PARAMETRIC STUDY ON PARETO, NASH MIN-MAX DIFFERENTIAL GAME

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Abstract

In parametric analysis, which often referrers to parametric optimization or parametric programming, a perturbation parameter is introduction to the optimization problem, which means that the coefficients in the objective function of the problem and on the other hand of the constraints are perturbed. In this paper, we present the qualitative and quantitative analysis of adapted approach of Min-Max differential game of fixed duration with general parameters in the cost functions and constrains between multiple players playing dependently (the Pareto concept) and others independently (the Nash concept), the solvability set and the stability sets of the first and the second kind are defined and algorithms for determining these sets are presented.

Keywords: Parametric Analysis, Dynamic game, Stability set of the first kind, Stability set of the second kind.

Introduction

Under the min-max (security) solution concept (Von Neumann and Morgenstern,1944) a parametric dynamic game study introduced between cooperative collation of players (Pareto,1896) and other players act independently (Nash,1951) (Starr and Ho,1967,1969) where the parameters presented in the objective functions and constraints, qualitative and quantitative analysis of some basic notions, such as the set of feasible parameters, the solvability set and the stability sets of the first and the second kind, are defined and analyzed qualitatively and quantitatively for some classes of parametric optimization problems(M.S. Osman,1977)(M.S. Osman, A. El-Bauna and E. Youness,1986) The same notions are redefined in this paper and applied to Pareto ,Nash min-max differential games of fixed duration.

Problem Formulation

Consider the differential game $P(\gamma,\varepsilon)$ with general parameters $\gamma \in R^k$ in the objective functions and general parameters $\varepsilon \in R^{k'}$ in the constraints between M of players that agree to form a coalition and cooperate as a single player I to minimize their collective costs playing a min - max differential game against the other N players outside the coalition which act independently to minimize their own cost.Let $u_i(t) \in R^{l_i}, i = 1,2,\cdots,M., l = \sum_{i=1}^{M} l_i$ be the composite control of the player i among the coalition, while $v_j(t) \in R^{m_j}, j = 1,2,\cdots,N., m = \sum_{j=1}^{N} m_j$ be the composite control of the player $j$outside the coalition, where the composite control $(u,v) \in R^s$ is an element
of the constrain set
$\Omega(\varepsilon) = \{(u,v) \in R^s \mid \dot{x}(t) = f(t,x,u,v,\varepsilon), x(t_0) = x_0, h(t,x,u,v,\varepsilon) \geq 0\}$,
where

$$\dot{x}(t) = f(t,x(t),u(t),v(t),\varepsilon),$$

represent the system of nonlinear differential equations that govern the game motion, the problem can be formulated as the following problems:

$$P_1(\gamma,\varepsilon): \min_u \overline{J}(t,\hat{u},v,\gamma) = \sum_{i=1}^{M} \overline{w}_i \overline{J}_i(t,\hat{u},v,\gamma)$$

$$= \overline{\phi}(x(t_f)) + \int_{t_0}^{t_f} \overline{I}(t,x,\hat{u},v,\gamma)\, dt$$

subject to

$$\Omega(\varepsilon) = \{(\hat{u},v) \in R^s \mid \dot{x}(t) = f(t,x,\hat{u},v,\varepsilon), x(t_0) = x_0, h(t,x,\hat{u},v,\varepsilon) \geq 0\} ,$$

$$P_2(\gamma,\varepsilon): \min_u \max_v \overline{J}(t,u^*,\overline{v},\gamma) = \sum_{i=1}^{M} \overline{w}_i \overline{J}_i(t,u^*,\overline{v},\gamma)$$

$$= \overline{\phi}(x(t_f)) + \int_{t_0}^{t_f} \overline{I}(t,x,u^*,\overline{v},\gamma)\, dt$$

subject to

$$\Omega(\varepsilon) = \{(u^*, \bar{v}) \in R^s \mid \dot{x}(t) = f(t, x, u^*, \bar{v}, \varepsilon), x(t_0) = x_0, h(t, x, u^*, \bar{v}, \varepsilon) \geq 0\}$$

where

$$\bar{J}_i(u(t), v(t), \gamma) = \bar{\phi}_i(x(t_f)) + \int_{t_0}^{t_f} \bar{I}_i(t, x(t), u(t), v(t), \gamma)dt, \quad i = 1, 2, \cdots, M$$

are the cost functions for the M cooperative players $\bar{\phi}(x(t_f)) = \sum_{i=1}^{M} \bar{w}_i \bar{\phi}_i(x(t_f))$ and $\bar{I}(t, x, u, v, \gamma) = \sum_{i=1}^{M} \bar{w}_i \bar{I}_i(t, x, u, v, \gamma)$ for each $\bar{w} \in \bar{W}, \bar{W} = \{\bar{w} \in R^M \mid \bar{w}_i \geq 0, \sum_{i=1}^{M} \bar{w}_i = 1\}$,

$P_3(\gamma, \varepsilon)$:

$$\min_{v_j} J_j(t, u, v_j, \hat{v}_{-j}, \gamma) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x, u, v_j, \hat{v}_{-j}, \gamma)\, dt \; j = 1, 2, \cdots, N.$$

subject to

$$\Omega(\varepsilon) = \{(u, \hat{v}) \in R^s \mid \dot{x}(t) = f(t, x, u, v_j, \hat{v}_{-j}, \varepsilon), x(t_0) = x_0, h(t, x, u, v_j, \hat{v}_{-j}, \varepsilon) \geq 0\}$$

and

$P_4(\gamma, \varepsilon)$:

$$\min_{v_j} \max_{u} J_j(t, \bar{u}, v_j, v^*_{-j}, \gamma) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x, \bar{u}, v_j, v^*_{-j}, \gamma)\, dt \quad , j = 1, 2, \cdots, N.$$

subject to

$$\Omega(\varepsilon) = \{(\bar{u}, v^*) \in R^s \mid \dot{x}(t) = f(t, x, \bar{u}, v_j, v^*_{-j}, \varepsilon), x(t_0) = x_0, h(t, x, \bar{u}, v_j, v^*_{-j}, \varepsilon) \geq 0\}$$

where

$J_j(t, u, v, \gamma) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x, u, v, \gamma)\, dt \quad , j = 1, 2, \cdots, N.$ are the cost functions for the N non-cooperative players.Here, $[t_0, t_f]$ denotes the fixed prescribed duration of the game,

x_0: the initial state known by all players,

$x(t) \in R^n$ is the state vector of the game,

$h(.) : [t_0, t_f] \times R^n \times R^s \times R^{k'} \to R^q$

$f(.) : [t_0, t_f] \times R^n \times R^s \times R^{k'} \to R^n$, $s = l + m$.

$I_j(.) : [t_0, t_f] \times R^n \times R^s \times R^k \to R$, $j = 1, 2, \cdots, N$.

$\bar{I}_i(.) : [t_0, t_f] \times R^n \times R^s \times R^k \to R$, $i = 1, 2, \cdots, M$.

$\phi_j(.) : R^n \to R$, $j = 1, 2, \cdots, N$.

$\bar{\phi}_i(.) : R^n \to R$, $i = 1, 2, \cdots, M$. are continuous differentiable functions.

In the following, we give the definition of the solvability and stability sets of problems.

The Solvability Set

Definition 1.1 The Solvability set B of problem $P(\gamma,\varepsilon)$ is defined as :

$$B=\left\{(\gamma,\varepsilon)\in R^{k+k'}\mid \text{the solution of } P(\gamma,\varepsilon) \text{ exist for some } \bar{w}\in\bar{W}\right\} \quad (1)$$

The Stability Set of The First Kind

Definition 1.2 Suppose that $(\gamma^*,\varepsilon^*)\in B$ with a corresponding set U^* of Pareto,Nash Min-Max solutions $(\hat{u},v),(u^*,\bar{v}),(u,\hat{v})$ and $(\bar{u}^j,v^*)$ of problems $P_1(\gamma^*,\varepsilon^*),P_2(\gamma^*,\varepsilon^*),P_3(\gamma^*,\varepsilon^*)$ and $P_4(\gamma^*,\varepsilon^*)$ respectively , then the stability set of first kind of problem $P(\gamma,\varepsilon)$ corresponding to U^* denoted by $S(U^*)$ is defined by

$$S(U^*)=\left\{(\gamma,\varepsilon)\in R^{k+k'}\mid U^* \text{ solve } P(\gamma,\varepsilon) \text{ for some } \bar{w}\in\bar{W}\right\} \quad (2)$$

Lemma 1.3 *If for each* $i=1,2,\cdots,M.$ *and* $j=1,2,\cdots,N.$ *the cost functions* $\bar{J}_i(t,u,v,\gamma)$ *and* $J_j(t,u,v,\gamma)$ *are linear in* γ *,* $f(t,x(t),u(t),v(t),\varepsilon)$ *and* $h(t,x(t),u(t),v(t),\varepsilon)$ *are linear functions in* ε*, then the set* $S(U^*)$ *is convex.*

Proof. Suppose that $(\gamma_1,\varepsilon_1),(\gamma_2,\varepsilon_2)\in S(U^*)$, then it follows that for

$P_1(\gamma_1,\varepsilon_1)$

$$\bar{J}(\hat{u},v,\gamma_1)\le\bar{J}(u,v,\gamma_1)$$
$$\dot{x}(t)=f(t,x(t),\hat{u},v,\varepsilon_1)$$
$$h(t,x(t),\hat{u},v,\varepsilon_1)\ge 0$$

$P_2(\gamma_1,\varepsilon_1)$

$$\bar{J}(u^*,\bar{v},\gamma_1)\le\bar{J}(u,\bar{v},\gamma_1)$$
$$\bar{J}(u^*,\bar{v},\gamma_1)\ge\bar{J}(u^*,v,\gamma_1)$$
$$\dot{x}(t)=f(t,x(t),u^*,\bar{v},\varepsilon_1)$$
$$h(t,x(t),u^*,\bar{v},\varepsilon_1)\ge 0$$

$P_3(\gamma_1,\varepsilon_1)$

$$J_j(t,u,\hat{v},\gamma_1)\le J_j(t,u,v,\gamma_1)$$
$$\dot{x}(t)=f(t,x(t),u,\hat{v},\varepsilon_1)$$
$$h(t,x(t),u,\hat{v},\varepsilon_1)\ge 0$$

$P_4(\gamma_1,\varepsilon_1)$

$$J_{\hat{j}}(t,\bar{u}^j,v^*,\gamma_1) \le J_{\hat{j}}(t,\bar{u}^j,v,\gamma_1)$$
$$J_j(t,\bar{u}^j,v^*,\gamma_1) \ge J_j(t,u,v^*,\gamma_1)$$
$$\dot{x}(t) = f(t,x(t),\bar{u}^j,v^*,\varepsilon_1)$$
$$h(t,x(t),\bar{u}^j,v^*,\varepsilon_1) \ge 0$$

(3)

similarly for

$P_1(\gamma_2,\varepsilon_2)$

$$\bar{J}(\hat{u},v,\gamma_2) \le \bar{J}(u,v,\gamma_2)$$
$$\dot{x}(t) = f(t,x(t),\hat{u},v,\varepsilon_2)$$
$$h(t,x(t),\hat{u},v,\varepsilon_2) \ge 0$$

$P_2(\gamma_2,\varepsilon_2)$

$$\bar{J}(u^*,\bar{v},\gamma_2) \le \bar{J}(u,\bar{v},\gamma_2)$$
$$\bar{J}(u^*,\bar{v},\gamma_2) \ge \bar{J}(u^*,v,\gamma_2)$$
$$\dot{x}(t) = f(t,x(t),u^*,\bar{v},\varepsilon_2)$$
$$h(t,x(t),u^*,\bar{v},\varepsilon_2) \ge 0$$

$P_3(\gamma_2,\varepsilon_2)$

$$J_j(t,u,\hat{v},\gamma_2) \le J_j(t,u,v,\gamma_2)$$
$$\dot{x}(t) = f(t,x(t),u,\hat{v},\varepsilon_2)$$
$$h(t,x(t),u,\hat{v},\varepsilon_2) \ge 0$$

$P_4(\gamma_2,\varepsilon_2)$

$$J_{\hat{j}}(t,\bar{u}^j,v^*,\gamma_2) \le J_{\hat{j}}(t,\bar{u}^j,v,\gamma_2)$$
$$J_j(t,\bar{u}^j,v^*,\gamma_2) \ge J_j(t,u,v^*,\gamma_2)$$
$$\dot{x}(t) = f(t,x(t),\bar{u}^j,v^*,\varepsilon_2)$$
$$h(t,x(t),\bar{u}^j,v^*,\varepsilon_2) \ge 0$$

(4)

now multiplying both sides of the sets of equations (3) by α and both sides of the sets of equations (4) by $(1-\alpha)$ and adding corresponding sequence of same problem together using the linearity property in γ and ε ,we get

P_1

$$\bar{J}(\hat{u},v,\alpha\gamma_1+(1-\alpha)\gamma_2)\le\bar{J}(u,v,\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$\dot{x}(t)=f(t,x(t),\hat{u},v,\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)$$
$$h(t,x(t),\hat{u},v,\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)\ge 0$$

P_2

$$\bar{J}(u^*,\bar{v},\alpha\gamma_1+(1-\alpha)\gamma_2)\le\bar{J}(u,\bar{v},\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$\bar{J}(u^*,\bar{v},\alpha\gamma_1+(1-\alpha)\gamma_2)\ge\bar{J}(u^*,v,\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$\dot{x}(t)=f(t,x(t),u^*,\bar{v},\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)$$
$$h(t,x(t),u^*,\bar{v},\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)\ge 0$$

P_3

$$J_j(t,u,\hat{v},\alpha\gamma_1+(1-\alpha)\gamma_2)\le J_j(t,u,v,\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$\dot{x}(t)=f(t,x(t),u,\hat{v},\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)$$
$$h(t,x(t),u,\hat{v},\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)\ge 0$$

P_4

$$J_{\hat{j}}(t,\bar{u}^j,v^*,\alpha\gamma_1+(1-\alpha)\gamma_2)\le J_{\hat{j}}(t,\bar{u}^j,v,\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$J_j(t,\bar{u}^j,v^*,\alpha\gamma_1+(1-\alpha)\gamma_2)\ge J_j(t,u,v^*,\alpha\gamma_1+(1-\alpha)\gamma_2)$$
$$\dot{x}(t)=f(t,x(t),\bar{u}^j,v^*,\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)$$
$$h(t,x(t),\bar{u}^j,v^*,\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)\ge 0 \qquad (5)$$

sets of equations (5) yields that $(\alpha\gamma_1+(1-\alpha)\gamma_2,\alpha\varepsilon_1+(1-\alpha)\varepsilon_2)\in S(U^*)$ and hence $S(U^*)$ is convex.

Lemma 1.4 *If for each* $i=1,2,\cdots,M.$ *and* $j=1,2,\cdots,N.$ *the cost functions* $\bar{J}_i(t,u,v,\gamma)$ *and* $J_j(t,u,v,\gamma)$ *are continuous on* R^k *,* $f(t,x(t),u(t),v(t),\varepsilon)$ *and* $h(t,x(t),u(t),v(t),\varepsilon)$ *are continuous on* $R^{k'}$ *, then the set* $S(U^*)$ *is closed.*

Proof. Let $\{\gamma_n,\varepsilon_n\}$ be a sequence in $S(U^*)$ such that $\{\gamma_n,\varepsilon_n\}$ converges to (γ_0,ε_0) as $n\to\infty$ for all $(u,v)\in\Omega(\varepsilon)$ then for

$P_1(\gamma_n,\varepsilon_n)$

$$\bar{J}(\hat{u},v,\gamma_n)\le\bar{J}(u,v,\gamma_n)$$
$$\dot{x}(t)=f(t,x(t),\hat{u},v,\varepsilon_n)$$

$$h(t, x(t), \hat{u}, v, \varepsilon_n) \geq 0$$

$P_2(\gamma_n, \varepsilon_n)$

$$\bar{J}(u^*, \bar{v}, \gamma_n) \leq \bar{J}(u, \bar{v}, \gamma_n)$$
$$\bar{J}(u^*, \bar{v}, \gamma_n) \geq \bar{J}(u^*, v, \gamma_n)$$
$$\dot{x}(t) = f(t, x(t), u^*, \bar{v}, \varepsilon_n)$$
$$h(t, x(t), u^*, \bar{v}, \varepsilon_n) \geq 0$$

$P_3(\gamma_n, \varepsilon_n)$

$$J_j(t, u, \hat{v}, \gamma_n) \leq J_j(t, u, v, \gamma_n)$$
$$\dot{x}(t) = f(t, x(t), u, \hat{v}, \varepsilon_n)$$
$$h(t, x(t), u, \hat{v}, \varepsilon_n) \geq 0$$

$P_4(\gamma_n, \varepsilon_n)$

$$J_{\hat{j}}(t, \bar{u}^j, v^*, \gamma_n) \leq J_{\hat{j}}(t, \bar{u}^j, v, \gamma_n)$$
$$J_j(t, \bar{u}^j, v^*, \gamma_n) \geq J_j(t, u, v^*, \gamma_n)$$
$$\dot{x}(t) = f(t, x(t), \bar{u}^j, v^*, \varepsilon_n)$$
$$h(t, x(t), \bar{u}^j, v^*, \varepsilon_n) \geq 0 \qquad (6)$$

Now taking the limit of both sides of the sets of equations (6) as $n \to \infty$ it yields that

P_1

$$\bar{J}(\hat{u}, v, \gamma_0) \leq \bar{J}(u, v, \gamma_0)$$
$$\dot{x}(t) = f(t, x(t), \hat{u}, v, \varepsilon_0)$$
$$h(t, x(t), \hat{u}, v, \varepsilon_0) \geq 0$$

P_2

$$\bar{J}(u^*, \bar{v}, \gamma_0) \leq \bar{J}(u, \bar{v}, \gamma_0)$$
$$\bar{J}(u^*, \bar{v}, \gamma_0) \geq \bar{J}(u^*, v, \gamma_0)$$
$$\dot{x}(t) = f(t, x(t), u^*, \bar{v}, \varepsilon_0)$$
$$h(t, x(t), u^*, \bar{v}, \varepsilon_0) \geq 0$$

P_3

$$J_j(t, u, \hat{v}, \gamma_0) \leq J_j(t, u, v, \gamma_0)$$
$$\dot{x}(t) = f(t, x(t), u, \hat{v}, \varepsilon_0)$$

$$h(t,x(t),u,\hat{v},\varepsilon_0)\geq 0$$

P_4

$$J_{\hat{j}}(t,\bar{u}^j,v^*,\gamma_0)\leq J_{\hat{j}}(t,\bar{u}^j,v,\gamma_0)$$
$$J_j(t,\bar{u}^j,v^*,\gamma_0)\geq J_j(t,u,v^*,\gamma_0)$$
$$\dot{x}(t)=f(t,x(t),\bar{u}^j,v^*,\varepsilon_0)$$
$$h(t,x(t),\bar{u}^j,v^*,\varepsilon_0)\geq 0 \qquad (7)$$

from equations (7) we deduce that $(\gamma_0,\varepsilon_0)\in S(U^*)$ and $S(U^*)$ is closed.

Theorem 1 *If* $(\hat{u},v)\in\Omega\subseteq R^s$ *is Pareto solution for the M players inside the coalition with state trajectory* x^* *corresponding to problem* $P_1(\gamma,\varepsilon)$ *,then there exist continuous costate functions* $p:[t_0,t_f]\to R^n$ *,* $\bar{\delta}\in R^q$ *such that the following relations are satisfied*

$$\dot{x}^*(t)=f(t,x^*,\hat{u},v,\varepsilon)\quad ,x^*(t_0)=x_0 \qquad (8)$$

$$\dot{p}(t)=-\frac{\partial\bar{H}(t,x^*,\hat{u},v,\bar{w}_1,\cdots,\bar{w}_M,p,\bar{\delta},\gamma,\varepsilon)}{\partial x(t)} \qquad (9)$$

$$p(t_f)=\frac{\partial\bar{\phi}(x^*(t_f))}{\partial x(t_f)},\quad \bar{\phi}=\sum_{i=1}^{M}\bar{w}_i\bar{\phi}_i(x^*(t_f)) \qquad (10)$$

$$\frac{\partial\bar{H}(t,x^*,\hat{u},v,\bar{w}_1,\cdots,\bar{w}_M,p,\bar{\delta},\gamma,\varepsilon)}{\partial u}=0 \qquad (11)$$

$$\bar{\delta}h(t,x^*,\hat{u},v,\varepsilon)=0 \qquad (12)$$

$$h(t,x^*,\hat{u},v,\varepsilon)\geq 0 \qquad (13)$$

$$\bar{\delta}\geq 0 \qquad (14)$$

where

$$\bar{H}(t,x,u,v,\bar{w}_1,\cdots,\bar{w}_M,p,\bar{\delta},\gamma,\varepsilon)=\sum_{i=1}^{M}\bar{w}_i\bar{I}_i(t,x,u,v,\gamma)+p^T f(t,x,u,v,\varepsilon)$$
$$-\bar{\delta}^T h(t,x,u,v,\varepsilon) \qquad (15)$$

is the Hamiltonian function of the players inside the coalition and its partial derivatives.Furthermore , If $(u^*,\bar{v})\in P\subseteq\Omega\subseteq R^s$ is Min-Max point for the M cooperative players with state trajectory x^* corresponding to problem $P_2(\gamma,\varepsilon)$,then there exist continuous costate functions $p:[t_0,t_f]\to R^n$, $\bar{\varsigma}\in R^q$ and $\bar{\eta}^j\in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t)=f(t,x^*,u^*,\bar{v},\varepsilon)\quad ,x^*(t_0)=x_0 \qquad (16)$$

$$\dot{p}(t) = -\frac{\partial \overline{H}(t, x^*, u^*, \overline{v}, \overline{w}_1, \cdots, \overline{w}_M, p, \overline{\varsigma}, \overline{\eta}^j, \gamma, \varepsilon)}{\partial x(t)} \quad (17)$$

$$p(t_f) = \frac{\partial \overline{\phi}(x^*(t_f))}{\partial x(t_f)}, \quad \overline{\phi} = \sum_{i=1}^{M} \overline{w}_i \overline{\phi}_i(x^*(t_f)) \quad (18)$$

$$\frac{\partial \overline{H}(t, x^*, u^*, \overline{v}, \overline{w}_1, \cdots, \overline{w}_M, p, \overline{\varsigma}, \overline{\eta}^j, \gamma, \varepsilon)}{\partial u} = 0 \quad (19)$$

$$\frac{\partial \overline{H}(t, x^*, u^*, \overline{v}, \overline{w}_1, \cdots, \overline{w}_M, p, \overline{\varsigma}, \overline{\eta}^j, \gamma, \varepsilon)}{\partial v_j} = 0 \quad (20)$$

$$\overline{\varsigma} h(t, x^*, u^*, \overline{v}, \varepsilon) = 0 \quad (21)$$

$$\overline{\eta}^j h(t, x^*, u^*, \overline{v}, \varepsilon) = 0 \quad (22)$$

$$h(t, x^*, u^*, \overline{v}, \varepsilon) \geq 0 \quad (23)$$

$$\overline{\varsigma} \geq 0 \quad (24)$$

$$\overline{\eta}^j \leq 0 \quad , j = 1, 2, \cdots, N. \quad (25)$$

where

$$\overline{H}(t, x, u, v, \overline{w}_1, \cdots, \overline{w}_M, p, \overline{\varsigma}, \overline{\eta}, \gamma, \varepsilon) = \sum_{i=1}^{M} \overline{w}_i \overline{I}_i(t, x, u, v, \gamma) + p^T f(t, x, u, v, \varepsilon)$$
$$-\overline{\varsigma}^T h(t, x, u, v, \varepsilon) \quad (26)$$

is the Hamiltonian function of the players inside the coalition and its partial derivatives evaluated using the two sets of multipliers $\overline{\varsigma}$ and $\overline{\eta}^j$.

For the players outside the coalition , If $(u, \hat{v}) \in \Omega \subseteq R^s$ is Nash equilibrium point for the N players corresponding to problem $P_3(\gamma, \varepsilon)$, then there exist continuous costate functions $q^j : [t_0, t_f] \to R^n$, $\delta^j \in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t) = f(t, x^*, u, \hat{v}, \varepsilon), x^*(t_0) = x_0 \quad (27)$$

$$\dot{q}_k^j(t) = -\frac{\partial H_j(t, x^*, u, \hat{v}, q^j, \delta^j, \gamma, \varepsilon)}{\partial x_k(t)} \quad (28)$$

$$q_k^j(t_f) = \frac{\partial \phi_j(x^*(t_f))}{\partial x_k(t_f)}, k = 1, 2, \cdots, n \quad , j = 1, 2, \cdots, N \quad (29)$$

$$\frac{\partial H_j(t, x^*, u, \hat{v}, q^j, \delta^j, \gamma, \varepsilon)}{\partial v_j} = 0, \quad j = 1, 2, \cdots, N \quad (30)$$

$$\delta^j h(t, x^*, u, \hat{v}, \varepsilon) = 0, \quad j = 1, 2, \cdots, N \quad (31)$$

$$h(t, x^*, u, \hat{v}, \varepsilon) \geq 0 \quad (32)$$

$$\delta^j \geq 0, \quad j = 1,2,\cdots,N \tag{33}$$

where

$$H_j(t,x,u,v,q^j,\delta^j,\gamma,\varepsilon) = I_j(t,x,u,v,\gamma) + q^{jT} f(t,x,u,v,\varepsilon)$$
$$-\delta^{jT} h(t,x,u,v,\varepsilon), \quad j = 1,2,\cdots,N \tag{34}$$

are the Hamiltonian functions of each of the N players and its partial derivatives.Furthermore ,if $(\bar{u}^j, v^*) \in \hat{D} \subseteq \Omega \subseteq R^s$ is a Min-Max point for player j of the Nash players corresponding to problem $P_4(\gamma,\varepsilon)$, then there exist continuous costate functions $q^j : [t_0, t_f] \to R^n$, $\varsigma^j \in R^q$ and $\eta^j \in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t) = f(t, x^*, \bar{u}^j, v^*, \varepsilon), x^*(t_0) = x_0 \tag{35}$$

$$\dot{q}_k^j(t) = -\frac{\partial H_j(t, x^*, \bar{u}^j, v^*, q^j, \varsigma^j, \eta^j, \gamma, \varepsilon)}{\partial x_k(t)} \tag{36}$$

$$q_k^j(t_f) = \frac{\partial \phi_j(x^*(t_f))}{\partial x_k(t_f)}, \quad k = 1,2,\cdots,n, \quad j = 1,2,\cdots,N \tag{37}$$

$$\frac{\partial H_{\hat{j}}(t, x^*, \bar{u}^j, v^*, q^j, \varsigma^j, \eta^j, \gamma, \varepsilon)}{\partial v_{\hat{j}}} = 0, \quad j = 1,2,\cdots,N., \quad \hat{j} = 1,2,\cdots,N \tag{38}$$

$$\frac{\partial H_j(t, x^*, \bar{u}^j, v^*, q^j, \varsigma^j, \eta^j, \gamma, \varepsilon)}{\partial u} = 0, \quad j = 1,2,\cdots,N \tag{39}$$

$$\varsigma^{\hat{j}} h(t, x^*, \bar{u}^j, v^*, \varepsilon) = 0, \quad j = 1,2,\cdots,N \quad \hat{j} = 1,2,\cdots,N \tag{40}$$

$$\eta^j h(t, x^*, \bar{u}^j, v^*, \varepsilon) = 0, \; j = 1,2,\cdots,N \tag{41}$$

$$h(t, x^*, \bar{u}^j, v^*, \varepsilon) \geq 0 \tag{42}$$

$$\varsigma^j \geq 0, \quad j = 1,2,\cdots,N \tag{43}$$

$$\eta^j \leq 0, \quad j = 1,2,\cdots,N \tag{44}$$

where

$$H_j(t,x,u,v,q^j,\varsigma^j,\eta^j,\gamma,\varepsilon) = I_j(t,x,u,v,\gamma) + q^{jT} f(t,x,u,v,\varepsilon)$$
$$-\varsigma^{jT} h(t,x,u,v,\varepsilon), \quad j = 1,2,\cdots,N \tag{45}$$

are the Hamiltonian functions of the Nash players and its partial derivatives evaluated using the two sets of multipliers ς^j and η^j .

Determination of The Stability Set of The First Kind

In this section we introduce method to determining the stability set of first kind $S(U^*)$ for Pareto,Nash Min-Max continuous differential game that can be summarized in the following steps:

1. Start with $(\gamma^*, \varepsilon^*) \in B$ corresponding to Pareto,Nash Min-Max solutions

U^*

2. Substituting in the system of equations given by theorem (2) we obtain

$$\sum_{i=1}^{M}\overline{w}_i \frac{\partial \overline{I}_i(t,x^*,\hat{u},v,\gamma)}{\partial u}+\sum_{j=1}^{n}p_j \frac{\partial f_j(t,x^*,\hat{u},v,\varepsilon)}{\partial u}-\sum_{r=1}^{q}\overline{\delta}_r \frac{\partial h_r(t,x^*,\hat{u},v,\varepsilon)}{\partial u}=0$$

$$\overline{\delta}h(t,x^*,\hat{u},v,\varepsilon)=0$$

$$\sum_{i=1}^{M}\overline{w}_i \frac{\partial \overline{I}_i(t,x^*,u^*,\overline{v},\gamma)}{\partial u}+\sum_{j=1}^{n}p_j \frac{\partial f_j(t,x^*,u^*,\overline{v},\varepsilon)}{\partial u}-\sum_{r=1}^{q}\overline{\varsigma}_r \frac{\partial h_r(t,x^*,u^*,\overline{v},\varepsilon)}{\partial u}=0$$

$$\sum_{i=1}^{M}\overline{w}_i \frac{\partial \overline{I}_i(t,x^*,u^*,\overline{v},\gamma)}{\partial v}+\sum_{j=1}^{n}p_j \frac{\partial f_j(t,x^*,u^*,\overline{v},\varepsilon)}{\partial v}-\sum_{r=1}^{q}\overline{\eta}_r^j \frac{\partial h_r(t,x^*,u^*,\overline{v},\varepsilon)}{\partial v}=0$$

$$\overline{\varsigma}h(t,x^*,u^*,\overline{v},\varepsilon)=0$$

$$\overline{\eta}^j h(t,x^*,u^*,\overline{v},\varepsilon)=0 \quad , j=1,2,\cdots,N.$$

$$\frac{\partial I_j(t,x^*,u,\hat{v},\gamma)}{\partial v_j}+\sum_{i=1}^{n}q_i^j \frac{\partial f_i(t,x^*,u,\hat{v},\varepsilon)}{\partial v_j}-\sum_{r=1}^{q}\delta_r^j \frac{\partial h_r(t,x^*,u,\hat{v},\varepsilon)}{\partial v_j}=0$$

$$\delta^j h(t,x^*,u,\hat{v},\varepsilon)=0 \quad j=1,2,\cdots,N.$$

$$\frac{\partial I_{\hat{j}}(t,x^*,\overline{u}^j,v^*,\gamma)}{\partial v_{\hat{j}}}+\sum_{i=1}^{n}q_i^{\hat{j}} \frac{\partial f_i(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial v_{\hat{j}}}-\sum_{r=1}^{q}\varsigma_r^{\hat{j}} \frac{\partial h_r(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial v_{\hat{j}}}=0$$

$$\varsigma^{\hat{j}}h(t,x^*,\overline{u}^j,v^*,\varepsilon)=0 \quad , j=1,2,\cdots,N., \quad \hat{j}=1,2,\cdots,N.$$

$$\frac{\partial I_j(t,x^*,\overline{u}^j,v^*,\gamma)}{\partial u}+\sum_{i=1}^{n}q_i^j \frac{\partial f_i(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial u}-\sum_{r=1}^{q}\eta_r^j \frac{\partial h_r(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial u}=0$$

$$\eta^j h(t,x^*,\overline{u}^j,v^*,\varepsilon)=0 \quad , j=1,2,\cdots,N. \qquad (46)$$

This system of equations (46) denoted by $F(\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j,\eta^j,\gamma,\varepsilon)$, represent $\overline{s}=2s+N(m+l)+2q+Nq(3+N)$ equations of the $M+k+k'+2q(1+2N)$ unknowns, which are nonlinear in $\gamma\in R^k$, $\varepsilon\in R^{k'}$ and linear in $\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j$ and $\eta^j(\in R^q)$. If $\overline{s}=M+k+k'+2q(1+2N)$ then we may obtain the unknowns explicitly.If $\overline{s}=k+k'$ and $\nabla_\gamma F(\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j,\eta^j,\gamma,\varepsilon)$,$\nabla_\varepsilon F(\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j,\eta^j,\gamma,\varepsilon)$ exist and continuous for every $(\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j,\eta^j,\gamma,\varepsilon)\in D_\alpha(\overline{w}^*,\overline{\delta}^*,\overline{\varsigma}^*,\overline{\eta}^{*j},\delta^{*j},\varsigma^{*j},\eta^{*j},\gamma^*,\varepsilon^*)$, where D_α is the neighborhood of the point of solution $(\overline{w}^*,\overline{\delta}^*,\overline{\varsigma}^*,\overline{\eta}^{*j},\delta^{*j},\varsigma^{*j},\eta^{*j},\gamma^*,\varepsilon^*)$ of the system F , then by the implicit function theorem , γ and ε can be expressed as function of $\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^{j},\delta^{j},\varsigma^{j},\eta^{j}$.

The Stability Set of The Second Kind

Definition 1.5 *Suppose that* $(\gamma^{*},\varepsilon^{*})\in B$ *with a corresponding set* U^{*} *of Pareto,Nash Min-Max solutions* $(\hat{u},v),(u^{*},\overline{v}),(u,\hat{v})$ *and* $(\overline{u}^{j},v^{*})$ *of problems* $P_1(\gamma^{*},\varepsilon^{*}),P_2(\gamma^{*},\varepsilon^{*})$, $P_3(\gamma^{*},\varepsilon^{*})$ *and* $P_4(\gamma^{*},\varepsilon^{*})$ *respectively,and*

$$\sigma(\gamma^{*},\varepsilon^{*},I)=\{(u,v)\in R^{s}\,|\,\dot{x}(t)=f(t,x,u,v,\varepsilon^{*}),x(t_0)=x_0,$$
$$h_r(t,x,u,v,\varepsilon^{*})=0,r\in I\subset\{1,2,\cdots,q\},h_r(t,x,u,v,\varepsilon^{*})\succ 0,r\notin I\} \qquad (47)$$

which denote either the unique side of σ or $int(\sigma)$ which contains U^{*} ,then the stability set of second kind of problem $P(\gamma,\varepsilon)$ corresponding to $\sigma(\gamma^{*},\varepsilon^{*},I)$ denoted by $Q(\sigma(\gamma^{*},\varepsilon^{*},I))$ is defined by

$$Q(\sigma(\gamma^{*},\varepsilon^{*},I))=\left\{(\gamma,\varepsilon)\in B\,|\,\Omega^{*}(\gamma,\varepsilon)\cap\sigma(\gamma^{*},\varepsilon^{*},I)\neq\phi\right\} \qquad (48)$$

where

$$\Omega^{*}(\gamma,\varepsilon)=\{(u^{*},v^{*})\in\Omega(\varepsilon)\,|\,(u^{*},v^{*})\ solve\ P(\gamma,\varepsilon)\ for\ some\ \overline{w}\in\overline{W}\} \qquad (49)$$

Lemma 1.6 *If for each* $i=1,2,\cdots,M$. *the cost function* $\overline{J}_i(t,u,v,\gamma)$ *is strictly convex with respect to* $u_i\in R^{l_i}$ *and concave with respect to* $v\in R^{m}$ *,and for each* $j=1,2,\cdots,N$. *the cost function* $J_j(t,u,v,\gamma)$ *is strictly convex with respect to* $v_j\in R^{m_i}$ *and concave with respect to* $u\in R^{l}$ *and let* $\sigma(\gamma_1,\varepsilon_1,I_1)$, $\sigma(\gamma_2,\varepsilon_2,I_2)$ *are two distinct sides of* $\Omega(\varepsilon)$ *, then*

$$Q(\sigma(\gamma_1,\varepsilon_1,I_1))\cap Q(\sigma(\gamma_2,\varepsilon_2,I_2))=\phi$$

Proof. Consider that $(\gamma^{*},\varepsilon^{*})\in Q(\sigma(\gamma_1,\varepsilon_1,I_1))\cap Q(\sigma(\gamma_2,\varepsilon_2,I_2))$, then we have

$$\Omega^{*}(\gamma^{*},\varepsilon^{*})\cap\sigma(\gamma_1,\varepsilon_1,I_1)\neq\phi,$$
$$\Omega^{*}(\gamma^{*},\varepsilon^{*})\cap\sigma(\gamma_2,\varepsilon_2,I_2)\neq\phi$$

which contradict that $\sigma(\gamma_1,\varepsilon_1,I_1)$, $\sigma(\gamma_2,\varepsilon_2,I_2)$ are two distinct sides.

Determination of the Stability Set of the Second Kind

To determine the stability set of second kind $Q(\sigma(\gamma,\varepsilon,I))$ for Pareto,Nash Min-Max continuous differential game that can be summarized in the following steps:

1. Start with $(\gamma^{*},\varepsilon^{*})\in B$ corresponding to Pareto,Nash Min-Max solutions U^{*} for problem $P(\gamma,\varepsilon)$.

2. Determining the side $\sigma(\gamma,\varepsilon,I)$ by substituting in the constraints to obtain the active set of constraints.
3. From the definition of the stability Set of The Second Kind by taking $I=\{1,2,\cdots,d\}\subset\{1,2,\cdots,q\}$ then we have the system of equations:

$$\sum_{i=1}^{M}\overline{w}_i\frac{\partial\overline{I}_i(t,x^*,\hat{u},v,\gamma)}{\partial u}+\sum_{j=1}^{n}p_j\frac{\partial f_j(t,x^*,\hat{u},v,\varepsilon)}{\partial u}-\sum_{r=1}^{d}\overline{\delta}_r\frac{\partial h_r(t,x^*,\hat{u},v,\varepsilon)}{\partial u}=0$$

$$h_r(t,x^*,\hat{u},v,\varepsilon)=0\quad r=1,2,\cdots,d.$$

$$\sum_{i=1}^{M}\overline{w}_i\frac{\partial\overline{I}_i(t,x^*,u^*,\overline{v},\gamma)}{\partial u}+\sum_{j=1}^{n}p_j\frac{\partial f_j(t,x^*,u^*,\overline{v},\varepsilon)}{\partial u}-\sum_{r=1}^{d}\overline{\varsigma}_r\frac{\partial h_r(t,x^*,u^*,\overline{v},\varepsilon)}{\partial u}=0$$

$$\sum_{i=1}^{M}\overline{w}_i\frac{\partial\overline{I}_i(t,x^*,u^*,\overline{v},\gamma)}{\partial v}+\sum_{j=1}^{n}p_j\frac{\partial f_j(t,x^*,u^*,\overline{v},\varepsilon)}{\partial v}-\sum_{r=1}^{d}\overline{\eta}_r^j\frac{\partial h_r(t,x^*,u^*,\overline{v},\varepsilon)}{\partial v}=0$$

$$h_r(t,x^*,u^*,\overline{v},\varepsilon)=0\quad r=1,2,\cdots,d.$$

$$\frac{\partial I_j(t,x^*,u,\hat{v},\gamma)}{\partial v_j}+\sum_{i=1}^{n}q_i^j\frac{\partial f_i(t,x^*,u,\hat{v},\varepsilon)}{\partial v_j}-\sum_{r=1}^{d}\delta_r^j\frac{\partial h_r(t,x^*,u,\hat{v},\varepsilon)}{\partial v_j}=0$$

$$h_r(t,x^*,u,\hat{v},\varepsilon)=0\quad r=1,2,\cdots,d.$$

$$\frac{\partial I_{\hat{j}}(t,x^*,\overline{u}^j,v^*,\gamma)}{\partial v_{\hat{j}}}+\sum_{i=1}^{n}q_i^{\hat{j}}\frac{\partial f_i(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial v_{\hat{j}}}-\sum_{r=1}^{d}\varsigma_r^{\hat{j}}\frac{\partial h_r(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial v_{\hat{j}}}=0$$

$$\frac{\partial I_j(t,x^*,\overline{u}^j,v^*,\gamma)}{\partial u}+\sum_{i=1}^{n}q_i^j\frac{\partial f_i(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial u}-\sum_{r=1}^{d}\eta_r^j\frac{\partial h_r(t,x^*,\overline{u}^j,v^*,\varepsilon)}{\partial u}=0$$

$$h_r(t,x^*,\overline{u}^j,v^*,\varepsilon)=0\quad r=1,2,\cdots,d.$$

(50)

This system of equations (50) represent $2s+N(m+l+d)+3d$ equations of the $M+k+k'+2d(1+2N)+s$ unknowns, which are nonlinear in $\gamma\in R^k,\varepsilon\in R^{k'},u\in R^l,v\in R^m$ and linear in $\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j$ and $\eta^j(\in R^q)$, from which we can obtain γ and ε as function of $\overline{w},\overline{\delta},\overline{\varsigma},\overline{\eta}^j,\delta^j,\varsigma^j,\eta^j,u$ and v.

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