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ABSTRACT

The principles of electrostatics are applied to dimensions both lower and higher than 3. Specifically, Laplace's equations are solved in n dimensions subject to hyper-spherical symmetry in order to obtain the electric potential and hence the electric field. The physical problems associated with these solutions in 3-dimensional space are identified. The radial dependences of the potential and electric field are scrutinized. The successively lower radial dependences of the multi-pole fields are obtained by differentiating those of the multi-poles of the immediately lower order. The same results are also obtained by considering the hyper-surfaces of hyper-spheres in n dimensions. This study reaffirms the principles of electrostatics and provides a glimpse of the notion of higher dimensions.

Key Words: Electrostatics, Dimensions, Laplace's Equation

INTRODUCTION

The physical space as we know is three-dimensional. This is evidenced by the fact that, in reality, one can draw at most three mutually perpendicular straight lines through a point. However, that has not stopped the human mind from envisaging the hyper-space (space having dimensions of greater than three.) In fact the geometry of hyper-space has already been worked out [*e.g.*, (1-3)]. The metaphysical aspects of this arcane subject were debated during the end of the nineteenth and beginning of the twentieth centuries (4).

In modern physics, time is regarded as the fourth dimension. But we are concerned with only spatial dimensions here. In the Kaluza-Klein and modern super-string theories, spatial dimensions higher than the third are assumed [*e.g.*, (5)]. However, such dimensions are said to be "compactified" into tiny circles smaller than the size of atoms (6). Multi-variate analysis and statistical mechanics are also conceptually based on higher dimensions. In this article, we venture into the higher dimensions via electrostatics. Specifically, Laplace's equations are written down and solved in n dimensions subject to hyper-spherical symmetry. The corresponding physical problems in three dimensions are identified. This will illustrate the relationship between electrostatics and dimensions of space and also provide us with a rare glimpse of the higher dimensions.

ELEMENTS OF ELECTROSTATICS

At the heart of electrostatics lies Coulomb's law. In accordance with this law, the electric field $\vec{E}$ due to a point charge q placed at the origin of a spherical coordinate system (r, θ, φ) $[0 \le r < \infty;\ 0 \le \theta \le \pi;\ 0 \le \varphi \le 2\pi]$ is expressed in Gaussian units by:

$$\vec{E} = \frac{q}{r^2}\hat{r} \tag{1}$$

The electric field is a conservative force field, and according to the potential theory, is obtained from the scalar potential Ψ:

$$\vec{E} = -\nabla\Psi \tag{2}$$

For the point charge,

$$\Psi = \frac{q}{r}, \quad r > 0 \tag{3}$$

In a region devoid of charges, the electric potential satisfies Laplace's equation:

$$\nabla^2\Psi = 0, \quad r > 0 \tag{4}$$

In order for the divergence theorem to hold to hold everywhere, including at the origin ($r = 0$), one must have

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r}), \tag{5}$$

where $\delta(\vec{r})$ is the 3-dimensional Dirac's delta function at the origin. The delta function gives a mathematical expression for the concept of the point charge at the origin. From Eqs. (3) and (4), one can now write for all space:

$$\nabla^2\Psi = -4\pi q\delta(\vec{r}) \tag{6}$$

which is equivalent to Poisson's equation in electrostatics [cf. (7)].

The point electric dipole consists of two equal and opposite charges $+q$ and $-q$ separated by an infinitesimal distance d apart. For a point dipole located at the origin, the potential is given by

$$\Psi = \frac{\vec{p}.\hat{r}}{r^2}, \tag{7}$$

where $\vec{p} = q\vec{d}$ is the dipole moment vector, $\vec{d}$ being the displacement vector from $+q$ to $-q$. The complete expression for the electric field as calculated from Eq. (2), by taking into account Eq. (5), is found in recent literature (8, 9):

$$\vec{E} = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} - \frac{4\pi}{3}\vec{p}\delta(\vec{r}) \tag{8}$$

This expression is valid everywhere in space, the last term on the right hand side giving the electric field at the origin. Eqs. (7) and (8) indicate that the angular dependences of the potential and electric field of the dipole depend upon the orientation of the dipole. However, away from the origin, the radial dependences are independent of the orientation, and are, therefore, characteristics of the electric dipole. The potential and the electric field of the dipole are inversely proportional to the square and cube of the radial distance, respectively. Thus, they fall off faster with the radial distance than those of the point charge. In this paper, we are concerned with radial dependences of the potentials and electric fields of point charge distributions outside the origin, only.

The electric quadrupole consists of two equal and opposite dipoles displaced from each other by another infinitesimal distance [*e.g.*, (10, 11)]. The linear and the square quadrupoles are two among many possible configurations of the quadrupole [*e.g.*, (11)]. However, the radial dependences of the potential and electric field of the quadrupole are independent of the construction of the quadrupole and are therefore characteristic features of the quadrupole. They fall off faster with distance than those of the dipole. By continuing this pattern, one can construct higher order multi-poles by displacing two equal and opposite multi-poles of the immediately lower order. Each time, the radial dependences of the potential and electric field fall off faster with the radial distance. The multi-poles are generally designated as 2^l-poles, where l is the order of the multi-pole: $l = 0$ represents the monopole (point charge); $l = 1$ gives the dipole; $l = 2$ the quadrupole; $l = 3$ the octupole; and so on.

LAPLACE'S EQUATION IN *n* DIMENSIONS

In this article, we investigate the radial part of Laplace's equations in dimensions both lower and higher than three and interpret what their solutions represent in 3-dimensional space. Consider an n-dimensional Euclidean space given by the linear coordinates $x_1, x_2, x_3, \ldots, x_n$ [$-\infty < x_1, x_2, x, \ldots, x_n < \infty$]. Define hyper-spherical coordinates ($r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \varphi$) [$0 \le r < \infty$; $0 \le \theta_1, \theta_2, \ldots, \theta_{n-2} \le \pi$; $0 \le \varphi \le 2\pi$] which are given by the relations [cf. (12)]:

$$\begin{gathered} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ \bullet \\ \bullet \\ \bullet \\ x_{n-1} = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\varphi \\ x_n = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\varphi \end{gathered} \tag{9}$$

Laplace's equation in n dimensions can be written in the form (13, 14)

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^2}\Lambda_n\Psi = 0 \tag{10}$$

where Λ_n is a second-order partial differential operator involving differentiations with respect to the angular coordinates only. For problems having "hyper-spherical symmetry", Ψ is independent of the angular coordinates, when we must have (13)

$$\frac{d}{dr}\left(r^{n-1}\frac{dR}{dr}\right) = 0 \tag{11}$$

By separation of variables, Eq. (11) integrates to

$$\Psi \propto \frac{1}{r^{n-2}}, \quad n \neq 2 \tag{12}$$

whence, by Eq. (2)

$$E \propto \frac{1}{r^{n-1}} \tag{13}$$

We can now examine the physical examples which the solutions to Laplace's equations represent in our familiar 3-dimensional space. First, for $n = 1$, Eqs. (12) and (13) give the radial dependences of the potential and electric field, respectively:

$$\Psi \propto r \qquad\qquad E = \text{const.} \tag{14}$$

Obviously, this represents the case of an infinite plane charge, where the electric field remains constant throughout space.

For $n = 2$, the integration of Eq. (11) leads to

$$\Psi \propto \ln r \qquad\qquad E \propto \frac{1}{r} \tag{15}$$

Clearly, this case belongs to an infinite line charge distribution in 3-dimensional space, where the electric field diminishes inversely as the radial distance from the line charge.

For the familiar 3-dimensional space ($n = 3$), Eqs. (12) and (13) give

$$\Psi \propto \frac{1}{r}, \quad r \neq 0 \qquad\qquad E \propto \frac{1}{r^2} \tag{16}$$

These represent the potential and electric field, respectively, of a point charge located at the origin.

For $n = 4$, Eqs. (12) and (13) give

$$\Psi \propto \frac{1}{r^{n-2}} \qquad\qquad E \propto \frac{1}{r^{n-1}} \tag{17}$$

These are identified as the potential and electric field, respectively, of the electric dipole in 3-dimensional space.

One can now extrapolate the results to the higher dimensions. For examples, $n = 5$ represents the quadrupole in 3-dimensional space; $n = 6$ the octupole; and so forth. The relationship between the order of the multi-pole and the dimensionality of space is thus established: $n = l + 3$.

Table I summarizes the solutions of the Laplace's equations in n dimensions and the physical examples they represent in 3-dimensional space. The radial dependences of the potential and electric field are shown for each case. They fall off faster with the radial distance in each successively higher dimension. Physical explanations are provided in the following sections.

Table I. Laplace's Equation in *n* Dimensions with Associated Examples

Dimensionality of Space n	Charge Distribution	Description	Radial Dependence of Potential	Radial Dependence of Field
1	Infinite Plane charge distribution	Monopole in 1-dimensional space	r	constant
2	Infinite Line charge distribution	Monopole in 2-dimensional space	$\log r$	$\frac{1}{r}$
3	Point Charge	Monopole in 3-dimensional space	$\frac{1}{r}$	$\frac{1}{r^2}$
4	Dipole	Equal and opposite charges separated by infinitesimal distance	$\frac{1}{r^2}$	$\frac{1}{r^3}$
5	Quadrupole	Equal and opposite dipoles separated by infinitesimal distance	$\frac{1}{r^3}$	$\frac{1}{r^4}$
6	Octupole	Equal and opposite quadrupoles separated by infinitesimal distance	$\frac{1}{r^4}$	$\frac{1}{r^5}$
7	16-pole	Equal and opposite octupoles separated by infinitesimal distance	$\frac{1}{r^5}$	$\frac{1}{r^6}$
⋮	⋮	⋮	⋮	⋮
n	2^{n-3}-pole	Equal and opposite 2^{n-4}-poles separated by infinitesimal distance	$\frac{1}{r^{n-2}}$	$\frac{1}{r^{n-1}}$
⋮	⋮	⋮	⋮	⋮

INTERPRETATION OF SOLUTIONS BY DIFFERENTIATION AND INTEGRATION

It may be noted that the dipole has only one configuration, since it is produced by a single displacement of two equal and opposite charges. But multi-poles of higher order can have an infinite number of configurations. The most interesting and insightful configurations are produced when all the successive displacements are equal and perpendicular to each of the previous ones. Fig. 1 shows the first 4 such displacements and the multi-poles resulting from them. The quadrupole is then a square, the octupole is a cube, and the 16-pole is a tesseract (4-dimensional cube) and so on. This demonstrates the inter-relation between electric multi-poles and the dimensionalities of space. It also allows one to open the doors to the hyper-space beyond the third dimension.

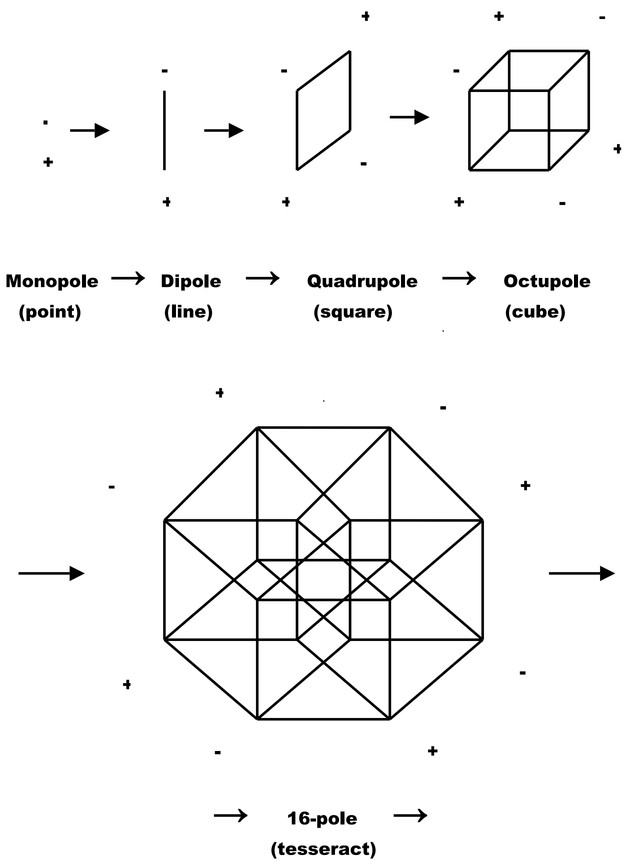

Figure 1. Multipole formation from point charge (monopole) to 16-pole by orthogonal displacements of equal and opposite multipoles of the immediately lower order. Each corner has opposite charge from adjacent corners. Arrows indicate the directions of the differentiation.

It has been shown in the literature that a displacement of two equal and opposite multi-poles of the lower order in the formation of a higher order multi-pole is equivalent to a differentiation when the potential of the resulting multi-pole is calculated (10, 11). Consider a displacement $d\vec{r} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi}$ in spherical polar coordinates. Then

$$\Psi_l = d\Psi_{l-1} = \frac{\partial\Psi_{l-1}}{\partial r}dr + \frac{\partial\Psi_{l-1}}{\partial\theta}d\theta + \frac{\partial\Psi_{l-1}}{\partial\varphi}d\varphi \qquad (18)$$

For a fixed , the radial dependence of Ψ_l is obtained by setting $d\theta$ and $d\varphi$ equal to zero:

$$\Psi_l \propto \frac{\partial\Psi_{l-1}}{\partial r} \qquad (19)$$

Thus, the radial dependence of the potential of a multi-pole is obtained by differentiating that of the multi-pole of the immediately lower order. The potentials and the resulting electric fields of all the multi-pole entries of Table I are easily accounted for by Eq. (19).

We can next investigate the remaining two entries of Table I. Going from n = 3 backwards, one is reminded that the inverse process of differentiation is taking the anti-derivative or indefinite integral. Thus the point charge integrates to the infinite line charge, and the latter in turn, integrates to the infinite plane charge distribution (Fig. 2). The relation $n = l + 3$ can be extended for $n < 3$ if we assign l values of -1 and -2 to the infinite line charge and the infinite plane charge distributions, respectively.

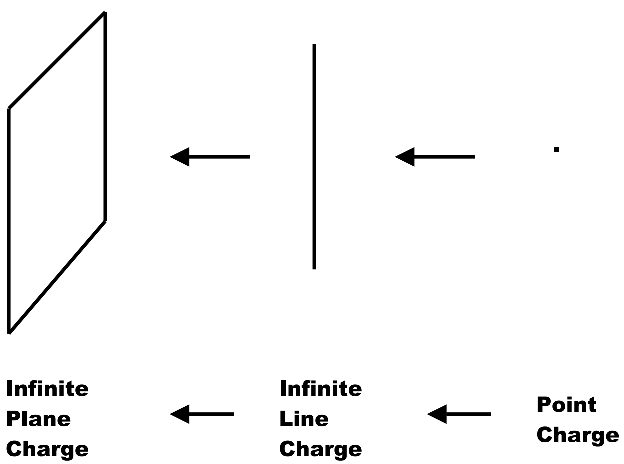

Figure 2. Infinite line charge and infinite plane charge distributions by integrations from a point charge. Arrows indicate the directions of the integration.

GEOMETRICAL INTERPRETATION OF SOLUTIONS

We next offer an alternative explanation of our results from a geometrical perspective. The solution of Laplace's equation in n dimensions with hyper-spherical symmetry belongs to that of the monopole in n dimensions, whose electric field is radial from the monopole at origin right up to infinity and is therefore independent of the angular coordinates. Examples given earlier are in compliance with this. Thus if we consider a hyper-sphere of radius r at origin, the electric field must be inversely proportional to the hyper-surface of the hyper-sphere. Expressions of the hyper-surface in n dimensions are readily found in the literature [e.g., (1, 2)]:

$$S_n = \frac{2\pi^{\frac{n}{2}} r^{n-1}}{\left(\frac{n}{2}-1\right)!} \qquad (20)$$

The electric field of the monopole in n dimensions is thus inversely proportional to r^{n-1}:

$$E_n \propto \frac{1}{S_n} \propto \frac{1}{r^{n-1}} \qquad (21)$$

The hyper-sphere corresponds to two points at a distance of r from the origin in 1-dimensional space, whereas it constitutes a circle of radius r in 2-dimensional plane, and of course, the familiar sphere in 3-dimensional space. S_n has values of 2 in 1 dimension, $2\pi r$ in 2 dimensions, $4\pi r^2$ in 3 dimensions; $2\pi^2 r^3$ in 4 dimensions; and so on. Thus, according to Eq. (21), the electric field is constant in 1 dimension; inversely proportional to r in 2 dimensions; inversely proportional to r^2 in 3 dimensions; inversely proportional to r^3 in 4 dimensions; and so on. In general

$$E_n \propto \frac{1}{r^{n-1}} = \frac{1}{r^{l+2}} \qquad (22)$$

The radial dependences of the electric fields are thus accounted for all the entries in Table I.

An interesting observation gleaned from this study is that a monopole in the 4-dimensional space corresponds to a dipole in the 3-dimensional space. This is only a special case of the general fact that a monopole in the n-dimensional hyper-space is equivalent to a dipole in the hyper-space of n-1 dimensions. This can be viewed as follows. The electric field of the monopole is directed radially outward from the origin up to infinity. As one dimension collapses, it diverts the electric field from infinity back to the origin, thus creating a dipolar field in the lower dimension. The same process produces a multi-pole of a higher order in the space of the lower dimension. This process is not unlike the compactification of the extra dimensions in the Kaluza-Klein and super-string theories.

CONCLUDING SUMMARY

The relationship between familiar charge distributions of electrostatics and dimensionality of space has been demonstrated. The electric fields due to these charge distributions correspond to the solution of Laplace's equations in the appropriate dimensions. From the point charge in three dimensions, successively higher-order multi-poles are generated by differentiating the potential with respect to the radial coordinate, or equivalently, displacing equal and opposite charge distributions in a higher dimension. Finally, all multi-poles in three dimensions are generated from a single monopole in a hyper-space of a higher dimension through successive collapse of the extra dimensions. Thus even though dimensions higher than 3 may not exist in reality, the physical principles applied to these higher dimensions do yield results which are real in our 3-dimensional universe.

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