	C_4	261.63	1.3187	F_4	349.23	0.9879	G_3	196.00	1.7602
6	π	600	C_4	261.63	1.3187	$F^\#_4/G^b_4$	369.99	0.9324	$F^\#_3/G^b_3$	185.00	1.8649
7	$7\pi/6$	700	C_4	261.63	1.3187	G_4	392.00	0.8801	F_3	174.61	1.9758
8	$4\pi/3$	800	C_4	261.63	1.3187	$G^\#_4/A^b_4$	415.30	0.8307	E_3	164.81	2.0933
9	$3\pi/2$	900	C_4	261.63	1.3187	A_4	440.00	0.7841	$D^\#_3/E^b_3$	155.56	2.2177
10	$5\pi/3$	1000	C_4	261.63	1.3187	$A^\#_4/B^b_4$	466.16	0.7401	D_3	146.83	2.3496
11	$11\pi/6$	1100	C_4	261.63	1.3187	B_4	493.88	0.6985	$C^\#_3/D^b_3$	138.59	2.4893
0	2π	1200	C_4	261.63	1.3187	C_5	523.25	0.6593	C_3	130.81	2.6374

Table 5: Frequencies and wavelengths of chromatic intervals spanning an octave above and below C_4.

a multiplicative sense (being stretched or compressed by some scalar factor). The velocity of any note can be given by multiplying its frequency and wavelength together:

$$v = f\lambda \tag{16}$$

This (Eq. (16)) is a common formula to a physics class, and gives a good platform to see how intonation is affected when considering temperature affects velocity. For example, sound travels faster in warmer areas (where molecules contain more energy, so can vibrate faster than in colder areas). Using the ideal gas law, it is possible to link this with basic chemistry study by defining the speed of sound in a pure gas. Suits (2015) shows this can be done by combining the heat ratio for the gas (adiabatic constant γ), absolute temperature T (Kelvins), the mass of one gas molecule M and Boltzmann's constant k_B (1.38064852 × 10-23 m^2 kg s^{-2} K^{-1}, which converts absolute temperature units into energy units) [8].

$$v = \sqrt{((\gamma * k_B * T)/M)} \tag{17}$$

The speed of sound is also affected by other factors such as humidity and air pressure, which lead to wider plateaus of learning, yet are beyond the scope of this paper. This section can also lead nicely into studies of the mathematics behind musical instrument creation: the story of musical structure underpins many intersecting areas of quantitative study. Pedagogical tangents aside, for music instrumentalists, temperature affects tuning (*intonation*), as a difference in temperature affects velocity, which affects f and λ. Hence, instruments are in tune with each other when all musicians play notes that have aligning f and λ. With skillful alignment, amplitude (*dynamics, volume*) also increases as waveforms are additive. Hence, the sound quality of any particular instrument (*timbre*) refers to the overall shape of the waveform being created by an instrument. Thus, tonal blending is the skill of matching waveforms; whereas intonation is the skill of matching periodicity.

This section has addressed many of the listed elements contained in the scaffold of section 4.. In sum:

- Pitches allow for octave equivalence ($\arg(z) = \text{Arg}(z) \pm 2\pi k$).
- Pitches can be interpreted as note or interval, signified by an integer n.
- Harmonic and melodic intervals are equivalent measurements of semitone distances between notes.
- Translations by intervals n can be made in cents, multiplicative intervals n for f and λ use $2^{\frac{n}{12}}$.
- Temperature affects velocity (Eq. 17), which affects intonation, since $\text{v} = \text{f}\lambda$.

The next section will add to deliberations by addressing the remaining elements listed in the scaffold of section 3.. In the next section, we will use a trigonometric approach toward unpacking interval quality and general inversional equivalence. We will continue deliberations in section 6. by outlining ways musical structures can be categorised and indexed using mathematical set theory approaches. Proceeding sections will unpack one of the main ideas behind the doctoral thesis[1]: that all musical structures can be represented using matrix form. Hence, we will adopt a mathematical approach to constructing a platform that can be used to explore musical structures that underpin any stylistic use of equal tempered music.

5. Trigonometric Representation and Inversional Equivalence

The last section described how any pitch n can be found around the circumference of a unit circle, so this section will show this using Pythagorean trigonometry. When we map θ on a unit circle, all pitches can be seen to contain inversional equivalence. Inversional equivalence is the name given to any set of two intervals whose spans sum to an octave in opposing directions. Rather, opposing directions are equal when interval spans total one octave. For example, the interval eight semitones above a given note (interval quality of a minor sixth) has an inversional equivalent four semitones below that same note (interval quality of a major third) as both intervals sum to an octave (8 + 4 = 12). Hence, moving down by four semitones yields the same note as moving up by eight semitones (via differing interval qualities). This holds for all values of *n* in Table 4..

n	θ°	*Radians*	$\cos(\theta), x$	$\sin(\theta), yi$	Quality Ascending	Quality Descending
0	0ºor360º	0or2π	1	0	Perfect Unison/Octave	
1	30ºor − 330º	$\pi/6$ or − 11π/6	½	$\sqrt{(3)}/2$	Minor 2nd	Major 7th
2	60ºor − 300º	$\pi/3$ or − 5π/3	$\sqrt{(3)}/2$	½	Major 2nd	Minor 7th
3	90ºor − 270º	$\pi/2$ or − 3π/2	0	1	Minor 3rd	Major 6th
4	120ºor − 240º	2π/3 or − 4π/3	$\sqrt{(3)}/2$	−1/2	Major 3rd	Minor 6th
5	150ºor − 210º	5π/6 or − 7π/6	½	$-\sqrt{(3)}/2$	Perfect 4th	Perfect 5th
6	±180º	$\pm\pi$	−1	0	Tritone	
7	210ºor − 150º	7π/6 or − 5π/6	−1/2	$-\sqrt{(3)}/2$	Perfect 5th	Perfect 4th
8	240ºor − 120º	4π/3 or − 2π/3	$-\sqrt{(3)}/2$	−1/2	Minor 6th	Major 3rd
9	270ºor − 90º	3π/2 or − $\pi/2$	0	−1	Major 6th	Minor 3rd
10	300ºor − 60º	5π/3 or − $\pi/3$	$-\sqrt{(3)}/2$	½	Minor 7th	Major 2nd
11	330ºor − 30º	11π/6 or − $\pi/6$	−1/2	$\sqrt{(3)}/2$	Major 7th	Minor 2nd

Table 6: Musical and mathematical equivalencies for the set C about a unit circle (n).

For sake of interlinkage, on the complex plane, $n = 3$ is found at (0, *i*); and at $n = 9$, (x, y) = (0, -*i*). Mapping *n* around a unit circle clearly shows all pitches. One advantage of tabulating or mapping the chromatic scale as per Table 5. or Figure 1 is pedagogical.

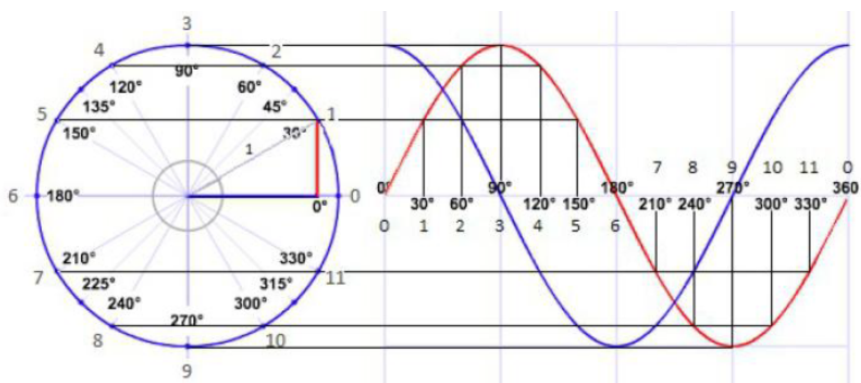

Figure 1. Values for *n* around a unit circle. Sinusoidally, each *n*-value represents a point on the sin wave. This unit circle can also be mapped in the complex plane.

In this visual setting, the inversional equivalence of intervals can be easily shown as equivalent distances spanning opposite directions. In other words, for any note, two interval qualities in opposite directions can be used to reach any other note. To a musician, the inversion of any pitch n is found by the simple equation ($n_2 = 12 – n_1$, mod 12). Alternatively, interval inversions can also be calculated using $n_1 \pm n_2 = 2\pi$ or 360^o, where n is in radians or degrees respectively.

Figure 1 shows that each value plotted around the unit circle corresponds to a point on a sine wave for θ moving anticlockwise. Looking at Table 5., sin(θ) is equivalent to the y-coordinate (or imaginary value yi on the complex plane). Likewise, cos(θ) represents the x-coordinate (or real values of θ on the complex plane). In Figure 1, all pitches on the unit circle would correspond to points on a cosine wave if values for n were ordered like a clock face (with its twelve replaced by a zero). This approach would consider each hour division still equal to each previous θ, except motion proceeds in the opposite direction (clockwise motion).

In Table 4., if we begin at cos(0) and look at each cos(θ) value going down, it has the same pattern as if we were to begin at sin($\pi/2$) and go up for each value of sin(θ). Comparing the properties of clock face motion from mathematical convention algebraically; θ would become negative motion; the starting point of cos differs from sin by $\pi/2$ where n remains consistent. We can write this difference as sin($\pi n/6$) = cos($\pi/2$-$\pi n/6$) (Eq. (18)).

One trigonometric identity is that cos(θ) equals the derivative (change over time) of sin(θ). This derivative signifies the rate of change of a pitch through time by calculating its gradient over a tiny nudge in time (Eq. (19)). When any note is played it also contains a rhythmic element that marks its place in time. This rhythmic element is called a note's *articulation*. When the articulation of a note is clear, musicians playing together are informed of exactly how time is being understood by those around them. See the following algebraic and derivative analogies

$$cos((\pi/2) - (\pi n/6)) = sin(\pi n/6) \tag{18}$$

$$\frac{d}{dn}(sin(\pi n/6)) = (\pi/6)(cos(\pi n/6)) \tag{19}$$

Here (Eqs. 18-19) demonstrate that both approaches are equivalent. A learner may find it easier to think of motions around the unit circle as clock motion, or as normal mathematical convention. Where the clock analogy refers to real values of y through time (as a derivative point on a cosine wave), the conventional approach refers to imaginary values of y (as a point on a sine wave) on the complex plane. Converting between the two approaches enable the mapping of pitches on either sine or cosine waveforms. These approaches are also inversionally equivalent as from any pitch, cos(π/2-πn/6) will reach the same value for n as sin(πn/6). Musically, the derivative comfortably signifies a melodic interval, and harmonic intervals span the length of time it takes to articulate their sound. Harmonic intervals are often thought of as instantaneous sounds, or sounds occurring over the time of its articulation, and hence can be measured as the limit as time approaches zero (when time equals zero, sounds are instantaneous). As harmonic and melodic intervals are equivalent measurements using values of n, harmonic pitches can be used to describe melodic operations through time and vice versa. It follows that we can simply interpret n as harmonic intervals to describe the construction of any greater musical structure occurring through or at any point in time. This means we are able to represent pitches related to each other in a way that does not need to include time.

The next section will focus on using set theory to categorize pitches into larger musical structures. Such structures are created using differing combinations of notes and intervals in different ways. Simply put, the formation of larger musical structures involves finding patterns of notes or intervals that contextualize any musical element. The proceeding section will use matrices to summarize deliberations.

6. Indexing Formal Orderings via Set Theory

Consider the set S of chromatic pitches situated between the number of musical notes n within an octave (8ve). For the ascending and descending versions of S,

$$S_{ascending} = \{0; 1; 2; \ldots ; n-1\} \wedge S_{descending} = \{n-1; n-2; n-3; \ldots ; n-n\}$$

Therefore,

$$S_{asc.} = \{0; 1; 2; \ldots ; 11\} \wedge S_{desc.} = \{11; 10; 9; \ldots ; 0\} \qquad (20)$$

These sets can be easily indexed. This will allow easy recall of any formalized harmonic interval combination (like scales, modes, pentatonics, tetrachords or chordal qualities to name a few). As musicians can use any combination of notes from the chromatic scale to create music, it is useful to map all possible ordered combinations within set S. Hence, we wish to find the Power Set of S; which is the set of all subsets of S. So, listing all subsets of S yields the Power Set. For instance, for the set {0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11}:

- The empty set { } is a subset of {0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11}.
- All single note combinations of C are subsets: 0, {1}, {2},. . . , {11}.
- All two-note combinations are also subsets: {0; 1}, {0; 2}, . . . ; and so on.
- In fact, any *n*-note combinations of values in the set S are also subsets of S.
- Including {0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11}, which is also a subset of S.

Every member of $P(S)$ is an ordered set, as each subset contains numbers grouped from smallest to largest. As pitches contain note / interval duality, subsets of $P(S)$ can be read as notes or intervals. Since all ordered sets allow for both octave and inversional equivalences they can be found within the span of an octave. The Power Set is found by raising 2 to the power of the number of members in the set.

$$|P(S)| = 2^n = 2^{12} = 4096 \tag{21}$$

Hence, there are 4,096 different formal orderings within $P(S)$. Another way to create a power set is to use the binary system, letting "1" mean "put the matching member into this subset". This gives a useful way to categorise all possible sets as a binary index.

Decimal	Binary (12-bit)	Subsets of *S*	Hexadecimal
0	000000000000	{ }	0_{16}
1	000000000001	{0}	1_{16}
2	000000000010	{1}	2_{16}
...	...	...	...
4094	011111111111	{0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10}	FFE_{16}
4095	111111111111	{0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11}	FFF_{16}

Table 7: The Power Set P(S) with Decimal, Binary and Hexadecimal indexes.

All subsets of $P(S)$ can have an assigned value in binary, decimal and hexadecimal, shown in Table 4.. This is a efficient way to index every formal note set individually. Each subset represents all *ordered* interval sets, such as chord or scale qualities. Here we represent both notes and intervals as pitches with numbers 0 to 11 in each subset. Arithmetic on those numbers performed modulo 12 keep the results with the defined set of 12 notes/intervals. Even though this representation can be used for notes and intervals, we treat them as separate things, just like a set of notes is different than a single (non-set) note. As such, we are implicitly defining the addition of an interval I, to a note, N_1, as: $N_2 = N_1 + I$, which is evaluated as $N_2 \equiv N_1 + I$ (modulo 12). Similarly, adding an interval I_2 to an initial interval I_1 gives a new interval I_3: $I_3 \equiv I_1 + I_2$ (modulo 12).

Two other forms of addition are available but have a less immediate real world meaning:

- Note + Note (more often used as a reference to specific notes in larger constructions)
- Interval + Note (this is the same as note + interval, yet is rarely used since musicians often start with a note (or notes / scale / chord / composition) and shift by an interval (addition); the note is referred to first and the interval second).

Now we can get back to looking at addition involving notes/intervals and sets of same. The obvious candidate is through performing a set union:

- $S_3 = S_1 + S_2$ when S_3 contains the combined elements of S_1 and S_2
- i. e., $\{2; 5; 9\} \cup \{0; 2; 4; 7; 11\} = \{0; 2; 4; 5; 7; 9; 11\}$

Similarly to the four ways we can define addition of notes and intervals, there are now multiple ways of defining addition of notes, intervals and sets of same (listing only the sensible ones):

- Note + Interval
- Interval + Interval
- Set of Notes + Set of Notes ≡ Union of Sets
- Set of Intervals + Set of Intervals ≡ Union of Sets
- Set of Notes + Interval
- Note + Set of Intervals

The last two are particularly interesting. A set of notes can represent a scale, a chord, a composition. That set of notes can be transposed by adding an interval to all included elements. Starting with the set of notes in a C major chord, $\{0; 4; 7\}$ we can add the A interval, 9, to get the set of notes in a A major chord, $\{9; 1; 4\}$. Likewise, adding the D interval to the set of notes in a C major scale yields the set of notes in a D major scale: {0; 2; 4; 5; 7; 9; 11} + 2 = {2; 4; 6; 7; 9; 11; 1} (Section 2.). A set of intervals can represent the the quality of a scale/chord/composition. For example, the quality of a major chord could be represented as {0; 4; 7}. Note that this is different than the specific C major chord. Such chordal quality representations will always contain the interval 0, which signifies the musical root of the chord. Musically, a chordal root simply points to the reference note of the chord. Now that we have the major chord quality we can make any major chord we want:

Note + major chord quality = major chord for note (i. e., the D major chord: 2 + {0; 4; 7} = {2; 6; 9}).

This discussion on addition was all modulo 12 but would work just as well with standard addition of integers positive and negative. Having expanded the range of notes, we would need to define which note is zero. In section 4., we described any point of interest as z^n (Eq. (9)); yet it could be the definition from a MIDI input; it could represent middle C on the keyboard or the A musicians use to tune. The note defined as zero is often a tonic, root of a chord or some other significant point of reference. In this way, sets of intervals become more interesting and zero is still well defined. Allowing intervals outside of 0 to 11, we can represent more larger musical structures, such as:

- A major chord with added ninth {0; 4; 7; 14}
- The harmonic series on a brass instrument with one set of fingerings {0; 12; 19; 24; 28; 31; 34; 36; ...}
- The three inversions of the major chord quality {0; 4; 7}, {-5; 0; 4} and {-8; -5; 0}

With such indexing, any patten of notes or intervals can be categorised as an ordered set. Such sets can emerge through the use of either melodic or harmonic intervals, and hence can describe any larger musical structure. Such a system for constructing ordered sets of melodic or harmonic intervals requires an understanding of the aforementioned musical convention based around enharmonic equivalence (Section 2.). The next section will tether this concept by using matrices to demonstrate the underlying system that defines musical atmospheres known as tonal centres or keys.

7. Matrix Representation

Recall that pitches are contextualized by *intervallic qualities* that form the basis of a musical atmosphere in the form of a melodic interval set known as a scale. Hence, the collection of pitches that define a particular scale quality also define a particular key. When reading musical scores, it is natural to see melodic interactions reading left-to-right and harmonic interactions occurring horizontally. Matrices make good use of displaying information this way. One can read columns as harmonic interval interactions and rows as melodic intervals. We can show all chromatic scales as a matrix, created by adding the values of $S_{asc.}$ and $S_{desc.}$ (Eq. (20)) in base twelve:

							$S_{asc.}$						
	$S(n+n)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
	0	0	1	2	3	4	5	6	7	8	9	10	11
	11	11	0	1	2	3	4	5	6	7	8	9	10
	10	10	11	0	1	2	3	4	5	6	7	8	9
	9	9	10	11	0	1	2	3	4	5	6	7	8
	8	8	9	10	11	0	1	2	3	4	5	6	7
	7	7	8	9	10	11	0	1	2	3	4	5	6
$S_{desc.}$	**6**	6	7	8	9	10	11	0	1	2	3	4	5
	5	5	6	7	8	9	10	11	0	1	2	3	4
	4	4	5	6	7	8	9	10	11	0	1	2	3
	3	3	4	5	6	7	8	9	10	11	0	1	2
	2	2	3	4	5	6	7	8	9	10	11	0	1
	1	1	2	3	4	5	6	7	8	9	10	11	0

Table 8: The circulant matrix S: all chromatic scales (tonic = column 0).

Table 7. outlines every possible chromatic scale. One property of the matrix S is that it is circulant, meaning each row 'i' has been rotated one element n to the right relative to the preceding row. More generally, we can recreate the matrix S through the method of cyclic permutation, which refers to the action of shifting all elements of a set by some fixed offset. Cyclic permutation allows for octave equivalence, with elements shifted off the end of a row being inserted back at the beginning. For the matrix S, with the first row made of elements $\{0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11\}$, a cyclic permutation of one place to the left would yield $\{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 0\}$; and a cyclic permutation of one place to the right would yield $\{11; 0; 1; 2; 3; 4; 5; 6; 7; 8; 9; 10\}$. The mapping of Table 4. can be described as each row 'i' of n is shifted to the right by one ($n_i \rightarrow n_{i+1}$ (mod 12)). Generally,

$$n_i \rightarrow n_{i+k}(mod\ 12) \tag{22}$$

for any shift of k places to the right.

$$n_i \rightarrow n_{i-k}(mod\ 12) \tag{23}$$

for any shift of k places to the left.

The matrix approach is not entirely different from conventional musical set theory approaches [4, 5, 9]. It is not my intention to give an exegesis of musical set theory techniques in this paper; we wish to show how the general method can be extended to yield all harmonic combinations in a manner aligned with tonal approaches to harmony. We can finally introduce the harmonic context to my deliberations. A fundamental harmonic reference is the cycle of fourths / fifths. Its ordering can be created by having a gap of five or seven semitones between each note. When five semitones are used, the order is called the cycle of fourths, when seven semitones are used, it is called the cycle of fifths. We do this to get the following definition of the set H:

$$H = \{0; 5; 10; 3; 8; 1; 6; 11; 4; 9; 2; 7; 0\} \ \& \ \{0; 7; 2; 9; 4; 11; 6; 1; 8; 3; 10; 5; 0\} \quad (24)$$

The previous keyboard visual (Table 2.) changes to a tonally ordered configuration:

The previous keyboard visual (Table 1) changes to a tonally ordered configuration:

n	0	5	10	3	8	1	6	11	4	9	2	7	0
Name	C / B#	F / E#	A#/ Bb	D#/ Eb	G#/ Ab	C#/ Db	F#/ Gb	B / Cb	E / Fb	A	D	G	C

Table 9: Cycle of Fourths ordering.

n	0	7	2	9	4	11	6	1	8	3	10	5	0
Name	C	G	D	A	E / Fb	B / Cb	F#/ Gb	C#/ Db	G#/ Ab	D#/ Eb	A#/ Bb	F / E#	C / B#

Table 10: Cycle of Fifths ordering.

Tables 7. and 10 are mirror images of each other, making them inversionally related. This inversional relationship has to do with the amount of raised or lowered notes found in each key with tonic n. To unpack, each subsequent key increases or decreases the amount of raised (sharp) or lowered (flat) notes in its pitch collection by one. We can show this by comparing Sets S and H additively,

Table 7. provides a very clear view of the underlying system to harmonic orderings. It is also circulant,

$$n_i \rightarrow n_{i+7} (mod\ 12) \quad (25)$$

for a shift of 7 places to the right.

$$n_i \rightarrow n_{i-5} (mod\ 12) \quad (26)$$

for a shift of 5 places to the left.

$H + S$		$SetS_{asc.}$												
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**0**
	0	0	1	2	3	4	5	6	7	8	9	10	11	0
	5	5	6	7	8	9	10	11	0	1	2	3	4	5
	10	10	11	0	1	2	3	4	5	6	7	8	9	10
	3	3	4	5	6	7	8	9	10	11	0	1	2	3
	8	8	9	10	11	0	1	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7	8	9	10	11	0	1
SetH	**6**	6	7	8	9	10	11	0	1	2	3	4	5	6
	11	11	0	1	2	3	4	5	6	7	8	9	10	11
	4	4	5	6	7	8	9	10	11	0	1	2	3	4
	9	9	10	11	0	1	2	3	4	5	6	7	8	9
	2	2	3	4	5	6	7	8	9	10	11	0	1	2
	7	7	8	9	10	11	0	1	2	3	4	5	6	7
	0	0	1	2	3	4	5	6	7	8	9	10	11	0

Table 11:The Matrix H: all chromatic scales in harmonic ordering.

As the cycles of fourths and fifths move in opposing directions and their shift values add to an octave (i. e., $5 + 7 = 12$), they are inversionally equivalent. The addition of enharmonics can be seen in Table 7.. Substituting the enharmonic names for each element n in H yields Table 7.. Recall from Section 2. that any major scale quality can be formed using the set $\{0; 2; 4; 5; 7; 9; 11\}$ added to any tonic note. In both Tables 7. and 7., the columns marked zero signify the tonics of each key (row). Hence, in Table 7., we have shaded columns that pertain to every major scale quality. As can be seen, each shaded column is simply a corresponding value of the scale quality set. Also notice that for each change of row, either one sharp or flat has been added or taken from the previous key signature. Hence, Table 7. demonstrates how each key is ordered using either cycle of fifths or fourths orderings. In table 7., we have labelled notes as per the convention outlined previously (Section 2.).

This process is generalizable. To see which keys contain which notes, shade the appropriate columns that pertain to the scale quality in question. For instance, a harmonic minor scale quality is created using the melodic interval pattern $\{0; 2; 3; 5; 7; 8; 11\}$. Shading these columns over Table 7. – or Table 7., using the appropriate enharmonic naming convention (Section 2.) – will yield all notes contained within every harmonic minor scale quality. In this way, all ordered sets of notes can be found in all keys. Looking back at the

$H + S$		$SetS_{asc.}$												
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**0**
	0	C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B	C
	5	F	F#/Gb	G	G#/Ab	A	Bb	B	C	C#/Db	D	D#/Eb	E	F
	10	Bb	B	C	C#/Db	D	Eb	E	F	F#/Gb	G	G#/Ab	A	Bb
	3	Eb	E	F	F#/Gb	G	Ab	A	Bb	B	C	C#/Db	D	Eb
	8	Ab	A	Bb	B	C	Db	D	Eb	E	F	F#/Gb	G	Ab
	1	C#/Db	D	D#/Eb	E	E#/F	F#/Gb	G	G#/Ab	A	A#/Bb	B	B#/C	C#/Db
$SetH$	**6**	F#/Gb	G	G#/Ab	A	A#/Bb	B/Cb	C	C#/Db	D	D#/Eb	E	E#/F	F#/Gb
	11	B/Cb	C	C#/Db	D	D#/Eb	E/Fb	F	F#/Gb	G	G#/Ab	A	A#/Bb	B/Cb
	4	E	F	F#	G	G#	A	A#/Bb	B	C	C#	D	D#	E
	9	A	A#/Bb	B	C	C#	D	D#/Eb	E	F	F#	G	G#	A
	2	D	D#/Eb	E	F	F#	G	G#/Ab	A	A#/Bb	B	C	C#	D
	7	G	G#/Ab	A	A#/Bb	B	C	C#/Db	D	D#/Eb	E	F	F#	G
	0	C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B	C

Table 12: All letter names of all major scale qualities (shaded).

ordered sets contained within Table 4., such indexed formal orderings tell which columns to shade in matrix H when its first "1" means "$n = 0$ is in this set." Where the first "1" to appear from the left in binary is not $n = 0$, the pattern maps to the row with that corresponding tonic. As each row can accommodate any binary lookup in this way, all 4096 formal orderings mentioned in Section 2..4. can be mapped to matrix H.

In sum, this paper has outlined the underpinning structure and intersecting concepts surrounding equal tempered music. For all that we have unpacked in this paper, there is still much remaining in order to proceed from this juncture. This entire process can be computerised. We have been working on such a project and have come to some interesting insights. Firstly, the approach we have outlined in this paper enables the visualisation of harmonic structures from a common point (the matrix H). In taking snapshots of musical structures at any given pulse of time, animations can be made that follow musical movements that occur over any durations of time in any piece. These structures can also be analysed and compared with one another. Moreover, linking this approach with the internals of a MIDI reader could efficiently and accurately translate scores into an analyzable structure. It follows that from the build up of such a database of mappings with a common underlying structure; a repository of music can be created which enables the quantitative measurement of any equal tempered music. Lastly, holographic representations of musical structure may be possible with the development of techniques in optics. Through such developments, the structure of music can be seen true to its mathematical underpinnings.

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