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On some models in linear thermo-elasticity with rational material laws

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Abstract
In the present work, we shall consider some common models in linear thermo-elasticity within a common structural framework. Due to the flexibility of the structural perspective we will obtain well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, etc.

Keywords
Evolutionary equations, Well-posedness, Non-Fourier heat conduction, Thermoelasticity, Lord-Shulman model, Green-Lindsay model, Green-Naghdi model, Dual-phase-lag model

1. Introduction

The coupled dynamical thermo-elasticity (CTE) theory was developed by Biot [1] to eliminate the drawback of the uncoupled theory of thermo-elasticity that the elastic changes in a material have no effects on temperature. Like other classical thermodynamical theories of continua, this theory is developed on the basis of the firm grounds of irreversible thermodynamics by employing Fourier's law and has been used to study the coupling effects of elastic and thermal fields over the years. However, this theory suffers from the paradox of infinite heat propagation speed and predicts unsatisfactory descriptions of a solid's response to some situations, like fast transient loading at low temperature, etc. Generalized thermo-elasticity theories have therefore been developed in last few decades with the aim of eliminating this drawback. Extended thermo-elasticity (ETE) theory was introduced by Lord and Shulman [2] by employing a modified Fourier law proposed by Catteneo [3] and Vernotte [4,5] that includes one thermal relaxation time parameter. Temperature-rate-dependent thermo-elasticity (TRDTE) theory by Green and Lindsay [6] and thermo-elasticity theories of types I, II and III by Green and Naghdi [7–9] are also advocated in this context. Later on, Chandrasekharaiah [10] modified the governing equations of thermo-elasticity on the basis of a so-called dual phase-lag heat conduction equation due to Tzou [11, 12] and proposed two different models of thermo-elasticity, namely dual phase-lag model I (DPL-I) and dual phase-lag model II (DPL-II). The dual phase-lag heat conduction law is supposed to be the macroscopic formulation of the microscopic effects in heat transport processes. A possible application of this generalized heat conduction law arises in the modeling of laser pulses. It has been found out that laser pulses can be shortened

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to the range of femtoseconds (10^{-15} s). When the response time becomes shorter, the non-equilibrium thermodynamic transition and the microscopic effects in the energy exchange during the heat transport procedure become pronounced (Tzou [13]). The formulation therefore becomes microscopic in nature. The dual phase-lag heat conduction law incorporates these microscopic effects in the heat transport process by introducing two macroscopic lagging (or delayed) responses as possible outcomes. A detailed history about the development of some well-established non-Fourier heat conduction models and their importance is available in [10, 14–23].

Recently, a structural formulation for linear material laws in classical mathematical physics was reported by Picard [24]. Here, a class of evolutionary problems is considered to cover a number of initial boundary value problems of classical mathematical physics and the solution theory is established. The well-posedness of classical thermo-elasticity and Lord–Shulman theory was shown to be covered by this model. The main objective of this present work is to show that the aforementioned models of generalized thermo-elasticity can be treated within the common structural framework of evolutionary equations. Due to the flexibility of the structural perspective we will obtain well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, etc. The solution strategy is not based on constructions involving fundamental solutions (semi-group theory), which will allow for even more general materials resulting for example in changes of type (e.g. from parabolic to hyperbolic) or for suitable non-local material properties involving for example spatial integral operators. It should be noted that evolutionary equations in the form just discussed have also been studied with regards to homogenization theory; see for example [25–27]. Hence, the general perspective on thermo-elasticity to be presented may also shed some new light on the theory of homogenization of such models.

The article is structured as follows. We begin by introducing the framework of evolutionary equations and recall the general well-posedness result. We will focus on so-called rational material laws defined as functions of the time-derivative ∂_0, which is established as a normal operator in a suitable exponentially weighted L^2-space. In the following sections we will show how the different models of generalized thermoelasticity can be incorporated into this framework and we will derive assumptions on the material coefficients yielding the well-posedness of the corresponding evolutionary equations.

2. Foundations

2.1. The framework of evolutionary equations

The family of Hilbert spaces $\left(H_{\varrho,0}(\mathbb{R},H)\right)_{\varrho\in\mathbb{R}}$, H a complex Hilbert space, with $H_{\varrho,0}(\mathbb{R},H) := L^2\left(\mathbb{R},\mu_\varrho,H\right)$, where the measure μ_ϱ is defined by $\mu_\varrho(S) := \int_S \exp(-2\varrho t)\ \mathrm{d}t$, $S \subseteq \mathbb{R}$ a Borel set, $\varrho \in \mathbb{R}$, provides the desired Hilbert space setting for evolutionary problems (see [28, 29]). The sign of ϱ is associated with the direction of causality, where the positive sign is linked to forward causality. Since we have a preference for forward causality, we shall usually assume that $\varrho \in]0,\infty[$. By construction of these spaces, we can establish

$$
\begin{aligned}
\exp(-\varrho \mathbf{m}_0) : H_{\varrho,0}(\mathbb{R},H) &\to H_{0,0}(\mathbb{R},H)\,(= L^2(\mathbb{R},H)) \\
\varphi &\mapsto \exp(-\varrho \mathbf{m}_0)\,\varphi,
\end{aligned}
$$

where $(\exp(-\varrho\mathbf{m}_0)\varphi)(t) := \exp(-\varrho t)\varphi(t)$, $t \in \mathbb{R}$, as a unitary mapping. We use $\mathbf{m}_0$ as notation for the multiplication-by-argument operator corresponding to the time parameter.

In this Hilbert space setting the time-derivative operation, defined as the closure of

$$
\begin{aligned}
\mathring{C}_\infty(\mathbb{R},H) \subseteq H_{\varrho,0}(\mathbb{R},H) &\to H_{\varrho,0}(\mathbb{R},H) \\
\varphi &\mapsto \dot{\varphi},
\end{aligned}
$$

where by $\mathring{C}_\infty(\mathbb{R},H)$ we denote the space of arbitrary differentiable functions from $\mathbb{R}$ to H having compact support, generates a normal operator $\partial_{0,\varrho}$ with[1]

$$
\begin{aligned}
\mathfrak{Re}\,\partial_{0,\varrho} &= \varrho, \\
\mathfrak{Im}\,\partial_{0,\varrho} &= \frac{1}{\mathrm{i}}\left(\partial_{0,\varrho} - \varrho\right).
\end{aligned}
$$

The skew-self-adjoint operator $\mathrm{i}\,\mathfrak{Im}\,\partial_{0,\varrho}$ is unitarily equivalent to the differentiation operator $\partial_{0,0}$ in $L^2(\mathbb{R},H) = H_0(\mathbb{R},H)$ with domain $H^1(\mathbb{R},H)$ (the space of weakly differentiable functions in $L^2(\mathbb{R},H)$) via

$$\mathrm{i}\,\mathfrak{Im}\,\partial_{0,\varrho} = (\exp(-\varrho\mathbf{m}_0))^{-1}\,\partial_{0,0}\exp(-\varrho\mathbf{m}_0)$$

and has the Fourier–Laplace transformation as its spectral representation, which is the unitary transformation

$$\mathcal{L}_\varrho := \mathcal{F}\ \exp(-\varrho\mathbf{m}_0) : H_{\varrho,0}(\mathbb{R},H) \to L^2(\mathbb{R},H),$$

where $\mathcal{F} : L^2(\mathbb{R},H) \to L^2(\mathbb{R},H)$ is the Fourier transformation given as the unitary extension of

$$\mathring{C}_\infty(\mathbb{R},H) \ni \varphi \mapsto \left(s \mapsto \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp(-\mathrm{i}s\cdot t)\varphi(t)\,\mathrm{d}t\right).$$

Indeed, this follows from the well-known fact that $\mathcal{F}$ is unitary in $L^2(\mathbb{R},H)$ and a spectral representation for $\frac{1}{\mathrm{i}}\partial_{0,0}$ in $L^2(\mathbb{R},H)$. In particular, we have

$$\mathfrak{Im}\,\partial_{0,\varrho} = \mathcal{L}_\varrho^*\mathbf{m}_0\mathcal{L}_\varrho$$

and thus,

$$\partial_{0,\varrho} = \mathcal{L}_\varrho^*(\mathrm{i}\mathbf{m}_0 + \varrho)\mathcal{L}_\varrho.$$

It is crucial to note that for $\varrho \neq 0$ we have that $\partial_{0,\varrho}$ has a bounded inverse. Indeed, for $\varrho > 0$ we find from $\mathfrak{Re}\,\partial_{0,\varrho} = \varrho$ that

$$\left\|\partial_{0,\varrho}^{-1}\right\|_{\varrho,0} = \frac{1}{\varrho}, \tag{1}$$

where $\|\cdot\|_{\varrho,0}$, $\varrho \in\,]0,\infty[$, denotes the operator norm on $H_{\varrho,0}(\mathbb{R},H)$. For continuous functions φ with compact support we find

$$\left(\partial_{0,\varrho}^{-1}\varphi\right)(t) = \int_{-\infty}^{t} \varphi(s)\ \mathrm{d}s,\ t \in \mathbb{R}, \varrho \in\,]0,\infty[\,, \tag{2}$$

which shows the causality of $\partial_{0,\varrho}^{-1}$ for $\varrho > 0$.[2] Since it is usually clear from the context which ϱ has been chosen, we shall, as it is customary, drop the index ϱ from the notation for the time derivative and simply use ∂_0 instead of $\partial_{0,\varrho}$.

We are now able to define operator-valued functions of ∂_0 via the induced function calculus of ∂_0^{-1} as

$$M\left(\partial_0^{-1}\right) := \mathcal{L}_\varrho^* M\left((\mathrm{i}\mathbf{m}_0 + \varrho)^{-1}\right)\mathcal{L}_\varrho.$$

Here, we require that $z \mapsto M(z)$ is a bounded, analytic function defined on $B_{\mathbb{C}}\left(\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right)$ for some $\varrho_1 \in\,]0,\infty[$ attaining values in $L(H)$, the space of bounded linear operators on H. Then, for $\varrho > \varrho_1$ the operator $M\left((\mathrm{i}\mathbf{m}_0 + \varrho)^{-1}\right)$ defined as

$$\left(M\left((\mathrm{i}\mathbf{m}_0 + \varrho)^{-1}\right)f\right)(t) := M\left((\mathrm{i}t + \varrho)^{-1}\right)f(t) \quad (t \in \mathbb{R}, f \in L^2(\mathbb{R},H))$$

is bounded and linear, and hence $M(\partial_0^{-1}) \in L(H_{\varrho,0}(\mathbb{R},H))$. Moreover, due to the analyticity of M we obtain that $M(\partial_0^{-1})$ becomes causal (see [24, Theorem 2.10]).

We recall from [24] (and the concluding chapter of [28]) that the common form of standard initial boundary value problems of mathematical physics is given by

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}U = F, \tag{3}$$

where A is the canonical skew-self-adjoint extension to $H_{\varrho,0}(\mathbb{R},H)$ of a skew-self-adjoint operator in H. We recall the well-posedness result for this class of problems.

Theorem 2.1 ([24], Solution Theory). *Let $A : D(A) \subseteq H \to H$ be a skew-self-adjoint operator and $M : B\left(\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right) \to L(H)$ an analytic and bounded mapping, where $\varrho_1 \in]0,\infty[$. Assume that there is $c \in]0,\infty[$ such that for all $z \in B\left(\frac{1}{2\varrho_1}, \frac{1}{2\varrho_1}\right)$ the estimate*

$$\mathfrak{Re}\, z^{-1}M(z) = \frac{1}{2}\left(z^{-1}M(z) + \left(z^{-1}\right)^* M(z)^*\right) \geq c \tag{4}$$

holds. For $\varrho > \varrho_1$ we denote the canonical extension of A to $H_{\varrho,0}(\mathbb{R}, H)$ again by A. Then the evolutionary problem

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)} U = F$$

is well-posed in the sense that $\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}$ has a bounded inverse on $H_{\varrho,0}(\mathbb{R}, H)$. Moreover, the inverse is causal.

For the models under consideration it, suffices to consider M as a rational, bounded-operator-valued function, which, possibly by eliminating removable singularities, is analytic at 0 (in [24] these material laws are called 0-analytic). This means in particular that M can be factorized in the form

$$M(z) = \prod_{k=0}^{s} Q_k(z)^{-1} P_k(z) \tag{5}$$

where P_k, Q_k are polynomials.[3] In this case, condition (4) simplifies to

$$\varrho M(0) + \mathfrak{Re}\, M'(0) \geq c \tag{6}$$

for some $c > 0$ and all sufficiently large $\varrho > 0$. Indeed, the only differences between the expressions in (6) and (4) are terms multiplied by a multiple of $|z| = \left|\frac{1}{it+\varrho}\right|$, which are small, if $\varrho > 0$ is chosen sufficiently large. A finer classification of these models can be obtained by looking at the (unbounded) linear operator A and the 'zero patterns' of $M(0)$ and $M'(0)$.

2.2. *The equations of thermo-elasticity*

We start with the classical equations of irreversible thermo-elasticity in an elastic body $\Omega \subseteq \mathbb{R}^3$ due to Biot [1]. Before we can formulate these equations properly, we need to define the spatial differential operators involved.

Definition 2.2. We define the operator $\mathring{\operatorname{grad}}$ as the closure of

$$\begin{aligned} \operatorname{grad}|_{\mathring{C}_\infty(\Omega)} : \mathring{C}_\infty(\Omega) \subseteq L^2(\Omega) &\to L^2(\Omega)^3 \\ \phi &\mapsto (\partial_1\phi, \partial_2\phi, \partial_3\phi), \end{aligned}$$

where we recall that $\mathring{C}_\infty(\Omega)$ denotes the space of smooth functions with compact support in Ω. Likewise we define $\mathring{\operatorname{div}}$ as the closure of

$$\begin{aligned} \operatorname{div}|_{\mathring{C}_\infty(\Omega)^3} : \mathring{C}_\infty(\Omega)^3 \subseteq L^2(\Omega)^3 &\to L^2(\Omega) \\ (\phi_1, \phi_2, \phi_3) &\mapsto \sum_{i=1}^{3} \partial_i\phi_i. \end{aligned}$$

Integration by parts yields $\mathring{\operatorname{grad}} \subseteq -\left(\mathring{\operatorname{div}}\right)^* =: \operatorname{grad}$ and, similarly, $\mathring{\operatorname{div}} \subseteq -\left(\mathring{\operatorname{grad}}\right)^* =: \operatorname{div}$.
Moreover, we define the operator

$$\begin{aligned} \operatorname{sym} : L^2(\Omega)^{3\times3} &\to L^2(\Omega)^{3\times3} \\ \Phi &\mapsto \left(x \mapsto \frac{1}{2}\left(\Phi(x) + \Phi(x)^\top\right)\right), \end{aligned}$$

which clearly is the orthogonal projector onto the closed subspace

$$L^2_{\mathrm{sym}}(\Omega)^{3\times 3} := \left\{\Phi \in L^2(\Omega)^{3\times 3} \mid \Phi(x) = \Phi(x)^\top \quad (x \in \Omega \text{ a.e.})\right\}$$

of $L^2(\Omega)^{3\times 3}$. Similar to the definition above, we define the operator $\mathring{\mathrm{Grad}}$ as the closure of

$$\begin{aligned} \mathrm{Grad}\,|_{\mathring{C}_\infty(\Omega)^3} : \mathring{C}_\infty(\Omega)^3 \subseteq L^2(\Omega)^3 &\to L^2_{\mathrm{sym}}(\Omega)^{3\times 3} \\ (\phi_1, \phi_2, \phi_3) &\mapsto \left(\frac{1}{2}\left(\partial_j\phi_i + \partial_i\phi_j\right)\right)_{i,j\in\{1,2,3\}} \end{aligned}$$

and $\mathring{\mathrm{Div}}$ as the closure of

$$\begin{aligned} \mathrm{Div}\,|_{\mathrm{sym}[\mathring{C}_\infty(\Omega)^{3\times 3}]} : \mathrm{sym}[\mathring{C}_\infty(\Omega)^{3\times 3}] \subseteq L^2_{\mathrm{sym}}(\Omega)^{3\times 3} &\to L^2(\Omega)^3 \\ \left(\phi_{ij}\right)_{i,j\in\{1,2,3\}} &\mapsto \left(\sum_{j=1}^{3} \partial_j\phi_{ij}\right)_{i\in\{1,2,3\}}. \end{aligned}$$

By integration by parts we again obtain $\mathring{\mathrm{Grad}} \subseteq -\left(\mathring{\mathrm{Div}}\right)^* =: \mathrm{Grad}$ as well as $\mathring{\mathrm{Div}} \subseteq -\left(\mathring{\mathrm{Grad}}\right)^* =: \mathrm{Div}$.

We are now able to formulate the equations of thermo-elasticity. Let $u \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega)^3)$ denote the displacement field of the thermoelastic body Ω and $\sigma \in H_{\varrho,0}(\mathbb{R}, L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ the stress. Then u and σ satisfy the balance of momentum equation

$$\varrho_0\partial_0^2 u - \mathrm{Div}\,\sigma = f, \tag{7}$$

where $\varrho_0 \in L^\infty(\Omega)$ denotes the mass density of Ω and $f \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega)^3)$ is an external forcing term. Furthermore, let $\eta \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega))$ denote the entropy and $q \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega)^3)$ the heat flux. Then, these quantities satisfy the conservation law

$$\varrho_0\partial_0\eta + \mathrm{div}(T_0^{-1}q) = T_0^{-1}h, \tag{8}$$

where T_0 denotes the reference temperature[4] and $h \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega))$ is a heating source term. The equations are completed by the following relations:

$$\sigma = C\varepsilon - \Gamma\theta, \tag{9}$$

$$\varrho_0\eta = \Gamma^*\varepsilon + \nu\theta, \tag{10}$$

$$q = -\kappa\,\mathrm{grad}\,\theta. \tag{11}$$

Here $\varepsilon = \mathrm{Grad}\,u$ is the strain, $\theta \in H_{\varrho,0}(\mathbb{R}, L^2(\Omega))$ denotes the temperature, $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ is the elasticity tensor, $\nu \in L^\infty(\Omega)$ stands for the specific heat, $\kappa \in L^\infty(\Omega)$ denotes the thermal conductivity and $\Gamma \in L(L^2(\Omega), L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ is the thermo-elasticity tensor that results from the Duhamel–Neumann law linking the stress to strain and temperature. Assuming that C is invertible, we may rewrite (9) as

$$\varepsilon = C^{-1}\sigma + C^{-1}\Gamma\theta. \tag{12}$$

Consequently, (10) can be written as

$$\varrho_0\eta = \Gamma^*C^{-1}\sigma + \left(\Gamma^*C^{-1}\Gamma + \nu\right)\theta \tag{13}$$

and, hence, with $v := \partial_0 u, \sigma, \theta$ and q as our basic unknowns, (7), (11), (12) and (13) can be combined to the following equations on $H_{\varrho,0}(\mathbb{R}, H)$, where $H := L^2(\Omega)^3 \oplus L^2_{\mathrm{sym}}(\Omega)^{3\times 3} \oplus L^2(\Omega) \oplus L^2(\Omega)^3$:

$$\left(\partial_0 \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\Gamma & 0 \\ 0 & \Gamma^* C^{-1} & \Gamma^* C^{-1}\Gamma + \nu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\operatorname{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & \operatorname{grad} & 0 \end{pmatrix}\right) \begin{pmatrix} v \\ \sigma \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ h \\ 0 \end{pmatrix}.$$

This system is, at least formally, of the form (3), where $M(\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1$ with

$$M_0 = \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\Gamma & 0 \\ 0 & \Gamma^* C^{-1} & \Gamma^* C^{-1}\Gamma + \nu & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^{-1} \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\operatorname{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & \operatorname{grad} & 0 \end{pmatrix}. \tag{14}$$

To make A become skew-self-adjoint, we need to impose boundary conditions on our unknowns. For instance, one could require homogeneous Dirichlet conditions for v and θ, which can be formulated by $v \in D(\mathring{\operatorname{Grad}})$ and $\theta \in D(\mathring{\operatorname{grad}})$. Then, A becomes

$$A = \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\mathring{\operatorname{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & \mathring{\operatorname{grad}} & 0 \end{pmatrix}, \tag{15}$$

which clearly is skew-self-adjoint. Of course, other boundary conditions can be imposed making A skew-self-adjoint; see for example [30].

As we shall see, the Lord–Shulman model [2], the two dual phase-lag models [10] and the three Green–Naghdi models [7–9] are based on the same relations (9), (10), differences only appearing in the modification of Fourier's law (11). In the case of the Green–Lindsay model [6], although of the same formal shape, the meaning of the temperature θ is replaced by the differential expression $(1 + n_0\partial_0)$ applied to temperature. Therefore, in order to avoid confusion, we shall use in this case

$$\Theta := \theta + n_0 \partial_0 \theta$$

instead of re-dedicating the symbol θ, where n_0 is the thermal relaxation time, a characteristic of this model.

3. Solution theory to some thermo-elastic models

In this section, we will show that the models of thermo-elasticity mentioned in the introduction can be written as evolutionary problems in the sense of Section 2.1 and thus, their well-posedness can be shown with the help of Theorem 2.1. In fact, we will show that a generalized model of the basic Green–Lindsay type allows to recover all other models as special cases. We will begin to formulate this abstract model and prove its well-posedness. In the subsequent subsection, we will show how the classical models can be recovered from the abstract one and which conditions yield their well-posedness.

3.1. A general rational material law for thermo-elasticity

We consider the following material law $M(\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1(\partial_0^{-1})$, where

$$M_0 = \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\Gamma & 0 \\ 0 & \Gamma^* C^{-1} & \nu + \Gamma^* C^{-1}\Gamma + \zeta_0^* a_0 \zeta_0 & \zeta_0^* a_0 \\ 0 & 0 & a_0\zeta_0 & a_0 \end{pmatrix}, \tag{16}$$

$$M_1(\partial_0^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_1(\partial_0^{-1}) & 0 \\ 0 & 0 & 0 & a_2(\partial_0^{-1}) \end{pmatrix}. \tag{17}$$

Here $a_0 \in L(L^2(\Omega)^3)$ is a self-adjoint operator, $\zeta_0 \in L(L^2(\Omega), L^2(\Omega)^3)$ and $a_1 : B(0,r) \to L(L^2(\Omega))$ and $a_2 : B(0,r) \to L(L^2(\Omega)^3)$ are rational functions for some $r > 0$. We recall from the previous section that $\varrho_0, \nu \in L^\infty(\Omega)$ denote the mass density and the specific heat, respectively, which will be assumed to be real and strictly positive, in other words $\varrho_0(x), \nu(x) \geq c$ for some $c > 0$ and almost every $x \in \Omega$. Moreover, the elasticity tensor $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ is assumed to be self-adjoint and strictly positive definite.

Theorem 3.1. *Let M_0 and $M_1(\partial_0^{-1})$ be as in* (16) *and* (17)*, respectively. We assume that $\varrho_0, \nu \in L^\infty(\Omega)$ are real-valued and strictly positive and $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ is self-adjoint and strictly positive definite. Moreover, we assume that a_0 is strictly positive definite on its range and $\mathfrak{Re}\, a_2(0)$ is strictly positive definite on the kernel of a_0. Then the evolutionary problem*

$$\left(\partial_0 M_0 + M_1(\partial_0^{-1}) + A\right) \begin{pmatrix} v \\ \sigma \\ \Theta \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ h \\ 0 \end{pmatrix} \tag{18}$$

is well-posed in the sense of Theorem 2.1, where A is given by[5] *(15) and $\Theta = (1 + n_0\partial_0)\theta$.*

Proof. According to Theorem 2.1, we need to verify condition (4) for $M(\partial_0^{-1}) = M_0 + \partial_0^{-1} M_1(\partial_0^{-1})$. Or, equivalently, by the structural properties assumed for M_1, we need to verify that there exists $\varrho_1 > 0$ such that for all $\varrho > \varrho_1$ we have that

$$\varrho M_0 + \mathfrak{Re}\, M_1(0) \geq c$$

for some $c > 0$. Indeed, the latter equation is precisely the reformulation of (6) for the particular M under consideration. For $\varrho > 0$, we compute

$$\begin{aligned}
&\varrho M_0 + \mathfrak{Re}\, M_1(0) \\
&= \varrho \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\Gamma & 0 \\ 0 & \Gamma^* C^{-1} & \nu + \Gamma^* C^{-1}\Gamma + \zeta_0^* a_0 \zeta & \zeta_0^* a_0 \\ 0 & 0 & a_0\zeta_0 & a_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{Re}\, a_1(0) & 0 \\ 0 & 0 & 0 & \mathfrak{Re}\, a_2(0) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \Gamma^* & 1 & \zeta_0^* \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \\
&\quad \left(\varrho \begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathfrak{Re}\, a_1(0) + \zeta_0^* \,\mathfrak{Re}\, a_2(0)\zeta_0 & -\zeta_0^* \,\mathfrak{Re}\, a_2(0) \\ 0 & 0 & -\mathfrak{Re}\, a_2(0)\zeta_0 & \mathfrak{Re}\, a_2(0) \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \zeta_0 & 1 \end{pmatrix}.
\end{aligned}$$

We read off that the latter is strictly positive definite if and only if the operator

$$\varrho\begin{pmatrix} \varrho_0 & 0 & 0 & 0 \\ 0 & C^{-1} & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & a_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Re\mathfrak{e}\, a_1(0) + \zeta_0^* \Re\mathfrak{e}\, a_2(0)\zeta_0 & -\zeta_0^* \Re\mathfrak{e}\, a_2(0) \\ 0 & 0 & -\Re\mathfrak{e}\, a_2(0)\zeta_0 & \Re\mathfrak{e}\, a_2(0) \end{pmatrix}$$

is strictly positive definite. As, by assumption, the operators ϱ_0 and C^{-1} are positive definite anyway, we only have to study the positive definiteness of the operator

$$\varrho\begin{pmatrix} \nu & 0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} \Re\mathfrak{e}\, a_1(0) + \zeta_0^* \Re\mathfrak{e}\, a_2(0)\zeta_0 & -\zeta_0^* \Re\mathfrak{e}\, a_2(0) \\ -\Re\mathfrak{e}\, a_2(0)\zeta_0 & \Re\mathfrak{e}\, a_2(0) \end{pmatrix}. \tag{19}$$

Now, decomposing the underlying (spatial) Hilbert space as

$$L^2(\Omega) \oplus L^2(\Omega)^3 = R\left(\begin{pmatrix} \nu & 0 \\ 0 & a_0 \end{pmatrix}\right) \oplus N\left(\begin{pmatrix} \nu & 0 \\ 0 & a_0 \end{pmatrix}\right),$$

which can be done, since both a_0 and ν are strictly positive definite on the respective ranges, we realize that (19) is strictly positive definite on $H_{\varrho,0}\left(\mathbb{R}, R\left(\begin{pmatrix} \nu & 0 \\ 0 & a_0 \end{pmatrix}\right)\right)$ with positive definiteness constant arbitrarily large, depending on the choice of $\varrho > 0$. By Euclid's inequality ($2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2$, $a, b \in \mathbb{R}$, $\varepsilon > 0$), the assertion follows, if we show that the operator in (19) is strictly positive definite on the nullspace of a_0. However, by assumption, $\Re\mathfrak{e}\, a_2(0)$ is strictly positive on $N(a_0)$. This yields the assertion. □

Remark 3.2. We write down equation (18) line by line. It is

$$\begin{aligned} \partial_0\varrho_0 v - \operatorname{Div}\sigma &= f, \\ \partial_0 C^{-1}\sigma + \partial_0 C^{-1}\Gamma\Theta - \mathring{\operatorname{Grad}} v &= 0, \\ \partial_0\Gamma^* C^{-1}\sigma + \partial_0\left(\nu + \Gamma^* C^{-1}\Gamma + \zeta_0^* a_0\zeta_0\right)\Theta + \partial_0\zeta_0^* a_0 q + a_1(\partial_0^{-1})\Theta + \operatorname{div} q &= h, \\ \partial_0 a_0\zeta_0\Theta + \partial_0 a_0 q + a_2(\partial_0^{-1})q + \mathring{\operatorname{grad}}\Theta &= 0. \end{aligned}$$

Defining $u := \partial_0^{-1}v$, $\varepsilon := \operatorname{Grad} u$ and $\eta := \varrho_0^{-1}\left(\Gamma^*\varepsilon + \left(\nu + \zeta_0^* a_0\zeta_0\right)\Theta + \zeta_0^* a_0 q + \partial_0^{-1}a_1(\partial_0^{-1})\Theta\right)$ we get from the second line

$$\sigma = C\varepsilon - \Gamma\Theta.$$

Moreover, the fourth line reads as

$$\partial_0 a_0 q + a_2(\partial_0^{-1})q = -\partial_0 a_0\zeta_0\Theta - \mathring{\operatorname{grad}}\Theta$$

and the first line is

$$\partial_0^2\varrho_0 u - \operatorname{Div}\sigma = f.$$

Finally, the third line reads as

$$\begin{aligned} \partial_0\varrho_0\eta + \operatorname{div} q &= \partial_0\left(\Gamma^*\varepsilon + \left(\nu + \zeta_0^* a_0\zeta_0\right)\Theta + \zeta_0^* a_0 q + \partial_0^{-1}a_1(\partial_0^{-1})\Theta\right) + \operatorname{div} q \\ &= \partial_0\Gamma^* C^{-1}\sigma + \partial_0\left(\Gamma^* C^{-1}\Gamma + \nu + \zeta_0^* a_0\zeta_0\right)\Theta + \partial_0\zeta_0^* a_0 q + a_1(\partial_0^{-1})\Theta + \operatorname{div} q \\ &= h, \end{aligned}$$

where we have used $\sigma = C\varepsilon - \Gamma\Theta$. Summarizing, our material relations are

$$\partial_0^2\varrho_0 u - \operatorname{Div}\sigma = f, \tag{20}$$

$$\partial_0\varrho_0\eta + \operatorname{div} q = h, \tag{21}$$

$$\sigma = C\varepsilon - \Gamma\Theta, \tag{22}$$

$$\varrho_0\eta = \Gamma^*\varepsilon + \left(\nu + \zeta_0^* a_0\zeta_0\right)\Theta + \zeta_0^* a_0 q + \partial_0^{-1}a_1(\partial_0^{-1})\Theta, \tag{23}$$

$$\partial_0 a_0 q + a_2(\partial_0^{-1})q = -\partial_0 a_0\zeta_0\Theta - \mathring{\operatorname{grad}}\Theta, \tag{24}$$

$$\Theta = (1 + n_0\partial_0)\theta. \tag{25}$$

We note that for $n_0 = a_0 = \zeta_0 = a_1(\partial_0^{-1}) = 0$ and $a_2(\partial_0^{-1}) = \kappa^{-1}$ we recover the equations of irreversible thermo-elasticity (see Section 2.2).

We will now discuss several models of thermo-elasticity and we will show that they all are covered by the model proposed above. Due to the importance of $M(0) = M_0$, $M'(0) = M_1(0)$ in the discussion of well-posedness (see (6)), we are first led to distinguish two classes of models:

- Generic models.
  These models are characterized by $M(0) = \mathfrak{Re} M(0)$ being strictly positive definite. For these (6) is always satisfied. Moreover, $M(0) + \varepsilon$ is then also strictly positive definite for any sufficiently small self-adjoint operator ε.
- Degenerate models.
  These models fail to have remarkable stability with regards to perturbations of the generic models. They are characterized by $M(0) = \mathfrak{Re} M(0)$ having a non-trivial null space. In these cases (6) can be ensured for example by assuming that $M(0) = \mathfrak{Re}\, M(0)$ is strictly positive definite on its own range $M(0)[H]$, in other words,
  $$\langle x | M(0)x \rangle_H \geq c_0 > 0 \text{ for all } x \in M(0)[H],$$
  $\mathfrak{Re}\, M'(0)$ being strictly positive definite on the null space $[\{0\}]\, M(0)$, in other words,
  $$\left\langle x \middle| \, \mathfrak{Re}\, M'(0)x \right\rangle_H \geq c_0 > 0 \text{ for all } x \in [\{0\}]\, M(0).$$

3.2. The generic case

3.2.1. Lord–Shulman model. In contrast to the model for irreversible thermo-elasticity (see Section 2.2), Lord and Shulman [2] proposed to replace Fourier's law (11) by the so-called Cattaneo-Vernotte modification of Fourier's law (see [3–5]), which reads as

$$\partial_0 \tau q + q = -\kappa \operatorname{\mathring{grad}} \theta,$$

where $\tau \in L^\infty(\Omega)$ is assumed to be real-valued and strictly positive definite. This results in a system of the form (18), where

$$a_0 = \tau\kappa^{-1}, a_1(\partial_0^{-1}) = \zeta_0 = n_0 = 0 \text{ and } a_2(\partial_0^{-1}) = \kappa^{-1}. \tag{26}$$

In consequence, we obtain the well-posedness for this model by Theorem 3.1.

Corollary 3.3. *Let M_0, $M_1(\partial_0^{-1})$ and A be given by* (16), (17), *and a skew-self-adjoint restriction of* (14), *respectively. Assume that $\tau, \varrho_0, \nu \in L^\infty(\Omega)$, $\kappa \in L(L^2(\Omega)^3)$, $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ are self-adjoint and strictly positive definite*[6] *as well as (26). Then* (18) *is well-posed in the sense of Theorem 2.1.*

Proof. It suffices to observe that the kernel of a_0 is trivial. □

Remark 3.4. If we mark possible non-zero entries in the operator matrix $M(0)$ by a star, we have the zero-pattern

$$M(0) = \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & \star & 0 \\ 0 & \star & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}.$$

The zero-pattern of M_1 is

$$M_1(\partial_0^{-1}) = M_1(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}.$$

We observe that there are no higher-order terms in the material law operator in the case of the Lord–Shulman model.

3.2.2. Green–Naghdi model of type II. In the models proposed by Green and Naghdi (see [7–9]) a modified heat flux of the form

$$\begin{aligned} q &= -\partial_0^{-1}\left(\widetilde{k} + k\partial_0\right)\mathring{\mathrm{grad}}\theta \\ &= -(\partial_0^{-1}\widetilde{k} + k)\mathring{\mathrm{grad}}\theta \end{aligned}$$

is assumed by considering $k, \widetilde{k} \in \mathbb{R}$ as the thermal conductivity and conductivity rate, respectively. Depending on $\widetilde{k}$ and k, we distinguish between three types of this model. If $k << \widetilde{k}$ and $\widetilde{k} \neq 0$, we speak about the Green–Naghdi model of type II. In this case, the above heat flux satisfies

$$\partial_0\left(\widetilde{k}\right)^{-1} q = -\mathring{\mathrm{grad}}\theta.$$

This system is covered by the abstract one if in (18)

$$n_0 = \zeta_0 = a_1(\partial_0^{-1}) = a_2(\partial_0^{-1}) = 0 \text{ and } a_0 = \left(\widetilde{k}\right)^{-1}. \tag{27}$$

Hence, $M(0)$ has the zero-pattern

$$M(0) = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix},$$

while

$$M_1(\partial_0^{-1}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding well-posedness result in a generalized fashion reads as

Corollary 3.5. *Let M_0, $M_1(\partial_0^{-1})$ and A be given by* (16), (17), *and a skew-self-adjoint restriction of* (14), *respectively. Assume that $\widetilde{k}, \varrho_0, \nu \in L^\infty(\Omega)$, $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ are self-adjoint and strictly positive definite as well as* (27). *Then* (18) *is well-posed in the sense of Theorem 2.1.*

Proof. Again, the assertion follows when applying Theorem 3.1 while observing that $N(a_0) = \{0\}$. □

3.2.3. The generic Green–Lindsay model. As indicated earlier in Section 2.2, here the material relations (9) to (11) are modified to

$$\begin{aligned} \sigma &= C\varepsilon - \Gamma\left(\theta + n_0\partial_0\theta\right), \\ \varrho_0\eta &= d\theta + d_1\partial_0\theta + \Gamma^*\varepsilon - b^*\mathring{\mathrm{grad}}\theta, \\ q &= -b\partial_0\theta - \kappa\mathring{\mathrm{grad}}\theta. \end{aligned}$$

Here b, d are material parameters, $d_1 = dn$ and n, $n_0(\neq 0)$ are the thermal relaxation times. Now, letting

$$\Theta := \theta + n_0\partial_0\theta$$

we get

$$\begin{aligned} \theta &= \left(1 + n_0\partial_0\right)^{-1}\Theta \\ &= \partial_0^{-1}\left(\partial_0^{-1} + n_0\right)^{-1}\Theta \end{aligned}$$

and the material relations as given above turn into

$$\begin{aligned} \varepsilon &= C^{-1}\sigma + C^{-1}\Gamma\Theta, \\ q &= -\left(\partial_0^{-1} + n_0\right)^{-1}\left(b\Theta + \kappa\partial_0^{-1}\mathring{\mathrm{grad}}\Theta\right), \end{aligned}$$

which yield

$$\partial_0 n_0 \kappa^{-1} q + \kappa^{-1} q = -\partial_0 \kappa^{-1} b\Theta - \mathring{\operatorname{grad}}\Theta.$$

Moreover, we have, using the Neumann series,

$$\begin{aligned}
\varrho_0 \eta &= d\theta + d_1 \partial_0 \theta + \Gamma^* \varepsilon - b^* \mathring{\operatorname{grad}}\theta \\
&= (d + d_1 \partial_0)\, \partial_0^{-1} \left(\partial_0^{-1} + n_0\right)^{-1} \Theta + \Gamma^* \varepsilon + b^* \kappa^{-1} b \left(\partial_0^{-1} + n_0\right)^{-1} \Theta + b^* \kappa^{-1} q \\
&= \Gamma^* \varepsilon + \left(\partial_0^{-1} d + d_1\right) n_0^{-1} \sum_{j=0}^{\infty} \left(-\partial_0^{-1} n_0^{-1}\right)^j \Theta + b^* \kappa^{-1} b n_0^{-1} \sum_{j=0}^{\infty} (-\partial_0^{-1} n_0^{-1})^j \Theta + b^* \kappa^{-1} q \\
&= \Gamma^* \varepsilon + \left(d_1 n_0^{-1} + b^* \kappa^{-1} b n_0^{-1}\right) \Theta + b^* \kappa^{-1} q \\
&\qquad + \partial_0^{-1} \left(d n_0^{-1} - \left(d_1 + b^* \kappa^{-1} b\right) n_0^{-2}\right) \sum_{j=0}^{\infty} \left(-\partial_0^{-1} n_0^{-1}\right)^j \Theta \\
&= \Gamma^* \varepsilon + \left(d_1 n_0^{-1} + b^* \kappa^{-1} b n_0^{-1}\right) \Theta + b^* \kappa^{-1} q + \partial_0^{-1} \left(d - \left(d_1 + b^* \kappa^{-1} b\right) n_0^{-1}\right) \left(n_0 + \partial_0^{-1}\right)^{-1} \Theta.
\end{aligned}$$

Thus, we are in our abstract situation with

$$\begin{aligned}
a_0 = n_0 \kappa^{-1},\ a_2(\partial_0^{-1}) = \kappa^{-1}, \zeta_0 = b n_0^{-1}, \nu = d_1 n_0^{-1} \text{ and} \\
a_1(\partial_0^{-1}) = \left(d - \left(d_1 + b^* \kappa^{-1} b\right) n_0^{-1}\right) \left(n_0 + \partial_0^{-1}\right)^{-1}
\end{aligned} \tag{28}$$

In this case the operator matrix $M(0)$ has the zero-pattern

$$M(0) = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & \bigstar \\ 0 & 0 & \bigstar & \bigstar \end{pmatrix}.$$

The zero-pattern of $M_1(0)$ is now

$$M_1(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \bigstar & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}.$$

Also, we note that there are higher-order terms in the material law operator. The corresponding well-posedness result is noted in the following corollary.

Corollary 3.6. *Let M_0, $M_1(\partial_0^{-1})$ and A be given by* (16), (17)*, and a skew-self-adjoint restriction of* (14)*, respectively. Let $n_0 > 0$ and assume that $h, \varrho_0 \in L^\infty(\Omega)$, $\kappa \in L(L^2(\Omega)^3)$, $C \in L(L^2_{\mathrm{sym}}(\Omega)^{3\times 3})$ are self-adjoint and strictly positive definite, $b \in L(L^2(\Omega), L^2(\Omega)^3)$ as well as* (28)*. Then* (18) *is well-posed in the sense of Theorem 2.1.*

Proof. Again, note that the kernel of a_0 is trivial. Apply Theorem 3.1. □

3.2.4. Dual-phase-lag model of type -II (DPL-II model). In case of the DPL-II model, apart from (9), (10) we have here the modified Fourier law as

$$\left(1 + n_1 \partial_0 + \frac{1}{2} n_1^2 \partial_0^2\right) q = -\kappa \left(1 + n_2 \partial_0\right) \mathring{\operatorname{grad}}\theta$$

where $n_1, n_2 \in \mathbb{R} \setminus \{0\}$ are called phase-lags. Assuming that κ is invertible, we can write the latter relation as

$$
\begin{aligned}
-\mathring{\mathrm{grad}}\theta &= (1+n_2\partial_0)^{-1}\left(1+n_1\partial_0+\frac{1}{2}n_1^2\partial_0^2\right)\kappa^{-1}q \\
&= \left(\partial_0^{-1}+n_1+\frac{1}{2}n_1^2\partial_0\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q \\
&= \frac{1}{2}n_1^2\partial_0\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q+\left(n_1+\partial_0^{-1}\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q \\
&= \frac{1}{2}n_1^2n_2^{-1}\partial_0\sum_{j=0}^{\infty}\left(-\partial_0^{-1}n_2^{-1}\right)^j\kappa^{-1}q+\left(n_1+\partial_0^{-1}\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q \\
&= \frac{1}{2}n_1^2n_2^{-1}\partial_0\kappa^{-1}q-\frac{1}{2}n_1^2n_2^{-2}\sum_{j=0}^{\infty}\left(-\partial_0^{-1}n_2^{-1}\right)^j\kappa^{-1}q+\left(n_1+\partial_0^{-1}\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q \\
&= \frac{1}{2}n_1^2n_2^{-1}\partial_0\kappa^{-1}q+\left(\left(n_1+\partial_0^{-1}\right)-\frac{1}{2}n_1^2n_2^{-1}\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}q.
\end{aligned}
$$

Thus, this corresponds to the abstract situation when in (18)

$$
\begin{aligned}
&n_0=\zeta_0=a_1(\partial_0^{-1})=0 \text{ and} \\
&a_0=\frac{1}{2}n_1^2n_2^{-1}\kappa^{-1},\ a_2(\partial_0^{-1})=\left(\left(n_1+\partial_0^{-1}\right)-\frac{1}{2}n_1^2n_2^{-1}\right)\left(\partial_0^{-1}+n_2\right)^{-1}\kappa^{-1}.
\end{aligned}
\tag{29}
$$

Therefore, the zero-pattern of $M(0)$ is

$$
M(0)=\begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}
$$

and the zero-pattern of $M_1(0)$ is

$$
M_1=\mathfrak{Re}\,M_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}.
$$

It is seen that this is similar to the case of the Lord–Shulman model. Thus, the well-posedness conditions are similar to the ones in Corollary 3.3 using (29) instead of (26). There are, however (different) higher-order terms in the material law.

3.3. *The p-degenerate case*

3.3.1. Green–Naghdi model of type I and type III. Recall that in the Green–Naghdi model (see Section 3.2.2), Fourier's law is replaced by

$$
q=-(\partial_0^{-1}\underset{\sim}{k}+k)\mathring{\mathrm{grad}}\theta.
$$

In the Green–Naghdi model of type I, it is assumed that $\widetilde{k}=0$, $k>0$. Thus, the above relation becomes $q=-k\mathring{\mathrm{grad}}\theta$, which is the classical Fourier's law and so we have

$$
M(0)=\begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

and

$$M_1(\partial_0^{-1}) = M_1(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix},$$

with no higher-order terms. This is the classical model of thermo-elasticity discussed in the introduction; see for example [31–33]. In the case of the Green–Naghdi model of type III, we have that $k, \widetilde{k} > 0$. This yields that the modified Fourier's law becomes

$$\left(\partial_0^{-1}\widetilde{k} + k\right)^{-1} q = -\mathring{\operatorname{grad}}\theta,$$

and hence, we are in the situation of Section 3.1 with

$$\begin{aligned} n_0 = \zeta_0 = a_0 = a_1(\partial_0^{-1}) = 0 \text{ and} \\ a_2(\partial_0^{-1}) = \left(\partial_0^{-1}\widetilde{k} + k\right)^{-1}. \end{aligned} \tag{30}$$

Thus, the zero-patterns of $M(0)$ and $M_1(0)$ look the same as above with the difference that higher-order terms appear (i.e. $M_1(\partial_0^{-1}) \neq M_1(0)$). The well-posedness result reads as follows.

Corollary 3.7. *Let M_0, $M_1(\partial_0^{-1})$ and A be given by* (16), (17)*, and a skew-self-adjoint restriction of* (14)*, respectively. Assume that $\varrho_0, \nu \in L^\infty(\Omega)$, $k \in L(L^2(\Omega)^3)$, $C \in L(L^2_{\text{sym}}(\Omega)^{3\times 3})$ are self-adjoint and strictly positive definite, $\widetilde{k} \in L\left(L^2(\Omega)^3\right)$ as well as* (30)*. Then* (18) *is well-posed in the sense of Theorem 2.1.*

Proof. By the strict positive definiteness of k, it follows that $\mathfrak{Re}\, a_2(0) = \mathfrak{Re}\, k^{-1}$ is strictly positive on $L^2(\Omega)^3 = N(0) = N(a_0)$. Now, apply Theorem 3.1 to obtain the required result. □

3.3.2. Dual-phase-lag model of type -I (DPL-I model). We conclude our considerations by the study of the DPL-I model. Here again, we assume (9) and (10) hold, while Fourier's law (11) is replaced by

$$(1 + n_1\partial_0)\, q = -\kappa\, (1 + n_2\partial_0)\, \mathring{\operatorname{grad}}\theta,$$

with two phase-lags $n_1, n_2 \in \mathbb{R} \setminus \{0\}$. The latter gives

$$\begin{aligned} -\mathring{\operatorname{grad}}\theta &= (1 + n_2\partial_0)^{-1}\, (1 + n_1\partial_0)\, \kappa^{-1} q \\ &= \left(\partial_0^{-1} + n_2\right)^{-1} \left(\partial_0^{-1} + n_1\right) \kappa^{-1} q, \end{aligned}$$

which shows that we are in the case

$$\begin{aligned} n_0 = \zeta_0 = a_0 = a_1(\partial_0^{-1}) = 0 \text{ and} \\ a_2(\partial_0^{-1}) = \left(\partial_0^{-1} + n_2\right)^{-1} \left(\partial_0^{-1} + n_1\right) \kappa^{-1}. \end{aligned} \tag{31}$$

Therefore, the zero-pattern of $M(0)$ is

$$M\,(0) = \begin{pmatrix} \bigstar & 0 & 0 & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & \bigstar & \bigstar & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the zero-pattern of $M_1(0)$ is

$$M_1(0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bigstar \end{pmatrix}.$$

Similarly to Corollary 3.7, using (31) instead of (30) and imposing $n_1 \cdot n_2 > 0$, we get the corresponding well-posedness result also for this type of equation.

4. Conclusion

Various models of thermo-elasticity are written as evolutionary problems and their well-posedness results are shown. We formulate an abstract model with rational material laws which is of basic Green–Lindsay-type and we prove its well-posedness. All other models are shown to be recovered from this abstract one and we find the conditions which yield their well-posedness. We also show that these models can be classified into two distinct classes, namely a generic model and a p-degenerate model. Due to the flexibility of the structural perspective, we obtain well-posedness results for a large class of generalized models allowing for more general material properties such as anisotropies, inhomogeneities, etc.

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Notes

1. Recall that for normal operators N in a Hilbert space H,

$$\mathfrak{Re}\, N := \frac{1}{2}\overline{(N+N^{*})},$$
$$\mathfrak{Im}\, N := \frac{1}{2\mathrm{i}}\overline{(N-N^{*})}$$

and

$$N = \mathfrak{Re}\, N + \mathrm{i}\, \mathfrak{Im}\, N.$$

It is

$$D(N) = D(\mathfrak{Re}\, N) \cap D(\mathfrak{Im}\, N).$$

2. If $\varrho < 0$ the operator $\partial_{0,\varrho}$ is also boundedly invertible and its inverse is given by

$$\left(\partial_{0,\varrho}^{-1}\varphi\right)(t) = -\int_t^\infty \varphi(s)\, ds \quad (t \in \mathbb{R})$$

for all $\varphi \in \mathring{C}_\infty(\mathbb{R}, H)$. Thus, $\varrho < 0$ corresponds to the backward causal (or anticausal) case.

3. The form $U = M\left(\partial_0^{-1}\right) V$ may be interpreted as coming from solving an integro-differential equation of the form

$$\partial_0^N Q\left(\partial_0^{-1}\right) U = \partial_0^N P\left(\partial_0^{-1}\right) V,$$

where $N \in \mathbb{N}$ is the degree of the operator polynomial Q.

4. For simplicity we have set the reference temperature T_0 in the introduction (and also later on) to $T_0 = 1$. In equation (8) we let $T_0 \in]0, \infty[$ be arbitrary to keep the formulation more comparable with the classically proposed models.
5. Or any other skew-self-adjoint restriction of (14).
6. For ease of formulation, note that we identified a_0, ϱ_0 and ν with the induced multiplication operators on L^2. In this way, self-adjointness is just the same as to say the respective L^∞-functions assume only real values and, thus, strict positivity coincides with strict positivity of the respective functions.

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