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MODELING THE TEMPERATURE BEHAVIOR OF AN RLC CIRCUIT

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ABSTRACT

A problem that occurs in the use of electrical appliances is overheating. Electrical device components require reasonable working temperatures to prevent damage and increase efficiency. To gain an understanding of overheating we worked with an RLC circuit (a circuit consisting of a resistor, an inductor, and a capacitor) to represent a simplified model of an electrical component. The behavior of this circuit is similar to that of many electrical appliance components because as current flows through the resistor there is a rise in temperature due to the resistance to the electric current. Therefore, by using the RLC circuit, we can possibly get a better understanding of an electrical component's temperature behavior. We first investigated the circuit's differential equation to find the solution for the current. The derived current can be used in the power loss of the resistor, which is equal to the heat dissipated from the circuit resistance. To model the cooling of the system we added radiation, conduction, and convection terms to the differential equation. With each added cooling term the temperature in our system was seen to decrease significantly.

Keywords: RLC, circuit, temperature, cooling, overheating, electrical appliance, electrical appliance component, power loss, Euler method, trapezoid method.

INTRODUCTION

Understanding overheating in electrical appliances is of general interest in science due to the amount of damage that is caused because of it. As an electrical device operates it also develops heating. Heating is a natural byproduct of mechanical work, and it is related to the dissipated energy associated with the electrical current drawn by a device's electrical circuit. Overheating can cause device damage (see, for example, Carson Dunlop & Associates 2012) and is detrimental to many electric circuits. It is for this reason that ways of cooling electric circuits are incorporated in devices. Cooling can be incorporated in a device by means of heat transfer mechanisms, such as radiation, conduction, and convection. In this study, we studied cooling by investigating an RLC circuit theoretically. Overheating can be a major factor in shortening the life of a device if it is not appropriately regulated. Sometimes, it is possible to provide adequate cooling, in which case an electrical device may operate efficiently and avoid any permanent damage.

A direct impact of the modeling we carried out here is the understanding as to how temperature run-away can be avoided in an electrical device and prevent overheating. In this respect, since the main source of heating takes place in a circuit component through which current flows, we modeled a single electric device component as a general RLC series circuit, as shown in Figure 1, where V(t) is the time dependent voltage, R is the resistance, L is the inductance, and C is the capacitance.

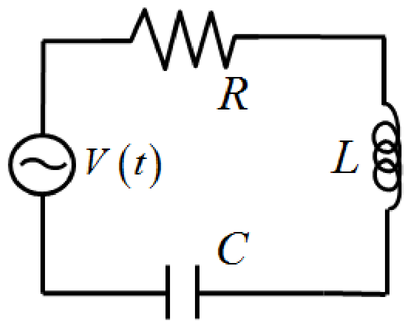

Figure 1. A typical RLC circuit

By understanding such component heating characteristics, we hoped to improve the circuit's design as well as its longevity, and, thereby, extend the life of a general electrical appliance. This circuit is viewed to have a behavior similar to an electrical device component because, due to the current flow, the circuit experiences electrical heat loss due to its resistance to current flow. To investigate how the RLC circuit cools off, we included cooling mechanisms in our study, such as radiation, conduction, and convection.

METHOD

The differential equation for an RLC circuit, not including any cooling mechanisms, is written, in the usual way, by the sum of the voltage across each component which is then set equal to the voltage source ($V(t) = V_o \sin(w_d t + \theta)$) , where the current (I) is the rate of change of the charge (q) over time (t). We have

$$L\frac{dI}{dt} + IR + \frac{q}{C} = V_o \sin(\omega_d t + \theta). \quad (1)$$

Since current is dq/dt, we can write this as a second order differential equation for q(t),

$$L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = V_o \sin(\omega_d t + \theta). \quad (2)$$

For later purposes, we are interested in solving Equation (2), analytically and numerically, to find the current behavior. The analytic solution is obtained as the sum of two parts, a particular solution (q_p) and a homogeneous solution (q_h), which will give us our full analytical solution (Hasbun 2009). The particular solution can be found by using Euler's identity for complex numbers,

$$e^{ix} = \cos(x) + i\sin(x). \tag{3}$$

Using Equation (3) we assume the particular solution for $q_p(t)$ in the form

$$q_p(t) = A\sin(\omega_d t + \delta_p) = \text{Im}\left(Ae^{i(\omega_d t+\delta_p)}\right). \tag{4}$$

We substitute our solution into Equation (2) and solve for the unknowns, to obtain

$$A = \frac{V_o}{L\sqrt{(2\gamma\omega_d)^2+(\omega_o^2-\omega_d^2)^2}}, \qquad \omega_d = 2\pi f, \qquad \gamma = \frac{R}{2L}, \qquad \omega_o = \frac{1}{\sqrt{LC}},$$

$$\delta_p = \theta - \tan^{-1}\left(\frac{2\gamma\omega_d}{\omega_o^2-\omega_d^2} \right).$$

We also need to find our homogeneous solution, so we set the right side of our differential equation from Equation (2) equal to zero,

$$L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = 0 \tag{5}$$

whose solution for q can be written in the form

$$q_h = e^{-\gamma t}\left(Ae^{i\omega t} + Be^{-i\omega t}\right) = e^{-\gamma t}B\cos(\omega t + \delta_h), \tag{6}$$

where

$$B = q_o, \quad \omega_o = \frac{1}{\sqrt{LC}}, \quad \delta_h = 0, \text{ and } \omega = \sqrt{\omega_o^2 + \gamma^2}.$$

So at t = 0, we have that $q_h(0) = q_o$. This solution assumes that the initial charge is q_o with an initial current of $\dot{q}_h(t = 0) = -\gamma q_o = I_{oh}$. The full analytical solution to Equation (1) is thus,

$$q_{full}(t) = q_p(t) + q_h(t) = A\sin(\omega_d t + \delta_p) + e^{-\gamma t}B\cos(\omega t + \delta_h). \tag{7}$$

For later and more complicated extensions of this problem, we wish to compare this analytic solution (Equation [7]) against a full numerical solution of Equation (2). To this end, we employ the Euler-Cromer method, which is effected, for a second order differential equation for q(t), as follows. We let the current and charge at time t_i be $I_i \equiv I(t_i)$ and $q_i \equiv q(t_i)$, respectively, to write

$$q_{i+1} = q_i + I_{i+1}\Delta t, \tag{8}$$

where Δt is some small time increment. The change in current over time isrepresented by β, which is a function of time, charge, and current; that is, we write

$$I_{i+1} = I_i + \beta_{i+1}\Delta t, \tag{9a}$$

β, referring back to Equation (2), is

$$\beta\left(t, q, \frac{dq}{dt}\right) = \frac{dI}{dt} = \frac{d^2q}{dt^2} = -\frac{q}{CL} - \frac{R}{L}\frac{dq}{dt} + \frac{V_0 \sin(\omega_d t + \theta)}{L}, \tag{9b}$$

or based on Equations (9), to effect the Euler-Cromer method, we have

$$\beta_{i+1} = -\frac{1}{CL} q_{i+1} - \frac{R}{L} I_{i+1} + \left(\frac{V_0 \sin(\omega_d t_{i+1} + \theta)}{L}\right). \tag{10}$$

As stated before, we compare the numerical solution against the analytical solution to check the validity of the numerical approach that will be employed later in more complicated cases. Thus, in Figure 2, a comparison between the analytic (Equation [7]) and the numeric solution (Equations [8–10]) is made by superimposing them on the same graph.

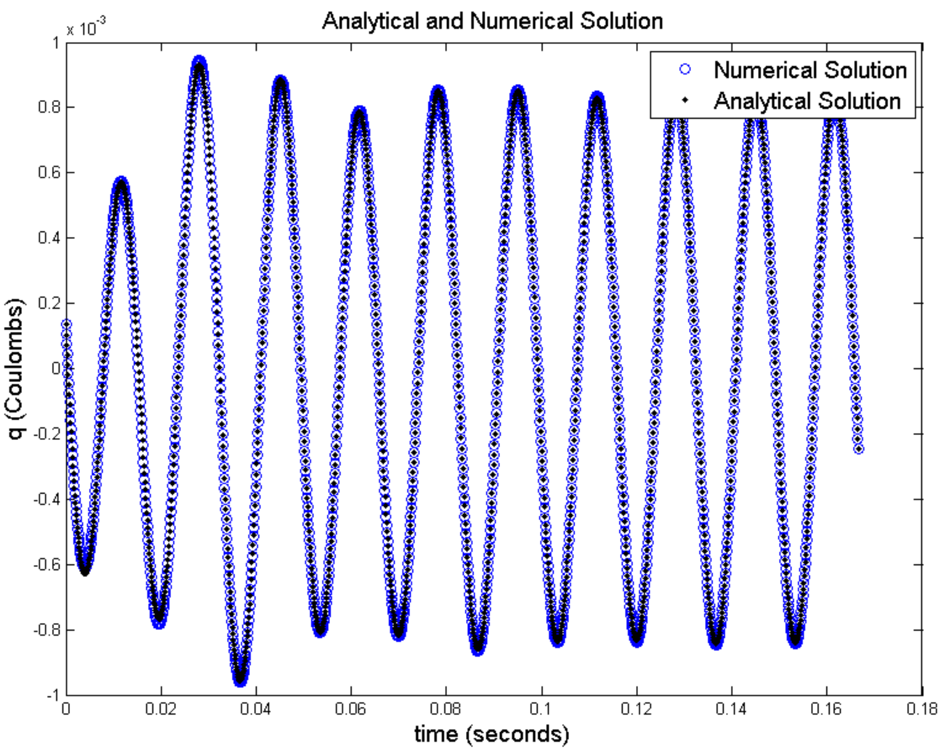

Figure 2. The charge, q, versus time for the analytical and the numerical solution of Equation (2). The plot of the analytical solution is used to check the accuracy of the numerical Euler-Cromer method. The calculation parameters used here are f = 60 Hz, ω = 2πf, R = 20 Ω, L = 0.3 H, C = 5e-5 F, Vo = 20 V.

It is seen that the numerical solution obtained using the Euler-Cromer method very closely encapsulates the analytic solution. This shows that the Euler-Cromer method is accurate enough to describe the RLC differential equation. In fact, a measure of the closeness between the two solutions is the measurement of the error

$$\text{Error}=\sqrt{\frac{1}{N}\sum_{i}^{N}(q_{full}(t_i)\text{-}q_i)^2}\ ,$$

which is a small number when the full analytic solution ($q_{full}(t_i)$) of Equation (7) and the numeric solution (q_i) of Equations (7–10) are very close to each other. Yet the error can be large when the two solutions deviate from each other drastically. In our case, the error has a value of about 6.6×10^{-6}, which is extremely good. In Figure (2) a total of over 1500 points were used in the calculation. Therefore, we employ the same numerical method to study a modified version of Equation (1) later below. For the benefit of the reader, the appendix shows the MATLAB code that reproduces Figure 2.

POWER LOSS

As mentioned earlier, we wished to model an electrical appliance component's behavior under an applied voltage. To this end, in a standard way (Halliday et al. 1997) to find the temperature behavior without cooling, we relate the power loss of the resistor to its heat gained,

$$mc_p\frac{dT}{dt}=(\text{power loss})=I^2R. \tag{11}$$

In this case we use our previously acquired full analytical solution from Equation (7) to substitute for the current, $I=\frac{dq_{full}}{dt}$, and write Equation (11) as

$$mc_p\frac{dT}{dt}=\left(\frac{dq_{full}}{dt}\right)^2R. \tag{12}$$

This equation can be written as

$$\int_{T_o}^{T}dT=T(t)-T_o=\frac{R}{mc_p}\int_0^t\left(\frac{dq_{full}}{dt}\right)^2dt.$$

This integral is cumbersome to solve analytically, but since we need to develop a numerical approach for more complicated situations later, we proceed with such an approach here as well. For this purpose, the numerical scheme adapted here is the trapezoidal method,

$$\int_{T_o}^{T}dT=T(t_n)-T_o\approx\frac{R\Delta t}{mc_p}\sum_{i=1}^{n}\left(\frac{dq_{full}}{dt}\right)_i^2, \tag{13}$$

where $\Delta t = \frac{T_f - T_0}{N-1}$, and $t_n = n\Delta t$ with n= 0,..., N. Here N is the maximum number of points (N~5000), and T_f is our final temperature value. The temperature of the resistor, $T(t_n)$,, is the temperature of our system at time t_n. Also T_o is the temperature of the environment, which is set at the room temperature of 298 K.

RADIATION

To improve on Equation (12) and describe our electrical appliance component more realistically, we include the resistor's blackbody properties as it radiates energy. The heat loss associated with radiation can be included through the Stefan-Boltzmann law (Halliday et al. 1997) or

$$\dot{Q} = e\sigma A(T^4 - T_o^4).$$

Adding this term to Equation (12) we now have

$$mc_p \frac{dT}{dt} = I^2R - e\sigma A(T^4 - T_o^4), \tag{14}$$

where,

- T = the temperature of the resistor circuit,
- T_o = the temperature of the environment,
- A = the area of the resistor,
- e = the emissivity coefficient,
- c_p = the specific heat of the resistor (carbon),
- m = the mass of the resistor, and
- σ = the Stefan-Boltzmann constant.

For our general purposes, we use a common carbon-composite resistor, with a specific heat (c_p) of 691 $\frac{J}{kg \cdot K}$ at 298.15 K (Halliday et al. 1997). To effect the numerical solution of Equation (14), we employ the previously mentioned Euler-Cromer Method, written as

$$T_{i+1} = T_i + \frac{1}{mc_p}[I_i^2R - e\sigma A(T_i^4 - T_o^4)]\Delta t, \tag{15}$$

where

$$\Delta t = \frac{t_f - t_i}{N-1}, \quad t_i = i\Delta t, \quad T_i = T(t_i),$$

and t_f is our final time of interest.

CONDUCTION

An even more realistic modeling of an electric appliance component should also include a term associated with heat loss due to contact with other devices, i.e. conductive heat loss. Here, we include this term in the form of a heat sink (aluminum). The heat sink is supposedly in thermal contact with the resistor to draw heat from it. However, it will also act as a blackbody and radiate energy. The conduction term is modeled by Fourier's law (Bergman et al. 2011),

$$\dot{Q} = \frac{kA(T-T_o)}{d},$$

where k is the conduction coefficient, d is the heat sink thickness, and A is the heat sink area. This equation is essentially an expression that says that heat flows from a region of higher temperature to a region of lower temperature. Our differential equation to include radiation and conduction thus becomes,

$$mc_p \frac{dT}{dt} = I^2R - e\sigma A_{RH}(T^4 - T_o^4) - \frac{kA_H(T-T_o)}{d}, \qquad (16)$$

where we have specified the fact that both the resistor and the heat sink radiate through their combined area ($A_{RH} = A_R + A_H$, with A_R and A_H the resistor and heat sink's surface areas, respectively) since heat is also conducted away through the heat sink. We still need to adjust the mass and specific heat term (mc_p); however, to reflect the system as a whole, we make the replacement

$$mc_p \rightarrow (mc_p)_{RH} = m_R c_{p(R)} + m_H c_{p(H)},$$

where
- m_R = the mass of the resistor,
- $c_{p(R)}$ = the specific heat of the resistor (carbon),
- m_H = the mass of the heat sink (aluminum), and
- $c_{p(H)}$ = the specific heat of heat sink (aluminum).

We use aluminum as the material for the heat sink, whose specific heat capacity is 900 $\frac{J}{kg \cdot K}$ at 298.15 K (Halliday et al. 1997). The conduction coefficient, k, is 235 $\frac{W}{m \cdot K}$ (Cutnell et al. 2004). The modified version of Equation (16) becomes

$$(mc_p)_{RH} \frac{dT}{dt} = [I^2R - e\sigma A_{RH}(T^4 - T_o^4) - \frac{kA_H(T-T_o)}{d}], \qquad (17)$$

which can be solved numerically by the Euler-Cromer method to write

$$T_{i+1} = T_i + \frac{1}{(mc_p)_{RH}}[I_i^2R - e\sigma A_{RH}(T_i^4 - T_o^4) - \frac{kA_H(T_i-T_o)}{d}]\Delta t\,, \qquad (18)$$

with,

$$\Delta t = \frac{t_f - t_i}{N-1}, \quad t_i = i\Delta t, \quad T_i = T(t_i),$$

where, as before, t_f is the final time, and N is the total number of steps used (N~5001).

CONVECTION

In addition to radiation and conduction for heat loss, an electrical appliance component can be further cooled through the use of a fan. Adding a fan will cause forced air over the surface of the system. In this case, as air passes over the surface of the component, it experiences heat loss due to convection. Here, the convection term can be written as

$$\dot{Q} = -hA_{RH}(T - T_A),$$

where,

T_A = the temperature of air forced over the surface,

T = the temperature of the system,

A_{RH} = the area of the entire system (the resistor component and the heat sink), and

h = the convection coefficient for forced air.

With the convection term included the full form of the differential equation is

$$(mc_p)_{RH}\frac{dT}{dt} = [I^2R - e\sigma A_{RH}(T^4 - T_o^4) - \frac{kA_H(T-T_o)}{d} - hA_{RH}(T - T_A)], \quad (19)$$

and the Euler-Cromer method is used to solve it. We write the time dependent temperature of the system as,

$$T_{i+1} = T_i + \frac{1}{(mc_p)_{RH}}[I_i^2R - e\sigma A_{RH}(T_i^4 - T_o^4) - \frac{kA_H(T_i-T_o)}{d} - hA_{RH}(T_i - T_A)]\Delta t, \quad (20)$$

with Δt, t_i, and T_i, as described before.

RESULTS

Figure 3 shows the obtained results as each cooling term is added. Initially the temperature of the resistor circuit and the temperature of the environment are both set to 298 K. For the case when the only term included is the power loss due to the resistor, the temperature of the system builds linearly at a constant rate, and this trend continues until the resistor would more than likely fail. This situation undoubtedly leads to overheating. This temperature behavior can be detrimental to any system, and it is the reason why the rise in temperature is arrested by transferring heat away from the resistor. The resistor radiates energy as well, thus losing some heat, and we can see this when the radiation is added. There, the temperature in the system rises temporarily but eventually plateaus as the circuit reaches equilibrium at about 298.3 K. Further cooling is achieved through conduction, so the temperature behavior with radiation and conduction terms included gives a similar trend as the plot with just radiation; however, the temperature does not rise as much and the system reaches equilibrium faster (at about 298.2 K). Further cooling is obtained when the convection term is included. Generally, the convection involves forced air through the circuit in which the air is cooler (room temperature) than operating temperature of the circuit. Adding such term to the system, along with the previous two terms, gives rise to a negative slope in the temperature, and the slope eventually plateaus when the system reaches equilibrium. Finally, and for completeness, we list here the parameters used in creating Figure 3: f = 60 Hz, ω = 2πf, R = 2Ω, L = 0.3H, C = 5e-5 F, Vo = 20 V, $m_R = 2x10^{-4}$ kg; carbon specific heat c_p = 691 J/kg*K, T_o = 298 K, σ = 5.67 x 10^{-8} J/m.s.K^4, e = 1; T_e = 298 K, $c_{p(R)}$ = 691 carbon, J/Kg.K, A_R = 3 x 10^{-2} m^2, c_p(Al) = 900 J/kg*K-Aluminum heat sink, A_H = 1.6e -5m^2, d = 1.327 x 10^{-2} m (heat sink), aluminum density = 2.7e-3 kg/m^3, conductivity coefficient k = 235 J/K.s.m, T_A = 297 K, $A_{RH} = A_R + A_H$, and convection coefficient h_A = 10*A_{RH}(J/m^2.K).

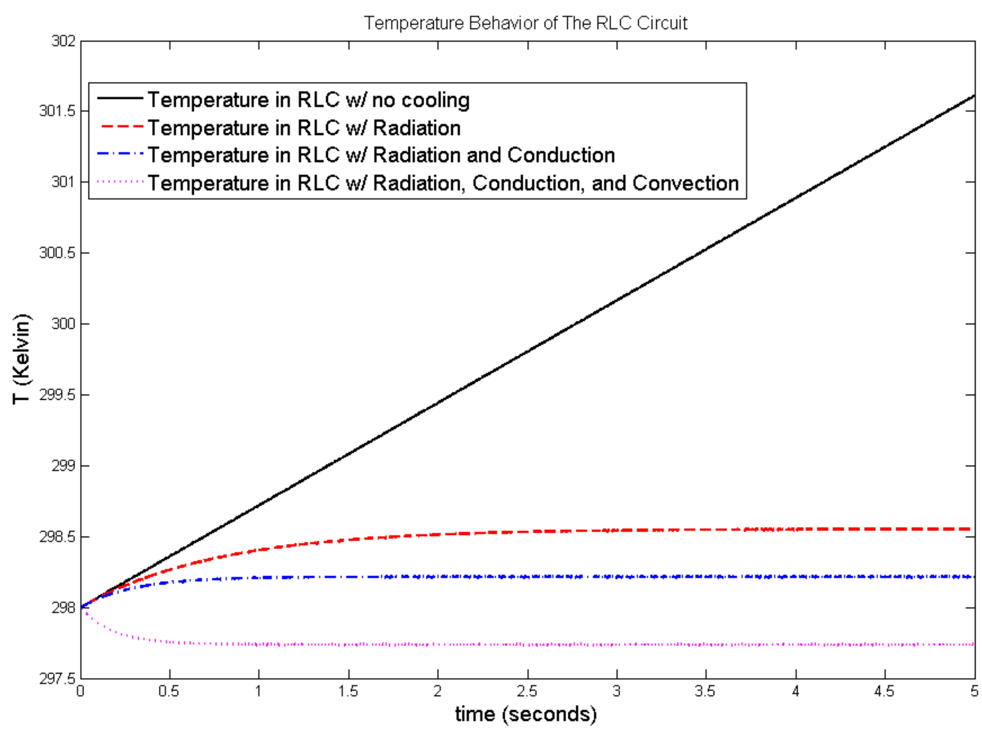

Figure 3. Temperature behaviors as time progresses for various forms of cooling.

DISCUSSION

In this paper, we have modeled an electrical appliance component as an RLC circuit. In the absence of any cooling the temperature of the circuit rises linearly at a constant rate due to current flow resistance. This increase in system temperature causes the temperature of the circuit to rise unabatedly. Adding the radiation, conduction, and convection heat transfer mechanisms provides enough cooling to lower the system's temperature. With each added term the system shows an improvement in temperature control. This study shows that the most effective way to ensure temperature regulation is to introduce multiple sources of cooling. The model presented here serves as a theoretical method to predict the temperature behavior of electrical appliance components. As has been shown, the model is amenable to various forms of cooling and the Euler-Cromer numerical method seems to be a useful approach to study the temperature behavior. In the appendix, we have provided a MATLAB script that reproduces the results of Figure 2, and which is amenable to modification as desired by the reader.

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Appendix: MATLAB Program

```
%Plots the Numerical solution, using Euler-Cromer Method, and Analytical
%solution (soln) for an RLC Circuit (by Kelly Ford & J.E Hasbun 4/19/2018)
clear;
t0=0;                                   %initial time
f=60;                                   %frequency (Hz)
w=2*pi*f;                               %angular frequency
R=20;                                   %resistance (ohms)
L=0.3;                                  %induction (Henry)
C=5e-5;                                 %capacitance (Farad)
V0=20;                                  %driving force amplitude
theta=0;                                %driving force initial phase angle
wo=sqrt(1/C/L);                         %SHO natural frequency
tau=2*pi/w;                             %force's period of rotation
tmax=10*tau;                            %maximum time in terms of tau
dt=0.0001;                              %step size
NPTS=(tmax-t0)/dt+1;                    %number of points
%============================================================================
%particular soln (qp) - Analytical
q0=4e-4;                                %initial charge-Analytic
I0=3.0;                                 %initial current(A)-Analytic
t=[t0:dt:tmax];                         %time array
gam=R/2/L;                              %gamma value
desc1=(2*gam*w).^2+(wo^2-w^2).^2;
A=V0/L/sqrt(desc1);                     %particular soln amplitude
den=wo^2-w^2;
```

```
if den==0, den=1.e-3; end
if(w < wo)
  ph=atan(2*gam*w/den);           %Phase difference between voltage and soln
else
  ph=pi+atan(2*gam*w/den);        %shift by pi needed if w > wo
end
delta=theta-ph;                       %phase shift for qp
qp=A*sin(w*t+delta);                  %the particular solution
%==========================================================================
%homogenous solution (qh) - Analytical
desc2=wo^2-gam^2;                     %must be positive
if desc2 <= 0;                        %ensure homogeneous problem conditions
    disp('gam needs to be smaller');
    break;
end
wu=sqrt(desc2);                       %underdamped homogeneous frequency
qh=q0*exp(-gam*t).*cos(wu*t);         %homogeneous solution
%==========================================================================
%Full Analytical solution (qA)
qA=qh+qp;                             %full Analytical solution
%==========================================================================
%Numerical Solution (qN) - Euler-Cromer Method
 q0n=q0+qp(1);                        %initial charge-Numerical
 I0n=(-q0*gam)+(A*w*cos(delta));      %that is [dqh/dt+dqp/dt](t=0)
 B0=(-q0/C-R*I0n+V0)/L;               %dI/dt (initial rate of current change)
 V=V0*sin(w*t+theta);                 %initial voltage-Numerical
 qN(1)=q0n;
 I(1)=I0n;
 B(1)=B0;
 for i=1:NPTS-1
     I(i+1)=I(i)+B(i)*dt;                        %new current
     qN(i+1)=qN(i)+I(i+1)*dt;                    %new charge
     B(i+1)=(-qN(i+1)/C-R*I(i+1)+V(i+1))/L;      %new rate of current change
 end;
plot(t,qN,'go',t,qA,'k-');
title('Analytical and Numerical Solution','FontSize',8)
ylabel('q (mA)');
xlabel('time (seconds)','FontSize',8);
legend('Numerical Solution','Analytical Solution')
```