

Planar Curves out of Their Curvatures in R^2

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Abstract

This research aims to introduce some of the main ideas of differential geometry. The research deals with the main concepts needed to understand this work. In this research properties of curves in R^2 are studied. The research is built on using the curvature of a curve in R^2 to derive a parametric formula for the *velocity* and *acceleration*. Also the geometry of focal points has been discussed. Examples are built to support the aim of this research.

Keywords: Curvature, Frenet Formulas, focal point, planner curves.

Introduction

The main content of this research is some differential geometry concerning curves in R^2 . a brief description of curves in R^2 is introduced. The geometry of curves in R^2 is described using Frenet formulas. The concept of a focal point is a base within this research, so definitions and theorems concerning focal points are included. As this research is about geometry of curves, the curvature of a curve in R^2 is studied in the field of focal points.

Differential geometry is a mathematical field which allows the study of geometrical concepts using calculus. Thus, calculus is used to explain how to extract planar curves out of their curvatures, an idea which supports the content of this research. Examples are built to support the aim of this research. Other geometrical concepts are included.

Curves in R^2

Definition 1.1. A curve C in R^2 is a differentiable function $\vec{f} : I \rightarrow R^2$ where I is an (open) interval.

Thus, if t is a parameter of \vec{f} , $t \in I$, then we write \vec{f} as

$$\vec{f}(t) = (f_1(t), f_2(t))$$

where f_1 and f_2 are differentiable real valued functions defined on I .

We say that \vec{f} is a parametrization of C or C is the trace of \vec{f} on I .

Definition 1.2. A curve C in \mathbb{R}^2 with parametrization $\vec{f}: I \rightarrow \mathbb{R}^2$ is *regular* provided that $\vec{f}'(t) \neq \vec{0}$ for all $t \in I$. If $\vec{f}'(t_1) = \vec{0}$ for some $t_1 \in I$, then $\vec{f}(t_1)$ is a *singular point* of C .

Definition 1.3. Let $\vec{f}: I \rightarrow \mathbb{R}^2$ be a curve in \mathbb{R}^2 . Let $\delta: J \rightarrow I$ be a differentiable real valued function where J is also an interval. Then $\vec{w} = \vec{f}(\delta): J \rightarrow \mathbb{R}^2$ is a *reparametrization* of \vec{f} by δ .

Definition 1.4. The reparametrization \vec{w} of \vec{f} is *orientation preserving* if $\delta' > 0$ on J and *orientation reversing* if $\delta' < 0$ on J .

Now we take $I = [a, b]$ and define the *arclength* along C by

$$s(t) = \int_a^t \|\vec{f}'(u)\| \, du \tag{1}$$

where $\|\vec{f}'(t)\|$ is the norm of $\vec{f}'(t)$, usually called the *speed* of \vec{f} . If C is a regular curve, then $\|\vec{f}'(t)\| > 0$ for all $t \in I$ and so s is a strictly increasing function of t which has an inverse. That is, Equation (1) can be solved for s and then C has a reparametrization by arclength.

Proposition 1.5. Let \vec{w} be an arclength reparametrization of \vec{f} . Then \vec{w} has unit speed.

Proof: Observe that \vec{w} is defined by $\vec{w}(s) = \vec{f}(t(s))$. Then differentiate with respect to s to get

$$\left\| \frac{d\vec{w}}{ds} \right\| = \left\| \frac{d\vec{f}}{dt} \frac{dt}{ds} \right\| = \left| \frac{dt}{ds} \right| \left\| \frac{d\vec{f}}{dt} \right\| = \frac{1}{\|\vec{f}'\|} \|\vec{f}'\| = 1. \diamond$$

Reparametrizing a regular curve by arclength (also called unit speed reparametrization) simplifies calculations regarding geometry of curves.

Curvature of a curve in \mathbb{R}^2

Now take \vec{w} as a unit speed curve, and then $\vec{w}'' \perp \vec{w}'$. For,

$$\|\vec{w}'\| = 1 \Rightarrow \vec{w}' \cdot \vec{w}' = 1 \Rightarrow \vec{w}' \cdot \vec{w}'' = 0.$$

If $\vec{T}(s)$ is the *unit tangent* of \vec{w} at s , then $\vec{T} = \vec{w}'$.

Also

$$\vec{T}' = \vec{w}''.$$

Definition 2.1. The *curvature* of a unit speed curve \vec{w} is the function $\kappa(s) = \|\vec{T}'(s)\|$ where \vec{T} is the unit tangent of \vec{w} .

Now let $\vec{N}(s)$ be the *unit normal* of \vec{w} at s ,

then

$$\bar{T}'(s) = \kappa(s) \bar{N}(s) \tag{2}$$

The orthonormal set of the vectors $\{\bar{T}, \bar{N}\}$ is called *Frenet frame*. So \bar{N}' can be written in terms of \bar{T} and \bar{N} . In fact, if $\bar{N}' = a\bar{T} + b\bar{N}$, then

$$b = \bar{N}' \cdot \bar{N} = 0$$

and by Equation (2),

$$a = \bar{N}' \cdot \bar{T} = -\bar{N} \cdot \bar{T}' = -\bar{N} \cdot \kappa\bar{N} = -\kappa.$$

Hence

$$\bar{N}'(s) = -\kappa(s)\bar{T}(s) \tag{3}$$

Equations (2) and (3) are called *Frenet formulas*.

Now let \bar{f} be an arbitrary regular curve with parameter t and unit speed reparametrization $\bar{w}(s)$. Let $\nu(t) = \|\bar{f}'(t)\|$.

Theorem 2.2. Frenet formulas of \bar{f} are

$$\bar{T}'(t) = \kappa(t) \nu(t) \bar{N}(t) \tag{4}$$

$$\bar{N}'(t) = -\kappa(t) \nu(t) \bar{T}(t) \tag{5}$$

Proof: It is clear that there is a correspondence between points on \bar{f} and \bar{w} in a way that $\exists t$ and s such that

$$\bar{f}(t) = \bar{w}(s), \quad \bar{T}_{\bar{f}}(t) = \bar{T}_{\bar{w}}(s), \quad \bar{N}_{\bar{f}}(t) = \bar{N}_{\bar{w}}(s) \quad \text{and}$$

$$\kappa_{\bar{f}}(t) = \kappa_{\bar{w}}(s).$$

Thus, Equation (2) implies that

$$\bar{T}'(t) = \frac{d\bar{T}}{ds} \frac{ds}{dt} = \kappa(s) \bar{N}(s) \frac{ds}{dt} = \kappa(t) \nu(t) \bar{N}(t).$$

Also Equation (3) implies that

$$\bar{N}'(t) = \frac{d\bar{N}}{ds} \frac{ds}{dt} = -\kappa(s) \bar{T}(s) \frac{ds}{dt} = -\kappa(t) \nu(t) \bar{T}(t). \quad \diamond$$

Theorem 2.3. The *velocity* and *acceleration* vectors of \bar{f} at t are respectively

$$\bar{f}'(t) = \nu(t) \bar{T}(t) \tag{6}$$

$$\bar{f}''(t) = \nu'(t) \bar{T}(t) + \kappa(t) \nu^2(t) \bar{N}(t) \tag{7}$$

Proof: The unit tangent of \bar{f} at t is $\bar{T}(t) = \frac{\bar{f}'(t)}{\|\bar{f}'(t)\|}$,

and so

$$\bar{f}'(t) = \|\bar{f}'(t)\| \bar{T}(t) = \nu(t)\bar{T}(t),$$

and so Equation (6) is derived.

Now differentiate Equation (6) with respect to t , then use Equation (4) to get Equation (7). \diamond

Theorem 2.4. Let \vec{f} be an arbitrary regular curve. Then the curvature of \vec{f} is given by

$$\kappa(t) = \frac{\|\vec{f}'(t) \times \vec{f}''(t)\|}{\|\vec{f}'(t)\|^3} \tag{8}$$

Proof: By Equations (6) and (7), we have

$$\vec{f}' \times \vec{f}'' = v\vec{T} \times (v\vec{T} + \kappa v^2\vec{N}) = \kappa v^3\vec{T} \times \vec{N} .$$

Thus,

$$\|\vec{f}' \times \vec{f}''\| = \kappa v^3 \|\vec{T} \times \vec{N}\| = \kappa v^3 .$$

Hence Equation (8) follows. \diamond

Definition 2.5. The *total curvature* of a regular curve \vec{f} on I is

$$K = \int_I v(t)\kappa(t) dt .$$

The focal curve of a curve in R^2

Definition 3.1. The *focal curve* of a regular curve \vec{f} is the curve

$$\vec{g} = \vec{f} + \frac{1}{\kappa}\vec{N} .$$

Similarly, the focal point with base $\vec{f}(t)$ is the point

$$\vec{g}(t) = \vec{f}(t) + \frac{1}{\kappa(t)}\vec{N}(t) .$$

Now let $\Lambda_{\vec{p}}(t) = \|\vec{f}(t) - \vec{p}\|^2$ be the distance function whose domain is the curve C with parametrization \vec{f} .

Theorem 3.2. The point \vec{p} is a focal point of C with base $\vec{f}(t)$ iff

$$\Lambda'_{\vec{p}}(t) = 0 \text{ and } \Lambda''_{\vec{p}}(t) = 0 .$$

Proof: Recall that $\Lambda_{\vec{p}}(t) = \|\vec{f}(t) - \vec{p}\|^2 = (\vec{f}(t) - \vec{p}) \cdot (\vec{f}(t) - \vec{p})$.

Thus,

$$\Lambda'_{\vec{p}}(t) = 2(\vec{f}(t) - \vec{p}) \cdot \vec{f}'(t)$$

$$\text{And } \Lambda''_{\vec{p}}(t) = 2(\vec{f}(t) - \vec{p}) \cdot \vec{f}''(t) + 2\vec{f}'(t) \cdot \vec{f}'(t) .$$

If \vec{p} is a focal point of C with base $\vec{f}(t)$, then $\vec{p} = \vec{f}(t) + \frac{1}{\kappa(t)}\vec{N}(t)$. Thus,

$$\begin{aligned} \Lambda'_{\bar{p}}(t) &= 2(\bar{f}(t) - \bar{f}(t)) - \frac{1}{\kappa(t)} \bar{N}(t) \cdot \bar{f}'(t) \\ &= \frac{-2}{\kappa(t)} \bar{N}(t) \cdot \bar{f}'(t) = 0. \end{aligned}$$

Also

$$\begin{aligned} \Lambda''_{\bar{p}}(t) &= 2(\bar{f}(t) - \bar{f}(t)) - \frac{1}{\kappa(t)} \bar{N}(t) \cdot \bar{f}''(t) + 2\bar{f}'(t) \cdot \bar{f}'(t) \\ &= \frac{2}{\kappa(t)} \bar{N}'(t) \cdot \bar{f}'(t) + 2\bar{f}'(t) \cdot \bar{f}'(t) \\ &= -2\bar{f}'(t) \cdot \bar{f}'(t) + 2\bar{f}'(t) \cdot \bar{f}'(t) = 0. \end{aligned}$$

Conversely, the equation $\Lambda'_{\bar{p}}(t) = 0$ implies that $\bar{f}(t) - \bar{p} = m\bar{N}$. Now put $\bar{p} = \bar{f}(t) - m\bar{N}$ in the equation $\Lambda''_{\bar{p}}(t) = 0$ to get

$$m\bar{N} \cdot \bar{f}''(t) + \bar{f}'(t) \cdot \bar{f}'(t) = 0,$$

which is equivalent to

$$-m\bar{N}' \cdot \bar{f}'(t) + \bar{f}'(t) \cdot \bar{f}'(t) = 0$$

or

$$m\kappa(t)\bar{f}'(t) \cdot \bar{f}'(t) + \bar{f}'(t) \cdot \bar{f}'(t) = 0.$$

Thus, $m = -\frac{1}{\kappa(t)}$. So $\bar{p} = \bar{f}(t) + \frac{1}{\kappa(t)}\bar{N}(t)$, i.e. \bar{p} is a focal point of C with

base $\bar{f}(t)$. \diamond

Example 3.3. Let $\bar{f}(t) = (t, t^2)$. Then

$$\bar{f}'(t) = (1, 2t) \quad \text{and} \quad \bar{f}''(t) = (0, 2).$$

So $\|\bar{f}'(t)\| = \sqrt{1 + 4t^2}$ and $\|\bar{f}'(t) \times \bar{f}''(t)\| = 2.$

Thus,

$$\kappa(t) = \frac{2}{(\sqrt{1 + 4t^2})^3}.$$

By solving the equation

$$(0, 2) = \frac{4t}{\sqrt{1 + 4t^2}} \frac{(1, 2t)}{\sqrt{1 + 4t^2}} + \frac{2}{(\sqrt{1 + 4t^2})^3} (1 + 4t^2)\bar{N}(t)$$

for $\bar{N}(t)$, we get

$$\bar{N}(t) = \frac{(-2t, 1)}{\sqrt{1 + 4t^2}}.$$

Thus,

$$\begin{aligned} \vec{g}(t) &= (t, t^2) + \frac{(\sqrt{1+4t^2})^3}{2} \frac{(-2t, 1)}{\sqrt{1+4t^2}} \\ &= (-4t^3, \frac{1}{2} + 3t^2). \end{aligned}$$

Now $\Lambda_{\vec{p}}(t) = (t - a)^2 + (t^2 - b)^2$ with $\vec{p} = (a, b)$. So

$$\Lambda'_{\vec{p}}(t) = (2 - 4b)t + 4t^3 - 2a$$

and

$$\Lambda''_{\vec{p}}(t) = 2 - 4b + 12t^2.$$

Equating the last two equations to zero, we get

$$b = 1/2 + 3t^2 \quad \text{and} \quad a = -4t^3.$$

Example 3.4. Consider the curve defined by :

$$\vec{f}(t) = (9\cos t + \cos t \cos 3t + 3\sin t \sin 3t, 9\sin t + \sin t \cos 3t - 3\cos t \sin 3t)$$

Then

$$\vec{f}'(t) = (9 - 8\cos 3t)(-\sin t, \cos t)$$

and

$$\vec{f}''(t) = (9 - 8\cos 3t)(-\cos t, -\sin t) + 24\sin 3t(-\sin t, \cos t).$$

So

$$\|\vec{f}'(t)\| = 9 - 8\cos 3t \quad \text{and} \quad \|\vec{f}'(t) \times \vec{f}''(t)\| = (9 - 8\cos 3t)^2.$$

Thus,

$$\kappa(t) = \frac{1}{9 - 8\cos 3t}.$$

By solving the equation

$$\vec{f}''(t) = 24\sin 3t(-\sin t, \cos t) + \frac{1}{9 - 8\cos 3t} (9 - 8\cos 3t)^2 \vec{N}(t)$$

for $\vec{N}(t)$, we get $\vec{N}(t) = (-\cos t, -\sin t)$.

Thus,

$$g(t) = (9\cos t \cos 3t + 3\sin t \sin 3t, 9\sin t \cos 3t - 3\cos t \sin 3t)$$

Let $\vec{p} = (a, b)$. Then

$$\Lambda_{\vec{p}}(t) = (9\cos t + \cos t \cos 3t + 3\sin t \sin 3t - a)^2 + (9\sin t + \sin t \cos 3t - 3\cos t \sin 3t - b)^2$$

. So $\Lambda'_{\vec{p}}(t) = 2(9 - 8\cos t)(-3\sin 3t + a\sin t - b\cos t)$,

$$\Lambda_p''(t) = 2(9 - 8\cos 3t)(-9\cos 3t + a\cos t + b\sin t) + 48\sin 3t(-3\sin 3t + a\sin t - b\cos t)$$

Equating the last two equations to zero, we get

$$-3\sin 3t + a\sin t - b\cos t = 0$$

and

$$-9\cos 3t + a\cos t + b\sin t = 0.$$

Solving the last two equations together, we get

$$a = 9\cos t \cos 3t + 3\sin t \sin 3t \quad \text{and} \quad b = 9\sin t \cos 3t - 3\cos t \sin 3t .$$

Planar curves out of their curvatures

Let \vec{f} be a planar curve with parameter s , the arclength. Let $\phi(s)$ be the slope angle of the tangent line at $\vec{f}(s)$. Then the unit tangent of \vec{f} at s is defined by

$$\tau(s) = (\cos\phi(s), \sin\phi(s)).$$

Now $k(s) = \left\| \frac{d\tau}{ds} \right\| = \frac{d\phi}{ds}$. Thus,

$$\phi(s) = \int_0^s k(s) ds \tag{9}$$

Now if $\vec{f}(s) = (x(s), y(s))$, then

$$x(s) = \int_0^s \cos\phi(s) ds \tag{10}$$

and

$$y(s) = \int_0^s \sin\phi(s) ds \tag{11}$$

Example 4.1. Let $k(s) = \frac{1}{3}$ with $\vec{f}(0) = (0,0)$. Then by Equation (9),

$$\phi(s) = \int_0^s k(s) ds = \frac{1}{3}s .$$

By Equation (10), we have

$$x(s) = \int \cos \frac{1}{3}s ds = 3\sin \frac{1}{3}s + c_1 .$$

But $x(0) = 0$, and so $x(s) = 3\sin \frac{1}{3}s$. Also by Equation (11), we have

$$y(s) = \int \sin \frac{1}{3} s ds = -3 \cos \frac{1}{3} s + c_2 .$$

But $y(0) = 0$, and so $y(s) = -3 \cos \frac{1}{3} s + 3$.

So $\vec{f}(s) = (3 \sin \frac{1}{3} s, -3 \cos \frac{1}{3} s + 3)$, which is a circle of radius 3.

Example 4.2. Let $k(s) = \frac{1}{s^2 + 1}$ with $\vec{f}(0) = (0,0)$.Then by Equation (9),

$$\phi(s) = \int_0^s k(s) ds = \int_0^s \frac{1}{s^2 + 1} ds = \tan^{-1} s .$$

Now by Equation (10), we have

$$\begin{aligned} x(s) &= \int \cos(\tan^{-1} s) ds \\ &= \int \frac{1}{\sqrt{s^2 + 1}} ds = \sinh^{-1} s + c_1 . \end{aligned}$$

But $x(0) = 0$, and so $x(s) = \sinh^{-1} s$. Also by Equation (11), we have

$$\begin{aligned} y(s) &= \int \sin(\tan^{-1} s) ds . \\ &= \int \frac{s}{\sqrt{s^2 + 1}} ds = \sqrt{s^2 + 1} + c_2 . \end{aligned}$$

But $y(0) = 0$, and so $y(s) = \sqrt{s^2 + 1} - 1$.So $\vec{f}(s) = (\sinh^{-1} s, \sqrt{s^2 + 1} - 1)$.

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