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# LESLIE MATRICES AND WOMEN POPULATION IN THE UNITED STATES OF AMERICA

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## ABSTRACT

This research tests the accuracy of the Leslie matrix, which is a discrete age-structured method that uses fertility and survival rates, as a tool for predicting women population. Based on available data for the year 2000, we have constructed a Leslie matrix that predicts female population in the United States for every five years from the years 2000 to 2020. To test the accuracy of this method, we compare the aforementioned obtained projected data for the year 2010 with the actual data for women population in the United States obtained by the 2010 U.S. Census.

***Key words:*** Leslie matrices, population.

## INTRODUCTION

In "Essay on the Principle of Population," Thomas Malthus was the first one to offer a scientific explanation and variables influencing long-term population projections (1). Such pioneering work prompted passing of Census Act 1800, the first national census in Britain that has been collected every decade since. In any projection, dimensions of population are the first to be considered. The first complex dimension accounts for age and sex of the species observed. In demography, a key dimension of population dynamics is the distinction between men and women. To clarify, the population growth almost completely correlates to the fertility rates of females. Thus, just by observing female population and the growth within this subgroup, it is possible to predict population growth.

## METHODS AND DATA

In 1945, P. H. Leslie constructed a matrix as a means to project population of different species. The Leslie matrix requires three different categories of data divided into equal age groups: survival rates for each age group of females $s_i$, fertility rates for each age group of female population $F_i$, and the initial female population for each age group $p_i$. The matrix is of dimension $n \times n$, where $n$ is number of age groups. Data is distributed and calculated as follows: fertility data occupies the first row of the matrix while survival rates occupy the first sub-diagonal. To project population data, the t-th power of the Leslie matrix is multiplied by the initial population vector $x_0$, where $t$ is the number of years between the initial and final year, *i.e.*,

$$x_t = L^t x_0,$$

where $L$ is the matrix

$$L = \begin{pmatrix} F_0 & F_1 & F_2 & \cdots & F_{n-2} & F_{n-1} \\ s_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & s_{n-2} & 0 \end{pmatrix}$$

For all $1 \leq i \leq n$, let $l_i = s_0 s_1 \ldots s_{i-1}$ and $m_i = F_{i-1}/s_{i-1}$, *i.e.*, $l_i$ is the fraction of females surviving from birth to age $i$ and $m_i$ is the number of females born, on average, to a female of age $i$. If $\lambda$ is a complex eigenvalue of the matrix $L$, then $\lambda$ satisfies the discrete Euler-Lotka equation

$$1 = \sum_{i=1}^{n} \lambda^{-i} l_i m_i.$$

For further information, see (2), (3) and (4).

We used Census-based data for the year 2000 (see (5)) and applied the Leslie matrix model to project population for the next two consecutive 5-year intervals by using fertility and survival rates from (6). To compare the effectiveness of the Leslie matrix, we also used simulated data provided in (7), which uses previous projections and modified data to predict population. Moreover, we also analyze the spectrum (*i.e.*, the set of eigenvalues) of each of the Leslie matrices obtained by using both sets of data, with the goal to compare our results with the conclusions of the Perron-Frobenius Theorem (see *e.g.* (3)), which claims that any matrix with nonnegative real entries has a unique largest real eigenvalue. The Leslie matrices, their eigenvalues and the predictions for each set of data were obtained by using the software MATHEMATICA 8.

## RESULTS

Projections for the year 2010 obtained by using the Leslie matrix constructed from fertility and survival rates in Table I versus the data provided for female population in (5) is shown in Figure 1. To better compare actual versus projected values, Figure 2 shows how unreliable the data is by presenting the percent error per age group. In comparison, simulated data published in (7) gives better results, as shown in Table II. This data has been modified to fit the anticipated projections and thus result in better accuracy. In Figures 3 and 4, the complex eigenvalues of the Leslie Matrices corresponding to each set of data are shown. We can see that the Leslie matrices corresponding to the data in Table I and Table II, respectively, have a real eigenvalue that is larger than the others, which satisfies the conclusion of the Perron-Frobenius Theorem. However, the magnitudes of the eigenvalues corresponding to the Leslie matrix obtained from Table II are very similar as shown in Figure 4, whereas the eigenvalues of the Leslie matrix obtained from Table I are significatively different as shown in Figure 3.

## DISCUSSION

Based on our results, we believe Leslie matrix projections are not reliable for the age groups 0-10 and age groups over 85 years old, but are reliable for the rest of the age groups. This claim is supported by our percent error calculations, which show maximum error in projection to be 8%, whereas at extremes it exceeds well above 50%. Furthermore, by the behavior of eigenvalues, we can see that available data is not enough to accurately predict the population. In these age categories, we believe this is caused by inaccurate data in fertility rates in the 0-10 and over 60 age groups. While the rates may be very small, they are far from insignificant and result in such extreme deviations of projections from accurate data. Simulated data used in (7), give more accurate projections, which leads to conclusion that not all data is true and/or available for age groups under 10 years old. When data for these age groups is manipulated, the results are closer to the true value, which is confirmed the data in Table II. Moreover, we have concluded that in order to give better predictions using manipulated data, the available information has to be manipulated in such a way that the corresponding Leslie matrix has complex eigenvalues whose magnitudes are very close to the magnitude of the unique largest real eigenvalue.

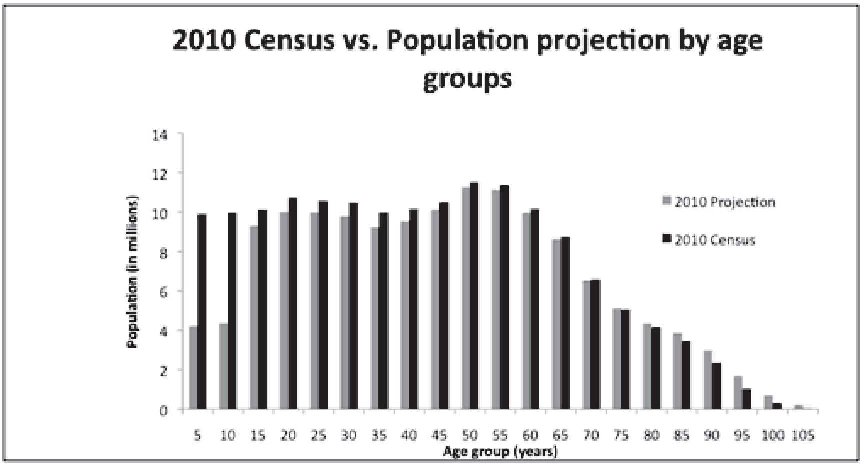


***Figure 1.*** Comparison between actual vs. projected values.

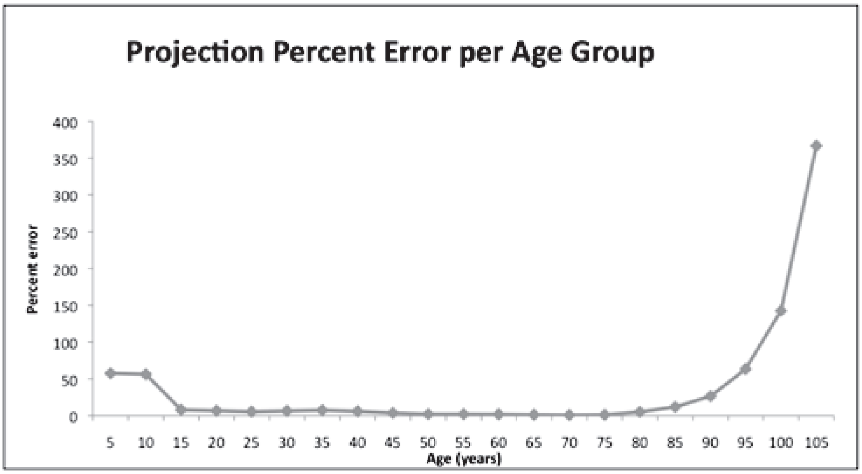


***Figure 2.*** Projection error between actual vs. projected values.

***Table I.*** Results using data from (6).

| | | | Actual Population | Estimated Population | | | | Actual Population |
|---|---|---|---|---|---|---|---|---|
| **Age Group** | **Fertility rates $F_i$** | **Survival rates $s_i$** | **2005** | **2005** | **2010** | **2015** | **2020** | **2010** |
| 5 | 0 | 0.993164 | 9.36507 | 4.39275 | 4.19878 | 4.11921 | 4.13708 | 9.88194 |
| 10 | 0.000001 | 0.998622 | 10.0262 | 9.30105 | 4.36273 | 4.17008 | 4.09105 | 9.95902 |
| 15 | 0.00005 | 0.99927 | 10.0079 | 10.0124 | 9.28823 | 4.35671 | 4.16433 | 10.0973 |
| 20 | 0.0009 | 0.998303 | 9.82889 | 10.0006 | 10.0051 | 9.28145 | 4.35353 | 10.7367 |
| 25 | 0.0485 | 0.996493 | 9.27619 | 9.81221 | 9.9836 | 9.98812 | 9.2657 | 10.5718 |
| 30 | 0.1123 | 0.997464 | 9.58258 | 9.24366 | 9.77779 | 9.94859 | 9.9531 | 10.4663 |
| 35 | 0.1214 | 0.996375 | 10.1886 | 9.55827 | 9.22021 | 9.753 | 9.92336 | 9.9656 |
| 40 | 0.0941 | 0.995024 | 11.388 | 10.1517 | 9.52363 | 9.18679 | 9.71764 | 10.1376 |
| 45 | 0.0404 | 0.993177 | 11.3128 | 11.3313 | 10.1012 | 9.47624 | 9.14108 | 10.497 |
| 50 | 0.0079 | 0.990092 | 10.2029 | 11.2356 | 11.254 | 10.0323 | 9.41158 | 11.4995 |
| 55 | 0.0005 | 0.984612 | 8.97782 | 10.1018 | 11.1243 | 11.1425 | 9.93285 | 11.3649 |
| 60 | 0.0005 | 0.975703 | 6.96051 | 8.83967 | 9.94636 | 10.9531 | 10.971 | 10.1412 |
| 65 | 0.0003 | 0.960272 | 5.66882 | 6.79139 | 8.6249 | 9.70469 | 10.6869 | 8.74042 |
| 70 | 0.0002 | 0.936197 | 5.13318 | 5.44361 | 6.52158 | 8.28225 | 9.31915 | 6.58272 |

***Table I.*** (Continued)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 75 | 0.0001 | 0.905493 | 4.95453 | 4.80567 | 5.09629 | 6.10548 | 7.75381 | 5.03419 |
| 80 | 0.00001 | 0.860207 | 4.37136 | 4.48629 | 4.3515 | 4.61466 | 5.52847 | 4.13541 |
| 85 | 0 | 0.789186 | 3.11047 | 3.76027 | 3.85914 | 3.74319 | 3.96956 | 3.44895 |
| 90 | 0 | 0.681708 | 1.91332 | 2.45474 | 2.96755 | 3.04558 | 2.95407 | 2.34659 |
| 95 | 0 | 0.537671 | 0.830206 | 1.30432 | 1.67342 | 2.02301 | 2.0762 | 1.02398 |
| 100 | 0 | 0.462403 | 0.228669 | 0.446378 | 0.701297 | 0.899747 | 1.08771 | 0.288981 |
| 105 | 0 | 0.236645 | 0.040397 | 0.105737 | 0.206406 | 0.324282 | 0.416046 | 0.044202 |

***Table II.*** Results using data from (7).

| | | | **Actual Population** | **Estimated Population** | | | | **Actual Population** |
|---|---|---|---|---|---|---|---|---|
| **Age Group** | **Fertility rates $F_i$** | **Survival rates $s_i$** | **2005** | **2005** | **2010** | **2015** | **2020** | **2010** |
| 5 | 0 | 0.993164 | 9.36507 | 9.15632 | 9.59477 | 9.61736 | 9.02188 | 9.88194 |
| 10 | 0 | 0.998622 | 10.0262 | 9.30105 | 9.09373 | 9.52918 | 9.55161 | 9.95902 |
| 15 | 0.0009 | 0.99927 | 10.0079 | 10.0124 | 9.28823 | 9.0812 | 9.51605 | 10.0973 |
| 20 | 0.0477 | 0.998303 | 9.82889 | 10.0006 | 10.0051 | 9.28145 | 9.07457 | 10.7367 |

***Table II.*** (Continued)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 25 | 0.1097 | 0.996493 | 9.27619 | 9.81221 | 9.9836 | 9.98812 | 9.2657 | 10.5718 |
| 30 | 0.1135 | 0.997464 | 9.58258 | 9.24366 | 9.77779 | 9.94859 | 9.9531 | 10.4663 |
| 35 | 0.0912 | 0.996375 | 10.1886 | 9.55827 | 9.22021 | 9.753 | 9.92336 | 9.9656 |
| 40 | 0.0397 | 0.995024 | 11.388 | 10.1517 | 9.52363 | 9.18679 | 9.71764 | 10.1376 |
| 45 | 0.008 | 0.993177 | 11.3128 | 11.3313 | 10.1012 | 9.47624 | 9.14108 | 10.497 |
| 50 | 0.5 | 0.990092 | 10.2029 | 11.2356 | 11.254 | 10.0323 | 9.41158 | 11.4995 |
| 55 | 0 | 0.984612 | 8.97782 | 10.1018 | 11.1243 | 11.1425 | 9.93285 | 11.3649 |
| 60 | 0 | 0.975703 | 6.96051 | 8.83967 | 9.94636 | 10.9531 | 10.971 | 10.1412 |
| 65 | 0 | 0.960272 | 5.66882 | 6.79139 | 8.6249 | 9.70469 | 10.6869 | 8.74042 |
| 70 | 0 | 0.936197 | 5.13318 | 5.44361 | 6.52158 | 8.28225 | 9.31915 | 6.58272 |
| 75 | 0 | 0.905493 | 4.95453 | 4.80567 | 5.09629 | 6.10548 | 7.75381 | 5.03419 |
| 80 | 0 | 0.860207 | 4.37136 | 4.48629 | 4.3515 | 4.61466 | 5.52847 | 4.13541 |
| 85 | 0 | 0.789186 | 3.11047 | 3.76027 | 3.85914 | 3.74319 | 3.96956 | 3.44895 |
| 90 | 0 | 0.681708 | 1.91332 | 2.45474 | 2.96755 | 3.04558 | 2.95407 | 2.34659 |
| 95 | 0 | 0.537671 | 0.830206 | 1.30432 | 1.67342 | 2.02301 | 2.0762 | 1.02398 |
| 100 | 0 | 0.462403 | 0.228669 | 0.446378 | 0.701297 | 0.899747 | 1.08771 | 0.288981 |
| 105 | 0 | 0.236645 | 0.040397 | 0.105737 | 0.206406 | 0.324282 | 0.416046 | 0.044202 |

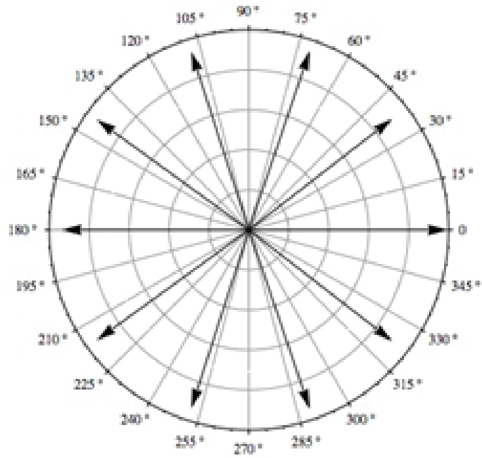


***Figure 3.*** Eigenvalues of the Leslie matrix corresponding to the fertility and survival rates from Table I.

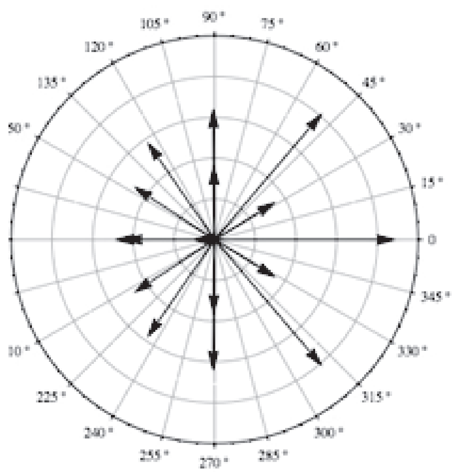


***Figure 4.*** Eigenvalues of the Leslie matrix corresponding to the fertility and survival rates from Table II.

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