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Introducing Abstract Mathematics through Digit Sums and Cyclic Patterns

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INTRODUCING ABSTRACT MATHEMATICS THROUGH DIGIT SUMS AND CYCLIC PATTERNS

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ABSTRACT

Using simple concepts that middle and high school students should be able to grasp, including "clock face arithmetic," the standard multiplication table, and adding the digits of a number together, more abstract concepts such as modular arithmetic and cyclic groups may be introduced at an early stage in the students' mathematical career. We find this approach to be organic and appealing to most students, encouraging them to think in different ways about familiar objects, and we encourage educators to test the concepts in their own classrooms.

Keywords: modular arithmetic, digit sum, cyclic patterns, middle school math education, secondary math education.

INTRODUCTION

Teaching an abstract algebra course in college to math majors or even to graduate students is often rather difficult. Students may get lost in unmotivated axioms, unfamiliar notations, and unconnected topics presented throughout the course. Of course, we as teachers have already internalized the motivation for the axioms; are familiar with the notation; and see the big picture connecting all of the topics, but it is difficult to convey all of this to the students. At least in a real analysis, linear algebra, or basic topology course, students' reasoning may be aided by pictures, but what pictures can be used in an abstract algebra course? *How can we motivate our students to learn abstract concepts that may be hard to visualize?*

We propose that the clarifying "images" of abstract algebra could be number patterns (Laughbaum 2014; Stump 2011). In this paper, we present many examples of patterns that could help introduce undergraduate students to the theory of cyclic groups in particular, but the authors believe that similar ideas may be replicated in some other abstract algebra topics, which may be a further research topic that the authors explore in subsequent work. Moreover we believe, in line with Kiernan (2004), that exposing a younger audience of primary or middle school students to "algebraic thinking" by way of patterns and functions in particular is a worthwhile goal in primary mathematics education.

This paper is organized loosely as an exposition of some methods for teaching an introductory lesson in the cyclic groups in a practical and computational way. What follows may be regarded as sample lecture notes. While the material may be used for teaching college students, the intended audience is actually high school or middle school teachers who may use this material for introducing abstract concepts as early as possible in the mathematics curriculum.

The Cyclic Groups and DIGITSUM Function

We first introduce the finite cyclic groups in an informal, nonaxiomatic way. If more rigor is needed, then the definitions found in any abstract algebra textbook, such as Dummit and Foote (2003), may be presented. For our purposes, the **finite cyclic group** of order n is a set G of n elements with a commutative operation called **addition**, denoted by +, and distinguished **identity** element, often denoted by e (but also denoted 0 in practice), which is **generated** by a single element g. In other words, $G = \{e, g, g + g = 2g, g + g + g = 3g, \dots\}$ such that $ng = e$. We often use the notation C_n to stand for a cyclic group of order n. The addition operation in a finite cyclic group is equivalent to **modular addition**, and so cyclic groups find use in various important fields of study, including cryptography and cryptanalysis (Bard 2009; Swenson 2008).

For example, the cyclic group C_5 has 5 elements, 0, 1, 2, 3, and 4, with the relation 1+1+1+1+1 = 0 (that is, 5 is equivalent to 0). In this group, we have 3 + 4 = 2, since when 3 + 4 = 7 is divided by 5, the remainder is equal to 2 (this is an example of modular addition). Moreover, each cyclic group C_n also admits a well-defined multiplication operation (which gives C_n the structure of a **ring**). For example, in C_5, we have $2 \times 3 = 1$, because the usual product of 2 and 3 is equal to 6, which has remainder 1 when divided by 5 (this is **modular multiplication**). Modular arithmetic can be used to reinforce number sense and give practice for division with remainders in primary and secondary students, helping students to master related Common Core Standards for Mathematics, NBT.B.6 and NS.B.2 (cf. NCTM 2012).

Cyclic means repetition or recurring; students may infer that the word *cyclic* might be related to the *cycle* in *bicycle*, and indeed the bicycle wheel is an excellent visual clue that cyclic groups must involve repetition of a pattern (as the wheel spins around, it returns to its starting position at some point). One of the best concrete examples of how cyclic groups work is a wall clock. Every number on a wall clock repeats after twelve hours, so counting by 1, we obtain the repeating pattern, $(1, 2, 3, \dots, 10, 11, 12, 1, 2, 3, \dots)$. Note, by relabeling 12 = 0, we have a model for C_{12}, the cyclic group of 12 elements). But interesting patterns are found by counting by different amounts or using different starting points. For example, starting at 3, and increasing by 3, the student obtains $(3, 6, 9, 12, 3, 6, 9, 12, 3, \dots)$. This could plant the seeds for **groups** and **subgroups** in the student's mind. More practically, these concepts may help to address Common Core Standard OA.B.3.

As another "real world" example, the teacher tells the student, "A doctor asked you to take four pills of a medication daily, at 6 hour intervals. If the first pill is taken at 8:00 am, then find the times for the other pills to be taken." Of course, we get a sequence of times, 8:00 am, 2:00 pm, 8:00 pm, which extends to the next day starting at 2:00 am, and the pattern continues thereafter. This simple pattern may be recalled when students are first exposed to **cosets** and **quotient groups** much later in their academic career, as the set {2, 8} is one of six cosets in the quotient group $C_{12}/\{0, 6\}$.

In what follows, the **digit sum** of a positive integer is found by adding the digits of the number, then adding the digits of the result until we obtain a single digit number. For example, the digit sum of 7965 is found by adding: 7 + 9 + 6 + 5 = 27, and then 2 + 7 = 9. This rule defines a *function*, which we call DIGITSUM, from the set of positive integers to the set of nonzero digits, {1, 2, 3, … , 8, 9}. For example, DIGITSUM(7965) = 9. This is an easy way to introduce or reinforce the idea of a function as a rule that associates output to any given input (cf. Common Core Standards IF.A).

Mathematicians and students throughout the ages have been fascinated by digit sums. An interesting nontrivial use of digit sums was found by Tartaglia (1506-1559) in the claim (later rediscovered and proven by various mathematicians) that DIGITSUM(p) = 1 for all perfect numbers p, except for 6 (Dickson 1999). We note that DIGITSUM is essentially the same thing as reduction modulo 9, however we find that students are quicker at understanding adding digits repeatedly than finding remainders after long division.

The domain of DIGITSUM is the set of all positive integers (or **natural** numbers). If we write an array of the first 150 natural numbers as shown in Table I, then it is a good exercise to observe the patterns as one applies DIGITSUM to each number (see Table II). Each column displays a **cyclic permutation** of the numbers (1, 2, 3, 4, 5, 6, 7, 8, 9). The range $R = \{1, 2, 3, \ldots, 8, 9\}$ of DIGITSUM may be identified with the cyclic group C_9. It is not difficult to prove that DIGITSUM respects the addition operation in the sense that DIGITSUM($a + b$) = DIGITSUM(DIGITSUM(a) + DIGITSUM(b)). However, the C_9 group structure is only apparent if one relabels "9" as "0." In fact, simply identifying the element 9 as the identity of the group suffices, leading to equations such as 5 + 9 = 5, and 9 + 9 = 9, which may be strange to students at first. Regardless of how we label the identity, the range of DIGITSUM is isomorphic to the cyclic group C_9. It is a useful exercise for students to write out the entire 9×9 addition table for the group. We note in passing that DIGITSUM is a monoid homomorphism from the set of positive integers to C_9, however we cannot call DIGITSUM a *group* homomorphism because the domain set is not a group.

Table I. An array of natural numbers from 1 to 150

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150

Figure 1. A clock face illustrating modular arithmetic with the numbers 1–9. Image courtesy of Pixabay.com.

Let us now use DIGITSUM to explore patterns in the table of numbers from 1 to 150. We organize the domain and range into tables of size 10 by 15. The cyclical pattern is more evident in the digit sums of the numbers in columns rather than in rows, but of course the pattern exists in both. For example, in the sixth column, we find the cyclic pattern, (6, 7, 8, 9, 1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2). Teachers could show this to a student on a wall clock that ranges from 1 to 9 instead of 1 to 12. On the "nine-hour" clock, numbers complete one cycle in nine hours instead of twelve hours, and then start over, as suggested by Figure 1.

Table II. DIGITSUM applied to the numbers in Table I

1	2	3	4	5	6	7	8	9	1
2	3	4	5	6	7	8	9	1	2
3	4	5	6	7	8	9	1	2	3
4	5	6	7	8	9	1	2	3	4
5	6	7	8	9	1	2	3	4	5
6	7	8	9	1	2	3	4	5	6
7	8	9	1	2	3	4	5	6	7
8	9	1	2	3	4	5	6	7	8
9	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	1
2	3	4	5	6	7	8	9	1	2
3	4	5	6	7	8	9	1	2	3
4	5	6	7	8	9	1	2	3	4
5	6	7	8	9	1	2	3	4	5
6	7	8	9	1	2	3	4	5	6

Arithmetic Sequences

Consider an **arithmetic sequence**, $(a, a + d, a + 2d, a + 3d, \dots)$, in which a and d are whole numbers. It is well known (and an interesting exercise for middle school students) to show that when each number in the sequence is divided by a then the sequence of remainders forms a cyclical pattern. What we are really doing is reducing each number modulo a. For example, the sequence with $a = 5, d = 7$, namely (5, 12, 19, 26, 33, 40, 47, 54, 61, 68...), when reduced modulo 5 results in: (0, 2, 4, 1, 3; 0, 2, 4, 1, 3...). When the same sequence is reduced modulo 6, we obtain: (5, 0, 1, 2, 3, 4; 5, 0, 1, 2...). We observe that for the first sequence, the cyclical pattern of the remainders is 0, 2, 4, 1, 3, and for the second sequence, the cyclical pattern of the remainders is 5, 0, 1, 2, 3,

4. Students may observe that the same initial sequence resulted in patterns that repeated after different numbers of terms (5 in the first case, and 6 in the second). Next, we might ask students to consider the following arithmetic sequence ($a = 5, d = 3$): (5, 8, 11, 14, 17, 20, 23, 26, 29, 32...). When reduced modulo 6, this sequence results in the following: (5, 2; 5, 2; 5, 2; 5, 2; 5, 2; 5, 2...), a cyclical pattern that repeats blocks of 2 as opposed to 6, even though we reduced modulo 6. Ask students why the repeated blocks might be different in size, and lead them toward examining the **GCD** (greatest common divisor) of the modulus and common difference of the sequence (cf. Common Core Standard NS.B.4).

As a follow-up exercise, consider the effect of reducing each column of Table I by its first element, that is, reduce column 1 modulo 1, column 2 modulo 2, column 3 modulo 3, etc., as shown in Table III. By doing this, students practice division and discover cyclic patterns. Remind the students that the only possible remainders under division by n are

Table III. Reduction of each column of Table I modulo its first element

Column	1	2	3	4	5	6	7	8	9	10
	0	0	0	0	0	0	0	0	0	0
	0	0	1	2	0	4	3	2	1	0
	0	0	2	0	0	2	6	4	2	0
	0	0	0	2	0	0	2	6	3	0
	0	0	1	0	0	4	5	0	4	0
	0	0	2	2	0	2	1	2	5	0
	0	0	0	0	0	0	4	4	6	0
	0	0	1	2	0	4	0	6	7	0
	0	0	2	0	0	2	3	0	8	0
	0	0	0	2	0	0	6	2	0	0
	0	0	1	0	0	4	2	4	1	0
	0	0	2	2	0	2	5	6	2	0
	0	0	0	0	0	0	1	0	3	0
	0	0	1	2	0	4	4	2	4	0
	0	0	2	0	0	2	0	4	5	0

the whole numbers less than n, however the columns headed by $a = 2, 4, 5, 6, 8$, and 10 do not contain all possible remainders. Students may reflect (and teachers will reinforce) that each of these numbers have nontrivial GCD with 10. Column 1 is fairly uninteresting because there is only one possible remainder: 0. Columns 2, 5, and 10 display only 0 because these three numbers are divisors of 10 (indeed, so is the number 1). Only columns 3, 7, and 9 display all possible remainders in cyclic sequence: (0, 1, 2) for $a = 3$; (0, 3, 6, 2, 5, 1, 4) for $a = 7$; and (0, 1, 2, 3, 4, 5, 6, 7, 8) for $a = 9$. More advanced students may discover that the pattern of numbers in each column is directly related to 10 mod a. For example, 10 mod 7 = 3, so this is why column 7 of Table III shows increase by 3 (modulo 7) from one term to the next.

DIGITSUM and the Multiplication Table

We now consider the multiplication table, up to 15 as shown in Table IV. It is obvious that the rows and columns of the table represent arithmetic sequences with the common difference d in each row or column being the same as the first term, that is, $d = a$. There are many wonderful patterns in the table that can spark the interest of younger students (see, e.g., Charlesworth 2012; Frye et al. 2013), but our purpose is to connect some of these patterns to more abstract ideas, including the cyclic groups. For example, students can see that multiplication (×) is a **binary operation**, taking two numbers, say 7 and 4, and producing a single output, 28. Moreover, the operation is **commutative**: $4 \times 7 = 7 \times 4 = 28$.

Table IV. The 15 x 15 multiplication table

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225

We point out that the digit sums of numbers in any column or row also form cyclical patterns. For example, the digit sums of the numbers in column 3 are (3, 6, 9; 3, 6, 9; …) – in fact columns 6, 12, and 15 also exhibit cyclic patterns of order 3 after applying DIGITSUM. All numbers that are not multiples of 3 show digit sum patterns containing all nine numbers 1, 2, … 9 in some order. This is easily checked by hand and makes for an engaging student activity. It is also fun to see students work on the nines column, as they quickly realize all of the digit sums will be 9; this is a wonderful opportunity to remind them that this gives a test for divisibility by 9.

Recall, if $(a_1, a_2, a_3, a_4, \ldots)$ is any sequence, then the sequence of **first differences** is: $(a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots)$. By examining first differences of certain diagonals of the multiplication table, another interesting pattern emerges. For example, starting at $1 \times 14 = 14$ and moving diagonally up and right in Table IV, we find the sequence (14, 26, 36, 44, 50, 54, 56, 56, 54, 50, 44, 36, 26, 14), whose first differences are: (12, 10, 8, 6, 4, 2, 0, −2, −4, −6, −8, −10, −12); the latter is an arithmetic sequence starting at $a = 12$ with common difference $d = -2$. Every up-right diagonal exhibits a sequence of first differences which is arithmetic with the same common difference. More advanced middle school or high school students should verify this claim, the proof of which is presented below:

Consider any three consecutive entries in an up-right diagonal on the multiplication table, say nk, $(n+1)(k-1)$, $(n+2)(k-2)$. The first differences are as follows: $(n+1)(k-1) - nk = k - n - 1$ and $(n+2)(k-2) - (n+1)(k-1) = k - n - 3$. Note that the difference between these two is as follows: $(k-n-3) - (k-n-1) = -2$. Thus, the sequence of first differences is arithmetic with common difference $d = -2$, regardless of which diagonal is chosen.

DIGITSUM and Sequences of Powers

After considering the multiplication tables, we look for cyclical patterns in the powers of numbers, a^n. There are patterns, however these do not become apparent until after we apply DIGITSUM. Once again except for multiples of three: 3, 6, 9, etc., the digit sums of powers of other numbers do exhibit cyclical patterns as shown in Table V. Students may observe that the patterns for 4 and 13 are identical. This is because $13 = 4 + 9$, and 9 acts as an identity under DIGITSUM, but it is worthwhile asking students why they believe the two patterns match.

Table V. Powers of whole numbers, a^n, and their digit sums

n	*a* = 2		*a* = 4		*a* = 5		*a* = 7		*a* = 8		*a* = 13	
1	2	2	4	4	5	5	7	7	8	8	13	4
2	4	4	16	7	25	7	49	4	64	1	169	7
3	8	8	64	1	125	8	343	1	512	8	2197	1
4	16	7	256	4	625	4	2401	7	4096	1	28561	4
5	32	5	1024	7	3125	2	…	4	…	8	…	7
6	64	1	4096	1	15625	1		1		1		1
7	128	2	…	4	…	5		7		8		4
8	256	4		7		7		4		1		7
9	512	8		1		8		1		8		1
10	1024	7		4		4		7		1		4

Finally, students may be shown a cyclic group in another form: with operation given by multiplication (and reduced via DIGITSUM). Each column of Table V exhibits a cyclic group structure: $(C_6, \times)$ for $a = 2$; $(C_3, \times)$ for $a = 5, 7, 13$; and $(C_2, \times)$ for $a = 8$. Mathematically, we are seeing the fact that DIGITSUM respects multiplication, a fact that may be proven in a college-level course. For completeness, a sketch of the proof follows.

Let a, b be whole numbers and write each in expanded form, $a = a_0 + a_1 10 + a_2 10^2 + \cdots + a_n 10^n$, and $b = b_0 + b_1 10 + b_2 10^2 + \cdots + b_m 10^m$. As a first step in finding the digit sum of either number, we sum the coefficients to obtain $a' = a_0 + a_1 + a_2 + \cdots + a_n$, and $b' = b_0 + b_1 + b_2 + \cdots + b_m$, respectively. If the result of either sum is greater than 9, then further reduction is necessary, however we first consider the case when both a', b' are less than 10, so that DIGITSUM(a) = a', and DIGITSUM(b) = b'. Elementary algebra then implies that DIGITSUM(a)×DIGITSUM(b) is equal to the sum of all terms of the form $a_i b_j$ for $1 \leq i \leq n, 1 \leq j \leq m$. Now consider DIGITSUM($ab$). Since $ab = \sum_{k\geq 0}^{n+m}(a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0)\, 10^k$, we find that the first reduction step to compute the digit sum of ab is again equal to the sum of all terms of the same form. By induction on the number of reduction steps required to find DIGITSUM, the general result follows.

CONCLUSION

We strongly believe that experimenting with the patterns in numbers with the help of DIGITSUM as outlined in this paper could help young students discover important mathematical concepts in a delightful and recreational way, while providing alternative, engaging methods for teachers to address certain Common Core standards. Moreover, early exposure to modular arithmetic may improve students' number sense and pave the way for eventual understanding of higher mathematics such as group theory.

Unfortunately, this paper is incomplete in the sense that we have not had the opportunity to test the methods described here in the classroom. The reason is that we, being a small university, do not have the desired resources to test the concepts in our paper. We strongly recommend the reader to test the concepts in this paper against the more conventional methods.

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