GOLDBACH CONJECTURE: METHOD TO DETERMINE PRIME PAIRS

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Abstract

Being the Goldbach Conjecture considered the most famous conjecture unsolved, concerning prime numbers, this paper is dedicated to prime numbers, seeking to create a method to determine prime pairs for a given even number. This method, described in the paper, allows to determine a Goldbach partition for an even number, sometimes even all, in a practical and effective way, without using complex algorithms.

Keywords: Goldbach Conjecture; prime numbers; even number; Goldbach partition

Introduction

Goldbach Conjecture is a topic that had always aroused curiosity through the centuries for its apparent simplicity. However, 273 years after its first appearance in a letter sent by Christian Goldbach to Leonhard Euler in June 1742, the conjecture has not yet been proved. Neither Goldbach nor Euler could prove it, and as them, many others tried, with no result. To be fair, mainly in the last century, Number Theory has evolved due to many mathematicians, whom, in seeking to prove Goldbach Conjecture, were able to grant results also important in Number Theory.

In this paper is described a method to determine Goldbach partitions of an even number, based on the concept of prime number and prime factors.

Goldbach Partition

Goldbach Conjecture is one of the oldest problems in number theory and in all mathematics. It states:

"Every even integer greater than 2 can be expressed as the sum of two prime numbers".

In other words:

Let a be an even number greater than 2. Then, there exist p, prime number, such that $a-p$ is a prime number.

The pair (p , $a-p$) is called a **Goldbach partition** of a.

The following are examples of Goldbach partitions for some even numbers:

$4=2+2$

$6=3+3$

$8=3+5$

$24=5+19=7+17=11+13$

...

The method we are going to present in this paper is based on a well-known primality test and in the prime factorization of an integer.

Proposition 1: *If a positive integer $a>1$ has no prime divisor lower or equal to $\sqrt{a}$, then a is prime number.*

Proof: By contraposition we must prove that: if a positive integer $a>1$ is composite then a admits at least one prime factor $p \leq \sqrt{a}$.

Let $a>1$ be a composite number, then it is possible to write $a=bc$, for some positive integers b and c, both higher than 1. Suppose, without loss of generality, that $b \leq c$. Then,

$$b^2 \leq bc = a \Rightarrow b \leq \sqrt{a}$$

Since $b>1$, the Fundamental Theorem of Arithmetic assures that b has at least one prime divisor p, with $p \leq b \leq \sqrt{a}$. Given that p divides b and b divides a, we must conclude that p divides a, i.e., $p \leq \sqrt{a}$ is a prime divisor of a. Thus, a admits at least one prime factor lower than $\sqrt{a}$.

$\square$

Every positive integer $a \geq 2$ can be written, in exactly one way, as a product of prime powers,

$$a = p_1^{s_1} p_2^{s_2} ... p_n^{s_n} \qquad (1)$$

where p_i is a prime factor, s_i a natural number, for $i=\{1,2,...,n\}$, and $p_1 < p_2 < ... < p_n$, by convention. This factorization is called *prime decomposition* of a.

Remark: Notice that, given a positive integer $a \geq 2$, with prime decomposition defined by (1), for any prime number p, such that $p \neq p_i$, $i = \{1,\ 2,\ 3,\ \ldots,\ n\}$ and $p < a-1$, we have that $a-p$ is not divisible by p_i, $i = \{1,\ 2,\ 3,\ \ldots,\ n\}$. In fact, suppose, by reduction to the absurd, that exists p_i such that p_i divides $a-p$, then p_i divides $(a-(a-p))$, that is p_i divides p, which is an absurd, since p and p_i are distinct prime numbers.

Example 1: Consider $a = 30 = 2 \times 3 \times 5$. Calculating $a-p$, for each prime number $p < 29$, distinct of 2, 3 and 5, we have:

$30-7=23$

$30-11=19$

$30-13=17$

$30-17=13$

$30-19=11$

$30-23=7$

We observe that $a-p$ is not divisible by 2, 3 or 5.

In this case, $a-p$ always results in a prime.

Example 2: Consider $a = 70 = 2 \times 5 \times 7$. Calculating $a-p$, for each prime number $p < 69$, distinct of 2, 5 and 7, we have:

$70-3=67$

$70-11=59$

$70-13=57$

...

$70-67=3$

We observe that $a-p$ is not divisible by 2, 5 or 7.

Moreover, we can verify that, in this case, $a-p$ is a prime number or multiple of 3.

The next result is the foundation of the method to determine Goldbach partitions.

Proposition 2: *Let $a \geq 8$ be an even number, which prime decomposition is written as,*

$a = p_1^{s_1} p_2^{s_2} \ldots p_n^{s_n}$, *with* p_i *prime,* $i \in \mathbb{N}$, $p_1 < p_2 < \ldots < p_n$, $s_i \in \mathbb{N}$ (2)

and let A *and* B *be sets defined by* $A = \{p_1, p_2, \ldots, p_n\}$ *and* $B = \{t \text{ prime} : p_1 < t < p_n \wedge t \notin A\}$.

If exists a prime number p, *with* $p \notin A$, *such that* $p \in \left]a - p_{n+1}^2, a-3\right]$ *and, for each prime* $t \in B$, $a-p$ *is not divisible by* t, *then* $a-p$ *is prime.*

Proof: Let p be a prime number in the conditions defined.

The upper limit, $a-3$, only limits superiorly the value of p, since if $p=a-3$, then $a-p$ assumes the lower possible value, 3. For the lower limit, we have:

$p > a - p_{n+1}^2 \Leftrightarrow a-p < p_{n+1}^2 \Leftrightarrow \sqrt{a-p} < p_{n+1}$

So, $a-p$ will be a prime number if $a-p$ is not divisible by any prime number lower or equal to $\sqrt{a-p} < p_{n+1}$.

Since $a-p$ is not divisible by $p_i \in A$, $i=\{1,2,...n\}$, as we already observed, and by hypothesis, $a-p$ is not divisible by any prime number $t \in B$, then $a-p$ is not divisible by any prime number lower or equal to $\sqrt{a-p}$, leading to conclude, according to ***proposition 1***, that $a-p$ is necessarily a prime number.

□

In case the prime decomposition of a include all prime numbers lower or equal to p_n, i.e., $B=\varnothing$, then $a-3,\ a-5,\ ...,\ a-p_n$ are divisible by 3, 5, …, p_n, respectively, then we can set the upper limit of the interval defined above to $a-p_{n+1}$.

We use this result to find Goldbach partitions of an even integer in the next examples.

Example 3: Consider $a=220=2^2\times5\times11$. We have $A=\{2,5,11\}$, $B=\{3,7\}$, $p_n=11$ and $p_{n+1}=13$. Therefore, $a-p_{n+1}^2=220-13^2=51$; $a-3=217$.

In the interval $]51,217]$, we can find the following prime numbers:

53	59	61	67	71	73	79	83	89	97	101	103	107	109	113	127
131	137	139	149	151	157	163	167	173	179	181	191	193	197	199	211

For each p in the previous list, $a-p$ is:

167	161	159	153	149	147	141	137	131	123	119	117	113	111	107	93
89	83	81	71	69	63	57	53	47	41	39	29	27	23	21	9

Removing the multiples of 3 and 7, i.e., 161, 159, 153, 147, 141, 123, 119, 117, 111, 93, 81, 69, 63, 57, 39, 27, 21 and 9, the remaining numbers are necessarily prime numbers: 23, 29, 41, 47, 53, 71, 83, 89, 107, 113, 131, 137, 149 and 167. Therefore, the following pairs:

(197,23)	(191,29)	(179,41)	(173,47)	(167,53)	(149,71)	(137,83)
(131,89)	(113,107)	(107,113)	(89,131)	(83,137)	(71,149)	(53,167)

are Goldbach partitions for 220.

Notice that some partitions are repeated: (167,53), (149,71), (137,83), (131,89) and (113,107).

Clearly, in case $a - p_{n+1}^2 < \frac{a}{2}$, we must consider the interval $\left]\frac{a}{2}, a - p_{n+1}\right]$, if $B = \emptyset$, and $\left]\frac{a}{2}, a - 3\right]$, if $B \neq \emptyset$.

In this case, all partitions of 220 were found. However, often not all the partitions of an even number are found using the interval $\left]a - p_{n+1}^2, a - 3\right]$.

Example 4: Consider $a = 60 = 2^2 \times 3 \times 5$. We have $A = \{2,3,5\}$, $B = \emptyset$, $p_n = 5$ and $p_{n+1} = 7$. Therefore, $a - p_{n+1}^2 = 60 - 7^2 = 11$, $\frac{a}{2} = 30$ and $a - 7 = 53$.

Since $B = \emptyset$ and $11 < 30$, the interval to consider is $]30, 53]$, which enables to determine all partitions of 60:

(31,29)	(37,23)	(41,19)	(43,17)	(47,13)	(53,7)

Example 5: Consider $a = 3150 = 2 \times 3^2 \times 5^2 \times 7$. We have $A = \{2,3,5,7\}$, $B = \emptyset$, $p_n = 7$ and $p_{n+1} = 11$. Therefore, $a - p_{n+1}^2 = 3150 - 11^2 = 3029$ and $a - p_{n+1} = 3150 - 11 = 3139$.

In this case, since $B = \emptyset$, all prime numbers in the interval $]3029, 3139]$ will give a Goldbach partition, and thus, the following pairs are Goldbach partitions of 3150:

(3037,113)	(3041,109)	(3049,101)	(3061,89)	(3067,83)	(3079,71)
(3083,67)	(3089,61)	(3109,41)	(3119,31)	(3121,29)	(3137,13)

Obviously, not all Goldbach partitions were found with the interval. For instance, 1571 and 1579 define a Goldbach partition for 3150.

In previous examples, the considered interval contained at least one prime number. In case the interval contains no prime numbers, we must set a new interval, with greater range, which contains at least one prime number p, such that $a - p$ is also prime.

Proposition 3: *Let $a \geq 8$ be an even number, with prime decomposition defined by* (2)*, and consider the associated sets A and B. Let also $k \in \mathbb{N}$.*

If exists p *prime, with* $p > p_n$*, such that* $p \in \left]a - p_{n+1+k}^2, a - p_{n+1+k}\right]$*,* $a - p$ *is not divisible by* $p_{n+1}, p_{n+2}, ..., p_{n+k}$ *and, for each prime* $t \in B$*,* $a - p$ *is not divisible by* t*, then* $a - p$ *is prime.*

Proof: Analogous of Proposition 2.

Example 6: Consider $a = 2048 = 2^{11}$. We have $A = \{2\}$, $B = \varnothing$, $p_n = 2$ and $p_{n+1} = 3$. Therefore, $a - p_{n+1}^2 = 2048 - 3^2 = 2039$ and $a - p_{n+1} = 2048 - 3 = 2045$.

In the interval $]2039, 2045]$, there are no prime numbers. So, according to the proposition above, one must define a new interval, with a bigger range, that contains prime numbers.

For example, assuming $k = 1$, we have $p_{n+2} = 5$ and the new interval is $]2023, 2043]$, which contains two prime numbers, 2029 and 2039. Then $a - p$ is 19 or 9. Therefore, we can define two possible Goldbach partitions: (2029,19) and (2039,9). Since 19 is not divisible by 3, the pair (2029,19) is a Goldbach partition for 2048.

As we verified, in the examples presented previously, sometimes all Goldbach partitions of an even number are determined, while in others, only a few are determined.

Given an even number $a \geq 4$ with prime decomposition defined by (2), $g(a)$ is the *number of Goldbach´s partitions determined by the interval* $]a - p_{n+1}^2, a - 3]$ and $G(a)$ is the *number of Goldbach´s partitions admitted by* a.

Example *7*: Consider $a = 300 = 2^2 \times 3 \times 5^2$. We have $A = \{2, 3, 5\}$, $B = \varnothing$, $p_n = 5$, $p_{n+1} = 7$ and therefore, the interval to consider is $]251, 293]$. The partitions determined by the interval are:

(257,43)	(263,37)	(269,31)	(271,29)	(277,23)	(281,19)	(283,17)	(293,7)

The missing partitions are:

(151,149)	(163,137)	(173,127)	(191,109)	(193,107)	(197,103)	(199,101)
(211,89)	(227,73)	(229,71)	(233,67)	(239,61)	(241,59)	

The missing partitions correspond to the interval $]150, 251]$.

In this case, 8 of 21 partitions were determined with the interval, so $g(300) = 8$ and $G(300) = 21$.

Clearly, if $a - p_{n+1}^2 > \frac{a}{2}$, there is no warranty that all the partitions are determined, missing the correspondent to $\left[\frac{a}{2}, a - p_{n+1}^2\right]$.

However, it is possible to indicate the upper limit of the number of partitions an even number can admit. Indeed, setting $\pi(a)$ the number of prime numbers lower than a, $\pi(a) - \# A$ represents the number of primes lower than a that are not prime factors of a, therefore, exists $\pi(a) - \# A$ prime numbers candidates to define a partition, which can determine, at most, $\frac{\pi(a) - \# A}{2}$ different partitions of a. Then, we have:

$$g(a) \leq G(a) \leq \frac{\pi(a) - \# A}{2} \qquad (3)$$

The method we described, to determine Goldbach partitions of an even number, can be summarized in the following diagrams:

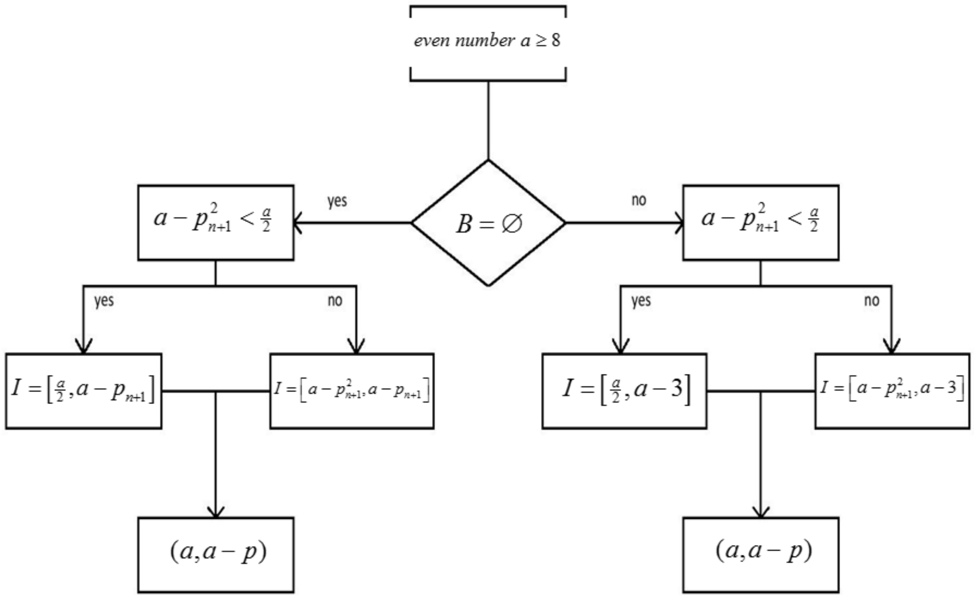

Figure 6 - Method to determine a Goldbach Partition for a given even number.

If no prime pair was found with the previous diagram:

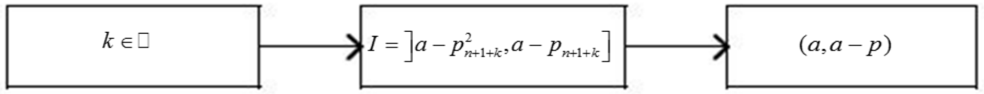

Figure 7 - Alternative interval to determine a Goldbach Partition.

We will now present examples to illustrate the different paths in the diagrams.

Example 8: Consider $a = 252 = 2^2 \times 3^2 \times 7$. We have $A = \{2,3,7\}$, $B = \{5\}$, $p_n = 7$ and $p_{n+1} = 11$. Therefore, $a - p_{n+1}^2 = 252 - 11^2 = 131$, $a - 3 = 252 - 3 = 249$ and $\frac{a}{2} = \frac{252}{2} = 126$.

Since $131 > 126$, the interval to consider is $]131, 249]$, where we can find the following prime numbers:

137	139	149	151	157	163	167	173	179	181	191
193	197	199	211	223	227	229	233	239	241	

For each p in the previous list, $a - p$ is:

115	113	103	101	95	89	85	79	73	71	61
59	55	53	41	29	25	23	19	13	11	

Removing the multiples of 5, i.e., 115, 95, 85, 55 and 25, the remaining numbers are necessarily prime numbers: 11, 13, 19, 23, 29, 41, 53, 59, 61, 71, 73, 79, 89, 101, 103, and 113. Therefore, the following pairs:

(139,113)	(149,103)	(151,101)	(163,89)	(173,79)	(179,73)	(181,71)	(191,61)
(193,59)	(199,53)	(211,41)	(223,29)	(229,23)	(233,19)	(239,13)	(241,11)

are Goldbach partitions for 252.

Example 9: Consider $a = 308 = 2^2 \times 7 \times 11$. We have $A = \{2,7,11\}$, $B = \{3,5\}$, $p_n = 11$ and $p_{n+1} = 13$. Therefore, $a - p_{n+1}^2 = 308 - 13^2 = 139$, $\frac{a}{2} = 154$ and $a - 3 = 305$.

In this case, since $139 < 154$, the interval to consider is $[154, 305]$, where we can find the following prime numbers:

157	163	167	173	179	181	191	193	197	199	211	223	227
229	233	239	241	251	257	263	269	271	277	281	283	293

For each p in the previous list, $a - p$ is:

151	145	141	135	129	127	117	115	111	109	97	85	81
79	75	69	67	57	51	45	39	37	31	27	25	15

Removing the multiples of 3 and 5, i.e., 145, 141, 129, 135, 117, 115, 111, 85, 81, 75, 69, 57, 51, 45, 39, 27, 25 and 15, the remaining numbers are necessarily prime numbers: 31, 37, 67, 79, 97, 109, 127, and 151. Therefore, the following pairs:

(157,151)	(181,127)	(199,109)	(211,97)	(229,79)	(241,67)	(271,37)	(277,31)

are Goldbach partitions for 308.

Example 10: Consider $a = 30 = 2 \times 3 \times 5$. We have $A = \{2,3,5\}$, $B = \varnothing$, $p_n = 5$ and $p_{n+1} = 7$. Therefore, $a - p_{n+1}^2 = 30 - 7^2 = -19$, $\frac{a}{2} = 15$ and $a - p_{n+1} = 30 - 7 = 23$.

In this case, since $-19 < 15$, the interval to consider is $[15,23]$, where we can find the following prime numbers: 17, 19 and 23. Therefore, $a - p$ is: 13, 11 or 7.

Since $B = \varnothing$, $a - p$ is prime for any p. Therefore, the following pairs:

(17,13)	(19,11)	(23,7)

are Goldbach partitions for 30.

Example 11: Consider $a = 390390 = 2 \times 3 \times 5 \times 7 \times 11 \times 13^2$. We have $A = \{2,3,5,7,11,13\}$, $B = \varnothing$, $p_n = 13$ and $p_{n+1} = 17$. Therefore, $a - p_{n+1}^2 = 390390 - 17^2 = 390101$, $\frac{a}{2} = 195195$ and $a - p_{n+1} = 390390 - 17 = 390373$.

In this case, since $390301 > 195195$, the interval to consider is $[390101, 390373]$, where we can find the following prime numbers:

390107	390109	390113	390119	390151	390157	390161	390191	390193
390199	390209	390211	390223	390263	390281	390289	390307	390323
390343	390347	390353	390359	390367	390373			

For each p in the previous list, $a - p$ is:

283	281	277	271	239	233	229	199	197	191	181	179
167	127	109	101	83	67	47	43	37	31	23	17

Since $B = \varnothing$, $a - p$ is prime for any p. Therefore, the following pairs:

(390107,283)	(390109,281)	(390113,277)	(390119,271)	(390151,239	(390157,233)
(390161,229)	(390191,199)	(390193,197)	(390199,191)	(390209,181)	(390211,179)
(390223,167)	(390263,127)	(390281,109)	(390289,101)	(390307,83)	(390323,67)
(390343,47)	(390347,43)	(390353,37)	(390359,31)	(390367,23)	(390373,17)

are Goldbach partitions for 390390.

Example 12: Consider $a = 995328 = 2^{12} \times 3^5$. We have $A = \{2,3\}$, $B = \varnothing$, $p_n = 3$ and $p_{n+1} = 5$.

Therefore $a - p_{n+1}^2 = 995328 - 5^2 = 995303$, $\frac{a}{2} = 497664$ and $a - p_{n+1} = 995328 - 5 = 995323$.

Since $995328 > 497664$, the interval to consider is $[995303, 995323]$, where is not possible to find a prime number. Assuming $k = 2$, we have $p_{n+3} = 11$ and the new interval to consider is $[995207, 995317]$. In this interval we can find the following prime numbers:

995219	995227	995237	995243	995273	995303

For each p in the previous list, $a - p$ is:

109	101	91	85	55	25

Removing the multiples of 5 and 7, i.e., 91, 85, 55 and 25, the remaining numbers are necessarily prime numbers: 109 and 101. Therefore, the following pairs:

(995219,109)	(995227,101)

are Goldbach partitions for 995328.

Conclusion

At first contact with Goldbach Conjecture, it appears like a problem that is simple to verify. Euler, in response to a Goldbach letter, affirmed the conjecture was probably true, but he was unable to prove it at that point. Certainly, Euler realized, before Goldbach, the apparent simplicity inherent to the conjecture, and probably thought Goldbach or other mathematician would, sooner or later, prove the conjecture, since it was reasonably simple to find a partition for a given even number. Still, despite the apparent simplicity, trying to prove has been fruitless, for several reasons, being the most relevant, the unpredictability and irregularity of prime numbers. Thus, Goldbach Conjecture has a hidden complexity, increased because an even number, with its prime factors, hinders any attempt to create a method or an algorithm, or even an approach to the problem, that could allow someone to prove the conjecture.

At last, the most important feature of the method described is that, for a given even number and a prime number lower to it, to verify if the difference is also a prime number, it is only necessary to verify if this difference is divisible by a set of prime numbers, which is a lot easier and faster than to verify if it is a prime number.

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