Approximate representations of groups

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Abstract

In this thesis, we consider various notions of approximate representations of groups. Loosely speaking, an approximate representation is a map from a group into the unitary operators on a Hilbert space that satisfies the homomorphism equation up to a small error. Maps that are close to actual representations are trivial examples of approximate representations, and a natural question to ask is whether all approximate representations of a given group arise in this way. A group with this property is called stable.

In joint work with Lev Glebsky, Alexander Lubotzky and Andreas Thom, we approach the stability question in the setting of local asymptotic representations. We provide sufficient condition in terms of cohomology vanishing for a finitely presented group to be stable. We use this result to provide new examples of groups that are stable with respect to the Frobenius norm, including the first examples of groups that are not Frobenius approximable.

In joint work with Narutaka Ozawa and Andreas Thom, we generalize a theorem by Gowers and Hatami about maps with non-vanishing uniformity norm. We use this to prove a very general stability result for uniform ε-representations of amenable groups which subsumes results by both Gowers-Hatami and Kazhdan.

Contents

Introduction

Unitary representations are fundamental objects of study in group theory and have been so since the birth of the subject. A unitary representation is simply a homomorphism from a group into the group of unitary operators on a Hilbert space. The main theme of this thesis is *approximate representations*, a broad notion that subsumes various generalizations of the definition of a unitary representation. The fundamental philosophy behind all kinds of approximate representations, which goes back to Turing [69] and Ulam [70], is to relax the defining equation of a homomorphism, that is, $\varphi(xy) = \varphi(x)\varphi(y)$, so that it only holds up to some error $\varepsilon \geq 0$. This error is measured in an appropriate metric, typically induced by a norm $\|\cdot\|$ on the bounded operators on the Hilbert space, that is,

$$\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \varepsilon. \tag{$\star$}$$

Depending on which version of an approximate representation we are dealing with, we require Equation ($\star$) to be satisfied for some, "most" or all x and y in the group. Loosely speaking, we can divide approximate representations into *local* ones, where we consider a sequence of maps such that Equation ($\star$) is satisfied on larger and larger finite subsets of the group and for smaller and smaller ε as n grows, and *global* ones, where we only consider a single map satisfying Equation ($\star$) for all (or most) $x, y \in \Gamma$. One (but not the only) central question of this thesis is the question of *stability*: given an approximate representation, is there a genuine representation which is "close" to the approximate representation in some sense? The precise formulation and answer to this question depend much on the particular notion of approximate representation we work with.

Let us make the above more precise and outline the main results of this thesis. In the local setting, we consider the notion of an *asymptotic representation*. This is a sequence of maps $\varphi_n \colon \Gamma \to \mathbf{U}(\mathscr{H}_n)$ from a discrete, countable group Γ into the unitary group $\mathbf{U}(\mathscr{H}_n)$ on a Hilbert space $\mathscr{H}_n$ such that $\|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\|_n \to 0$, for $n \to \infty$, all $x, y \in \Gamma$ and some family of norms $\|\cdot\|_n$ on the bounded operators on $\mathscr{H}_n$. In this setting, the stability question becomes the following: For a given asymptotic representation $(\varphi_n)_{n\in\mathbb{N}}$, can one find a sequence of genuine representations $\pi_n \colon \Gamma \to \mathbf{U}(\mathscr{H}_n)$ such that

$\|\varphi_n(x) - \pi_n(x)\| \to 0$, for $n \to \infty$ and all $x \in \Gamma$? This question is treated in the literature for various groups and norms, maybe most famously by Voiculescu in [71], where he provides an example of asymptotically commuting matrices that are not close to any commuting matrices with respect to the operator norm. Our main contribution to this topic is a stability result for asymptotic representations with respect to the Frobenius norm (also called the unnormalized Hilbert-Schmidt norm). In a joint work with Lev Glebsky, Alexander Lubotzky and Andreas Thom [18] we provide non-trivial examples of asymptotic representations and give a sufficient condition for a group to be stable in terms of cohomology vanishing.

Another notion that emerges in the context of asymptotic representations is that of *approximable groups*. It is an open problem to determine whether *all* groups are approximable, both with respect to the operator norm and the normalized Hilbert-Schmidt norm. This is related to more general open problems stated by Kirchberg [9] and Connes [17]. Using our stability result, we provide examples of groups that are not approximable with respect to the Frobenius norm.

In the global setting, we generalize an inverse theorem for the uniformity norm, proven by Gowers and Hatami [36]. They prove that any matrix-valued map from a finite group with non-vanishing uniformity norm is correlated to a representation in a certain sense. A map with "big" uniformity norm can be interpreted as a rather weak form of global approximate representation. In joint work with Narutaka Ozawa and Andreas Thom [19], we prove a similar result for maps from amenable groups that take values in the bounded operators on a Hilbert space. As a corollary, we prove a stability theorem for so-called *uniform ε-representations*: Any map from an amenable group that satisfies Equation ($\star$) for *all* $x, y \in \Gamma$ is uniformly close to a genuine representation. This generalizes earlier stability results of both [45] and [36].

The thesis is divided into three parts. The first part consists of all the preliminary theory that is needed in order to prove the main theorems. The second part is devoted to the local notion of asymptotic representations. The third part is about global approximate representations. For convenience of the reader, we provide an overview of the contents.

Part I

- In Chapter 1, we review various elements of group theory. We define the notion of an amenable group, we discuss the special unitary group of quaternion algebras and review the basics of group cohomology.

- In Chapter 2, we discuss the basic operator algebra we need, with focus on von Neumann algebras and the notion of unitarily invariant norms together with positive definite maps.
- In Chapter 3, we recall the definition of an ultrafilter and review the notion of ultraproducts, both in the context of metric groups and Banach algebras.

Part II

- In Chapter 4, the objects of study are the asymptotic representations, in particular for finitely presented groups. This chapter also has an introductory flavor as we provide the background knowledge necessary to understand our main theorems in the following chapter.
- In Chapter 5, we prove our main results about asymptotic representations. That is, we show that groups that have vanishing second cohomology with Hilbert space coefficients are Frobenius stable and use this characterization together with facts about algebraic groups to provide examples of groups that are not approximable with respect to the Frobenius norm.

Part III

- In Chapter 6, we turn our attention from the local to the global picture. We discuss and prove an inverse theorem for maps with non-vanishing uniformity norm for amenable groups.
- In Chapter 7, we discuss the notion of uniform ε-representations. The main theorem from the previous chapter is used to prove a stability result in this context.

The thesis also includes a section on further research and a list of equations that are used throughout the thesis and can be used a quick reference for the convenience of the reader.

Notation and conventions

We use the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ for the natural numbers, the integers, the rational numbers, the real numbers and the complex numbers, respectively. The cardinality of a any set S is written $|S|$. We also use $|\cdot|$ to denote the absolute value on $\mathbb{C}$. The imaginary unit is denoted $i \in \mathbb{C}$ and complex conjugation of

$\lambda \in \mathbb{C}$ is denoted $\bar{\lambda}$. The complex unit circle is denoted $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Given a group or ring A, we use 1_A to denote the (multiplicative) identity. In the special case where A is the bounded operators $\mathbf{B}(\mathscr{H})$ on a Hilbert space, we use the notation $1_{\mathscr{H}}$ for the identity and if $\mathscr{H}$ has finite dimension $n \in \mathbb{N}$, we write 1_n instead. We let $\mathbf{M}_n(A)$ to denote the the $n \times n$-matrices with entries in A, for $n \in \mathbb{N}$. We shall write $\mathrm{diag}(a_1, \dots, a_n)$ or for the diagonal matrix in $\mathbf{M}_n(A)$ with entries $a_1, \dots, a_n$ in A. For a non-empty set S with $|S| = n$ we let $\mathbb{F}_S$ or $\mathbb{F}_n$ denote the free group generated by S, and for $R \subseteq \mathbb{F}_S$ we let $\langle\!\langle R \rangle\!\rangle$ denote the normal subgroup generated by R. We define $\langle S \mid R \rangle = \mathbb{F}_S / \langle\!\langle R \rangle\!\rangle$. Depending on the context, we denote the trivial group by 0 or 1. For a prime number $p \in \mathbb{N}$ we let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote the p-adic rationals and integers, respectively. We use $|\cdot|_p$ to denote the p-adic absolute value on $\mathbb{Q}_p$. Throughout the thesis, ω will denote a fixed, free ultrafilter on $\mathbb{N}$, and for a sequence $(x_n)_{n \in \mathbb{N}}$ in a topological space X, we shall write $\lim_{n \to \omega} x_n = x$ or $x_n \to x$ for $n \to \omega$ to indicate that x_n converges to some x in X along ω. We also adopt the Landau notation to this setting, $O(x_n)$ and $o(x_n)$, for $n \to \omega$, as explained in Chapter 3. If $A \subseteq B$ is an inclusion of sets, we let $\chi_A \colon B \to \{0, 1\}$ denote the characteristic function on A, which takes the value 1 on A and 0 on $B \backslash A$. For a normed space V over $\mathbb{C}$, we let V^* be the dual space consisting of bounded linear maps $\varphi \colon V \to \mathbb{C}$. Given a set Γ and $p \in [1, \infty)$ we let $\ell^p(\Gamma)$ be the space of p-summable complex-valued functions $f \colon \Gamma \to \mathbb{C}$, and, similarly, we let $\ell^\infty(\Gamma)$ denote the space of bounded, complex-valued functions on Γ. Given $\Phi \in (\ell^\infty(\Gamma))^*$ we shall write $\Phi_x f(x)$ for $\Phi(f)$, where it is understood that x is an element of Γ.

We want to emphasize some standing assumptions and conventions that we follow throughout this thesis: The natural numbers *do not* contain zero, that is $\mathbb{N} = \{1, 2, \dots\}$. The word countable means finite or countably infinite. All Hilbert spaces are assumed to be separable, that is, they have a countable basis. We allow vector space norms to attain the value ∞. By a metric group, we mean a group G equipped with a bi-invariant metric d. Being bi-invariant means that for all $g, h, k \in G$, it holds that $d(g, h) = d(kg, kh) = d(gk, hk)$. In the construction of the ultraproduct of metric groups in Chapter 3, we do not require sequences to be bounded. Given a statement $P(n)$, for $n \in \mathbb{N}$, we use the phrasing $P(n)$ *holds for most* $n \in \mathbb{N}$ to signify $\{n \in \mathbb{N} \mid P(n)\} \in \omega$, where ω is the fixed ultrafilter from above.

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Part I

Preliminary theory

Chapter 1

Groups

In this preliminary chapter we review the group theory that we need in the thesis. The topics are quite diverse ranging from the theory of discrete, amenable groups to the unitary group of a quaternion algebra and group cohomology. We do not try to give a complete picture of any of these topics but we introduce the necessary notions and fundamental theorems so to make the thesis more or less self-contained. In the beginning of every section we provide general references to the topics.

1.1 Amenable groups

Amenability of discrete groups is an important notion with a wide range of applications. It has its roots in measure theory and the definition was given in 1929 by von Neumann [48]. We recall this definition and provide some basic examples of amenable groups. For a thorough treatment of amenability, we refer the reader to [42], or [16, Chapter 4] for a somewhat shorter but very good introduction to the subject. For a group Γ we let $\ell^\infty(\Gamma)$ denote the space of complex-valued functions $\varphi\colon \Gamma \to \mathbb{C}$ that are *bounded*, that is,

$$\|\varphi\|_{\ell^\infty(\Gamma)} = \sup_{x\in\Gamma}|\varphi(x)| < \infty,$$

and let $1_{\ell^\infty(\Gamma)}$ denote the constant function 1 on Γ. For a linear map $\Phi\colon \ell^\infty(\Gamma) \to \mathbb{C}$, we shall write $\Phi_x f(x) = \Phi(f)$ whenever convenient. Here it is understood that x runs through elements in Γ.

Definition 1.1. A *mean* on a group Γ is a linear functional $\mathbb{E}\colon \ell^\infty(\Gamma) \to \mathbb{C}$ such that $\mathbb{E}(1_{\ell^\infty(\Gamma)}) = 1$ and $\mathbb{E}(f) \geq 0$ for all $f \in \ell^\infty(\Gamma)$ with $f \geq 0$. A mean $\mathbb{E}$ is called *left-invariant*, if

$$\mathbb{E}_y f(xy) = \mathbb{E}_y f(y), \tag{a}$$

for all $x \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Similarly, $\mathbb{E}$ is *right-invariant*, if $\mathbb{E}_y f(yx) = \mathbb{E}_y f(y)$ for all $x \in \Gamma$. A mean that is both left- and right-invariant is called *bi-invariant*. A mean which satisfies

$$\mathbb{E}_x f(x^{-1}) = \mathbb{E}_x f(x), \tag{b}$$

for all $f \in \ell^\infty(\Gamma)$, is called *symmetric*.

Definition 1.2. A group Γ is *amenable* if there exists a left-invariant mean on Γ.

Proposition 1.3. *A group Γ is amenable if and only if there exists a bi-invariant, symmetric mean on Γ.*

Proof. It is easy to see that $\mathbb{E} \in \ell^\infty(\Gamma)$ is a left-invariant mean if and only if $f \mapsto \mathbb{E}_x f(x^{-1})$, for $f \in \ell^\infty(\Gamma)$, is a right-invariant mean. Furthermore, the mean $f \mapsto \mathbb{E}_x \mathbb{E}_y f(xy^{-1}), f \in \ell^\infty(\Gamma)$ is bi-invariant. Also, if $\mathbb{E}$ is already bi-invariant, then

$$f \mapsto \tfrac{1}{2}\mathbb{E}_x(f(x) + f(x^{-1}))$$

is symmetric and bi-invariant, which concludes the proof. □

For the most part, we are content with left-invariance, but as right-invariance and symmetry might come in handy, we usually work with means enjoying all these properties, and the above proposition ensures that we lose no generality in the context of amenable groups.

Amenable groups have many interesting properties, but an invariant mean is in itself a powerful tool which we shall use a lot. In fact we shall not delve too deep into the theory of amenable groups. We shall, however, provide some basic examples of amenable groups.

Example 1.4. Any finite group Γ is amenable. The functional $\mathbb{E}\colon \ell^\infty(\Gamma) \to \mathbb{C}$ given by

$$\mathbb{E}(f) = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x),$$

is a bi-invariant symmetric mean. In fact, it is the only left-invariant mean on Γ. Observe that in this case it holds that

$$\mathbb{E}_x \mathbb{E}_y F(x, y) = \mathbb{E}_y \mathbb{E}_x F(x, y),$$

for all $F \in \ell^\infty(\Gamma \times \Gamma)$. As we shall see, this is *not* the case in general and this fact turns out to pose a technical issue when stating the main results in Chapters 6 and 7.

Example 1.5. The additive group $\mathbb{Z}$ is amenable. Indeed, we can define a symmetric, bi-invariant mean $\mu\colon \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ as a limit along the ultrafilter ω (see Chapter 3):

$$\mu_n(f) = \lim_{n\to\omega} \frac{1}{2n+1} \sum_{x=-n}^{n} f(x),$$

for $f \in \ell^\infty(\mathbb{Z})$.

The technical issue mentioned in the first example is already apparent for $\mathbb{Z}$. To see this, first note that if $f \in \ell^\infty(\mathbb{Z})$ is finitely supported, i.e. $f(x) = 0$ for all but finitely many $x \in \mathbb{Z}$, then $\mathbb{E}(f) = 0$ for all left-invariant means $\mathbb{E}$ on $\mathbb{Z}$. Consider the function $F\colon \mathbb{Z}^2 \to \mathbb{R}$ given by

$$F(x,y) = \begin{cases} 1, & |x| \le |y|, \\ 0, & |x| > |y|. \end{cases}$$

Using the above fact, it is easy to see that we have that $\mathbb{E}_x\mathbb{E}_y F(x,y) = 1$, whereas $\mathbb{E}_y\mathbb{E}_x F(x,y) = 0$ for all left-invariant means on $\mathbb{Z}$. In fact, it is not hard to generalize this to any infinite amenable group.

The class of amenable groups is stable under taking subgroups, quotients, extensions and inductive limits, so from the above examples one can construct plenty of new ones. In particular, all solvable groups are amenable. A basic example of a non-amenable group is the free group on two generators $\mathbb{F}_2$. Thus, also every group containing $\mathbb{F}_2$ is non-amenable.

1.2 Quaternions and their unitary group

We now turn our attention to something completely different. For the main results in Chapter 5.3, we consider the special unitary group over quaternion algebras. In this section, we introduce the relevant notions. For more information on quaternion algebras, we refer the reader to [31]. For a unital, commutative ring A, we let $A\langle x_1, \ldots, x_n\rangle$ denote the free unital A-algebra (of non-commutative polynomials) in the variables $x_1, \ldots, x_n$. Given elements $a_1, \ldots, a_n$ in the algebra $A\langle x_1, \ldots, x_n\rangle$, we let $(a_1, \ldots, a_n) \subseteq A\langle x_1, \ldots, x_n\rangle$ be the ideal generated by the set $\{a_1, \ldots, a_n\}$. We consider the *quaternions over* A, defined by

$$D(A) = A\langle \mathbf{i}, \mathbf{j}, \mathbf{k}\rangle / (\mathbf{i}^2 + 1, \mathbf{j}^2 + 1, \mathbf{k}^2 + 1, \mathbf{ijk} + 1),$$

and we define a map $\tau = \tau_A\colon D(A) \to D(A)$ by $\tau(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}) = x - y\mathbf{i} - z\mathbf{j} - w\mathbf{k}$. This map is an *involution*, that is, an A-module map such that

$\tau^2 = \mathrm{id}_{D(A)}$ and $\tau(\xi\eta) = \tau(\eta)\tau(\xi)$ for all $\xi, \eta \in D(A)$. For $n \in \mathbb{N}$ we can define a non-degenerate, hermitian sequilinear form $h_n \colon D(A)^n \times D(A)^n \to D(A)$ by

$$h((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{j=1}^{n} \tau(y_j)x_j,$$

for $x_j, y_j \in D(A)$ and $j = 1, \ldots, n$. We are mostly interested in the case, where $A = K$. We have the following dichotomy (see [31, Proposition 1.1.7]):

Proposition 1.6. *Let K be a field of characteristic different from 2. Then either*

- *$D(K)$ is a division algebra or*
- *$D(K) \simeq \mathbf{M}_2(K)$.*

If the latter holds, we say that $D(K)$ *splits*. Even if $D(K)$ does not split, the quadratic extension $K' = K(\sqrt{-1})$ of K is a *splitting field* for $D(K)$, which simply means that

$$D(K) \otimes_K K' \simeq \mathbf{M}_2(K').$$

In particular, there is an embedding of K-algebras from $D(K)$ to $\mathbf{M}_2(K')$.

Proposition 1.7. *Let K be a field of characteristic different from 2. Assume that $D(K)$ splits and let $\varphi \colon D(K) \to \mathbf{M}_2(K)$ be any K-algebra isomorphism. Then*

$$\varphi \circ \tau \circ \varphi^{-1}(T) = J_2^{-1} T^t J_2, \qquad T \in \mathbf{M}_2(K),$$

where T^t is the transposed matrix of T and $J_2 = \begin{pmatrix} 0 & 1_K \\ -1_K & 0 \end{pmatrix}$.

Proof. Let $\sigma = \varphi \circ \tau \circ \varphi^{-1}$. The map $T \mapsto \sigma(T^t), T \in \mathbf{M}_2(K)$ is a unital automorphism of $\mathbf{M}_2(K)$ and hence inner, that is, there exists $S \in \mathrm{GL}_2(K)$ such that

$$\sigma(T^t) = STS^{-1},$$

for $T \in \mathbf{M}_2(K)$. In particular, we get that

$$T = \sigma^2(T) = \sigma(ST^tS^{-1}) = (S^tS^{-1})^{-1}TS^tS^{-1},$$

for $T \in \mathbf{M}_2(K)$, which entails that $S^tS^{-1} \in K1_{\mathbf{M}_2(K)}$. In other words $S = xS^t$ for some $x \in K$, but since $S = (S^t)^t = (xS)^t = x^2S$, we get $x = \pm 1$. That is, S is either symmetric or skew-symmetric. It is easy to see that S cannot be symmetric. Indeed, if $S = S^t$, then $\sigma(S) = S$, which entails $S \in K1_{\mathbf{M}_2(K)}$, so $\sigma(T) = T^t$, for $T \in \mathbf{M}_2(K)$. This is not the case, as the transposition map has non-central fixed points, e.g. the diagonal matrix $\operatorname{diag}(1_K, 0)$, whereas the only fixed points of σ are $K1_{\mathbf{M}_2(K)}$. Thus S is skew-symmetric, that is

$$S = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} = xJ_2^{-1}$$

for some $x \in K\backslash\{0\}$, and the result follows. □

For any odd prime number $p \in \mathbb{N}$, we have that $D(\mathbb{Q}_p)$ splits. Indeed, by Proposition 1.6 we need only to show that $D(\mathbb{Q}_p)$ contains zero-divisors. A basic counting argument shows that the equation

$$x^2 + y^2 = -1$$

is solvable modulo p, which implies, by Hensel's lemma, that it is solvable in $\mathbb{Q}_p$ as well. We take $x, y \in \mathbb{Q}_p$ satisfying the equation. The element $\xi = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$ is then a zero divisor since $\tau(\xi)\xi = h_1(\xi) = 0$. By the way, an explicit isomorphism $D(\mathbb{Q}_p) \to \mathbf{M}_2(\mathbb{Q}_p)$ is given by

$$\mathbf{i} \mapsto \begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \qquad \mathbf{j} \mapsto \begin{pmatrix} -y & x \\ x & y \end{pmatrix}, \qquad \mathbf{k} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $x, y \in \mathbb{Q}_p$ satisfy $x^2 + y^2 = -1$ as above. In contrast, the algebra $D(\mathbb{R})$ is a division algebra, which is usually called the Hamiltonian quaternions and sometimes denoted $\mathbb{H}$. Indeed, one checks that h_1 is positive definite, that is, $h_1(\xi, \xi) > 0$ for all $\xi \in D(\mathbb{R})\backslash\{0\}$ so $\frac{\tau(\xi)}{h_1(\xi)}$ is an inverse to ξ.

Now fix $n \in \mathbb{N}$ and let K be a field of characteristic zero. Let $K' \supseteq K$ be any field such that $D(K) \otimes_K K' \simeq \mathbf{M}_2(K')$ and define the *determinant* of an element of $\mathbf{M}_n(D(K)) \simeq D(K) \otimes_K \mathbf{M}_n(K)$ as the determinant of the corresponding element in $\mathbf{M}_{2n}(K') \simeq \mathbf{M}_2(K') \otimes_{K'} \mathbf{M}_n(K')$. It can be shown that this definition is independent of the splitting field K' [31, Section 2.6]. Thus we can define

$$\mathbf{G}(K) = \mathbf{SU}_n(D(K), h_n),$$

the group of $n \times n$-matrices with entries in $D(K)$ of determinant 1 such that the form h_n is preserved. Let $p \in \mathbb{N}$ be an odd prime number. We can use the discussion above to describe $\mathbf{G}(\mathbb{Q}_p)$. Since $D(\mathbb{Q}_p) \simeq \mathbf{M}_2(\mathbb{Q}_p)$, it easily follows from Proposition 1.7 that $\mathbf{G}(\mathbb{Q}_p)$ is the *symplectic group*. That is

$$\mathbf{G}(\mathbb{Q}_p) \simeq \mathbf{Sp}_{2n}(\mathbb{Q}_p) = \{T \in \mathbf{M}_{2n}(\mathbb{Q}_p) \mid T^t J_{2n} T = J_{2n}\},$$

where, $J_{2n} \in \mathbf{M}_{2n}(\mathbb{Q}_p)$ is the block matrix

$$J_{2n} = \begin{pmatrix} 0 & 1_{\mathbf{M}_n(\mathbb{Q}_p)} \\ -1_{\mathbf{M}_n(\mathbb{Q}_p)} & 0 \end{pmatrix}.$$

Now, we consider $\mathbf{G}(\mathbb{R})$. As an $\mathbb{R}$-vector space, we can identify $D(\mathbb{R})^n$ and $\mathbb{C}^n \oplus \mathbb{C}^n$. Concretely, we identify the entries using the map $D(\mathbb{R}) \to \mathbb{C} \oplus \mathbb{C}$, given by

$$1_{D(K)} \mapsto (1,0), \quad \mathbf{i} \mapsto (i,0), \qquad \mathbf{j} \mapsto (0,1), \qquad \mathbf{k} \mapsto (0,i).$$

It is now straightforward to check that, under this identification, h_n splits into a hermitian form $h_{n1}\colon \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to \mathbb{C}$ and a non-degenerate symplectic form $h_{n2}\colon \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to \mathbb{C}$ in the sense that

$$h_n(\xi,\eta) = (h_{n1}(\xi,\eta), h_{n2}(\xi,\eta)) \in \mathbb{C} \oplus \mathbb{C},$$

for all $\xi, \eta \in \mathbb{C}^n \oplus \mathbb{C}^n$. It follows that $\mathbf{G}(\mathbb{R})$ can be identified with the matrices in $\mathbf{M}_2(\mathbb{C})$ that preserve both a Hermitian and a symplectic form. In other words, $\mathbf{G}(\mathbb{R}) \simeq \mathbf{U}(2n) \cap \mathbf{Sp}_{2n}(\mathbb{C})$, where $\mathbf{U}(2n)$ denotes the unitary group of complex $2n \times 2n$-matrices.

1.3 Group cohomology

In Chapter 4, we shall associate an element of the second cohomology group to a local asymptotic representation. We recall a fairly concrete construction of group cohomology that will suffice for our needs. For more information on the topic, we refer to [11]. Let Γ be any group. A Γ-*module* is an abelian group V equipped with a left action π of Γ. For the definition of cohomology we consider both Γ and V as abstract groups without any topology. Consider the chain complex

$$0 \longrightarrow V \xrightarrow{d^0} C^1(\Gamma,V) \xrightarrow{d^1} C^2(\Gamma,V) \xrightarrow{d^2} C^3(\Gamma,V) \xrightarrow{d^3} \cdots,$$

where the n-*cochains* $C^n(\Gamma,V)$, for $n \in \mathbb{N}$, is the space of functions from Γ^n to V and $d^0(v)(x_1) = \pi(x)v - v$ for $v \in V$ and $x \in \Gamma$, and

$$\begin{aligned} d^n(f)(x_1,\dots,x_{n+1}) = {} & \pi(x_1)f(x_2,\dots,x_{n+1}) \\ & + \sum_{j=1}^{n} (-1)^j f(x_1,\dots,x_j x_{j+1},\dots,x_{n+1}) \\ & + (-1)^{n+1} f(x_1,\dots,x_n), \end{aligned}$$

for $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_{n+1} \in \Gamma$. The image of d^{n-1}, denoted $B^n(\Gamma,V)$, is called the n-*coboundaries*, is contained in the kernel of d^n, denoted $Z^n(\Gamma,V)$, called the n-*cocycles*, for all $n \geq 0$, so we can compute the quotient groups, denoted $H^n(\Gamma, V)$, which we call the .cohomology of Γ with coefficients in V. More precisely, $H^0(\Gamma, V) = Z^0(\Gamma, V)$ and

$$H^n(\Gamma, V) = Z^n(\Gamma, V)/B^n(\Gamma, V),$$

for $n \in \mathbb{N}$. Given an n-cocycle c in $Z^n(\Gamma, V)$ we write $[c]$ for the corresponding cohomology class in $H^n(\Gamma, V)$. As we shall focus on the second cohomology, let us spell out what the above means in this case. Cocycles $c \in Z^2(\Gamma, V)$ are functions $c\colon \Gamma \times \Gamma \to V$ such that

$$\pi(x)c(y,z) - c(xy,z) + c(x,yz) - c(x,y) = 0,$$

for all $x, y, z \in \Gamma$, and $c \in B^2(\Gamma, V)$ if and only if there exists $b\colon \Gamma \to V$ with

$$c(x,y) = (d^1 b)(x,y) = \pi(x)b(y) - b(xy) + b(y),$$

for all $x, y \in \Gamma$.

Cohomology vanishing for Hilbert space coefficients

We now consider the special case where $V = \mathscr{H}$ is a Hilbert space and Γ is a (countable, discrete) group that acts on $\mathscr{H}$ through a unitary representation π. In this setting, we want to examine groups for which the n'th cohomology groups vanish.

Definition 1.8. Let $n \in \mathbb{N}$. A group Γ is called *n-Kazhdan* if $H^n(\Gamma, \mathscr{H}_\pi) = 0$ for all Hilbert spaces $\mathscr{H}_\pi$ and unitary representations $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H}_\pi)$. We call Γ *strongly n-Kazhdan* if Γ is k-Kazhdan for all $k \in \{1, 2, \dots, n\}$.

We recall that a finitely generated group has Kazhdan's Property (T) if and only if it is 1-Kazhdan (see [8] for more information on the subject). Thus, being n-Kazhdan can be thought of as a higher dimensional version of Property (T). The study of higher dimensional cohomology vanishing has been carried out in many contexts, and we refer the interested reader to [4, 5, 10, 23, 24, 50, 52].

Example 1.9. Finite groups are n-Kazhdan for any $n \in \mathbb{N}$. Indeed, if Γ is a finite group and $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ is a unitary representation, then for every $c \in Z^n(\Gamma, \mathscr{H})$ we can define $b \in C^{n-1}(\Gamma, \mathscr{H})$ by

$$b(x_1, \dots, x_{n-1}) = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} c(x_1, \dots, x_{n-1}, x), \qquad x_1, \dots, x_{n-1} \in \Gamma.$$

It is now straightforward to check that $d^{n-1} b = c$, which implies $H^n(\Gamma, \mathscr{H}) = 0$.

Let us examine some properties of n-Kazhdan groups. We note that if Γ is countable, then $C^n(\Gamma, \mathscr{H})$ has some extra structure; it is a Fréchet space. Indeed, the countable family of seminorms defined by

$$\|f\|_F = \max_{x \in F} \|f(x)\|_{\mathscr{H}},$$

for $f \in C^n(\Gamma, \mathscr{H})$ and finite $F \subseteq \Gamma^n$, is clearly separating, and since $\|\cdot\|_{\mathscr{H}}$ is complete, the metric associated to the above family of seminorms is complete as well. We recall that between Fréchet spaces a version of the open mapping theorem applies (see e.g. [61]).

Proposition 1.10. *Let $n \in \mathbb{N}$ and let Γ be a countable n-Kazhdan group. Then for every finite set $F \subseteq \Gamma^{n-1}$ there are a finite set $F_0 \subseteq \Gamma^n$ and a constant $K > 0$ such that for all unitary representations $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ and all cocycles $c \in Z^n(\Gamma, \mathscr{H})$ there is an element $b \in C^{n-1}(\Gamma, \mathscr{H})$ such that $c = d^{n-1}b$ and $\|b\|_F \leq K\|c\|_{F_0}$.*

Proof. Throughout the proof, we fix $F \subseteq \Gamma^{n-1}$. We call a triple (π, F_0, K) *good* if it satisfies the conclusion of the proposition we are about to prove and *bad* otherwise. We first prove that for every fixed π, there exist F_0 and K such that (π, F, K) is good. By definition of the topology on $C^n(\Gamma, \mathscr{H})$, the basic open sets are given by

$$U_{\delta,F'} = \{f \in C^n(\Gamma, \mathscr{H}) \mid \|f\|_{F'} < \delta\},$$

for a finite $F' \subseteq \Gamma^n$ and $\delta > 0$. By assumption, the map $d^{n-1}\colon C^{n-1}(\Gamma, \mathscr{H}) \to Z^n(\Gamma, \mathscr{H})$ is surjective and since π is unitary, it follows that d^{n-1} is bounded as well, so by the open mapping theorem there are $K \geq 0$ and $F_0 \subseteq \Gamma^n$, such that

$$U_{K^{-1},F_0} \cap Z^n(\Gamma, \mathscr{H}) \subseteq d^{n-1}(U_{1,F}).$$

By rescaling by $K\|c\|_{F_0}$ on both sides, this can be rephrased as

$$U_{\|c\|_{F_0},F_0} \cap Z^n(\Gamma, \mathscr{H}) \subseteq d^{n-1}(U_{K_\pi\|c\|_{F_0},F}),$$

for $c \in Z^n(\Gamma, \mathscr{H})$ and it follows that (π, F_0, K) is good.

We now want to prove the existence of F_0 and K such that (π, F_0, K) is good for all π simultaneously. Note that if (π_1, F_1, K_1) is good, then so is (π_2, F_2, K_2) for all subrepresentations π_2 of π_1, $F_2 \supseteq F_1$ and $K_2 \geq K_1$. Assume for contradiction that there exist bad triples (π_j, F_j, K_j), for $j \in \mathbb{N}$, such that $F_1 \subseteq F_2 \subseteq \cdots \subseteq \bigcup_{j=1}^{\infty} F_j = \Gamma^n$ and $K_j \to \infty$ for $j \to \infty$. Consider the direct sum $\pi' = \bigoplus_{j \in \mathbb{N}} \pi_j$ which is a unitary representation on the Hilbert space direct sum. We just saw that π' is part of a good triple, say (π', F', K'), but this contradicts the fact that π_j is a subrepresentation of π' for all $j \in \mathbb{N}$ and that $F_j \supseteq F'$ and $K_j \geq K$ for most $j \in \mathbb{N}$. □

We need an extension theorem for n-Kazhdan groups, which follows immediately from the Hochschild-Serre spectral sequence in group cohomology. For convenience, we extracted the content of the spectral sequence that we need in the following proposition. To make sense of this proposition, we recall that if $1 \to \Lambda \to \tilde{\Gamma} \to \Gamma \to 1$ is a short exact sequence of groups and V is a $\tilde{\Gamma}$-module, then there is a natural action of Γ on $H^k(\Lambda, V)$, for $k \in \mathbb{N}$, which is induced by the conjugation action of $\tilde{\Gamma}$ on Λ.

Proposition 1.11. *Let $1 \to \Lambda \to \tilde{\Gamma} \to \Gamma \to 1$ be a short exact sequence of group and let V be a Γ-module. Then there exist subgroups $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = H^n(\tilde{\Gamma}, V)$ such that*

$$H^k(\Gamma, H^{n-k}(\Lambda, V)) = M_k / M_{k-1}$$

for $k = 0, \ldots, n$.

We shall not prove the above here; an exposition of spectral sequences can be found in [11, Chapter VII].

Theorem 1.12. *Consider a short exact sequence of groups $1 \to \Lambda \to \tilde{\Gamma} \to \Gamma \to 1$. If Λ is strongly n-Kazhdan and Γ is n-Kazhdan, then $\tilde{\Gamma}$ is also n-Kazhdan. In particular, this applies if Λ is finite.*

Proof. By Proposition 1.11 it is enough to show $H^k(\Gamma, H^{n-k}(\Lambda, \mathscr{H})) = 0$ for $k = 0, \ldots n-1$. If $k < n$, then $H^{n-k}(\Lambda, \mathscr{H}) = 0$ by assumption. For $k = n$, we have that $H^0(\Lambda, \mathscr{H})$ is the closed subspace of $\pi|_\Lambda$-fixed points in $\mathscr{H}$ and the Γ-action is simply induced from the unitary action of $\tilde{\Gamma}$ on $\mathscr{H}$, so we conclude that $H^n(\Gamma, H^0(\Lambda, \mathscr{H})) = 0$. □

Remark 1.13. For simplicity, we have chosen to work with Hilbert space coefficients. We note that most statements are true in the more general setting where $V = \mathscr{H}$ is a Banach space and π is a linear isometric action of Γ on V. The only place where one has to be careful is in Proposition 1.10, since it is a statement about the whole class of actions on Hilbert spaces, but the corollary still holds true if we consider any class of Banach spaces that allow for taking direct sums of isometric actions. For instance, it holds for the class of C^*-algebras.

Chapter 2

Von Neumann algebras

We also need some preliminaries about operator algebras, especially the basic theory of von Neumann algebras. We provide the definitions we need and cite some fundamental results. Since unitarily invariant norms are central to the thesis, we devote an entire section to this topic. We end the chapter with a section about positive definite functions, where the notion of an amenable group from last chapter will come into play. As in the previous chapter, we do not aim to treat the topic exhaustively and many theorems are given without proofs. For more information on the theory of von Neumann algebras, we refer the reader to [53], [65] and [12].

2.1 Definitions and fundamental results

Fix a separable Hilbert space $\mathscr{H}$, that is $\mathscr{H} \simeq \ell^2(\mathbb{N})$ or $\mathscr{H} \simeq \mathbb{C}^n$ for some $n \in \mathbb{N}$. We denote the inner product of $\mathscr{H}$ by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|_{\mathscr{H}}$. In the context of operator algebras, we always work with the complex numbers $\mathbb{C}$ as base field. We let $\mathbf{B}(\mathscr{H})$ denote the bounded linear operators on $\mathscr{H}$, that is, linear maps $T \colon \mathscr{H} \to \mathscr{H}$ so that

$$\|T\|_{\mathrm{op}} = \sup\{\|T\xi\|_{\mathscr{H}} \mid \xi \in \mathscr{H}, \|\xi\|_{\mathscr{H}} = 1\} < \infty.$$

Given $T \in \mathbf{B}(\mathscr{H})$ we denote the *adjoint operator* of T by T^*. This is the unique linear operator on $\mathscr{H}$ satisfying $\langle T^*\xi, \eta \rangle = \langle \xi, T\eta \rangle$ for $\xi, \eta \in \mathscr{H}$. Note that $T \mapsto T^*$ is an involution, i.e. $(T^*)^* = T$ and $(TS)^* = S^*T^*$. We call $T \in \mathbf{B}(\mathscr{H})$

- *normal*, if $T^*T = TT^*$,
- *self-adjoint*, if $T = T^*$,

- *skew-Hermitian*, if $T = -T^*$,
- *postive*, if there is $S \in \mathbf{B}(\mathscr{H})$ such that $T = S^*S$,
- a *projection*, if $T = T^* = T^2$,
- a *partial isometry*, if T^*T (or, equivalently TT^*) is a projection,
- an *isometry*, if $T^*T = 1_{\mathcal{H}}$,
- a *co-isometry*, if T^* is an isometry and
- *unitary*, if $T^*T = TT^* = 1_{\mathscr{H}}$.

We let $\sigma(T) \subseteq \mathbb{C}$ denote the spectrum of T, that is, $\lambda \in \sigma(T)$ if and only if $T - \lambda 1_{\mathscr{H}}$ is not invertible in $\mathbf{B}(\mathscr{H})$. Recall that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$ and if T is normal, then T is self-adjoint if and only if $\sigma(T)$ is contained in $\mathbb{R}$, T is positive if and only if $\sigma(T)$ is contained in $[0, \infty)$, and T is unitary if and only if $\sigma(T)$ is contained in $\mathbb{T}$. Recall that the *operator norm* $\|\cdot\|_{\mathrm{op}}$ defined above turns $\mathbf{B}(\mathscr{H})$ into a Banach $*$-algebra. Furthermore, recall the *C^*-identity* which we will use again and again: $\|T^*T\|_{\mathrm{op}} = \|T\|_{\mathrm{op}}^2$. Also, recall that an operator T is positive if and only if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathscr{H}$. For self-adjoint $T, S \in \mathbf{B}(\mathscr{H})$ we write $T \leq S$ if $S - T$ is positve. This is a partial order on the set of self-adjoint operators. We also recall that if $T \leq S$ and $R \in \mathbf{B}(\mathscr{H})$ then $RTR^* \leq RSR^*$ and $\|T\|_{\mathrm{op}} \leq \|S\|_{\mathrm{op}}$. The last claim follows from he fact that if $T = A^*A$, $S = B^*B$ and $\xi \in \mathscr{H}$, then

$$\|A\xi\|_{\mathscr{H}}^2 = \langle T\xi, \xi\rangle \leq \langle S\xi, \xi\rangle = \|B\xi\|_{\mathscr{H}}^2,$$

so $\|A\|_{\mathrm{op}} \leq \|B\|_{\mathrm{op}}$ and, using the C^*-identity, we get

$$\|T\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}^2 \leq \|B\|_{\mathrm{op}}^2 = \|S\|_{\mathrm{op}}.$$

Definition 2.1. The *weak operator topology* is the topology on $\mathbf{B}(\mathscr{H})$ generated by the seminorms

$$T \mapsto |\langle T\xi, \eta\rangle|, \qquad \xi, \eta \in \mathscr{H}, T \in \mathbf{B}(\mathscr{H}).$$

The *strong operator topology* is the topology on $\mathbf{B}(\mathscr{H})$ generated by the seminorms

$$T \mapsto \|T\xi\|_{\mathscr{H}}, \qquad \xi \in \mathscr{H}, T \in \mathbf{B}(\mathscr{H}).$$

The weak operator topology is weaker than the strong operator topology which is weaker than the topology induced by $\|\cdot\|_{\mathrm{op}}$, and the topologies are equivalent if and only if $\mathscr{H}$ is finite dimensional. A feature of the weak operator topology is that the unit ball of $\mathbf{B}(\mathscr{H})$ is compact in this topology. A reason to work with the strong operator topology is that if $(T_j)_{j\in J}$ is an operator norm bounded, increasing net (with respect to the abovementioned order), it has a least upper bound T and $(T_j)_{j\in J}$ converges to T in the strong operator topology.

Definition 2.2. A *C^*-algebra* is a $*$-subalgebra $\mathcal{A} \subseteq \mathbf{B}(\mathscr{H})$ which is closed in the topology coming from $\|\cdot\|_{\mathrm{op}}$. A *von Neumann algebra* is a $*$-subalgebra $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$, such that $1_{\mathscr{H}} \in \mathcal{M}$, which is closed in the strong operator topology

Since the strong operator topology is weaker than the norm topology, every von Neumann algebra is a C^*-algebra. Let $\mathcal{A}$ be a C^*-algebra. We let $\mathcal{A}_+$ be the set of positive elements, let $\mathcal{A}^*$ denote the dual vector space of bounded linear functionals and we let $\mathrm{U}(\mathcal{A})$ denote the unitary group of $\mathcal{A}$, that is, the group of unitary elements. In the special case $\mathcal{A} = \mathbf{B}(\mathscr{H})$ we write $\mathbf{U}(\mathscr{H})$ instead and if $\mathscr{H} = \mathbb{C}^n$ we simply write $\mathbf{U}(n)$. A fundamental fact about von Neumann algebras, which goes back to the eponymous von Neumann, is the *Bicommutant Theorem*. For a subset $\mathcal{S} \subseteq \mathbf{B}(\mathscr{H})$ we denote the *commutant* of S by $\mathcal{S}' = \{T \in \mathbf{B}(\mathscr{H}) \mid ST = TS, \text{ for all } S \in \mathcal{S}\}$ and $\mathcal{S}'' = (\mathcal{S}')'$.

Theorem 2.3. *Let $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ be a unital $*$-subalgebra. The following are equivalent:*

- *$\mathcal{M}$ is a von Neumann algebra,*
- *$\mathcal{M}$ is closed in the weak operator topology and*
- *$\mathcal{M} = \mathcal{M}''$.*

Given elements S and T in a von Neumann algebra $\mathcal{M}$, we define

$$S\mathcal{M}T = \{SRT \in \mathcal{M} \mid R \in \mathcal{M}\}.$$

If $P \in \mathcal{M}$ is a projection, then the *corner* $P\mathcal{M}P$ is again a von Neumann algebra when viewed as a subalgebra of $\mathbf{B}(P\mathscr{H})$. Let $\mathscr{K}$ be another (separable) Hilbert space. Given two von Neumann algebras $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ and $\mathcal{N} \subseteq \mathbf{B}(\mathscr{K})$ we define the *von Neumann algebra tensor product*

$$\mathcal{M}\bar{\otimes}\mathcal{N} \subseteq \mathbf{B}(\mathscr{H} \otimes \mathscr{K})$$

as the von Neumann algebra generated by the operators $T \otimes S \in \mathbf{B}(\mathscr{H} \otimes \mathscr{K})$ for $T \in \mathcal{M}$ and $S \in \mathcal{N}$. We define $\mathcal{M}_\infty = \mathcal{M}\bar{\otimes}\mathbf{B}(\ell^2(\mathbb{N}))$, and view $\mathcal{M}$ as a corner of $\mathcal{M}_\infty$. More precisely, we implicitly fix a rank 1-projection $E \in \mathbf{B}(\ell^2(\mathbb{N}))$ and identify

$$\mathcal{M} \simeq (1_\mathcal{M} \otimes E)\mathcal{M}_\infty(1_\mathcal{M} \otimes E),$$

where $1_\mathcal{M}$ is the unit of $\mathcal{M}$. Consistent with this identification, we write $1_\mathcal{M}$ instead of $1_\mathcal{M} \otimes E$.

Definition 2.4. Let $\mathcal{A}$ be a C^*-algebra. A linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ is called *positive* if $\varphi(T) \geq 0$ for all $T \in \mathcal{A}$ with $T \geq 0$. A positive linear functional φ is called a *state* if $\varphi(1_\mathcal{A}) = 1$.

We recall the Cauchy-Schwarz inequality $|\varphi(S^*T)|^2 \leq \varphi(S^*S)\varphi(T^*T)$, for $S, T \in \mathcal{A}$ which is valid for all positive linear functionals $\varphi\colon \mathcal{A} \to \mathbb{C}$.

Definition 2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. A linear map $\varphi\colon \mathcal{M} \to \mathcal{N}$ is called *normal* if for all operator norm bounded, increasing nets $(T_j)_{j\in J} \subseteq \mathcal{M}$ of self-adjoint operators with least upper bound T, we have that $\varphi(T_j)$ converges and

$$\lim_{j\to\infty} \varphi(T_j) = \varphi(T).$$

For a von Neumann algebra $\mathcal{M}$, we let $\mathcal{M}_* \subseteq \mathcal{M}^*$ denote the the subspace of normal functionals on $\mathcal{M}$.

Let us mention a small caveat: Not all representations $\mathcal{M} \to \mathbf{B}(\mathscr{H})$ are normal! However, we defined $\mathcal{M}$ as a subalgebra of $\mathbf{B}(\mathscr{H})$ and fortunately the inclusion map is a normal representation. Henceforth, whenever we write $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ we tacitly assume that the inclusion map is normal. The following fundamental theorem is due to Sakai [62].

Theorem 2.6. *The space $\mathcal{M}_*$ is a predual of $\mathcal{M}$ in the sense that the map $\Phi\colon \mathcal{M} \to (\mathcal{M}_*)^*$ given by*

$$\Phi(T)(\xi) = \xi(T),$$

for $T \in \mathcal{M}, \xi \in \mathcal{M}_*$, is an isometric isomorphism. Furthermore the predual of $\mathcal{M}$ is unique up to isometric isomorphism.

Definition 2.7. The weak*-topology on $\mathcal{M}$ coming from the unique predual is called *the ultraweak topology*.

The ultraweak toplogy is stronger than the weak topology and weaker than the strong topology. We recall that in the ultraweak topology the unit ball of $\mathcal{M}$ is compact (just as for the weak topology). Now, we let recall the Borel functional calculus. For a subset $X \subseteq \mathbb{C}$ let $C(X)$ denote the $*$-algebra of complex-valued, continuous functions on X and let $\mathcal{B}_\infty(X)$ denote the algebra of complex-valued, bounded Borel measurable functions on X. Given $T \in \mathbf{B}(\mathscr{H})$ we let $C^*(T)$ denote the C^*-algebra generated by T and $1_{\mathscr{H}}$, and $\mathrm{vNalg}(T)$ the von Neumann algebra generated by T (which, by definition, contains $1_{\mathscr{H}}$). Also, recall that χ_X

Theorem 2.8 (Borel functional calculus)**.** *Let $T \in \mathbf{B}(\mathscr{H})$ be a normal operator. There is a unique $*$-isomorphism $C(\sigma(T)) \to C^*(T)$ such that $\mathrm{id}_{\sigma(T)}(T) = T$ and $\chi_{\sigma(T)}(T) = 1_{\mathscr{H}}$, which extends to a $*$-homomorphism $\mathcal{B}_\infty(\sigma(T)) \to \mathrm{vNalg}(T)$. For $f \in \mathcal{B}_\infty(\sigma(T))$, we write $f(T)$ for the image of f under this $*$-homomorphism. If $(f_j)_{j\in J}$ is a bounded, increasing net of real-valued Borel measurable functions with $\lim_{j\to\infty} f_j(x) = f(x)$ for (almost) all $x \in X$, then $f(T) = \lim_{j\to\infty} f_j(T)$, where the limit is taken in the strong operator topology.*

We also recall the polar decomposition:

Proposition 2.9. *Let $T \in \mathbf{B}(\mathscr{H})$ and let $|T| = \sqrt{T^*T} \in C^*(T)$ by the continuous functional calculus. Then there exists a partial isometry $U \in \mathrm{vNalg}(T)$ such that*

$$T = U|T|, \qquad |T| = U^*T.$$

If $T \in P\mathcal{M}Q$ for some projections $P, Q \in \mathcal{M}$, then we can assume that $U \in P\mathcal{M}Q$ as well.

As an illustration of how to utilize the above proposition, we recall the proof of the following basic lemmata, which we shall need later.

Lemma 2.10. *Let $T = U|T|$ be the polar decomposition of $T \in \mathbf{B}(\mathscr{H})$ and let $P \in \mathrm{vNalg}(|T|)$ be a projection. Then UP is a partial isometry.*

Proof. Since $U^*U|T| = |T| = |T|^* = |T|U^*U$ we have that $PU^*U = U^*UP$ by the Bicommutant Theorem. It follows that $(UP)^*UP = PU^*UP$ is a projection, so UP is a partial isometry by definition. □

Lemma 2.11. *Let $\mathcal{M}$ be a von Neumann algebra, let $P, Q \in \mathcal{M}$ be projections and let $S \in P\mathcal{M}Q$ with $\|S\|_{\mathrm{op}} \leq 1$. Then there are partial isometries $V_1, V_2 \in P\mathcal{M}Q$ such that*

$$S = \frac{1}{2}(V_1 + V_2).$$

Proof. We write $S = U|S|$ where $U \in P\mathcal{M}Q$ is a partial isometry. It follows by our assumptions that $|S|^2 \leq Q$ so we can define $V_\pm = |S| \pm i\sqrt{Q - |S|^2}$ using functional calculus. One checks that $V_\pm$ are unitaries in $Q\mathcal{M}Q$ and

$$S = \frac{1}{2}(UV_+ + UV_-).$$

It is now easy to check that $V_1 = UV_+$ and $V_2 = UV_-$ are partial isometries with the desired properties. $\square$

We need a strengthening of the Russo-Dye Theorem. The following statement was proved in [43] (and later elaborated on in [39]).

Theorem 2.12. *Let $\mathcal{M}$ be a von Neumann algebra, let $n \in \mathbb{N}, n \geq 3$ and let $T \in \mathcal{M}$ with $\|T\|_{\mathrm{op}} < 1 - \frac{2}{n}$. Then there are unitaries $U_1, \ldots, U_n \in \mathcal{M}$ such that*

$$T = \frac{1}{n}(U_1 + \cdots + U_n).$$

Now it is time to recall the representation theorem for von Neumann algebras, which describes the structure of normal $*$-homomorphisms $\mathcal{M} \to \mathbf{B}(\mathscr{K})$. Recall that we defined $\mathcal{M}_\infty = \mathcal{M}\bar{\otimes}\mathbf{B}(\ell^2(\mathbb{N}))$ for a von Neumann algebra $\mathcal{M}$. For a proof of the following, see e.g. [65, Theorem IV.5.5].

Theorem 2.13. *Let $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ be a von Neumann algebra, and let $\pi\colon \mathcal{M} \to \mathbf{B}(\mathscr{K})$ be a normal representation on a separable Hilbert space $\mathscr{K}$. Then there is an isometry $V\colon \mathscr{K} \to \mathscr{H} \otimes \ell^2(\mathbb{N})$ such that $P = VV^* \in (\mathcal{M} \otimes 1_{\ell^2(\mathbb{N})})'$ and*

$$\pi(T) = V^*(T \otimes 1_{\ell^2(\mathbb{N})})V, \qquad T \in \mathcal{M}.$$

Lastly, we briefly recall the lingo of unbounded traces. Addition and multiplication of positive real numbers is extended to $[0, \infty]$ in the standard way with $x + \infty = \infty$ for all $x \in [0, \infty]$, $\infty \cdot x = \infty$ if $x > 0$ and $\infty \cdot 0 = 0$.

Definition 2.14. A *trace* on a von Neumann algebra $\mathcal{M}$ is a map $\tau\colon \mathcal{M}_+ \to [0, \infty]$ such that $\tau(S + T) = \tau(S) + \tau(T)$, $\tau(\lambda T) = \lambda\tau(T)$ and $\tau(T^*T) = \tau(TT^*)$ for all $S, T \in \mathcal{M}_+$ and $\lambda \in [0, \infty)$.

A trace τ is called *faithful* if $\tau(T) > 0$ for all $T \gneq 0$, τ is called *semi-finite* if for all $T \in \mathcal{M}_+$ there is $0 \lneq S \leq T$ such that $\tau(S) < \infty$ and τ is called *finite* if $\tau(1_{\mathcal{M}}) < \infty$. The notion of a normal trace is an obvious modification of the definition for functionals: a trace is called *normal* if for all operator norm bounded, increasing nets $(T_j)_{j\in J} \subseteq \mathcal{M}_+$ with least upper bound T, we have that

$$\sup_{j\in J} \tau(T_j) = \tau(T).$$

All traces that we consider are semi-finite and normal. We provide the most basic example of a trace.

Example 2.15. We define the trace $\mathrm{Tr}\colon \mathbf{B}(\mathscr{H})_+ \to \mathbb{C}$ by

$$\mathrm{Tr}(T) = \sum_{j \in J} \langle T\xi_j, \xi_j \rangle,$$

for $T \in \mathcal{M}_+$ and some countable orthonormal basis $(\xi_j)_{j\in J}$ of $\mathscr{H}$. This definition is independent of the choice of basis. The trace Tr is normal, faithful and semi-finite and, up to multiplication with a positive scalar, this is the only trace on $\mathbf{B}(\mathscr{H})$. This example works in particular if $\mathscr{H} \simeq \mathbb{C}^n$ is finite dimensional, where $\mathrm{Tr}_n = \mathrm{Tr}$ is a finite trace. We might find use of the *normalized trace on* $\mathbf{M}_n(\mathbb{C})$ which is defined as $\mathrm{tr}_n = \mathrm{tr} = \frac{1}{n}\mathrm{Tr}$.

2.2 Unitarily invariant norms

Unitarily invariant norms play a major role in all of the thesis. In this section, we provide the basic examples of such norms and state and prove quite a few fundamental facts about them. We already used the word norm in the previous sections as something well-known, but note that in this thesis we allow norms to attain the value ∞ in order to cover the "norms" arising from semi-finite traces on von Neumann algebras.

Definition 2.16. A *norm* on a C^*-algebra $\mathcal{A}$ is a map $\|\cdot\|\colon \mathcal{A} \to [0,\infty]$, such that for all $\lambda \in \mathbb{C}$ and $S, T \in \mathcal{A}$, the following holds: $\|\lambda T\| = |\lambda|\|T\|$,

$$\|S + T\| \leq \|S\| + \|T\|, \tag{c}$$

and $\|T\| = 0$ if and only if $T = 0$. A norm is called *unitarily invariant* if for all $U, V \in \mathrm{U}(\mathcal{A})$ and $T \in \mathcal{A}$, it holds that

$$\|UTV\| = \|T\|. \tag{d}$$

A norm is called *submultiplicative* if $\|ST\| \leq \|S\|\|T\|$ for all $S, T \in \mathcal{A}$. If $\mathcal{A}$ is a von Neumann algebra, the norm $\|\cdot\|$ is called *ultraweakly lower semi-continuous* if the unit ball $\{T \in \mathcal{M} \mid \|T\| \leq 1\}$ is closed in the ultraweak toplogy.

All norms considered in this thesis will be unitarily invariant. Submultiplicativity plays an important role in Part II of this thesis and we shall discuss this property briefly at the end of this section (Proposition 2.23). Lower semi-continuity will be useful in Part III, more precisely, we need this property in Lemma 2.29, which will be used extensively in Part III. Note that this property is trivially satisfied for norms on finite dimensional von Neumann algebras.

Example 2.17. As already mentioned, the *operator norm* on $\mathbf{B}(\mathscr{H})$ is defined as

$$\|T\|_{\mathrm{op}} = \sup\{\|T\xi\|_{\mathscr{H}} \mid \xi \in \mathscr{H}, \|\xi\|_{\mathscr{H}} = 1\},$$

for $T \in \mathbf{B}(\mathscr{H})$. This norm is unitarily invariant, submultiplicative and ultraweakly lower semi-continuous. Note that by definition of $\mathbf{B}(\mathscr{H})$, we have that $\|T\|_{\mathrm{op}} < \infty$ for all $T \in \mathbf{B}(\mathscr{H})$.

Example 2.18. Let $\mathcal{M}$ be a von Neumann algebra with a semifinite faithful trace $\tau\colon \mathcal{M} \to \mathbb{C}$ and let $p \in [1, \infty)$. The *p-norm* is given by

$$\|T\|_{p,\tau} = \tau(|T|^p)^{1/p}, \qquad T \in \mathcal{M}.$$

Since τ is faithful, this is actually a norm (otherwise it is a *seminorm*). By the trace property this norm is unitarily invariant, and if τ is infinite it might attain the value ∞. If τ is assumed to be normal, the norm is ultraweakly lower semi-continuous. Of particular interest to us are the norms coming from the normalized and unnormalized trace on $\mathbf{M}_n(\mathbb{C})$ for $p = 2$. We mention them explicitly and give them names as to be able to distinguish them easily. The norm

$$\|T\|_{2,\mathrm{tr}} = \sqrt{\frac{1}{n}\sum_{i,j=1}^{n} |T_{ij}|^2}, \qquad T = [T_{ij}]_{i,j=1}^{n} \in \mathbf{M}_n(\mathbb{C}),$$

associated to the normalized trace we call *normalized Hilbert-Schmidt norm*, and the norm

$$\|T\|_{2,\mathrm{Tr}} = \sqrt{\sum_{i,j=1}^{n} |T_{ij}|^2}, \qquad T = [T_{ij}]_{i,j=1}^{n} \in \mathbf{M}_n(\mathbb{C}),$$

associated to the unnormalized trace we call the *Frobenius norm*. As it turns out, $\|\cdot\|_{2,\mathrm{Tr}}$ is submultiplicative, whereas $\|\cdot\|_{2,\mathrm{tr}}$ is not. These claims can be proven directly without much of a hassle, but it will also follow from Proposition 2.23.

There are more interesting examples of unitarily invariant, ultraweakly lower semi-continuous norms that we shall not delve into, but it is worth mentioning that there are nice structure results for such norms on von Neumann algebras (see for instance [28, 27]). We want to prove some basic properties of unitarily invariant norms that we need throughout the whole thesis. For this, we need a basic fact about positive operators in von Neumann algebras.

Proposition 2.19. *Let $R, S \in \mathcal{M}$ and assume $0 \leq R \leq S$. Then there exists $T \in \mathcal{M}$ with $\|T\|_{\mathrm{op}} \leq 1$ and $R = S^{1/2}TS^{1/2}$.*

Proof. Let $\mathscr{H}_1$ be the closure of the subspace $S^{1/2}\mathscr{H}$ in $\mathscr{H}$. Since we have that $\|R^{1/2}\xi\|_{\mathrm{op}} \leq \|S^{1/2}\xi\|_{\mathrm{op}}$ for all $\xi \in \mathscr{H}$, the linear map $S^{1/2}\xi \mapsto R^{1/2}\xi$, for $\xi \in \mathscr{H}$, is bounded and thus extends to a operator $A\colon \mathscr{H}_0 \to \mathscr{H}$. We define $A\eta = 0$ for η in the orthogonal complement of $\mathscr{H}_0$. Thus we get an operator $A\colon \mathscr{H} \to \mathscr{H}$. Evidently $\|A\|_{\mathrm{op}} \leq 1$ and $AS^{1/2} = R^{1/2}$, and it follows that $S^{1/2}A^*AS^{1/2} = (R^{1/2})^*R^{1/2} = R$. By the bi-commutant theorem, we have that $A \in \mathcal{M}$. Indeed if $B \in \mathcal{M}'$, then $BAS^{1/2}\xi = BR^{1/2}\xi = R^{1/2}B\xi = AS^{1/2}B\xi = ABS^{1/2}\xi$ for all $\xi \in \mathscr{H}$, so AB and BA agree on $\mathscr{H}_0$. On the other hand, if η is orthogonal to $\mathscr{H}_0$, then so is $B\eta$, so $AB\eta = 0 = BA\eta$. Thus $A \in \mathcal{M}'' = \mathcal{M}$. Letting $T = A^*A$, we reach the desired conclusion. □

The following proposition is used a lot in this thesis, especially in Part III. The equations are contained in the list on page 87 for quick reference.

Proposition 2.20. *Let $\mathcal{M}$ be a von Neumann algebra and let $\|\cdot\|\colon \mathcal{M} \to [0,\infty]$ be a unitarily invariant norm on $\mathcal{M}$. Then, for all $R, S, T \in \mathcal{M}$, we have that*

$$\|RTS\| \leq \|R\|_{\mathrm{op}}\|T\|\|S\|_{\mathrm{op}}, \tag{e}$$

$$\|T\| = \|T^*\| = \||T|\|, \tag{f}$$

$$\|T^*T\| = \|TT^*\|. \tag{g}$$

If, furthermore, $0 \leq R \leq S$, then

$$\|R\| \leq \|S\|. \tag{h}$$

Proof. We begin with the proof of Equation (e). Assume that $\|R\|_{\mathrm{op}}, \|S\|_{\mathrm{op}} < 1$. By Theorem 2.12, for some $n \in \mathbb{N}$ big enough, R and S are convex combinations of n unitaries, $R = \frac{1}{n}\sum_{i=1}^n U_i$ and $S = \frac{1}{n}\sum_{j=1}^n V_j$, with $U_i, V_j \in \mathbf{U}(\mathcal{M}), i, j = 1, \ldots, n$, so

$$\|RTS\| \leq \frac{1}{n^2}\sum_{i,j=1}^n \|U_iTV_j\| = \frac{1}{n^2}\sum_{i,j=1}^n \|T\| = \|T\|.$$

Now let R and S be general and let $\varepsilon > 0$. Define

$$R_\varepsilon = \frac{1}{\|R\|_{\mathrm{op}} + \varepsilon}R, \qquad S_\varepsilon = \frac{1}{\|S\|_{\mathrm{op}} + \varepsilon}S.$$

Then $\|R_\varepsilon\|_{\mathrm{op}}, \|S_\varepsilon\|_{\mathrm{op}} < 1$, so

$$\|RTS\| = (\|R\|_{\mathrm{op}} + \varepsilon)(\|S\|_{\mathrm{op}} + \varepsilon)\|R_\varepsilon TS_\varepsilon\| \leq (\|R\|_{\mathrm{op}} + \varepsilon)(\|S\|_{\mathrm{op}} + \varepsilon)\|T\|.$$

Since this holds for all $\varepsilon > 0$, the result follows.

Now, for Equation (f), by the polar decomposition, we have that $T = U|T|$ and $|T| = U^*T$ for a partial isometry $U \in \mathcal{M}$. Thus, according to Equation (e), we have that
$$\|T\| = \|U|T|\| \leq \||T|\| = \|U^*T\| \leq \|T\|,$$
so $\|T\| = \||T|\|$. By taking adjoints on both sides of the equations, one proves that $\|T^*\| = \||T|\|$.

Proceeding with Equation (g), using the polar decomposition as above, we get
$$\begin{aligned}\|T^*T\| &= \||T|^2\| = \||T||T|^*\| = \|U^*TT^*U\| \\ &\leq \|TT^*\| = \|U|T||T|U^*\| \leq \|T^*T\|.\end{aligned}$$

Finally, we prove Equation (h). Assuming that $R \leq S$, we use Proposition 2.19 to determine $T \in \mathcal{M}$ with $\|T\|_{\mathrm{op}} \leq 1$ such that $R = S^{1/2}TS^{1/2}$. Since $(S^{1/2}T^{1/2})^* = T^{1/2}S^{1/2}$, it follows from Equation (g) and Equation (e) that
$$\|R\| = \|S^{1/2}TS^{1/2}\| = \|T^{1/2}ST^{1/2}\| \leq \|S\|. \qquad \square$$

We note that Equation (e) is really a characterization of unitary invariance.

Corollary 2.21. *Let $\mathcal{M}$ be a von Neumann algebra and let $\|\cdot\|\colon \mathcal{M} \to [0,\infty]$ be a unitarily invariant norm on $\mathcal{M}$. Let $S, T, P \in \mathcal{M}$ with $\|S\|_{\mathrm{op}}, \|T\|_{\mathrm{op}} \leq 1$, such that $P \geq S^*S$ and $P \geq T^*T$. Then*
$$\max\{\|P - S^*S\|, \|P - T^*T\|\} \leq 2\|P - S^*T\|. \tag{i}$$

Proof. Since $P - S^*S, P - T^*T$ and $(S-T)^*(S-T)$ are positive, we get that
$$\begin{aligned}0 \leq P - S^*S &\leq P - S^*S + P - T^*T + (S-T)^*(S-T) \\ &= P - S^*S + P - T^*T + S^*S + T^*T - S^*T - T^*S \\ &= P - S^*T + P - T^*S.\end{aligned}$$

Similarly, we have
$$0 \leq P - T^*T \leq P - S^*T + P - T^*S.$$

By Equation (h) and Equation (f), using that $(P - T^*S)^* = P - S^*T$ (since P is positive), the result follows. $\square$

Corollary 2.22. *Let $\mathcal{M}$ be a von Neumann algebra, let $\|\cdot\|\colon \mathcal{M} \to [0,\infty]$ be a unitarily invariant norm on $\mathcal{M}$ and let $S, T \in \mathcal{M}$. Then*
$$\|S^*T\| \leq \frac{1}{2}(\|S^*S\| + \|T^*T\|) = \frac{1}{2}(\|SS^*\| + \|TT^*\|). \tag{j}$$

Proof. By the polar decomposition and Equation (g), we can find an operator U with $\|U\|_{\mathrm{op}} \leq 1$ so that $U^*S^*T \geq 0$ and $\|S^*T\| = \|U^*S^*T\|$. Note that this implies that $U^*S^*T = T^*SU$. Thus

$$\begin{aligned} 0 &\leq (SU-T)^*(SU-T) = U^*S^*SU + T^*T - U^*S^*T - T^*SU \\ &= U^*S^*SU + T^*T - 2U^*S^*T, \end{aligned}$$

so $U^*S^*T \leq \frac{1}{2}(U^*S^*SU + T^*T)$, and the inequality follows from Equation (h) and Equation (e). The last equality is Equation (g). □

The following proposition characterizes submultiplicative norms among the unitarily invariant ones.

Proposition 2.23. *Let $\mathcal{M}$ be a von Neumann algebra let $\|\cdot\|\colon \mathcal{M} \to [0,\infty]$ be a unitarily invariant norm. The following are equivalent:*

(1) $\|\cdot\|$ *is submultiplicative,*

(2) $\|T\| \geq \|T\|_{\mathrm{op}}$ *for all $T \in \mathcal{M}$ and*

(3) $\|P\| \geq 1$ *for all non-zero projections $P \in \mathcal{M}$.*

Proof. We prove $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. The first implication $(1) \Rightarrow (3)$ is immediate, since $\|P\| = \|P^2\| \leq \|P\|^2$.

To prove $(3) \Rightarrow (2)$, first note that we may assume that $T \geq 0$ by Equation (f). We might also assume that $\|T\|_{\mathrm{op}} = 1$. Under these assumptions, let $\varepsilon > 0$ and consider the projection $P_\varepsilon = \chi_{[1-\varepsilon,1]}(T)$, which is non-zero and $(1-\varepsilon)P_\varepsilon \leq T$, so by Equation (h) it follows that

$$1 - \varepsilon \leq \|(1-\varepsilon)P_\varepsilon\| \leq \|T\|.$$

Since $\varepsilon > 0$ was arbitrary, $1 \leq \|T\|$ which is the desired conclusion.

The implication $(2) \Rightarrow (1)$ follows from from Equation (e). Indeed

$$\|ST\| \leq \|S\|_{\mathrm{op}}\|T\| \leq \|S\|\|T\|.$$ □

In particular, it proves the claims made at the end of Example 2.18 that $\|\cdot\|_{2,\mathrm{Tr}}$ is submultiplicative and $\|\cdot\|_{2,\mathrm{tr}}$ is not (for $n \geq 2$). For instance, one can apply (3) above and the fact that $\|P\|_{2,\mathrm{Tr}} = \sqrt{\operatorname{rank} P} \geq 1$ and $\|P\|_{2,\mathrm{tr}} = \sqrt{\frac{\operatorname{rank}(P)}{n}} < 1$ for all projections $P \in \mathbf{M}_n(\mathbb{C})$ that are not zero or the identity.

2.3 Positive definite functions

We now combine the theory of amenable groups with operator algebras. We still consider $\mathscr{H}$ a fixed separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Definition 2.24. Let Γ be a group and let $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ be a von Neumann algebra. A map $\varphi\colon \Gamma \to \mathcal{M}$ is called *positive definite* if for all finite sets $F \subseteq \Gamma$, the matrix $[\varphi(x^{-1}y)]_{x,y\in F} \in \mathbf{M}_n(\mathcal{M})$, where $n = |F|$, is positive as an operator on $\mathscr{H}^n$. Specifically,

$$\sum_{x,y\in F} \langle \varphi(x^{-1}y)\xi_y, \xi_x\rangle \geq 0,$$

for all $\xi_x \in \mathscr{H}$ and $x \in F$.

This innocent-looking definition turns out to impose a lot of structure on the map φ in question. Indeed, we recall the Stinespring Dilation Theorem below which essentially says that all positive definite maps come as cut-downs of representations. The precise statement that we need is a bit more subtle as we want to control in which algebra the representation lives. The construction is classical (see [64, 44]), but for convenience we include it here. Recall our notation $\mathcal{M}_\infty = \mathcal{M}\bar{\otimes}\mathbf{B}(\ell^2(\mathbb{N}))$, where we regard $\mathcal{M}$ as the corner $1_{\mathcal{M}}\mathcal{M}_\infty 1_{\mathcal{M}}$.

Theorem 2.25. *Let Γ be a countable group and let $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ be a von Neumann algebra. For every positive definite map*

$$\varphi\colon \Gamma \to \mathcal{M}$$

there exist $U \in \mathcal{M}_\infty 1_{\mathcal{M}}$ and a representation $\pi\colon \Gamma \to \mathbf{U}(\mathcal{M}_\infty)$ such that

$$\varphi(x) = U^*\pi(x)U,$$

for $x \in \Gamma$. In particular $\|U\|_{\mathrm{op}}^2 = \|\varphi(1)\|_{\mathrm{op}}$.

Proof. Consider the vector space $\mathcal{A} = C_{\mathrm{fin}}(\Gamma, \mathscr{H})$ of finitely supported maps $\Gamma \to \mathscr{H}$ equipped with the sequilinear form

$$\langle f, g\rangle_\varphi = \sum_{x,y\in\Gamma} \langle \varphi(x^{-1}y)f(y), g(x)\rangle,$$

for $f, g \in \mathcal{A}$. Since φ is positive definite, this is a positive semidefinite sequilinear form, so $\mathcal{A}/\mathcal{N}_\varphi$ is a pre-Hilbert space where

$$\mathcal{N}_\varphi = \{f \in \mathcal{A} \mid \langle f, f\rangle_\varphi = 0\}.$$

By completion, we get a Hilbert space $\tilde{\mathscr{H}}$ where Γ acts as unitaries by the formula

$$\pi_0(x)[f] = [x.f],$$

for $f \in \mathcal{A}$, where $x.f(y) = f(x^{-1}y)$, for $x, y \in \Gamma$, is the left translation action and $[f] = f + \mathcal{N}_\varphi$. Furthermore, let $U_0 \colon \mathscr{H} \to \tilde{\mathscr{H}}$ be given by

$$U_0(\xi) = [\chi_{\{1_\Gamma\}}\xi],$$

for $\xi \in \mathscr{H}$, where $\chi_{\{1_\Gamma\}}$ is the indicator function on the one-point set $\{1_\Gamma\}$. Note that since φ is positive definite, it holds that $\varphi(x^{-1}) = \varphi(x)^*$ for all $x \in \Gamma$. Thus, for all $\xi \in \mathscr{H}$ and $f \in \mathcal{A}$, we have

$$\langle U_0\xi, f\rangle_\varphi = \sum_{x\in\Gamma} \langle \varphi(x^{-1})\xi, f(x)\rangle_{\mathscr{H}} = \langle \xi, \sum_{x\in\Gamma} \varphi(x)f(x)\rangle_{\mathscr{H}},$$

and we conclude $U_0^*([f]) = \sum_{x\in\Gamma} \varphi(x)f(x)$ for $f \in \mathcal{A}$. It follows that

$$\varphi(x) = U_0^*\pi_0(x)U_0, \qquad x \in \Gamma.$$

We also define an action of the commutant $\mathcal{M}' \subseteq \mathbb{B}(\mathscr{H})$ on $\mathcal{A}$ by

$$(\rho(T)f)(x) = T(f(x)), \qquad T \in \mathcal{M}', f \in \mathcal{A}, x \in \Gamma.$$

In order to extend ρ to a normal representation of $\mathcal{M}'$ on $\tilde{\mathscr{H}}$, we have to check that $\rho(\mathcal{N}_\varphi) = \{0\}$ and that ρ is bounded. Let $T \in \mathcal{M}'$, let $F \subseteq \Gamma$ be finite and let $\operatorname{diag}(T)$ and $\operatorname{diag}(T^*)$ be the operators on $\mathscr{H}^n$ (where $n = |F|$) that act diagonally as T and T^*, respectively. Since $[\varphi(x^{-1}y)]_{x,y\in F}$ is positive for any F, the operator $S = \operatorname{diag}(T^*)[\varphi(x^{-1}y)]_{x,y\in F}\operatorname{diag}(T)$ is positive; in fact, since T commutes with $\varphi(x)$, we get that $0 \le S \le \|T\|_{\mathrm{op}}^2[\varphi(x^{-1}y)]_{x,y\in F}$, so

$$\begin{aligned}\|[Tf]\|_{\tilde{\mathscr{H}}}^2 &= \sum_{x,y\in\Gamma} \langle T^*\varphi(x^{-1}y)Tf(y), f(x)\rangle \\ &= \langle Sf, f\rangle_{\mathscr{H}^n} \le \|T\|_{\mathrm{op}}^2 \sum_{x,y\in\Gamma} \langle \varphi(x^{-1}y)f(y), f(x)\rangle = \|T\|_{\mathrm{op}}^2\|f\|_{\tilde{\mathscr{H}}}^2,\end{aligned}$$

which implies both desired properties of ρ. (Here, with a slight abuse of notation, we consider f as a vector in $\mathscr{H}^n$ where n is the cardinality of the support of f.) It is now easy to see that ρ is a normal representation of $\mathcal{M}'$. Thus, by Theorem 2.13, there is an isometry $V : \tilde{\mathscr{H}} \to \mathscr{H} \otimes \ell^2(\mathbb{N})$ such that $\rho(T) = V^*(T \otimes 1_{\ell^2(\mathbb{N})})V$ and $VV^* \in (\mathcal{M}' \otimes 1_{\ell^2(\mathbb{N})})'$. Clearly $\pi_0(\Gamma) \subseteq \rho(\mathcal{M}')'$, so

$$\pi(x) = V\pi_0(x)V^* + 1_{\mathcal{M}_\infty} - VV^* \in (\mathcal{M}' \otimes 1_{\ell^2(\mathbb{N})})' = \mathcal{M}_\infty,$$

for $x \in \Gamma$, is a unitary representation of Γ which, together with the map $U = VU_0$, has the desired properties. $\square$

Note that all maps $x \mapsto U^*\pi(x)U, x \in \Gamma$ are positive definite, where $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ and $U \in \mathbf{B}(\mathscr{H})$, so the above theorem really characterizes positive definite functions. As we see in the next proposition, amenable groups have a lot of positive definite maps. For any bounded map we can relate a positive definite map by means of a mean. For this, we recall a basic construction, which, in particular, allows us to take the mean of a map from an amenable group into a von Neumann algebra. Let Γ be any discrete group, let V be a Banach space, let $\mu \in (\ell^\infty(\Gamma))^*$ and let $\varphi\colon \Gamma \to V^*$ be a map which is bounded in the sense $\sup_{x\in\Gamma}\|\varphi(x)\|_{V^*} < \infty$. We define $\mu(\varphi) \in V^*$ by the formula

$$\mu(\varphi)(v) = \mu_x(\varphi(x)(v)),$$

for all $v \in V$. We shall use this construction in two particular settings. First, given map $\varphi\colon \Gamma \to \mathcal{M}$, which is bounded with respect to the operator norm, we use the unique predual $V = \mathcal{M}_*$ to realize the above construction. The defining formula can be expressed like

$$f(\mu(\varphi)) = \mu(f \circ \varphi), \qquad f \in \mathcal{M}_*,$$

since $\mathcal{M}_*$ consists of normal linear functionals on $\mathcal{M}$. Note that, given bounded maps $\varphi, \psi\colon \Gamma \to \mathcal{M}$ and $S, T \in \mathcal{M}$, we have

$$\mu_x(S\varphi(x)T + \psi(x)) = S\mu_x(\varphi(x))T + \mu_x(\psi(x)).$$

This follows from the fact that the maps $T \mapsto f(STR), S, R, T \in \mathcal{M}$ are normal whenever $f \in \mathcal{M}_*$.

Alternatively, if $\varphi\colon \Gamma \to \mathscr{H}$ takes values in our Hilbert space $\mathscr{H}$, we let $V = \bar{\mathscr{H}}$ be the conjugate Hilbert space which is a predual of $\mathscr{H}$ and the formula looks like

$$\langle \mu(\varphi), \eta \rangle = \mu_x(\langle \varphi(x), \eta \rangle), \qquad \eta \in \mathscr{H}.$$

Proposition 2.26. *Let Γ be an amenable group with left-invariant mean $\mathbb{E}$, let $\mathcal{M} \subseteq \mathbf{B}(\mathscr{H})$ be a von Neumann algebra and let $\varphi\colon \Gamma \to \mathcal{M}$ be given such that $\sup_{x\in\Gamma}\|\varphi(x)\|_{\mathrm{op}} < \infty$. Then the map $\tilde{\varphi}\colon \Gamma \to \mathcal{M}$ defined by*

$$\tilde{\varphi}(x) = \mathbb{E}_z \varphi(xz)\varphi(z)^*,$$

for $x \in \Gamma$, is positive definite.

Proof. The functionals $T \mapsto \langle T\xi, \eta\rangle$ are normal for all $\xi, \eta \in \mathscr{H}$ so they commute with the mean $\mathbb{E}$. Let $F \subseteq \Gamma$ be finite and let $\xi_x \in \mathscr{H}$ for $x \in F$. Then

$$\begin{aligned}
\sum_{x,y\in F} \langle \tilde{\varphi}(x^{-1}y)\xi_y, \xi_x\rangle &= \sum_{x,y\in F} \mathbb{E}_z\langle \varphi(x^{-1}yz)\varphi(z)^*\xi_y, \xi_x\rangle \\
&= \sum_{x,y\in F} \mathbb{E}_z\langle \varphi(x^{-1}z)\varphi(y^{-1}z)^*\xi_y, \xi_x\rangle \\
&= \sum_{x,y\in F} \mathbb{E}_z\langle \varphi(y^{-1}z)^*\xi_y, \varphi(x^{-1}z)^*\xi_x\rangle \\
&= \mathbb{E}_z\Big\langle \sum_{y\in F} \varphi(y^{-1}z)^*\xi_y, \sum_{x\in F} \varphi(x^{-1}z)^*\xi_x\Big\rangle \geq 0.
\end{aligned}$$

Here we used that $\mathbb{E}$ is left-invariant and positive and that $\langle \xi, \xi\rangle \geq 0$, for all $\xi \in \mathscr{H}$. □

At one place we need the notion of a positive definite kernel. There is a representation theorem for such kernels in the spirit of the GNS-construction or Theorem 2.25. Also the proof goes along the same line, and we shall not present it here.

Definition 2.27. Let Γ be a group. A map $\kappa\colon \Gamma \times \Gamma \to \mathbb{C}$ is called a *kernel*. A kernel κ is called *positive definite* if $[k(x,y)]_{x,y\in F} \in \mathrm{M}_n(\mathbb{C})$ is a positive as an operator on $\mathbb{C}^n$ for all finite $F \subseteq \Gamma$ with $n = |F|$.

Theorem 2.28. *Let Γ be a countable group. For any positive definite kernel $\kappa\colon \Gamma \times \Gamma \to \mathbb{C}$ there exists a Hilbert space $\mathscr{H}$ and a map $\alpha\colon \Gamma \to \mathscr{H}$ such that*

$$\kappa(x,y) = \langle \alpha(x), \alpha(y)\rangle, \qquad x, y \in \Gamma.$$

The only place where we use ultraweak lower semi-continuity of a norm is in the following lemma. In fact, this extra assumption on the norm $\|\cdot\|$ is only necessary for this thesis if the group Γ is infinite, since Equation (k) follows directly from the triangle inequality in the case of finite groups.

Lemma 2.29. *Let Γ be an amenable group with left-invariant mean $\mathbb{E}$, let $\mathcal{M}$ be a von Neumann algebra, let $\|\cdot\|\colon \mathcal{M} \to [0,\infty]$ be a ultraweakly lower semi-continuous norm on $\mathcal{M}$ and let $\varphi\colon \Gamma \to \mathcal{M}$ such that $\sup_{x\in\Gamma}\|\varphi(x)\|_{\mathrm{op}} < \infty$. Then*

$$\|\mathbb{E}_x\varphi(x)\| \leq \mathbb{E}_x\|\varphi(x)\|. \tag{k}$$

Proof. Note that for $\mu \in \ell^1(\Gamma) \subseteq \ell^\infty(\Gamma)^*$ we have that $\mu(\varphi) = \sum_{x\in\Gamma} \mu(x)\varphi(x)$, where the sum converges in operator norm. If furthermore $\mu(x) \geq 0$ for all

$x \in \Gamma$, then for all finite $F \subseteq \Gamma$, we have that

$$\| \sum_{x \in F} \mu(x)\varphi(x) \| \leq \sum_{x \in F} \mu(x) \|\varphi(x)\| \leq \mu_x(\|\varphi(x)\|).$$

Using lower semi-continuity of the norm, it follows that

$$\|\mu(\varphi)\| \leq \mu_x(\|\varphi(x)\|).$$

Now let $\mu_j \in \ell^1(\Gamma)$, for $j \in J$, be a net of positive functions with $\|\mu_j\|_1 = 1$ converging to $\mathbb{E}$ in the weak*-topology on $\ell^\infty(\Gamma)^*$. This implies that $\mu_j(\varphi)$ converges to $\mathbb{E}(\varphi)$ in the ultraweak topology, whence we conclude that

$$\|\mathbb{E}(\varphi)\| \leq \liminf_{j \to \infty} \|\mu_j(\varphi)\| \leq \liminf_{j \to \infty} (\mu_j)_x(\|\varphi(x)\|) = \mathbb{E}_x\|\varphi(x)\|. \qquad \square$$

Chapter 3

Ultrafilters and ultraproducts

As mentioned in the notation section (page vii), we fix a free ultrafilter ω on $\mathbb{N}$ throughout the thesis and this chapter is devoted to the task of explaining what this means. We recall the definition of filters, ultrafilters, limits of sequences along filters and ultraproducts of groups and Banach spaces. For more information, we refer the reader to [14, Appendix B].

3.1 Filters

In this thesis we work only with (ultra)filters on $\mathbb{N}$ and we shall only provide the relevant definitions in this setting.

Definition 3.1. A *filter* on $\mathbb{N}$ is a subset $\mathcal{F} \subseteq \mathrm{P}(\mathbb{N})$ of the power set of $\mathbb{N}$ such that

(1) $\emptyset \notin \mathcal{F}$,

(2) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ and

(3) if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

An *ultrafilter* is a filter $\mathcal{F}$ with the additional property that

(4) $A \in \mathcal{F}$ if and only if $\mathbb{N}\backslash A \notin \mathcal{F}$.

Example 3.2. The *Fréchet filter* (or *co-final filter*) on $\mathbb{N}$ is the collection

$$\mathcal{F}_{cofin} = \{I \subseteq \mathbb{N} \mid \mathbb{N}\backslash I \text{ is finite}\}$$

is easily seen to be a filter which is not an ultrafilter.

Example 3.3. Let $m \in \mathbb{N}$. The *principal ultrafilter* generated by m is the collection

$$\mathcal{F}_m = \{I \subseteq \mathbb{N} \mid m \in I\}.$$

An ultrafilter which is not principal is called *free*. Note that an ultrafilter is free in and only if it contains the Fréchet filter.

Given a filter $\mathcal{F}$, we think of the sets $I \in \mathcal{F}$ as "big" sets. Given a statement $P(n)$ for $n \in \mathbb{N}$, we use the wording $P(n)$ *holds for most* $n \in \mathbb{N}$ as slang for $\{n \in \mathbb{N} \mid P(n)\} \in \mathcal{F}$. Ultrafilters are characterized by being *maximal* among filters, that is, a filter $\mathcal{F}$ is an ultrafilter if and only if for all filters $\mathcal{V}$ on $\mathbb{N}$ the inclusion $\mathcal{F} \subseteq \mathcal{V}$ implies $\mathcal{F} = \mathcal{V}$. Thus, existence of free ultrafilters on $\mathbb{N}$ is an easy application of Zorn's lemma. One can think of an ultrafilter $\mathcal{F}$ as a consistent way of declaring which of the two sets I and $\mathbb{N}\backslash I$ is in the majority.

Definition 3.4. Let X be a topological space. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and $x \in X$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in X *converges to x along $\mathcal{F}$* if for all neighborhoods U of x we have $x_n \in U$ for most $n \in \mathbb{N}$. We use the notation $\lim_{n\to\mathcal{F}} x_n = x$, or $x_n \to x$ for $n \to \omega$.

Let us say some words about this definition. First, one observes that in Hausdorff spaces, sequences converge to at most one point. In the case where $\mathcal{F} = \mathcal{F}_m$ is a principal ultrafilter, the above definition is simply $\lim_{n\to\mathcal{F}_n} x_n = x_m$. On the other hand, it is easy to see that convergence along the Fréchet filter is the same as convergence in the usual sense, and we shall use the usual the notation $n \to \infty$ instead of $n \to \mathcal{F}_{cofin}$. It follows that if ω is a free ultrafilter, then all sequences $(x_n)_{n\in\mathbb{N}}$ that converge in the usual sense also converge along ω and $\lim_{n\to\infty} x_n = \lim_{n\to\omega} x_n$. More generally, the limit along a free ultrafilter belongs to the cluster points of $(x_n)_{n\in\mathbb{N}}$. Supposing that the limit exists, of course, à propos of which we state a central fact about ultrafilters.

Theorem 3.5. *Let X be a compact Hausdorff space. Then every sequence in X converges along any ultrafilter on* $\mathbb{N}$.

In view of the above discussion and this theorem, one can think of the choice of free ultrafilter ω as a consistent way of assigning convergent subsequences to all sequences (in any compact Hausdorff space). As a special, but useful case, we consider sequences $(x_n)_{n\in\mathbb{N}}$ a sequence of real numbers; then there is a limit $\lim_{n\to\omega} x_n \in [-\infty, \infty]$. We adopt the Landau notation to the setting of ultrafilters. Given two non-negative real sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ we write

$$x_n = O(y_n),$$

for $n \to \omega$, if there exists a constant $C > 0$ such that $x_n \leq Cy_n$, for most $n \in \mathbb{N}$, and we write

$$x_n = o(y_n),$$

for $n \to \omega$, if there exists a third non-negative real sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\omega} \varepsilon_n = 0$ such that $x_n = \varepsilon_n y_n$. Evidently $x_n = o(y_n)$ implies $x_n = O(y_n)$ for $n \to \omega$.

3.2 Ultraproducts

Ultraproducts of metric groups

We need the construction of ultraproducts of metric groups. Let $(G_n, d_n)_{n\in\mathbb{N}}$ be a family of metric groups. Recall our standing assumption that d_n are all bi-invariant, that is $d_n(g, h) = d_n(kg, kh) = d_n(gk, hk)$ for all $g, h, k \in G_n$. Using this assumption, the subgroup

$$N_\omega = \Big\{(g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n \mid \lim_{n\to\omega} d_n(g_n, 1_{G_n}) = 0\Big\}$$

of the (group-theoretic) direct product $\prod_{n\in\mathbb{N}} G_n$ is normal, so we can define the *metric ultraproduct*

$$\prod_{n\to\omega} (G_n, d_n) = \prod_{n\in\mathbb{N}} G_n \Big/ N_\omega.$$

Note that, as a set, the direct product is just the usual cartesian product, and *we do not require the sequences to be bounded*, which is in contrast to the Banach space ultraproduct as we will see shortly. This entails that the natural candidate for a metric on $\prod_{n\to\omega}(G_n, d_n)$, namely the metric induced by the pseudo-metric

$$d((g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}}) = \lim_{n\to\omega} d_n(g_n, h_n),$$

for $g_n, h_n \in G_n$, on $\prod_{n\in\mathbb{N}} G_n$ will be infinite in general. Since we already allow our norms to take infinite values this might not scare us off, but as a matter of fact we don't need the metric on the ultraproduct, so it is not important. It is worth noting though (albeit also irrelevant for our purposes), that the bounded, bi-invariant pseudo-metric

$$d'((g_n)_{n\in\mathbb{N}}, (h_n)_{n\in\mathbb{N}}) = \lim_{n\to\omega} \min\{d_n(g_n, h_n), 1\},$$

for $g_n, h_n \in G_n$ on $\prod_{n\in\mathbb{N}} G_n$ induces a bi-invariant metric on $\prod_{n\to\omega}(G_n, d_n)$.

Ultraproducts of Banach spaces

We now turn our attention to ultraproduct of Banach spaces. Consider a sequence of Banach spaces $(V_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ and denote the ℓ^∞-direct product by

$$\ell^\infty(\omega, (V_n)_{n\in\mathbb{N}}) = \left\{(v_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} V_n \mid \sup_{n\in\mathbb{N}} \|v_n\|_n < \infty\right\}$$

and the closed subspace of null-sequences

$$c_0(\omega, (V_n)_{n\in\mathbb{N}}) = \left\{(v_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, (V_n)_{n\in\mathbb{N}}) \mid \lim_{n\to\omega} \|v_n\|_{V_n} = 0\right\}.$$

We define the *ultraproduct Banach space* by

$$\prod_{n\to\omega} (V_n, \|\cdot\|_n) = \ell^\infty(\omega, (V_n)_{n\in\mathbb{N}}) \Big/ c_0(\omega, (V_n)_{n\in\mathbb{N}}).$$

The ultraproduct Banach space is itself a Banach space with the norm induced by $\|(x_n)_{n\in\mathbb{N}}\| = \lim_{n\to\omega} \|x_n\|_{V_n}$ for $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} V_n$. Moreover, if the V_n are all Banach algebras, C^*-algebras or Hilbert spaces, so is the ultraproduct.

Ultraproducts of unitary groups acting on ultraproducts of matrices

We now consider a particular situation which will be relevant for us later on. We let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers and consider a family of submultiplicative, unitarily invariant norms $\|\cdot\|_{k_n}$ on $\mathbf{M}_{k_n}(\mathbb{C}), n \in \mathbb{N}$. We usually omit the index and denote all norms by $\|\cdot\|$. Since the norms are unitarily invariant, they induce bi-invariant metrics on the unitary groups $\mathbf{U}(k_n)$ by

$$\mathrm{dist}_{\|\cdot\|}(U, V) = \|U - V\|, \qquad U, V \in \mathbf{U}(k_n).$$

We denote the metric ultraproduct respectively the ultraproduct Banach space as

$$\mathbf{U}(\omega, \|\cdot\|) = \prod_{n\to\omega} (\mathbf{U}(k_n), \mathrm{dist}_{\|\cdot\|}), \quad \text{resp.,} \quad \mathbf{M}(\omega, \|\cdot\|) = \prod_{n\to\omega} (\mathbf{M}_{k_n}(\mathbb{C}), \|\cdot\|).$$

Since the norms are submultiplicative, the group $\mathbf{U}(\omega, \|\cdot\|)$ acts on $\mathbf{M}(\omega, \|\cdot\|)$ from the left by the action induced by

$$(U_n)_{n\in\mathbb{N}}(T_n)_{n\in\mathbb{N}} = (U_n T_n)_{n\in\mathbb{N}},$$

for $U_n \in \mathbf{U}(k_n)$ and $T_n \in \mathbf{M}_{k_n}(\mathbb{C})$. Similarly, we can define a right action and a conjugation action. By unitary invariance, we see that these actions are isometric. Note that submultiplicativity is quite essential here. Indeed, in the context of the normalized Hilbert-Schmidt norm we consider the following diagonal matrices:

$$U_n = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathbf{U}(n), \qquad T_n = \mathrm{diag}(\sqrt{n}, 0, \ldots, 0) \in \mathbf{M}_n(\mathbb{C}).$$

Obviously

$$\|U_n - 1_n\|_{2,\mathrm{tr}} = \frac{2}{\sqrt{n}} \to 0,$$

for $n \to \infty$, and $\|T_n\|_{2,\mathrm{tr}} = 1$ for all $n \in \mathbb{N}$, but

$$\|(U_n - 1_n)T_n\|_{2,\mathrm{tr}} = \|\mathrm{diag}(-2\sqrt{n}, 0, \ldots, 0)\|_{2,\mathrm{tr}} = 2,$$

for all $n \in \mathbb{N}$. This shows that $(U_n)_{n\in\mathbb{N}}$ does not act trivially on $\mathbf{M}(\omega, \|\cdot\|)$, although it defines the identity element of $\mathbf{U}(\omega, \|\cdot\|)$.

Part II

Local asymptotic representations

Chapter 4

Local approximations

In this chapter, we give an introduction to approximable and stable groups with respect to a class of metric groups $\mathcal{C}$. We provide all the relevant terminology together with some examples of approximable groups and (non-trivial) examples of asymptotic representations. We focus on asymptotic representations with respect to the Frobenius norm, and we shall provide some basic observations and examples in this setting. The reason for choosing to emphasize the Frobenius norm is that some of our main results are specifically about this norm as we shall see in the next chapter. For the most part, we restrict our attention to finitely presented groups. This is partly out of convenience and partly because some definitions, e.g. the notion of defect (Definition 4.1 below), only make sense in this special case.

4.1 Asymptotic homomorphisms

For the purpose of this section, we fix finite sets S and R where $R \subseteq \mathbb{F}_S$ and denote the finitely presented group with generators S and relations R by Γ. Any map $\varphi\colon S \to G$, uniquely determines a homomorphism $\mathbb{F}_S \to G$ which we shall also denote by φ.

Definition 4.1. Let (G, d) be a metric group and let $\varphi, \psi\colon S \to G$ be maps. The *defect* of φ is defined by

$$\operatorname{def}(\varphi) = \max_{r \in R} d(\varphi(r), 1_G).$$

The *distance* between φ and ψ is defined by

$$\operatorname{dist}(\varphi, \psi) = \max_{s \in S} d(\varphi(s), \psi(s)).$$

The *homomorphism distiance* of φ is defined by

$$\operatorname{HomDist}(\varphi) = \inf\{\operatorname{dist}(\varphi, \pi|_S) \mid \pi \in \operatorname{Hom}(\Gamma, G)\}.$$

These notions obviously depend on the metric d. Sometimes we work with several metrics on the same group G, so to avoid confusion, we might write $\operatorname{def}_d, \operatorname{dist}_d$ and $\operatorname{HomDist}_d$. If the distance d comes from a norm $\|\cdot\|$ (see below), we write $\operatorname{def}_{\|\cdot\|}, \operatorname{dist}_{\|\cdot\|}$ and $\operatorname{HomDist}_{\|\cdot\|}$. Recall that ω denotes a fixed, free ultrafilter on $\mathbb{N}$.

Definition 4.2. Let $(G_n, d_n)_{n\in\mathbb{N}}$ be a sequence of metric groups. A sequence of maps $\varphi_n \colon S \to G_n$, for $n \in \mathbb{N}$, is called an *asymptotic homomorphism* if $\lim_{n\to\omega} \operatorname{def}(\varphi_n) = 0$.

If an asymptotic homomorphism $(\varphi_n)_{n\in\mathbb{N}}$ is equivalent to a sequence of genuine representations, that is, $\lim_{n\to\omega} \operatorname{HomDist}(\varphi_n) = 0$, we call $(\varphi_n)_{n\in\mathbb{N}}$ *trivial*. In accordance with our usual nomenclature, we talk about asymptotic *representations* if the target groups G_n are unitary groups of some sort. In this part of the thesis, we are mostly interested in asymptotic representations with respect to the class of unitary groups $\mathbf{U}(n)$ on finite dimensional Hilbert spaces, equipped with the metrics

$$d_n(U, V) = \|T - S\|_n, \qquad U, V \in \mathbf{U}(n),$$

coming from some family of unitarily invariant norms $\|\cdot\|_n$ on $\mathbf{M}_n(\mathbb{C})$. As a warm-up, we prove the following basic lemma. Recall that $\langle\!\langle R \rangle\!\rangle$ denotes the normal subgroup of $\mathbb{F}_S$ generated by R and that metric groups are equipped with *bi-invariant* metrics according to our teminology.

Lemma 4.3. *For all $r \in \langle\!\langle R \rangle\!\rangle$ there is a constant K_r such that, for all metric groups (G, d) and maps $\varphi \colon S \to G$, it holds that*

$$d(\varphi(r), 1_G) \le K_r \operatorname{def}(\varphi).$$

Proof. Given $r \in \langle\!\langle R \rangle\!\rangle$, we determine $r_1, \dots, r_k \in R \cup R^{-1}$ and $x_1, \dots, x_k \in \mathbb{F}_S$ such that

$$r = x_1 r_1 x_1^{-1} x_2 r_2 x_2^{-1} \cdots x_k r_k x_k^{-1}.$$

Note that by bi-invariance

$$d(\varphi(r_j), 1_G) = d(\varphi(r_j^{-1}), 1_G) \le \operatorname{def}(\varphi)$$

for all j. Thus, using bi-invariance again, together with the triangle inequality, we get

$$\begin{aligned} d(\varphi(r), 1_{k_n}) &= d(\varphi(x_1 r_1 x_1^{-1}) \cdots \varphi(x_k r_k x_k^{-1}), 1_G) \\ &\leq \sum_{j=1}^{k} d(\varphi(x_j)\varphi(r_j)\varphi(x_j)^{-1}, 1_G) \\ &= \sum_{j=1}^{k} d(\varphi(r_j), 1_G) \\ &\leq k \cdot \operatorname{def}(\varphi), \end{aligned}$$

So letting $K_r = k$, we are done. □

Let us provide one last definition in this section.

Definition 4.4. Let (G_n, d_n) be a sequence of metric groups. Two sequences $\varphi_n, \psi_n \colon S \to G_n$ are called *(asymptotically) equivalent* if

$$\lim_{n \to \omega} \operatorname{dist}(\varphi_n, \psi_n) = 0.$$

Observe that there is a correspondence between asymptotic homomorphisms and (genuine) homomorphisms into ultraproducts. Indeed, a homomorphism $\varphi \colon \Gamma \to \prod_{n \to \omega} (G_n, d_n)$ induces an asymptotic homomorphism by taking any sequence $(\varphi_n(x))_{n \in \mathbb{N}}$ representing $\varphi(x)$, for $x \in S$. Conversely, a sequence $\varphi_n \colon S \to G_n$ is an asymptotic homomorphism if and only if the induced homomorphism

$$\varphi_\omega \colon \mathbb{F}_S \to \prod_{n \to \omega} (G_n, d_n),$$

where $\varphi_\omega(x)$ is represented by $(\varphi_n(x))_{n \in \mathbb{N}}$, for $x \in \mathbb{F}_S$, factors through the group $\Gamma = \mathbb{F}_S / \langle\langle R \rangle\rangle$. In this setup, two asymptotic homomorphisms $(\varphi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ are equivalent if and only if they give rise to the same homomorphism, that is $\varphi_\omega = \psi_\omega$.

4.2 Approximable groups

We can use the concepts from the previous section to define approximable, finitely presented groups. From now on, we let $\mathcal{C}$ be a fixed class of metric groups.

Definition 4.5. A finitely presented group $\Gamma = \langle S \mid R \rangle$ is called *$\mathcal{C}$-approximable* if there exists an asymptotic homomorphism $\varphi_n \colon S \to G_n$, with $(G_n, d_n) \in \mathcal{C}$ for $n \in \mathbb{N}$, such that, for all $x \in \mathbb{F}_S \backslash \langle\!\langle R \rangle\!\rangle$, we have that

$$\lim_{n \to \omega} d_n(\varphi_n(x), 1_{G_n}) > 0.$$

We remarked that asymptotic homomorphisms correspond to genuine homomorphisms into ultraproducts. From this observation, it is easy to prove the following characterization of $\mathcal{C}$-approximability.

Proposition 4.6. *Let $\mathcal{C}$ be a class of metric groups. A finitely presented group Γ is $\mathcal{C}$-approximable if and only if there is a family $(G_n, d_n)_{n \in \mathbb{N}}$ in $\mathcal{C}$ and an injective homomorphism*

$$\Gamma \hookrightarrow \prod_{n \to \omega} (G_n, d_n).$$

We recall the following definition, which is clearly stronger than $\mathcal{C}$-approximability.

Definition 4.7. A group Γ is called *residually* $\mathcal{C}$ if for all $x \in \Gamma \backslash \{1_\Gamma\}$ there is a homomorphism $\pi \colon \Gamma \to G$ for some $G \in \mathcal{C}$ such that $\pi(x) \neq 1_G$.

Having these definitions out of the way, we are ready to look at some general examples. The notion of $\mathcal{C}$-approximability has been studied in many cases and we shall only look at a few examples. We refer the reader to [1] for an overview.

Example 4.8. A *sofic* group is a Sym-approximable group in the case where Sym is the class of finite symmetric groups $\mathrm{Sym}(n)$ for $n \in \mathbb{N}$, equipped with the *normalized Hamming metric*,

$$d_{\mathrm{ham},n}(g,h) = \frac{|\{i \in \{1,2,\ldots,n\} \mid g(i) \neq h(i)\}|}{n}, \qquad g,h \in \mathrm{Sym}(n).$$

For a thorough treatment of sofic groups, we refer to [16].

Example 4.9. A more general example is *weakly sofic* group [33], where $\mathcal{C}$ consists of *all* finite groups equipped with any bi-invariant metric. See [49] for recent advances in this setting.

Example 4.10. A group is called *linear sofic* if it is approximable with respect to the class general linear groups $\mathbf{GL}_n(\mathbb{C})$, for $n \in \mathbb{N}$, equipped with the normalized rank metric $d_{\mathrm{rank}}(g,h) = \frac{1}{n}\operatorname{rank}(g-h)$ for $g,h \in \mathbf{GL}_n(\mathbb{C})$. This notion was introduced in [3], where it was proven that sofic groups are linear sofic, and linear sofic groups are weakly sofic.

Example 4.11. Turning the attention to the finite dimensional unitary groups, we consider the so-called MF groups [15]. Those are the $\mathbf{U}_{\mathrm{op}}$-approximable groups, where $\mathbf{U}_{\mathrm{op}}$ is the class of unitary groups $\mathbf{U}(n)$ equipped with the metric coming from the operator norm.

Example 4.12. The class $\mathbf{U}_{2,\mathrm{tr}}$, consisting of finite dimensional unitary groups equipped with the metric coming from normalized Hilbert-Schmidt norm, gives rise to what is usually called *hyperlinear* groups. Every sofic group is hyperlinear. This follows from the following relation between the Hamming distance the normalized Hilbert-Schmidt norm:

$$d_{\mathrm{ham},n}(g,h) = \frac{1}{2}\|P_g - P_h\|_{2,\mathrm{tr}}^2, \qquad g,h \in \mathrm{Sym}(n).$$

Here P_g and P_h denote the permutation matrices associated to g and h. We refer the reader to [54, 14] for more information.

Example 4.13. An example of particular interest to us is the $\mathbf{U}_{2,\mathrm{Tr}}$-approximable groups. Here $\mathbf{U}_{2,\mathrm{Tr}}$ denotes the class of finite dimensional unitary groups, this time equipped with the metric coming from the Frobenius norm, that is, the *unnormalized* Hilbert-Schmidt norm. We call such groups *Frobenius approximable*.

One sees immediately that all finitely presented, residually finite groups enjoy all of the above properties. It is also elementary to prove that all amenable groups are sofic, and thus also hyperlinear and weakly sofic. Furthermore, it is known that all amenable groups are MF, but the proof is highly non-trivial as it relies on the recent breakthrough in classification of nuclear C^*-algebras [67] (see also [51]). We are unaware of any similar results in the context of Frobenius approximable groups. In fact, we do not know whether all solvable groups are Frobenius approximable, let alone whether amenable groups are. On the other hand, actual examples of non-approximable groups are scarce, and one might be tempted to ask the following question:

Question 4.14. *Is every finitely presented group $\mathcal{C}$-approximable?*

Remarkably enough, this question remains open in many important cases. For instance, it is not known whether all (finitely presented) groups are sofic. This question was posed by Gromov [37], and if it would turn out to have a positive answer, it would have various interesting implications, including Kaplansky's direct finiteness conjecture and the surjunctivity conjecture (see [14]). As the notions of linear sofic, weakly sofic groups and hyperlinear groups subsume the notion of a sofic group, it is also unknown whether all groups enjoy each of these properties. The case of hyperlinearity is connected to the celebrated Connes Embedding Problem about embeddability of II_1-factors into the (tracial) ultraproduct of the hyperfinite II_1-factor [17, 54]. More precisely, the existence a non-hyperlinear group would provide a negative solution to this problem. The situation is similar for MF groups; it is unknown whether all groups are MF, and this constitutes a test case for a more general question of Kirchberg [9], namely, whether all stably finite C^*-algebras are embeddable in an ultraproduct of finite dimensional C^*-algebras. Only in few cases, the above question is answered. One example is in [66], where it was shown that the Higman group,

$$\langle x_1, x_2, x_3, x_4 \mid x_i^{-1}[x_{i+1}, x_i], i \in \mathbb{Z}/4\mathbb{Z}\rangle,$$

is not approximable with respect to a certain class of finite metric groups $\mathcal{F}_c$. To the best of our knowledge it is still unknown if the Higman group is (weakly) sofic, hyperlinear, MF or Frobenius approximable. One of our main results (Corollary 5.21) proves the existence of a finitely presented group which is not Frobenius approximable.

Approximability for general groups

Until now, we have considered finitely presented groups, but the notion of approximability can be defined for general groups.

Definition 4.15. Let Γ be a discrete group and let (G, d) be a metric group. Let $\varepsilon > 0$ and $F \subseteq \Gamma$. An (ε, F)-approximate homomorphism is a map $\varphi\colon \Gamma \to G$ such that, for all $x, y \in F$, it holds that

$$d(\varphi(x)\varphi(y), \varphi(xy)) < \varepsilon.$$

In some sense, this is a quantitative version of what we call an asymptotic homomorphism. Indeed, given a finitely presented group $\Gamma = \langle S \mid R\rangle$ with a fixed section $\sigma\colon \Gamma \to \mathbb{F}_S$ of the quotient map $\mathbb{F}_S \to \Gamma$, we see that a map $\varphi_n\colon S \to G_n$ is an asymptotic homomorphism if and only if $\tilde{\varphi}_n = \varphi_n \circ \sigma$ is (ε_n, F_n)-approximate homomorphisms for some sequence ε_n tending to zero and finite subsets $F_n \subseteq \Gamma$ satisfying $\Gamma = \bigcup_{n\in\mathbb{N}} F_n$.

Definition 4.16. We say that a discrete group Γ is *$\mathcal{C}$-approximable* if, for all $x \in \Gamma\backslash\{1_\Gamma\}$, there is $\eta_\Gamma(x) > 0$ such that the following holds: For all $\varepsilon > 0$ and finite $F \subseteq \Gamma$, there exists $(G, d) \in \mathcal{C}$ and an (ε, F)-approximate homomorphism $\varphi\colon \Gamma \to G$, such that $\varphi(1_\Gamma) = 1_G$ and $d(\varphi(x), 1_\Gamma) > \eta_\Gamma(x)$ for all $x \in F\backslash\{1_\Gamma\}$.

It is not to hard to prove that this definition is the same as the one provided previously for finitely presented groups. The characterization of approximability in terms of ultraproducts, Proposition 4.6, also generalizes to general groups. However one has to allow for ulrafilters on arbitrary index sets instead of $\mathbb{N}$.

Let us discuss some permanence properties of the class of $\mathcal{C}$-approximable groups. It is straightforward to see that being $\mathcal{C}$-approximable is a *local* property in the sense that Γ is $\mathcal{C}$-approximable if and only if all finitely generated subgroups of Γ are $\mathcal{C}$-approximable. Along the same lines it is easy to see that $\mathcal{C}$-approximability passes to subgroups. Furthermore, in many cases $\mathcal{C}$-approximability is also preserved by taking direct products over an arbitrary index set. This holds for all examples of classes provided above, but we shall only prove it for Frobenius approximable groups, since, to the best of our knowledge, this observation is new (albeit proven in a more or less standard and straightforward way, see [41]). The rest of the proofs can be found in the literature (see e.g. [16, 41]).

Proposition 4.17. *Let* $(\Gamma_i)_{i\in I}$ *be a family of Frobenius approximable groups. Then* $\prod_{i\in I}\Gamma_i$ *is Frobenius approximable.*

Proof. For a subset $J\subseteq I$ we let $\pi_J\colon \prod_{i\in I}\Gamma_i\to\prod_{j\in J}\Gamma_j$ denote the projection map. We write $\pi_i=\pi_{\{i\}}$. Furthermore, since all Γ_i are Frobenius approximated, we determine $\eta_{\Gamma_i}(x)>0$, for all $i\in I$ and $x\in\Gamma_i\backslash\{1_{\Gamma_i}\}$ as in Definition 4.16. Furthermore, for all $x\in\Gamma\backslash\{1_\Gamma\}$, choose $i(x)\in I$ such that $\pi_{i(x)}(x)\neq 1_{\Gamma_{i(x)}}$, and define $\eta_\Gamma(x)=\eta_{\Gamma_{i(x)}}(\pi_{i(x)}(x))$, which is strictly positive by the choice of $i(x)$.

Let $\varepsilon>0$ and let $F\subseteq\prod_{i\in I}\Gamma_i$ be finite. For each $y\in F$ we can find an $(\varepsilon|F|^{-1/2},\pi_{i(y)}(F))$-approximate representation $\varphi_y\colon\Gamma_{i(y)}\to\mathbf{U}(n_y)$ such that

$$\|\varphi_y(x)-1_{n_y}\|_{2,\mathrm{Tr}_{n_y}}>\eta_{\Gamma_i}(x),$$

for all $x\in\pi_{i(y)}(F)$. We consider the direct sum

$$\varphi=\bigoplus_{y\in F}(\varphi_y\circ\pi_{i(y)})\colon\Gamma\to\bigoplus_{y\in F}\mathbf{U}(n_y)\subseteq\mathbf{U}(n)$$

where $n=\sum_{y\in F}n_y$. Using the equality

$$\|T\|^2_{2,\mathrm{Tr}_n}=\sum_{y\in F}\|T_y\|^2_{2,\mathrm{Tr}_{n_y}},$$

where $T=\bigoplus_{y\in F}T_y$ is the block-diagonal matrix in $\mathbf{M}_n(\mathbb{C})$ with blocks $T_y\in\mathbf{M}_{n_y}(\mathbb{C})$, we conclude that for all $x,y\in F$, it holds that

$$\|\varphi(xy)-\varphi(x)\varphi(y)\|_{2,\mathrm{Tr}_n}\leq\sqrt{|F|\frac{\varepsilon^2}{|F|}}=\varepsilon.$$

By definition of $\eta_\Gamma(x)$, we also conclude that

$$\|\varphi(x)-1_n\|_{2,\mathrm{Tr}_n}\geq\eta_\Gamma(x).$$

□

As we are going to answer Question 4.14 in the negative for the Frobenius norm, one could argue that it is only a neat extra feature of our counterexample that it admits a finite presentation. It is worth mentioning though, that, under mild assumptions on the class $\mathcal{C}$, one is always able to find finitely presented counterexamples, as noted in the following proposition.

Proposition 4.18. *Let $\mathcal{C}$ be a class of metric groups and assume that $\mathcal{C}$-approximability is preserved under products. Then all groups are $\mathcal{C}$-approximable if and only if all fintely presented groups are $\mathcal{C}$-approximable.*

Proof. As mentioned, we need only consider finitely generated groups since $\mathcal{C}$-approximability is a local property. Also, since $\mathcal{C}$-approximability passes to arbitrary direct products, it also passes to projective limits since they are subgroups of a direct product. This concludes the proof, since every finitely generated group is a projective limit of finitely presented groups. □

4.3 Stability of asymptotic homomorphisms

We now define what it means for a group to be stable with respect to our class of metric groups $\mathcal{C}$.

Definition 4.19. A finitely presented group $\Gamma = \langle S \mid R \rangle$ is called *$\mathcal{C}$-stable* if for all $\varphi_n \colon S \to G_n$, where $(G_n, d_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ it holds that

$$\lim_{n\to\omega} \operatorname{def}(\varphi_n) = 0 \Longrightarrow \lim_{n\to\omega} \operatorname{HomDist}(\varphi_n) = 0.$$

In other words, a group is stable if every asymptotic representation is trivial. We use the term *Frobenius stable* for groups that are stable with respect to our favorite class of metric groups $\mathbf{U}_{2,\mathrm{Tr}}$ (see Example 4.13). As with approximability, there is a characterization in terms of ultraproducts.

Proposition 4.20. *A finitely presented group Γ is $\mathcal{C}$-stable if and only if all homomorphisms into an ultraproduct*

$$\varphi\colon \Gamma \to \prod_{n\to\omega} (G_n, d_n),$$

for $(G_n, d_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$, are liftable, that is, there is a homomorphism $\tilde{\varphi}\colon \Gamma \to \prod_{n\in\mathbb{N}} G_n$ representing φ.

A simple but crucial observation is the fact that all finitely presented groups Γ that are both $\mathcal{C}$-approximable and $\mathcal{C}$-stable are residually $\mathcal{C}$. In particular, if the groups of $\mathcal{C}$ are all linear groups, then Γ is residually finite by Mal'cev's Theorem. This applies for instance to the classes $\mathbf{U}_{\mathrm{op}}$, $\mathbf{U}_{2,\mathrm{tr}}$ and $\mathbf{U}_{2,\mathrm{Tr}}$.

The stability question has been investigated in many different settings, see for instance [71, 32, 2, 25, 40, 7]. A famous example of a non-trivial asymptotic representation of $\mathbb{Z}^2$ with respect to the operator norm, is the *asymptotically commuting matrices* of Voiculescu [71]. As we shall see below, the same construction provides us with a non-trivial asymptotic representation with respect to the Frobenius norm.

Example 4.21. We recall the construction of Voiculescu's asymptotically commuting matrices and use it to show that $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ is not Frobenius stable. We consider the sequence $\varphi_n \colon \{a, b\} \to \mathbf{U}(n)$ given by

$$\varphi_n(a) = \begin{pmatrix} 1 & & & & \\ & \lambda_n & & & \\ & & \lambda_n^2 & & \\ & & & \ddots & \\ & & & & \lambda_n^{n-1} \end{pmatrix}, \qquad \varphi_n(b) = \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & 0 & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix},$$

where $\lambda_n = \exp(2\pi i/n)$ is the n'th root of unity. A direct calculation shows that

$$\varphi_n(a)\varphi_n(b)\varphi_n(a)^*\varphi_n(b)^* = \bar{\lambda}_n 1_n,$$

so that, as $n \to \infty$, we have

$$\operatorname{def}_{\|\cdot\|_{2,\mathrm{Tr}}}(\varphi_n) = \|\bar{\lambda}_n 1_n - 1_n\|_{2,\mathrm{Tr}} = \sqrt{n}|\bar{\lambda}_n - 1| = O\big(\tfrac{1}{\sqrt{n}}\big).$$

Thus, we conclude that φ_n is an asymptotic representation with respect to $\|\cdot\|_{2,\mathrm{Tr}}$. As mentioned, Voiculescu [71] (see also [26]) proved that this asymptotic representation is non-trivial with respect to the operator norm, more precisely,

$$\operatorname{HomDist}_{\|\cdot\|_{\mathrm{op}}}(\varphi_n) \geq \sqrt{2 - |1 - \lambda_n|} - 1,$$

for $n \geq 7$. Since $\operatorname{HomDist}_{\|\cdot\|_{2,\mathrm{Tr}}}(\varphi_n) \geq \operatorname{HomDist}_{\|\cdot\|_{\mathrm{op}}}(\varphi_n)$ we conclude that $(\varphi_n)_{n\in\mathbb{N}}$ is non-trivial with respect to the Frobenius norm.

We now turn our attention to the Baumslag-Solitar group

$$\mathrm{BS}(2,3) = \langle a, b \mid b^{-1}a^2ba^{-3} \rangle.$$

It is well-known that $\mathrm{BS}(2,3)$ is not residually finite, and we shall exploit this fact to show that it is not Frobenius stable either. For this reason, we need the following.

Lemma 4.22 ([6]). *Let Γ be a finite group and assume that $x, y \in \Gamma$ satisfy $y^{-1}x^2y = x^3$. Then $xy^{-1}xy = y^{-1}xyx$.*

Proof. Let $k \in \mathbb{N}$ be such that $x^{2k} = 1_\Gamma$. Then

$$x^k = x^{3k} = y^{-1}x^{2k}y = 1_\Gamma,$$

so it follows that the order of x is odd, say $2m+1$, for $m \in \mathbb{N}$. From this we get that

$$y^{-1}xy = y^{-1}x^{2m+2}y = x^{3(m+1)},$$

proving that $y^{-1}xy$ commutes with x. □

By Mal'cev's Theorem we get the following consequence.

Corollary 4.23. *Let $U, V \in \mathbf{U}(n)$ satsifsying $V^*U^2V = U^3$. Then it follows that $UV^*UV = V^*UVU$.*

For a linear algebraic proof of this fact, see [29]. The above corollary implies that, if $\varphi_n \colon \mathrm{BS}(2,3) \to \mathbf{U}(k_n)$ is an asymptotic representation with

$$\lim_{n\to\omega} \|\varphi_n(a)\varphi_n(b)^*\varphi_n(a)\varphi_n(b) - \varphi_n(b)^*\varphi_n(a)\varphi_n(b)\varphi_n(a)\|_{2,\mathrm{Tr}} > 0,$$

then $(\varphi_n)_{n\in\mathbb{N}}$ is non-trivial. Thus we aim to construct such a sequence of maps and we shall do so with $k_n = 6n$. The construction is due to Glebsky [18] and somewhat similar to one by Rădulescu [60], who studied approximation properties of $\mathrm{BS}(2,3)$ with respect to the normalized Hilbert-Schmidt norm.

Example 4.24. We first need to fix some notation. Given $n \in \mathbb{N}$, we consider a $6n$-dimensional Hilbert space $\mathscr{H}$ with orthonormal basis $\xi_0, \ldots, \xi_{6n-1}$. We decompose $\mathscr{H}$ into a sum of 6-dimensional subspaces in two different ways, $\mathscr{H} = \bigoplus_{j=0}^{n-1} \mathcal{S}_j = \bigoplus_{j=0}^{n-1} \mathcal{C}_j$, where

$$\mathcal{S}_j = \mathrm{span}\{\xi_{3j}, \xi_{3j+1}, \xi_{3j+2}, \xi_{3j+3n}, \xi_{3j+3n+1}, \xi_{3j+3n+2}\}$$

and

$$\mathcal{C}_j = \mathrm{span}\{\xi_{2j}, \xi_{2j+2n}, \xi_{2j+4n}, \xi_{2j+1}, \xi_{2j+2n+1}, \xi_{2j+4n+1}\}$$

for $j = 0, \ldots, n$. We shall use the bases of $\mathcal{S}_j$, respectively $\mathcal{C}_j$, in the order that they appear above. Also, let $\lambda = \exp(\frac{2\pi i}{6n})$. We define $A \in \mathbf{U}(6n)$ as the diagonal operator $A\xi_j = \lambda^j \xi_j, j = 1, \ldots, n$ and let S_j and C_j denote the restriction of A to $\mathcal{S}_j$ and $\mathcal{C}_j$, respectively. In the ordered bases as above, we have that

$$S_j = \lambda^{3j}\,\mathrm{diag}(1, \lambda, \lambda^2, -1, -\lambda, -\lambda^2)$$

and

$$C_j = \lambda^{2j}\operatorname{diag}(1, \exp(\tfrac{2\pi i}{3}), \exp(\tfrac{4\pi i}{3}), \lambda, \lambda\exp(\tfrac{2\pi i}{3}), \lambda\exp(\tfrac{4\pi i}{3})).$$

This entails that the diagonal operators S_j^2 and C_j^3 both act approximately as multiplication by λ^{6j}, more precisely,

$$\|S_j^2 - \lambda^{6j}1_{\mathcal{S}_j}\|_{2,\mathrm{Tr}}^2 = 2|\lambda^2-1|^2 + 2|\lambda^4-1|^2 = O(\tfrac{1}{n^2})$$

and

$$\|C_j^3 - \lambda^{6j}1_{\mathcal{C}_j}\|_{2,\mathrm{Tr}}^2 = 3|\lambda^3-1|^2 = O(\tfrac{1}{n^2}),$$

for $n\to\infty$. Thus, given any unitary $U\colon \mathcal{C}_j \to \mathcal{S}_j$ between the 6-dimensional subspaces of $\mathscr{H}$, it holds that

$$\|U^*S_j^2U - C_j^3\|_{2,\mathrm{Tr}}^2 = O(\tfrac{1}{n^2}),$$

for $n\to\infty$, so that, if $B\in \mathbf{U}(\mathscr{H})$ is any unitary with $B(\mathcal{C}_j)=\mathcal{S}_j$, it follows that

$$\|B^*A^2B - A^3\|_{2,\mathrm{Tr}}^2 = O(\tfrac{1}{n}),$$

for $n\to\infty$. This shows that the assignment $\varphi_n(a)=A$ and $\varphi_n(b)=B$ defines an asymptotic representation of $\mathrm{BS}(2,3)$. We claim that if we let $U_j\colon \mathcal{C}_j\to\mathcal{S}_j$ be the unitary defined by the matrix

$$U_j = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

in the above ordered bases and let $B=\bigoplus_{j=0}^{n-1}U_j\in\mathbf{U}(6n)$, the asymptotic representation is non-trivial. To see this, let $\tilde{S}_j\colon\mathcal{S}_j\to\mathcal{S}_j$ be given by

$$\tilde{S}_j = \operatorname{diag}(1,1,1,-1,-1,-1)$$

and $\tilde{C}_j\colon\mathcal{C}_j\to\mathcal{C}_j$ be given by

$$\tilde{C}_j = \operatorname{diag}(1, \exp(\tfrac{2\pi i}{3}), \exp(\tfrac{4\pi i}{3}), 1, \exp(\tfrac{2\pi i}{3}), \exp(\tfrac{4\pi i}{3})).$$

We immediately get that

$$\|S_j - \lambda^{3j}\tilde{S}_j\|_{2,\mathrm{Tr}} = O(\tfrac{1}{n}), \qquad \|C_j - \lambda^{2j}\tilde{C}_j\|_{2,\mathrm{Tr}} = O(\tfrac{1}{n}),$$

for $n \to \infty$. Furthermore, we have that

$$B_j^* \tilde{S}_j B_j = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

whence it easily follows that

$$\|\tilde{C}_j B_j^* \tilde{S}_j B_j - B_j^* \tilde{S}_j B_j \tilde{C}_j\|_{2,\mathrm{Tr}} = \sqrt{2}|1 - \exp(\tfrac{4\pi i}{3})| = \sqrt{6}.$$

Combining these facts, we get that

$$\begin{aligned} \|C_j B_j^* S_j B_j - B_j^* S_j B_j C_j\|_{2,\mathrm{Tr}} &\geq \|\lambda^{2j} \tilde{C}_j B_j^* \lambda^{3j} \tilde{S}_j B_j - B_j^* \lambda^{3j} \tilde{S}_j B_j \lambda^{2j} \tilde{C}_j\|_{2,\mathrm{Tr}} \\ &\quad - 2\|S_j - \lambda^{3j} \tilde{S}_j\|_{2,\mathrm{Tr}} - 2\|C_j - \lambda^{2j} \tilde{C}_j\|_{2,\mathrm{Tr}} \\ &= \|\tilde{C}_j B_j^* \tilde{S}_j B_j - B_j^* \tilde{S}_j B_j \tilde{C}_j\|_{2,\mathrm{Tr}} - O(\tfrac{1}{n}) \\ &= \sqrt{6} - O(\tfrac{1}{n}). \end{aligned}$$

Now, we conclude that

$$\begin{aligned} \|AB^*AB - B^*ABA\|_{2,\mathrm{Tr}}^2 &= \sum_{j=0}^{n-1} \|C_j B_j^* S_j B_j - B_j^* S_j B_j C_j\|_{2,\mathrm{Tr}}^2 \\ &= 6n - O(1), \end{aligned}$$

for $n \to \infty$. Thus, when $\varphi_n(a) = A$ and $\varphi_n(b) = B$ as above, we get

$$\lim_{n\to\infty} \|\varphi_n(a)\varphi_n(b)^*\varphi_n(a)\varphi_n(b) - \varphi_n(b)^*\varphi_n(a)\varphi_n(b)\varphi_n(a)\|_{2,\mathrm{Tr}} = \infty.$$

It follows by the remark after Corollary 4.23 that $(\varphi_n)_{n\in\mathbb{N}}$ is non-trivial.

Chapter 5

Stability results for asymptotic representations

In order to approach the stability question, we investigate the asymptotic behavior of the defect of an asymptotic representation more closely. We associate a 2-cocycle to an asymptotic representation and show that this cocycle determines whether the asymptotic representation is "improvable", in the sense that there exists an equivalent asymptotic representation with defect tending faster to zero. Using this, we prove that, if all relevant cocycles are trivial, then Γ is Frobenius stable. More precisely, if the group Γ is 2-Kazhdan (Definition 1.8), then all asymptotic representations can be improved to representations. In the last section we use this result to provide examples of Frobenius stable groups. Among these examples, there are groups that are not Frobenius approximable.

5.1 Diminishing the defect of asymptotic representations

We fix a finitely presented group $\Gamma = \langle S \mid R \rangle$, a family of *submultiplicative*, unitarily invariant norms on $\mathbf{M}_k(\mathbb{C})$, for $k \in \mathbb{N}$, all of which we denote $\|\cdot\|$, and an asymptotic representation $\varphi_n \colon S \to \mathbf{U}(k_n)$, for $n \in \mathbb{N}$, with respect to these norms. The maps φ_n are defined on S, but we want to associate maps $\tilde{\varphi}_n$ defined on Γ. For this, we fix a section $\sigma \colon \Gamma \to \mathbb{F}_S$ of the quotient map $\mathbb{F}_S \to \Gamma$. We may assume that $\sigma(1_\Gamma) = 1_{\mathbb{F}_S}$ and $\sigma(x^{-1}) = \sigma(x)^{-1}$, for all x such that $x^2 \neq 1_\Gamma$.

Lemma 5.1. *There exists a sequence $\tilde{\varphi}_n \colon \Gamma \to \mathbf{U}(k_n)$ such that $\tilde{\varphi}(1_\Gamma) = 1_{k_n}$, $\tilde{\varphi}_n(x^{-1}) = \tilde{\varphi}_n(x)^*$ and*

$$\|\varphi_n(\sigma(x)) - \tilde{\varphi}_n(x)\| = O(\operatorname{def}(\varphi_n)), \tag{1}$$

for $n \to \omega$ and all $x \in \Gamma$.

Proof. We define $\tilde{\varphi}_n(x) = \varphi_n(\sigma(x))$, for all x with $x^2 \neq 1_\Gamma$. By our assumptions on σ, it follows that $\tilde{\varphi}_n(x^{-1}) = \tilde{\varphi}_n(x)^*$ for such x. Given $x \in \Gamma$ with $x^2 = 1_\Gamma$, we first consider the map $f\colon \mathbb{C} \to \mathbb{C}$ given by

$$f(\lambda) = \begin{cases} 1, & \mathrm{Re}(\lambda) \geq 0, \\ -1, & \mathrm{Re}(\lambda) < 0, \end{cases}$$

for $\lambda \in \mathbb{C}$ and define self-adjoint unitaries $T_n = f(\varphi_n(\sigma(x)))$ by the functional calculus. Since $|\lambda - f(\lambda)| \leq |\lambda^2 - 1|$, for $\lambda \in \mathbb{C}$, and since $\sigma(x)^2 \in \langle\!\langle R \rangle\!\rangle$, it follows by Lemma 4.3 that

$$\|\varphi_n(\sigma(x)) - T_n\| \leq \|\varphi_n(\sigma(x))^2 - 1_{k_n}\| = O(\mathrm{def}(\varphi_n)),$$

for $n \to \omega$. Now, letting $\tilde{\varphi}_n(x) = T_n$, we conclude that $\tilde{\varphi}$ has the desired properties. □

We fix $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ provided by this lemma and define maps $c_n\colon \Gamma\times\Gamma \to \mathbf{M}(k_n)$ by

$$c_n(x,y) = \frac{\tilde{\varphi}_n(x)\tilde{\varphi}_n(y) - \tilde{\varphi}_n(xy)}{\mathrm{def}(\varphi_n)},$$

for all $n \in \mathbb{N}$ such that $\mathrm{def}(\varphi_n) > 0$ and $c_n(x,y) = 0$ if $\mathrm{def}(\varphi_n) = 0$, for all $x, y \in \Gamma$.

Proposition 5.2. *Let $x, y, z \in \Gamma$. The maps c_n satisfy the following equations*

$$\tilde{\varphi}_n(x)c_n(y,z) - c_n(xy,z) + c_n(x,yz) - c_n(x,y)\tilde{\varphi}_n(z) = 0,$$
$$c_n(x,x^{-1}) = c_n(1_\Gamma, x) = c_n(x, 1_\Gamma) = 0 \qquad \textit{and} \qquad c_n(x,y)^* = c_n(y^{-1}, x^{-1}).$$

Furthermore, we have that

$$\|c_n(x,y)\| = O(1) \qquad \textit{for } n \to \omega \tag{m}$$

Proof. For all $x, y, z \in \Gamma$ and $n \in \mathbb{N}$ we see that

$$\begin{aligned}
&\mathrm{def}(\varphi_n) \cdot (\tilde{\varphi}_n(x)c_n(y,z) - c_n(xy,z) + c_n(x,yz) - c_n(x,y)\tilde{\varphi}_n(z)) \\
&\quad = \tilde{\varphi}_n(x)(\tilde{\varphi}_n(y)\tilde{\varphi}_n(z) - \tilde{\varphi}_n(yz)) - (\tilde{\varphi}_n(xy)\tilde{\varphi}_n(z) - \tilde{\varphi}_n(xyz)) \\
&\qquad + (\tilde{\varphi}_n(x)\tilde{\varphi}_n(yz) - \tilde{\varphi}_n(xyz)) - (\tilde{\varphi}_n(x)\tilde{\varphi}_n(y) - \tilde{\varphi}_n(xy))\tilde{\varphi}_n(z) \\
&\quad = \tilde{\varphi}_n(x)\tilde{\varphi}_n(y)\tilde{\varphi}_n(z) - \tilde{\varphi}_n(x)\tilde{\varphi}_n(yz) - \tilde{\varphi}_n(xy)\tilde{\varphi}_n(z) + \tilde{\varphi}_n(xyz) \\
&\qquad + \tilde{\varphi}_n(x)\tilde{\varphi}_n(yz) - \tilde{\varphi}_n(xyz) - \tilde{\varphi}_n(x)\tilde{\varphi}_n(y)\tilde{\varphi}_n(z) + \tilde{\varphi}_n(xy)\tilde{\varphi}_n(z) \\
&\quad = 0,
\end{aligned}$$

which proves the first equation. The second line of equations is immediate from the definition of c_n and the fact that $\tilde{\varphi}_n(x^{-1}) = \tilde{\varphi}_n(x)^*$. For the last assertion, note that, since $\sigma(x)\sigma(y)\sigma(xy)^{-1} \in \langle\!\langle R \rangle\!\rangle$, it follows from Lemma 4.3 that

$$\|\varphi_n(\sigma(x)\sigma(y)\sigma(xy)^{-1}) - 1_{k_n}\| = O(\operatorname{def}(\varphi_n)), \qquad \text{for } n \to \omega,$$

and thus we get (by using Equation (l) if necessary) that

$$\operatorname{def}(\varphi_n)\|c_n(x,y)\| = O(\operatorname{def}(\varphi_n)), \qquad \text{for } n \to \omega. \qquad \square$$

Remark 5.3. The constants that hide in Equations (l) and (m) are coming from Lemma 4.3 and thus depend on $x, y \in \Gamma$, the presentation $\Gamma = \langle S \mid R \rangle$ and on the choice of section σ, *but not on* $(\varphi_n)_{n\in\mathbb{N}}$. This fact will be important in the proof of Lemma 5.10.

We need the various notions of ultraproducts discussed in Chapter 3 and we use the shorthand

$$\mathbf{U}(\omega, \|\cdot\|) = \prod_{n\to\omega} (\mathbf{U}(k_n), \operatorname{dist}_{\|\cdot\|}), \quad \text{and} \quad \mathbf{M}(\omega, \|\cdot\|) = \prod_{n\to\omega} (\mathbf{M}_{k_n}(\mathbb{C}), \|\cdot\|).$$

As we have seen, the asymptotic representation $(\varphi_n)_{n\in\mathbb{N}}$ induces a homomorphism $\varphi_\omega : \Gamma \to \mathbf{U}(\omega, \|\cdot\|)$ on the level of the group Γ. In fact, the element $\varphi_\omega(x)$ is represented by the sequence $(\tilde{\varphi}_n(x))_{n\in\mathbb{N}}$, for $x \in \Gamma$. Since $\|\cdot\|$ is submultiplicative, Γ acts on $\mathbf{M}(\omega, \|\cdot\|)$ through φ_ω. By Equation (m) it follows $c_n(x, y)$ is a bounded sequence, so there is an induced map

$$c\colon \Gamma \times \Gamma \to \mathbf{M}(\omega, \|\cdot\|).$$

This is almost the cocycle we want. However, it does not quite satisfy the cocycle equations with respect to the conjugation action of Γ by φ_ω, which is what we want. In order to correct for that, we define $\alpha(x, y) = c(x, y)\varphi_\omega(xy)^*$. The reason for defining c rather than just α is to make the following calculations more natural. It is also worth noting that c is a cocycle in the different, but equivalent, picture of *Hochschild* cohomology.

Corollary 5.4. *The map* $\alpha\colon \Gamma \times \Gamma \to \mathbf{M}(\omega, \|\cdot\|)$ *is a 2-cocycle with respect to the isometric action* ρ *given by* $\rho(x)T = \varphi_\omega(x)T\varphi_\omega(x)^*$, *for* $x \in \Gamma$ *and* $T \in \mathbf{M}(\omega, \|\cdot\|)$.

Proof. Given $x, y, z \in \Gamma$ we have that

$$\begin{aligned}
&\varphi_\omega(x)\alpha(y,z)\varphi_\omega(x)^* - \alpha(xy,z) + \alpha(x,yz) - \alpha(x,y) \\
&\qquad = \varphi_\omega(x)c(y,z)\varphi_\omega(yz)^*\varphi_\omega(x)^* - c(xy,z)\varphi_\omega(xyz)^* \\
&\qquad\quad + c(x,yz)\varphi_\omega(xyz)^* - c(x,y)\varphi_\omega(xy)^* \\
&\qquad = (\varphi_\omega(x)c(y,z) - c(xy,z) + c(x,yz) - c(x,y)\varphi_\omega(z))\varphi_\omega(xyz)^* \\
&\qquad = 0,
\end{aligned}$$

where we used that φ_ω is a homomorphism. $\square$

Proposition 5.5. *Assume that α represents the trivial cohomology class, i.e. there exists a map $\beta\colon \Gamma \to \mathbf{M}(\omega, \|\cdot\|)$ satisfying*

$$\alpha(x, y) = \varphi_\omega(x)\beta(y)\varphi_\omega(x)^* - \beta(xy) + \beta(x),$$

for $x, y \in \Gamma$. Then the following holds:

$$\beta(1_\Gamma) = 0, \tag{n}$$

$$\beta(x) = -\varphi_\omega(x)\beta(x^{-1})\varphi_\omega(x)^*, \tag{o}$$

$$c(x, y) = \varphi_\omega(x)\beta(y)\varphi_\omega(y) - \beta(xy)\varphi_\omega(xy) + \beta(x)\varphi_\omega(xy). \tag{p}$$

Furthermore, we can choose $\beta(x)$ to be skew-Hermitian for all $x \in \Gamma$.

Proof. Equation (p) is immediate from the equation $c(x, y) = \alpha(x, y)\varphi_\omega(xy)$. Equation (n) follows from (p) and Proposition 5.2 with $x = y = 1_\Gamma$ and (o) follows from (n), (p) and Proposition 5.2 with $y = x^{-1}$. For the last claim, we possibly need to alter β a little. Note that $\beta'(x) = -\beta(x)^* = \varphi_\omega(x)\beta(x^{-1})^*\varphi_\omega(x)^*$ also satisfies Equations (n), (o) and (p). Indeed,

$$\begin{aligned}
c(x, y) &= c(y^{-1}, x^{-1})^* \\
&= (\varphi_\omega(y^{-1})\beta(x^{-1})\varphi_\omega(x^{-1}) - \beta(y^{-1}x^{-1})\varphi_\omega(y^{-1}x^{-1}) \\
&\quad + \beta(y^{-1})\varphi_\omega(y^{-1}x^{-1}))^* \\
&= \varphi_\omega(x)\beta(x^{-1})^*\varphi_\omega(y) - \varphi_\omega(xy)\beta((xy)^{-1})^* + \varphi_\omega(xy)\beta(y^{-1})^* \\
&= \beta'(x)\varphi_\omega(xy) - \beta'(xy)\varphi_\omega(xy) + \varphi_\omega(x)\beta'(y)\varphi_\omega(x)
\end{aligned}$$

for $x, y \in \Gamma$, which proves (p), whence the other two follow. Thus, replacing β with

$$\beta^\sharp(x) = \frac{\beta(x) - \beta(x)^*}{2}, \qquad x \in \Gamma, n \in \mathbb{N},$$

we see that $\beta^\sharp(x)$ is skew-Hermitian and that Equations (n), (o) and (p) are still satisfied. $\square$

For the rest of the section, we assume that α is trivial and that β is a skew-Hermitian associated 1-cochain as above. We furthermore let $\beta_n\colon \Gamma \to \mathrm{M}_{k_n}(\mathbb{C})$ be a skew-Hermitian lift of β. It is easy to check that the matrix

$$\exp(-\operatorname{def}(\varphi_n)\beta_n(x)) = \sum_{j=0}^{\infty} \frac{(-\operatorname{def}(\varphi_n))^j}{j!}\beta_n(x)^j$$

is unitary, for every $x \in \Gamma$, so we may define a sequence of maps $\psi_n\colon \Gamma \to \mathrm{U}(k_n)$ by

$$\psi_n(x) = \exp(-\operatorname{def}(\varphi_n)\beta_n(x))\tilde{\varphi}_n(x).$$

As we know that $\tilde{\varphi}_n(1_\Gamma) = 1_{k_n}$ and $\beta_n(1_\Gamma) = 0$, we also get that $\psi_n(1_\Gamma) = 1_{k_n}$.

Proposition 5.6. *With the notation from above, we have that* $\|\tilde{\varphi}_n(x) - \psi_n(x)\| = O(\operatorname{def}(\varphi_n))$, *for* $n \to \omega$ *and all* $x \in \Gamma$. *More precisely, it holds that*

$$\|\tilde{\varphi}_n(x) - \psi_n(x)\| \leq 2\|\beta_n(x)\| \operatorname{def}(\varphi_n),$$

for most $n \in \mathbb{N}$.

Proof. Let $x \in \Gamma$. Using submultiplicativity, it is easy to see that

$$\|1_{k_n} - \exp(T)\| \leq \|T\| \exp(\|T\|)$$

for any $T \in \mathbf{M}(k_n)$. It now follows by unitary invariance that

$$\begin{aligned}\|\tilde{\varphi}_n(x) - \psi_n(x)\| &= \|1_{k_n} - \exp(-\operatorname{def}(\varphi_n)\beta_n(x))\| \\ &\leq \operatorname{def}(\varphi_n)\|\beta_n(x)\| \exp(\operatorname{def}(\varphi_n)\|\beta_n(x)\|),\end{aligned}$$

so since $\|\beta_n(x)\|$ is a bounded sequence and $\lim_{n\to\omega} \operatorname{def}(\varphi_n) = 0$, we conclude that $\exp(\operatorname{def}(\varphi_n)\|\beta_n(x)\|) \leq 2$, for most n, whence the desired conclusion follows. ∎

Lemma 5.7. *For any* $x, y \in \Gamma$, *it holds that*

$$\|\psi_n(xy) - \psi_n(x)\psi_n(y)\| = o(\operatorname{def}(\varphi_n)),$$

for $n \to \omega$.

Proof. Fix $x, y \in \Gamma$ and let $\xi_n(z) = (1_{k_n} - \operatorname{def}(\varphi_n)\beta_n(z))\tilde{\varphi}_n(z)$, for an $z \in \Gamma$. Let $C = 2\max\{\|\beta(x)\|, \|\beta(y)\|, \|\beta(xy)\|\}$. We note that every operator $T \in \mathbf{M}_k(\mathbb{C})$ satisfies

$$\|1_k - T - \exp(-T)\| \leq \|T\|^2 \exp(\|T\|).$$

It follows that, for most $n \in \mathbb{N}$,

$$\|\psi_n(z) - \xi_n(z)\| \leq C \cdot \operatorname{def}(\varphi_n)^2,$$

when $z \in \{x, y, xy\}$. By this (and submultiplicativity) it follows that

$$\|\psi_n(xy) - \psi_n(x)\psi_n(y)\| = \|\xi_n(xy) - \xi_n(x)\xi_n(y)\| + o(\operatorname{def}(\varphi_n)),$$

for $n \to \omega$. Hence it suffices to show that

$$\|\xi_n(xy) - \xi_n(x)\xi_n(y)\| = o(\operatorname{def}(\varphi_n)),$$

for $n \to \omega$. However, this is clearly a consequence of the following calculation:

$$\begin{aligned}\xi_n(xy) - \xi_n(x)\xi_n(y) &= \tilde{\varphi}_n(xy) - \tilde{\varphi}_n(x)\tilde{\varphi}_n(y) \\ &\quad - \operatorname{def}(\varphi_n)(\beta_n(x)y\tilde{\varphi}_n(xy) \\ &\quad - \tilde{\varphi}_n(x)\beta_n(y)\tilde{\varphi}_n(y) \\ &\quad - \beta_n(x)\tilde{\varphi}_n(x)\tilde{\varphi}_n(y)) \\ &\quad - \operatorname{def}(\varphi_n)^2\beta_n(x)\tilde{\varphi}_n(x)\beta_n(y)\tilde{\varphi}(y) \\ &= \operatorname{def}(\varphi_n)(-c_n(x,y) \\ &\quad + \tilde{\varphi}_n(x)\beta_n(y)\tilde{\varphi}_n(y) \\ &\quad - \beta_n(xy)\tilde{\varphi}_n(xy) \\ &\quad + \beta_n(x)\tilde{\varphi}_n(x)\tilde{\varphi}_n(y)) \\ &\quad - \operatorname{def}(\varphi_n)^2\beta_n(x)\tilde{\varphi}_n(x)\beta_n(y)\tilde{\varphi}_n(y).\end{aligned}$$

Indeed, combining Equation (p) with the fact that $\|\beta_n(x)\tilde{\varphi}_n(x)\beta_n(y)\tilde{\varphi}_n(y)\| \leq \|\beta_n(x)\|\|\beta_n(y)\|$ is bounded, the desired result follows. □

Finally we are ready to define the asymptotic representation $\varphi'_n\colon S \to \mathrm{U}(k_n)$ by $\varphi'_n = \psi_n|_S$. It follows from Proposition 5.6 that φ'_n is an asymptotic representation equivalent to φ_n. Moreover, from Lemma 5.7 we can prove the φ'_n has an effectively smaller defect. In conclusion, we get the following theorem.

Theorem 5.8. *Let $\Gamma = \langle S \mid R\rangle$ be a finitely presented group and let $\varphi_n\colon S \to \mathrm{U}(k_n)$ be an asymptotic representation with respect to a family of submultiplicative, unitarily invariant norms $\|\cdot\| = \|\cdot\|_n$ on $\mathbf{M}(k_n)$. Assume that the 2-cocycle α associated to $(\varphi_n)_{n\in\mathbb{N}}$ is trivial in $H^2(\Gamma, \mathbf{M}(\omega, \|\cdot\|))$. Then there exists an asymptotic representation $\varphi'_n\colon S \to \mathrm{U}(k_n)$ such that, as $n \to \omega$,*

(1) $\operatorname{dist}(\varphi_n, \varphi'_n) = O(\operatorname{def}(\varphi_n))$ *and*

(2) $\operatorname{def}(\varphi'_n) = o(\operatorname{def}(\varphi_n))$*,*

Proof. Assertion (1) follows from Proposition 5.6. For Assertion (2), we let $r = x_1x_2\cdots x_m \in R$ be written as a reduced word, with $x_j \in S \cup S^{-1}$, where $j = 1, \ldots, m$. By iteration of Lemma 5.7, using that ψ_n takes unitary values and that $\|\cdot\|$ is unitarily invariant, we see that

$$\begin{aligned}\|\varphi'_n(r) - 1_{k_n}\| &= \|\psi_n(x_1)\psi_n(x_2)\cdots\psi_n(x_m) - 1_{k_n}\| \\ &= \|\psi_n(x_1x_2)\psi_n(x_3)\cdots\psi_n(x_m) - 1_{k_n}\| + o(\operatorname{def}(\varphi_n)) \\ &\;\;\vdots \\ &= \|\psi(1_\Gamma) - 1_{k_n}\| + o(\operatorname{def}(\varphi_n)),\end{aligned}$$

for $n \to \omega$. Since $\psi(1_\Gamma) = 1_{k_n}$, Assertion (2) follows. □

It is worth noting that there is also a converse to Theorem 5.8 in the following sense.

Proposition 5.9. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented group and let $\varphi_n, \psi_n : S \to \mathrm{U}(k_n)$ be asymptotic representations with respect to some family of submultiplicative, unitarily invariant norms $\|\cdot\| = \|\cdot\|_n$ on $\mathbf{M}(k_n)$. Suppose that*

(1) $\mathrm{dist}(\varphi_n, \psi_n) = O_\omega(\mathrm{def}(\varphi_n))$ *and*

(2) $\mathrm{def}(\psi_n) = o_\omega(\mathrm{def}(\varphi_n))$.

Then the 2-cocycle α associated to $(\varphi_n)_{n\in\mathbb{N}}$ is trivial in $H^2(\Gamma, \mathbf{M}(\omega, \|\cdot\|))$. In particular, if $\mathrm{HomDist}(\varphi_n) = O(\mathrm{def}(\varphi_n))$, for $n \to \omega$, then α is trivial.

Proof. If $\mathrm{def}(\varphi_n) = 0$ for most $n \in \mathbb{N}$, there is nothing to prove, so let us assume this is not the case. Let $\tilde{\varphi}_n, \tilde{\psi}_n \colon \Gamma \to \mathbf{U}(k_n)$ be the induced maps given by Lemma 5.1. We note that the sequences $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ induce the same map φ_ω. Define

$$\gamma_n(x) = \frac{\tilde{\varphi}_n(x) - \tilde{\psi}_n(x)}{\mathrm{def}(\varphi_n)}$$

for n with $\mathrm{def}(\varphi_n) > 0$ and $\gamma_n(x) = 0$ otherwise. By Assumption (1), $\gamma_n(x)$ is essentially bounded in n, so it defines an element $\gamma(x) \in \mathbf{M}(\omega, \|\cdot\|)$. Now, if we prove that

$$c(x,y) = \varphi_\omega(x)\gamma(y) - \gamma(xy) + \gamma(x)\varphi_\omega(y),$$

it will easily follow that $\beta(x) = \gamma(x)\varphi_\omega(x)^*$ will satisfy $d^1\beta = \alpha$. To prove this, note that

$$\|\tilde{\psi}_n(xy) - \tilde{\psi}_n(x)\tilde{\psi}_n(y)\| = o(\mathrm{def}(\varphi_n)), \qquad n \to \omega,$$

for all $x, y \in \Gamma$. Indeed, this follows from Assumption (2). As a consequence, we see that

$$\begin{aligned}
\mathrm{def}(\varphi_n) \cdot (\tilde{\varphi}_n(x)\gamma_n(y) - \gamma_n(xy) + \gamma_n(x)\tilde{\psi}_n(y))& \\
= \tilde{\varphi}_n(x)\tilde{\varphi}_n(y) - \tilde{\varphi}_n(x)\tilde{\psi}_n(y) - \tilde{\varphi}_n(xy) + \tilde{\psi}_n(xy)& \\
+ \tilde{\varphi}_n(x)\tilde{\psi}_n(y) - \tilde{\psi}_n(x)\tilde{\psi}_n(y)& \\
= \tilde{\varphi}_n(x)\tilde{\varphi}_n(y) - \tilde{\varphi}_n(xy) + \tilde{\psi}_n(xy) - \tilde{\psi}_n(x)\tilde{\psi}_n(y)& \\
= \mathrm{def}(\varphi_n) \cdot c_n(x,y) + o(\mathrm{def}(\varphi_n)),&
\end{aligned}$$

for $n \to \omega$. By dividing the above equation by $\mathrm{def}(\varphi_n)$ (which is possible for most n) and taking the limit, we reach the desired conclusion. $\square$

5.2 The Frobenius-stability of 2-Kazhdan groups

We now consider the Frobenius norm $\|\cdot\|_{2,\mathrm{Tr}}$. This norm is submultiplicative, so the techniques developed in last section apply. Moreover, since the norm comes from the inner product $(T,S) \mapsto \mathrm{Tr}(TS^*)$ for $T,S \in \mathbf{M}_k(\mathbb{C})$, it follows that the ultraproduct $\mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}})$ is a Hilbert space. In particular, if the group Γ is 2-Kazhdan, then $H^2(\Gamma, \mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}})) = 0$ for any isometric action of Γ on $\mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}})$, since invertible isometries are unitaries. Thus Theorem 5.8 applies to all asymptotic representations $(\varphi_n)_{n\in\mathbb{N}}$, and we could use the theorem repeatedly to get a sequence of equivalent asymptotic representations $\varphi_n, \varphi_n', \varphi_n'', \ldots$ with better and better defect. This approach, however, seems too naïve to prove stability as there is no reason this process would ever give us a genuine representation. Luckily, a slightly more careful argument works. First, we need to overcome a small technicality. In Theorem 5.8, we saw that $\mathrm{dist}(\varphi_n, \varphi_n') = O(\mathrm{def}(\varphi_n))$ for $n \to \omega$, but we a more precise statement. The following uses the notation from last section.

Lemma 5.10. *Let $\Gamma = \langle S \mid R\rangle$ be a finitely presented, 2-Kazhdan group. There exists a constant $K \geq 0$ such that, for all asymptotic representations $\varphi_n \colon S \to \mathbf{U}(k_n)$ with respect to the Frobenius norm, the associated maps φ_n' satisfy*

$$\mathrm{dist}(\varphi_n, \varphi_n') \leq K \,\mathrm{def}(\varphi_n).$$

Proof. It follows from Proposition 1.10 applied to $F = S \subseteq \Gamma$ that there is a constant K_1 and a finite set $F_0 \subseteq \Gamma^2$ such that, for all 2-cocycles $\alpha \in Z^2(\Gamma, \mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}}))$, there is a 1-cochain $\beta \in C^1(\Gamma, \mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}}))$ satisfying $d^1\beta = \alpha$ and

$$\max_{s\in S}\|\beta(s)\| \leq K_1 \max_{x,y\in F_0}\|\alpha(x,y)\|.$$

This, in particular, applies to the 2-cocycle α associated to any asymptotic representation $(\varphi_n)_{n\in\mathbb{N}}$. We note that replacing $\beta(x)$ with the skew-Hermitian element $2^{-1}(\beta(x) - \beta(x)^*)$, as is done in Proposition 5.5, does not increase the norm, so the above inequality also holds for the modified cochain. Furthermore, from Equation (m) we get a constant K_2 such that for all asymptotic representations $\varphi_n \colon \Gamma \to \mathrm{U}(k_n)$ with respect to the Frobenius norm the associated 2-cocycle α satisfies

$$\|\alpha(x,y)\| = \|c(x,y)\| \leq K_2,$$

for all $x, y \in F_0$. As mentioned in Remark 5.3, the same K_2 works for all asymptotic representations. In conclusion, we get that

$$\max_{s\in S}\|\beta(s)\| \leq K_1K_2.$$

Letting $K = 2K_1K_2$, it now follows from Proposition 5.6, since $\varphi_n' = \psi_n|_S$ by definition, that $\operatorname{dist}(\varphi_n, \varphi_n') \leq K \operatorname{def}(\varphi_n)$. □

Theorem 5.11. *Every finitely presented, 2-Kazhdan group is Frobenius stable.*

Proof. Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented, 2-Kazhdan group and determine K from Lemma 5.10. Now, define the quantity

$$\theta(\varphi) = \operatorname{HomDist}(\varphi) - 2K \operatorname{def}(\varphi),$$

for any map $\varphi\colon S \to \mathbf{U}(k)$ and any $k \in \mathbb{N}$. We note that, if $\varphi_n : S \to \mathrm{U}(k_n)$ is any asymptotic representation, then $\lim_{n\to\omega} \theta(\varphi_n) \geq 0$ and equality holds if and only if φ_n is equivalent to a sequence of homomorphisms. Now, fix a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of strictly positive real numbers tending to zero along ω and let $(k_n)_{n\in\mathbb{N}}$ be a sequence of natural numbers. By the above, we need to prove that, for all sequences of maps $\psi_n\colon S \to \mathrm{U}(k_n)$ with $\operatorname{def}(\psi_n) \leq \varepsilon_n$, the quantity $\theta(\psi_n)$ tends to zero. The space of maps $\varphi\colon S \to \mathrm{U}(k_n)$ such that $\operatorname{def}(\varphi) \leq \varepsilon_n$ is compact for each $n \in \mathbb{N}$, and since θ is continuous, there is $\varphi_n\colon S \to \mathrm{U}(k_n)$ with $\operatorname{def}(\varphi_n) \leq \varepsilon_n$ such that φ_n maximizes θ, for all n. Evidently $(\varphi_n)_{n\in\mathbb{N}}$ is an asymptotic representation. Thus, by Theorem 5.8 and Lemma 5.10, there exists an asymptotic representation $\varphi_n'\colon S \to \mathrm{U}(k_n)$ with $\operatorname{dist}(\varphi_n, \varphi_n') \leq K \operatorname{def}(\varphi_n)$ and

$$\operatorname{def}(\varphi_n') \leq \frac{1}{4} \operatorname{def}(\varphi_n),$$

for most $n \in \mathbb{N}$. In particular, $\operatorname{def}(\varphi_n') \leq \varepsilon_n$, and it follows that, for most n, we have the following inequality

$$\operatorname{HomDist}(\varphi_n) \leq \operatorname{HomDist}(\varphi_n') + K \operatorname{def}(\varphi_n).$$

Combining what we know so far, we get that

$$\begin{aligned} \operatorname{HomDist}(\varphi_n') - \frac{1}{2} K \operatorname{def}(\varphi_n) &\leq \operatorname{HomDist}(\varphi_n') - 2K \operatorname{def}(\varphi_n') \\ &= \theta(\varphi_n') \leq \theta(\varphi_n) \\ &= \operatorname{HomDist}(\varphi_n) - 2K \operatorname{def}(\varphi_n) \\ &\leq \operatorname{HomDist}(\varphi_n') - K \operatorname{def}(\varphi_n), \end{aligned}$$

or, in other words,

$$\operatorname{def}(\varphi_n) \leq \frac{1}{2} \operatorname{def}(\varphi_n).$$

This can only be the case if $\operatorname{def}(\varphi_n) = 0$ for most n, but then φ_n is an actual representation for most $n \in \mathbb{N}$. This means that $\operatorname{HomDist}(\varphi_n) = 0$, so we conclude that $\lim_{n\to\omega} \theta(\varphi_n) = 0$. Since $\theta(\varphi_n)$ was chosen maximal, we conclude that $\lim_{n\to\omega} \theta(\psi_n) = 0$ for all ε_n-almost representations ψ_n. □

This theorem has an immediate corollary, which follows from the observation right below Proposition 4.20.

Corollary 5.12. *Let Γ be a finitely presented 2-Kazhdan group. Then either*

- *Γ is residually finite, or*
- *Γ is not Frobenius approximable.*

In the next section, we shall prove the existence of non-residually finite 2-Kazhdan groups and thus provide the first examples of groups which are not Frobenius approximable. It is worth noting that the proof of Theorem 5.11 still works if one replaces the Frobenius norm with any submultiplicative norm and changes the cohomology vanishing assumption accordingly. For instance, if $H^2(\Gamma, \mathcal{A}) = 0$ for any isometric Γ-action on a C^*-algebra $\mathcal{A}$, then Γ is stable with respect to the operator norm. We are, however, not aware of any cohomology vanishing results in this setting. For the normalized Hilbert-Schmidt norm, we run into some problems of a more fundamental nature; the norm is not submultiplicative so Theorem 5.8 does not apply. Although we can say little about stability with respect to either the operator norm or the normalized Hilbert-Schmidt norm, we can still use Theorem 5.11 to prove a partial stability result in these cases.

Corollary 5.13. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely presented 2-Kazhdan group and let $\varphi_n \colon S \to \mathrm{U}(k_n)$ be a sequence of maps such that*

$$\operatorname{def}(\varphi_n) = o(k_n^{-1/2}),$$

for $n \to \omega$, where the defect is measured with respect to either the operator norm or the Hilbert-Schmidt norm. Then φ_n is equivalent to a sequence of representations with respect to the same norm.

Proof. Let $\|\cdot\|$ be the norm in question. Recall that $\|T\| \leq \|T\|_{2,\mathrm{Tr}} \leq \sqrt{k}\|T\|$, for all $T \in \mathbf{M}(k)$. Thus, by assumption

$$\operatorname{def}_{\|\cdot\|_{2,\mathrm{Tr}}}(\varphi_n) \leq \sqrt{k_n}\operatorname{def}_{\|\cdot\|}(\varphi_n) = o(1),$$

for $n \to \omega$, in other words, φ_n is an asymptotic representation with respect to $\|\cdot\|_{2,\mathrm{Tr}}$. By Theorem 5.11 there are representations $\pi_n \colon \Gamma \to \mathrm{U}(k_n)$ with

$$\|\varphi_n(s) - \pi_n(s)\| \leq \|\varphi_n(s) - \pi_n(s)\|_{2,\mathrm{Tr}} = o(1),$$

for $s \in S$ and $n \to \omega$. $\square$

5.3 Examples of stable and non-approximable groups

We use the results from last section to provide examples of stable and non-approximable groups with respect to the Frobenius norm. As we alluded to earlier, there are two reasons to choose this norm: First, it is submultiplicative and as we saw this is an essential prerequisite to applying Theorem 5.8. Second, $\mathbf{M}(\omega, \|\cdot\|_{2,\mathrm{Tr}})$ is a Hilbert space and, as we shall see next, there are nice cohomology vanishing results in this context. Indeed, the existence of strongly n-Kazhdan groups for all $n \in \mathbb{N}$ is already known. They arise as lattices of certain p-adic Lie groups, so we recall the following definition.

Definition 5.14. Let G be a topological group. A *lattice* Γ in G is a subgroup $\Gamma \subseteq G$ which is discrete as a subspace such that the space G/Γ admits a G-invariant, finite (σ-additive) Borel measure. A lattice Γ is called *uniform* if G/Γ is compact.

For the next theorem, we refer the reader to the original proofs by Garland [30] for finite dimensional Hilbert spaces and Ballmann-Świątkowski [5] in the general case.

Theorem 5.15. *Fix $n \in \mathbb{N}$ with $n \geq 2$. There exists some $p_0(n) \in \mathbb{N}$ such that all lattices $\Gamma \subseteq \mathbf{Sp}_{2n}(\mathbb{Q}_p)$ are strongly $(n-1)$-Kazhdan, whenever $p \geq p_0(n)$.*

It is standard fact that all lattices in $\mathbf{Sp}_{2n}(\mathbb{Q}_p)$ are uniform and that uniform lattices are finitely presented. In the case where $p \geq p_0(3)$, the above theorem states that lattices in $\mathbf{Sp}_{2n}(\mathbb{Q}_p)$ are 2-Kazhdan and by Theorem 5.11 they are, in fact, Frobenius stable. It is also known that lattices in $\mathbf{Sp}_{2n}(\mathbb{Q}_p)$ are residually finite (if the rank $n \geq 3$) and in this way we get plenty examples of Frobenius approximable, stable groups. In order to construct stable groups that are not Frobenius approximable, we have to work some more. The following construction is essentially known and follows ideas of Deligne [20] accommodated to the setting of p-adic Lie groups. See [72] for a short exposition of this result.

In the following, fix $n \in \mathbb{N}$ with $n \geq 2$ and a prime number $p \geq 3$. For a field K of characteristic 0, we define

$$\mathbf{G}(K) = \mathbf{SU}_n(D(K), h_n),$$

where $D(K)$ is the quaternions over K and h_n is the canonical hermitian sesquilinear form on $D(K)^n$ explained in Section 1.2. The group $\mathbf{G}(K)$ is the K-points of an absolutely almost simple, simply connected $\mathbb{Q}$-algebraic group $\mathbf{G}$ of type C_n. For an exposition of the general theory of algebraic groups, we refer the reader to [55] or [46]. More generally, if A is a unital subring of a field K, we define

$$\mathbf{G}(A) = \mathbf{SU}_n(D(K), h_n) \cap \mathbf{M}_n(D(A)).$$

An mentioned in Section 1.2 that $\mathbf{G}(\mathbb{R})$ is isomorphic to the compact group $\mathbf{U}(2n) \cap \mathbf{Sp}_{2n}(\mathbb{C})$ and that $\mathbf{G}(\mathbb{Q}_p)$ is isomorphic to $\mathbf{Sp}_{2n}(\mathbb{Q}_p)$, the symplectic group. We consider the abstract subgroup $\mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ of $\mathbf{G}(\mathbb{Q})$. A standard fact (see [55, Chapter 5.4]) shows that the diagonal embedding,

$$\mathbf{G}(\mathbb{Z}[\tfrac{1}{p}]) \subseteq \mathbf{G}(\mathbb{Q}) \subseteq \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{Q}_p),$$

is a lattice embedding. This implies that $\mathbf{G}(\mathbb{Z}[\frac{1}{p}]) \subseteq \mathbf{G}(\mathbb{Q}_p)$ is also a lattice. Indeed, the existence of an invariant measure is immediate and since $\mathbf{G}(\mathbb{R})$ is compact, it follows that the embedding $\mathbf{G}(\mathbb{Z}[\frac{1}{p}]) \subseteq \mathbf{G}(\mathbb{Q}_p)$ must be discrete. It is worth noting that $\mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ is not a lattice in $\mathbf{G}(\mathbb{R})$, since the latter, being compact, has only finite lattices. We can, however, use the embedding (as an abstract group) $\mathbf{G}(\mathbb{Z}[\frac{1}{p}]) \subseteq \mathbf{G}(\mathbb{R})$ to describe the former group somewhat explicitly. Indeed, the embedding $D(\mathbb{R}) \to \mathbf{M}_n(\mathbb{C})$ maps $D(\mathbb{Z}[\frac{1}{p}])$ into the subring $\mathbf{M}_2(\mathbb{Z}[i, \frac{1}{p}]) \subseteq \mathbf{M}_2(\mathbb{C})$, and as in the case of real coefficients, we get a group isomorphism

$$\mathbf{G}(\mathbb{Z}[\tfrac{1}{p}]) \simeq \mathbf{U}(2n) \cap \mathrm{Sp}_{2n}(\mathbb{Z}[i, \tfrac{1}{p}]),$$

of abstract groups.

We use the inclusion $\mathbf{G}(\mathbb{Z}[\frac{1}{p}]) \subseteq \mathbf{G}(\mathbb{Q})$ to introduce two *completions* of $\mathbf{G}(\mathbb{Q})$: the *arithmetic* and the *congruence* completion. The arithmetic topology can be defined for a general inclusion of groups. Recall that two subgroups Γ and Λ of the same group G are *commensurable* if the subgroup $\Gamma \cap \Lambda$ has finite index in both Γ and Λ.

Definition 5.16. Let G be a group and let $\Gamma \subseteq G$ be a subgroup. The *arithmetic completion on G with respect to* Γ is the Hausdorff completion $\widehat{G}$ of G with respect to the subgroups

$$\mathcal{B}(\Gamma) = \{\Lambda \subseteq G \mid \Lambda \text{ commensurable to } \Gamma\}.$$

That is, $\widehat{G}$ is the inverse limit of the directed system G/Λ, where $\Lambda \in \mathcal{B}(\Gamma)$.

The group $\widehat{G}$ is a subgroup of the direct product $\prod_{\Lambda \in \mathcal{B}(\Gamma)} G/\Lambda$ and thus has a natural topology induced by the product topology, where G/Λ is viewed as a discrete group. Note that the kernel of the natural map $\iota\colon G \to \widehat{G}$ is exactly the intersection of all finite index subgroups of Γ. Thus, ι is injective if and only if Γ is residually finite. Furthermore, the profinite completion $\hat{\Gamma}$ of Γ is isomorphic to the closure of $\iota(\Gamma)$ inside $\widehat{G}$.

In order to define the second completion, we consider some specific finite index subgroups of $\mathbf{G}(\mathbb{Z}[\frac{1}{p}])$, namely the *congruence subgroups*. Given $m \in \mathbb{N}$ not divisible by p, we have that p is invertible modulo m and thus there is a surjection $\varphi_m \colon \mathbb{Z}[\frac{1}{p}] \to \mathbb{Z}/m\mathbb{Z}$. The congruence subgroups $E(m) \subseteq \mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ are defined as the elements of $\mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ whose matrix entries lie in the kernel of the map φ_m. The *congruence completion* $\overline{\mathbf{G}(\mathbb{Q})}$ is the Hausdorff completion with respect to the system

$$\mathcal{B}' = \{\Lambda \in \mathcal{B}(\mathbf{G}(\mathbb{Z}[\tfrac{1}{p}])) \mid \exists m : E(m) \subseteq \Lambda\}.$$

By definition, there is a surjection $\pi \colon \widehat{\mathbf{G}(\mathbb{Q})} \to \overline{\mathbf{G}(\mathbb{Q})}$. Rapinchuk [58] and Tomanov [68] proved that Γ has the so-called congruence subgroup property, which, by a result of Prasad-Rapinchuk [57] proves that the surjection π defined above is an isomorphism. This last statement has a certain interpretation in the language of *adeles*. Let $\mathcal{P}$ be the set of all prime numbers. For a subset $S \subseteq \mathcal{P}$, we can define the *restricted product*

$$\mathbf{G}(\mathbb{A}_S) = \Big\{(x_p)_{p\in S} \in \prod_{p\in S} \mathbf{G}(\mathbb{Q}_p) \mid x_p \notin \mathbf{G}(\mathbb{Z}_p) \text{ for a finite number of } p \in S\Big\}.$$

We also define $\mathbf{G}(\mathbb{A}) = \mathbf{G}(\mathbb{A}_{\mathcal{P}}) \times \mathbf{G}(\mathbb{R})$, where $\mathcal{P}$ denotes the set of all primes. This group carries a natural topology, called the *restricted product topology* which turns $\mathbf{G}(\mathbb{A})$ into a locally compact topological group. The strong approximation theorem (see [55, Chapter 7]) states that $\mathbf{G}(\mathbb{Q})$ is dense in $\mathbf{G}(\mathbb{A}_{\mathcal{P}\setminus\{p\}})$ for any prime p, which can be formulated in the following way:

Theorem 5.17 (Strong approximation)**.** *For any prime number $p \in \mathcal{P}$ we have that $\overline{\mathbf{G}(\mathbb{Q})} \simeq \mathbf{G}(\mathbb{A}_{\mathcal{P}\setminus\{p\}})$.*

Combining the above facts, we conclude that

$$\mathbf{G}(\mathbb{A}) \simeq \mathbf{G}(\mathbb{Q}_p) \times \mathbf{G}(\mathbb{R}) \times \widehat{\mathbf{G}(\mathbb{Q})}.$$

Now we are almost ready to define the candidate for a non-approximable group. The only thing we need is the existence of a certain universal extension of $\mathbf{G}(\mathbb{Q}_p) \simeq \mathbf{Sp}_{2n}(\mathbb{Q}_p)$. An extension of topological groups is nothing but a short exact sequence where the homomorphisms are continuous. The following construction is due to Deligne [21] and universality by Prasad [56] (see also [22]).

Theorem 5.18. *Let $p \in \mathbb{N}$ be a prime and let $C(p)$ be the cyclic group of order $p-1$. There exists a central extension*

$$1 \longrightarrow C(p) \longrightarrow \widetilde{\mathbf{G}(\mathbb{Q}_p)} \longrightarrow \mathbf{G}(\mathbb{Q}_p) \longrightarrow 1$$

of topological groups which is universal *in the sense that if*

$$1 \longrightarrow C \longrightarrow E \longrightarrow \mathbf{G}(\mathbb{Q}_p) \longrightarrow 1$$

is another central extension of topological groups with discrete kernel C, then the extensions fit into a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & C \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{Q}_p) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & C(p) \arrow[r] & \widetilde{\mathbf{G}(\mathbb{Q}_p)} \arrow[r] & \mathbf{G}(\mathbb{Q}_p) \arrow[r] & 1
\end{tikzcd}$$

We define $\tilde{\Gamma}$ to be the pre-image of $\Gamma = \mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ in the group $\widetilde{\mathbf{G}(\mathbb{Q}_p)}$. This is evidently an extension of $\mathbf{G}(\mathbb{Z}[\frac{1}{p}])$ with finite kernel $C(p)$, so we can invoke Theorem 1.12 to conclude that $\tilde{\Gamma}$ is 2-kazhdan if $p \geq p_0(n)$. We prove that $\tilde{\Gamma}$ is not residually finite. For this, we need a last theorem in the spirit of Theorem 5.18. The proof works for all almost absolutely simple, simply connected algebraic groups and is due to Moore [47] in the special case of *split* groups and Prasad-Rapinchuk [57] in general.

Theorem 5.19. *Let Z be a finite group and assume that there is an extension*

$$1 \longrightarrow Z \longrightarrow E \xrightarrow{\eta} \mathbf{G}(\mathbb{A}) \longrightarrow 1$$

that splits *over $\mathbf{G}(\mathbb{Q})$, i.e. there exists a homomorphism $\psi\colon \mathbf{G}(\mathbb{Q}) \to E$ such that $\eta \circ \psi = \mathrm{id}_{\mathbf{G}(\mathbb{Q})}$. Then $|Z| \leq 2$.*

Theorem 5.20. *If $p \geq 5$ then $\tilde{\Gamma}$, defined above, is not residually finite.*

Proof. Our aim is to prove that every finite index subgroup of $\tilde{\Gamma}$ contains the unique subgroup of $C(p)$ of index 2. Let $\widehat{\widetilde{\mathbf{G}(\mathbb{Q}_p)}}$ denote the arithmetic completion of $\widetilde{\mathbf{G}(\mathbb{Q}_p)}$ with respect to the subgroup $\tilde{\Gamma}$. It follows from the definition that there is commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & C(p) \arrow[r] \arrow[d, "\mu"'] & \tilde{\Gamma} \arrow[r] \arrow[d, "\nu"'] & \Gamma \arrow[r] \arrow[d, "\iota"'] & 1 \\
1 \arrow[r] & Z \arrow[r] & \widehat{\widetilde{\mathbf{G}(\mathbb{Q}_p)}} \arrow[r] & \widehat{\mathbf{G}(\mathbb{Q}_p)} \arrow[r] & 1
\end{tikzcd}$$

where ν and ι are the natural maps explained after Definition 5.16 and μ is surjective. Since Γ is residually finite, it follows that ι is injective, and we conclude that the kernel of μ coincides with the kernel of ν. In other words, $\ker(\mu)$ is the intersection of all finite index subgroups of $\tilde{\Gamma}$, so if we can prove that $\ker(\mu)$ is non-trivial, we are done. We observed that

$$\mathbf{G}(\mathbb{A}) \simeq \mathbf{G}(\mathbb{R}) \times \mathbf{G}(\mathbb{Q}_p) \times \widehat{\mathbf{G}(\mathbb{Q})},$$

and thus the group

$$\tilde{E} = \mathbf{G}(\mathbb{R}) \times \widetilde{\mathbf{G}(\mathbb{Q}_p)} \times \widetilde{\widehat{\mathbf{G}(\mathbb{Q})}}$$

is an extension of $\mathbf{G}(\mathbb{A})$ with kernel $\tilde{F} = \{1_{G(\mathbb{R})}\} \times C(p) \times Z$. Now define the subgroup $F = \{(1, a, b) \in \tilde{E} \mid a \in C(p), b = \mu(a)\}$ of $\tilde{F}$ and let $E = \tilde{E}/F$. Thus E is also a central extension of $\mathbf{G}(\mathbb{A})$, and the kernel is isomorphic to Z. Note that $\widetilde{\mathbf{G}(\mathbb{Q})}$ embeds diagonally into the last two factors of $\tilde{E}$. This embedding maps $C(p)$ into F, it follows that the map $\widetilde{\mathbf{G}(\mathbb{Q})} \to \tilde{E} \to E$ factors through $\mathbf{G}(\mathbb{Q}) \simeq \widetilde{\mathbf{G}(\mathbb{Q})}/C(p)$. This shows that the exact sequence

$$1 \to Z \to E \to \mathbf{G}(\mathbb{A}) \to 1$$

splits over $\mathbf{G}(\mathbb{Q})$. By Theorem 5.19 it follows that $|Z| \leq 2$. Thus $\ker(\mu)$ contains the subgroup in $C(p)$ of index 2, and the proof is complete. $\square$

From the Theorem 5.20 and Corollary 5.12 we can now formulate what one could call our main theorem, or punchline, of this part of the thesis.

Corollary 5.21. *There exists a finitely presented, Frobenius stable group, which is not Frobenius approximable.*

Part III

Global approximate representations

Chapter 6

Inverse theorem for the uniformity norm

We now turn our attention from the local to the global picture. In this chapter, specifically, we shall consider a rather weak notion of global approximate representations connected to the uniformity norm, more precisely the U^2-norm. The uniformity norm of complex-valued maps on finite abelian groups were introduced by Gowers in his work on arithmetic progressions [34, 35] and later generalized by Gowers and Hatami [36] to cover maps from a finite (possibly non-abelian) group into the $n \times n$-matrices. There are U^k-norms for all $k \in \mathbb{N}$, but in the following discussion we content ourselves with $k = 2$. The definition is as follows.

Definition 6.1. Let Γ be a finite group and let $n \in \mathbb{N}$. For a map $\varphi\colon \Gamma \to \mathrm{M}_n(\mathbb{C})$ we define the *uniformity norm* as

$$\|\varphi\|_{U^2}^4 = \frac{1}{|\Gamma|^3} \sum_{xy^{-1}zw^{-1}=1_\Gamma} \mathrm{tr}(\varphi(x)\varphi(y)^*\varphi(z)\varphi(w)^*).$$

Here tr is the canonical normalized trace on $\mathrm{M}_n(\mathbb{C})$ (see Example 2.15). The third power in the denominator on the right hand side normalizes the norm so that all representations of Γ have uniformity norm 1. The 4'th power on the left hand side is there so that $\|\cdot\|_{U^2}$ is actually a norm. That this is true is not evident from the definition although not hard to prove either, but, as we shall not make use of this fact, we shall not prove it.

Let us explain why the uniformity norm is relevant to the topic of approximate representations. For a finite group Γ, one observes that if $\varphi\colon \Gamma \to \mathrm{M}_n(\mathbb{C})$ is a map such that $\|\varphi(x)\|_{\mathrm{op}} \leq 1$, for all $x \in \Gamma$, then the statement $\|\varphi\|_{U^2}^4 \geq 1-\varepsilon$

for some $\varepsilon \in [0,1]$ is equivalent to the statement

$$\frac{1}{|\Gamma|^3} \sum_{xy^{-1}zw^{-1}=1_\Gamma} \|1_n - \varphi(x)\varphi(y)^*\varphi(z)\varphi(w)^*\|_{2,\mathrm{tr}}^2 \leq 2\varepsilon.$$

In other words, if $\|\varphi\|_{U^2}$ is "big" (meaning close to 1) if and only if the relation $xy^{-1}zw^{-1} = 1_\Gamma$ is preserved by the map φ on average, up to a small error. In particular, if $\varphi(xy) = \varphi(x)\varphi(y)$ for "most" $x, y \in \Gamma$, then $\|\varphi\|_{U^2}$ is big. Another typical situation where $\|\varphi\|_{U^2}$ is big is if φ is *correlated* to a representation, in the sense of the following proposition (see [36] for a proof).

Proposition 6.2. *Let $n \in \mathbb{N}$ and $c \in [0,1]$. Let Γ be a finite group and consider a map $\varphi\colon \Gamma \to \mathbf{M}_n(\mathbb{C})$, such that $\|\varphi(x)\|_{\mathrm{op}} \leq 1$. Assume that there is a representation $\pi\colon \Gamma \to \mathbf{M}_n(\mathbb{C})$ such that*

$$\Big|\frac{1}{|\Gamma|} \sum_{x\in\Gamma} \mathrm{tr}(\varphi(x)\pi(x)^*)\Big| \geq c.$$

Then $\|\varphi\|_{U^2} \geq c$.

We are interested in the *inverse* statement of the above proposition; given a map φ with big uniformity norm, we want to prove that there exists a representation π which is correlated to φ. In general, this is too much to hope for, so we have to accommodate the statement a little. The idea is to allow the dimension of π to differ slightly from the dimension of φ. The first version of such an inverse theorem for matrix-valued functions was proven by Gowers and Hatami.

Theorem 6.3 (Gowers-Hatami, [36])**.** *Let $c \in (0,1]$, let Γ be a finite group, let $n \in \mathbb{N}$ and let $\varphi\colon \Gamma \to \mathbf{M}_n(\mathbb{C})$. Assume that $\|\varphi(x)\|_{\mathrm{op}} \leq 1$ and $\|\varphi\|_{U^2}^4 \geq c$. Then there are $m \in [\frac{c}{2-c}n, \frac{2-c}{c}n]$, $\pi\colon \Gamma \to \mathbf{U}(m)$ and maps $U, V\colon \mathbb{C}^n \to \mathbb{C}^m$ such that*

$$\frac{1}{|\Gamma|} \sum_{x\in\Gamma} \mathrm{tr}(\varphi(x)V^*\pi(x)^*U) \geq t(c),$$

where $t(c) = \max\left\{\frac{c^8}{(2-c)^8}, \frac{c^2}{4}\right\}$. Moreover, if $n \leq m$, we can take U and V to be isometries and if $n \geq m$ we can take U and V to be co-isometries.

Letting $\varepsilon = 1 - c$, this Theorem 6.3 fits in the topic of this thesis as a stability question. Our notion of approximate representations is now *maps with "big" U^2-norm* and being "close" to a representation means *being correlated to a representation* of roughly the same dimension, twisted by isometries or co-isometries. Note that if $c = 1$ then $t(c) = 1$ and $n = m$. Thus the conclusion is

$$\frac{1}{|\Gamma|} \sum_{x\in\Gamma} \mathrm{tr}(\varphi(x)V^*\pi(x)^*U) = 1,$$

where U and V are unitaries. It is not too hard to see that this entails that $\varphi(x) = U\pi(x)V^*$ for $x \in \Gamma$. If c is big, that is, as $\varepsilon = 1 - c$ approaches 0, then $t(1-\varepsilon) = \frac{(1-\varepsilon)^8}{(1+\varepsilon)^8} = 1 - O(\varepsilon)$ and as c approaches 0, then $t(c) = c^2/4 = O(c^2)$.

The main theorems of this chapter (Theorems 6.4, 6.6 and 6.7) constitute generalizations of Theorem 6.3 in more ways: we allow for infinite, amenable groups Γ, we allow the target $\mathrm{M}_n(\mathbb{C})$ to be replaced by a general von Neumann algebra and we accommodate the assumption $\|\varphi\|_{U^2} \geq c$ to a more general setting, involving unitarily invariant, ultraweakly lower semi-continuous norms. We shall refrain from defining *the* uniformity norm for an (infinite) amenable group and write the estimates we need explicitly every time, the reason being that there is no natural way to define the mean over all quadruples (x, y, z, w) satisfying $xy^{-1}zw^{-1} = 1_\Gamma$ for an infinite amenable group. In fact, most of the time we shall need estimates on two different expressions involving iterated means.

6.1 The general inverse theorem

We state and prove the very general version of the inverse theorem for c close to 1. The proof is very conceptual; the crux of the proof is to apply the Stinespring Dilation Theorem (Proposition 2.25) to a certain positive definite map, which is provided by Proposition 2.26. The rest of the proof is basically repeated applications of the equations on page 87. The idea to use the Stinespring Dilation Theorem in connection with global approximate representations is not new; we draw heavy inspiration from Shtern's work on uniform ε-representations [63].

For the proof, recall some notation from Chapter 2. For a von Neumann algebra $\mathcal{M}$, we defined $\mathcal{M}_\infty = \mathcal{M}\bar{\otimes}\mathbf{B}(\ell^2(\mathbb{N}))$ and view $\mathcal{M}$ as the corner $1_\mathcal{M}\mathcal{M}_\infty 1_\mathcal{M}$ of $\mathcal{M}_\infty$. Furthermore, recall that we defined what it means to take the mean of a bounded map from an amenable group to a von Neumann algebra (see Section 2.3). As mentioned, proof uses a lot of equations from Chapters 1 and 2. For convenience, we compiled all that we need in a list on page 87. The usage of an equation will be indicated by the corresponding letter to the right of the place where it is used.

Theorem 6.4. *Let $\varepsilon \geq 0$, let Γ be a countable, amenable group with a bi-invariant, symmetric mean $\mathbb{E}$, let $\mathcal{M}$ be a von Neumann algebra and let $\|\cdot\|$ be a unitarily invariant, ultraweakly lower semi-continuous norm on $\mathcal{M}_\infty$. Let $\varphi\colon \Gamma \to \mathcal{M}$ be any map and assume that $\|\varphi(x)\|_{\mathrm{op}} \leq 1$ for all $x \in \Gamma$ and that*

$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_\mathcal{M} - \varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*\| \leq \varepsilon,$$
$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_\mathcal{M} - \varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*\| \leq \varepsilon.$$

Then there exists a projection $P \in \mathcal{M}_\infty$, partial isometries $U, V \in P\mathcal{M}_\infty 1_\mathcal{M}$ and a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ such that

$$\mathbb{E}_x \|1_\mathcal{M} - \varphi(x)V^*\rho(x)^*U\| \leq 44\varepsilon,$$

and

$$\|1_\mathcal{M} - U^*U\| \leq 20\varepsilon, \qquad \|P - UU^*\| \leq 15\varepsilon, \qquad \|P - VV^*\| \leq 85\varepsilon.$$

Proof. Define $\tilde{\varphi}\colon \Gamma \to \mathcal{M}$ by

$$\tilde{\varphi}(x) = \mathbb{E}_y \varphi(xy)\varphi(y)^*,$$

for $x \in \Gamma$, which is positive definite by Proposition 2.26. Thus, by Proposition 2.25, there exists a unitary representation $\pi\colon \Gamma \to \mathbf{U}(\mathcal{M}_\infty)$ together with $U \in \mathcal{M}_\infty 1_\mathcal{M}$ with $\|U\|_{\mathrm{op}} \leq 1$ such that

$$\tilde{\varphi}(x) = U^*\pi(x)U,$$

for $x \in \Gamma$. Define $V = \mathbb{E}_x \pi(x)^* U \varphi(x) \in \mathcal{M}_\infty 1_\mathcal{M}$ and

$$A = \mathbb{E}_x \pi(x)UU^*\pi(x)^* = \mathbb{E}_x \pi(x)^*UU^*\pi(x),$$

where the last equality uses symmetry of the mean. Evidently A commutes with $\pi(y)$ for any $y \in \Gamma$, that is, $A \in \pi(\Gamma)'$. Also, since $0 \leq UU^* \leq 1_{\mathcal{M}_\infty}$, it easily follows that $0 \leq A \leq 1_{\mathcal{M}_\infty}$. We see that

$$\begin{aligned}
&\mathbb{E}_x \|1_\mathcal{M} - \varphi(x)V^*\pi(x)^*U\| \\
&\quad = \mathbb{E}_x \|\mathbb{E}_y(1_\mathcal{M} - \varphi(x)\varphi(y)^*U^*\pi(yx^{-1})U)\| \\
&\quad \leq \mathbb{E}_x\mathbb{E}_y \|1_\mathcal{M} - \varphi(x)\varphi(y)^*U^*\pi(yx^{-1})U\| && \text{(k)} \\
&\quad = \mathbb{E}_x\mathbb{E}_y \|1_\mathcal{M} - \varphi(x)\varphi(y)^*\tilde{\varphi}(yx^{-1})\| \\
&\quad = \mathbb{E}_x\mathbb{E}_y \|\mathbb{E}_z(1_\mathcal{M} - \varphi(x)\varphi(y)^*\varphi(yx^{-1}z)\varphi(z)^*)\| \\
&\quad \leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z \|1_\mathcal{M} - \varphi(x)\varphi(y)^*\varphi(yx^{-1}z)\varphi(z)^*\| && \text{(k)} \\
&\quad = \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z \|1_\mathcal{M} - \varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*\| \leq \varepsilon. && \text{(a)}
\end{aligned}$$

Thus, U and V satisfy the desired inequality, but they need not be partial isometries and we have *a priori* no control over their range projections. In order to correct for that, we start by noting that

$$\begin{aligned}
\|U^*(1_{\mathcal{M}_\infty} - A)U\| &\leq \mathbb{E}_x \|U^*U - U^*\pi(x)UU^*\pi(x)^*U\| && \text{(k)} \\
&= \mathbb{E}_x \|U^*U - \tilde{\varphi}(x)\tilde{\varphi}(x)^*\| \\
&\leq \mathbb{E}_x \|1_\mathcal{M} - \tilde{\varphi}(x)\tilde{\varphi}(x)^*\| && \text{(h)} \\
&\leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z \|1_\mathcal{M} - \varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*\| \leq \varepsilon. && \text{(k)}
\end{aligned}$$

Since $\pi(x)$ and $(1_{\mathcal{M}_\infty} - A)^{1/2}$ commute, it follows that

$$\begin{aligned}
\|A - A^2\| &= \|(1_{\mathcal{M}_\infty} - A)^{1/2} A (1_{\mathcal{M}_\infty} - A)^{1/2}\| \\
&\le \mathbb{E}_x \|(1_{\mathcal{M}_\infty} - A)^{1/2} \pi(x) U U^* \pi(x)^* (1_{\mathcal{M}_\infty} - A)^{1/2}\| && \text{(k)} \\
&= \mathbb{E}_x \|\pi(x)(1_{\mathcal{M}_\infty} - A)^{1/2} U U^* (1_{\mathcal{M}_\infty} - A)^{1/2} \pi(x)^*\| \\
&= \|(1_{\mathcal{M}_\infty} - A)^{1/2} U U^* (1_{\mathcal{M}_\infty} - A)^{1/2}\| && \text{(d)} \\
&= \|U^*(1_{\mathcal{M}_\infty} - A)U\| \le \varepsilon. && \text{(g)}
\end{aligned}$$

Now, let $P = \chi_{[1/2,1]}(A)$, which is a projection in the commutant $\pi(\Gamma)'$, so the map $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ given by $\rho(x) = P\pi(x)P$ is a representation. Since $|\chi_{[1/2,1]}(t) - t| \le 2(t - t^2)$ for all $t \in [0,1]$, we have that

$$\|P - A\| \le 2\|A - A^2\| \le 2\varepsilon. \tag{h}$$

Letting $P^\perp = 1_{\mathcal{M}_\infty} - P$, we see that, for all $x \in \Gamma$, it follows that

$$\begin{aligned}
&\|V^* P^\perp \pi(x)^* P^\perp U\| \\
&\quad \le \mathbb{E}_y \|\varphi(y)^* U^* P^\perp \pi(yx^{-1}) P^\perp U\| && \text{(k)} \\
&\quad \le \mathbb{E}_y \|U^* P^\perp \pi(yx^{-1}) P^\perp U\| && \text{(e)} \\
&\quad \le \frac{1}{2}(\mathbb{E}_y \|\pi(y)^* P^\perp U U^* P^\perp \pi(y)\| + \|\pi(x)^* P^\perp U U^* P^\perp \pi(x)\|) && \text{(j)} \\
&\quad = \|P^\perp U U^* P^\perp\| && \text{(d)} \\
&\quad = \|U^* P^\perp U\| && \text{(g)} \\
&\quad \le \|U^*(1_{\mathcal{M}_\infty} - A)U\| + \|U^*(A - P)U\| \le \varepsilon + 2\varepsilon = 3\varepsilon. && \text{(c)}
\end{aligned}$$

Since $\pi(x) = \rho(x) + P^\perp \pi(x) P^\perp$ for $x \in \Gamma$, we get that

$$\mathbb{E}_x \|1_{\mathcal{M}} - \varphi(x) V^* \rho(x)^* U\| \le \mathbb{E}_x \|1_{\mathcal{M}} - \varphi(x) V^* \pi(x)^* U\| + 3\varepsilon \le 4\varepsilon. \tag{c,e}$$

We define $U_0 = PU$ and $V_0 = PV$. These are elements of $P\mathcal{M}_\infty 1_{\mathcal{M}}$, but not necessarily partial isometries. Write the polar decomposition of $U_0 = S|U_0|$ and define $U_1 = S\chi_{[1/2,1]}(|U_0|)$. By Lemma 2.10, it follows that U_1 is a partial isometry, and we calculate

$$\begin{aligned}
\|U_1 - U_0\| &\le \|\chi_{[1/2,1]}(|U_0|) - |U_0|\| && \text{(e)} \\
&\le 2\||U_0| - |U_0|^2\| && \text{(h)} \\
&\le 2\|1_{\mathcal{M}} - U^* P U\| && \text{(h)} \\
&\le 2\|1_{\mathcal{M}} - U^* A U\| + 4\varepsilon && \text{(c)} \\
&\le 2\mathbb{E}_x \|1_{\mathcal{M}} - \tilde{\varphi}(x)\tilde{\varphi}(x)^*\| + 4\varepsilon \le 6\varepsilon. && \text{(k)}
\end{aligned}$$

This, in turn, allows us to estimate the following,

$$\mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_0^*\rho(x)^*U_1\| \le \mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_0^*\rho(x)U_0\| + 6\varepsilon \le 10\varepsilon, \quad \text{(c,e)}$$

whence we conclude

$$\|1_{\mathcal{M}} - U_1^*U_1\| \le 2\mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_0^*\rho(x)^*U_1\| \le 20\varepsilon. \quad \text{(i,k)}$$

We proceed by estimating

$$\begin{aligned}
\|P - U_0U_0^*\| &= \|P - PUU^*P\| \\
&= \|(1_{\mathcal{M}_\infty} - UU^*)^{1/2}P(1_{\mathcal{M}_\infty} - UU^*)^{1/2}\| && \text{(g)} \\
&\le \|(1_{\mathcal{M}_\infty} - UU^*)^{1/2}A(1_{\mathcal{M}_\infty} - UU^*)^{1/2}\| + 2\varepsilon && \text{(c,e)} \\
&\le \mathbb{E}_x\|(1_{\mathcal{M}_\infty} - UU^*)^{1/2}\pi(x)^*UU^*\pi(x)(1_{\mathcal{M}_\infty} - UU^*)^{1/2}\| + 2\varepsilon && \text{(k)} \\
&= \mathbb{E}_x\|U^*\pi(x)(1_{\mathcal{M}_\infty} - UU^*)\pi(x)^*U\| + 2\varepsilon && \text{(g)} \\
&= \mathbb{E}_x\|U^*U - U^*\pi(x)UU^*\pi(x)^*U\| + 2\varepsilon \\
&\le \mathbb{E}_x\|1_{\mathcal{M}} - \tilde{\varphi}(x)\tilde{\varphi}(x)^*\| + 2\varepsilon \le 3\varepsilon, && \text{(h)}
\end{aligned}$$

and, combining the above estimates, we get that

$$\begin{aligned}
\|P - U_1U_1^*\| &\le \|P - U_0U_0^*\| + \|(U_0 - U_1)U_0^*\| + \|U_1(U_0 - U_1)^*\| && \text{(c)} \\
&\le \|P - U_0U_0^*\| + 6\varepsilon + 6\varepsilon \le 15\varepsilon. && \text{(e,f)}
\end{aligned}$$

Now, in a similar fashion as above, we use the polar decomposition of $V_0^* = T|V_0^*|$ to define $V_1^* = T\chi_{[1/2,1]}(|V_0^*|)$ and get a partial isometry. Then V_1 is a partial isometry as well, and it holds that

$$\begin{aligned}
\|P - V_0V_0^*\| &= \mathbb{E}_x\|P - \rho(x)V_0V_0^*\rho(x)^*\| && \text{(d)} \\
&\le \mathbb{E}_x\|U_0U_0^* - U_0U_0^*\rho(x)V_0V_0^*\rho(x)^*U_0U_0^*\| + 9\varepsilon && \text{(c)} \\
&\le \mathbb{E}_x\|1_{\mathcal{M}} - U_0^*\rho(x)V_0V_0^*\rho(x)^*U_0\| + 9\varepsilon && \text{(e)} \\
&\le 2\mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_0^*\rho(x)^*U_0\| + 9\varepsilon && \text{(i)} \\
&\le 17\varepsilon.
\end{aligned}$$

This entails that

$$\|V_1 - V_0\| \le 2\||V_0^*| - |V_0^*|^2\| \le 2\|P - V_0V_0^*\| \le 34\varepsilon, \quad \text{(e,h)}$$

and it follows that

$$\|P - V_1V_1^*\| \le \|P - V_0V_0^*\| + 68\varepsilon \le 85\varepsilon. \quad \text{(c,e)}$$

Finally, we conclude that

$$\begin{aligned}
&\mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_1^*\rho(x)^*U_1\| \\
&\quad \le \mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V_0^*\rho(x)^*U_0\| + 6\varepsilon + 34\varepsilon \le 44\varepsilon. && \text{(c,e)}
\end{aligned}$$

Now the proof is complete by renaming U_1 and V_1 to U and V. □

Remark 6.5. We note for later use that if one does not require V to be a partial isometry, then we can end the above proof earlier and get the better estimate

$$\mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V^*\rho(x)^*U\| \leq 10\varepsilon.$$

Also note that it follows from the estimates that

$$\begin{aligned}
\mathbb{E}_x\|\rho(x) - U\varphi(x)V^*\| &= \mathbb{E}_x\|P - U\varphi(x)V^*\rho(x)^*\| && \text{(d)}\\
&\leq \mathbb{E}_x\|UU^* - U\varphi(x)V^*\rho(x)^*UU^*\| + 30\varepsilon && \text{(c)}\\
&\leq \mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V^*\rho(x)^*U\| + 30\varepsilon \leq 74\varepsilon. && \text{(e)}
\end{aligned}$$

In other words, after twisting φ by U and V, it is approximated on average by ρ.

6.2 The inverse theorem for the normalized Hilbert-Schmidt norm

We shall now concentrate on 2-norm of some suitable normal trace τ on $\mathcal{M}_\infty$. In this case, our Theorem 6.4 subsumes Theorem 6.3, at least if $\varepsilon = 1 - c$ is sufficiently small. For general c we will have to accommodate the proof, although it will go along the same line of reasoning. Note that the following theorems not only generalize the *statement* of Theorem 6.3; they also provide a conceptual proof using operator algebras which differs from the one provided in [36]. The estimates, however, are different than the ones in Theorem 6.3; in Theorem 6.6 they are worse, that is, $O(\varepsilon^{1/2})$ instead of $O(\varepsilon)$ for $\varepsilon \to 0$, and in Theorem 6.7 they are better, $O(c)$ instead of $O(c^2)$ for $c \to 0$.

Theorem 6.6. *Let $\varepsilon > 0$, let Γ be a countable, amenable group with a symmetric, bi-invariant mean $\mathbb{E}$ and let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal trace τ on $\mathcal{M}_\infty$ such that $\tau(1_{\mathcal{M}}) = 1$. Furthermore, let $\varphi\colon \Gamma \to \mathcal{M}$ be a map with $\|\varphi(x)\|_{\mathrm{op}} \leq 1$ for all $x \in \Gamma$. Assume that*

$$\begin{aligned}
\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*) \geq 1 - \varepsilon,\\
\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*) \geq 1 - \varepsilon.
\end{aligned}$$

Then there exist a projection $P \in \mathcal{M}_\infty$, partial isometries $U, V \in P\mathcal{M}_\infty 1_{\mathcal{M}}$ and a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ satisfying

$$\mathbb{E}_x\tau(\varphi(x)V^*\rho(x)^*U) \geq 1 - 63\varepsilon^{1/2},$$

and the following inequalities $\|1_{\mathcal{M}} - U^*U\|_{2,\tau} \leq 29\varepsilon^{1/2}$, $\|P - UU^*\|_{2,\tau} \leq 22\varepsilon^{1/2}$ and $\|P - VV^*\|_{2,\tau} \leq 121\varepsilon^{1/2}$.

Proof. Since τ is normal, the 2-norm is an ultraweakly lower semi-continuous, unitarily invariant norm on $\mathcal{M}_\infty$. By the Cauchy-Schwarz inequality applied to $\mathbb{E}$, it follows that

$$\begin{aligned}
&\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_{\mathcal{M}} - \varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*\|_{2,\tau} \\
&\quad \le (\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z(\|1_{\mathcal{M}} - \varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*\|_{2,\tau}^2))^{1/2} \\
&\quad \le (\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z(2 - 2\operatorname{Re}\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*))^{1/2} \le (2\varepsilon)^{1/2}.
\end{aligned}$$

In a similar fashion, we get

$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_{\mathcal{M}} - \varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*\|_{2,\tau} \le (2\varepsilon)^{1/2}.$$

Hence, from Theorem 6.4 there exist a projection $P \in \mathcal{M}_\infty$ and partial isometries $U, V \in P\mathcal{M}_\infty 1_{\mathcal{M}}$ together with a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ such that

$$\begin{aligned}
\|1_{\mathcal{M}} - \mathbb{E}_x\varphi(x)V^*\rho(x)^*U\|_{2,\tau} &\le \mathbb{E}_x\|1_{\mathcal{M}} - \varphi(x)V^*\rho(x)^*U\|_{2,\tau} \\
&\le 44(2\varepsilon)^{1/2},
\end{aligned} \tag{k}$$

and, moreover, $\|1_{\mathcal{M}} - U^*U\|_{2,\tau} \le 20(2\varepsilon)^{1/2}$, $\|P - UU^*\|_{2,\tau} \le 15(2\varepsilon)^{1/2}$ and $\|P - VV^*\|_{2,\tau} \le 85(2\varepsilon)^{1/2}$. It now follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\Big|1 - |\tau(\mathbb{E}_x\varphi(x)V^*\rho(x)^*U)|\Big| &\le |1 - \tau(\mathbb{E}_x\varphi(x)V^*\rho(x)^*U)| \\
&= |\tau(1_{\mathcal{M}} \cdot (1_{\mathcal{M}} - \mathbb{E}_x\varphi(x)V^*\rho(x)^*U))| \\
&\le \|1_{\mathcal{M}}\|_{2,\tau}\|1_{\mathcal{M}} - \mathbb{E}_x\varphi(x)V^*\rho(x)^*U\|_{2,\tau} \\
&\le 44(2\varepsilon)^{1/2}.
\end{aligned}$$

By multiplying U with a complex number of modulus 1, we can assume that $\tau(\mathbb{E}_x\varphi(x)V^*\rho(x)^*U) \ge 0$ so we get the desired inequality:

$$\tau(\mathbb{E}_x\varphi(x)V^*\rho(x)^*U) > 1 - 44(2\varepsilon)^{1/2} > 1 - 63\varepsilon^{1/2}. \qquad \square$$

There are various comments that are appropriate to make about this theorem. We start out by one of the technical sort. One might wonder whether the expressions

$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*), \qquad \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*)$$

define non-negative real numbers, as it would otherwise be appropriate to take the real part. As it turns out, this is indeed the case. To prove it for the first expression, note that

$$\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*) = \tau(\varphi(y)^*\varphi(yz)\varphi(xz)^*\varphi(x))$$

by the trace property. Now, one can prove that $K\colon \Gamma\times\Gamma\to\mathcal{M}$ given by

$$K(x,y)=\mathbb{E}_z\varphi(y)^*\varphi(yz)\varphi(xz)^*\varphi(x)$$

is a positive definite operator-valued kernel using the same line of reasoning as in Proposition 2.26. It follows that the composition $\tau\circ K\colon \Gamma\times\Gamma\to\mathbb{C}$ is a positive definite kernel as well. Using the representation theorem for positive definite kernels (Theorem 2.28), there exists a Hilbert space valued function $\alpha\colon \Gamma\to\mathcal{H}$ such that for all $x,y\in\Gamma$, we have $(\tau\circ K)(x,y)=\langle\alpha(x),\alpha(y)\rangle$ and we can conclude that

$$\begin{aligned}
&\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*)\\
&=\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(y)^*\varphi(yz)\varphi(xz)^*\varphi(x)))\\
&=\mathbb{E}_x\mathbb{E}_y((\tau\circ K)(x,y))=\mathbb{E}_x\mathbb{E}_y\langle\alpha(x),\alpha(y)\rangle=\langle\mathbb{E}_x\alpha(x),\mathbb{E}_y\alpha(y)\rangle\geq 0.
\end{aligned}$$

The proof for the second expression is almost immediate, since

$$\begin{aligned}
&\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*)\\
&=\mathbb{E}_x\tau(\mathbb{E}_y(\varphi(xy)\varphi(y)^*)(\mathbb{E}_z(\varphi(xz)\varphi(z)^*))^*)\geq 0.
\end{aligned}$$

Here we used that τ is assumed to be normal and thus commutes with the mean.

For convenience we assume τ to be defined on all of $\mathcal{M}_\infty$ (so that expressions like $\|P-UU^*\|_{2,\tau}$ make sense), but any normal tracial state τ' on $\mathcal{M}$ extends canonically to $\mathcal{M}_\infty$ by $\tau=\tau'\otimes\mathrm{Tr}$, where Tr is the canonical trace on $\mathbf{B}(\mathscr{H})$. It is also worth noting that τ *will never be finite on* $\mathcal{M}_\infty$, which is a reason why we insist on letting our norms take the value ∞. One could fix this by only working with finite inflations $\mathcal{M}\bar{\otimes}\mathrm{M}_n(\mathbb{C})$, but this seems like an unnatural and unneccesary extra technicality to impose. Note also that the trace is automatically semifinite by the assumption $\tau(1_\mathcal{M})=1$.

As mentioned, the above theorem is only a generalization of Theorem 6.3 if ε is small enough. For instance, if $\varepsilon\geq\frac{1}{3969}$ then $1-63\varepsilon^{1/2}\leq 0$ and the inequality

$$\left|\mathbb{E}_x\tau(\varphi(x)V^*\rho(x)^*U)\right|\geq 1-63\varepsilon^{1/2}$$

is trivially satisfied for any choice of ρ,U and V, so the conclusion is virtually an empty statement. The next theorem will be an appropriate variant for $\varepsilon=1-c$ in the interval $[\frac{3968}{3969},1)$. The theorem is specific for the trace and we cannot use Theorem 6.4. The proof, however, is similar and even a bit shorter. Note that in this case we need less assumptions; we only assume one inequality and that the lower bound $\frac{c}{2}$ is bigger than Gowers and Hatami's bound, which is $\frac{c^2}{4}$.

Theorem 6.7. *Let $c \in (0,1]$, let Γ be a countable, amenable group with left-invariant mean $\mathbb{E}$, let $\mathcal{M}$ be a von Neumann algebra and let τ be a normal trace on $\mathcal{M}_\infty$ such that $\tau(1_{\mathcal{M}}) = 1$. Let $\varphi\colon \Gamma \to \mathcal{M}$ be a map with $\|\varphi(x)\|_{\mathrm{op}} \leq 1$ for all $x \in \Gamma$. Assume that*

$$\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*) \geq c.$$

Then there exists a projection $P \in \mathcal{M}_\infty$, partial isometries $U, V \in P\mathcal{M}_\infty 1_{\mathcal{M}}$ and a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ such that

$$\mathbb{E}_x\tau(\varphi(x)V^*\rho(x)^*U) \geq \frac{c}{2},$$
$$\frac{c}{2} \leq \tau(UU^*) \leq \tau(P) \leq \frac{2}{c}, \qquad \frac{c}{2} \leq \tau(VV^*) \leq \tau(P) \leq \frac{2}{c}.$$

Proof. The proof begins as that of Theorem 6.4. Define the positive definite $\tilde{\varphi}(x) = \mathbb{E}_y\varphi(xy)\varphi(y)^*$, for $x \in \Gamma$, and use Proposition 2.25 to determine $U \in \mathcal{M}_\infty 1_{\mathcal{M}}$ and a representation $\pi\colon \Gamma \to \mathcal{M}_\infty$ such that $\tilde{\varphi}(x) = U^*\pi(x)U$. Let $V = \mathbb{E}_x\pi(x)^*U\varphi(x) \in \mathcal{M}_\infty 1_{\mathcal{M}}$. Now the proof diverges a little from the previous, as we let $A = \mathbb{E}_x\pi(x)VV^*\pi(x)^*$. This is a positive element of $\mathcal{M}_\infty$, and we can define $P = \chi_{[c/2,1]}(A^{1/2})$. Since A lies in the commutant $\pi(\Gamma)'$, so does P, and therefore $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ given by $\rho(x) = P\pi(x)P$ for $x \in \Gamma$ is a representation. We have that $\|A^{1/2}\|_{\mathrm{op}}^2 = \|A\|_{\mathrm{op}} \leq \|V\|_{\mathrm{op}}^2 \leq \|U\|_{\mathrm{op}}^2 = \|\tilde{\varphi}(1)\| \leq 1$ and by the Cauchy-Schwarz inequality

$$\tau(A^{1/2})^2 \leq \tau(A)\tau(1_{\mathcal{M}}) = \tau(VV^*) = \tau(V^*V) \leq \tau(1_{\mathcal{M}}) = 1.$$

Thus, by the inequality $\chi_{[c/2,1]}(t) \leq \frac{2}{c}t$ for $t \in [0,1]$, we have

$$\tau(P) \leq \frac{2}{c}\tau(A^{1/2}) \leq \frac{2}{c}.$$

Similarly, letting $P^\perp = 1_{\mathcal{M}_\infty} - P$, we use the inequality $t^2(1 - \chi_{[c/2,1]}(t)) \leq \frac{c}{2}t$, for $t \in [0,1]$, to get

$$\tau(AP^\perp) \leq \frac{c}{2}\tau(A^{1/2}) \leq \frac{c}{2}.$$

Since τ is normal and thus commutes with the mean, we conclude

$$\begin{aligned}\mathbb{E}_x\tau(\varphi(x)V^*P^\perp\pi(x)^*U) &= \mathbb{E}_x\tau(\pi(x)^*U\varphi(x)V^*P^\perp)\\ &= \tau(VV^*P^\perp) = \mathbb{E}_y\tau(\pi(y)VV^*P^\perp\pi(y)^*)\\ &= \mathbb{E}_y\tau(\pi(y)VV^*\pi(y)^*P^\perp) = \tau(AP^\perp) \leq \frac{c}{2}.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}\mathbb{E}_x\tau(\varphi(x)V^*\pi(x)^*U) &= \mathbb{E}_x\mathbb{E}_y\tau(\varphi(x)\varphi(y)^*U^*\pi(yx^{-1})U)\\ &= \mathbb{E}_x\mathbb{E}_y\tau(\varphi(x)\varphi(y)^*\tilde{\varphi}(yx^{-1}))\\ &= \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yx^{-1}z)\varphi(z)^*)\\ &= \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\tau(\varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*) \geq c,\end{aligned}$$

so we conclude

$$\begin{aligned}\mathbb{E}_x\tau(\varphi(x)V^*\rho(x)^*U) &= \mathbb{E}_x\tau(\varphi(x)V^*\pi(x)^*PU)\\ &= \mathbb{E}_x\tau(\varphi(x)V^*\pi(x)^*U) - \mathbb{E}_x\tau(\varphi(x)V^*\pi(x)^*P^{\perp}U)\\ &\geq c - \frac{c}{2} = \frac{c}{2}.\end{aligned}$$

As in the proof of Theorem 6.4, note that U and V are not partial isometries and they also fail to map into the right Hilbert space, so we have to correct for that. The latter problem is again solved by replacing U and V with $U_0 = PU$ and $V_0 = PV$ which both lie in $P\mathcal{M}_\infty 1_{\mathcal{M}}$. Of course, since $\rho(x) = P\rho(x)P$, we still have

$$\mathbb{E}_x\tau(\varphi(x)V_0^*\rho(x)^*U_0) = \mathbb{E}_x\tau(\varphi(x)V^*\rho(x)^*U) \geq \frac{c}{2}.$$

Now since $\|U_0\|_{\mathrm{op}}, \|V_0\|_{\mathrm{op}} \leq 1$, by Lemma 2.11 there are partial isometries $U_1, U_2, V_1, V_2 \in P\mathcal{M}_\infty 1_{\mathcal{M}}$ such that $U_0 = \frac{1}{2}(U_1 + U_2)$ and $V_0 = \frac{1}{2}(V_1 + V_2)$ Thus, there must be at least one combination of partial isometries, say, U_1 and V_1 such that

$$\left|\mathbb{E}_x\tau(\varphi(x)V_1^*\rho(x)^*U_1)\right| \geq \frac{c}{2}.$$

By multiplying U_1 with a complex number of modulus 1, we can assume

$$\mathbb{E}_x\tau(\varphi(x)V_1^*\rho(x)^*U_1) \geq \frac{c}{2}.$$

Let $B = \mathbb{E}_x\varphi(x)V_1^*\rho(x)^*$. Then $\|B\|_{\mathrm{op}} \leq 1$, so $BB^* \leq 1_{\mathcal{M}}$, and hence $\tau(BB^*) \leq 1$, which gives us

$$\tau(U_1U_1^*) \geq \tau(BB^*)\tau(U_1^*U_1) \geq |\tau(BU_1)| = \mathbb{E}_x\tau(\varphi(x)V_1^*\rho(x)^*U_1) \geq \frac{c}{2}.$$

A similar calculation gives us that $\tau(V_1V_1^*) \geq \frac{c}{2}$, and the proof is complete with U_1 and V_1 as U and V. $\square$

Chapter 7

Stability results for uniform ε-representations

This chapter is devoted to the notion of a uniform ε-representation which is a global notion of an approximate representation. We review earlier results on uniform ε-representations and use Theorem 6.4 to prove Theorem 7.7, which is a stability theorem uniform ε-representations of amenable groups. In the context of uniform ε-representations, we sometimes talk about *Ulam stability* or *strong Ulam stability* since the stability question goes back to Ulam [70] in this setting.

7.1 Uniform ε-representations

We start out by the central definition.

Definition 7.1. Let $\varepsilon \geq 0$, let Γ be a group and let (G, d) be a metric group. A *uniform ε-homomorphism* is a map $\varphi \colon \Gamma \to G$ such that

$$d(\varphi(xy), \varphi(x)\varphi(y)) \leq \varepsilon,$$

for all $x, y \in \Gamma$.

The above definition coincides with what we called an (ε, F)-approximate homomorphism in the special case where $F = \Gamma$ (see Definition 4.15). As usual, we talk about uniform *ε-representations* if $G \subseteq \mathrm{U}(\mathscr{H})$. Stability questions for uniform ε-representations have been studied in different settings with respect to different norms. Grove, Karcher and Ruh [38] and later Kazhdan [45] investigated this question for the operator norm. The result of Kazhdan reads as follows.

Theorem 7.2. *Let $0 \leq \varepsilon \leq \frac{1}{200}$, let Γ be a countable, amenable group and let $\mathscr{H}$ be a Hilbert space. Let $\varphi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ be an ε-representation with respect to the operator norm. Then there exists a representation $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ such that*

$$\|\varphi(x) - \pi(x)\|_{\mathrm{op}} \leq 2\varepsilon, \qquad x \in \Gamma.$$

This theorem answers the stability question for amenable groups and the operator norm in a very clean way; the 2ε-bound in the conclusion is the same for all amenable groups and all Hilbert spaces. On the other hand, for many non-amenable groups there are more or less explicit constructions of non-trivial ε-representations, i.e. ε-representations are not close in the operator norm to any genuine representations. The prime example being one of Rolli. Recall that $\mathbb{F}_2$ denotes the free group on two generators.

Theorem 7.3 ([59])**.** *For every $\varepsilon > 0$ there exists a uniform ε-representation $\varphi\colon \mathbb{F}_2 \to \mathbf{U}(1)$, such that for all representations $\pi\colon \mathbb{F}_2 \to \mathbf{U}(1)$ there is $x \in \Gamma$ such that*

$$\|\varphi(x) - \pi(x)\|_{\mathrm{op}} \geq 1.$$

Burger, Ozawa and Thom later used this to prove non-stability of many non-amenable groups.

Theorem 7.4 ([13])**.** *Let Γ be a group and assume that $\mathbb{F}_2 \subseteq \Gamma$ is a subgroup. For all $\varepsilon > 0$ there exists a uniform ε-representation $\varphi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ on some Hilbert space $\mathscr{H}$ such that for all representations $\pi\colon \Gamma \to \mathbf{U}(\mathscr{H})$ there is $x \in \Gamma$ such that*

$$\|\varphi(x) - \pi(x)\|_{\mathrm{op}} \geq \frac{1}{10}.$$

They actually show a similar result for a slightly larger class of non-amenable groups. It remains open whether all non-amenable groups have non-trivial uniform ε-representations. Now, we turn our attention uniform ε-representations with respect to the 2-norm coming from a normal faithful trace on a von Neumann algebra. First thing to note is that if $\mathcal{M} = \mathbf{M}_n(\mathbb{C})$ with normalized trace $\mathrm{tr} = \mathrm{tr}_n$ and we consider the Hilbert-Schmidt norm $\|\cdot\|_{2,\mathrm{tr}}$, then the inequalities $\|\cdot\|_{2,\mathrm{tr}} \leq \|\cdot\|_{\mathrm{op}} \leq \sqrt{n}\|\cdot\|_{2,\mathrm{tr}}$ together with Theorem 7.2 show that all ε-representations $\varphi\colon \Gamma \to \mathbf{U}(n)$ from an amenable group Γ are $2\sqrt{n}\varepsilon$-close to a genuine representation. This estimate, however, depends on the dimension n of the Hilbert space and the argument does not work if $\mathcal{M}$ is infinite dimensional. This begs the question: is it possible to prove a "dimension independent" stability theorem for ε-representations of amenable groups with respect to the 2-norm? At first, the answer seems to be "no", as we run into some difficulties illustrated by the following example.

Example 7.5. Let $0 < \varepsilon < \frac{1}{2}$, let Γ be a countable, amenable group with bi-invariant mean $\mathbb{E}$ and let $\mathcal{M}$ be a factor with a faithful normal tracial state τ. Assume that there is an representation $\pi\colon \Gamma \to \mathbf{U}(\mathcal{M})$, such that $\varphi(\Gamma)'' = \mathcal{M}$, and let $P \in \mathcal{M}$ be a projection with $\tau(P) = 1 - \varepsilon^2$. Define $\varphi\colon \Gamma \to P\mathcal{M}P$ by $\varphi(x) = P\pi(x)P$. It is easy to see that

$$\|\varphi(xy) - \varphi(x)\varphi(y)\|_{2,\tau} \leq \|1_{\mathcal{M}} - P\|_{2,\tau} = \varepsilon,$$

so φ is an ε-representation. Now assume that there exists $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}P)$ such that $\|\varphi(x) - \rho(x)\| \leq \frac{1}{2}$. Define $T = \mathbb{E}_x \rho(x)\pi(x)^* \in P\mathcal{M}$. One checks that $\rho(y)T = T\pi(y)$ for all $y \in \Gamma$ and that $\|1_{\mathcal{M}} - T\|_{2,\tau} \leq 2\|1_{\mathcal{M}} - P\|_{2,\tau} < 1$. The last shows, in particular that T is non-zero, but since π is a factor representation this means that ρ and π are unitarily equivalent. In particular $P = 1_{\mathcal{M}}$ which is impossible as we assumed $\varepsilon > 0$. (Note that φ does not take unitary values and $\|\cdot\|_{2,\tau}$ is not normalized on $P\mathcal{M}P$, but both can be corrected for by taking the unitary part of $\varphi(x)$ for all x and replacing $\|\cdot\|_{2,\tau}$ by $\frac{1}{1-\varepsilon^2}\|\cdot\|_{2,\tau}$.)

The above example shows that if Γ is amenable and either

- Γ has finite dimensional representations of arbitrarily high dimension, or
- Γ has a representation generating a (necessarily hyperfinite) II_1-factor,

then there are non-trivial uniform ε-representations of Γ for all $\varepsilon > 0$. It also shows that even if we restrict our attention to the class of *finite* groups, there is no hope to get a stability result à la Theorem 7.2 since there are finite groups with irreducible representations of arbitrarily high dimension. The heart of the problem is that $\|\cdot\|_{2,\tau}$ is insensible to small dimensional perturbations like cut-down by a projection P with big trace. But, as it turns out, this is the only problem. As soon as we allow the approximating representation to live on a slightly larger Hilbert space, we can use Theorem 6.4 to prove a stability result that holds in broad generality. This strategy was already used in [36] to prove a stability result for the class of finite groups and $\mathcal{M} = \mathbf{M}_n(\mathbb{C})$ and the proof is basically the same in the more general case.

7.2 The stability theorem

The proof actually works for a slightly larger class of maps.

Definition 7.6. Let $\varepsilon \geq 0$, let Γ be an amenable group with bi-invariant mean $\mathbb{E}$, and let (G, d) be a metric group. A map $\varphi\colon \Gamma \to G$ is called a *mean ε-homomorphism* if for all $x \in \Gamma$

$$\mathbb{E}_y d(\varphi(xy), \varphi(x)\varphi(y)) \leq \varepsilon.$$

Note that the notion of a mean ε-homomorphism also covers the case of maps from a finite group to a discrete group which satisfy for all $x \in \Gamma$ that the equality $\varphi(xy) = \varphi(x)\varphi(y)$ holds for "most" y. Again we use the terminology *a mean ε-representation* if the group G consists of operators.

Theorem 7.7. *Let $\varepsilon > 0$, let Γ be a countable, amenable group with bi-invariant mean $\mathbb{E}$, let $\mathcal{M}$ be a von Neumann algebra and let $\|\cdot\|$ be a unitarily invariant, ultraweakly lower semi-continuous norm on $\mathcal{M}_\infty$. Let $\varphi\colon \Gamma \to \mathbf{U}(\mathcal{M})$ be a mean ε-representation with respect to $\|\cdot\|$. Then there is a projection $P \in \mathcal{M}_\infty$, a partial isometry $U \in P\mathcal{M}_\infty 1_\mathcal{M}$ and a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ such that*

$$\|\varphi(x) - U^*\rho(x)U\| \leq 71\varepsilon,$$

for all $x \in \Gamma$, and

$$\|1_\mathcal{M} - U^*U\| \leq 40\varepsilon, \qquad \|P - UU^*\| \leq 30\varepsilon.$$

Proof. We show that φ satisfies the conditions of Theorem 6.4 with 2ε instead of ε. Indeed,

$$\begin{aligned}
&\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_\mathcal{M} - \varphi(x)\varphi(y)^*\varphi(yz)\varphi(xz)^*\| \\
&\quad\leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\Big(\|1_\mathcal{M} - \varphi(xy^{-1})\varphi(yz)\varphi(xz)^*\| \\
&\qquad + \|(\varphi(xy^{-1}) - \varphi(x)\varphi(y)^*)\varphi(xy)\varphi(xz)^*\|\Big) && \text{(c)} \\
&\quad\leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_\mathcal{M} - \varphi(xy^{-1})\varphi(yz)\varphi(xz)^*\| + \varepsilon && \text{(d,a)} \\
&\quad\leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|\varphi(xz) - \varphi(xy^{-1})\varphi(yz)\| + \varepsilon && \text{(d)} \\
&\quad= \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|\varphi(xz) - \varphi(x)\varphi(z)\| + \varepsilon \leq 2\varepsilon && \text{(a)}
\end{aligned}$$

and, in a similar fashion, we get that

$$\begin{aligned}
&\mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\|1_\mathcal{M} - \varphi(xy)\varphi(y)^*\varphi(z)\varphi(xz)^*\| \\
&\quad\leq \mathbb{E}_x\mathbb{E}_y\mathbb{E}_z\Big(\|1_\mathcal{M} - \varphi(x)\varphi(z)\varphi(xz)^*\| \\
&\qquad + \|(\varphi(x) - \varphi(xy)\varphi(y)^*)\varphi(z)\varphi(xz)^*\|\Big) && \text{(c,d)} \\
&\quad\leq 2\varepsilon.
\end{aligned}$$

It follows from Remark 6.5 that there are a projection $P \in \mathcal{M}_\infty$, operators $U, V \in P\mathcal{M}_\infty 1_\mathcal{M}$, such that U is a partial isometry and $\|V\|_{\mathrm{op}} \leq 1$, and a representation $\rho\colon \Gamma \to \mathbf{U}(P\mathcal{M}_\infty P)$ so that

$$\|1_\mathcal{M} - \mathbb{E}_x\varphi(x)V^*\rho(x)^*U\| \leq \mathbb{E}_x\|1_\mathcal{M} - \varphi(x)V^*\rho(x)^*U\| \leq 20\varepsilon \qquad \text{(k)}$$

and

$$\|1_{\mathcal{M}} - U^*U\| \leq 40\varepsilon, \qquad \|P - UU^*\| \leq 30\varepsilon.$$

From this, we derive the desired estimate:

$$\begin{aligned}
&\|\varphi(x) - U^*\rho(x)U\| \\
&\qquad \leq \|\varphi(x)\mathbb{E}_y\varphi(y)V^*\rho(y)^*U - U^*\rho(x)U\| + 20\varepsilon && \text{(c,d)} \\
&\qquad \leq \mathbb{E}_y\|\varphi(x)\varphi(y)V^*\rho(y)^*U - U^*\rho(x)U\| + 20\varepsilon && \text{(k)} \\
&\qquad \leq \mathbb{E}_y\|\varphi(xy)V^*\rho(y)^*U - U^*\rho(x)U\| + 21\varepsilon && \text{(c,e)} \\
&\qquad = \mathbb{E}_y\|\varphi(y)V^*\rho(x^{-1}y)^*U - U^*\rho(g)U\| + 21\varepsilon && \text{(a)} \\
&\qquad \leq \mathbb{E}_y\|\varphi(y)V^*\rho(y)^*UU^*\rho(x)U - U^*\rho(x)U\| + 51\varepsilon && \text{(c,e)} \\
&\qquad \leq 20\varepsilon + 51\varepsilon = 71\varepsilon. && \square
\end{aligned}$$

This theorem is quite general in its setting and the statement subsumes Theorem 7.2. Indeed, if $\mathcal{M} = \mathbf{B}(\mathscr{H})$ and $\|\cdot\| = \|\cdot\|_{\mathrm{op}}$ then for $\varepsilon < \frac{1}{40}$ it follows that U is actually a unitary (between $\mathscr{H}$ and the image of P) and the map $x \mapsto U^*\rho(x)U$ is a representation on $\mathscr{H}$ which approximates φ, so we recover Theorem 7.2. Furthermore, we recover a stability result of Gowers and Hatami [36] about the 2-norm.

Theorem 7.8. *Let $\varepsilon > 0$, let Γ a countable, amenable group and let $\varphi\colon \Gamma \to \mathbf{U}(n)$ be a mean ε-representation with respect to the normalized Hilbert-Schmidt norm. Then there is $m \in [n, (1 + 2500\varepsilon^2)n]$, an isometry $V\colon \mathbb{C}^n \to \mathbb{C}^m$ and a representation $\rho\colon \Gamma \to \mathbf{U}(m)$, such that*

$$\|\varphi(x) - V^*\rho(x)V\|_{2,\mathrm{tr}} \leq 211\varepsilon,$$

for all $x \in \Gamma$.

Proof. Let $\mathcal{M} = \mathbf{M}_n(\mathbb{C})$ and let $\|\cdot\|$ be the 2-norm on $\mathcal{M}_\infty \simeq \mathbf{B}(\ell^2(\mathbb{N}))$, normalized in such a way that $\|1_{\mathcal{M}}\| = 1$. We note that the inequalities

$$\|1_{\mathcal{M}} - U^*U\| \leq 40\varepsilon \qquad \text{and} \qquad \|P - UU^*\| \leq 30\varepsilon$$

translate into

$$|\operatorname{rank}(P) - \operatorname{rank}(1_{\mathcal{M}})| \leq (40^2 + 30^2)\varepsilon^2 n = 2500\varepsilon^2 n.$$

First assume $\operatorname{rank}(P) \geq \operatorname{rank}(1_{\mathcal{M}})$. We let $Q = P - UU^*$ and $R = 1_{\mathcal{M}} - U^*U$. Since $U \in P\mathcal{M}_\infty 1_{\mathcal{M}}$ is a partial isometry, we can find a partial isometry, say

$U_0 \in Q\mathcal{M}_\infty R$, such that $V = U + U_0 \in P\mathcal{M}_\infty 1_\mathcal{M}$ satisfies $V^*V = 1_\mathcal{M}$, that is, V is an isometry from $1_\mathcal{M}(\ell^2(\mathbb{N})) \simeq \mathbb{C}^n$ into $P(\ell^2(\mathbb{N}))$. It follows that

$$\begin{aligned}\|\varphi(x) - V^*\rho(x)V\| &\leq \|\varphi(x) - U^*\rho(x)U\| + \|U_0^*\rho(x)U\| \\ &\quad + \|U^*\rho(x)U_0\| + \|U_0^*\rho(x)U_0\| \\ &\leq 71\varepsilon + 3\|Q\| < 161\varepsilon,\end{aligned}$$

for $x \in \Gamma$, and we are done by identifying the image of P with $\mathbb{C}^m$, where $m = \operatorname{rank}(P)$.

If $\operatorname{rank}(P) \leq \operatorname{rank}(1_\mathcal{M})$, we pick any projection $P' \geq P$ with $\operatorname{rank}(P') = \operatorname{rank}(1_\mathcal{M})$ and consider the representation $\rho' \colon \Gamma \to \mathbf{U}(P'\mathcal{M}_\infty P')$, given by $\rho'(x) = \rho(x) + P' - P$, for $x \in \Gamma$. As above, we can extend U to $V \in P'\mathcal{M}_\infty 1_\mathcal{M}$ with $V^*V = 1_\mathcal{M}$. It follows that V is a unitary between the spaces $1_\mathcal{M}(\ell^2(\mathbb{N}))$ and $P'(\ell^2(\mathbb{N}))$ and a calculation similar to the above gives us

$$\begin{aligned}\|\varphi(x) - V^*\rho'(x)V\| &\leq \|\varphi(x) - V^*\rho(x)V\| + \|(P' - P)\| \\ &< (161 + \sqrt{2500})\varepsilon = 211\varepsilon,\end{aligned}$$

for $x \in \Gamma$. Identifying the image of P' with $\mathbb{C}^n$, we are done. $\square$

Further thoughts and open problems

We end this thesis with some ideas and suggestions for further research.

In the context of Theorem 5.11 it would be interesting to investigate cohomology vanishing results for other coefficients than Hilbert spaces. It was pointed out to us (by Tim de Laat via Andreas Thom) that Oppenheim's generalization [50] of Ballmann and Świątkowski's techniques can be used to prove cohomology vanishing with coefficients in $\mathrm{M}(\omega, \|\cdot\|)$ where $\|\cdot\|$ is the p-norm on $\mathrm{M}_n(\mathbb{C})$ for any $p \in [1, \infty)$. This result can be used to prove that the group $\tilde{\Gamma}$ in Theorem 5.20 is not approximable with respect to the unnormalized p-norm for any $p \in [1, \infty)$. The operator norm (that is $p = \infty$), however, remains intractable, and it is unclear to us whether our approach could provide examples of groups that are not MF, if such groups exist.

Furthermore, we would like to understand the class of Frobenius approximable groups better from the inside. For instance, are all amenable groups Frobenius approximable? (This is, however, a bold question from somebody who does not even know if all solvable groups are Frobenius approximable.)

In connection with uniform ε-representations (Theorem 7.7), we would like to understand the rôle of amenability better. As mentioned (Theorem 7.4), many non-amenable groups admit non-trivial ε-representations with respect to the operator norm. To the best of our knowledge, it is still open whether this kind of operator norm-stability (à la Theorem 7.2) really characterizes amenability or whether stability can be proved for certain non-amenable groups.

We also would like to know whether the estimates in the assumptions of Theorem 6.4 are both necessary. The iterated means have different values in general, but it could happen that there is some relationship between the two expressions. Understanding this better, could possibly provide is with a natural way of defining *the* mean over the relation $xy^{-1}zw^{-1} = 1_\Gamma$.

Equation index

Throughout the thesis we make use various basic equations and inequalities from the introductory Chapters 1 and 2. Especially the proofs in Chapters 6 and 7 make heavy use of these, so we collected these equations for an easy and quick reference.

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Declaration of authorship

I hereby declare that present thesis is written by myself and is based on my own research. No other person's work – published or unpublished – has been used without due acknowledgment. The main results of this thesis are based on the articles [A] and [B], where my contributions are as follows:

In article [A], I have proven and written everything under supervision of Andreas Thom, except Sections 4.1 and 5.2. Sections 2 and 3.1 were written after a draft of Lev Glebsky. Some of the results of Section 3 and 5.1 were proven independently by our coauthors Lev Glebsky and Alexander Lubotzky.

The statements of [B] were proven and written by me under supervision of Andreas Thom and in collaboration with Narutaka Ozawa.

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[A] M. De Chiffre, L. Glebsky, A. Lubotzky, and A. Thom. Stability, cohomology vanishing, and non-approximable groups. *Preprint*, November 2017. Available at: https://arxiv.org/abs/1711.10238v2. Submitted to *Annals of Mathematics*.

[B] M. De Chiffre, N. Ozawa, and A. Thom. Operator algebraic approach to inverse and stability theorems for amenable groups. *Preprint*, June 2017. Available at: https://arxiv.org/abs/1706.04544v2. Submitted to *Mathematika*.