k-irreducible triangulations of 2-manifolds

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Abstract

This thesis deals with k-irreducible triangulations of closed, compact 2-manifolds without boundary. A triangulation is k-irreducible, if all its closed cycles of length less than k are nullhomotopic and no edge can be contracted without losing this property. k-irreducibility is a generalization of the well-known concept of irreducibility, and can be regarded as a measure of how closely the triangulation approximates a smooth version of the underlying surface.

Research follows three main questions: What are lower and upper bounds for the minimum and maximum size of a k-irreducible triangulation? What are the smallest and biggest explicitly constructible examples? Can one achieve complete classifications for specific 2-manifolds, and fixed k?

We address all three of these questions. In particular, we

- present the state of the art in k-irreducible triangulations in detail,
- prove new properties and equivalent definitions of k-irreducible triangulations, and use that to extend the concept to the 2-sphere,
- extend the known lower size bound of $f_0 \geq \frac{k^2}{2}$ for a minimal k-irreducible triangulation from orientable to arbitrary 2-manifolds,
- prove a new lower size bound of $f_0 \notin o(g^{\frac{2}{3}})$ for at least 5-irreducible triangulations,
- construct the first infinite series of 5-irreducible triangulations of polynomial growth in the Euler genus of the underlying 2-manifold, with a rational exponent smaller than 1 ($f_0 \in O(g^{\frac{8}{9}})$), contradicting the conjectured minimum of $f_0 \in O(\frac{g}{\log g})$,
- present algorithms achieving a complete classification of 4-irreducible tori and Klein bottles, of 5-irreducible projective planes, and of the smallest 4-irreducible N_3, 5-irreducible tori, and 6-irreducible projective planes,
- present a heuristic algorithm that produced the smallest known up to 8-irreducible triangulations of Euler genus up to 20, and
- construct explicit infinite series of k-irreducible triangulations of fixed 2-manifolds, containing what we conjecture to be the minimal ($f_0 = \lceil \frac{3}{4}k^2 \rceil$) and maximal ($f_0 = 2k^2 - 4k + 4$) k-irreducible triangulations of the torus, and the maximal k-irreducible triangulations of the projective plane ($f_0 = k^2 - k + 1$). We also conjecture to have found the minimal growth rate of k-irreducible triangulations of the projective plane ($f_0 = \frac{2}{3}k^2 + O(k)$) and of the Klein bottle ($f_0 = \frac{7}{8}k^2 + O(k)$). The largest growth rate of arbitrary k-irreducible 2-manifolds we found is $f_0 = \frac{9}{8}gk^2 + O(g)$ in the orientable, and $f_0 = h\frac{3k^2-2k}{4} + O(h)$ in the non-orientable case.

Contents

0. Introduction

Triangulations are fascinating mathematical objects at the verge of different disciplines. The way we use them, they consist of a set of triangles and some information on how to put these triangles together to form a 2-dimensional surface. An algebraic topologist might call them pure simplicial 2-complexes, while a graph theoretician might regard them as simple, undirected graphs, embedded in a particular way into 2-dimensional surfaces. Applications range from land surveying to finite element methods.

The first theoretical interest in triangulations was motived graph theoretically, in the 19^{th} century. The question was: For a fixed surface, what is the maximum number of colours needed to colour any graph embedded into it, such that no two adjacent vertices have the same colour? Any embedded graph can be completed to a triangulation by adding diagonals into its non-triangular faces, which does not simplify the problem. Thus, one could also ask: What is the maximum number of colours needed to colour an arbitrary triangulation of a fixed surface? Heawood [8] gave an upper bound for that number in 1890, and the arguments in his proof also showed that this number is a lower bound for the number of vertices in a triangulation of the 2-manifold. This bound is tight for all but three 2-manifolds, though it took another 80 years and several papers to actually construct such minimal triangulations. The last cases were solved in 1968 by Ringel and Youngs [18]. For a nice, comprehensive summary of the whole proof and its development, see [17].

Some time earlier, in 1934, Steinitz and Rademacher [20] had shown that all infinitely many triangulations of the 2-sphere can be generated from the tetrahedron by a simple local operation called vertex split. The inverse operation, called an edge contraction, iteratively reduces any arbitrary triangulation of the 2-sphere to the tetrahedron, as that is the only such triangulation that an edge contraction cannot be applied to: The tetrahedron is irreducible. That raises the questions: What are the irreducible triangulations of other 2-manifolds? By definition, they form a generating set as well, but is it also finite?

These questions were, apparently, not posed immediately. With the above breakthrough of Ringel et al., though, interest was sparked, and a lot of research was done. Following the example of Steinitz and Rademacher, complete classifications of all irreducible triangulations with Euler genus up to 4 were created (by hand, in [2], [12], [13], [23], and using heavy computations in [21]). In 1989, Barnette and Edelson [3] gave the first upper bound for the maximal size of an irreducible triangulation of a fixed 2-manifold – and with it the proof that the class of these triangulations is finite for any 2-manifold. The best bound to date is from 2010, given by Joret and Wood [11]. The best counterpart, explicit series of large irreducible triangulations, was given by Sulanke [22] in 2006. Surprisingly, the exact size of maximal irreducible triangulations remains unknown for all 2-manifolds without a complete classification (that is, of Euler genus greater than 4), though only a factor of $\frac{26}{11}$ separates the largest explicit series from the upper bound.

To generalize irreducibility, an additional property is imposed on the triangulations: A *k-irreducible* triangulation has edge width at least k, a property we call *ew-k*, and is irreducible with respect to that. In other words: Any essential (non-nullhomotopic) cycle in the triangulation has length at least k, and any edge contraction either produces a shorter essential cycle or destroys the triangulation. As any cycle in a triangulation has length at least 3, any 3-irreducible triangulation is also irreducible and vice versa. By construction, the k-irreducible triangulations also form a generating set for all ew-k triangulations. The parameter k can be interpreted as a measure of average smoothness, on all 2-manifolds but the sphere. The greater the k, the better a k-irreducible triangulation approximates a smooth version of its underlying 2-manifold. On the sphere, there are no essential cycles. Thus, the only k-irreducible triangulation there, for any k, is the tetrahedron, an odd and unintuitive exception.

Different motivations lead to the study of k-irreducible triangulations: Brehm (personal communication, 2013) showed that any k-irreducible projective plane corresponds to *universal triangulations* of a $2k$-gon, that is, triangulations in which any straight line embedding of its boundary into $\mathbb{R}^2$ can be completed to a straight line embedding of the whole triangulation. Albertson and Hutchinson [1] were the first to raise the question of a shortest essential cycle in a triangulation of an arbitrary 2-manifold. They gave a simple general lower bound for the size of an orientable k-irreducible triangulation, albeit without any influence from the genus of the 2-manifold. Different approaches were directed at ew-4 triangulations: Hartsfield and Ringel [7] introduced *clean triangulations*, in which every 3-cycle is a face, and every face is a 3-cycle. In parallel, ew-4 triangulations were also studied as *locally cyclic* graphs, in which the subgraph induced by the neighbours of any vertex forms a cycle. Fisk et al. [5] constructed all five minimal locally cyclic (4-irreducible) triangulations of the projective plane. From Clark et al. [4], a bound for the minimal size of 4-irreducible triangulations of arbitrary 2-manifolds can be derived, which is asymptotically tight due to a series by Seress and Szabó [19]. The smallest ew-k triangulations of orientable 2-manifolds for arbitrarily large k and genus are due to Przytycka and Przytycki [16], who called them *high representativity triangulations*. To the best of our knowledge, no series of large k-irreducible triangulations has been published. The first upper size bound was given by Malnič and Nedela [14] (*k-minimal triangulations*), the best one by Gao et al. [6], albeit with the authors themselves stating that they expect the actual maximum to grow at a much lower rate, both in k and in the genus of the underlying 2-manifold.

Thus, the three main questions are: What are lower and upper bounds for the minimum and maximum size of k-irreducible triangulations? What are the smallest and biggest explicitly constructible examples? Can one achieve a complete classification for specific 2-manifolds, and fixed k?

This thesis is devoted to the study of k-irreducible triangulations of closed, compact 2-manifolds without boundary. We address all three main questions, pushing the boundaries of knowledge on this intriguing subject on various points, using various techniques and giving an in-depth overview on where exactly these boundaries are.

Chapter 1 gives a basic formal introduction of ew-k and k-irreducible triangulations.

Chapter 2 presents the state of the art in k-irreducible and ew-k triangulations in detail. Our contributions are highlighted and put into perspective.

Chapter 3 examines general properties of k-irreducible and ew-k triangulations in detail. We introduce an equivalent definition of k-irreducibility that avoids essential cycles, and use this to redefine k-irreducible 2-spheres. This new class contains all triangulations associated with platonic solids, fitting the intuitive view of k-irreducibility as a measure for smoothness.
Furthermore, we extend the general lower bound of $f_0 \geq \frac{k^2}{2}$ for the minimal size of ew-k triangulations, given in [1], to arbitrary 2-manifolds, and we prove a new lower bound of $f_0 \notin o(g^{\frac{2}{3}})$ for the minimal size of ew-5 triangulations.

Chapter 4 presents a new construction principle for vertex-transitive ew-k triangulations, transforming the problem into a group theoretic setting. We prove the existence of a series of groups that produces vertex-transitive ew-k triangulations of size $f_0 \in o(g)$ for an arbitrary k, with g denoting the Euler genus of the underlying 2-manifold. While [16] provides a smaller series for an arbitrary k with $f_0 \in O(\frac{g}{\log g})$, the potential of our construction is shown in Section 4.3, where we give explicit groups producing the smallest series of 5-irreducible triangulations known at this point with $f_0 \in O(g^{\frac{8}{9}})$.

Chapter 5 presents algorithmic approaches. The first one takes the k-irreducible triangulations of a 2-manifold, and generates all its $(k+1)$-irreducible triangulations. We achieved a complete classification of all 1197 4-irreducible tori, all 2303 4-irreducible Klein bottles, all 63 5-irreducible projective planes, the 15 minimal 6-irreducible projective planes, the unique minimal 5-irreducible torus, and the 3 minimal 4-irreducible non-orientable 2-manifolds of Euler genus 3. The second algorithm heuristically generates small k-irreducible triangulations of an arbitrary 2-manifold, for an arbitrary k. Its implementation yielded the smallest known up to 8-irreducible triangulations of 2-manifolds with Euler genus up to 20. All these results have been obtained by extensive parallel computations on the computing cluster of the Faculty of Mathematics at Technische Universität Dresden.

In Chapter 6, we construct explicit, infinite series of k-irreducible triangulations of fixed 2-manifolds, with the smallest and largest known growth rates. We conjecture to have found the minimal ($f_0 = \lceil \frac{3}{4}k^2 \rceil$) and maximal ($f_0 = 2k^2 - 4k + 4$) k-irreducible triangulations of the torus, the maximal k-irreducible triangulations of the projective plane ($f_0 = k^2 - k + 1$), and the minimal growth rate of k-irreducible triangulations of the projective plane ($f_0 = \frac{2}{3}k^2 + O(k)$) as well as of the Klein bottle ($f_0 = \frac{7}{8}k^2 + O(k)$). The largest growth rate of arbitrary k-irreducible 2-manifolds we found is $f_0 = \frac{9}{8}gk^2 + O(g)$ in the orientable, and $f_0 = h\frac{3k^2-2k}{4} + O(h)$ in the non-orientable case.

1. Preliminaries

The basic objects of this thesis are *k-irreducible triangulations*. In this section, we introduce them briefly, and step by step.

1.1. What is a triangulation?

In our context, it is a pure 2-dimensional simplicial complex. In other words:

Definition 1.1 (2-manifolds, triangulations)**.**
A compact, connected topological Hausdorff space M is called a 2-manifold*, if any point in M has a neighbourhood homeomorphic to $\mathbb{R}^2$.*
A triangulation *T of a 2-manifold M is a (pure, 2-dimensional) simplicial complex $\mathcal{K}$, homeomorphic to M, together with a homeomorphism $h : \mathcal{K} \hookrightarrow M$.*
We denote the number of vertices V of a triangulation T by f_0, the number of edges E by f_1, and the number of faces F by f_2.

Remark 1.2.
In particular, this thesis deals with closed 2-dimensional triangulations without boundary and without singularities. The following holds:

- *For every triangle ABC in F, its three edges AB, BC and CA are contained in E.*
- *For every edge AB in E, its two vertices A and B are contained in V.*
- *The intersection of any two objects in T is either empty or a common face.*
- *Every vertex is contained in at least three edges.*
- *Every edge is contained in exactly two triangles.*

Triangulations appear in a number of different contexts. Another way to think about them is as a special case of a polyhedral map, where every face is a triangle. Note that every polyhedral map can be transformed into a triangulation by adding diagonals to any non-triangular face. Or, we can take a simple, undirected graph without duplicate edges and add surface information by inserting triangles into some of its 3-cycles. As we do not want to have dangling edges or vertices, the graph we start with should also fulfil the last two bullet points of the remark above. That graph then is the 1-skeleton of the created triangulation. In general, it is not possible to recreate a triangulation given just its 1-skeleton. For k-irreducible triangulations, though, it is, as we will see in Section 3.5.

Definition 1.3 (Euler characteristic, orientability, (Euler) genus)**.**
Let T be a triangulation of a 2-manifold M. The Euler characteristic *χ of M is defined by* Euler's formula*:*

$$\chi = f_0 - f_1 + f_2 \tag{1.1}$$

M is called orientable*, if the triangles ABC of T can be oriented (given an orientation $\overrightarrow{ABC}$ or $\overrightarrow{ACB}$) in a way such that any two triangles with a common edge AB induce different orientations ($\overrightarrow{AB}$ and $\overrightarrow{BA}$) in that edge. Otherwise, it is called* non-orientable*.*
We denote by M_g the (orientable) 2-manifold created by glueing g handles to a sphere, with g being an integer greater than or equal to 0, and by N_h the (non-orientable) 2-manifold created by glueing h Möbius strips to a sphere, with h being an integer greater than 0. g and h are called the genus *of M_g or N_h, respectively. The* Euler genus *g_e of M_g is set to twice the genus g, $g_e(M_g) = 2g$, while the* Euler genus *g_e of N_h equals its genus h, $g_e(N_h) = h$.*

The most important examples are the 2-sphere (M_0), the torus (M_1), the double torus (M_2), the projective plane $\mathbb{R}P^2$ (N_1), and the Klein bottle (N_2).
Note that, in a triangulated 2-manifold, every edge $AB \in E$ is contained in exactly two triangles $ABC, ABD \in F$. Thus, as every triangle contains exactly three edges, this yields the following formula, which we will refer to as the *edge-to-face ratio of triangulated 2-manifolds*:

$$2f_2 = 3f_3 \tag{1.2}$$

Combining this with Euler's formula (1.1) yields what we refer to as *Euler's formula for 2-manifolds:*

$$f_0 - \frac{f_1}{3} = \chi \tag{1.3}$$

It is a well-known fact that both Euler characteristic and orientability are topological invariants. Thus, any surface homeomorphic to a manifold M – and thus, any triangulation T of M – has the same Euler characteristic and orientability. Thus, every 2-manifold belongs to one of the following two classes:

Theorem 1.4.
Let M be a 2-manifold. If M is orientable, there is an integer g greater than or equal to 0 such that M is homeomorphic to M_g, and the Euler characteristic $\chi(M)$ is given by $2 - 2g = 2 - 2g_e$.
If M is non-orientable, there is an integer h greater than 0 such that M is homeomorphic to N_h, and the Euler characteristic $\chi(M)$ is given by $2 - h = 2 - g_e$.

This has been known since the 19th century and there are multiple proofs. One nice and comprehensive example of those can be found in [17]. Note that this yields an easy way of determining the topological type of a triangulation, by checking its orientability and calculating its Euler characteristic χ, simply by counting the vertices, edges and triangles of the triangulation and applying Euler's formula (1.1).

1.2. A short detour on minimal triangulations.

In graph theory, there is the question of determining the smallest genus of a 2-manifold that a given graph can be embedded into. Turning this around, the related question concerning triangulations is the following:

Given a 2-manifold, what are its smallest triangulations?

This thesis refers to the number of vertices when mentioning the size of a triangulation. Some literature uses the number of edges or the number of triangles. As any one of these determines the other two by Euler's formula (1.1) and the edge-to-face ratio of triangulated 2-manifolds (1.2), there is really no difference.
Also note that we do not distinguish between combinatorially isomorphic triangulations. As such, every statement regarding the number of certain triangulations is to be read up to a relabelling of its vertices, edges and triangles.
Obviously, the tetrahedron is the smallest triangulation of the 2-sphere. It is also well-known that an embedding of the complete graph K_6 yields the smallest triangulation of the projective plane, and that the Moebius torus is the unique minimal torus triangulation:

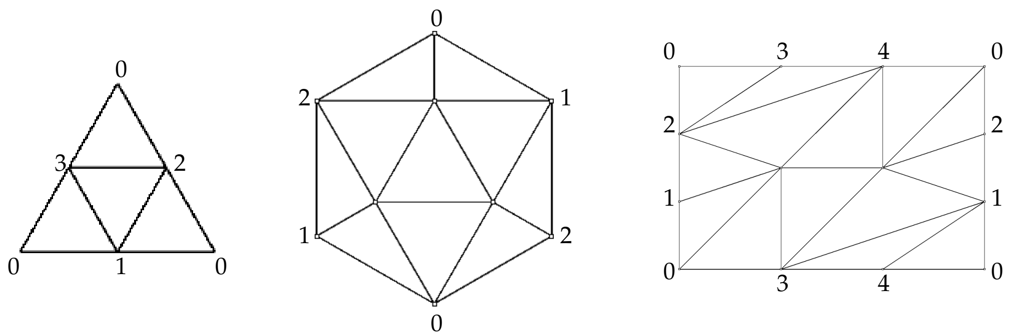

Figure 1.1.: The smallest triangulations of the 2-sphere, the projective plane and the torus

It is also not hard to give a lower bound for the number of vertices in a minimal triangulation. As the maximum number of edges is achieved if every vertex is connected to all other vertices, the number of edges f_1 is bounded from above by the number $\binom{f_0}{2}$ of combinations of two vertices.

$$f_1 \leq \binom{f_0}{2} = \frac{f_0(f_0-1)}{2}$$

Inserting this into Euler's formula for 2-manifolds (1.3), we get

$$\chi = f_0 - \frac{f_1}{3} \geq f_0 - \frac{f_0(f_0-1)}{6},$$

which leads to

$$f_0^2 - 7f_0 + 6\chi \geq 0.$$

Thus, as $\chi \leq 2$ and $f_0 \geq 4$, we get

$$f_0 \geq \left\lceil \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rceil,$$

a lower bound called the Heawood number, which has been known since the 19th century [8]. Surprisingly, this bound is actually achieved for all but three 2-manifolds, as Ringel showed 1971 in his famous Map Color Theorem:

Theorem 1.5 (Ringel, [17]).
Let T be a minimal triangulation of a 2-manifold M with Euler characteristic χ. Then, its number of vertices f_0 is given by

$$f_0 = \begin{cases} 8, & M = N_2, \\ 9, & M = N_3, \\ 10, & M = M_2, \\ \left\lceil \frac{7+\sqrt{49-24\chi}}{2} \right\rceil, & \textit{for any other manifold.} \end{cases} \tag{1.4}$$

1.3. What are irreducible triangulations?

The short answer: Irreducibility is another notion of extremality. In particular, it contains minimality, but is not limited to it and is defined locally, in a comprehensive and very natural way.

The local operation used is an *edge contraction*. Given a triangulation, imagine physically contracting an edge by taking both its endpoints and pushing them towards each other, shrinking the edge. Once they touch, the edge is gone – contracted – as well as both the triangles that the edge was contained in, which have been reduced to a single edge each. As such, we get a new triangulation, that has one fewer vertices, three fewer edges and two fewer triangles (see Figure 1.2).

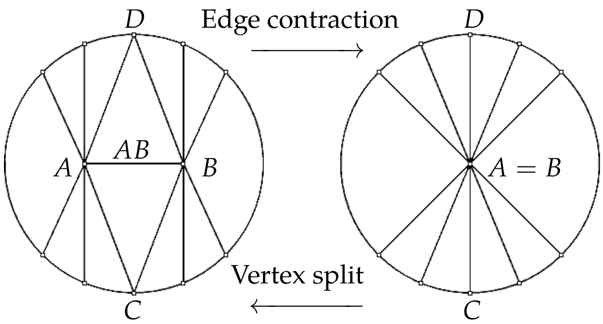

Figure 1.2.: Edge contraction vs. vertex split

The inverse operation is called a *vertex split*: We fix a vertex in the triangulation, and we fix two of its neighbours. Now we pull the vertex apart into two new vertices, adding an edge between them, and two triangles, one for each of the neighbours we fixed. Here is a more formal definition:

Definition 1.6 (Edge contractions, vertex splits)**.**
Let T be a triangulation, let AB be an edge of T, and let ABC and ABD be the two triangles in T containing AB. The process of contracting the edge *AB consists of the following steps (see Figure 1.2):*

- *Remove AB, ABC and ABD.*
- *Identify $A = B$, $AC = BC$ and $AD = BD$.*

The inverse operation is called a vertex split*:*

- *Duplicate a vertex A and two of its neighbouring edges AC, AD.*
- *Insert an edge AB between the vertex A and its duplicate B.*
- *Insert triangles ABC and ABD between the new edge AB and the old edges with their respective duplicates (AC, BC and AD, BD).*

Note that, while it is always possible to perform an edge contraction on the abstract sets $< V, E, F >$ defining a triangulation T of a 2-manifold M, the result might no longer be a triangulation of that 2-manifold. In particular, the contraction might lead to two incidences of the same edge AB, creating a singularity. Thus, there are triangulations that cannot be reduced by edge contractions without changing their topological type. These triangulations are called *irreducible*.

Definition 1.7 (Irreducible triangulations, (non-)contractability)**.**
Let T be a triangulation. T is called irreducible*, if none of its edges can be contracted such that the result is still a triangulation.*
An edge is called non-contractible*, if it cannot be contracted. Otherwise, it is called* contractible.

Obviously, every minimal triangulation is irreducible. Otherwise, we could contract an edge and get an even smaller triangulation. The inverse implication is wrong for all 2-manifolds but the 2-sphere. Figure 1.3 shows some examples of irreducible triangulations of the torus.

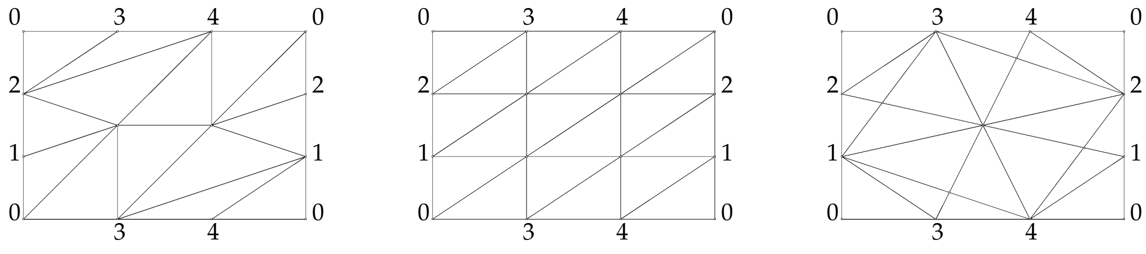

Figure 1.3.: Examples of irreducible torus triangulations

1.4. What is ew-k? What is a k-irreducible triangulation?

Until now, irreducibility has not been connected to any additional properties. We refine that now with respect to the length of the shortest essential cycle. A triangulation is irreducible in that stronger sense, or *k-irreducible*, if its shortest essential cycle has length k and no edge can be contracted without creating a shorter essential cycle or destroying the triangulation.

Definition 1.8 (Adjacency, (essential) cycles, edge width, k-irreducible triangulations)**.**
Let T be a triangulation of a 2-manifold M, and let $n \geq 1$ be a natural number. Two vertices of T are adjacent, *if they are connected by an edge. A series of n vertices $(A_1, A_2, \ldots, A_n, A_1)$ with $A_j, A_{j+1} (j \in \{1, \ldots, n-1\})$ and A_n, A_1 being adjacent is called a (closed)* n-cycle. *For convenience, we also denote by c the curve on M consisting of the connecting edges of a closed cycle c.*
A cycle c in T is called nullhomotopic, *if it is homotopic to a point on M. Otherwise, it is called* essential, *or* non-nullhomotopic.
The edge width *of a triangulation is the length of its shortest essential cycle.*
Let $k \geq 3$ be a natural number.
T is called an ew-k triangulation, *if it does not contain essential cycles of length less than k.*
A contraction-minimal ew-k triangulation is called k-irreducible.

Note that ew-k is just a lower edge width limit on all 2-manifolds but the sphere. It can be interpreted as a lower limit for the average smoothness of the triangulation. The larger the k, the more edges are in any essential cycle of the triangulation, and thus, the closer it gets to an actual smooth cycle, if one were to ideally embed it. Figure 1.4 contains some examples to illustrate this.

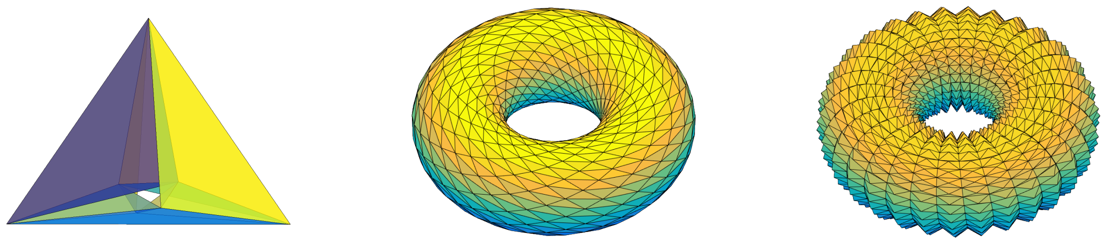

Figure 1.4.: One 3- and two 30-irreducible tori – embedded

Figure 1.4 also illustrates another thing: In addition to the edge width limit, there is also the condition that no edge can be contracted without losing that limit. While the former gives us a minimum level of smoothness, the combination of both also gives us a maximum. The middle image of Figure 1.4 shows what we believe to be a smallest 30-irreducible torus, the right image shows what we believe to be a largest one, the basic idea being to fit as many vertices as possible onto the surface, while still staying k-irreducible. Thus, k-irreducible triangulations are finite triangular approximations of 2-manifolds, living in a small corridor of smoothness close to, but also always a certain distance away, from their smooth analogues.

2. Embedding our results into the state of the art

k-irreducible triangulations have been studied from multiple perspectives, and under multiple names. Before going into the literature, note that k-irreducible triangulations are a natural generalization of irreducible triangulations. As the minimum size of any essential cycle is 3, any triangulation is ew-3. Thus, a 3-irreducible triangulation is the same as an irreducible one.

2.1. The basis: Irreducible triangulations

Obviously, there is a lower size limit for irreducible triangulations, as they cannot be smaller than the minimal ones. What is more interesting – and not obvious at all – is that there is also an upper size limit. First proven by Barnette and Edelson in 1989 [3], it was greatly improved only recently, in 2010.

Theorem 2.1 (Joret & Wood, [11]).
Let T be an irreducible triangulation of a 2-manifold other than S^2 with Euler characteristic χ. Then, an upper bound for its number of vertices is given by

$$f_0 \leq 22 - 13\chi.$$

Interestingly, it is still open whether this bound is asymptotically sharp. The largest known explicit series of irreducible triangulations were given by Sulanke in 2006, and all known biggest irreducible triangulations of any 2-manifold are of this type.

Theorem 2.2 (Sulanke, [22]).
Let M be a 2-manifold without boundary, and with Euler characteristic $\chi < 2$. Then, there are irreducible triangulations of M with vertex number

$$f_0 = \begin{cases} \left\lfloor \frac{17(2-\chi)}{4} \right\rfloor, & \textit{if } M \textit{ is orientable,} \\ \left\lfloor \frac{11(2-\chi)}{2} \right\rfloor, & \textit{if } M \textit{ is non-orientable.} \end{cases}$$

What else do we know? Steinitz and Rademacher proved in 1934 that the tetrahedron is actually the only irreducible triangulation of the 2-sphere [20]. There are two irreducible projective planes (Barnette, [2], see Figure 2.1). Lawrencenko [12] determined the 21 irreducible triangulations of the torus (three of which we have already seen in Figure 1.3, including the unique minimal one with 7 and the unique maximal one with 10 vertices). The 29 irreducible triangulations of the Klein bottle (N_2) were determined by Lawrencenko and Negami [13] and Sulanke [23]. Sulanke also generated all 9708 irreducible triangulations of N_3, all 6297982 irreducible triangulations of N_4 and all 396784 irreducible triangulations of M_2 [21].

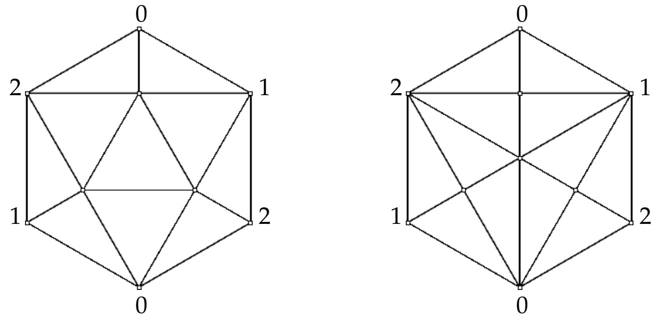

Figure 2.1.: The two irreducible triangulations of the projective plane

One reason to study irreducible triangulations at all is that any triangulation can be reduced to an irreducible one by a series of edge contractions. As such, the inverse is also true:

Corollary 2.3.
For any triangulation T of a 2-manifold M, there is at least one irreducible triangulation T_0 of M such that T can be created from T_0 by performing a finite series of vertex splits.

Proof. Let T be a triangulation of a 2-manifold M. T can be transformed into an irreducible triangulation T_0 by inductively contracting an arbitrary contractible edge, as long as such an edge exists. As T is finite, so is this process. The inverse procedure generates T from T_0 in a finite number of vertex splits. $\square$

Thus, the irreducible triangulations of a specific 2-manifold are a *finite* generating set of the *infinite* set of all triangulations of that 2-manifold. As such, important information about all triangulations can be found in a relatively small number of objects, allowing for the following proof technique:

$$\left.\begin{array}{l} \rightarrow \text{All irreducible triangulations of a} \\ \qquad \text{2-manifold } M \text{ have property } (P) \\ \rightarrow (P) \text{ is not lost under vertex splits} \end{array}\right\} \Rightarrow (P) \text{ holds for all triangulations of } M$$

One example of an actual usage of this abstract proof technique has been given by Sulanke [22], who used his complete classification of all irreducible triangulations of 2-manifolds with Euler genus less than or equal to 4 to show that all triangulations of N_2, N_3, N_4 and M_2 contain an essential separating cycle, as this property is not lost under vertex splits. In the general case, this was conjectured by Barnette, and is still open.

Remark 2.4. *Corollary 2.3 is well known. To our knowledge, the first to introduce it in this form, including the fact that the number of irreducible triangulations is finite for any fixed 2-manifold, were Barnette and Edelson in 1989 [3].*

2.2. k-irreducible triangulations

Similar to irreducible triangulations generating all (ew-3) triangulations, the k-irreducible triangulations are a generating set for all ew-k triangulations:

Corollary 2.5.
For any ew-k triangulation T of a 2-manifold M, there is at least one k-irreducible triangulation T_0 of M such that T can be created from T_0 by performing a finite series of vertex splits.

Proof. Let T be an ew-k triangulation of a 2-manifold M. T can be transformed into a k-irreducible triangulation T_0 by inductively contracting an arbitrary contractible edge that is not part of an essential k-cycle. T is finite. Thus, after a finite number of steps, every edge is contained in an essential k-cycle, rendering the triangulation k-irreducible. The inverse procedure generates T from T_0 in a finite number of vertex splits. □

2.2.1. Lower size bounds and infinite series with few vertices

Recall that the exact size of the minimal 3-irreducible triangulations have been given by Ringel (see Theorem 1.5). The first lower bound for ew-k triangulations was indirectly given by Albertson and Hutchinson in 1978. They raised the question of the length of a minimal non-separating essential cycle in a triangulation, with a cycle being called *non-separating*, if the triangulation remains connected after cutting it along the cycle.

Theorem 2.6 (Albertson, Hutchinson [1])**.**
All triangulations of a surface M_g with genus g at least 1 contain a non-separating essential cycle c of length $|c|$ at most $\sqrt{2f_0}$.

In an ew-k triangulation, all essential cycles, and in particular the non-separating ones, have length larger than or equal to k. Thus, k is also bounded from above by $\sqrt{2f_0}$,

$$k \leq |c| \leq \sqrt{2f_0},$$

which yields a lower bound of $\frac{k^2}{2}$ for the number of vertices f_0 in an ew-k triangulation, and thus also for k-irreducible ones.

Corollary 2.7.
An ew-k triangulation of an orientable 2-manifold other than the sphere has at least $\frac{k^2}{2}$ vertices.

In Section 3.6, we extend this result to arbitrary 2-manifolds, with a new, simple proof that uses the local geodesity of k-irreducible triangulations.

Theorem 3.19.
Let $k \geq 3$ be some integer, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. Then, T contains at least $f_0 \geq \frac{k^2}{2}$ vertices.

Note that, as ew-k triangulations are generated from k-irreducible ones by vertex-splitting, this result is directly applicable to them as well. Surprisingly, to our knowledge, this is the only lower size bound for arbitrary k. The other known bounds are special cases for fixed k, the first one being the Heawood bound introduced in Section 1.2,

$$\min_{T\ k\text{-irreducible}} f_0 \geq \min_{T\ 3\text{-irreducible}} f_0 = \min_{T} f_0 \geq \frac{7+\sqrt{49-24\chi}}{2}.$$

Using the fact that any k-irreducible triangulation can be reduced to a 3-irreducible one by edge contractions, this is extended to an arbitrary k. Note that $k = 3$ is only indirectly contained in the constants of this bound. The only other known lower bound deals with

---

$$k = 4.$$

Theorem 2.8 (Clark et al. [4]).
Suppose G is a graph with n vertices and e edges, such that each edge belongs to at least one triangle, but at most d. Then $e = o(n^2)$.

A triangle in a graph is just a 3-cycle. A k-irreducible triangulation (ew-k does not suffice) does not contain any non-face triangles (Corollary 3.2). Thus, the only 3-cycles any edge of a k-irreducible triangulation of a 2-manifold is contained in, are its two neighbouring triangles. Thus, we can apply Theorem 2.8 with $d = 2$, and Euler's formula for 2-manifolds (1.3) yields that the Euler characteristic χ grows slower than quadratic in the number of vertices f_0,

$$\chi = f_0 - \frac{f_1}{3} \in o(f_0^2).$$

Solving this for f_0 yields the promised bound for the growth rate of the number of vertices f_0 of k-irreducible triangulations with respect to the Euler genus g, for k at least 4.

Corollary 2.9.
Let $k \geq 4$ be an integer, and let T be a k-irreducible triangulation of a 2-manifold with Euler genus g. Then, the number of vertices f_0 of T grows faster than the square root of g.

In particular, 4-irreducible triangulations are distinctly larger than 3-irreducible ones. The bound also holds for ew-k triangulations, as any ew-k triangulation can be reduced to a k-irreducible one by a series of edge contractions.
This result is asymptotically best possible: For any $\epsilon > 0$, Seress and Szabó constructed an infinite series of ew-4-triangulations with $f_0 \in O(g^{\frac{1}{2}+\epsilon})$.

Theorem 2.10 (Seress, Szabó [19]).
There exists an infinite sequence S of integers and an absolute constant $c > 0$ such that for each $n \in S$, there is a locally cyclic graph on n vertices with $> n^{2-c/\sqrt{\log\log n}}$ edges.

Locally cyclic graphs are graphs in which the direct neighbourhood of every vertex, the subgraph induced by its neighbours, is homeomorphic to a cycle. Thus, for every vertex of such a graph, there is an ordering of its neighbours such that any two of them are adjacent in that ordering if and only if they are also adjacent in the graph. This yields a triangulation whose 1-skeleton is the given graph, and which is uniquely determined. Moreover, it is also ew-4, as, without diagonals in any direct neighbourhood of a graph, its only 3-cycles are its triangles, and those are non-essential. Inserting Theorem 2.10 into Euler's formula for 2-manifolds (1.3) yields

$$\chi = f_0 - \frac{f_1}{3} < f_0 - \frac{{f_0}^{2-\epsilon}}{3},$$

with $\epsilon = c/\sqrt{\log\log f_0}$. f_0 is taken from the infinite sequence S of integers. Thus:

Corollary 2.11.
For any fixed $\epsilon > 0$, there is an infinite series of ew-4 triangulations with their number of vertices f_0 growing at most as fast as $g^{\frac{1}{2}+\epsilon}$, with g denoting the Euler genus of the underlying (orientable or non-orientable) 2-manifold.

$k = 5$

In Section 3.7, we give the first lower size bound for ew-5-triangulations.

Theorem 3.20.
Let T be an ew-5-triangulation of a 2-manifold with Euler characteristic χ. Then, the following holds:

$$f_0^3 - 19f_0^2 + 54\chi f_0 - 36\chi^2 \geq 0$$

In particular, in any infinite series of ew-5-triangulations, the number of vertices f_0 of the triangulations grows at least as fast as $g^{\frac{2}{3}}$, with g denoting the Euler genus of the underlying 2-manifold.

A summary on the known limits for the smallest size of 5-irreducible triangulations of 2-manifolds with Euler genus up to 10 is given in Table 3.1. Note that there is still no general lower size bound involving both the Euler genus g and the length k of the shortest essential cycle. In the case of vertex-regular triangulations, where all vertices have the same valency, we prove the following.

Theorem 3.21.
Let $k \geq 3$ be an integer, and let T be a vertex-regular k-irreducible triangulation of a 2-manifold with Euler genus $g > 0$. Then, the number of vertices f_0 of T grows at least as fast as $g^{\frac{k-1}{k+1}}$, if k is odd, and at least as fast as $g^{\frac{k-2}{k}}$, if k is even.

Unfortunately, the ideas at hand seem to be too weak to yield this in the general case. Our intuition is that the smallest k-irreducible triangulations should be close to vertex-regular, in line with all known small examples. Thus, we conjecture this bound to be true in general.

Conjecture 3.22.
Let k be an integer, and let T be an ew-k triangulation of a 2-manifold with Euler genus g. Then, the number of vertices f_0 of T grows at least as fast as $g^{\frac{k-1}{k+1}}$, if k is odd, and at least as fast as $g^{\frac{k-2}{k}}$, if k is even.

Even if this bound holds, it is still open whether it can be tight. The ideas of the proof of Theorem 3.21 suggest that ew-$(2n)$-triangulations grow similarly to ew-$(2n$–$1)$- triangulations. However, the result of Clark et al. (Theorem 2.8) contradicts this, at least for $n = 2$, as smallest 4-irreducible triangulations grow faster than the square root of the Euler genus of their underlying manifold, which is the minimal growth rate of 3-irreducible triangulations.
In Section 4.3, we construct the smallest series of ew-5-triangulations known at this point. Their number of vertices f_0 grows at the rate of $g^{\frac{8}{9}}$, rendering it the first series to achieve a polynomial growth in g, with an exponent strictly less than 1. Of course, there is still a certain gap to the bound of $g^{\frac{2}{3}}$ of Theorem 3.20.

Theorem 4.5.
Let i be a positive integer such that $12i + 7$ is prime. Then, there is a vertex-transitive ew-5 triangulation T with $f_0 = 4(12i + 7)^8$ vertices, every one of which has valency $\overline{v} = 12i + 6$, of an underlying 2-manifold with Euler characteristic $\chi = -8i(12i + 7)^8$.

Corollary 4.6.
There is an infinite series of 5-irreducible triangulations with $f_0 \in O(|\chi|^{\frac{8}{9}})$ vertices, χ denoting the Euler characteristic of the underlying 2-manifolds.

We believe the minimum to be much closer to the bound of $g^{\frac{2}{3}}$. The two main contributions of Chapter 4 are:

- There are ew-5-triangulations with polynomial growth rate and a rational exponent strictly less than 1.
- The construction principle is applicable to finding vertex-transitive ew-k- triangulations for an arbitrarily large k. It reduces the problem of determining such a triangulation to the problem of finding an explicit, finite group that fulfils certain conditions. Also, for a large class of groups, the resulting ew-k triangulation would contain $O(g^{\frac{n}{n+1}})$ vertices, supporting our Conjecture 3.22. Our intuition is that small k-irreducible triangulations should be close to regular, rendering a group theoretic approach very natural.

$k > 5$

The smallest series of ew-k triangulations for an arbitrary k greater than 5 consists of orientable 2-manifolds.

Theorem 2.12 (Przytycka, Przytycki [16]).
There is an infinite series of triangulations of orientable 2-manifolds M_g with f_0 vertices that have edge width at least $\sqrt{3.6\frac{f_0}{g}\log g}$, if $f_0 \leq 4.5g\log g$, and at least $\sqrt{36\frac{f_0}{g}\log g}$, if $f_0 > 4.5g\log g$.

To translate this into our context, set $k = \sqrt{c\frac{f_0}{g}\log g}$ and solve for f_0.

Corollary 2.13.
There is an infinite series of ew-k triangulations of orientable 2-manifolds M_g with their number of vertices f_0 growing quadratically in k and at the rate of $\frac{g}{\log g}$ with respect to the genus g. More precisely:

$$f_0 \leq ck^2\frac{g}{\log g}\,, \begin{cases} c = 3.6, & f_0 \leq 4.5g\log g, \\ c = 36, & f_0 > 4.5g\log g. \end{cases}$$

We believe the actual minimum to be much closer to the bound of Conjecture 3.22. However, due to a cycle of unfortunate citations, there is literature claiming a lower bound of $ck^2\frac{g}{\log^2 g}$ for the number f_0 of vertices in an ew-k triangulation (e.g. [15], Section 5.4, the origin being [10]). That bound does not apply. Counter examples include our ew-5-series with growth rate $g^{\frac{8}{9}}$, Seress' and Szabós' ew-4-series with growth rate $g^{\frac{1}{2}+\epsilon}$, and of course Ringel's 3-irreducible series with growth rate $\sqrt{g}$. The mistake occured when the following result of Hutchinson was cited without the restriction on the number of vertices f_0.

Theorem 2.14 (Hutchinson [9]).
All triangulations of M_g, with $g > 1$, contain an essential cycle of length at most $c_1\sqrt{\frac{f_0}{g}}\log g$, if f_0 is larger than or equal to g, and of length at most $c_2\log g$, if g is strictly larger than f_0.

In the case that the number of vertices f_0 of the triangulation is larger than or equal to the genus g, this cannot yield an asymptotic lower size bound for large g, as there are examples with growth rate $f_0 \in o(g)$. In the case that g is strictly larger than f_0, the theorem does not yield a connection between the number of vertices f_0 and the edge width of the triangulation.

Another approach at creating infinite series is to fix the underlying 2-manifold, and let k tend to infinity. In Section 6.1, we present a construction principle for such series. We apply the principle to orientable 2-manifolds of genus up to 3, as well as to non-orientable 2-manifolds of genus up to 4, and get the smallest growth rates for k-irreducible triangulations known at this point, for these manifolds (see Table 6.1).

Starting point for these series are small 3- and 4-irreducible triangulations of the manifolds, gained by a heuristic algorithm described in Section 5.3. The results include a list of smallest known up to 8-irreducible triangulations of orientable and nonorientable 2-manifolds with Euler genus up to 20 (see Table 5.5).

For the projective plane, the torus and the Klein bottle, we refine the above series by giving explicit constructions for arbitrary k (Sections 6.1.2 to 6.1.4). The resulting k-irreducible triangulations of the projective plane have $f_0 = \lceil \frac{2}{3}k^2 \rceil$ vertices. The torus series has size $f_0 = \lceil \frac{3}{4}k^2 \rceil$, and the Klein bottle has $f_0 = \lfloor \frac{7}{8}k^2 \rfloor$ for k even, and $f_0 = \lceil \frac{7}{8}k^2 \rceil$ for k odd. We conjecture these growth rates and, in the case of the torus, the triangulations themselves, to be minimal.

2.2.2. Upper size bounds and infinite series with many vertices

Compared to the case of 3-irreducible triangulations, the results on largest k-irreducible triangulations for $k > 3$ are very limited. Recall that, for 3-irreducible triangulations, the factor between the largest known series and the best upper bounds is only $\frac{26}{11}$ (see Theorems 2.1 and 2.2). There are only finitely many k-irreducible triangulations of any fixed 2-manifold and for any fixed k. The first to prove this were Malnič and Nedela in 1995.

Theorem 2.15 (Malnič, Nedela [14])**.**
The class of k-irreducible triangulations ($k \geq 3$) is finite for each closed surface $M \not\approx S^2$.

The best upper size bound is due to Gao et al., from 1996.

Theorem 2.16 (Gao et al. [6])**.**
Let T be a k-irreducible triangulation of M, for some $k \geq 3$. If M has Euler genus g, then

$$|E(T)| \leq 3k \cdot k!(6k)^k g^2.$$

The authors state themselves that they expect the true bound to be $c_k g$, instead of $c_k g^2$, which would fit the best understood case of $k = 3$. Also, the super-exponential bound in k seems way too large. Our expectation would be something like $O(k^2 g)$, but we lack any evidence at all to conjecture that. The idea is that, while small k-irreducible triangulations of a 2-manifold with Euler genus g consist of a tight entanglement of their g handles or cross-caps, the big ones consist of clearly separable big building blocks – big k-irreducible tori in the orientable case, big k-irreducible projective planes in the non-orientable case. Again, this fits the knowledge we have at this moment for the 3-irreducible case (see Section 2.1).

Surprisingly, we did not find any explicit constructions of series of large k-irreducible triangulations in the literature. We fill this gap in Section 6.2: Following the intuition given above, we first construct large k-irreducible tori and projective planes, and then glue these basic building blocks together to create large k-irreducible triangulations of arbitrary 2-manifolds, all for an arbitrary k greater than 3. Thus, we get lower bounds for the maximum size of such a triangulation.

Lemma 6.12.
Let $k > 3$ be some integer, and let T be a largest k-irreducible triangulation of a 2-manifold M. Then, T has at least

$$f_0 \geq \begin{cases} k^2 - k + 1, & M = N_1, \\ 2k^2 - 4k + 4, & M = M_1, \\ \frac{9}{8}gk^2 - g, & M = M_g,\ k = 4m, \\ \left\lfloor \frac{9}{8}gk^2 - \frac{5}{8}g \right\rfloor, & M = M_g,\ k = 4m+1, \\ \frac{9}{8}gk^2 - \frac{12}{8}g, & M = M_g,\ k = 4m+2, \\ \left\lfloor \frac{9}{8}gk^2 - \frac{13}{8}g \right\rfloor, & M = M_g,\ k = 4m+3, \\ \frac{3}{4}hk^2 - \frac{hk}{2}, & M = N_h,\ k \text{ even}, \\ \left\lfloor \frac{3}{4}hk^2 - \frac{hk}{2} + \frac{h}{4} \right\rfloor, & M = N_h,\ k \text{ odd}, \end{cases}$$

vertices.

For the projective plane and the torus, we conjecture this to be best-possible.

Conjecture 6.13.
Let $k \geq 3$ be an integer. Then, a largest k-irreducible triangulation of the projective plane has $f_0 = k^2 - k + 1$, and a largest k-irreducible triangulation of the torus has $f_0 = 2k^2 - 4k + 4$ vertices.

For all other 2-manifolds, we believe that our glueing construction is the right way to create maximal k-irreducible triangulations. More research needs to be done on k-irreducible triangulations with boundary to get a good intuition on the basic building blocks, though.

2.2.3. Full classifications

As mentioned in Section 2.1, all 3-irreducible triangulations are known up to an Euler genus of 4. For larger k, the results are much thinner: Hartsfield and Ringel [7] were the first to explicitly construct a smallest 4-irreducible projective plane and the smallest 4-irreducible torus. They also gave one 4-irreducible Klein bottle with 14 vertices, which they conjectured to be smallest possible. The only other result in this respect is by Fisk et al. [5], who constructed all five 4-irreducible triangulations of the projective plane.
In Chapter 5, we present an algorithm producing all k-irreducible triangulations by splitting vertices in $\tilde{k}$-irreducible triangulations, with $\tilde{k} < k$. Building on the full classifications of all 3-irreducible triangulations of 2-manifolds with Euler genus up to 4 mentioned above, we confirm the five 4-irreducible triangulations of the projective plane, and give a full classification of all 63 5-irreducible projective planes, all 1197 4-irreducible tori and all 2303 4-irreducible Klein bottles. In particular, this contains the 29 minimal 4-irreducible

Klein bottles, confirming Hartsfield and Ringels conjecture that the minimal size of such a triangulation should be 14.
Furthermore, we give all three minimal 4-irreducible non-orientable 2-manifolds with Euler genus 3 ($f_0 = 15$), confirm that there is exactly one smallest 5-irreducible triangulation of the torus (the one given in Section 6.1.3, $f_0 = 19$), and also exactly one 5-irreducible triangulation of the torus with 20 vertices. For the Klein bottle, we furthermore prove that the smallest 5-irreducible triangulation has at least 21 vertices (with the smallest known examples having 22 vertices, see Section 6.1.4), and, finally, we provide all 15 smallest 6-irreducible triangulations of the projective plane ($f_0 = 24$).

2.2.4. k-irreducibility on the 2-sphere

As there are no essential cycles on the 2-sphere, with the current definition, the tetrahedron is the only k-irreducible triangulation of the 2-sphere, for any k. Thus, most results on k-irreducible triangulations exclude the 2-sphere. As k-irreducibility can be understood as a measure for how closely the triangulation approximates a smooth surface, this seems highly unnatural. In Chapter 3, we study properties of k-irreducible and ew-k triangulations and prove that some of them are, on any 2-manifold but the sphere, equivalent to the common definition. One of these properties, the exact $(k-1)$-local disc property, is both applicable and meaningful on triangulations of the 2-sphere, allowing us to fill the aforementioned gap in a natural way, and redefine k-irreducibility on 2-spheres.
With this redefinition, all triangulations associated with platonic solids become k-irreducible, with the cube and the dodecahedron being completed to a triangulation by inserting a single central vertex into every face (see Section 3.4). Moreover, these are the only k-irreducible 2-spheres with at most 17 vertices, and for k at most 8, which was proven with the help of complete classification algorithms, see Section 5.5. We created another big class of $2k$-irreducible 2-spheres by glueing together two copies of a k-irreducible projective plane. While we have not examined this in detail, and in particular have no proof that this always yields a $2k$-irreducible 2-sphere, it works for all the 11225 up to 6-irreducible projective planes known to us (see Section 5.2).

3. Local and global properties in detail

k-irreducible triangulations by definition contain an interesting mix of local and global properties. While a triangulation being ew-k is clearly a global feature, irreducibility is based on the local operation edge contraction. In this chapter, we examine the definitions of ew-k and k-irreducibility in detail, and give clear reformulations, both from a purely global and from a purely local perspective.

We start with the non-contractibility of edges with respect to ew-k, and give a direct, equivalent, global alternative definition for k-irreducibility on all 2-manifolds but the sphere. Next, we present *local planarity*, an equivalent view on the lower edge width limit. While defined locally, testing for local planarity still requires information about the whole triangulation. This is overcome with the next iteration, the *local disc* property, which we prove k-irreducible triangulations to have on all 2-manifolds but the sphere, but which in turn does not imply k-irreducibility yet. Equivalence is achieved by extending to *exact local disc*, a purely local redefinition of k-irreducibility.

As the exact local disc property does not use essential cycles, it is also applicable to the 2-sphere. There, it yields a more natural definition of k-irreduciblity, that contains not only the tetrahedron as 3-irreducible, but also all other triangulations associated with platonic solids as 4-, 6- and 10-irreducible triangulations, respectively.

Another local feature of k-irreducibility is local geodesity, which we use to add a simple, geometric proof to the general lower size bound of $f_0 \leq \frac{k^2}{2}$ given in [1], and extend the bound to arbitrary k-irreducible 2-manifolds.

Finally, in Section 3.7, we prove a new lower bound of $f_0 \notin o(g^{\frac{2}{3}})$ for general ew-5 triangulations, using the local disc property. We extend our proof to vertex-regular ew-k triangulations, with size bounds $f_0 \notin o(g^{\frac{k-1}{k+1}})$, if k is odd, and of $f_0 \notin o(g^{\frac{k-2}{k}})$, if k is even. We conjecture these bounds to hold for ew-k triangulations in general. Note that the smallest known series of these triangulations, presented in Chapter 4, has a growth rate of $f_0 \in O(g^{\frac{8}{9}})$.

3.1. Redefining edge contractibility

Note that the general definition of k-irreducibility is quite indirect. What exactly does it mean that no edge is contractible with respect to a lower limit on the edge width? A more direct approach:

Lemma 3.1.
Let T be a triangulation of a 2-manifold M that is not the sphere, and let $k \geq 3$ be an integer. T is k-irreducible, if and only if it is ew-k and every edge of T is contained in an essential k-cycle.

Proof.
Note that both sides of this equivalence contain ew-k. What we have to prove is that an edge is non-contractible with respect to ew-k if and only if it is contained in an essential k-cycle.
$'' \Leftarrow ''$: Let every edge of T be contained in an essential k-cycle. Then, contracting an edge shortens any such essential cycle by one. Thus, the resulting triangulation contains an essential cycle of length $k-1 < k$, and thus, no edge is contractible with respect to ew-k.
$'' \Rightarrow ''$: Let T be a k-irreducible triangulation of a 2-manifold M that is not the sphere. Thus, T is non-contractible ew-k. For an edge of T to be non-contractible, the contraction would either lead to T no longer being a triangulation, or to T containing an essential cycle of length strictly less than k. As an edge contraction shortens any cycle by at most one, the only way an edge contraction can produce an essential cycle of length less than k is if the original edge was contained in an essential k-cycle, as claimed. On the other hand, the only way the contraction does not yield another triangulation of M is if contracting the edge produces a double-edge. In that case, the edge is part of a 3-cycle that does not border a face in T. If that 3-cycle is essential, k cannot be larger than 3 to begin with, and thus that 3-cycle is the essential k-cycle containing the edge as claimed. The only open case is that there is an edge AB in T that is contained in a non-essential non-face triangle ABC (see Figure 3.1).

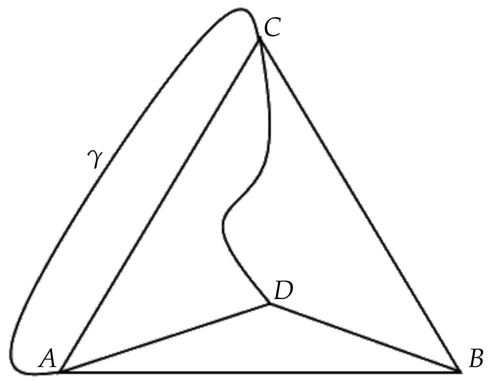

Figure 3.1.: The neighbourhood of a non-face triangle

A non-essential cycle is the boundary of a subtriangulation of T homeomorphic to a disc. As M is not the sphere, we can choose ABC to be the innermost such cycle. As T is a triangulation and ABC is not a face, there is a common neighbour D of AB inside ABC. The edge AD is non-contractible. It cannot be contained in a non-essential non-face triangle, as that would be contained in ABC, contradicting our choice of ABC as the innermost such non-face triangle. Thus, AD is contained in an essential k-cycle γ. As ABC is non-

essential, γ crosses ABC a second time, w.l.o.g. at C. Replacing the part of γ inside ABC with the edge AC creates another essential cycle with length strictly less than k, a contradiction to T being ew-k.
Thus, every edge is contained in an essential k-cycle. □

As shown in the proof, a k-irreducible triangulation does not contain any non-face non-essential triangles, and thus, for k at least 4, no non-face triangles at all. This is a property Hartsfield and Ringel [7] called *clean*, as in such a triangulation, every face is a triangle, and every triangle is a face.

Corollary 3.2.
In a k-irreducible triangulation, there are no non-face non-essential triangles.

In particular, if there was a 3-valent vertex in a k-irreducible triangulation, its three neighbours would have to form a triangle, closing the triangulation and forming a tetrahedron.

Corollary 3.3.
The only k-irreducible triangulation containing a 3-valent vertex is the tetrahedron.

3.2. Local planarity

The next step is an alternative view of ew-k triangulations. Note that limiting the length of essential cycles from below is the same as claiming that all shorter cycles are non-essential. Thus, any neighbourhood of any edge within the triangulation that has diameter less than k cannot contain an essential cycle. In other words, local neighbourhoods are planar within the triangulation. With that goal in mind, the metric we use is essentially an extension of the graph distance in the 1-skeleton of the triangulation, with edges and triangles included for the sake of an easier neighbourhood definition. In particular, this means that shortest paths always move along edges and vertices, and crossing a triangle directly does not shorten a path.

Definition 3.4 (Distance)**.**
Let $T = < V, E, F >$ be a triangulation. The distance *of any two elements $Z_1, Z_2 \in T$ (vertices, edges or triangles) is defined as half the graph distance in the graph created from the triangulation by inserting a vertex in each of its triangles, connected to the three vertices of each triangle, and replacing each edge by a vertex, connected to the two end points of the former edge.*

Note that, as the graph distance is a metric, so is this one. Also note that the distance of a triangle to any of its edges is 1. We now use this to define neighbourhoods of any element of the triangulation.

Definition 3.5 (Neighbourhoods)**.**
Let Z be an element of the triangulation (a vertex, an edge or a triangle), $D \in \mathbb{N}, r = \frac{D}{2}$. The neighbourhood *$B_r(Z) =< V_{B_r}, E_{B_r}, F_{B_r} >$ of the centre Z with radius r (and diameter D) is*

the subtriangulation $B_r(Z) \leq T$ that consists of all vertices $A \in V$ with distance $d(A,Z)$ at most r from the centre Z, as well as all edges and triangles that contain only such vertices and that themselves have at most distance r from the centre, as well as any edge contained in such a triangle.

Thus, $B_r(Z)$ consists of the following vertices, edges and triangles:

$$\begin{aligned} V_{B_r} &= \{A \in V | d(A,Z) \leq r\}, \\ E_{B_r} &= \{AB \in E | A,B \in V_{B_r}, d(AB,Z) \leq r\} \cup \{AB \in E | \exists C \in V_{B_r} : ABC \in F_{B_r}\}, \\ F_{B_r} &= \{ABC \in F | A,B,C \in V_{B_r}, d(ABC,Z) \leq r\}. \end{aligned}$$

Note that, for a triangle $ABC \in F$ to be contained in the radius-r-neighbourhood of a centre $Z \in T$, one of its vertices has to have distance at most $r - \frac{1}{2}$ from the centre, and all three of it have to have distance at most r. This also means that the radius-0-neighbourhood of a triangle is the empty set, as is the radius-0-neighbourhood of an edge, while the radius-0-neighbourhood of a vertex contains the vertex itself. Figure 3.2 shows the local neighbourhoods appearing in the 7-vertex 3-irreducible projective plane. Note that the radius-$\frac{3}{2}$-neighbourhood of the vertex A in this triangulation contains all but the other two 4-valent vertices, and all edges and triangles that are not incident with the other two 4-valent vertices. In particular, all essential 3-cycles containing A are completely contained in its radius-$\frac{3}{2}$-neighbourhood.

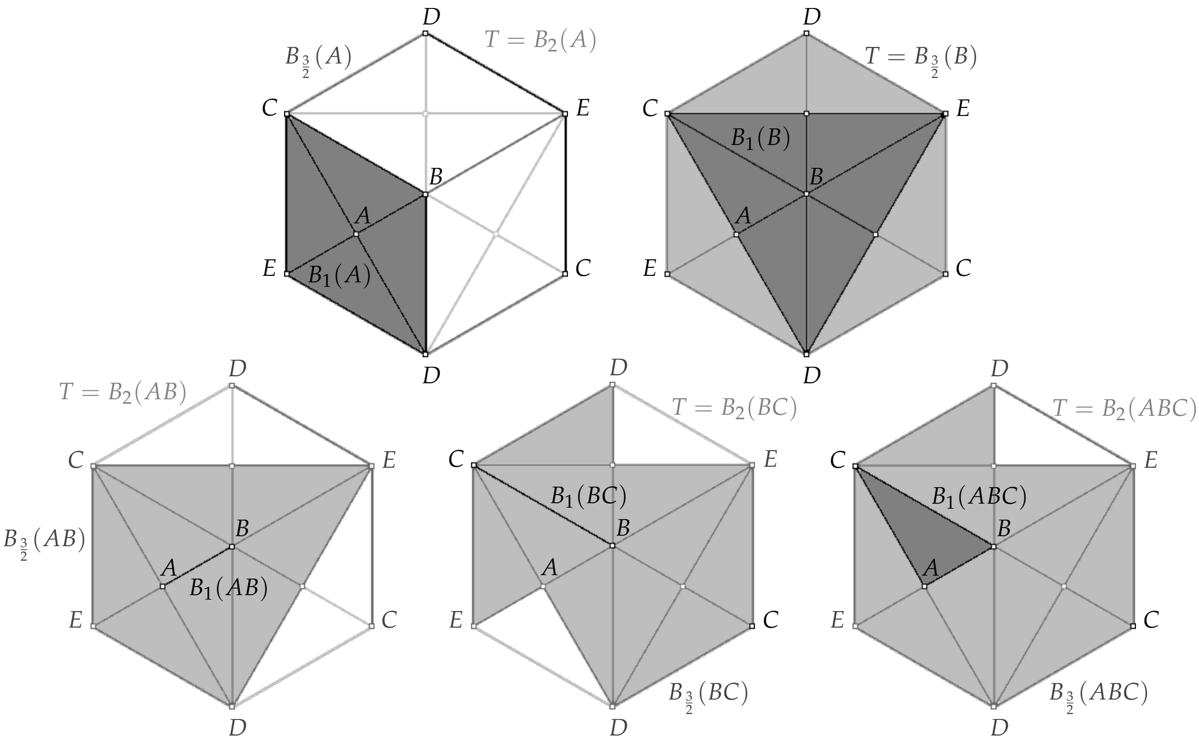

Figure 3.2.: Local neighbourhoods in the 7-vertex 3-irreducible projective plane

We now give an explicit definition of local planarity, which is an intrinsic property of ew-k triangulations and yields an equivalent definition of ew-k on all 2-manifolds but the sphere.

Definition 3.6 (k-local planarity)**.**
*Let T be a triangulation, and let $k \in \mathbb{N}$ be a positive integer. T is called k-*locally planar*, if any local neighbourhood in T with diameter k contains no essential cycles of T.*

Lemma 3.7.
Let T be a triangulation of a 2-manifold M other than the sphere, and let $k \geq 3$ be an integer. Then, T is ew-k if and only if it is (k-1)-locally planar.

Proof.
$'' \Rightarrow ''$: Let T be an ew-k triangulation, and let $Z \in T$ be an arbitrary vertex, edge or triangle. We prove that $B_{\frac{k-1}{2}}(Z)$ does not contain any essential cycles of T.
Let $i \in \mathbb{N}$ be the smallest integer such that the diameter-i neighbourhood $B_{\frac{i}{2}}(Z)$ <u>does</u> contain an essential cycle γ of T. As the next-smaller neighbourhood, $B_{\frac{i-1}{2}}(Z)$, does not, γ can always be chosen to contain a vertex of the centre Z and to only contain one connected path in the part $B_{\frac{i}{2}}(Z) \setminus B_{\frac{i-1}{2}}(Z)$ of the diameter-i neighbourhood that differs from the smaller one. As the distance of any two vertices is an integer, such a connected path is just a single edge, if i is odd and Z is a vertex, or if i is even and Z is an edge or a triangle. If under all such cycles, γ is chosen to be shortest possible, the length of γ is bounded from above by the sum of twice the distance from a vertex in the centre Z to the farthest possible vertex in $B_{\frac{i-1}{2}}(Z)$ and the number of edges γ contains outside that smaller neighbourhood. If Z is not a vertex, γ can additionally contain an edge in Z.

$$|\gamma| \leq \begin{cases} 2(\frac{i-1}{2} - \frac{1}{2}) + |\gamma \cap B_{\frac{i}{2}}(Z) \setminus B_{\frac{i-1}{2}}(Z)|, & i \text{ even},\ Z \in V, \\ 2(\frac{i-1}{2}) + 1, & i \text{ odd},\ Z \in V, \\ 1 + 2(\frac{i-1}{2} - \frac{1}{2}) + 1, & i \text{ even},\ Z \in E \cup F, \\ 1 + 2(\frac{i-1}{2} - 1) + |\gamma \cap B_{\frac{i}{2}}(Z) \setminus B_{\frac{i-1}{2}}(Z)|, & i \text{ odd},\ Z \in E \cup F. \end{cases}$$

In the first or the last case, assume that γ contains more than two edges on the part $B_{\frac{i}{2}}(Z) \setminus B_{\frac{i-1}{2}}(Z)$ of the neighbourhood not contained in the smaller one. Thus, γ also contains an edge AB such that neither A nor B is contained in the smaller neighbourhood $B_{\frac{i-1}{2}}(Z)$ (see Figure 3.3).

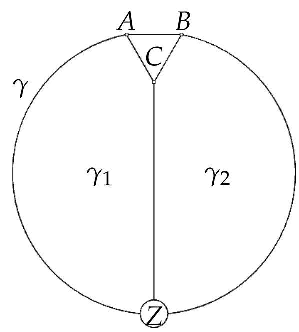

Figure 3.3.: Local configuration described in the proof of Lemma 3.7

A and B have distance $d(A, Z) = d(B, Z) = \frac{i}{2}$ from the centre Z. Thus, AB can only be contained in $B_{\frac{i}{2}}(Z)$, if one of its neighbouring triangles ABC is contained as well. For that, a common neighbour C of A and B has to have distance $d(C, Z) = \frac{i}{2} - 1$ from Z.
$\gamma = Z \dots AB \dots Z$ connects Z to A, crosses AB, and goes back to Z. Let $\gamma_1 = Z \dots AC \dots Z$ be the cycle that consists of the first connection $Z \dots A$, as well as the direct connection back via C, and let $\gamma_2 = Z \dots CB \dots Z$ be the cycle consisting of the other side. As γ is essential, at least one of γ_1 or γ_2 is essential as well, and both are at least one edge shorter than γ. This is a contradiction, as γ was chosen to be shortest possible. Thus, γ contains at most two edges on the outer part of the neighbourhood and thus, γ has at least length i, in any of the four cases above.
As T has edge width at least k, the length $|\gamma|$ is at least k, and thus, the same is true for the diameter i. In other words: Any integer-diameter neighbourhood $B_{\frac{i}{2}}(Z)$ of any centre $Z \in V$ that contains an essential cycle γ has diameter at least k. In particular, $B_{\frac{k-1}{2}}(Z)$ contains no essential cycles of T, rendering T (k-1)-locally planar.

$'' \Leftarrow ''$: Let T be (k-1)-locally planar and let γ be the shortest essential cycle of T. γ is completely contained in the diameter-$|\gamma|$ neighbourhood $B_{\frac{|\gamma|}{2}}(A)$ of any one of its vertices $A \in \gamma$. As T is (k-1)-locally planar, only local neighbourhoods with a radius strictly greater than $\frac{k-1}{2}$ contain essential cycles. Thus, $\frac{|\gamma|}{2}$ is strictly greater than $\frac{k-1}{2}$, and thus, the length of the shortest essential cycle of T is at least k, concluding the proof. □

k-irreducibility is ew-k with contraction-minimality. Thus, the immediate consequence of Lemma 3.7 is the following.

Corollary 3.8.
On every 2-manifold but the sphere, a triangulation is k-irreducible, if and only if it is contraction-minimal (k-1)-locally planar.

3.3. Local disc

Another way to describe local planarity is: Every local neighbourhood is homeomorphic to a planar disc with holes, and each of the boundary curves are nullhomotopic as a cycle in the whole triangulation. They are not necessarily nullhomotopic in the local neighbourhood itself. In particular, local planarity is not locally decidable. For a general ew-k triangulation, it cannot be guaranteed that any of its local neighbourhoods is homeomorphic to a disc without holes. Counter examples can be constructed to any given ew-k triangulation with just a little local addition. In contrast, this condition does hold in k-irreducible triangulations.

Definition 3.9 (Local disc property)**.**
*Let T be a triangulation, and let $k \in \mathbb{N}$ be a positive integer. T is called k-*locally disc*, if every local neighbourhood in T with diameter at most k is homeomorphic to a disc.*

Note that k-local disc implies k-local planarity. As mentioned above, the inverse implication does not hold. It does, however, if we add contraction-minimality to the local planarity. More precisely: k-irreducible triangulations are (k-1)-locally disc.

Lemma 3.10.
Let $k \geq 3$ be an integer, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. Then, T is also (k-1)-locally disc.

Proof. Let T be k-irreducible, let $Z \in T$ be some element of T (a vertex, an edge or a triangle), and let $i \in \mathbb{N}$ be the smallest integer such that the neighbourhood $B_{\frac{i}{2}}(Z)$ of Z with diameter i is not homeomorphic to a planar disc. We show that i is greater than or equal to k by contradiction.
Assume i is strictly less than k. As T is k-irreducible, it is also (k-1)-locally planar. Thus, $B_{\frac{i}{2}}(Z)$ does not contain any essential cycles of T and thus, for it to not be homeomorphic to a disc, its boundary $\delta B_{\frac{i}{2}}(Z)$ consists of multiple simple closed cycles in T, possibly connected via common vertices. All of them are non-essential in T, and thus bound a subtriangulation homeomorphic to a disc. Fix one such simple closed cycle δB_0 that does not contain $B_{\frac{i}{2}}(Z)$ in its disc B_0. As $\delta B_{\frac{i}{2}}(Z)$ contains other components as well, there is a closed non-essential cycle δT_0 connecting Z to δB_0 and containing B_0 in the disc T_0 that it bounds. Choose δT_0 as short as possible. Then, if i is even and Z is a vertex, or if i is odd and Z is an edge or a triangle, δT_0 contains an edge $PQ \in E$ that is contained in $B_{\frac{i}{2}}(Z)$, but not in $B_{\frac{i-1}{2}}(Z)$, with both its end points being contained in the smaller neighbourhood. In the other two cases (i odd, Z a vertex, and i even, Z an edge or a triangle), δT_0 contains a single vertex P that is not contained in the smaller neighbourhood (see Figure 3.4).

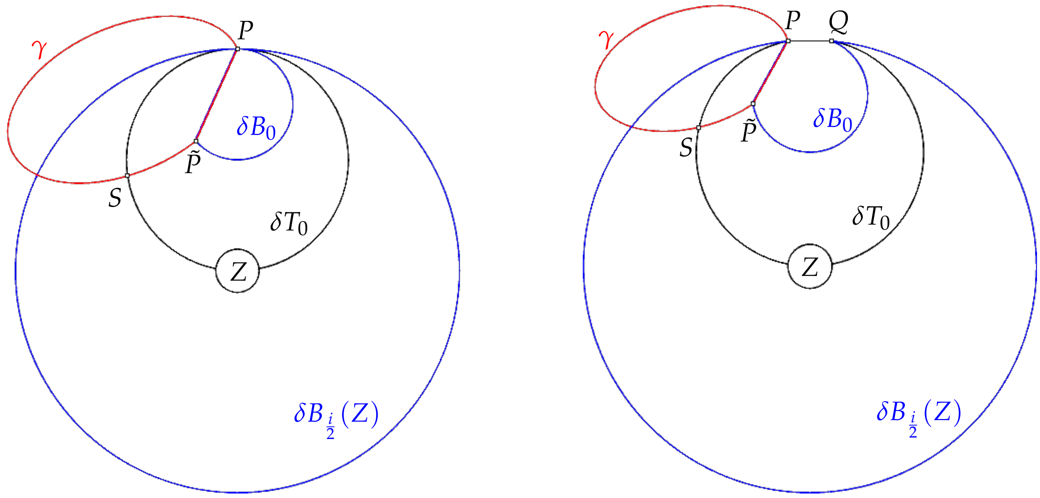

Figure 3.4.: Local configurations described in the proof of Lemma 3.10

As δB_0 is a simple closed cycle bounding a disc, it contains a (second) neighbour $\tilde{P}$ of P, which, as it is on δB_0, has distance $d(P, Z)$ from the centre Z at least $\frac{i-1}{2}$. Thus, as any vertex other than P (or Q) on δT_0 has distance strictly less than $\frac{i-1}{2}$ from Z, $\tilde{P}$ is part of the interior of T_0. As T is k-irreducible, the edge $P\tilde{P}$ is part of an essential k-cycle γ. δT_0

is non-essential, so γ intersects it in at least two points. One of them is P. Denote the other one with S. Then, the path $S \dots \tilde{P}$ on γ is at least as long as the path $S \dots P$ on δT_0. Otherwise, $\tilde{P}$ would be closer to Z than $\frac{i-1}{2}$. Thus, the part $S \dots \tilde{P}P$ of γ is at least one edge longer than the direct connection $S \dots P$ on δT_0. As all this is inside the planar disc T_0, replacing that part of γ by the direct connection on δT_0 yields another essential cycle of T, with length strictly less than the length of γ, which was k. That is a contradiction to T being ew-k, and thus, the diameter i of the smallest neighbourhood of an arbitrary centre element $Z \in T$ that is not homeomorphic to a disc is at least k.
Thus, for all elements Z of T, all their neighbourhoods $B_{\frac{i}{2}}(Z)$ with diameter at most k-1 are homeomorphic to a planar disc, rendering T (k-1)-locally disc. □

In Section 3.7, we use the local disc property of k-irreducible triangulations to give a new lower size bound for the number of vertices in ew-5 triangulations, as well as conjecturing a new lower size bound for arbitrary k-irreducible triangulations.
Note that testing a triangulation for local disc is locally decidable. More importantly, it makes no use of essential cycles. Thus, triangulations of the 2-sphere can be tested for this property as well, yielding different results for different k. Below, we give an extension to the definition of (k-1)-locally disc triangulations and show that this yields equivalence to k-irreducibility. With that, we have found a definition of k-irreducibility that yields meaningful results on all 2-manifolds, including the sphere, and is equivalent to the common definition for all previously covered surfaces.

Definition 3.11 (Exact local disc property)**.**
Let T be a triangulation, and let $k \in \mathbb{N}$ be a positive integer. T is called exactly k-locally disc, *if it is k-locally disc and if for every edge and every triangle in T, its radius-$\frac{k+1}{2}$-neighbourhood is not homeomorphic to a disc.*

Note that k-local disc behaves distinctly different than k-local planarity with respect to edge contractibility. There are triangulations that are not k-locally disc, but can be transformed into a k-local disc triangulation by a series of edge contractions. This is not possible for k-local planarity. Also, there are k-local disc triangulations that lose this property after a series of edge contractions, and then get it back after some more contractions, without losing k-local planarity at any point. In an exact k-local disc triangulation, though, no edge can be contracted without losing k-local disc.

Theorem 3.12.
Let T be a triangulation of a 2-manifold other than the sphere, and let $k \geq 3$ be an integer. T is k-irreducible, if and only if it is exactly (k-1)-locally disc.

Proof.
"$\Rightarrow$": Let T be a k-irreducible triangulation. By Lemma 3.10, T is also (k-1)-locally disc. In addition, every edge $AB \in T$ of a k-irreducible triangulation is contained in an essential k-cycle γ. The neighbourhood $B_{\frac{|\gamma|}{2}}(AB)$ contains γ completely, and is thus not

homeomorphic to a planar disc. The same holds for the neighbourhood of any adjacent triangle ABC, concluding exactness.

$'' \Leftarrow ''$: Let T be exactly (k-1)-locally disc. In particular, T is also (k-1)-locally planar. Lemma 3.7 yields that T is ew-k. We show that every edge is contained in an essential k-cycle.

Assume there are edges $AB \in T$ that are not contained in an essential k-cycle. As T is exactly (k-1)-locally disc, the neighbourhood $B_{\frac{k}{2}}(AB)$ of such an edge is not homeomorphic to a disc. As the next smaller neighbourhood centred at AB, $B_{\frac{k-1}{2}}(AB)$, is homeomorphic to a disc and the underlying 2-manifold is not the sphere, the boundary $\delta B_{\frac{k}{2}}(AB)$ consists of multiple simple closed cycles, possibly connected via common vertices. Recall that $B_{\frac{k-1}{2}}(A)$ and $B_{\frac{k-1}{2}}(B)$ are homeomorphic to a disc, and that $B_{\frac{k}{2}}(AB) \setminus B_{\frac{k-1}{2}}(AB)$ does not contain any vertices if k is even. Thus, in that case, there is a vertex $P_{AB} \in B_{\frac{k-1}{2}}(A) \setminus B_{\frac{k-1}{2}}(B)$ and an adjacent vertex $P_{BA} \in B_{\frac{k-1}{2}}(B) \setminus B_{\frac{k-1}{2}}(A)$, with their connecting edge $P_{AB}P_{BA}$ being contained in neither of the vertex-neighbourhoods, but in $B_{\frac{k}{2}}(AB)$ (compare with the left image of Figure 3.5). If k is odd, any edge added in $B_{\frac{k}{2}}(AB) \setminus B_{\frac{k-1}{2}}(AB)$ is adjacent to a new vertex. Thus, in that case, there is a vertex $P_{AB} = P_{BA} \in \delta B_{\frac{k-1}{2}}(A) \cap \delta B_{\frac{k-1}{2}}(B)$ and a pair of edges connecting this vertex to $B_{\frac{k-1}{2}}(AB)$. One of these edges is contained in $B_{\frac{k-1}{2}}(A) \setminus B_{\frac{k-1}{2}}(B)$, the other one in $B_{\frac{k-1}{2}}(B) \setminus B_{\frac{k-1}{2}}(A)$ (compare with the right image of Figure 3.5).

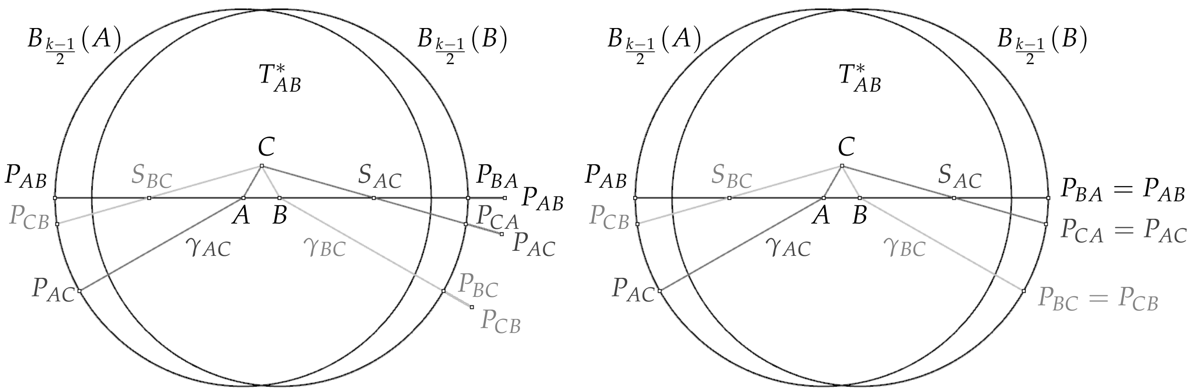

Figure 3.5.: Configurations for even and odd k, described in the proof of Theorem 3.12

Combining two shortest connections from AB to these vertices yields a simple k-cycle $\gamma_{AB} = P_{AB} \dots AB \dots P_{BA}P_{AB}$ (with $P_{BA}P_{AB}$ denoting just a single vertex if k is odd). As we assumed AB to not be contained in an essential k-cycle, γ_{AB} is nullhomotopic. Thus, it splits the whole triangulation T into two components, one of which is homeomorphic to a disc, and will be denoted by T_{AB}.

Fix AB and a containing nullhomotopic k-cycle γ^*_{AB} such that T^*_{AB} does not contain any other non-essential k-cycle $\gamma_{\tilde{A}\tilde{B}}$ constructed in this way around an edge $\tilde{A}\tilde{B}$ not contained in an essential k-cycle.

Let ABC be the triangle containing the edge AB that is contained in T^*_{AB}. C does not lie on the boundary γ^*_{AB}. Otherwise, there would be an edge AB_1 from (w.l.o.g.) A to the second vertex of the path $B \dots P_{AB}$. But then, $B_{\frac{k-1}{2}}(A)$ would contain all of γ^*_{AB} and, as $B_{\frac{k}{2}}(AB)$ is not homeomorphic to a planar disc, neither would be $B_{\frac{k-1}{2}}(A)$, a contradiction to T being (k-1)-locally disc.

Analogous to the situation around AB, the diameter-k neighbourhood of the edge AC is not homeomorphic to a disc, as T is exactly (k-1)-locally disc. If k is even, there are adjacent vertices P_{AC}, P_{CA}, with their connecting edge in $B_{\frac{k}{2}}(AC) \setminus \left(B_{\frac{k-1}{2}}(A) \cup B_{\frac{k-1}{2}}(C)\right)$, and there are cycles $\gamma_{AC} = P_{AC} \dots AC \dots P_{CA}P_{AC}$ of length k. Again, if k is odd, P_{CA} and P_{AC} coincide. Note that there might be several choices for P_{AC} and P_{CA}, and for γ_{AC}. If both P_{AC} and P_{CA} are contained in T^*_{AB}, any according γ_{AC} is nullhomotopic, as it can be homotopically moved to be contained in T^*_{AB} completely. If all possible P_{AC} and P_{CA} are of this type, AC is not contained in an essential k-cycle. That is a contradiction to the choice of AB and γ^*_{AB} not containing any other such construction. Thus, there are pairs of vertices P_{AC} and P_{CA} that are not both contained in T^*_{AB}. As C is contained in the interior of T^*_{AB}, the shortest connection $C \dots P_{CA}$ in any according γ_{AC} intersects γ^*_{AB} at least once. Denote one such intersection by S_{AC}.

Assume that S_{AC} is contained in the shortest connection $A \dots P_{AB} \subset \gamma^*_{AB}$. Then, as $A \dots P_{AB}$ and $C \dots P_{CA}$ have the same length and are both shortest possible, the distance $d(A, S_{AC})$ equals $d(C, S_{AC})$, and subsequently, P_{CA} is as close to A as it is to C. Thus, $B_{\frac{k-1}{2}}(A)$ contains all of γ_{AC} and is not homeomorphic to a disc, a contradiction. Thus, S_{AC} is contained in the shortest connection $B \dots P_{BA} \subset \gamma^*_{AB}$.

The distance $d(C, S_{AC})$ is at least as great as the distance $d(B, S_{AC})$. Otherwise, A would be as close or closer to S_{AC} as B, and $B_{\frac{k-1}{2}}(A)$ would contain all of γ^*_{AB}, rendering it not homeomorphic to a disc. Similarily, $d(B, S_{AC}) \geq d(C, S_{AC})$. Otherwise, $B_{\frac{k-1}{2}}(A)$ would contain all of γ_{AC}. Thus, $d(C, S_{AC}) = d(B, S_{AC})$. By moving the part of γ_{AC} that is inside T^*_{AB} to the boundary γ^*_{AB}, we get a cycle that is as long as γ_{AC}, contains the edge AB and is homotopic to γ_{AC}. As AB is not contained in an essential k-cycle, γ_{AC} is non-essential. Recall that γ_{AC} was chosen arbitrarily with P_{AC} and P_{CA} not both contained in T^*_{AB}, and that γ_{AC} is non-essential as well if both P_{AC} and P_{CA} are contained in T^*_{AB}. Thus, AC is not contained in an essential k-cycle. By the minimality of T^*_{AB}, all γ_{AC} have at least one of P_{AC} and P_{CA} outside of T^*_{AB}, and they intersect γ^*_{AB} in a vertex S_{AC} as close to C as it is to B.

The same arguments hold for BC. It is not contained in an essential k-cycle, and any locally essential k-cycle $\gamma_{BC} = P_{BC} \dots BC \dots P_{CB}P_{BC}$ connects BC to vertices P_{BC}, P_{CB} that are not both contained in T^*_{AB} ($P_{BC} = P_{CB}$ if k is odd), and γ_{BC} intersects γ^*_{AB} in a vertex S_{BC} that is as close to A as it is to C.

Due to the relative positioning of γ_{AC} and γ_{BC}, there is another intersection S of γ_{AC} and γ_{BC} outside of T^*_{AB}. Otherwise, at least one of the cycles would have to be essential. Assume first that S is contained in the part $A \dots P_{AC}$ of γ_{AC}, and in the part $C \dots P_{CB}$ of γ_{BC}. Again, both parts have the same length, and both are shortest possible. Thus,

$d(A,S) = d(C,S)$ and, subsequently, P_{AC} is as close to C as it is to A. Thus, $B_{\frac{k-1}{2}}(C)$ contains all of γ_{AC} and is not homeomorphic to a disc, a contradiction. Similarly, the parts $C \ldots P_{CA} \subset \gamma_{AC}$ and $B \ldots P_{BC} \subset \gamma_{BC}$ cannot intersect, and neither can $S_{AC} \ldots P_{CA} \subset \gamma_{AC}$ and $S_{BC} \ldots P_{CB} \subset \gamma_{BC}$ (recall that $d(C, S_{AC}) = d(B, S_{AC})$ and $d(C, S_{BC}) = d(A, S_{BC})$).
Thus, γ_{AC} and γ_{BC} intersect in a vertex S that is contained in the part $A \ldots P_{AC}$ of γ_{AC}, and in the part $B \ldots P_{BC}$ of γ_{BC}. Similar to the intersection S_{AC} between γ^*_{AB} and γ_{AC}, the distances $d(S,A)$ and $d(S,B)$ are the same.
C is as close to P_{AB} as A, and as close to P_{BA} as B. Thus, $B_{\frac{k-1}{2}}(C)$ contains all of γ^*_{AB}. As $B_{\frac{k-1}{2}}(C)$ is homeomorphic to a disc, it has to also contain all of T^*_{AB}. Similarly, $B_{\frac{k-1}{2}}(A)$ contains all of T_{BC}, and $B_{\frac{k-1}{2}}(B)$ contains all of T_{AC}. Note that these arguments hold for any choice of γ_{AC} and γ_{BC}. Also note that for any such choice, B is contained in the interior of T_{AC}, and A is contained in the interior of T_{BC}. Otherwise, $B_{\frac{k-1}{2}}(B)$ (resp. $B_{\frac{k-1}{2}}(A)$) would contain P_{BA} (resp. P_{AB}) and its neighbouring edges, a contradiction. For the final contradiction concluding the proof, it remains to show that $B_{\frac{k-1}{2}}(C)$ contains all of T_{AB} for any choice of γ_{AB}. Figure 3.6 visualizes the remaining arguments.

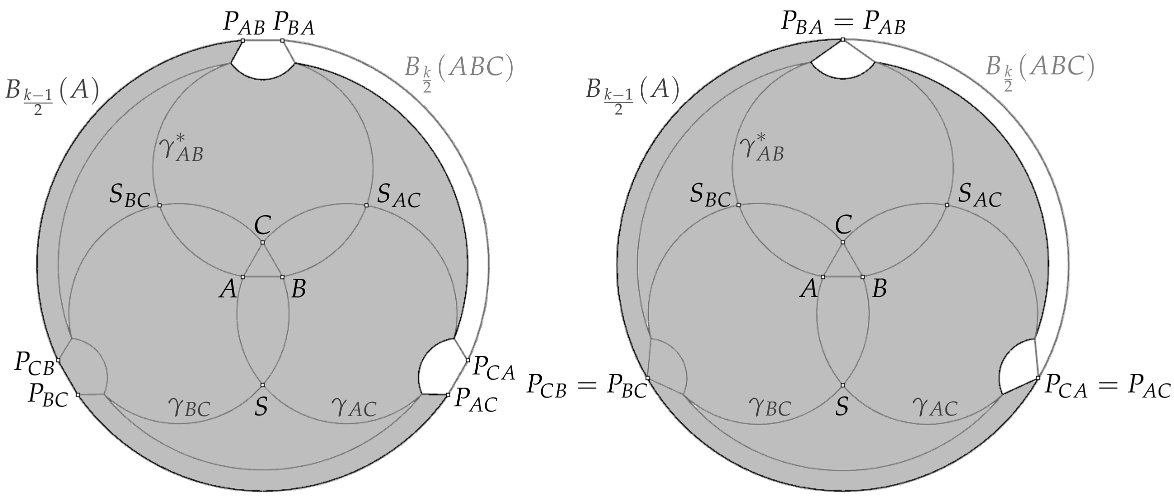

Figure 3.6.: Local configurations described in the proof of Theorem 3.12, refined

Let γ_{AB} once again be an arbitrary non-essential cycle of T that is contained in $B_{\frac{k}{2}}(AB)$, that is essential in that subtriangulation and that contains the edge AB. Thus, we no longer require T_{AB} to be minimal. As we have seen before, for any fixed γ_{AC} and for any fixed γ_{BC}, B is contained in the interior of T_{AC}, and $A \in \mathrm{int}(T_{BC})$. γ_{AB} is essential in $B_{\frac{k}{2}}(AB)$, T_{AC} is contained in $B_{\frac{k-1}{2}}(B)$, and T_{BC} in $B_{\frac{k-1}{2}}(A)$. Thus, $B \ldots P_{BA} \subset \gamma_{AB}$ intersects γ_{AC}, and $A \ldots P_{AB}$ intersects γ_{BC}. In analogy to the discussion of the positioning of S, these intersections cannot occur in the part $A \ldots P_{AC}$ of γ_{AC}, nor in the part $B \ldots P_{BC}$ of γ_{BC}. Otherwise, either $B_{\frac{k-1}{2}}(A)$ or $B_{\frac{k-1}{2}}(B)$ contain all of γ_{AB} and are not homeomorphic to a disc. Thus, the intersections behave similar to S_{AC} and S_{BC}. In particular, C has the same distance to each of them as A, respectively B. Thus, $B_{\frac{k-1}{2}}(C)$ once again contains all of γ_{AB}, and thus all of T_{AB}, for an arbitrary γ_{AB}.

This produces the final contradiction. On the one hand, the diameter-k-neighbourhood of the triangle ABC is not homeomorphic to a disc, as T is exactly (k-1)-locally disc. On the other hand, that neighbourhood consists of the union of the diameter-k-neighbourhoods of the three edges of ABC, the only other addition being triangles at maximum distance of ABC, with every one of their vertices being contained in exactly one radius-$\frac{k-1}{2}$-neighbourhood of A, B, or C. In particular, any cycle that is essential in $B_{\frac{k}{2}}(ABC)$ is also essential in one of the corresponding edge neighbourhoods, and any such cycle is homotopic to a γ_{AB}, γ_{AC} or γ_{BC}. As we have just seen, these are nullhomotopic in the diameter-(k-1)-neighbourhood of the opposite vertex, and thus, in $B_{\frac{k}{2}}(ABC)$. Thus, $B_{\frac{k}{2}}(ABC)$ does not contain locally essential cycles, a contradiction.
In conclusion, there are no edges $AB \in T$ that are not contained in an essential k-cycle.

□

3.4. A natural definition of k-irreducible 2-spheres

Note that the exact local disc property is not only equivalent to k-irreducibility on any 2-manifold but the sphere, it also produces meaningful triangulations on the 2-sphere. Recall that, with the classic definition of k-irreducibility, the tetrahedron is the only k-irreducible 2-sphere, for any k. This is in stark contrast to the intuition that increasing the parameter k should yield a better approximation of the smooth surface. Now, we are able to amend this, by giving a natural redefinition of k-irreducibility on the 2-sphere.

Definition 3.13 (k-irreducible 2-spheres)**.**
Let $k \geq 3$ be an integer. A triangulation T of S^2 is called k-irreducible*, if it is exactly (k-1)-locally disc.*

With this definition, all triangulations associated with platonic solids become k-irreducible, with the cube and the dodecahedron being completed to a triangulation by adding a central vertex into every face. Up to a maximum size of 17 vertices, these are actually the only up to 8-irreducible 2-spheres, as shown by a complete classification algorithm (see Section 5.5). Also, glueing together two copies of any one of the 11255 up to 6-irreducible triangulations of the projective plane known to us (see Section 5.2) yields a 2k-irreducible 2-sphere, further supporting the intuitiveness of the definition.
Note that the proof of Theorem 3.12 implies that, in an exactly k-locally disc triangulation of the 2-sphere, the diameter-(k+1)-neighbourhood of any edge contains the whole triangulation.

3.5. Local geodesity

When talking about a triangulation being locally disc, the question is whether certain subtriangulations are homeomorphic to a disc. One could also ask the other way around: Given a k-irreducible triangulation and a subtriangulation that is homeomorphic to a disc, what properties does it have?

Definition 3.14 (Local geodesity, max-k local geodesity)**.**
Let T be a triangulation of a 2-manifold other than the sphere. T is called locally geodesic*, if for every edge $AB \in E$ and every subtriangulation $AB \in T_0 \leq T$ homeomorphic to a disc, there is a path $w = P_1 \ldots AB \ldots P_2$ in that subtriangulation that connects two boundary vertices $P_1, P_2 \in \delta T_0$, contains the edge AB and is shortest possible under all paths connecting P_1 and P_2 in T_0.*
Let k be a positive integer. T is called max-k locally geodesic*, if it is locally geodesic and the length of all shortest paths w containing the edge in question is at most k.*

Thus, in a max-k locally geodesic triangulation, every edge is geodesic in every disc it is contained in, and the diameter of every disc is limited twice. One limit is due to the number of vertices on the boundary of that disc, the other limit is k. As such, the smaller the k, the stronger the condition max-k local geodesity, leading to the following implications.

$$\forall \tilde{k} \geq k : \text{max-}k \text{ local geodesity} \Rightarrow \text{max-}\tilde{k} \text{ local geodesity} \Rightarrow \text{local geodesity}$$

Note that there are no locally geodesic triangulations on the 2-sphere. There, every triangle bounds two discs. For the triangulation to be locally geodesic, both of them would have to be triangles (as the maximum distance of two boundary vertices in a disc bounded by three vertices is 1). Thus, the whole triangulation would consist of only these two triangles, a contradiction.

Lemma 3.15.
Let $k \geq 3$ be an integer, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. Then, T is also max-(k-1) locally geodesic.

Proof. Let T be k-irreducible. In particular, it is ew-k and every one of its edges is contained in an essential k-cycle. We show that it is max-(k-1) locally geodesic.
Let $AB \in E$ be an edge, and let γ be an essential k-cycle containing AB. Any subtriangulation $T_0 \leq T$ that contains AB and is homeomorphic to a disc has a nullhomotopic boundary δT_0. Thus, as γ contains an edge in T_0 and is essential, it is not completely contained in T_0 and intersects the boundary δT_0 in at least two vertices. Let P_1, P_2 be two of these intersections such that the part $w = P_1 \ldots AB \ldots P_2$ of γ does not contain any other vertices of the boundary δT_0. As γ has length k and is not completely contained in T_0, w has length at most k-1. Replacing w in γ by any other connection $P_1 \ldots P_2$ inside T_0 yields another essential cycle, homotopic to γ. As T is ew-k, there are no essential cycles shorter than k and thus, there is no such other connection of P_1 and P_2 that is shorter than w.

Thus, w is shortest-possible. As both the edge $AB \in E$ and the containing disc $T_0 \leq T$ were chosen arbitrarily, T is max-(k-1) locally geodesic. $\square$

With this, we can prove two nice, local properties of k-irreducible triangulations:

Corollary 3.16.
Let $k \geq 3$ be an integer, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. Then, any neighbour of a 4-valent vertex in T has degree at least 6.

Proof. This follows directly from Lemma 3.15. Assume that there is a neighbour B of a 4-valent vertex A with degree less than 6. Then, the radius-$\frac{3}{2}$-neighbourhood of their connecting edge AB, which is a disc if k is greater than 3, contains at most five outer vertices. As any cycle containing AB connecting two boundary vertices of that disc has length at least 3, there is no such cycle that cannot be homotopically shortened by moving it to the boundary, a contradiction to the max-(k-1) local geodesity, if k is greater than 3. For $k = 3$, the neighbourhood cannot be a disc, as AB has to be contained in an essential 3-cycle. But then, two of the at most five outer vertices of the neighbourhood have to be identical, either creating a forbidden double edge, or a 3-valent vertex, contradicting Corollary 3.3. $\square$

Corollary 3.17.
Let $k \geq 3$ be an integer, and let M be a 2-manifold other than the sphere. Then, there is a minimal k-irreducible triangulation of M that does not contain 4-valent vertices.

Proof. Let $k \geq 3$ be an integer, let T be a k-irreducible triangulation of a 2-manifold other than the sphere, and let A be a 4-valent vertex in T. Then, any edge connecting two neighbouring vertices of A can be flipped without creating an essential cycle shorter than k: Let BC be that edge, denote the other neighbours of A by D and E, and the other vertex forming a triangle with BC by F (see Figure 3.7).

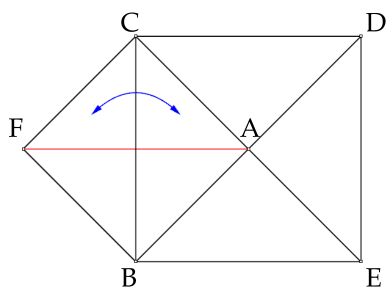

Figure 3.7.: Flipping an edge near a 4-valent vertex

Flipping BC creates a new edge AF, and any essential cycle shorter than k created by the flip would have to contain AF. Let γ be a shortest essential cycle containing AF after the flip. Then, it intersects the disc $FCDEB$ in two vertices. The part of γ inside the disc can be pulled to its boundary without increasing its length, creating a new essential cycle $\tilde{\gamma}$ homotopic to and not longer than γ. As $\tilde{\gamma}$ does not contain AF, it is already contained in T before the flip, and thus, as T is k-irreducible, $\tilde{\gamma}$, and with it γ itself, have length at least k. Thus, any such flip preserves the ew-k property of T.

The flip increases the valency of A and F, and decreases that of B and C. As a k-irreducible triangulation that is not the tetrahedron contains no 3-valent vertices (Corollary 3.3), both A and F are at least 5-valent after the flip, and by Corollary 3.16, so are B and C.

If, after the flip, there are edges no longer contained in an essential k-cycle, they can be contracted. Thus, for any k-irreducible triangulation T of a 2-manifold other than the sphere that contains 4-valent vertices, there is another k-irreducible triangulation of the same 2-manifold that has less vertices or that contains less 4-valent vertices than T. □

As max-k local geodesity implies max-$\tilde{k}$ local geodesity for any $\tilde{k}$ greater than or equal to k, the inverse implication of Lemma 3.15 – max-(k-1) locally geodesity implying k-irreduciblity – cannot be true. Our intuition is that it should hold when adding ew-k as a lower bound. Unfortunately, the ideas at hand only seem strong enough to prove this for the case that k is odd.

Lemma 3.18.
Let $k \geq 3$ be an <u>odd</u> integer, and let T be an ew-k triangulation of a 2-manifold other than the sphere. If T is max-(k-1) locally geodesic, then it is also k-irreducible.

Proof. Let $k \geq 3$ be odd and let T be ew-k and max-(k-1) locally geodesic. We show that every one of its edges is contained in an essential k-cycle.

Let $AB \in E$ be an edge. In its local neighbourhood $B_{\frac{k}{2}}(AB)$, any path w containing AB and connecting two boundary vertices $P_1, P_2 \in \delta B_{\frac{k}{2}}(AB)$ has length $|w|$ at least k. Thus, $B_{\frac{k}{2}}(AB)$ is not homeomorphic to a disc. Stronger still, it must contain an essential cycle γ. Otherwise, it could be completed to a subtriangulation $T_0 \leq T$ homeomorphic to a disc, without reducing the distance of AB to the boundary, a contradiction to T being max-(k-1) locally geodesic.

As T is ew-k, it is also (k-1)-locally planar. Thus, γ is not contained in $B_{\frac{k-1}{2}}(AB)$. Thus, similar to the arguments in the proof of Lemma 3.7, there is an essential cycle $\tilde{\gamma} = P \ldots AB \ldots P$, with a vertex $P \in B_{\frac{k}{2}}(AB) \setminus B_{\frac{k-1}{2}}(AB)$. $\tilde{\gamma}$ has length k and contains AB, concluding the proof. □

For even k, this argument fails at the beginning. In that case, $B_{\frac{k}{2}}(AB)$ does not contain any more vertices than $B_{\frac{k-1}{2}}(AB)$. Thus, the distance of a shortest path containing AB and connecting two boundary vertices can be k-1. The first neighbourhood to definitely contain an essential cycle is $B_{\frac{k+1}{2}}(AB)$, guaranteeing an essential cycle containing AB of length k+1. Alternatively, similar arguments hold for $B_{\frac{k}{2}}(A)$ and $B_{\frac{k}{2}}(B)$, yielding essential k-cycles containing the vertices A and B, though not necessarily the edge AB.

3.6. A simple, genus-independent, lower size bound

Using that k-irreducible triangulations are both max-(k-1) locally geodesic and (k-1)-locally disc, we give a genus-independent lower bound for the number of vertices in such a triangulation. The result is new only for non-orientable 2-manifolds. For orientable 2-manifolds, this repeats a result Albertson and Hutchinson [1] gave in 1978, and adds a nice, short, triangulation-oriented proof to it.

Theorem 3.19.
Let $k \geq 3$ be an integer, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. Then, T contains at least $f_0 \geq \frac{k^2}{2}$ vertices.

Proof. Let k be an integer greater than or equal to 3, and let T be a k-irreducible triangulation of a 2-manifold other than the sphere. In particular, T is also (k-1)-locally disc. Thus, for any vertex $A \in V$, its neighbourhoods $B_{\frac{i}{2}}(A)$ are homeomorphic to a disc, for all diameters $1 \leq i \leq k$-1. Thus, their boundaries $\delta B_{\frac{i}{2}}(A)$ are simple closed cycles and, as the vertices on the boundaries have distance $\lfloor \frac{i}{2} \rfloor$ from the central vertex A, the vertices of these cycles are distinct for all even diameters $i = 2j$.
On the other hand, T is also max-(k-1) locally geodesic. Thus, for any neighbouring edge $AB \in E$ of the central vertex A and any neighbourhood $B_{\frac{i}{2}}(A)$ homeomorphic to a disc, there is a shortest connection of two boundary vertices $P_1, P_2 \in \delta B_{\frac{i}{2}}(A)$ containing AB. The length of any such connection is at least i. Thus, for the connection of the boundary vertices to be at least as long as the shortest connection containing AB, there have to be at least $2i$ vertices on the boundary.
The total number of vertices of T is at least as big as the sum of the numbers of vertices of all distinct cycles $\delta B_{\frac{i}{2}}(A)$ with even diameters $2 \leq i \leq k$-1. Thus, we get a bound of

$$f_0 \geq 1 + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} |\delta B_{\frac{2j}{2}}(A)| \geq 1 + \sum_{j=1}^{\lfloor \frac{k-1}{2} \rfloor} 4j = \begin{cases} 1 + 4\frac{\frac{k-1}{2}\frac{k+1}{2}}{2} = \frac{k^2+1}{2}, & k \text{ odd}, \\ 1 + 4\frac{(\frac{k}{2}-1)\frac{k}{2}}{2} = \frac{(k-1)^2+1}{2}, & k \text{ even}. \end{cases}$$

This concludes the proof for odd k. The same arguments hold for neighbourhoods of an arbitrary edge $AB \in E$. There, the vertices of the surrounding simple cycles are distinct for all odd diameters $i = 2j + 1$. Summing up yields bounds of

$$f_0 \geq 2 + \sum_{j=1}^{\lfloor \frac{k-2}{2} \rfloor} |\delta B_{\frac{2j+1}{2}}(AB)| \geq 2 + \sum_{j=1}^{\lfloor \frac{k-2}{2} \rfloor} 2(2j+1) = \begin{cases} 2 + 4\frac{\frac{k-3}{2}\frac{k-1}{2}}{2} + k - 3 = \frac{(k-1)^2}{2}, & k \text{ odd}, \\ 2 + 4\frac{(\frac{k}{2}-1)\frac{k}{2}}{2} + k - 2 = \frac{k^2}{2}, & k \text{ even}, \end{cases}$$

concluding the proof for even k as well. □

This bound renders the series constructed in Section 6.1 asymptotically minimal. Other than that, it is mainly of theoretical interest, especially since it is not dependent on the genus. The smallest series of k-irreducible triangulations of the projective plane has growth rate $\frac{2}{3}k^2$ (see Section 6.1.2) and is known to be optimal up to $k = 6$ (see Section 5.2). The smallest orientable series, k-irreducible tori with a growth rate of $\frac{3}{4}k^2$, is known to be optimal up to $k = 5$ (see Section 6.1.3 for the construction, and Section 5.2 again for the computational proof of minimality).

3.7. New lower bounds for fixed k

Theorem 3.20.
Let T be an ew-5-triangulation of a 2-manifold with Euler characteristic χ. Then, the following holds:

$$f_0^3 - 19f_0^2 + 54\chi f_0 - 36\chi^2 \geq 0$$

In particular, in any infinite series of ew-5-triangulations, the number of vertices f_0 of the triangulations grows at least as fast as $g^{\frac{2}{3}}$, with g denoting the Euler genus of the underlying 2-manifold.

Proof. Any ew-5-triangulation of a 2-manifold can be reduced to a 5-irreducible triangulation of that same 2-manifold by a finite series of edge contractions. Thus, w.l.o.g., let T be a 5-irreducible triangulation of a 2-manifold with Euler genus g. In particular, T is 4-locally disc. Thus, for any vertex $Z \in V$, its diameter-4 neighbourhood $B_2(Z)$ is homeomorphic to a planar disc.
Denote the valency of a vertex $A \in V$ by v_A. All vertices in $B_2(Z)$ are adjacent to a direct neighbour of Z. Thus, when summing up the valencies of the direct neighbours of Z, we count all the vertices in $B_2(Z)$, though some of them more than once: Z is adjacent to all its neighbours, so it itself is counted v_Z times. The v_Z direct neighbours of Z are themselves adjacent to two other neighbours of Z, each. Thus, they are counted twice. There are no additional edges between direct neighbours of Z. Otherwise, since T is 4-locally disc, there would be 3-valent neighbour of Z, a contradiction to T being locally geodesic. In addition, every two neighbours of Z that are adjacent have a second common neighbour in $B_2(Z) \setminus B_1(Z)$. Those common outer neighbours are thus counted once too often for all v_Z edges connecting neighbours of Z. Thus, in total, $B_2(Z)$ contains $1 + \sum_{A \in \delta B_1(Z)} (v_A - 3)$ vertices.
T contains at least as many vertices as the largest diameter-2 neighbourhood of any one of its vertices, and this largest neighbourhood is at least as large as the average size of all diameter-2 neighbourhoods, $\overline{|B_2|} = \frac{1}{f_0} \sum_{Z \in V} |B_2(Z)| = \frac{1}{f_0} \sum_{Z \in V} \Big(1 + \sum_{A \in \delta B_1(Z)} (v_A - 3)\Big)$. A vertex A is a direct neighbour of a vertex Z if and only if Z is also a direct neighbour of A. Thus, the last term, $(v_A - 3)$, appears a total number of v_A times in this sum. Thus, we get $\overline{|B_2|} = \frac{1}{f_0} \sum_{A \in V} (1 + v_A(v_A - 3))$. If we denote the average valency by $\overline{v}$ and fragment the valency v_A into its average part and its deviation from it, $\overline{v} + (v_A - \overline{v})$, we get

$$\overline{|B_2|} = \frac{1}{f_0} \sum_{A \in V} \left(\overline{v}^2 - 3\overline{v} + 1\right) + \frac{2\overline{v} - 3}{f_0} \sum_{A \in V} (v_A - \overline{v}) + \frac{1}{f_0} \sum_{A \in V} (v_a - \overline{v})^2 .$$

The first sum contains a constant term, f_0 times. The second sum contains the deviations from the average valency. The sum of all these deviations is zero. The last sum contains only quadratic terms, and is thus greater than or equal to zero. Thus, we can conclude that the average size $\overline{|B_2|}$ of a diameter-2 neighbourhood grows at least quadratic in the average valency $\overline{v}$, and that average size was a lower limit for the number of vertices f_0 of T.

$$f_0 \geq \overline{|B_2|} \geq \overline{v}^2 - 3\overline{v} + 1 = \left(\overline{v} - \frac{3}{2}\right)^2 - \frac{5}{4} \tag{3.1}$$

With the Handshaking lemma, $2f_1 = \overline{v} f_0$, and Euler's formula for 2-manifolds, $f_0 - \frac{f_1}{3} = \chi$ (1.3), we can express the average valency $\overline{v}$ as

$$\overline{v} = 6(1 - \frac{\chi}{f_0}).$$

Inserting this into (3.1), we get

$$f_0 \geq \left(6(1 - \frac{\chi}{f_0}) - \frac{3}{2}\right)^2 - \frac{5}{4} = 19 - 54\left(\frac{\chi}{f_0}\right) + 36\left(\frac{\chi}{f_0}\right)^2,$$

and thus,

$$f_0^3 - 19f_0^2 + 54\chi f_0 - 36\chi^2 \geq 0.$$

With Euler characteristic χ and genus g being linked by $\chi = 2 - 2g$ for orientable, and $\chi = 2 - g$ for non-orientable 2-manifolds, this yields the claimed lower bound for the number of vertices f_0 in a 5-irreducible triangulation, and thus, in any triangulation with edge width at least 5. □

In particular, this yields explicit lower bounds for the size of ew-5 triangulations of small-genus 2-manifolds, as shown in Table 3.1. These bounds are known to be tight for the projective plane and the torus (see the constructions in Section 6.1), and improved for the Klein bottle by the complete classification (Section 5.2). The results of the heuristic calculations presented in Section 5.4 add upper bounds for the minimal size of such triangulations.

χ	orientable	non-orientable
1		$f_0 = 16$
0	$f_0 = 19$	$21 \leq f_0 \leq 22$
-1		$22 \leq f_0 \leq 25$
-2	$24 \leq f_0 \leq 27$	$24 \leq f_0 \leq 26$
-3		$26 \leq f_0 \leq 29$
-4	$28 \leq f_0 \leq 31$	$28 \leq f_0 \leq 31$
-5		$30 \leq f_0 \leq 33$
-6	$31 \leq f_0 \leq 36$	$31 \leq f_0 \leq 35$
-7		$33 \leq f_0 \leq 37$
-8	$34 \leq f_0 \leq 39$	$34 \leq f_0 \leq 39$

Table 3.1.: Size bounds for minimal ew-5 triangulations with Euler genus up to 10

Theorem 3.21.
Let $k \geq 3$ be an integer, and let T be a vertex-regular k-irreducible triangulation of a 2-manifold with Euler genus $g > 0$. Then, the number of vertices f_0 of T grows at least as fast as $g^{\frac{k-1}{k+1}}$, if k is odd, and at least as fast as $g^{\frac{k-2}{k}}$, if k is even.

Proof. Let T be a vertex-regular k-irreducible triangulation for some integer k greater than or equal to 3 with an underlying 2-manifold with Euler genus $g > 0$. Thus, all vertices of T have a fixed number $\overline{v}$ of neighbours. As the underlying 2-manifold is not the sphere, $\overline{v}$ is at least 5.

As T is k-irreducible, it is also (k-1)-locally disc. Thus, for any vertex $Z \in V$, all its neighbourhoods $B_{\frac{i}{2}}(Z)$ with diameter i less than or equal to k-1 are homeomorphic to a disc. $B_0(Z)$ is just the vertex itself. $B_1(Z)$ contains Z and its direct neighbours, so, $\overline{v} + 1$ vertices. As shown in the proof of Theorem 3.20, $B_2(Z)$ contains $\overline{v}(\overline{v} - 3) + 1$ vertices, if k is at least 5. Any bigger neighbourhood $B_i(Z)$ with radius i at most $\frac{k-1}{2}$ contains the vertices of $B_{i-1}(Z)$ and adds $|\delta B_{i-1}(Z)|(\overline{v} - 4) - |\delta B_{i-2}(Z)|$ new ones:

Every vertex in the boundary $\delta B_{i-1}(Z)$ of the next-smaller neighbourhood has $\overline{v}$ neighbours. At least one of them is a connection to the inside of the neighbourhood, another two of them are its neighbours within the boundary, and two are common outer neighbours, shared with its direct neighbours in $\delta B_{i-1}(Z)$. The first three are no new addition of $B_i(Z)$, the latter two are counted twice. Thus, for any boundary vertex of the smaller neighbourhood, at most $\overline{v} - 4$ new vertices are added in $B_i(Z)$. In this count, every vertex has only one interior neighbour, so, any additional ones have yet to be subtracted. For any vertex in $\delta B_{i-1}(Z)$ with more than one interior neighbour, there is an edge in $\delta B_{i-2}(Z)$ such that that edge, together with the outer vertex, forms a triangle of T. On the other hand, any edge in $\delta B_{i-2}(Z)$ corresponds to an outer vertex with more than one interior neighbour. Thus, the number of additional interior neighbours that has to be distracted, is $|\delta B_{i-2}(Z)|$. Note that, as $\overline{v}$ is at least 5, every vertex in the boundary of such a neighbourhood has at least one neighbour not contained in the neighbourhood itself.

This yields that the number of vertices in a neighbourhood $B_i(Z)$ of any vertex $Z \in V$ with radius i at most $\frac{k-1}{2}$ is a polynomial in $\overline{v}$, with degree i. As T contains at least the vertices of $B_{\frac{k-1}{2}}(Z)$, f_0 grows at least as fast as $\overline{v}^{\lfloor \frac{k-1}{2} \rfloor}$.

$$f_0 \notin o(\overline{v}^{\lfloor \frac{k-1}{2} \rfloor})$$

As demonstrated in the proof of Theorem 3.20, Handshaking lemma and Euler's formula for 2-manifolds yield

$$\overline{v} = 6 - 6\frac{\chi}{f_0},$$

and thus, with respect to g, f_0 grows at least polynomial, with degree $\frac{\lfloor \frac{k-1}{2} \rfloor}{\lceil \frac{k+1}{2} \rceil}$, which equals $\frac{k-1}{k+1}$ if k is odd, and $\frac{k-2}{k}$, if k is even. □

Without vertex-regularity, there are two problems with this proof strategy. The first is that it cannot be guaranteed that a vertex on the boundary of a neighbourhood has neighbours on the outside. This can be overcome by giving upper and lower bounds for the size of the neighbourhoods, which are potentially not optimal, but still polynomial in the local vertex valencies with the right degrees. The second problem is that it is unclear how the valencies of the triangulation are distributed within it. Is it not true - in general - that a large valency of a central vertex Z yields a large neighbourhood $B_i(Z)$ for some arbitrary radius i. What is unclear, though, is if there is some general relationship between the minimal size of the maximal neighbourhood $B_i(Z)$ and the average valency $\overline{v}$, in k-irreducible triangulations. Our intuition is that it should be polynomial with degree i. Thus, we conjecture Theorem 3.21 to be true in the general case, that is, without vertex-regularity.

Conjecture 3.22.
Let k be an integer, and let T be an ew-k triangulation of a 2-manifold with Euler genus g. Then, the number of vertices f_0 of T grows at least as fast as $g^{\frac{k-1}{k+1}}$, if k is odd, and at least as fast as $g^{\frac{k-2}{k}}$, if k is even.

Note that, for $k = 5$, Theorem 3.20 confirms the conjecture.

4. A construction principle for vertex-transitive ew-k triangulations

We construct vertex-transitive polyhedral maps of a certain face width by finding certain subgroups in the fundamental group of a basic manifold, and creating the maps by identifying edges in the universal covering of that basic manifold. In a second step, the maps are triangulated, preserving both the vertex-transitivity and the face width, which, in triangulations, coincides with the edge width.

Section 4.1 describes the general construction. In Section 4.2, we show that such subgroups actually exist for an arbitrary k, yielding the existence of another series of ew-k triangulations with size $f_0 \in o(g)$, for any k. The smallest such series known for k greater than 5 remains to be the one of Przytycka and Przytycki [16], with size $f_0 \in O(\frac{g}{\log g})$.

In Section 4.3, we present a specific series of groups that yield an infinite series of ew-5 triangulations with size $f_0 \in O(g^{\frac{8}{9}})$. This is the smallest known such series at this point and, in particular, the first series of at least 5-irreducible triangulations with polynomial growth rate in g, with a rational exponent less than 1. We believe the actual minimum to be closer to our bound of Theorem 3.20 ($f_0 \notin o(g^{\frac{2}{3}})$). Still, this series proves that the series of Przytycka and Przytycki is not asymptotically best possible for all k greater than 4, which, as mentioned in Section 2.2.1, seems to have been common believe due to a cycle of unfortunate citations.

4.1. Moving edge width checks into a group theoretic setting

We start with a $2n$-gon, and choose a pairwise identification of its outer edges such that the induced identification only leaves a single vertex. This yields a polyhedral map P_0, consisting of a single vertex, a single face, and n edges. Thus, it is homeomorphic to a 2-manifold M_0 with Euler characteristic $\chi_0 = 1 - n + 1$, which might be non-orientable. Any edge on the boundary of the $2n$-gon represents a basic essential cycle of M_0. Moving once around the $2n$-gon and noting the edges and their orientation yields a rule of length $2n$. The fundamental group $\pi_1(M_0)$ of M_0 is the free group on n generators, factorized by that rule. Every edge of the $2n$-gon appears twice in the rule. If these appearances are oriented in opposite directions, for all edges, M_0 is orientable. Otherwise, it is non-orientable.

Every 2-manifold has a universal covering. That of M_0 is a regular tessellation of the hyperbolic plane into $2n$-gons, with $2n$ of them meeting in every vertex. In particular, it is an infinite, vertex- and face-transitive polyhedral map P_∞, with its deck transformation group being isomorphic to the fundamental group $\pi_1(M_0)$ of M_0.

A normal subgroup $N \trianglelefteq \pi_1(M_0)$ of the fundamental group of M_0, with a finite quotient group $G = \pi_1(M_0)/N$ yields, in this setting, a set of identifications of edges in the universal covering. If N is chosen in a way that it does not imply any additional identifications of the n generating elements of $\pi_1(M_0)$ (the n pairwise identified edges of the original $2n$-gon), the result is again a vertex- and face-transitive polyhedral map P, which is finite. It consists of $|G|$ $2n$-gons, with $2n$ of them meeting in every one of the $|G|$ vertices. The Handshaking lemma yields $2f_1 = \overline{v} f_0 = 2m|G|$, and with Euler's formula, we get an Euler characteristic $\chi = f_0 - f_1 + f_2 = |G|(2-m)$ for the underlying 2-manifold M of P. Note that there are orientable covers M of non-orientable manifolds M_0.

Depending on the identifications implicated by N, P has a certain face width w, that is, the number of vertices in the shortest essential cycle in P not crossing any edges (but instead moving from face to face through a common vertex). Note that the face width is always lower than or equal to the edge width, as any cycle containing only edges and vertices does not technically cross an edge. In triangulations, or triangular polyhedral maps, the inverse also holds, as any path crossing a triangle from one vertex to another can be pulled to one of the edges of the triangle. Thus, in triangular polyhedral maps, edge width and face width are the same.

The last step: Any triangulation of the original $2n$-gon is lifted to a triangular polyhedral map T of M, as P is a finite cover of M_0. Triangulating the faces of P might increase the face width, but cannot decrease it. Thus, T has face width at least w, and thus is ew-w. If the underlying triangulation of the $2n$-gon does not introduce new vertices, and thus only adds $2n-3$ diagonals, the resulting triangular polyhedral map T also keeps the vertex-transitivity of P. In that case, T has $f_0 = |G|$ vertices with valency $\overline{v} = 2n + 2(2n-3) = 6n-6$ each, and it has $f_1 = |G|(3n-3)$ edges as well as $f_2 = |G|(2n-2)$ triangles.

Thus, we can construct vertex-transitive ew-k triangulations by finding a finite quotient group G of the fundamental group $\pi_1(M_0)$ of some arbitrary 2-manifold M_0 that has a 1-vertex representation, with G fulfilling certain properties (not inducing essential face cycles of length less than k in P). Moreover, any diagonal-only triangulation of that 1-vertex representation yields such a triangulation with the same G, though some of those might be isomorphic.

Theorem 4.1.
Let P_0 be a 1-vertex representation of a 2-manifold M_0 with a single face and n edges, and let $N \trianglelefteq \pi_1(M_0)$ be a normal subgroup of the fundamental group of M_0 such that the polyhedral map P created by identifying edges in the universal covering of P_0 according to N is finite and has face width at least k, for some integer $k \geq 3$, and let $G = \pi_1(M_0)/N$ denote the quotient group created by N. Then, there are vertex-transitive ew-k triangulations T of a 2-manifold M with Euler characteristic $\chi = |G|(2-n)$, with $f_0 = |G|$ vertices of valency $\overline{v} = 2n + 2(2n-3) = 6n-6$ each, with $f_1 = |G|(3n-3)$ edges as well as $f_2 = |G|(2n-2)$ triangles.

Proof. Follows directly from the construction described above. □

Note that there is a high degree of freedom to work with here, as the underlying 2-manifold M_0 as well as the triangulation of its 1-vertex representation as well as that 1-vertex representation itself can be chosen arbitrarily. The difficult part is finding a suitable group G. The condition for P having face width at least k is that N does not contain an element not naturally closing in the universal covering that is created by combining up to $k-1$ diagonals of the basic $2n$-gon. Otherwise, that element will be a non-essential cycle of length at most $k-1$ in T. A diagonal, on the other hand, is any combination of up to n of the n generators of $\pi_1(M_0)$ or their inverses such that they are adjacent as edges of the basic $2n$-gon.

Another benefit of this construction is the size of the resulting triangulations. If a series of groups G_n is found that fulfils the conditions and that have a size $|G_n| \in O(d^n)$ growing polynomially in n, then this yields a series of ew-k triangulations with size growth $f_0 \in O(\chi^{\frac{d}{d+1}})$ for some constant d, possibly dependent on k. This, on the other hand, is exactly the type of vertex growth we conjecture to be optimal for minimal k-irreducible triangulations (and thus, in particular, for minimal ew-k triangulations), see Conjecture 3.22.

4.2. Such groups exist for any edge width limit

Theorem 4.2.
Let P_0 be a 1-vertex representation of a 2-manifold M_0 with a single face, and let k be some positive integer. Then, there is a normal subgroup $N \trianglelefteq \pi_1(M_0)$ of the fundamental group of M_0 such that the polyhedral map P created by identifying edges in the universal covering of P_0 according to N is finite and has face width at least k.

Proof. Let P_0 be a 1-vertex representation of a 2-manifold M_0 with a single face, let k be some positive integer, and let $\pi_1(M_0)$ denote the fundamental group of M_0.

It is well-known that the fundamental groups of 2-manifolds are residually finite, which is a strong and heavily researched group theoretic property. We only need the following implication: For any finite subset $\Sigma \subset \pi_1(M_0)$, there is a normal subgroup $N \trianglelefteq \pi_1(M_0)$ with a finite quotient group $\pi_1(M_0)/N$, such that N does not contain any element of Σ, $N \cap \Sigma = \emptyset$.

Let P_N be the polyhedral map constructed as described in Section 4.1, for some normal subgroup $N \trianglelefteq \pi_1(M_0)$. For any fixed, positive integer k, the condition for P_N having face width at least k is that any path of length less than k that is not closed in the universal covering of P_0 is not closed in P_N, either. In other words: That path, regarded as an element in $\pi_1(M_0)$, cannot be identified in the factorization of $\pi_1(M_0)$ by N. The set $\Sigma \subset \pi_1(M_0)$ of all these paths is large, even for small k, but it is finite. Thus, by the residual finiteness of $\pi_1(M_0)$, there is a normal subgroup N that does not contain any of these paths, with a finite quotient group $\pi_1(M_0)/N$, concluding the proof. □

As described in the previous section, Theorem 4.2 can immediately be extended to triangulations with a given edge width limit by lifting an arbitrary triangular subdivision of P_0, which does not introduce new vertices, up to P:

Corollary 4.3.
Let P_0 be a 1-vertex representation of a 2-manifold M_0 with a single face and n edges, and let $k \geq 3$ be some integer. Then, there are vertex-transitive ew-k triangulations T of a 2-manifold M with Euler characteristic χ, with $\frac{\chi}{f_0} = 2 - n$ and with valency $6n - 6$.

As there are such representations P_0 for arbitrarily large n, and as any ew-k triangulation can be reduced to a k-irreducible one by a series of edge contractions (possibly losing vertex-transitivity), this yields the following existence statement:

Corollary 4.4.
For any integer k at least 3, there is an infinite series of k-irreducible triangulations of 2-manifolds with Euler genus g and with size $f_0 \in o(g)$.

Two things to note: On the one hand, as mentioned before, this is weaker than the result of Przytycka and Przytycki [16] with growth rate $f_0 \in O(\frac{g}{\log g})$. On the other hand, the conclusion step towards Corollary 4.3 is quite rough. When checking the face width condition for the polyhedral map P, the generating elements of any possibly essential cycle are all diagonals of the faces of P, as well as their outer edges, as well as the inverses of these elements. Those faces are $2n$-gons, thus, there are $n(2n - 3)$ diagonals. Adding the outer edges and the inverses, this yields $2n(2n - 2)$ elements. Any combination of up to $k - 1$ of those has to be checked for being non-essential or non-closed. If we introduce fixed diagonals in P_0, thus triangulating P in the lift, there are only $2n - 3$ diagonals in any $2n$-gon, yielding $6n - 6$ elements to be combined into $(k - 1)$-paths. Thus, checking the face width of P is roughly $\left(\frac{2n}{3}\right)^{k-1}$ times as complicated as checking the edge width of any triangulation T constructed in this way. In fact, the explicit construction of ew-5 triangulations we give in the next section makes use of this by checking the edge width of the resulting triangulation T directly, rather than checking the face width of the intermediate configuration P.

4.3. Vertex-transitive ew-5 triangulations with few vertices

In an application of the presented construction, we fix a specific series of configurations and prove that the resulting structures are ew-5 triangulations with a size of $f_0 \in O(g^{\frac{8}{9}})$. In particular, for an infinite set of integers $n \geq 2$, we fix a 1-vertex, 1-face polyhedral map P_0 of a 2-manifold M_0, we fix a triangular subdivison T_0 of P_0 and we give a subgroup $G \leq \pi_1(M_0)$ of the fundamental group $\pi_1(M_0)$, such that the triangulation T resulting from lifting T_0 to the part P of the universal cover of P_0, on which G operates, is ew-5. In particular, P does not necessarily have a face width of at least 5.
For convenience, we introduce a new index variable i with $n = 2i + 2$.

Theorem 4.5.
Let i be a positive integer such that $12i + 7$ is prime. Then, there is a vertex-transitive ew-5 triangulation T with $f_0 = 4(12i + 7)^8$ vertices, every one of which has valency $\overline{v} = 12i + 6$, of an underlying 2-manifold with Euler characteristic $\chi = -8i(12i + 7)^8$.

Note that, with a group of size $|G| = 4(12i + 7)^8$, and with the new index variable i, these are exactly the values given in Corollary 4.3. Similarly, we can deduce the following statement with regard to 5-irreducible triangulations.

Corollary 4.6.
There is an infinite series of 5-irreducible triangulations with $f_0 \in O(|\chi|^{\frac{8}{9}})$ vertices, χ denoting the Euler characteristic of the underlying 2-manifolds.

At the moment, there is no known series of ew-5 triangulations with a smaller growth of their number of vertices, with respect to the genus of the underlying 2-manifolds. Still, again, we do not believe this construction to be optimal, and believe the true minimum to be much closer to the bound of $g^{\frac{2}{3}}$, see Theorem 3.20.
The series at hand shows the power of the general construction introduced in this chapter, while also illustrating some general ideas of group constructions satisfying the ew-5 conditions. Note that the groups were crafted by hand using only elementary group operations. It might be worthwhile to invest some more research in finding other groups, possibly yielding better results. Of particular interest would also be a generalization for higher k, in particular at first $k = 6$ or $k = 7$.

4.3.1. The basic triangulation and the generating group

Following the outline given in Section 4.1, we start with a $(4i + 4)$-gon, and transform it to a 1-vertex, 1-face polyhedral map of an orientable 2-manifold M_0 with Euler characteristic $\chi = -2i$ by identifying opposite edges. The group G is generated by $2i + 2$ elements that are identified with those pairs of opposite edges. Figure 4.1 shows the specific triangulation T_0 of the $(4i + 4)$-gon chosen here as well as the nomination of the group elements identified with their oriented edges. Note that the inner edges $\overrightarrow{I_k}, \overrightarrow{II_k}, \overrightarrow{III_k}, \overrightarrow{IV_k} (k = 1, \dots, i)$ as well as $\overrightarrow{III_0}$ are generated by the outer edges $\overrightarrow{O_k} (k = 1, \dots 2i + 1)$ and $\overrightarrow{I_0}$.
Before presenting the actual group G, note that this exact setup also generates the minimal 3-irreducible triangulations generated by Ringel [17], for the right choice of G:

Remark 4.7.
If we were to set $G = \mathbb{Z}_{12i+7}$ and set the generators to

$$\begin{aligned} O_j &:= \quad (-1)^{j+1} j \in \mathbb{Z}_{12i+7}, \quad j = 1, \dots, 2i + 1, \\ I_0 &:= \quad 3i + 2 \in \mathbb{Z}_{12i+7}, \end{aligned}$$

the resulting triangulations would be the minimal 3-irreducible ones that Ringel constructed for the orientable case 7 in his famous Map Color Theorem ([17]). In fact, our construction contains this

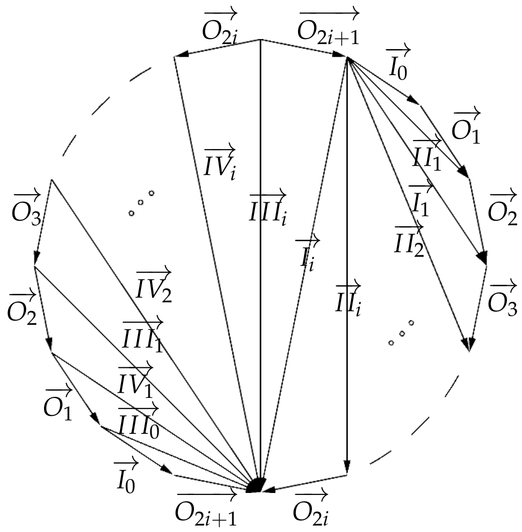

Figure 4.1.: Basic triangulation T_0 and nomination of group elements

group as a subgroup (component C_1), and was originally found by careful extensions of Ringel's work.

Also note that, in order to generate ew-5 triangulations, G has to be non-abelian. Otherwise, there are essential 4-cycles of the form

$$ABA^{-1}B^{-1} = ABB^{-1}A^{-1} = e.$$

Now for the actual group G: It consists of nine components $C_i,\ i \in \{0, \dots, 8\}$, and uses the following group action.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \\ a_4 + b_4 + a_1 b_3 - a_3 b_1 \\ a_5 + b_5 \\ a_6 + b_6 + a_1 b_5 - a_5 b_1 \\ a_7 + b_7 \\ a_8 + b_8 + a_1 b_7 - a_7 b_1 \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}_4 \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \\ \mathbb{Z}_{12i+7} \end{pmatrix}$$

It is an easy excercise to show that G really is a group. Note that the neutral element is $\overrightarrow{e} = \overrightarrow{0}$, the element that has 0 in every component. The inverse elements are componentwise negatives $(\overrightarrow{a})^{-1} = -\overrightarrow{a}$. The generators $\overrightarrow{I_0}$ and $\overrightarrow{O_k},\ k \in \{1, \dots, 2i+1\}$, are:

$$
\underset{\overrightarrow{O_1}}{\begin{pmatrix}0\\1\\1\\1\\0\\1\\0\\0\\0\end{pmatrix}},
\underset{\overrightarrow{O_{2r}}}{\begin{pmatrix}0\\-2r\\(2r)^2\\1\\0\\0\\0\\0\\0\end{pmatrix}},
\underset{\overrightarrow{O_{2r+1}}}{\begin{pmatrix}0\\2r+1\\(2r+1)^2\\1\\0\\0\\0\\0\\0\end{pmatrix}},
(r=1,\dots,i-1),
\underset{\overrightarrow{O_{2i}}}{\begin{pmatrix}0\\-2i\\4i^2\\1\\0\\0\\0\\0\\0\end{pmatrix}},
\underset{\overrightarrow{O_{2i+1}}}{\begin{pmatrix}0\\a:=2i+1\\a^2\\1\\0\\1\\0\\1\\0\end{pmatrix}},
\underset{\overrightarrow{I_0}}{\begin{pmatrix}1\\b:=3i+2\\0\\d\\0\\f\\0\\h\\0\end{pmatrix}}
$$

The inner edges of T_0 can be directly calculated from these generators:

$$
\begin{aligned}
\overrightarrow{I_k} &= \overrightarrow{II_k}\cdot\overrightarrow{O_{2k}} = \overrightarrow{I_0}\cdot\overrightarrow{O_1}\cdot\dots\cdot\overrightarrow{O_{2k}}, && k=1,\dots,i, && (4.1)\\
\overrightarrow{II_k} &= \overrightarrow{I_{k-1}}\cdot\overrightarrow{O_{2k-1}} = \overrightarrow{I_0}\cdot\overrightarrow{O_1}\cdot\dots\cdot\overrightarrow{O_{2k-1}}, && k=1,\dots,i, && (4.2)\\
\overrightarrow{III_0} &= \overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}, && && (4.3)\\
\overrightarrow{III_k} &= \overrightarrow{O_{2k}}\cdot\overrightarrow{IV_k} = \overrightarrow{O_{2k}}\cdot\overrightarrow{O_{2k-1}}\cdot\dots\cdot\overrightarrow{O_1}\cdot\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}, && k=1,\dots,i, && (4.4)\\
\overrightarrow{IV_k} &= \overrightarrow{O_{2k-1}}\cdot\overrightarrow{III_{k-1}} = \overrightarrow{O_{2k-1}}\cdot\overrightarrow{O_{2k-2}}\cdot\dots\cdot\overrightarrow{O_1}\cdot\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}, && k=1,\dots,i. && (4.5)
\end{aligned}
$$

Note that, in order for the outer cycle – and thus all triangles – to close, and thus for G to really be a subgroup of $\pi_1(M_0)$, the generators must fulfill the following condition:

$$\overrightarrow{O_{2i+1}}\cdot\overrightarrow{I_i} = \overrightarrow{O_{2i+1}}\cdot\overrightarrow{I_0}\cdot\overrightarrow{O_1}\cdot\dots\cdot\overrightarrow{O_{2i}} \overset{!}{=} \overrightarrow{O_{2i}}\cdot\dots\cdot\overrightarrow{O_1}\cdot\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}} = \overrightarrow{III_i} \qquad (4.6)$$

Note that this is trivial in an abelian world. The only non-abelian parts of the group action are found in components C_4, C_6 and C_8, where (4.6) yields

$$
\begin{aligned}
9d+10 &\equiv 0 \ (mod\ 12i+7),\\
9f+4 &\equiv 0 \ (mod\ 12i+7),\\
9h+10 &\equiv 0 \ (mod\ 12i+7).
\end{aligned}
$$

for our choice of the outer edges $\overrightarrow{O_j}$ and $C_1(\overrightarrow{I_0})$. As the inverse element to 3 in $\mathbb{Z}_{12i+7}$ is $8i+5$, $3^{-1}\equiv(-4i-2)\ (mod\ 12i+7)$, this yields:

$$
\begin{aligned}
d &\equiv \tfrac{-10}{9} \equiv -10(-4i-2)^2 \ (mod\ 12i+7),\\
f &\equiv \tfrac{-4}{9} \equiv -4(-4i-2)^2 \ (mod\ 12i+7),\\
h &\equiv \tfrac{-10}{9} \equiv -10(-4i-2)^2 \ (mod\ 12i+7).
\end{aligned}
$$

Hence, while $\overrightarrow{I_0}$ cannot be completely calculated from the outer edges, it does depend on

them. Also, while $C_0(\overrightarrow{O_j}) = 0$ for all outer edges $\overrightarrow{O_j}$, $C_0(\overrightarrow{I_0}) = 1$. We will thus count $\overrightarrow{I_0}$ along the inner edges from now on.
For the other inner edges, we get:

$$\underbrace{\begin{pmatrix} 1 \\ b-k \\ \frac{k(2k+1)(4k+1)}{3} \\ d+2k \\ k(2b+1+d+2k) \\ 1+f \\ b+1+k(1+f) \\ h \\ kh \end{pmatrix}}_{\overrightarrow{I_k}}, \underbrace{\begin{pmatrix} 1 \\ b+k \\ \frac{k(2k-1)(4k-1)}{3} \\ d+2k-1 \\ k(2b+2-d-2k)-b \\ 1+f \\ b+1-k(1+f) \\ h \\ -kh \end{pmatrix}}_{\overrightarrow{II_k}}, \underbrace{\begin{pmatrix} 1 \\ a+b \\ a^2 \\ 1+d \\ b-ad \\ 1+f \\ b-af \\ 1+h \\ b-ah \end{pmatrix}}_{\overrightarrow{III_0}},$$

$$\underbrace{\begin{pmatrix} 1 \\ a+b-k \\ a^2+\frac{k(2k+1)(4k+1)}{3} \\ d+2k+1 \\ k(a-2-d-2k)-ad+b \\ 2+f \\ -2(1+i+k)-f(a+k) \\ 1+h \\ b-ah-k(1+h) \end{pmatrix}}_{\overrightarrow{III_k}}, \underbrace{\begin{pmatrix} 1 \\ a+b+k \\ a^2+\frac{k(2k-1)(4k-1)}{3} \\ d+2k \\ k(a-1+d+2k)-ad+2b+a \\ 2+f \\ -2(1+i-k)-f(a-k) \\ 1+h \\ b-ah+k(1+h) \end{pmatrix}}_{\overrightarrow{IV_k}}, (k=1,\ldots,i).$$

Note that $\overrightarrow{I_0}$ and $\overrightarrow{III_0}$ differ from $\overrightarrow{I_{k=0}}$ and $\overrightarrow{III_{k=0}}$ only in C_5 and C_6. For simplicity, we will only list them separately, if those two components are concerned, and thus imply $k \in \{0,\ldots,i\}$ whenever writing $\overrightarrow{I_k}$, $\overrightarrow{III_k}$, and $k \in \{1,\ldots,i\}$ for $\overrightarrow{II_k}$, $\overrightarrow{IV_k}$.
A brief motivation for the different components – this might be more clear after reading the whole chapter:

- C_0 distinguishes outer and inner edges, laying the ground for some very useful case distinctions when considering closed essential and non-essential cycles later on.

- C_1 is the Ringel component. Essentially, it guarantees that every edge of the basic complex and their inverses are distinct. Thus, the construction yields a triangulation and not some object with duplicate edges.

- C_2 is a squared version of the Ringel component, used to distinguish purely outer 4-cycles of the type $\overrightarrow{O_j} \cdot \overrightarrow{O_l} \cdot \overrightarrow{O_{l+\Delta}}^{-1} \cdot \overrightarrow{O_{j-\Delta}}^{-1}$ for different values of Δ.

- The remaining components are three pairs, each operating with a semidirect product with C_1. Together, they yield enough non-commutativity to prevent all essential 4-cycles of type $ABA^{-1}B^{-1}$, as well as some more complex cases of essential 4-cycles that, in an abelian world, are closed because of this specific construction.

- In particular, C_3 and C_4 render products of outer edges non-abelian, making use of the fact that C_1 contains every element of $\mathbb{Z}_{12i+7}$ at most once and that $C_3(\overrightarrow{O_j}) = 1$ for all outer edges $\overrightarrow{O_j}$.

- C_5, C_6, C_7 and C_8 provide distinctions between the different types of inner edges, most noticeably between the ones on the left side ($\overrightarrow{III_k}, \overrightarrow{IV_k}$) and the ones on the right side ($\overrightarrow{III_k}, \overrightarrow{IV_k}$). Note that the main difference between both sides is the inclusion of $\overrightarrow{O_{2i+1}}$ in their construction.

For a quick overview and to further clarify the construction, Figures 4.2 to 4.4 show the commutative components of G and their identified edges in T_0.

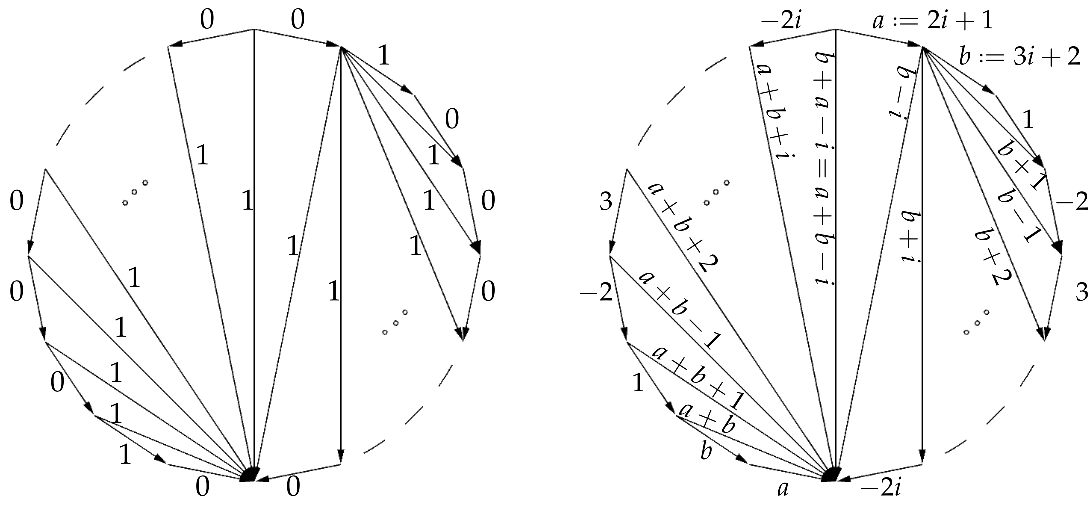

Figure 4.2.: $C_0 =< Z_4, + >$ and $C_1 =< Z_{12i+7}, + >$

4.3.2. Proving ew-5

To prove that T has edge width at least 5, we need to show that it does not contain essential cycles of length 3 or 4. A cycle of T corresponds to a series of the basic elements in G, the ones identified with the oriented edges in T_0, and their inverses. For the cycle to be closed, the multiplication of that series has to yield the neutral element $\overrightarrow{e} = \overrightarrow{0} \in G$. We will thus classify all possible 3- and 4-cycles in G consisting of those central $12i+6$ group elements and show that they either do not add up to the neutral element or correspond to a non-essential cycle of T, and thus are closed in the universal covering.

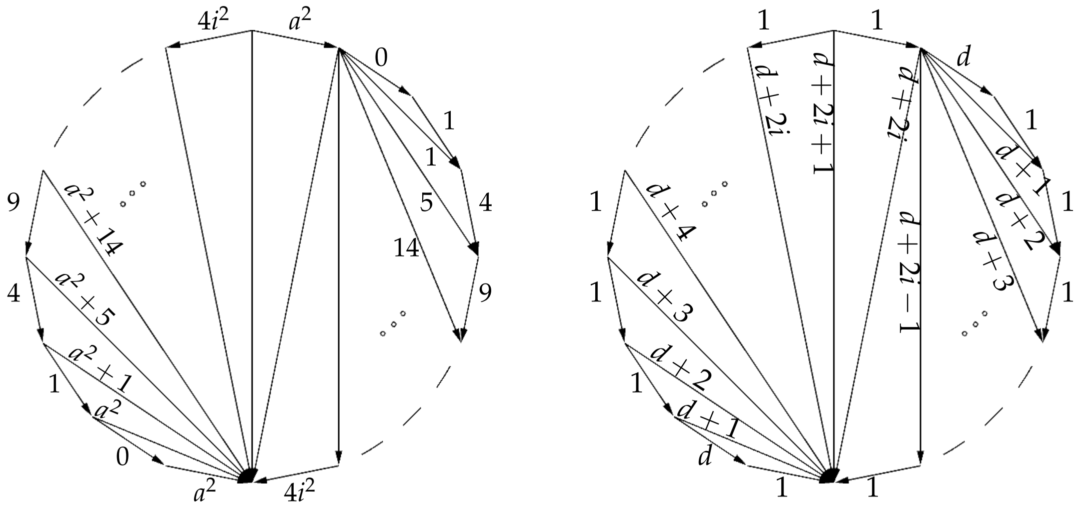

Figure 4.3.: $C_2 =< Z_{12i+7}, + >$ and $C_3 =< Z_{12i+7}, + >$

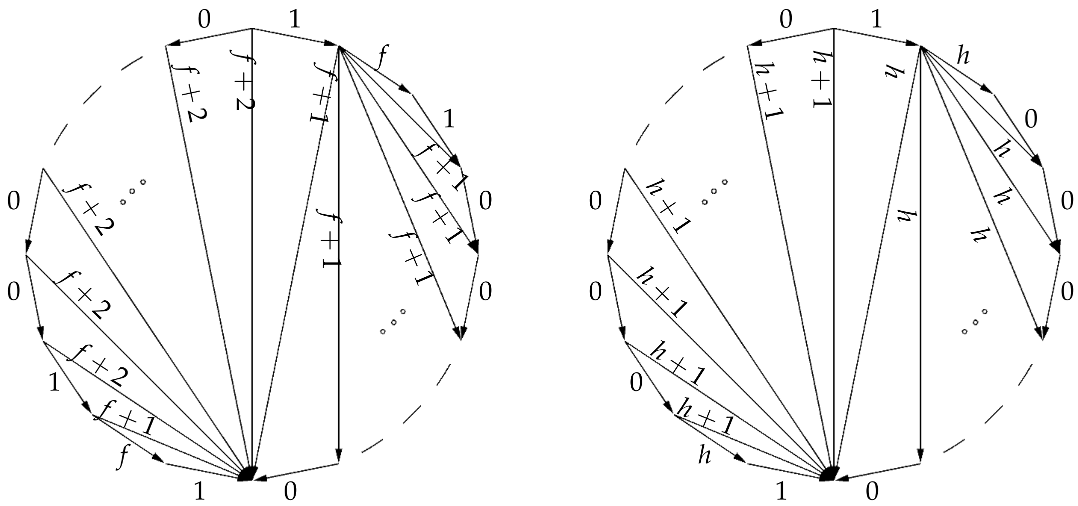

Figure 4.4.: $C_5 :< Z_{12i+7}, + >$ and $C_7 :< Z_{12i+7}, + >$

Possible 3-cycles

Lemma 4.8.
The only closed 3-cycles of T are its triangles.

Proof.
C_0 equals 0 for outer edges, and 1 for inner edges. Thus, there are only two cases of 3-cycles to study: Those consisting of three outer edges (or their inverses) and those consisting of one outer edge, one inner edge and one inversed inner edge.

Three outer edges:
C_3 equals 1 for outer elements, and -1 for their inverses. Thus, no combination of three of these elements can add up to 0 in $\mathbb{Z}_{12i+7}$ and thus, there is no closed 3-cycle consisting only of outer elements or their inverses.

One outer edge, two inner edges:

W.l.o.g., start the 3-cycle with a positively oriented outer edge $\overrightarrow{O}$. As the aim is to add up to $\overrightarrow{0}$, C_0 leaves two choices for the orientation of the interior edges $\overrightarrow{A}$, $\overrightarrow{B}$: both pointing towards $\overrightarrow{O}$ or both pointing away (see Figure 4.5).

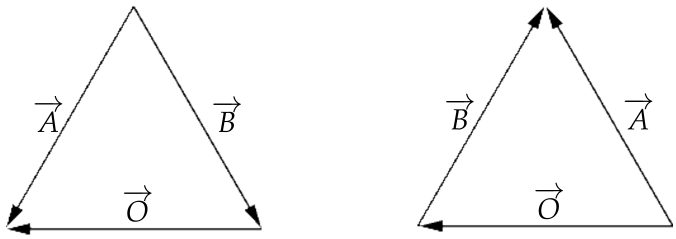

Figure 4.5.: Nomination of group elements in both types of mixed closed 3-cycles

With this notation, the conditions for the two types of cycles to be closed are $\overrightarrow{B} \cdot \overrightarrow{O} \stackrel{!}{=} \overrightarrow{A}$ and $\overrightarrow{O} \cdot \overrightarrow{B} \stackrel{!}{=} \overrightarrow{A}$. In an abelian world, they are the same. Thus, we will evaluate the abelian conditions for $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{O}$ for both types simultaneously.

Assume first that $\overrightarrow{O} \neq \overrightarrow{O_{2i+1}}$:

Then, $C_7(\overrightarrow{O}) = 0$, and thus $C_7(\overrightarrow{A}) \stackrel{!}{=} C_7(\overrightarrow{B})$, which is the case if and only if either both $\overrightarrow{A}$ and $\overrightarrow{B}$ are on the right side of T_0, and thus are of type $\overrightarrow{I_k}$ or $\overrightarrow{II_k}$, or if both are on the left side of T_0, and thus are of type $\overrightarrow{III_k}$ or $\overrightarrow{IV_k}$ (see Figure 4.4). Also, $C_3(\overrightarrow{O}) = 1$, and thus, $C_3(\overrightarrow{A}) = C_3(\overrightarrow{B}) + 1$. As $\overrightarrow{A}$ and $\overrightarrow{B}$ are on the same side of T_0, this forces them to be neighbours (see Figure 4.3). This leaves us with the following possible configurations:

$$(\overrightarrow{A}, \overrightarrow{B}) = \begin{cases} (\overrightarrow{I_k}, \overrightarrow{II_k}), \\ (\overrightarrow{II_k}, \overrightarrow{I_{k-1}}), \\ (\overrightarrow{III_k}, \overrightarrow{IV_k}), \\ (\overrightarrow{IV_k}, \overrightarrow{III_{k-1}}). \end{cases}$$

For each of these configurations, there is exactly one outer edge $\overrightarrow{O}$ such that $(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{O})$ fulfills the closing condition in C_1:

$$(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{O}) = \begin{cases} (\overrightarrow{I_k}, \overrightarrow{II_k}, \overrightarrow{O_{2k}}), \\ (\overrightarrow{II_k}, \overrightarrow{I_{k-1}}, \overrightarrow{O_{2k-1}}), \\ (\overrightarrow{III_k}, \overrightarrow{IV_k}, \overrightarrow{O_{2k}}), \\ (\overrightarrow{IV_k}, \overrightarrow{III_{k-1}}, \overrightarrow{O_{2k-1}}). \end{cases}$$

Note that these are exactly the configurations of the triangles in T_0. As we started with two differently oriented types of 3-cycles, it remains to show that only the correctly oriented one closes in all components. In C_8, we get:

$$0 = \begin{cases} C_8(\overrightarrow{O_{2k}}) & \stackrel{!}{=} \begin{cases} C_8(\overrightarrow{II_k}^{-1} \cdot \overrightarrow{I_k}) = 0, \\ C_8(\overrightarrow{I_k} \cdot \overrightarrow{II_k}^{-1}) = -\frac{40}{9}k, \end{cases} \\ C_8(\overrightarrow{O_{2k-1}}) & \stackrel{!}{=} \begin{cases} C_8(\overrightarrow{I_{k-1}}^{-1} \cdot \overrightarrow{II_k}) = 0, \\ C_8(\overrightarrow{II_k} \cdot \overrightarrow{I_{k-1}}^{-1}) = \frac{20}{9}(2k-1), \end{cases} \\ C_8(\overrightarrow{O_{2k}}) & \stackrel{!}{=} \begin{cases} C_8(\overrightarrow{IV_k}^{-1} \cdot \overrightarrow{III_k}) = \frac{4}{9}k, \\ C_8(\overrightarrow{III_k} \cdot \overrightarrow{IV_k}^{-1}) = 0, \end{cases} \\ C_8(\overrightarrow{O_{2k-1}}) & \stackrel{!}{=} \begin{cases} C_8(\overrightarrow{III_{k-1}}^{-1} \cdot \overrightarrow{IV_k}) = \frac{2}{9}(1-2k), \\ C_8(\overrightarrow{IV_k} \cdot \overrightarrow{III_{k-1}}^{-1}) = 0. \end{cases} \end{cases}$$

Note that $k \in \{1, \ldots, i\}$ and that all these equations operate in $\mathbb{Z}_{12i+7}$ with $12i + 7$ prime. Thus, for all of the four cases, only one of the conditions is fulfilled. That one is the properly oriented triangle. The other cycle does not close and can thus be discarded.

Thus, the only closed 3-cycles that do not contain $\overrightarrow{O_{2i+1}}$ correspond to the triangles of T_0.

$\underline{\overrightarrow{O} = \overrightarrow{O_{2i+1}}}$:

With $C_7(\overrightarrow{O_{2i+1}}) = 1$, $\overrightarrow{A}$ is positioned on the left side of T_0, and $\overrightarrow{B}$ is positioned on the right side (see Figure 4.4). In C_3, they again differ by one, fixing the configurations:

$$(\overrightarrow{A}, \overrightarrow{B}) = \begin{cases} (\overrightarrow{III_k}, \overrightarrow{I_k}), \\ (\overrightarrow{IV_k}, \overrightarrow{II_k}). \end{cases}$$

Both these configurations add up to 0 in the abelian components. Checking equations 4.1 to 4.5, we see that such pairs of $\overrightarrow{A}, \overrightarrow{B}$ consist of nearly the same outer generators. They only differ in the order these generators are multiplied in, and in the fact that $\overrightarrow{A}$ contains $\overrightarrow{O_{2i+1}}$, which is applied to $\overrightarrow{B}$ in our 3-cycle. Thus, to prove that there are no essential 3-cycles of this type, we are restricted to the non-abelian components. In C_8, we get:

$$0 = C_8(\overrightarrow{O_{2i+1}}) \stackrel{!}{=} \begin{cases} C_8(\overrightarrow{I_k}^{-1} \cdot \overrightarrow{III_k}) = \frac{20}{9}k, & k \in \{0, \ldots, i\}, \\ C_8(\overrightarrow{III_k} \cdot \overrightarrow{I_k}^{-1}) = \frac{2}{9}(28 + 47i + k) = \frac{2}{9}(k - i), & k \in \{0, \ldots, i\}, \\ C_8(\overrightarrow{II_k}^{-1} \cdot \overrightarrow{IV_k}) = -\frac{20}{9}k, & k \in \{1, \ldots, i\}, \\ C_8(\overrightarrow{IV_k} \cdot \overrightarrow{II_k}^{-1}) = \frac{2}{9}(28 + 47i - k) = \frac{2}{9}(-i - k), & k \in \{1, \ldots, i\}. \end{cases}$$

Note that all these equations once again operate in $\mathbb{Z}_{12i+7}$. Thus, the only closed 3-cycles that contain $\overrightarrow{O_{2i+1}}$ are $(\overrightarrow{O_{2i+1}}, \overrightarrow{I_i}, \overrightarrow{III_i}^{-1})$ and $(\overrightarrow{O_{2i+1}}, \overrightarrow{III_0}^{-1}, \overrightarrow{I_0})$, which correspond to the two triangles it is contained in.

Thus, the only closed 3-cycles of T correspond to triangles in T_0, and are thus non-essential, and closed in the universal covering. □

Possible 4-cycles

A cycle of the form $\gamma\gamma^{-1}$ for some path γ is trivially closed and non-essential. We show that the only other closed 4-cycles in T are pairs of adjacent triangles.

Lemma 4.9.
The only non-trivial closed 4-cycles of T are pairs of adjacent triangles.

Proof. Again, C_0 equals 1 for outer edges $\overrightarrow{O}$, and it equals 0 for inner edges. Thus, there are four possible types of closed 4-cycles: Four outer edges, a combination of two outer and two inner edges, and four inner edges.

Four outer edges:
W.l.o.g., start the 4-cycle with a positively oriented outer edge $\overrightarrow{A}$. If the second edge is also positively oriented, C_3 yields that the remaining two edges have to be inversed (the left configuration in Figure 4.6). If the second edge is inversed, there are two choices for the remaining edges: They can point towards each other (the right configuration in Figure 4.6) or away from each other. The latter is just a rotation of the first configuration, though, and will thus be omitted.

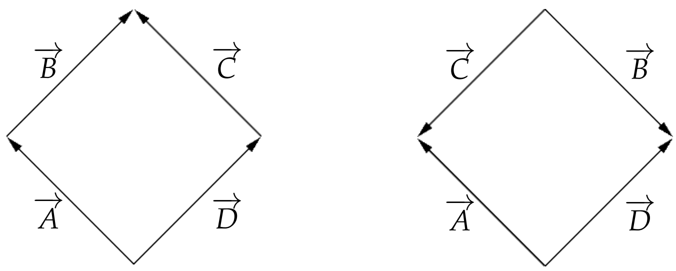

Figure 4.6.: Nomination of group elements in both types of outer 4-cycles.

Once again, the edges are labelled in a way that the closing conditions $\overrightarrow{A} \cdot \overrightarrow{B} \overset{!}{=} \overrightarrow{D} \cdot \overrightarrow{C}$ and $\overrightarrow{A} \cdot \overrightarrow{C}^{-1} \overset{!}{=} \overrightarrow{D} \cdot \overrightarrow{B}^{-1}$ are the same in an abelian world.
If we denote the first component C_1 of the edges $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overrightarrow{D}$ by X_A, X_B, X_C, X_D and note that the third component C_3 of an outer edge is just the square of its first component, the abelian closing condition in C_1 and C_3 yields:

$$\begin{aligned} X_A + X_B &\overset{!}{=} X_C + X_D, \\ X_A^2 + X_B^2 &\overset{!}{=} X_C^2 + X_D^2. \end{aligned}$$

and thus

$$\begin{aligned} X_D &\overset{!}{=} X_A + X_B - X_C, \\ X_A^2 + X_B^2 &\overset{!}{=} X_C^2 + X_A^2 + X_B^2 + X_C^2 + 2X_AX_B - 2X_AX_C - 2X_BX_C, \\ &= X_A^2 + X_B^2 + 2(X_C - X_B)(X_C - X_A). \end{aligned}$$

All these equations operate in $\mathbb{Z}_{12i+7}$. As $12i+7$ is prime and an edge is uniquely determined by its first component C_1, it follows that either $\overrightarrow{C} = \overrightarrow{B}$ (and thus, $\overrightarrow{A} = \overrightarrow{D}$) or $\overrightarrow{C} = \overrightarrow{A}$ (and thus, $\overrightarrow{B} = \overrightarrow{D}$). Hence, with Figure 4.6, we get the cycles $\overrightarrow{A}\,\overrightarrow{B}\,\overrightarrow{B}^{-1}\overrightarrow{A}^{-1}$ (left type, trivial), $\overrightarrow{A}\,\overrightarrow{B}^{-1}\overrightarrow{B}\,\overrightarrow{A}^{-1}$ (right type, also trivial), $\overrightarrow{A}\,\overrightarrow{B}\,\overrightarrow{A}^{-1}\overrightarrow{B}^{-1}$ (left type, still to be checked) and $\overrightarrow{A}\,\overrightarrow{A}^{-1}\overrightarrow{B}\,\overrightarrow{B}^{-1}$ (right type, trivial again).
The only remaining outer 4-cycle to check is $\overrightarrow{A}\,\overrightarrow{B}\,\overrightarrow{A}^{-1}\overrightarrow{B}^{-1}$. If that adds up to the neutral element, we have $\overrightarrow{A}\,\overrightarrow{B} \stackrel{!}{=} \overrightarrow{B}\,\overrightarrow{A}$. In C_4, that yields $X_A - X_B \stackrel{!}{=} X_B - X_A$ and thus, $\overrightarrow{A} = \overrightarrow{B}$ and this cycle is trivial as well.
Thus, we have shown that T does not contain any non-trivial closed 4-cycles consisting only of outer edges.

Two outer and two inner edges:
W.l.o.g., start the 4-cycle with a positively oriented inner edge $\overrightarrow{A}$. The cycle contains one more inner edge $\overrightarrow{B}$, which is inversed due to C_0. There are three possible placements of $\overrightarrow{B}$ in the cycle, relative to $\overrightarrow{A}$ (second, third and fourth $\rightarrow$ the three rows in Figure 4.7). The remaining two edges are the outer ones $\overrightarrow{O}, \overrightarrow{O'}$. They can either be oriented in the same direction (w.l.o.g. the direction of $\overrightarrow{A}$), point towards each other or point away from each other. The latter two are the same if $\overrightarrow{A}$ and $\overrightarrow{B}$ are not adjacent. Hence, there are eight types of mixed 4-cycles to check (Figure 4.7).

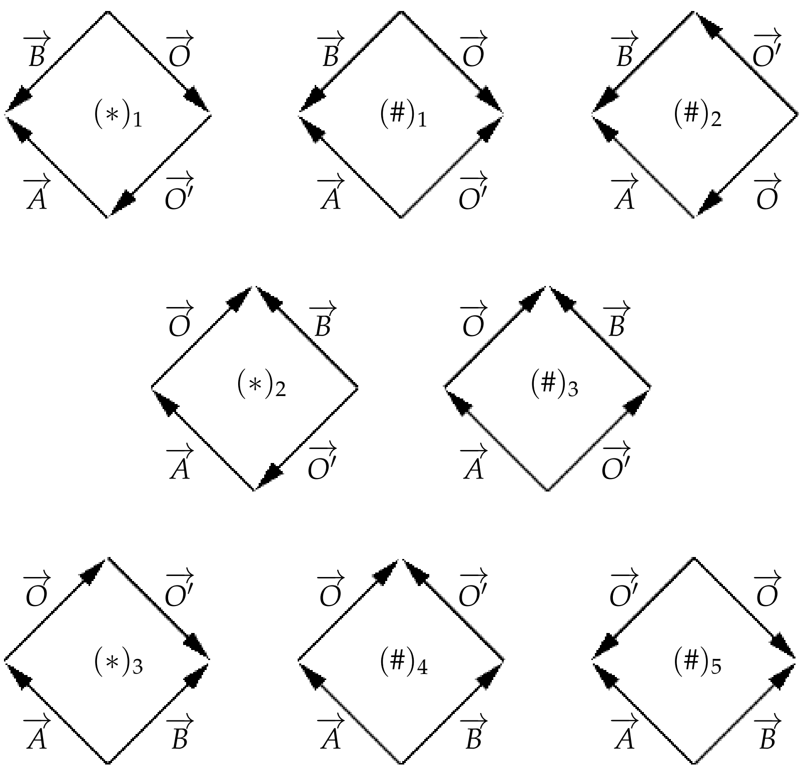

Figure 4.7.: Nomination of group elements in the eight types of mixed 4-cycles, the two abelian classes marked $(*)$ and $(\#)$.

Again, the edges are nominated in a way that the conditions for closing are as similar as possible in an abelian sense. There are two cases of abelian conditions:

$$\begin{aligned} &(*) \quad \overrightarrow{B} \stackrel{!}{=} \overrightarrow{O} + \overrightarrow{O'} + \overrightarrow{A}, \\ &(\#) \quad \overrightarrow{O} + \overrightarrow{A} \stackrel{!}{=} \overrightarrow{O'} + \overrightarrow{B}. \end{aligned}$$

Consider C_7: It equals h for right inner edges, $h+1$ for left inner edges, 0 for all but the last outer edges $\overrightarrow{O_{j\neq 2i+1}}$ and 1 for $\overrightarrow{O_{2i+1}}$. Thus, we will not only distinguish between $(*)$ and $(\#)$, but also between the number of appearances of $\overrightarrow{O_{2i+1}}$.

$\underline{(*)\, \overrightarrow{B} \stackrel{!}{=} \overrightarrow{O} + \overrightarrow{O'} + \overrightarrow{A},\, \overrightarrow{O} = \overrightarrow{O'} = \overrightarrow{O_{2i+1}}}$:

As $C_7(\overrightarrow{O_{2i+1}}) = 1$, we get a contradiction:

$$\{h, h+1\} \ni C_7(\overrightarrow{B}) \stackrel{!}{=} C_7(\overrightarrow{A}) + 2 \in \{h+2, h+3\}$$

$\underline{(*)\, \overrightarrow{B} \stackrel{!}{=} \overrightarrow{O} + \overrightarrow{O'} + \overrightarrow{A},\, \overrightarrow{O} \neq \overrightarrow{O_{2i+1}} \neq \overrightarrow{O'}}$:

The closing condition in C_7 yields:

$$\{h, h+1\} \ni C_7(\overrightarrow{B}) \stackrel{!}{=} C_7(\overrightarrow{A}) + 0 \in \{h, h+1\}$$

Thus, $\overrightarrow{A}$ and $\overrightarrow{B}$ are on the same side of T_0. Furthermore, as C_3 equals 1 for outer edges, we get that $C_3(\overrightarrow{B}) = C_3(\overrightarrow{A}) + 2$. As they are on the same side, this is the case if and only if $\overrightarrow{A}$ and $\overrightarrow{B}$ are neighbours of the same type (see Figure 4.3). The closing condition in C_1 yields:

$$C_1(\overrightarrow{O}) + C_1(\overrightarrow{O'}) \stackrel{!}{=} C_1(\overrightarrow{B}) - C_1(\overrightarrow{A}) = \pm 1$$

Thus, $\overrightarrow{O}$ and $\overrightarrow{O'}$ are also neighbours. With $C_2(\overrightarrow{B}) - C_2(\overrightarrow{A})$ being the sum of two neighbouring squares, the following configurations remain:

$$(\overrightarrow{A}, \overrightarrow{B}, \{\overrightarrow{O}, \overrightarrow{O'}\}) = \begin{cases} (\overrightarrow{I_k}, \overrightarrow{I_{k+1}}, \{\overrightarrow{O_{2k+1}}, \overrightarrow{O_{2k+2}}\}), & k \in \{0, \dots, i-1\}, \\ (\overrightarrow{II_k}, \overrightarrow{II_{k+1}}, \{\overrightarrow{O_{2k}}, \overrightarrow{O_{2k+1}}\}), & k \in \{1, \dots, i-1\}, \\ (\overrightarrow{III_k}, \overrightarrow{III_{k+1}}, \{\overrightarrow{O_{2k+1}}, \overrightarrow{O_{2k+2}}\}), & k \in \{0, \dots, i-1\}, \\ (\overrightarrow{IV_k}, \overrightarrow{IV_{k+1}}, \{\overrightarrow{O_{2k}}, \overrightarrow{O_{2k+1}}\}), & k \in \{1, \dots, i-1\}. \end{cases}$$

Note that all these configurations add up to 0 in the abelian components. Also note that, in these cases, $C_7(\overrightarrow{A} \cdot \overrightarrow{B}^{-1}) = C_7(\overrightarrow{B}^{-1} \cdot \overrightarrow{A}) = C_7(\overrightarrow{O}) = C_7(\overrightarrow{O'}) = 0$ and thus, $C_8(*_1) = C_8(\overrightarrow{A} \cdot \overrightarrow{B}^{-1} \cdot \overrightarrow{O} \cdot \overrightarrow{O'}) = C_8(\overrightarrow{A} \cdot \overrightarrow{B}^{-1})$, and $C_8(*_3) = C_8(\overrightarrow{B}^{-1} \cdot \overrightarrow{A} \cdot \overrightarrow{O} \cdot \overrightarrow{O'}) = C_8(\overrightarrow{B}^{-1} \cdot \overrightarrow{A})$. Thus, the closing conditions in C_8 are:

$$\begin{aligned} &(*)_1 : 0 \stackrel{!}{=} && C_8\left(\overrightarrow{I_k} \cdot \overrightarrow{I_{k+1}}^{-1}\right) && = -2h, \\ &(*)_1 : 0 \stackrel{!}{=} && C_8\left(\overrightarrow{II_k} \cdot \overrightarrow{II_{k+1}}^{-1}\right) && = 2h, \end{aligned}$$

$$
\begin{aligned}
(*)_1 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{III_k} \cdot \overrightarrow{III_{k+1}}^{-1}\right) &= 0, \\
(*)_1 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{IV_k} \cdot \overrightarrow{IV_{k+1}}^{-1}\right) &= 0, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{I_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{I_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+2}}\right) &= -4h(1+k),\ k \in \{0,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{I_k} \cdot \overrightarrow{O_{2k+2}} \cdot \overrightarrow{I_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}}\right) &= 2h(1+2k),\ k \in \{0,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{II_k \cdot \overrightarrow{O_{2k}}} \cdot \overrightarrow{II_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}}\right) &= 2h(1+2k),\ k \in \{1,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{II_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{II_{k+1}}^{-1} \cdot \overrightarrow{O_{2k}}\right) &= -4hk,\ k \in \{1,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{III_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{III_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+2}}\right) &= -2(h+1)(1+2k),\ k \in \{0,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{III_k} \cdot \overrightarrow{O_{2k+2}} \cdot \overrightarrow{III_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}}\right) &= 4(h+1)(1+k),\ k \in \{0,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{IV_k} \cdot \overrightarrow{O_{2k}} \cdot \overrightarrow{IV_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}}\right) &= 4(h+1)k,\ k \in \{1,\dots,i-1\}, \\
(*)_2 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{IV_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{IV_{k+1}}^{-1} \cdot \overrightarrow{O_{2k}}\right) &= -2(h+1)(1+2k),\ k \in \{1,\dots,i-1\}, \\
(*)_3 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{I_{k+1}}^{-1} \cdot \overrightarrow{I_k}\right) &= 0, \\
(*)_3 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{II_{k+1}}^{-1} \cdot \overrightarrow{II_k}\right) &= 0, \\
(*)_3 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{III_{k+1}}^{-1} \cdot \overrightarrow{III_k}\right) &= 2(h+1), \\
(*)_3 : 0 &\overset{!}{=} & C_8\left(\overrightarrow{IV_{k+1}}^{-1} \cdot \overrightarrow{IV_k}\right) &= -2(h+1).
\end{aligned}
$$

That leaves contradictions for all but four cases that need to be checked further. Note that the order of $\overrightarrow{O}, \overrightarrow{O'}$ is also not fixed yet. Taking both possibilities into consideration, C_4 yields:

$$
\begin{aligned}
(*)_1 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{III_k} \cdot \overrightarrow{III_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{O_{2k+2}}\right) &= 6+8k,\ k \in \{0,\dots,i-1\}, \\
(*)_1 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{III_k} \cdot \overrightarrow{III_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+2}} \cdot \overrightarrow{O_{2k+1}}\right) &= 0,\ k \in \{0,\dots,i-1\}, \\
(*)_1 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{IV_k} \cdot \overrightarrow{IV_{k+1}}^{-1} \cdot \overrightarrow{O_{2k}} \cdot \overrightarrow{O_{2k+1}}\right) &= -2-8k,\ k \in \{1,\dots,i-1\}, \\
(*)_1 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{IV_k} \cdot \overrightarrow{IV_{k+1}}^{-1} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{O_{2k}}\right) &= 0,\ k \in \{1,\dots,i-1\}, \\
(*)_3 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{I_{k+1}}^{-1} \cdot \overrightarrow{I_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{O_{2k+2}}\right) &= 0,\ k \in \{0,\dots,i-1\}, \\
(*)_3 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{I_{k+1}}^{-1} \cdot \overrightarrow{I_k} \cdot \overrightarrow{O_{2k+2}} \cdot \overrightarrow{O_{2k+1}}\right) &= -6-8k,\ k \in \{0,\dots,i-1\}, \\
(*)_3 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{II_{k+1}}^{-1} \cdot \overrightarrow{II_k} \cdot \overrightarrow{O_{2k}} \cdot \overrightarrow{O_{2k+1}}\right) &= 0,\ k \in \{1,\dots,i-1\}, \\
(*)_3 : 0 &\overset{!}{=} & C_4\left(\overrightarrow{II_{k+1}}^{-1} \cdot \overrightarrow{II_k} \cdot \overrightarrow{O_{2k+1}} \cdot \overrightarrow{O_{2k}}\right) &= 2+8k,\ k \in \{1,\dots,i-1\}.
\end{aligned}
$$

Again, four cases remain. Comparing them with Figure 4.1, we see that they, and thus the only non-trivial 4-cycles of type $(*)$ that do not contain $\overrightarrow{O_{2i+1}}$, are pairs of adjacent triangles with a common inner edge:

$$\begin{aligned}
\left(\overrightarrow{III_k}\cdot\overrightarrow{III_{k+1}}^{-1}\cdot\overrightarrow{O_{2k+2}}\cdot\overrightarrow{O_{2k+1}}\right) &\rightarrow \text{two triangles around } \overrightarrow{IV_{k+1}},\\
\left(\overrightarrow{IV_k}\cdot\overrightarrow{IV_{k+1}}^{-1}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{O_{2k}}\right) &\rightarrow \text{two triangles around } \overrightarrow{III_k},\\
\left(\overrightarrow{I_{k+1}}^{-1}\cdot\overrightarrow{I_k}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{O_{2k+2}}\right) &\rightarrow \text{two triangles around } \overrightarrow{II_{k+1}},\\
\left(\overrightarrow{II_{k+1}}^{-1}\cdot\overrightarrow{II_k}\cdot\overrightarrow{O_{2k}}\cdot\overrightarrow{O_{2k+1}}\right) &\rightarrow \text{two triangles around } \overrightarrow{I_k}.
\end{aligned}$$

$\underline{(*)\,\overrightarrow{B}\overset{!}{=}\overrightarrow{O}+\overrightarrow{O'}+\overrightarrow{A},\{\overrightarrow{O},\overrightarrow{O'}\}=\{\overrightarrow{O_{2i+1}},\overrightarrow{O_{j\neq 2i+1}}\}}$:

As $C_7(\overrightarrow{O_{2i+1}})=1, C_7(\overrightarrow{O_{j\neq 2i+1}})=0$, the closing condition in C_7 yields:

$$\{h,h+1\}\ni C_7(\overrightarrow{B})\overset{!}{=}C_7(\overrightarrow{A})+1\in\{h+1,h+2\}$$

Thus, $\overrightarrow{B}$ is limited to the left side of T_0, and $\overrightarrow{A}$ to the right side. Again, we have $C_3(\overrightarrow{B})=C_3(\overrightarrow{A})+2$. This leaves the following configurations:

$$\{\overrightarrow{O},\overrightarrow{O'}\}=\{\overrightarrow{O_{2i+1}},\overrightarrow{O_{j\neq 2i+1}}\}:(\overrightarrow{A},\overrightarrow{B})=\begin{cases}(\overrightarrow{I_k},\overrightarrow{IV_{k+1}}), & k\in\{0,\dots,i-1\},\\ (\overrightarrow{II_k},\overrightarrow{III_k}), & k\in\{1,\dots,i\}.\end{cases}$$

In C_1, we get

$$C_1(\overrightarrow{O})+C_1(\overrightarrow{O'})\overset{!}{=}\begin{cases}C_1(\overrightarrow{IV_{k+1}})-C_1(\overrightarrow{I_k})=a+2k+1, & k\in\{0,\dots,i-1\},\\ C_1(\overrightarrow{III_k})-C_1(\overrightarrow{II_k})=a-2k, & k\in\{1,\dots,i\}.\end{cases}$$

and thus:

$$\left(\overrightarrow{A},\overrightarrow{B},\{\overrightarrow{O},\overrightarrow{O'}\}\right)=\begin{cases}(\overrightarrow{I_k},\overrightarrow{IV_{k+1}},\{\overrightarrow{O_{2i+1}},\overrightarrow{O_{2k+1}}\}), & k\in\{0,\dots,i-1\},\\ (\overrightarrow{II_k},\overrightarrow{III_k},\{\overrightarrow{O_{2i+1}},\overrightarrow{O_{2k}})\}, & k\in\{1,\dots,i\}.\end{cases}$$

Again, all these configurations add up to 0 in the abelian components.

$$\begin{array}{lll}
(*)_1 & \left(\overrightarrow{I_0}\cdot\overrightarrow{IV_1}^{-1}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_1}\right): & -\frac{2}{9}(22+17i)=C_6\overset{!}{=}0,\\
(*)_1 & \left(\overrightarrow{I_0}\cdot\overrightarrow{IV_1}^{-1}\cdot\overrightarrow{O_1}\cdot\overrightarrow{O_{2i+1}}\right): & -\frac{2}{9}(28+47i)=C_8\overset{!}{=}0,\\
(*)_1 & \left(\overrightarrow{I_k}\cdot\overrightarrow{IV_{k+1}}^{-1}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_{2k+1}}\right): & -\frac{5+24k}{27}=C_6\overset{!}{=}0\overset{!}{=}C_8=-\frac{115+228k}{54},\\
(*)_1 & \left(\overrightarrow{I_k}\cdot\overrightarrow{IV_{k+1}}^{-1}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{O_{2i+1}}\right): & \frac{7}{27}(7+12k)=C_6\overset{!}{=}0,\\
(*)_1 & \left(\overrightarrow{II_k}\cdot\overrightarrow{III_k}^{-1}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_{2k}}\right): & \frac{49+24k}{27}=C_6\overset{!}{=}0\overset{!}{=}C_8=\frac{228k-7}{54},\\
(*)_1 & \left(\overrightarrow{II_k}\cdot\overrightarrow{III_k}^{-1}\cdot\overrightarrow{O_{2k}}\cdot\overrightarrow{O_{2i+1}}\right): & -\frac{7}{27}(12k-7)=C_6\overset{!}{=}0,\\
\\
(*)_2 & \left(\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{IV_1}^{-1}\cdot\overrightarrow{O_1}\right): & \text{nullhomotopic},\\
(*)_2 & \left(\overrightarrow{I_0}\cdot\overrightarrow{O_1}\cdot\overrightarrow{IV_1}^{-1}\cdot\overrightarrow{O_{2i+1}}\right): & \frac{20}{9}i=C_6\overset{!}{=}0,
\end{array}$$

$$
\begin{array}{lll}
(*)_2 & \left(\overrightarrow{I_k}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{IV_{k+1}}^{-1}\cdot\overrightarrow{O_{2k+1}}\right): & -\frac{20}{9}k=C_8\overset{!}{=}0,\\
(*)_2 & \left(\overrightarrow{I_k}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{IV_{k+1}}^{-1}\cdot\overrightarrow{O_{2i+1}}\right): & -\frac{7}{27}(5+12k)=C_6\overset{!}{=}0,\\
(*)_2 & \left(\overrightarrow{II_k}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{III_k}^{-1}\cdot\overrightarrow{O_{2k}}\right): & \frac{20}{9}k=C_8\overset{!}{=}0,\\
(*)_2 & \left(\overrightarrow{II_k}\cdot\overrightarrow{O_{2k}}\cdot\overrightarrow{III_k}^{-1}\cdot\overrightarrow{O_{2i+1}}\right): & \frac{7}{27}(12k+7)=C_6\overset{!}{=}0,
\end{array}
$$

$$
\begin{array}{lll}
(*)_3 & \left(\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_1}\cdot\overrightarrow{IV_1}^{-1}\right): & \frac{2}{9}=C_8\overset{!}{=}0,\\
(*)_3 & \left(\overrightarrow{I_0}\cdot\overrightarrow{O_1}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{IV_1}^{-1}\right): & \frac{20}{9}=C_8\overset{!}{=}0,\\
(*)_3 & \left(\overrightarrow{I_k}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{IV_{k+1}}^{-1}\right): & -\frac{2}{9}(8k-1)=C_8\overset{!}{=}0,\\
(*)_3 & \left(\overrightarrow{I_k}\cdot\overrightarrow{O_{2k+1}}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{IV_{k+1}}^{-1}\right): & \frac{20}{9}(k+1)=C_8\overset{!}{=}0,\\
(*)_3 & \left(\overrightarrow{II_k}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{O_{2k}}\cdot\overrightarrow{III_k}^{-1}\right): & \frac{16}{9}k=C_8\overset{!}{=}0,\\
(*)_3 & \left(\overrightarrow{II_k}\cdot\overrightarrow{O_{2k}}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{III_k}^{-1}\right): & -\frac{20}{9}k=C_8\overset{!}{=}0.
\end{array}
$$

The only remaining 4-cycles of this type are $\left(\overrightarrow{I_0}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{IV_1}^{-1}\cdot\overrightarrow{O_1}\right)$, boundary of the two adjacent triangles around $\overrightarrow{III_0}$ and $\left(\overrightarrow{II_i}\cdot\overrightarrow{O_{2i}}\cdot\overrightarrow{III_i}^{-1}\cdot\overrightarrow{O_{2i+1}}\right)$, boundary of the two adjacent triangles around $\overrightarrow{I_i}$ (see Figure 4.1).

Thus, the only non-trivial 4-cycles of type $(*)$ are pairs of adjacent triangles with a common inner edge, and the only edges whose adjacent triangles have not been covered yet are $\overrightarrow{I_0}$ and $\overrightarrow{III_i}$.

<u>$(\#)\,\overrightarrow{O}+\overrightarrow{A}\overset{!}{=}\overrightarrow{O'}+\overrightarrow{B}$, $(\overrightarrow{O}=\overrightarrow{O_{2i+1}}=\overrightarrow{O'}$ or $\overrightarrow{O}\neq\overrightarrow{O_{2i+1}}\neq\overrightarrow{O'})$:</u>

In both these cases, $C_7(\overrightarrow{O})=C_7(\overrightarrow{O'})$. Thus, $C_7(\overrightarrow{A})=C_7(\overrightarrow{B})$, restricting them to the same side. Also, $C_3(\overrightarrow{O})=C_3(\overrightarrow{O'})=1$, and thus, $C_3(\overrightarrow{A})=C_3(\overrightarrow{B})$. Combining these conditions, we get that $\overrightarrow{A}=\overrightarrow{B}$ (see Figure 4.3). But then, with C_1, we get that $\overrightarrow{O}=\overrightarrow{O'}$ as well. Comparing with Figure 4.7, we see that, in this case, the types $(\#)_1$, $(\#)_2$, $(\#)_4$ and $(\#)_5$ are trivial. The only remaining case is

$$(\#)_3:\ \overrightarrow{A}\cdot\overrightarrow{O}\cdot\overrightarrow{A}^{-1}\cdot\overrightarrow{O}^{-1}.$$

If $\overrightarrow{O}\neq\overrightarrow{O_{2i+1}}$, C_8 yields a contradiction:

$$2hC_1(\overrightarrow{O})\overset{!}{=}0$$

For $\overrightarrow{O}=\overrightarrow{O_{2i+1}}$ (and $\overrightarrow{A}\in\{\overrightarrow{I_k},\overrightarrow{II_k},\overrightarrow{III_k},\overrightarrow{IV_k}\}$), C_8 yields:

$$(\#)_3:C_8(\overrightarrow{A}\cdot\overrightarrow{O_{2i+1}}\cdot\overrightarrow{A}^{-1}\cdot\overrightarrow{O_{2i+1}}^{-1})=\frac{2}{9}(28+47i\mp 9k)\overset{!}{=}0,$$

and thus, $9k\overset{!}{=}\mp i$ (eliminating $\overrightarrow{A}=\overrightarrow{I_k},\overrightarrow{III_k}$). For $\overrightarrow{A}\in\{\overrightarrow{II_k},\overrightarrow{IV_k}\}$, adding C_6 yields the contradiction:

$$(\#)_3 : C_6(\overrightarrow{A} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{A}^{-1} \cdot \overrightarrow{O_{2i+1}}^{-1}) = \frac{2}{9}(13+17i+9k) = \frac{2}{9}(13+18i) = \frac{2}{9}(6+6i) \stackrel{!}{=} 0$$

$\underline{(\#)\, \overrightarrow{O} + \overrightarrow{A} \stackrel{!}{=} \overrightarrow{O'} + \overrightarrow{B}, \{\overrightarrow{O}, \overrightarrow{O'}\} = \{\overrightarrow{O_{2i+1}}, \overrightarrow{O_{j\neq 2i+1}}\}}$:

With $\overrightarrow{O_{2i+1}}$ appearing exactly once, we also have C_7 equalling 1 for exactly one of the outer edges, forcing $\overrightarrow{A}$ and $\overrightarrow{B}$ on different sides of T_0:

$$\begin{cases} \{h+1, h+2\} \ni C_7(\overrightarrow{B}) + 1 & \stackrel{!}{=} C_7(\overrightarrow{A}) + 0 \in \{h, h+1\}, \\ \{h, h+1\} \ni C_7(\overrightarrow{B}) + 0 & \stackrel{!}{=} C_7(\overrightarrow{A}) + 1 \in \{h+1, h+2\}. \end{cases}$$

Also, we have $C_3(\overrightarrow{O}) = C_3(\overrightarrow{O'}) = 1$, yielding $C_3(\overrightarrow{A}) \stackrel{!}{=} C_3(\overrightarrow{B})$ again. Together, this yields the following cases:

$$\{\overrightarrow{A}, \overrightarrow{B}\} = \begin{cases} \{\overrightarrow{I_k}, \overrightarrow{IV_k}\}, & k \in \{1, \dots, i\}, \\ \{\overrightarrow{II_k}, \overrightarrow{III_{k-1}}\}, & k \in \{1, \dots, i\}. \end{cases}$$

The closing condition in C_1 yields:

$$C_1(\overrightarrow{O'}) - C_1(\overrightarrow{O}) \stackrel{!}{=} C_1(\overrightarrow{A}) - C_1(\overrightarrow{B}) = \begin{cases} C_1(\overrightarrow{I_k}) - C_1(\overrightarrow{IV_k}) & = -a - 2k, \\ C_1(\overrightarrow{IV_k}) - C_1(\overrightarrow{I_k}) & = a + 2k, \\ C_1(\overrightarrow{II_k}) - C_1(\overrightarrow{III_{k-1}}) & = 2k - 1 - a, \\ C_1(\overrightarrow{III_{k-1}}) - C_1(\overrightarrow{II_k}) & = a - 2k + 1, \end{cases}$$

fixing the following configurations:

$$(\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{O}, \overrightarrow{O'}) = \begin{cases} (\overrightarrow{I_k}, \overrightarrow{IV_k}, \overrightarrow{O_{2i+1}}, \overrightarrow{O_{2k}}), & k \in \{1, \dots, i\}, \\ (\overrightarrow{IV_k}, \overrightarrow{I_k}, \overrightarrow{O_{2k}}, \overrightarrow{O_{2i+1}}), & k \in \{1, \dots, i\}, \\ (\overrightarrow{II_k}, \overrightarrow{III_{k-1}}, \overrightarrow{O_{2i+1}}, \overrightarrow{O_{2k-1}}), & k \in \{1, \dots, i\}, \\ (\overrightarrow{III_{k-1}}, \overrightarrow{II_k}, \overrightarrow{O_{2k-1}}, \overrightarrow{O_{2i+1}}), & k \in \{1, \dots, i\}. \end{cases}$$

Note that all these configurations add up to 0 in the abelian components, by construction. Also note that of the five types of (#)-cycles, only $(\#)_3$ is not symmetric with respect to switching $\overrightarrow{A}, \overrightarrow{B}$ and $\overrightarrow{O}, \overrightarrow{O'}$. Thus, only for $(\#)_3$, all four types of configurations have to be checked. C_6 and C_8 yield (note that $C_6(\overrightarrow{III_0}) \neq C_6(\overrightarrow{III_{k,k=0}})$):

$$\begin{array}{lll} (\#)_1 & \left(\overrightarrow{I_k} \cdot \overrightarrow{IV_k}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{O_{2k}}^{-1}\right): & \frac{4}{9}(7+5i-2k) = C_6 \stackrel{!}{=} 0, \\ (\#)_1 & \left(\overrightarrow{II_k} \cdot \overrightarrow{III_{k-1}}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{O_{2k-1}}^{-1}\right): & \frac{2}{9}(5+10i+4k) = \frac{4}{9}(-1-i+2k) = C_6 \stackrel{!}{=} 0, \\ (\#)_1 & \left(\overrightarrow{II_1} \cdot \overrightarrow{III_0}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{O_1}^{-1}\right): & -\frac{16}{9}i = C_6 \stackrel{!}{=} 0, \end{array}$$

$$
\begin{array}{lll}
(\#)_2 & \left(\overrightarrow{I_k} \cdot \overrightarrow{IV_k}^{-1} \cdot \overrightarrow{O_{2k}}^{-1} \cdot \overrightarrow{O_{2i+1}}\right): & \frac{4}{9}(7+5i+7k) = \frac{4}{9}(-7i+7k) = C_6 \stackrel{!}{=} 0, \\
(\#)_2 & \left(\overrightarrow{II_k} \cdot \overrightarrow{III_{k-1}}^{-1} \cdot \overrightarrow{O_{2k-1}}^{-1} \cdot \overrightarrow{O_{2i+1}}\right): & \frac{4}{9}(7+5i-7k) = \frac{4}{9}(-7i-7k) = C_6 \stackrel{!}{=} 0, \\
(\#)_2 & \left(\overrightarrow{II_1} \cdot \overrightarrow{III_0}^{-1} \cdot \overrightarrow{O_1}^{-1} \cdot \overrightarrow{O_{2i+1}}\right): & \frac{20}{9}i = C_6 \stackrel{!}{=} 0, \\
 & & \\
(\#)_3 & \left(\overrightarrow{I_k} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{IV_k}^{-1} \cdot \overrightarrow{O_{2k}}^{-1}\right): & -\frac{20}{9}k = C_8 \stackrel{!}{=} 0, \\
(\#)_3 & \left(\overrightarrow{IV_k} \cdot \overrightarrow{O_{2k}} \cdot \overrightarrow{I_k}^{-1} \cdot \overrightarrow{O_{2i+1}}^{-1}\right): & \frac{2}{9}(28+47i-k) = \frac{2}{9}(-i-k) = C_8 \stackrel{!}{=} 0, \\
(\#)_3 & \left(\overrightarrow{II_k} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{III_{k-1}}^{-1} \cdot \overrightarrow{O_{2k-1}}^{-1}\right): & \frac{20}{9}k = C_8 \stackrel{!}{=} 0, \\
(\#)_3 & \left(\overrightarrow{III_{k-1}} \cdot \overrightarrow{O_{2k-1}} \cdot \overrightarrow{II_k}^{-1} \cdot \overrightarrow{O_{2i+1}}^{-1}\right): & \frac{2}{9}(27+47i+k) = \frac{2}{9}(-1-i+k) = C_8 \stackrel{!}{=} 0, \\
 & & \\
(\#)_4 & \left(\overrightarrow{I_k} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{O_{2k}}^{-1} \cdot \overrightarrow{IV_k}^{-1}\right): & -\frac{16}{9}k = C_8 \stackrel{!}{=} 0, \\
(\#)_4 & \left(\overrightarrow{II_k} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{O_{2k-1}}^{-1} \cdot \overrightarrow{III_{k-1}}^{-1}\right): & \frac{2}{9}(1+8k) = C_8 \stackrel{!}{=} 0, \\
 & & \\
(\#)_5 & \left(\overrightarrow{I_k} \cdot \overrightarrow{O_{2k}}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{IV_k}^{-1}\right): & \frac{20}{9}k = C_8 \stackrel{!}{=} 0, \\
(\#)_5 & \left(\overrightarrow{II_k} \cdot \overrightarrow{O_{2k-1}}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{III_{k-1}}^{-1}\right): & -\frac{20}{9}(-1+k) = C_8 \stackrel{!}{=} 0, \\
(\#)_5 & \left(\overrightarrow{II_1} \cdot \overrightarrow{O_1}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{III_0}^{-1}\right): & \text{nullhomotopic.}
\end{array}
$$

Thus, the only remaining 4-cycles of this type are $\left(\overrightarrow{I_i} \cdot \overrightarrow{IV_i}^{-1} \cdot \overrightarrow{O_{2i}}^{-1} \cdot \overrightarrow{O_{2i+1}}\right)$, boundary of the two adjacent triangles around $\overrightarrow{III_i}$ and $\left(\overrightarrow{II_1} \cdot \overrightarrow{O_1}^{-1} \cdot \overrightarrow{O_{2i+1}} \cdot \overrightarrow{III_0}^{-1}\right)$, boundary of the two adjacent triangles around $\overrightarrow{I_0}$ (see Figure 4.1).

In total: The only non-trivial 4-cycles that contain both inner and outer edges are the boundaries of adjacent triangles with a common inner edge.

Four inner edges:

Assume first that all four edges are positively oriented. As C_0 equals 1 for inner edges and operates on $\mathbb{Z}_4$, this is a possibility. Denote the number of left edges (edges of type $\overrightarrow{III}$ or $\overrightarrow{IV}$) by x. For the cycle to be closed in C_7, which equals h for right and $h+1$ for left inner edges, we get:

$$
\begin{aligned}
4h + x &\equiv 0 \ (mod\ 12i+7), x \in \{0,1,2,3,4\} \\
\overset{\cdot 9,\ h \equiv \frac{-10}{9}}{\Leftrightarrow} \quad 9x &\equiv 40 \ (mod\ 12i+7) \\
\Rightarrow \quad x = 1, &\quad i = 2
\end{aligned}
$$

Thus, such a cycle can only be closed if it contains one edge from the left side and three edges from the right side, and if $i = 2$. In that case, C_5 yields a contradiction (note that C_5

equals $f+1$ or $f+2$ for left inner edges and $f+1$ for right inner edges):

$$\begin{aligned} 4(f+1) + \begin{cases} 0 \\ 1 \end{cases} & \equiv 0 \; (mod\; 31) \\ \overset{\cdot 9,\; f \equiv \frac{-4}{9}}{\Leftrightarrow} \qquad \begin{cases} 36 \\ 45 \end{cases} & \equiv 16 \; (mod\; 31) \end{aligned}$$

Thus, as C_0 equals 1 for inner edges, the cycles consist of two positively oriented and two inverted edges. W.l.o.g., start with a positively oriented inner edge $\overrightarrow{A}$. For the other positively oriented edge $\overrightarrow{B}$, there are three possible placements relative to $\overrightarrow{A}$. Placing $\overrightarrow{B}$ fourth yields a rotated version of placing it second, leaving two types of 4-cycles consisting only of inner edges (see Figure 4.8).

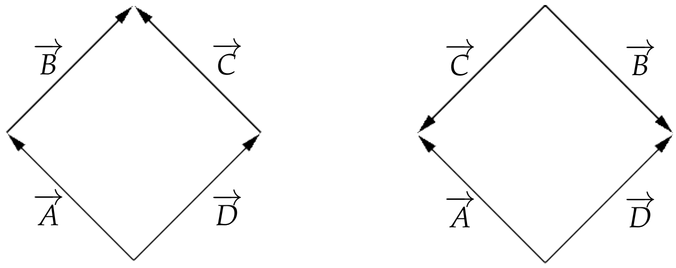

Figure 4.8.: Nomination of group elements in both types of inner 4-cycles.

Again, the edges are nominated in a way that the closing conditions are the same in an abelian world:

$$\overrightarrow{A} + \overrightarrow{B} \overset{!}{=} \overrightarrow{C} + \overrightarrow{D}$$

With $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}, \overrightarrow{D} \in \{\overrightarrow{I_k}, \overrightarrow{II_k}, \overrightarrow{III_k}, \overrightarrow{IV_k}\}$, this is the most extensive case yet. Some observations to narrow it down:
First, note that C_3 is even for $\overrightarrow{I_k}$ and $\overrightarrow{IV_k}$, and odd for $\overrightarrow{II_k}$ and $\overrightarrow{III_k}$. Thus, the combined C_3 of a pair $\{\overrightarrow{A}, \overrightarrow{B}\}$ is

$$\begin{cases} \text{odd, } \{\overrightarrow{A}, \overrightarrow{B}\} \in \left\{\{\overrightarrow{I_{k_1}}, \overrightarrow{II_{k_2}}\}, \{\overrightarrow{I_{k_1}}, \overrightarrow{III_{k_2}}\}, \{\overrightarrow{II_{k_1}}, \overrightarrow{IV_{k_2}}\}, \{\overrightarrow{III_{k_1}}, \overrightarrow{IV_{k_2}}\}\right\}, \\ \text{even, } \{\overrightarrow{A}, \overrightarrow{B}\} \in \left\{\{\overrightarrow{I_{k_1}}, \overrightarrow{I_{k_2}}\}, \{\overrightarrow{II_{k_1}}, \overrightarrow{II_{k_2}}\}, \{\overrightarrow{III_{k_1}}, \overrightarrow{III_{k_2}}\}, \{\overrightarrow{IV_{k_1}}, \overrightarrow{IV_{k_2}}\}, \{\overrightarrow{I_{k_1}}, \overrightarrow{IV_{k_2}}\}, \{\overrightarrow{II_{k_1}}, \overrightarrow{III_{k_2}}\}\right\}. \end{cases}$$

Thus, if $\{\overrightarrow{A}, \overrightarrow{B}\}$ is a combination of the first line, so is $\{\overrightarrow{C}, \overrightarrow{D}\}$ and vice versa.
On the other hand, C_7 equals $h+1$ for left inner edges ($\overrightarrow{III_k}, \overrightarrow{IV_k}$) and h for right inner edges ($\overrightarrow{I_k}, \overrightarrow{II_k}$). Thus, the number of left and right inner edges in $\{\overrightarrow{A}, \overrightarrow{B}\}$ must be the same as the number of left and right inner edges in $\{\overrightarrow{C}, \overrightarrow{D}\}$. This narrows it down to the following cases:

$$\begin{aligned} \{\overrightarrow{A}, \overrightarrow{B}\} \in \left\{\{\overrightarrow{I_{k_1}}, \overrightarrow{I_{k_2}}\}, \{\overrightarrow{II_{k_1}}, \overrightarrow{II_{k_2}}\}\right\} & \Leftrightarrow \{\overrightarrow{C}, \overrightarrow{D}\} \in \left\{\{\overrightarrow{I_{k_3}}, \overrightarrow{I_{k_4}}\}, \{\overrightarrow{II_{k_3}}, \overrightarrow{II_{k_4}}\}\right\} \\ \{\overrightarrow{A}, \overrightarrow{B}\} \in \left\{\{\overrightarrow{III_{k_1}}, \overrightarrow{III_{k_2}}\}, \{\overrightarrow{IV_{k_1}}, \overrightarrow{IV_{k_2}}\}\right\} & \Leftrightarrow \{\overrightarrow{C}, \overrightarrow{D}\} \in \left\{\{\overrightarrow{III_{k_3}}, \overrightarrow{III_{k_4}}\}, \{\overrightarrow{IV_{k_3}}, \overrightarrow{IV_{k_4}}\}\right\} \end{aligned}$$

$$\{\overrightarrow{A},\overrightarrow{B}\}\in\left\{\{\overrightarrow{I_{k_1}},\overrightarrow{II_{k_2}}\}\right\} \quad\Leftrightarrow\quad \{\overrightarrow{C},\overrightarrow{D}\}\in\left\{\{\overrightarrow{I_{k_3}},\overrightarrow{II_{k_4}}\}\right\}$$
$$\{\overrightarrow{A},\overrightarrow{B}\}\in\left\{\{\overrightarrow{III_{k_1}},\overrightarrow{IV_{k_2}}\}\right\} \quad\Leftrightarrow\quad \{\overrightarrow{C},\overrightarrow{D}\}\in\left\{\{\overrightarrow{III_{k_3}},\overrightarrow{IV_{k_4}}\}\right\}$$
$$\{\overrightarrow{A},\overrightarrow{B}\}\in\left\{\{\overrightarrow{I_{k_1}},\overrightarrow{III_{k_2}}\},\{\overrightarrow{II_{k_1}},\overrightarrow{IV_{k_2}}\}\right\} \quad\Leftrightarrow\quad \{\overrightarrow{C},\overrightarrow{D}\}\in\left\{\{\overrightarrow{I_{k_3}},\overrightarrow{III_{k_4}}\},\{\overrightarrow{II_{k_3}},\overrightarrow{IV_{k_4}}\}\right\}$$
$$\{\overrightarrow{A},\overrightarrow{B}\}\in\left\{\{\overrightarrow{I_{k_1}},\overrightarrow{IV_{k_2}}\},\{\overrightarrow{II_{k_1}},\overrightarrow{III_{k_2}}\}\right\} \quad\Leftrightarrow\quad \{\overrightarrow{C},\overrightarrow{D}\}\in\left\{\{\overrightarrow{I_{k_3}},\overrightarrow{IV_{k_4}}\},\{\overrightarrow{II_{k_3}},\overrightarrow{III_{k_4}}\}\right\}$$

With C_7, the closing condition in C_1 reduces to $\pm k_A \pm k_B \overset{!}{=} \pm k_C \pm k_D$ (as $2b$ can be subtracted on both sides, as can any appearance of a resulting from left inner edges). The sign is determined by the types of inner edges, with $\overrightarrow{I_k}, \overrightarrow{III_k}$ yielding negative signs and $\overrightarrow{II_k}, \overrightarrow{IV_k}$ yielding positive signs. As we cannot have all positive signs on one side, and all negative signs on the other, this reduces our cases further. It is easy to check that the closing condition in C_3 of all remaining cases reduces to

$$k_A + k_B \overset{!}{=} k_C + k_D, \tag{4.7}$$

and that only two cases of sign-distributions in C_1 are left: Either both sides being mixed, or both sides having the same signs. Assume for a moment that k_A and k_C assume the positive sign, with k_B and k_D assuming the negative sign. Then, C_1 yields:

$$k_A - k_B \overset{!}{=} k_C - k_D$$

But then, (4.7) yields that $k_A = k_C$ and $k_B = k_D$. Thus, for any combination of positive and negative signs, the elements with the positive signs have the same index, as have the elements with the negative index.
On the other hand, if both sides have the same signs in C_1, C_2 yields one of the two conditions ($3\cdot C_2$ with either all $\pm = +$ or all $\pm = -$):

$$k_A(2k_A\pm1)(4k_A\pm1)+k_B(2k_B\pm1)(4k_B\pm1) \overset{!}{=} k_C(2k_C\pm1)(4k_C\pm1)+k_D(2k_D\pm1)(4k_D\pm1)$$

With (4.7), this simplifies to

$$-12(\pm1+2k_A+2k_B)(k_A-k_C)(k_B-k_C) \overset{!}{=} 0,$$

and thus, as $12i+7$ is prime and $2k_A+2k_B \neq \mp1$, either $k_A = k_C$ and thus $k_B = k_D$, or $k_B = k_C$ and thus $k_A = k_D$.
Omitting the trivially nullhomotopic ones $\left((\overrightarrow{A}\cdot\overrightarrow{B}\cdot\overrightarrow{B}^{-1}\cdot\overrightarrow{A}^{-1}) \text{ or } (\overrightarrow{A}\cdot\overrightarrow{A}^{-1}\cdot\overrightarrow{B}\cdot\overrightarrow{B}^{-1})\right)$, this leaves the following possible cycles:

$$\begin{array}{lll}
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{I_{k_2}}\cdot\overrightarrow{I_{k_1}}^{-1}\cdot\overrightarrow{I_{k_2}}^{-1}): & \frac{20}{9}(k_1-k_2) = C_8 \overset{!}{=} 0, & \to 1 \\
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{II_{k_2}}\cdot\overrightarrow{I_{k_1}}^{-1}\cdot\overrightarrow{II_{k_2}}^{-1}): & \frac{20}{9}(k_1+k_2) = C_8 \overset{!}{=} 0, & \to 2 \\
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{III_{k_2}}\cdot\overrightarrow{I_{k_1}}^{-1}\cdot\overrightarrow{III_{k_2}}^{-1}): & \frac{2}{9}(28+47i+k_1-10k_2) = C_8 \overset{!}{=} 0, & \to 3 \\
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{III_{k_2}}\cdot\overrightarrow{I_{k_2}}^{-1}\cdot\overrightarrow{III_{k_1}}^{-1}): & \frac{2}{9}(28+47i-10k_1+k_2) = C_8 \overset{!}{=} 0, & \to 3 \\
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{I_{k_2}}^{-1}\cdot\overrightarrow{III_{k_2}}\cdot\overrightarrow{III_{k_1}}^{-1}): & -\frac{20}{9}(k_1-k_2) = C_8 \overset{!}{=} 0, & \to 1 \\
(\overrightarrow{I_{k_1}}\cdot\overrightarrow{III_{k_2}}\cdot\overrightarrow{III_{k_1}}^{-1}\cdot\overrightarrow{I_{k_2}}^{-1}): & (k_1-k_2)(-5-12i+4k_1+4k_2) = C_4 \overset{!}{=} 0, & \to 1
\end{array}$$

$$
\begin{array}{lll}
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1}) : & -\frac{2}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{I_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}) : & \frac{2}{9}(28 + 47i + k_1 + 10k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 4 \\
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{III_{k_1}}^{-1}) : & \frac{2}{9}(28 + 47i - 10k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 5 \\
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1}) : & -\frac{20}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2 \\
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}}^{-1}) : & -(k_1 - k_2)(3 + 12i - 4k_1 + 4k_2) = C_4 \stackrel{!}{=} 0, & \rightarrow 6 \\
(\overrightarrow{I_{k_1}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1}) : & -\frac{2}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2
\end{array}
$$

$$
\begin{array}{lll}
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}}^{-1}) : & -\frac{20}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}}^{-1}) : & = (\overrightarrow{I_{k_2}} \cdot \overrightarrow{II_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{II_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}) : & \frac{2}{9}(28 + 47i - k_1 + 10k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 7 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1}) : & \frac{2}{9}(28 + 47i + 10k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 7 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1}) : & \frac{20}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}}^{-1}) : & -(k_1 - k_2)(5 + 12i + 4k_1 + 4k_2) = C_4 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1}) : & \frac{2}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}}^{-1}) : & \frac{2}{9}(28 + 47i - k_1 - 10k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 5 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1}) : & \frac{2}{9}(28 + 47i + 10k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 4 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1}) : & \frac{20}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2 \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}}^{-1}) : & = (\overrightarrow{I_{k_2}} \cdot \overrightarrow{IV_{k_1}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{II_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1}) : & = (\overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_1}}^{-1})^{-1}, &
\end{array}
$$

$$
\begin{array}{lll}
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}}^{-1}) : & \frac{2}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}) : & \frac{2}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2 \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}}^{-1}) : & = (\overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{I_{k_1}}^{-1}) : & = (\overrightarrow{I_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{I_{k_1}}^{-1}) : & = (\overrightarrow{I_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{I_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}}^{-1}) : & \frac{22}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_2}}^{-1}) : & \frac{2}{9}(k_1 - k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}}^{-1}) : & = (\overrightarrow{II_{k_2}} \cdot \overrightarrow{III_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_2}}^{-1} \cdot \overrightarrow{I_{k_1}}^{-1}) : & = (\overrightarrow{I_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{IV_{k_2}}^{-1} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{I_{k_1}}^{-1}) : & = (\overrightarrow{I_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{III_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{I_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}) : & \frac{22}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2 \\
(\overrightarrow{III_{k_1}} \cdot \overrightarrow{I_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_2}}^{-1}) : & \frac{2}{9}(k_1 + k_2) = C_8 \stackrel{!}{=} 0, & \rightarrow 2
\end{array}
$$

$$
\begin{array}{lll}
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}): & -\frac{2}{9}(k_1 - k_2) = C_8 \overset{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}}^{-1}): & = (\overrightarrow{III_{k_2}} \cdot \overrightarrow{IV_{k_1}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}}^{-1}): & = (\overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_2}}^{-1} \cdot \overrightarrow{II_{k_1}}^{-1}): & = (\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{IV_{k_2}}^{-1} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1}): & = (\overrightarrow{II_{k_1}} \cdot \overrightarrow{II_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}): & -\frac{22}{9}(k_1 - k_2) = C_8 \overset{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{II_{k_2}} \cdot \overrightarrow{IV_{k_2}}^{-1}): & -\frac{2}{9}(k_1 - k_2) = C_8 \overset{!}{=} 0, & \rightarrow 1 \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}}^{-1}): & = (\overrightarrow{I_{k_2}} \cdot \overrightarrow{IV_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{II_{k_1}}^{-1}): & = (\overrightarrow{II_{k_1}} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{III_{k_2}}^{-1} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1}): & = (\overrightarrow{II_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{III_{k_2}} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{III_{k_2}}^{-1}): & = (\overrightarrow{III_{k_2}} \cdot \overrightarrow{II_{k_1}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}, & \\
(\overrightarrow{IV_{k_1}} \cdot \overrightarrow{II_{k_1}}^{-1} \cdot \overrightarrow{I_{k_2}} \cdot \overrightarrow{III_{k_2}}^{-1}): & = (\overrightarrow{III_{k_2}} \cdot \overrightarrow{I_{k_2}}^{-1} \cdot \overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{k_1}}^{-1})^{-1}. &
\end{array}
$$

The following cases appear:

1 The condition in C_8 (or C_4) reduces to $k_1 \overset{!}{=} k_2$, which in these cases renders the cycles trivially nullhomotopic. Note that there are two cases where $(k_1 - k_2)$ is just one non-constant factor, with the other one being

$$(\pm(5 + 12i) + 4k_1 + 4k_2) = (\mp 2 + 4k_1 + 4k_2) \neq 0,\ k_1, k_2 \in \{0, \ldots, i\}.$$

2 The condition in C_8 reduces to $k_1 \overset{!}{=} -k_2$, which is a contradiction, as in every one of these cases, $k_1 \geq 1$ or $k_2 \geq 1$.

3 C_8 yields that $\begin{cases} k_1 \overset{!}{=} i + 10k_2, \\ k_2 \overset{!}{=} i + 10k_1, \end{cases}$ and thus $\begin{cases} k_1 = i, k_2 = 0, \\ k_2 = i, k_1 = 0, \end{cases}$ leaving the cycles

$$
\begin{array}{ll}
(\overrightarrow{I_i} \cdot \overrightarrow{III_0} \cdot \overrightarrow{I_i}^{-1} \cdot \overrightarrow{III_0}^{-1}): & -\frac{10}{9}(1 + 3i) = C_6 \overset{!}{=} 0,\ \lightning, \\
(\overrightarrow{I_0} \cdot \overrightarrow{III_i} \cdot \overrightarrow{I_i}^{-1} \cdot \overrightarrow{III_0}^{-1}): & \text{adjacent triangles around } \overrightarrow{O_{2i+1}}.
\end{array}
$$

4 C_8 yields that $\begin{cases} k_1 \overset{!}{=} i - 10k_2, \\ k_2 \overset{!}{=} i - 10k_1, \end{cases}$ leaving the cycles

$$
\begin{array}{rl}
(\overrightarrow{I_{i-10k_2}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{I_{i-10k_2}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}): & \frac{26}{9} + \frac{2}{3}i + 30k_2 = \frac{5}{2}(1 + 12k_2) = C_6 \overset{!}{=} 0,\ \lightning, \\
(\overrightarrow{I_0} \cdot \overrightarrow{IV_{\frac{i}{10}}} \cdot \overrightarrow{I_0}^{-1} \cdot \overrightarrow{IV_{\frac{i}{10}}}^{-1}): & \frac{16}{45}(25 + 39i) = \frac{16}{45}(4 + 3i) = C_6 \overset{!}{=} 0,\ \lightning, \\
(\overrightarrow{II_{k_1}} \cdot \overrightarrow{III_{i-10k_1}} \cdot \overrightarrow{I_{i-10k_1}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1}): & \frac{26}{9} + \frac{2}{3}i + 30k_1 = \frac{5}{2}(1 + 12k_1) = C_6 \overset{!}{=} 0,\ \lightning, \\
(\overrightarrow{II_{\frac{i}{10}}} \cdot \overrightarrow{III_0} \cdot \overrightarrow{I_0}^{-1} \cdot \overrightarrow{IV_{\frac{i}{10}}}^{-1}): & \frac{98+123i}{9} = \frac{35}{12} = C_6 \overset{!}{=} 0,\ \lightning.
\end{array}
$$

5 C_8 yields that $\begin{cases} -k_1 - 10k_2 \overset{!}{=} i, \\ -k_2 - 10k_1 \overset{!}{=} i, \end{cases}$ a contradiction.

6 C_4 yields that $k = \begin{cases} k_1 = k_2, \\ k_1 = k_2 - 1, \end{cases}$ $k \in \{1, \dots, i\}$, leaving the cycles

$$\begin{aligned} (\overrightarrow{I_k} \cdot \overrightarrow{IV_k} \cdot \overrightarrow{III_k}^{-1} \cdot \overrightarrow{II_k}^{-1}) &: \quad \text{adjacent triangles around } \overrightarrow{O_{2k}}, \\ (\overrightarrow{I_{k-1}} \cdot \overrightarrow{IV_k} \cdot \overrightarrow{III_{k-1}}^{-1} \cdot \overrightarrow{II_k}^{-1}) &: \quad \text{adjacent triangles around } \overrightarrow{O_{2k-1}}. \end{aligned}$$

7 C_8 yields that $\begin{cases} k_1 \overset{!}{=} 10k_2 - i, \\ k_2 \overset{!}{=} 10k_1 - i, \end{cases}$ leaving the cycles

$$\begin{aligned} (\overrightarrow{II_{10k_2-i}} \cdot \overrightarrow{IV_{k_2}} \cdot \overrightarrow{II_{10k_2-i}}^{-1} \cdot \overrightarrow{IV_{k_2}}^{-1}) &: \quad \tfrac{26}{9} + \tfrac{2}{3}i + 30k_2 = \tfrac{5}{2}(1 + 12k_2) = C_6 \overset{!}{=} 0, \ \lightning, \\ (\overrightarrow{II_{k_1}} \cdot \overrightarrow{IV_{10k_1-i}} \cdot \overrightarrow{II_{10k_1-i}}^{-1} \cdot \overrightarrow{IV_{k_1}}^{-1}) &: \quad \tfrac{26}{9} + \tfrac{2}{3}i + 30k_1 = \tfrac{5}{2}(1 + 12k_1) = C_6 \overset{!}{=} 0, \ \lightning. \end{aligned}$$

Thus, the only remaining inner 4-cycles that are not trivially nullhomotopic are pairs of adjacent triangles with a common outer edge.

In total: We have shown that there are only three types of closed 4-cycles, all of which only yield trivial cycles or pairs of adjacent triangles. □

Proof of Theorem 4.5.
Inserting T_0, M_0 and G in the construction described in Section 4.1 yields a vertex-transitive triangulation T with as many vertices as there are elements in G, which is $4(12i+7)^8$. Note that G is a subgroup of the fundamental group $\pi_1(M_0)$ (see the arguments following Equation (4.6)) The valency of every vertex is given by the size of the $(4i+4)$-gon, which has $4i+1$ diagonals. Adding their inverses, this yields $\overline{v} = 12i+6$. The Handshaking lemma and Euler's formula for 2-manifolds yield $\chi = f_0(1 - \frac{\overline{v}}{6}) = -8i(12i+7)^8$. Lemma 4.8 yields that T is ew-4. Together with Lemma 4.9, T is ew-5. □

5. Algorithmic investigations

The k-irreducible triangulations are a generating set for all ew-k triangulations (Corollary 2.5). In particular, all $(k+1)$-irreducible triangulations of a fixed 2-manifold can be generated from the set of all its k-irreducible ones. In addition, for the 2-manifolds of Euler genus up to 4, the complete list of all 3-irreducible triangulations is known (see Section 2.1). Thus, it is natural to ask if we can produce the full list of all k-irreducible triangulations, at least up to a certain Euler genus, and up to some k greater than 3. In other words: Can we find an algorithm and implement it in a computer program such that it produces all $(k+1)$-irreducible triangulations from the set of all k-irreducible ones? Can we do so in a way that the program terminates in a feasible time, using the technical infrastructure available to us?

The answer is yes. In Section 5.1, we present such an algorithm. Its implementation was done in C++, with parallelization added using the MPI-framework. In Section 5.2, we present the results obtained by heavy parallel computations on the computing clusters of the Faculty of Mathematics at Technische Universität Dresden, which consist of 24 cores each and, more importantly, provide around 6 GB of RAM per core.

Another natural algorithmic problem is the heuristic generation of k-irreducible triangulations with certain properties, for an arbitrary 2-manifold and an arbitrarily large k. In Section 5.3, we present an algorithm that aims at finding such triangulations with few vertices. Some results produced by said algorithm are presented in Section 5.4, in particular the size of the smallest known k-irreducible triangulations for 2-manifolds of Euler genus up to 20, both orientable and non-orientable, and k at most 8. These heuristic examples are used in Section 6.1.1 to produce infinite series of k-irreducible triangulations of small 2-manifolds. Those series yield the smallest known growth rates for the size of k-irreducible triangulations with respect to k, for those fixed 2-manifolds. An interesting question that remains open is to find an algorithm that produces large k-irreducible triangulations of a fixed 2-manifold.

Finally, in Section 5.5, we present all up to 8-irreducible triangulations of the 2-sphere with up to 17 vertices, using the natural redefinition given in Section 3.3.

All triangulations presented in this chapter and the specific implementation used to obtain them can be found on the CD accompanying this thesis. Appendix A contains a short description of the implementation.

5.1. Algorithms for complete enumerations

While the general idea is simple, the challenge here is to deal with the algorithmic complexity, both regarding time and storage. An intuitive first approach:

5.1.1. ew-k triangulations

Fix some integer k greater than or equal to 3, and fix a 2-manifold M. Denote by $\mathbf{T}_n$ the set of all ew-k triangulations of M with $f_0 = n$ vertices. Thus, for every triangulation $T \in \mathbf{T}_n$, there is an integer $\tilde{n} \leq n$ and a k-irreducible triangulation $T_0 \in \mathbf{T}_{\tilde{n}}$ such that T is generated from T_0 by $n - \tilde{n}$ consecutive vertex splits. Denote by $f_{0,k,\min}$ the size of the smallest k-irreducible triangulation of M.

Algorithm 1 (Generating all ew-k triangulations $\mathbf{T}_n$ of M with n vertices)**.**
If $n < f_{0,k,\min}$, $\mathbf{T}_n$ is empty.
If $n = f_{0,k,\min}$, $\mathbf{T}_n$ consists only of the minimal k-irreducible triangulations of M.
If $n > f_{0,k,\min}$, generate $\mathbf{T}_n$ inductively by performing every possible vertex split on every triangulation $T \in \mathbf{T}_{n-1}$ and adding the k-irreducible triangulations of M with n vertices.

For every triangulation $T \in \mathbf{T}_n$, there are only finitely many such possible vertex splits. As the number of k-irreducible triangulations is finite for any k and any fixed 2-manifold M, so is $\mathbf{T}_n$.
In particular, the $(k+1)$-irreducible triangulations of M are ew-k. Thus, in theory, one could generate all ew-k triangulations of M up to a certain size, and get all $(k+1)$-irreducible triangulations of M by checking those final lists for ew-$(k+1)$ and irreducibility. As there is an upper bound for the number of vertices in a $(k+1)$-irreducible triangulation of any 2-manifold, and for any arbitrary k, this is a finite algorithm. In reality, this approach is not feasible. Note that a vertex split is fixed by choosing the vertex to be split as well as two of its neighbours – those that will be adjacent to both duplicate vertices created by the split. Thus, the number of possible splits for a triangulation T is roughly given by

$$\#_{\text{splits}} \approx \sum_{A \in V} \binom{v_A}{2},$$

with v_A denoting the valency of the vertex $A \in V$. Even when somehow omitting isomorphic ones, this quickly produces too many triangulations to be handled by any computing system available to us.
Thus, refinements are needed to generate all $(k+1)$-irreducible triangulations. For that, we have a closer look at the inverse procedure: How exactly are the $(k+1)$-irreducible triangulations reduced to k-irreducible ones by edge contractions?

5.1.2. k-irreducible triangulations

Let T be a $(k+1)$-irreducible triangulation. Thus, T can be reduced to a k-irreducible triangulation T_0 by repeatedly contracting edges that are not contained in an essential k-

cycle. Now, on the one hand, every edge of T is contained in an essential $(k+1)$-cycle and there are no shorter essential cycles in T. On the other hand, an edge contraction reduces the length of any cycle containing that edge by one, and does not change the length of any other cycle. Thus, any intermediate triangulation T' that is generated from T on the way to T_0 is ew-k, and every edge is contained in an essential cycle that has length at most $k+1$. Thus, in the inverse procedure – generating all $(k+1)$-irreducible triangulations from the k-irreducible ones by vertex-splitting – any intermediate triangulation containing an edge that is not part of an essential cycle of length at most $k+1$ is a dead end and can be discarded. No further splits of such a triangulation can yield a $(k+1)$-irreducible one.

Furthermore, note that edge contractions are commutative, with one exception: If one edge contraction removes a 3-valent vertex, and the next contracts an edge of the resulting triangle. In this case, switching their order would create a double edge. Using a less strict definition of triangulations would solve that, but we will also see later that this case does not pose a problem in our algorithms.

Denote by Σ the set of edges of the $(k+1)$-irreducible triangulation T that are contracted when transforming T into the k-irreducible triangulation T_0. These edges are contracted in a specific order. Any other ordering (that does not produce double edges) yields the same output T_0, but will most likely produce different intermediate triangulations. Thus, in the inverse procedure – splitting T_0 to get to T – there are also multiple paths that reach T. As we are not interested in the intermediate triangulations, it is sufficient to pursue one of those paths, greatly reducing the number of intermediate triangulations that need to be checked. The question is: How do we fix one path?

Let γ be an essential k-cycle in T_0. As T is ew-$(k+1)$, it does not contain such cycles. Thus, there is an edge in Σ whose contraction transforms an essential $(k+1)$-cycle of T into γ. In the one exception to commutativity mentioned above, this is true for both edge contractions. W.l.o.g., pick the ordering of those two that does not produce a double edge. The inverse of that contraction is a split of a vertex of γ such that the newly created edge, together with γ, forms an essential $(k+1)$-cycle. As the edge contractions are commutative, in the process of contracting T to T_0, any arbitrary essential k-cycle of T_0 can be chosen to be created last. Thus, in the inverse procedure, when considering splits in T_0, it is sufficient to consider only those that extend on one arbitrary, fixed, essential k-cycle. The same holds in any intermediate triangulation between T and T_0. Note that extending γ can mean prolonging it, but can also mean that one edge of γ is just pulled to one side. Thus, that edge, together with γ, forms an essential $(k+1)$-cycle, but γ itself remains of length k (see Figure 5.1).

Thus, we can greatly reduce the number of necessary splits in any intermediate triangulation to

$$\#_{\text{splits}} = \sum_{A\in\gamma} \#_{A\text{-splits extending }\gamma} \leq \sum_{A\in\gamma} \frac{{v_A}^2}{4} + v_A - 2 \ll \sum_{A\in V} \binom{v_A}{2},$$

for some arbitrarily chosen essential k-cycle γ. There are two things to note about the

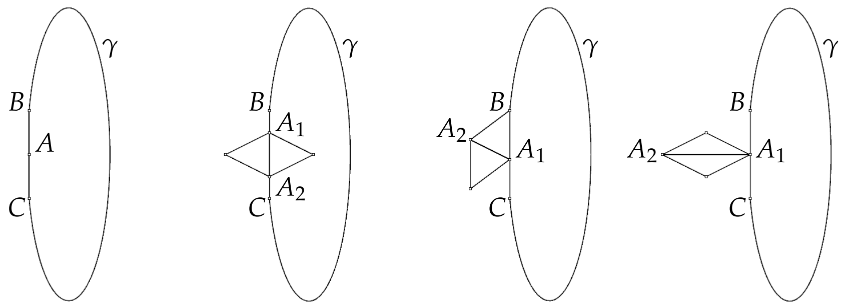

Figure 5.1.: Extending an essential cycle γ at vertex A: Before the split, an extension that prolongs γ, an extension that does not, and a split that is not an extension of γ

strength of this simplification: Firstly, not every vertex $A \in V$ is split, but only the k vertices of γ. Secondly, for any fixed vertex $A \in \gamma$, not every possible split is done, but only those extending γ. The exact number of those depends on the relative positioning of the predecessor and the successor of A in γ inside the neighbours of A. It is maximal, if predecessor and successor are opposite each other, and then that number is $\lfloor \frac{v_A{}^2}{4} + v_A - 2 \rfloor$ (any split of A with a combination of two neighbours on the same side of γ does not extend it). This yields the following improved algorithm.

Fix some integer k greater than or equal to 3, and fix a 2-manifold M. Denote the set of intermediate triangulations with n vertices that are created by the algorithm, with $\mathbf{T}_n$. Thus, for every triangulation $T \in \mathbf{T}_n$, there is a k-irreducible triangulation $T_0 \in \mathbf{T}_{\tilde{n}}$ such that T is generated from T_0 by $n - \tilde{n}$ consecutive vertex splits. As argued above, these intermediate triangulations are ew-k, though not every ew-k triangulation of M with n vertices is contained in $\mathbf{T}_n$, in contrast to Algorithm 1. Also, every edge of such an intermediate triangulation is contained in an essential cycle of length at most $k+1$. Again, let $f_{0,k,\min}$ denote the size of the smallest k-irreducible triangulation of M.

Algorithm 2 (Generating all $(k+1)$-irreducible triangulations of M)**.**
If $n < f_{0,k,\min}$, $\mathbf{T}_n$ is empty.
If $n = f_{0,k,\min}$, $\mathbf{T}_n$ consists only of the minimal k-irreducible triangulations of M.
If $n > f_{0,k,\min}$, $\mathbf{T}_n$ consists of all k-irreducible triangulations of M with n vertices, and of all triangulations generated inductively by performing the following steps on every next-smaller intermediate triangulation $T \in \mathbf{T}_{n-1}$:

- *Pick an arbitrary essential k-cycle γ in T. If T does not contain such a cycle, it is $(k+1)$-irreducible. In that case, add it to the output.*
- *For any split extending γ to an essential $(k+1)$-cycle, do the following steps:*
  - *Perform the split.*
  - *If the resulting triangulation contains an edge that is not contained in an essential cycle of length at most $k+1$, discard it. Otherwise, add it to $\mathbf{T}_n$.*

By inductively generating the sets of intermediate triangulations $\mathbf{T}_n$ up to a certain size $f_0 = n$, all the $(k+1)$-irreducible triangulations of M with at most n vertices are returned as a byproduct. Furthermore, if the algorithm terminates, which is the case if $\mathbf{T}_n$ is empty for an n bigger than the number of vertices in the largest k-irreducible triangulation of M, all $(k+1)$-irreducible triangulations of M have been found. Unfortunately, we don't have a proof that the algorithm is finite for all 2-manifolds M and all k. In theory, the algorithm can also manually be stopped when reaching the upper bound for the number of vertices in a k-irreducible triangulation of M. In practice, even for small k and M, that bound is too high to handle triangulations in any explicit implementation we know of. Fortunately, we did not encounter this problem in our implementations. They always either terminated or had exceedingly large complexity before even reaching the vertex numbers of the biggest $(k+1)$-irreducible triangulations known to us.

5.1.3. A note on isomorphic triangulations

As mentioned above, even when contracting a single $(k+1)$-irreducible triangulation T to a single k-irreducible triangulation T_0, there are multiple contraction paths that yield the same result. Moreover, for a fixed T, in general, T_0 is not unique. There are multiple k-irreducible triangulations that T can be reduced to. In the inverse process of Algorithm 2, every one of them is split inductively to yield T. Thus, there is a huge number of splitting paths, intersecting in a smaller, but still huge, number of triangulations that appear multiple times in the algorithm. Thus, there is a big number of isomorphic intermediate triangulations created in Algorithm 2 (as well as in Algorithm 1). Not only is there nothing to gain from processing isomorphic triangulations multiple times. Due to the arbitrary choice of the essential k-cycle γ that is to be extended, isomorphic copies in different representations might produce entirely new paths with entirely different intermediate triangulations, without adding new $(k+1)$-irreducible triangulations to the final output. Thus, any productive implementation of the algorithms needs to find a way to deal with isomorphic triangulations efficiently.

In our implementation, every triangulation is transformed into a pseudo-standardized form, and encoded in a sequence of characters. That way, most isomorphic triangulations have the same encoding, and are easily comparable. To standardize a triangulation, its vertices are sorted by their degree and by the sum of the degree of their neighbours. Within these restrictions, they are sorted in a way that aims at a lexicographically maximal encoding of the triangulation. For triangulations of high symmetry and, in particular, regular triangulations with all vertices having the same number of neighbours, actually determining the maximum here would require $O(f_0!)$ time, which has to be avoided. Thus, our implementations use a heuristic that we found to work well for most triangulations. At the very end, the output – the complete list of $(k+1)$-irreducible triangulations presented in the next section – was checked separately and the few remaining duplicates were removed by hand. Thus, these lists do not contain isomorphic duplicates.

5.2. Complete classification of small k-irreducible triangulations

The following tables show the exact numbers of k-irreducible triangulations determined by our programs, with Sulanke's enumeration of small-genus irreducible triangulations being added to complete the overview.

f_0	M_1	M_2	N_1	N_2	N_3	N_4
6			1			
7	1		1			
8	4			6		
9	15			19	133	37
10	1	865		2	2521	10347
11		26276		2	4638	370170
12		117047			1320	1891557
13		159205			946	2067817
14		54527			93	956967
15		38195			50	700733
16		664			7	186999
17		5				89036
18						19427
19						3975
20						832
21						79
22						6
$\sum$	21	396784	2	29	9708	6297982

Table 5.1.: 3-irreducible triangulations by size as given by Sulanke [22]

f_0	M_1	N_1	N_2	N_3
11		2		
12	1	1		
13	2	2		
14	16		29	
15	157		380	3
16	947		1290	?
17	60		237	?
18	13		179	?
19			126	?
20	1		62	?
21				?
$\sum$	1197	5	2303	3+?

Table 5.2.: 4-irred. triangulations by size
(for N_1, this confirms [5])

In addition to the known 4-irreducible triangulations of the projective plane, our programs produced the complete list of all 4-irreducible torus and Klein bottle triangulations. In particular, the maximal 4-irreducible torus triangulation has 20 vertices and is

unique. Note that there are no 19-vertex 4-irreducible torus triangulations, a size gap that is unique in any of the few complete series known.
Furthermore, the three smallest 4-irreducible non-orientable 2-manifolds of genus 3 where produced, confirming that there are no such triangulations with less than 15 vertices. One of them is shown in Figure 5.2.

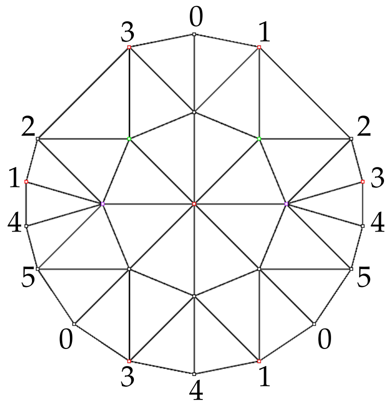

Figure 5.2.: A 4-irreducible N_3 with 15 vertices

f_0	M_1	N_1	N_2
16		1	
17		3	
18		30	
19	1	14	
20	1	8	
21	48+?	7	?
22	558+?		796+?
23	9673+?		?
24	?		?
$\sum$	10281+?	63	796+?

Table 5.3.: 5-irred. triangulations by size

f_0	N_1
24	15
25	307+?
26	2200+?
27	5901+?
28	1785+?
29	616+?
30	229+?
31	102+?
32	?
$\sum$	11155+?

Table 5.4.: 6-irred. N_1 by size

There are 63 different 5-irreducible triangulations of the projective plane, ranging from a unique 16-vertex one to a maximal size of 21 vertices. For the torus, our programs proved that there is a unique minimal 5-irreducible triangulation, with 19 vertices, and a unique 20-vertex 5-irreducible triangulation. In addition, we produced 5-irreducible torus triangulations with up to 23 vertices, albeit with a slight modification of Algorithm 2, that runs quicker, but which we cannot prove to produce a complete list. For the Klein bottle, our programs guarantee that there is no 5-irreducible triangulation with less than 21 vertices. The smallest known such triangulations have 22 vertices, and 796 of them were produced by the aforementioned variation.
There are at least 11155 different 6-irreducible triangulations of the projective plane. Exactly 15 of them are minimal with size $f_0 = 24$.
Note that all extremal triangulations found by the algorithm support the conjectured op-

timality of the series we present in Chapter 6. The complete lists of all triangulations mentioned in Tables 5.2 to 5.4 as well as the specific implementation used to obtain these results can be found on the CD accompanying this thesis, and a description of the structure of the implementation is given in Appendix A.

5.3. Generating random k-irreducible triangulations

While it would be preferable to have a complete enumeration of all k-irreducible triangulations, the limits imposed by complexity are quite strong. An alternative to at least produce some results, and to gain an intuition, is to create some k-irreducible triangulations at random, for a given k and for a specific 2-manifold. We present a simple algorithm aiming at finding such triangulations that are as small as possible. The results of our program are presented in the next section, and used in Section 6.1.1 as a starting point for an inductive generation of relatively small k-irreducible triangulations, for a fixed 2-manifold and arbitrarily large k.
At the core of the main algorithm is an alteration of existing k-irreducible triangulations.

Algorithm 3 (Heuristically shrinking a k-irreducible triangulation T)**.**
Up to a maximum number of iterations, repeat the following steps:

1. *Gather all edges of T that can be flipped without creating an essential $(k-1)$-cycle.*

2. *Choose one of those edges at random. Perform the flip, creating a new triangulation $\tilde{T}$ that is still ew-k, though not necessarily k-irreducible.*

3. *Contract all edges of $\tilde{T}$ that are not contained in an essential k-cycle. If there are such edges, the heuristic was successful. In that case, restart the algorithm with $\tilde{T}$, resetting the counter for the maximum number of iterations (unsuccessful flips).*

To produce a small k-irreducible triangulation T of a given 2-manifold M of genus g, Algorithm 3 is applied three times:

Algorithm 4 (Generating a random, small, k-irreducible triangulation of M)**.**

1. ***Create some triangulation T_3 of M:***
   Let T_B denote a basic building block. If M is orientable, T_B is the Möbius torus. Otherwise, it is the 6-vertex triangulation of the projective plane. Create a triangulation of M by glueing together g copies of T_B, removing a triangle on both sides, then identifying the boundaries. With every added copy of T_B, the genus of the created triangulation is increased by 1. As the basic building blocks are 3-irreducible, every edge of the T_3 is contained in an essential 3-cycle, rendering it 3-irreducible.

2. ***Randomize T_3:***
   Apply Algorithm 3 on T_3, with a low number of maximum iterations, creating $\tilde{T}_3$.

3. ***Iteratively increase the edge width to*** k***:***
   For every $3 < \tilde{k} \leq k$*, do the following steps.*
   - ***Split vertices in*** $\tilde{T}_{\tilde{k}-1}$***, until it is ew-***$\tilde{k}$***:***
     Find an essential $(\tilde{k}-1)$*-cycle, and destroy it by performing an arbitrary split that does so. Repeat this process until the shortest essential cycle has length* $\tilde{k}$*. Denote the resulting triangulation by* $T_{\tilde{k}}$*.*
   - ***Contract edges in*** $T_{\tilde{k}}$ ***until it is*** $\tilde{k}$***-irreducible:***
     Recursively look for edges in $T_{\tilde{k}}$ *that are not contained in an essential* $\tilde{k}$*-cycle, and contract them. Repeat this process until there is no such edge. As an edge contraction only reduces the length of cycles containing the edge, and that only by 1, the resulting triangulation is* $\tilde{k}$*-irreducible.*
   - ***Randomize*** $T_{\tilde{k}}$***:***
     Apply Algorithm 3 on $T_{\tilde{k}}$*, with a low number of maximum iterations, creating* $\tilde{T}_{\tilde{k}}$*.*
4. ***Heuristically try to shrink*** $\tilde{T}_k$***:***
   Finally, apply Algorithm 3 again, with a larger number of maximum iterations, to produce a small k-irreducible triangulation T of M.

It is still not well understood what kind of triangulation produces the best results, if inserted as an intermediate triangulation into the algorithm. We approached this gap by randomizing the intermediate triangulations. In our experiments, the algorithm was run in parallel by 24 processes, each of them restarting the algorithm three times at each of the steps 2)-4), yielding a total number of $648 = 24 \cdot 27$ heuristically generated small triangulations for every approached combination of 2-manifold and k. The results are presented in Table 5.5.

There are numerous ways to further improve this algorithm. E.g., one could look for specific cycles in the splitting step, split vertices there chosen by a specific criterion, perform the split in a specific way, or contract specifically chosen edges in the contraction step, e.g. by defining some weight. We have implemented some of these suggestions, albeit without improving the results. One alternative we did not implement is a simulated annealing approach, modifying Algorithm 3 to allow the edge flips to destroy ew-k, and then either to keep flipping edges or to perform splits to get it back.

5.4. The smallest known up to 8-irreducible triangulations of 2-manifolds with Euler genus up to 20

All triangulations listed in Table 5.5 are smallest known in their respective class, and all but one have been found by our heuristic program. The one exception is the 3-irreducible N_{15}, where the best result of our program had 14 vertices, instead of the optimal 13. In particular, the heuristic has found 36 out of the 37 known minimal k-irreducible triangulations, including the Ringel series and the minimal 4- and 5-irreducible triangulations

Euler genus	orientable	Euler genus	orientable	non-orientable	orientable	non-orientable	orientable	non-orientable	orientable	non-orientable	orientable	non-orientable
1		$\underline{6}^*$		$\underline{11}^*$		$\underline{16}^*$		$\underline{24}^*$		32		41
2	$\underline{7}^*$	$\underline{8}^*$	$\underline{12}^*$	$\underline{14}^*$	$\underline{19}^*$	$\underline{22}$	$\underline{27}$	$\underline{31}$	$\underline{37}$	$\underline{43}$	$\underline{48}$	55
3		9*		$\underline{15}^*$		25		35		49		$\underline{63}$
4	10*	$\underline{9}^*$	$\underline{16}$	16	27	26	39	37	54	52	$\underline{70}$	66
5		9*		18		29		41		58		75
6	10*	10*	19	18	31	31	44	44	63	62	$\underline{80}$	80
7		10*		20		33		48		67		88
8	11*	11*	21	21	36	35	51	50	72	71	95	92
9		11*		22		37		54		77		101
10	12*	12*	22	23	39	39	57	56	83	80	106	104
11		12*		24		41		60		85		113
12	12*	12*	24	25	42	42	62	60	87	89	115	117
13		13*		26		43		67		93		123
14	13*	13*	26	26	45	45	66	68	95	95	127	127
15		13*		27		47		73		102		134
16	14*	14*	28	28	49	49	74	75	104	105	137	139
17		14*		29		50		76		107		142
18	14*	14*	30	30	52	51	79	79	111	111	148	146
19		15*		30		53		80		113		152
20	15*	15*	30	31	52	55	78	86	109	118	145	158
k	3		4		5		6		7		8	

Table 5.5.: Size of smallest known k-irreducible triangulations by topological type

known due to the complete classification. The configurations proven to be minimal have been marked with a star. The underlined configurations are those where the result of the heuristic coincides with the explicit series given in Chapter 6. Note that the 7- and 8-irreducible projective plane, as well as the 8-irreducible Klein bottle found by the program are even smaller than those of our series.

5.5. All up to 8-irreducible triangulations of the 2-sphere with up to 17 vertices

We used Algorithm 1 to generate all triangulations of the 2-sphere with up to 17 vertices. These lists were then checked for the exact $(k-1)$-local disc property, following our redefinition 3.13 of k-irreducibility on the 2-sphere. Table 5.6 shows the results. Surprisingly, the only k-irreducible triangulations found are associated with platonic solids, with the cube being completed to a triangulation by adding a 4-valent central vertex into every square. We believe it is natural that these triangulations should appear, though did not expect them to be the only results. Also, there are no 5- or 7-irreducible 2-spheres with at most 17 vertices.

k	k-irreducible S_2 with $f_0 \leq 17$
3	The tetrahedron
4	The octahedron
5	-
6	The icosahedron, and the cube with a 4-valent vertex added in every square
7	-

Table 5.6.: Up to 8-irreducible 2-spheres with up to 17 vertices

It is easy to check that the last remaining platonic solid, the dodecahedron, yields a 10-irreducible triangulation of the 2-sphere, if one adds a central vertex into every 5-gon.

6. Explicit series for fixed 2-manifolds

6.1. ew-k triangulations with few vertices

We present explicit constructions for infinite series of ew-k triangulations with few vertices, for fixed underlying 2-manifolds. In Section 6.1.1, we introduce a general way of expanding ew-k_0 seed triangulations with specific properties to ew-nk_0 triangulations, for an arbitrary factor $n \in \mathbb{N}$. With this construction, infinite series of ew-k triangulations can be generated for an arbitrary 2-manifold, given that one of its triangulations is known and has the required properties. For Euler genus up to 4, the series with the smallest known growth rates are generated and presented in Table 6.1 at the end of that section.

In sections 6.1.2 to 6.1.4, these series are extended to arbitrary k for the projective plane ($f_0 = \lceil \frac{2}{3}k^2 \rceil$), the torus ($f_0 = \lceil \frac{3}{4}k^2 \rceil$) and the Klein bottle ($f_0 \geq \lfloor \frac{7}{8}k^2 \rfloor$). We conjecture these growth rates to be minimal, and the triangulations themselves to be minimal in the case of the torus.

6.1.1. Construction by natural subdivision

One natural way to construct infinite series of ew-k triangulations for a fixed 2-manifold is to take a k_0-irreducible triangulation and alter it in a way that increases the minimal edge width, while keeping the general structure. E.g., one can try to create ew-k triangulations with few vertices by starting with a k_0-irreducible triangulation that is known to be small. If we follow the intuition that small ew-k triangulations should be close to regular, for 2-manifolds of small genus, we have to look for an alteration that creates a lot of 6-valent vertices. One way to achieve this is *natural subdivision:*

Definition 6.1 (Natural subdivision)**.**
*Let T be a triangulation, and let $n \in \mathbb{N}$ be a positive integer. The n-*fold natural subdivision of *T is created in the following steps:*

- *Insert $(n-1)$ new vertices on every edge of T.*
- *In every triangle of T, connect the new vertices previously inserted on the edge by three families of lines parallel to the original edges of the triangle. Insert a (6-valent) vertex on every intersection point of those lines (see Figure 6.1).*

This construction keeps the valencies of the original vertices, while increasing their distance by adding 6-valent vertices in between. Thus, there are three types of vertices in an

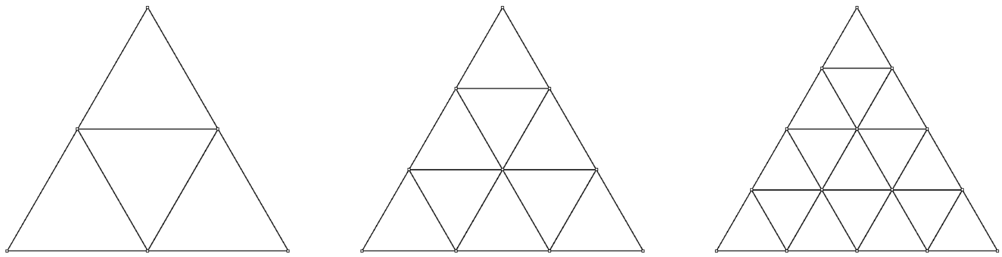

Figure 6.1.: 2-, 3- and 4-fold natural subdivision of a triangle.

n-fold natural subdivision: The $f_{0,0}$ original vertices of T, the $f_{1,0}(n-1)$ vertices inserted on its edges, and the $f_{2,0}\frac{(n-2)(n-1)}{2}$ vertices inserted in its triangles.

$$f_{0,n} = f_{0,0} + f_{1,0}(n-1) + f_{2,0}\frac{(n-2)(n-1)}{2}$$

Applying the edge-to-face ratio of triangulated 2-manifolds (1.2), we get

$$f_{0,n} = f_{0,0} + f_{1,0}\left((n-1) + \frac{2}{3}\frac{(n-2)(n-1)}{2}\right),$$

and with Euler's formula for 2-manifolds (1.3), the number of vertices $f_{0,n}$ in the n-fold natural subdivision is given by

$$f_{0,n} = n^2 f_{0,0} + (n^2-1)(-\chi), \tag{6.1}$$

with $f_{0,0}$ denoting the number of vertices in the original triangulation, and χ denoting the Euler characteristic of the underlying 2-manifold.

Note that performing an n-fold natural subdivision transforms every essential k-cycle of T into an essential (nk)-cycle. Thus, if T has edge width k, its n-fold natural subdivision has edge width at most nk. The goal is for the subdivision to produce ew-$\tilde{k}$ triangulations with few vertices. Thus, the best we can get is $\tilde{k} = nk$, which is not achieved by all basic triangulations T. Figure 6.2 shows that the 2-fold natural subdivision of the 6-vertex triangulation of the projective plane contains an essential 5-cycle, and is thus not ew-6.

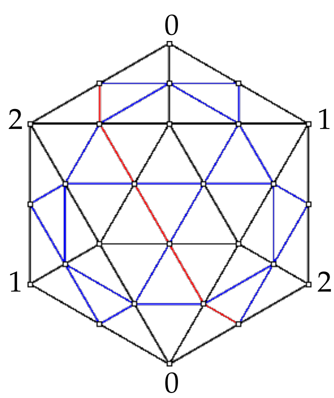

Figure 6.2.: 2-fold natural subdivision of the 6-vertex projective plane

To secure ew-(nk), we need T to not only be ew-k, but also dually ew-$2k$:

Definition 6.2 (Dual cycles, dual edge width, dual ew-k)**.**
Let T be a triangulation of a 2-manifold M, and let $n \geq 1$ be a positive integer. A series $c^* = (\Delta_1, \Delta_2, \ldots, \Delta_n, \Delta_1)$ of n triangles with $\Delta_j, \Delta_{j+1} (j \in \{1, \ldots, n-1\})$ and Δ_n, Δ_1 sharing an edge is called a (closed) dual n-cycle*. For convenience, the curve on M connecting the middle points of every triangle with the middle points of its shared edges is also denoted by c^*.*
A dual cycle c^* of T is called nullhomotopic*, if it is homotopic to a point on M. Otherwise, it is called* essential*, or* non-nullhomotopic*.*
The dual edge width *of a triangulation is the length of its shortest essential dual cycle.*
Let $k \geq 3$ be a natural number. T is called dually ew-k*, if it has dual edge width at least k.*

Lemma 6.3.
Let $k \geq 3$ be an integer, and let T be a triangulation which is both ew-k and dually ew-$(2k)$. Then, any n-fold natural subdivision of T is ew-(nk).

$$\text{ew-}k \text{ and dually ew-}(2k) \Rightarrow \text{naturally } n\text{-subdivided ew-}(nk)$$

Proof. Let T be both ew-k and dually ew-$(2k)$. Thus, every essential cycle of T^n that consists only of original edges of T has length at least nk. We show that there are only two other classes of essential cycles of T^n, and that both of them have the same minimal length. Let γ be an essential cycle of T^n containing edges not part of an original edge of T. Thus, at some point, γ crosses an original edge. There are two possible types of crossing through the two adjacent original faces (see Figure 6.3). In the first case, γ intersects three adjacent edges of one original vertex, w.l.o.g. at distances a, m and b. Then, the length of the crossing is at least $\max(a, m) + \max(b, m)$. The shortest such cycle has crossing length $a + b$. Thus, there is a cycle homotopic to and not longer than γ that avoids this crossing by moving on the bordering original edges.

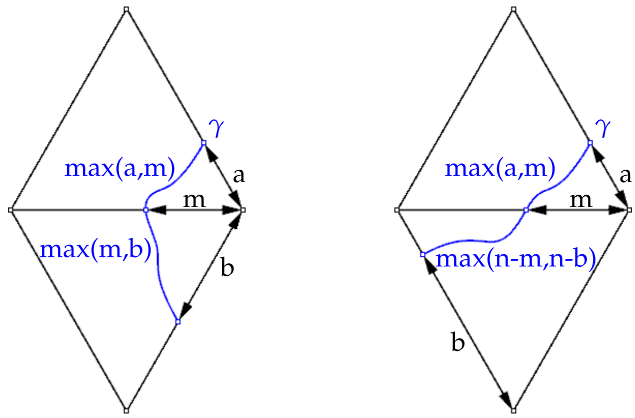

Figure 6.3.: Two types of crossing two adjacent original triangles

Note that, if γ contains an original vertex, the crossings next to it always are of the first type. Thus, γ can inductively be moved completely onto original edges without prolonging it. As these cycles have length at least nk, so does γ.

Assume now that γ does not contain crossings of the first type, instead traversing every pair of neighbouring original triangles diagonally. With the distances a, m and b defined as in Figure 6.3, these crossings have length at least $\max(a,m) + \max(n-m, n-b) = \max(a,m) + n - \min(m,b)$, which is minimal for m between a and b. Thus, the length of the crossing is at most n:

$$|\text{crossing}| \geq \begin{cases} a+n-b & \geq n,\ a \geq b, \\ n & \geq n,\ a < b. \end{cases}$$

Thus, γ consists of pairs of triangle crossings with a total length of at most n each. As it does not contain an original vertex, it corresponds to a dual cycle of T. As γ is essential and T is dually ew-$(2k)$, there are at least $2k$ triangles crossings, and thus, at least k crossing pairs. Thus, γ has length at least nk.
Every essential cycle of T^n is of one of these three types. Thus, T^n is ew-(nk). □

Thus, if a triangulation is both ew-k and dually ew-k, we can use the natural subdivision to create infinite series of ew-$\tilde{k}$ triangulations. As the goal is to create those with few vertices, it is only natural to start with the smallest ew-k triangulations, the k-irreducible ones.

Definition 6.4 (Strong k-irreducibility)**.**
Let T be a triangulation, and let $k \geq 3$ be an integer. T is called strongly k-irreducible, *if T is k-irreducible and dually ew-$(2k)$.*

Note that, for orientable 2-manifolds, ew-k implies dual ew-$2k$. Any essential dual cycle can be pulled to the next edges on either side, replacing each pair of triangle crossings with at most a single edge. Thus, k-irreducibility and strong k-irreducibility are the same in the case of orientable manifolds. As we have seen in Figure 6.2, there is a difference in the non-orientable case.
Also note that, even though strong k-irreducibility implies ew-(nk) in the n-fold natural subdivision, there is no guarantee that every edge there is contained in an essential (nk)-cycle. In total, we get:

Corollary 6.5.
Let T be a strongly k_0-irreducible triangulation of a 2-manifold M with Euler characteristic χ, and let T have $f_{0,0}$ vertices. Then, there is an infinite series of k-irreducible triangulations of M, $k_0 \mid k$ with at most

$$f_0 \leq \Big(\frac{f_{0,0} - \chi}{k_0^2}\Big)k^2 + \chi$$

many vertices.

Proof. This follows immediately from Lemma 6.3, with the number of vertices in an n-fold natural subdivision given in Equation (6.1) and $n = \frac{k}{k_0}$. As there might be contractible edges in the subdivision with respect to ew-k, there is no guarantee for equality to hold. □

Checking small examples of k_0-irreducible manifolds for their dual edge width, we get infinite series of ew-k triangulations, with k being multiples of k_0. The series with the smallest known growth rates for some fixed 2-manifolds are shown in Table 6.1.

top. type	$f_0 \leq$	k_0	$f_{0,0}$
N_1	$\frac{2}{3}k^2 + 1$	3	7
M_1	$\frac{3}{4}k^2$	4	12
N_2	$\frac{7}{8}k^2$	4	14
N_3	$k^2 - 1$	4	15
M_2	$\frac{9}{8}k^2 - 2$	4	16
N_4	$\frac{11}{9}k^2 - 2$	3	9
M_3	$\frac{21}{16}k^2 - 4$	8	80

Table 6.1.: Smallest known growth rates for infinite series of k-irreducible triangulations for fixed 2-manifolds

The first such subdivision-created series consists of $(3n)$-irreducible triangulations of the projective plane with octahedral symmetry. They are constructed from the 3-irreducible triangulation projective plane with 7 vertices, inheriting its symmetry. Figure 6.4 shows the 30-irreducible triangulation of the series, constructed by a 10-fold natural subdivision.

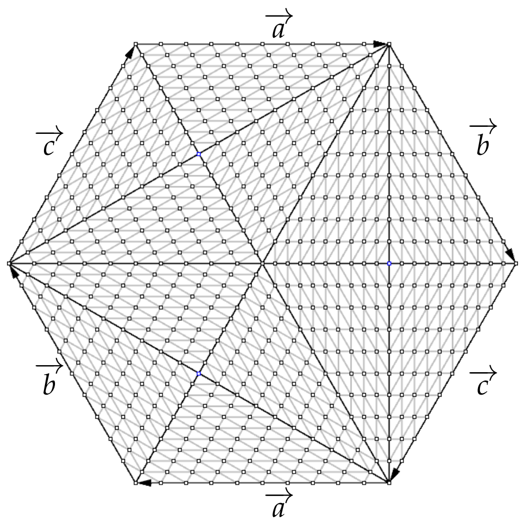

Figure 6.4.: Subdivision-constructed 30-irreducible projective plane

6.1.2. Explicit constructions for the projective plane

We extend the subdivision-created series of $(3n)$-irreducible triangulations of the projective plane in two ways: First, we give slight variations to produce $(3n+1)$- and $(3n-1)$-irreducible such triangulations, albeit losing some symmetry. Second, we use Corollary 3.17 to alter the $(3n)$-irreducible series itself to lose one additional vertex. In total, we prove the following:

Lemma 6.6.
For an arbitrary $k \geq 3$, there are k-irreducible triangulations of the projective plane with

$$f_0 = \left\lceil \frac{2}{3}k^2 \right\rceil$$

vertices.

This series is known to be minimal for $k \in \{3,4,6\}$, and known to not be minimal for $k \in \{5,7,8,9,10\}$. The unique smallest 5-irreducible projective plane is half a dodecahedron, with opposite edges identified and a central 5-valent vertex inserted into every face (see Figure 6.5). For $7 \leq k \leq 10$, the heuristic program given in Section 5.3 produced triangulations beating the above series by one to two vertices. We conjecture $\frac{2}{3}k^2$ to be the correct growth order.

Conjecture 6.7.
For an arbitrary $k \geq 3$, the minimal k-irreducible triangulations of the projective plane have

$$f_0 = \frac{2}{3}k^2 + O(k)$$

vertices.

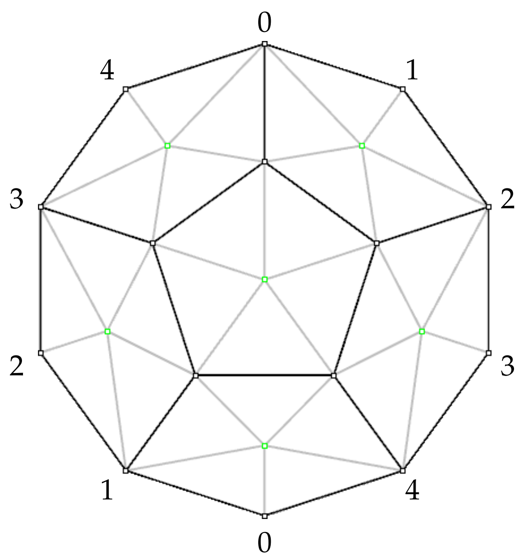

Figure 6.5.: Unique smallest 5-irreducible projective plane ($f_0 = 16$)

Small $(3n+1)$- and $(3n-1)$-irreducible projective planes

Figure 6.6 shows the construction for $k = 29$ (and $k = 31$). The $(3n)$-irreducible projective plane created by subdivision consists of three rotated, triangulated 4-gons of equal edge length n, with a central 4-valent vertex each (see Figure 6.4). Two of those are altered such that two of their outer edges have length $n-1$ (or $n+1$, respectively), and the central 4-valent vertex is replaced by a pair of 5-valent vertices. All other vertices remain of degree 6.

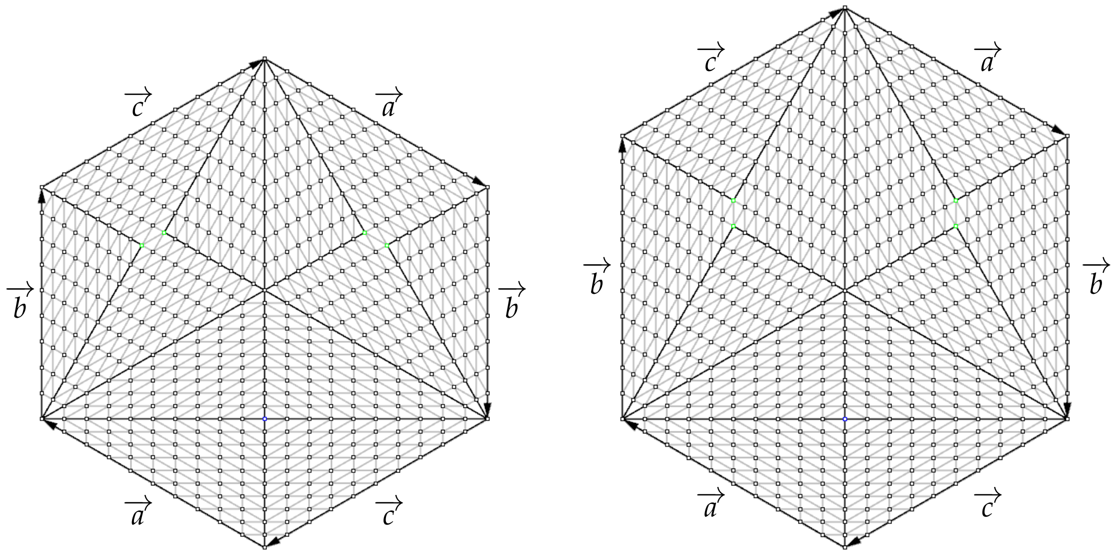

Figure 6.6.: 29- and 31-irreducible projective plane, created by altering the subdivision-created 30-irreducible one

This modification can also be expressed in terms of vertex splits/ edge contractions. In the right image of Figure 6.6, the contraction of the following edges reproduces the 30-irreducible triangulation: The edge between the left pair of 5-valent vertices, and all its vertical neighbours, going to the end of vector $\overrightarrow{b}$, slightly to the upper left. Then all its vertical right neighbours, going to the centre, slightly to the lower right. That last set mirrored on the right side, going from the centre to the right pair of 5-valent vertices. And, lastly, the horizontal neighbours of that pair, going steeply down to the lower right end of vector $\overrightarrow{b}$.

Note that any essential cycle of the triangulation has to cross the corridor formed by this set of edges. Thus, any such cycle is one edge longer than its counterpart in the 30-irreducible triangulation, rendering the bigger one ew-31. It is not hard to see that every edge is also contained in an essential 31-cycle. Thus, we really have 31-irreducibility. A similar series of contractions transforms the 30-irreducible triangulation into the 29-irreducible one.

Loosing one last vertex in the $(3n)$-irreducible projective planes

By Corollary 3.17, for every 2-manifold but the sphere, there is always a minimal k-irreducible triangulation without 4-valent vertices. The proof works by flipping an edge connecting two of the neighbours of a 4-valent vertex. In fact, with the same arguments, two of those edges that are adjacent can be flipped simultaneously. In our case, the 4-valent vertex is surrounded by 6-valent vertices. Thus, flipping these two edges essentially moves the 4-valent vertex.

In the $(3n)$-irreducible projective planes created by subdivision, there are three 4-valent vertices. By three series of double flips (see the left image in Figure 6.7), we move them into the middle of the triangulation. This creates a triangle of three 5-valent vertices with a central 3-valent vertex. As a k-irreducible triangulation other than the tetrahedron does not contain 3-valent vertices, it can be contracted. Note that this procedure does not produce shorter essential cycles at any point. The right image in Figure 6.7 shows the resulting triangulation in the case of $k = 30$.

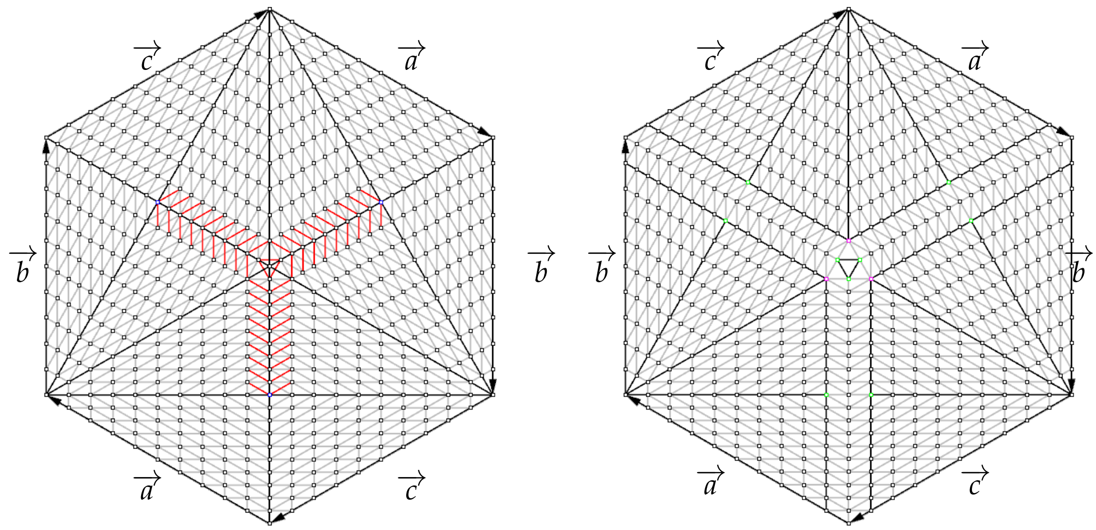

Figure 6.7.: Flips in subdivision-created 30-irreducible projective plane, and resulting triangulation after contracting the central 3-valent vertex

What is the size of the triangulations?

Each of the triangulations consists of 12 subdivided triangles of edge length n, and each of those has $\frac{(n-1)(n-2)}{2}$ internal vertices. Piecing them together accordingly, we get a size of

$$f_0 = 12\frac{(n-1)(n-2)}{2} + \begin{cases} 15(n-1)+4+(3n), & k = 3n \\ 19(n-1)+7+(3n+1), & k = 3n+1 \\ 11(n-1)-3+(3n-1), & k = 3n-1 \end{cases} = \left\lceil \frac{2}{3}k^2 \right\rceil$$

vertices.

6.1.3. Explicit constructions for the torus

We present the smallest known k-irreducible triangulations of the torus, for an arbitrary $k \geq 3$, prove that they are minimal under all regular k-irreducible torus triangulations (at the end of the section), and conjecture them to be minimal without the restriction of regularity.

Lemma 6.8.
For an arbitrary $k \geq 3$, there are regular k-irreducible triangulations of the torus with

$$f_0 = \left\lceil \frac{3}{4}k^2 \right\rceil$$

vertices. Under all such triangulations, the ones given in this section are minimal.

Conjecture 6.9.
The triangulations constructed in this section are minimal under all k-irreducible triangulations of the torus. In particular, the minimal number of vertices in such a triangulation is

$$f_0 = \left\lceil \frac{3}{4}k^2 \right\rceil.$$

Let k be even, and start with a single vertex. Add six neighbours, creating a triangulated 6-gon with a central vertex. Successively add layers around that disc, keeping all vertices 6-valent and not introducing essential cycles. Repeat this process, until the outer vertices have distance $\frac{k}{2}$ from the centre. The result is a regular triangulated 6-gon with each of its six edges being subdivided into $\frac{k}{2}$ parts. By identifying the three pairs of opposite edges such that the result is orientable, we get a triangulation of the torus (see Figure 6.8).

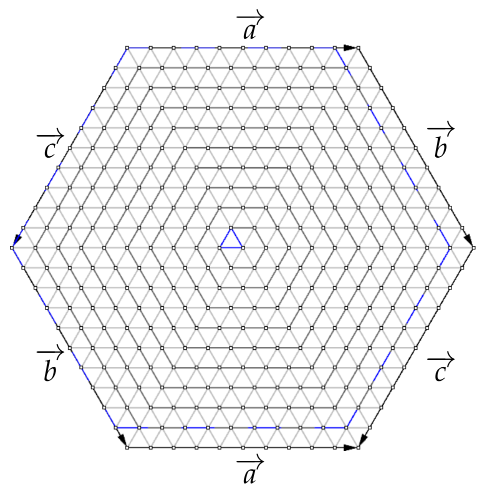

Figure 6.8.: Conjectured minimal 20-irreducible triangulation of the torus

This triangulation is vertex- and edge-transitive, which is easy to see by embedding it into its universal covering. Thus, w.l.o.g., any shortest essential cycle contains the centre of the construction, and thus, has length at least k. Also, every edge is contained in an essential

k-cycle, as the concatenation of any two straight neighbouring boundary components is essential, has length k and can, by symmetry, moved to cover any edge of the triangulation. Thus, this is in fact a k-irreducible triangulation of the torus.

For odd k, the construction is similar, yet centred around a triangle. Start with that and add nine neighbours, rendering the three central vertices 6-valent. Again, successively add layers around that disc, keeping valency 6 and not introducing essential cycles, until the outer vertices have distance $\frac{k}{2}$ from the central triangle. That is, for any outer vertex, the closest of the three central vertices is at distance $\frac{k-1}{2}$. The resulting triangulation again is a triangulated 6-gon, though not a regular one. Half of its edges are subdivided into $\frac{k-1}{2}$ parts, the other half into $\frac{k+1}{2}$. In order to identify boundary components and create a torus triangulation, we glue one extra triangle to the end of one of the long components, prolonging the neighbouring short component to $\frac{k+1}{2}$ as well. Then, again, opposite components are identified (see Figure 6.9). The prolonged straight component is identified with its opposite counterpart, all but the last edge of the next component is identified with its shorter vis-à-vis, and the last pair of opposites has length $\frac{k+1}{2}$ again, and is straight apart from one edge.

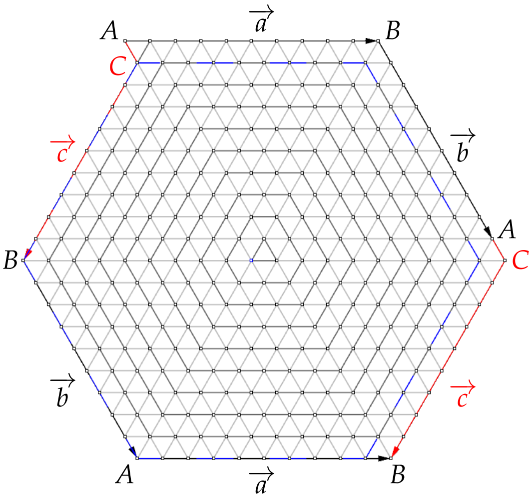

Figure 6.9.: Conjectured minimal 19-irreducible triangulation of the torus

The resulting triangulation is again edge- and vertex-transitive and thus, w.l.o.g., any shortest essential cycle contains one of the three central edges. Thus, the triangulation is ew-k. Having found one essential k-cycle, it immediately follows from the edge-transitivity that every edge is contained in such a cycle. Thus, the triangulation is k-irreducible as well.

What is the size of the triangulations?

Start with k odd, and note that all the vertices of the triangulation are contained in the radius $\frac{k-1}{2}$-neighbourhood of any one of its vertices. In Figure 6.9, that neighbourhood is marked in blue for the lower left vertex of the central triangle. As all vertices are 6-valent, such a neighbourhood consists of $\frac{k-1}{2}$ rings, each of length 6 times its radius, plus the central vertex. Thus, we get a total size of

$$f_0 = \sum_{i=1}^{\frac{k-1}{2}} 6i + 1 = 1 + 3\frac{k-1}{2}\frac{k+1}{2} = \frac{3k^2+1}{4} = \left\lceil \frac{3}{4}k^2 \right\rceil .$$

vertices. For k even, the triangulations consist of a similar vertex-neighbourhood, with radius $\frac{k}{2}$, and with $3\frac{k}{2}+1$ of its outer vertices counted too much due to the identification. Replacing $k-1$ by k in the above formula, we get

$$f_0 = \frac{3(k+1)^2+1}{4} - \frac{3k}{2} - 1 = \frac{3}{4}k^2$$

vertices in the case of an even k.

Proof of Lemma 6.8. We have already shown the existence, regularity and k-irreducibility of such triangulations. It remains to show that they are smallest-possible:
Any k-irreducible triangulation is also $(k-1)$-locally disc. Thus, any local neighbourhood with radius at most $k-1$ is homeomorphic to a disc. As all vertices have degree 6, the size of such a local neighbourhood is fixed for a fixed radius, and the size of a maximal such neighbourhood is a lower bound for the number of vertices in such a k-irreducible triangulation. All vertices of both constructions described above are contained in such a maximal neighbourhood. If k is even, it is centred around a triangle. If k is odd, around a vertex. In Figures 6.8 and 6.9, an example of such a containing maximal neighbourhood and its centre is marked in blue.
Thus, there are no smaller k-irreducible triangulations of the torus with constant valency 6, and the given triangulations achieve the lower bound. □

Our intuition is that these triangulations should also be minimal under all k-irreducible triangulations of the torus. Unfortunately, we lack the means to prove this.

6.1.4. Explicit constructions for the Klein bottle

By glueing together two triangulated Möbius strips, we provide another approach at generating small k-irreducible triangulations of the Klein bottle. As for the projective plane and the torus, the resulting growth rate is the same as in the construction by natural subdivision (Section 6.1.1), but works for any k. In total, we prove the following:

Lemma 6.10.
For an arbitrary $k \geq 3$, there are k-irreducible triangulations of the Klein bottle with

$$f_0 = \begin{cases} \left\lfloor \frac{7}{8}k^2 \right\rfloor , & k \text{ even} \\ \left\lceil \frac{7}{8}k^2 \right\rceil , & k \text{ odd} \end{cases}$$

vertices.

This series is known to be minimal for $k = 3$ and $k = 4$, and coincides with the results of the heuristic programs for $k \leq 7$. The 8-irreducible Klein bottle found by the heuristic programs beats it by one vertex (see Section 5.4). We conjecture the growth rate of the series to be optimal.

Conjecture 6.11.
For an arbitrary $k \geq 3$, the minimal k-irreducible triangulations of the Klein bottle have

$$f_0 = \frac{7}{8}k^2 + O(k)$$

vertices.

Note that the smallest known growth rate of k-irreducible torus triangulations ($\frac{3}{4}k^2$) is distinctly smaller than that of k-irreducible Klein bottles. The results of the heuristic programs support this discrepancy, which seems to be unique under all pairs of orientable/ non-orientable 2-manifolds of the same Euler characteristic.

Triangulated Möbius strips

Start the Möbius strip with an outer k-cycle γ. All but the last two vertices of γ are 6-valent, and have two neighbours on either side of γ. The last two vertices have degree 5. The first of them has two neighbours on the one side of γ, and one on the other, and the second one vice versa. The remaining inner vertices of the Möbius strip are all of degree 6, and every boundary vertex of the Möbius strip has distance $\left\lfloor \frac{k}{4} \right\rfloor$ from γ (see the top left image in Figure 6.10).
Note that the boundary ϕ of this triangulation is geodesic up to distance $2 \cdot \left\lfloor \frac{k}{4} \right\rfloor$: Without crossing γ, the shortest connection of any two boundary vertices can be found on ϕ itself. Also note that the triangulation is ew-k.
Assume first that k is a multiple of 4. Then, glueing two such Möbius strips together along their boundaries ϕ_1 and ϕ_2 yields a triangulation T of a Klein bottle. In Figure 6.10, the identified cycles are marked red. Note that all arguments of this section hold for any arbitrary rotation of the Möbius strips prior to the identification. Let ξ be a shortest essential cycle of T. If ξ is completely contained in one of the Möbius strips, it has length at least k, as those partial triangulations are ew-k. If ξ contains edges of both Möbius strips, but does not cross one of the outermost rings γ_1 or γ_2, it can be homotopically pulled to only one Möbius strip without prolonging it, due to the geodesity property of ϕ_1 and ϕ_2. Thus, the length of ξ again is at least k. Finally, if ξ crosses the outermost ring of both strips, it has length at least $4 \cdot \left\lfloor \frac{k}{4} \right\rfloor = k - (k \mod 4)$. Thus, if k is a multiple of 4, T is ew-k. If not, modifications are needed.
For $k = 4m + 1$ ($m \in \mathbb{N}$), we add a half ring of $3m + 1$ vertices to one of the Möbius strips, pushing ϕ_1 inwards. This is done in a way ensuring that the distance of any vertex of γ_1 to ϕ_1 on that side of γ_1 is at least $m + 1$. Thus, any connection from ϕ_1 crossing γ_1 has length at least $2m + 1$ and the critical shortest essential cycles described above are also

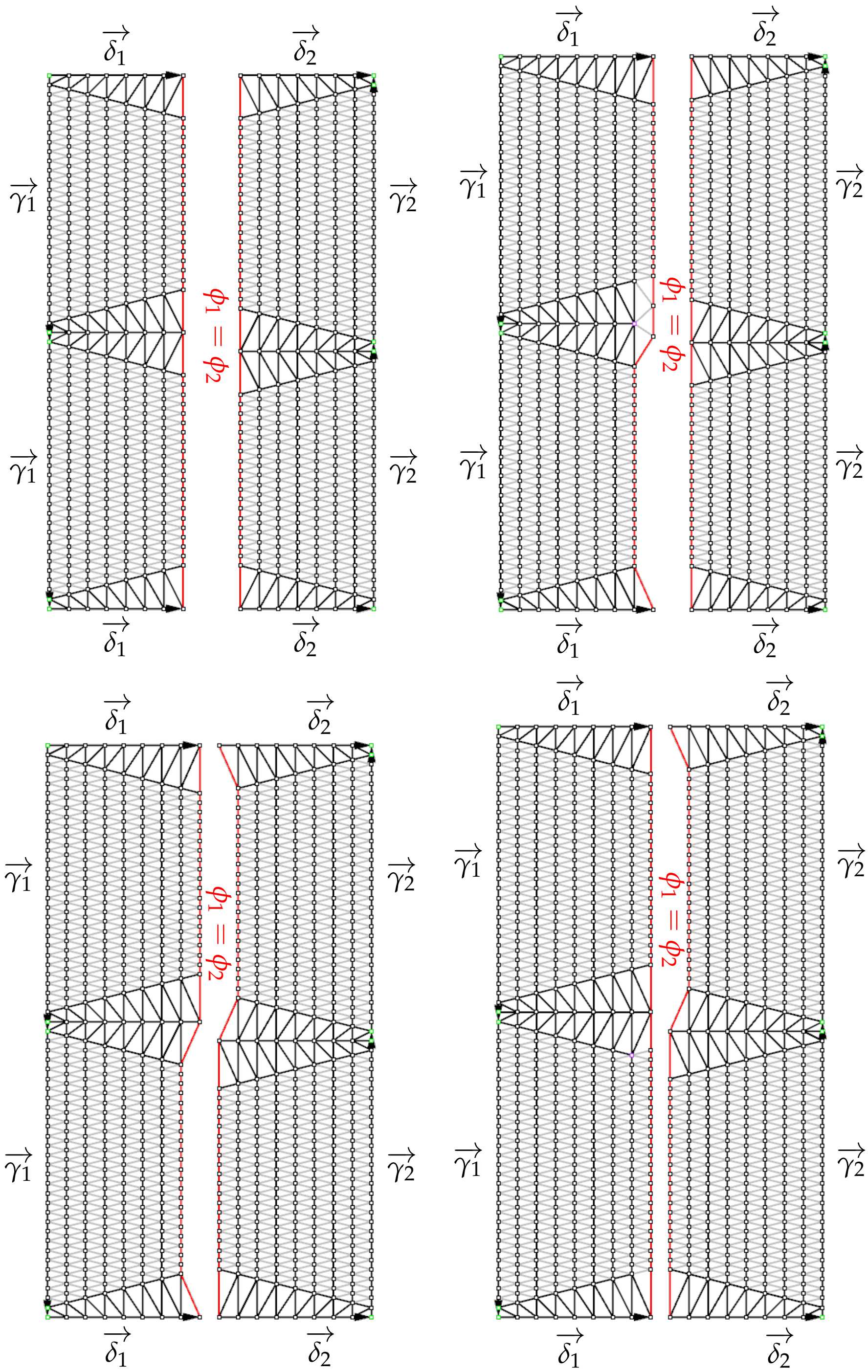

Figure 6.10.: Small 28−, 29−, 30− and 31-irreducible Klein bottles

prolonged by one to a minimum length of k. At the same time, the geodesity property of ϕ_1 is kept, as well as the ew-k of the single Möbius strip. The length of the identified cycles ϕ_1 and ϕ_2 stays the same at $6m + 2$. Similar constructions are carried out for $k = 4m + 2$ and $k = 4m + 3$. As both Möbius strips are changed, the length of the identified cycles ϕ_1 and ϕ_2 can be reduced by one to $6m + 3$ and $6m + 5$, respectively.

Thus, we get ew-k, for all k. With the ring edges covered by the separated Möbius strips, and the edges between rings covered by essential cycles crossing both strips, it is not hard to see that every edge is actually contained in an essential k-cycle. Thus, the triangulations are really k-irreducible.

What is the size of the triangulations?

The basic Möbius strip triangulations consist of k outer vertices, and $\left\lfloor \frac{k}{4} \right\rfloor$ rings of size $2(k-i)$, with i denoting the index of the ring. The basic Klein bottle triangulations consist of two such copies, with their innermost ring identified, but without the modifications depending on $(k \mod 4)$. Thus, they contain

$$\begin{aligned} f_0 = \;& 2\left(k + \sum_{i=1}^{\lfloor \frac{k}{4} \rfloor -1} 2(k-i)\right) + 2\left(k - \left\lfloor \tfrac{k}{4} \right\rfloor\right) = 2\left\lfloor \tfrac{k}{4} \right\rfloor \left(2k - \left\lfloor \tfrac{k}{4} \right\rfloor\right) \\ = \;& \begin{cases} 14m^2, & k = 4m, \\ 14m^2 + 4m, & k = 4m+1, \\ 14m^2 + 8m, & k = 4m+2, \\ 14m^2 + 12m, & k = 4m+3 \end{cases} \end{aligned}$$

vertices. With the additional vertices from the modifications, we get a final size of

$$f_0 = \begin{cases} 14m^2, & k = 4m, \\ 14m^2 + 7m + 1, & k = 4m+1, \\ 14m^2 + 14m + 3, & k = 4m+2, \\ 14m^2 + 21m + 8, & k = 4m+3 \end{cases} = \begin{cases} \left\lfloor \frac{7}{8} k^2 \right\rfloor, & k \text{ even}, \\ \left\lceil \frac{7}{8} k^2 \right\rceil, & k \text{ odd}. \end{cases}$$

These are the asymptotically smallest k-irreducible triangulations of the Klein bottle known at this point. Note that they are far from unique. Even in this simple construction, there are multiple ways to identify ϕ_1 and ϕ_2, and the construction of the separate Möbius strips is not unique, either.

6.2. k-irreducible triangulations with many vertices

We present explicit constructions for infinite series of large k-irreducible triangulations, for arbitrary fixed 2-manifolds. In Sections 6.2.1 and 6.2.2, we construct what we conjecture to be largest k-irreducible triangulations of the projective plane and the torus, for an arbitrary k. In Section 6.2.3, we extend these ideas to create infinite series of large k-irreducible triangulations of the same 2-manifolds, but with a boundary component of length k. These are then used in a glueing construction to yield large k-irreducible triangulations of an arbitrary 2-manifold without boundary, similar to the construction of the largest known irreducible triangulations given in [22]. We believe that our glueing construction is the right way to create maximal k-irreducible triangulations. In total, we prove the following.

Lemma 6.12.
Let $k > 3$ be an integer, and let M be a 2-manifold other than the sphere. Then, there is a k-irreducible triangulation of M with

$$f_0 \geq \begin{cases} k^2 - k + 1, & M = N_1, \\ 2k^2 - 4k + 4, & M = M_1, \\ \frac{9}{8}gk^2 - g, & M = M_g,\ k = 4m, \\ \left\lfloor \frac{9}{8}gk^2 - \frac{5}{8}g \right\rfloor, & M = M_g,\ k = 4m+1, \\ \frac{9}{8}gk^2 - \frac{12}{8}g, & M = M_g,\ k = 4m+2, \\ \left\lfloor \frac{9}{8}gk^2 - \frac{13}{8}g \right\rfloor, & M = M_g,\ k = 4m+3, \\ \frac{3}{4}hk^2 - \frac{hk}{2}, & M = N_h,\ k \text{ even}, \\ \left\lfloor \frac{3}{4}hk^2 - \frac{hk}{2} + \frac{h}{4} \right\rfloor, & M = N_h,\ k \text{ odd}, \end{cases}$$

vertices.

For the projective plane and the torus, we conjecture this to be best-possible.

Conjecture 6.13.
Let $k \geq 3$ be an integer. Then, a largest k-irreducible triangulation of the projective plane has $f_0 = k^2 - k + 1$, and a largest k-irreducible triangulation of the torus has $f_0 = 2k^2 - 4k + 4$ vertices.

In order to yield a similar conjecture for arbitrary 2-manifolds, more research needs to be done on the properties required in the basic building blocks, and potentially on k-irreducible triangulations with boundary in general.

6.2.1. Explicit constructions for the projective plane

In order to develop specific large series of k-irreducible triangulations of the projective plane, we present an inductive way of increasing any k-irreducible projective plane to $(k+1)$-irreducible, while adding as many vertices as possible.

Lemma 6.14.
Let $k \geq 3$ be an integer, and let T be a k-irreducible triangulation of the projective plane with f_0 vertices. Then, there is a $(k+1)$-irreducible triangulation $\tilde{T}$ with $f_0 + 2k$ vertices.

Proof. Let T be k-irreducible, and let γ_0 be an essential k-cycle of T. Thicken γ_0 by replacing each of its edges with a quadrilateral containing a single central vertex. Note that this adds two vertices for every one of the k vertices of γ_0. In Figure 6.11, T is cut open along another essential k-cycle γ, then the quadrilaterals are added, dividing γ_0 into two parts, γ_0' and γ_0''. Note that any other essential cycle of T crosses γ_0 after the thickening. Thus, any essential k-cycle is prolonged to an essential $(k+1)$-cycle. As the series of added quadrilaterals adds no new shorter essential cycles, $\tilde{T}$ is ew-$(k+1)$.

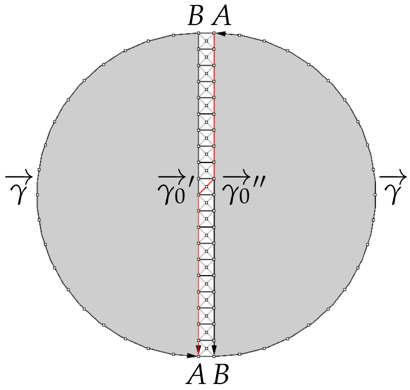

Figure 6.11.: Transforming a 20-irreducible projective plane into a 21-irreducible one

Also, every original edge of T was contained in an essential k-cycle. As that cycle is prolonged by one edge, every original edge in $\tilde{T}$ is contained in an essential $(k+1)$-cycle. The newly added edges of the quadrilaterals are covered by variations of γ_0. In Figure 6.11, one of these variations is highlighted in red. □

Thus, we can construct large k-irreducible triangulations of the projective plane for an arbitrary k by starting with a large k_0-irreducible one (with $k_0 < k$) and repeatedly adding quadrilaterals along a shortest essential cycle.

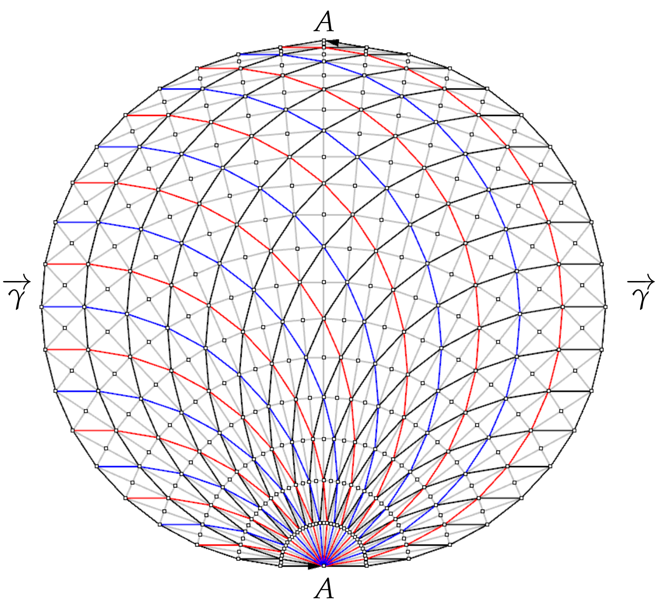

Figure 6.12.: 20-irreducible triangulation of the projective plane with 381 vertices

Figure 6.12 shows a biggest known 20-irreducible triangulation of the projective plane. Note that the quadrilaterals have always been added to the essential k-cycle on the outside of the image, thus, γ_0 and γ coincide. Thus, the image also contains biggest known k-irreducible projective planes for all k up to 20. They can be obtained by cutting along one of the blue or red lines, keeping the part on the left and reidentifying opposite vertices of the new boundary.

What is the size of the triangulations?

The largest known such examples all stem from the 3-irreducible 7-vertex triangulation of the projective plane. To get to k-irreducible, $2i$ vertices are added, for every intermediate stage i of i-irreducibility. Thus, the total number of vertices in a large k-irreducible triangulation of the projective plane is

$$f_0 = 7 + \sum_{i=3}^{k-1} 2i = 1 + \sum_{i=1}^{k-1} = k(k-1) + 1.$$

6.2.2. Explicit constructions for the torus

We present the largest known k-irreducible triangulations of the torus, for an arbitrary k, and conjecture them to be maximal.

The construction starts with a square of $(k-1) \times (k-1)$ quadrilaterals, with a central vertex each. One additional triangle is added at every corner of the square, as depicted in Figure 6.13. Then, opposite sides are identified, starting and ending with one edge of an extra triangle on all sides.

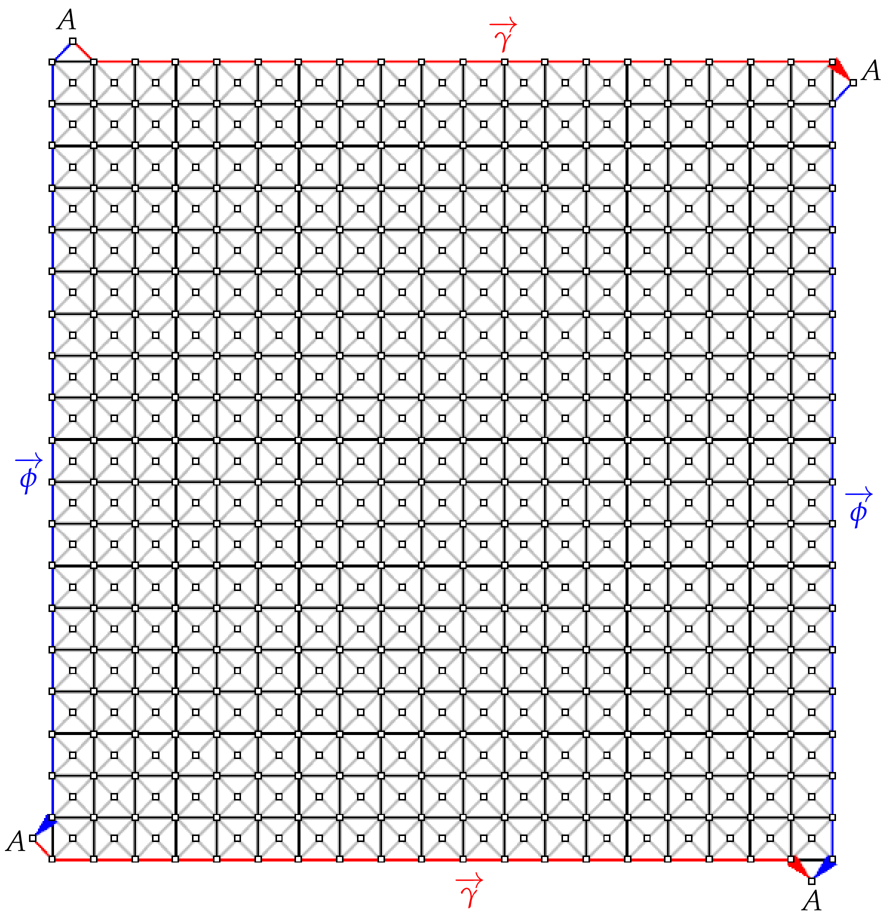

Figure 6.13.: 20-irreducible triangulation of the torus with 724 vertices

The resulting triangulation of the torus contains two types of shortest essential cycles. Taking orientation and notation from Figure 6.13, the first one is homotopic to $\overrightarrow{\gamma}$. It consists either of $k-2$ horizontal and two diagonal edges, or of $k-1$ horizontal and one vertical edge. The second class of shortest essential cycles is homotopic to $\overrightarrow{\phi}$, and consists either of $k-2$ vertical and two diagonal edges, or of $k-1$ vertical and one horizontal edge. Either way, their length is k, rendering the triangulation ew-k. By homotopic, parallel shifts of $\overrightarrow{\gamma}$ and $\overrightarrow{\phi}$, every edge of the triangulation is covered, and thus contained in an essential k-cycle. Hence, the triangulation is also k-irreducible.

What is the size of the triangulations?

As the four extra triangles form one additional quadrilateral, the triangulation contains a total number of $(k-1)^2+1$ quadrilaterals. Every such quadrilateral consists of a central vertex and four boundary vertices, with every one of the boundary vertices being contained in four quadrilaterals. Thus, in total, there are

$$f_0 = 2\left((k-1)^2+1\right) = 2k^2-4k+4$$

vertices, half of them with valency 4 and half of them with valency 8.

6.2.3. All other 2-manifolds

The largest known 3-irreducible triangulations for any fixed orientable 2-manifold are created by cutting a triangle from a large 3-irreducible triangulation of the torus, and then glueing copies of that triangulation together until the desired genus is reached. The same construction works in the non-orientable case with copies of large 3-irreducible projective planes (see [22]). Note that the removal of a single triangle always preserves the 3-irreducibility.

We present a similar approach for k-irreducible triangulations, with k greater than 3. As the cycle cut out of the basic building block is itself essential in the final glued triangulation, the smallest size of such a cycle is k. Note that there is no guarantee for a triangulation to still be k-irreducible after the removal of a k-gon, as there might be edges no longer contained in an essential k-cycle.

In a first step, we present large basic building blocks, both for non-orientable and orientable manifolds, building on the ideas of Sections 6.2.1 and 6.2.2. That is, for an arbitrary integer k greater than 3, we present k-irreducible triangulations of the projective plane and the torus with a boundary component of length k. We then introduce a glueing template, that is, a polygonal map of the 2-sphere that contains g k-gons. These k-gons are replaced with g copies of our basic building blocks, glueing the boundary of the building block to the boundary of the k-gon. In total, this yields a large k-irreducible triangulation of a – orientable or non-orientable – 2-manifold with genus g.

Non-orientable basic building blocks

Assume first that k is even. Take one of the large k-irreducible projective planes introduced in Section 6.2.1, and cut out a $\frac{k}{2}$-irreducible triangulation of the same series that is contained in its interior. Due to the inductive construction of these triangulations, the result is automatically k-irreducible, with a single boundary component of length k (see the left image in Figure 6.14).

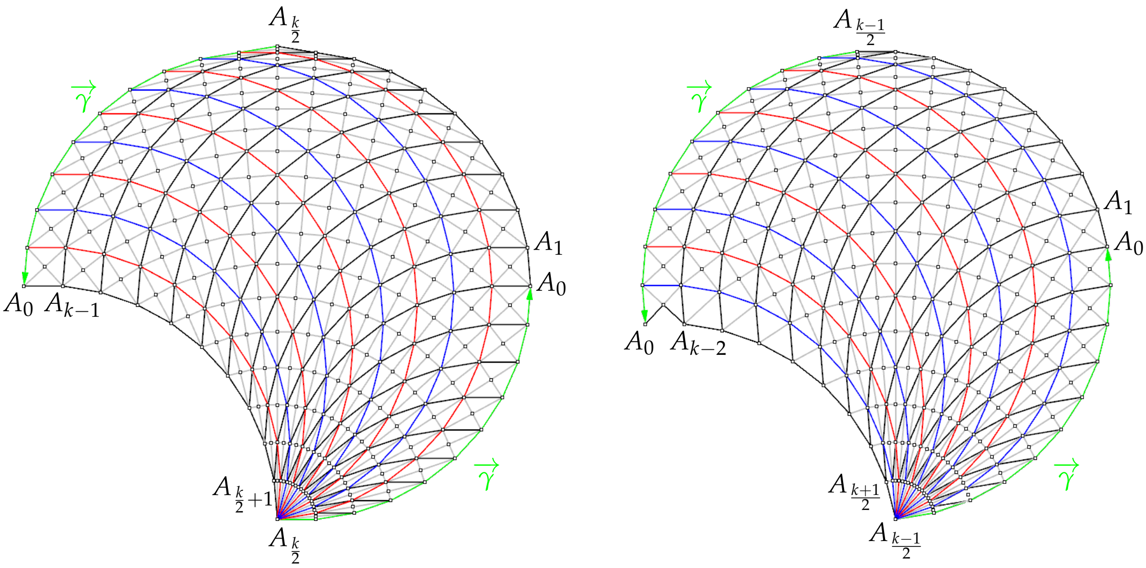

Figure 6.14.: 20- and 19–irreducible projective plane with boundary of length 20 and 19

For k odd, instead cut a $\frac{k-1}{2}$-irreducible triangulation out of a k-irreducible one. While, again, every edge is contained in an essential k-cycle, the resulting boundary component only has length $k-1$. Thus, we prolong it by removing one of its boundary edges, which results in $\frac{k-3}{2}$ edges connecting the boundary with a 4-valent vertex no longer being contained in an essential k-cycle. Contracting them yields the desired k-irreducible projective plane with a boundary component of length k. Note that the same result can be obtained by removing a $\frac{k+1}{2}$-irreducible triangulation in the beginning, and then glueing additional vertices onto the boundary to reduce its length by 1. The result of either one of these methods can be seen in the right image of Figure 6.14. Note that the contracted edges leave their respective quadrilaterals without a central vertex.

What is the size of the non-orientable building blocks?

The size of the big k-irreducible triangulations of the projective plane without boundary is $f_0 = k(k-1)+1$. Of that, we subtract the size of the $\left\lfloor \frac{k}{2} \right\rfloor$-irreducible triangulation that was cut out. The outer vertices of that smaller triangulation form the new boundary component, and thus, should not have been subtracted. As opposite sides were identified in the smaller triangulation, this yields an additional $\left\lfloor \frac{k}{2} \right\rfloor$ vertices. Lastly, for k odd, $\frac{k-3}{2}$

edge contractions were performed. Thus, the total number of vertices is

$$f_0 = \begin{cases} \Big(2n(2n-1)+1\Big) - \Big(n(n-1)+1\Big) + n, & k = 2n, \\ \Big((2n+1)2n+1\Big) - \Big(n(n-1)+1\Big) + n - (n-1), & k = 2n+1, \end{cases} = \left\lceil \frac{3}{4}k^2 \right\rceil.$$

Orientable basic building blocks

In the orientable case, while it is possible to again cut out a k-gon from our big series of k-irreducible tori, this involves a big number of edge contractions afterwards. Instead, we present a direct construction of a k-irreducible torus triangulation with one boundary component of length k, which has the same number of vertices as the complicated cut-and-contraction construction.

For k even, take a rectangle of $\left\lceil \frac{3k}{4} \right\rceil \times \left\lfloor \frac{3k}{4} \right\rfloor$ squares with one central vertex each. Identify the last $\frac{k}{2}$ upper horizontal with the first $\frac{k}{2}$ lower horizontal edges, and the first $\frac{k}{2}$ left vertical edges with the last $\frac{k}{2}$ right vertical ones, creating a triangulation of the torus with a single boundary component of length k (see Figure 6.15). That boundary component is rectangular, with one of its sides starting at each of the corners of the original rectangle. Other than the boundary component, there are two classes of shortest essential cycles. One consists of $\left\lceil \frac{3k}{4} \right\rceil$ side steps and $\left\lfloor \frac{k}{4} \right\rfloor$ vertical steps, with diagonal edges counting $\frac{1}{2}$ in both directions. The other one consists of $\left\lfloor \frac{3k}{4} \right\rfloor$ vertical and $\left\lceil \frac{k}{4} \right\rceil$ horizontal steps. As any edge of the triangulation is covered by these two classes, we get k-irreducibility.

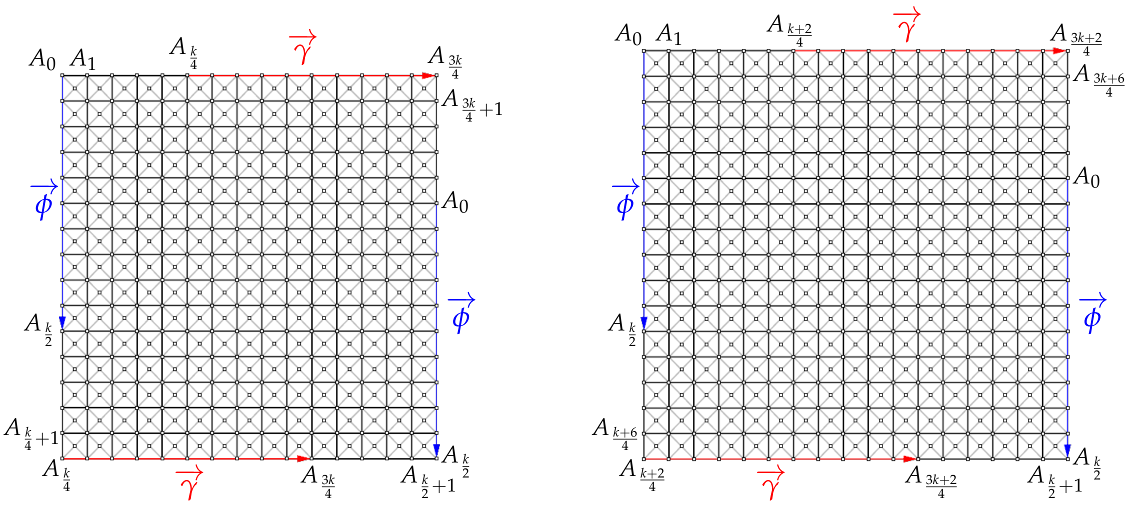

Figure 6.15.: 20− and 22-irreducible torus with boundary of according length

For odd k, the construction is similar. In the case of $k = 4m+1 (m \in \mathbb{N})$, the triangulation initially consists of $(3m+1) \times (3m+1)$ squares with a central vertex. For $k = 4m+3$, there are $(3m+3) \times (3m+2)$ squares instead. In both cases, the identified parts have length $\frac{k+1}{2}$, leaving a single boundary component of length $k-1$. To prolong it, the horizontal edge in the top left corner of the rectangle is removed. As a consequence, the shortest essential cycle containing any but the first downward diagonal in the first row has length $k + 1$. Thus, m (or $m + 2$, if $k = 4m + 3$) edges have to be contracted. The result again is a k-irreducible triangulation of the torus with a single boundary component of length k (see Figure 6.16).

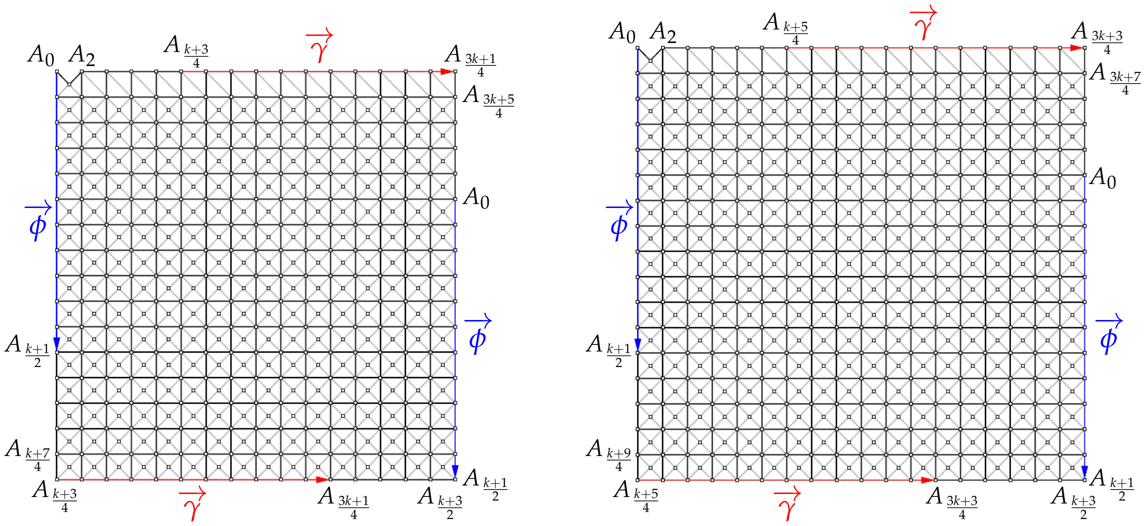

Figure 6.16.: 21− and 23-irreducible torus with boundary of according length

What is the size of the orientable building blocks?

For every square with a central vertex contained in the triangulations, there are two vertices: The central one and, w.l.o.g., the one in the bottom left corner. This count misses only the boundary vertices on the top (left and right) and, in the case of an odd k, the last row of vertices in the top which are the bottom left corner of a square without a central vertex. Thus, we get a total number of

$$f_0 = \begin{cases} 2(3m)^2 + (2m-1) = & \frac{9}{8}k^2 + \frac{k}{2} - 1,\ k = 4m, \\ 2(3m+1)3m + (2m) + (3m+1) = & \frac{9}{8}k^2 + \frac{k}{2} - \frac{5}{8},\ k = 4m+1, \\ 2(3m+2)(3m+1) + (2m) = & \frac{9}{8}k^2 + \frac{k}{2} - \frac{12}{8},\ k = 4m+2, \\ 2(3m+3)(3m+1) + (2m+1) + (3m+3) = & \frac{9}{8}k^2 + \frac{k}{2} - \frac{13}{8},\ k = 4m+3, \end{cases}$$

vertices.

How does the glueing work?

The general idea is to glue the basic building blocks in a way that keeps them as far apart as possible, and thus losing as few vertices as possible to the identification. For a 2-manifold of genus 2, we just glue together two of the basic building blocks, along their boundary. For any bigger genus, we use the following routine.

Figure 6.17 shows four polygonal maps of the 2-sphere, with the last polygon being the backside. The first two maps represent the case of an odd k, with the red lines consisting of $\frac{k-3}{2}$ edges. In the other two maps, k is even, and the length of the red lines is $\frac{k-4}{2}$. Thus, the first and the third map contain four k-gons, while the second and the last map only contain three of them. Note that the shortest simple closed cycle that is not a triangle has length k. Also note that every vertex, with a single exception on the bottom of the second map, is contained in exactly two k-gons. These properties are kept if we remove one of the k-gons of the first or third map, and glue two copies of it together, yielding a polygonal map with six k-gons. Repeating this, we can, for an arbitrary k, produce such a map with an arbitrary even number g of k-gons. To get an odd number g of k-gons, we stop that process at $g-1$ and glue one instance of the second map, if k is odd, or of the last map, if k is even.

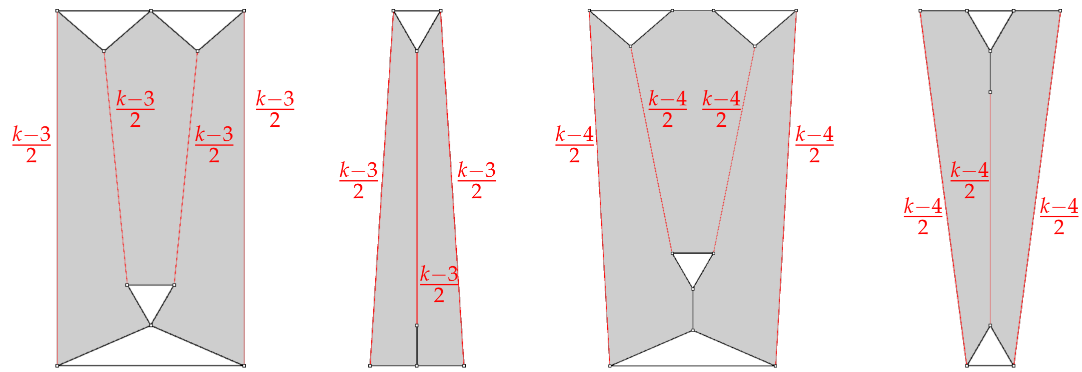

Figure 6.17.: Glueing templates

Replacing every k-gon with a copy of the non-orientable basic building blocks given above, we get a triangulation of N_g. If instead we choose the orientable blocks, we get a triangulation of M_g.

Let γ be a shortest essential cycle in one of these triangulations. If γ is completely contained in a basic building block, it has length k, as the single blocks are k-irreducible. If γ does not intersect any of the basic building blocks, it has length k as well, as any closed cycle in the glueing templates that is not a triangle has length at least k. The remaining case is that γ intersects a basic building block, but is not completely contained in it. Then, the part of γ inside the block can be homotopically moved to the boundary of that block, without prolonging γ. Otherwise, there would be a cycle inside that block, homotopic to and shorter than its boundary, a contradiction to the k-irreducibility of the constructed basic building blocks.

Also, every edge of the final triangulations is part of one of the basic building blocks. As they are k-irreducible, that edge is contained in an essential k-cycle. Thus, the created triangulations are k-irreducible themselves.

What is the size of the final triangulations?

The final triangulation consists of g copies of the basic building block, with every boundary vertex of those being counted twice, a single exception occuring if both g and k are odd, where there is one vertex counted thrice. Thus, we get a final number of

$$f_0 = g \cdot f_{0,\text{Block}} - \left\lceil \frac{gk}{2} \right\rceil$$

vertices.

In the orientable case, this means that the biggest known k-irreducible triangulations of M_g, with an arbitrary genus g and an arbitrary integer k greater than 3, have a total size of

$$f_0 = \begin{cases} g\left(\frac{9}{8}k^2 - 1\right), & k = 4m, \\ \left\lfloor g\left(\frac{9}{8}k^2 - \frac{5}{8}\right)\right\rfloor, & k = 4m+1, \\ g\left(\frac{9}{8}k^2 - \frac{12}{8}\right), & k = 4m+2, \\ \left\lfloor g\left(\frac{9}{8}k^2 - \frac{13}{8}\right)\right\rfloor, & k = 4m+3. \end{cases}$$

The biggest known k-irreducible triangulations of the non-orientable N_h, with an arbitrary genus h and an arbitrary integer k greater than 3, consist of

$$f_0 = \begin{cases} h\left(\frac{3}{4}k^2 - \frac{k}{2}\right), & k \text{ even}, \\ \left\lfloor h\left(\frac{3}{4}k^2 - \frac{k}{2} + \frac{1}{4}\right)\right\rfloor, & k \text{ odd}, \end{cases}$$

vertices.

In particular, the maximal known non-orientable k-irreducible triangulations grow quicker than their orientable counterparts with respect to Euler genus. Thus, when looking to create big triangulations, it is not feasible to mix orientable and non-orientable basic building blocks.

Note that we give no specification on how exactly to glue the k-cycle of a specific basic building block into the k-cycle that is the boundary of a k-gon in the glueing template. Thus, this construction yields a huge number of large k-irreducible triangulations. This might also open up room for further improvements. While the k-irreducibility in the basic building blocks guarantees the k-irreducibility of the whole triangulation, maybe a weaker condition on these blocks would be sufficient, if one were to choose a particular way of glueing.

7. Final remarks

This thesis lays the groundwork for future research on k-irreducible triangulations. In the following, we present some interesting questions that should be further pursued.

What is the order of growth of minimal 5-irreducible triangulations of arbitrary 2-manifolds? The smallest known explicit examples stem from our series in Section 4.3, and have a size of $f_0 \in O(g^{\frac{8}{9}})$. Note that this is the first series of greater-than-4-irreducible triangulations with a polynomial growth rate and an exponent less than 1. Still, there is a gap to the exponent in the best lower bound for that number ($f_0 \notin o(g^{\frac{2}{3}})$, Theorem 3.20). We believe the actual minimum to grow polynomially in the Euler genus, but closer to the bound than to our series. **Can one construct examples of 5-irreducible triangulations smaller than $\mathbf{O(g^{\frac{8}{9}})}$?** We believe this to be possible, but can one really get down to $O(g^{\frac{2}{3}})$? Similar questions arise for greater k, beginning with an extension of our lower bound to non-vertex-regular triangulations: **Does the lower bound of $\mathbf{f_0 \notin o(g^{\frac{k-1}{k+1}})}$ for odd, and $\mathbf{f_0 \notin o(g^{\frac{k-2}{k}})}$ for even k, hold for arbitrary k-irreducible 2-manifolds?** And, of course, one should approach the minimum from the other side. In particular, it would be very interesting to **find an infinite series of k-irreducible triangulations of arbitrary 2-manifolds, for arbitrarily large k, that grows polynomially in g, with an exponent less than 1.** Our intuition is that small k-irreducible triangulations should be close to vertex-regular. Thus, it is very natural to approach this group theoretically, e.g. with the construction principle we gave in Chapter 4.

A related problem is the question of small k-irreducible triangulations of fixed 2-manifolds. **What are the minimal k-irreducible triangulations of small 2-manifolds, in particular the projective plane, the torus and the Klein bottle?** This is an ambitious question. Even the correct growth rate would be very interesting. In Section 6.1, we presented infinite series with growth rates $\frac{2}{3}k^2$, $\frac{3}{4}k^2$ and $\frac{7}{8}k^2$, respectively, and conjectured them to be minimal. In the case of the torus, we proved that the triangulations themselves are minimal under all vertex-regular k-irreducible torus triangulations, and strongly believe them to be minimal without this restriction. For the projective plane and the Klein bottle, there are minimal triangulations smaller than the series (see Table 5.5). Maybe these examples can be used to **construct an explicit series of minimal k-irreducible projective planes/ Klein bottles?** Also, the heuristic results seem to suggest an anomaly for the growth rate of the Klein bottle: **What is the difference in growth rates of minimal orientable/ non-orientable k-irreducible triangulations? Is it true that the Klein bottle grows quicker than the torus, and is it the only non-orientable 2-manifold to outgrow its orientable counterpart?**

Similar questions can be asked for large k-irreducible triangulations. A bold formulation: **What is the largest k-irreducible triangulation of an arbitrary 2-manifold?** We conjecture the series constructed in Section 6.2 to be maximal for the torus ($f_0 = 2k^2 - 4k + 4$) and the projective plane ($f_0 = k^2 - k + 1$). **Can one improve upon our maximal growth rate of k-irreducible arbitrary 2-manifolds ($\frac{9}{8}gk^2$ in the orientable, $h\frac{3k^2-2k}{4}$ in the non-orientable case)?** Given that the best upper bound for the size of such a triangulation is super-exponential in k and quadratic in g, a more realistic first step would be to **provide better upper bounds for the maximal size of a k-irreducible triangulation**, preferably linear in g and quadratic in k.

The algorithmic approaches might also be pushed further. We believe that careful improvements in our algorithms and implementations should be able to **create a complete classification of all 6-irreducible projective planes, all 5-irreducible tori and the smallest 5-irreducible Klein bottles.** This could be used to gain new insights into these objects, and possibly create new series. It would also be interesting to **investigate the given full classifications in detail,** e.g. by determining which k-irreducible triangulation is reducible to which (k-1)-irreducible one, and vice versa. An interesting question concerning randomized algorithms might be to **heuristically determine large k-irreducible triangulations of fixed 2-manifolds.** Our intuition is that such triangulations consist of large basic building blocks, though. If that is true, a local heuristic algorithm similar to the one we gave in Section 5.3 might be hard-pressed to find a global maximum.

We also gave a new definition of k-irreducible 2-spheres, naturally raising the questions: **What properties of arbitrary k-irreducible triangulations are kept on the sphere? What are minimal/ maximal k-irreducible 2-spheres? How can one construct explicit series?** All triangulations associated with the platonic solids are k-irreducible 2-spheres, with the new definition. Also, glueing together two copies of any one of the known 11255 up to 6-irreducible projective planes yields a 2k-irreducible 2-sphere. Is this a general property? It might be worthwhile to invest more research into these objects.

This thesis restricted itself to 2-manifolds without boundary. **What properties of k-irreducible triangulations remain, if we regard 2-manifolds with boundary?** In a way, we already used such objects when glueing together large k-irreducible triangulations in Section 6.2. Or, one could leave the area of 2-manifolds altogether: **Is there a meaningful extension of k-irreducibility to arbitrary surfaces,** in particular those with singularities?

An analogue version of k-irreducibility could also be regarded for more general objects than triangulations. **Is there a good generalization of *k*-irreducibility for polygonal maps?** To achieve similar results, one would probably have to include some restriction on dual edge width. Or, even more boldly: **Is there a good smooth analogon to k-irreducibility, in differential geometry, maybe letting k tend to infinity?** Answering these questions might yield interesting results, both transferring knowledge from triangulations to the general case, and vice versa.

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A. A brief description of the implementation

The implementation of the algorithms described in Chapter 5 can be found in the CD accompanying this thesis. In particular, this includes the following:

- A program using Algorithm 2 to compute all k-irreducible triangulations that can be reduced by edge contractions to an ew-$(k-1)$ triangulation given in an input set. In particular, when given all $(k-1)$-irreducible triangulations of a 2-manifold, this program produces all k-irreducible triangulations of that 2-manifold.
- A program checking a given set of triangulations for ew-k.
- A program using Algorithm 1 to generate all triangulations up to a specific size that can be reduced by edge contractions to one out of a given input set of triangulations. In particular, when given all k-irreducible triangulations of a 2-manifold, this program produces all ew-k triangulations of that manifold up to a given maximum number of vertices.
- A program checking a given set of triangulations for exact $(k-1)$-local disc property. The two last programs were used to produce all k-irreducible triangulations of the 2-sphere with up to 17 vertices and up to $k = 7$. Recall that the first program cannot be applied to the 2-sphere, as it involves essential cycles.
- A program heuristically trying to find small k-irreducible triangulations of a given 2-manifold using Algorithm 4.

A.1. Overall structure of the programs

Every program is stored in a separate folder and can be compiled without referencing the others. All programs have been implemented in C++, and can be run in parallel on an arbitrary number of processes using the MPI-framework, with the interaction of the parallel instances being reduced to the division of the input and the merging of the output. Thus, the main computations of every process run independently from all other processes. If parallel execution is not wanted, e.g. for debugging purposes, it can be switched off with a single comment at the top of the main file of every program. In that case, a single process alternative is compiled instead.
Every program comes with two shell scripts (*create* and *create_sp*) highlighting how to compile both the parallel and the single process version. In addition, a commented minimal running example is given, including two more shell scripts (*startTEST* and *startTEST_sp*) highlighting how to use the compiled program. In general, the compiled executable has to be called inductively for the overall program to reach its goal.

The following C++ files appear:

- A .cpp-file starting with a capital letter, its name describing the overall purpose of the program. This is the main file, initiating the computation. Here, it can be specified if the program should run in parallel or not, and if the input should be hard coded for debugging purposes. Lastly, at the core of most programs, the parameter k is also set here (hard coded for performance).

- singleprocess.cpp and singleprocess.h: If it is specified in the main file that parallel execution is NOT wanted, this is where the execution of the main algorithm takes place – in the function single_process::execute(…). A big part of this is to handle large numbers of triangulations in the output and check them for duplicates. Up to a maximal storage size specified at the very top of the .cpp-file, this is handled in the working memory. This size limit needs to be set according to your platform. If set too high, the program might exceed the working memory, and crash. A number of routines is given to merge sets of triangulations, both in the working memory and, once the limit is exceeded, on the hard drive itself. Note that the latter slows down the program considerably.

- multipleprocesses.cpp and multipleprocesses.h: These two files are used when parallel execution was specified. They serve the same purpose as their single process counterparts, and add routines to split the input to different parallel processes, as well as merging their outputs at the very end. Again, take caution in setting an appropriate maximum size of working memory every process is allowed to use according to your platform.

- triangulation.cpp and triangulation.h: Here, the class *Triangulation* is defined. This class it at the core of every program. Out of the given input, an object of type *Triangulation* is created (by the execute(…)-function of singleprocess.cpp, respectively multipleprocesses.cpp), and the majority of the computations are done in member functions of this class, directly operating on that object. Every program contains these two files, and they are identical at the core, with the more complex functions only appearing in the programs they are needed in.

- multiinput.cpp and multiinput.h: These two files are identical in all programs that use them, which are all but the heuristic. They define a small helper class used to read input from multiple files that contain the same class of triangulations, that is, the same size and 2-manifold. They also provide a simple routine giving the total number of triangulations in these files, used to divide the work load (heuristically) equally between a number of processes working in parallel.

- multioutput.cpp and multioutput.h: Again, these two files are identical in all programs that use them, which are again all but the heuristic. They define a helper class dividing serially given output to multiple files line by line, so as to divide the workload of the next iteration of the program equally between a number of parallel working processes.

All programs but one share this general structure, and thus contain all these files. The one exception is the program heuristically creating small k-irreducible triangulations: As, there, no input and no big output has to be handled, the main execution, parallel or not, has been moved to the main file completely. The only additional files required are triangulation.cpp and triangulation.h, with their purpose remaining as explained above.

Shared basic structure of the *Triangulation* class

An object of the class *Triangulation* always contains an (ordered) vector of vertices. A *Vertex* is a miniature class in itself, containing an ordered vector of its adjacent vertices. There is

no separate storage of edges or triangles. All routines work with the ordered adjacency information of the vertices. Note that reordering the vertices or rotating/reverting the order of the neighbours of some vertices yields a different object describing the same abstract triangulation.
The basic representation of a triangulation, when written to a file, roughly follows the representation given by Sulanke in his complete classification of 3-irreducible triangulations of 2-manifolds up to Euler genus 4 ([21]). That is, the number of vertices is given, followed by a list of ordered adjacencies, with the vertices being labelled alphabetically, and their adjacencies given in that order. This representation is called a *sulankeDescription* in the code. As an example, this is the tetrahedron in a sulankeDescription:

$$4\ bcd, acd, abd, abc$$

For debugging purposes, and to make big triangulations ($f_0 > 26$) humanly readable, there is another method of printing out a triangulation object: *Triangulation::ToString()*. It also prints the adjacency lists, but without replacing the indices with letters, and introducing a lot more space to make it readable. Note that this method also needs additional separators between adjacent neighbours, to distinguish, e.g., 111 from 1-1-1. As memory is one of the main limitations for the classification programs, internally, the first description is used.
The most complex routine shared by most programs is *Standardize()*, which transforms the triangulation object into a lexicographically pseudo-maximal representation. This method is at the very core of the classification algorithms, as it enables the program to quickly eliminate most isomorphically duplicate triangulations. Without that, these programs would not be feasibly runnable. A more detailed explanation of the reasoning and rough structure of the routine itself can be found in Section 5.1.3, and a more in-depth explanation of the implementation as a comment in the code itself (see any of the files *triangulation.cpp*).

A.2. The different programs in more detail

A.2.1. Generating all k-irreducible triangulations

This program (*FindAllKIrreducibleTriangulations*) follows Algorithm 2 in Section 5.1. Specifically, the executable created from the C++ code takes the set of intermediate triangulations $\mathbf{T}_n$, and transforms it, as described in the algorithm, to the set of intermediate triangulations $\mathbf{T}_{n+1}$. All k-irreducible triangulations of a fixed 2-manifold M are generated, if this program is iterated, starting with n_0 the smallest size of a $(k-1)$-irreducible triangulation of M, and continuing until there are no more intermediate triangulations ($\mathbf{T}_{n_{\max}} = \emptyset$). The executable creating T_{n+1} expects the following input:

- A directory, in which the output of the execution of index n is stored.
- A directory, into which the current execution (index $(n+1)$) should store its output.
- Optionally, files containing the n-vertex $(k-1)$-irreducible triangulations of M.

As a minimal running example, the creation of all 4-irreducible triangulations of the projective plane is prepared.
A note on a depth-first approach: We implemented this width-first, creating all intermediate triangulations of a fixed size before increasing the size up to the first k-irreducible

triangulations. As the main limitation of the implementation is the handling of an abundance of intermediate triangulations, we do not think a depth-first approach would be feasible.

A.2.2. Checking a given set of triangulations for ew-k

This program (*TestAllForEWK*) takes the same input as *FindAllKIrreducibleTriangulations*, and performs a simple check for ew-k on both the separate input files, if they exist, as well as the intermediate files that might be created by an instance of the other program. The reason is simple: That other program does its own ew-k check, but only in combination with doing the next iteration of splits. At some point, the next iteration of splits will no longer be feasible on any platform, while a simple test might still be doable. At that point, there is one series of intermediate triangulations that might contain k-irreducible triangulations, but has yet to be checked, the intended use case of this program. Therefore, the input is the same as the other program:

- A directory, in which the intermediate triangulations of Program 1 might be stored.
- A directory, into which the output should be stored.
- Optionally, files whose contents should be checked for ew-k separately.

A.2.3. Generating all triangulations

This program (*PerformAllSplits*) follows Algorithm 1 in Section 5.1. Specifically, the executable created from the C++ code takes the set of intermediate triangulations $\mathbf{T}_n$, and transforms it, by performing, one by one, every possible split, to the set of intermediate triangulations $\mathbf{T}_{n+1}$. Thus, when iterated, all triangulations are generated that can be reduced by edge contractions to the given input set of triangulations. In particular, if that input set are all k-irreducible triangulations of a fixed 2-manifold M, the program produces all ew-k triangulations up to the given maximal size. The executable creating T_{n+1} expects the following input:

- A directory, in which the output of the execution of index n is stored.
- A directory, into which the current execution (index $(n+1)$) should store its output.
- Optionally, files containing any additional n-vertex triangulations that should be used for the generation.

As a minimal running example, all triangulations of the 2-sphere with up to 8 vertices are generated, using the tetrahedron as the sole input.

A.2.4. Checking input triangulations for exact $(k-1)$-local disc property

This program (*TestForExactDiscProperty*) takes the same input as *PerformAllSplits*, and performs a simple check for triangulations being exactly $(k-1)$-locally disc on the intermediate triangulations created from that program, as well as on any separate files, if such files are given. The overall structure is the same as in *TestAllForEWK*.
The combination of *PerformAllSplits* and *TestForExactDiscProperty* was applied to the 2-sphere to produce all of its up to 8-irreducible triangulations with up to 17 vertices.

A.2.5. Heuristically generating small k-irreducible triangulations

This program (*HeuristicallyFindSmallKIrreducibleTriangulations*) follows Algorithm 4 to try and find small k-irreducible triangulations of 2-manifolds that are not the sphere. It does so handling a range of k and g, distinguishing between orientable and non-orientable manifolds. It expects the following input:

- The minimal k it should create k-irreducible triangulations for.
- The maximal k.
- The minimal genus g for orientable 2-manifolds M_g.
- The maximal genus g for the orientable case.
- The minimal genus h for the non-orientable 2-manifolds N_h.
- The maximal genus h for that case.
- A directory, in which the output will be stored.

In the output directory, every parallel process creates a log file, containing information on the size of the triangulations used in the central steps of the algorithm: At the end of the flips and contractions after the glueing, as well as after increasing k to the wanted amount, as well as the final size found in every iteration.
For every 2-manifold dealt with, a separate directory is created inside the output directory, containing the minimal triangulation found by every parallel process. The file names start with the found minimal size, allowing for a quick overview.