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ASSOCIATE RING GRAPHS

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ABSTRACT

R is a commutative ring with unity. The associate ring graph *AG*(*R*) is the graph with the vertex set $V = R - \{0\}$ and edge set $E = \{(a,b) / a, b$ are associates and $a \neq b\}$. Since the relation of being associate is an equivalence relation, this graph is an undirected graph and also each component is complete. In this paper, I present some of the interesting results majority of which are for the ring of integers modulo n, n is a positive integer.

1) *AG*(*R*) is an empty graph if *R* is a Boolean ring.
2) $AG(Z_n)$ is complete if and only if n is prime.
3) If n is even then $AG(Z_n)$ has an isolated vertex n/2.
4) If p is prime and $p \neq 2$, then $AG(Z_{2p}) = K_1 \cup K_{p-1} \cup K_{p-1}$.
5) $AG(Z_{p^2}) = K_{p-1} \cup K_{p(p-1)}$.
6) $AG(Z_{pq}) = K_{p-1} \cup K_{q-1} \cup K_{pq-p-q+1}$.
7) A C-program to find the components of $AG(Z_n)$.

1. Introduction

The motivation for associate ring graphs is from zero-divisor graphs defined by I.Beck in the year 1988. He introduced the idea of these graphs for commutative rings *R* with unity 1. He defined $\Gamma_0(R)$ to be the graph whose vertices are elements of *R* and in which two vertices x and y are adjacent if and only if $xy = 0$. Beck was

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mostly concerned with coloring $\Gamma_0(R)$. In his paper [1] he studied the subgraph $\Gamma(R)$ whose set of vertices is $Z(R)^* = Z(R) - \{0\}$ where $Z(R)$ is the set of zero-divisors of R. $\Gamma(R)$ is non empty unless R is an integral domain and, by a result of G.Ganesan, $Z(R)$ and hence (R) is finite if and only if R is finite. It is shown that $\Gamma(R)$ is connected with $diam(\Gamma(R))$ d≤3. Lot of results were subsequently developed (Some of them can be seen in [2] and [3]) by several authors for zero-divisor graphs. If R is a field then (R) is empty or (R) has no edges when all non-zero elements are used as vertices. Since a field is very rich with respect to algebraic structure, it is quite reasonable to associate a graph which is also rich graph theoretically. We know that complete graphs take this place. So I thought of defining a graph from a ring R so that it is complete when R is a field. This graph is nothing but the so called ASSOCIATE RING GRAPH.

2. Preliminaries

All the fundamental concepts of ALGEBRA are from [4] and of GRAPH THEORY are from [5].

3. Associate Ring Graphs

3.1 Associate ring graph: Let R be a ring with unity 1 (not necessarily commutative). The associate ring graph of R denoted by $AG(R)$ is the graph (V,E) where the vertex set $V = R-\{0\}$ and the edge set $E = \{(a,b) / a$ is an associate of b and $a \neq b\}$.

Note: Throughout this paper a ring always means a ring with unity 1.

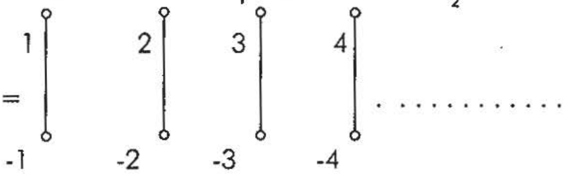
3.2 Orbit of an element of a ring: If a is an element of a ring R then the orbit of a denoted by $Or(a)$ is defined as $Or(a) = \{a.u \mid u \text{ is a unit in } R\}$.

3.3 Theorem: The orbits of elements of a ring are either identical or disjoint.
Proof: Let R be a ring and a , b are two elements of R.
If $Or(a)$ and $Or(b)$ are disjoint we have nothing to prove.
Suppose that $Or(a) \cap Or(b) \neq \emptyset$.
Let $c \in Or(a) \cap Or(b)$. Then $c = a.u$ and $c = b.v$ for some units u,v in R.
$\therefore a.u = b.v \Rightarrow a = b.(v.u^{-1})$ and $b = a.(u.v^{-1})$ and so a and b are associates.
Let x be an arbitrary element in $Or(a)$. Then $x = a.s$, s is a unit in R.
So $x = b.(v.u^{-1}).s$
i.e., $x = b.(v.u^{-1}.s)$

i.e., $x = b.$(a unit in R).
i.e., $x \in Or(b)$ and so $Or(a) \subseteq Or(b)$. Similarly we can show that $Or(b) \subseteq Or(a)$.
Thus $Or(a) = Or(b)$.
Hence the Orbits of any two elements of a ring are either disjoint or identical. ***

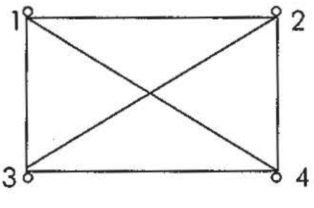
3.4 Observation: Since the relation of being associative is an equivalence relation it partitions R in to disjoint sets and it can be easily seen that the equivalence class containing an element a is nothing but $Or(a)$. Thus our graph contains connected components equal in number to the number of disjoint equivalence classes except $\{0\}$.

3.5 Example 1. Consider the ring $(Z, +, \cdot)$ of integers. We know that 1 and -1 are the only units of Z. Therefore for any $0 \neq a$ in Z, $Or(a) = \{ a, -a \}$. Hence $AG(Z)$ consists of infinite number of components each is a K_2.

AG(Z) =


Therefore $AG(Z) = K_2 \cup K_2 \cup K_2 \ K_2 \ldots\ldots\ldots\ldots$

Example 2. Consider $(Z_5, +_5, X_5)$. This is a field. Every non zero element is a unit and so any two non-zero elements are associates. Hence the graph is a complete graph with four vertices 1, 2, 3, 4.

The graph $AG(Z_5)$ is


Hence $AG(Z_5) = K_4$.

Example 3. Consider $(Z_6, +_6, \times_6)$. Here $Z_6 = \{ 0, 1, 2, 3, 4, 5 \}$.

The Units of Z_6 are 1, 5.

$Or(1) = \{1 \times_6 1, 1 \times_6 5\} = \{1, 5\}$.

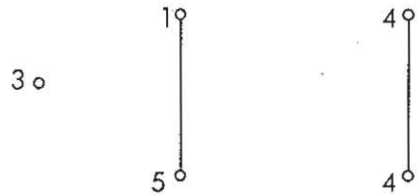
$Or(2) = \{2, 4\}$

$Or(3) = \{3\}$

$Or(4) = \{2, 4\}$

$Or(5) = \{1, 5\}$.

The graph $AG(Z_6)$ is



Hence $AG(Z_6) = K_1 \cup K_2 \cup K_2$.

3.6 Theorem: $AG(R)$ is an empty graph(without edges) if R is a Boolean ring.

Proof: Let R be a Boolean ring with unity 1.

We show that R has no units other than the unity 1.

Let $0 \neq a$ be a unit in R. i.e., $a.b = 1$ for some $0 \neq b$ in R. Since R is Boolean, $a^2 = a$.

Now $a.b = 1 \Rightarrow a.(a.b) = a.1 \Rightarrow a^2.b = a \Rightarrow a.b = a \Rightarrow 1 = a$.

Hence 1 is the only unit in R. Therefore the orbit of every non-zero element of R contains only itself.

Hence $AG(R)$ has no edges. ***

3.7 Theorem: $AG(Z_n) = K_{n-1}$ (the complete graph with n-1 vertices) if and only if n is prime.

Proof: Suppose that $AG(Z_n)$ is complete.

i.e., every pair of non zero elements of Z_n are connected by an edge.

We know that Z_n is a commutative ring with unity 1.

If a is any non-zero element of Z_n then a and 1 are joined by an edge. i.e., a and 1 are associates.

i.e., $1 = u \cdot a$ for some unit u in Z_n.

i.e., a is an invertible element in Z_n.

i.e., every non zero element in Z_n is invertible.

Thus Z_n is a field and hence n is prime.

Conversely suppose that n is prime.

Therefore Z_n is a field.

Let x and y be two non zero elements of Z_n.

Since Z_n is a field x and y are units.

So $x^{-1}.y$ is also a unit in Z_n.

We have $x.(x^{-1}.y) = y$.

$\Rightarrow x$ is an associate of y.

$\Rightarrow x$ and y are joined by an edge.

Thus every pair of non-zero elements of Z_n are joined by an edge.

Hence $AG(Z_n)$ is complete. ***

3.8 Theorem: If n is even then $AG(Z_n)$ has an isolated vertex namely $n/2$.

Proof: Suppose n is even.

i.e., $n = 2m$ for some m in $N = \{1,2,3,.....\}$.

We show that $m = n/2$ is an isolated vertex in $AG(Z_n)$.

We know that the units of Z_n are the non-zero elements of Z_n which are relatively prime to n. Since n is even these units must be odd.

Let $a = 2k+1$ be a unit in Z_n.

Then we have $m.a = m.(2k+1) = 2mk + m = nk + m = m$ (Since $nk = 0$ in Z_n).

Thus the only associate of m is m itself.

Since $AG(Z_n)$ has no self loops m is an isolated vertex of $AG(_n)$. ***

3.9 Theorem: If $n = 2p$ where p is a prime ($\neq 2$) then $AG(Z_n) = K_1 \cup K_{p-1} \cup K_{p-1}$.

Proof: Let $n = 2p$. By **3.8**, $AG(Z_n)$ has an isolated vertex $n/2 = p$. So $AG(Z_n)$ contains K_1. Also $AG(Z_n)$ has a component $K_{\varphi(n)} = K_{\varphi(2p)} = K_{\varphi(2)\varphi(p)} = K_{p-1}$.

It is enough to prove that the graph has only one component left and that is also a K_{p-1}.

We show that the remaining vertices other than p and the units in $K_{\varphi(n)} = K_{p-1}$ forms the vertices of the other K_{p-1}.

Clearly the number of vertices remaining are $[(n-1)-(p-1)-1] = p-1$.

We have m is a unit if and only if $(m, 2p) = 1$.

If and only if m is odd and not a multiple of p.

If and only if m is odd and $m \neq p$.

If and only if $m = 1,3,5,\ldots,(p-2),(p+2),\ldots,(2p-1)$.

Therefore the set of remaining elements is $D = \{2,4,\ldots(p-1),(p+1),\ldots(2p-2)\}$.

We show that the orbit of any general element $2k$ of D is D. The associates of $2k$ are $2k(1)$, $2k(3)$, ..., $2k(p-2)$, $2k(p+2)$, ..., $2k(2p-1)$. These products are all even and so are elements of D. We show that that these products are distinct.

Suppose that $2k(2m-1) = 2k(2s-1)$ where $m \neq s$ and $m > s$.

So $2p$ divides $2k(2m-1) - 2k(2s-1) = 4k(m-s)$.

So p divides $2k(m-s)$.

Since p does not divide 2 and k, we must have $p \mid (m-s)$.

Since $(m-s) < p$ we must have $m = s$, a contradiction.

Thus the orbit of $2k$ is D. Therefore every element of D is an associate to every other element of D. This shows that the elements in D form the required K_{p-1}.

Hence $AG(Z_{2p}) = K_1 \cup K_{p-1} \cup K_{p-1}$. ***

3.10 Theorem: $AG(Z_{p2}) = K_{p-1} \cup K_{p(p-1)}$.

Proof: Let p be a prime number.

We have $Z_{p2} = \{0, 1, 2, ..., (p^2-1)\}$.

For any $0 \neq a$ in Z_{p2}, $(a, p^2) = 1$ if and only if p does not divide a.

if and only if a is not a multiple of p.

Hence $Or(1)$ = units of $Z_{p2} = \{1, 2, ..., (p-1), (p+1), ...(2p-1), (2p+1), ... (p-1)p-1, (p-1)p+1, ...(p^2-1)\}$.

The remaining non-zero elements of Z_{p2} are $p, 2p, 3p, ...(p-1)p$.

Obviously the number of elements in $Or(1)$ = number of units = $(p^2-1) - (p-1) = p(p-1)$.

Thus $AG(_{p2})$ has $K_{p(p-1)}$ as a component.

To prove the theorem it is enough to show that the remaining $(p-1)$ non-units(zero-divisors) forms a K_{p-1}.

Let $D = \{p, 2p, ..., (p-1)p\}$.

We have $Or(p) = \{p.1, p.2, ..., p.(p-1), p.(p+1), ...\}$.

Clearly the first $(p-1)$ elements of $Or(p)$ are elements of D.

So D is a subset of $Or(p)$. $\longrightarrow$ (1)

Since p is a non-unit, all elements of $Or(p)$ are non-units.

So $Or(p) \cap Or(1) = \varnothing$

Therefore $Or(p)$ is a subset of $\{Or(1)\}^c = D$. $\longrightarrow$ (2)

From (1) and (2) we get $Or(p) = D$.

Thus the elements $p, 2p, 3p, ..., (p-1)p$ of $Or(p)$ forms the vertices of the required $K_{(p-1)}$.

Hence $AG(Z_{p2}) = K_{p-1} \cup K_{p(p-1)}$. ***

3.11 Theorem: $AG(Z_{pq}) = K_{(p-1)} \cup K_{(q-1)} \cup K_{pq-p-q+1}$.

Proof: Without loss of generality we assume that $p < q$. The cases when $p = 2$ and $p = q$ are already dealt in 3.9 and 3.10 respectively.

Now n is a unit in Z_{pq} if and only if $(n, pq) = 1$.

If and only if n is neither a multiple of p nor a multiple of q.

Also n is not a unit if and only if n is either a multiple of p or a multiple of q.

We have $Or(1) = \{1, 2, \ldots, (p-1), (p+1), \ldots(q-1), (q+1), \ldots\ldots, (pq-1)\}$.

Obviously $n[Or(1)] = j\,(pq) = j\,(p).j\,(p) = (p-1).(q-1) = pq - p - q + 1$.

Thus $K_{pq-p-q+1}$ is a component of $AG\,(Z_{pq})$.

Since p, q are distinct primes they are not associates.

For let $p = u.q$ where u is a unit in Z_{pq}.

i.e., $p - u.q$ is divisible by pq.

i.e., $p - u.q = k.pq$ where k is an integer.

i.e., $p = q(u + kp)$.

i.e., p is divisible by q, a contradiction.

Hence $Or(p) \cap Or(q) = \emptyset$.

Here $(p + q)$ is neither a multiple of p nor a multiple of q.

So $(p + q)$ is a unit in Z_{pq} and hence $p(p + q)$ is an associate of p.

But $p(p + q) = p^2 + pq = p^2$ (since $pq = 0$ in Z_{pq}).

Thus p^2 is an associate of p.

Similarly we can show that $p^3, p^4, \ldots$ are associates of p.

Thus $1.p$, $2.p$, ..., $p.p$, ..., $(q-1).p$ are distinct elements in $Or(p)$.

Therefore $n[Or(p)] \geq q-1$ and similarly $n[Or(q)] \geq p-1$.

We have $Or(1)\ Or(p)\ Or(q) \subseteq Z_{pq}^{*}$ ———————————— (1)

Also $n[Or(1)\ Or(p)\ Or(q)\,] = n[Or(1)] + n[Or(p)] + n[Or(q)]$ (the union is disjoint)

$$\geq (pq-p-q+1) + (p-1) + (q-1)$$
$$= pq-1$$
$$= n\,[Z_{pq}^{*}]$$

Therefore $n[Or(1) \cup Or(p) \cup Or(q)] \geq n[Z_{pq}^{*}]$ ——————(2)

From (1) and (2) we get $Or(1) \cup Or(p) \cup Or(q) = Z_{pq}$.

Now $Or(p)$ cannot contain more than $(q-1)$ elements otherwise $Or(q)$ contains less than $(p-1)$ elements which is not true. Thus $n[Or(p)] = (q-1)$ and so $n[Or(q)] = (p-1)$.

Hence Z_{pq} has only three distinct orbits namely $Or(1)$, $Or(p)$ and $Or(q)$ with elements $(pq-p-q+1)$, $(q-1)$ and $(p-1)$ respectively.

Hence $AG(Z_{pq}) = K_{(p-1)} \cup K_{(q-1)} \cup K_{pq-p-q+1}$. ***

3.12 C- program to find the components of $AG(Z_n)$: A c-programming is prepared to find the components of $AG(Z_n)$ for a given positive integer n.

Example:

Enter '*n*' value: 50

ORBIT 1: { 1, 3, 7, 9, 11, 13, 17, 19,
21, 23, 27, 29, 31, 33, 37, 39, 41, 43
47, 49 }.

No. of elements is : 20

ORBIT 2: { 2, 4, 6, 8, 12, 14, 16, 18,
22, 24, 26, 28, 32, 34, 36, 38, 42, 44,
46, 48 }.

No. of elements is : 20

ORBIT 5: { 5, 15, 35, 45 }.

No. of elements is : 4

ORBIT 10: { 10, 20, 30, 40 }.

No. of elements is : 4

ORBIT 25: { 25 }.

No. of elements is : 1

FINAL SET is: { 1, 4, 4, 20, 20}=49.

Thus $AG(Z_{50}) = K_1 \cup K_4 \cup K_4 \cup K_{20} \cup K_{20}$.

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