ISSN 0975-3303
Mapana J Sci, **10**, 2(2011), 53-62
https://doi.org/10.12725/mjs.19.5

Chromatic Excellence in Graphs

Kulrekha Mudartha, * R. Sundareswaran** and V. Swaminathan***

Abstract

Excellence in graphs introduced by G.H. Fricke is extended to partitions of the vertex set with respect to a parameter. A graph G is said to be Chromatic excellent if {v} appears in a chromatic partition of G for every v∈V(G). This paper is devoted to the study of chromatic excellence in graphs.

Keywords: β_0-excellent, Chromatic Excellence(χ-excellence), Excellence in Graphs, Chromatic just excellence, Chromatic Partition.

Introduction

G.H. Fricke et al [1] introduced the concept of excellence in graphs with respect to graph parameters. A graph is excellent with respect to a parameter λ (maximum or minimum), if every vertex of the graph belongs to a λ-set of the graph (a λ-set is a maximum/minimum subset of the vertex set with respect to the parameter λ). For example, if we consider the domination

* Ramanujan Research Centre in Mathematics, Saraswathi Narayanan College, Madurai; kmudartha@gmail.com
** Rajalakshmi Engineering College, Chennai; neyamsundar@yahoo.com
***Ramanujan Research Centre in Mathematics, Madurai; sulanesri@yahoo.com

Received: March 2011; Revised: August 2011

parameter, then a graph is said to be domination excellent if every vertex belongs to a minimum domination set of a graph. This concept can be extended to partitions defined by parameters. E. Sampathkumar [4] introduced the concept of fixed, free and totally free points with respect to the parameter χ in a graph. Inspired by this paper, the concept of excellence with respect to partitions defined by a parameter is studied. Let λ be a parameter (minimum/maximum). Let P_λ be a λ-partition of G. (P_λ has minimum/maximum cardinality according as λ is a maximum/minimum parameter). A vertex $v \in V$ (G) is called P_λ-good if {v} belongs to a λ-partition of G. Otherwise, v is said to be P_λ-bad. A graph G is said to be P_λ-excellent if every vertex is P_λ-good. We can define P_λ -commendable, P_λ -fair, P_λ-poor graphs (according as number of P_λ-good vertices in G is greater than, equal to or less than P_λ -bad vertices in G). For our present discussion, we take β_0 as the parameter. Then $P_{\beta 0}$ will be a chromatic partition of G (that is λ -partition of G). In this paper we make a study of chromatically excellent graphs. We show that χ-excellence coincides with criticality with respect to proper colorings in graphs. Adopting the concept of just excellence introduced by N. Sridharan and M. Yamuna [5], the concept of chromatic just excellence is introduced and interesting properties are derived.

Definition 1 G is chromatically excellent if for every vertex v, there exists a chromatic partition Π such that $\{v\} \in \Pi$.

Example 1 K_n is chromatically excellent.

Example 2 C_{2n} is not chromatically excellent. C_{2n+1} $(n \geq 1)$ is chromatically excellent.

Example 3 W_{2n} $(n \geq 2)$ is chromatically excellent.

Example 4 Let G be the graph obtained from C_5 by adding vertices $u_1, u_2, \cdots, u_k$ and making each u_i adjacent with all u_j , $j \geq i$ and also adjacent with all the vertices of C_5. Then $\delta(G)=k+2$, $\chi(G)=k+3$ and G is chromatically excellent.

Remark 1 A graph is β_0-excellent if every vertex belongs to a maximum independent set of the graph. β_0-excellence and χ-excellence have no relationship. That is, a graph may be β_0-excellent but not χ-excellent and vice versa. For example, P_{2n} ($n \geq 2$) is β_0-excellent but not χ-excellent. K_n is both β_0-excellent and χ-excellent. $K_{1,n}$ ($n \geq 2$) is neither β_0-excellent nor χ-excellent. The graph in Figure 1 is χ-excellent but not β_0-excellent. ($\beta_0(G)=3$ and 1, 5, 6, 7 are β_0-bad vertices).

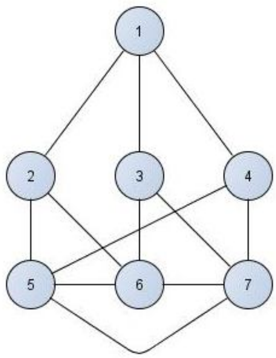

Figure 1: χ-excellent but not β_0-excellent

Proposition 1 Let $\chi(G)=2$ (Then G is a bipartite graph and vice-versa). Let $|V(G)| \geq 3$. Then G is not χ-excellent.

Proof Suppose G is χ-excellent. Let $v \in V(G)$. Then there exists a chromatic partition $\Pi=\{V_1, V_2\}$ of G such that $V_1=\{v\}$. Therefore $|V_2|=|G-\{v\}|$ is totally disconnected. Clearly $|V_2| \geq 2$ and v is adjacent to some vertex of V_2. Let $w \in V_2$ be such that $wv \in E(G)$. Let $w_1 \neq w \in V_2$. Since G is χ-excellent, there exists a chromatic partition $\Pi_1=\{\{w_1\}, V_3\}$. Therefore $v,w \in V_3$, a contradiction since $vw \in E(G)$. Therefore G is not χ-excellent.

Corollary 1 Any tree of order greater than or equal to 3 is not χ-excellent.

Remark 2 χ-excellent graphs have no isolates. For, suppose G is a χ- excellent graph which has an isolate v. If G= $\overline{K_n}$, (n ≥ 2), then χ(G)=1 and no vertex v ∈ V (G) can appear as a singleton in a chromatic partition. Therefore G ≠ $\overline{K_n}$. Therefore χ(G) ≥ 2. Then there exists a χ-partition Π={{v}, V_2, · · ·, $V_{\chi(G)}$ }. Therefore deg(v) ≥ χ(G)−1 ≥ 1. But v is an isolate, a contradiction. Therefore any χ-excellent graph has no isolates.

Proposition 2 Let G_1 and G_2 be two graphs. Then $G_1 \cup G_2$ is not χ-excellent.

Proof If V(G_1) (or V(G_2)) is a singleton then clearly $G_1 \cup G_2$ is not χ-excellent.

Let |V (G_1)| ≥ 2 and |V(G_2)| ≥ 2.

Case (i): $\chi(G_1)= \chi(G_2)=k$. Suppose there exists a χ-partition of $G_1 \cup$G2 such that {v} is an element of the partition for some v∈V(G_1).

Let Π={{v}, V_2, · · ·, V_k} be a χ-partition of $G_1 \cup G_2$. Then {V_2 −V(G_1) , · · ·, V_k−V(G_1)} is a proper color partition of G_2 and hence $\chi(G_2) \leq k-1$, a contradiction. A similar reasoning shows that for any v ∈V(G_2), {v} can not appear in any χ-partition of $G_1 \cup G_2$. Therefore $G_1 \cup G_2$ is not χ-excellent.

Case (ii): Let $\chi(G_1) \leq \chi(G_2)$. Let $\chi(G_2)=k$. Then $\chi(G_1) \cup \chi(G_2)=\chi(G_2)=k$. Suppose there exists a χ-partition of $G_1 \cup G_2$ such that {v} is an element of the partition for some v ∈ V(G_1). Let Π={{v}, V_2, · · ·, V_k} be a χ-partition of $G_1 \cup G_2$. Then {V_2 −V (G_1), · · ·

, $V_k - V(G_1)\}$ is a proper color partition of G_2 and hence $\chi(G_2) \leq k-1$, a contradiction. Therefore $G_1 \cup G_2$ is not χ-excellent.

Corollary 2 If G is χ -excellent then G is connected.

Remark 3 If G_1 and G_2 have same chromatic number, then no vertex of $G_1 \cup G_2$ can appear as a singleton in any χ -partition of $G_1 \cup G_2$. If $\chi(G_1) \leq \chi(G2)$ then no vertex of G_1 can appear as a singleton in any χ-partition of $G_1 \cup G_2$. But a vertex of G_2 may appear as a singleton in a χ-partition of $G_1 \cup G_2$. For example, consider $G_1=C_6$ and $G_2=C_5$. $\chi(G_1 \cup G_2)=3$. Let $V(G_1)=\{u_1, u_2, \cdots, u_6\}$ and $V(G_2)=\{v_1, v_2, \cdots, v_5\}$. Let $\Pi=\{\{v_1\}, \{v_2, v_4, u_1, u_3, u_5\}, \{v_3, u_2, u_4, u_6\}\}$ is a χ-partition of $G_1 \cup G_2$. In fact, as G_2 is χ-excellent, every vertex of G_2 is χ-free.

Corollary 3 If G is χ-excellent then G is connected, $\delta \geq \chi - 1$ and G has no pendent vertices.

Remark 3 P_n $(n \geq 3)$ is not χ-excellent but it is an induced subgraph of an odd cycle which is χ_-excellent (P_n is an induced subgraph of C_{n+1} if n is even and C_{n+2} if n is odd).

Proposition 3 If G is χ-excellent then $\mu(G)$ is χ-excellent.

Proof Let $V(G)=\{u_1, u_2, \cdots, u_n\}$ and $V(\mu(G))=\{u_1, u_2, \cdots, u_n, u'_1, u'_2, \cdots, u'_n, v\}$. Let G be λ-excellent. Let $\Pi=\{\{u_i\}, V_2, \cdots, V_k\}$ be a χ-partition of G where $k=\chi$, $\chi(\mu(G))=k+1$. Let $\Pi_i=\{\{u_i\}, V_2 \cup \{v\}, V_3, \cdots, V_k, \{u'_1, u'_2, \cdots, u'_n\}\}$. Π_i is a χ-partition of $\mu(G)$. Let $\Pi'_i=\{\{u'_i\}, V_2 \cup V'_2, \cdots, V_k \cup V'_k, \{u_i, v\}\}$. Π'_i is a χ-partition of

μ(G). Let Π_v={{v}, {u_i, u'_i}, $V_2 \cup V'_2$, · · · , $V_k \cup V'_k$}. Then Π_v is a χ-partition of μ(G). Therefore μ(G) is χ-excellent.

Proposition 4 Any critical graph is χ-excellent.

Proof Let G be a critical graph with chromatic number χ. Let u $\in$ V (G). Then $\chi(G-u) < \chi(G)$. Suppose $\chi(G-u)=\chi(G)-k$, (k ≥ 1). Let {V_1, V_2, · · · , $V_{\chi(G)-k}$} be a χ-partition of G−u. Then {{u}, V_1, V_2, · · · , $V_{\chi(G)-k}$ } is a proper color partition of G. Therefore $\chi(G) \leq \chi(G)-k+1$. Therefore, k≤1. Therefore k=1. Therefore {{u}, V_1, V_2, · · ·, $V_{\chi(G)-1}$} is a χ-partition of G. Therefore G is χ-excellent.

Proposition 5 If a graph G is χ-excellent, then it is critical.

Proof Suppose G is χ-excellent. Then for any u $\in$ V (G), u is either fixed or free and the end vertices of any edge in the graph are both fixed or free. But, $\chi(G-u) < \chi(G)$ for every u $\in$ V(G) and $\chi(G-e)<\chi(G)$ for every e$\in$E(G).[4] Therefore for any proper subgraph H(G), $\chi(H) < \chi(G)$. Therefore G is critical.

Proposition 6 Let G be a vertex transitive graph with a chromatic partition containing a singleton. Then G is χ-excellent.

Proof Let Π be a chromatic partition containing {u}, say. Let Π={{u}, S_2, · · ·, S_χ}. Let v $\in$ V (G), v ≠ u. Since G is vertex transitive there exists an automorphism φ such that $\varphi(u)=v$. Let Π={{$\varphi(u)$}, $\varphi(S_2)$, · · · , $\varphi(S_\chi)$}. since φ is an automorphism, φ(S2), · · · , $\varphi(S_\chi)$ are all independent. Therefore there exists a chromatic partition containing {v}. Hence, the result.

Observation 1 There exists a vertex transitive graph which is not complete in which there exists a chromatic partition containing a singleton, for example C_5.

Observation 2 There exists a vertex transitive graph which is not complete in which there exists no chromatic partition containing a singleton, for example, Peterson graph.

Definition 2 A graph G is just χ-excellent if every vertex appears as a singleton in exactly one χ-partition.

Example 5 K_n and C_{2n+1} are just χ-excellent.

Property 1 Every just χ-excellent graph is χ-excellent and hence connected.

Property 2 Let G be any χ-excellent graph. Add a vertex u and make it adjacent with every vertex of G. Let H be the resulting graph. Then H is not just χ-excellent, but H is χ-excellent. For; In any χ-partition of H, {u} appears as an element. Let $v \in V(G)$.Then there exists a χ-partition Π of G such that $\{v\} \cup \Pi$. Then $\Pi \cup \{u\}$ is a χ-partition of H.

Property 3 If G is χ-excellent, then G has exactly one χ – partition (that is G is uniquely colorable) if and only if G is complete.

Property 4 Let $G \neq K_n$, be a χ-excellent graph with a full degree vertex . Then G is not just χ-excellent.

Remark 3 Let G be a non- complete χ-excellent graph. Suppose u is not a full degree vertex in G. Then u is not χ-fixed.

Proof Let Π be a χ-partition of G. Let u be not a full degree vertex. There exists $v \in V(G)$ such that u and v are not adjacent. Suppose u is χ-fixed, then {u} appears in any χ-partition. Let $\Pi_1=\{\{u\}, V_2, \cdots, V_\chi\}$ be a χ-partition. Let $v \in V_i$, $2 \leq i \leq \chi$. Then $\Pi_2=\{V_i - \{v\}, \{u, v\}, V_3, \cdots, V_\chi\}$ is also a χ-partition not containing {u}, a contradiction. Therefore u is not fixed.

Remark 4 The following is a family of graphs which are χ-excellent but not just χ-excellent. Consider C_5. Replace each vertex by K_{2n+1}, $n \geq 1$ and join every vertex of K_{2n+1} with every vertex of another K_{2n+1} if the vertices for which these are replaced graphs are adjacent. The resulting graph has chromatic number 5n + 3, is χ-excellent but not just χ-excellent.

Remark 5 If G is a just excellent graph and $G \neq K_n$, then any χ-partition of G can contain exactly one singleton.

Proof Suppose there exists a χ-partition Π of G containing more than one singleton. Let $\Pi=\{\{u_1\}, \{u_2\}, V_3, \cdots, V_\chi\}$ be a χ-partition of G. Since G is just χ-excellent and $G \neq K_n$, no vertex of V(G) is a full degree vertex. Therefore there exists $v_1 \in V(G)$ such that u_1 and v_1 are not adjacent. Let $v_1 \in V_i$, $3 \leq i \leq \chi$. Clearly $|V_i| \geq 2$ (for if $V_i=\{v_1\}$, then u_1 and v_1 are adjacent). Let $\Pi_2=\{\{u_1, v_1\}, \{u_2\}, V_3, \cdots, V_i-\{v_1\}, \cdots, V_\chi\}$. Then Π_2 is a χ-partition containing $\{u_2\}$ a contradiction, since G is just χ-excellent.

Corollary 2 If G is just χ-excellent and $G \neq K_n$, then $\chi \leq \lfloor n+1/2 \rfloor$.

Remark 6 W_6 has chromatic number $4 > \lfloor (n+1)/2 \rfloor$ and W_6 is χ-excellent. Clearly, W_6 is not just χ-excellent.

Remark 7 The bound is sharp as seen in C_5 ($\chi(C_5)=3=5+1/2$) and C_5 is just χ-excellent.

Proposition 6 Let G be a just χ-excellent graph which is not complete. Let $u \in V(G)$. Let $\Pi=\{\{u\}, V_2, \cdots, V_\chi\}$ be a χ-partition. If $|V_i| \geq 3$, for some i, $2 \leq i \leq \chi$ then there exist at least some V_j with $|V_j| \geq 3$ containing a vertex not adjacent to u.

Proof Suppose u is adjacent to every vertex in every V_i with $|V_i| \geq 3$ ($2 \leq i \leq \chi$).

Case 1: $|V_i| \geq 3$ for all i, $2 \leq i \leq \chi$. Then u is a full degree vertex in G, a contradiction since G is just _-excellent and $G \neq K_n$.

Case 2: Let $|V_i| \geq 3$ for all i, $2 \leq i \leq t$ and $|V_{t+1}|=2$. Let $V_{t+1}=\{v_1, v_2\}$. Suppose there exists $V_{t+2}, \cdots, V_\chi$ such that $|V_{t+j}|=2$, $2 \leq j \leq \chi - t$ (Note that no V_i is a singleton since G is just χ-excellent). Since Π is a χ-partition, u is adjacent with at least one vertex in each of $V_{t+1}, \cdots, V_\chi$. Suppose u is adjacent with v_1 and not adjacent with v_2 in V_{t+1}. Then u is adjacent with every vertex in V_{t+j}, $2 \leq j \leq \chi-t$. For, otherwise, there exists some $w \in V_{t+j}$ with which u is not adjacent. Therefore $\Pi_1=\{\{u, v_2, w\}, V_2, \cdots, V_t, \{v_1\}, \cdots, V_{t+j}-\{w\}, \cdots$

, V_χ } a contradiction since G is just χ-excellent. Therefore u is adjacent with every vertex in V−{v_2} (Observe that if $V_{t+1}=V_\chi$ then also u is adjacent with every vertex in V− {v_2}). Since G is just χ-excellent there exists a chromatic partition $\Pi_2=\{\{v_2\}, V'_2, \cdots, V'_\chi\}$. Therefore $u \in V'_i$ a contradiction since u is adjacent with every vertex in V−{v_2}. Therefore the proposition is true.

Remark 8 Let G be a graph which is just χ-excellent. If there exists a χ-partition in which one of the element is a singleton say {u} and some other element with cardinality greater than or equal to 3. Then there exists a χ-partition in which none of the elements is a singleton.

Proof Let G be a just χ-excellent graph satisfying the hypothesis. Then there exists a χ-partition $\Pi=\{\{u\}, V_2, \cdots, V_\chi\}$ in which $|V_i| \geq 3$ for some i, $2 \leq i \leq \chi$ and V_i contains a non-neighbourhood say v of u. Then $\Pi_1=\{\{u, v\}, V_2, \cdots, V_i-\{v\}, \cdots, V_\chi\}$ is a χ-partition of G in which each class contains at least 2 vertices of G.

Remark 9 If G is just χ-excellent, $G \neq K_n$ and $\beta_0(G)= 2$, then G has exactly 'n' χ-partitions.

Remark 10 If G is just χ-excellent and $G \neq K_n$ then G has exactly 'n' χ-partitions if and only if in those χ-partitions in which one element is a singleton and the cardinality of any other element of the partition is 2.

Conclusion

Chromatic excellence is a new concept defined on Chromatic Partitions of Graphs. Usually the excellence is defined with respect to parameters. In this paper excellence is defined with respect to partitions. It paves a way for study of excellence with respect to different partitions.

References

[1] G.H. Fricke, T. W. Haynes, S.T. Hedetniemi, S.M. Hedetniemi and R.C. Laskar. "Excellent trees," *Bulletin of the Institute of Combinatorics and its Applications*, vol 34 pp 27-38. 2002

[2] M.A. Henning and T.W. Haynes. "Total domination excellent trees." *Discrete Mathematics*, vol. 263, pp 93-104, 2003.

[3] T.W. Haynes and Michael A. Henning. "A characterization of i-excellent trees." *Discrete Mathematics*, vol. 248, pp 69-77, 2002.

[4] E. Sampathkumar. "Chromatically fixed, free and totally free vertices in a graph." *Journal of Combinatorics, Inforamtion and System Sciences*, vol. 17, nos. 1-2, pp 130-138, 1992.

[5] N. Sridharan and M. Yamuna. "Excellent-Just Excellent -Very Excellent Graphs." *Journal of Mathematics and Physical Sciences* vol.14, no.5, pp. 471-475, 1980.

[6] N. Sridharan and M. Yamuna. "A Note on Excellent graphs." *Ars Combinatoria,* vol. 78, pp. 267-276, 2006.

[7] M. Yamuna. "Excellent- Just Excellent -Very Excellent Graphs." Ph. D Thesis, Alagappa University, India, 2003.