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The Graphs Whose Sum of Global Connected Domination Number and Chromatic Number is 2n-5

G Mahadevan,* A Selvam Avadayappan† and Twinkle Johns‡

Abstract

A subset S of vertices in a graph G = (V,E) is a dominating set if every vertex in V-S is adjacent to atleast one vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced sub graph < S > is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. In this paper we characterize the classes of graphs for which $\gamma_{gc}(G) + \chi(G) = 2n-5$ and 2n-6 of global connected domination number and chromatic number and characterize the corresponding extremal graphs.

Keywords: Global connected domination number, chromatic number AMS subject Classification: 05C (primary)

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* Department of Mathematics, Anna University, Tirunelveli, gmaha2003@yahoo.co.in

† Department of Mathematics, VHNSN College, Virdhunagar, selvamavadayappan@yahoo.co.in

‡ Department of Mathematics, VPMM Engineering College for Women, Krishnankoil, twinkle.johns@gmail.com

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1. Introduction

Graphs discussed in this paper are simple, finite and undirected graphs. A subset S of vertices in a graph G = (V,E) is a dominating set if every vertex in V-S is adjacent to atleast one vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced sub graph < S > is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. Note that any global connected dominating set of a graph G has to be connected in G (but not necessarily in $\overline{G}$). Here global connected domination number γ_{gc} is well defined for any connected graph. For a cycle C_n of order n ≥ 6, $\gamma_g(C_n) = \lceil n/3 \rceil$ while $\gamma_{gc}(C_n)$ = n-2 for n ≥ 4 and $\gamma_g(Kn)$ = 1,while $\gamma_{gc}(K_n)$ = n. The chromatic number $\chi(G)$ is defined as the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color.

Notations

$K_n(P_k)$ is the graph obtained from K_n by attaching the end vertex of P_k to any one vertices of K_n. $K_n(mP_k)$ is the graph obtained from K_n by attaching the end vertices of m copies of P_k to any one vertices of K_n. The graph $(m_1, m_2, \ldots, m_n)$ denote the graph obtained from K_n by pasting m_1 edges to any one vertex u_i of K_n m_2 edges to any vertex u_j of K_n for i ≠ j m_3 edges to any vertex u_k of K_n for i ≠ j ≠ k ≠1, ………… m_n edges to all the distinct vertices of K_n. $C_n(P_k)$ is the graph obtained from C_n by attaching the end vertex of P_k to any one vertices of C_n. $S^*(K_{1,n})$ is a graph obtained from $K_{1,n}$ by subdividing n-1 edges.

2. Preliminary Results

Theorem 2.1 [1] Let G be a graph of order n ≥ 2. Then, (i) 2 ≤ $\gamma_{gc}(G)$ ≤ n (ii) $\gamma_{gc}(G)$ = n if and only if $G \cong K_n$.

Corollary 2.2 [1] For all positive integers p and q, $\gamma_{gc}(K_{p,q}) = 2$.

Theorem 2.3 [1] For any graph G of order $n \geq 3$, $\gamma_{gc}(G) = n-1$ if and only if $G \cong K_n - e$, where e is an edge of K_n.

Theorem 2.4 For any connected graph G of order $n \geq 1$, $\gamma_{gc}(G) + \chi(G) < 2n-1$.

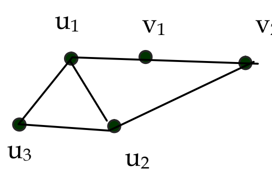
Theorem 2.5 For any connected graph G of order $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-2$ if and only if $G \cong K_n - e$, where e is any edge of K_n.

Theorem 2.6 For any connected graph G of order $n > 2$, $\gamma_{gc}(G) + \chi(G) = 2n-3$ if and only if $G \cong K_3(P_2)$, $K_n - \{e_1, e_2\}$,where e is edge in outside the cycle of graph, $n \geq 5$.

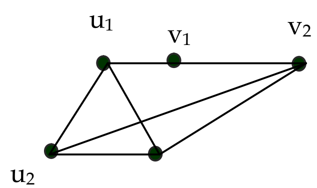
Theorem 2.7 For any connected graph G of order $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-4$ if and only if $G \cong P_4$, C_4, $K_2(2P_2)$, $K_4(P_2)$, $K_n - \{e_1, e_2, e_3\}$ for $n \geq 6$ and e is an edge in outside the cycle of graph.

3. Main Result

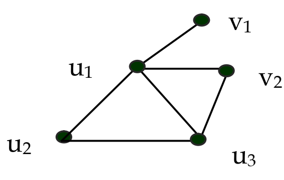
Theorem 3.1 For any connected graph G for $n \geq 3$, $\gamma_{gc}(G) + \chi(G) = 2n-5$ if and only if $G \cong G_1, G_2, G_3, G_4, G_5, G_6, P_5, K_3(P_3)$, $K_3(2P_2)$, $K_4(P_2,P_2,0)$, $K_5(P_2)$ and $K_n - \{e_1, e_2, e_3, e_4\}$, where e_1, e_2, e_3, e_4 are consecutive edges in outside the cycle of K_n of order $n \geq 7$.



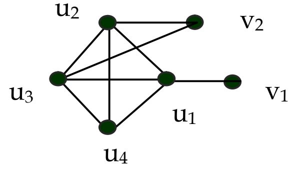
G_1



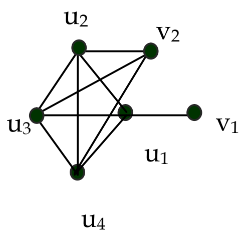
G_2



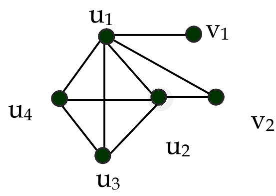
G_3



G_4



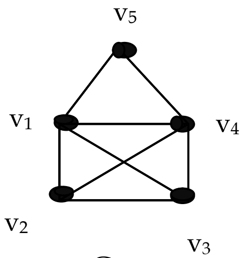
G5



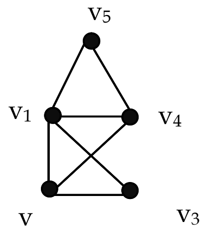
G6

Proof: Assume that $\gamma_{gc}(G) + \chi(G) = 2n-5$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n-5$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n-4$ (or) $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n-3$ (or) $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n-2$ (or) $\gamma_{gc}(G) = n - 4$ and $\chi(G) = n-1$ (or) $\gamma_{gc}(G) = n - 5$ and $\chi(G) = n$.

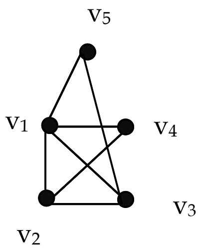
Case 1: Let $\gamma_{gc}(G) = n$ and $\chi(G) = n-5$. Since $\chi(G) = n-5$, G contains a clique K on n-5 vertices or does not contain a clique K on n-5 vertices. Let G contains a clique K on n-5 vertices. Let $S = \{v_1, v_2, v_3, v_4, v_5\} \in V-K$. Then the induced sub graph < S > has the following possible cases. < S > = K_5, $\overline{K_5}$, $C_3(C_3)$, $K_4 \cup K_1$, $K_3(P_3)$, C_5, $C_3(P_2, P_2, 0)$, $C_3(C_3)$, $C_4(P_2)$, P_5, $K_3 \cup K_2$, $K_{1,4}$, $C_4 \cup K_1$, $K_3(P_2) \cup K_1$, $P_3 \cup K_2$, $K_1 \cup K_2 \cup \overline{K_2}$, $K_{1,3} \cup K_1$, $K_3 \cup \overline{K_2}$, $K_2 \cup \overline{K_3}$, $K_2 \cup K_2 \cup K_1$, $P_3 \cup \overline{K_2}$, $C_3(2P_2)$, $K_4(P_2)$, $K_4 - e \cup K_1$, Petersen graph, $K_4 - e(P_2)$, $K_5 - e$, $K_5 - 2e$, where e is any edge on the cycle of K_5.



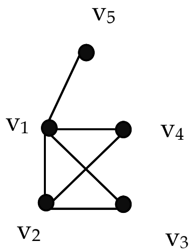
G1



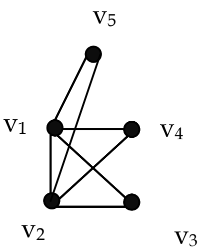
G2



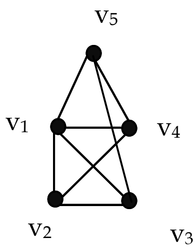
G_3



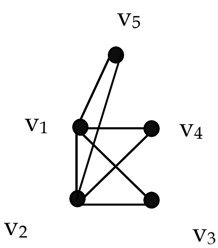
G_4



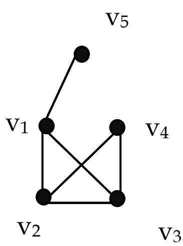
G_5



G_6



G_7



G_8

It can be verified that for all the above cases no graph exist.

If G does not contain the clique K on n-5 vertices, then it can be verified that no new graph exists.

Case 2: Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-4$. Since $\gamma_{gc}(G) = n-1$, then by theorem 2.3, $G \cong K_n-e$. But for K_n-e, $\chi(G) = n-1$, which is a contradiction.

Case 3: Let $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-3$. Since $\chi(G) = n-3$, G contains a clique K on n-3 vertices or does not contain a clique K on n-3 vertices. Let G contains a clique K on n-3 vertices. Let

$S = \{v_1, v_2, v_3\} \in V\text{-}K$. Then the induced sub graph < S > has the following possible cases: < S > = K_3, $\overline{K}_3$, P_3, $K_2 \cup K_1$.

Subcase (i): Let < S > = K_3. Since G is connected, there exist a vertex u_i of K_{n-3} adjacent to anyone of $\{v_1, v_2, v_3\}$. Without loss of generality, let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ is a global connected dominating set, hence $\gamma_{gc}(G) = 2$ so that $K \cong K_1$ which is a contradiction.

Subcase (ii): Let < S > = $\overline{K}_3$. Since G is connected, let all the vertices of $\overline{K}_3$ be adjacent to vertex u_i. Then u_i and anyone of the vertices of $\overline{K}_3$ forms a global connected dominating set. Without loss of generality v_1 and u_i forms a global connected dominating set. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. If two vertices of $\overline{K}_3$ are adjacent to u_i and the third vertex adjacent to u_j for some $i \neq j$, Then $\{u_i , u_j\}$ forms a global connected dominating set. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. If all the three vertices of $\overline{K}_3$ are adjacent to three distinct vertices of K_{n-3} say $\{u_i, u_j, u_k\}$ for some $i \neq j \neq k$. Then $\{u_i, u_j, u_k\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$, then n=5, which is a contradiction.

Subcase (iii): Let < S > = P_3. Since G is connected there exist a vertex u_i of K_{n-3} which is adjacent to any one of the pendent vertices of P_3 say v_1 or v_3.Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, v_2, u_i\}$ forms global connected dominating set. Hence $\gamma_{gc}(G) = 3$, so that $K = K_2$ then $G \cong P_5$. On the increasing the degree of u_i, $\gamma_{gc}(G) = 2$, which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to v_2 then $\{v_2, u_i\}$ forms global connected dominating set. Hence $\gamma_{gc}(G) = 2$ which is a contradiction.

Subcase (iv): Let < S > = $K_2 \cup K_1$. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to anyone of $\{v_1, v_2\}$ and v_3. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$ so that $K = K_1$ which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to anyone of $\{v_1, v_2\}$ and u_j for some $i \neq j$ in K_{n-3} adjacent to v_3. Without loss of generality let u_i be adjacent to v_1. Then $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ so that $K \cong K_2$, then $G \cong P_5$. On increasing the degree of u_i, $K \cong K_3$, which is a contradiction.

If G does not contain the clique K on n-3 vertices, then it can be verified that no new graph exists.

Case 4: Let $\gamma_{gc}(G)$ = n-3 and $\chi(G)$ = n-2. Since $\chi(G)$ = n-2, G contains a clique K on n-2 vertices or does not contain a clique K on n-2 vertices. Let S = $\{v_1, v_2\} \in$ V-K. Then the induced subgraph < S > has the following possible cases <S> = K_2 and $\overline{K_2}$.

Subcase (i): Let <S> = K_2. Since G is connected, there exist a vertex u_i of K_{n-2} which is adjacent to anyone of $\{v_1, v_2\}$. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ forms a global connected dominating set in G so that $\gamma_{gc}(G) = 2$ hence $K \cong K_3$. Then $G \cong K_3(P_3)$. On increasing the degree, $G \cong G_1, G_2$.

Subcase (ii): Let < S > = $\overline{K_2}$. Since G is connected. Let both the vertices of $\overline{K_2}$ be adjacent to vertex u_i for some i in K_{n-2}. Then anyone of the vertices of $\overline{K_2}$ and u_i forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$ so that $K \cong K_3$.Then $G \cong K_3$ $(2P_2)$.On increasing the degree of u_i $G \cong G_3$. If both the vertices of $\overline{K_2}$ are adjacent to two distinct vertices of K_{n-2} say u_i and u_j for $i \neq j$ in K_{n-2}. $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G)=3$. Then $K \cong K_4$ hence $G \cong K_4(P_2,P_2,0,0)$.On increasing the degree, $G \cong G_4, G_5, G_6$.

If G does not contain the clique K on n-2 vertices, then it can be verified that no new graph exist.

Case 5: Let $\gamma_{gc}(G)$ = n-4 and $\chi(G)$ = n-1. Since $\chi(G)$ = n-1, G contains a clique K on n-1 vertices or does not contain a clique K on n-1 vertices. Let v be the vertex not in K_{n-1}. Since G is connected the vertex v is adjacent to vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. Then $\gamma_{gc}(G) = 2$, so that $K \cong K_5$ hence $G \cong K_5$ (P_2). On increasing the degree, $G \cong K_n$-$\{e_1, e_2, e_3, e_4\}$ where e_1, e_2, e_3, e_4 are consecutive edges in outside the cycle of K_n of order $n \geq 7$.

If G does not contain the clique K on n-1 vertices, then it can be verified that no new graph exists.

Case 6 Let $\gamma_{gc}(G)$ = n-5 and $\chi(G)$ = n. Since $\chi(G)$ = n, $G \cong K_n$. But for K_n, $\gamma_{gc}(G)$ = n, which is a contradiction.

Conversely if G is anyone of the graph G_1,G_2, G_3, G_4, G_5, G_6 P_5, $K_3(P_3)$ $K_3(2P_2)$, $K_4(P_2,P_2,0)$, $K_5(P_2)$ and K_n-{e_1, e_2, e_3, e_4} where e_1, e_2, e_3, e_4 are edges in outside the cycle of K_n of order n ≥ 7, then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n-5$ for which $\gamma_{gc}(G) + \chi(G) = 2n-7$, 2n-8, which will be reported later.

References

[1] D Delic and C Wang, The global connected domination in graphs (print).

[2] F Harary, *Graph Theory*, Addison Wesley, Reading Mass, 1972.

[3] J Clark and D A Holton, *A First Look at Graph Theory*, Allied Publishers Ltd., 1995

[4] J P Joseph and S Arumugam, Domination and connectivity in graphs, *International Journal of Management and Systems,* vol. 8, pp. 233-236, 1992.

[5] J P Joseph and S Arumugam, Domination and colouring in graphs, *International Journal of Management and Systems*, vol. 8, pp. 37-44, 1997.

[6] J P Joseph and G Mahadevan, *Complementary Connected Domination Number Andchromatic Number of a Graph: Mathematical and Computational Models*, Allied Publications, India, pp. 342-349, 200

[7] J P Joseph, G Mahadevan and A Selvam, On complementary perfect domination number of a graph, *Acta Ciencia Indica,* Vol XXXI M, pp. 847, 2006.

[8] G Mahadevan, A Selvam and M Hajmeeral, On efficient domination number and chromatic number of a graph I, *International Journal of Physical Sciences*, vol. 21 M, pp. 1-8, 2009.

[9] G Mahadevan, "On domination theory and related topics in graphs", Ph.D. Thesis, Manomaniam Sundaranar University, Tirunelveli, 2005.

[10] E Sampathkumar, Global domination number of a graph, *Journal Math. Phy. Sci.*, vol. 23, pp. 377-385, 1939.