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Sum Labeling for Some Star and Cycle Related Special Graphs

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Abstract

A *sum labeling* is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, v ϵ V (G) with labels λ(u) and λ(v), respectively, (uv) is an edge iff λ(u) + λ(v) is the label of another vertex in V (G). Any graph supporting such a labeling is called a *sum graph*. It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *isolates* and the labeling scheme that requires the fewest isolates is termed *optimal*. The number of isolates required for a graph to support a sum labeling is known as the *sum number* of the graph. In this paper, we will obtain *optimal sum labeling scheme* for some star and cycle related special graphs.

Keywords: Sum labeling, sum number, sum graph, path union, arbitrary supersubdivision

AMS Subject Classification (2010): 05C78

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1. Introduction

All the graphs considered here are simple, finite and undirected. For all terminologies and notations we follow Harary [1] and graph labeling as in [2]. Sum labeling of graphs was introduced by Harary [3] in 1990. Following definitions are useful for the present study.

Definition 1.1 A *Sum Labeling* is a mapping λ from the vertices of G into the positive integers such that, for any two vertices u, v ϵ V (G) with labels λ(u) and λ(v), respectively, (uv) is an edge iff λ(u) + λ(v) is the label of another vertex in V (G). Any graph supporting such a labeling is called a *Sum Graph*.

Definition 1.2 It is necessary to add (as a disjoint union) a component to sum label a graph. This disconnected component is a set of isolated vertices known as *Isolates* and the labeling scheme that requires the fewest isolates is termed *Optimal*.

Definition 1.3 The number of isolates required for a graph G to support a sum labeling is known as the *Sum Number* of the graph. It is denoted as $\sigma(G)$.

Definition 1.4 Let G be a graph with q edges. A graph H is called a *Super subdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i then super subdivision is called arbitrary super subdivision of G.

Definition 1.5 A *caterpillar* is a graph which has the property that if we remove all the vertices of degree 1 then what remains is a path. Such a path is called the *spine* of the caterpillar. The two end points of a spine are the *head* and *tail* respectively. Other vertices of the spine are called the *internal vertices.* The vertices of degree 1 of a caterpillar, other than head and tail, will be called the *feet*. These vertices are attached to the internal vertices of spine with edge called the *legs* of the caterpillar.

Definition 1.6 A shrub is a tree which has at most one inner vertex. This special vertex is called the root of the shrub. All neighbours of this vertex are leaves or near leaves.

Definition 1.7 (Chung et. al [4]) A tree is called a *spider* if it has a center vertex c of degree k > 1 and each other vertex either is a leaf (pendent vertex) or has degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x_1 paths of length a_1, x_2 paths of length a_2, ..., x_n paths of length a_m, we denote the spider by $SP(a_1^{x_1}, a_2^{x_2}, \ldots, a_m^{x_n})$ where $a_1 < a_2 < \ldots < a_m$ and $x_1 + x_2 + \ldots + x_n = k$

Definition 1.8 (Shee and Ho. et al [5]) Let $G_1, G_2, \ldots, G_n$, $n \geq 2$ be *n* copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1} for i =1, 2,..., n-1 is called *path union* of G.

2. Optimal Sum Labeling Scheme for Arbitrary Super Subdivision of Star Related Graphs

Sethuraman et. al. [6], introduced a new method of construction called Supersubdivision of graph and proved that arbitrary supersubdivision of any path and cycle C_n are graceful. Kathiresan et.al [7], proved that arbitrary supersubdivision of any star is graceful. In [8], Gerard Rozario et.al proved that arbitrary super subdivision of path, cycle and star are sum graph with sum number 2.

In this section, we prove that graphs obtained by arbitrary super subdivision of comb, caterpillar, shrubs and path union of spider are sum graph with sum number 2.

Theorem 2.1 Arbitrary supersubdivision of comb $P_n \odot K_1$ is sum graph with sum number 2.

Proof: Let G be a comb $P_n \odot K_1$. Let v_i $(1 \leq i \leq n)$ be the vertices of path and w_i be the pendent vertex adjacent to v_i $(1 \leq i \leq n)$. Let H be the arbitrary supersubdivision of G which is obtained by replacing every edge of G with K_{2,m_i}. Let $m = \sum_1^{n-1} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision of G where $1 \leq j \leq m$. Let x and y be two isolated vertices. Therefore, the vertex set of H is V(H) = { $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots w_n, u_1, u_2, \ldots, u_m$}.

Define $f : V(H) \rightarrow N$

$$f(v_1) = 1 \;; \quad f(v_i) = f\left(v_{(i-1)}\right) + 2 \quad ; 2 \leq i \leq n$$

$f(w_1) = 2$; $f(w_i) = f(w_{(i-1)}) + 2$; $2 \leq i \leq n$

$f(u_1) = m + n$ $f(u_j) = f(u_{j-1}) - 1$; $2 \leq j \leq m$

Then $f(x) = f(u_1) + 1$ *and* $f(y) = f(u_1) + 2$

Thus, arbitrary supersubdivision of comb $P_n \odot K_1$ is sum graph with sum number 2.

Theorem 2.2 Arbitrary supersubdivision of caterpillar is sum graph with sum number 2.

Proof: Let G be a caterpillar with n vertices. Let v_i ($1 \leq i \leq n$) be the vertices of G. First we need to identify the vertices of G in order to super subdivide G. We present an algorithm to identify the vertices of caterpillar. The algorithm starts from the head of the caterpillar (1st vertex of the spine). Whenever the algorithm finds a new internal vertex of the spine having degree ≥ 3, it first visits all vertices of degree 1 adjacent to it before proceeding to the next vertex of spine. The head of the caterpillar is named as v_1 and every new vertex identified by the algorithm is named as $v_2, v_3, \ldots\ldots, v_n$ respectively.

Let H be the arbitrary supersubdivision of caterpillar G which is obtained by replacing every edge of G with K_{2,m_i}. Let $m = \sum_1^{n-1} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision of G where $1 \leq j \leq m$. Let x and y be two isolated vertices. Therefore, the vertex set of H is $V(H) = \{ v_1, v_2, \ldots\ldots, v_n, u_1, u_2, \ldots\ldots, u_m\}$.

Define $f : V(H) \to N$

$f(v_i) = i$; $1 \leq i \leq n$

$f(u_1) = m + n$ $f(u_j) = f(u_{j-1}) - 1$; $2 \leq j \leq m$

Then $f(x) = f(u_1) + 1$ *and* $f(y) = f(u_1) + 2$

Hence, arbitrary supersubdivision of caterpillar is sum graph with sum number 2.

Illustration: Sum labeling for arbitrary supersubdivision of caterpillar is given in figure 2.1

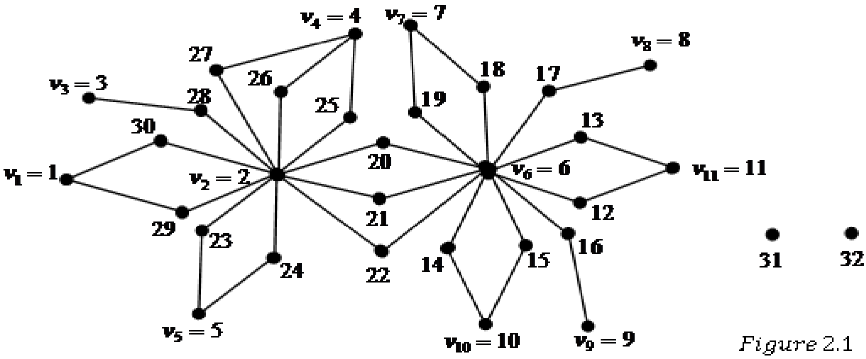

Figure 2.1

Theorem 2.3 Arbitrary supersubdivision of shrub is optimal summable with sum number 2.

Proof: Let G be a shrub with root r and $b_1, b_2, \ldots\ldots, b_k$ be the vertices adjacent to r. Let $d_i = deg\,(b_i) - 1\,; i = 1, 2, \ldots, k$. Let c_{ij} be the vertices adjacent to b_i other than r for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, d_i$. In order to identify the vertices of shrub we follow the following algorithm. The algorithm starts from the root of the shrub and then it identifies an adjacent vertex (say b_1) of r. Before it proceeds to the next adjacent vertex of r, the algorithm first visits all neighbour vertices c_{ij} of b_1 with degree 1 (if any) and proceeds to branches of c_{ij} of higher order (if any). The algorithm renames the root r as v_1 and assigns $v_2, v_3, \ldots, v_n$ respectively to the new vertex it visits first. Now, the vertices of shrubs are renamed as $v_1, v_2, v_3, \ldots, v_n$ where n is the total number of vertices of the shrub.

Let H be the arbitrary supersubdivision of shrub G which is obtained by replacing every edge of G with K_{2,m_i}. Let $m = \sum_1^{n-1} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision of G where $1 \leq j \leq m$. Let x and y be two isolated vertices. Therefore, the vertex set of H is $V(H) = \{ v_1, v_2, \ldots\ldots, v_n, u_1, u_2, \ldots\ldots, u_m\}$.

Define f : V(H) → N

$$f(v_i) = i \quad ; 1 \leq i \leq n;$$

$$f(u_1) = m + n \ ; \quad f(u_j) = f(u_{j-1}) - 1 \ ; 2 \leq j \leq m.$$

Then $f(x) = f(u_1) + 1$ *and* $f(y) = f(u_1) + 2$

Hence, arbitrary supersubdivision of shrub is optimal summable with sum number 2.

Illustration: Sum labeling for arbitrary supersubdivision of shrub is given in figure 2.2

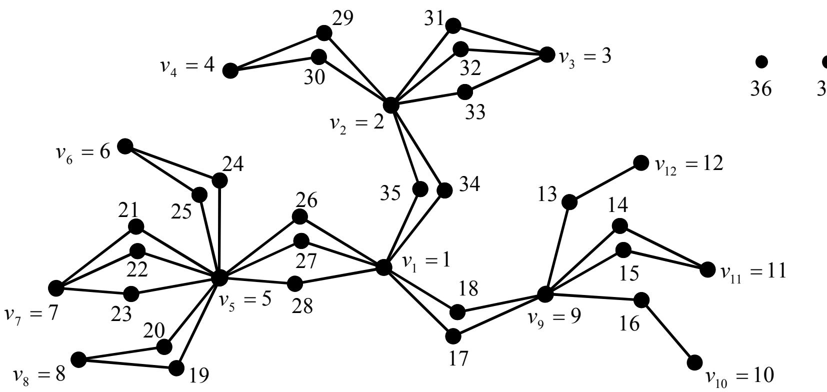

Figure 2.2

Theorem 2.4 Arbitrary supersubdivision of path union of spider is sum graph with sum number 2.

Proof: Let G be the path union of spiders obtained from n spiders (not necessarily unique) by adding an edge between centre vertex of i^{th} spider and $(i+1)^{th}$ spider for i =1, 2,..., (n-1). Let $c_1, c_2, \ldots, c_n$ be the center vertices of the spider. Let $k_1, k_2, \ldots, k_n$ be the number of vertices in i^{th} spider excluding the center vertex where $1 \leq i \leq n$. Let $p_{11}, p_{12}, \ldots, p_{1k_1}, \ p_{21}, p_{22}, \ldots, p_{2k_2}, \ldots\ldots, \ p_{n1}, p_{n2}, \ldots, p_{nk_n}$ be the vertices of the i^{th} spider respectively. Therefore, V(G) = { $c_1, c_2, \ldots, c_n, p_{11}, p_{12}, \ldots, p_{1k_1}, p_{21}, p_{22}, \ldots, p_{2k_2}, \ldots\ldots, p_{n1}, p_{n2}, \ldots, p_{nk_n}$}. Let t be the total number of vertices of G. i.e., $t = n + k_1 + k_2 + \cdots + k_n$. Let G* be the arbitrary supersubdivision of G which is obtained by replacing every edge of G with K_{2,m_i}. Let $m = \sum_1^{n-1} m_i$. Let u_j be the vertices which are used for arbitrary supersubdivision of G where $1 \le j \le m$. Let x and y be two isolated vertices. Therefore, the vertex set of G* is

$$V(G^*)=\{c_1, c_2, \dots, c_n, p_{11}, p_{12}, \dots, p_{1k_1}, \dots, p_{n1}, p_{n2}, \dots, p_{nk_n}, u_1, u_2, \dots, u_m\}.$$

Define $f : V(G^*) \to N$

$$f(c_1) = 1 \quad ; f(c_i) = f(c_{i-1}) + k_{(i-1)} + 1 \;\; for\ 2 \le i \le n$$

$$for\ 1 \le i \le n$$

$$\begin{cases} f(p_{i1}) = f(p_1) + 1 \\ for\ 2 \le j \le k_i \\ \{ f(p_{ij}) = f(p_{i(j-1)}) + 1 \end{cases}$$

$$f(u_1) = m + t \qquad f(u_j) = f(u_{j-1}) - 1 \;\; ; 2 \le j \le m$$

Then $f(x) = f(u_1) + 1$ and $f(y) = f(u_1) + 2$

Hence, arbitrary supersubdivision of path union of spiders is optimal summable with sum number 2.

Illustration: Sum labeling for arbitrary supersubdivision of path union of spiders is given in figure 2.3

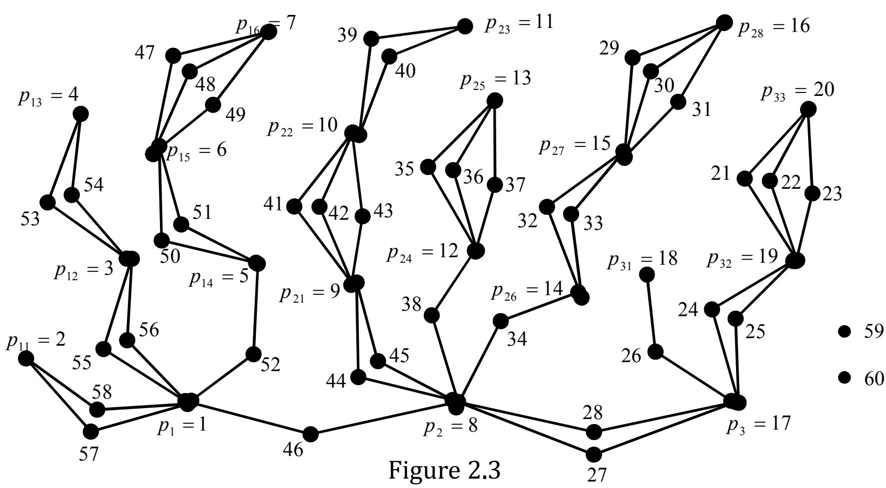

Figure 2.3

3. Optimal Sum Labeling Scheme for Cycle Related Graphs

In [9], Ponraj et.al proved that the graph G with $V(G) = V(C_n) \cup \{v\}$ and $E(G) = E(C_n) \cup \{c_1v, c_3v\}$ and the graph [C_n, P_m] are pair sum graph. In this section we prove that the above said two graphs are sum graphs.

Theorem 3.1 Let G be a graph with $V(G) = V(C_n) \cup \{v\}$ and $E(G) = E(C_n) \cup \{c_1v, c_3v\}$. Then G is a sum graph with sum number 2.

Proof: Let $c_1, c_2, c_3, \dots\dots\dots, c_n$ be the vertices of the cycle. Let x and y be two isolated vertices. The vertex set of G is given by V(G) = { $c_1, c_2, c_3, \dots\dots\dots, c_n, v, x, y$}

Define f : V(G) → N

Case (i) : ***n is odd number***

Case (i) (a): n = 3

For C_3, $f(c_1) = 1;\ f(c_2) = 2;\ f(c_3) = 3$

$$f(v) = f(c_1) + f(c_3)$$

$$f(x) = f(c_1) + f(v)\ \ and\ f(y) = f(c_3) + f(v)$$

Thus, G is sum graph with sum number 2 when n = 3.

Case (i) (b): n = 5, 7

For C_5, C_7 $f(c_1) = 2;\ f(c_2) = 1;\ f(c_3) = 3$

$$f(c_n) = f(c_1) + f(c_3)$$

$$f(c_i) = f(c_{i-1}) + f(c_{i-2})\ \ ; 4 \le i \le (n-1)$$

$$f(v) = \begin{Bmatrix} f(c_n) + f(c_{n-1}) & if\ \ f(c_n) < f(c_{n-1}) \\ f(c_{n-1}) + f(c_{n-2}) & if\ \ f(c_n) > f(c_{n-1}) \end{Bmatrix}$$

$$f(x) = f(c_1) + f(v)\ \ and\ f(y) = f(c_3) + f(v)$$

Thus, G is sum graph with sum number 2 when n = 5, 7.

Case (i) (c): n ≥ 9

For C_n, $f\left(c_{\left(\frac{n-3}{2}\right)}\right) = 1;\quad f\left(c_{\left(\frac{n-3}{2}-1\right)}\right) = 2;\quad f\left(c_{\left(\frac{n-3}{2}+1\right)}\right) = 3;$

$$f\left(c_{\left(\frac{n-3}{2}-2\right)}\right) = 5; f\left(c_{\left(\frac{n-3}{2}+2\right)}\right) = 4$$

$$for\ 1 \leq i \leq \frac{n-9}{2}$$

$$\left\{\begin{array}{c} f\left(c_{\left(\frac{n-3}{2}+i+2\right)}\right) = f\left(c_{\left(\frac{n-3}{2}-i\right)}\right) + f\left(c_{\left(\frac{n-3}{2}-i-1\right)}\right) \\ f\left(c_{\left(\frac{n-3}{2}-i-2\right)}\right) = f\left(c_{\left(\frac{n-3}{2}+i+1\right)}\right) + f\left(c_{\left(\frac{n-3}{2}+i+2\right)}\right) \end{array}\right\}$$

$$f(c_{n-3}) = f(c_1) + f(c_2);\ f(c_n) = f(c_{n-3}) + f(c_{n-4})$$

$$f(c_{n-2}) = f(c_1) + f(c_n);\ f(c_{n-1}) = f(c_{n-3}) + f(c_{n-2})$$

$$f(v) = f(c_n) + f(c_{n-1})$$

$$f(x) = f(c_1) + f(v)\ and\ f(y) = f(c_3) + f(v)$$

Thus, G is sum graph with sum number 2 when $n \geq 9$

Case (ii): n is even number

Case (ii) (a): n = 4

For C_4, $f(c_1) = 1; f(c_2) = 2; f(c_4) = 3; f(c_3) = f(c_1) + f(c_4)$

$$f(v) = f(c_2) + f(c_3)$$

$$f(x) = f(c_1) + f(v)\ and\ f(y) = f(c_3) + f(v)$$

Thus, G is sum graph with sum number 2 when n = 4.

Case (ii) (b): $n \geq 6$

For C_n, $f(c_1) = 1; f(c_2) = 2; f(c_n) = 3; f(c_3) = f(c_2) + f(c_n)$

$$f(c_{n-1}) = f(c_1) + f(c_n)$$

$$for\ 4 \leq i \leq n/2$$

$$\left\{\begin{array}{l} \text{If i is even, } \left\{\begin{array}{c} f(c_i) = f(c_{i-1}) + f(c_{i-2}) - 1 \\ f\left(c_{i+(n-2i+2)}\right) = f\left(c_{i+(n-2i+2)+1}\right) + f\left(c_{i+(n-2i+2)+2}\right) \end{array}\right\} \\ \text{If i is odd, } \left\{\begin{array}{c} f(c_i) = f(c_{i-1}) + f(c_{i-2}) \\ f\left(c_{i+(n-2i+2)}\right) = f\left(c_{i+(n-2i+2)+1}\right) + f\left(c_{i+(n-2i+2)+2}\right) - 1 \end{array}\right\} \end{array}\right.$$

$$f\left(c_{\left(\frac{n}{2}+1\right)}\right) = f\left(c_{\left(\frac{n}{2}\right)}\right) + f\left(c_{\left(\frac{n}{2}-1\right)}\right)$$

$$f(v) = \begin{Bmatrix} f\left(c_{\left(\frac{n}{2}\right)}\right) + f\left(c_{\left(\frac{n}{2}+1\right)}\right) \; if\, f\left(c_{\left(\frac{n}{2}\right)}\right) < f\left(c_{\left(\frac{n}{2}+2\right)}\right) \\ f\left(c_{\left(\frac{n}{2}+1\right)}\right) + f\left(c_{\left(\frac{n}{2}+2\right)}\right) \; if\, f\left(c_{\left(\frac{n}{2}\right)}\right) > f\left(c_{\left(\frac{n}{2}+2\right)}\right) \end{Bmatrix}$$

$$f(x) = f(c_1) + f(v) \;\; and \; f(y) = f(c_3) + f(v)$$

Thus, G is sum graph with sum number 2 when $n \geq 6$

Hence, G is sum graph with sum number 2 for all $n \geq 3$.

Illustration: Sum labeling of G with n = 5 and n = 8 is given in figure 3.1

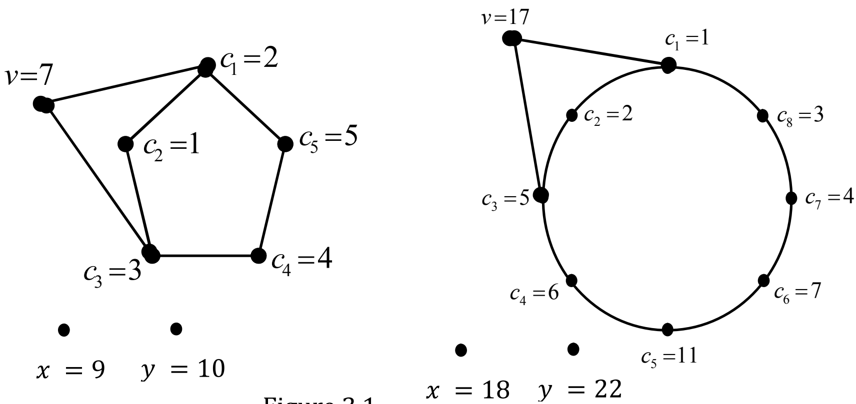

Figure 3.1

Notation: Two copies of the cycle C_n connected by the path P_m is denoted by $[C_n, P_m]$

Theorem 3.2 The graph $[C_n, P_m]$ is a sum graph with sum number 2.

Proof: Let $c_1, c_2, c_3, \ldots\ldots\ldots, c_n$ be the vertices of the first copy of cycle C_n and $v_1, v_2, v_3, \ldots\ldots\ldots, v_n$ be the vertices of the second copy of cycle C_p. Let $p_1, p_2, \ldots\ldots, p_m$ be the vertices of path connecting the two copies of the cycle. Let x and y be two isolated vertices.

Define f : V(G) → N

Case (i): *n is odd number*

Case (i) (a): n = 3

For C_3, $f(c_1) = 1;\ f(c_2) = 2;\ f(c_3) = 3$

$f(p_1) = f(c_1) + f(c_3)$; $f(p_2) = f(p_1) + f(c_1)$

$f(p_i) = f(p_{(i-1)}) + f(p_{(i-2)})\ for\ 3 \leq i \leq m$

$f(v_1) = f(p_m) + f(p_{(m-1)})$ $f(v_2) = f(v_1) + f(p_m)$

$f(v_i) = f(v_{(i-1)}) + f(v_{(i-2)})\ for\ 3 \leq i \leq n$

$f(x) = f(v_n) + f(v_1)\ and\ f(y) = f(v_n) + f(v_{(n-1)})$

Thus, G is sum graph with sum number 2 when n = 3.

Case (i) (b): n = 5, 7

For C_5, C_7 $f(c_1) = 2;\ f(c_2) = 1;\ f(c_3) = 3$

$f(c_n) = f(c_1) + f(c_3)$

$f(c_i) = f(c_{i-1}) + f(c_{i-2})\ ;4 \leq i \leq n-1$

$$f(p_1) = \begin{Bmatrix} f(c_n) + f(c_{n-1}) & if \quad f(c_n) < f(c_{n-1}) \\ f(c_{n-1}) + f(c_{n-2}) & if \quad f(c_n) > f(c_{n-1}) \end{Bmatrix}$$

$f(p_2) = f(p_1) + f(c_1)$

$f(p_i) = f(p_{(i-1)}) + f(p_{(i-2)})\ for\ 3 \leq i \leq m$

$f(v_1) = f(p_m) + f(p_{(m-1)})$
$f(v_2) = f(v_1) + f(p_m)$

$f(v_i) = f(v_{(i-1)}) + f(v_{(i-2)})\ for\ 3 \leq i \leq n$

$f(x) = f(v_n) + f(v_1)\ and\ f(y) = f(v_n) + f(v_{(n-1)})$

Thus, G is sum graph with sum number 2 when n = 5, 7.

Case (i) (c): $\boldsymbol{n \geq 9}$

For C_n, $f\left(c_{\left(\frac{n-3}{2}\right)}\right) = 1$; $f\left(c_{\left(\frac{n-3}{2}-1\right)}\right) = 2$; $f\left(c_{\left(\frac{n-3}{2}+1\right)}\right) = 3$;

$$f\left(c_{\left(\frac{n-3}{2}-2\right)}\right) = 5; f\left(c_{\left(\frac{n-3}{2}+2\right)}\right) = 4$$

$$for\ 1 \leq i \leq \frac{n-9}{2}$$

$$\left\{\begin{matrix} f\left(c_{\left(\frac{n-3}{2}+i+2\right)}\right) = f\left(c_{\left(\frac{n-3}{2}-i\right)}\right) + f\left(c_{\left(\frac{n-3}{2}-i-1\right)}\right) \\ f\left(c_{\left(\frac{n-3}{2}-i-2\right)}\right) = f\left(c_{\left(\frac{n-3}{2}+i+1\right)}\right) + f\left(c_{\left(\frac{n-3}{2}+i+2\right)}\right) \end{matrix}\right\}$$

$$f(c_{n-3}) = f(c_1) + f(c_2);\ f(c_n) = f(c_{n-3}) + f(c_{n-4})$$

$$f(c_{n-2}) = f(c_1) + f(c_n);\ f(c_{n-1}) = f(c_{n-3}) + f(c_{n-2})$$

$$f(p_1) = f(c_n) + f(c_{n-1})$$

$$f(p_2) = f(p_1) + f(c_1)$$

$$f(p_i) = f\left(p_{(i-1)}\right) + f\left(p_{(i-2)}\right)\ for\ 3 \leq i \leq m$$

$$f(v_1) = f(p_m) + f\left(p_{(m-1)}\right)\quad f(v_2) = f(v_1) + f(p_m)$$

$$f(v_i) = f\left(v_{(i-1)}\right) + f\left(v_{(i-2)}\right)\ for\ 3 \leq i \leq n$$

$$f(x) = f(v_n) + f(v_1)\ and\ f(y) = f(v_n) + f\left(v_{(n-1)}\right)$$

Thus, G is sum graph with sum number 2 when $n \geq 9$

Case (ii): n is even number

Case (ii) (a): n = 4

For C_4, $f(c_1) = 1$; $f(c_2) = 2$; $f(c_4) = 3$; $f(c_3) = f(c_1) + f(c_4)$

$f(p_1) = f(c_2) + f(c_3)$; $f(p_2) = f(p_1) + f(c_1)$

$$f(p_i) = f\left(p_{(i-1)}\right) + f\left(p_{(i-2)}\right)\ for\ 3 \leq i \leq m$$

$f(v_1) = f(p_m) + f\left(p_{(m-1)}\right)$; $f(v_2) = f(v_1) + f(p_m)$

$$f(v_i) = f\left(v_{(i-1)}\right) + f\left(v_{(i-2)}\right)\ for\ 3 \leq i \leq n$$

$$f(x) = f(v_n) + f(v_1)\ and\ f(y) = f(v_n) + f\left(v_{(n-1)}\right)$$

Thus, G is sum graph with sum number 2 when n = 4.

Case (ii) (b): $\boldsymbol{n} \geq \mathbf{6}$

For C_n,

$$f(c_1) = 1;\ f(c_2) = 2;\ f(c_n) = 3;\ f(c_3) = f(c_2) + f(c_n)$$

$$f(c_{n-1}) = f(c_1) + f(c_n)\ \ for\ 4 \leq i \leq {}^{n}/_{2}$$

$$\begin{cases} \text{If i is even, } \begin{cases} f(c_i) = f(c_{i-1}) + f(c_{i-2}) - 1 \\ f\left(c_{i+(n-2i+2)}\right) = f\left(c_{i+(n-2i+2)+1}\right) + f\left(c_{i+(n-2i+2)+2}\right) \end{cases} \\ \text{If i is odd, } \begin{cases} f(c_i) = f(c_{i-1}) + f(c_{i-2}) \\ f\left(c_{i+(n-2i+2)}\right) = f\left(c_{i+(n-2i+2)+1}\right) + f\left(c_{i+(n-2i+2)+2}\right) - 1 \end{cases} \end{cases}$$

$$f\left(c_{\left(\frac{n}{2}+1\right)}\right) = f\left(c_{\left(\frac{n}{2}\right)}\right) + f\left(c_{\left(\frac{n}{2}-1\right)}\right)$$

$$f(p_1) = \begin{cases} f\left(c_{\left(\frac{n}{2}\right)}\right) + f\left(c_{\left(\frac{n}{2}+1\right)}\right)\ if\ f\left(c_{\left(\frac{n}{2}\right)}\right) < f\left(c_{\left(\frac{n}{2}+2\right)}\right) \\ f\left(c_{\left(\frac{n}{2}+1\right)}\right) + f\left(c_{\left(\frac{n}{2}+2\right)}\right)\ if\ f\left(c_{\left(\frac{n}{2}\right)}\right) > f\left(c_{\left(\frac{n}{2}+2\right)}\right) \end{cases}$$

$$f(p_2) = f(p_1) + f(c_1)$$

$$f(p_i) = f\left(p_{(i-1)}\right) + f\left(p_{(i-2)}\right)\ for\ 3 \leq i \leq m$$

$$f(v_1) = f(p_m) + f\left(p_{(m-1)}\right)\quad f(v_2) = f(v_1) + f(p_m)$$

$$f(v_i) = f\left(v_{(i-1)}\right) + f\left(v_{(i-2)}\right)\ for\ 3 \leq i \leq n$$

$$f(x) = f(v_n) + f(v_1)\ \ and\ f(y) = f(v_n) + f(v_{(n-1)})$$

Thus, G is sum graph with sum number 2 when $n \geq 6$

Hence, G is sum graph with sum number 2 for all $n \geq 3$.

Illustration: Sum labeling for the graph [C_6, P_2] *is given in figure 3.2*

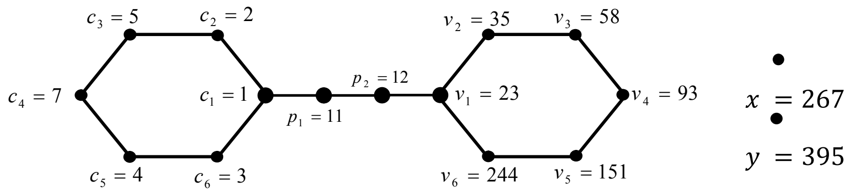

Figure 3.2

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