Topological Conjugacies Between Cellular Automata

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Für Hella, Amos und Josua

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Preface

But thou hast ordered all things in
measure and number and weight.

Wisdom of Solomon, 11:20b

Sometimes, when one sets out on a journey to a specific destination and one has a rough plan how to get there, one ends up at a completely different place than originally anticipated. But this place might turn out to be at least as beautiful as the original goal. Anybody who ever went hiking with me can confirm this.

This thesis started as a project to better understand the dynamics of some simple neuron models. But soon a lack of theory for discrete dynamical systems on graphs became apparent.

The action of cellular automata on spatially periodic points provided a particular simple instance of this phenomenon. Hence the question became: Which dynamical properties of cellular automata can be derived from the dynamics of their restriction to spatially periodic points of a fixed period? In this context it is natural to study a notion of isomorphism which we call conjugacy on all tori. But again, almost nothing was known about topological conjugacies between cellular automata and there was a lot to investigate there.

Thus a few years, some conferences, many arcane handwritten notes, the birth of our sons and a hundreds of questions later, this thesis mostly deals with periodic cellular automata and tools to investigate topological conjugacies between them.

Writing this thesis would not have been possible without the support of many people. First of all, I thank my wife Hella for all her love, her support and her interest in my work.

I am greatly indebted to my advisor Professor Stefan Siegmund. He gave me a lot of freedom to pursue my ideas and follow my interests. He was always interested in my results and had an open ear for all my problems.

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Room C218 provided a lively and stimulating working environment, thanks to everybody who shared this office with me. Many thanks to the organizers of Automata 2015, Automata 2016 and ICDEA 2017 for giving me the possibility to present my work.

All computations were done using SageMath [Dev16] and NetworkX [HSS08] using the Python programming language [Ros+16]. I am very grateful to everybody who invested his time in these great open-source projects.

Last but not least, I thank everybody who went hiking, climbing, canoeing and cycling with me despite my creative route planning.

Dresden, *Jeremias Epperlein*
February 1, 2017

Chapter 1.

Introduction

Classifying structures up to some notion of isomorphism, i.e., describing and deciding when one should consider the structures as “the same”, is a fundamental goal of modern mathematical research. The structures in which we are interested are so called cellular automata. They were introduced by von Neumann in [VNB66] as a model for self-replication. Consider a grid or a string of cells each having a state from the same finite set. Each cell sees its own state and the state of its neighbors that are at most a fixed distance away. At every discrete time step, every cell updates its state according to a rule, taking into account only the states it can see. All cells update simultaneously and all use the same rule. It was soon realized that the discrete nature of cellular automata make them an attractive modeling tool in many situations.

The seminal paper [Hed69] by Hedlund gave cellular automata a firm place in the theory of discrete topological dynamical systems. One can characterize cellular automata as the continuous, shift-commuting dynamical systems on symbolic spaces. Continuity ensures that each cell can only see the neighbors that are closer than a fixed distance. The fact that they commute with the shift ensures that every cell uses the same rule. This framework allows to talk about dynamical properties such as equicontinuity, expansivity or entropy and the relations between them, see for example the works of Boyle and Kitchens [BK99] and Blanchard [Bla00] or the surveys of Blanchard, Kurka and Maass [BKM97], Kari [Kar05] and Kurka [Kur09].

The straightforward way to define the notion of isomorphism for discrete dynamical systems is to say that two systems have the same dynamics after a “relabeling” of the state space in a structure preserving way. More precisely, we say that two dynamical systems $f : X \to X$ and $g : Y \to Y$ are isomorphic or conjugate if there is a bijective map $\varphi : X \to Y$ such that both φ and φ^{-1} are structure preserving and such that $\varphi \circ f = g \circ \varphi$. The precise meaning of structure preserving depends on the systems under consideration. In topological dynamics one assumes a topology on X and Y and considers continuous maps f and g, hence one naturally wants φ to be continuous too. In measurable dynamics, on the other hand, one might only demand that φ is measurable, whereas in smooth dynamics one might ask for some smoothness of φ. For cellular automata the natural structure that can be preserved is given by the topology and the shift.

Very few things are known in general about topological conjugacies between cellular automata. To our knowledge the only cases which were investigated before are subshifts, see for example Chapter 2 in the book of Kitchens [Kit98], and expansive or positively expansive cellular automata, see the survey of Nasu [Nas04].

A conjugacy between topological dynamical systems f and g must map the p-periodic points of f homeomorphically onto the p-periodic points of g. This gives rise to a simple invariant for topological conjugacy. A seemingly stronger invariant is obtained if one considers the restrictions of f and g to the p-periodic points which again must be conjugate. For many dynamical systems these restrictions turn out to be rather simple finite systems. For cellular automata, however, the set of p-periodic points is a subshift of finite type, whose topological structure can be more involved. The major result in this thesis shows that, nevertheless, the two invariants just described are equivalent. More precisely, two p-periodic cellular automata on two-sided sofic shifts f and g are topologically conjugate if and only if there is a homeomorphism between the state spaces mapping q-periodic points onto q-periodic points for all $q \in \{1, \dots, p\}$. To prove this, we have to analyze the topological structure of intersecting sofic shifts. The main tool to do this will be an algebraic structure which we call a derivative algebra. Pierce in the 1970s (see [Pie70] and [Pie72]) introduced what he called topological Boolean algebras to analyze the topological structure of compact metrizable zero-dimensional spaces. Topological Boolean algebras are closure algebras with an additional operation satisfying a rather complex axiom system. We show that derivative algebras are equivalent to Pierce's concept. They need, however, only one operation additional to the Boolean ones and have a very simple axiom system. The existence of a homeomorphism between two compact metrizable zero-dimensional spaces is then equivalent to the existence of an isomorphism between associated derivative algebras, as long as these algebras are finite. We will show that topological conjugacy and the weaker notion of topological orbit equivalence between periodic dynamical systems can also be characterized in terms of these algebras, again under the assumption that the algebras involved are finite. Showing that the derivative algebra generated by multiple sofic shifts is finite will then be done by graph theoretical methods. For one sofic subshift this was shown by Head in a series of papers [Hea85], [Hea86] and [Hea91]. For multiple sofic shifts however, we need some new ideas. Although this approach seems to be quite involved at first sight, we show that the result is not merely an effect of the zero-dimensionality of the state space. We construct two homeomorphic non-sofic two-sided subshifts and on each of them we build a cellular automaton for which every point has minimal period two. We will show that these cellular automata are, nevertheless, not topologically conjugate.

We apply these methods to classify the 256 elementary cellular automata with alphabet $\{0, 1\}$ and radius one up to topological conjugacy (for the curious reader, there are 83 equivalence classes). A particular interesting pair of conjugate cellular automata is given by rule 90 and rule 150. Both of them are left- and right-permutive and hence are known to be conjugate. When one actually calculates this conjugacy, one notices the phenomenon that the conjugacy uses local rules of larger and larger radius. Although there exist many conjugacies between these cellular automata, we will show that this phenomenon necessarily occurs. Every topological conjugacy between rule 90 and rule 150 uses an infinite number of local rules. Finally we investigate another equivalence relation between cellular automata, namely conjugacy on all tori.

We start Chapter 2 by introducing our notations and the basic objects of our study: subshifts, cellular automata and topological conjugacies. We also introduce some definitions and basic lemmas from graph theory that are crucial for the analysis of subshifts in dimension one. We conclude this chapter by an overview of the literature on the conjugacy problem for subshifts and some known theorems on periodic cellular automata.

Chapter 3 then introduces derivative algebras. After giving the definition we turn to the class of derivative algebras that we need most, namely the powerset algebra of a topological space together with the Cantor-Bendixson derivative.

In Chapter 4, we apply the results about derivative algebras to characterize the conjugacy of periodic dynamical systems on certain topological spaces. More precisely, these are the periodic dynamical systems on metrizable Stone spaces for which the derivative algebra generated by their sets of periodic points is finite.

To apply these results to cellular automata, it remains to show that indeed the derivative algebra generated by periodic cellular automata on sofic two-sided shifts is finite. This is the focus of Chapter 5. As we will show, this is equivalent to showing that the derivative algebras generated by a finite number of sofic two-sided shifts is finite. We do so by defining a large derivative algebra whose finiteness immediately follows from its definition and which contains the derivative algebra we are interested in. We start by considering the derivative algebras generated by subshifts of finite type and by one-sided path spaces in graphs. Since this algebra is quite huge and therefore somewhat impractical for calculations, we also introduce simpler derivative algebras that at least allow us to extract certain information about the subshifts under investigation. As a side result, we show that every metrizable Stone space, whose derivative algebra is finite, is homeomorphic to a one-sided subshift of finite type. Finally we show how to reduce the problem of determining the derivative algebra generated by multiple two-sided sofic shifts to the cases already considered.

In Chapter 6, we apply our results to classify the elementary cellular automata with alphabet $\{0, 1\}$ and radius one. It is well-known that these elementary cellular automata fall into 88 equivalence classes up to spatial symmetry and permutations of the alphabet. The systematically calculated invariants allow us to differentiate between 77 classes up to topological conjugacy. The data for these invariants is presented in Appendix A. Specific investigations of these classes then allow us to show that there are precisely 83 different elementary cellular automata up to topological conjugacy. One particular interesting class of elementary cellular automata is that of the left- and right-permuting ones. The systems in this class are known to be conjugate to the one-sided 4-shift and are therefore also pairwise conjugate. We show that rule 90 and rule 150 are only conjugate by homeomorphisms using an infinite number of local rules. This shows the complexity of the homeomorphisms that can arise as conjugacies between cellular automata. To do so, we use results by Thomas et al. [TSL06] and Goldwasser et al. [GKT97] to explicitly calculate the minimal preperiod for the action of the cellular automata on spatially periodic points.

Chapter 7 deals further with the action of cellular automata on spatially periodic points and the sequences of finite dynamical systems that arise from this action. These systems define yet another notion of isomorphism between cellular automata and we investigate the connection between this notion, topological conjugacy and various dynamical properties.

We conclude with a mixture of open problems in Chapter 8.

Before we delve into proper mathematics, a few words on notation are needed. The natural numbers start with one, so $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. The non-negative integers are denoted by $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. The powerset of a set M is denoted by $\mathscr{P}(M) = \{ K \mid K \subseteq M \}$.

If $\varphi : X \to Y$ is some function, we denote by $D(\varphi) := X$ the domain of φ. If $M \subseteq X$ is any subset of the domain of φ, we denote by $\varphi_{|M}$ the restriction of φ to M and by $\varphi[M]$ the image of M under φ. For $N \subseteq Y$ we denote by $\varphi^{-1}[N]$ the preimage of N. For two sets X and Y we denote by Y^X the set of all functions from X to Y. If X is a topological space, we denote by $\overline{M}$ the closure of $M \subseteq X$. By $\mathrm{id}_X : X \to X$ we denote the identity map on a set X. The field with two elements is denoted by $\mathbb{F}_2$.

Chapter 2.

Subshifts, Graphs and Cellular Automata

In this chapter, we introduce most of the objects that will play a part in the later parts of this thesis. We start with Stone spaces as the topological spaces on which our dynamical systems will act. Next we use graphs to describe the structure of certain symbolic spaces. After introducing dynamical systems and subshifts, we turn to their endomorphisms and arrive at cellular automata. Finally, we survey some known results about topological conjugacies between special classes of cellular automata and collect some facts about periodic cellular automata.

2.1. Stone Spaces and Symbolic Spaces

The biggest part of this thesis plays in the principality of symbolic dynamics, occupying a beautiful part of the huge kingdom of discrete dynamical systems. We therefore start by introducing some notation that will allow us to speak easily about symbolic spaces.

Throughout this thesis, A will be a non-empty finite set, usually containing at least two elements, which we call the *alphabet*. A finite sequence of symbols from A is called a *word*. For a word $w \in A^n$ we denote by $|w| := n$ the *length of this word*. The unique word of length zero is called the *empty word*. For $I \subseteq \mathbb{Z}$ and $k, \ell \in \mathbb{Z}$ with $\{k, k+1, \dots, \ell\} \subseteq I$ and $x \in A^I$ we denote by $x_{[k,\ell]} := x_k x_{k+1} \dots x_\ell$ the *subword of x from position k to position ℓ*. We also write $x_{[k,\ell+1)} = x_{(k-1,\ell]} = x_{(k-1,\ell+1)} := x_{[k,\ell]}$. For $I \subseteq \mathbb{Z}$ and $x \in A^I$ we call a word $w \in \bigcup_{n\in\mathbb{N}} A^n$ a *subword of x* if there is $i \in I$ such that $x_{[i,i+|w|)} = w$. The *concatenation* $v := w_1 w_2 \dots w_n \in A^{|w_1|+\dots+|w_n|}$ of a finite number of words $w_1, \dots, w_n$ is defined by

$$v_\ell = (w_k)_{\ell-(|w_1|+\dots+|w_{k-1}|)} \tag{2.1}$$

where $k \in \mathbb{N}$ is the unique positive integer for which $|w_1|+\dots+|w_{k-1}| < \ell \leq |w_1|+\dots+|w_k|$. Concatenating the same word w k-times gives the *k-th power* of w, denoted by

$$w^k := \underbrace{w \dots w}_{k \text{ times}}.$$

We also identify the symbols in A with the words of length one over A. This allows us to write, e.g., $w = a_1 a_2 a_3 = (a_1, a_2, a_3)$ for $a_1, a_2, a_3 \in A$.

For an infinite sequence of words $(w_k)_{k\in\mathbb{N}}$ with $|w_k| \geq 1$ for $k \in \mathbb{N}$ we define the concatenation $w_1w_2\cdots \in A^{\mathbb{N}}$ by the same formula (2.1) as in the case of a finite number of words. Finally for a two-sided infinite sequence $(w_k)_{k\in\mathbb{Z}}$ we define the concatenation $v = \ldots w_{-2}w_{-1}.w_0w_1\ldots$ by

$$v_\ell = \begin{cases} (w_0w_1w_2\ldots)_{\ell+1} & \text{if } \ell \geq 0 \\ (\tilde{w}_{-1}\tilde{w}_{-2}\tilde{w}_{-3}\ldots)_{-\ell} & \text{otherwise} \end{cases} \tag{2.2}$$

where $\tilde{w_k}$ denotes the reversal of w_k, i.e., $(\tilde{w}_k)_\ell = (w_k)_{|w_k|-\ell+1}$ for $k < 0$ and $\ell \in \{1,\ldots,|w_k|\}$.

If $(w_i)_{i\in\mathbb{N}}$ is an eventually constant sequence of non-empty words, i.e., there is $k_0 \in \mathbb{N}$ with $w_k = w_{k_0}$ for all $k \geq k_0$, we write $w_1w_2\ldots w_{k_0}^\infty$ for the concatenation of these words. The same holds for bi-infinite sequences of words which are eventually constant to the left. We represent them by ${}^\infty w_{\ell_0}w_{\ell_0+1}\ldots w_{-1}.w_0w_1\ldots$. For example

$$({}^\infty 0.12^\infty)_i = ({}^\infty(00).1(22)^\infty)_i = \begin{cases} 0 & \text{if } i < 0 \\ 1 & \text{if } i = 0 \\ 2 & \text{if } i > 0 \end{cases}.$$

Finally we define ${}^\infty w^\infty := {}^\infty w.w^\infty$. By σ we denote the *left shift* on $A^{\mathbb{N}}$, $A^{\mathbb{N}_0}$ and $A^{\mathbb{Z}}$. This map is given by $\sigma(x)_i = x_{i+1}$.

Now we turn to some topological notions. A topological space X is *Hausdorff* if for every pair of different points $x, y \in X$ there are disjoint open neighborhoods of x and y. A topological space is *metrizable* if there exists a metric generating the topology. It is *separable* if it contains a countable dense subset. A point in a topological space is called *isolated* if it has an open neighborhood consisting only of this point. A topological space is called *perfect* if it does not contain any isolated point. A subset of a topological space is called a *perfect subset* if it is closed and perfect when considered as a topological space with the induced topology. A subset is *clopen* if it is simultaneously open and closed. A topological space is called *zero-dimensional* if it has a base consisting of clopen subsets. If we want to emphasize the topology, we write $(X, \mathcal{T}_X)$ for a topological space X whose open sets are $\mathcal{T}_X$. If topological spaces X and Y are homeomorphic, we write $X \cong Y$.

All results in this section are real classics and can be found in most textbooks on topology, for example the one by [Eng89]. We nevertheless give proofs for some of them in order to get a feeling for the topological spaces on which our dynamical systems live.

A result that we will use over and over again is the following theorem.

Theorem 2.1. *Let X be a compact topological space and let Y be a topological space. If $\varphi\colon X \to Y$ is continuous, then $\varphi[X]$ is compact. If, additionally, φ is bijective and Y is Hausdorff, then φ is a homeomorphism.*

We also need two classical results which describe how certain topological properties transfer to product spaces, see for example Theorem 3.2.4 and Theorem 6.2.14 in the book by ENGELKING [Eng89].

Theorem 2.2 (Tychonoff's theorem). *For every family $(X_i)_{i\in I}$ of compact topological spaces the product space $\prod_{i\in I} X_i$ endowed with the product topology is again compact.*

Theorem 2.3. *For every family $(X_i)_{i\in I}$ of zero-dimensional topological spaces the product space $\prod_{i\in I} X_i$ endowed with the product topology is again zero-dimensional.*

Besides product spaces we will also use disjoint unions to build new topological spaces from old ones.

Definition 2.4 (Disjoint union of topological spaces). *For two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ we define the* disjoint union $X \oplus Y$ *as the topological space whose set of points is given by the set theoretic disjoint union of the set X and Y and whose topology is given by $\mathscr{T}_{X\oplus Y} = \{ U \oplus V \mid U \in \mathscr{T}_X, V \in \mathscr{T}_Y \}$.*

With these notations in place, we can now turn to the topological spaces which will serve as the state spaces of our dynamical systems.

Definition 2.5 (Stone space). *A zero-dimensional, compact Hausdorff space is called a* Stone space. *A perfect, metrizable Stone space is called a* Cantor space.

The name Stone space is due to the fact that these spaces arise as the topological spaces in the Stone duality theorem, see for example Chapter 34 in the book by GIVANT and HALMOS [GH09]. Many sources, for example the book just mentioned, use the term *Boolean space* instead of *Stone space*. We will almost exclusively talk about metrizable Stone spaces but we will nevertheless always mention the metrizability assumption.

Example 2.6. *The classical example of a Cantor space is the middle thirds Cantor set $\mathfrak{C} := \bigcap_{k=0}^{\infty} C_k$ where $C_0 := [0,1] \subseteq \mathbb{R}$, $C_{k+1} := \frac{1}{3}(C_k \cup (2 + C_k))$.*

Example 2.7. *The Cantor spaces, which we will need in our analysis of cellular automata, are constructed as follows. First we endow our finite alphabet A of cardinality at least two with the discrete topology. If we then consider the product space A^M for a countable set M, we obtain a Cantor space. Let us first check that A^M is metrizable. Without loss of generality we may assume that $M = \mathbb{N}$ or $M = \mathbb{Z}$. Define a metric d on A^M by*

$$d(x,y) = 2^{-\min\{\,|k| \;\mid\; k\in M,\, x_k \neq y_k\,\}} \text{ if } x \neq y.$$

Since

$$\left\{ y \in A^{\mathbb{N}} \;\middle|\; d(x,y) < \frac{1}{2^n} \right\} = \{ y \in A^{\mathbb{N}} \mid y_{[1,n]} = x_{[1,n]} \}$$

and

$$\left\{ y \in A^{\mathbb{Z}} \;\middle|\; d(x,y) < \frac{1}{2^n} \right\} = \{ y \in A^{\mathbb{Z}} \mid y_{[-n,n]} = x_{[-n,n]} \},$$

this metric generates the product topology on A^M. Two points are close in this metric if they agree on a large ball in the index set around the origin. Our space A^M is zero-dimensional by Theorem 2.3 and it is compact by Theorem 2.2. It is also easily seen to be perfect because A contains at least two elements.

Every closed subset of a Cantor space is again compact, zero-dimensional and metrizable, hence a metrizable Stone space, but not necessarily perfect. Based on the following lemma we show that all Cantor spaces are homeomorphic and that the metrizable Stone spaces are up to homeomorphism precisely the closed subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$.

Lemma 2.8. *For every $\varepsilon > 0$ and every compact, zero-dimensional and perfect metric space X there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ the space X is the disjoint union of 2^k non-empty clopen sets of diameter at most ε.*

Proof. Since X is zero-dimensional, we can cover it with clopen sets of diameter at most ε. Since X is compact, this cover has a finite subcover $V_1, \ldots, V_n$. The set system

$$\mathscr{U} := \left\{ \bigcap_{k\in M} V_k \cap \bigcap_{k\notin M} X \setminus V_k \;\middle|\; M \subseteq \{1,\ldots,n\} \right\} \setminus \{\emptyset\}$$

forms a partition of X into clopen sets of diameter at most ε. Choose $k_0 \in \mathbb{N}$ such that $2^{k_0} \geq |\mathscr{U}|$. By further splitting the sets in $\mathscr{U}$ into clopen parts, we obtain a partition of X into 2^k clopen sets of diameter at most ε for every $k \geq k_0$. □

Theorem 2.9. *Every Cantor space is homeomorphic to the Cantor space $\{0,1\}^{\mathbb{N}}$.*

Proof. Let X be a Cantor space. Let d be a metric generating the topology on X. Our goal is to define recursively a sequence of non-negative integers $(n_k)_{k\in\mathbb{N}}$ and a family of spaces X_w index by words $w \in W_\ell := \{1,\ldots,2^{n_1}\} \times \{1,\ldots,2^{n_2}\} \times \cdots \times \{1,\ldots,2^{n_\ell}\}$ for $\ell \in \mathbb{N}$ such that

(a) $n_1 = 0$, $X_1 = X$,
(b) $X_{wa} \subseteq X_w$ for all $w \in W_\ell, a \in \{1,\ldots,2^{n_{\ell+1}}\}, \ell \in \mathbb{N}$,
(c) $X_w = \biguplus_{a\in\{1,\ldots,2^{n_{\ell+1}}\}} X_{wa}$,
(d) $\operatorname{diam}(X_w) \leq \frac{1}{\ell}$ for $w \in W_\ell, \ell \in \mathbb{N}$.

Here $\operatorname{diam}(M) := \sup\{\, d(x,y) \mid x,y \in M \,\}$ denotes the diameter of the set $M \subseteq X$. Assume we defined n_k for $k \leq m$ and X_w for $w \in W_1 \cup \cdots \cup W_m$. By Lemma 2.8 there is $n_{m+1} \in \mathbb{N}$ such that we can split each of the sets X_w for $w \in \{1,\ldots,2^{n_m}\}$ into $2^{n_{m+1}}$ non-empty clopen sets $X_{w1},\ldots,X_{w2^{n_{m+1}}}$ of diameter at most $\frac{1}{m+1}$. These sets fulfill conditions (b) to (d).

For each element $x \in \prod_{k=1}^{\infty}\{1,\ldots,2^{\ell_k}\}$, the set $X_x := \bigcap_{k=1}^{\infty} X_{x_{[1,\ldots,k]}}$ is the intersection of a nested sequence of closed sets whose diameter tends to zero. Because X is a compact space, each set X_x therefore contains exactly one element which we denote by $\varphi(x)$. This defines a bijective map $\varphi\colon Y := \prod_{k=1}^{\infty}\{1,\ldots,2^{\ell_k}\} \to X$. Since $x_{[1,\ldots,n]} = y_{[1,\ldots,n]}$ implies $d(\varphi(x),\varphi(y)) \leq \frac{1}{n}$, we see that φ is a bijective continuous map between compact Hausdorff spaces, hence a homeomorphism. Finally $\{1,\ldots,2^k\}$ is homeomorphic to $\{0,1\}^k$, so Y, and therefore also X, are homeomorphic to $\{0,1\}^{\mathbb{N}}$. □

Corollary 2.10. *Every metrizable Stone space is homeomorphic to a closed subset of the Cantor space $\{0,1\}^{\mathbb{N}}$.*

Proof. Let X be a metrizable Stone space. Endow $Y := X \times \mathfrak{C}$ with the product topology. As a product of zero-dimensional metrizable compact spaces, Y is again zero-dimensional, metrizable and compact. Since $\mathfrak{C}$ is perfect, no point in Y is isolated and hence Y is also perfect. All these properties together imply that Y is a Cantor space. The set X is a closed subspace of Y which is homeomorphic to $\{0,1\}^{\mathbb{N}}$ by Theorem 2.9. □

2.2. Graphs

Most of our combinatorial constructions are based on graphs in one form or the other. We need very few results from graph theory, so this section mainly gives the relevant definitions. The terminology varies a lot between sources, especially when it comes to terms such as path, walk or trail. As a reference for directed graphs we recommend the book by Bang-Jensen and Gutin [BJG08]. All results mentioned in this section can be found in this book. The book by Lind and Marcus on symbolic dynamics [LM95] also contains most of the results on graphs that we need.

Definition 2.11 (Graph, terminal and initial vertex, loop)**.** *A* graph G *is a tuple consisting of a set* $V(G)$ *of* vertices*, a set* $E(G)$ *of* edges *and two maps* $\mathrm{i}_G\colon E(G) \to V(G)$ *and* $\mathrm{t}_G\colon E(G) \to V(G)$*. For an edge* $e \in E(G)$ *we call* $\mathrm{i}_G(e)$ *the* initial *and* $\mathrm{t}_G(e)$ *the* terminal *vertex of* e*. We also say that* e *starts in* $\mathrm{i}_G(e)$ *and ends in* $\mathrm{t}_G(e)$*. We often omit the graph in the index and simply write* $\mathrm{i}(e)$ *and* $\mathrm{t}(e)$*. This means that all our graphs are directed, that they might have multiple edges starting and ending in the same vertex and that we allow* loops*, i.e., there might be edges* $e \in E(G)$ *with* $\mathrm{i}_G(e) = \mathrm{t}_G(e)$*. The edges* $e \in E(G)$ *with* $\mathrm{i}_G(e) = i$ *are called the* out-going edges *of* i*. Those with* $\mathrm{t}_G(e) = i$ *are called the* in-going edges *of* i*.*

If the edge set is a subset of $V(G)^2$*, the* canonical initial and terminal maps *are given by the projection onto the first and second coordinate, i.e.,* $\mathrm{i}_G(i,j) = i$ *and* $\mathrm{t}_G(i,j) = j$*.*

If not mentioned otherwise, all our graphs are finite, i.e., the vertex set as well as the edge set of our graphs are finite.

Example 2.12. *Consider the graph* G *given by* $V(G) := \{1,2,3\}$*,* $E(G) := \{a,b,c,d,e\}$ *and*

$$\begin{aligned} &\mathrm{i}(a) = 1, \mathrm{t}(a) = 1, && \mathrm{i}(b) = 1, \mathrm{t}(b) = 2, && \mathrm{i}(c) = 1, \mathrm{t}(c) = 2, \\ &\mathrm{i}(d) = 2, \mathrm{t}(d) = 3, && \mathrm{i}(e) = 3, \mathrm{t}(e) = 2. \end{aligned}$$

This graph is shown in Figure 2.1. The edge a *is a loop since its initial and terminal vertex is the vertex* 1*.*

Definition 2.13 (Isomorphism of graphs)**.** *Let* K *and* G *be graphs. We call* K *and* G isomorphic*, if there are bijective maps* $\theta_V : V(K) \to V(G)$ *and* $\theta_E : E(K) \to E(G)$ *such that* $\mathrm{i}_G(\theta_E(e)) = \theta_V(\mathrm{i}_K(e))$ *and* $\mathrm{t}_G(\theta_E(e)) = \theta_V(\mathrm{t}_K(e))$ *for all* $e \in E(K)$*. The pair* (θ_V, θ_E) *is called a graph isomorphism. If every vertex in* K *and* G *has at least one out-going or in-going edge, then* θ_E *already determines* θ_V *and we simply call* θ_E *the graph isomorphism.*

It will sometimes be convenient to group some graphs together into a single graph. This is accomplished by the following definition.

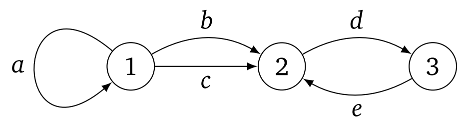

Figure 2.1.: Example of a graph with a loop and multiple edges between the same pair of nodes.

Definition 2.14 (Disjoint union of graphs)**.** *Let $G_1, \ldots, G_n$ be graphs. The* disjoint union *of $G_1, \ldots, G_n$ is defined as the graph G whose vertex and edge set is the disjoint union of the vertex and edge sets of $G_1, \ldots, G_n$. More formally,*

$$\begin{aligned} V(G) &= \bigcup_{j=1}^{n} V(G_j) \times \{ j \}, \\ E(G) &= \bigcup_{j=1}^{n} E(G_j) \times \{ j \}, \\ \mathrm{i}(e, j) &:= (\mathrm{i}(e), j), \\ \mathrm{t}(e, j) &:= (\mathrm{i}(e), j). \end{aligned}$$

We often treat $G_1, \ldots, G_n$ as subgraphs of their disjoint union.

Definition 2.15 (Path)**.** *Let I be a set of consecutive integers. A* path *or more specifically an I-*path *in a graph G is a sequence of edges $(\gamma_k)_{k \in I}$ of G such that $\mathrm{t}(\gamma_k) = \mathrm{i}(\gamma_{k+1})$ for all $k \in I$ with $k + 1 \in I$. If $I = \mathbb{N}$, we call γ an* infinite path*. If $I = \mathbb{Z}$, we call γ a* bi-infinite *path. We denote the set of all infinite paths in G by $\mathrm{Path}_{\mathbb{N}}(G)$, the set of all bi-infinite paths by $\mathrm{Path}_{\mathbb{Z}}(G)$ and the set of all finite paths simply by $\mathrm{Path}(G)$. Normally we index finite paths with the index set $I = \{1, \ldots, n\}$. If I is bounded from below, we call $\mathrm{i}(\gamma) := \mathrm{i}(\gamma_{\min(I)})$ the* initial vertex *of γ. If I is bounded from above, we call $\mathrm{t}(\gamma) := \mathrm{t}(\gamma_{\max(I)})$ the* terminal vertex *of γ. The set of finite or infinite paths whose initial vertex lies in a set $S \subseteq V(G)$ is denoted by $\mathrm{Path}(G, S)$ and $\mathrm{Path}_{\mathbb{N}}(G, S)$, respectively.*

For convenience, it is sometimes useful to also consider empty paths. For every vertex there is precisely one empty path starting and ending in this vertex. Appending or prepending an empty path to another path does not change this path. Formally it might therefore be desirable to add the initial and terminal vertex to the data that defines a path, but this is only necessary if the path is empty and creates problems with (bi-)infinite paths. In practice this slight imprecision does not pose any problems.

Sometimes we do not care which edges a path traverses and are only interested in the vertices it passes by. For this case we make the following definition.

Definition 2.16 (Vertex path)**.** *A* vertex path *in a graph G is a finite sequence of vertices $(v_k)_{k \in \{1, \ldots, n\}}$ such that there is an edge from v_k to v_{k+1} for all $k \in \{1, \ldots, n-1\}$.*

Example 2.17. *Consider again the graph G from Example 2.12. We have*

$$\begin{aligned}\mathrm{Path}_{\mathbb{Z}}(G) = &\left\{ \sigma^k({}^{\infty}a.b(de)^{\infty}) \;\middle|\; k \in \mathbb{Z} \right\} \cup \left\{ \sigma^k({}^{\infty}a.c(de)^{\infty}) \;\middle|\; k \in \mathbb{Z} \right\} \cup \\ &\{ {}^{\infty}a^{\infty}, {}^{\infty}(de)^{\infty}, {}^{\infty}(ed)^{\infty} \}\end{aligned}$$

and, for example, $abded \in \mathrm{Path}(G)$ and $ab(de)^{\infty} \in \mathrm{Path}_{\mathbb{N}}(G, \{1\})$. A vertex path is, for example, given by $(1,2,3,2,3)$.

We often treat paths as words over the alphabet $E(G)$. Everything said about the concatenation of words also applies to paths.

Definition 2.18 (Essential graph)**.** *A graph is called* essential *if for every vertex $i \in V(G)$ there is an edge starting in i and an edge ending in i. These are precisely the graphs for which every vertex is traversed by an bi-infinite path. Every graph G contains a maximal essential subgraph $\tilde{G}$ and $\mathrm{Path}_{\mathbb{Z}}(G) = \mathrm{Path}_{\mathbb{Z}}(\tilde{G})$. One can obtain $\tilde{G}$ for example by successively removing vertices from G that either lack in-going or out-going edges.*

Definition 2.19 (Closed path, directed cycle, acyclic)**.** *A finite path $(\gamma_j)_{j\in\{1,\dots,n\}}, n \in \mathbb{N}$ is called* closed, *if $\mathrm{i}(\gamma) = \mathrm{t}(\gamma)$. A graph G is called a* (directed) cycle *if one can enumerate its vertices by $i_0, \dots, i_{n-1}$ and its edges by $e_0, \dots, e_{n-1}$ such that $\mathrm{i}(e_\ell) = i_\ell$ and $\mathrm{t}(e_\ell) = i_{(\ell+1) \bmod n}$ for $\ell \in \{0, \dots, n-1\}$. A graph is called* acyclic *if it does not contain a closed path.*

To prove results about acyclic graphs it is often useful to order their vertices such that edges only pass from higher to lower vertices. The following lemma shows that this is always possible.

Lemma 2.20. *Let G be an acyclic graph. There exists a total order $\leq$ on the vertices of G such that $\mathrm{i}_G(e) > \mathrm{t}_G(e)$ for every edge $e \in E(G)$.*

Proof. The result is trivial if the graph contains at most one vertex. We proceed by induction. Since G is acyclic, there can be no path of length larger then $|V|+1$. Otherwise one vertex would be traversed at least twice by the path and the path would contain a cycle. For any path $\gamma \in \mathrm{Path}(G)$ of maximal length, the vertex $i := \mathrm{i}(\gamma)$ has no incoming edge. Removing i from G gives an acyclic graph $\tilde{G}$ whose vertices can be ordered such that the initial vertices of edges are greater then their terminal vertices by the induction hypothesis. We extend this order to G by demanding i to be larger than all other vertices. There is no edge whose terminal vertex is i and the assertion holds for all edges starting in i as well as for all edges in $\tilde{G}$. □

Definition 2.21 (Subgraph, induced subgraph)**.** *Let G be a graph. A graph H is called a* subgraph *of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, $i_H(e) = i_G(e)$ and $t_H(e) = t_G(e)$ for all $e \in E(H)$. If $M \subseteq V(G)$, we call a subgraph H the* subgraph induced by M *if it is the maximal subgraph of G having vertex set M, in other words,*

$$E(H) = \{ e \in E(G) \mid \mathrm{i}_G(e) \in M, \mathrm{t}_G(e) \in M \}.$$

Definition 2.22 (Strongly connected, strongly connected components, condensation)**.** *A directed graph G is* strongly connected *if for each pair of vertices $(i,j) \in V(G)^2$ there is a* *path starting in i and terminating in j, i.e.,*

$$\forall i, j \in V(G)\, \exists \gamma \in \mathrm{Path}(G)\colon\; \mathrm{i}(\gamma) = i, \mathrm{t}(\gamma) = j.$$

We define an equivalence relation on $V(G)$ by

$$i \equiv j \iff \exists \gamma_1, \gamma_2 \in \mathrm{Path}(G) : \mathrm{i}(\gamma_1) = i = \mathrm{t}(\gamma_2) \text{ and } \mathrm{i}(\gamma_2) = j = \mathrm{t}(\gamma_1).$$

Let $S(G)$ be the set of equivalence classes with respect to $\equiv$. We call elements of $S(G)$ the strongly connected components *of G. Indeed this terminology is justified as the subgraphs induced by the strongly connected components are precisely the maximal strongly connected subgraphs of G.*

Based on this equivalence relation we define a graph $\mathscr{S}(G)$, called the strong component graph *or* condensation *of G. The vertices of $\mathscr{S}(G)$ are the subgraphs of G induced by the elements of $S(G)$. For strongly connected components H and K with $H \neq K$ there is an edge from H to K in $\mathscr{S}(G)$ if there is an edge $e \in E(G)$ starting in a vertex in H and ending in a vertex in K.*

Example 2.23. *For the graph G from Example 2.12 let K_1 be the subgraph induced by the vertex 1 and let K_2 be the subgraph induced by the vertices 2 and 3. The condensation $\mathscr{S}(G)$ of G has these two subgraphs as vertices and contains a single edge from K_1 to K_2.*

Lemma 2.24. *The condensation $\mathscr{S}(G)$ of a graph G is acyclic.*

Proof. Assume there is a closed path γ of length n in $\mathscr{S}(G)$. By definition $\mathscr{S}(G)$ contains no loops, thus n is at least two. There must be edges e_k in $E(G)$ such that $\mathrm{i}(e_k) \in V(\mathrm{i}(\gamma_k))$ and $\mathrm{t}(e_k) \in V(\mathrm{t}(\gamma_k))$ for every $k \in \{1, \dots, n\}$. Since the vertices of $\mathscr{S}(G)$ are strongly connected subgraphs, there are also paths α_k from $\mathrm{t}(e_k)$ to $\mathrm{i}(e_{k+1})$ for $k \in \{1, \dots, n-1\}$ and a path α_n from $\mathrm{t}(e_n)$ to $\mathrm{i}(e_1)$. Set $K_1 := \mathrm{i}_{\mathscr{S}(G)}(\gamma_1)$, $K_2 := \mathrm{i}_{\mathscr{S}(G)}(\gamma_2)$, $i_1 := \mathrm{i}_G(e_1) \in V(K_1)$ and $i_2 := \mathrm{t}_G(e_1) \in V(K_2)$. The edge e_1 starts in i_1 and ends in i_2. The path $\alpha_1 e_2 \alpha_2 \dots e_n \alpha_n$ starts in i_2 and ends in i_1. Therefore K_1 and K_2 are the same strongly connected component. Hence γ must have length one, contradicting $n \geq 2$. □

Definition 2.25 (Order on the strongly connected components)**.** *Let G be a graph. We define a relation $<$ on the strongly connected components of G by setting $K < H$ if there is a finite path in G which starts in $K \in V(\mathscr{S}(G))$ and ends in $H \in V(\mathscr{S}(G))$. By Lemma 2.24 this relation is a partial order.*

For every path $(\gamma_\ell)_{\ell \in I}$ there exists a unique vertex path $K_1, \dots, K_n$ in $\mathscr{S}(G)$ and uniquely determined indices $k_1 < k_2 < \dots < k_{n-1}$ in I such that

$$\begin{aligned}
&\mathrm{i}(\gamma_k) \in V(K_1),\ \mathrm{t}(\gamma_k) \in V(K_1) && \text{for } k \in I, k < k_1,\\
&\mathrm{i}(\gamma_{k_\ell}) \in V(K_\ell),\ \mathrm{t}(\gamma_{k_\ell}) \in V(K_{\ell+1}) && \text{for } \ell \in \{1, \dots, n-2\},\\
&\mathrm{i}(\gamma_k) \in V(K_\ell),\ \mathrm{t}(\gamma_k) \in V(K_\ell) && \text{for } \ell \in \{1, \dots, n-2\}, k \in I, k_\ell < k < k_{\ell+1},\\
&\mathrm{t}(\gamma_k) \in V(K_n) && \text{for } k \in I, k > k_{n-1}.
\end{aligned}$$

We say that γ *starts in the component K_1* and *ends in the component K_n*.

For any graph G with at least two edges, the spaces $E(G)^{\mathbb{N}}$ and $E(G)^{\mathbb{Z}}$ are Cantor spaces. For any $S \subseteq V(G)$, the sets $\mathrm{Path}_{\mathbb{N}}(G,S)$, $\mathrm{Path}_{\mathbb{N}}(G)$ and $\mathrm{Path}_{\mathbb{Z}}(G)$ are closed subspaces of $E(G)^{\mathbb{N}}$ and $E(G)^{\mathbb{Z}}$. In particular they are metrizable Stone spaces.

2.3. Topological Dynamics and Subshifts

As mentioned in the introduction, topological dynamics provides a natural framework to study the behavior of cellular automata. We thus introduce in this section the necessary notions from topological and symbolic dynamics and give proofs for some known results about subshifts which we will need later on. As a reference for topological dynamics we recommend the book by KURKA [Kur03]. The standard references for subshifts are the book by LIND and MARCUS [LM95] and the book by KITCHENS [Kit98].

For us a *topological dynamical system* is a compact topological space X together with a continuous map $f: X \to X$. Since the domain and codomain of f coincide, we can repeatedly apply f and obtain the k-th iterate of f,

$$f^k = \underbrace{f \circ \cdots \circ f}_{k \in \mathbb{N}}.$$

We can think of X as a set of states and of f as describing the transition from a state at one point in time to the state one time step later. The sequence $(f^k(x))_{k \in \mathbb{N}_0}$ then describes the evolution of the state x as time progresses. Iterating f those gives rise to an action of the monoid $\mathbb{N}_0$ on X. It will sometimes be useful to generalize this and consider actions of an arbitrary monoid Λ on a compact space X. Recall that a monoid is a set with an associative binary operation and a neutral element which we denote by 1_Λ.

Definition 2.26 (Monoid action). *Let Λ be a monoid and let X be a topological space. A family of continuous maps $(f_i)_{i \in \Lambda}$ from X to X is called a* (continuous left) action of Λ on *X if*

$$\begin{aligned} f_{1_\Lambda} &= id, \\ f_{ij} &= f_i \circ f_j \quad \text{for all } i,j \in \Lambda. \end{aligned}$$

The set of all points which can be reached via the monoid action from a given starting point x is called the orbit of x.

An invertible dynamical system can be seen either as an action of $\mathbb{N}$ or as an action of $\mathbb{Z}$. Both perspectives give rise to a different notion of orbit.

Definition 2.27 (Orbit). *Let $f: X \to X$ be a dynamical system. We denote by* $\mathrm{Orb}^+(f,x) := \{ f^k(x) \mid k \in \mathbb{N}_0 \}$ *the* forward orbit of x under f. *If f is invertible, we denote by* $\mathrm{Orb}(f,x) := \{ f^k(x) \mid k \in \mathbb{Z} \}$ *the* orbit of x under f. *The set of all orbits of f is denoted by* $\mathrm{Orb}(f) := \{ \mathrm{Orb}(f,x) \mid x \in X \}$.

Finite orbits are of particular interest to us. They give rise to the notion of periodic and preperiodic points.

Definition 2.28 (Periodic points)**.** *Let $f : X \to X$ be a dynamical system. For $p \in \mathbb{N}$ we denote by*

$$Per_p(f) := \{\, x \in X \mid f^p(x) = x \,\}$$

the set of periodic points of f with period p. *The set of* fixed points *of f is denoted by $Fix(f) := Per_1(f)$. The set of* periodic points of f with minimal period p *is denoted by*

$$\widetilde{Per}_p(f) := Per_p(f) \setminus \bigcup \{\, Per_\ell(f) \mid \ell \in \{1, \dots, p-1\} \,\}\,.$$

The set of all periodic points of f is denoted by $Per(f) := \bigcup_{p \in \mathbb{N}} Per_p(f)$.

Definition 2.29 (Preperiodic points)**.** *For a dynamical system $f : X \to X$ and $p \in \mathbb{N}, q \in \mathbb{N}_0$ we define*

$$\begin{aligned} Pre_{q,p}(f) &:= \left\{\, x \in X \;\middle|\; f^q(x) = f^{q+p}(x) \,\right\}, \\ \widetilde{Pre}_{q,p}(f) &:= \{\, x \in X \mid \operatorname{pre}(x) = q \text{ and } \operatorname{per}(x) = p \,\} \end{aligned}$$

where

$$\operatorname{pre}(x) := \min\{\, q \in \mathbb{N}_0 \mid f^q(x) \in Per(f) \,\} \in \mathbb{N}_0 \cup \{\infty\}$$

is the minimal preperiod *and where*

$$\operatorname{per}(x) := \min\left\{\, p \in \mathbb{N} \;\middle|\; f^{\operatorname{pre}(x)}(x) \in Per_p(f) \,\right\} \in \mathbb{N} \cup \{\infty\}$$

is the minimal period *of x with respect to f. We call $Pre_{q,p}(f)$ the* preperiodic points with period p and preperiod q *and we call $\widetilde{Pre}_{q,p}(f)$ the* preperiodic points with minimal period p and minimal preperiod q.

Let $f : X \to X$ be a dynamical system. If $f^p = id_X$, we call f p-periodic. If $f^{q+p} = f^q$, then we call f preperiodic.

The natural notion of isomorphism between dynamical systems is that of a topological conjugacy.

Definition 2.30 (Topological conjugacy)**.** *Two topological dynamical systems $f : X \to X$ and $g : Y \to Y$ are called* topologically conjugate *if there is a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. More generally let Λ be a monoid and $(f_i)_{i\in\Lambda}$ and $(g_i)_{i\in\Lambda}$ be two continuous actions of Λ on the topological spaces X and Y. We call $(f_i)_{i\in\Lambda}$ and $(g_i)_{i\in\Lambda}$ topologically conjugate if there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi \circ f_i = g_i \circ \varphi$ for all $i \in \Lambda$.*

We now look at a particularly simple class of dynamical systems, namely subshifts. For the remainder of this section let Γ be a countable group and let $\Lambda \subseteq \Gamma$ be a submonoid. Denote the neutral element by $1_\Gamma = 1_\Lambda$. While we almost exclusively will present results on subshifts over the group $\mathbb{Z}$, considering the more general case often leads to interesting questions and conjectures. Furthermore, proving results in the more general context highlights the places where one really uses the structure of $\mathbb{Z}$.

Definition 2.31 (Shift maps, shift-invariant, subshifts over groups). *For $i \in \Gamma$ we define the* shift map $\sigma_i : A^\Gamma \to A^\Gamma$ *by*

$$(\sigma_i(x))_j := x_{i^{-1}j}.$$

These maps define an action of Γ on A^Γ. A set $X \subseteq A^\Gamma$ is called shift-invariant *if $\sigma_i[X] \subseteq X$ for all $i \in \Gamma$.*

Endowed with the product topology, A^Γ is a compact space. If A has cardinality at least two, A^Γ is a Cantor space. A closed, shift-invariant subset $X \subseteq A^\Gamma$ is called a subshift. *We call the elements of A^Γ* configurations.

Example 2.32. *Consider the alphabet $A = \{0,1\}$. The set of all configurations $x \in A^{\mathbb{Z}}$ with no pair of consecutive ones is a subshift, called the* golden-mean subshift. *We can define a similar subshift over the group $\Gamma = \mathbb{Z}^2$. Consider again the set X of all configurations $x \in A^\Gamma$ with no two adjacent ones. More precisely*

$$X := \left\{ x \in A^{\mathbb{Z}^d} \;\middle|\; \forall k,\ell \in \mathbb{Z} : x_{k,\ell} = 1 \implies x_{k+1,\ell} = 0 \text{ and } x_{k,\ell+1} = 0 \right\}$$

This subshift is known in statistical physics as the hard square model.

We sometimes need a slight generalization of the concept of a subshift to the case where Γ is not a group but only a submonoid of a group. All results about subshifts on monoids in this section are folklore and more or less straightforward generalizations of the corresponding results over $\mathbb{Z}$ or $\mathbb{N}_0$. However, they seem to be not explicitly stated in the literature.

Definition 2.33 (Subshifts over submonoids). *Let Γ be countable group and let $\Lambda \subseteq \Gamma$ be a submonoid. A closed subset $X \subseteq A^\Lambda$ is called a* subshift *if $\sigma_i[X] \subseteq X$ for all $i \in \Lambda^{-1} = \left\{ i^{-1} \;\middle|\; i \in \Lambda \right\}$. The subshift A^Λ is called the* full A-shift *or the* full k-shift, *where k is the cardinality of A.*

Remark 2.34. We stick here with the convention of the book by Ceccherini-Silberstein and Coornaert [CSC10] to define $\sigma_i(x)_j$ as $x_{i^{-1}j}$ and not x_{ij}. This has the following advantages. First of all we get $\sigma_i \circ \sigma_j = \sigma_{ij}$ instead of $\sigma_i \circ \sigma_j = \sigma_{ji}$, i.e., we get a left action instead of a right action. Our definition also coincides with the intuitive understanding of what it means to shift a configuration by i. If a configuration x over the alphabet $\{0,1\}$ has a single one at the origin, the configuration $\sigma_i(x)$ has a single one at position i. The obvious disadvantage is that this definition does not carry over to monoids in general and we have the somewhat artificial restriction to submonoids of groups. However, the only monoid we have to deal with will be $\mathbb{N}_0$ and products of $\mathbb{N}_0$ with groups.

Subshifts can more combinatorially be described by so-called forbidden patterns.

Definition 2.35 (Pattern, $\mathbb{X}_M$). *A* pattern *w is an element of A^H for some finite subset $H \subseteq \Lambda$. For a pattern w and $i \in \Lambda$ we denote by $[w]_i$ the set of all configurations, in which w*

appears at position i, i.e.,

$$[w]_i := \left\{ x \in A^\Lambda \;\middle|\; \sigma_{i^{-1}}(x)_{|D(w)} = w \right\}.$$

Let M be a set of patterns. Denote by $\mathbb{X}_M$ the set of all configurations in which no pattern from M appears, i.e.,

$$\begin{aligned}\mathbb{X}_M &:= \left\{ x \in A^\Lambda \;\middle|\; \forall w \in M\ \forall i \in \Lambda \colon x \notin [w]_i \right\} \\ &= A^\Lambda \setminus \left(\bigcup_{w \in M} \bigcup_{i \in \Lambda} [w]_i \right).\end{aligned}$$

We call M the set of forbidden patterns *of $\mathbb{X}_M$.*

Lemma 2.36. *For every set of patterns M, the space $\mathbb{X}_M$ is a subshift.*

Proof. The set $A^\Lambda \setminus \mathbb{X}_M = \bigcup_{w \in M} \bigcup_{i \in \Lambda} [w]_i$ is open, hence $\mathbb{X}_M$ is closed. Assume there was $x \in \mathbb{X}_M$ and $i \in \Lambda^{-1}$ with $\sigma_i(x) \notin \mathbb{X}_M$. Then there must be $w \in M$ and $j \in \Lambda$ such that $\sigma_i(x) \in [w]_j$ and therefore $x \in [w]_{i^{-1}j}$, contradicting $x \in \mathbb{X}_M$. Thus $\mathbb{X}_M$ is also shift-invariant. □

Lemma 2.37. *Let $X \subseteq A^\Lambda$ be a subshift. If M is the set of all patterns not appearing in X, i.e.,*

$$M = \left\{ w \in A^H \;\middle|\; H \subseteq \Lambda \text{ finite}, \forall x \in X\ \forall i \in \Lambda \colon x \notin [w]_i \right\},$$

then $X = \mathbb{X}_M$.

Proof. Since by definition no pattern in M appears in any configuration in X, $X \subseteq \mathbb{X}_M$. Since $A^\Lambda \setminus X$ is open, there is a set of patterns N such that $A^\Lambda \setminus X = \bigcup_{w \in N} [w]_{1_\Lambda}$. Because X is shift-invariant, we also have $A^\Lambda \setminus X = \bigcup_{j \in \Lambda} \bigcup_{w \in N} [w]_j = A^\Lambda \setminus \mathbb{X}_N$, hence $X = \mathbb{X}_N$. Now $w \in N$ implies that for all $x \in X$ and for all $j \in \Lambda$ we have $x \notin [w]_j$. Therefore $N \subseteq M$ and $X = \mathbb{X}_N \supseteq \mathbb{X}_M$. □

Continuous, shift-commuting maps between subshifts have a nice combinatorial description. This result goes back to the seminal paper [Hed69] by Hedlund where it is additionally credited to Curtis and Lyndon.

Lemma 2.38. *Let A and B be finite alphabets and let X be a closed subset of A^Λ. A map $h \colon X \to B$ is continuous if and only if there is a finite subset $H \subseteq \Lambda$ and a map $h_{loc} \colon A^H \to B$ such that*

$$h(x) = h_{loc}(x_{|H}). \tag{2.3}$$

Proof. Let h be continuous. For every element $b \in B$ the set $h^{-1}[\{b\}]$ is clopen, hence the union of clopen cylinder sets. Since X is compact, this union can be assumed to be finite. Additionally there are only finitely many symbols in B. Hence there exists a finite set $H \subseteq \Lambda$ and sets of patterns $W_b \subseteq A^H$ for $b \in B$ such that $h^{-1}[\{b\}] = \bigcup_{w \in W_b} [w]_{1_\Lambda}$.

Let $h_{\mathrm{loc}}(w)$ be the unique element in $h[\{\, x \in X \mid x_{|H} = w \,\}]$, which by definition fulfills $h(x) = h_{\mathrm{loc}}(x_{|H})$.

If, on the other, h is defined by (2.3), then $x_{|H} = y_{|H}$ implies $h(x) = h(y)$ for all $x, y \in X$. Hence h is continuous. $\square$

Theorem 2.39 (Curtis-Lyndon-Hedlund on monoids)**.** *Let Γ be a countable group and $\Lambda \subseteq \Gamma$ a submonoid. Let $X \subseteq A^\Lambda$ and $Y \subseteq B^\Lambda$ be subshifts. A map $f : X \to Y$ is continuous and shift-commuting if and only if there is a finite subset $H \subseteq \Lambda$ and a* block map $f_{loc} : A^H \to B$, *called the* local rule*, such that*

$$f(x)_i = f_{loc}(\sigma_{i^{-1}}(x)_{|H}) \quad \text{for all } i \in \Lambda. \tag{2.4}$$

Proof. Assume that f is continuous and shift-commuting. By Lemma 2.38 there is $H \subseteq \Gamma$ and a local rule $f_{\mathrm{loc}} : A^H \to A$ such that

$$f(x)_{1_\Lambda} = f_{\mathrm{loc}}(x_{|H}). \tag{2.5}$$

Since f commutes with the shift,

$$\begin{aligned} f(x)_i &= (\sigma_{i^{-1}}(f(x)))_{1_\Lambda} \\ &= f_{\mathrm{loc}}(\sigma_{i^{-1}}(x)_{|H}). \end{aligned}$$

If, on the other hand, f is defined by (2.4), then $x \mapsto f(x)_i$ is continuous for every $i \in \Lambda$ and

$$\begin{aligned} \sigma_j(f(x))_i &= f(x)_{j^{-1}i} \\ &= f_{\mathrm{loc}}((\sigma_{i^{-1}j}(x)_{|H}) \\ &= f(\sigma_j(x))_i. \end{aligned}$$

$\square$

Definition 2.40 (Subshift of finite type, sofic subshift)**.** *Let $X \subseteq A^\Lambda$ be a subshift. If there is a finite set of patterns $M = \{w_1, \dots, w_n\}$ such that $X = \mathbb{X}_M$, we call X a* subshift of finite type*. We can always assume that there is a single finite set $H \subseteq \Lambda$ such that $M \subseteq A^H$. Subshifts that are images of subshifts of finite type under continuous, shift-commuting maps are called* sofic subshifts*. More precisely, a subshift $X \subseteq A^\Lambda$ is sofic if there is an alphabet B, a subshift of finite type $Y \subseteq B^\Lambda$ and a continuous surjective map $\varphi : Y \to X$ with $\varphi \circ \sigma_i = \sigma_i \circ \varphi$ for all $i \in \Lambda$.*

Remark 2.41. If $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{N}_0$, we can always find a set of forbidden patterns M consisting of patterns whose domain is of the form $0, \dots, n-1$ for some n. We can identify these patterns with words in A^n. If X is a one- or two-sided subshift of finite type defined by a finite set of words of length at most n, we say that X has *window width n*.

A subshift conjugate to a sofic subshift is almost by definition again a sofic subshift. The same holds for subshifts of finite type.

Lemma 2.42. *Let $X \subseteq A^\Lambda$ be a subshift of finite type. If $Y \subseteq B^\Lambda$ is a subshift which is topologically conjugate to X, then Y is also a subshift of finite type.*

Proof. Since shift-commuting homeomorphisms between X and Y can be represented by block maps as shown in Theorem 2.39, we can find continuous shift-commuting maps $\varphi\colon A^\Lambda \to B^\Lambda$ and $\psi\colon B^\Lambda \to A^\Lambda$ such that $(\varphi\circ\psi)_{|Y} = \mathrm{id}_Y$ and $(\psi\circ\varphi)_{|X} = \mathrm{id}_X$. The subshift X is of finite type, hence it is defined by a finite set of forbidden patterns $M \subseteq A^H$ for some finite subset $H \subseteq \Lambda$. Since φ and ψ are continuous, there is a finite set $\tilde{H} \subseteq \Lambda$ with $1_\Lambda \in \tilde{H}$ such that for all $y_1, y_2 \in B^\Lambda$ with $(y_1)_{|\tilde{H}} = (y_2)_{|\tilde{H}}$ both $\psi(y_1)_{|H} = \psi(y_2)_{|H}$ and $\varphi(\psi(y_1))_{1_\Lambda} = \varphi(\psi(y_2))_{1_\Lambda}$ hold.

Let $\tilde{Y}$ be the set of all configurations in B^Λ that contain only patterns over $\tilde{H}$ which also appear in configurations in Y. More precisely,

$$\begin{aligned}\tilde{Y} &:= \left\{\, \tilde{y} \in B^\Lambda \;\middle|\; \forall i \in \Lambda^{-1}\, \exists y \in Y : \sigma_i(\tilde{y})_{|\tilde{H}} = y_{|\tilde{H}} \,\right\}\\ &= \mathbb{X}_{\tilde{M}},\end{aligned}$$

where $\tilde{M} := B^{\tilde{H}} \setminus \left\{\, y_{|\tilde{H}} \;\middle|\; y \in Y \,\right\}$. This set $\tilde{Y}$ is a subshift of finite type. All that is left to do is showing that $Y = \tilde{Y}$. By definition, $Y \subseteq \tilde{Y}$ holds. For every $\tilde{y} \in \tilde{Y}$ and $i \in \Lambda^{-1}$ there is $y \in Y$ with $\sigma_i(\tilde{y})_{|\tilde{H}} = y_{|\tilde{H}}$. Then $\sigma_i(\psi(\tilde{y}))_{|H} = \psi(\sigma_i(\tilde{y}))_{|H} = \psi(y)_{|H} \notin M$, hence $\psi(\tilde{y}) \in X$ and $\psi[\tilde{Y}] \subseteq X$. Additionally,

$$\begin{aligned}\varphi(\psi(\tilde{y}))_{i^{-1}} &= \varphi(\psi(\sigma_i(\tilde{y})))_{1_\Lambda}\\ &= \varphi(\psi(y))_{1_\Lambda}\\ &= y_{1_\Lambda} = \sigma_i(\tilde{y})_{1_\Lambda} = \tilde{y}_{i^{-1}}.\end{aligned}$$

Thus $\tilde{y} = \varphi(\psi(\tilde{y})) \in \varphi[\psi[\tilde{Y}]] \subseteq \varphi[X] = Y$. □

Most of the time we focus our attention on the situation where $\Gamma = \Lambda = \mathbb{Z}$. A subshift $X \subseteq A^{\mathbb{Z}}$ is called a *two-sided subshift*. Since $\mathbb{Z}$ is generated by one element, two-sided subshifts are precisely the closed subsets of $X \subseteq A^{\mathbb{Z}}$ for which $\sigma_{-1}[X] = X$. In the case of $\Gamma = \mathbb{Z}$ we therefore omit the index -1 and simply write σ or σ_X for the *left shift*. For $\Lambda = \mathbb{N}_0$, the shift maps are also all powers of the left shift. Closed subsets $X \subseteq A^{\mathbb{N}_0}$ that are invariant under σ are called *one-sided subshifts*. Notice, however, that in this case $\sigma\colon A^{\mathbb{N}_0} \to A^{\mathbb{N}_0}$ is no longer bijective, as for example $\sigma(10^\infty) = \sigma(0^\infty) = 0^\infty$. It is sometimes convenient to identify one-sided subshifts with subsets of $A^{\mathbb{N}}$ although $\mathbb{N}$ is not a monoid. The left shift, however, acts on $A^{\mathbb{N}}$ the same way as it does in $A^{\mathbb{N}_0}$, hence all results from this section carry over to $A^{\mathbb{N}}$.

Let G be a graph. The set $\mathrm{Path}_{\mathbb{Z}}(G)$ is a closed, shift-invariant subset of $E(G)^{\mathbb{Z}}$, in other words, a two-sided subshift over the alphabet $E(G)$. Since $\gamma \in \mathrm{Path}_{\mathbb{Z}}(G)$ if and only if $\mathrm{i}_G(\gamma_i) = \mathrm{t}_G(\gamma_{i+1})$ for all $i \in \mathbb{Z}$, $\mathrm{Path}_{\mathbb{Z}}(G)$ is a subshift of finite type with window width 2, called the *(two-sided) edge shift* of G. By the same reasoning, the set $\mathrm{Path}_{\mathbb{N}}(G)$ is a one-sided subshift of finite type over the alphabet $E(G)$. We will show that, up to topological conjugacy, all one-sided and two-sided subshifts of finite type arise in that way.

Definition 2.43 (Higher block representation)**.** *Let $X \subseteq A^{\mathbb{Z}}$ be a two-sided subshift and let $n \in \mathbb{N}$. Define a map $\varphi_n\colon A^{\mathbb{Z}} \to (A^n)^{\mathbb{Z}}$ by*

$$\varphi_n(x)_i = x_{[i,i+n)} \quad \textit{for } i \in \mathbb{Z}.$$

We call $\varphi_n[X]$ *the* higher block representation of X with block-length n. *We can apply the same construction to one-sided subshifts* $X \subseteq A^{\mathbb{N}}$ *and obtain a subshift of* $(A^n)^{\mathbb{N}}$.

Lemma 2.44. *The higher block representation* $\varphi_n[X]$ *is a subshift which is topologically conjugate to* X *via the conjugacy* φ_n*. In particular* $\varphi_n[X]$ *is of finite type if and only if* X *is of finite type and* $\varphi_n[X]$ *is sofic if and only if* X *is sofic.*

Proof. The map φ_n is continuous, injective and commutes with the shift, hence it is a topological conjugacy. $\square$

Theorem 2.45. *Let* X *be a one-sided or two-sided subshift of finite type with window width* n *over the alphabet* A*. If* φ_n *is the higher block map with block length* n*, there is an essential graph* G *such that* $\varphi_n[X] = \mathrm{Path}_{\mathbb{N}}(G)$ *or* $\varphi_n[X] = \mathrm{Path}_{\mathbb{Z}}(G)$*, respectively. In particular every one-sided or two-sided subshift of finite type is conjugate to the edge shift of some graph.*

Proof. Define the graph G by

$$\begin{aligned} V(G) &:= \left\{ w \in A^{n-1} \;\middle|\; [w]_0 \cap X \neq \emptyset \right\}, \\ E(G) &:= \{ w \in A^n \mid [w]_0 \cap X \neq \emptyset \}. \end{aligned}$$

The initial and terminal maps are given by $\mathrm{i}_G(x_1,\dots,x_n) = x_1,\dots,x_{n-1}$ and $\mathrm{t}_G(x_1,\dots,x_n) = x_2,\dots,x_n$. For every $x \in X$ and $k \in \mathbb{Z}$ we have $\varphi_n(x)_k \in E(G)$. Additionally $\mathrm{t}_G(\varphi_n(x)_k) = x_{[k+1,k+n)} = \mathrm{i}_G(\varphi_n(x)_{k+1})$, hence $\varphi_n[X] \subseteq \mathrm{Path}_{\mathbb{Z}}(G)$. Define a map $\psi\colon \mathrm{Path}_{\mathbb{Z}}(G) \to A^{\mathbb{Z}}$ by $\psi(\gamma)_k = (\gamma_k)_1$. Then $\psi(\gamma)_{[k,k+n)} = (x_k, x_{k+1}, \dots, x_{k+n-1}) = \gamma_k$, hence $\varphi_n(\psi(\gamma)) = \gamma$. If there was $x = \psi(\gamma) \notin X$, there would have to be $k \in \mathbb{Z}$ such that $[x_{[k,\dots,k+n-1)}]_0 \cap X = \emptyset$ since X is a subshift of finite type with window width n. But $x_{[k,\dots,k+n)} = \gamma_k \in E(G)$, hence $[x_{[k,\dots,k+n)}]_0 \cap X \neq \emptyset$, a contradiction. Therefore $\psi[\mathrm{Path}_{\mathbb{Z}}(G)] \subseteq X$ and $\varphi_n[X] \supseteq \varphi_n[\psi[\mathrm{Path}_{\mathbb{Z}}(G)]] = \mathrm{Path}_{\mathbb{Z}}(G)$. Hence finally $\varphi_n[X] = \mathrm{Path}_{\mathbb{Z}}(G)$. The one-sided version is proved in the same way. $\square$

The graphs obtained by applying this construction to an one-sided or two-sided full shift are called *De Bruijn graphs*.

Definition 2.46 (Edge and vertex labeling, right-resolving)**.** *A* vertex *or* edge labeling *of a graph* G *is simply a map from the vertex or edge set into a finite set. An edge labeling* $\mathscr{L}$ *is called* right-resolving *if and only if for every vertex all out-going edges carry different labels, i.e., if for all pairs of different edge* $e_1, e_2 \in E(G)$ *with* $\mathrm{i}_G(e_1) = \mathrm{i}_G(e_2)$ *we have* $\mathscr{L}(e_1) \neq \mathscr{L}(e_2)$*.*

Let $\mathscr{L}: E(G) \to A$ be an edge labeling of G with values in a finite alphabet A. There is a natural extension $\mathscr{L}\colon \mathrm{Path}_{\mathbb{Z}}(G) \to A^{\mathbb{Z}}$ given by

$$\mathscr{L}(\gamma)_k := \mathscr{L}(\gamma_k).$$

This map is continuous and shift-commuting, hence $\mathscr{L}[\mathrm{Path}_{\mathbb{Z}}(G)]$ is a sofic shift. The next lemma shows that every sofic subshift in dimension one arises in that way as the image of an edge shift under an edge labeling.

Lemma 2.47. *If $X_1, \dots, X_n$ are one-sided or two-sided sofic shifts over the alphabet A, then there exists a graph G, subgraphs $G_1, \dots, G_n$ and an edge labeling $\mathcal{L}: V(G) \to A$ such that $\mathcal{L}[\mathrm{Path}_{\mathbb{Z}}(G_k)] = X_k$ or $\mathcal{L}[\mathrm{Path}_{\mathbb{N}}(G_k)] = X_k$ for all $k \in \{1, \dots, n\}$. The pair $(G, \mathcal{L})$ is called* a cover *of $X_1, \dots, X_n$.*

Proof. By the definition of sofic shifts there are subshifts of finite type $Y_1, \dots, Y_n$ and factor maps $\varphi_k: X_k \to Y_k$. By Theorem 2.45 we may assume that these subshifts of finite type are given as $\mathrm{Path}_K(G_k)$ for some graphs G_k and that φ_k is given by an edge labeling of G_k. We can now simply set G to be the disjoint union of these graphs. □

To conclude this section, we investigate how subshifts of finite type and sofic subshifts behave under intersection and union.

Lemma 2.48. *If $X \subseteq A^\Lambda$ and $Y \subseteq A^\Lambda$ are subshifts of finite type, then $X \cap Y$ is a subshift of finite type.*

Proof. Let W_X and W_Y be finite sets of forbidden patterns of X and Y, respectively. A configuration x is contained in $X \cap Y$ if it contains no pattern from W_X or W_Y. Hence $X \cap Y$ is a subshift of finite type with forbidden patterns $W_X \cup W_Y$. □

Remark 2.49. The union of two subshifts of finite type must not be a subshift of finite type. For example every two-sided subshift of finite type containing ${}^\infty 01^\infty$ and ${}^\infty 10^\infty$ must also contain ${}^\infty 1.0^k 1^\infty$ for some $k \in \mathbb{N}$.

Lemma 2.50. *If $X \subseteq A^\Lambda$ and $Y \subseteq A^\Lambda$ are sofic subshifts, then $X \cap Y$ and $X \cup Y$ are again sofic.*

Proof. By the definition of sofic subshifts, there are subshifts of finite type $\tilde{X} \subseteq B^\Lambda$ and $\tilde{Y} \subseteq C^\Lambda$ and continuous, shift-commuting, surjective maps $f: \tilde{X} \to X$ and $g: \tilde{Y} \to Y$. By Theorem 2.39, these maps are defined by block maps $f_{\mathrm{loc}}: B^H \to B$ and $g_{\mathrm{loc}}: C^H \to C$ for some finite set $H \subseteq \Lambda$. By enlarging H if necessary, we may assume that sets of forbidden patterns of $\tilde{X}$ and $\tilde{Y}$ are contained in B^H and C^H. Let $W_{\tilde{X}}$ and $W_{\tilde{Y}}$ be these sets of forbidden patterns. Define a subshift of finite type $Z \subseteq (B \times C)^\Lambda$ with forbidden patterns $W_Z \subseteq B^H \times C^H$ by

$$\begin{aligned} W_Z := &\left\{ (v, w) \;\middle|\; v \in W_{\tilde{X}}, w \in C^H \right\} \\ &\cup \left\{ (v, w) \;\middle|\; v \in B^H, w \in W_{\tilde{Y}} \right\} \\ &\cup \left\{ (v, w) \;\middle|\; v \in B^H, w \in C^H, f_{\mathrm{loc}}(v) \neq g_{\mathrm{loc}}(w) \right\}. \end{aligned}$$

A pair $(x, y) \in B^\Lambda \times C^\Lambda$ is contained in Z if and only if $x \in \tilde{X}$, $y \in \tilde{Y}$ and $f(x) = g(y)$. Define a map $h: Z \to A^\Lambda$ by $h(x, y) := f(x) = g(y)$. An element $z \in A^\Lambda$ is in $h[Z]$ if and only if there is $x \in \tilde{X}$, $y \in \tilde{Y}$ with $h(x, y) = z = f(x) = g(y)$, in other words, if $z \in X \cap Y$.

For the union of X and Y consider the subshift of finite type $Z := \tilde{X} \times \{0\}^\Lambda \,\dot\cup\, \tilde{Y} \times \{1\}^\Lambda$. The map $h: Z \to X \cup Y$, $h(x, 0) := f(x)$, $h(y, 1) := g(y)$ is surjective, hence $X \cup Y$ is also sofic. □

2.4. Cellular Automata

In this section, we introduce the central object of our study, cellular automata. We will only give a very brief introduction and discuss results about them whenever we need them. Good introductions to the theory of cellular automata in the one-dimensional setting are given by the surveys of Kari [Kar05] and Kurka [Kur09] as well as the book by the same author [Kur03]. For cellular automata on other groups, the standard reference is the book by Ceccherini-Silberstein and Coonaert [CSC10]. Again the generalization to monoids is almost always straightforward, but to our knowledge it is not written down explicitly in the literature.

Throughout this section let Γ be a countable group and $\Lambda \subseteq \Gamma$ a submonoid. We will almost exclusively need $\Gamma = \mathbb{Z}$ and $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{N}_0$.

Definition 2.51 (Cellular automaton)**.** *Let Λ be a submonoid of a countable group. Let $X \subseteq A^\Lambda$ be a subshift. A continuous map $f : X \to X$ that commutes with all shifts, i.e., $f \circ \sigma_i = \sigma_i \circ f$ for all $i \in \Lambda^{-1}$, is called a* cellular automaton on X.

Cellular automata are thus the natural endomorphisms of subshifts. We already saw in Theorem 2.39 that continuous, shift commuting maps between subshifts over a monoid Λ can be represented by local rules $f_{\text{loc}} : A^H \to A$. If $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{N}_0$ we can always enlarge H so that it consists of consecutive integers. This allows us to view the local rule in this case as a map from words into the alphabet. Theorem 2.39 therefore yields the following corollary.

Corollary 2.52 (Curtis-Lyndon-Hedlund)**.** *Let $X \subseteq A^{\mathbb{Z}}$ be a two-sided subshift. A map $f : X \to X$ is a cellular automaton if and only if there are $\ell, r \in \mathbb{N}_0$ and a map $f_{loc} : A^{\ell+r+1} \to A$ such that*

$$f(x)_i = f_{loc}(x_{[i-\ell,i+r]}) \text{ for } i \in \mathbb{Z}.$$

We call ℓ the left *radius and r the* right *radius of f. If $\ell = r$, we simply call it the* radius.

Example 2.53. *Consider the alphabet $A = \{0, 1, 2\}$. We define two block maps $f_{loc} : A^2 \to A$ and $g_{loc} : A^2 \to A$ by*

$$f_{loc}(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 = 0 \text{ and } x_1 = 2 \\ 2 & \text{if } x_2 = 0 \text{ and } x_1 = 1 \\ x_1 & \text{otherwise} \end{cases}$$

$$g_{loc}(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 = 2 \text{ and } x_1 = 1 \\ 1 & \text{if } x_2 = 2 \text{ and } x_1 = 0 \\ x_1 & \text{otherwise} \end{cases}.$$

By setting $f(x)_i = f_{loc}(x_i, x_{i+1})$ we obtain a cellular automaton with left radius zero and right radius one on $A^{\mathbb{Z}}$. It exchanges the symbols 1 and 2 if they are followed by a zero. It is *easy to see that f is an involution, i.e., $f^2 = \mathrm{id}_{A^{\mathbb{Z}}}$. The same holds for $g : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $g(x)_i = g_{loc}(x_i, x_{i+1})$.*

The concatenation $h = g \circ f$, however, is not periodic any more. To see this consider $k \in \mathbb{N}$ and $x \in A^{\mathbb{Z}}$ with $x_{[1,2k+4]} = 1^{2k+2}02$. We have $f(x)_{[1,2k+3]} = 1^{2k+1}20$ and $g(f(x))_{[1,2k+2]} = 1^{2k}02$. Therefore $h(x), h^2(x), \dots, h^k(x)$ are all pairwise different. Since k was arbitrary, this shows that no two powers of h coincide.

The local rules f_{loc} and g_{loc} also define a cellular automaton on $A^{\mathbb{N}}$ by the same formula as in the two-sided case. If X is the subshift consisting of all configurations that have a zero at every second position, then f can be considered as a cellular automaton on X, but this is not possible for g. For example $g({}^{\infty}(10).(20)^{\infty}) = {}^{\infty}(10)11.(21)^{\infty}$.

To visualize the dynamics of a cellular automaton, one typically plots the sequence of states with different colors representing the symbols and time going downwards, see Figure 2.2. for an example. This visualization can be formalized by the following definition.

Definition 2.54 (Space-time diagram)**.** *Let $X \subseteq A^{\Lambda}$ be a subshift and let $f : X \to X$ be a cellular automaton. For $x \in X$ we define the* one-sided space-time diagram

$$Y_f^{\mathbb{N}_0}(x) = (f^k(x))_{k \in \mathbb{N}_0} \in A^{\mathbb{N} \times \Lambda}.$$

If f is invertible, we can also define a two-sided space-time diagram

$$Y_f^{\mathbb{Z}}(x) = (f^k(x))_{k \in \mathbb{Z}} \in A^{\mathbb{Z} \times \Lambda}.$$

The set of all one-sided and two-sided space time diagrams form a subshift of, respectively, $A^{\mathbb{N}_0 \times \Lambda}$ and $A^{\mathbb{Z} \times \Lambda}$, which we denote by $\mathbb{Y}_f^{\mathbb{N}_0}$ and $\mathbb{Y}_f^{\mathbb{Z}}$.

Lemma 2.55. *Let $X \subseteq A^{\Lambda}$ be a subshift and let $f : X \to X$ be a cellular automaton. If X is a subshift of finite type, then $\mathbb{Y}_f^{\mathbb{N}_0}$ and $\mathbb{Y}_f^{\mathbb{Z}}$ are also subshifts of finite type. If X is a sofic subshift, then $\mathbb{Y}_f^{\mathbb{N}_0}$ and $\mathbb{Y}_f^{\mathbb{Z}}$ are sofic subshifts, too.*

Proof. If X is a subshift of finite type, there is a finite set $H \subseteq \Lambda$, such that $X = \mathbb{X}_M$ for some set of forbidden patterns $M \subseteq A^H$ and such that f has a local rule $f_{\mathrm{loc}} : A^M \to A$. The subshifts $\mathbb{Y}_f^{\mathbb{N}}$ and $\mathbb{Y}_f^{\mathbb{Z}}$ are both defined by the set $\tilde{M} \subseteq A^{\{0,1\} \times H}$ of forbidden patterns, given by

$$\tilde{M} := A^{\{0,1\} \times H} \setminus \left\{ (w^0, w^1) \in A^{\{0,1\} \times H} \;\middle|\; w^0 \in A^H \setminus M \text{ and } (f_{\mathrm{loc}}(w^0))_{1_\Lambda} = w^1_{1_\Lambda} \right\}.$$

If X is sofic, there is a subshift of finite type $Z \subseteq B^{\Lambda}$ for some finite alphabet B and a shift-commuting continuous map $g : Z \to X$ with $g[Z] = X$. Define

$$\tilde{Z} := \left\{ (z_n)_{n \in \mathbb{N}} \in Z^{\mathbb{N}_0} \;\middle|\; f(g(z_n)) = g(z_{n+1}) \text{ for all } n \in \mathbb{N}_0 \right\}.$$

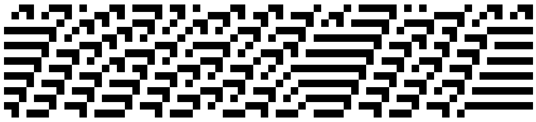

Figure 2.2.: Space-time diagram of the cellular automaton $w_{67} : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ for a random initial state. The cellular automaton is given by $(w_{67})(x)_i = x_{i-1}x_i x_{i+1} + x_{i-1} + x_i + 1$. Zeros are represented by white squares, ones by black squares.

This space is clearly a subshift of finite type. The map g naturally induces a continuous shift-commuting map $\tilde{g} : \tilde{Z} \to A^{\mathbb{N}_0 \times \Lambda}$ by $\tilde{g}((z_n)_{n\in\mathbb{N}_0}) := (g(z_n))_{n\in\mathbb{N}_0}$ and we have

$$\tilde{g}[\tilde{Z}] = \left\{ x_n \in X^{\mathbb{N}_0} \;\middle|\; f(x_n) = x_{n+1} \text{ for all } n \in \mathbb{N}_0 \right\} = \mathbb{Y}_f^{\mathbb{N}_0}.$$

Therefore $\mathbb{Y}_f^{\mathbb{N}_0}$ is a sofic shift. Replacing all occurrences of $\mathbb{N}_0$ by $\mathbb{Z}$ in this argument gives the same result for $\mathbb{Y}_f^{\mathbb{Z}}$. □

Cellular automata give rise to subshifts in another way as the following lemma shows.

Lemma 2.56. *Let $X \subseteq A^\Lambda$ be a subshift of finite type. If $f, g : X \to X$ are two cellular automata, then the set $Y := \{ x \in X \mid f(x) = g(x) \}$ is a subshift of finite type.*

Proof. By Corollary 2.52 there exists a finite subset $H \subseteq \Lambda$ such that f and g are generated by block maps $f_{\text{loc}} : A^H \to A$ and $g_{\text{loc}} : A^H \to A$, and such that X has a set of forbidden patterns in A^H. Let W be the set of all patterns w in A^H which are not forbidden in X and for which $f_{\text{loc}}(w) = g_{\text{loc}}(w)$. Let Z be the subshift of finite type whose set of forbidden patterns is $A^H \setminus W$. A configuration $x \in X$ is contained in Y if and only if for every $i \in \Lambda$

$$f_{\text{loc}}(\sigma_{i^{-1}}(x)_{|H}) = f(x)_i = g(x)_i = g_{\text{loc}}(\sigma_{i^{-1}}(x)_{|H}),$$

in other words, if $\sigma_{i^{-1}}(x)_{|H} \in W$. Therefore $Y = Z$. □

Corollary 2.57. *Let $X \subseteq A^\Lambda$ be a subshift of finite type or a sofic subshift. Let $f : X \to X$ be a cellular automaton. If $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, then $\mathrm{Pre}_{q,p}(f) = \left\{ x \in X \;\middle|\; f^{q+p}(x) = f^q(x) \right\}$ is again, respectively, a subshift of finite type or a sofic subshift.*

Proof. By Corollary 2.52 f^q and f^{p+q} are defined by block maps. They therefore extend to cellular automata g_1, g_2 on the full shift A^Λ. By Lemma 2.56 $Y := \{y \in A^\Lambda \mid g_1(x) = g_2(x)\}$ is a subshift of finite type. Finally, by Lemma 2.48 and Lemma 2.50 the set $\mathrm{Pre}_{q,p}(f) = Y \cap X$ is a subshift of finite or a sofic subshift as the intersection of a subshift of finite type with a subshift of finite type or a sofic subshift. □

The following lemma shows that every cellular automaton on a subshift with periodic points also has periodic points. In the cases which we will consider, we will therefore always have periodic points available to construct conjugacy invariants.

Lemma 2.58. *Let $f : X \to X$ be a cellular automaton on a subshift $X \subseteq A^{\mathbb{Z}}$. If $x \in Per_k(\sigma_X)$ for some $k \in \mathbb{N}$, then there are $p \in \mathbb{N}, q \in \mathbb{N}_0$ such that $x \in Pre_{q,p}(f)$.*

Proof. Every element $x \in \mathrm{Per}_k(\sigma_X)$ is uniquely determined by $x_{[0,\ldots,k)}$, hence $\mathrm{Per}_k(\sigma_X)$ is finite. Additionally $\sigma_X^k(f(x)) = f(\sigma_X^k(x)) = f(x)$, so $\mathrm{Per}_k(\sigma_X)$ is invariant under f. By the pigeonhole principle there must be $q, r \in \{0, \ldots, |A|^k\}$ with $q < r$ such that $f^q(x) = f^r(x)$ and therefore $x \in \mathrm{Pre}_{q,r-q}(f)$. □

We already introduced topological conjugacy as the natural isomorphism notion between topological dynamical systems. Since the composition of cellular automata is yet another cellular automaton, the conjugation of a cellular automaton by an invertible cellular automaton is also a cellular automaton. The simplest instance of this is conjugacy by a symbol permutation ("exchanging black and white"). Another way of getting a conjugate cellular automaton from a given one is to reflect the local rule ("exchanging left and right"). This is equivalent to conjugation by the reflection map

$$\tau : A^{\mathbb{Z}} \to A^{\mathbb{Z}}, \quad \tau(x)_k := x_{-k}.$$

See the paper by Cattaneo et al. [Cat+97] for further properties of these conjugacies. The following theorem shows that, at least on two-sided full shifts, this are in a sense the only general methods to turn cellular automaton into cellular automata by conjugation.

Theorem 2.59. *Denote by CA_A the set of all cellular automata over the two-sided full A-shift. If $\varphi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a homeomorphism, the following are equivalent.*

(a) $\left\{ \varphi \circ f \circ \varphi^{-1} \;\middle|\; f \in CA_A \right\} \subseteq CA_A$,

(b) $\left\{ \varphi \circ f \circ \varphi^{-1} \;\middle|\; f \in CA_A \right\} = CA_A$,

(c) $\exists h \in CA_A : \varphi = h$ *or* $\varphi = h \circ \tau$.

Proof. The implications (c) ⇒ (b) and (b) ⇒ (a) are trivial.

(a) ⇒ (c) Let f be an arbitrary cellular automaton. The map $g := \varphi \circ f \circ \varphi^{-1}$ is again a cellular automaton by the assumption and therefore commutes with σ. Hence

$$g = \sigma \circ g \circ \sigma^{-1} = \sigma \circ \varphi \circ f \circ \varphi^{-1} \circ \sigma^{-1} \text{ and}$$
$$f = \varphi^{-1} \circ \sigma \circ \varphi \circ f \circ \varphi^{-1} \circ \sigma^{-1} \circ \varphi.$$

By setting $f := \sigma$, we see that $\varphi^{-1} \circ \sigma \circ \varphi$ is a cellular automaton. Now Ryan's theorem, see [Rya72], tells us that the center of the group $\{f \in \mathrm{CA}_A \mid f \text{ is invertible}\}$ consists only of powers of the shift, i.e., if an invertible cellular automaton commutes with all other invertible cellular automata, it must be a power of the shift. Hence $\varphi^{-1} \circ \sigma \circ \varphi = \sigma^k$ for some $k \in \mathbb{Z}$ or equivalently $\sigma \circ \varphi = \varphi \circ \sigma^k$. This first of all implies that $k \neq 0$. Take any point $y \in \mathrm{Fix}(\sigma^k)$. Then $(\sigma \circ \varphi)(y) = (\varphi \circ \sigma^k)(y) = \varphi(y)$. Hence $\varphi(y) \in \mathrm{Fix}(\sigma)$ and therefore φ defines an injective mapping from $\mathrm{Fix}(\sigma^k)$ into $\mathrm{Fix}(\sigma)$. Having a look at the

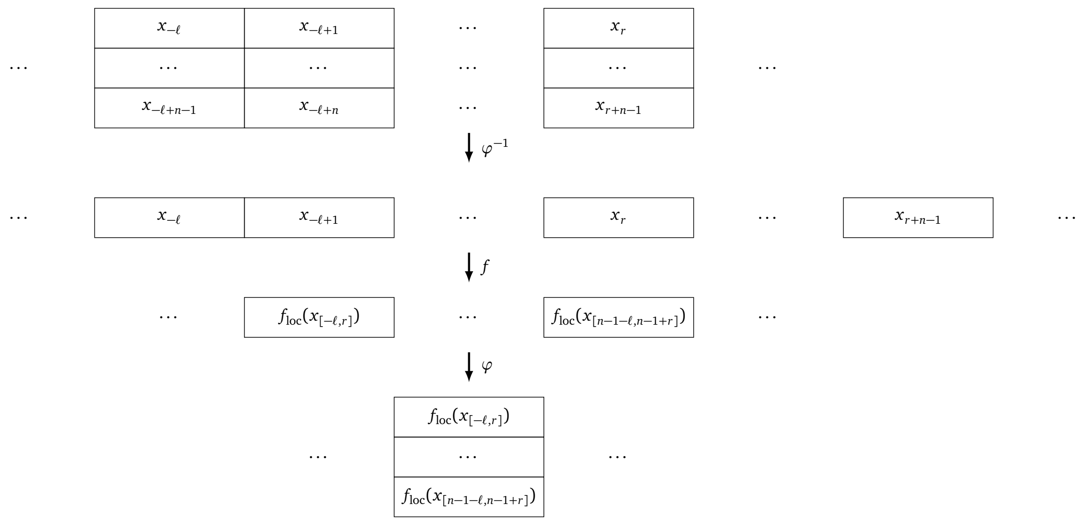

Figure 2.3.: The local rule of $\varphi \circ f \circ \varphi^{-1}$ in Lemma 2.60.

cardinalities, we see that $|A|^{|k|} = |\text{Fix}(\sigma^k)| \le |\text{Fix}(\sigma)| = |A|$, implying $k = \pm 1$. In the case of $k = 1$ we are done. In the other case

$$\tau \circ \varphi^{-1} \circ \sigma \circ \varphi \circ \tau^{-1} = \tau \circ \sigma^{-1} \circ \tau^{-1} = \sigma,$$

hence $\varphi \circ \tau^{-1}$ is a cellular automaton. □

If $X \subseteq A^{\mathbb{Z}}$ and $Y \subseteq A^{\mathbb{Z}}$ are subshifts that are conjugate via a conjugacy $\varphi : X \to Y$ and $f : X \to X$ is a cellular automaton, then clearly $\varphi \circ f \circ \varphi^{-1} : Y \to Y$ is also a cellular automaton. We record a slightly stronger version of this result for the case where Y is a higher block representation of X.

Lemma 2.60. *Let $X \subseteq A^{\mathbb{Z}}$ be a two-sided subshift and let $Y \subseteq (A^n)^{\mathbb{Z}}$ be its higher block representation with block length n. Let $\varphi : X \to Y$ be the corresponding higher block map. If $f : X \to X$ is a cellular automaton with left radius $\ell \in \mathbb{N}_0$ and right radius $r \in \mathbb{N}_0$, then $\varphi \circ f \circ \varphi^{-1} : Y \to Y$ is a cellular automaton with left radius ℓ and right radius r as well.*

Proof. Let $f_{\text{loc}} : A^{\ell+r+1} \to A$ be the local rule of f. Let $W \subseteq A^n$ be the set of all words of length n appearing in X. The block map $h_{\text{loc}} : W^{\ell+r+1} \to A^n$ defined by

$$h_{\text{loc}}((x_1,\dots,x_n),\dots,(x_{\ell+r+1},\dots,x_{\ell+r+n})) := (f_{\text{loc}}(x_1,\dots,x_{\ell+r+1}),\dots,f_{\text{loc}}(x_n,\dots,x_{\ell+r+n}))$$

is a block map generating $\varphi \circ f \circ \varphi^{-1}$ with left radius ℓ and right radius r as can be seen in Figure 2.3. □

We now introduce a weaker and a stronger isomorphism notion for cellular automata, namely topological orbit equivalence and strong (topological) conjugacy. Together with

Theorem 2.45 it allows to always assume that the state space of a cellular automaton on a subshift of finite type is an edge shift.

Definition 2.61 (Topological orbit equivalence)**.** *Two invertible dynamical systems $f : X \to X$ and $g : Y \to Y$ are called* topologically orbit equivalent *if there exists a homeomorphism $\varphi : X \to Y$ mapping every orbit of f onto an orbit of g. More formally,*

$$\forall x \in X \exists y \in Y : \varphi[\mathrm{Orb}(f, x)] = \mathrm{Orb}(g, y).$$

This notion has important applications in group theory for example in the study of topological full groups, see for example the paper by Bezuglyi and Medynets [BM08].

Notice that one can make a similar definition for non-invertible dynamical systems by demanding that φ maps every forward orbit of f onto a forward orbit of g. This notion might be called *forward orbit equivalence* but this terminology is almost absent from the literature. It is easy to see that two forward orbit equivalent invertible systems are also orbit equivalent. The converse, however, is not true.

Example 2.62. *Consider the subshift of finite type X generated by $\{{}^\infty 1.0^\infty, {}^\infty 2.0^\infty\}$. The left shift σ on X is orbit equivalent to the right shift σ^{-1}, since every orbit of σ is also an orbit of σ^{-1} and vice versa. They are, however, not forward orbit equivalent. There are two points in X with infinite forward orbit under σ for which the closures of these forward orbits have finite non-empty intersection, namely $x = {}^\infty 1.0^\infty$ and $y = {}^\infty 2.0^\infty$. More formally, $\overline{\mathrm{Orb}^+(\sigma, x)} \cap \overline{\mathrm{Orb}^+(\sigma, y)} = \{{}^\infty 0^\infty\}$. This can not happen for σ^{-1}. The closures of forward orbits under σ^{-1} of two points either have infinite intersection, if one of the orbits is contained in the other, or have empty intersection.*

For periodic dynamical systems, and this is basically the only situation where we use topological orbit equivalence, the forward orbit coincides with the orbit and the difference between these notions of isomorphism vanishes.

Besides the topology there is a second structure on a subshift, namely the shift. It might thus be desirable to also preserve this additional structure.

Definition 2.63 (Strong conjugacy)**.** *Let $X \subseteq A^\Lambda$ and $Y \subseteq B^\Lambda$ be subshifts. Two cellular automata $f : X \to X$ and $g : Y \to Y$ are called* strongly conjugate *if they are conjugate by a homeomorphism $\psi : X \to Y$ that commutes with the shift in the sense that $\psi \circ \sigma_i = \sigma_i \circ \psi$ for every $i \in \Lambda^{-1}$.*

However, we will put our focus on not necessarily strong topological conjugacies between cellular automata for several reasons. First of all, most dynamical properties which are investigated in the literature on topological dynamics of cellular automata can be defined without mentioning the subshift.

Next, a prerequisite for a classification of cellular automata up to strong conjugacy is a classification of subshifts of finite type up to topological conjugacy. In the next section we will see that this problem has a very special flavor different from the conjugacy problem for general cellular automata.

There is even a stronger connection between strong conjugacy and topological conjugacy of subshifts. Recall that a cellular automata $f : A^\Lambda \to A^\Lambda$ gives rise to a subshift of finite type $\mathbb{Y}_f^{\mathbb{N}_0}$ via its space-time diagrams. Two such subshifts are topologically conjugate as $\mathbb{N}_0 \times \Lambda$ actions if and only if the corresponding cellular automata are strongly conjugate. Thus strong conjugacy of cellular automata can be seen as a special case of conjugacy between subshifts of finite type.

2.5. Topological Conjugacies Between Subshifts of Finite Type

For subshifts, and in particular for one-sided or two-sided subshifts of finite type, the conjugacy problem is well investigated. See for example Chapter 2 in the book by Kitchens [Kit98] on which this section is based. Another good reference is, as always, the book by Lind and Marcus [LM95], Chapter 7. We give an outline of the results and questions in this area, mainly to highlight the very special nature of this subclass of cellular automata.

We already saw in Theorem 2.45 that every two-sided subshift of finite type is topologically conjugate to the edge shift of an essential graph. Since isomorphisms of graphs lift to topological conjugacies of the corresponding edge shifts, we can describe the edge shift of a graph up to conjugacy by its *adjacency matrix*, i.e., the matrix $L \in \mathbb{N}_0^{V(G)\times V(G)}$ given by $L_{ij} = |\{e \in E(G) \mid \mathrm{i}_G(e) = i, \mathrm{t}_G(e) = j\}|$. Criteria for the existence of a topological conjugacy between subshifts of finite type as well as invariants ruling out its existence can therefore often be expressed in terms of linear algebra.

For example, consider an essential graph G with adjacency matrix L and let X be the edge shift of G. We may assume that $V(G) = \{1,\ldots,n\}$. The number of closed paths of length k starting and ending in $i \in V(G)$ equals $(L^k)_{ii}$. This implies $\mathrm{Per}_k(\sigma_X) = (L^k)_{11} + \cdots + (L^k)_{nn} = \mathrm{tr}(L^k)$, where $\mathrm{tr}(L)$ is the trace of L. Since $|\mathrm{Per}_k(\sigma_X)| \le |A|^k$ for every $k \in \mathbb{N}$, one can combine the information about the number of periodic points into a single object, the *zeta function* of the shift X, defined by

$$\begin{aligned}\zeta_X(t) &= \exp\Big(\sum_{k=1}^{\infty} \frac{|\mathrm{Per}_k(\sigma_X)|}{k} t^k\Big)\\ &= \exp\Big(\sum_{k=1}^{\infty} \frac{\mathrm{tr}(L^k)}{k} t^k\Big).\end{aligned}$$

This definition directly implies that two subshifts have the same number of k-periodic points for all k if and only if their zeta functions agree. For irreducible subshifts of finite type Bowen and Lanford showed that

$$\zeta_X(t) = \det(I - tL)^{-1},$$

see [Wil73]. The term irreducible here means that the subshift is conjugate to the edge shift of a strongly connected graph. By calculating the polynomial $\det(I - tL)$ one can therefore decide if two subshifts of finite type have the same number of k-periodic points for all $k \in \mathbb{N}$.

Rufus Bowen asked if two irreducible two-sided subshifts of finite type are topologically conjugate if they have the same zeta function. This question was answered negatively by Williams in the very remarkable article [Wil73]. Based on his earlier work on Axiom A attractors, Williams introduced the notions of shift equivalence and strong shift equivalence in order to characterize topological conjugacy between subshifts of finite type. We are going to explain these notions now.

Besides graph automorphisms, there are two canonical ways to get from one edge shift to a conjugate one, *out-splitting* and *out-amalgamation* as well as *in-splitting* and *in-amalgamation*. In both cases we consider a partition $\mathscr{E}$ of the edges $E(G)$. For an out-splitting, each set in $\mathscr{E}$ has to consist of edges starting at the same vertex. For each edge $e \in E(G)$ denote by $\mathscr{E}_e$ the unique set in $\mathscr{E}$ such that $e \in \mathscr{E}_e$. Define a map $\varphi_{\mathrm{out}}\colon \mathrm{Path}_{\mathbb{Z}}(G) \to (E(G) \times \mathscr{E})^{\mathbb{Z}}$ by $\varphi_{\mathrm{out}}(\gamma)_i = (\gamma_i, \mathscr{E}_{\gamma_{i+1}})$, in other words, φ_{out} annotates each edge in γ by the set in $\mathscr{E}$ to which the following edge belongs. The map φ is injective, continuous and shift-commuting, hence the image $Y := \varphi_{\mathrm{out}}[\mathrm{Path}_{\mathbb{Z}}(G)]$ is a subshift conjugate to $\mathrm{Path}_{\mathbb{Z}}(G)$. It is even an edge shift of the graph $\tilde{G}$ given by

$$\begin{aligned}
V(\tilde{G}) &:= \{\, (\mathrm{i}_G(e), \mathscr{E}_e) \mid e \in E(G) \,\} \subseteq V(G) \times \mathscr{E}, \\
E(\tilde{G}) &:= \left\{\, (e_1, \mathscr{E}_{e_2}) \;\middle|\; e_1, e_2 \in E(G), \mathrm{t}_G(e_1) = \mathrm{i}_G(e_2) \,\right\} \subseteq E(G) \times \mathscr{E}, \\
\mathrm{i}_{\tilde{G}}(e_1, \mathscr{E}_{e_2}) &:= (\mathrm{i}_G(e_1), \mathscr{E}_{e_1}), \\
\mathrm{t}_{\tilde{G}}(e_1, \mathscr{E}_{e_2}) &:= (\mathrm{t}_G(e_1), \mathscr{E}_{e_2}) = (\mathrm{i}_G(e_2), \mathscr{E}_{e_2}).
\end{aligned}$$

The edge shift Y is then called the out-splitting of X, and X is called the out-amalgamation of Y.

For an in-splitting, each set in $\mathscr{E}$ must consist of edges ending in the same vertex. The map φ_{in} defined by $\varphi_{\mathrm{in}}(\gamma)_i := (\gamma_i, \mathscr{E}_{\gamma_{i-1}})$ is again a conjugacy between the edge shift of G and the edge shift Z of a graph similar to $\tilde{G}$, called the in-splitting of X. X is called the in-amalgamation of Z.

Somewhat surprisingly, Williams [Wil73] showed that each conjugacy between two-sided subshifts of finite type is the concatenation of in- and out-splittings, in- and out-amalgamations and graph isomorphisms. On one hand, we therefore have a complete understanding how we can get from one subshift to any of its conjugates. On the other hand, there are still pairs of relatively small graphs for which it is not known if there edge shifts are conjugate, see, e.g., the example due to Ashley given in Example 2.2.7 in [Kit98].

On the level of adjacency matrices, out- and in-splittings take the following form. If G is an essential graph with adjacency matrix $L \in \mathbb{N}_0^{n\times n}$ and $\mathscr{E}$ is a partition of the edges of G fulfilling the conditions of an in- or out-splitting, then there are matrices $R \in \mathbb{N}_0^{n\times|\mathscr{E}|}$ and $S \in \mathbb{N}_0^{|\mathscr{E}|\times n}$ such that $L = RS$ and $\tilde{L} = SR$ where $\tilde{L}$ is the adjacency matrix of the in-splitting or out-splitting of G.

On the other hand if there is $m \in \mathbb{N}$ and matrices $R \in \mathbb{N}_0^{n\times m}, S \in \mathbb{N}_0^{m\times n}$ with $L = RS$ and $K = SR$, the matrices L and K are called *elementary strongly shift equivalent*. Two matrices L and K are then called *strongly shift equivalent* if there are matrices $H_1 = L, H_2, \dots, H_{\ell-1}, H_\ell = K$ each elementary strongly shift equivalent to the next one, in other words, L and K are strongly shift equivalent if and only if there are matrices $R_1, R_2, \dots, R_\ell, S_1, \dots, S_\ell$ such that $L = R_1S_1, S_1R_1 = R_2S_2, \dots, S_{\ell-1}R_{\ell-1} = R_\ell S_\ell, S_\ell R_\ell = K$. We already mentioned that the adjacency matrices of two essential graphs with topologically conjugate edge shifts are strong shift equivalence. Williams [Wil73] showed that also the converse holds. Hence the problem of finding a conjugacy between two subshifts of finite type completely reduces to a problem in linear algebra over the non-negative integers.

Williams also introduced a weakening of the notion of strong shift equivalence which he called *shift equivalence*. Two matrices L and K are called shift equivalent if there is $\ell \in \mathbb{N}$ and two matrices R, S over the non-negative integers such that $L^\ell = RS, K^\ell = SR, LR = RK$ and $SL = KS$. By taking $R = R_1R_2 \dots R_\ell$ and $S = S_\ell S_{\ell-1} \dots S_1$ one sees that strongly shift equivalent matrices are also shift equivalent. Williams stated in [Wil73] that shift equivalence and strong shift equivalence are equivalent but later an error in the proof was found by Parry, see the erratum [Wil74]. Thus the *Williams Conjecture* was born. In 1999 Kim and Roush [KR99] gave an example of a pair of shift equivalent but not strongly shift equivalent irreducible subshifts of finite type. The proof is based on earlier work of these two authors and Wagoner on the so called "sign-gyration-compatibility-condition". There seems to have been no significant progress on this question since then. In particular it is open if topological conjugacy of two-sided subshifts is decidable. In contrast Kim and Roush showed in [KR88] that shift equivalence of matrices is decidable. For a comprehensive overview of the ideas surrounding shift equivalence see the survey [Wag99] by Wagoner. What remains from the Williams Conjecture is now known as the "Little shift equivalence conjecture", see Conjecture 3.1 in the famous list of open problems in symbolic dynamics by Boyle [Boy08].

Another classical invariant for topological conjugacy is topological entropy $\mathrm{h}_{\mathrm{top}}$ as introduced by Adler, Konheim and McAndrew. Unfortunately for cellular automata in general this quantity can not even be approximated. This was shown by Hurd, Kari and Culik in [HKC92]. For irreducible subshifts, however, the topological entropy can be derived from the knowledge of the number of k-periodic points, since

$$h_{\mathrm{top}}(\sigma_X) = \lim_{\ell\to\infty} \frac{1}{\ell} \log|\mathrm{Per}_\ell(\sigma_X)|.$$

Finally one can associate various groups to a subshift of finite type such that these groups must be isomorphic for conjugate subshifts. For example let R be a commutative ring, thought of as a $\mathbb{Z}$-module. Let $p \in \mathbb{Z}[\lambda]$ be a polynomial with coefficients in $\mathbb{Z}$. Then A as well as $p(A)$ act on R^n and one can form the *generalized Bowen-Franks-group*

$$R^n/R^n p(A).$$

Each group obtained in that way is an invariant for topological conjugacy. The classical Bowen-Franks group corresponds to $R=\mathbb{Z}$ and $p(\lambda)=1-\lambda$. It was introduced by Bowen and Franks in [BF77]. Another important case is $R=\mathbb{Z}[t]$ and $p(\lambda)=1-t\lambda$. The group obtained that way is the so called *dimension group*. Krieger constructed this group in [Kri80] based on ideas from the theory of operator algebras. There are many isomorphic ways to build it. Consider L as a linear map from $\mathbb{Q}^n \to \mathbb{Q}^n$ acting on vectors on the right and let $\mathscr{R}_L$ be the eventual image of L, i.e., $\mathscr{R}_L = \bigcap_{k=0}^{n} L^k\mathbb{Q}^n = L^n\mathbb{Q}^n$. Let Δ_L be the abelian group defined by

$$\Delta_L := \left\{ v \in \mathscr{R}_L \;\middle|\; \exists k \in \mathbb{N} : vL^k \in \mathbb{Z}^n \right\}.$$

This group is isomorphic to the dimension group. L acts on this group by multiplication from the right, denote this action by δ_L. Furthermore let

$$\Delta_L^+ := \left\{ v \in \mathscr{R}_L \;\middle|\; \exists k \in \mathbb{N} : vL^k \in \mathbb{N}^n \right\}$$

be the set of eventually positive elements in $\Delta(L)$. The triple $(\Delta(L), \delta_L, \Delta_L^+)$ is called the *dimension triple* of L. One can show that shift equivalent non-negative matrices have isomorphic dimension triples in the sense that there exists a group isomorphism between their dimension groups which intertwines the action of the matrices on the dimension group and which maps eventually positive elements to eventually positive elements. Furthermore, the dimension triple is a complete invariant in the sense that non-negative matrices with isomorphic dimension triples are shift equivalent, see Theorem 7.5.8 in [LM95]. There are further invariants for strong shift equivalence often using algebraic number theory, see for example Section 12.3 in [LM95] or the paper by Eilers and Kiming [EK12].

Away from the case of irreducible subshifts of finite type, Salo and Törmä [ST14] showed that topological conjugacy is decidable for pairs of countable subshifts of finite type with Cantor-Bendixson rank two (see Section 4.1 for more about the Cantor-Bendixson rank).

Some of the above results were extended to sofic shifts or shifts of finite type over larger groups. For example Nasu in [Nas86] defined a notion of strong shift equivalence for sofic shifts and showed that it is equivalent to topological conjugacy. Schraudner defined in [Sch08] a notion of state splittings and amalgamations for shifts of finite type over $\mathbb{Z}^d$. This allowed him to decompose topological conjugacies between these systems into a series of matrix transformations similarly as in the one-dimensional setting. Finally Jeandell and Vanier in [JV15b] obtained some results on the hardness of the conjugacy problem for higher-dimensional subshifts of finite type, which by the considerations at the end of Section 2.4 also apply to strong conjugacy of cellular automata.

2.6. Expansive and Non-Expansive Cellular Automata

For cellular automata, we lack such linear algebraic invariants. Furthermore, most decision problems related to asymptotic invariants such as entropy, limit sets or attractors turn out to be undecidable.

There is one class of cellular automata whose conjugacy problem received a lot of attention because of its connection to subshifts, namely *expansive* and *positively expansive* cellular automata. There is a very comprehensive survey on the relevant results by Nasu [Nas04].

Definition 2.64 (Positively expansive). *Let (X,d) be a metric space. A dynamical system $f : X \to X$ is called* positively expansive *if and only if there is $\varepsilon > 0$ such that for all $x, y \in X$ there is $n \in \mathbb{N}_0$ with $d(f^n(x), f^n(y)) \geq \varepsilon$. An invertible dynamical system $f : X \to X$ is called* expansive *if and only if there is $\varepsilon > 0$ such that for all $x, y \in X$ with $x \neq y$ there is $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) \geq \varepsilon$. We call ε the* expansivity constant *of f.*

The following theorem is a classical result due to Hedlund [Hed69] and Reddy [Red68].

Theorem 2.65. *Let X be a metrizable Stone space. A map $f : X \to X$ is invertible and expansive if and only if it is topologically conjugate to a two-sided subshift. A map $f : X \to X$ is surjective and positively expansive if and only if it is topologically conjugate to a one-sided subshift.*

The goal is now to identify the subshifts arising from this theorem. Under which conditions is, for example, an expansive cellular automaton on a two-sided subshift conjugate to a two-sided subshift of finite type or to a full shift. In dimension one there are basically four cases. The subshift X can be one-sided or two-sided and the cellular automaton $f : X \to X$ can be surjective and positively expansive or injective and expansive. The results in these four cases differ a lot with respect to completeness. To state them we need some standard definitions from topological dynamics.

Definition 2.66 (Wandering point, transitive, mixing). *Let X be a compact space. Let $f : X \to X$ be a dynamical system. A point $x \in X$ is called* wandering *if there is a neighborhood U of x and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $f^n[U] \cap U = \emptyset$. f is called* non-wandering *if it has no wandering point. f is called* topologically transitive *if for every pair of open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ with $f^{-n}[U] \cap V \neq \emptyset$. Finally f is called* topologically mixing*, if for every pair of open sets $U, V \subseteq X$ there is $n_0 \in \mathbb{N}$ such that $f^{-n}[U] \cap V \neq \emptyset$ for all $n \geq n_0$.*

We give a summary of the most relevant results based on Nasu's survey.

X is one-sided, f is positively expansive: If X is a transitive subshift of finite type, then f is conjugate to a one-sided subshift of finite type. Surjectivity follows automatically. The ideas for the proof are due to Kurka in [Kur97] and independently Nasu, see [Nas04]. The precise result and a nice presentation of the proof can be found Section 6 of a paper by Boyle and Kitchens [BK99]. If X is a full-shift, then sufficiently large powers of f are conjugate to a full shift. This is shown in Corollary 3.14 in the article [BKM97] by Blanchard and Maas. This result is best possible as Boyle, Fiebig and Fiebig present in Example 5.6 in [BFF97] a positively expansive cellular automaton on a full shift not topologically conjugate to a one-sided full shift.

X is two-sided, f is surjective and positively expansive: In this case one has the more or less complete picture. If X is a mixing subshift of finite type, then f is conjugate to a one-sided full shift. This was shown by Nasu, see [Nas04], combining the results of Kurka [Kur97], which we already mentioned, and earlier results of himself from [Nas95]. According to [Nas04], Fiebig showed in unpublished work that mixing can be replaced by non-wandering and that there are no positively expansive cellular automata on two-sided subshift of finite type with a wandering point.

X is one-sided, f is injective and expansive: If X is a full shift, then f is conjugate to a subshift of finite type as Nasu showed in [Nas02]. By an earlier result of Boyle and Maas in [BM00], this implies that all sufficiently large powers of f are conjugate to a full shift. They made the conjecture that f must actually be conjugate to a full shift and this problem is still open.

X is two-sided, f is injective and expansive: While this case received a lot of attention, the results obtained are the weakest of all four cases. Fiebig gave an example of a non-transitive subshift of finite type X and an injective, expansive cellular automaton on X which is not conjugate to any subshift of finite type. It is open if every injective expansive cellular automaton on a transitive two-sided subshift X is conjugate to a two-sided subshift of finite type. See for example Problem 12.1 in [Boy08]. According to Nasu in [Nas04], it is likewise open if every injective expansive cellular automaton on a two-sided full shift is conjugate to a two-sided full shift.

From Theorem 2.65 it is easy to see that every expansive or positively expansive dynamical system on a metrizable Stone Space can only have finitely many p-periodic points for every $p \in \mathbb{N}$. This is true in general as the following easy lemma shows.

Lemma 2.67. *Let X be a compact metrizable space, let $f : X \to X$ be a dynamical system and let $p \in \mathbb{N}$. If f is expansive or positively expansive, then $\mathrm{Per}_p(f)$ is finite.*

Proof. Let d be a compatible metric on X and let $\varepsilon > 0$ be an expansivity constant of f with respect to d. By continuity of f and compactness of X, there is $\delta > 0$ such that for all points $x, y \in X$ with $d(x, y) < \delta$ we also have $d(f^k(x), f^k(y)) < \varepsilon$ for all $k \in \{0, \dots, p-1\}$. In particular there can be no two points in $\mathrm{Per}_p(f)$ which are δ-close. We can cover X by finitely many balls of radius $\frac{\delta}{2}$ and each of these balls can contain at most one point from $\mathrm{Per}_p(f)$. Hence f has only finitely many p-periodic points. □

Contrary to this, the p-periodic points of a cellular automaton in general form a subshift of possibly infinite cardinality. These subshifts will be an object of central interest to us.

If a dynamical system f has only finitely many periodic points of each period length, it does not matter if one considers $(|\mathrm{Per}_p(f)|)_{p\in\mathbb{N}}$ or $(|\widetilde{\mathrm{Per}}_p(f)|)_{p\in\mathbb{N}}$. Both sequences can easily be expressed in terms of each other. More precisely,

$$
\begin{aligned}
|\mathrm{Per}_p(f)| &= \sum_{k|p} |\widetilde{\mathrm{Per}}_k(f)|,\\
|\widetilde{\mathrm{Per}}_p(f)| &= \sum_{k|p} \mu(\frac{p}{k})|\mathrm{Per}_k(f)|
\end{aligned}
$$

where $\mu : \mathbb{N} \to \mathbb{N}$ is the Möbius function. For this Möbius inversion formula see any textbook on number theory, for example the one by Ireland and Rosen [IR82].

We now want to clarify in which ways the periodic and preperiodic points of a dynamical system intersect. Denote by $\gcd(k,\ell)$ the greatest common divisor of k and ℓ and by $\mathrm{lcm}(k,\ell)$ their least common multiple.

Lemma 2.68. *Let X be a set and $f : X \to X$. For $p_1, p_2 \in \mathbb{N}$ we have*

$$
Per_{p_1}(f) \cap Per_{p_2}(f) = Per_{\gcd(p_1,p_2)}(f).
$$

Proof. Set $p := \gcd(p_1, p_2)$. If $x \in X$ is in $\mathrm{Per}_p(f)$, then it is clearly also in $\mathrm{Per}_{p_1}(f)$ and $\mathrm{Per}_{p_2}(f)$. On the other hand, there are $k_1, k_2 \in \mathbb{Z}$ such that $p = k_1p_1 + k_2p_2$. When restricted to its periodic points, the function f is invertible. Hence for $x \in \mathrm{Per}_{p_1}(f) \cap \mathrm{Per}_{p_2}(f)$ we have

$$
\begin{aligned}
f^p(x) &= f^{k_1p_1}(f^{k_2p_2}(x))\\
&= f^{k_1p_1}(x)\\
&= x.
\end{aligned}
$$

In other words, $\mathrm{Per}_p(f) \supseteq \mathrm{Per}_{p_1}(f) \cap \mathrm{Per}_{p_2}(f)$. □

We can slightly generalize these investigations and instead of the periodic points also look at the preperiodic points.

Lemma 2.69. *Let X be a set and $f : X \to X$. For $p_1, p_2 \in \mathbb{N}$ and $q_1, q_2 \in \mathbb{N}_0$ we have*

$$
Pre_{q_1,p_1}(f) \cap Pre_{q_2,p_2}(f) = Pre_{\min(q_1,q_2),\gcd(p_1,p_2)}(f).
$$

Proof. Set $q := \min(q_1, q_2)$ and $p := \gcd(p_1, p_2)$. If $x \in \mathrm{Pre}_{q,p}(f)$, then $f^{q+p}(x) = f^q(x)$. Thus also $f^{q_1+p_1}(x) = f^{q_1-q}(f^{q+p\frac{p_1}{p}}(x)) = f^{q_1-q}(f^q(x)) = f^{q_1}(x)$. In other words, $x \in \mathrm{Pre}_{q_1,p_1}(f)$ and, by the same reasoning, $x \in \mathrm{Pre}_{q_2,p_2}(f)$.

On the other hand consider $x \in \mathrm{Pre}_{q_1,p_1}(f) \cap \mathrm{Pre}_{q_2,p_2}(f)$. Without loss of generality we assume that $q_1 \le q_2$, in other words, that $q = q_1$. For all $m \ge q_2$ we have $f^m(x) \in \mathrm{Per}_{p_1}(f) \cap \mathrm{Per}_{p_2}(f) = \mathrm{Per}_p(f)$. In particular

$$
f^{q_1+q_2p_1p+p}(x) = f^{q_1+q_2p_1p}(x) = f^{q_1}(x).
$$

But we also have

$$f^{q_1+q_2p_1p+p}(x) = f^{q_1+p}(x),$$

since $x \in \mathrm{Pre}_{q_1,p_1}(x)$. Combining these equations gives $f^{q_1+p}(x) = f^{q_1}(x)$, hence $x \in \mathrm{Pre}_{q,p}(f)$. $\square$

Theorem 2.70. *Let X be a set and $f : X \to X$. For $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ we have*

$$\mathit{Pre}_{q,p}(f) = \biguplus \left\{ \widetilde{\mathit{Pre}}_{k,\ell}(f) \;\middle|\; k \le q,\ \ell \mid p \right\} \tag{2.6}$$

and in particular

$$\widetilde{\mathit{Pre}}_{q,p}(f) = \mathit{Pre}_{q,p}(f) \setminus \bigcup \left\{ \mathit{Pre}_{k,\ell}(f) \;\middle|\; k \le q,\ \ell \mid p,\ (k,\ell) \ne (q,p) \right\}. \tag{2.7}$$

Proof. The right hand side of (2.6) is clearly contained in the left hand side. If $x \in \mathrm{Pre}_{q,p}(f)$, there must be $k \le q$ and $\ell \in \mathbb{N}$ such that $x \in \widetilde{\mathrm{Pre}}_{k,\ell}(f)$. By Lemma 2.69 and the minimality of ℓ, we have $\ell = \gcd(\ell, p)$, hence $\ell \mid p$. Equation (2.7) follows directly from (2.6) and the observation that

$$\left\{ \widetilde{\mathrm{Pre}}_{k,\ell}(f) \;\middle|\; k \le q,\ \ell \mid p,\ (k,\ell) \ne (q,p) \right\} = \left\{ \mathrm{Pre}_{k,\ell}(f) \;\middle|\; k \le q,\ \ell \mid p,\ (k,\ell) \ne (q,p) \right\}. \square$$

2.7. Periodic Cellular Automata

As we discussed before, restricting cellular automata to their p-periodic points gives rise to periodic cellular automata. We can use those to obtain obstructions for the existence of conjugacies for the original systems. In this context it is also very natural to consider cellular automata on subshifts of finite type, as the p-periodic points of cellular automata have this structure. Even if one starts with a full shift as the state space, restricting a cellular automaton to its periodic points of minimal period at most p gives rise to a periodic cellular automaton on a sofic shift. This section discusses two known theorems that characterize periodic cellular automata in different ways.

We showed that every two-sided subshift of finite type is conjugate to the edge shift of a graph G. Every graph automorphism of G gives rise to a map $\mathrm{Path}_{\mathbb{Z}}(G) \to \mathrm{Path}_{\mathbb{Z}}(G)$ which is continuous and commutes with the shift. In other words, it is a cellular automaton. Since G is finite, the automorphism must have finite order and the induced cellular automaton must be periodic.

The next result shows that the converse is also true. For every periodic cellular automaton f on a subshift of finite type X, we find a graph G, whose edge shift is conjugate to X, such that the representation of f on the edge shift of G is induced by a graph automorphism. The result can be found in the paper [BLR88, Prop. 2.9] by Boyle, Lind and Rudolph, where it is attributed to John Franks. With trivial modifications, the prove yields the stronger result that every finite subgroup of the automorphism group of a subshift of finite type arises in that way. We follow the proof presented in [Kit98, Lemma 3.3.8].

Theorem 2.71. *Let $X \subseteq A^{\mathbb{Z}}$ be a two-sided subshift. If $f : X \to X$ is a periodic cellular automaton, then there exists an alphabet B, a subshift $Y \subseteq B^{\mathbb{Z}}$ such that σ_Y is conjugate to σ_X and a permutation $\pi\colon B \to B$ such that f is conjugate to the cellular automaton $g : Y \to Y$ defined by*

$$g(x)_i = \pi(x_i) \text{ for } i \in \mathbb{Z}.$$

Proof. Define a partition $\mathscr{M} := \{ [a]_0 \mid a \in A \}$ of X. Applying f to this partition multiple times and taking the common refinement, we get

$$\mathscr{N} := \bigvee_{k=0}^{p-1} f^k(\mathscr{M}) = \left\{ M_0 \cap f[M_1] \cap \cdots \cap f^{p-1}[M_{p-1}] \;\middle|\; M_0, \ldots, M_{p-1} \in \mathscr{M} \right\} \setminus \{\emptyset\}.$$

$\mathscr{N}$ is a partition of X into a finite number of clopen sets and since f is p-periodic, for each $M \in \mathscr{N}$ there is a unique $N \in \mathscr{N}$ such that $f[M] = N$. Therefore there is a uniquely determined permutation $\pi\colon \mathscr{N} \to \mathscr{N}$ with $\pi(N) := f[N]$ for all $N \in \mathscr{N}$. We now define a map $\tilde{\psi}\colon X \to \mathscr{N}^{\mathbb{Z}}$ by mapping each point to the sequence of partition elements which $(\sigma^i(x))_{i \in \mathbb{Z}}$ traverses, i.e., $\tilde{\psi}(x)_i$ is the unique element of $\mathscr{N}$ in which $\sigma^i(x)$ lies. Set $Y := \tilde{\psi}[X]$ and let $\psi\colon X \to Y$ be the map obtained from $\tilde{\psi}$ by restricting the range to Y. Since $\mathscr{N}$ is a refinement of $\mathscr{M}$, for each element $N \in \mathscr{N}$ there is a unique symbol $a_N \in A$ such that $N \subseteq [a_N]_0$. Therefore ψ is bijective and its inverse is given by

$$\psi^{-1}((N_i)_{i \in \mathbb{Z}}) = (a_{N_i})_{i \in \mathbb{Z}}.$$

The map ψ^{-1} is defined by a one-block map, hence continuous and shift-commuting. The map ψ is therefore a conjugacy between σ_X and σ_Y. Define the cellular automaton $g : Y \to Y$ by

$$g := \psi \circ f \circ \psi^{-1}.$$

Finally for $y := (N_i)_{i \in \mathbb{Z}} \in Y$ and $j \in \mathbb{Z}$ we have

$$\begin{aligned} \sigma^j(\psi^{-1}(y)) &\in N_j, \\ f(\psi^{-1}(\sigma^j(y))) &\in f[N_j] = \pi(N_j), \\ g(y)_j &= \psi(f(\psi^{-1}(y)))_j = \pi(y_j). \end{aligned}$$

□

Corollary 2.72. *Let X by a subshift of finite type. If $f : X \to X$ is a p-periodic cellular automaton with $p \in \mathbb{N}$, then there exists a graph G, a graph automorphism $\pi\colon E(G) \to E(G)$ and a conjugacy ψ from X to $\mathrm{Path}_{\mathbb{Z}}(G)$ such that f is conjugate to the cellular automaton g defined by*

$$g(x)_i := \pi(x_i) \text{ for } i \in \mathbb{Z}$$

via the conjugacy ψ.

Proof. Applying Theorem 2.71 to f, we obtain a subshift Y over the alphabet B which is conjugate to X via the conjugacy ψ, such that the cellular automaton $h := \psi^{-1} \circ f \circ \psi$ has left and right radius 0.

X is a subshift of finite type, hence Y, too, is a subshift of finite type by Lemma 2.42. Passing to a higher block representation, we find an essential graph G such that Y is conjugate to $\mathrm{Path}_{\mathbb{Z}}(G)$ via $\varphi : Y \to \mathrm{Path}_{\mathbb{Z}}(G)$. By Lemma 2.60 the map $g := \varphi \circ h \circ \varphi^{-1}$ can be defined by a local map with left and right radius 0. This means, there is a map $\pi\colon E(G) \to E(G)$ with $g(x)_i = \pi(x_i)$. Since h maps $\mathrm{Path}_{\mathbb{Z}}(G)$ to itself, π is a graph automorphism. □

Example 2.73. *Consider again the cellular automaton f from Example 2.53. If we apply f to the partition $\mathscr{E} = \{[0]_0, [1]_0, [2]_0\}$ of $A^{\mathbb{Z}}$ and take the common refinement with $\mathscr{E}$, we obtain*

$$\{[0]_0, [10]_0, [11]_0 \cup [12]_0, [20], [21]_0 \cup [22]_0\}.$$

The corresponding shift is represented as an edge shift in Figure 2.4. The graph automorphism corresponding to f is given by the exchange of the vertices $[10]_0$ and $[20]_0$.

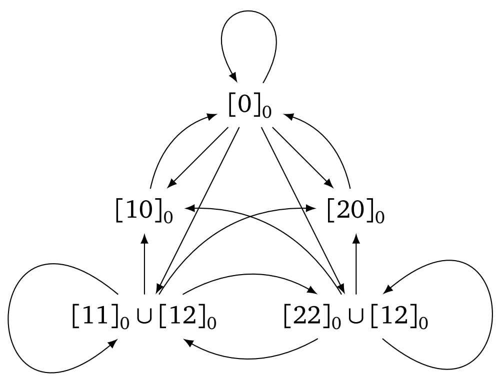

Figure 2.4.: Representation of the subshift from Example 2.73 as an edge shift.

We now take a look at another source of periodic cellular automata. Sensitivity is a property that received a lot of attention in the dynamical systems literature. It measures how much small perturbations in the initial state influence the dynamics. A class of systems showing very little sensitivity are the equicontinuous systems.

Definition 2.74 (Equicontinuous dynamical system). *Let (X, d) be a compact metric space. A dynamical system $f : X \to X$ is called* equicontinuous *if the set of its iterates is equicontinuous, i.e.,*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in X\, \forall \ell \in \mathbb{N}_0 : d(x, y) < \delta \implies d(f^{\ell}(x), f^{\ell}(y)) < \varepsilon.$$

It is easy to see that equicontinuity of a dynamical system does not depend on the metric but only on the topology.

The following theorem shows that on two-sided sofic shifts the equicontinuous cellular automata are precisely the preperiodic ones. It is due to KURKA [Kur97, Theorem 4] for the case of full shifts.

Theorem 2.75. *Let X be a two-sided sofic subshift. For every cellular automaton $f : X \to X$ the following are equivalent.*

(a) *f is preperiodic.*

(b) *f is equicontinuous.*

Proof. (a) $\implies$ (b) If f is preperiodic, the lengths of its orbits are uniformly bounded, hence $\left\{ f^\ell \;\middle|\; \ell \in \mathbb{N} \right\}$ is finite. Every finite set of continuous functions is equicontinuous.

(b) $\implies$ (a) Let d be the standard metric on X as defined in Example 2.7. Since f is equicontinuous, there is $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies

$$d(f^\ell(x), f^\ell(y)) < 1.$$

This translates into the combinatorial statement that there is $n \in \mathbb{N}$ such that for all $x, y \in X$ with $x_{[-n,n]} = y_{[-n,n]}$ we also have the equality $f^\ell(x)_0 = f^\ell(y)_0$ for all $\ell \geq 0$. Let W be the set of all words which appear as subwords in elements of X.

Let $w \in W$ with $|w| = 2n + 1$. As a sofic shift, X is the image of an edge shift under an edge labeling. Therefore there exists a positive integer $m \in \mathbb{N}$, words $u, v \in W$, whose lengths are a multiple of m, and a configuration $x = x^w \in X$ such that $x_{[-n,n]} = w$, $x = {}^\infty u x_{[-m,0)} . x_{[0,m)} v^\infty$, $y := {}^\infty u^\infty \in X$ and $z := {}^\infty v^\infty \in X$. In particular $x_{-i} = y_{-i}$ and $x_i = z_i$ for all sufficiently large $i \in \mathbb{N}$.

Since f is equicontinuous and commutes with the shift, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} f^\ell(x)_{(-\infty,-k]} &= f^\ell(y)_{(-\infty,-k]}, \\ f^\ell(x)_{[k,\infty)} &= f^\ell(z)_{[k,\infty)} \end{aligned}$$

for all $\ell \in \mathbb{N}_0$. Since y and z are σ-periodic, they are eventually periodic under f by Lemma 2.58. In other words, there are positive integers q_x and p_x with $f^{q_x+p_x}(y) = f^{q_x}(y)$ and $f^{q_x+p_x}(z) = f^{q_x}(z)$.

By the pigeon hole principle, there are $\ell_1, \ell_2 \in \mathbb{N}$ with $\ell_1 < \ell_2$ such that

$$f^{q_x+p_x\ell_2}(x)_{[-k,k]} = f^{q_x+p_x\ell_1}(x)_{[-k,k]}.$$

We also have

$$\begin{aligned} f^{q_x+p_x\ell_2}(x)_{(-\infty,-k]} &= f^{q_x+p_x\ell_2}(y)_{(-\infty,-k]} \\ &= f^{q_x+p_x\ell_1}(y)_{(-\infty,-k]} \\ &= f^{q_x+p_x\ell_1}(x)_{(-\infty,-k]} \end{aligned}$$

and $f^{q_x+p_x\ell_2}(x)_{[k,\infty)} = f^{q_x+p_x\ell_1}(x)_{[k,\infty)}$. Together this shows that x is eventually periodic under f with preperiod $q_w := q_x + p_x\ell_1$ and period $p_w := p_x(\ell_2 - \ell_1)$. Define $q := \max(\{ q_w \mid w \in W, |w| = 2n+1 \})$ and $p := \operatorname{lcm}(\{ p_w \mid w \in W, |w| = 2n+1 \})$.

Let $y \in X$ be an arbitrary configuration. Set $w := y_{[i-n,i+n]}$ and consider x^w as defined above. We have $f^{q+p}(y)_0 = f^{q+p}(x^w)_0 = f^q(x^w)_0 = f^q(y)_0$, since $q \geq q_w$ and $p_w \mid p$. For $i \in \mathbb{Z}$ we also have $f^{q+p}(y)_i = f^{q+p}(\sigma^i(y))_0 = f^q(\sigma^i(y))_0 = f^q(y)_i$. This shows that f has preperiod q and period p. □

Corollary 2.76. *A cellular automaton $f : X \to X$ on a sofic subshift $X \subseteq A^{\mathbb{Z}}$ is periodic if and only if it is invertible and equicontinuous.*

Despite these characterizations, periodic cellular automata can nevertheless be quite complex. Kari and Ollinger showed in [KO08] that it is undecidable, in the algorithmic sense, if a given invertible cellular automaton on a two-sided full shift is periodic.

Chapter 3.

Derivative Algebras

For our investigation of topological conjugacy of dynamical systems on metrizable Stone spaces we have to understand the topological structure of these spaces. To do so, a more algebraic approach to topology is very beneficial. Already in the beginning of set theoretic topology, topological spaces were studied by looking at operations on their power set. See for example the works of Sierpinsk [Sie27] and Riesz [Rie09] for early applications of this method.

Moore [Moo10] and later McKinsey and Tarski [MT44] looked at these structures in an abstract algebraic framework. Starting with a Boolean algebra together with a closure operation fulfilling certain axioms one defines what it means for an element of the Boolean algebra to be closed and therefore gets a topology on the space underlying the Boolean algebra. In the appendix of [MT44], McKinsey and Tarski also mention that a structure similar to closure algebras can be obtained by considering the operation of taking the set of accumulation points, the so called Cantor-Bendixson-derivative.

Later, in several works Pierce [Pie70; Pie72] investigated closure algebras together with the additional operation of taking the inner accumulation points. He called these algebras topological Boolean algebras and used them to classify certain Stone spaces.

We will study Boolean algebras with a derivative operation as suggested in [MT44] and call them derivative algebras. It will be shown that these structures are actually equivalent to the topological Boolean algebras of Pierce but have a much shorter and more intuitive axiom system.

3.1. Boolean Algebras

We begin by recapitulating some notions about Boolean algebras. A classical introduction to Boolean algebras is the book [Hal63] by Halmos. There exists a newer extended version due to Givant and Halmos[GH09] which we will normally cite. For an outlook on the vast theory of Boolean algebras, of which we only need a very modest part, one can consult the handbook [MK89]. In particular, we will mostly be concerned with finite Boolean algebras.

Definition 3.1 (Boolean algebra)**.** *A* Boolean algebra $\mathscr{D}$ *is a set D equipped with two binary and one unary operation* $\vee$, $\wedge$, *and* ^{c} *on D and two constants* 0 *and* 1 *in D, that fulfill the following axioms.*

(i) *(Idempotence)* $a \vee a = a$, $a \wedge a = a$,

(ii) *(Commutativity)* $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

(iii) *(Associativity)* $a \vee (b \vee z) = (a \vee b) \vee z$,

(iv) *(Absorption)* $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$,

(v) *(Distributivity)* $a \vee (b \wedge z) = (a \vee b) \wedge (a \vee z), a \wedge (b \vee z) = (a \wedge b) \vee (a \wedge z)$,

(vi) *(De Morgan laws)* $(a \vee b)^c = a^c \wedge b^c$, $(a \wedge b)^c = a^c \vee b^c$,

(vii) *(Complement)* $a \vee a^c = 1, a \wedge a^c = 0$,

(viii) *(Non Triviality)* $1 \neq 0$.

The three operations $\vee$, $\wedge$, *and* ^{c} *are called* join, meet *and* complement.

.

Notice that some of these axioms can be derived from the others and that there are different possibilities of giving a minimal axiom system (see for example [Hal63, p. 5]).

Example 3.2. *The canonical example of a Boolean algebra is the power set $\mathscr{P}(X)$ of some set X together with the operations union, intersection and complement. The constants are defined as* $0_{\mathscr{P}(X)} = \emptyset$ *and* $1_{\mathscr{P}(X)} = X$.

Boolean algebras are therefore algebras in the sense of universal algebra. Concepts like subalgebra and homomorphism carry over from this abstract setting. For the sake of completeness we nevertheless present them here in our concrete situation.

Definition 3.3 (Subalgebra)**.** *A Boolean algebra $\mathscr{E}$ is a* subalgebra *of the Boolean algebra $\mathscr{D}$, if $E \subseteq D$ and all operations in $\mathscr{E}$ are just the restrictions of the operations in $\mathscr{D}$. In particular $0_{\mathscr{D}} \in E$ and $1_{\mathscr{D}} \in E$ must hold.*

Definition 3.4 (Homomorphism)**.** *A* homomorphism *between Boolean algebras $\mathscr{D}$ and $\mathscr{E}$ is a map $\varrho : D \to E$ that commutes with the Boolean operations, i.e. it satisfies*

(i) $\varrho(a \vee b) = \varrho(a) \vee \varrho(b)$,

(ii) $\varrho(a \wedge b) = \varrho(a) \wedge \varrho(b)$,

(iii) $\varrho(a^c) = \varrho(a)^c$,

(iv) $\varrho(0) = 0, \varrho(1) = 1$.

A homomorphism $\varrho : \mathscr{D} \to \mathscr{E}$ is an isomorphism *of Boolean algebras if it is bijective.*

Every Boolean algebra carries a natural partial order defined by

$$a \le b \Leftrightarrow a \wedge b = a. \tag{3.1}$$

Lemma 3.5. *For a Boolean algebra $\mathscr{D}$ the relation $\le$ defined in (3.1) is a partial order. Furthermore $a \le b \Leftrightarrow a \vee b = b$ for all $b \in \mathscr{D}$.*

Proof. Reflexivity follows from idempotency of $\wedge$. If $a \wedge b = a$ and $b \wedge c = b$, then $a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge b = a$, proving transitivity. Finally if $a \wedge b = a$ and $a \wedge b = b$, we get $a = b$ and therefore antisymmetry.

If $a \le b$, then $a \vee b = (a \wedge b) \vee b = b$ by the absorption axiom. On the other hand if $a \vee b = b$, then $a \wedge b = a \wedge (a \vee b) = a$, again by absorption. □

Definition 3.6 (Atom)**.** *An element $a \neq 0$ in a Boolean algebra for which $b \le a$ implies $b = 0$ or $b = a$ is called an* atom .

It follows immediately that for two different atoms a and b we have $a \wedge b = 0$. At least for finite Boolean algebras the atoms form the basic building blocks of all elements of the algebra as the following two lemmas show.

Lemma 3.7. *Let $b \neq 0$ be an element of a finite Boolean algebra $\mathscr{D}$. Then there are atoms $a_1, \dots, a_k$ in $\mathscr{D}$ such that $b = a_1 \vee a_2 \vee \cdots \vee a_k$. These atoms are unique up to permutation.*

Proof. Let b be an element of $\mathscr{D}$ and let M be the set of all atoms a for which $a \le b$. Clearly $\bigvee M \le b$, so assume $b \neq \bigvee M$. Let N be the set of elements $c \neq 0$ for which $c \le b \wedge (\bigvee M)^c$. Since N has only finitely many elements, it must have a minimal element, which is necessarily an atom. But this atom is less or equal b, but not in M, contradicting the definition of M. So $b = \bigvee M$. Since $b = a_1 \vee \cdots \vee a_k$ implies $a_i \le b$ for $i = 1, \dots, k$, we also showed uniqueness. □

Lemma 3.8. *Let $\mathscr{D}$ and $\mathscr{E}$ be finite Boolean algebras with atom sets I and J, respectively. Every map $\varrho : I \to J$ can be extended uniquely to a homomorphism from $\mathscr{D}$ to $\mathscr{E}$. If ϱ is bijective, this extension is an isomorphism.*

In particular a finite Boolean algebra $\mathscr{D}$ with atom set I is isomorphic to the power set algebra $\mathscr{P}(I)$.

Proof. Since every element $a \in \mathscr{D}$ has a unique representation as $a = c_1 \vee \cdots \vee c_k$ with $c_1, \dots, c_k \in I$, we define

$$\varrho(a) := \varrho(c_1) \vee \cdots \vee \varrho(c_k). \tag{3.2}$$

Using the finiteness of $\mathscr{D}$ it is easy to check that this is indeed a homomorphism and since every homomorphism is compatible with $\vee$, this is the only possibility to define an extension of ϱ to a homomorphism.

If ϱ is a bijection on I all atoms on the right hand side of (3.2) are pairwise different and so ϱ is injective. Surjectivity follows directly by Lemma 3.7. □

3.2. Derivative Algebras and Topological Boolean Algebras

In order to incorporate topological information into our algebra we equip it with yet another operation.

Definition 3.9 (Derivative algebras)**.** *A* derivative algebra *is a Boolean algebra $\mathscr{D}$ equipped with a unary operation * that fulfills the following three axioms.*

(i-a) $0^* = 0$,

(i-b) $a^{**} \leq a^*$,

(i-c) $(a \vee b)^* = a^* \vee b^*$.

This structure is defined in Appendix I in the paper [MT44] by McKinsey and Tarski. However, it is not studied further there. The name "derivative algebra" follows the terminology of "closure algebras" or "topological Boolean algebra". The term "derived set"[1] goes back already to Cantor's seminal article [Can72], which was one of the starting points of point set topology. Esakia uses the name "derivative algebra" in a paper [Esa04] for a very similar structure where Axiom (i-b) is replaced by $a^{**} \leq a^* \vee a$. The focus in Esakia's work is, however, on logical aspects.

The following lemma is an immediate consequence of the axioms.

Lemma 3.10. *Let $\mathscr{D}$ be a derivative algebra and let $a, b \in \mathscr{D}$. If $a \leq b$, then $a^* \leq b^*$.*

Proof. Since $a \leq b$, we have $b = a \vee b$. Hence $b^* = (a \vee b)^* = a^* \vee b^*$, in other words, $a^* \leq b^*$. □

Example 3.11. *Let G be a graph. Consider the Boolean algebra $B = \mathscr{P}(V(G))$. We can define a derivative operation on B by setting $M^* := \{\, \mathrm{t}_G(\gamma) \mid \gamma \in \mathrm{Path}(G), \mathrm{i}_G(\gamma) \in M \text{ and } |\gamma| \geq 1 \,\}$ for $M \subseteq \mathscr{P}(V(G))$. The derived set of M is thus the set of all vertices reachable from M by non-empty paths. Axioms (i-a) and (i-c) are fulfilled by definition. Let i be a vertex in M^{**}. There are non-empty paths $\alpha, \beta \in \mathrm{Path}(G)$ with $\mathrm{i}_G(\alpha) \in M$, $\mathrm{t}_G(\alpha) = \mathrm{i}_G(\beta)$ and $\mathrm{t}_G(\beta) = i$. Taking the concatenation $\alpha\beta$ shows that $i \in M^*$, hence (i-b) is also satisfied.*

[1] Cantor, writing in German, uses the term "abgeleitete Menge".

We will not use this example further on, but it useful to keep it in mind in order to have a simple finite example of a non trivial derivative algebra. The class of derivative algebras that we will use over and over are given by the powerset of suitable topological spaces, where the derived set of M consists of the accumulation points of M. This will be dealt with in detail in Definition 3.17.

Subalgebras and homomorphisms of derivative algebras are defined as usual for algebras in the sense of universal algebra.

Definition 3.12 (Derivative subalgebra, homomorphism)**.** *A* derivative subalgebra *$\mathscr{B}$ of a derivative algebra $\mathscr{D}$ is a Boolean subalgebra of $\mathscr{D}$ such that the derivative operation on $\mathscr{B}$ is the restriction of the derivative operation on $\mathscr{D}$ and such that $\mathscr{B}$ is closed under taking derivatives. A* homomorphism *$\rho : \mathscr{B} \to \mathscr{D}$ between derivative algebras is a homomorphism of Boolean algebras, that additionally commutes with the derivative in the sense that $\rho(a)^* = \rho(a^*)$ for all $a \in \mathscr{B}$. A bijective homomorphism between derivative algebras is called an* isomorphism between derivative algebras.

Lemma 3.13. *Let $\mathscr{B}$ be a finite derivative algebra. A Boolean subalgebra $\mathscr{C}$ of $\mathscr{B}$ is also a derivative subalgebra of $\mathscr{B}$ if and only if $a^* \in \mathscr{C}$ for every atom $a \in \mathscr{C}$.*

Proof. We have to show that $b^* \in \mathscr{C}$ for every $b \in \mathscr{C}$. Since $\mathscr{C}$ is finite, we can decompose b as the join of pairwise different atoms $a_1, \dots, a_n$ of $\mathscr{C}$. Since $\mathscr{C}$ is closed under joins, we get $b^* = a_1^* \vee \cdots \vee a_n^* \in \mathscr{C}$. □

Lemma 3.14. *Let $\mathscr{B}$ and $\mathscr{C}$ be finite derivative algebras. A homomorphism of Boolean algebras $\varphi : \mathscr{B} \to \mathscr{C}$ is also a homomorphism of derivative algebras if and only if*

$$\varphi(a^*) = \varphi(a)^* \tag{3.3}$$

for all atoms a of $\mathscr{B}$.

Proof. For every homomorphism of derivative algebras the equation holds (3.3) for all $a \in \mathscr{B}$, in particular for the atoms. Since $\mathscr{B}$ is finite, every element of $\mathscr{B}$ is the join of atoms by Lemma 3.7, hence for $b \in \mathscr{B}$ there are atoms $a_1, \dots, a_n$ such that $b = a_1 \vee \dots a_n$. Applying (3.3) and the fact that ρ is a homomorphism of Boolean algebras

$$\begin{aligned}
\rho(b^*) &= \rho(a_1^* \vee \cdots \vee a_n^*) \\
&= \rho(a_1^*) \vee \cdots \vee \rho(a_n^*) \\
&= \rho(a_1)^* \vee \cdots \vee \rho(a_n)^* \\
&= (\rho(a_1) \vee \cdots \vee \rho(a_n))^* \\
&= \rho(b)^*.
\end{aligned}$$

□

Derivative algebras are quite similar to topological Boolean algebras introduced by Pierce, which we define next. Notice, however, that the axiom system for these algebras is much more complex.

Definition 3.15 (Topological Boolean algebras). *A topological Boolean algebra $\mathscr{D}$ as defined in [Pie72] is a Boolean algebra equipped with two unary operations $'$ and $^-$ that fulfill the following axioms.*

(ii-a) $(a \vee b)^- = a^- \vee b^-$,

(ii-b) $a \wedge a^- = a$, *i.e.,* $a \leq a^-$,

(ii-c) $a^{--} = a^-$,

(ii-d) $0^- = 0$,

(ii-e) $a^{-\prime} \wedge a = a'$,

(ii-f) $a' = (a \wedge b)' \vee (a \wedge b^c)' \vee ((a \wedge b)^- \wedge a \wedge b^c) \vee ((a \wedge b^c)^- \wedge a \wedge b)$.

The remainder of this section will be used to show that derivative algebras and topological Boolean algebras are equivalent concepts in the sense that one can define the corresponding operations in terms of each other.

For a derivative algebra we can define

(i-op) $$a' = a^* \wedge a \text{ and } a^- = a^* \vee a,$$

and for a topological Boolean algebra we can define a derivative operation as

(ii-op) $$a^* = a^{-\prime},$$

These resulting structures are equivalent in the sense of the following lemma.

Theorem 3.16. *Let $\mathscr{D}$ be Boolean algebra together with unary operations $^*, ', ^-$. The algebra $\mathscr{D}$ fulfills the axioms (i-a)–(i-c) and (i-op) if and only if it fulfills the axioms (ii-a)–(ii-f) and (ii-op).*

Proof. Let $a, b, c, d \in \mathscr{D}$ be arbitrary elements. All implications will be shown just by algebraic manipulations which, however, will get quite involved at times.

($\Rightarrow$) Let $\mathscr{D}$ fulfill the axioms (i-a)–(i-c) and (i-op). We show that this implies the identities (ii-a)–(ii-f) and (ii-op)

(ii-a) $(a \vee b)^- = a^- \vee b^-$:

$$\begin{aligned}
(a \vee b)^- &= (a \vee b)^* \vee (a \vee b) && \text{(Definition of } a^-\text{)} \\
&= a^* \vee a \vee b^* \vee b && \text{(i-c)} \\
&= a^- \vee b^-. && \text{(Definition of } a^-\text{)}
\end{aligned}$$

(ii-b) $a \wedge a^- = a$:

$$\begin{aligned} a \wedge a^- &= a \wedge (a^* \vee a) && \text{(Definition of } a^-\text{)} \\ &= a. \end{aligned}$$

(ii-c) $a^{--} = a^-$:

$$\begin{aligned} a^{--} &= (a^* \vee a)^- && \text{(Definition of } a^-\text{)} \\ &= a^{*-} \vee a^- && \text{(ii-a)} \\ &= (a^{**} \vee a^*) \vee (a^* \vee a) && \text{(Definition of } a^-\text{)} \\ &= a^* \vee a = a^-. && \text{(i-b)} \end{aligned}$$

(ii-d) $0^- = 0$:

$$\begin{aligned} 0^- &= 0 \vee 0^* && \text{(Definition of } a^-\text{)} \\ &= 0. && \text{(i-a)} \end{aligned}$$

(ii-e) $a^{-\prime} \wedge a = a'$:

$$\begin{aligned} a^{-\prime} \wedge a &= (a^-)^* \wedge a^- \wedge a && \text{(Definition of } a'\text{)} \\ &= (a^* \vee a)^* \wedge a && \text{(Definition of } a^-\text{), (ii-b)} \\ &= (a^{**} \vee a^*) \wedge a && \text{(i-c)} \\ &= a^* \wedge a = a'. && \text{(i-b), (Definition of } a'\text{)} \end{aligned}$$

(ii-f) $a' = (a \wedge b)' \vee (a \wedge b^c)' \vee ((a \wedge b)^- \wedge a \wedge b^c) \vee ((a \wedge b^c)^- \wedge a \wedge b)$:

$$\begin{aligned} &(a \wedge b)' \vee (a \wedge b^c)' \vee ((a \wedge b)^- \wedge a \wedge b^c) \vee ((a \wedge b^c)^- \wedge a \wedge b) \\ &= \big((a \wedge b)^* \wedge a \wedge b\big) \vee \big((a \wedge b^c)^* \wedge a \wedge b^c\big) \vee \\ &\quad \big(((a \wedge b)^* \vee (a \wedge b)) \wedge a \wedge b^c\big) \vee \big(((a \wedge b^c)^* \vee (a \wedge b^c)) \wedge a \wedge b\big) && \text{(Def. of } a', a^-\text{)} \\ &= \big(a \wedge (((a \wedge b)^* \wedge b) \vee ((a \wedge b^c)^* \wedge b^c))\big) \vee && \text{(Distributivity)} \\ &\quad \big((a \wedge b)^* \wedge a \wedge b^c) \vee ((a \wedge b^c)^* \wedge a \wedge b\big) && (b \wedge b^c = 0) \\ &= a \wedge \big(((a \wedge b)^* \wedge b) \vee ((a \wedge b^c)^* \wedge b^c) \vee \\ &\qquad ((a \wedge b)^* \wedge b^c) \vee ((a \wedge b^c)^* \wedge b)\big) && \text{(Distributivity)} \\ &= a \wedge ((a \wedge b)^* \vee (a \wedge b^c)^*) && \text{(Distributivity)} \\ &= a \wedge ((a \wedge b) \vee (a \wedge b^c))^* && \text{(i-c)} \\ &= a \wedge a^* = a'. && \text{(Distributivity)} \end{aligned}$$

(ii-op) $a^{-\prime} = a^*$:

$$\begin{aligned} a^{-\prime} &= (a^* \vee a)' && \text{(Definition of } a^-\text{)} \\ &= (a^* \vee a)^* \wedge (a^* \vee a) && \text{(Definition of } a'\text{)} \\ &= (a^{**} \vee a^*) \wedge (a^* \vee a) && \text{(i-c)} \\ &= a^* \wedge (a^* \vee a) && \text{(i-b)} \\ &= a^*. \end{aligned}$$

($\Leftarrow$) Let $\mathscr{D}$ fulfill the axioms (ii-a)–(ii-f) and (ii-op). We show that this implies the identities (i-a)–(i-c) and (i-op). As a starting point we give an alternative expression for a^*. Axiom (ii-f) applied to the case $b = a$ gives

$$\begin{aligned} a^* = a^{-\prime} &= (a \wedge a^-)' \vee (a^- \wedge a^c)' \vee ((a \wedge a^-)^- \wedge a^- \wedge a^c) \vee ((a^- \wedge a^c)^- \wedge a \wedge a^-) \\ &= a' \vee (a^- \wedge a^c)' \vee (a^- \wedge a^- \wedge a^c) \vee ((a^- \wedge a^c)^- \wedge a) \\ &= a' \vee (a^- \wedge a^c) \vee ((a^- \wedge a^c)^- \wedge a). \end{aligned}$$

In particular this means that

$$\begin{aligned} &(a^- \wedge a^c)^- \wedge a \leq a^{-\prime}, \\ &(a^- \wedge a^c)^- \wedge a = (a^- \wedge a^c)^- \wedge a \wedge a \leq a^{-\prime} \wedge a = a'. \end{aligned}$$

Together this gives

$$a^* = a^{-\prime} = a' \vee (a^- \wedge a^c). \tag{3.4}$$

(i -a) $0^* = 0$:

$$\begin{aligned} 0' &= 0^{-\prime} \wedge 0 = 0, \\ 0^* &= 0^{\prime -} = 0^- = 0. \end{aligned}$$

(i -b) $a^{**} \leq a^*$: To show this, we first have to establish that taking the closure of elements preserves the order $\leq$. For any pair of elements b, c with $b \leq c$ we have $b \vee c = c$ as shown in Lemma 3.5. By Axiom (ii-a) this implies $b^- \vee c^- = c^-$, in other words $b^- \leq c^-$. Now notice that $a' \leq a$, since $a' \wedge a = (a^{-\prime} \wedge a) \wedge a = a'$. Furthermore $a^- \wedge a^c \leq a^-$ and hence

$$(a^- \wedge a^c)^- \leq a^{--} = a^-. \tag{3.5}$$

Taking the intersection with a^c and using (3.4) gives $(a^- \wedge a^c)^- \wedge a^c \leq a^- \wedge a^c \leq a^*$. On the other hand (ii-f) also shows $(a^- \wedge a^c)^- \wedge a \leq a^{-\prime} = a^*$. Taking the join of these two inequalities gives

$$(a^- \wedge a^c)^- \leq a^*. \tag{3.6}$$

Other consequences of (3.4) are $a^{\prime -} \wedge a^{\prime c} \leq a^{\prime *} \leq a^*$ and $a^{\prime -} \wedge a' \leq a' \leq a^*$. Again, taking the join of these inequalities gives

$$a'^{-} \leq a^*. \tag{3.7}$$

Plugging (3.6) and (3.7) into (3.4) finally gives

$$a^{*-} = (a' \vee (a^- \wedge a^c))^- = a'^- \vee (a^- \wedge a^c)^- \leq a^*$$

and

$$a^{**} = a^{*\prime} \vee (a^{*-} \wedge a^{*c}) = a^{*\prime} \vee 0 \leq a^*.$$

(i -c) $(a \vee b)^* = a^* \vee b^*$: First we notice that by Axiom (ii-f) with $a := c^- \vee d^-$ and $b := c^-$ we have

$$\begin{aligned}(c^- \vee d^-)' &= (c^-)' \vee (d^- \wedge (c^-)^c)' \vee (c^- \wedge (c^- \vee d^-) \wedge (c^-)^c) \vee ((d^- \wedge (c^-)^c)^- \wedge c^-)\\ &= c^* \vee (d^- \wedge (c^-)^c)' \vee ((d^- \wedge (c^-)^c)^- \wedge c^-)\end{aligned}$$

and with $a := c^- \vee d^-$ and $b := d^-$ we have

$$(c^- \vee d^-)' = d^* \vee (c^- \wedge (d^-)^c)' \vee ((c^- \wedge (d^-)^c)^- \wedge d^-).$$

Taking the join of these equations gives

$$(c^- \vee d^-)' = c^* \vee d^* \vee ((d^- \wedge (c^-)^c)^- \wedge c^-) \vee ((c^- \wedge (d^-)^c)^- \wedge d^-). \tag{3.8}$$

Now we want to show that the third and fourth term on the right hand side are contained in the first two terms. To see this, notice that $c' = c^{-\prime} \wedge c \leq c \leq c^-$, which together with $c^- \wedge c = c$ and (3.4) implies

$$\begin{aligned}c^* \vee c &= \big(c' \vee (c^- \wedge c^c)\big) \vee (c^- \wedge c)\\ &= c' \vee c^- = c^-.\end{aligned} \tag{3.9}$$

For $b \leq a$ we have $a' \geq (a \wedge b)' = b'$ and $a^- = (a \vee b)^- = a^- \vee b^- \geq b^-$. Therefore both mappings $a \mapsto a'$ and $a \mapsto a^-$ preserve the order $\leq$, hence this is also true for $a \mapsto a^* = a^{-\prime}$. Plugging (3.9) into the third term on the right hand side of (3.8) gives

$$\begin{aligned}(d^- \wedge (c^-)^c)^- \wedge c^- &\leq ((d^- \wedge (c^-)^c)^* \vee (d^- \wedge (c^-)^c)) \wedge c^-\\ &= (d^- \wedge (c^-)^c)^* \wedge c^-\\ &\leq d^{-*} = d^{--\prime} = d^{-\prime} = d^*.\end{aligned}$$

Exchanging c and d we also get

$$(c^- \wedge (d^-)^c)^- \wedge d^- \leq c^*.$$

Finally applying these two inequalities to (3.8)

$$\begin{aligned}(c \vee d)^* &= (c \vee d)^{-\prime} \\ &= (c^- \vee d^-)' \\ &= c^* \vee d^*.\end{aligned}$$ □

To finish our analysis of the relationship between topological Boolean algebras and derivative algebras, we show that the definitions (i-op) and (ii-op) are compatible with each other. More precisely, we show that if one starts with a topological Boolean algebra and defines a derivative operation by (ii-op), applying (i-op) to this derivative gives back the two original operations $'$ and $^-$. For $'$ this follows directly from Axiom (ii-e), since $a^* \wedge a = a^{-\prime} \wedge a = a'$ is precisely what we have to show. For $^-$ we use (3.4) to conclude that

$$\begin{aligned}a^* \vee a &= a' \vee (a^- \wedge a^c) \vee a \\ &= a \vee (a^- \wedge a^c) \\ &= (a^- \wedge a) \vee (a^- \wedge a^c) \\ &= a^-.\end{aligned}$$

Similarly, starting with a derivative algebra and defining a new operation by plugging (i-op) into (ii-op) gives back the derivative operation. To see this, consider

$$\begin{aligned}a^{-\prime} &= (a^* \vee a)' \\ &= (a^* \vee a)^* \wedge (a^* \vee a) \\ &= (a^{**} \vee a^*) \wedge (a^* \vee a) \\ &= a^* \wedge (a^* \vee a) \\ &= a^*.\end{aligned}$$

3.3. Derivative Algebras and Topology

Definition 3.17 (Accumulation Points). *Let $(X, \mathcal{T})$ be a topological space and let $M \subseteq X$ be a subset. The set of* accumulation points *of M in X is defined as*

$$ac_{\mathcal{T}}(M) := \{\, x \in X \mid \forall U \in \mathcal{T} : x \in U \implies \exists z \in U \cap M : z \neq x \,\}.$$

If there is only one topology under consideration, we omit the index and simply write $ac(M)$ or $ac_X(M)$.

For a discussion of accumulation points and their properties see for example [Eng89].

Remark 3.18. Notice that accumulation points are sometimes also called *limit points* in the literature, and there are the closely related notions of *condensation points*, *proximal points* and *cluster points*. Of these terms, we will only use cluster points for the set of limits of convergent subsequences of a sequence. Some authors, however, use limit points in connection with sets and accumulation point in connection with sequences. See [Voi] for a discussion of this case of Babylonian confusion.

We can characterize the set of accumulation points by the following identity.

Lemma 3.19 (Approximating the accumulation points from the inside). *Let X be a topological space. If $M \subseteq X$, then*

$$ac(M) = \bigcup_{S \subseteq M} \overline{S} \setminus S.$$

Proof. ($\supseteq$) Let $x \in \overline{S} \setminus S$ for some $S \subseteq M$ and let U be an open neighborhood of x. Then $U \cap S \neq \emptyset$, since otherwise $X \setminus U$ is a closed superset of S not containing x and hence x would not lie in $\overline{S}$. Therefore there is $z \in S \cap U \subseteq M \cap U$ and since $x \notin S$, we have $z \neq x$ and thus $x \in \mathrm{ac}(M)$.

($\subseteq$) Let $x \in \mathrm{ac}(M)$. Let $\mathcal{U}(x)$ be the set of neighborhoods of x. For each open neighborhood $U \in \mathcal{U}(x)$ there is $z_U \neq x$ with $z_U \in U \cap M$. We will show that $x \in \overline{\{ z_U \mid U \in \mathcal{U}(x) \}} \setminus \{ z_U \mid U \in \mathcal{U}(x) \}$. Since $x \neq z_U$ for all U, it is enough to show $x \in \overline{\{ z_u \mid U \in \mathcal{U}(x) \}}$. Now assume that this is not the case. Then $V := X \setminus \overline{\{ z_U \mid U \in \mathcal{U}(x) \}}$ is an open neighborhood of x. But this implies $z_V \in V \cap \{ \overline{z_U} \mid U \in \mathcal{U}(x) \} = \emptyset$, a contradiction. □

Similarly as we do with the closure of a set, we can give an alternative definition of accumulation points by approximating them from the outside as

$$\mathrm{ac}_{\mathrm{out}}(M) := \bigcap_{a \in M} \overline{M \setminus \{a\}}. \tag{3.10}$$

This approach has some nice properties but in general we only get a subset of the accumulation points as the following lemma and example shows.

Lemma 3.20 (Approximating the accumulation points from the outside). *Let X be a topological space. If $M \subseteq X$, then*

$$ac(M) \supseteq ac_{out}(M).$$

Proof. Let $x \in \bigcap_{y \in M} \overline{M \setminus \{y\}}$. Either $x \notin M$ and hence $x \in \overline{M} = \overline{M \setminus \{x\}}$, or $x \in M$, in which case $x \in \overline{M \setminus \{x\}}$, too. In both cases we have $x \in \overline{M \setminus \{x\}} \setminus (M \setminus \{x\}) \subseteq \mathrm{ac}(M)$. □

Example 3.21 (A topological space with $\mathrm{ac}_{out} \neq \mathrm{ac}$). *Consider the set $X = \{1,2\}$ together with the topology $\{ \emptyset, \{1\}, \{1,2\} \}$. The closed sets of X are $\{\emptyset, \{2\}, \{1,2\}\}$. Obviously $ac_{out}(\{1\}) = \emptyset$ whereas we have $ac(\{1\}) = \{2\}$.*

We can, however, rectify the situation by taking the union with $\overline{M} \setminus M$ or by imposing stronger conditions on our space X.

Lemma 3.22 (Approximating the accumulation points from the outside under restrictions). *For a topological space X the following assertions hold.*

(a) *$ac(M) = \overline{M} \setminus M \cup ac_{out}(M)$ for all $M \subseteq X$.*

(b) *$ac(M) = ac_{out}(M)$ for all $M \subseteq X$ if and only if $(X, \mathscr{T})$ is T_1, i.e., if every set containing exactly one element is closed in X.*

Proof. (a) By Lemmas 3.19 and 3.20 we have to show that $\mathrm{ac}(M) \subseteq \overline{M} \setminus M \cup \mathrm{ac}_{\mathrm{out}}(M)$. It is enough to show that for $S \subsetneq M$ and $y \in \overline{S} \setminus S$ we have $y \in \overline{M} \setminus M \cup \mathrm{ac}_{\mathrm{out}}(M)$. For $y \notin M$ we have $y \in \overline{S} \setminus M \subseteq \overline{M} \setminus M$. On the other hand let $y \in M$ and let $a \in M$ be an arbitrary element. If $y \neq a$, then $y \in M \setminus \{a\} \subseteq \overline{M \setminus \{a\}}$. If $y = a$, then $y \in \overline{S} \subseteq \overline{M \setminus \{y\}} = \overline{M \setminus \{a\}}$, since $S \subseteq M \setminus \{y\}$. Therefore $y \in \bigcap_{a \in M} \overline{M \setminus \{a\}} = \mathrm{ac}_{\mathrm{out}}(M)$.

(b)
($\Leftarrow$) By (a) it is enough to show that $\overline{M} \setminus M \subseteq \overline{M \setminus \{x\}}$ for all $x \in M$. Let $y \in \overline{M} \setminus M$. Assume there was $x \in M$ and an open neighborhood U of y with $U \cap (M \setminus \{x\}) = \emptyset$. Then $U \setminus \{x\}$ is an open neighborhood of y, since $\{x\}$ is closed and we have $(U \setminus \{x\}) \cap M = U \cap (M \setminus \{x\}) = \emptyset$. This, however, contradicts $y \in \overline{M}$.

($\Rightarrow$) If $\mathrm{ac}(M) = \mathrm{ac}_{\mathrm{out}}(M)$, then by (a) we have $\overline{M} \setminus M \subseteq \overline{M \setminus \{x\}}$ for all $x \in M$. In particular we have $\overline{\{x\}} \setminus \{x\} \subseteq \overline{\{x\} \setminus \{x\}} = \emptyset$ and therefore $\overline{\{x\}} = \{x\}$ for all $x \in X$. Hence $(X, \mathscr{T})$ is T_1. □

As already mentioned, a better known structure similar to derivative algebras is that of (topological) closure algebras. These are Boolean algebras with one additional binary operation $^-$ fulfilling axioms (ii-a) - (ii-d). One of their useful properties is that they can be used to define topological spaces, see [KJ66].

To be more precise, let $\mathscr{D} = \mathscr{P}(X)$ be the power set algebra of some non-empty set X. Defining a set M as closed if $M^- = M$, we get a topology

$$\mathscr{T} = \left\{ M \subseteq X \;\middle|\; (X \setminus M)^- = X \setminus M \right\}.$$

Moreover, if we then proceed and define the operation of taking the topological closure

$$M^\sim := \bigcap \{ S \supseteq M \mid S \text{ is closed with respect to } \mathscr{T} \}$$

we get our original closure operation back in the sense that $M^\sim = M^-$ for all $M \subseteq X$.

Given a derivative algebra one can of course repeat this construction, since a derivative operation * gives rise to a closure operation $^-$ by (i-op). Thus, if $\mathscr{D} = \mathscr{P}(X)$ is the power set algebra of some non empty set X and $*$ is a derivative operation on X, we get a topology by defining

$$\mathscr{T}_* := \{ M \subseteq X \mid (X \setminus M)^* \subseteq X \setminus M \}.$$

Lemma 3.23. *Let X be a set. If $*$ is a derivative operation on $\mathscr{P}(X)$, i.e., a binary operation fulfilling axioms (i-a) - (i-c), then $\mathscr{T}_*$ is a topology on X.*

Proof. By Axiom (i-a), $\mathscr{T}_*$ contains X and the empty set. We have to show that $\mathscr{T}_*$ is closed under arbitrary unions and finite intersections, or equivalently, that $\tilde{\mathscr{T}}_* = \{X \setminus M \mid M \in \mathscr{T}_*\} = \{M \subseteq X \mid M^* \subseteq M\}$ is closed under arbitrary intersections and finite unions. First, let $M_i, i \in I$ be a family of subsets of X with $M_i^* \subseteq M_i$ for all $i \in I$. The inclusion $(\bigcap_{i\in I} M_i)^* \subseteq M_i^* \subseteq M_i$ holds for every $i \in I$ by Lemma 3.10, hence also $(\bigcap_{i\in I} M_i)^* \subseteq \bigcap_{i\in I} M_i$. Second, for every $M, N \in \tilde{\mathscr{T}}_*$ we have $(M \cup N)^* = M^* \cup N^* \subseteq M \cup N$ by Axiom (i-c). □

However, we can in general not hope to recover the operation * from this topology. For every derivative operation $*$, the operation $a \mapsto a^- := a^* \vee a$ is a closure operation and in particular again a derivative operation. Both of these derivative operations generate the same topology, i.e., $\mathscr{T}_* = \mathscr{T}_-$. Therefore there can be no universal way to recover $*$ from this topology.

In the case of T_1-spaces, however, we have the following theorem.

Theorem 3.24 (Topology from derivative algebras). *(a) For a T_1-space $(X, \mathscr{T})$ the map $M \mapsto ac(M)$ is a derivative operation.*

(b) If X is a set and $\mathscr{P}(X)$ together with * is a derivative algebra, such that

$$\forall x \in X : \{x\}^* = \emptyset,$$

then $(X, \mathscr{T}_*)$ is a T_1-space and we have $M^* = ac(M)$ for all $M \subseteq X$.

Proof. (a) Let $(X, \mathscr{T})$ be a T_1-space and let $M \subseteq X$. By Lemma 3.22 $\mathrm{ac}(M)$ is an intersection of closed sets and therefore closed. Moreover, we have

$$\mathrm{ac}_{\mathscr{T}}(\mathrm{ac}_{\mathscr{T}}(M)) = \bigcap_{a \in \mathrm{ac}(M)} \overline{\mathrm{ac}_{\mathscr{T}}(M) \setminus \{a\}} \subseteq \overline{\mathrm{ac}_{\mathscr{T}}(M)} = \mathrm{ac}_{\mathscr{T}}(M).$$

Therefore $M \mapsto \mathrm{ac}_{\mathscr{T}}(M)$ fulfills Axiom (i-b). That $\mathrm{ac}_{\mathscr{T}}(\emptyset) = \emptyset$ and that $M_1 \subseteq M_2$ implies $\mathrm{ac}_{\mathscr{T}}(M_1) \subseteq \mathrm{ac}_{\mathscr{T}}(M_2)$ follows by Lemma 3.19. This also gives us $\mathrm{ac}_{\mathscr{T}}(M_1 \cup M_2) \supseteq \mathrm{ac}_{\mathscr{T}}(M_1) \cup \mathrm{ac}_{\mathscr{T}}(M_2)$. For Axiom (i-c) to hold, it remains to show that $\mathrm{ac}_{\mathscr{T}}(M_1 \cup M_2) \subseteq \mathrm{ac}_{\mathscr{T}}(M_1) \cup \mathrm{ac}_{\mathscr{T}}(M_2)$. We can derive this as follows.

$$\begin{aligned}
\mathrm{ac}_{\mathscr{T}}(M_1 \cup M_2) &= \bigcup_{S \subseteq M_1 \cup M_2} \overline{S} \setminus S \\
&= \bigcup_{S_1 \subseteq M_1} \bigcup_{S_2 \subseteq M_2} \overline{S_1 \cup S_2} \cap (S_1 \cup S_2)^c \\
&= \bigcup_{S_1 \subseteq M_1} \bigcup_{S_2 \subseteq M_2} (\overline{S_1} \cup \overline{S_2}) \cap S_1^c \cap S_2^c \\
&\subseteq \left(\bigcup_{S_1 \subseteq M_1} \overline{S_1} \cap S_1^c \right) \cup \left(\bigcup_{S_2 \subseteq M_2} \overline{S_2} \cap S_2^c \right) \\
&= \mathrm{ac}(M_1) \cup \mathrm{ac}(M_2).
\end{aligned}$$

(b) Let $\mathscr{P}(X)$ together with * be a derivative algebra such that for all $x \in X$ the equality $\{x\}^* = \emptyset$ holds. If X is endowed with the topology $\mathscr{T}_*$, a set M is closed, if $M^* \subseteq M$. Hence

$$\overline{\{x\}} = \bigcap \{ M \subseteq X \mid x \in M, M \text{ is closed} \} = \bigcap \{ M \subseteq X \mid x \in M, M^* \subseteq M \} = \{x\},$$

so $(X, \mathscr{T}_*)$ is a T_1-space.

Let B be an arbitrary subset of X. For any superset $C \supseteq B$ with $C^* \subseteq C$ we have $B^* \subseteq C^* \subseteq C$ and thus $B \cup B^* \subseteq C$. Together with $(B^* \cup B)^* = B^{**} \cup B^* = B^*$ we have

$$\begin{aligned}
\overline{B} &= \bigcap \{ S \subseteq X \mid B \subseteq S, S^* \subseteq S \} \\
&= B^* \cup B, \\
B^* &= ((B \setminus \{x\}) \cup \{x\})^* \\
&= (B \setminus \{x\})^* \cup \{x\}^* \\
&= (B \setminus \{x\})^*.
\end{aligned}$$

Since $(X, \mathscr{T}_*)$ is a T_1-space, we can apply Lemma 3.22.

$$\begin{aligned}
\mathrm{ac}(B) &= \bigcap_{a \in B} \overline{B \setminus \{a\}} \\
&= \bigcap_{a \in B} (B \setminus \{ a \})^* \cup (B \setminus \{a\}) \\
&= \bigcap_{a \in B} B^* \cup (B \setminus \{a\}) \\
&= B^* \cup \bigcap_{a \in B} (B \setminus \{a\}) \\
&= B^*.
\end{aligned}$$

□

Definition 3.25 (Derivative algebra of a topological space)**.** *Let $(X, \mathscr{T})$ be a T_1-space. The derivative algebra obtained by endowing the Boolean algebra $\mathscr{P}(X)$ with the derivative operation $M \mapsto ac_{\mathscr{T}}(M)$ is denoted by $\mathscr{D}(X)$. We call $\mathscr{D}(X)$* the derivative algebra generated by $(X, \mathscr{T})$. *Whenever we encounter a subset M of a T_1 $(X, \mathscr{T})$ we denote by $M^* = ac_{\mathscr{T}}(M)$ the derivative in the derivative algebra $\mathscr{D}(X)$, if not mentioned otherwise.*

Notice that for a non T_1-space X the map $M \mapsto \mathrm{ac}(M)$ is in general not a derivative operation, since Axiom (i-b) does not necessarily hold as the following example shows.

Example 3.26. *Consider the space $\{1, 2\}$ with the trivial topology $\{\emptyset, \{1, 2\}\}$. Then $\overline{\{1\}} = \overline{\{2\}} = \{1, 2\}$, hence $ac(\{1\}) = \{2\}$ and $ac(ac(\{1\})) = ac(\{2\}) = \{1\} \not\subseteq ac(\{1\})$.*

In the light of the close connection between derivative algebras and topology we make the following definition.

Definition 3.27 (Closed element)**.** *An element a of a derivative algebra $\mathscr{D}$ is called* closed*, if $a^* \leq a$. It is called* open *if a^c is closed.*

While the operation of taking the closure of a set behaves reasonably nice under continuous mappings, the same is not true for the operation of taking the accumulation points. For example for the map $f : \mathbb{R} \to \mathbb{R}$ that maps every element to zero, we have $f[\mathrm{ac}(\mathbb{R})] = f[\mathbb{R}] = \{0\} \not\subseteq \mathrm{ac}(f[\mathbb{R}]) = \mathrm{ac}(\{0\}) = \emptyset$.

The following lemma, however, shows that taking the derivative commutes with homeomorphisms.

Lemma 3.28. *Let $f : X \to Y$ be a homeomorphism between topological spaces X and Y. Then $f[ac(M)] = ac(f[M])$.*

Proof. A simple calculation based on Lemma 3.19 shows

$$\begin{aligned} f[\mathrm{ac}(M)] &= f[\bigcup_{S\subseteq M} \overline{S} \setminus S] \\ &= \bigcup_{S\subseteq M} \overline{f[S]} \setminus f[S] \\ &= \bigcup_{Q\subseteq f[M]} \overline{Q} \setminus Q \\ &= \mathrm{ac}(f[M]). \end{aligned}$$

□

3.4. Derivative Algebras and the Order on their Atoms

Since a finite Boolean algebra is generated by its atoms by Lemma 3.5, it is nice to have a characterization, when a set of elements of a Boolean algebra forms the atom set of a derivative subalgebra. Such a characterization is provided by the next lemma.

Lemma 3.29 (Atoms of derivative subalgebras)**.** *Let $\mathscr{D}$ be a finite derivative algebra. A non-empty set $T \subseteq \mathscr{D}$ of disjoint elements is the set of atoms of a derivative subalgebra of $\mathscr{D}$ if and only if for every element $a \in T$ the derivative a^* is the join of elements of T.*

Proof. If T is the set of atoms of a derivative subalgebra of $\mathscr{D}$, then for every $a \in T$ the derivative a^* must also lie in T and is therefore the join of elements of T.

On the other hand let $\mathscr{E}$ be the Boolean subalgebra of $\mathscr{D}$ generated by T. Every element $b \in \mathscr{E}$ can be represented as the join of elements $a_1, \dots, a_n \in T$. Therefore $b^* = a_1^* \vee \cdots \vee a_n^*$ and every element on the right-hand-side is the join of elements of T, hence $b^* \in \mathscr{E}$. This shows, that $\mathscr{E}$ is derivative subalgebra of $\mathscr{D}$, whose set of atoms is T. □

We will capture the structure of our derivative algebra by a relation on its atoms.

Definition 3.30 (An order on the atoms)**.** *Let $\mathscr{D}$ be a finite derivative algebra and let T be its set of atoms. We define a binary relation $\preceq$ on T by*

$$a \preceq b :\Longleftrightarrow (a \leq b^* \text{ or } a = b). \tag{3.11}$$

In the cases where we will apply our theory, namely finite derivative algebras generated by their closed elements, this relation is a partial order. In Theorem 3.33 we will see that this order together with some cardinality data characterizes the derivative algebra up to homomorphism.

Lemma 3.31 (Order on the atoms for finite derivative algebras generated by their closed elements)**.** *For any derivative algebra $\mathscr{D}$ the relation $\preceq$ is a quasiorder, i.e., it is reflexive and transitive. If $\mathscr{D}$ is finite and generated by its closed elements, $\preceq$ is a (partial) order.*

Proof. Since $\preceq$ is reflexive by definition, we only have to show its transitivity. For this let $a \preceq b$ and $b \preceq c$. Then either $a = b$ or $b = c$, in which case we have $a \preceq c$ immediately, or $a \leq b^*$ and $b \leq c^*$. But then $a \leq b^* \leq c^{**} \leq c^*$.

For the second part assume that $\mathscr{D}$ is generated by its closed elements. The quasiorder $\preceq$ defines an equivalence relation $\equiv$ by

$$a \equiv b :\Longleftrightarrow a \preceq b \text{ and } b \preceq a$$

The equivalence class of $a \in D$ under $\equiv$ is denoted by $[a]_\equiv$. Notice that for $a \equiv b$ we either have $a = b$ and thus $a^* = b^*$ or we have $a^* \leq b^{**} \leq b^*$ and $b^* \leq a^{**} \leq a^*$, hence also $a^* = b^*$.

Let J be the set of atoms of $\mathscr{D}$. Consider the Boolean subalgebra $\mathscr{E}$ of $\mathscr{D}$ with atom set I obtained by merging all atoms equivalent under $\equiv$, that is

$$I = \left\{ \bigvee [a]_\equiv \;\middle|\; a \in J \right\}.$$

Since the equivalence classes of $\equiv$ form a partition of J, this is the atom set of a Boolean algebra. To see that it is also the atom set of a derivative algebra by Lemma 3.29 we have to show that for any atom $b \in I$ the derivative b^* is the join of atoms in I. For $b \in I$ there is $a \in J$ with $b = \bigvee [a]_\equiv$. As shown above, $b^* = \bigvee \{ c^* \in I \mid c \equiv a \} = a^*$. Now there are $c_1, \ldots, c_k \in J$ with $a^* = c_1 \vee \cdots \vee c_k$. For an atom $d \in J$ with $d \equiv c_i, i = 1, \ldots, k$ we have, however, that either $d = c_i$ or $d \leq c_1^* \leq a^{**} \leq a^*$.

On I we can also define the relation $\preceq$ by (3.11). This relation on I is clearly an order, since it is antisymmetric by the definition of I. It remains to show that all closed elements in $\mathscr{D}$ are also contained in $\mathscr{E}$. Because $\mathscr{D}$ is generated by its closed elements this implies $\mathscr{D} = \mathscr{E}$ and in particular that $I = J$. Let b be a closed element of $\mathscr{D}$. We have to show that for each atom $a \in J$ with $a \leq b$ we also have $\bigvee [a]_\equiv \leq b$. But for $c \equiv a$ we either have $a = c$ or we have $c \leq a^* \leq b^* \leq b$, since b is closed. $\square$

Lemma 3.32. *Let $\mathscr{B}$ be a finite derivative algebra generated by its closed elements. Let $T_\mathscr{B}$ be the atoms of $\mathscr{B}$ order by $\preceq$ defined in Definition 3.30. Let $a \in T_\mathscr{B}$ be an atom. If $a \npreceq a^*$, the equation*

$$a^* = \bigvee \{ b \in T_\mathscr{B} \mid b \preceq a \}$$

holds and if $a \leq a^*$, the equation

$$a^* = \bigvee \{ b \in T_{\mathscr{B}} \mid b \preceq a, a \neq b \}$$

holds.

Proof. Let $a \in T_{\mathscr{B}}$ be an atom. Since $\mathscr{B}$ is finite, we can represent a^* as the join of atoms $b_1, \ldots, b_n$ and hence $a^* = \bigvee \{ b \in T_{\mathscr{B}} \mid b \leq a^* \}$. If $a \leq a^*$, then $b \leq a^*$ if and only if $b \preceq a$. If on the other hand $a \not\leq a^*$, then $b \leq a^*$ if and only if $b \preceq a$ and $b \neq a$. □

Theorem 3.33 (Equivalence of derivative algebras and the order on their atoms)**.** *Let $\mathscr{B}$ and $\mathscr{C}$ be derivative algebras generated by their closed elements and let $(T_{\mathscr{B}}, \preceq)$ and $(T_{\mathscr{C}}, \preceq)$ be the corresponding atom sets together with the order $\preceq$ defined in Definition 3.30. Let $S_{\mathscr{B}} \subseteq T_{\mathscr{B}}$ and $S_{\mathscr{C}} \subseteq T_{\mathscr{B}}$ be the sets of those atoms a of $\mathscr{B}$ and $\mathscr{C}$, respectively, for which $a \leq a^*$. If $\mathscr{B}$ and $\mathscr{C}$ are finite, then every isomorphism of derivative algebras $\rho : \mathscr{B} \to \mathscr{C}$ induces an isomorphism of ordered sets from $T_{\mathscr{B}}$ to $T_{\mathscr{C}}$ and every isomorphism of ordered sets $\tau : T_{\mathscr{B}} \to T_{\mathscr{C}}$ with $\tau[S_{\mathscr{B}}] = S_{\mathscr{C}}$ extends to an isomorphism of derivative algebras from $\mathscr{B}$ to $\mathscr{C}$.*

Proof. Let $\rho : \mathscr{B} \to \mathscr{C}$ be an isomorphism of derivative algebras. For $a, b \in T_{\mathscr{B}}$ with $a \preceq b$ we either have $a = b$ implying $\rho(a) = \rho(b)$ or $a \leq b^*$ implying $\rho(a) \leq \rho(b)^*$ and hence $\rho(a) \preceq \rho(b)$.

For the other direction let $\tau : T_{\mathscr{B}} \to T_{\mathscr{C}}$ be an isomorphism of ordered sets with $\tau[S_{\mathscr{B}}] = S_{\mathscr{C}}$. As a bijective map between the atoms of $\mathscr{B}$ and $\mathscr{C}$, there is a unique extension of τ to an isomorphism of Boolean algebras ρ. More precisely, every element $a \in \mathscr{B}$ can be uniquely represented as the union of atoms $a_1, \ldots, a_k$ and then $\rho(a) = \tau(a_1) \vee \ldots \tau(a_n)$.

For an atom $a \notin S_{\mathscr{B}}$ also $\tau(a) \notin S_{\mathscr{C}}$ and hence by Lemma 3.32

$$\begin{aligned}
\rho(a^*) &= \rho(\bigvee \{ b \in T_{\mathscr{B}} \mid b \preceq a \}) \\
&= \bigvee \{ \tau(b) \mid b \in T_{\mathscr{B}}, b \preceq a \} \\
&= \bigvee \{ c \mid c \in T_{\mathscr{C}}, c \preceq \tau(a) \} \\
&= \tau(a).
\end{aligned}$$

For $a \in S_{\mathscr{B}}$ also $\tau(a) \in S_{\mathscr{C}}$ and $\rho(a^*) = \bigvee \{ \tau(b) \mid b \in T_{\mathscr{B}}, b \preceq a, b \neq a \} = \tau(a)$. □

3.5. Properties of Derivative Algebras

In this section we prove some properties of derivative algebras which we use in the next chapter. All of the remaining results in this section appear in [Pie72] but are formulated there in terms of topological Boolean algebras instead of derivative algebras.

Lemma 3.34. *Let $\mathscr{D}$ be a derivative algebra and let $a, b \in \mathscr{D}$. The following implications hold.*

(a) *If $a \geq 1^*$, then a is closed.*

(b) *If a is open, $b \leq a$ and $(a \cap b^c)^* \cap a \leq a \cap b^c$, then b is open.*

Proof. (a) $a^* \leq 1^* \leq a$, hence a is closed.

(b) This is again a simple algebraic manipulation.

$$\begin{aligned}(a^c \cap b^c)^* = a^{c*} \leq a^c \leq b^c,\\ b^{c*} = (a^c \cap b^c)^* \cup (a \cap b^c)^* \leq b^c.\end{aligned}$$ □

A subalgebra of a derivative algebra $\mathscr{D}$ always has the same maximal element as $\mathscr{D}$. Taking the set of all elements less than or equal to a given element $b \in \mathscr{D}$ does therefore not even give a Boolean subalgebra of $\mathscr{D}$. However, with appropriately defined operations this set is again a derivative algebra.

Definition 3.35 (Restriction of a derivative algebra)**.** *Let $\mathscr{D}$ be a derivative algebra and let $a \in \mathscr{D}$. The* restriction of $\mathscr{D}$ to a *is defined by*

$$\mathscr{D} \wedge a := \{ c \wedge a \mid c \in \mathscr{D} \} = \{ c \in \mathscr{D} \mid c \leq a \}.$$

Lemma 3.36. *For any derivative algebra $\mathscr{D}$ and any element a the restriction $\mathscr{E} := \mathscr{D} \wedge a$ of $\mathscr{D}$ to a is again a derivative algebra where the Boolean operations are given by*

$$\begin{aligned}c \vee_{\mathscr{E}} d &:= (c \vee d) \wedge a = (c \vee d),\\ c \wedge_{\mathscr{E}} d &:= c \wedge d \wedge a = (c \wedge d),\\ c^{c_{\mathscr{E}}} &:= c^c \wedge a.\end{aligned}$$

The constants are

$$\begin{aligned}0_{\mathscr{E}} &:= 0_{\mathscr{D}},\\ 1_{\mathscr{E}} &:= a.\end{aligned}$$

The derivative operation is given by

$$c^{*_{\mathscr{E}}} := c^* \wedge a.$$

Proof. Checking that $\mathscr{E}$ is a Boolean algebra is routine, see for example Chapter 12 in [GH09]. The three additional axioms of a derivative algebra are also fulfilled, since for

$b, c \in \mathscr{D}$

$$\begin{aligned} 0_{\mathscr{E}}^{*_{\mathscr{E}}} &= 0_{\mathscr{D}} \wedge a = 0_{\mathscr{E}}, \\ b^{*_{\mathscr{E}}*_{\mathscr{E}}} &= (b^* \wedge a)^* \wedge a \\ &\leq b^{**} \wedge a \\ &\leq b^* \wedge a = b^{*_{\mathscr{E}}}, \\ (b \vee c)^{*_{\mathscr{E}}} &= (b \vee c)^* \wedge a \\ &= (b^* \wedge a) \vee (c^* \wedge a) = b^{*_{\mathscr{E}}} \vee c^{*_{\mathscr{E}}}. \end{aligned}$$ □

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Some of the results on derivative algebras depend on certain generators. In order to carry out inductive proofs on finite derivative algebras, we need information of how generating sets behave under restriction. While for Boolean algebras, the meet with an element always gives a homomorphism onto the restriction to this element, this is not always true for derivative algebras.

Lemma 3.37. *If $\mathscr{D}$ is a derivative algebra and $a \in \mathscr{D}$ is open, then*

$$\rho : \mathscr{D} \to \mathscr{D} \wedge a, \quad b \mapsto a \wedge b$$

is a homomorphism and $M \wedge a = \rho[M]$ is a generating set of $\mathscr{D} \wedge a$ for any generating set M of $\mathscr{D}$.

Proof. Clearly ρ is a homomorphism of Boolean algebras. Since a is open, $(a^c)^* \leq a^c$. Hence for $b \in \mathscr{D}$

$$\begin{aligned} \rho(b^*) &= b^* \wedge a \\ &= ((b \wedge a)^* \vee (b \wedge a^c)^*) \wedge a \\ &= (b \wedge a)^* \wedge a \\ &= \rho(b)^{*_{\mathscr{E}}}. \end{aligned}$$

This shows that $\mathscr{D}$ is also a derivative algebra. That the image of a generating set under an algebra-homomorphism is again a generating set, holds for all algebras in the sense of universal algebra. To prove this let $\mathscr{E}$ be the subalgebra generated by $\rho[M]$. Its preimage under ρ is again a subalgebra, see for example Theorem 3 in the classical book by Birkhoff [Bir67]. Since M is contained in this algebra and generates $\mathscr{D}$, we have $\rho^{-1}[\mathscr{E}] = \mathscr{D}$ and $\mathscr{E} = \rho[\mathscr{D}]$. □

While the above result does not hold for closed a, the following result will be sufficient for our applications.

Lemma 3.38. *Let $\mathscr{D}$ be a derivative algebra and let M be the set of its closed elements. Assume that $\mathscr{D}$ is generated by M. For a closed set $a \in M$ denote by $\mathscr{D}_a$ the subalgebra of $\mathscr{D} \wedge a$ generated by $M_a := \{\, m \wedge a \mid m \in M \,\}$ and by $\mathscr{D}_{a^c}$ the subalgebra of $\mathscr{D} \wedge a^c$ generated by $M_{a^c} := \{\, m \wedge a^c \mid m \in M \,\}$. Then $\mathscr{D} = \mathscr{D}_a \oplus \mathscr{D}_{a^c}$, that is, each element in $\mathscr{D}$ has a unique* *representation as the disjoint join of elements in $\mathscr{D}_a$ and $\mathscr{D}_{a^c}$. In particular $\mathscr{D} \wedge a = \mathscr{D}_a$ is generated by M_a, a set consisting of closed elements.*

Proof. Since $\mathscr{D} \wedge a$ and $\mathscr{D} \wedge a^c$ are contained in $\mathscr{D}$, so are $\mathscr{D}_a$ and $\mathscr{D}_{a^c}$. It remains to show that $\mathscr{D} \subseteq \mathscr{D}_a \oplus \mathscr{D}_{a^c}$. To do so, we have to show that the right hand side is a subalgebra of $\mathscr{D}$ containing M. For each $m \in M$ we have $m \wedge a \in \mathscr{D}_a$ and $m \wedge a^c \in \mathscr{D}_{a^c}$, hence $M \subseteq \mathscr{D}_a \oplus \mathscr{D}_{a^c}$. Clearly $\mathscr{D}_a \oplus \mathscr{D}_{a^c}$ is a Boolean subalgebra of $\mathscr{D}$. Finally, for $b_1 \in \mathscr{D}_a$ and $b_2 \in \mathscr{D}_{a^c}$ consider $(b_1 \vee b_2)^* = b_1^* \vee b_2^* c = (b_1^* \wedge a) \vee (b_1^* \wedge a^c) \vee (b_2^* \wedge a) \vee (b_2^* \wedge a^c)$. Since b_1^* and b_2^* are closed, they are contained in M and hence $(b_1 \vee b_2)^*$ is contained in $\mathscr{D}_a \oplus \mathscr{D}_{a^c}$, which therefore is a derivative subalgebra of $\mathscr{D}$. □

Lemma 3.39 (Open atom in finite derivative algebras)**.** *Every finite derivative algebra $\mathscr{D}$ which is generated by its closed elements contains an open atom.*

Proof. Let I be the set of atoms of $\mathscr{D}$. Consider a maximal element $a \in I$ with respect to the order $\preceq$ on I introduced in Definition 3.30. Then $(a^c)^*$ is the join of atoms in I by Lemma 3.5. But a can not be among those atoms, since otherwise there would be an atom $b \leq a^c$ such that $a \leq b^*$, i.e., $a \prec b$, contradicting the maximality of a. Therefore $(a^c)^* \leq a^c$, in other words, a^c is closed and a is open. □

Chapter 4.

Topological Conjugacy of Periodic Cellular Automata

Restricting any cellular automaton to its p-periodic points gives rise to a p-periodic cellular automaton. For shifts these are rather boring finite invertible systems which are completely described up to conjugacy by the number of their q-periodic orbits for $q \in \{1, \dots, p\}$. It is therefore easy to decide if the restrictions of two shifts to their p-periodic points are conjugate. We already saw that for cellular automata the structure of their p-periodic points might be significantly more complex. In this chapter we will analyze p-periodic dynamical systems on metrizable Stone spaces and show that in the case of systems with a finite derivative algebra, conjugacy is equivalent to a natural necessary condition on the q-periodic points for $q \in \{1, \dots, p\}$.

4.1. The Cantor-Bendixson Rank

While examining the structure of metrizable Stone spaces, it is often necessary to repeatedly take the derivative of a subset. Sometimes one even wants to do this α-times where α is any ordinal number. The definition of this operation using transfinite induction is straightforward. For any set M in a T_1 space and α an ordinal number define

$$M^{(\alpha+1)} := (M^{(\alpha)})^* = \mathrm{ac}(M^{(\alpha)}) \text{ for any successor ordinal } \alpha + 1,$$
$$M^{(\alpha)} := \bigcap_{\beta<\alpha} M^{(\beta)} \text{ for any limit ordinal } \alpha.$$

All results in this section are classics in set theory and can be found for example in Chapter 4 in the book by Jech [Jec03]. By Corollary 2.10 we know that all metrizable Stone spaces are homeomorphic to closed subsets of the reals. Hence we will only formulate the results for this situation.

Lemma 4.1. *For every subset $X \subseteq \mathbb{R}$ there is a minimal ordinal number $\alpha =:$ rank(X), called its* Cantor-Bendixson rank*, such that $X^{(\alpha)} = X^{(\alpha+1)}$.*

Proof. We proved in Theorem 3.24 that $M \mapsto \mathrm{ac}(M) = M^*$ is a derivative operation. In particular $M^{**} \subseteq M^*$ for all $M \subseteq X$, hence $X \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \cdots \supseteq X^{(\omega_0)} \supseteq X^{(\omega_0+1)} \supseteq \dots$.

Assume we would have $X^{(\alpha)} \neq X^{(\alpha+1)}$ for all ordinals α. Then $X^{(\alpha+1)} \subseteq X$, α an ordinal, would be pairwise different sets, giving us a surjective map from a subset of the powerset of X onto the ordinal numbers. By the axiom of replacement, the ordinals would form a set. But this is false, see [Jec03, Chap. 2] . Therefore there exists an ordinal number α with $X^{(\alpha)} = X^{(\alpha+1)}$ and there is also a minimal one, since the ordinals are well-ordered [Jec03, Chap. 2]. $\square$

Definition 4.2. *For a topological space X denote by* $\mathrm{Iso}(X) := X \setminus X^*$ *the isolated points of X. Denote by* $\mathrm{Dense}(X) := \bigcap_{\alpha \in Ord} X^{(\alpha)} = X^{(\mathrm{rank}(X))}$ *the perfect part of X.*

Example 4.3. *Consider the space $I := \{ 0 \} \cup \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$. All points in I except 0 are isolated and we have $I^{(1)} = \{0\}$ and $I^{(2)} = \emptyset$. The Cantor-Bendixson rank of I is therefore 2. For $k \in \mathbb{N}$ consider the set $Z_k := I^k \subseteq \mathbb{R}^k$. See Figure 4.1 for an illustration of Z_2. A point x is contained in the ℓ-th derivative of Z_k if and only it is zero in at least ℓ components, i.e., for $x \in Z_k$ we have*

$$x \in Z_k^{(\ell)} \iff |\{i \in \{1,\ldots,k\} \mid x_i = 0\}| \geq \ell.$$

In particular the space Z_k has Cantor-Bendixson rank $k+1$, as $Z_k^{(k)} = \{ (0,\ldots,0) \}$ and $Z_k^{(k+1)} = \emptyset = Z_k^{(k+2)}$. Therefore $\mathrm{Dense}(Z_k) = \emptyset$ and $\mathrm{Iso}(Z_k) = I \setminus \{0\}^k$. To get a space with the same rank as Z_k and non-empty perfect part, we glue an interval to the point $(0,\ldots,0)$ and consider the space $\tilde{Z}_k := Z_k \cup [-1,0] \times \{0\}^{k-1}$. We get

$$\begin{aligned}\tilde{Z}_k^{(\ell)} &= Z_k^{(\ell)} \cup \{0\} \times [-1,0] \times \{ 0 \}^{k-1}, \\ \mathrm{Dense}(\tilde{Z}_k) &= [-1,0] \times \{0\}^{k-1}.\end{aligned}$$

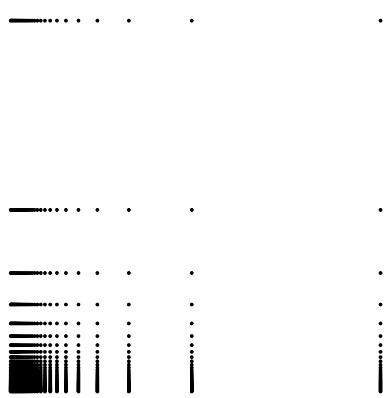


Figure 4.1.: The space Z_2 from Example 4.3.

Lemma 4.4. *For every closed subset $X \subseteq \mathbb{R}$ the set* $\mathrm{Dense}(X)$ *is either empty or a perfect subset of $\mathbb{R}$. Every perfect subset of X is contained in* $\mathrm{Dense}(X)$.

Proof. By the definition of the Cantor-Bendixson rank, we have $\mathrm{Dense}(X) = X^{(\mathrm{rank}(X))} = \mathrm{Dense}(X)^*$. If $M \subseteq X$ is perfect, we have $M^* \supseteq M$ and therefore $\mathrm{Dense}(X) = X^{(\mathrm{rank}(X))} \supseteq M^{(\mathrm{rank}(X))} \supseteq M$. $\square$

Theorem 4.5 (Cantor-Bendixson)**.** *For every subset $X \subseteq \mathbb{R}$ the set $X \setminus \mathrm{Dense}(X)$ is at most countable.*

Proof. Let $\{J_k \mid k \in \mathbb{N}\}$ be a countable base of the topology of X. Consider $x \in X \setminus \mathrm{Dense}(X)$. Since $\mathrm{Dense}(X) = \bigcap_{\alpha \in \mathrm{Ord}} X^{(\alpha)}$, there must be a minimal ordinal α such that $x \notin X^{(\alpha)}$. It must be a successor ordinal, for otherwise $x \in \bigcap_{\beta \leq \alpha} X^{(\beta)} = X^{(\alpha)}$. Hence there is an ordinal $\beta =: \beta_x$ such that $\alpha = \beta + 1$ and $x \in X^{(\beta)}$ but $x \notin (X^{(\beta)})^*$. In other words, x is an isolated point of $X^{(\beta)}$ and there exists a minimal index $k =: k_x \in \mathbb{N}$ such that $X^{(\beta)} \cap J_{k_x} = \{x\}$. We want to show that the mapping $\varphi : X \setminus \mathrm{Dense}(X) \to \mathbb{N}$, $x \mapsto k_X$ is injective, because then $X \setminus \mathrm{Dense}(X)$ must be countable. Let $x, y \in X \setminus \mathrm{Dense}(X)$ be two different points such that $\beta_x \leq \beta_y$. Then $J_{k_x} \cap X^{(\beta_y)} \subseteq J_{k_x} \cap X^{(\beta_x)} = \{x\}$, hence $y \notin J_{k_x} \cap X^{(\beta_y)}$ and $k_x \neq k_y$. □

Lemma 4.6. *Let $[a, b] \subseteq \mathbb{R}$ be a compact interval and let $X \subseteq [a, b]$ be a perfect subset. For every $\varepsilon > 0$ there exist disjoint intervals $[c_1, d_1] \subseteq [a, b]$ and $[c_2, d_2] \subseteq [a, b]$ of diameter at most ε such that $X \cap [c_1, d_1]$ and $X \cap [c_2, d_2]$ are perfect.*

Proof. Since X is perfect, there are at least two points y, z in $X \cap (a, b)$. There is $\delta < \varepsilon$ such that $[y - \delta, y + \delta] \subseteq [a, b]$ and $[z - \delta, z + \delta] \subseteq [a, b]$ and such that $[y - \delta, y + \delta] \cap [z - \delta, z + \delta] = \emptyset$. If one of the interval borders is an isolated point of $[y - \delta, y + \delta]$, we can slightly shrink the interval such that only this point is removed. The same holds for $[z - \delta, z + \delta]$. That way we obtain two disjoint intervals $[c_1, d_1]$ and $[c_2, d_2]$ without isolated points which are non-empty because they contain y and z, respectively. □

Theorem 4.7. *If X is a perfect subset of $\mathbb{R}$, then $|X| = |\mathbb{R}|$.*

Proof. By Lemma 4.6 we can construct a family of non-empty compact intervals $I_w, w \in \bigcup_{n=1}^{\infty} \{0, 1\}^n$ such that $X \cap I_w \neq \emptyset$ is perfect, $I_{w1} \subseteq I_w, I_{w0} \subseteq I_w$ and $I_{w0} \cap I_{w1} = \emptyset$ for all $w \in \bigcup_{n=1}^{\infty} \{0, 1\}^n$. Then $\varphi : \{0, 1\}^{\mathbb{N}} \to X, \varphi(x) := \bigcup_{\ell \in \mathbb{N}} I_{x_{[1,\ell]}}$ is an injective map, showing $|\{0, 1\}^{\mathbb{N}}| = |\mathbb{R}| \leq |X| \leq |\mathbb{R}|$. □

Corollary 4.8. *A compact topological space which is homeomorphic to a subset of $\mathbb{R}$ is either at most countable or has the same cardinality as $\mathbb{R}$.*

Theorem 4.9. *Let X be a compact subset of $\mathbb{R}$. If $Y \subseteq X$ is non-empty, open and does not contain any isolated point, then $|Y| = |\mathbb{R}|$.*

Proof. Let y be an arbitrary point of Y. Because Y is open, we can find a small compact interval M around y such that $X \cap M$ is contained in Y. Since Y has no isolated points, the only points in $X \cap M$ which can be isolated are the two boundary points of M. If one of them is isolated, we can make M slightly smaller on this side so as to exclude only this point. After this modification $X \cap M$ is a perfect subset of $\mathbb{R}$ whose cardinality equals that of $\mathbb{R}$ by Theorem 4.7. Since M is contained in Y, $|Y| = |\mathbb{R}|$, too. □

4.2. Extension of Homeomorphisms

Before we deal with conjugacies of dynamical systems on metrizable Stone spaces, we first have to clarify when two such spaces are homeomorphic. As Pierce showed in [Pie70] and [Pie72], derivative algebras are a very helpful tool to answer this question. In particular he showed that if the derivative algebra of a space is finite, then this space is composed of pretty simple building blocks and we can inductively build a homeomorphism by defining extensions along these building blocks. The presentation in this section follows the first two chapters in the small book of Pierce [Pie72]. The difference is that we use derivative algebras instead of topological Boolean algebras. Furthermore, there are many details omitted in Pierce's book, some of which can be found in [Pie70].

Theorem 4.10 (Extension of homeomorphisms)**.** *Let X, Y be metrizable Stone spaces. Let $M \subseteq X$ and $N \subseteq Y$ be closed sets such that either*

(a) $(X \setminus M)^* \cap (X \setminus M) = (Y \setminus N)^* \cap (Y \setminus N) = \emptyset$ *(i.e. $X \setminus M$ and $Y \setminus N$ both consist of isolated points) and $|X \setminus M| = |Y \setminus N|$, or*

(b) $(X \setminus M)^* = (X \setminus M)$ *and* $(Y \setminus N)^* = (Y \setminus N)$ *(i.e. $X \setminus M$ and $Y \setminus N$ are both perfect).*

If $\theta : M \to N$ is a homeomorphism for which $\theta[M \cap (X \setminus M)^*] = N \cap (Y \setminus N)^*$ holds, then θ can be extended to a homeomorphism from X to Y.

To prepare for the proof of this theorem we prove some lemmas.

Lemma 4.11. *On every Cantor space X, there exists a compatible metric d, such that for every $x \in X$ the map $y \mapsto d(x, y)$ is injective.*

Proof. Since all Cantor spaces are homeomorphic, if suffices to show the existence of one metric space with this property that is compact, perfect and zero-dimensional. Consider the set $\{0,1\}^{\mathbb{N}}$ with the metric

$$d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{3^n} \delta(x_n, y_n) \text{ where } \delta(a, b) = \begin{cases} 0 & \text{if } a = b \\ 1 & \text{otherwise} \end{cases}.$$

This metric generates the product topology on $\{0,1\}^{\mathbb{N}}$. We will show that $x \mapsto d(x, 0)$ is injective. For $d(x, 0) = d(y, 0)$ we have

$$0 = \sum_{n=1}^{\infty} \frac{1}{3^n}(x_n - y_n).$$

Assume that $x \neq y$ and let $\ell \in \mathbb{N}$ be the smallest index such that $x_\ell \neq y_\ell$. Without loss of generality we may assume $x_\ell = 0$ and $y_\ell = 1$. Then $0 = \frac{1}{3^\ell}(-1 + \sum_{n=\ell+1}^{\infty} \frac{1}{3^n}(x_n - y_n))$ and

$$1 = \sum_{n=\ell+1}^{\infty} \frac{1}{3^n}(x_n - y_n) \leq \sum_{n=1}^{\infty} \frac{1}{3^n}|x_n - y_n| \leq \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}.$$

Since this is a contradiction, we have $x = y$.

Furthermore it holds that for every $x, y \in X$ we have $d(x, y) = d(0, x \oplus y)$ where $\oplus$ is coordinate wise addition in $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. Thus $d(x, y) = d(x, z)$ implies $x \oplus y = x \oplus z$, hence $y \oplus z = 0$ and finally $y = z$. □

Lemma 4.12. *Let X be a Cantor space with metric d such that for every $x \in X$ the map $y \mapsto d(x, y)$ is injective. If M is a closed subset of X, then the map $\Phi : X \to M$ defined by $\Phi(x) := argmin_{y \in M} d(x, y)$ is a retraction of X onto M, i.e., it is continuous and surjective and it fixes all points in M.*

Proof. The equality $\Phi_{|M} = \mathrm{id}_M$ holds by definition. To show that Φ is continuous, let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to $a \in X$ and assume that $(\Phi(x_n))_{n\in\mathbb{N}}$ does not converge to $\Phi(a)$. Since X is compact, we may choose a subsequence and assume that $(\Phi(x_n))_{n\in\mathbb{N}}$ converges to $b \neq \Phi(a)$. By the definition of Φ, we have $d(a, b) \geq d(a, \Phi(a))$. Since $y \mapsto d(x, y)$ is injective, even $d(a, b) > d(a, \Phi(a))$ holds. We have $d(x_n, \Phi(x_n)) - d(a, x_n) \to d(a, b)$ as $n \to \infty$. Hence for sufficiently large $n \in \mathbb{N}$ the inequality $d(x_n, \Phi(x_n)) > d(a, x_n) + d(a, \Phi(a)) \geq d(x_n, \Phi(a)) \geq d(x_n, \Phi(x_n))$ holds. This contradiction shows that Φ is continuous. □

Lemma 4.13. *Let (X, d) be a compact metric space. If $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ are two sequences such that every point in X is a cluster point of both (x_n) and (y_n), then there is a bijection $\tau : \mathbb{N} \to \mathbb{N}$ such that*

$$d(x_n, y_{\tau(n)}) \to 0 \text{ as } n \to \infty.$$

Proof. Since X is compact, there is a sequence $(B_n)_{n\in\mathbb{N}}$ of non-empty balls of radius r_n in X such that $(r_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence tending to 0 and such that every point in X is contained in infinitely many of these balls.

Define two maps from $\mathbb{N}$ to itself by

$$\begin{aligned}
\pi_1(n) &:= \min\{\, \ell \in \mathbb{N} \mid x_\ell \in B_n \text{ and } \ell \notin \pi_1[\{1, \dots, n-1\}]\,\},\\
\pi_2(n) &:= \min\{\, \ell \in \mathbb{N} \mid y_\ell \in B_n \text{ and } \ell \notin \pi_2[\{1, \dots, n-1\}]\,\}.
\end{aligned}$$

These maps are injective by definition. Since every point is contained in infinitely many balls, these maps are surjective as well. Since $d(x_{\pi_1(n)}, y_{\pi_2(n)}) \leq 2r_n \to 0$ as $n \to \infty$, for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_{\pi_1(n)}, y_{\pi_2(n)}) < \varepsilon$ for all $n \geq n_0$. Furthermore there is $n_1 \in \mathbb{N}$ such that $\pi_1^{-1}(n) \geq n_0$ for all $n \geq n_1$. Thus $d(x_{\pi_1(\pi_1^{-1}(n))}, y_{\pi_2(\pi_1^{-1}(n))}) < \varepsilon$ for all $n \geq n_1$. Therefore the bijection $\tau = \pi_2 \circ \pi_1^{-1}$ satisfies $d(x_n, y_{\tau(n)}) \to 0$ as $n \to \infty$. □

Lemma 4.14 (Splitting Cantor spaces)**.** *Let X be a Cantor space with a compatible metric d. For every $\varepsilon > 0$ there is a finite partition of X into clopen subsets $V_1, \dots, V_k$ of diameter at most ε.*

Proof. Since X is zero-dimensional, for each $x \in X$ we find a clopen set U_x which is contained in the ball of radius $\frac{\varepsilon}{2}$ around x. Since X is compact, there exist points $x_1, \dots, x_n$

such that $U_{x_1}, \dots U_{x_n}$ cover X. Let $\mathscr{E}$ be the Boolean subalgebra of $\mathscr{P}(X)$ generated by $U_{x_1}, \dots, U_{x_n}$. The atoms of $\mathscr{E}$ are a partition of X into clopen sets and each of them has diameter at most ε. □

Lemma 4.15 (Open subsets of Cantor spaces)**.** *Let $U \neq \emptyset$ be an open subset of a Cantor space X with metric d. If U is non-compact, there are non-empty clopen subsets $(V_k)_{k\in\mathbb{N}} \subseteq U$ such that*

(a) $V_k \cap V_\ell = \emptyset$ *for all* $k, \ell \in \mathbb{N}, k \neq \ell$,

(b) $\bigcup_{k\in\mathbb{N}} V_k = U$,

(c) *the diameter of V_k goes to zero as k goes to infinity.*

Proof. Since all Cantor spaces are homeomorphic, we may assume that $U \subseteq \{0,1\}^{\mathbb{N}}$. Consider a set of cylinder sets $\mathscr{Z} = \left\{ [u]_1 \;\middle|\; u \in \bigcup_{n\in\mathbb{N}} \{0,1\}^n \right\}$ and let M_U be the set of maximal cylinder sets from $\mathscr{Z}$ contained in U, i.e.,

$$M_U := \{D \in \mathscr{Z} \mid D \subseteq U \text{ and } \nexists E \in \mathscr{Z} : D \subsetneq E \subseteq U\}.$$

The set M_U is countable, since $\mathscr{Z}$ is countable and each of the sets in M_U is clopen and non-empty. Additionally M_U is not finite because X is not compact. Finally for each $x \in U$ there is a minimal $r \in \mathbb{N}$ and $u \in \{0,1\}^r$ such that $x \in [u]_1 \subseteq U$. Hence U is the countable disjoint union of the sets in M_U and any enumeration of these sets gives us a sequence of sets $(V_k)_{k\in\mathbb{N}}$ fulfilling condition (a) and condition (b). By Lemma 4.14 we can further split up these sets to also fulfill condition (c). □

To simplify checking, if an extension of a continuous map is again continuous, we prove yet another lemma.

Lemma 4.16. *Let X, Y be metric spaces and let $M_1, \dots, M_k$ be a cover of X, i.e., $M_1 \cup \dots \cup M_k = X$. For every element $x \in X$ and every function $f : X \to Y$ the following are equivalent.*

(a) *f is continuous in x.*

(b) *For every $\ell \in \{1, \dots, k\}$ and every sequence of distinct points $(x_n)_{n\in\mathbb{N}}$ in M_ℓ with $\lim_{n\to\infty} x_n = x$ the limit $\lim_{n\to\infty} f(x_n)$ exists and equals $f(x)$.*

Proof. If f is continuous in x, (b) holds for any sequence in X converging to x and we are done. On the other hand assume that (a) does not hold. In that case there must be a sequence of distinct points $(x_n)_{n\in\mathbb{N}}$ converging to x for which $\lim_{n\to\infty} f(x_n)$ either does not exist or is unequal to $f(x)$. In both cases there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and $\varepsilon > 0$ such that $f(x_{n_k})$ always has distance at least ε from $f(x)$. There is at least one index $\ell \in \{1, \dots, k\}$ for which M_ℓ contains infinitely many elements of this subsequence. Thus (b) does not hold for this set M_ℓ. □

Proof of Theorem 4.10. First we treat case (a). If $(X \setminus M)$ and $(Y \setminus N)$ are both finite, we can extend θ with any bijection between these two sets. Therefore assume that $(X \setminus M)$ and $(Y \setminus N)$ are both infinite. Because X and Y are compact metrizable spaces and thus second countable, $(X \setminus M)$ and $(Y \setminus M)$ must both be enumerable. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be an enumeration of $X \setminus M$ and $Y \setminus N$. Let d be a compatible metric on X with $y \mapsto d(x,y)$ injective for all $x \in X$, whose existence is guaranteed by Lemma 4.11. Define a map $\varphi : X \to (X \setminus M)^* = (X \setminus M)^* \cap M$ by $x \mapsto \operatorname{argmin}_{y\in(X\setminus M)^*} d(x,y)$. By Lemma 4.12 this is a retraction of X onto $(X \setminus M)^*$. Assume there would be $\varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$\min\left\{ d(x_{n_k}, y) \;\middle|\; y \in (X \setminus M)^* \right\} = d(x_{n_k}, \varphi(x_{n_k})) \geq \varepsilon \text{ for } k \in \mathbb{N}. \tag{4.1}$$

Since X is compact, (x_{n_k}) would have a convergent subsequence whose limit x is by definition in $(X \setminus M)^*$. But this contradicts (4.1). Therefore we have

$$d(x_n, \varphi(x_n)) \to 0 \text{ as } n \to \infty.$$

In particular, for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to $x \in X$, the sequence $(\varphi(x_{n_k}))_{k\in\mathbb{N}}$ also converges to x. By the same reasoning we can get a metric d on Y and a retraction $\psi : Y \to (Y \setminus N)^* = (Y \setminus N)^* \cap N$ with

$$d(y_n, \psi(y_n)) \to 0 \text{ as } n \to \infty.$$

Every point in $(Y \setminus N)^*$ is by definition a condensation point of the sequence $(y_n)_{n\in\mathbb{N}}$ and thus also a condensation point of the sequence $(\psi(y_n))_{n\in\mathbb{N}}$. By the assumptions of our theorem we have $(Y \setminus N)^* = \theta[(X \setminus M)^*]$, hence every point in $(Y \setminus N)^*$ is also a condensation point of the sequence $(\theta(\varphi(x_n)))_{n\in\mathbb{N}}$. The pair of sequences $(\theta(\varphi(x_n))_{n\in\mathbb{N}}$ and $(\psi(y_n))_{n\in\mathbb{N}}$ in the compact space $(Y \setminus N)^*$ fulfill the assumptions of Lemma 4.13. Thus there is a bijection $\tau : \mathbb{N} \to \mathbb{N}$ such that

$$0 = \lim_{n\to\infty} d(\theta(\varphi(x_n)), \psi(y_{\tau(n)})).$$

We can now define the required extension of θ by $\theta(x_n) := y_{\tau(n)}$. By Lemma 4.16 we have to show that for every convergent sequence of distinct points $(\tilde{x}_n)_{n\in\mathbb{N}}$ in $X \setminus M$ we have $\lim_{n\to\infty} \theta(\tilde{x}_n) = \theta(\lim_{n\to\infty} \tilde{x}_n)$. Since Y is compact, we may assume that $\lim_{n\to\infty} \theta(\tilde{x}_n)$ exists. There must be an injective mapping $\pi : \mathbb{N} \to \mathbb{N}$ such that $\tilde{x}_n = x_{\pi(n)}$ for all $n \in \mathbb{N}$. There is a subsequence $(\tilde{x}_{n_k})_{k\in\mathbb{N}}$ such that $(\pi(\tilde{n}_k))_{k\in\mathbb{N}}$ is strictly increasing, in other words, $(\tilde{x}_{n_k})_{k\in\mathbb{N}} = (x_{\pi(n_k)})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$. We know that $x := \lim_{n\to\infty} \tilde{x}_n \in (X \setminus M)^* \subseteq M$. Furthermore $\varphi(x_{\pi(n_k)}) \to x$ and $\theta(\varphi(x_{\pi(n_k)})) \to \theta(x)$ for $k \to \infty$. Finally this implies

$$\begin{aligned}\theta(x) &= \lim_{k\to\infty} \theta(\varphi(x_{\pi(n_k)})) = \lim_{k\to\infty} \psi(y_{\tau(\pi(n_k))}) = \lim_{k\to\infty} y_{\tau(\pi(n_k))} \\ &= \lim_{k\to\infty} \theta(x_{\pi(n_k)}) = \lim_{n\to\infty} \theta(\tilde{x}_n).\end{aligned}$$

Now consider case (b). If $(X \setminus M)^* \cap M = \emptyset = (Y \setminus N)^* \cap N$, $X \setminus M$ and $Y \setminus N$ are both Cantor spaces and any homeomorphism between them is an extension of θ. Therefore suppose that $(X \setminus M)^* \cap M$ and $(Y \setminus N)^* \cap N$ are both non-empty. Hence $(X \setminus M)$ and $Y \setminus N$ are both open and non-compact. By Lemma 4.15 they can be decomposed into disjoint clopen subsets $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ which are homeomorphic to the Cantor set and whose diameter goes to zero. From each set V_n and W_n choose one point x_n resp. y_n and define $X_0 := M \cup \{ x_n \mid n \in \mathbb{N} \}$ and $Y_0 := N \cup \{ y_n \mid n \in \mathbb{N} \}$. We see that $X_0^* \subseteq M$, so X_0 is closed in X and similarly Y_0 is closed in Y. Furthermore $(X_0 \setminus M)^* \cap (X_0 \setminus M) = \emptyset = (Y_0 \setminus N)^* \cap (Y_0 \setminus M)$, $|X_0 \setminus M| = |\mathbb{N}| = |Y_0 \setminus N|$, $(X_0 \setminus M)^* \cap M = (X \setminus M)^* \cap M$ and $(Y_0 \setminus N)^* \cap N = (Y \setminus N)^* \cap N$. Therefore by case (a) of Theorem 4.10 we get a mapping $\tau : \mathbb{N} \to \mathbb{N}$ with the property that $x_n \mapsto y_{\tau(n)}$ extends the homeomorphism θ. Let $\psi_n : V_n \to W_{\tau(n)}$ be homeomorphisms and define $\tilde{\theta} = \theta \cup \bigcup_{n=1}^{\infty} \psi_n$. Then $\tilde{\theta}$ is a bijective extension of θ. It remains to show that $\tilde{\theta}$ is continuous.

Let $(z_n)_{k\in\mathbb{N}}$ be any convergent sequence of distinct points in $X \setminus M$ with limit z. Again we may assume that $\lim_{n\to\infty} \tilde{\theta} z_n$ exists. If $z \in V_\ell$ for some $\ell \in \mathbb{N}$, then there exists $k_0 \in \mathbb{N}$ such that $z_k \in V_\ell$ for all $k \geq k_0$, since V_ℓ is open. But then

$$\lim_{k\to\infty} \tilde{\theta}(z_k) = \lim_{k\to\infty} \theta_\ell(z_k) = \theta_\ell(z) = \tilde{\theta}(z).$$

Therefore we are left with the case that $z \in M$. There is a sequence of positive integers n_k such that $z_k \in V_{n_k}$. Because the sets V_ℓ for $\ell \in \mathbb{N}$ are closed, every integer appears only a finite number of times in n_k. Since the diameter of V_ℓ tends to 0 as ℓ goes to infinity, we have $z = \lim_{k\to\infty} z_k = \lim_{k\to\infty} x_{n_k}$. Because $x_n \mapsto y_{\tau(n)}$ extends θ homeomorphically and $z = \lim x_{n_k} \in M$, we have $\lim_{k\to\infty} y_{\tau(n_k)} = \theta(\lim_{k\to\infty} x_{n_k}) = \theta(z)$.

Since the diameter of W_ℓ goes to zero as ℓ goes to infinity, we finally get $\lim_{k\to\infty} y_{\tau(n_k)} = \lim_{k\to\infty} \theta_{n_k}(z_{n_k})$ and thus

$$\tilde{\theta}(z) = \theta(z) = \lim_{k\to\infty} y_{\tau(n_k)} = \lim_{k\to\infty} \theta_{n_k}(z_k) = \lim_{k\to\infty} \tilde{\theta}(z_k). \qquad \square$$

Lemma 4.17. *Let X be a metrizable Stone space and let $\mathscr{B}$ be a finite derivative subalgebra of $\mathscr{P}(X)$, which is generated by its closed elements. If M is an atom of $\mathscr{B}$ with $M \subseteq M^*$, then there is a compact subset Y of X such that M is an open set in Y.*

Proof. We prove this by induction on the size of $\mathscr{B}$. If $\mathscr{B}$ contains only one element, then M is already an open subset of X. Otherwise $\mathscr{B}$ contains an open atom N by Lemma 3.39. If $M = N$ we are done. Otherwise $X \setminus N$ is a metrizable Stone space and $\mathscr{B} \wedge N^c$ is a finite derivative subalgebra of $\mathscr{P}(X \setminus N)$ by Lemma 3.36. By Lemma 3.38 it is generated by closed elements. Furthermore $\mathscr{B} \wedge N^c$ contains fewer elements than $\mathscr{B}$ and M is an atom of $\mathscr{B} \cap N^c$. Therefore by induction M is contained as an open set in a compact subset of $X \setminus N \subseteq X$. $\square$

Lemma 4.18. *Let X be a metrizable Stone space. Let $\mathscr{B}$ be a finite derivative subalgebra of $\mathscr{P}(X)$, which is generated by its closed elements. For every atom M of $\mathscr{B}$, one of the following three cases holds.*

(a) $M^* \cap M = M \iff |M| = |\mathbb{R}|$.

(b) $M^* \cap M = \emptyset$ *and* $M^* \cap M^c \neq \emptyset \iff |M| = |\mathbb{N}|$.

(c) $M^* = \emptyset \iff |M| \in \mathbb{N}$.

Proof. Since $\mathscr{B}$ is a derivative algebra, $\mathscr{B} \ni M \cap M^* \subseteq M$. Because M is an atom of $\mathscr{B}$, either $M \cap M^* = M$ or $M \cap M^* = \emptyset$. In the first case M is contained as an open subset in the compact subset of X by Lemma 4.17. Since X is homeomorphic to a compact subset of $\mathbb{R}$ and M does not contain any isolated point, Theorem 4.9 implies that $|M| = |\mathbb{R}|$. In the second case M consists of isolated points. Since M is second-countable as a metrizable Stone space, we have $|M| \leq |\mathbb{N}|$. If $M^* = \emptyset$, then M must be finite for otherwise we can find a sequence of pairwise different points in M that converge in the compact space X. In the last remaining case where $M^* \cap M = \emptyset$ and $M^* \cap M^c \neq \emptyset$ we must have $|M| = |\mathbb{N}|$ because a finite set in a metrizable space can have no accumulation points. □

The following is Theorem 4.3 in [Pie72] rephrased in our terminology of derivative algebras.

Theorem 4.19 (Characterization of homeomorphic metrizable Stone spaces with finite derivative algebras)**.** *Let X and Y be metrizable Stone spaces. Let $\mathscr{B}$ and $\mathscr{C}$ be finite derivative subalgebras of, respectively, $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ which are both generated by their closed elements. If $\rho : \mathscr{B} \to \mathscr{C}$ is an isomorphism of derivative algebras such that $|\rho(M)| = |M|$ for all $M \in \mathscr{B}$ with $M^* = \emptyset$, then there exists a homeomorphism $\varphi : X \to Y$ such that $\varphi[M] = \rho(M)$ for all $M \in \mathscr{B}$.*

Proof. If $\mathscr{B}$ and $\mathscr{C}$ both only have one atom, either X and Y are perfect and hence homeomorphic to the Cantor set, or both spaces are finite and have the same cardinality. In both cases, X and Y are homeomorphic by a homeomorphism φ and the condition $\varphi[M] = \rho(M)$ for $M \in \mathscr{B}$ is trivially fulfilled, since $\mathscr{B} = \{\emptyset, X\}$. From here we proceed by induction.

Let $\mathscr{B}$ and $\mathscr{C}$ be derivative algebras having at least two atoms. By Lemma 3.39 there is an open atom M of $\mathscr{B}$. Since ρ is an isomorphism of derivative algebras, $N := \rho(M)$ is also an open atom of $\mathscr{C}$. The complements $\tilde{X} = X \setminus M$ and $\tilde{Y} = Y \setminus N$ are both closed. Since $\rho(\tilde{X}) = \tilde{Y}$, ρ induces an isomorphism of the derivative algebras $\mathscr{B} \wedge \tilde{X}$ and $\mathscr{C} \wedge \tilde{Y}$. We have to check that $|\rho(Q)| = Q$ for all atoms Q of $\mathscr{B} \wedge \tilde{X}$ with $Q^* \wedge \tilde{X} = \emptyset$. Since $\tilde{X}$ is closed, we have $Q^* = Q^* \cap \tilde{X} = \emptyset$, hence by the assumptions. By Lemma 3.38 both $\mathscr{B} \cap \tilde{X}$ and $\mathscr{C} \cap \tilde{Y}$ are generated by their closed elements. By induction we obtain a homeomorphism $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ such that $\tilde{\varphi}[Q] = \rho(Q)$ for all atoms Q of $\mathscr{B} \wedge \tilde{X}$. In particular $\tilde{\varphi}[M^* \cap \tilde{X}] = \rho(M^* \cap \tilde{X}) = \rho(M)^* \cap \rho(\tilde{X}) = N^* \cap \tilde{Y}$.

By Lemma 4.18 either $M^* \cap M = M$ or $M^* \cap M = \emptyset$. In the second case either $M^* \cap M^c \neq \emptyset$, then also $N^* \cap N^c \neq \emptyset$ and $|M| = |N| = |\mathbb{N}|$ or $M^* = \emptyset = N^*$, but then by the assumptions $|M| = |N|$. In both cases we can apply Theorem 4.10 and get a homeomorphism $\varphi : X \to Y$ extending $\tilde{\varphi}$. For every set $Q \in \mathscr{B}$ we have $\varphi[Q] = \tilde{\varphi}[Q \cap \tilde{X}] \cup \varphi[Q \cap M] = \rho(Q \cap \tilde{X}) \cup \rho(Q \cap M) = \rho(Q)$.

□

Theorem 4.20 (Isomorphism of metrizable Stone spaces, derivative algebras and the order on the atoms)**.** *Let X, Y be metrizable Stone spaces with derivative algebras $\mathscr{B}$ resp. $\mathscr{C}$. Let $(T_{\mathscr{B}}, \preceq)$ and $(T_{\mathscr{C}}, \preceq)$ be the atoms of, respectively, $\mathscr{B}$ and $\mathscr{C}$ together with the order $\preceq$ as defined in Definition 3.30. If $\mathscr{B}$ and $\mathscr{C}$ are finite, then the following are equivalent.*

(a) *X is homeomorphic to Y.*

(b) *There exists an isomorphism of derivative algebras $\rho : \mathscr{B} \to \mathscr{C}$ such that $|M| = |\rho(M)|$ for all atoms $M \in T_{\mathscr{B}}$.*

(c) *There exists an isomorphism of ordered sets $\tau : T_{\mathscr{B}} \to T_{\mathscr{C}}$ such that $|M| = |\tau(M)|$ for all atoms $M \in T_{\mathscr{B}}$.*

Proof. $(a) \implies (b)$: If $\varphi : X \to Y$ is a homeomorphism, then $\rho : \mathscr{B} \to \mathscr{P}(Y)$, $\rho(M) := \varphi[M]$ is an injective homomorphism between derivative algebras.

The image $\rho[\mathscr{B}]$ is a derivative subalgebra of $\mathscr{P}(Y)$ and therefore contains $\mathscr{C}$. On the other hand there is a homomorphism $\tilde{\rho} : \mathscr{C} \to \mathscr{P}(X)$ between derivative algebras defined by $\tilde{\rho}(M) := \varphi^{-1}[M]$. Again $\mathscr{B}$ is a derivative subalgebra of $\rho[\mathscr{C}]$. Putting both homomorphism together we obtain $\mathscr{C} \subseteq \rho[\mathscr{B}] \subseteq \rho[\tilde{\rho}[\mathscr{C}]] = \mathscr{C}$, in other words $\mathscr{C} = \rho[\mathscr{B}]$, hence ρ is an isomorphism of derivative algebras.

$(b) \iff (c)$ follows immediately from Theorem 3.33.

$(b) \implies (a)$: Both $\mathscr{B}$ and $\mathscr{C}$ are generated by a single closed element. Hence by Theorem 4.19 there is a homeomorphism $\varphi : X \to Y$ such that $\rho(M) = \varphi[M]$ for every atom $M \in T_{\mathscr{B}}$. In particular this implies $|\rho(M)| = |\varphi[M]| = |M|$. □

4.3. Topological Orbit Equivalence and Conjugacy

Constructing a conjugacy between two periodic dynamical systems comes basically down to picking one element from each orbit of the dynamical systems in the right way. To do so, we use the fact that each metrizable Stone space is homeomorphic to a subset of the reals where we have a natural order and so we can just pick the smallest element in each orbit.

Definition 4.21 (Closed total order)**.** *Let X be a topological space and let $\leq$ be a total order on X that is closed as a subset of $X \times X$ with the product topology. In the following we simply call such an order a* closed total order. *For a homeomorphism $f : X \to X$ we define*

$$L_{\leq}(f, 0) := \left\{ x \in X \;\middle|\; x \leq f^k(x) \text{ for all } k \in \mathbb{Z} \right\}.$$

We also give names to the images of this set under the iterates of f, namely $L_{\leq}(f, m) := f^m[L(f, 0)]$.

Example 4.22 (Examples of closed total orders). *Let $\leq$ be the usual order on $\mathbb{R}$. For two converging sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $x_n \leq y_n$ for all $n \in \mathbb{N}$ the same inequality also holds in the limit, i.e., $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$. Hence $\leq$ is closed as a relation in $\mathbb{R}\times\mathbb{R}$. Since every metrizable Stone space is homeomorphic to a subset of the reals by Corollary 2.10, there exists such an order on every metrizable Stone space. This is what we will need for our applications.*

Lemma 4.23 (Properties of $L_{\leq}(f,m)$). *Let X be a compact metrizable space and let $\leq$ be a closed total order on X. If $f : X \to X$ is a p-periodic dynamical system, the following properties hold. The sets $L_{\leq}(f,k), k \in \{0,\dots,p-1\}$ form a covering of X by closed sets. For $k_1, k_2 \in \{0,\dots,p-1\}, k_1 < k_2$ the intersection $L_{\leq}(f,k_1)\cap L_{\leq}(f,k_2)$ is contained in $\mathrm{Per}_{k_2-k_1}(f)$. Every periodic orbit of f with minimal period q contains exactly one point from each of the sets $L_{\leq}(f,0),\dots,L_{\leq}(f,q-1)$.*

Proof. Given a point $x \in X$ denote by $x_{\min}$ the point in $\mathrm{Orb}(f,x)$ which is minimal with respect to $\leq$. Such a point always exists, since $\mathrm{Orb}(f,x)$ is finite. By definition, we have $x_{\min} \in L_{\leq}(f,0)$. There is $k \in \{0,\dots,p-1\}$ such that $x_{\min} = f^k(x)$. This implies $x = f^{p-k}(x_{\min}) \in L_{\leq}(f,p-k)$, therefore the sets $L_{\leq}(f,0),\dots,L_{\leq}(f,p-1)$ cover X. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $L_{\leq}(f,0)$ which converges in X, we have $x_n \leq f^k(x_n)$ for all $k \in \mathbb{N}$. Since $\leq$ is closed and f is continuous, we also have $\lim_{n\to\infty} x_n \leq f^k(\lim_{n\to\infty} x_n)$. Therefore $L_{\leq}(f,0)$ is closed and hence also compact. As the images of a compact set under a continuous function the sets $L_{\leq}(f,k)$ for $k \in \{1,\dots,p-1\}$ are also compact.

If $x \in L_{\leq}(f,k_1)\cap L_{\leq}(f,k_2)$, then $f^{-k_1}(x) \in L_{\leq}(f,0)$ and $f^{-k_2}(x) \in L_{\leq}(f,0)$, hence $f^{-k_2}(x) \leq f^{-k_1}(x) \leq f^{-k_2}(x)$. Applying f^{k_2} to the resulting equation gives $x = f^{k_2-k_1}(x)$.

We already saw that each orbit of f has exactly one point in $L_{\leq}(f,0)$. The images of this point then lie in the sets $L_{\leq}(f,1),\dots,L_{\leq}(f,q-1)$, respectively. By the previous argument there is exactly one point from the orbit in each of these sets. □

Theorem 4.24 (Characterization of conjugacies in the simple case). *Let X and Y be Cantor spaces. If $f : X \to X$ and $g : Y \to Y$ are strictly p-periodic dynamical systems, i.e., $\widetilde{\mathrm{Per}}_p(f) = X$ and $\widetilde{\mathrm{Per}}_p(f) = Y$ for some $p \in \mathbb{N}$, then f and g are conjugate.*

Proof. We may assume that both X and Y are the Cantor middle thirds set. The standard order $\leq$ on the reals is closed as a subset of $X\times X = Y\times Y$. For every $a \in L_{\leq}(f,\ell)\cap L_{\leq}(f,m)$ with $0 \leq \ell \leq m < p$ we have $a \in \mathrm{Per}_{m-\ell}(f)$. However, the point a has minimal period p under f, thus $m = \ell$. This shows that the sets $L_{\leq}(f,0),\dots,L_{\leq}(f,p-1)$ and $L_{\leq}(g,0),\dots,L_{\leq}(g,p-1)$ are pairwise disjoint and therefore not only closed but also open. Hence $L_{\leq}(f,0)$ and $L_{\leq}(g,0)$ are clopen subsets of the Cantor set and are therefore themselves homeomorphic to the Cantor set. In other words, there is a homeomorphism $\theta : L_{\leq}(f,0) \to L_{\leq}(g,0)$. Since $L_{\leq}(f,0)$ and its images under f are closed and disjoint, we can extend θ to $f^{\ell}[L_{\leq}(f,0)]$, $\ell \in \{1,\dots,p-1\}$ such that $\theta\circ f = g\circ\theta$. This function θ is the conjugacy between f and g that we are looking for. □

We already met topological orbit equivalence as a weakening of topological conjugacy in Section 2.4. For a periodic dynamical system $f : X \to X$ the set $\mathrm{Orb}(f)$ is a partition of X.

It will be useful to factor out the induced equivalence relation and consider the resulting quotient space. We start with the relevant definition and a classical result which can be found for example in Section 2.4 in the book by Engelking [Eng89].

Definition 4.25 (Quotient space). *Let X be a topological space and Q a partition of X. We define a new topological space X/Q which we call the* quotient of X by Q. *The set underlying X/Q is just Q and we have a surjective map $\pi : X \to X/Q$ mapping every element of x to the unique element of Q in which it lies. A set M in X/Q is defined as open, if and only if $\pi^{-1}(M) = \bigcup M$ is open in X. The resulting topology is called the* quotient topology.

The quotient topology satisfies a nice universal property.

Lemma 4.26. *Let X and Y be topological spaces, let Q be a partition of X and let $\pi : X \to X/Q$ be the natural projection. A map $f : X/Q \to Y$ is continuous if and only if $f \circ \pi : X \to Y$ is continuous.*

For general invertible dynamical system f it is often problematic to factor out $\mathrm{Orb}(f)$ because the resulting space might be very unpleasant from a topological point of view. For example it is not necessarily Hausdorff. In the case of periodic dynamical systems the resulting factor space, however, is well-behaved, as the next lemma shows.

Lemma 4.27. *Let X be a compact Hausdorff space and let $f : X \to X$ be continuous and p-periodic for some $p \in \mathbb{N}$. The space $X/\mathrm{Orb}(f)$ endowed with the quotient topology is a compact Hausdorff space.*

Proof. Let $M, N \in \mathrm{Orb}(f)$ be two orbits of f. Both sets contain at most p points of X, so they are closed subsets of X. Every compact Hausdorff space is normal, see for example Theorem 32.3 in the book of Munkres [Mun00]. Therefore there are disjoint open sets $U, V \subseteq X$ with $M \subseteq U$ and $N \subseteq V$. These sets are, however, not necessarily invariant under f so we can not use them to separate M and N in $X/\,\mathrm{Orb}(f)$. To do this we define

$$\tilde{U} := \bigcap_{n=0}^{p-1} f^n[U] \text{ and}$$
$$\tilde{V} := \bigcap_{n=0}^{p-1} f^n[V].$$

These sets are open and invariant under f. For $x \in M$ and $k \in \mathbb{Z}$ we have $f^{-k}(x) \in M$ and so $x \in f^k[M] \subseteq f^k[U]$. Therefore $\tilde{U}$ and $\tilde{V}$ are disjoint open and f-invariant sets containing M and N, respectively. Let $\pi : X \to X/\mathrm{Orb}(f)$ be the natural projection. Then $\pi^{-1}[\pi[\tilde{U}]] = \tilde{U}$ and $\pi^{-1}[\pi[\tilde{V}]] = \tilde{V}$, therefore $\pi[\tilde{U}]$ and $\pi[\tilde{V}]$ are disjoint open subsets of $X/\,\mathrm{Orb}(f)$ containing M respectively N. All in all this shows that $\mathrm{Orb}(f)$ is a Hausdorff space. Now $X/\mathrm{Orb}(f)$ is the continuous image of a compact space, hence it is itself compact, see Theorem 2.1. □

The next result shows that the space obtained by factoring out $\mathrm{Orb}(f)$ is an invariant of topological orbit equivalence.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \hat{X} & \xrightarrow[\hat{\varphi}]{} & \hat{Y} \end{array}$$

Figure 4.2.: Orbit equivalences and the induced map on the factor space.

Lemma 4.28. *Let $f : X \to X$ and $g : Y \to Y$ be periodic dynamical systems on compact Hausdorff spaces. Define $\hat{X} := X/\operatorname{Orb}(f)$ and $\hat{Y} := Y/\operatorname{Orb}(g)$. If f and g are topologically orbit equivalent, then $\hat{X} \cong \hat{Y}$.*

Proof. Let $\varphi : X \to Y$ be the orbit equivalence between f and g. We define maps $\hat{\varphi} : \hat{X} \to \hat{Y}$ and $\hat{\psi} : \hat{Y} \to \hat{X}$ by

$$\hat{\varphi}(\operatorname{Orb}(f,x)) = \varphi[\operatorname{Orb}(f,x)] = \operatorname{Orb}(g,\varphi(x)),$$
$$\hat{\psi}(\operatorname{Orb}(g,y)) = \varphi^{-1}[\operatorname{Orb}(g,y)] = \operatorname{Orb}(g,\varphi^{-1}(y))$$

for $x \in X$ and $y \in Y$. These maps are well-defined, since ψ is an orbit equivalence and we have $\hat{\varphi} \circ \hat{\psi} = \operatorname{id}_{\hat{X}}$ and vice versa. It remains to show, that $\hat{\varphi}$ and $\hat{\psi}$ are continuous.

The diagram in Figure 4.2 commutes and by the universal property of the quotient topology stated in Lemma 4.26 $\hat{\varphi}$ is continuous if and only if $\hat{\varphi} \circ \pi = \pi \circ \varphi$ is continuous. But φ is a homeomorphism and π is continuous by the definition of the quotient topology. Therefore $\hat{\varphi}$ is continuous and so is $\hat{\psi}$ by similar reasoning. □

Theorem 4.29 (Characterizations of topological conjugacy)**.** *Let X, Y be compact metrizable spaces with a closed total order $\leq$. If $f : X \to X$ and $g : Y \to Y$ are two p-periodic dynamical systems, the following are equivalent.*

(a) *f and g are topologically conjugate.*

(b) *f and g are topologically orbit equivalent.*

(c) *There is a homeomorphism $\psi : X/\operatorname{Orb}(f) \to Y/\operatorname{Orb}(g)$ with $\psi[\pi_X[Per_k(f)]] = \pi_Y[Per_k(g)]$ for $k \in \{1,\dots,p\}$.*

(d) *There is a homeomorphism $\varphi : L_{\leq}(f,0) \to L_{\leq}(g,0)$ with $\varphi[Per_k(f) \cap L_{\leq}(f,0)] = Per_k(g) \cap L_{\leq}(g,0)$ for $k \in \{1,\dots,p\}$.*

Proof. The implication (a) $\Longrightarrow$ (b) holds by definition. The implication (b) $\Longrightarrow$ (c) is the content of Lemma 4.28.

(c) $\Longrightarrow$ (d). Consider the embedding $\iota : L(f,0) \to X$ and the natural projection $\pi : X \to X/\operatorname{Orb}(f)$. Both maps are continuous, hence $\psi = \pi \circ \iota$ is continuous. As we saw in

Lemma 4.23, the set $L(f,0)$ is closed and contains exactly one point from each orbit of f, hence ψ is also bijective. We also know from Lemma 4.27 that $L(f,0)$ is compact, hence ψ is a homeomorphism. For the same reason $L(g,0)$ is homeomorphic to $Y/\operatorname{Orb}(g)$. The desired implication now follows at once.

(d) $\implies$ (a). Let $\theta : L(f,0) \to L(g,0)$ be a homeomorphism with $\theta[L(f,0) \cap \operatorname{Per}_k(f)] = L(g,0) \cap \operatorname{Per}_k(f)$ for all $k \in \{1,\dots,p\}$. Define $\tilde{\theta} : X \to Y$ by $\tilde{\theta}(x) = g^{\ell} \circ \theta \circ f^{-\ell}$ for $x \in L(f,\ell)$. If $x \in L(f,\ell_1) \cap L(f,\ell_2)$ with $\ell_2 > \ell_1$, then $f^{\ell_2-\ell_1}(x) = x$, hence $f^{-\ell_1}(x) = f^{-\ell_2}(x) \in \operatorname{Per}_{\ell_2-\ell_1}(x)$ and hence also $\theta(f^{-\ell_1}(x)) = \theta(f^{-\ell_2}(x)) \in \operatorname{Per}_{\ell_2-\ell_1}(g)$. Together this shows

$$\begin{aligned} g^{\ell_1}(\theta(f^{-\ell_1}(x)) &= g^{\ell_1}(\theta(f^{-\ell_2}(x))) \\ &= g^{\ell_2-\ell_1}(g^{\ell_1}(\theta(f^{-\ell_2}(x)))) \\ &= g^{\ell_2}(\theta(f^{-\ell_2}(x)))). \end{aligned}$$

Therefore $\tilde{\theta}$ is well-defined. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in one of the sets $L(f,\ell), \ell \in \{0,\dots,p-1\}$ converging in X. Since $L(f,\ell)$ is closed, the limit of $(x_n)_{n\in\mathbb{N}}$ lies also in $L(f,\ell)$. Since $\tilde{\theta}_{|L(f,\ell)}$ is continuous, by Lemma 4.16 the whole map $\tilde{\theta}$ is continuous. Finally $\tilde{\theta}$ intertwines f and g almost by definition, as for $x \in L(f,\ell)$ we have

$$\begin{aligned} g(\tilde{\theta}(x)) &= g^{\ell+1}(\theta(f^{-\ell}(x))) \\ &= g^{\ell+1}(\theta(f^{-(\ell+1)}(f(x)))) \\ &= \tilde{\theta}(f(x)). \end{aligned}$$

□

Remark 4.30. While the implication from (a) to (b) always holds regardless of the assumptions on the space or the dynamical systems in question, the converse is false for p-periodic dynamical systems if we remove the existence of a closed total order from our assumptions. Circle rotations provide an example. Let $f,g : S^1 \to S^1$ be the rotations of the circle S^1 by, respectively, $2\pi/5$ and $4\pi/5$. Both maps are 5-periodic and the idendity map is an orbit equivalence between them. However, their rotation numbers $2\pi/5$ and $4\pi/5$ have different absolute values and hence they are not conjugate (see for example the book by Katok and Hasselblatt [KH95]). This argument also shows that one can not find a closed total order on S^1.

On the other hand topological orbit equivalence and topological conjugacy also become distinct notions for dynamical systems on Cantor spaces if we drop the periodicity assumption. There are mixing subshifts of finite type which are not conjugate to their inverse, for an example due to Köllmer see Proposition 30 in the book by Parry and Tuncel [PT82]. On the other hand every invertible dynamical system is orbit equivalent to its inverse, the necessary homeomorphism being simply the identity map.

In the situation of metrizable Stone spaces, a seemingly even weaker condition turns out to be equivalent to topological conjugacy. In order to prove this, we need some further preparations.

Definition 4.31 (Periodic point algebra). *Let X be a metrizable Stone space. For a p-periodic dynamical system $f : X \to X$ we denote by $\mathscr{D}(f)$ the derivative subalgebra of $\mathscr{P}(X)$*

generated by the sets $\mathrm{Per}_k(f), k \in \{1,\dots,p\}$*. We call this derivative algebra the* periodic point algebra of f.

Lemma 4.32. *Let X be a Stone space and let $f : X \to X$ be a p-periodic dynamical systems on X. If $\mathscr{D}(f)$ is finite, then the automorphism of $\mathscr{D}(f)$ induced by f is trivial, i.e., $f[b] = b$ for all $b \in \mathscr{D}(f)$.*

Proof. Let M be the set of atoms of $\mathscr{D}(f)$. Consider the derivative algebra $\mathscr{C}$ with atoms $T = \left\{ \bigcup_{k=0}^{p-1} f^k[b] \;\middle|\; b \in M \right\}$. We want to show that $\mathscr{C}$ is a derivative subalgebra of $\mathscr{P}(X)$ containing $\mathscr{D}(f)$.

Since the elements of T are pairwise disjoint sets and their union is X, they are indeed the atoms of a Boolean subalgebra of $\mathscr{P}(X)$. Furthermore for $f \in \{1,\dots,p\}$

$$\mathrm{Per}_k(f) = \bigcup \left\{ \bigcup_{k=0}^{p-1} f^k[a] \;\middle|\; a \in M, a \subseteq \mathrm{Per}_k(f) \right\} \in \mathscr{C}.$$

We have to show that for every atom $b \in T$ its derivative b^* is in $\mathscr{C}$. By definition there is an element $c \in M$ with $b = \bigcup_{k=0}^{p-1} f^k[c]$. There are atoms $a_1,\dots,a_\ell \in M$ with $c^* = a_1 \cup \dots \cup a_\ell$. The derivative of b equals

$$\begin{aligned}
b^* &= \bigcup_{k=0}^{p-1} f^k[c^*] \\
&= \bigcup_{k=0}^{p-1} f^k[a_1 \cup \dots \cup a_\ell] \\
&= \bigcup_{k=0}^{p-1} f^k[a_1] \cup \dots \cup f^k[a_\ell] \\
&= \underbrace{\bigcup_{k=0}^{p-1} f^k[a_1]}_{\in T} \cup \dots \cup \underbrace{\bigcup_{k=0}^{p-1} f^k[a_\ell]}_{\in T}.
\end{aligned}$$

Therefore $\mathscr{C}$ is a derivative algebra containing $\mathrm{Per}_k(f)$ for all $k \in \{1,\dots,k-1\}$ and thus $\mathscr{D}(f) = \mathscr{C}$. Since f acts trivially on $\mathscr{C}$ by definition, this proves the theorem. □

Lemma 4.33. *Let X be a Stone space and let $f : X \to X$ be a p-periodic dynamical system. If the derivative algebra $\mathscr{D}(f)$ is finite, then $\mathscr{D}(f) \cap L(f,0) = \{ M \cap L(f,0) \mid M \in \mathscr{D}(f) \}$ is a derivative algebra with*

$$(M \cap L(f,0))^* := ac_{L(f,0)}(M \cap L(f,0)) = M^* \cap L(f,0)$$

for $M \in \mathscr{D}(f)$ and the map $\rho : \mathscr{D}(F) \to \mathscr{D}(f) \cap L(f,0)$, defined by $M \mapsto M \cap L(f,0)$, is an isomorphism of derivative algebras. Furthermore for every atom M of $\mathscr{D}(f)$ we have

$$\begin{aligned}
|\rho(M)| &= \frac{|M|}{k} && \text{for } |M| \in \mathbb{N}, M \subseteq \widetilde{Per}_k(f), \\
|\rho(M)| &= |M| && \text{for } |M| \notin \mathbb{N}.
\end{aligned}$$

Proof. We start by showing that $\mathrm{ac}_{L(f,0)}(M \cap L(f,0)) = M^* \cap L(f,0)$. If $x \in \mathrm{ac}_{L(f,0)}(M \cap L(f,0))$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $(M \cap L(f,0)) \setminus \{x\}$ with $\lim_{n\to\infty} x_n = x$. This implies $x \in M^* \cap L(f,0)$. On the other hand for every $x \in M^* \cap L(f,0)$ there is a sequence x_n in $M \setminus \{x\}$ converging to x. Since the sets $L(f,\ell), \ell \in \{0,\dots,p-1\}$ cover X, there must be $m \in \{0,\dots,p-1\}$ such that $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ in $L(f,m)$. Since $L(f,m)$ is closed, the limit of this subsequence x is also in $L(f,m)$. By Lemma 4.23 this implies $f^{-m}(x) = x$. Applying f^{-m} to $(x_{n_k})_{k\in\mathbb{N}}$ gives a sequence in $f^{-m}[L(f,m)] \setminus \{f^{-m}(x)\} = L(f,0) \setminus \{f^{-m}(x)\}$ converging to $f^{-m}(x) = x$. Hence $x \in \mathrm{ac}_{L(f,0)}(M \cap L(f,0))$.

By Lemma 4.32 f acts trivially on $\mathscr{D}(f)$, hence every set $M \in \mathscr{D}(f)$ is a union of orbits of f and in particular $M \cap L(f,0) \neq \emptyset$ if $M \neq \emptyset$. Therefore $\mathscr{D}(f) \cap L(f,0)$ is a Boolean algebra with atoms $\mathscr{M} = \{ K \cap L(f,0) \mid K \text{ is an atom of } \mathscr{D}(f) \}$. Let $K \cap L(f,0)$ be such an atom. We have

$$\begin{aligned}
(K \cap L(f,0))^* &= \mathrm{ac}_{L(f,0)}(K \cap L(f,0)) \\
&= K^* \cap L(f,0) \\
&= \bigcup \{ H \mid H \subseteq K^*, H \text{ is an atom of } \mathscr{D}(f) \} \cap L(f,0) \\
&= \bigcup \{ H \cap L(f,0) \mid H \subseteq K^*, H \cap L(f,0) \text{ is an atom of } \mathscr{D}(f) \cap L(f,0) \}
\end{aligned}$$

The Boolean algebra $\mathscr{D}(f) \cap L(f,0)$ is therefore a derivative algebra by Lemma 3.13. That ρ is a homomorphism of Boolean algebras follows directly from the axioms. Since $\rho(M)^* = (M \cap L(f,0))^* = M^* \cap L(f,0) = \rho(M^*)$, it is also a homomorphism of derivative algebras. The map ρ is surjective by definition. It is injective, since the only set $M \in \mathscr{D}(f)$, for which $\rho(M) = M \cap L(f,0) = \emptyset$, is the empty set itself.

Since the sets $\widetilde{\mathrm{Per}}_\ell(f), \ell \in \{1,\dots,p\}$ are all contained in $\mathscr{D}(f)$, for every atom $M \in \mathscr{D}(f)$ there is a unique $k \in \{1,\dots,p\}$ such that $M \subseteq \widetilde{\mathrm{Per}}_k(f)$. The sets $M \cap L(f,0),\dots,M \cap L(f,k-1)$ partition M and by the invariance of M under f the sets $M \cap L(f,0),\dots,M \cap L(f,k-1)$ all have the same cardinality. Hence $k \cdot |\rho(M)| = |M|$ and the last conclusion of the lemma follows. □

Theorem 4.34 (Conjugacy of periodic dynamical systems on metrizable Stone spaces). *Let X, Y be metrizable Stone spaces. Let $f : X \to X$ and $g : Y \to Y$ be p-periodic dynamical systems. Let $\mathscr{D}(f)$ and $\mathscr{D}(g)$ be the periodic point algebras of f and g and let $T_{\mathscr{D}(f)}$ and $T_{\mathscr{D}(g)}$ be the atoms of these algebras, ordered by $\preceq$ as defined in Definition 3.30. If $\mathscr{D}(f)$ and $\mathscr{D}(g)$ are finite, then the following are equivalent.*

(a) *f is topologically conjugate to g.*

(b) *f is topologically orbit equivalent to g.*

(c) *There exists a homeomorphism $\theta : X \to Y$ with $\theta[\mathrm{Per}_k(f)] = \mathrm{Per}_k(g)$ for all $k \in \{0,\dots,p-1\}$.*

(d) *There exists an isomorphism of derivative algebras $\rho : \mathscr{D}(f) \to \mathscr{D}(g)$ with $|M| = |\rho(M)|$ for all atoms M of $\mathscr{D}(f)$ such that $\rho(\mathrm{Per}_k(f)) = \mathrm{Per}_k(g)$ for all $k \in \{0,\dots,p-1\}$.*

(e) *There exists an isomorphism of ordered sets* $\tau : T_{\mathscr{D}(f)} \to T_{\mathscr{D}(g)}$ *with* $|M| = |\tau(M)|$ *for all atoms* $M \in T_{\mathscr{D}(f)}$, *such that* $\tau(Per_k(f)) = Per_k(g)$ *for all* $k \in \{0, \ldots, p-1\}$.

Proof. By Theorem 2.9 every metrizable Stone space is homeomorphic to a closed subset of the reals. As shown in Example 4.22 this implies the existence of a closed order on X and Y.

The equivalence of topological conjugacy and orbit equivalence for f and g thus follows from Theorem 4.29.

Since each orbit of length k of f must be mapped to an orbit of length k of g by an orbit equivalence, every orbit equivalence θ between f and g fulfills (c).

Let $\theta : X \to Y$ be a homeomorphism satisfying (c). As a map between the underlying spaces θ induces an injective homomorphism of Boolean algebras from $\mathscr{D}(f)$ into $\mathscr{P}(Y)$. Since θ is a homeomorphism, $\theta(a^*) = \theta(a)^*$ for all $a \in \mathscr{D}(f)$ by Lemma 3.28. Therefore $\theta(\mathscr{D}(f))$ is a derivative subalgebra of $\mathscr{P}(Y)$. By the assumptions on θ we have $\theta(\mathrm{Per}_k(f)) = \mathrm{Per}_k(g) \in \mathscr{D}(g)$, hence $\theta(\mathscr{D}(f)) \subseteq \mathscr{D}(g)$. By the same reasoning $\theta^{-1}(\mathscr{D}(g)) \subseteq \mathscr{D}(f)$. But this gives $\theta(\mathscr{D}(f)) \subseteq \mathscr{D}(g) = \theta(\theta^{-1}(\mathscr{D}(g))) \subseteq \theta(\mathscr{D}(f))$. Since $|M| = |\theta(M)|$ for all subsets M of X, this shows that (d) holds.

Let $\rho : \mathscr{D}(f) \to \mathscr{D}(g)$ be an isomorphism of derivative algebras fulfilling (d). Set $\tilde{\mathscr{D}}(f) := \mathscr{D}(f) \cap L(f,0)$ and $\tilde{\mathscr{D}}(g) := \mathscr{D}(g) \cap L(g,0)$ as in Lemma 4.33. By this lemma, the derivative algebras $\mathscr{D}(f)$, $\mathscr{D}(g)$, $\tilde{\mathscr{D}}(f)$ and $\tilde{\mathscr{D}}(g)$ are all isomorphic. Furthermore

$$|M \cap L(f,0)| = |\rho(M) \cap L(g,0)| \text{ for all } M \in \mathscr{D}(f).$$

By Theorem 4.19 there is a homeomorphism $\varphi : L(f,0) \to L(g,0)$ such that $\varphi[\mathrm{Per}_k(f) \cap L(f,0)] = \mathrm{Per}_k(g) \cap L(g,0)$. Finally Theorem 4.29 implies that f and g are topologically conjugate. Thus we showed that (d) $\implies$ (a).

The equivalence (d) $\Leftrightarrow$ (e) is a consequence of Theorem 3.33. □

4.4. A Strange Pair of Shift Spaces

Our goal in this section is to show that the conclusion of Theorem 4.34 is not valid if we drop the assumption that the derivative algebra generated by $\{\mathrm{Per}_k(f) \mid k \in \{1, \ldots, p\}\}$ is finite. More specifically, we will construct a pair of 2-periodic cellular automata f and g on two-sided subshifts X and Y and a homeomorphism $\varphi : X \to Y$ such that

(a) f and g are not topologically orbit equivalent and in particular not topologically conjugate,

(b) $\varphi[\mathrm{Per}_k(f)] = \mathrm{Per}_k(g)$ for $k \in \mathbb{N}$.

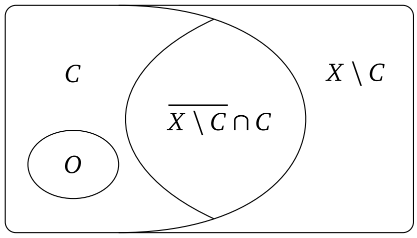


Figure 4.3.: Structure of the Stone space in Theorem 4.35.

The basis of our construction is the existence of a pair of metrizable Stone spaces Y, Z such that

$$Y \oplus Y \cong Z \oplus Z, \text{ but } Y \ncong Z. \tag{4.2}$$

Remember, that $\oplus$ denotes the disjoint union of topological spaces as defined in Definition 2.4.

Chapter 45 in [GH09] is devoted to the construction of such a pair of spaces. The original construction [Han58] is due to Hanf, the version presented by Givant and Halmos in [GH09] is based on simplifications by Dana Scott. More precisely, Hanf constructed metrizable Stone spaces Y, U such that

$$Y \ncong Y \oplus U, \text{ but } Y \cong Y \oplus U \oplus U. \tag{4.3}$$

Setting $Z := Y \oplus U$ gives the desired properties.

To use such spaces as the state space of a cellular automaton, we have to realize them as shift spaces. We will not repeat their construction here, but will realize such space as two-sided subshifts which will require some substantial modifications. Notice, however, that the spaces in Chapter 45 of [GH09] can be realized as one-sided subshifts, because they fulfill the assumption of the following theorem.

Theorem 4.35. *Let X be a metrizable Stone space. If X contains a set $C \subseteq X$ homeomorphic to the Cantor set with $C \nsubseteq \overline{X \setminus C}$, then X is homeomorphic to a one-sided subshift.*

Proof. The obstacle to finding a subshift homeomorphic to X is shift-invariance. With every subset of the full shift, all shifted versions of this subset have to be included as well. We will use the Cantor space C to "soak up this garbage". Since all Cantor spaces are homeomorphic, we can without loss of generality assume that $X \subseteq \{0,1\}^{\mathbb{N}}$. The set $Y := C \setminus \overline{X \setminus C} \neq \emptyset$ is an open subset of C and C is zero-dimensional. Hence we can find a subset $O \subseteq Y$ which is clopen in C, see Figure 4.3 for an illustration. Since C is compact, and hence closed in X, and O is closed in C, O is also closed in X. On the other hand $X \setminus O = (C \setminus O) \cup \overline{X \setminus C}$ is a closed subset of X. Thus O is open in X. Therefore X can be partitioned into the two clopen subsets O and $X \setminus O$ and we have $X \cong (X \setminus O) \oplus O$. Since O is a clopen subset of the Cantor space C, it is itself homeomorphic to C. Together this implies

$$\begin{aligned} X &\cong (X \setminus O) \oplus C \\ &\cong (X \setminus O) \oplus (C \oplus C) \\ &\cong X \oplus C. \end{aligned}$$

Set $S := \{ 2x \mid x \in X \} \cup \{0,1\}^{\mathbb{N}} \subseteq \{ 0,1,2 \}^{\mathbb{N}}$. We have

$$\begin{aligned} S &\cong \{ 2x \mid x \in X \} \cup \{ 0,1 \}^{\mathbb{N}} \\ &\cong X \oplus C \\ &\cong X. \end{aligned}$$

Since S is clearly shift invariant, we realized X as a one-sided subshift. □

While the assumptions in the previous theorem are fulfilled in our application, it is not clear, if they can be omitted. This gives rise to the following question.

Question 4.36. *Is every metrizable Stone space homeomorphic to a one-sided subshift?*

The answer is no in the case of two-sided subshifts as we will see soon.

What makes it hard to construct spaces Y and U with $Y \oplus U \ncong Y \cong Y \oplus U \oplus U$ is the large set of endomorphisms that, e.g., the Cantor set has. This makes it very easy to make room in Y in order to fit the additional space U into that space. One therefore desires a space Y with rigid subsets. This can be done by defining in topological terms points of type α for α an ordinal. We say that a point x in a topological space X has *type* α, if

$$x \in (\mathrm{Iso}(X)^{(\alpha)} \setminus \mathrm{Iso}(X)^{(\alpha+1)}) \cap \mathrm{Dense}(X).$$

Example 4.37. *Consider the following space $U := \bigcup_{\ell \in \mathbb{N}} U_\ell \cup 1^\infty \subseteq \{0,1,2\}^{\mathbb{N}}$, which appears as a building block in Chapter 45 in [GH09]. The sets U_ℓ consist of one-sided sequences over the alphabet $\{0,1,2\}$. Each of them starts with a block of ℓ ones. Either they contain at most ℓ blocks of ones of length ℓ and no other blocks of ones, or the initial block of ones is followed by an arbitrary sequence of zeros and twos, i.e.,*

$$U_\ell := \{1^\ell 0^{k_1} 1^\ell 0^{k_2} \dots 0^{k_m} 1^\ell 0^\infty \mid 1 \le m \le \ell, k_1, \dots, k_m \in \mathbb{N}\} \cup \left\{ 1^\ell x \;\middle|\; x \in \{0,2\}^{\mathbb{N}} \right\}.$$

The sets U_ℓ are disjoint closed subsets of the compact space U. Each U_ℓ contains exactly one point of type ℓ, namely $1^\ell 0^\infty$, and no other point of type $k \in \mathbb{N}$. The point 1^∞ is of type ω_0.

The space U in Example 4.37 contains exactly one point of type ℓ for each $\ell \in \mathbb{N}$. Since every point of type ℓ has to be mapped to a point of the same type, every automorphism of U has to fix the set of all points of type ℓ. In particular every automorphism of U must have infinitely many fixed points. But a two-sided subshift over the alphabet A has an automorphism with at most $|A|$ fixed points, namely the shift map. Hence U is not homeomorphic to a two-sided subshift. This leads directly to the following question.

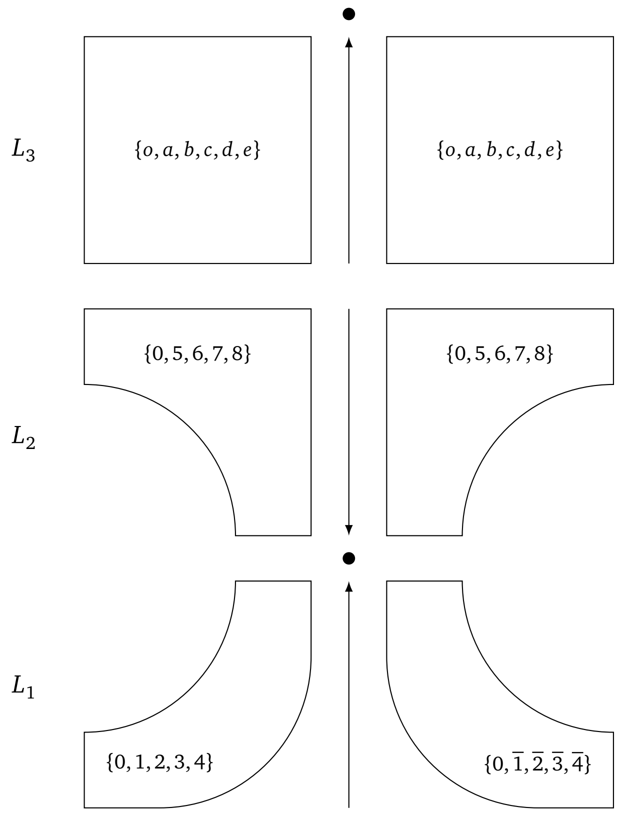


Figure 4.4.: Structure of the space Y.

Question 4.38. *Which metrizable Stone spaces are homeomorphic to two-sided subshifts? The answer also seems to be unknown for subshifts of finite type or sofic subshifts in the two-sided setting as well as in higher dimensions or over groups other than* $\mathbb{Z}^d$.

As we just saw, not all spaces occurring in the construction in [GH09] can be realized as two-sided subshifts. We therefore give an explicit construction of two two-sided subshifts Y and Z with $Y \ncong Z$ and $X \oplus X \cong Z \oplus Z$.

The shift will use the following alphabet $\{0, o, 1, 2, 3, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}, 5, 6, 7, 8, a, b, c, d, e\}$. These symbols fall into three groups: the zero symbols $\{0, o\}$, the marker symbols $\{1, \overline{1}, 5, a, c\}$ and the separator symbols $\{2, 3, 4, \overline{2}, \overline{3}, \overline{4}, 6, 7, 8, b, d, e\}$.

The space Y is horizontally[1] divided into 3 parts L_1, L_2 and L_3, each having a left and a right half L_k^{left} and L_k^{right}, and a middle part L_k^{middle} for $k \in \{1, 2, 3\}$. Figure 4.4 illustrates the general structure of Y and the symbols used in the respective parts. Each of the halves $L_k^{\text{left}}, L_k^{\text{right}}, k \in \{1, 2\}$ is the union of a countable number of sofic shifts $L_{k,\ell}^{\text{left}}$ and $L_{k,\ell}^{\text{right}}$, respectively. The halves L_3^{left} and L_3^{right} are the union of a countable number of sofic shifts $L_{3,\ell}^{\text{left}}$ respectively $L_{3,\ell}^{\text{right}}$ with one point ${}^{\infty}o^{\infty}$ removed.

[1] All spatial notions here have no topological meaning. They just allow us to refer for example to something “on the left” and hopefully help to provide some mental picture of the space.

The sofic shifts forming the left halves are given by edge-labeled graphs in Figure 4.5. They are concatenations of zero symbols and blocks of the form $\alpha\gamma^\ell\beta\gamma^{\ell+1}$ where α is a marker, β a separator and γ a zero symbol. The columns on the right side are obtained by replacing each occurrence of such a block $\alpha\gamma^\ell\beta\gamma^{\ell+1}$ by $\alpha\gamma^{\ell+1}\beta\gamma^\ell$, or in other words, by shifting each separator symbol one position to the left. Additionally, to obtain L_1^{right} from L_1^{left} one replaces all occurrences of non-zero symbols by their counterparts with a line on top. For example ${}^\infty(10200)10300.(10200)^\infty \in L_{1,1}^{\text{left}}$ is turned into ${}^\infty(\overline{1}00\overline{2}0)\overline{1}00\overline{3}0.(\overline{1}00\overline{2}0)^\infty \in L_{1,1}^{\text{right}}$. The result of this procedure is depicted in Figure 4.6.

The middle part is the closure of the left and the right half minus these two parts. All in all this means

$$\begin{aligned}
L_k^{\text{left}} &:= \dot{\bigcup_{\ell\in\mathbb{N}}} L_{k,\ell}^{\text{left}},\\
L_k^{\text{right}} &:= \dot{\bigcup_{\ell\in\mathbb{N}}} L_{k,\ell}^{\text{right}},\\
L_k^{\text{middle}} &:= \overline{L_k^{\text{left}} \cup L_k^{\text{right}}} \setminus (L_k^{\text{left}} \cup L_k^{\text{right}}),\\
L_k &:= L_k^{\text{left}} \cup L_k^{\text{middle}} \cup L_k^{\text{right}}.
\end{aligned}$$

The middle part thus consists of all points having at most one non-zero symbol and

$$\begin{aligned}
L_3^{\text{middle}} &= \{\, \sigma^m({}^\infty o.\beta o^\infty) \mid m \in \mathbb{Z}, \beta \in \{a,b,c,d,e\}\,\} \cup \{{}^\infty o^\infty\},\\
L_2^{\text{middle}} &= \{\, \sigma^m({}^\infty 0.\beta 0^\infty) \mid m \in \mathbb{Z}, \beta \in \{5,6,7,8\}\,\} \cup \{{}^\infty 0^\infty\},\\
L_1^{\text{middle}} &= \left\{\, \sigma^m({}^\infty 0.\beta 0^\infty) \;\middle|\; m \in \mathbb{Z}, \beta \in \{1,2,3,4,\overline{1},\overline{2},\overline{3},\overline{4}\}\,\right\} \cup \{{}^\infty 0^\infty\}.
\end{aligned}$$

All three parts L_1, L_2 and L_3 are compact. While L_3 and $L_1 \cup L_2$ are disjoint, we have $L_1 \cap L_2 = \{{}^\infty 0^\infty\}$.

For the definition of Z and some intermediate steps in our proof we also need modified versions $\tilde{L}_1, \tilde{L}_2$ of L_1 and L_2 obtained by replacing every block of the form $\alpha 0^\ell \beta 0^{\ell+1}$, where α is a marker symbol and β is a separator symbol, by $\alpha 0^{\ell+1}\beta 0^{\ell+1}$ in $L_{k,\ell}^{\text{left}}$ and by replacing $\alpha 0^{\ell+1}\beta 0^\ell$ by $\alpha 0^{\ell+2}\beta 0^\ell$ in $L_{k,\ell}^{\text{right}}$. The middle parts of L_k and $\tilde{L}_k$ are the same. Additionally, we set

$$\begin{aligned}
U^{\text{left}} &:= \left\{\, x \in \tilde{L}_1 \;\middle|\; x_{-1} = 1 \,\right\},\\
U^{\text{right}} &:= \left\{\, x \in \tilde{L}_1 \;\middle|\; x_{-1} = \overline{1} \,\right\},\\
V_1 &:= \bigcup_{k\in\mathbb{N}} \left\{\, x \in \tilde{L}_{1,k}^{\text{left}} \;\middle|\; x_{-\lceil\frac{k+3}{2}\rceil} = 1 \,\right\} \cup \bigcup_{k\in\mathbb{N}} \left\{\, x \in \tilde{L}_{1,k}^{\text{right}} \;\middle|\; x_{-\lceil\frac{k+4}{2}\rceil} = \overline{1} \,\right\},\\
V_2 &:= \bigcup_{k\in\mathbb{N}} \left\{\, x \in \tilde{L}_{2,k}^{\text{left}} \;\middle|\; x_{-\lceil\frac{k+3}{2}\rceil} = 5 \,\right\} \cup \bigcup_{k\in\mathbb{N}} \left\{\, x \in \tilde{L}_{2,k}^{\text{right}} \;\middle|\; x_{-\lceil\frac{k+4}{2}\rceil} = 5 \,\right\},\\
W &:= \left\{\, x \in \tilde{L}_2 \;\middle|\; x_{-1} = 5 \,\right\}
\end{aligned}$$

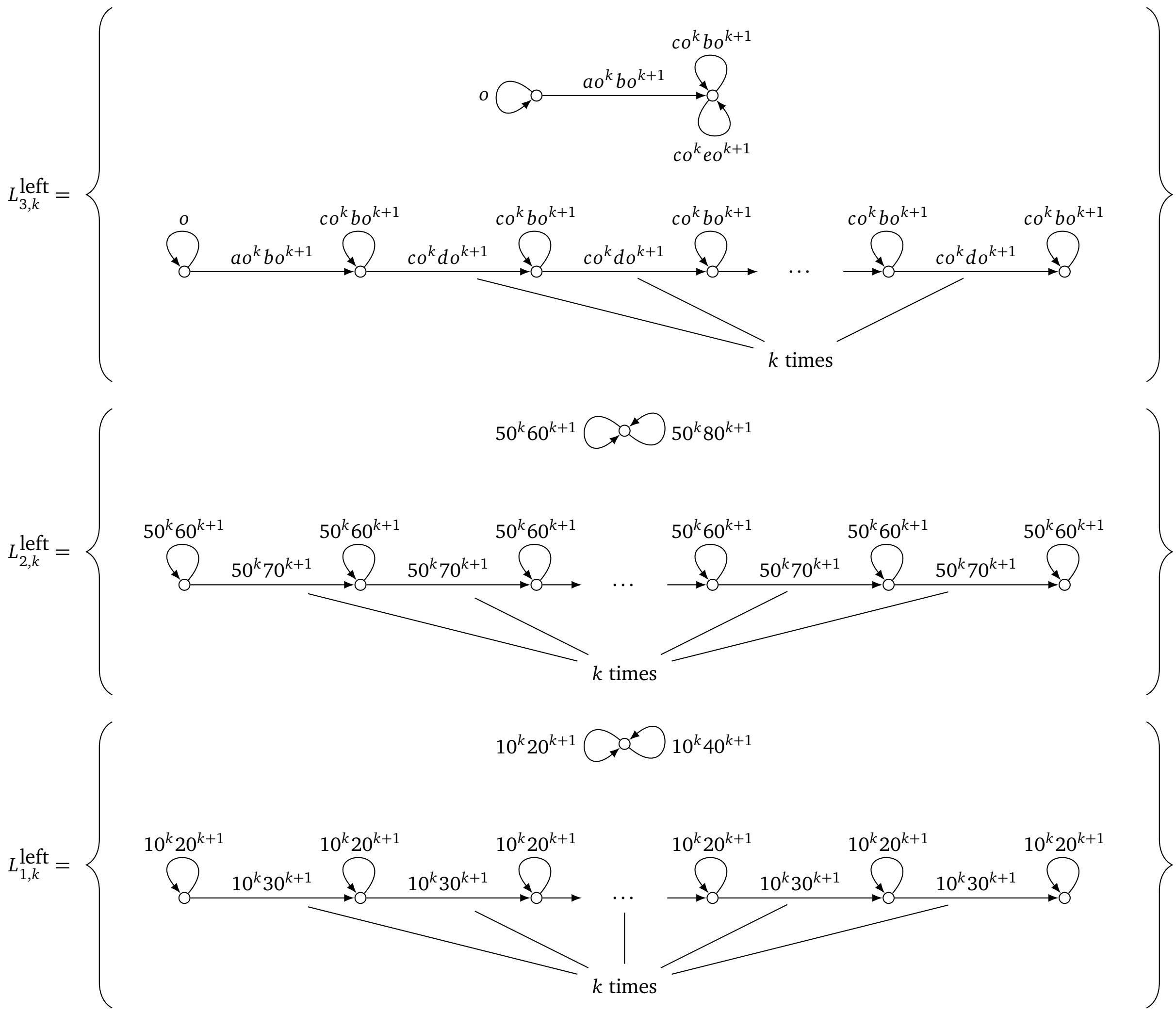


Figure 4.5.: The left half of the space Y.

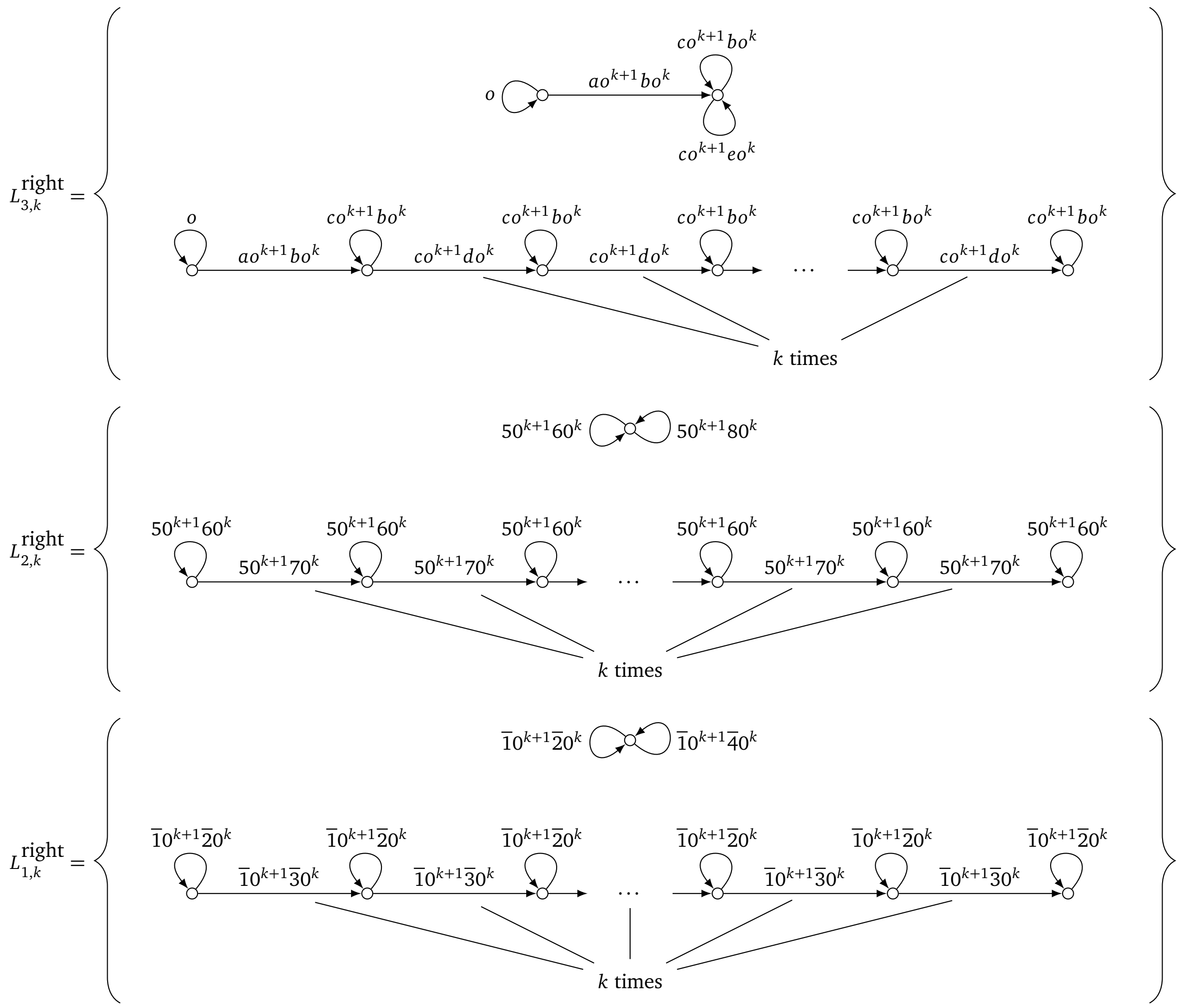


Figure 4.6.: The right half of the space Y.

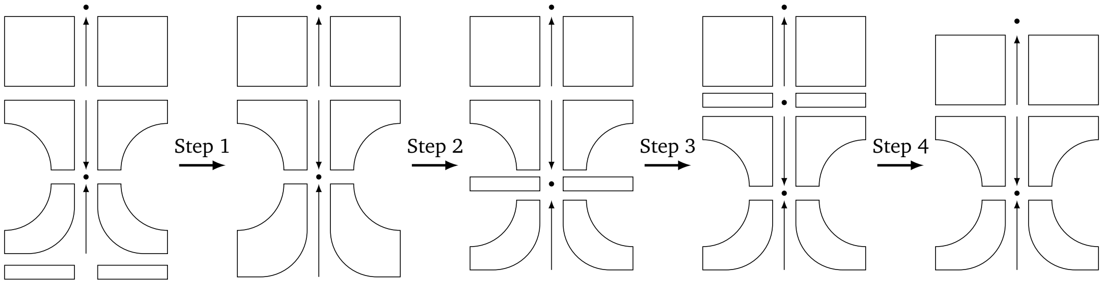


Figure 4.7.: Proving $Y \oplus Y \cong Z \oplus Z$ by a series of homeomorphisms.

We can now define

$$Y := L_1 \cup L_2 \cup L_3,$$
$$Z := \tilde{L}_1^{\text{left}} \cup L_1^{\text{middle}} \cup L_1^{\text{right}} \cup L_2 \cup L_3.$$

Since each of the columns and each of the three middle parts is shift invariant, so are Y and Z.

We show $Y \oplus Y \cong Z \oplus Z$ by constructing a sequence of homeomorphisms as shown in Figure 4.7.

Step 0: $Z \cong Y \oplus U^{\text{left}} \cong Y \oplus U^{\text{right}}$.
Define a map $\varphi_0 : Y \oplus U^{\text{left}} \to Z$. This map fixes all points in $L_2 \cup L_3$. For a point $x \in L_1$, φ_0 inserts the symbol 0 to the right of each marker symbol 1 appearing in x. If the marker symbol is immediately left of the origin, the symbol 0 is also inserted to the left of the origin. For example

$$x = \ldots 300010020001, 002000100\ldots$$

is mapped to

$$\varphi_0(x) = \ldots 3000\mathbf{1}0\mathbf{0}0200\mathbf{0}1\mathbf{0}, 00200\mathbf{0}1\mathbf{0}00\ldots$$

This map is clearly continuous and injective on Y. It is not surjective on Y as

$$Z \setminus \varphi_0[Y] = \left\{ x \in \tilde{L}_1 \;\middle|\; x_{-1} = 1 \right\}.$$

This remainder is precisely U^{left}, hence fixing the points in U^{left} turns φ_0 into a bijective map. Furthermore, φ_0 is continuous as the knowledge of $x_{[-k,k]}$ uniquely determines $\varphi_0(x)_{[-k,k]}$, in other words, the map is Lipschitz-continuous with Lipschitz-constant 1 with respect to the standard metric as defined in Example 2.7. Since Y and U^{left} are compact, their disjoint union is compact too. Therefore φ_0 is a homeomorphism by Theorem 2.1. The same arguments also establish that the maps constructed in the following steps are homeomorphisms once we know that they are bijective.

Since U^{left} and U^{right} are clearly homeomorphic, we also have $Z \cong Y \oplus U^{\text{right}}$.

Step 1: $(L_1 \cup L_2) \oplus U^{\text{left}} \oplus U^{\text{right}} \cong \tilde{L}_1 \cup L_2$.
This step is almost the same as Step 0. Define a map $\varphi_1 : (L_1 \cup L_2) \oplus U^{\text{left}} \uplus U^{\text{right}} \to \tilde{L}_1 \cup L_2$ as follows. First of all, φ_1 fixes all points in L_2. For a point $x \in L_1$, φ_1 inserts the symbol 0 to the right of each marker symbol $\{1, \overline{1}\}$ appearing in x. If the marker symbol is immediately left of the origin, the symbol 0 is also inserted to the left of the origin. This map restricted to $L_1 \cup L_2$ is clearly injective and maps L_1 into $\tilde{L}_1$, but it is not surjective as $\tilde{L}_1 \setminus \varphi_1(L) = \left\{ x \in \tilde{L}_1 \;\middle|\; x_{-1} \in \{1, \overline{1}\} \right\}$. But this remainder is precisely $U^{\text{left}} \cup U^{\text{right}}$, hence fixing the points in these sets turns φ_1 into a homeomorphism from $(L_1 \cup L_2) \oplus U^{\text{left}} \oplus U^{\text{right}}$ into $\tilde{L}_1 \cup L_2$.

Step 2: $\tilde{L}_1 \cup L_2 \cong L_1 \cup L_2 \cup V_1$.
Similarly as in the first step, we construct a homeomorphism $\varphi_2 : L_1 \cup L_2 \cup V_1 \to \tilde{L}_1 \cup L_2$. The map φ_2 fixes all points in L_2 and all points in the middle part of L_1. It inserts the symbol 0 between every occurrence of a marker symbol and a separator symbol if they appear in that order and are only separated by zeros. As in the description of φ_1, we have to take care how blocks overlapping the origin are dealt with. Let x be an element of one of the two halves of L_1. Let $i \in \{-1, -2, \dots\}$ be the position of the first occurrence of a non-zero symbol in x left of the origin and let $j \in \{0, 1, \dots\}$ be the position of the first non-zero symbol in x right of the origin.

If x_i is a marker symbol, then x_j is a separator symbol and φ_2 adds a zero between them. If $|i| - 1 > |j|$, we add the zero left of the origin, otherwise it is added to the right of the origin. In other words, the side with the longer block of zeros next to the origin gets an additional zero. If there is a draw, the zero is added on the right side.

For example for

$$x = \dots 30000100002000010, 00200001000 \dots$$

we have $i = -2$ and $j = 2$, hence the zero is added right of the origin and we get

$$\varphi_2(x) = \dots 300001\mathbf{0}0002000010, \mathbf{0}0020000010\mathbf{0}000 \dots .$$

This method ensures that φ_2 is continuous. Clearly φ_2 maps L_1 into $\tilde{L}_1$. However, not every point in $\tilde{L}_1$ appears as the image of $(\varphi_2)_{|L_1}$. If $\varphi_2(x) = y \in \tilde{L}_1$ has a marker symbol at position $i \in \{\dots, -2, -1\}$ as the first non-zero symbol left of the origin and a separator symbol at position $j \in \{0, 1, \dots\}$ as the first non-zero symbol right of the origin, x also must have a marker symbol as the first non-zero symbol left of the origin and a separator symbol as the first non-zero symbol right of the origin.

If y is in the left half of $\tilde{L}_1$, there is some $k \in \mathbb{N}$ with $y \in \tilde{L}_{1,k}^{\text{left}}$ and the equation $|i| - 1 + |j| = k + 1$ holds. Since the zero symbol is always added to the side already having more zeros and to the right side in case of a draw, it is impossible that $|i| - 1 = |j|$ or $|i| - 1 = |j| + 1$. Hence it can never happen that $2|i| = k + 3$ or $2|i| = k + 4$. In other words, it can not happen that $|i| = \lceil \frac{k+3}{2} \rceil$.

On the other hand for $y \in \tilde{L}_{1,k}^{\text{right}}$ we have $|i| - 1 + |j| = k + 2$ and it can never happen that $|i| = \lceil \frac{k+4}{2} \rceil$. All in all this means $\tilde{L}_1 \setminus \varphi_2[L_1] = V_1$. Fixing the points in V_1 therefore gives us a homeomorphism $\varphi_2 : L_1 \cup L_2 \cup V_1 \to \tilde{L}_1 \cup L_2$.

Step 3: $L_1 \cup L_2 \cup V_1 \cong (L_1 \cup L_2) \oplus W$.
The third step is basically equivalent to the composition of Step 2 and Step 1 in that order.

Define a homeomorphism $\varphi_3 : L_1 \cup L_2 \cup V_1 \to L_1 \cup \tilde{L}_2$ by fixing all points in L_1 and all points in the middle part of L_2. In points in L_2^{left} and L_2^{right} a zero is inserted between every marker and separator symbol.

If the origin lies between the marker symbol and the following separator symbol, the zero is added on the side of the origin with the larger block of zeros as in φ_2. In case of a draw, it is added to the right side. This gives

$$\varphi_3[L_1 \cup L_2] = L_1 \cup \tilde{L}_2 \setminus V_2$$

The set V_1 is then mapped by φ_3 bijectively into V_2 by replacing the marker symbols 1 and $\overline{1}$ by the marker symbol 5 and replacing the separator symbols 2 and $\overline{2}$ by the separator symbols 6, replacing 3 and $\overline{3}$ by 7 and by replacing 4 and $\overline{4}$ by 8. The resulting map is a homeomorphism $\varphi_3 : L_1 \cup L_2 \cup V_1 \to L_1 \cup \tilde{L}_2$.

Define a homeomorphism $\varphi_4 : (L_1 \cup L_2) \oplus W \to L_1 \cup \tilde{L}_2$ by first of all fixing all points in L_1. In points in L_2^{left} and L_2^{right} a zero is inserted between every marker and separator symbol. If the marker symbol is immediately left of the origin, the zero is also inserted left of the origin. This gives

$$\varphi_4[L_1 \cup L_2] = L_1 \cup \tilde{L}_2 \setminus W.$$

By fixing all points in W we therefore get a homeomorphism $\varphi_4 : (L_1 \cup L_2) \oplus W \to L_1 \cup \tilde{L}_2$.

Step 4: $L_3 \oplus W = L_3$.
Define a homeomorphism $\varphi_5 : L_3 \oplus W \to L_3$ as follows. For $x \in L_3$ insert the zero symbol o in front of every a. If the a is immediately to the right of the origin, insert the o to the right of the origin, too. Therefore φ_5 maps L_3 into L_3 and $L_3 \setminus \varphi_5[L_3] = \{ x \in L_3 \mid x_0 = a \} =: R$. We now have to map W bijectively onto R.

Let x be a point in $W \cap \tilde{L}_{2,k}^{\text{left}}$ for some $k \in \mathbb{N}$. We can partition x into blocks of length $2k+4$, all starting with the marker symbol 5 and containing exactly one separator symbol s. Such a block has the form $50^{k+1}s0^{k+1}$. By removing all zero and marker symbols from x, we obtain a two-sided sequence $\alpha(x) \in \{6,7,8\}^{\mathbb{Z}}$ and it is possible to reconstruct x from this sequence if we know that $x \in W \cap \tilde{L}_{2,k}^{\text{left}}$. We turn this into a one-sided sequence $\beta(x) \in \{6,7,8\}^{\mathbb{N}}$ by defining

$$(\beta(x)_i)_{i \in \mathbb{N}} := (\alpha(x)_0, \alpha(x)_{-1}, \alpha(x)_1, \alpha(x)_{-2}, \alpha(x)_2, \dots).$$

From $\beta(x)$ we create a point $\varphi_5(x)$ by defining

$$\varphi_5(x) := {}^{\infty}o.(ao^k bo^{k+1}) w_{\beta(x)_1} w_{\beta(x)_2} \cdots$$

where

$$\begin{aligned} w_6 &:= co^k bo^{k+1}, \\ w_7 &:= co^k do^{k+1}, \\ w_8 &:= co^k eo^{k+1}. \end{aligned}$$

For $x \in \tilde{L}_{2,k}^{\text{left}}$ the sequence $\beta(x)$ is either a sequence of symbols in $\{6,7\}$ with at most k occurrences of 7 or it is an arbitrary element of $\{6,8\}^{\mathbb{N}}$. Hence $\varphi_5(x)$ is a point in R and φ_5 is injective, since w_6, w_7 and w_8 are pairwise different. As an illustration consider the following example.

$$\begin{aligned} x &= \dots 50006000\,50006000\,50006000\,5.0007000\,50007000\,50006000 \dots \in \tilde{L}_{2,2}^{\text{left}} \\ \alpha(x) &= \dots 666.776\dots \\ \beta(x) &= 767666\dots \\ \varphi_5(x) &= {}^{\infty}o, aooboooo\,coodooo\,coodooo\,cooeooo\,\dots \end{aligned}$$

By the same procedure we can map $W \cap \tilde{L}_2^{\text{right}}$ bijectively onto $R \cap L_3^{\text{right}}$. Finally, the unique point in $W \cap \tilde{L}_2^{\text{middle}}$, ${}^{\infty}0.50^{\infty}$, is mapped to ${}^{\infty}o, ao^{\infty}$ by φ_5.

All in all this means

$$\begin{aligned} Z \oplus Z &\cong (Y \oplus U^{\text{left}}) \oplus (Y \oplus U^{\text{right}}) \\ &\cong \tilde{L}_1 \cup L_2 \oplus L_3 \oplus Y \\ &\cong L_1 \cup L_2 \cup V_1 \oplus L_3 \oplus Y \\ &\cong L_1 \cup L_2 \oplus W \oplus L_3 \oplus Y \\ &\cong L_1 \cup L_2 \oplus L_3 \oplus Y \\ &\cong Y \oplus Y \end{aligned}$$

After we showed that $Y \oplus Y \cong Z \oplus Z$, we want to show that Y and Z are not homeomorphic. The proof is basically the same as in [GH09]. The additional twist mainly consists in showing the existence of an involution φ of X fixing a certain topologically defined subset. The existence of such an involution is obvious in the non-shift-invariant construction in Halmos's proof, but requires more work in our setting. We first identify the points of type ℓ in the spaces Y and Z. A point y is isolated in Y if and only if it is contained in $L_{k,\ell}^{\text{left}}$ or $L_{k,\ell}^{\text{right}}$ and contains ℓ occurrences of blocks in $M := \{10^k 30^{k+1}, \overline{1}0^{k+1}\overline{3}0^k, 50^k 70^{k+1}, 50^{k+1}70^k, co^k do^{k+1}, co^{k+1}do^k, ao^k bo^{k+1}, ao^{k+1}bo^k\}$. Therefore all points in the middle of Y are contained in $\mathrm{Iso}(Y)^{(\omega_0)}$, so they are in particular not of type ℓ for $\ell \in \mathbb{N}$. The points y in $L_{k,\ell}^{\text{left}}$ and $L_{k,\ell}^{\text{right}}$ which are contained in $\mathrm{Iso}(Y)^{(m)}$ for $m \geq 1$ are therefore precisely those having at most $k-m$ occurrences of blocks in M. A point is in $\mathrm{Dense}(Y)$ if and only if its does not contain any of the symbols $3, \overline{3}, 7$ or d.

Together this implies for $\ell \geq 2$

$$\begin{aligned}
\mathrm{Type}_\ell(Y) = &\left\{ \sigma^k({}^\infty(10^\ell 20^{\ell+1})^\infty) \;\middle|\; k \in \{0,\dots,2\ell+2\} \right\} && (\text{in } L^{\text{left}}_{1,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty(\overline{1}0^{\ell+1}\overline{2}0^\ell)^\infty) \;\middle|\; k \in \{0,\dots,2\ell+2\} \right\} && (\text{in } L^{\text{right}}_{1,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty(50^\ell 60^{\ell+1})^\infty) \;\middle|\; k \in \{0,\dots,2\ell+2\} \right\} && (\text{in } L^{\text{left}}_{2,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty(50^{\ell+1}60^\ell)^\infty) \;\middle|\; k \in \{0,\dots,2\ell+2\} \right\} && (\text{in } L^{\text{right}}_{2,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty o.(ao^\ell bo^{\ell+1})(co^\ell bo^{\ell+1})^\infty) \;\middle|\; k \in \mathbb{Z} \right\} && (\text{in } L^{\text{left}}_{3,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty(co^{\ell-1}bo^\ell)^\infty) \;\middle|\; k \in \{0,\dots,2\ell\} \right\} && (\text{in } L^{\text{left}}_{3,\ell-1}) \\
&\cup \left\{ \sigma^k({}^\infty o.(ao^{\ell+1}bo^\ell)(co^{\ell+1}bo^\ell)^\infty) \;\middle|\; k \in \mathbb{Z} \right\} && (\text{in } L^{\text{right}}_{3,\ell}) \\
&\cup \left\{ \sigma^k({}^\infty o.(co^\ell bo^{\ell-1})^\infty) \;\middle|\; k \in \{0,\dots,2\ell\} \right\}. && (\text{in } L^{\text{right}}_{3,\ell-1})
\end{aligned}$$

Almost the same characterization holds for the points of type ℓ in Z and we have

$$\begin{aligned}
\mathrm{Type}_\ell(Z) = &\,\mathrm{Type}_\ell(Y) \setminus L^{\text{left}}_{1,\ell} \\
&\cup \left\{ \sigma^k({}^\infty(10^{\ell+1}20^{\ell+1})^\infty) \;\middle|\; k \in \{0,\dots,2\ell+3\} \right\} && (\text{in } \tilde{L}^{\text{left}}_{1,\ell}).
\end{aligned}$$

Based on this we can further characterize certain subsets of Z. For a topological space X, let $\mathrm{Lim}(X)$ be the set of all points $x \in X$, for which there is $k_0 \in \mathbb{N}$ and a sequence $(y_k)_{k\in\mathbb{N}}$ converging to x with $y_k \in \mathrm{Type}_k(X)$ for all $k \geq k_0$. Let $\mathrm{Sing}(X)$ be the set of all points $x \in \mathrm{Lim}(X)$ for which there exists a k_0 and a neighborhood U of x such that $|U \cap \mathrm{Type}_k(X)| = 1$ for all $k \geq k_0$. Finally define $\mathrm{Fixed}(X) := \mathrm{Lim}(X) \setminus \mathrm{Sing}(X)$. It is easy to see that $\mathrm{Lim}(Y)$ as well as $\mathrm{Lim}(Z)$ are precisely those points that contain at most one non-zero symbol and for which this symbol is in $\{1,2,\overline{1},\overline{2},5,6,a,b,c\}$. Furthermore,

$$\mathrm{Sing}(Y) = \mathrm{Sing}(Z) = \bigcup_{k\in\mathbb{Z}} \sigma^k[\{{}^\infty 0.s0^\infty \mid s \in \{1,2,,\overline{1},\overline{2}\}]$$

and

$$\begin{aligned}
\mathrm{Fixed}(Y) = \mathrm{Fixed}(Z) = &\bigcup_{k\in\mathbb{Z}} \sigma^k[\{\, {}^\infty 0.s0^\infty \;\mid s \in \{0,5,6\}\,\}] \\
&\cup \bigcup_{k\in\mathbb{Z}} \sigma^k[\{\, {}^\infty o.so^\infty \;\mid s \in \{o,a,b,c\}\,\}].
\end{aligned}$$

Now we want to show that there exists an involution $\varphi : Y \to Y$, i.e., a homeomorphism with $\varphi^2 = \mathrm{id}_Y$, such that $\mathrm{Fix}(\varphi) \supseteq \mathrm{Fixed}(Y)$ and such that $\mathrm{Fix}(\varphi) \cap \bigcup_{k\in\mathbb{N}} \mathrm{Type}_k(Y) = \emptyset$. The involution φ does the following. All points in the middle, i.e., those that contain at most one non-zero symbol, are fixed by φ. Consider a point x in one of the two halves of Y. Let i be the position of the first non-zero symbol in x left of the origin and let j be the position of the first non-zero symbol in x right of the origin. To get to a point in the other half, we will either move all marker symbols or all separator symbols. If x is in the left half, either all marker symbols are moved to the left or all separator symbols are moved to the right. If x is in the right half, we do the opposite and either move all marker symbols to the right or all separator symbols to the left. Moving a symbol left or right means that we exchange it with the zero symbol on its left or right side, respectively. To determine if marker or separator symbols should be moved, consider the symbol x_i if $|i| > |j|$ or the symbol x_j if $|i| \leq |j|$. If this symbol is a separator symbol, move these, if it is a marker symbol, move those. Finally we exchange $1, 2, 3, 4$ with $\overline{1}, \overline{2}, \overline{3}, \overline{4}$.

For example the point

$$x = \ldots 00001000.02000001000030 \ldots$$

has a block of three zeros left of the origin and a block of one zero on the right, hence $i = -4$ and $j = 1$. Since the block of zeros left of the origin is larger than the block of zeros on the right and it is terminated by a marker symbol, we are going to move the marker symbols. The point x lies in the left half, hence we are going to move all marker symbols to the left and the image of x is

$$\varphi(x) = \ldots 000\overline{1}0000.0\overline{2}000010000 0\overline{3}0 \ldots$$

It is easy to see that φ is an involution with $\varphi[L_{i,k}^{\text{left}}] = L_{i,k}^{\text{right}}$ which fixes all points with at most one non-zero symbol different from $1, 2, 3, 4, \overline{1}, \overline{2}, \overline{3}, \overline{4}$. Hence no point of type $k \in \mathbb{N}$ is fixed and $\text{Fixed}(Y) \subseteq \text{Fix}(\varphi)$.

Restricted to one of the sets $L_{i,k}^{\text{left}}$ and $L_{i,k}^{\text{right}}$, the map φ is clearly continuous, as the knowledge of $x_{[-\ell,\ell]}$ allows one to deduce $\varphi(x)_{[-\ell,\ell]}$. To see that φ is continuous, let x be a point in the middle and let y be some point such that $x_{[-\ell,\ell]} = y_{[-\ell,\ell]}$ for some $\ell \in \mathbb{N}$. We know that x consists of zero symbols and at most one non-zero symbol. If $x_{[-\ell,\ell]}$ contains only zero symbols, then so does $\varphi(x)_{[-\ell+1,\ell-1]}$ and $\varphi(y)_{[-\ell+1,\ell-1]}$. Now assume there is a non-zero symbol in x at position $i \in \{-\ell, \ldots, \ell\}$. If $i > 0$, then the block of zeros left of the origin in y is larger than the block of zeros right of the origin, hence the symbol at position i in y is not moved and we have $\varphi(y)_{[-\ell+1,\ell-1]} = \varphi(y)_{[-\ell+1,\ell+1]}$. The same holds for $i < 0$, hence φ is a continuous map.

We now have to show that Z does not have such an involution. Assume there would be an involution $\psi : Z \to Z$, fixing no point of type k and fixing all points in $\text{Fixed}(Z)$.

The set $M = \tilde{L}_1^{\text{left}} \cup L_1^{\text{middle}} \cup L_1^{\text{right}} \cup L_2 \subseteq Z$ contains an odd number of points of type k for every $k \in \mathbb{N}$. Since ψ is an involution and non of these points is fixed by ψ, there is a sequence $(y_k)_{k\in\mathbb{N}}$ of points of type k in M whose image under ψ lies in $Z \setminus M = L_3$. Since M is compact, $(y_k)_{k\in N}$ has a subsequence $(y_{k_m})_{m\in\mathbb{N}}$ converging to $y \in M \cap \text{Lim}(Z)$. Since ψ is continuous and L_3 is a closed subset of Z, $\psi(y) \in L_3$. In particular $y \neq \psi(y)$ and hence $y \notin \text{Fixed}(Z)$. Therefore $y \in \text{Lim}(Z) \setminus \text{Fixed}(Z) = \text{Sing}(Z)$. Since points of type k as well as $\text{Sing}(Z)$ are defined in purely topological terms, they are invariant under homeomorphisms, in particular $\psi(y) \in \text{Sing}(Z) \subseteq M$. But this contradicts $\psi(y) \in L_3$. □

We have now reached the goal of this section in the following theorem.

Theorem 4.39. *There exist two-sided subshifts $\tilde{Y}$ and $\tilde{Z}$, cellular automata $f : \tilde{Y} \to \tilde{Y}, g : \tilde{Z} \to \tilde{Z}$ and a homeomorphism $\theta : \tilde{Y} \to \tilde{Z}$ such that*

(a) *f and g are not topologically orbit equivalent,*

(b) $\theta[Per_k(f)] = Per_k(g)$ *for* $k \in \mathbb{N}$.

Proof. Using the spaces Y and Z defined above, define the spaces $\tilde{Y} := \{-1,1\} \times Y$ and $\tilde{Z} := \{-1,1\} \times Z$ and the functions $f\colon \tilde{Y} \to \tilde{Y}, f(k,x) = (-k,x)$ and $g\colon \tilde{Z} \to \tilde{Z}, g(k,x) = (-k,x)$. Both functions are two-periodic and have no fixed points, therefore $\mathrm{Per}_2(f) = \tilde{Y} \cong \tilde{Z} = \mathrm{Per}_2(g)$ and condition (c) of Theorem 4.34 is fulfilled. Now $\tilde{Y}/\mathrm{Orb}(f) \cong Y$ and $\tilde{Z}/\mathrm{Orb}(g) \cong Z$. Since $Y \ncong Z$ holds, the dynamical systems f and g can not be orbit equivalent by Theorem 4.29. □

Based on this example we can go further and construct a pair of non-conjugate periodic dynamical systems on the full Cantor space fulfilling property (c) of Theorem 4.34. As an additional ingredient, the construction uses the following lemma.

Lemma 4.40. *Let $f : X \to X$ be a p-periodic dynamical system on a metrizable Stone space X with $p \in \mathbb{N}$. If $q \geq 2$ is coprime to p, then there exists a pq-periodic dynamical systems $\tilde{f} : \tilde{X} \to \tilde{X}$ on the Cantor space $\tilde{X}$ and a subspace $M \subseteq \tilde{X}$, such every point in X either has minimal period pq or minimal period p with respect to $\tilde{f}$, such that $Per_p(\tilde{f}) = M$ and such that $\tilde{f}_{|M}$ is conjugate to f. The space $\tilde{X}$ and the subspace M depend only on X, p and q, but not on f.*

Proof. Let $C \subseteq \mathbb{R}$ be the middle third Cantor set. Consider the sets $C_k = e^{k\frac{2\pi i}{q}} C \subseteq \mathbb{C}$ for $k = 0, \ldots, q-1$. Define $\tilde{X} := X \times (\bigcup_{k=0}^{q-1} C_k)$. This space is compact as the product of compact spaces by Tychonoff's theorem, Theorem 2.2. As the product of zero-dimensional spaces it is also zero-dimensional, see Theorem 2.3, and it is also perfect because C and hence all of the sets C_k are perfect. All these properties together imply that X is a Cantor space. Define $\tilde{f} : \tilde{X} \to \tilde{X}$ by $\tilde{f}(x,c) := (f(x), e^{\frac{2\pi i}{q}} c)$. Clearly this map is continuous and pq-periodic.

Since q is coprime to p, the only p-periodic points (x,c) of $\tilde{f}$ are those with $c = 0$, hence $\mathrm{Per}_p(\tilde{f}) = X \times \{0\}$ and $\tilde{f}_{|\mathrm{Per}_p(\tilde{f})}$ is conjugate to f via the projection onto the first coordinate. □

Corollary 4.41. *There exists a pair $\tilde{f}, \tilde{g}$ of 6-periodic dynamical systems on the Cantor set, such that f and g are not topologically orbit equivalent but such that condition (c) of Theorem 4.34 holds.*

Proof. Let $f : \tilde{Y} \to \tilde{Y}$ and $g : \tilde{Z} \to \tilde{Z}$ be the pair of cellular automata constructed in 4.39. There is a homeomorphism $\varphi : \tilde{Y} \to \tilde{Z}$. Applying Lemma 4.40 to $f : \tilde{Y} \to \tilde{Y}$ and $\varphi^{-1} \circ g \circ \varphi : \tilde{Y} \to \tilde{Y}$ with $q := 3$ we obtain a Cantor space X with a subspace M and 6-periodic dynamical systems $\tilde{f} : X \to X$ and $\tilde{g} : X \to X$. Since $\tilde{f}_{|\mathrm{Per}_2(f)}$ is conjugate to f and $\tilde{g}_{|\mathrm{Per}_2(g)}$ is conjugate to $\varphi^{-1} \circ g \circ \varphi$, the dynamical systems f and g are not topologically orbit equivalent. On the other hand, $X = \widetilde{\mathrm{Per}}_2(\tilde{f}) \cup \widetilde{\mathrm{Per}}_6(\tilde{f}) = \widetilde{\mathrm{Per}}_2(\tilde{g}) \cup \widetilde{\mathrm{Per}}_6(\tilde{g})$. Since $\widetilde{\mathrm{Per}}_2(\tilde{f}) =$

$\mathrm{Per}_2(\tilde{f}) = M = \widetilde{\mathrm{Per}}_2(\tilde{g}) = \mathrm{Per}_2(\tilde{g})$, we also have $\mathrm{Per}_6(\tilde{f}) = \mathrm{Per}_6(\tilde{g})$ and therefore condition (c) of Theorem 4.34 is fulfilled with $\theta = \mathrm{id}_X$. $\square$

The constructions in this section rely heavily on spaces with infinite Cantor-Bendixson rank. In the next section we will see that in dimension one subshifts of finite type always have finite Cantor-Bendixson rank. This is, however, not true in higher dimensions as constructions by Jeandel and Vanier [JV11] as well as Salo and Törmäa [ST13] show.

Question 4.42. *Are there two subshifts of finite type $Y \subseteq A^{\mathbb{Z}^2}$ and $Z \subseteq A^{\mathbb{Z}^2}$ such that $Y \oplus Y \cong Z \oplus Z$ but $Y \ncong Z$?*

Chapter 5.

Topological Structure of Subshifts

In this chapter, we show that the derivative algebra generated by a finite number of sofic shifts is finite. As we already know, sofic shifts can be represented by edge-labeled graphs. Let G be a graph with an edge labeling $\mathscr{L}$ and a set of distinguished vertices S. We saw that $\mathscr{L}(\mathrm{Path}_{\mathbb{Z}}(G))$ is a two-sided sofic shift and that every two-sided sofic shift arises that way. If we replace $\mathbb{Z}$ by $\mathbb{N}$, we obtain the same result for one-sided sofic shifts. If one applies the labeling to $\mathrm{Path}(G,S)$, one obtains a regular language and if one does the same for $\mathrm{Path}_{\mathbb{N}}(G,S)$ one obtains the adherence of this language. Head showed in [Hea85] and [Hea86] that the adherence of a regular language generates a finite derivative algebra[1]. He later used this result in [Hea91] to show that two-sided sofic shifts generate finite derivative algebras. We extend these results to the derivative algebras generated by multiple two-sided sofic shifts.

We first analyze the simpler special case of a finite collection of subshifts of finite type in Section 5.1, in other words, we ignore the edge labels and concentrate on the structure of the graphs. Besides the space $\mathrm{Path}_{\mathbb{Z}}(G)$ we will also look at the space $\mathrm{Path}_{\mathbb{N}}(G,S)$. Somewhat surprisingly, introducing this distinguished set S of initial vertices allows us to get rid of the edge labels also in the sofic case, as long as we are only constructing homeomorphisms and not conjugacies. This will be shown in Section 5.4.

In between we will show in Section 5.2 how to use the condensation of a graph to effectively obtain information about our derivative algebras. Furthermore, in Section 5.3 we show how to realize abstract derivative algebras as the derivative algebras of one-sided subshifts of finite type.

The results on the derivative algebras of multiple sofic shifts obtained in this chapter allow us to apply Theorem 4.34 to periodic cellular automata. As we already know, the set of p-periodic points of a cellular automaton on a one-sided or two-sided sofic shift is again a sofic shift. The results in this chapter therefore apply in particular to the periodic point algebra of periodic cellular automata on such shifts.

[1] More precisely he used the framework of topological Boolean algebras.

5.1. Derivative Algebras of Multiple Subshifts of Finite Type

Our investigation starts by analyzing the topological structure of multiple intersecting subshifts of finite type. We know that every two-sided subshift of finite type is conjugate to the edge shift of some graph. Our goal is therefore to partition the edge shift of a graph G into the atoms of a derivative algebra. This partition is given by the sets $\mathrm{Full}_{\mathbb{Z}}(K)$ for subgraphs K of G.

Definition 5.1 ($\mathrm{Full}_{\mathbb{Z}}(G)$)**.** *Let G be a graph. We denote by $\mathrm{Full}_{\mathbb{Z}}(G) \subseteq \mathrm{Path}_{\mathbb{Z}}(G)$ the set of all bi-infinite paths in G that contain all edges of G, i.e.,*

$$\mathrm{Full}_{\mathbb{Z}}(G) := \{\, \gamma \in \mathrm{Path}_{\mathbb{Z}}(G) \mid \forall e \in E(G)\, \exists k \in \mathbb{Z} : \gamma_k = e \,\}.$$

The sets $\mathrm{Full}(G) \subseteq \mathrm{Path}(G)$, $\mathrm{Full}_{\mathbb{N}}(G) \subseteq \mathrm{Path}_{\mathbb{N}}(G)$ and $\mathrm{Full}_{\mathbb{N}}(G,S) \subseteq \mathrm{Path}_{\mathbb{N}}(G,S)$ for $S \subseteq V(G)$ are defined analogously.

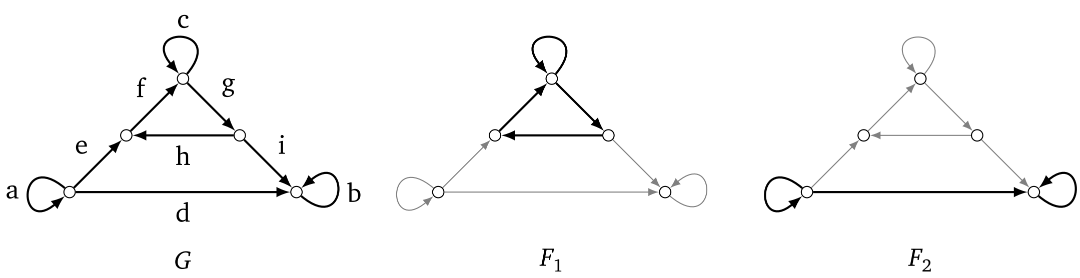


Figure 5.1.: The graph G of Example 5.2 together with its subgraphs F_1 and F_2.

Example 5.2. *Consider the graph G with its two subgraphs F_1 and F_2 in Figure 5.1. No path in G can contain both the edge d and the edge e, hence $\mathrm{Full}_{\mathbb{Z}}(G) = \emptyset$. On the other hand,*

$$\begin{aligned}
\mathrm{Full}_{\mathbb{Z}}(F_1) &= \mathrm{Path}_{\mathbb{Z}}(F_1) \setminus (\{\, {}^{\infty}c^{\infty} \,\} \cup \left\{\, \sigma^k({}^{\infty}(fgh)^{\infty}) \;\middle|\; k \in \{0,1,2\} \,\right\}),\\
\mathrm{Full}_{\mathbb{Z}}(F_2) &= \left\{\, \sigma^k({}^{\infty}a.db^{\infty}) \;\middle|\; k \in \mathbb{Z} \,\right\}.
\end{aligned}$$

The following two lemmas will provide building blocks to show that subsets of our edge shifts have a certain cardinality.

Lemma 5.3. *Let G be a strongly connected graph with at least one edge. For every pair of vertices $i_1, i_2 \in V(G)$, there is at least one finite path $\gamma \in \mathrm{Full}(G)$ starting in i_1 and ending in i_2.*

Proof. Let $e_1, \ldots, e_n$ be an enumeration of the edges of G. Since G is strongly connected, for every pair of vertices $j_1, j_2 \in V(G)$ there is a path $\alpha_{j_1,j_2} \in \mathrm{Path}(G)$ from j_1 to j_2. Setting

$$\gamma := \alpha_{i_1,\mathrm{i}(e_1)} e_1 \alpha_{\mathrm{t}(e_1),\mathrm{i}(e_2)} e_2 \alpha_{\mathrm{t}(e_2),\mathrm{i}(e_3)} \cdots e_{n-1} \alpha_{\mathrm{t}(e_{n-1}),\mathrm{i}(e_n)} e_n \alpha_{\mathrm{t}(e_n),i_2}$$

gives a path starting in i_1, traversing all edges of G and ending in i_2. $\square$

Lemma 5.4. *Let G be a strongly connected graph which has at least one edge and which is not a cycle, i.e., there are two different edges $e_1, e_2 \in E(G)$ with $\mathrm{i}(e_1) = \mathrm{i}(e_2)$. For every vertex $i \in V(G)$ there are two different paths $\gamma_1, \gamma_2 \in \mathrm{Full}(G)$ of equal length starting and ending in i.*

Proof. Let $e_1, e_2 \in E(G)$ be two different edges with $\mathrm{i}(e_1) = \mathrm{i}(e_2) =: j$. For $j_1, j_2 \in V(G)$ let α_{j_1,j_2} be a path in Full(G) staring in j_1 and ending in j_2, whose existence is guaranteed by Lemma 5.3. Set $\tilde{\gamma}_1 := \alpha_{i,j} e_1 \alpha_{t(e_1),i}$ and $\tilde{\gamma}_2 := \alpha_{i,j} e_2 \alpha_{t(e_2),i}$. Set $\gamma_1 := \tilde{\gamma}_1^{|\tilde{\gamma}_2|}$ and $\gamma_2 := \tilde{\gamma}_2^{|\tilde{\gamma}_1|}$. Both of these paths start and end in i, contain all edges of G and have length $|\tilde{\gamma}_1| \cdot |\tilde{\gamma}_2|$. They are, however, different from each other, as $(\gamma_1)_{|\alpha^{i,j}|+1} = e_1 \neq e_2 = (\gamma_2)_{|\alpha^{i,j}|+1}$. □

The fact that $\mathrm{Full}_{\mathbb{N}}(G)$ or $\mathrm{Full}_{\mathbb{Z}}(G)$ is non-empty for a graph G puts severe restrictions on the structure of G. It must have the form defined in the following definition.

Definition 5.5. *Let G be a graph, let $G_1, \dots, G_n$ be subgraphs of G with pairwise disjoint vertex sets and let $e_1, \dots, e_{n-1} \in E(G)$ be edges. We say that G has the form*

$$G_1 \xrightarrow{e_1} G_2 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} G_n$$

if the following conditions hold.

$$\begin{aligned} V(G) &= V(G_1) \cup \dots \cup V(G_n), \\ E(G) &= E(G_1) \cup \dots \cup E(G_n) \cup \{e_1, \dots, e_{n-1}\}, \\ \mathrm{i}_G(e_k) &\in V(G_k) \textit{ for } k \in \{1, \dots, n-1\}, \\ \mathrm{t}_G(e_k) &\in V(G_{k+1}) \textit{ for } k \in \{1, \dots, n-1\}. \end{aligned}$$

Lemma 5.6. *For every graph G without isolated vertices and for every subset $S \subseteq V(G)$, the following are equivalent.*

(a) *The set $\mathrm{Full}_{\mathbb{N}}(G, S)$ is non-empty.*

(b) *The graph G has the form $K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} K_n$, where $K_1, \dots, K_n$ are strongly connected graphs with $S \cap V(K_1) \neq \emptyset$ and $|E(K_n)| \geq 1$.*

Proof. (a) $\Longrightarrow$ (b) Assume there is a path $\gamma \in \mathrm{Full}_{\mathbb{N}}(G, S)$. Since G has no isolated vertices, each strongly connected component of G is traversed by γ. After γ leaves a component, it can never visit it again. Hence there exists an ordering of the strongly connected components $K_1, \dots, K_n$ and indices $\ell_1, \dots, \ell_{n-1}$ such that

$$\begin{aligned} &\gamma_\ell \in E(K_1) \text{ if } \ell < \ell_1, \\ &\gamma_\ell \in E(K_m) \text{ if } \ell_{m-1} < \ell < \ell_m,\ m \in \{2, \dots, n-1\}, \\ &\gamma_\ell \in E(K_n) \text{ if } \ell_{n-1} < \ell \end{aligned} \tag{5.1}$$

and such that G has the form

$$K_1 \xrightarrow{\gamma_{\ell_1}} K_2 \xrightarrow{\gamma_{\ell_2}} \dots \xrightarrow{\gamma_{\ell_{n-1}}} K_n. \tag{5.2}$$

Since γ starts in K_1, $V(K_1) \cap S \neq \emptyset$.

(b) $\Longrightarrow$ (a) Let i be a vertex in $S \cap V(K_1)$. If G fulfills (b), by Lemma 5.3 there are paths $\beta_\ell \in \mathrm{Full}(K_\ell)$ for $\ell \in \{1, \ldots, n\}$, which are empty if $E(K_\ell) = \emptyset$, such that

$$\begin{aligned}
&\mathrm{i}_G(\beta_1) = i && \mathrm{t}_G(\beta_1) = \mathrm{i}(e_1), && \\
&\mathrm{i}_G(\beta_\ell) = \mathrm{t}(e_{\ell-1}) && \mathrm{t}_G(\beta_\ell) = \mathrm{i}(e_\ell) && \text{for } \ell \in \{2, \ldots, n-1\} \text{ with } E(K_\ell) \neq \emptyset, \\
&\mathrm{i}_G(\beta_n) = \mathrm{t}(e_{n-1}) && \mathrm{t}_G(\beta_n) = \mathrm{t}(e_{n-1}). &&
\end{aligned}$$

The path $\beta_1 e_1 \beta_2 e_2 \ldots e_{n-2} \beta_{n-1} e_{n-1} \beta_n^\infty$ is then an element of $\mathrm{Full}_{\mathbb{N}}(G, S)$. $\square$

The characterization of graphs which contain bi-infinite paths containing all edges is very similar to the one-sided case.

Lemma 5.7. *For every graph G without isolated vertices the following are equivalent.*

(a) *The set $\mathrm{Full}_{\mathbb{Z}}(G)$ is non-empty.*

(b) *The graph G has the form $K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \ldots \xrightarrow{e_{n-1}} K_n$, where $K_1, \ldots, K_n$ are strongly connected graphs, $|E(K_1)| \geq 1$ and $|E(K_n)| \geq 1$.*

Proof. (a) $\Longrightarrow$ (b) Assume there is $\gamma \in \mathrm{Full}_{\mathbb{Z}}(G)$. As in the previous lemma, there is an ordering $K_1, \ldots, K_n$ of the strongly connected components of G such that γ fulfills (5.1) and such that G has the form (5.2). Since $\gamma_{\ell_1 - 1} \in E(K_1)$ and $\gamma_{\ell_{n-1}+1} \in E(K_n)$, both of these components contain at least one edge.

(b) $\Longrightarrow$ (a) If G fulfills (b), there are paths $\beta_\ell \in \mathrm{Path}(K_\ell)$ for $\ell \in \{1, \ldots, n\}$, which are empty if $E(K_\ell) = \emptyset$, such that

$$\begin{aligned}
&\mathrm{i}(\beta_1) = \mathrm{i}(e_1) && \mathrm{t}(\beta_1) = \mathrm{i}(e_1), && \\
&\mathrm{i}(\beta_\ell) = \mathrm{t}(e_{\ell-1}) && \mathrm{t}(\beta_\ell) = \mathrm{i}(e_\ell) && \text{for } \ell \in \{2, \ldots, n-1\} \text{ with } E(K_\ell) \neq \emptyset, \\
&\mathrm{i}(\beta_n) = \mathrm{t}(e_{n-1}) && \mathrm{t}(\beta_n) = \mathrm{t}(e_{n-1}). &&
\end{aligned}$$

The path ${}^\infty\beta_1 . e_1 \beta_2 e_2 \ldots e_{n-2} \beta_{n-1} e_{n-1} \beta_n^\infty$ is then an element of $\mathrm{Full}_{\mathbb{Z}}(G)$. $\square$

For every bi-infinite path γ in a graph G there is exactly one subgraph K for which $\gamma \in \mathrm{Full}_{\mathbb{Z}}(K)$, namely the subgraph induced by all the edges traversed by γ. This subgraph must have the form defined in Definition 5.5 as we just proved. To construct a derivative subalgebra of $\mathscr{P}(\mathrm{Path}_{\mathbb{Z}}(G))$ from these subgraphs, we have to calculate the derivative of $\mathrm{Full}_{\mathbb{Z}}(K)$. This is accomplished in the next two lemmas, first in the one-sided and afterwards in the two-sided setting. Remember that for $M \subseteq \mathrm{Path}_{\mathbb{Z}}(G)$ we have $\gamma \in M^*$ if for every $k \in \mathbb{N}$ there is a path in $M \setminus \{\gamma\}$ which agrees with γ on $\{-k, \ldots, k\}$.

Lemma 5.8. *Let G be a graph and let K be a subgraph of G that has the form*

$$K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \ldots \xrightarrow{e_{n-1}} K_n, \quad n \in \mathbb{N},$$

where $K_1, \dots, K_n$ are strongly connected components of K. Let $S \subseteq V(K)$. Let $\mathscr{H}$ be the set of all subgraphs of G of the form

$$K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \dots \xrightarrow{e_{\ell-2}} K_{\ell-1} \xrightarrow{e_{\ell-1}} H_\ell$$

where $1 \leq \ell \leq n$ and where H_ℓ is a (not necessarily strongly connected) subgraph of K_ℓ.

The set of accumulation points of $\mathrm{Full}_{\mathbb{N}}(K,S)$ in the space $\mathrm{Path}_{\mathbb{N}}(G,S) \subseteq E(G)^{\mathbb{N}}$ is given by

$$\mathrm{Full}_{\mathbb{N}}(K,S)^* = \begin{cases} \bigcup \{ \mathrm{Full}_{\mathbb{N}}(H, S \cap V(H)) \mid H \in \mathscr{H} \} & \textit{if } K_n \textit{ is not a cycle} \\ \bigcup \{ \mathrm{Full}_{\mathbb{N}}(H, S \cap V(H)) \mid H \in \mathscr{H} \} \setminus \mathrm{Full}_{\mathbb{N}}(K,S) & \textit{otherwise} \end{cases}.$$

Proof. ($\subseteq$) Let γ be a path in $\mathrm{Full}_{\mathbb{N}}(K,S)^*$. Every edge of γ is contained in $E(K)$, for otherwise there is $j \in \mathbb{N}$ with $\gamma_j \notin E(K)$ and there is no path $\alpha \in \mathrm{Full}_{\mathbb{N}}(K,S)$ with $\gamma_j = \alpha_j$. Hence $\gamma \in \mathrm{Path}_{\mathbb{N}}(K,S)$ and there is $\ell \in \{1,\dots,n\}, m \in \mathbb{N}$ such that all the edges in $\{\gamma_k \mid k \geq m\}$ lie in K_ℓ. Let H_ℓ be the subgraph of K_ℓ determined by these edges. The finite path $\gamma_{[1,\dots,m]}$ can be extended to an infinite path in $\mathrm{Full}_{\mathbb{N}}(K,S)$, hence it must contain all the edges in $K_1,\dots,K_{\ell-1}$ and all the edges $e_1,\dots,e_{\ell-1}$. There exists a subgraph $H \in \mathscr{H}$ with $\gamma \in \mathrm{Full}_{\mathbb{N}}(H, S \cap V(H))$. If K_n is a cycle, then γ can not be in $\mathrm{Full}_{\mathbb{N}}(K,S)$, for otherwise there would be no other path in $\mathrm{Path}_{\mathbb{N}}(K,S)$ agreeing with γ on $\{1,\dots,m\}$.

($\supseteq$) By Lemma 5.3 there are paths $\alpha_\ell \in \mathrm{Full}(K_\ell)$ for $\ell \in \{1,\dots,n\}$, which again might be empty, such that

$$\begin{aligned} &\mathrm{i}(\alpha_1) \in V(K_1) \cap S, && \mathrm{t}(\alpha_1) = \mathrm{i}(e_1), && \\ &\mathrm{i}(\alpha_\ell) = \mathrm{t}(e_{\ell-1}), && \mathrm{t}(\alpha_\ell) = \mathrm{i}(e_\ell) && \text{for } \ell \in \{2,\dots,n-1\}, \\ &\mathrm{i}(\alpha_n) = \mathrm{t}(e_{n-1}), && \mathrm{t}(\alpha_n) = \mathrm{t}(e_{n-1}), && \end{aligned}$$

and $\mathrm{t}(\alpha_1) = \mathrm{i}(\alpha_1) \in V(K_1) \cap S$ if $n = 1$. If K_n is not a cycle, then by Lemma 5.4 there is another path $\tilde{\alpha}_n$, sharing the properties of α_n stated above, such that $\alpha_n^\infty \neq \tilde{\alpha}_n^\infty$.

Let γ be a path in $\mathrm{Full}_{\mathbb{N}}(H, S \cap V(H))$ with $H \in \mathscr{H}$. There is $\ell \in \mathbb{N}$ and a subgraph H_ℓ of K_ℓ such that H has the form

$$K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \dots \xrightarrow{e_{\ell-2}} K_{\ell-1} \xrightarrow{e_{\ell-1}} H_\ell.$$

Let $m \in \mathbb{N}$ be sufficiently large such that $\mathrm{t}(\gamma_m) \in V(H_\ell)$. There is a path $\beta \in \mathrm{Path}(K)$ from $\mathrm{t}(\gamma_m)$ to $\mathrm{i}(\alpha_\ell)$.

If $\ell < n$, then

$$\tilde{\gamma} := \gamma_{[1,m]} \beta \alpha_\ell e_\ell \alpha_{\ell+1} e_{\ell+1} \dots e_{n-1} \alpha_n^\infty$$

is a path in $\mathrm{Full}_{\mathbb{N}}(K,S)$ different from γ with $\tilde{\gamma}_{[1,m]} = \gamma_{[1,m]}$.

If $\ell = n$ and K_n is not a cycle, then

$$\tilde{\gamma}^1 := \gamma_{[1,m]}\beta\alpha_n^\infty,$$
$$\tilde{\gamma}^2 := \gamma_{[1,m]}\beta\tilde{\alpha}_n^\infty$$

are two different paths in $\mathrm{Full}_{\mathbb{N}}(K,S)$ with $\tilde{\gamma}^1_{[1,m]} = \tilde{\gamma}^2_{[1,m]} = \gamma_{[1,m]}$. At least one of these two paths is different from γ, hence $\gamma \in \mathrm{Full}_{\mathbb{N}}(K,S)^*$. □

Lemma 5.9. *Let G be a graph and K a subgraph of G that has the form*

$$K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \dots \xrightarrow{e_{n-1}} K_n, \quad n \in \mathbb{N},$$

where $K_1,\dots,K_n$ are strongly connected components of K.

Let $\mathscr{H}$ be the set of all subgraphs of K of the form

$$H_{\ell_1} \xrightarrow{e_{\ell_1}} K_{\ell_1+1} \xrightarrow{e_{\ell_1+1}} \dots \xrightarrow{e_{\ell_2-2}} K_{\ell_2-1} \xrightarrow{e_{\ell_2-1}} H_{\ell_2}$$

where $1 \le \ell_1 \le \ell_2 \le n$ and where H_{ℓ_1} and H_{ℓ_2} are (not necessarily strongly connected) subgraphs of K_{ℓ_1} and K_{ℓ_2}, respectively.

The set of accumulation points of $\mathrm{Full}_{\mathbb{Z}}(K)$ *in the space* $\mathrm{Path}_{\mathbb{Z}}(G) \subseteq E(G)^{\mathbb{Z}}$ *is given by*

$$\mathrm{Full}_{\mathbb{Z}}(K)^* = \begin{cases} \bigcup\{\,\mathrm{Full}_{\mathbb{Z}}(H) \mid H \in \mathscr{H}\,\} & \text{if } K_1 \text{ or } K_n \text{ is not a cycle} \\ \bigcup\{\,\mathrm{Full}_{\mathbb{Z}}(H) \mid H \in \mathscr{H}\,\} \setminus \mathrm{Full}_{\mathbb{Z}}(K) & \text{otherwise} \end{cases}.$$

Proof. ($\subseteq$) Let γ be a path in $\mathrm{Full}_{\mathbb{Z}}(K)^*$. As in the one-sided setting, every edge in γ must be contained in $E(K)$, in other words, $\gamma \in \mathrm{Path}_{\mathbb{Z}}(K)$. There is $m \in \mathbb{N}$ such that $\{\,\gamma_k \mid k \in \{-m,\dots,m\}\,\} = \{\,\gamma_k \mid k \in \mathbb{Z}\,\}$, hence there are indices $\ell_1,\ell_2 \in \{1,\dots,n\}$ such that all the edges in $\{\,\gamma_k \mid k \le -m\,\}$ lie in K_{ℓ_1} and all edges in $\{\,\gamma_k \mid k \ge m\,\}$ lie in K_{ℓ_2}. Let H_{ℓ_1} and H_{ℓ_2} be the subgraphs of K_{ℓ_1} and K_{ℓ_2} determined by these edges. Since $\gamma_{[-m,\dots,m]}$ can be extended to a path in $\mathrm{Full}_{\mathbb{Z}}(K)$, it must contain all the edges in $K_{\ell_1+1},\dots,K_{\ell_2-1}$ and all the edges $e_{\ell_1},\dots,e_{\ell_2-1}$. Hence $\gamma \in \mathrm{Full}_{\mathbb{Z}}(H)$ for some $H \in \mathscr{H}$.

If K_1 and K_n are both cycles, then γ can not be an element of $\mathrm{Full}_{\mathbb{Z}}(K)$, because in that case there would be no second path in $\mathrm{Path}_{\mathbb{Z}}(K)$ agreeing with γ on $\{-m,\dots,m\}$.

($\supseteq$) As in the one-sided setting, choose paths $\alpha_\ell \in \mathrm{Full}(K_\ell), \ell \in \{1,\dots,n\}$ such that

$$\begin{aligned} &\mathrm{i}(\alpha_1) = \mathrm{i}(e_1), && \mathrm{t}(\alpha_1) = \mathrm{i}(e_1) && \\ &\mathrm{i}(\alpha_\ell) = \mathrm{t}(e_{\ell-1}), && \mathrm{t}(\alpha_\ell) = \mathrm{i}(e_\ell) && \text{for } \ell \in \{2,\dots,n-1\}, \\ &\mathrm{i}(\alpha_n) = \mathrm{t}(e_{n-1}), && \mathrm{t}(\alpha_n) = \mathrm{t}(e_{n-1}). && \end{aligned}$$

Let γ be a path in $\mathrm{Full}_{\mathbb{Z}}(H)$ with $H \in \mathscr{H}$. By definition, there are indices $\ell_1,\ell_2 \in \{1,\dots,n\}$

and subgraphs H_{ℓ_1} and H_{ℓ_2} of K_{ℓ_1} and K_{ℓ_2} such that H has the from

$$H_{\ell_1} \xrightarrow{e_{\ell_1}} K_{\ell_1+1} \xrightarrow{e_{\ell_1+1}} \dots \xrightarrow{e_{\ell_2-2}} K_{\ell_2-1} \xrightarrow{e_{\ell_2-1}} H_{\ell_2}.$$

Let m be sufficiently large such that $\{\gamma_k \mid k \in \{-m,\dots,m\}\} = \{\gamma_k \mid k \in \mathbb{Z}\}$. There are paths $\beta_1 \in \mathrm{Path}(K_{\ell_1})$ from $\mathrm{t}(\alpha_{\ell_1})$ to $\mathrm{i}(\gamma_{-m})$ and $\beta_2 \in \mathrm{Path}(K_{\ell_2})$ from $\mathrm{t}(\gamma_m)$ to $\mathrm{i}(\alpha_{\ell_2})$.

The path

$$\tilde{\gamma} := {}^{\infty}\alpha_1 e_1 \alpha_2 e_2 \dots e_{\ell_1-1}\alpha_{\ell_1}\beta_1\gamma_{[-m,0)}\cdot\gamma_{[0,m]}\beta_2\alpha_{\ell_2}e_{\ell_2}\dots e_{n-1}\alpha_n^{\infty} \in \mathrm{Full}_{\mathbb{Z}}(K)$$

agrees with γ on $\{-m,\dots,m\}$. Assume $\gamma = \tilde{\gamma}$. In this case $\ell_1 = 1$ and $\ell_2 = n$. Either K_1 and K_n are both cycles and $\gamma \in \mathrm{Full}_{\mathbb{Z}}(K)$, or at least one of K_1 and K_n is not a cycle. Then either

$${}^{\infty}\tilde{\alpha}_1 e_1 \alpha_2 e_2 \dots e_{\ell_1-1}\alpha_{\ell_1}\beta_1\gamma_{[-m,0)}\cdot\gamma_{[0,m]}\beta_2\alpha_{\ell_2}e_{\ell_2}\dots e_{n-1}\alpha_n^{\infty} \in \mathrm{Full}_{\mathbb{Z}}(K)$$

or

$${}^{\infty}\alpha_1 e_1 \alpha_2 e_2 \dots e_{\ell_1-1}\alpha_{\ell_1}\beta_1\gamma_{[-m,0)}\cdot\gamma_{[0,m]}\beta_2\alpha_{\ell_2}e_{\ell_2}\dots e_{n-1}\tilde{\alpha}_n^{\infty} \in \mathrm{Full}_{\mathbb{Z}}(K)$$

is a path different from γ, which agrees with γ on $\{-m,\dots,m\}$. □

We now have all the ingredients to construct our derivative algebras on $\mathrm{Path}_{\mathbb{N}}(G,S)$ and $\mathrm{Path}_{\mathbb{Z}}(G)$.

Theorem 5.10. *For any graph G and any subset $S \subseteq V(G)$ the sets*

$$T_{\mathbb{N}} := \{\mathrm{Full}_{\mathbb{N}}(K, V(K)\cap S) \mid K \text{ is a subgraph of } G,\ \mathrm{Full}_{\mathbb{N}}(K,V(K)\cap S) \neq \emptyset\} \text{ and}$$
$$T_{\mathbb{Z}} := \{\mathrm{Full}_{\mathbb{Z}}(K) \mid K \text{ is a subgraph of } G,\ \mathrm{Full}_{\mathbb{Z}}(K) \neq \emptyset\}$$

are the sets of atoms of finite derivative subalgebras of, respectively, $\mathscr{P}(\mathrm{Path}_{\mathbb{N}}(G,S))$ and $\mathscr{P}(\mathrm{Path}_{\mathbb{Z}}(G))$.

Proof. As we already mentioned in the two-sided setting, for every path $\gamma \in \mathrm{Path}_{\mathbb{N}}(G,S)$ there is exactly one subgraph K of G with $\gamma \in \mathrm{Full}_{\mathbb{N}}(K, V(K)\cap S)$. In other words, the sets in $T_{\mathbb{N}}$ form a partition of $\mathrm{Path}_{\mathbb{N}}(G,S)$ and are therefore the atoms of a Boolean subalgebra of $\mathscr{P}(\mathrm{Path}_{\mathbb{N}}(G,S))$. By Lemma 5.8, the derivative of every element of $T_{\mathbb{N}}$ is the union of elements in $T_{\mathbb{N}}$, hence $T_{\mathbb{N}}$ is the atom set of a derivative subalgebra of $\mathscr{P}(\mathrm{Path}_{\mathbb{N}}(G,S))$ by Lemma 3.13. The same result for $T_{\mathbb{Z}}$ is implied by Lemma 5.9. □

Corollary 5.11. *If Y and $Y_1,\dots,Y_n$ are one-sided or two-sided subshifts of finite type with $Y_1,\dots,Y_n \subseteq Y$, then the derivative subalgebra of $\mathscr{P}(Y)$ generated by $Y_1,\dots,Y_n$ is finite.*

Proof. Set $I := \mathbb{N}$ or $I := \mathbb{Z}$ such that $Y \subseteq A^I$. By the proof of Theorem 2.45 there is a graph G with subgraphs $G_1,\dots,G_n$ and a conjugacy $\varphi : \mathrm{Path}_I(G) \to Y$ such that

$$\varphi[\mathrm{Path}_I(G_k)] = Y_k \text{ for } k \in \mathbb{N}.$$

By Theorem 5.10 the derivative subalgebra of $\mathrm{Path}_I(G)$ generated by $\mathrm{Path}_I(G_1),\dots,\mathrm{Path}_I(G_n)$ is finite. The homeomorphism φ induces an isomorphism between this subalgebra and the derivative subalgebra of Y generated by $Y_1,\dots,Y_n$, hence also the later one is finite. □

We now know how to construct the derivative algebra we are interested in. In order to apply Theorem 4.34 we additionally need information about the cardinality of its atoms. We thus want to calculate the cardinalities of $\mathrm{Full}_{\mathbb{Z}}(G)$ and $\mathrm{Full}_{\mathbb{N}}(G,S)$.

To unify the treatment of the one-sided and two-sided case somewhat, we first prove the following lemma.

Lemma 5.12. *For every strongly connected graph G the following are equivalent.*

(a) *G is not a cycle.*

(b) *$\mathrm{Path}_{\mathbb{N}}(G,\{j\})$ is perfect for all $j\in G$.*

(c) *$|\mathrm{Path}_{\mathbb{N}}(G,\{j\})|=|\mathbb{R}|$ for all $j\in G$.*

Proof. If G is not a cycle, there must be a vertex i with two distinct out-going edges e_1,e_2. Since G is strongly connected, there exists a path α_1 from $\mathrm{t}(e_1)$ to $\mathrm{i}(e_1)$ and a path α_2 from $\mathrm{t}(e_2)$ to $\mathrm{i}(e_2)$. Then $\alpha_1=\gamma_{[1,n]}(e_1\alpha_1)^\infty$ and $\alpha_2=\gamma_{[1,n]}(e_2\alpha_2)^\infty$ are two different paths starting in j and at least one of them is different from γ. The set $\{\,\gamma\in\mathrm{Path}_{\mathbb{N}}(G)\mid i(\gamma)=j\,\}$ is thus a closed perfect subset of the Cantor space $E(G)^{\mathbb{N}}$. By Theorem 4.7 the cardinality of such a set equals the cardinality of the reals. If, on the other hand, G is a cycle, then every path in $\mathrm{Path}_{\mathbb{N}}(G)$ is uniquely determined by its initial vertex, hence there is exactly one path in $\mathrm{Path}_{\mathbb{N}}(G,\{j\})$. □

Theorem 5.13. *If K is a graph of the form*

$$K_1\xrightarrow{e_1}K_2\xrightarrow{e_2}\dots\xrightarrow{e_{n-2}}K_{n-1}\xrightarrow{e_{n-1}}K_n,\quad n\in\mathbb{N},$$

where $K_1,\dots,K_n$ are strongly connected subgraphs with $|E(K_n)|\geq 1$ and $S\cap V(K_1)\neq\emptyset$, then

$$|\mathrm{Full}_{\mathbb{N}}(K,S)|=\begin{cases}|V(K_1)\cap S| & \text{if } K_n \text{ is a cycle, } E(K_\ell)=\emptyset \text{ for all } \ell\in\{1,\dots,n-1\}\\ |\mathbb{N}| & \text{if } K_n \text{ is a cycle and } \exists \ell\in\{1,\dots,n-1\}: E(K_\ell)\neq\emptyset\\ |\mathbb{R}| & \text{otherwise}\end{cases}.$$

Proof. If K_n is a cycle, there is a path $\alpha\in\mathrm{Full}(K_n)$ with $\mathrm{i}(\alpha)=\mathrm{t}(\alpha)$, such that every path in $\mathrm{Full}_{\mathbb{N}}(K,S)$ is of the form $\beta\alpha^\infty$ for some $\beta\in\mathrm{Path}(K,S)$. Hence $\mathrm{Full}_{\mathbb{N}}(K,S)$ is at most countable. If furthermore $E(K_\ell)=\emptyset$ for all $\ell\in\{1,\dots,n-1\}$, then the initial vertex of a path $\gamma\in\mathrm{Full}_{\mathbb{N}}(K,S)$ uniquely determines this path. There are $|V(K_1)\cap S|$ many possibilities for this initial vertex. If, on the other hand, K_n is a cycle and $E(K_\ell)\neq\emptyset$ for some $\ell\in\{1,\dots,n-1\}$, then there is a path $\beta\in\mathrm{Full}(K_\ell)$ with $\mathrm{i}(\beta)=\mathrm{t}(\beta)$, a path $\gamma_1\in\mathrm{Path}(K)$ starting in a vertex in S and ending in $\mathrm{i}(\beta)$ and a path γ_2 starting in $\mathrm{t}(\beta)$ and ending in $\mathrm{i}(\alpha)$. Combining these paths, one obtains countably many paths $\gamma_1.\beta^k\gamma_2\alpha^\infty, k \in \mathbb{N}$ in $\mathrm{Full}_{\mathbb{N}}(G,S)$.

Finally, if K_n is not a cycle and $i \in V(K_n)$, then $|\mathrm{Path}_{\mathbb{N}}(K_n,\{i\})| = |\mathbb{R}|$ by Lemma 5.12. By Lemma 5.6 and Lemma 5.3 there is also a path $\gamma \in \mathrm{Full}(K,S)$ ending in i. The concatenation of γ with a path in $\mathrm{Path}_{\mathbb{N}}(K_n,\{i\})$ gives a path in $\mathrm{Full}_{\mathbb{N}}(K,S)$, hence $|\mathrm{Full}_{\mathbb{N}}(K,S)| = |\mathbb{R}|$. □

Theorem 5.14. *If K is a graph of the form*

$$K_1 \xrightarrow{e_1} K_2 \xrightarrow{e_2} \dots \xrightarrow{e_{n-2}} K_{n-1} \xrightarrow{e_{n-1}} K_n, \quad n \in \mathbb{N},$$

where $K_1,\dots,K_n$ are strongly connected subgraphs with $E(K_1) \neq \emptyset$, $E(K_n) \neq \emptyset$, then

$$|\mathrm{Full}_{\mathbb{Z}}(K)| = \begin{cases} |V(K)| & \text{if } K \text{ is a cycle} \\ |\mathbb{N}| & \text{if } n \geq 2 \text{ and } K_1 \text{ and } K_n \text{ are cycles} \\ |\mathbb{R}| & \text{otherwise} \end{cases}.$$

Proof. If K is a cycle, then $\mathrm{Full}_{\mathbb{Z}}(K) = \mathrm{Path}_{\mathbb{Z}}(K)$ and hence $|\mathrm{Full}_{\mathbb{Z}}(K)| = |E(K)|$. Consider the case that $n \geq 2$ and K_1 and K_n are both cycles. There are paths $\alpha_1 \in \mathrm{Full}(K_1)$ and $\alpha_2 \in \mathrm{Full}(K_n)$ such that every path $\gamma \in \mathrm{Full}_{\mathbb{Z}}(K)$ has the form ${}^\infty\alpha_1\beta_1.\beta_2\alpha_2^\infty$ with $\beta_1,\beta_2 \in \mathrm{Path}(K)$. Hence $|\mathrm{Full}_{\mathbb{Z}}(K)| \leq |\mathbb{N}|$. On the other hand for any $\gamma \in \mathrm{Full}_{\mathbb{Z}}(K)$ the set $\left\{ \sigma^k(\gamma) \;\middle|\; k \in \mathbb{Z} \right\}$ is a countable subset of $\mathrm{Full}_{\mathbb{Z}}(K)$, thus $|\mathrm{Full}_{\mathbb{Z}}(K)| = |\mathbb{N}|$.

In the case that K_n is not a cycle, choose a path $\alpha \in \mathrm{Full}(K_1)$ with $\mathrm{i}(\alpha) = \mathrm{t}(\alpha)$. By Theorem 5.13 $|\mathrm{Full}_{\mathbb{N}}(K,\{\mathrm{t}(\alpha)\})| = |\mathbb{R}|$ and for every $\gamma \in \mathrm{Full}_{\mathbb{N}}(K,\{\mathrm{t}(\alpha)\})$, the concatenation ${}^\infty\alpha.\gamma$ is in $\mathrm{Full}_{\mathbb{Z}}(K)$, which therefore also has the same cardinality as the reals. If K_1 is not a cycle, repeat the same argument with $\alpha \in \mathrm{Full}(K_n)$. □

With the information about the cardinality of the atoms at our disposal, Theorem 4.19 and Theorem 4.34 immediately yield the following corollaries.

Corollary 5.15. *Let $Y, Y_1,\dots,Y_n$ and $\tilde{Y}, \tilde{Y}_1,\dots,\tilde{Y}_n$ be one-sided or two-sided subshifts of finite type with $Y_1,\dots,Y_n \subseteq Y$ and $\tilde{Y}_1,\dots,\tilde{Y}_n \subseteq \tilde{Y}$ given by finite lists of forbidden patterns. It is decidable if there exists a homeomorphism $\varphi : Y \to \tilde{Y}$ with $\varphi[Y_i] = \tilde{Y}_i$ for all $i \in \{1,\dots,n\}$.*

Corollary 5.16. *Let X, Y be one-sided or two-sided subshifts of finite type. If $f : X \to X$ and $g : Y \to Y$ are p-periodic cellular automata, then f is topologically conjugate to g if and only if there is a homeomorphism $\rho : X \to Y$ such that $\rho[Per_k(f)] = Per_k(g)$ for all $k \in \{1,\dots,p\}$. If the subshifts are given by finite lists of forbidden blocks and the cellular automata are given by their local rules, the existence of such a homeomorphism is decidable.*

Corollary 5.17. *The conjugacy of periodic cellular automata on one- or two-sided subshifts of finite type is decidable.*

As it is often the case in symbolic dynamics, the situation in higher dimensions or over groups other than $\mathbb{Z}^d$ is drastically different from the one-dimensional situation. We will show that it is in general undecidable, if there exist a conjugacy between two given periodic cellular automata over full shifts. The famous domino problem asks if a subshift of finite type, given by a list of forbidden patterns, is empty. Berger showed in [Ber66] that the domino problem is undecidable over $\mathbb{Z}^d, d \geq 2$. Therefore we only have to prove that if one is able to decide the conjugacy problem for periodic cellular automata over full shifts, one is also able to decide the domino problem.

Theorem 5.18. *Let Γ be a countable group. If the emptiness of a subshift of finite type given by a finite set of forbidden patterns is undecidable over Γ, then topological conjugacy of 2-periodic cellular automata on full-shifts over Γ is also undecidable. In particular topological conjugacy of 2-periodic cellular automata on full shifts over $\mathbb{Z}^d, d \geq 2$, is undecidable.*

Proof. For a subshift of finite type $X \subseteq A^\Gamma$, given by a finite list $M \subseteq A^H$ of forbidden patterns over a finite subset $H \subseteq \Gamma$, define a cellular automaton f_X as follows. The domain of f_X is the full shift $Y := (A \times \{0,1\})^\Gamma$. Let π_A and $\pi_{\{0,1\}}$ be the projections of Y on the full shifts with alphabet A and $\{0,1\}$. Define

$$f_X(y)_i := \begin{cases} \pi_A(y)_i, 1 - \pi_{\{0,1\}}(y)_i & \text{if } \pi_A(\sigma_{i^{-1}}(y))_{|H} \in M \\ \pi_A(y)_i, \pi_{\{0,1\}}(y)_i & \text{otherwise} \end{cases}.$$

We can think of Y as consisting of a control layer A^Γ and a dynamic layer $\{0,1\}^\Gamma$. The control layer remains fixed under f_X and the bit at index i in the dynamic layer is flipped if and only if there is a forbidden pattern in the control layer at position i. The map f_X is two-periodic by definition, and $\mathrm{Fix}(f_X) = \pi_A^{-1}(X)$. In particular f_X has fixed points if and only if $X \neq \emptyset$. By Theorem 4.24 f_X is conjugate to $f_\emptyset$, if and only if X is empty. Therefore, being able to decide the existence of this conjugacy allows one to decide the emptiness of subshifts of finite type. $\square$

There are a number of reasons to believe that the conjugacy problem for cellular automata without the periodicity assumption is undecidable also in dimension one. First of all, if we allow cellular automata over subshifts of finite type, the problem includes as a "simple" case the conjugacy of subshifts of finite type themselves. As we saw in Section 2.5, the decidability of this problem is still open and already withstands attacks for quite some time. Next, for many dynamical properties, which are invariant under topological conjugacy, it is undecidable if a given cellular automaton possess this property. Finally, it is undecidable if two cellular automata are conjugate on their eventual image, i.e., the maximal subset on which they are surjective. This is due to the fact that it is even undecidable if the eventual image consists only of a single point, see the paper by Kari [Kar92] and Section 7 in the survey [Kar05]. The fact that sometimes every conjugacy between two cellular automata has to use infinitely many local rules, as we will show in Section 6.4, gives even more credibility to the conjecture.

Conjecture 5.19. *It is undecidable if two cellular automata (without the periodicity assumption) over one-sided or two-sided full shifts are conjugate.*

5.2. The Strong Component Graph

In the preceding section we constructed a derivative algebra containing the derivative algebra generated by multiple edge shifts as a subalgebra. This algebra, however, was really huge as we got one atom for every subgraph with a certain structure. Even if we are dealing with strongly connected graphs, the size of the algebra constructed that way grows exponentially with the size of the graph.

In this section, we will show how to use the condensation of a graph to recover information about our derivative algebra. First we show that if we are only dealing with the edge shift of a single graph, the condensation is actually enough to calculate the derivative algebra it generates. In particular, this allows us to decide when the sets of p-periodic points of two cellular automata are homeomorphic.

Afterwards we will introduce a special labeling of the condensation to calculate at least the cardinality of sets of the form $Y \setminus (Y_1 \cup \cdots \cup Y_n)$, where Y and $Y_1, \ldots, Y_n$ are edge shifts. This allows us to calculate the cardinality of $\mathrm{Pre}_{q,p}(f)$ for cellular automata f by Theorem 2.70. We will use both methods in Chapter 6 to obtain conjugacy invariants for elementary cellular automata. In this section, we only prove the results in the two-sided setting because this is what we will use later.

We start by defining a coarser partition of $\mathrm{Path}_{\mathbb{Z}}(G)$ based on the strongly connected components which a path traverses.

Definition 5.20 ($\mathrm{Trav}_{\mathbb{Z}}(K_1, \ldots, K_n)$). *Let G be a graph and let $K_1, K_2, \ldots, K_n$ be strongly connected components of G with $K_1 \leq K_2 \leq \cdots \leq K_n$. We denote by*

$$\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2, \ldots, K_n)$$

the set of all bi-infinite paths, that start in K_1, end in K_n and traverse a vertex in each of the components $K_1, \ldots, K_n$. More formally, $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1, \ldots, K_n)$ if there are indices $k_1, \ldots, k_n$ such that $\mathrm{i}_G(\gamma_{k_\ell}) \in V(K_\ell)$ for $\ell \in \{1, \ldots, n\}$, $\mathrm{i}_G(\gamma_k) \in V(K_1)$ for $k \leq k_1$ and $\mathrm{i}_G(\gamma_k) \in V(K_n)$ for $k \geq k_n$.

Example 5.21. *Consider the graph G shown in Figure 5.2. We have*

$$\begin{aligned}
\mathrm{Trav}_{\mathbb{Z}}(K_1, K_3) &= \emptyset = \mathrm{Trav}_{\mathbb{Z}}(K_1, K_4, K_2), \\
\mathrm{Trav}_{\mathbb{Z}}(K_4, K_4) &= \{{}^{\infty}g^{\infty}\}, \\
\mathrm{Trav}_{\mathbb{Z}}(K_2, K_4) &= \left\{ \sigma^k({}^{\infty}(d_3 d_1 d_2).e g^{\infty}) \;\middle|\; k \in \mathbb{Z} \right\}, \\
\mathrm{Trav}_{\mathbb{Z}}(K_1, K_4) &= \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2, K_4) \cup \mathrm{Trav}_{\mathbb{Z}}(K_1, K_3, K_4) \\
&= \mathrm{Path}_{\mathbb{Z}}(G) \setminus (\mathrm{Trav}_{\mathbb{Z}}(K_1) \cup \mathrm{Trav}_{\mathbb{Z}}(K_2) \cup \mathrm{Trav}_{\mathbb{Z}}(K_4) \cup \\
&\qquad \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2) \cup \mathrm{Trav}_{\mathbb{Z}}(K_2, K_4)).
\end{aligned}$$

For now we only need to consider sets of the form $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$. The need for more components will only arise when we introduce the labeling of the condensation. The fact

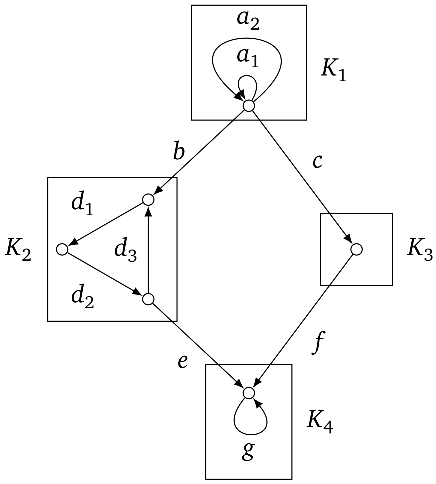


Figure 5.2.: The graph with strongly connected components $K_1, \dots, K_4$ from Example 5.21.

that sets of this form are the atoms of a derivative algebra is established by the following theorem.

Theorem 5.22. *Let G be a graph. Let K_1, K_2 be strongly connected components of G, both containing at least one edge. If $K_1 \le K_2$, then*

$$\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)^* = \bigcup \{ \mathrm{Trav}_{\mathbb{Z}}(L_1, L_2) \mid K_1 \le L_1 \le L_2 \le K_2 \} \setminus M$$

where $M := \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$ if K_1 and K_2 are both cycles and $M := \emptyset$ otherwise.

Proof. ($\subseteq$) Let γ be path in $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)^*$. There are indices $\ell_1, \ell_2 \in \mathbb{Z}$ and strongly connected components L_1, L_2 of G such that $\gamma_\ell \in E(L_1)$ for $\ell \le \ell_1$ and $\gamma_\ell \in E(L_2)$ for $\ell \ge \ell_2$. Since γ is an accumulation point of $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$, there is a path $\alpha \in \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$ with $\alpha_{[\ell_1, \ell_2]} = \gamma_{[\ell_1, \ell_2]}$. Thus there is a path from K_1 to L_1 and a path from L_2 to K_2, in other words, $K_1 \le L_1 \le L_2 \le K_2$. If K_1 and K_2 are both cycles, there is exactly one path α in $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$ with $\alpha_{[\ell_1, \ell_2]} = \gamma_{[\ell_1, \ell_2]}$, namely γ, hence $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2) \cap \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)^* = \emptyset$.

($\supseteq$) Now let γ be a path in $\mathrm{Trav}_{\mathbb{Z}}(L_1, L_2)$ with $K_1 \le L_1 \le L_2 \le K_2$. Let $n \in \mathbb{N}$ be sufficiently large such that $\mathrm{i}_G(\gamma_{-n}) \in V(L_1)$ and $\mathrm{t}_G(\gamma_n) \in V(L_2)$. There are paths $\alpha_1 \in \mathrm{Path}(K_1), \alpha_2 \in \mathrm{Path}(K_2)$ with $\mathrm{i}_G(\alpha_1) = \mathrm{t}_G(\alpha_1)$ and $\mathrm{i}_G(\alpha_2) = \mathrm{t}_G(\alpha_2)$. Since $K_1 \le L_1$ and $L_2 \le K_2$, there are also paths $\beta_1, \beta_2 \in \mathrm{Path}(G)$ with $\mathrm{i}_G(\beta_1) = \mathrm{t}_G(\alpha_1)$, $\mathrm{t}_G(\beta_1) = \mathrm{i}_G(\gamma_{-n})$, $\mathrm{i}_G(\beta_2) = \mathrm{t}_G(\gamma_n)$ and $\mathrm{t}_G(\beta_2) = \mathrm{i}_G(\alpha_2)$. If $K_1 < L_1$ or $L_2 < K_2$, the path ${}^{\infty}\alpha_1 \beta_1 \gamma_{[-n,0)} . \gamma_{[0,n]} \beta_2 \alpha_2^{\infty} \in \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)$ is different from γ but agrees with it on $\{-n, \dots, n\}$. This shows that $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)^*$.

Now consider the case that $K_1 = L_1$ and $K_2 = L_2$. Assume that K_1 is not a cycle. By Lemma 5.4 we can replace α by a different path $\tilde{\alpha}_1 \in \mathrm{Path}(K_1)$ such that

$${}^{\infty}\tilde{\alpha}_1 \beta_1 \gamma_{[-n,0)} . \gamma_{[0,n]} \beta_2 \alpha_2^{\infty}$$

is another path agreeing with γ on $\{-n, \dots, n\}$. Again this shows that $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1, K_2)^*$.

If K_2 is not a cycle, repeat the same argument with a path $\tilde{\alpha}_2$ replacing α_2. □

Again we also need the cardinality of $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2, \dots, K_n)$ in order to apply Theorem 4.19.

Lemma 5.23. *Let G be a graph. Let $K_1, \dots, K_n$ be strongly connected components of G with $K_1 \leq K_2 \leq \cdots \leq K_n$. If K_1 and K_n both contain at least one edge, then*

$$|\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)| = \begin{cases} |E(K_1)| & \text{if } K_1 = \cdots = K_n \text{ and } K_1 \text{ is a cycle} \\ |\mathbb{R}| & \text{if } K_1 \text{ or } K_n \text{ is not a cycle} \\ |\mathbb{N}| & \text{otherwise} \end{cases},$$

otherwise $\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)$ *is empty.*

Proof. If $K_1 \leq K_2 \leq \cdots \leq K_n$ and K_1 and K_n both contain at least one edge, there exists a path $\gamma \in \mathrm{Trav}(K_1, \dots, K_n)$ and there is an index $m \in \mathbb{N}$ such that $\gamma_{-m} \in E(K_1)$ and $\gamma_m \in E(K_n)$.

If $K_1 = K_n$ is a cycle, there are exactly $|E(K_1)|$ possibilities to choose an edge from K_1 and a path in $\mathrm{Trav}_{\mathbb{Z}}(K_1)$ is completely determined by the edge at index 0.

If K_1 and K_n are both cycles, then every path in $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2, \dots, K_n)$ is uniquely determined by its edges not contained in $E(K_1) \cup E(K_n)$. Since each path contains only finitely many of these edges, the set $\mathrm{Trav}_{\mathbb{Z}}(K_1, K_2, \dots, K_n)$ is at most countable. It is not finite, since it contains the non-periodic path γ and all its shifts, which are pairwise different.

Assume that K_n is not a cycle. By Theorem 5.13 there are uncountably many different paths in $\mathrm{Path}_{\mathbb{N}}(K_n)$ which start in $\mathrm{t}_G(\gamma_m)$. Each of one of them gives rise to a different path in $\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)$, namely $\gamma_{(-\infty,m]}.\alpha$, hence $|\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)| = |\mathbb{R}|$. A symmetric argument establishes this fact in the case that K_1 is not a cycle.

If K_1 or K_n contains no edge, there can be no path starting or ending in this component, in other words, $\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)$ is empty in that case. □

We now build an algorithm to calculate $\widetilde{\mathrm{Pre}}_{q,p}(f)$ for cellular automata $f : X \to X$ on subshifts of finite type X. We already showed in Theorem 2.70 that we can represent $\widetilde{\mathrm{Pre}}_{q,p}(f)$ by

$$\widetilde{\mathrm{Pre}}_{q,p}(f) = Y \setminus (Y_1 \cup \cdots \cup Y_m).$$

where $Y, Y_1, \dots, Y_m$ are subshifts of finite type all contained in Y. We take ℓ large enough such that the higher block-representations with window size ℓ of $Y, Y_1, \dots, Y_m$ can all be represented as the edge shift of some graphs $G, G_1, \dots, G_m$ which are all subgraphs of G. Let $\varphi : X \to \mathrm{Path}_{\mathbb{Z}}(E(G))$ be the higher block map. We now label all edges in G by the subshifts from $Y_1, \dots, Y_m$ in which they are contained. More precisely, we define

$I := \{1, \dots, m\}$ and $\mathscr{L} : E(G) \to \mathscr{P}(I)$ by

$$\mathscr{L}(e) := \{\, k \in I \mid e \in E(G_k) \,\}\,.$$

With these definitions we have

$$\varphi[Y] \setminus (\varphi[Y_1] \cup \cdots \cup \varphi[Y_m]) = \Big\{ \gamma \in \mathrm{Path}_{\mathbb{Z}}(G) \,\Big|\, \bigcap_{\ell \in \mathbb{Z}} \mathscr{L}(\gamma_\ell) = \emptyset \Big\}$$

and therefore

$$|\widetilde{\mathrm{Pre}}_{q,p}(f)| = \Big| \Big\{ \gamma \in \mathrm{Path}_{\mathbb{Z}}(G) \,\Big|\, \bigcap_{\ell \in \mathbb{Z}} \mathscr{L}(\gamma_\ell) = \emptyset \Big\} \Big|.$$

Together with these considerations, the next lemma shows that all we have to do is to find those strongly connected components $K_1, \dots, K_n$ of G, for which there is a path in $\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n) \cap Y \setminus (Y_1 \cup \cdots \cup Y_m)$.

Lemma 5.24. *Let I be a finite set and let G be a graph with edge labeling $\mathscr{L} : E(G) \to \mathscr{P}(I)$. Let $K_1, \dots, K_n$ be strongly connected components of a graph G with $K_1 \le K_2 \le \cdots \le K_n$. If there is at least one path $\alpha \in \mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)$ with $\bigcap_{k \in \mathbb{Z}} \mathscr{L}(\alpha_k) = \emptyset$, then*

$$\begin{aligned} \Big| \Big\{ \gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n) \,\Big|\, \bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset \Big\} \Big| &= |\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)| \\ &= \begin{cases} |E(K_1)| & \text{if } K_1 = K_n \text{ is a cycle} \\ |\mathbb{R}| & \text{if } K_1 \text{ or } K_n \text{ is not a cycle}\,. \\ |\mathbb{N}| & \text{otherwise} \end{cases} \end{aligned}$$

Proof. To shorten notation, define $M := \big\{ \gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n) \,\big|\, \bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset \big\}$. Choose $m \in \mathbb{N}$ large enough such that $\{\, \alpha_k \mid k \in \{-m, \dots, m\} \,\} = \{\, \alpha_k \mid k \in \mathbb{Z} \,\}$.

It is enough to show that $|\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)| \le |M|$. If $n = 1$ and $K_1 = K_n$ is a cycle, then all paths $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1)$ are just shifts of the path α, hence $\bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset$. If $n \ne 1$, then α is not periodic under the shift, hence $\big\{ \sigma^k(\alpha) \,\big|\, k \in \mathbb{Z} \big\}$ is an infinite subset of M. If K_n is not a cycle, there are uncountably many paths γ in $\mathrm{Path}_{\mathbb{N}}(K_n, \{\mathrm{t}_G(\alpha_m)\})$ by Theorem 5.13 and every one of these paths gives rise to a different path $\alpha_{(-\infty,m]}.\gamma \in M$. The case that K_1 is not a cycle is treated in an analogous way. The result then follows from Lemma 5.23. $\square$

Based on a labeling $\mathscr{L} : E(G) \to \mathscr{P}(I)$ we now define a vertex and an edge labeling of $\mathscr{S}(G)$ that allows us to determine the components $K_1, \dots, K_n$ for which $\mathrm{Trav}_{\mathbb{Z}}(K_1, \dots, K_n)$ contains a path γ with $\bigcap_{\ell \in \mathbb{Z}} \mathscr{L}(\gamma_\ell) = \emptyset$.

Definition 5.25. *Let I be a finite set. Let G be graph with edge labeling $\mathscr{L} : E(G) \to \mathscr{P}(I)$.*

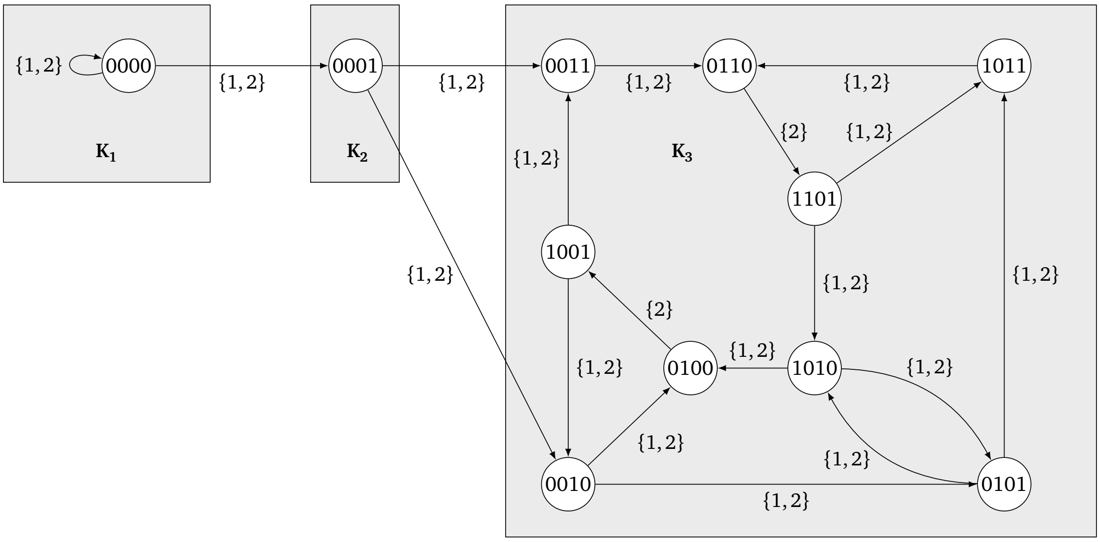


Figure 5.3.: The subgraph G of the De Bruijn graph for the cellular automaton w_{28} with $n = 2$.

We label the edges and vertices of $\mathscr{S}(G)$ by the following maps

$$\mathscr{L}_V : V(\mathscr{S}(G)) \to \mathscr{P}(I), \qquad K \mapsto \bigcap_{e \in E(K)} \mathscr{L}(e),$$
$$\mathscr{L}_E : E(\mathscr{S}(G)) \to \mathscr{P}(\mathscr{P}(I)), \quad (K, L) \mapsto \{\mathscr{L}(e) \mid e \in E(G), \mathrm{i}_G(e) \in V(K), \mathrm{t}_G(e) \in V(L)\}.$$

We use the convention that $\bigcap_{e \in E(K)} \mathscr{L}(e) = I$ whenever $E(K) = \emptyset$.

Example 5.26. *Figure 5.3 shows the graph whose edge shift corresponds to the higher block representation with block-length 5 of the 2-periodic points of the elementary cellular automaton $w_{28} : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$, whose local rule is $(w_{28})_{loc}(x_{-1}, x_0, x_1) = x_{-1} + x_0 + x_{-1}x_1 + x_{-1}x_0x_1$ (see Section 6.1). The edge $e = (a_1, a_2, \ldots, a_5)$ is labeled by $k \in \{1, 2\}$ if the corresponding pattern can appear in a configuration in $\mathrm{Per}_k(w_{90})$. The condensation of this graph has the form $K_1 \to K_2 \to K_3$. The labelings are given by*

$$\begin{aligned}
\mathscr{L}_V^C(K_1) &= \{1, 2\}, \\
\mathscr{L}_V^C(K_2) &= \{1, 2\}, \\
\mathscr{L}_V^C(K_2) &= \{2\}, \\
\mathscr{L}_E^C(K_1, K_2) &= \{\{1, 2\}\}, \\
\mathscr{L}_E^C(K_2, K_3) &= \{\{1, 2\}\}.
\end{aligned}$$

Example 5.27. *Figure 5.4 shows a graph G on the left side, whose condensation on the right side is more complex. For example the vertex D of $\mathscr{S}(G)$ corresponds to the subgraph of G induced by $\{d, e\}$. It is labeled by $\{1, 2, 3\} \cap \{1, 2, 3, 4\} = \{1, 2, 3\}$.*

Lemma 5.28. *Let I be a finite set and let G be a graph with edge labeling $\mathscr{L} : E(G) \to \mathscr{P}(I)$. Let $C := \mathscr{S}(G)$ be the condensation of G together with the edge and vertex labelings defined*

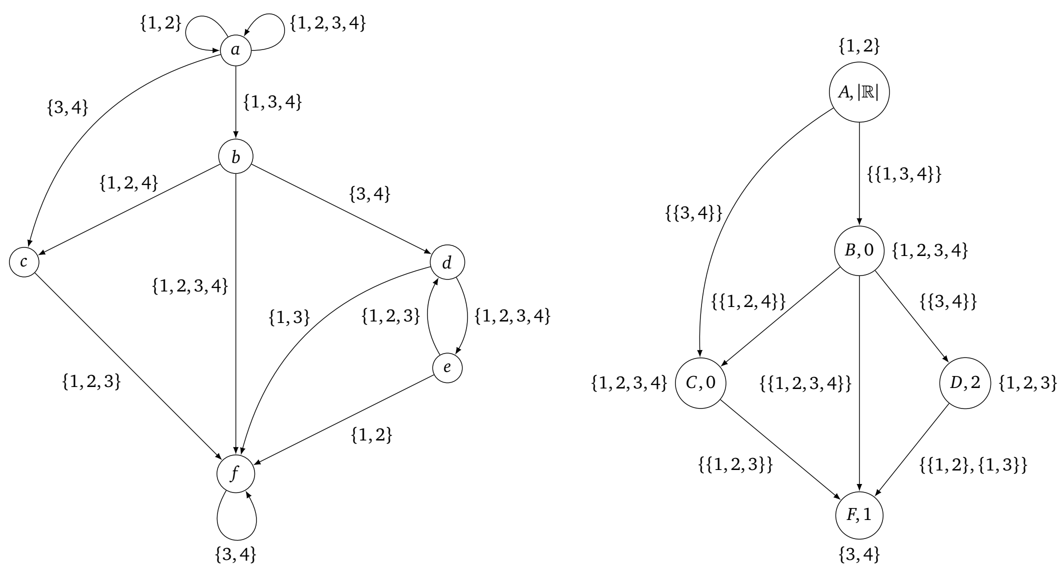


Figure 5.4.: A graph G with edge labels in $\mathscr{P}(\{1,2,3,4\})$ and the corresponding vertex and edge labeling of $\mathscr{S}(G)$, the cardinality of $\mathrm{Path}_{\mathbb{Z}}(K)$ is shown next to the name of the component K.

in Definition 5.25. If $K_1,\dots,K_n$ is a vertex path in C, there is a path $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n)$ with $\bigcap_{k\in\mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset$ if and only if

$$\emptyset \in \Big\{ \bigcap_{\ell=1}^{n} \mathscr{L}_V^C(K_\ell) \cap \bigcap_{\ell=1}^{n-1} L_\ell \,\Big|\, L_\ell \in \mathscr{L}_E^C(K_\ell, K_{\ell+1}) \text{ for } \ell \in \{1,\dots,n-1\} \Big\}.$$

Proof. Assume there is a path $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n)$ with $\bigcap_{k\in\mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset$. There must be indices $k_1 \le \dots \le k_{n-1}$ such that $\mathrm{i}_G(\gamma_{k_\ell}) \in K_\ell$ and $\mathrm{t}_G(\gamma_{k_\ell}) \in K_{\ell+1}$ for every $\ell \in \{1,\dots,n-1\}$. Then $\mathscr{L}(\gamma_{k_\ell}) \in \mathscr{L}_V^C(K_\ell, K_{\ell+1})$ and

$$\emptyset = \bigcap_{k\in\mathbb{Z}} \mathscr{L}(\gamma_k) \supseteq \bigcap_{\ell=1}^{n} \bigcap_{e\in E(K_\ell)} \mathscr{L}(e) \cap \bigcap_{\ell=1}^{n-1} \mathscr{L}(\gamma_{k_\ell}) = \bigcap_{\ell=1}^{n} \mathscr{L}_V^C(K_\ell) \cap \bigcap_{\ell=1}^{n-1} \mathscr{L}(\gamma_{k_\ell}).$$

To prove the reverse implication, assume there are sets $L_\ell \in \mathscr{L}_E^C(K_\ell, K_{\ell+1})$, $\ell \in \{1,\dots,n-1\}$ such that $\emptyset = \bigcap_{\ell=1}^{n} \mathscr{L}_V^C(K_\ell) \cap \bigcap_{\ell=1}^{n-1} L_\ell$. By the definition of the edge labeling $\mathscr{L}_E^C$, for every $\ell \in \{1,\dots,n-1\}$ there is an edge $e_\ell \in E(G)$ starting in a vertex in K_ℓ and ending in a vertex in $K_{\ell+1}$ such that $\mathscr{L}(e_\ell) \in L_\ell$. By Lemma 5.3 there are paths $\alpha_\ell \in \mathrm{Full}(K_\ell)$ such that

$$\begin{aligned}
&\mathrm{i}_G(\alpha_1) = \mathrm{t}_G(\alpha_1) = \mathrm{i}_G(e_1),\\
&\mathrm{i}_G(\alpha_\ell) = \mathrm{t}_G(e_{\ell-1}) \text{ for } \ell \in \{2,\dots,n-1\},\\
&\mathrm{t}_G(\alpha_\ell) = \mathrm{i}_G(e_\ell) \text{ for } \ell \in \{2,\dots,n-1\},\\
&\mathrm{i}_G(\alpha_n) = \mathrm{t}_G(\alpha_n) = \mathrm{t}_G(e_{n-1}).
\end{aligned}$$

Since α_ℓ contains all edges of K_ℓ, $\mathscr{L}_V^C(K_\ell) = \bigcap_{e\in E(K_\ell)} \mathscr{L}(e) = \bigcap_{k=1}^{|\alpha_\ell|} \mathscr{L}((\alpha_\ell)_k)$. Combining these paths into $\gamma := {}^\infty\alpha_1.e_1\alpha_2 e_2 \dots e_n \alpha_n^\infty \in \mathrm{Trav}(K_1,\dots,K_n)$ we get

$$\emptyset = \bigcap_{\ell=1}^{n} \mathscr{L}_V^C(K_\ell) \cap \bigcap_{\ell=1}^{n-1} L_\ell = \bigcap_{\ell=1}^{n}\bigcap_{k=1}^{|\alpha_\ell|} \mathscr{L}((\alpha_\ell)_k) \cap \bigcap_{\ell=1}^{n-1} \mathscr{L}(e_\ell) = \bigcap_{k\in\mathbb{Z}} \mathscr{L}(\gamma_k). \qquad \square$$

In a typical strong component graph arising in our application (see Chapter 6), many components will contain no edge. We can significantly reduce the size of our condensation if we remove these components with the help of the following definition and the lemma following it.

Definition 5.29. *Let C be an acyclic graph whose vertices are themselves graphs (for example a strong component graph of another graph) and whose edges are elements of $V(C)^2$ with the canonical initial and terminal map. Let $\mathscr{L}_V^C$ and $\mathscr{L}_E^C$ be vertex and edge labelings. Let K be a vertex of C with $|V(K)| = 1$. Define a new graph $\tilde{C}$ by first removing the vertex K from C and then adding a new edge (L_1, L_2) for each pair of edges of the form $(L_1, K), (K, L_2)$ in C, i.e.,*

$$\begin{aligned}
V(\tilde{C}) &:= V(C) \setminus K,\\
E(\tilde{C}) &:= E_{old} \cup E_{new}, \textit{ where}\\
E_{old} &:= \{\, e \in E(C) \mid \mathrm{i}_C(e) \neq K, \mathrm{t}_C(e) \neq K \,\} \textit{ and}\\
E_{new} &:= \left\{ (L_1,L_2) \in V(C)^2 \;\middle|\; (L_1,K),(K,L_2) \in E(C) \right\}.
\end{aligned}$$

We label the edges and vertices of this graph by

$$\begin{aligned}
&\mathscr{L}_V^{\tilde{C}} : V(\tilde{C}) \to \mathscr{P}(I), \quad \mathscr{L}_V^{\tilde{C}}(K) := \mathscr{L}_V^C(K),\\
&\mathscr{L}_E^{\tilde{C}} : E(\tilde{C}) \to \mathscr{P}(\mathscr{P}(I)),\\
&\mathscr{L}_E^{\tilde{C}}(L_1,L_2) := \begin{cases} \mathscr{L}_E^C(L_1,L_2) & \text{if } (L_1,L_2) \notin E_{new}\\ \mathscr{L}_E^C(L_1,L_2) \cup \{M_1 \cap M_2 \mid M_1 \in \mathscr{L}_E^C(L_1,K), & \text{otherwise}\\ \qquad\qquad M_2 \in \mathscr{L}_E^C(K,L_2)\} & \end{cases}.
\end{aligned}$$

We call $\tilde{C}$ an elementary contraction *of C. A graph obtained from C by a finite sequence of elementary contractions is called a* contraction *of C.*

Example 5.30. *Consider again the graph $C := \mathscr{S}(G)$ from Example 5.27. Figure 5.5 shows a sequence of two elementary contractions. In the first step, for example, the vertex C is contracted. The edge from A to F is labeled by $\{\{3\}\} = \{\{3,4\}\cap\{1,2,3\}\}$ after the contraction, as the path from A to F via C is labeled by $\{\{3,4\}\}$ and $\{\{1,2,3\}\}$. The edge from B to F is labeled by $\{\{1,2\},\{1,2,3,4\}\}$ since there was an edge-labeled $\{\{1,2,3,4\}\}$ before and the*

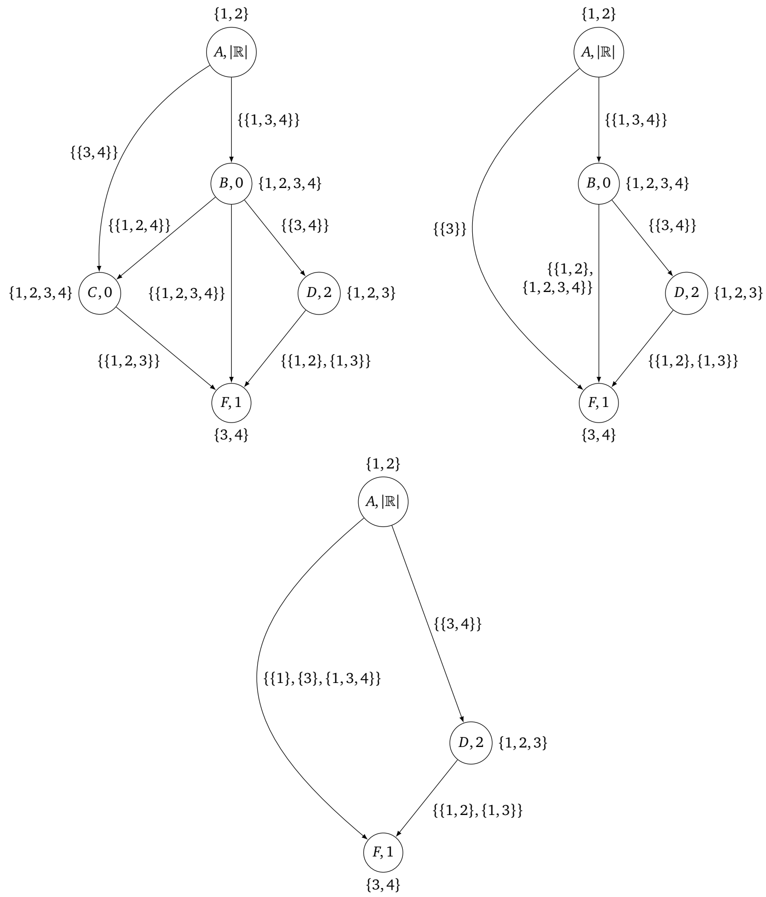


Figure 5.5.: A sequence of elementary contractions of the condensation of the graph from Example 5.27.

path from B to F traversing C is labeled by $\{\{1,2,4\}\}$ *and* $\{\{1,2,3\}\}$.

Lemma 5.31. *Let I be a finite set and let G be a graph with edge labeling $\mathscr{L} : E(G) \to \mathscr{P}(I)$. Let $C := \mathscr{S}(G)$ be the condensation of G together with the edge and vertex labelings defined in Definition 5.25. If D is a contraction of C, then for every vertex path $K_1,\dots,K_n$ in D there is a path $\gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n)$ with $\bigcap_{\ell=1}^{n} \mathscr{L}(\gamma_\ell) = \emptyset$ if and only if*

$$\emptyset \in \Big\{ \bigcap_{\ell=1}^{n} \mathscr{L}_V^D(K_\ell) \cap \bigcap_{\ell=1}^{n-1} L_\ell \,\Big|\, L_\ell \in \mathscr{L}_E^D(K_\ell, K_{\ell+1}) \text{ for } \ell \in \{1,\dots,n-1\} \Big\}.$$

Proof. Let $K_1,\dots,K_n$ be a vertex path in D. If $K_1,\dots,K_n$ is also a vertex path in C then we have nothing to show. Therefore assume that there is an index $k \in \{1,\dots,n-1\}$ and a strongly connected component $H \in V(C)$ of G such that D is obtained from C by contracting H and such that $K_1,K_2,\dots,K_k,H,K_{k+1},\dots,K_n$ is a vertex path in C. Then $\mathscr{L}_V^C(H) = \mathscr{P}(I)$ and for every $\tilde{L}_1 \in \mathscr{L}_E^C(K_k,H)$ and $\tilde{L}_2 \in \mathscr{L}_E^C(H,K_{k+1})$ we have

$\tilde{L}_1 \cap \tilde{L}_2 \in \mathscr{L}_E^D(K_k, K_{k+1})$. Therefore

$$\Big\{ \bigcap_{\ell=1}^{n} \mathscr{L}_V^D(K_\ell) \cap \bigcap_{\ell=1}^{n-1} L_\ell \,\Big|\, L_\ell \in \mathscr{L}_E^D(K_\ell, K_{\ell+1}) \text{ for } \ell \in \{1,\dots,n-1\} \Big\}$$

is contained in the set

$$\Bigg\{ \bigcap_{\ell=1}^{n} \mathscr{L}_V^C(K_\ell) \cap \mathscr{L}_V^C(H) \cap \bigcap_{\ell=1}^{k} L_\ell \cap \tilde{L}_1 \cap \tilde{L}_2 \cap \bigcap_{\ell=k+1}^{n-1} L_\ell \\ \Big|\, L_\ell \in \mathscr{L}_E^C(K_\ell, K_{\ell+1}) \text{ for } \ell \in \{1,\dots,n-1\}, \tilde{L}_1 \in \mathscr{L}_E^C(K_k, H), \tilde{L}_1 \in \mathscr{L}_E^C(H, K_{k+1}) \Bigg\}.$$

Together with Lemma 5.28 this proves the result for elementary contractions. Since every contraction is obtained by a sequence of elementary contractions, this proves the result. □

Combining the results, we obtained an algorithm to calculate the cardinality of

$$\Big\{ \gamma \in \mathrm{Path}_{\mathbb{Z}}(G) \,\Big|\, \bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset \Big\}.$$

Let C be the strong component graph of G. Apply elementary contractions to C until this is no longer possible and call the graph thus obtained D. The sets of the form $\mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n)$, where $K_1,\dots,K_n$ is a vertex path in D, partition $\mathrm{Path}_{\mathbb{Z}}(G)$. Thus with the help of Lemma 5.28 and Lemma 5.31 we determine all vertex paths $K_1,\dots,K_n$ in D for which there is a path $\alpha \in \mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n)$ with $\bigcap_{k \in \mathbb{Z}} \mathscr{L}(\alpha_k) = \emptyset$. Following this, we determine the cardinality of $\{ \gamma \in \mathrm{Trav}_{\mathbb{Z}}(K_1,\dots,K_n) \mid \bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset \}$ by Lemma 5.24. Adding up all these cardinalities gives us $|\{ \gamma \in \mathrm{Path}_{\mathbb{Z}}(G) \mid \bigcap_{k \in \mathbb{Z}} \mathscr{L}(\gamma_k) = \emptyset \}|$.

5.3. Representing Metrizable Stone Spaces by Subshifts

Before we proceed to derivative algebras generated by multiple sofic shifts, we present a second interlude. We show that every metrizable Stone space X which generates a finite derivative algebra is homeomorphic to a one-sided subshift of finite type.

The following theorem appears in a similar form in the paper of Head [Hea85, Thm. 4], but there only a slightly weaker statement, namely that X is homeomorphic to the adherence of a regular language, is shown.

Theorem 5.32. *Let $\mathscr{B}$ be a finite derivative algebra. Let T be the set of atoms of $\mathscr{B}$ and let $c : T \to \mathbb{N} \cup \{|\mathbb{N}|, |\mathbb{R}|\}$ be a function. The following are equivalent.*

(a) *$\mathscr{B}$ contains no smaller derivative subalgebra and for every $a \in T$ we have $c(a) \in \mathbb{N}$ if and only if $a^* = 0_{\mathscr{B}}$.*

(b) *There exists a metrizable Stone space X and an isomorphism of derivative algebras $\rho : \mathscr{B} \to \mathscr{D}(X)$ such that $|\rho(a)| = c(a)$ for all $a \in T$.*

(c) *There exists a one-sided subshift of finite type Y and an isomorphism of derivative algebras $\rho : \mathscr{B} \to \mathscr{D}(Y)$ such that $|\rho(a)| = c(a)$ for all $a \in T$.*

Proof. (c) $\Longrightarrow$ (b): Every one-sided subshift of finite type over the alphabet A is a closed subset of the Cantor space $A^{\mathbb{N}}$ and therefore a metrizable Stone space.

(b) $\Longrightarrow$ (a): The derivative algebra of a topological space X is by definition the smallest derivative subalgebra of $\mathscr{P}(X)$. Hence it contains no smaller derivative subalgebra. Since $\mathscr{D}(X)$ and $\mathscr{B}$ are isomorphic, $\mathscr{B}$ can not contain a smaller derivative subalgebra either. Finally for every subset $M \subseteq X$ of a metrizable space the derivative M^* is empty if and only if $|M| \in \mathbb{N}$. For $a \in T$ we have $c(a) = |\rho(a)|$, hence $a^* = 0_{\mathscr{B}}$ if and only if $c(a) \in \mathbb{N}$.

(a) $\Longrightarrow$ (c): We construct a graph G, whose edge shift generates a derivative algebra isomorphic to $\mathscr{B}$. Roughly speaking, we take the Hasse diagram of the partial order $\prec$ on T, direct edges upwards, add a loop to every vertex a with $c(a) = |\mathbb{N}|$, two loops if $c(a) = |\mathbb{R}|$ and a cycle of length $c(a)$ if $c(a) \in \mathbb{N}$. More precisely we define G as follows.

$$
\begin{aligned}
V(G) &:= \{\, (a,k) \mid a \in T, k \in \{0,\dots,k_V(a)-1\}\,\},\\
E(G) &:= \{\, (a,b) \mid a,b \in T, a \prec b \text{ and } \nexists c \in T : a \prec c \prec b \,\}\\
&\quad\ \cup \{\, (a,k) \mid a \in T, k \in \{0,\dots,k_E(a)-1\}\,\},\\
k_V(a) &:= \begin{cases} c(a) & \text{if } c(a) \in \mathbb{N}\\ 1 & \text{if } c(a) = |\mathbb{N}| \\ 1 & \text{if } c(a) = |\mathbb{R}| \end{cases} \text{ for } a \in T,\\
k_E(a) &:= \begin{cases} c(a) & \text{if } c(a) \in \mathbb{N}\\ 1 & \text{if } c(a) = |\mathbb{N}| \\ 2 & \text{if } c(a) = |\mathbb{R}| \end{cases} \text{ for } a \in T,\\
\mathrm{i}_G(a,b) &:= (a,0) \text{ for } a,b \in T,\\
\mathrm{t}_G(a,b) &:= (b,0) \text{ for } a,b \in T,\\
\mathrm{i}_G(a,k) &:= (a,k \mod k_V(a)) \text{ for } a \in T, k \in \{0,\dots,k_E(a)-1\},\\
\mathrm{t}_G(a,k) &:= (a,(k+1) \mod k_V(a)) \text{ for } a \in T, k \in \{0,\dots,k_E(a)-1\}.
\end{aligned}
$$

See Example 5.33 for an example of this construction. $\mathscr{B}$ contains no smaller derivative subalgebra, in other words, it is generated by the closed element $1_{\mathscr{B}}$. The relation $\prec$ on T is therefore a partial order by Lemma 3.31. The strongly connected components of G are the subgraphs K_a for $a \in T$ induced by the vertex sets $V_a := \{\,(a,k) \in V(G) \mid k \in \{0,\dots,k_V(a)-1\,\}$. Set

$$\mathrm{Ter}(a) := \{\, \gamma \in \mathrm{Path}_{\mathbb{N}}(G) \mid \gamma \text{ eventually lies in } K_a \,\}.$$

Then $\{\, \mathrm{Ter}(a) \mid a \in T \,\}$ is a partition of $\mathrm{Path}_{\mathbb{N}}(G)$.

If $c(a) \in \mathbb{N}$, then $a^* = 0_{\mathscr{B}}$, in particular there is no atom $b \in T$ with $b \prec a$. Therefore no edge enters K_a and $\mathrm{Ter}(a) = \mathrm{Path}_{\mathbb{N}}(K_a)$. Furthermore $|\mathrm{Ter}(a)| = c(a)$ and $\mathrm{Ter}(a)^* = \emptyset$. If $c(a) = |\mathbb{N}|$, then $\mathrm{Ter}(a) = \{\, \gamma \in \mathrm{Path}_{\mathbb{N}}(G) \mid \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \gamma_n = (a,0) \,\}$, hence $|\mathrm{Ter}(a)| \leq |\mathbb{N}|$. Since in this case $a^* \neq \emptyset$, there is at least one atom $b \in T$ with $b \prec a$ and therefore there are infinitely many paths in $\mathrm{Ter}(a)$ starting in K_b, hence $|\mathrm{Ter}(a)| = |\mathbb{N}|$. We also have $\mathrm{Ter}(a)^* = \bigcup \{\, \mathrm{Ter}(b) \mid b \in T, b \prec a \,\}$. Finally if $c(a) = |\mathbb{R}|$, then $\{(a,0),(a,1)\}^{\mathbb{N}} \subseteq \mathrm{Ter}(a)$, hence $|\mathrm{Ter}(a)| = |\mathbb{R}|$ and $\mathrm{Ter}(a)^* = \mathrm{Ter}(a) \cup \bigcup \{\, \mathrm{Ter}(b) \mid b \in T, b \prec a \,\}$. For every $a \in T$ the set $\mathrm{Ter}(a)$ is non-empty, hence the map $\rho : a \mapsto \mathrm{Ter}(a)$ is injective. In all three possible cases for $c(a)$ we showed that $\rho(a^*) = \rho(a)^*$. By Lemma 3.8 and Lemma 3.14 ρ is therefore an isomorphism of derivative algebras. □

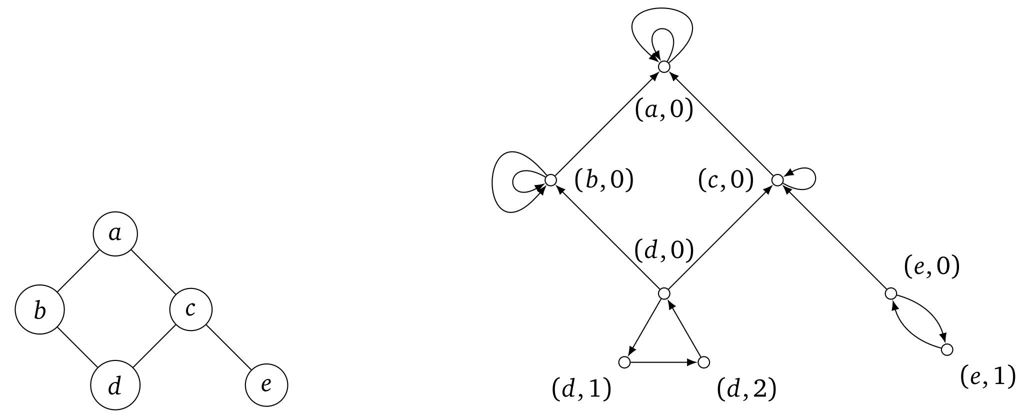


Figure 5.6.: Realization of a derivative algebra by a one-sided subshift of finite type.

Example 5.33. *The left side of Figure 5.6 shows the 5 atoms of a derivative algebra. The cardinalities of the atoms are $|a| = |b| = |\mathbb{R}|, |c| = |\mathbb{N}|, |d| = 3, |e| = 2$. The order $\preceq$ on the atoms is shown as a Hasse diagram, in other words, one atom is smaller than another, if there is a path from the first to the second going only upwards. For example $d \prec a$, while b and c are incomparable.*

The right side of Figure 5.6 shows a subshift of finite type constructed as in the proof of Theorem 5.32. There is a cardinality preserving isomorphism between the derivative algebra generated by the one-sided shift of finite type defined by this graph and the derivative algebra defined by the order on the atoms given on the left side.

Remark 5.34. While Corollary 5.11 together with Theorem 4.19 shows that every two-sided subshift of finite type is homeomorphic to a one-sided subshift of finite type, the converse is not true. Consider for example the one-sided subshift X generated by $\{\, 0^k 1^\infty \mid k \in \mathbb{N} \,\}$. It contains exactly one accumulation point, 0^∞. Assume there would be a two-sided subshift of finite type Y homeomorphic to X. Then Y would be homeomorphic to $\mathrm{Path}_{\mathbb{Z}}(G)$ for some graph G. Since X is countably infinite, G must contain at least two strongly connected components, both of which are cycles. But then Y contains two distinct accumulation points.

Question 5.35. *Is every metrizable Stone space which generates a finite derivative algebra homeomorphic to a two-sided sofic shift?*

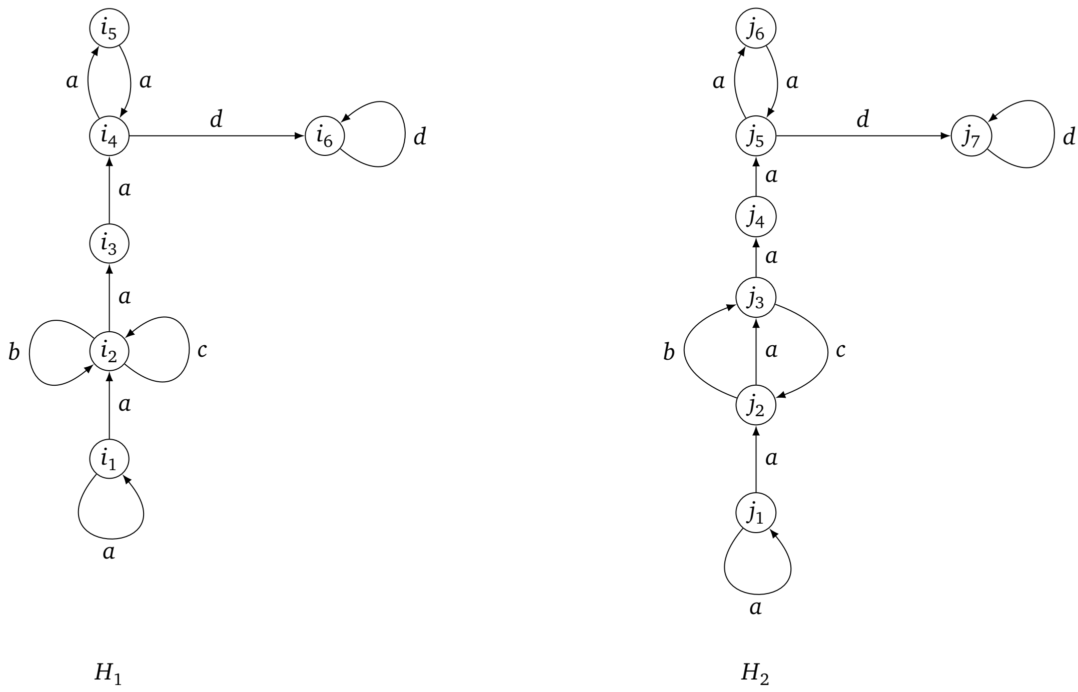


Figure 5.7.: Two sofic shifts whose union generates a complicated derivative algebra.

5.4. Derivative Algebras of Multiple Sofic Shifts

After having shown that multiple one-sided or two-sided subshifts of finite type generate a finite derivative algebra, we extend this result to multiple sofic shifts. By Lemma 2.47 we can reduce this problem to the study of the images of edge shifts under an edge labeling. The following example, however, shows that the methods from Section 5.1 do not apply straightforwardly. In particular, for a graph G with edge labeling $\mathscr{L}$, the Boolean algebra generated by $\{ \mathscr{L}(\mathrm{Full}(H)) \mid H \text{ is a subgraph of } G \}$ must not be a derivative algebra.

Example 5.36. *Consider the sofic shift given by the graph G with edge labeling $\mathscr{L}$ depicted in Figure 5.7. The underlying graph consists of two parts, H_1 and H_2. Set $X := \mathscr{L}(\mathrm{Full}_{\mathbb{Z}}(H_1))$ and $Y := \mathscr{L}(\mathrm{Full}_{\mathbb{Z}}(H_2))$. It is easy to see that ${}^{\infty}a.bcba^{\infty} \in (X \cap Y)^*$, but ${}^{\infty}a.bca^{\infty} \notin (X \cap Y)^*$.*

Let $x \in \{{}^{\infty}a.bcba^{\infty}, {}^{\infty}a.bca^{\infty}\}$ and let H be a subgraph of G such that $x \in \mathscr{L}(\mathrm{Full}_{\mathbb{Z}}(H))$. Either H contains the subgraph induced by $i_1, \ldots, i_5$ or it contains the subgraph induced by $j_1, \ldots, j_6$. In both cases $\{ {}^{\infty}a.bcba^{\infty}, {}^{\infty}a.bca^{\infty} \} \subseteq \mathrm{Full}(H)$. This implies that a set M in the Boolean algebra generated by $\{ \mathscr{L}(\mathrm{Full}(H)) \mid H \text{ is a subgraph of } G \}$ either contains $\{{}^{\infty}a.bcba^{\infty}, {}^{\infty}a.bca^{\infty}\}$ as a subset or is disjoint from it. This Boolean algebra is therefore not a derivative algebra.

Nevertheless we can reduce the problem of determining the derivative algebra generated by a finite number of two-sided sofic subshifts to the cases we already covered in the previous sections. The main tool to do so is the folding of a two-sided subshift.

Definition 5.37 (Folding of a subshift). *For $x \in A^{\mathbb{Z}}$ define the* folding $\mathscr{F}(x) \in (A \times A)^{\mathbb{N}}$ *of x by*

$$\mathscr{F}(x)_i := (x_{i-1}, x_{-i}).$$

The folding $\mathscr{F}(X)$ of a subshift $X \subseteq A^{\mathbb{Z}}$ *is then simply the image $\mathscr{F}[X]$ of X under $\mathscr{F}$.*

For example we have $\mathscr{F}({}^{\infty}(01)2.34(56)^{\infty}) = (3,2)(4,1)((5,0),(6,1))^{\infty}$. This notion was introduced without a name by Head in [Hea91]. The map $\mathscr{F} : A^{\mathbb{Z}} \to (A \times A)^{\mathbb{N}}$ is continuous and bijective, hence $\mathscr{F}(X)$ is homeomorphic to X for all subshifts X. Notice, however, that the folding of a subshift is in general not shift invariant and in particular not a one-sided subshift.

Definition 5.38 (Folding of a graph). *Let G be a graph. Define the* folding $\mathscr{F}(G)$ *of G as the graph with*

$$\begin{aligned}
V(\mathscr{F}(G)) &:= V(G)^2, \\
E(\mathscr{F}(G)) &:= E(G)^2, \\
\mathrm{i}_{\mathscr{F}(G)}(e_1, e_2) &:= (\mathrm{i}_G(e_1), \mathrm{t}_G(e_2)), \\
\mathrm{t}_{\mathscr{F}(G)}(e_1, e_2) &:= (\mathrm{t}_G(e_1), \mathrm{i}_G(e_2)).
\end{aligned}$$

In other words, $\mathscr{F}(G)$ is obtained as the Cartesian product of two copies of G with the direction of every edge being reversed in the second copy.

If $\mathscr{L} : E(G) \to A$ is an edge labeling of G, then $\mathscr{F}(G)$ also has a naturally defined edge labeling $\widetilde{\mathscr{L}} : E(\mathscr{F}(G)) \to A^2$ given by

$$\widetilde{\mathscr{L}}(e_1, e_2) := (\mathscr{L}(e_1), \mathscr{L}(e_2)).$$

We call $\widetilde{\mathscr{L}}$ the folding *of $\mathscr{L}$.*

The next lemma gives a connection between the folding of a graph and the folding of its edge shift.

Lemma 5.39. *Let G be a graph. If $\tilde{G} = \mathscr{F}(G)$ is the folding of G, then*

$$\mathscr{F}(\mathrm{Path}_{\mathbb{Z}}(G)) = \mathrm{Path}_{\mathbb{N}}(\tilde{G}, \Delta_{V(G)}),$$

where $\Delta_{V(G)} = \{(v, v) \mid v \in V(G)\}$. If $\mathscr{L}$ is an edge labeling and $\widetilde{\mathscr{L}}$ is the folding of $\mathscr{L}$, then

$$\mathscr{F}(\mathscr{L}[\mathrm{Path}_{\mathbb{Z}}(G)]) = \widetilde{\mathscr{L}}[\mathscr{F}(\mathrm{Path}_{\mathbb{Z}}(G))] = \widetilde{\mathscr{L}}[\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \Delta_{V(G)})].$$

Proof. For every path $\gamma \in \mathrm{Path}_{\mathbb{Z}}(G)$, we have $\mathrm{i}_G(\gamma_{\ell+1}) = \mathrm{t}_G(\gamma_\ell)$ for all $\ell \in \mathbb{Z}$. Let $\tilde{\gamma}$ be the folding of γ. For every $\ell \in \mathbb{N}$

$$\begin{aligned} \mathrm{i}_{\tilde{G}}(\tilde{\gamma}_{\ell+1}) &= \mathrm{i}_{\tilde{G}}(\gamma_\ell, \gamma_{-\ell-1}) \\ &= (\mathrm{i}_G(\gamma_\ell), \mathrm{t}_G(\gamma_{-\ell-1})) \\ &= (\mathrm{t}_G(\gamma_{\ell-1}), \mathrm{i}_G(\gamma_{-\ell})) \\ &= \mathrm{t}_{\tilde{G}}(\gamma_{\ell-1}, \gamma_{-\ell}) \\ &= \mathrm{t}_{\tilde{G}}(\tilde{\gamma}_\ell) \end{aligned}$$

and $\mathrm{i}_{\tilde{G}}(\tilde{\gamma}_1) = \mathrm{i}_{\tilde{G}}(\gamma_0, \gamma_{-1}) = (\mathrm{i}_G(\gamma_0), \mathrm{t}_G(\gamma_1)) \in \Delta_{V(G)}$. Hence $\tilde{\gamma} \in \mathrm{Path}_{\mathbb{N}}(\tilde{G}, \Delta_{V(G)})$.

On the other hand, let $\tilde{\gamma} = (e^1_\ell, e^2_\ell)_{\ell\in\mathbb{N}} \in \mathrm{Path}_{\mathbb{N}}(\tilde{G}, \Delta_{V(G)})$. We can unfold $\tilde{\gamma}$ by defining $\gamma \in E(G)^{\mathbb{Z}}$ by

$$\gamma_\ell = \begin{cases} e^1_{\ell+1} & \text{if } \ell \geq 0 \\ e^2_{-\ell} & \text{if } \ell < 0 \end{cases}.$$

Clearly $\mathscr{F}(\gamma) = \tilde{\gamma}$, $\mathrm{i}_G(e^1_{\ell+1}) = \mathrm{t}_G(e^1_\ell)$ and $\mathrm{t}_G(e^2_{\ell+1}) = \mathrm{i}_G(e^2_\ell)$. Hence for $\ell \geq 0$, $\mathrm{i}_G(\gamma_{\ell+1}) = \mathrm{i}_G(e^1_{\ell+2}) = \mathrm{t}_G(e^1_{\ell+1}) = \mathrm{t}_G(\gamma_\ell)$, for $\ell < 0$, $\mathrm{i}_G(\gamma_{\ell+1}) = \mathrm{i}_G(e^2_{-\ell-1}) = \mathrm{t}_G(e^2_{-\ell}) = \mathrm{t}_G(\gamma_\ell)$, and $\mathrm{i}_G(\gamma_0) = \mathrm{i}_G(e^1_1) = \mathrm{t}_G(e^2_1) = \mathrm{t}_G(\gamma_1)$. All in all, this shows that $\gamma \in \mathrm{Path}_{\mathbb{Z}}(G)$. For $\gamma \in \mathrm{Path}_{\mathbb{Z}}(G)$, $\mathscr{F}(\mathscr{L}(\gamma)) = (\mathscr{L}(\gamma_{\ell-1}), \mathscr{L}(\gamma_{-\ell}))_{\ell\in\mathbb{N}} = \widetilde{\mathscr{L}}((\gamma_{\ell-1}, \gamma_{-\ell})_{\ell\in\mathbb{N}}) = \widetilde{\mathscr{L}}(\mathscr{F}(\gamma))$. □

The following theorem is well-known in automata theory as the subset construction, showing that the languages recognized by deterministic and non-deterministic automata coincide. See for example the book by Sakarovitch [Sak09, Prop. 3.2]. In symbolic dynamics it is used to construct right-resolving presentations of sofic shifts, see [LM95, Section 3.4].

Theorem 5.40. *Let G be a graph with edge labeling $\mathscr{L}$ and let $S \subseteq V(G)$. There is a graph $\tilde{G}$, a right-resolving edge labeling $\widetilde{\mathscr{L}}$ and a subset $\tilde{S} \subseteq V(G)$ such that*

$$\mathscr{L}[\mathrm{Path}_{\mathbb{N}}(G,S)] = \widetilde{\mathscr{L}}[\mathrm{Path}_{\mathbb{N}}(\tilde{G},\tilde{S})].$$

Proof. Define the graph $\tilde{G}$, the edge labeling $\widetilde{\mathscr{L}}$ and the set $\tilde{S}$ as follows.

$$\begin{aligned} V(\tilde{G}) &:= \mathscr{P}(V(G)) \text{ (hence the name subset construction)}, \\ E(\tilde{G}) &:= \{(M, \delta_a(M), a) \mid M \subseteq V(G), a \in A\} \setminus \{(M, \emptyset, a) \mid M \subseteq V(G), a \in A\}, \text{ where} \\ \delta_a(M) &:= \{\mathrm{t}_G(e) \in V(G) \mid e \in E(G), \mathrm{i}(e) \in M, \mathscr{L}(e) = a\}, \\ \mathrm{i}_{\tilde{G}}(M,N,a) &:= M, \\ \mathrm{t}_{\tilde{G}}(M,N,a) &:= N, \\ \widetilde{\mathscr{L}}(M,N,a) &:= a, \\ \tilde{S} &:= \{S\}. \end{aligned}$$

The edge labeling $\mathscr{L}$ is by definition right-resolving. If γ is a path in $\mathrm{Path}_{\mathbb{N}}(G,S)$ and

$x := \mathscr{L}(\gamma)$, then

$$
\begin{aligned}
&\mathrm{i}_G(\gamma_1) \in S := M_1,\\
&\mathrm{i}_G(\gamma_2) \in \delta_{x_1}(M_1) := M_2,\\
&\mathrm{i}_G(\gamma_3) \in \delta_{x_2}(M_2) := M_3,\\
&\dots\\
&\mathrm{i}_G(\gamma_k) \in \delta_{x_{k-1}}(M_{k-1}) := M_k,\\
&\dots
\end{aligned}
$$

Hence $\tilde{\gamma} := (M_1, M_2, x_1)(M_2, M_3, x_2)(M_3, M_4, x_3)\dots$ is a path in $\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \tilde{S})$ with $\widetilde{\mathscr{L}}(\tilde{\gamma}) = x = \mathscr{L}(\gamma)$.

On the other hand, let $\tilde{\gamma}$ be a path in $\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \tilde{S})$ and let $n \in \mathbb{N}$. Set $\tilde{x} := \widetilde{\mathscr{L}}(\tilde{\gamma})$. There must be an edge $e_n \in E(G)$ labeled by $\mathscr{L}$ with x_n, starting in a vertex in $\mathrm{i}_{\tilde{G}}(\tilde{\gamma}_n) \subseteq V(G)$ and ending in a vertex in $\mathrm{t}_{\tilde{G}}(\tilde{\gamma}_n)$. Next, there must be an edge $e_{n-1} \in E(G)$ labeled by $\mathscr{L}$ with x_{n-1}, ending in $\mathrm{i}_G(e_n)$ and starting in a vertex in $\mathrm{i}_{\tilde{G}}(\tilde{\gamma}_{n-1})$. Continuing this way, one obtains a finite path γ in $\mathrm{Path}(G)$ starting in a vertex in $\mathrm{i}_{\tilde{G}}(\tilde{\gamma}_1) = S$ with $\mathscr{L}(\gamma) = \widetilde{\mathscr{L}}(\tilde{\gamma}_{[1,n]})$. A straightforward compactness argument then shows the existence of a path $\beta \in \mathrm{Path}_{\mathbb{N}}(G, S)$ with $\mathscr{L}(\beta) = \widetilde{\mathscr{L}}(\tilde{\gamma})$. □

The following theorem, while pretty simple, is the key tool to get rid of the edge labels.

Theorem 5.41. *Let G be a graph, let $\mathscr{L} : E(G) \to A$ be an edge labeling and let $i \in V(G)$ be a distinguished vertex. If $\mathscr{L}$ is right-resolving, then the extension of $\mathscr{L}$ to $\mathrm{Path}_{\mathbb{N}}(G, \{i\})$, $\mathscr{L} : \mathrm{Path}_{\mathbb{N}}(G, \{i\}) \to \mathscr{L}[\mathrm{Path}_{\mathbb{N}}(G, \{i\})]$, is a homeomorphism.*

Proof. Since the edge labeling is right-resolving, its extension to the edge shift is bijective. Additionally $\mathscr{L} : \mathrm{Path}_{\mathbb{N}}(G, \{i\}) \to \mathscr{L}[\mathrm{Path}_{\mathbb{N}}(G, \{i\})]$ is continuous since it is induced by an edge labeling, hence it is a homeomorphism by Theorem 2.1. □

At the moment we are able to represent the folding of sofic subshifts by graphs with right resolving edge labelings. To apply the previous theorem, we have to merge them into one graph with one distinguished initial vertex. This is accomplished by a construction that is very similar to the one used to construct the intersection of sofic subshifts, see [LM95, Prop. 3.4.10].

Theorem 5.42. *For $k \in \{1, \dots, n\}$ let G_k be a graph, let $\mathscr{L}_k : E(G_k) \to A$ be a right-resolving edge labeling and let $S_k \subseteq V(G_k)$ be a set of vertices. There exists a graph $\tilde{G}$ with a right-resolving edge labeling $\widetilde{\mathscr{L}}$, a distinguished vertex $i \in V(\tilde{G})$ and subgraphs $\tilde{G}_k$ such that*

$$\widetilde{\mathscr{L}}[\mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \{i\})] = \mathscr{L}_k[\mathrm{Path}_{\mathbb{N}}(G_k, S_k)]$$

and such that

$$\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \{i\}) = \bigcup_{k=1}^{n} \mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \{i\}).$$

Proof. Let $G_1,\dots,G_n$ be graphs with disjoint vertex sets, right-resolving edge labelings $\mathscr{L}_1,\dots,\mathscr{L}_n$ with values in A and distinguished sets of vertices $S_1,\dots,S_n$. Let 0 be a vertex not contained in the vertex sets of $G_1,\dots,G_n$. Let G be the disjoint union of $G_1,\dots,G_n$. The edge labelings $\mathscr{L}_k, k\in\{1,\dots,n\}$ naturally define an edge labeling $\mathscr{L}$ on G. Now we construct a graph $\hat{G}$ by first adding a vertex 0 to G. Next, we add an edge from i to 0 labeled with $a\in A$ for every vertex in G which has no out-going edge labeled with a by $\mathscr{L}$. Finally, we add a loop from 0 to itself for every symbol in A. Formally this means

$$\begin{aligned}
V(\hat{G}) &:= V(G)\uplus\{0\},\\
E(\hat{G}) &:= E(G)\uplus\{\,(i,a)\mid i\in V(G),\forall e\in E(G):\mathscr{L}(e)\neq a\,\}\uplus\{0\}\times\{a\},\\
\mathrm{i}_{\hat{G}}(e) &:= \mathrm{i}_G(e) \text{ for } e\in E(G),\\
\mathrm{t}_{\hat{G}}(e) &:= \mathrm{t}_G(e) \text{ for } e\in E(G),\\
\hat{\mathscr{L}}(e) &:= \mathscr{L}_k(e) \text{ for } e\in E(G_k), k\in\{1,\dots,n\},\\
\mathrm{i}_{\hat{G}}(i,a) &:= i \text{ for } i\in V(G)\cup\{0\}, a\in A,\\
\mathrm{t}_{\hat{G}}(i,a) &:= 0 \text{ for } i\in V(G)\cup\{0\}, a\in A,\\
\hat{\mathscr{L}}(i,a) &:= a \text{ for } i\in V(G)\cup\{0\}, a\in A.
\end{aligned}$$

We identify the sets $S_1,\dots,S_k$ with the corresponding vertex sets in G and $\hat{G}$ and set $S:=S_1\cup\cdots\cup S_k$.

Since each of the edge labelings $\mathscr{L}_1,\dots,\mathscr{L}_k$ is right-resolving, each vertex in $\hat{G}$ has exactly one out-going edge labeled with a for every symbol $a\in A$. Hence for every $i\in V(G)$ and $x\in A^{\mathbb{N}}$ there is a unique path $\gamma(x,i)\in\mathrm{Path}_{\mathbb{N}}(\hat{G},\{i\})$ with $\mathscr{L}(\gamma(x,i))=x$.

Finally we define a new graph $\tilde{G}$ by

$$\begin{aligned}
V(\tilde{G}) &:= V(\hat{G})^S\setminus\{0\}^S,\\
E(\tilde{G}) &:= \left\{\,(e_i)_{i\in S}\in E(\hat{G})^S \;\middle|\; \exists i\in S:\mathrm{t}_{\hat{G}}(e_i)\neq 0 \text{ and } \forall i,j\in S:\hat{\mathscr{L}}(e_i)=\hat{\mathscr{L}}(e_j)\,\right\},\\
\mathrm{i}_{\tilde{G}}((e_i)_{i\in S}) &:= (\mathrm{i}_{\hat{G}}(e_i))_{i\in S},\\
\mathrm{t}_{\tilde{G}}((e_i)_{i\in S}) &:= (\mathrm{t}_{\hat{G}}(e_i))_{i\in S}.
\end{aligned}$$

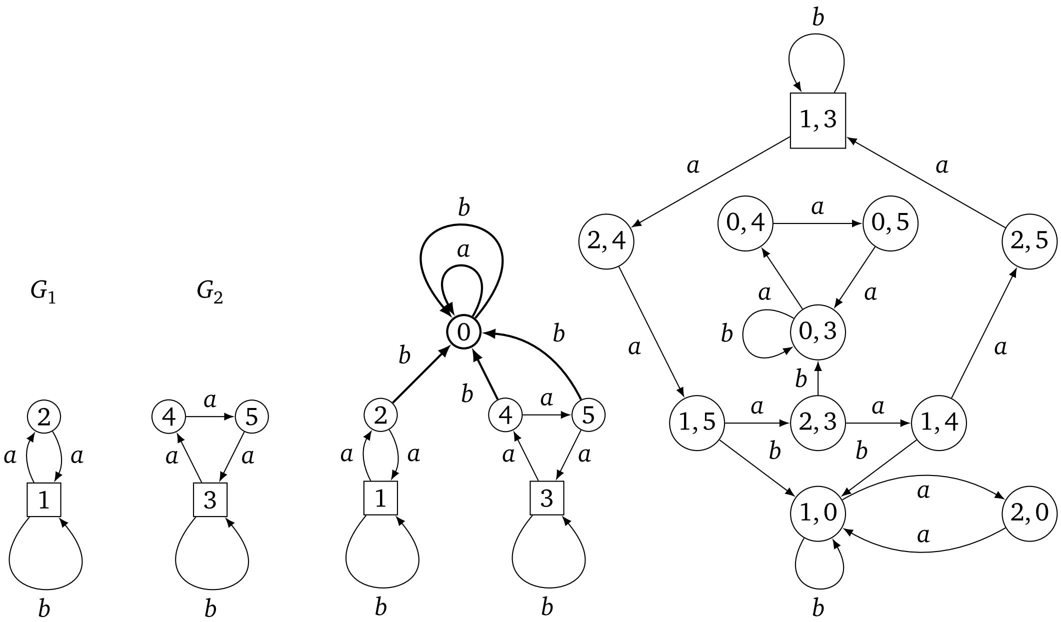
We also define a labeling $\widetilde{\mathscr{L}}:E(\tilde{G})\to A$ by $\widetilde{\mathscr{L}}((e_i)_{i\in S}):=\hat{\mathscr{L}}(e_j)$ where $j\in S$ is any fixed vertex. By the definition of $E(\tilde{G})$, the choice of this vertex does not influence $\widetilde{\mathscr{L}}$.

Set $\tilde{S}=\{(i)_{i\in S}\}$ and and let $\tilde{G_1},\dots,\tilde{G_n}$ be the induced subgraphs defined by

$$V(\tilde{G}_k):=\{\,(j_i)_{i\in S}\mid \exists i\in S_k: j_i\neq 0\,\}.$$

If two edges $(e_i^1)_{i\in S}$ and $(e_i^2)_{i\in S}$ of $\tilde{G}$ start in the same vertex $(j_i)_{i\in S}$ and have the same label a under $\widetilde{\mathscr{L}}$, then $\mathrm{i}_G(e_i^1)=\mathrm{i}_G(e_i^2)=j_i$ and $\mathscr{L}(e_i^1)=\mathscr{L}(e_i^2)=a$. Because $\hat{\mathscr{L}}$ is right-resolving, this implies $e_i^1=e_i^2$ for every $i\in S$, in other words, $(e_i^1)_{i\in S}=(e_i^2)_{i\in S}$. But this shows that $\widetilde{\mathscr{L}}$, too, is right-resolving.

Let $\tilde{\gamma}=((e_i^\ell)_{i\in S})_{\ell\in N}$ be an infinite path in $\mathrm{Path}_{\mathbb{N}}(\tilde{G},\tilde{S})$. If $\mathrm{t}_{\tilde{G}}(e_i^\ell)=0$ for some $i\in S,\ell\in\mathbb{N}$, then $\mathrm{t}_{\hat{G}}(e_i^m)=0$ for all $m\geq\ell$, as the only edges leaving the vertex 0 in $\hat{G}$ are loops



(a) The graphs G_1 and G_2. (b) The graph $\hat{G}$. (c) The graph $\tilde{G}$.

Figure 5.8.: The three steps in the construction of the graph $\tilde{G}$ in example Example 5.43.

ending in 0 again. Hence there must be a vertex $i \in S$ such that $e_i^\ell \neq 0$ for all $\ell \in \mathbb{N}$. Since $S = S_1 \cup \cdots \cup S_n$, there exists an index $k \in \{1, \ldots, n\}$ such that $i \in S_k$ and hence $\tilde{\gamma} \in \mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \tilde{S})$. Since $\tilde{\gamma}$ was an arbitrary infinite path in $\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \tilde{S})$, we have $\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \tilde{S}) = \bigcup_{k=1}^n \mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \tilde{S})$. Additionally $\gamma := (e_i^\ell)_{\ell \in \mathbb{N}}$ is a path in $\mathrm{Path}_{\mathbb{N}}(G_k, S_k)$ and $\mathscr{L}_k(\gamma) = \widetilde{\mathscr{L}}(\tilde{\gamma})$. This implies $\widetilde{\mathscr{L}}(\tilde{\gamma}) \in \mathscr{L}_k[\mathrm{Path}_{\mathbb{N}}(G_k, S_k)]$.

On the other hand, let x be an element of $\mathscr{L}_k[\mathrm{Path}_{\mathbb{N}}(G_k, S_k)]$ for some $k \in \{1, \ldots, n\}$. Define $\tilde{\gamma} := ((\gamma(x, i)_\ell)_{i \in S})_{\ell \in \mathbb{N}}$. This is a path in $\hat{G}$, starting in $(i)_{i \in S}$, whose label under $\widetilde{\mathscr{L}}$ is x. Since $x \in \mathscr{L}_k[\mathrm{Path}_{\mathbb{N}}(G_k, S_k)]$, there must be a vertex $j \in S_k$ such that $\mathrm{t}_{\hat{G}}(\gamma(x, j)_\ell) \neq 0$ for all $\ell \in \mathbb{N}$, hence $\tilde{\gamma} \in \mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \tilde{S})$. Together this shows

$$\widetilde{\mathscr{L}}[\mathrm{Path}_{\mathbb{N}}(\tilde{G}_k, \tilde{S})] = \mathscr{L}_k[\mathrm{Path}_{\mathbb{N}}(G_k, S_k)]. \qquad \square$$

Example 5.43. *We will apply the construction from Theorem 5.42 to the two edge-labeled graphs G_1 and G_2 depicted in Figure 5.8(a). The edge labelings define two subsets $X_1 = \mathscr{L}_1[\mathrm{Path}_{\mathbb{N}}(G_1, \{1\})]$ and $X_2 = \mathscr{L}_2[\mathrm{Path}_{\mathbb{N}}(G_2, \{3\})]$. A sequence is contained in X_1 if every block of "a"s it contains has even length and a sequence is contained in X_2 if every block of "a"s it contains has length divisible by three.*

By taking the disjoint union of G_1 and G_2 and adding the vertex 0 together with the corresponding edges, one obtains the graph $\hat{G}$ depicted in the middle.

With $S = \{1, 3\}$, the final output of the construction is the graph $\tilde{G}$ shown on the right side. For space reasons, all vertices not reachable from $(1, 3)$ are omitted. This does clearly not change $\mathrm{Path}_{\mathbb{N}}(\tilde{G}, \{(1, 3)\})$. It is easy to check that $\widetilde{\mathscr{L}}$ is right-resolving. Taking the subgraphs induced by all vertices having no zero in the first or second coordinate gives rise to the *subgraphs $\tilde{G}_1$ and $\tilde{G}_2$, respectively.*

To recapitulate what we proved, we introduce some further notation. For topological spaces X, Y and subsets $X_1, \ldots, X_n \subseteq X, Y_1, \ldots, Y_n \subseteq Y$, we write $(X_1, \ldots, X_n) \cong (Y_1, \ldots, Y_n)$ if there is a homeomorphism $\varphi : X_1 \cup \cdots \cup X_n \to Y_1 \cup \cdots \cup Y_n$ such that $\varphi[X_k] = Y_k$ for $k \in \{1, \ldots, n\}$.

We showed that for two-sided sofic shifts $Y_1, \ldots, Y_n \subseteq A^{\mathbb{Z}}$ there are graphs $G_1, \ldots, G_n$, $H_1, \ldots, H_n$, $F_1 \ldots F_n$ and a graph $\tilde{F}$ with subgraphs $\tilde{F}_1, \ldots \tilde{F}_n$, edge labelings $\mathscr{L}_{G_1}, \ldots, \mathscr{L}_{G_n}$, $\mathscr{L}_{H_1}, \ldots, \mathscr{L}_{H_n}$, $\mathscr{L}_{F_1}, \ldots, \mathscr{L}_{F_n}$, $\mathscr{L}_{\tilde{F}}$, sets of vertices $S_{H_1} \subseteq V(H_1), \ldots, S_{H_n} \subseteq V(H_n)$, $S_{F_1} \subseteq V(F_1), \ldots, S_{F_n} \subseteq V(F_n)$, and a vertex $i \in V(\tilde{F})$, such that $\mathscr{L}_{F_1}, \ldots, \mathscr{L}_{F_n}$ and $\mathscr{L}_{\tilde{F}}$ are right-resolving and such that

$$\begin{aligned}
(Y_1, \ldots, Y_n) &= (\mathscr{L}_{G_1}[\mathrm{Path}_{\mathbb{Z}}(G_1)], \ldots, \mathscr{L}_{G_n}[\mathrm{Path}_{\mathbb{Z}}(G_n)]) && \text{(Lem. 2.47)} \\
&\cong (\mathscr{L}_{H_1}[\mathrm{Path}_{\mathbb{N}}(H_1, S_{H_1})], \ldots, \mathscr{L}_{H_n}[\mathrm{Path}_{\mathbb{N}}(H_n, S_{H_n})]) && \text{(Lem. 5.39)} \\
&= (\mathscr{L}_{F_1}[\mathrm{Path}_{\mathbb{N}}(F_1, S_{F_1})], \ldots, \mathscr{L}_{F_n}[\mathrm{Path}_{\mathbb{N}}(F_n, S_{F_n})]) && \text{(Thm. 5.40)} \\
&= (\mathscr{L}_{\tilde{F}}[\mathrm{Path}_{\mathbb{N}}(\tilde{F}_1, \{i\})], \ldots, \mathscr{L}_{\tilde{F}}[\mathrm{Path}_{\mathbb{N}}(\tilde{F}_n, \{i\})]) && \text{(Thm. 5.42)} \\
&\cong (\mathrm{Path}_{\mathbb{N}}(\tilde{F}_1, \{i\}), \ldots, \mathrm{Path}_{\mathbb{N}}(\tilde{F}_n, \{i\})). && \text{(Thm. 5.41)}
\end{aligned}$$

By the results in Section 5.1, in particular Theorem 5.10, this shows that the derivative algebra generated by a finite number of two-sided sofic shifts is finite. Since all the theorems used in the chain of homeomorphisms above give explicit constructions for the graphs involved, one also obtains an algorithm to calculate this finite derivative algebra and the cardinality of its atoms and hence by Theorem 4.34 an algorithm to decide topological conjugacy of periodic cellular automata on two-sided sofic shifts.

We finish this section by explicitly stating the results analogues to Corollary 5.15 and Corollary 5.16 for sofic subshifts.

Corollary 5.44. *Let $Y, Y_1, \ldots, Y_n$ and $\tilde{Y}, \tilde{Y}_1, \ldots, \tilde{Y}_n$ be one-sided or two-sided sofic subshifts with $Y_1, \ldots, Y_n \subseteq Y$ and $\tilde{Y}_1, \ldots, \tilde{Y}_n \subseteq \tilde{Y}$ given as edge labelings of finite graphs. It is decidable if there is a homeomorphism $\varphi : Y \to \tilde{Y}$ with $\varphi[Y_k] = \tilde{Y}_k$ for all $k \in \{1, \ldots, n\}$.*

Corollary 5.45. *Let X and Y be one-sided or two-sided sofic shifts. If $f : X \to X$ and $g : Y \to Y$ are p-periodic cellular automata, then f is topologically conjugate to g if and only if there is a homeomorphism $\rho : X \to Y$ such that $\rho[\mathrm{Per}_k(f)] = \mathrm{Per}_k(g)$ for all $k \in \{1, \ldots, p\}$. If the subshifts are given as edge labelings and the cellular automata are given by their local rules, the existence of such a homeomorphism is decidable.*

Chapter 6.

Elementary Cellular Automata

In this chapter we apply the methods developed in the previous chapters to classify the 256 cellular automata with radius 1 over the binary alphabet, which are also called the *elementary* ones. There have been many proposed classifications for cellular automata, see for example the survey by Martinez [Mar13]. Usually the equivalence classes consist of cellular automata showing similar behavior, for example, with respect to sensitivity. Our equivalence classes in contrast consist of cellular automata having identical behavior in the sense of topological dynamics.

The results in this chapter, together with Theorem 2.59 and a very brief version of Section 5.2, were presented by the author at Automata 2015, see [Epp15]. Notice that this article contains an error in the formula for $|\operatorname{Trav}(K_1,\dots,K_n)|$, see Lemma 5.23 for the correct version. Consequently the data calculated for w_{156} was wrong. The error does, however, not affect any other result or computation presented there and in particular it changes nothing with respect to the classification.

6.1. Data for the 256 Elementary Cellular Automata

There are $2^3 = 8$ possible inputs for a binary function with three variables and therefore there are $2^8 = 256$ cellular automata with radius 1 on the two-sided full 2-shift. We enumerate them according to their Wolfram code and denote by w_c the cellular automaton with Wolfram code c. To do so, consider 8 positions enumerated from right to left and write at position k the output of the local rule with the binary expansion of k as the input. The result is the binary expansion of the Wolfram code of the cellular automaton. More formally,

$$c = \sum_{x_{-1}\in\{0,1\}} \sum_{x_0\in\{0,1\}} \sum_{x_1\in\{0,1\}} (w_c)_{\mathrm{loc}}(x_{-1},x_0,x_1)\cdot 2^{4x_{-1}+2x_0+x_1}.$$

This naming scheme was introduced by Wolfram in [Wol83].

In Section 2.4 it was shown that we can always get a conjugate cellular automaton by conjugation with the automorphisms of $\{0,1\}^{\mathbb{Z}}$ induced by

$$\begin{aligned} \upsilon\colon \{0,1\} \to \{0,1\}, &\quad \upsilon(a) = 1-a, \\ \tau\colon \mathbb{Z} \to \mathbb{Z}, &\quad \tau(k) = -k. \end{aligned}$$

Each equivalence class of cellular automata up to conjugation with these two homeomorphisms contains at most four elements (it contains less if, e.g., $f = \upsilon \circ f \circ \upsilon^{-1}$), see Appendix A. It is well known that 88 of these equivalence classes remain see for example the survey by Martinez [Mar13]. Let $\mathscr{E}$ be the set of these equivalence classes. In the following, each class will be represented by its element with the lowest Wolfram code.

For each of these representatives we calculate the cardinalities of $\widetilde{\mathrm{Pre}}_{q,p}$ for $q, p = 1, \ldots, n$ with $q + p \leq 5$ as outlined in Section 5.2. In order to also include some topological information, we calculate the cardinality of the derived set of the p-periodic points and the cardinality of the complement of this set for $p = 1, \ldots, 6$. The results are shown in Appendix A.

6.2. The Special Cases

There are 8 tuples of cellular automata in $\mathscr{E}$ that can not be distinguished by the invariants in Appendix A.

The following pairs of cellular automata are conjugate by the map $\vartheta : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ that flips every second bit, i.e.,

$$\vartheta\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}, \quad \vartheta(x)_k := \begin{cases} 1 - x_k & \text{if } k \text{ is odd} \\ x_k & \text{if } k \text{ is even} \end{cases}$$

(a) $(15, 170), w_{15} = \sigma \circ \upsilon$, $w_{170} = \sigma$. Notice that w_{15} and w_{170} can not be strongly conjugate because every cellular automaton commutes with σ.

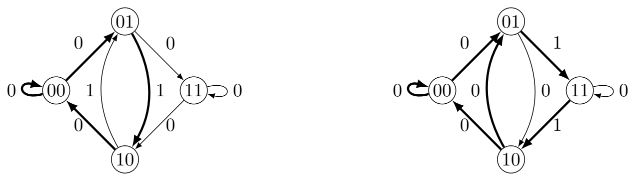
(b) $(23, 178)$,

(c) $(77, 232)$.

Next we have the three rules $w_{90}, w_{105}, w_{150} : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ with

$$\begin{aligned} w_{90,\mathrm{loc}}(x_{-1}, x_0, x_1) &= x_{-1} + x_1, \\ w_{105,\mathrm{loc}}(x_{-1}, x_0, x_1) &= 1 + x_{-1} + x_0 + x_1, \\ w_{150,\mathrm{loc}}(x_{-1}, x_0, x_1) &= x_{-1} + x_0 + x_1. \end{aligned}$$

These, together with their conjugates with respect to υ, are exactly the left- and right-permutive elementary cellular automata. Therefore, by a result of Kurka [Kur03], they are conjugate to the one-sided full shift with alphabet $\{1, \ldots, 4\}$ and in particular they



(a) w_{36} (b) w_{72}

Figure 6.1.: De Bruijn graphs for w_{36} and w_{72}.

are conjugate to each other. A conjugacy between w_{105} and w_{150} is given by

$$\varphi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}, \quad \varphi(x)_k := \begin{cases} 1 - x_k & \text{if } k \in \{\, 4\ell \mid \ell \in \mathbb{Z} \,\} \cup \{\, 4\ell + 1 \mid \ell \in \mathbb{Z} \,\} \\ x_k & \text{otherwise} \end{cases},$$

while the conjugacy between w_{90} and w_{150} is significantly more complicated. We will discuss these cellular automata in more detail in Section 6.3 and Section 6.4.

We will show on a case by case basis that all cellular automata in the remaining classes are pairwise non-conjugate. To do so, we define a further invariant of topological conjugacy.

Definition 6.1 (Fixed points with k preimages). *For a map $f : X \to X$ let $\mathit{Fix}_k(f)$ to be the set of all fixed points of f with k preimages, i.e.,*

$$\mathit{Fix}_k(f) := \left\{\, x \in \mathit{Fix}(f) \;\middle|\; |f^{-1}[\{x\}]| = k \,\right\}.$$

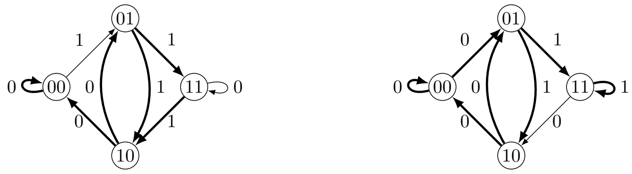
For every cellular automaton f with local rule $f_{\text{loc}} \colon \{\,0,1\,\}^3 \to \{\,0,1\,\}$ we draw the De Bruijn graph over the alphabet $\{0,1\}$ with block length 3 and annotate its edges by f_{loc}. An edge is drawn with a bold line if $f(x_{-1}, x_0, x_1) = x_0$. The edge shift of the subgraph defined by the bold edges thus equals $\psi[\text{Fix}(f)] \cong \text{Fix}(f)$ where $\psi : \{0,1\}^{\mathbb{Z}} \to (\{0,1\}^3)^{\mathbb{Z}}$ is the higher-block map with block length 3.

Rules 36 and 72

Because of the horizontal symmetry of the annotated De Bruijn graph in Figure 6.1(a), we see that $\text{Fix}_1(w_{36}) = \emptyset$. On the other hand ${}^{\infty}(011)^{\infty} \in \text{Fix}_1(w_{72})$.

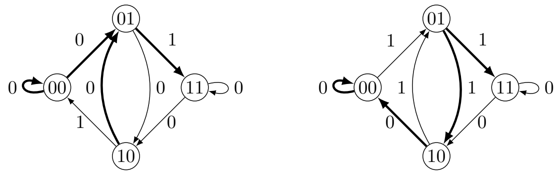
Rules 78 and 140

From Figure 6.2 we derive that ${}^{\infty}1^{\infty} \in \text{Fix}_1(w_{140})$. On the other hand, $\text{Fix}_1(w_{78}) = \emptyset$ since $w_{78}^{-1}[\{{}^{\infty}0^{\infty}\}] = \{{}^{\infty}0^{\infty}, {}^{\infty}1^{\infty})$ and each occurrence of 01010 or 10110 might be replaced by, respectively, 01110 or 10010 in fixed points of w_{78} without changing the image.



(a) w_{78} (b) w_{140}

Figure 6.2.: De Bruijn graphs for w_{78} and w_{140}.



(a) w_{24} (b) w_{46}

Figure 6.3.: De Bruijn graphs for w_{24} and w_{46}.

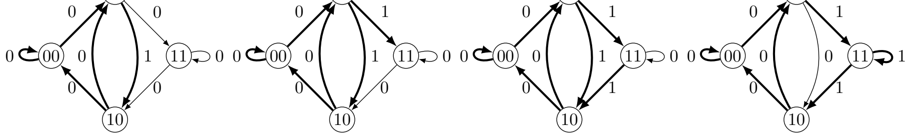
Rules 24 **and** 46

Both cellular automata agree with the shift, either σ or σ^{-1}, on their eventual image. Consider the sets $M_{24} := w_{24}^{-1}[\mathrm{Fix}(W_{24})] = w_{24}^{-1}[\{{}^{\infty}0^{\infty}\}]$ and $M_{46} := w_{46}^{-1}[\mathrm{Fix}(w_{46})] = w_{46}^{-1}[\{{}^{\infty}0^{\infty}\}]$. Both are countable subshifts of finite type. M_{24} is generated by ${}^{\infty}0.(10)^{\infty}$ and ${}^{\infty}1.(01)^{\infty}$, while M_{46} is generated by ${}^{\infty}1.0^{\infty}$. Therefore M_{24} has the four accumulation points ${}^{\infty}0^{\infty}$, ${}^{\infty}1^{\infty}$, ${}^{\infty}(01)^{\infty}$ and ${}^{\infty}(10)^{\infty}$, while M_{46} has only two, namely ${}^{\infty}0^{\infty}$ and ${}^{\infty}1^{\infty}$.

Rules 4, 12, 76 **and** 200

These cellular automata are all equal to the identity on their eventual image, or, more specifically, $\mathrm{Fix}(f) = f[\{0,1\}^{\mathbb{Z}}]$ for $f \in \{w_4, w_{12}, w_{76}, w_{200}\}$. The set of fixed points of each of these four cellular automata is a Cantor space. Notice that even $\mathrm{Fix}(w_4) = \mathrm{Fix}(w_{12})$.

As a last invariant we look at the possible cardinalities of the preimage of a point and



(a) w_4 (b) w_{12} (c) w_{76} (d) w_{200}

Figure 6.4.: De Bruijn graphs for w_4, w_{12}, w_{76} and w_{200}.

(a) w_{76} (b) w_{200}

Figure 6.5.: Space-Time-Diagrams of w_{76}, w_{200} with random initial condition and periodic boundary, black represents 0 and gray represents 1.

define $\mathfrak{P}(f) := \left\{ |f^{-1}[\{x\}]| \;\middle|\; x \in A^{\mathbb{Z}} \right\}$. Let Fib be the set of *Fibonacci numbers* defined by $a_1 = 1, a_2 = 2, a_{k+2} = a_{k+1} + a_k$ for $k \in \mathbb{N}$. We will show that

$$\begin{aligned} \mathfrak{P}(w_{200}) &= \mathfrak{P}(w_{12}) \\ &= |\mathbb{R}| \cup \{\, b_1 b_2 \dots b_k \mid k \in \mathbb{N}, b_\ell \in \mathrm{Fib} \text{ for } \ell \in \{1, \dots, k\}\}. \end{aligned}$$

In the case of w_{200}, the ambiguity in forming the preimage comes from blocks of the form 110^k11, see Figure 6.5(b). Since isolated ones are erased by w_{200}, the number of preimages of ${}^{\infty}1.0^k1^{\infty}$ equals the number of words of length $k-2$ containing no two consecutive ones, which equals $a_{k-1} \in \mathrm{Fib}$. If more than one block of the form 110^k11 occurs, one can independently put isolated ones in these blocks without changing the image, hence the number of preimages is the product of those for the single blocks. The same is true for w_{76}, but here we look at blocks which are terminated by 11 on both sides and which contain only isolated ones, e.g., 11001001010001011. We can replace 010^k10 by $01^{k+2}0$ without changing the image. But since we can not do this for adjacent occurrences of 010^k10, again the number of preimages of ${}^{\infty}10w01^{\infty}$ with w containing ℓ isolated ones is a_ℓ.

On the other hand,

$$w_{12}^{-1}[\{{}^{\infty}(01).0^{\infty}\}] = \left\{ {}^{\infty}(01).1^k0^{\infty} \;\middle|\; k \in \mathbb{N}_0 \right\} \cup \{ {}^{\infty}(01).1^{\infty} \},$$

so $|\mathbb{N}| \in \mathfrak{P}(w_{12})$. But $|\mathbb{N}| \notin \mathfrak{P}(w_4)$. Any point having infinitely many preimages with respect to w_4 must contain infinitely many occurrences of blocks of the form 10^k1 with $k \geq 3$ or it must start with ${}^{\infty}0$ or end with 0^{∞}. Thus it already has uncountably many preimages. Consequently, w_{12} is not conjugate to any of w_4, w_{76} and w_{200}.

This leaves us with these three cellular automata. Next we look at $w_4^{-1}[\{x\}]$ for

$$x = {}^{\infty}(01).0\underline{0000}0(10)^{\infty}).$$

Each element of this set has to coincide with x everywhere except for the underlined block of four consecutive zeros. In this block we only have to ensure that no isolated ones occur. So we have to determine the number of $\{0,1\}$ blocks of length 4 where ones only occur in blocks of length at least two. There can be only either zero or one block of ones of length from 2 to 4. This gives $1+3+2+1 = 7$ possibilities, thus $w_4^{-1}(\{x\}) = 7$. But 7 is not a product of Fibonacci numbers, hence w_4 is not conjugate to either w_{76} or w_{200}.

Finally we differentiate between these two cellular automata. Notice that $\mathrm{Fix}_3(w_{200})$ consists of all configurations in $\mathrm{Fix}(w_{200})$ containing the block 11000011 but no other block of zeros of length greater than two. Hence the closure of $\mathrm{Fix}_3(w_{200})$ is contained in

$\mathrm{Fix}_3(w_{200}) \cup \mathrm{Fix}_1(w_{200})$. On the other hand we have $({}^\infty 0.10^\infty) \in \mathrm{Fix}_3(w_{76})$, hence there is $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{Fix}_3(w_{76})$ with $x_n \to {}^\infty 0^\infty \in \mathrm{Fix}_2(w_{76})$. With that we have finally shown that w_{200} and w_{76} are not topologically conjugate.

Notice, however, that $|\mathrm{Fix}_k(w_{76})| = |\mathrm{Fix}_k(w_{200})| = |\mathbb{R}|$ for all $k \in \mathfrak{P}(w_{200}) = \mathfrak{P}(w_{76})$. Since w_{76} and w_{200} are idempotent, they are thus conjugate when $\{0,1\}^{\mathbb{Z}}$ is endowed with the discrete topology instead of the product topology.

6.3. Permutive Cellular Automata

As promised, we will now compute the conjugacy between w_{90} and w_{150}.

Definition 6.2 (Left- and right-permutive)**.** *A block map $f_{loc} : A^n \to A$ for $n \in \mathbb{N}$ is called* left-permutive *if the map $A \ni a \mapsto f(ax) \in A$ is a permutation for every $x \in A^{n-1}$. In other words, for $u, v \in A^n$ with $f(u) = f(v)$ and $u_{[2,n]} = v_{[2,n]}$ we also have $u_1 = v_1$. The map f_{loc} is called* right-permutive*, if the map $A \ni a \mapsto f(xa) \in A$ is a permutation for every $x \in A^{n-1}$.*

The following theorem is due to Nasu in [Nas95] and independently Shereshevsky and Afraimovich in [SA92], see for example the book by Kurka [Kur03, Prop. 5.8]. Compare this with the results mentioned at the beginning of Section 2.6.

Theorem 6.3. *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with left radius $\ell \in \mathbb{N}$ and right radius $r \in \mathbb{N}$ that is defined by the local rule $f_{loc} : A^{\ell+r+1} \to A$. If f_{loc} is left- and right-permutive, then f is conjugate to the one-sided full shift over the alphabet $A^{\ell+r}$, where the conjugacy φ is given by*

$$\varphi_i(x) = f^i(x)_{[-\ell,r)} \quad \textit{for } i \in \mathbb{N}_0.$$

The proof is based on the following lemma.

Lemma 6.4. *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton with left radius ℓ and right radius r that is defined by a left- and right-permutive local rule $f_{loc} : A^{\ell+r+1} \to A$. Let $k, m \in \mathbb{N}$, $k \leq m$ and $m - k \geq \ell + r$ and let $x, y \in A^{\mathbb{Z}}$. If $x_{[k,m)} = y_{[k,m)}$ and $f(x)_{[k,m)} = f(y)_{[k,m)}$, then $x_{[k-\ell,m+r)} = y_{[k-\ell,m+r)}$.*

Proof. Since $m - k \geq \ell + r$, the assumptions imply $f(x)_{k+\ell} = f(y)_{k+\ell}$ and $x_{[k,k+\ell+r)} = y_{[k,k+\ell+r)}$. By the definition of left-permutivity this implies $x_{k-1} = y_{k-1}$. Therefore we also have $x_{[k-1,k+\ell+r-1)} = y_{[k-1,k+\ell+r-1)}$, hence $x_{k-2} = y_{k-2}$. Continuing that way we get $x_{[k-\ell,m)} = y_{[k-\ell,m)}$. Using the right-permutivity of f_{loc} we get $x_{[k,m+r)} = y_{[k,m+r)}$ by the same reasoning. □

Proof of Theorem 6.3. Set $B := A^{\ell+r}$. Define a continuous map $\varphi : A^{\mathbb{Z}} \to B^{\mathbb{N}_0}$ by

$$\varphi_i(x) = f^i(x)_{[-\ell,r)} \quad \text{for } i \in \mathbb{N}_0.$$

$$\begin{tikzcd}
\mathbb{F}_2^{\mathbb{Z}} \arrow[rrr, "w_{90}"] \arrow[dd, "\varphi_{90,150}"'] \arrow[dr, "\psi_{90}"] & & & \mathbb{F}_2^{\mathbb{Z}} \arrow[dd, "\varphi_{90,150}"] \arrow[dl, "\psi_{90}"'] \\
& \mathbb{F}_2^{\mathbb{N}_0 \times \{0,1\}} \arrow[r, "\sigma"] \arrow[dl, "\psi_{150}^{-1}"] & \mathbb{F}_2^{\mathbb{N}_0 \times \{0,1\}} \arrow[dr, "\psi_{150}^{-1}"'] & \\
\mathbb{F}_2^{\mathbb{Z}} \arrow[rrr, "w_{150}"] & & & \mathbb{F}_2^{\mathbb{Z}}
\end{tikzcd}$$

Figure 6.6.: The conjugacies between w_{90}, w_{150} and the one-sided 4-shift.

We have

$$\sigma(\varphi(x))_i = \varphi(x)_{i+1} = f^i(f(x))_{[-\ell,r)} \quad \text{for } i \in \mathbb{N}_0.$$

The only thing left to show is that φ is bijective. Let $x, y \in A^{\mathbb{Z}}$ with $\varphi(x) = \varphi(y)$, in other words, we have $f^i(x)_{[-\ell,r)} = f^i(y)_{[-\ell,r)}$ for all $i \in \mathbb{N}_0$. By repetitively applying Lemma 6.4 we get $f^i(x)_{[-k\ell,kr)} = f^i(y)_{[-k\ell,kr)}$ for all $k, i \in \mathbb{N}_0$. In particular we have $x = y$, hence φ is injective.

Let W_n be the set of all possibles words that can occur as $x_{[-n\ell,nr)}$ and let S_n be the set of all possible n-tuples that can occur as $(f^i(x)_{[-\ell,r)})_{i=0}^{n-1}$. Since f has left radius ℓ and right radius r, it induces a well-defined map g_n from W_n to S_n. This map is injective because applying Lemma 6.4 n-times implies that for all $x, y \in A^{\mathbb{Z}}$ with $f^i(x)_{[-\ell,r)} = f^i(y)_{[-\ell,r)}$ for $i \in \{0, \ldots, n-1\}$ one also has $x_{[-n\ell,nr)} = y_{[-n\ell,nr)}$. Now comparing cardinalities gives $|W_n| = |A|^{n(\ell+r)} \geq |S_n|$, thus g_n is also surjective. In particular for every n-tuple $w \in B^n$ there is $x \in A^{\mathbb{Z}}$ with $(f^i(x)_{[-\ell,r)})_{i=0}^{n-1} = w$. Since $A^{\mathbb{Z}}$ is compact, this shows that φ is surjective. □

While Theorem 6.3 shows that w_{90} and w_{150} are conjugate, it does not give a concrete description of the conjugacy. We are going to rectify this now.

Let ψ_c be the conjugacy between $(\{0,1\}^{\mathbb{Z}}, w_c)$ and $(\{0,1\}^{\mathbb{N}_0 \times 2}, \sigma)$ described in the proof of Theorem 6.3 for $c \in \{90, 150\}$. Set $\varphi_{90,150} := \psi_{150}^{-1} \circ \psi_{90}$. Figure 6.6 gives an overview of the maps involved in this construction.

Let M be a set. The space $(\mathbb{F}_2)^M$ is a vector space over the field $\mathbb{F}_2$. We say that a vector $x \in \mathbb{F}_2^M$ has *finite support* if the set of indizes where x differs from 0 is finite. For a vector $w \in (\mathbb{F}_2)^M$ with finite support we can define the scalar product with any vector $x \in (\mathbb{F}_2)^M$ by $\langle w, x \rangle := \langle x, w \rangle := \sum_{i \in M} x_i w_i$. Note that the values of $\langle w, x \rangle$ lie in $\mathbb{F}_2$ contrary to the normal definition of a scalar product.

For $x \in \mathbb{F}_2^{\mathbb{Z}}$ and $c \in \{90, 150\}$ let $S^c(x) \in \mathbb{F}_2^{\mathbb{N}_0 \times \mathbb{Z}}$ be the space time diagram of x under w_c, i.e.,

$$S^c(x)_{i,j} := (Y_{w_c}^{\mathbb{N}_0}(x))_{i,j} = (w_c)^i(x)_j.$$

The definition of w_c then translates to the following recurrence relations for $i \in \mathbb{N}_0, j \in \mathbb{Z}$.

$$\begin{aligned} S^{90}(x)_{i+1,j} &= S^{90}(x)_{i,j-1} + S^{90}(x)_{i,j+1}, \\ S^{150}(x)_{i+1,j} &= S^{150}(x)_{i,j-1} + S^{150}(x)_{i,j} + S^{150}(x)_{i,j+1}. \end{aligned}$$

We have

$$\begin{aligned} &\psi_c(x)_{i,j} = S^c(x)_{i,j} \text{ for } i \in \mathbb{N}_0, j \in \{0,1\}, \\ &\psi_c^{-1}((S^c(x)_{i,j})_{i\in\mathbb{N}_0, j\in\{0,1\}}) = x. \end{aligned}$$

Our goal is to express x in terms of $S^c(x)_{i,j}$ and vice versa. To do so we define vectors $v^c_{i,j} \in \mathbb{F}_2^{\mathbb{N}_0\times\{0,1\}}$ and vectors $w^c_{i,j} \in \mathbb{F}_2^{\mathbb{Z}}$ for $i \in \mathbb{N}_0, j \in \mathbb{Z}$ according to the following recursive relations.

$$\begin{aligned} (v^{90}_{i,j})_{k,\ell} = (v^{150}_{i,j})_{k,\ell} &= \begin{cases} 1 & \text{if } k = i \text{ and } \ell = j \\ 0 & \text{otherwise} \end{cases} && \text{for } i,k \in \mathbb{N}_0 \text{ and } j,\ell \in \{0,1\}, \\ v^{90}_{i,j\pm1} &= v^{90}_{i+1,j} + v^{90}_{i,j\mp1} && \text{for } j \notin \{0,1\}, \\ v^{150}_{i,j\pm1} &= v^{150}_{i,j} + v^{150}_{i+1,j} + v^{150}_{i,j\mp1} && \text{for } j \notin \{0,1\}, \\ (w^{90}_{0,j})_\ell = (w^{150}_{0,j})_\ell &= \begin{cases} 1 & \text{if } \ell = j \\ 0 & \text{otherwise} \end{cases} && \text{for } j,\ell \in \mathbb{Z}, \\ w^{90}_{i+1,j} &= w^{90}_{i,j-1} + w^{90}_{i,j+1} && \text{for } i \in \mathbb{N}, \\ w^{150}_{i+1,j} &= w^{150}_{i,j-1} + w^{150}_{i,j} + w^{150}_{i,j+1} && \text{for } i \in \mathbb{N}. \end{aligned}$$

All of these vectors have finite support since this property is preserved under sums.

It is straightforward to prove by induction that for $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$

$$\begin{aligned} S^c(x)_{i,j} &= \langle x, w^c_{i,j}\rangle \\ &= \langle \psi_c(x), v_{i,j}\rangle. \end{aligned}$$

Consider the linear maps $\sigma : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ and $\tilde{\sigma} : \mathbb{F}_2^{\mathbb{N}_0\times\{0,1\}} \to \mathbb{F}_2^{\mathbb{N}_0\times\{0,1\}}$ defined by

$$\begin{aligned} \sigma(x)_j &= x_{j+1} && \text{for } j \in \mathbb{Z}, \\ \tilde{\sigma}(y)_{i,j} &= \begin{cases} 0 & \text{if } i = 0 \\ y_{i-1,j} & \text{if } i > 0 \end{cases} && \text{for } i \in \mathbb{N}_0 \text{ and } j \in \{0,1\}. \end{aligned}$$

Again, by induction we see that for $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$ we have

$$\begin{aligned} w^c_{i,j+1} &= \sigma^{-1}(w^c_{i,j}), \\ v^c_{i+1,j} &= \tilde{\sigma}(v^c_{i,j}). \end{aligned}$$

Together with the definition of the vectors $w^c_{i,j}$ and $v^c_{i,j}$ we get the following recursive relations

$$\begin{aligned}
w_{i+1,j}^{90} &= \sigma^{-1}(w_{i,j}^{90}) + \sigma(w_{i,j}^{90}),\\
w_{i+1,j}^{150} &= \sigma^{-1}(w_{i,j}^{150}) + w_{i,j}^{150} + \sigma(w_{i,j}^{150}),\\
v_{i,j\pm1}^{90} &= v_{i,j\mp1}^{90} + \tilde{\sigma}(v_{i,j}^{90}),\\
v_{i,j\pm1}^{150} &= v_{i,j}^{150} + v_{i,j\mp1}^{150} + \tilde{\sigma}(v_{i,j}^{150}).
\end{aligned}$$

We can now write our maps $\psi_c : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{N}_0\times\{0,1\}}$ and $\psi_c^{-1} : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{N}_0\times\{0,1\}}$ with the help of $v_{i,j}^c$ and $w_{i,j}^c$ as

$$\begin{aligned}
\psi_c(x)_{i,j} &= S^c(x)_{i,j} = \langle x, w_{i,j}^c\rangle,\\
\psi_c^{-1}(y)_j &= S^c(\psi^{-1}(y))_{0,j} = \langle y, v_{0,j}^c\rangle.
\end{aligned}$$

Composing ψ_{90} and ψ_{150}^{-1} gives

$$\begin{aligned}
\varphi_{90,150}(x)_j &= \psi_{150}^{-1}(\psi_{90}(x))_j\\
&= \langle(\langle x, w_{k,\ell}^{90}\rangle)_{k\in\mathbb{N}_0,\ell\in\{0,1\}}, v_{0,j}^{150}\rangle\\
&= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k,\ell}\langle x, w_{k,\ell}^{90}\rangle\\
&= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}\sum_{m\in\mathbb{Z}}(v_{0,j}^{150})_{k,\ell}(w_{k,\ell}^{90})_m x_m.
\end{aligned}$$

We therefore define vectors $u_i^{90,150} \in \mathbb{F}_2^{\mathbb{Z}}$ by

$$(u_j^{90,150})_m = \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k,\ell}(w_{k,\ell}^{90})_m.$$

Since $v_{0,j}$ has finite support for every $j \in \mathbb{Z}$, the vector $u_j^{90,150}$ has the same property. Using $u_j^{90,150}$ we can rewrite our formula for $\varphi_{90,150}$, which we derived above, as

$$\varphi_{90,150}(x)_j = \langle x, u_j^{90,150}\rangle.$$

The recursion relations for $w_{i,j}^c$ and $v_{i,j}^c$ translate to recursion relations for u_j, namely

$$\begin{aligned}
(u_{j\pm1}^{90,150})_m &= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j\pm1}^{150})_{k,\ell}(w_{k,\ell}^{90})_m\\
&= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}\Big((v_{0,j\mp1}^{150})_{k,\ell} + (v_{0,j}^{150})_{k,\ell} + (\tilde{\sigma}(v_{0,j}^{150}))_{k,\ell}\Big)(w_{k,\ell}^{90})_m\\
&= (u_{j\mp1}^{90,150})_m + (u_j^{90,150})_m + \sum_{k\in\mathbb{N}}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k-1,\ell}(w_{k,\ell}^{90})_m.
\end{aligned}$$

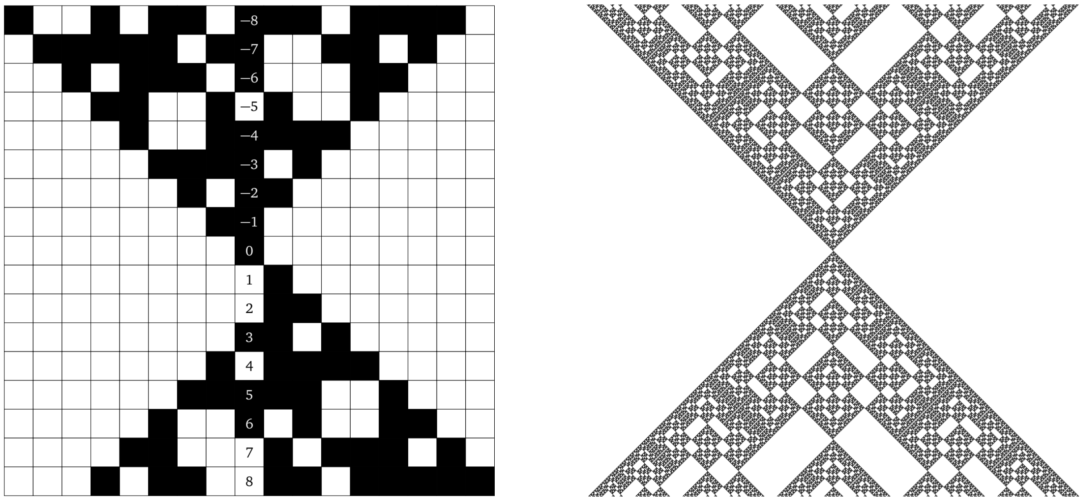


Figure 6.7.: Local rules of the conjugacy $\varphi_{90,150}$, the k-th row (see the numbers in the middle column in the plot on the left) corresponds to $u_k^{90,150}$, the plot on the right side shows $u_{-500}^{90,150}$ at the top up to $u_{500}^{90,150}$ at the bottom. Black squares correspond to ones, white squares to zeros.

The sum on the right side can be further simplified,

$$\begin{aligned}\sum_{k\in\mathbb{N}}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k-1,\ell}(w_{k,\ell}^{90})_m &= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k,\ell}(w_{k+1,\ell}^{90})_m\\ &= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k,\ell}(\sigma^{-1}(w_{k,\ell}^{90})+\sigma(w_{k,\ell}^{90}))_m\\ &= \sum_{k\in\mathbb{N}_0}\sum_{\ell\in\{0,1\}}(v_{0,j}^{150})_{k,\ell}\left((w_{k,\ell}^{90})_{m-1}+(w_{k,\ell}^{90})_{m+1}\right)\\ &= (u_j^{90,150})_{m-1}+(u_j^{90,150})_{m+1}.\end{aligned}$$

Together we finally get

$$(u_{j\pm1}^{90,150})_m = (u_{j\mp1}^{90,150})_m + (u_j^{90,150})_m + (u_j^{90,150})_{m-1} + (u_j^{90,150})_{m+1}.$$

The vectors defining the local rules of $\varphi_{90,150}$ can therefore be calculated by a cellular automaton with memory. The result is depicted in Figure 6.7.

The conjugacy just constructed is by far not the only conjugacy between w_{90} and w_{150}. Let $\varphi:\mathbb{F}_2^{\mathbb{Z}}\to\mathbb{F}_2^{\mathbb{Z}}$ be a conjugacy between w_{90} and w_{150}. Then

$$\begin{aligned}\psi_{150}\circ\varphi\circ\psi_{90}^{-1}\circ\sigma &= \psi_{150}\circ\varphi\circ w_{90}\circ\psi_{90}^{-1}\\ &= \psi_{150}\circ w_{150}\circ\varphi\circ\psi_{90}^{-1}\\ &= \sigma\circ\psi_{150}\circ\varphi\circ\psi_{90}^{-1},\end{aligned}$$

hence $\psi_{150}\circ\varphi\circ\psi_{90}^{-1}$ is an automorphism of the one-sided full 4-shift. On the other hand let θ be such an automorphism. Then $\psi_{150}^{-1} \circ \theta \circ \psi_{90}$ is a conjugacy between w_{90} and w_{150}. Thus there is a one-to-one correspondence between conjugacies of w_{90} and w_{150} and automorphisms of the one-sided full 4-shift X_4.

It is easy to see that the automorphism group of X_4 is large. For example consider the following automorphisms. For every word w over $\{0, \ldots, 3\}$ and symbols $a, b \in \{0, \ldots, 3\}$ we can define a cellular automaton $g_{w,a,b} : \{0, \ldots, 3\}^{\mathbb{N}_0} \to \{0, \ldots, 3\}^{\mathbb{N}_0}$ that exchanges the symbols a and b if they are followed by w, i.e.,

$$g_{w,a,b}(x)_i = \begin{cases} a & \text{if } x_i = b \text{ and } x_{[i+1,i+1+|w|)} = w \\ b & \text{if } x_i = a \text{ and } x_{[i+1,i+1+|w|)} = w \\ x_i & \text{otherwise} \end{cases} .$$

If $a \neq b$ and both a and b do not appear in w, then $g_{w,a,b}$ is an involution, in particular it is invertible. These maps are in general non-linear, hence there are also non-linear conjugacies between w_{90} and w_{150}.

6.4. Weak Conjugacy

The results in this section show that conjugacies between cellular automata can be quite complicated homeomorphisms. In particular we show that the elementary cellular automata with Wolfram code 90 and 150 are topologically conjugate but every conjugacy between them has to use infinitely many local rules in the sense of the following definition.

Definition 6.5 (Definable by a set of local rules, finite number of local rules)**.** *Let $X \subseteq A^{\mathbb{Z}}$ and $Y \subseteq B^{\mathbb{Z}}$ be two-sided subshifts. Let $M \subseteq \bigcup_{r \in \mathbb{N}} \{f_{loc} : A^{2r+1} \to B\}$ be a set of block maps. We say that a continuous map $\varphi : X \to Y$ is* definable by M *if for each index $i \in \mathbb{Z}$ there is a local rule $f_{loc} \in M$, $f_{loc} : A^{2r+1} \to B$ with $\varphi_i(x) = f_{loc}(x_{[i-r,i+r]})$.*

We say that a homeomorphism $\varphi : X \to Y$ has a finite number of local rules*, if there exist finite sets $M \subseteq \bigcup_{r \in \mathbb{N}} \{f_{loc} : A^{2r+1} \to B\}$ and $N \subseteq \bigcup_{r \in \mathbb{N}} \{f_{loc} : B^{2r+1} \to A\}$ such that φ is definable by M and φ^{-1} is definable by N.*

Remark 6.6. Notice that there are homeomorphisms between full shifts which are definable by a finite set of local rules but whose inverses are not. Consider for example the following $n \times n$ matrix over $\mathbb{F}_2$,

$$J_n = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} .$$

Its inverse is

$$J_n^{-1} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Now consider the two local rules

$$\begin{aligned} \xi : \mathbb{Z}_2^3 \to \mathbb{Z}_2, &\quad \xi(x_{-1}, x_0, x_1) = x_0 + x_1, \\ \iota : \mathbb{Z}_2^3 \to \mathbb{Z}_2, &\quad \iota(x_{-1}, x_0, x_1) = x_0. \end{aligned}$$

Let $(a_i)_{i\in\mathbb{Z}} \in \{\,\xi, \iota\,\}^{\mathbb{Z}}$ be a sequence of local rules that contains the block $\iota\xi^k\iota$ for each $k \in \mathbb{N}$. Then the global map φ corresponding to $(a_i)_{i\in\mathbb{Z}}$ is linear and can be represented by an infinite block matrix with blocks of the form J_n and I_n where $I_n \in \{0,1\}^{n\times n}$ is the identity matrix. Thus φ is invertible and its inverse φ^{-1} can be represented by an infinite block matrix with blocks of the form J_n^{-1} and I_n. In particular φ^{-1} is not definable by a finite set of local rules while φ is definable by $\{\,\xi, \iota\,\}$.

Lemma 6.7. *Let X and Y be the two-sided full shifts over the alphabets A and B. Let $r \in \mathbb{N}$ and let $M \subseteq \{f : A^{2r+1} \to B\}$ and $N \subseteq \{f : B^{2r+1} \to A\}$ be finite sets of local rules. Let $f : X \to X$ and $g : Y \to Y$ be cellular automata. Consider the set $\mathscr{C}$ of all sequences $(a_i, b_i)_{i\in\mathbb{Z}} \in (M \times N)^{\mathbb{Z}}$ such that the global function φ defined by $(a_i)_{i\in\mathbb{Z}}$ is a conjugacy between f and g and the global function defined by $(b_i)_{i\in\mathbb{Z}}$ is φ^{-1}. Then $\mathscr{C}$ is either empty or a subshift of finite type.*

Proof. There is a radius $R \in \mathbb{N}$ such that f and g both have local rules f_{loc} and g_{loc} with radius R. Let $(\xi_i, \zeta_i)_{i\in\mathbb{Z}} \in (M \times N)^{\mathbb{Z}}$ and let φ and ψ be the global rules defined by $(\xi_i)_{i\in\mathbb{Z}}$ respectively $(\zeta_i)_{i\in\mathbb{Z}}$. The condition $\psi = \varphi^{-1}$ is equivalent to

$$\begin{aligned} &\big((x_{-2r}, \dots, x_{2r}) \mapsto \zeta_i(\xi_{i-r}(x_{-2r}, \dots, x_0), \dots, \xi_{i+r}(x_0, \dots, x_{2r}))\big) = \mathrm{id}_{A,2r} \\ &\big((y_{-2r}, \dots, y_{2r}) \mapsto \xi_i(\zeta_{i-r}(y_{-2r}, \dots, y_0), \dots, \zeta_{i+r}(y_0, \dots, y_{2r}))\big) = \mathrm{id}_{B,2r}. \end{aligned} \tag{6.1}$$

for all $i \in \mathbb{Z}$ where $\mathrm{id}_{C,r} : C^{2r+1} \to C$ is defined by $\mathrm{id}_{C,r}(x_{-r}, \dots, x_r) = x_0$.

The condition that φ is a conjugacy between f and g is equivalent to

$$\begin{aligned} &\big((x_{-r-R}, \dots, x_{r+R}) \mapsto \xi_i(f(x_{-r-R}, \dots, x_{-r+R}), \dots, f(x_{r-R}, \dots, x_{r+R}))\big) \\ =&\big((x_{-r-R}, \dots, x_{r+R}) \mapsto g(\xi_{i-R}(x_{-R-r}, \dots, x_{-R+r}), \dots, \xi_{i+R}(x_{R-r}, \dots, x_{R+r}))\big) \end{aligned} \tag{6.2}$$

for all $i \in \mathbb{Z}$.

Set $q := max(R, r)$. We may characterize $\mathscr{C}$ as the set of all configurations in $(M \times N)^{\mathbb{Z}}$ not containing a block of the form $((\xi_{-q}, \zeta_{-q}), \dots, (\xi_q, \zeta_q))$ violating (6.1) or (6.2) for $i = 0$. Therefore $\mathscr{C}$ is a subshift of finite type with window width $2q + 1$. □

Theorem 6.8. *Let X be a two-sided full shift. For cellular automata $f, g : X \to X$ the following are equivalent.*

(a) *f and g are conjugate by a homeomorphism φ with a finite number of local rules.*

(b) *f and g are conjugate by a homeomorphism ψ commuting with a power of the shift, i.e., there exists $k \in \mathbb{N}$ such that $\psi \circ \sigma^k = \sigma^k \circ \psi$.*

Proof. (a) $\implies$ (b) Let M and N be the finite sets of local rules by which φ respectively φ^{-1} can be defined. There is a sequence $(c_i)_{i\in\mathbb{Z}} \in M^{\mathbb{Z}}$ of local rules for φ and a sequence $(d_i)_{i\in\mathbb{Z}} \in N^{\mathbb{Z}}$ of local rules for φ^{-1}.

Define $\mathscr{C}$ as in Lemma 6.7. Then $(c_i, d_i)_{i\in\mathbb{Z}} \in \mathscr{C}$ is a subshift of finite type and contains a periodic configuration $(\xi_i, \zeta_i)_{i\in\mathbb{Z}}$ with period $k \in \mathbb{N}$. Let ψ be the global map corresponding to $(\xi_i)_{i\in\mathbb{Z}}$. Then ψ is a conjugacy between f and g and

$$\begin{aligned}
\sigma^k(\psi(x))_i &= \psi(x)_{k+i} \\
&= \xi_{k+i}(x_{[k+i-r,k+i+r]}) \\
&= \xi_i(\sigma^k(x)_{[i-r,i+r]}) \\
&= \psi(\sigma^k(x))_i.
\end{aligned}$$

(b) $\implies$ (a) Since ψ and ψ^{-1} are continuous, there are sequences of local rules $\xi_i : A^{2r_i+1} \to B$ and $\zeta_i : B^{2r_i+1} \to A$ for ψ respectively ψ^{-1}. Since ψ and ψ^{-1} commute with σ^k, both of these sequences are k-periodic and therefore contain only at most k-different elements. Hence ψ has a finite number of local rules. $\square$

Lemma 6.9. *Let X and Y be subshifts and let $f : X \to X$ and $g : Y \to Y$ be cellular automata. If f and g are conjugate by a homeomorphism φ commuting with σ^k, i.e., $\varphi \circ \sigma_X^k = \sigma_Y^k \circ \varphi$ for some $k \in \mathbb{N}$, then $f_{|Per_k(\sigma_X)}$ and $g_{|Per_k(\sigma_Y)}$ are also conjugate by φ.*

Proof. Since φ commutes with σ^k, we get an induced map $\varphi : \mathrm{Per}_k(\sigma_X) \to \mathrm{Per}_k(\sigma_Y)$, which is the desired conjugacy between $f_{|\mathrm{Per}_k(\sigma_X)}$ and $g_{|\mathrm{Per}_k(\sigma_Y)}$. $\square$

With these results we established that two cellular automata which are conjugate by a homeomorphism with a finite number of local rules are also conjugate when restricted to $\mathrm{Per}_k(\sigma)$ for some $k \in \mathbb{N}$. The next theorem shows that this is never the case for w_{90} and w_{150}.

Theorem 6.10. *For any $n \in \mathbb{N}$ the finite dynamical systems $(w_{150})_{|Per_n(\sigma)}$ and $(w_{90})_{|Per_n(\sigma)}$ are not conjugate. In particular w_{150} and w_{90} are not conjugate by a homeomorphism with a finite number of local rules.*

We will show this by calculating the transient time for these dynamical systems. This will take up the rest of this section. For w_{90} this was already done in [Wol83] using the representation of linear cellular automata as the multiplication of Laurent polynomials by a fixed polynomial. The calculation for w_{150}, however, seems to be new. For reflective boundary conditions instead of periodic boundary conditions which we need, some small partial results can be found in a paper by Akin, Siap and Köroglu [ASK12]. All results for w_{90} are already contained in [TSL06] and we follow the proof presented there very closely. The information obtained there for the minimal polynomial, in which we will be interested in, is, however, not enough to also treat w_{150}. The crucial ingredient missing is an identity between Fibonacci polynomials involving composition and not only multiplication. This identity can for example be found in the paper [GKT97, Lemma 4] by Goldwasser, Klostermeyer and Trap on switch-setting games.

We start with some linear algebra over finite fields. Given a finite field F and a linear map $f : F^k \to F^k$ the annihilating polynomials of f at the point $y \in F^k$ are defined by $\operatorname{Ann}(f, y) := \{ p \in F[\lambda] \mid p(f)(y) = 0 \}$ and the set of *annihilating polynomials* of f is defined by $\operatorname{Ann}(f) = \bigcap_{y \in F^k} \operatorname{Ann}(f, y)$.

Both sets $\operatorname{Ann}(f, y)$ and $\operatorname{Ann}(f)$ are ideals and since the polynomial ring $F[\lambda]$ is a principal ideal domain, they are generated by a single polynomial. We denote these polynomials by $\mu_{f,y}$ and μ_f. The polynomial μ_f is called the *minimal polynomial* of f.

For a polynomial $q \neq 0$ the *multiplicity of the zero at* z is defined by

$$\operatorname{mul}_z(q) := \max\left\{ k \in \mathbb{N}_0 \;\middle|\; (x-z)^k \mid q(x) \right\}.$$

If $\operatorname{mul}_z(q) = 0$, then z is not a zero of q. Since we are going to calculate a lot with mul_z, we collect some useful properties of it.

Lemma 6.11. *If $g, h \in F[x]$ are polynomials over the field F and $z \in F$, then the following hold.*

(a) $\operatorname{mul}_z(g) = k \in \mathbb{N}_0$ *if and only if there is a polynomial* $\tilde{g} \in F[x]$ *with* $g(x) = (x - z)^k \tilde{g}(x)$ *and* $\tilde{g}(z) \neq 0$.

(b) $\operatorname{mul}_z(g(x)) = \operatorname{mul}_0(g(x+z))$.

(c) *If* $h(0) = 0$, *then* $\operatorname{mul}_0(g \circ h) = \operatorname{mul}_0(g)\operatorname{mul}_0(h)$.

(d) $\operatorname{mul}_z(gh) = \operatorname{mul}_z(g) + \operatorname{mul}_z(h)$.

Proof. (a) Set $k := \operatorname{mul}_z(g)$. By the definition of divisibility of polynomials there is $\tilde{g}(x)$ with $g(x) = (x-z)^k \tilde{g}(x)$ and by maximality of k, $\tilde{g}(z) \neq 0$. On the other hand, if $\tilde{g}(z) \neq 0$, then the maximal power of $(x - z)$ dividing $(x-z)^\ell \tilde{g}(x)$ is $(x-z)^\ell$.

(b) If $g \neq 0$ and $\operatorname{mul}_z(g(x)) = k \in \mathbb{N}_0$, by part (a) there is $\tilde{g} \in F[x]$ with $g(x) = (x-z)^k \tilde{g}(x)$ and $\tilde{g}(z) \neq 0$ Therefore $g(x+z) = x^k \tilde{g}(x+z)$ and again by part (a) $\operatorname{mul}_0(g(x+z)) = k$.

(c) We decompose g and h as $g(x) = x^{\ell}\tilde{g}(x)$ and $h(x) = x^k\tilde{h}(x)$ with $\ell = \mathrm{mul}_0(g)$, $k = \mathrm{mul}_0(h) > 0$, $\tilde{g}(0) \neq 0$ and $\tilde{h}(0) \neq 0$. Composition gives

$$\begin{aligned} g(h(x)) &= \tilde{g}(\tilde{h}(x)x^k)(\tilde{h}(x)x^k)^{\ell} \\ &= \tilde{g}(\tilde{h}(x)x^k)(\tilde{h}(x))^{\ell}x^{k\ell}. \end{aligned}$$

Defining $q(x) := \tilde{g}(\tilde{h}(x)x^k)(\tilde{h}(x))^{\ell}$ and using the fact that $k > 0$, we calculate

$$q(0) = \underbrace{\tilde{g}(0)}_{\neq 0}\underbrace{(\tilde{h}(0))}_{\neq 0}^{\ell} \neq 0.$$

By part (a) we can conclude $\mathrm{mul}_0(g \circ h) = k\ell = \mathrm{mul}_0(g)\,\mathrm{mul}_0(h)$.

(d) Again we have the decompositions $g(x) = (x-z)^{\mathrm{mul}_z(g)}\tilde{g}(x)$ and $h(x) = (x-z)^{\mathrm{mul}_z(h)}\tilde{h}(x)$. Then $(gh)(x) = (x-z)^{\mathrm{mul}_z(g)+\mathrm{mul}_z(h)}\tilde{g}(x)\tilde{h}(x)$ and $\tilde{g}(z)\tilde{h}(z) \neq 0$.

□

Lemma 6.12. *Let F be a finite field and let $f : F^k \to F^k$ be linear. For every $y \in F^k$ the minimal preperiod of y under f equals* $\mathrm{mul}_0(\mu_{f,y})$.

Proof. Since F^k is finite, every point is preperiodic under f. For every $y \in F^k$ there is a minimal preperiod $q \in \mathbb{N}_0$ and a minimal period $p \in \mathbb{N}$. Since $f^{p+q}(y) = f^q(y)$, the polynomial $\lambda^q(\lambda^p - 1)$ is in the annihilator of f at y and we have $\mu_{f,y} \mid \lambda^q(\lambda^p - 1)$. In particular $\mathrm{mul}_0(\mu_{f,y}) \leq q$. On the other hand, consider the decomposition $\mu_{f,y}(\lambda) = \lambda^{\mathrm{mul}_0(\mu_{f,y})}\nu(\lambda)$ as in Lemma 6.11 (a). By the pigeonhole principle there are positive integers k and ℓ with $k < \ell$ such that λ^k and λ^{ℓ} have the same remainder when divided by $\nu(\lambda)$, hence $\nu(\lambda) \mid \lambda^k(\lambda^{\ell-k}-1)$. By definition $\nu(0) \neq 0$, hence $\nu(\lambda) \mid \lambda^{\ell-k}-1$. Therefore $\lambda^{\mathrm{mul}_0(\mu_{f,y})}(\lambda^{\ell-k} - 1) \in \mathrm{Ann}(f, y)$ and $f^{\mathrm{mul}_0(\mu_{f,y})+\ell-k}(y) = f^{\mathrm{mul}_0(\mu_{f,y})}(y)$. By the minimality of q we have $q \leq \mathrm{mul}_0(\mu_{f,y})$. □

Theorem 6.13. *Let F be a finite field, let $f : F^k \to F^k$ be linear. For $y \in F^k$ let q_y be the minimal preperiod of y under f. Let $t := \max\{\, q_y \mid y \in F^k \,\}$ be the* transient time *of f. If μ_f is the minimal polynomial of f, then $t = \mathrm{mul}_0(\mu_f)$.*

Proof. By Lemma 6.12 we have $q_y = \mathrm{mul}_0(\mu_{f,y})$. By definition,

$$\begin{aligned} \mathrm{mul}_0(\mu_f) &= \mathrm{mul}_0(\mathrm{lcm}(\{\, \mu_{f,y} \mid y \in F^k \,\})) \\ &= \max\{\, \mathrm{mul}_0(\mu_{f,y}) \mid y \in F^k \,\} \\ &= \max\{\, q_y \mid y \in F^k \,\} = t. \end{aligned}$$

□

Corollary 6.14. *If F is a finite field and $f, g : F^k \to F^k$ are linear maps that are conjugate by a not necessarily linear bijection $\psi : F^k \to F^k$, then $\mathrm{mul}_0(\mu_f) = \mathrm{mul}_0(\mu_g)$.*

Notice that the minimal polynomials of conjugate linear maps $f, g : F^k \to F^k$ must not coincide as the following example shows. This is due to the fact that we allow non-linear conjugacies.

Example 6.15. *Consider the cellular automata $f, g : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ given by $f(x)_i = x_{i+1} + x_{i+2}$ and $g(x)_i = x_i + x_{i+1}$. Their restrictions to $Per_5(\sigma)$ are conjugate, but have minimal polynomials $\mu_f(\lambda) = \lambda^5 + \lambda^2 + \lambda$ respectively $\mu_g(\lambda) = \lambda^5 + \lambda^4 + \lambda$.*

To apply Corollary 6.14 to $(w_{90})_{|\mathrm{Per}_n(\sigma)}$ and $(w_{150})_{|\mathrm{Per}_n(\sigma)}$, we calculate their minimal polynomials. For this we need three additional lemmas and some facts about the Smith normal form of a matrix.

Lemma 6.16. *Let W be a vector space and let $f : V \to V$ be a linear map. Let W be a subspace of V that is invariant under f. Assume there are linear maps $g_1, \dots, g_k : V \to V$ commuting with f such that $g_1(W) + \cdots + g_k(W) = V$. Then $\deg \mu_f \le \dim W$ where μ_f is the minimal polynomial of f.*

Proof. Since W is invariant under f, we can restrict the domain and range of f to W to get a linear map $\tilde{f} : W \to W$. Set $h := \mu_{\tilde{f}}(f)$. The map h annihilates all vectors in W and commutes with g_ℓ for $\ell = 1, \dots, k$. For every $v \in V$ we can find vectors $w_1, \dots, w_k \in W$ such that $v = g_1(w_1) + \cdots + g_k(w_k)$. A short calculation gives

$$\begin{aligned} h(v) &= h(g_1(w_1)) + \cdots + h(g_k(w_k)) \\ &= g_1(h(w_1)) + \cdots + g_k(h(w_k)) \\ &= 0. \end{aligned}$$

Since v was arbitrary, this shows that $\mu_{\tilde{f}} \in \mathrm{Ann}(f)$, hence $\mu_f \mid \mu_{\tilde{f}}$ and $\deg \mu_f \le \deg \mu_{\tilde{f}} \le \dim W$. ☐

Lemma 6.17. *Let $f : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ be a linear cellular automaton. Let $\tilde{f}$ be the restriction of f to $Per_n(\sigma)$. If f is left-right-symmetric, i.e., $f \circ \tau = \tau \circ f$, then $\deg(\mu_{\tilde{f}}) \le \lceil \frac{n}{2} \rceil$.*

Proof. Define the vector $e_0 \in \mathbb{F}_2^{\mathbb{Z}}$ by

$$e_0(k) = \begin{cases} 1 & \text{if } n \mid k \\ 0 & \text{otherwise} \end{cases}.$$

Its iterates $e_0, \sigma^1(e_0), \dots, \sigma^{n-1}(e_0)$ under σ form a basis of $\mathrm{Per}_n(\sigma)$. Thus every F-invariant subspace W of $\mathrm{Per}_n(\sigma)$ that contains e_0 fulfills the assumptions of Lemma 6.16.

If $n = 2k+1$, we consider $W := \mathrm{Fix}(\tau) \cap \mathrm{Per}_n(\sigma)$. Since f commutes with τ and σ, this is an f invariant subspace. The vectors $e_0, \sigma^1(e_0) + \tau(\sigma^1(e_0)), \dots, \sigma^k + \tau(\sigma^k(e_0))$ generate W, hence $\dim(W) \le k+1$. Thus Lemma 6.16 applies and we get $\deg(\mu_{\tilde{f}}) \le k+1$.

If $n = 2k$, we consider the slightly smaller subspace $W = \{\, x \in \mathrm{Fix}(\tau) \cap \mathrm{Per}_{2k}(\sigma) \mid x_k = 0 \,\}$. This subspace is generated by $e_0, \sigma^1(e_0) + \tau(\sigma^1(e_0)), \dots, \sigma^{k-1}(e_0) + \tau(\sigma^{k-1}(e_0))$, hence

$\dim(W) \le k$. Since f is a cellular automaton with some radius $r \in \mathbb{N}_0$, there are coefficients $a_{-r}, \ldots, a_r \in \mathbb{F}_2$ such that $f(x)_k = a_{-r}x_{k-r} + \cdots + a_0 x_k + \cdots + a_r x_{k+r}$. Because f commutes with τ, $a_{-\ell} = a_\ell$ for all $\ell \in \{1, \ldots, r\}$. For every $x \in W$ we have $x_{k-\ell} = x_{\ell-k} = x_{k+\ell}$ for all $\ell \in \mathbb{Z}$. Since the entries of x lie in $\mathbb{F}_2$, this implies $f(x)_k = a_0 x_0 = 0$. Therefore W is again invariant under f and Lemma 6.16 gives $\deg(\mu_{\tilde{f}}) \le k$. □

Remark 6.18. Notice that it is not enough that $\tilde{f}$ commutes with τ. Consider the linear cellular automaton $f : \mathbb{F}_2^{\mathbb{Z}} \to \mathbb{F}_2^{\mathbb{Z}}$ defined by $f(x)_i = x_{i-1} + x_{i+1} + x_{i+5}$. The matrix representation of $\tilde{f} = f_{|\mathrm{Per}_{10}(\sigma)}$ with respect to the basis used in the proof of Lemma 6.17 is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and we have $\tau \circ \tilde{f} = \tilde{f} \circ \tau$. The minimal polynomial of $\tilde{f}$, however, is $x^6 + 1$.

Lemma 6.19. *If ν and η are polynomials over $\mathbb{F}_2$ with $\nu^2 = \eta^2$, then $\nu = \eta$.*

Proof. Both polynomials must have the same degree. If $\nu(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $\eta(x) = b_0 + b_1 x + \cdots + b_n x^n$, then $\nu(x)^2 = (a_0 + a_1 x + \cdots + a_n x^n)^2 = a_0^2 + a_1^2 x^2 + \cdots + a_n^2 x^{2n}$ and $\eta(x)^2 = b_0^2 + b_1^2 x^2 + \cdots + b_n^2 x^{2n}$. Comparing coefficients gives $a_0^2 = b_0^2, \ldots, a_n^2 = b_n^2$. Since the coefficients lie in $\mathbb{F}_2$, this implies $a_0 = b_0, \ldots, a_n = b_n$. □

The following theorem collects some facts about the Smith normal form of a matrix. See for example Chapter 6 in the book by Serre [Ser02], in particular Theorem 6.2.1.

Theorem 6.20. *If $K \in R^{n \times n}$ is a square matrix over a principal ideal domain R, then there are elements $\ell_1, \ldots, \ell_n \in R$, called the* invariant factors *of K, such that the product $\ell_1 \cdot \cdots \cdot \ell_k$ equals the greatest common divisor of all minors of K (not only the principal ones) of order k for all $k \in \{1, \ldots, n\}$. Furthermore $\ell_1 \mid \ell_2, \ell_2 \mid \ell_3, \ldots$. The diagonal matrix L with diagonal elements $L_{kk} = \ell_k$ is called the* Smith normal form *of K.*

Since the ring of polynomials over a field is a principal ideal domain, we can consider the invariant factors of the characteristic matrix of $M \in F^{n \times n}$. These factors allow us to calculate the characteristic and minimal polynomial of M as the following theorem shows, see Theorem 6.3.5 in [Ser02].

Theorem 6.21. *Let $M \in F^{n \times n}$ be a square matrix over a field F and let $K = xI - M \in F[x]^{n \times n}$ be its characteristic matrix. If $\ell_1, \ldots, \ell_n \in F[x]$ are the invariant factors of K, then $\ell_1 \cdot \cdots \cdot \ell_n$ equals the characteristic polynomial of M and ℓ_n equals the minimal polynomial of M.*

Let M_n be the matrix representation of $(w_{90})_{|\mathrm{Per}_n(\sigma)}$ with respect to the standard basis, i.e.,

$$M_1 = \begin{pmatrix}0\end{pmatrix}, M_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, M_n = \left.\begin{pmatrix}0 & 1 & 0 & \cdots & 0 & 0 & 1\\ 1 & 0 & 1 & \cdots & 0 & 0 & 0\\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1\\ 1 & 0 & 0 & \cdots & 0 & 1 & 0\end{pmatrix}\right\} n.$$

This matrix corresponds to applying w_{90} on $\mathbb{F}_2^{\{0,\dots,n-1\}}$ with periodic boundary conditions. For our calculations we will also need the matrices corresponding to the application of w_{90} with null-boundary conditions. Therefore define the matrices $\tilde{M}_n$ as the $n \times n$ matrices with

$$\tilde{M}_1 = \begin{pmatrix}0\end{pmatrix}, \tilde{M}_2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$(\tilde{M}_n)_{ij} = \begin{cases}0 & \text{if } (i,j) = (1,n) \text{ or } (i,j) = (n,1)\\ (M_n)_{ij} & \text{otherwise}\end{cases}.$$

Therefore

$$\tilde{M}_n = \left.\begin{pmatrix}0 & 1 & 0 & \cdots & 0 & 0 & \mathbf{0}\\ 1 & 0 & 1 & \cdots & 0 & 0 & 0\\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 1\\ \mathbf{0} & 0 & 0 & \cdots & 0 & 1 & 0\end{pmatrix}\right\} n.$$

Let χ_n and $\tilde{\chi}_n$ be the characteristic polynomial of M_n and $\tilde{M}_n$ and let μ_n and $\tilde{\mu}_n$ be the minimal polynomial of M_n and $\tilde{M}_n$.

Definition 6.22 (Fibonacci polynomials). *Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of polynomials in $\mathbb{F}_2[x]$ recursively defined by*

$$f_0(x) = 0, \quad f_1(x) = 1,$$
$$f_{n+2}(x) = x f_{n+1}(x) + f_n(x),$$

the so called Fibonacci polynomials.

We now prove various relations between the polynomials we defined. As already mentioned, the identities for the Fibonacci polynomials are proven for example in [GKT97]. The identities involving the characteristic and minimal matrices of M_n and $\tilde{M}_n$ can be found in [TSL06]. Only the connection between those two kinds of polynomials was missing.

Theorem 6.23. *The following equations hold.*

(i) $\chi_n(x) = x\tilde{\chi}_{n-1}(x)$ *for* $n \geq 1$,

(ii) $\chi_n(x) = x f_n(x)$ *for* $n \geq 2$*,*

(iii) $f_{2n}(x) = x f_n(x)^2$ *for* $n \geq 0$*,*

(iv) $f_{2n+1}(x) = f_n(x)^2 + f_{n+1}(x)^2$ *for* $n \geq 0$*,*

(v) $f_{n+t}(x) + f_{n-t}(x) = x f_n(x) f_t(x)$ *for* $n \geq t \geq 0$*,*

(vi) $f_{mn}(x) = f_m(x) f_n(x f_m(x))$ *for* $m, n \geq 0$*,*

(vii) $\mu_{2k}(x) = x f_k(x)$*,*

(viii) $\mu_{2k+1}(x) = x(f_{k+1}(x) + f_k(x))$*.*

Proof. (i) Since we are working over $\mathbb{F}_2$, the characteristic matrix of M_n is

$$M_n + xI = \begin{pmatrix} x & 1 & 0 & \cdots & 0 & 1 \\ 1 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & x & 1 \\ 1 & 0 & 0 & \cdots & 1 & x \end{pmatrix}.$$

Laplace expansion of its determinant along the first row gives

$$\det(M_n + xI) = x \begin{vmatrix} x & 1 & \cdots & 0 & 0 \\ 1 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 0 & 0 & \cdots & 1 & x \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 1 & 0 & \cdots & 1 & x \end{vmatrix} + \begin{vmatrix} 1 & x & 1 & \cdots & 0 \\ 0 & 1 & x & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & x \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix}. \tag{6.3}$$

By first moving the last row to the top and then forming the transpose we have the following equality.

$$\begin{vmatrix} 1 & x & 1 & \cdots & 0 \\ 0 & 1 & x & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & x \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & x & 1 & \cdots & 0 \\ 0 & 1 & x & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & x \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 1 & 0 & \cdots & 1 & x \end{vmatrix}.$$

Therefore the second and third term in (6.3) cancel out and we have

$$\chi_n(x) = \det(M_n + xI) = x \det(\tilde{M}_{n-1} + xI) = x \tilde{\chi}_{n-1}(x).$$

(ii) This time we expand the characteristic matrix of $\tilde{M}_n$ along the first row and get

$$\begin{aligned}
\tilde{\chi}_n(x) = \det(\tilde{M}_n + xI) &= \begin{vmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ 1 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & 1 & x \end{vmatrix} \\
&= x \begin{vmatrix} x & 1 & \cdots & 0 & 0 \\ 1 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 0 & 0 & \cdots & 1 & x \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 0 & 0 & \cdots & 1 & x \end{vmatrix} \\
&= x \begin{vmatrix} x & 1 & \cdots & 0 & 0 \\ 1 & x & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & x & 1 \\ 0 & 0 & \cdots & 1 & x \end{vmatrix} + \begin{vmatrix} x & \cdots & 0 & 0 \\ & \ddots & & \\ 0 & \cdots & x & 1 \\ 0 & \cdots & 1 & x \end{vmatrix} \\
&= x\tilde{\chi}_{n-1}(x) + \tilde{\chi}_{n-2}(x).
\end{aligned}$$

Together with $\tilde{\chi}_1(x) = x$ and $\tilde{\chi}_2(x) = x^2 + 1$ we have $\tilde{\chi}_n(x) = f_{n+1}(x)$ and $\chi_n(x) = x\tilde{\chi}_{n-1}(x) = xf_n(x)$.

(iii)+(iv) We prove these identities by induction. We have $f_0(x) = 0 = xf_0(x)$ and $f_1(x) = 1 = f_0(x)^2 + f_1(x)^2$. Furthermore

$$\begin{aligned}
f_{2n+2}(x) &= xf_{2n+1}(x) + f_{2n}(x) \\
&= x(f_n(x)^2 + f_{n+1}(x)^2) + xf_n(x)^2 \\
&= xf_{n+1}(x)^2, \\
f_{2n+3}(x) &= xf_{2n+2}(x) + f_{2n+1}(x) \\
&= x^2 f_{n+1}(x)^2 + f_n(x)^2 + f_{n+1}(x)^2 \\
&= (xf_{n+1}(x) + f_n(x))^2 + f_{n+1}(x)^2 \\
&= f_{n+2}(x)^2 + f_{n+1}(x)^2.
\end{aligned}$$

(v) We prove this identity for fixed n by induction on t. For $t = 0$ we have zero on both sides of the equation and the case $t = 1$ is just a transformation of the recurrence relation defining f_n. For the induction step we calculate

$$\begin{aligned} f_{n-t-1}(x) + f_{n+t+1}(x) &= (xf_{n-t}(x) + f_{n-t+1}(x)) + xf_{n+t}(x) + f_{n+t-1}(x) \\ &= x(f_{n-t}(x) + f_{n+t}(x)) + (f_{n-(t-1)}(x) + f_{n+t-1}) \\ &= x^2 f_n(x) f_t(x) + x f_n(x) f_{t-1}(x) \\ &= x f_n(x)(x f_t(x) + f_{t-1}(x)) \\ &= x^2 f_n(x) f_{t+1}(x). \end{aligned}$$

(vi) This identity is proved by fixing m and using induction on n. The case $n = 0$ corresponds to $f_0(x) = 0 = f_m(x)f_0(xf_m(x))$ and the case $n = 1$ corresponds to $f_m(x) = f_m(x)f_1(xf_m(x))$. For the induction step we use (v) and calculate

$$\begin{aligned} f_{m(n+1)}(x) &= x f_{mn}(x) f_m(x) + f_{m(n-1)}(x) \\ &= x f_m(x)\,(f_m(x) f_n\,(x f_m(x))) + f_m(x) f_{n-1}\,(x(f_m(x))) \\ &= f_m(x)(x f_m(x) f_n(x f_m(x)) + f_{n-1}(x f_m(x))) \\ &= f_m(x) f_{n+1}(x f_m(x)). \end{aligned}$$

(vii+viii) The characteristic matrix of M_n has a minor of order $(n-2)$ with value 1. The corresponding $(n-2)\times(n-2)$ matrix is highlighted in the following illustration.

$$\begin{pmatrix} x & 1 & 0 & \cdots & 0 & 1 \\ 1 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & x & 1 \\ 1 & 0 & 0 & \cdots & 1 & x \end{pmatrix}$$

Therefore the Smith normal form of $M_n + xI$ has the form $\operatorname{diag}(1,\dots,1,s(x),\mu_n(x))$ with $s \mid \mu_n$, $s\mu_n = \chi_n$ and hence $\deg(s) + \deg(\mu_n) = n$. By Lemma 6.17 we have $\deg(\mu_n) \le \lceil \frac{n}{2} \rceil$ and $\deg(s) \ge \lfloor \frac{n}{2} \rfloor$.

If $n = 2k$, this gives $s = \mu_n$ and we have $\mu_{2k}(x)^2 = s\mu_{2k}(x) = \chi_{2k}(x) = x^2 f_{2k}(x)^2$. By Lemma 6.19 this shows that $\mu_n(x) = x f_n(x)$.

If $n = 2k+1$, this implies $\mu_n(x) = u(x)s(x)$ where $u(x) \in \{x, x+1\}$ is a polynomial of degree one. Since

$$\begin{aligned} \chi_{2k+1}(x) &= x f_{2k+1}(x) \\ &= x(f_k(x) + f_{k+1}(x))^2, \end{aligned}$$

the multiplicity of the zero at 0 of χ_{2k+1} is odd, hence $\operatorname{mul}_0(u(x)s(x)^2) = \operatorname{mul}_0(u(x)) + 2\operatorname{mul}_0(s(x))$ is also odd and we have $u(x) = x$. Therefore $x(f_k(x) + f_{k+1}(x))^2 = xs(x)^2$. Again by Lemma 6.19 we have $s(x) = f_k(x) + f_{k+1}(x)$ and $\mu(x) = x(f_k(x) + f_{k+1}(x))$. $\square$

The matrix representation of $(w_{150})_{|\mathrm{Per}_n(\sigma)}$ with respect to the standard basis is given by $M_n + I$, hence the minimal polynomial of $(w_{150})_{|\mathrm{Per}_n(\sigma)}$ is given by $\mu_n(x+1)$. By Lemma 6.11 (d) we therefore know that the transient time of $(w_{150})_{|\mathrm{Per}_n(\sigma)}$ is given by $\mathrm{mul}_0(\mu_n(x+1)) = \mathrm{mul}_1(\mu_n(x))$. It is Theorem 6.23 that allows us to calculate mul_0 and mul_1 for the Fibonacci polynomials f_n and therefore also for the minimal polynomials μ_n.

Theorem 6.24. *For $n \geq 1$ we have*

$$\begin{aligned}
\mathrm{mul}_0(f_n) &= \max\left\{ 2^\ell - 1 \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid n \right\}, \\
\mathrm{mul}_1(f_n) &= \begin{cases} \max\left\{ 2^{\ell+1} \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid n \right\} & \text{if } 3 \mid n \\ 0 & \text{otherwise} \end{cases}, \\
\mathrm{mul}_0(f_{n+1} + f_n) &= 0, \\
\mathrm{mul}_1(f_{n+1} + f_n) &= \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Proof. We start with $\mathrm{mul}_0(f_n)$. Since $f_1(x) = 1$ and $f_2(x) = x$, we have $\mathrm{mul}_0(f_1) = 0$ and $\mathrm{mul}_0(f_2) = 1$. Thus the assumption is valid for $n \in \{1,2\}$. We proceed by induction. For even n, i.e., $n = 2k$, we have

$$\begin{aligned}
\mathrm{mul}_0(f_{2k}(x)) &= \mathrm{mul}_0(x f_k(x)^2) \\
&= 1 + 2\,\mathrm{mul}_0(f_k(x)) \\
&= 1 + 2\max\left\{ 2^\ell - 1 \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid n \right\} \\
&= \max\left\{ 2^{\ell+1} - 1 \;\middle|\; \ell \in \mathbb{N}_0,\ 2^{\ell+1} \mid 2n \right\} \\
&= \max\left\{ 2^\ell - 1 \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid 2n \right\}.
\end{aligned}$$

If n is odd, we have to show that $\mathrm{mul}_0(f_n) = 0$. Let $n = 2k+1$. Either k is odd and by induction $\mathrm{mul}_0(f_k) = 0$ and $\mathrm{mul}_0(f_{k+1}) > 0$ or k is even and by induction $\mathrm{mul}_0(f_k) > 0$ and $\mathrm{mul}_0(f_{k+1}) = 0$. In both cases we can conclude from Theorem 6.23 (iv) that $f_{2k+1}(0) = 1$, hence $\mathrm{mul}_0(f_{2k+1}(x)) = 0$.

Now we have a look at $\mathrm{mul}_1(f_n)$. Again we proceed by induction on n. We have $\mathrm{mul}_1(f_0) = 0 = \mathrm{mul}_1(f_1)$. For the induction step consider the equation $f_n(1) = f_{n-1}(1) + f_{n-2}(1)$. If n is not divisible by 3, exactly one of the summands on the right hand side is zero by the induction hypothesis. Therefore $f_n(1) = 1$ and $\mathrm{mul}_1(f_n) = 0$. Now assume n is divisible by 3, so there is $k \in \mathbb{N}$ with $n = 3k$. By Theorem 6.23 (vi) we have

$$\begin{aligned}
f_{3k}(x) &= f_3(x) f_k(x f_3(x)), \\
\mathrm{mul}_1(f_{3k}(x)) &= \mathrm{mul}_0(f_{3k}(x+1)) \\
&= \mathrm{mul}_0(f_3(x+1) f_k((x+1) f_3(x+1))) \\
&= \mathrm{mul}_0(x^2 f_k((x+1)x^2)) \\
&= 2 + \mathrm{mul}_0(f_k((x+1)x^2)).
\end{aligned}$$

By Lemma 6.11 we have $\mathrm{mul}_0(f_k((x+1)x^2)) = \mathrm{mul}_0((x+1)x^2)\,\mathrm{mul}_0(f_k(x)) = 2\,\mathrm{mul}_0(f_k(x))$ and hence $\mathrm{mul}_1(f_{3k}(x)) = 2 + 2\,\mathrm{mul}_0(f_k(x)) = \max\left\{ 2^{\ell+1} \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid n \right\}$.

We have already shown that $f_n(0) = 1$ for odd n and $f_n(0) = 0$ for n even. Therefore $f_{n+1}(0) + f_n(0) = 1$ and $\mathrm{mul}_0(f_{n+1} + f_n) = 0$.

We also saw that $f_n(1) = 0$ if $3 \mid n$ and $f_n(1) = 1$ otherwise. Hence we have

$$(f_{n+1} + f_n)(1) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}.$$

Finally we see that for $n = 3k + 1$ with $k \geq 0$ we have

$$\begin{aligned}
(f_{3k+2} + f_{3k+1})(x+1) &= (x+1) f_{3k+1}(x+1) + f_{3k}(x+1) + f_{3k+1}(x+1) \\
&= x f_{3k+1}(x+1) + f_{3k}(x+1), \\
\mathrm{mul}_1((f_{3k+2} + f_{3k+1})(x)) &= \mathrm{mul}_0((f_{3k+2} + f_{3k+1})(x+1)) \\
&= \mathrm{mul}_0(\underbrace{x f_{3k+1}(x+1)}_{\mathrm{mul}_0(\cdot)=1} + \underbrace{f_{3k}(x+1)}_{\mathrm{mul}_0(\cdot)\geq 2}) \\
&= 1.
\end{aligned}$$

□

Corollary 6.25.

$$\begin{aligned}
\mathrm{mul}_0(\mu_{2k}) &= \max\left\{ 2^\ell \;\middle|\; \ell \in \mathbb{N}_0, 2^\ell \mid k \right\}, \\
\mathrm{mul}_0(\mu_{2k+1}) &= 1, \\
\mathrm{mul}_1(\mu_{2k}) &= \begin{cases} \max\left\{ 2^{\ell+1} \;\middle|\; \ell \in \mathbb{N}_0, 2^\ell \mid k \right\} & \text{if } 3 \mid k \\ 0 & \text{otherwise} \end{cases}, \\
\mathrm{mul}_1(\mu_{2k+1}) &= \begin{cases} 1 & \text{if } k \equiv 1 \pmod 3 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

With all calculations in place, we can now proof the main result of this section.

Proof of Theorem 6.10. Set $w_{90,n} := (w_{90})_{|\mathrm{Per}_n(\sigma)}$ and $w_{150,n} := (w_{150})_{|\mathrm{Per}_n(\sigma)}$. We first consider the kernels of w_{90} and w_{150}. It is easy to see that

$$\begin{aligned}
\ker(w_{90}) &= \{{}^\infty 0^\infty, {}^\infty 1^\infty, {}^\infty(01)^\infty, {}^\infty(10)^\infty\}, \\
\ker(w_{150}) &= \{{}^\infty 0^\infty, {}^\infty(100)^\infty, {}^\infty(010)^\infty, {}^\infty(001)^\infty\}.
\end{aligned}$$

Therefore for every $y \in \mathrm{Per}_n(\sigma)$ we have

$$\begin{aligned}
|w_{90,n}^{-1}(y)| &= \begin{cases} 2 & \text{if } n \equiv 1 \pmod 2 \\ 4 & \text{if } n \equiv 0 \pmod 2 \end{cases}, \\
|w_{150,n}^{-1}(y)| &= \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \text{ or } n \equiv 2 \pmod 3 \\ 4 & \text{if } n \equiv 0 \pmod 3 \end{cases}.
\end{aligned}$$

Therefore if $w_{90,n}$ and $w_{150,n}$ are conjugate, n must be divisible by 6. Conjugate finite dynamical systems must also have the same transient time, but for $n = 6k$ the transient time of $w_{90,n}$ is

$$\mathrm{mul}_0(\mu_{6k}) = \max\left\{ 2^\ell \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid k \right\}$$

while the transient time of $w_{150,n}$ is

$$\mathrm{mul}_1(\mu_{6k}) = \max\left\{ 2^{\ell+1} \;\middle|\; \ell \in \mathbb{N}_0,\ 2^\ell \mid k \right\}.$$

In particular these transient times are never equal. $\square$

Chapter 7.

Conjugacy on Tori

At the end of the last chapter we analyzed w_{90} and w_{150} by studying their action on σ-periodic points. In this chapter we continue this study and take a look at another notion of isomorphism for cellular automata over $\mathbb{Z}^d$. In contract to many other dynamical systems, such cellular automata have an abundance of finite subsystems. Let $(e_k)_{k\in\{1,\dots,d\}}$ be the standard generators of $\mathbb{Z}^d$. Given $v \in \mathbb{Z}^d$, we call the set of points which are v_k periodic with respect to σ_{e_k} for all $k \in \{1,\dots,d\}$ spatially periodic with period v. These points form an invariant set for every cellular automaton. This subsystem corresponds to the action of the cellular automaton on the discrete torus $\mathbb{Z}^d/v\mathbb{Z}^d = (\mathbb{Z}/v_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/v_d\mathbb{Z})$. From this perspective, every cellular automaton over $\mathbb{Z}^d$ gives rise to a cellular automaton over the finite group $\mathbb{Z}^d/v\mathbb{Z}^d$. We now try to recover as much information about the cellular automaton as possible from these finite systems.

One can visualize a finite dynamical systems $f : X \to X$ by its state transition graph. This graph has vertex set X and an edge from x to y if $f(x) = y$. A conjugacy between finite dynamical systems is then nothing more than an isomorphism between their state transition graphs. Drawing the state transition graphs of cellular automata on tori yields beautiful pictures such as Figure 7.1. There is even a book by Wuensche and Lesser [WL92] consisting almost entirely of drawings of these graphs.

The action of cellular automata on spatially periodic points received a lot of attention, especially in connection with the automorphism group of subshifts, see for example the paper by Boyle and Krieger [BK87]. The questions studied here, however, seem to be new. The results in this chapter were presented by the author at Automata 2016, see [Epp16].

7.1. Cellular Automata on Tori

We start be making the correspondence between the invariant subsystem and the torus more precise.

Definition 7.1 (Torus). *For $v \in \mathbb{N}^d$ denote by $\mathbb{T}_v := \mathbb{Z}^d/v\mathbb{Z}^d$ the quotient of the ring $\mathbb{Z}^d$ by the ideal generated by v and call $\mathbb{T}_v$ a* torus*. Denote by $\pi : \mathbb{Z}^d \to \mathbb{T}_v$ the natural projection. Every configuration $x \in A^{\mathbb{T}_v}$ corresponds to a unique configuration $\tilde{x} \in A^{\mathbb{Z}^d}$*

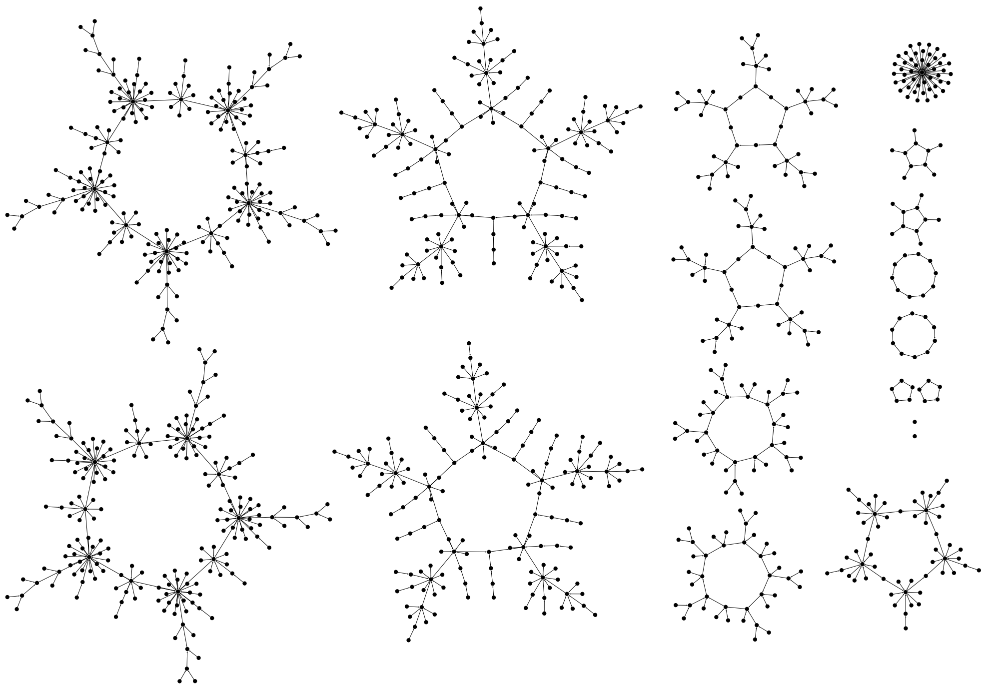


Figure 7.1.: State transition graph of the action of w_6 on the torus $\mathbb{Z}/10\mathbb{Z}$.

with $\tilde{x}_i = x_{\pi(i)}$ *for all* $i \in \mathbb{Z}^d$ *and* $\sigma_{e_k}^{v_k}(\tilde{x}) = \tilde{x}$ *for all* $k \in \{1,\dots,d\}$. *Conversely, every* $\tilde{x} \in A^{\tilde{\mathbb{T}}_v} := \left\{ y \in A^{\mathbb{Z}^d} \;\middle|\; \sigma_{e_k}^{v_k}(y) = y \text{ for all } k \in \{1,\dots,d\} \right\}$ *corresponds to a unique element in* $A^{\mathbb{T}_v}$.

Definition 7.2 (Induced cellular automata on tori)**.** *Let* $f : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ *be a cellular automaton. Since* f *commutes with all shifts,* $\tilde{\mathbb{T}}_v$ *is invariant under* f. *By the preceding considerations* f *induces a shift-commuting map* $f_{\mathbb{T}_v} : A^{\mathbb{T}_v} \to A^{\mathbb{T}_v}$. *The set* $A^{\mathbb{T}_v}$ *is finite and carries the discrete topology, hence* $f_{\mathbb{T}_v}$ *is automatically continuous, in other words, a cellular automaton. We call* $f_{\mathbb{T}_v}$ *the* cellular automaton on the torus $\mathbb{T}_v$ induced by f.

Remark 7.3. From yet another perspective, what we do is to consider f on the finite box $\{1,\dots,v_1\} \times \cdots \times \{1,\dots,v_d\}$ with periodic boundary conditions.

Definition 7.4 (Conjugate on a torus)**.** *Let* $v \in \mathbb{N}^d$ *and let* $f, g : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ *be cellular automata. We call* f *and* g conjugate on the torus $\mathbb{T}_v$ *if* $f_{\mathbb{T}_v}$ *and* $g_{\mathbb{T}_v}$ *are conjugate. We call* f *and* g conjugate on all tori *if* f *is conjugate to* g *on the torus* $\mathbb{T}_v$ *for every* $v \in \mathbb{N}^d$.

Equivalently, f and g are conjugate on $\mathbb{T}_v$ if the restrictions $f_{|\tilde{\mathbb{T}}_v} : A^{\tilde{\mathbb{T}}_v} \to A^{\tilde{\mathbb{T}}_v}$ and $g_{|\tilde{\mathbb{T}}_v} : A^{\tilde{\mathbb{T}}_v} \to A^{\tilde{\mathbb{T}}_v}$ are conjugate. Since $A^{\mathbb{T}_v}$ is finite, the topology plays no role here and the systems $f_{\mathbb{T}_v}$ and $g_{\mathbb{T}_v}$ are conjugate if there is a bijective map $\varphi : A^{\mathbb{T}_v} \to A^{\mathbb{T}_v}$ with $\varphi \circ f_{\mathbb{T}_v} = g \circ \varphi$.

The easiest example of cellular automata conjugate on a torus are those which are strongly conjugate.

Lemma 7.5. *Let* $f : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ *and* $g : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ *be cellular automata. If* f *and* g *are strongly conjugate, then* f *and* g *are conjugate on all tori.*

Proof. Let $v \in \mathbb{N}^d$. Since f and g are strongly conjugate, there is an invertible cellular automaton $\varphi : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ with $\varphi \circ f = g \circ \varphi$. All of the maps f, g and φ induce cellular automata on $A^{\mathbb{T}_v}$ and we have $\varphi_{\mathbb{T}_v} \circ f_{\mathbb{T}_v} = g_{\mathbb{T}_v} \circ \varphi_{\mathbb{T}_v}$, hence f and g are conjugate are conjugate on all tori. □

Invertible finite systems are conjugate if the numbers of their p-periodic points agree for all $p \in \mathbb{N}$. Thus if f and g are invertible cellular automata and $|\mathrm{Per}_p(f) \cap \tilde{\mathbb{T}}_v| = |\mathrm{Per}_p(g) \cap \tilde{\mathbb{T}}_v|$ for all $p \in \mathbb{N}$ and $v \in \mathbb{N}^d$, then f and g are conjugate on all tori. Based on this, we now construct two cellular automata f, g on a two-sided full shift which are conjugate on all tori, but which are not even topologically conjugate let alone strongly conjugate.

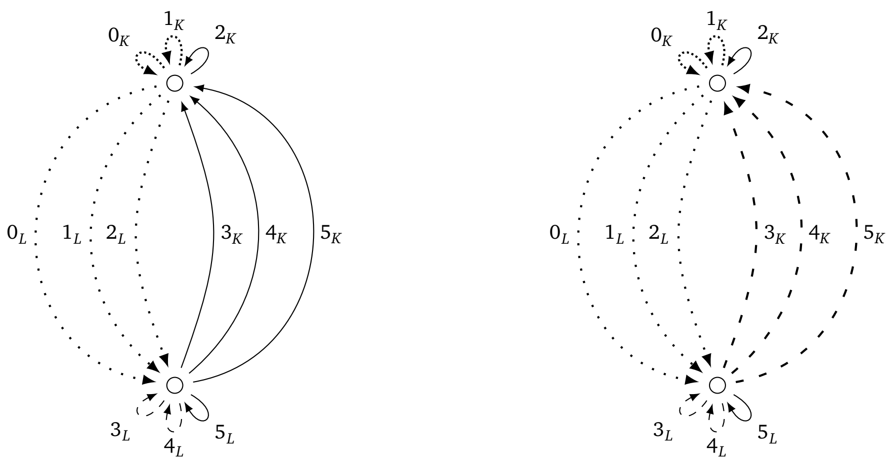


Figure 7.2.: Two cellular automata on the full 6-shift.

Example 7.6. *Consider the graph G depicted in Figure 7.2. Its edge shift is obtained by an out-splitting of the full 6-shift. For this the set $\{0, \ldots, 5\}$ is partitioned into the sets $L = \{0, 1, 2\}$ and $K := \{3, 4, 5\}$. The dashed and dotted lines define two graph-automorphisms $\tilde{f}$ and $\tilde{g}$ of G. Edges with the same dash pattern are permuted cyclically and edges which are not dashed are fixed. Written as elements of the permutation group of $E(G)$ the two automorphisms are defined by*

$$\tilde{f} = (0_L, 1_L, 2_L)(0_K, 1_K)(3_L, 4_L)$$
$$\tilde{g} = (0_L, 1_L, 2_L)(3_K, 4_K, 5_K)(0_K, 1_K)(3_L, 4_L)$$

The graph automorphisms induce automorphisms f, g of $\mathrm{Path}_{\mathbb{Z}}(G)$ and since this edge shift is conjugate to $\{1, \ldots, 6\}^{\mathbb{Z}}$, they also define two automorphisms $f, g : \{1, \ldots, 6\}^{\mathbb{Z}} \to \{1, \ldots, 6\}^{\mathbb{Z}}$. Both automorphisms have order 6. For example the cellular automaton f exchanges 0, 1 and 2 cyclically if they are followed by a symbol in L, it exchanges 0 and 1 if they are followed by a symbol in K and it exchanges 3 and 4 if these symbols are followed by a symbol from L.

We have

$$
\begin{aligned}
Fix(f) \cap Per(\sigma) &= Fix(g) \cap Per(\sigma) = \{\, 2_K, 5_L \,\}^{\mathbb{Z}} \cap Per(\sigma),\\
Per_2(f) \cap Per(\sigma) &= Per_2(g) \cap Per(\sigma) = \{\, 0_K, 1_K, 2_K, 3_L, 4_L, 5_L \,\}^{\mathbb{Z}} \cap Per(\sigma),\\
Per_3(f) \cap Per(\sigma) &= Per_3(g) \cap Per(\sigma) = \{\, 2_K, 5_L, 0_L, 1_L, 2_L, 3_K, 4_K, 5_K \,\}^{\mathbb{Z}} \cap Per(\sigma),\\
Per_6(f) \cap Per(\sigma) &= Per_6(g) \cap Per(\sigma) = Per(\sigma).
\end{aligned}
$$

By the above considerations this shows that f and g are conjugate on all tori.

If we now widen our focus and not only look at the σ-periodic points, we see that

$$
\begin{aligned}
Fix(f) &= \{\, 2_K, 5_L, 3_K, 4_K, 5_K \,\}^{\mathbb{Z}},\\
Fix(g) &= \{\, 2_K \,\}^{\mathbb{Z}} \cup \{\, 5_L \,\}^{\mathbb{Z}}.
\end{aligned}
$$

In particular $Fix(f)$ is infinite while $Fix(g)$ is finite. Therefore f and g are not topologically orbit equivalent let alone topologically conjugate, not even in the discrete topology.

7.2. Preimage Entropy

We now want to show that surjectivity is invariant under conjugacy on all tori. In dimension one this follows simply from the fact the surjectivity is equivalent to boundedness-to-one on periodic points, see Theorem 7.11, but in higher dimensions we have to work a little bit more.

The following construction is based on counting the preimages of single elements. It will be convenient to write $g^{-1}(x)$ instead of $g^{-1}[\{x\}]$ to simplify notation.

Definition 7.7 (Preimage entropy). *Let S, T be non-empty finite sets and let $g : A^S \to A^T$ be a map. We define the* preimage entropy *of g by*

$$
\begin{aligned}
h(g) &:= -\frac{1}{|T|} \sum_{x \in A^T} \frac{|g^{-1}(x)|}{|A^S|} \log \frac{|g^{-1}(x)|}{|A^S|}\\
&= \frac{|S|}{|T|} \log(A) - \sum_{x \in A^T} \frac{|g^{-1}(x)| \log |g^{-1}(x)|}{|T| \cdot |A|^{|S|}}.
\end{aligned}
$$

In other words, it equals the Shannon entropy of the image under g of the uniform probability measure on A^S averaged over T. As usual in the definition of entropies we have the convention that "$0 \cdot \infty = 0$".

Preimage entropy is not just a property of the image $g[A^S]$ but crucially depends on the map g. The key tool to bound this entropy is given by the following classical result.

Lemma 7.8. *If $a_1, \dots, a_n$ are non-negative real numbers with sum s, then*

$$-s \log s \le -\sum_{i=1}^{n} a_i \log a_i \le -s(\log s - \log n).$$

Proof. We can rewrite this inequality as

$$0 \le -\sum_{i=1}^{n} \frac{a_i}{s} \log \frac{a_i}{s} \le \log n.$$

This is the well-known fact that the Shannon entropy of a discrete probability distribution is non-negative and is maximized by the uniform distribution. The result follows directly from the concavity of the map $t \mapsto -t \log t$ on $[0,1]$, see for example the book by PETERSEN [Pet89]. □

Example 7.9. *Consider the following cellular automaton $f : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$ given by*

$$f(x)_i = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{if } x_i \in \{1,2\} \end{cases}.$$

For the torus $T := \mathbb{Z}/n\mathbb{Z} = \{0, \dots, n-1\}$ we have

$$|f_T^{-1}(x)| = 2^{|\{i \in T \mid x_i = 1\}|}$$

for $x \in \{0,1\}^T$ and $|f_T^{-1}(x)| = 0$ for $x \in \{0,1,2\}^T \setminus \{0,1\}^T$. Putting this into the definition of $h(f_T)$ gives

$$\begin{aligned} h(f_T) &= \log 3 - \frac{1}{n \cdot 3^n} \sum_{k=0}^{n} \binom{n}{k} 2^k \log 2^k \\ &= \log 3 - \frac{2 \log 2}{n \cdot 3^n} \sum_{k=1}^{n} \binom{n}{k} 2^{k-1} k \\ &= \log 3 - \frac{2 \log 2}{n \cdot 3^n} n \cdot 3^{n-1} = \log 3 - \frac{2}{3} \log 2. \end{aligned}$$

Notice that in this case $h(f_T)$ is independent of T. This is due to the fact that the state of $f(x)$ at index i is independent of the state of $f(x)$ at any other index. Hence the image of the uniform measure on $|A|^T$ is the product measure of the image of the uniform measure on $|A|^{\{0\}}$ and $h(f_T) = h(f_{\{0\}})$.

Apart from such easy examples, it seems hard to give closed expressions for $h(f_T)$ for non-surjective cellular automata. Even in dimension one, determining the number of preimages of a configuration leads to intricate combinatorial questions.

7.3. Surjectivity and Conjugacy on Tori

In this section, we show that surjectivity is invariant under conjugacy on all tori. This is not obvious, as two cellular automata f and g which are conjugate on all tori are not necessarily conjugate when considered on $\mathbb{Z}^d$. It is even not clear if they must be conjugate when restricted to the set of all spatially periodic points. Even if this turned out to be true, it would not immediately help us to show that surjectivity is invariant under conjugacy on all tori, as it seems to be still unknown if surjectivity of a cellular automaton implies surjectivity of its restriction to spatially periodic points (see Open Problem 2 in [Kar05]).

Definition 7.10 (Induced map on boxes)**.** *For $R \in \mathbb{N}$ let $B_R := \{-R, \ldots, R\}^d$ be the box with side lengths $2R+1$ centered at the origin. Let $f : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ be a cellular automaton with radius $r > 0$, i.e., f has a local rule $f_{loc} : A^{B_r} \to A$ with $f(x)_i = f_{loc}(x_{|B_r+i})$. Here we identify patterns in A^{B_r+i} with the corresponding patterns in A^{B_r}. For every $R \geq r$, f induces a map $\tilde{f}_R : A^{B_R} \to A^{B_{R-r}}$ by $\tilde{f}_R(x)_i = f_{loc}(x_{|B_r+i})$ for $i \in B_{R-r}$.*

There are many properties of cellular automata equivalent to surjectivity. The following theorem is one of the earliest results on cellular automata in the context of dynamical systems and goes back to Hedlund [Hed69] in dimension one and Maruoka and Kimura [MK76] in the higher-dimensional case.

Theorem 7.11. *For every cellular automaton $f : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ with radius $r > 0$ the following are equivalent.*

(a) *f is surjective.*

(b) *$\tilde{f}_R$ is surjective for every $R \geq r$.*

(c) *$\tilde{f}_R$ is balanced for every $R \geq r$, i.e., $|\tilde{f}_{R+r}^{-1}(x)| = |A|^{|B_{R+r}|-|B_R|}$ for all $x \in A^{B_R}$.*

In dimension one these properties are furthermore equivalent to the following properties

(d) *There is $D \in \mathbb{N}$ such that $|f^{-1}(x)| \leq D$ for all $x \in A^{\mathbb{Z}}$.*

(e) *There is $D \in \mathbb{N}$ such that $|f_{\mathbb{T}_v}^{-1}(x)| \leq D$ for all tori $\mathbb{T}_v, v \in \mathbb{N}^d$ and $x \in A^{\mathbb{T}_v}$.*

Notice that we can immediately conclude that in dimension one surjectivity is invariant under conjugacy on all tori because boundedness-to-one on tori, i.e., property (e), is obviously preserved by conjugacy on all tori.

In the following let $T := \mathbb{Z}^d / v\mathbb{Z}^d$ be a torus and let $R, r \in \mathbb{N}$. We cover a large part of T by translates of B_{R+r} as follows. Start by defining the vector $v' = ((2R+2r+1)\lfloor \frac{v_1}{2R+2r+1} \rfloor, \ldots, (2R+2r+1)\lfloor \frac{v_d}{2R+2r+1} \rfloor)$ and set

$$T' = \{0, \ldots, v'_1 - 1\} \times \cdots \times \{0, \ldots, v'_d - 1\} \subseteq T.$$

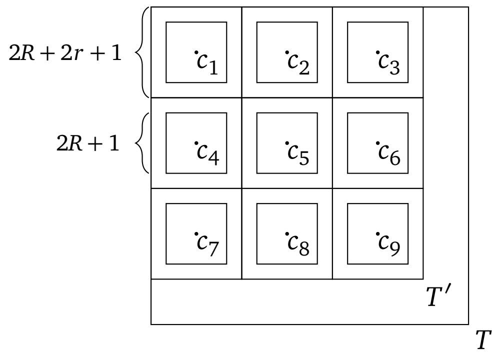


Figure 7.3.: Partitioning the torus T.

We can partition T' into n translates $C_1, \dots, C_n$ of B_{R+r} centered at $c_1, \dots, c_n$. See Figure 7.3 for an illustration. Finally set $W_R := A^{B_R}$.

Lemma 7.12. *With the above notation the following estimates hold.*

$$h(f_T) \geq \frac{|T'|}{|T|}\left(\log|A| - \sum_{w\in W_R} \frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}}\right) \geq 0,$$

$$h(f_T) \leq \frac{|T|+|T'|-n|B_R|}{|T|}\log|A| - \frac{|T'|}{|T|}\sum_{w\in W_R} \frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}} \leq \log|A|.$$

Proof. For patterns $w_1, \dots, w_n \in W = A^{B_R}$ we denote by $X_{w_1,\dots,w_n}$ the set of all configurations $x \in A^T$ for which $x_{|B_R+c_i} = w_i$ for $i = 1, \dots, n$. Partitioning A^T into these sets we get

$$\begin{aligned} h(f_T) &= \log|A| - \frac{1}{|A|^{|T|}|T|}\sum_{x\in A^T} |f_T^{-1}(x)|\log|f_T^{-1}(x)| \\ &= \log|A| - \frac{1}{|A|^{|T|}|T|}\sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R}\sum_{x\in X_{w_1,\dots,w_n}} |f_T^{-1}(x)|\log|f_T^{-1}(x)|. \end{aligned}$$

Because $\sum_{x\in X_{w_1,\dots,w_n}} |f_T^{-1}(x)| = |f_T^{-1}[X_{w_1,\dots,w_n}]|$, we can bound $h(f_T)$ from below by Lemma 2 and get

$$h(f_T) \geq \log|A| - \frac{1}{|A|^{|T|}|T|}\sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R} |f_T^{-1}[X_{w_1,\dots,w_n}]|\log|f_T^{-1}[X_{w_1,\dots,w_n}]|.$$

Using the fact that $|f_T^{-1}[X_{w_1,\dots,w_n}]| = |A|^{|T|-|T'|}\prod_{i=1}^n |\tilde{f}_R^{-1}(w_i)|$, we can compute the sum on the right-hand side. We get

$$\begin{aligned} L &:= \sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R} |f_T^{-1}[X_{w_1,\dots,w_n}]|\log|f_T^{-1}[X_{w_1,\dots,w_n}]| \\ &= \sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R} |A|^{|T|-|T'|}\prod_{i=1}^{n}|\tilde{f}_R^{-1}(w_i)|\Big((|T|-|T'|)\log|A|+\sum_{j=1}^{n}\log|\tilde{f}_R^{-1}(w_j)|\Big) \\ &= |A|^{|T|}\big(|T|-|T'|\big)\log|A| + \sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R}\frac{|A|^{|T|}}{|A|^{|T'|}}\prod_{i=1}^{n}|\tilde{f}_R^{-1}(w_i)|\sum_{j=1}^{n}\log|\tilde{f}_R^{-1}(w_j)|. \end{aligned}$$

Setting $K := \sum_{w'\in W_R}|\tilde{f}_R^{-1}(w')|$ and rearranging the summands on the right side, we get

$$L = |A|^{|T|}\big(|T|-|T'|\big)\log|A| + n\frac{|A|^{|T|}}{|A|^{|T'|}}\sum_{w\in W_R}K^{n-1}|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|.$$

Putting this back into our estimate of $h(f_T)$ and replacing K^{n-1} by the term $|A|^{|B_{R+r}|(n-1)} = |A|^{|T'|-|B_{R+r}|}$ we obtain

$$\begin{aligned} h(f_T) &\geq \log|A| - \left(1-\frac{|T'|}{|T|}\right)\log|A| - \frac{n|A|^{|T|-|B_{R+r}|}}{|T||A|^{|T|}}\sum_{w\in W_R}|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)| \\ &= \frac{|T'|}{|T|}\left(\log|A| - \sum_{w\in W_R}\frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}}\right). \end{aligned}$$

From Lemma 7.8 we also get an upper bound on $h(f_T)$, namely

$$h(f_T) \leq \log|A| - \frac{1}{|A|^{|T|}|T|}\left(L - \sum_{w_1\in W_R}\cdots\sum_{w_n\in W_R}|f_T^{-1}[X_{w_1,\dots,w_n}]|\log|X_{w_1,\dots,w_n}|\right).$$

Plugging $|X_{w_1,\dots,w_n}| = |A|^{|T|-n|B_R|}$ into this inequality gives

$$\begin{aligned} h(f_T) &\leq \log|A| - \frac{L}{|T|\cdot|A|^{|T|}} + \frac{|T|-n|B_R|}{|T|}\log|A| \\ &\leq \frac{|T|+|T'|-n|B_R|}{|T|}\log|A| - \frac{|T'|}{|T|}\sum_{w\in W_R}\frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}}. \end{aligned}$$ □

Theorem 7.13. *Let $(T_n)_{n\in\mathbb{N}} = (\mathbb{Z}^d/v_n\mathbb{Z}^d)$ be a sequence of growing tori, i.e.,*

$$\min(\{\, v_{n,i} \mid i\in\{1,\dots,d\}\,\}) \to \infty \text{ as } n\to\infty.$$

If $f : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ is a cellular automaton, then f is surjective if and only if $\lim_{n\to\infty} h(f_{T_n}) = \log|A|$.

Proof. Let r be a radius of f. Let $R\in\mathbb{N}$ and let T_n' and W_R be defined as above. Since

$(T_n)_{n\in\mathbb{N}}$ is a sequence of growing tori, $\frac{|T_n'|}{|T_n|}$ tends to one as n goes to infinity and we get

$$0 \le \log|A| - \sum_{w\in W_R} \frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}} \le \liminf_{n\to\infty} h(f_{T_n}),$$
$$\log|A| \ge (2-\frac{|B_R|}{|B_{R+r}|})\log|A| - \sum_{w\in W_R} \frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}} \ge \limsup_{n\to\infty} h(f_{T_n}).$$

For $R\to\infty$ we have $\frac{|B_R|}{|B_{R+r}|}\to 1$ and we get

$$\lim_{n\to\infty} h(f_{T_n}) = \lim_{R\to\infty} \log|A| - \sum_{w\in W_R} \frac{|\tilde{f}_R^{-1}(w)|\log|\tilde{f}_R^{-1}(w)|}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}}.$$

In particular the limit in the assertion exists.

If f is surjective, we have $|\tilde{f}_R^{-1}(w)| = |A|^{|B_{R+r}|-|B_R|}$ for every pattern $w\in W_R$ by Theorem 7.11 (c). Hence

$$\begin{aligned}\lim_{n\to\infty} h(f_{T_n}) &= \lim_{R\to\infty} \log|A| - |W| \frac{|A|^{|B_{R+r}|}(\log|A|^{|B_{R+r}|)} - \log|A|^{|B_R|})}{|A|^{|B_R|}\cdot|B_{R+r}|\cdot|A|^{|B_{R+r}|}}\\ &= \lim_{R\to\infty} \frac{|B_R|}{|B_{R+r}|}\log|A| = \log|A|.\end{aligned}$$

If, on the other hand, f is not surjective, then by Theorem 7.11 (b) there is $R\in\mathbb{N}$ such that

$$|\{\, x\in A^{B_R} \mid \tilde{f}_R^{-1}(x) \ne \emptyset \,\}| \le |A|^{|B_R|} - 1.$$

By Lemma 7.8 we therefore get the estimate

$$\begin{aligned}\lim_{n\to\infty} h(f_{T_n}) &\le (2-\frac{|B_R|}{|B_{R+r}|})\log|A| - \frac{|A|^{|B_{R+r}|}(\log(|A|^{|B_{R+r}|}) - \log(|A|^{|B_R|}-1))}{|B_{R+r}|\cdot|A|^{|B_{R+r}|}}\\ &= \log|A| - \frac{\log(|A|^{|B_R|}) - \log(|A|^{|B_R|}-1)}{|B_{R+r}|} < \log|A|.\end{aligned}$$ □

Corollary 7.14. *Surjectivity is invariant under conjugacy on all tori.*

Proof. If f and g are conjugate on a torus T, then clearly $h(f_T) = h(g_T)$ and the assertion follows from the previous theorem. □

Interestingly one can, however, not deduce surjectivity from the knowledge of the dynamics on a finite number of tori, as the following theorem shows.

Theorem 7.15. *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a cellular automaton. If $T_1,\dots,T_n$ are tori, there are cellular automata $g,h : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that g is surjective while h is not surjective and such that f_{T_k}, g_{T_k} and h_{T_k} are pairwise conjugate for all $k\in\{1,\dots,n\}$.*

Proof. We start with the easier part, namely the construction of h. Let r be the radius of f. There are integers $m_1, \dots, m_n$ such that $T_k = \mathbb{Z}/m_k\mathbb{Z}$ for $k \in \{1, \dots, n\}$. Set

$$m := \operatorname{lcm}(\{\, m_k \mid k \in \{1, \dots, n\}\,\}).$$

Let a be a fixed symbol in A. Define $h_{\text{loc}} : A^{2m+1} \to A$ by

$$h_{\text{loc}}(x_{-m}, \dots, x_m) = \begin{cases} f_{\text{loc}}(x_{[-r,r]}) & \text{if } x_{-m} = x_m \\ a & \text{otherwise} \end{cases}$$

Let h be the cellular automaton with local rule h_{loc} and radius m. We have $|h_{\text{loc}}^{-1}(a)| \geq |A|^{2m+1} - |A|^{2m} = (|A| - 1)|A|^{2m} > |A|^{2m}$. By Theorem 7.11 (c) h is not surjective. On the other hand, every configuration $x \in A^{T_k}$ fulfills $x_{i-m} = x_{i+m}$ for all $i \in \{0, \dots, m_k - 1\}$, hence $f_{T_i} = h_{T_i}$. The existence of g follows from a theorem of Ashley [Ash91, Theorem 3.5]. The theorem states that a cellular automaton g with the desired properties exists and that it is not only surjective but even 1-1 almost everywhere and right-closing (see [LM95] or [Kit98] for the meaning of these notions). Two years later, Ashley proved in [Ash93] a much stronger extension result. One can replace $\bigcup_{k=1}^{n} A^{T_k}$ by any subshift of finite type on which f is right-closing and one obtains a surjective cellular automaton g that coincides with f on this subshift. □

7.4. Injectivity and Further Properties

It is well known that in dimension one a cellular automaton is injective if and only if its restriction to the spatially periodic points is injective, see for example [Kar05].

Theorem 7.16. *A cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is injective if and only if its restriction to $\operatorname{Per}(\sigma)$ is injective.*

Proof. If f is injective, its restriction to any invariant set is also injective.

On the other hand, consider $x, y \in A^{\mathbb{Z}}$ with $f(x) = f(y)$. Let $r > 0$ be a radius of f. We distinguish two cases. In the first case x and y are left- and right-asymptotic, i.e., there is $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $|n| \geq n_0$ we have $x_n = y_n$. Define $\tilde{x} := {}^{\infty}x_{[-n_0-2r, n_0+2r]}{}^{\infty}$ and $\tilde{y} := {}^{\infty}y_{[-n_0-2r, n_0+2r]}{}^{\infty}$. For every $i \in \mathbb{Z}$ we either have $\tilde{x}_{[i-r,i+r]} = \tilde{y}_{[i-r,i+r]}$ or there is $j \in \mathbb{Z}$ such that $\tilde{x}_{[i-r,i+r]} = x_{[j-r,j+r]}$ and $\tilde{y}_{[i-r,i+r]} = y_{[j-r,j+r]}$. Therefore $f(\tilde{x}) = f(\tilde{y})$, hence $\tilde{x} = \tilde{y}$ and also $x = y$. Now assume that x and y are not left- and right-asymptotic. By the pigeon hole principle there are indices $i, j \in \mathbb{Z}, i + r < j - r$ such that $x_{[i-r,j-r)} \neq y_{[i-r,j-r)}, x_{[i-r,i+r]} = x_{[j-r,j+r]}$ and $y_{[i-r,i+r]} = y_{[j-r,j+r]}$. Then $\tilde{x} := {}^{\infty}x_{[i-r,j-r)}{}^{\infty}$ and $\tilde{y} := {}^{\infty}y_{[i-r,j-r)}{}^{\infty}$ have the same image under f. Since f is injective on $\operatorname{Per}(\sigma)$, we have $\tilde{x} = \tilde{y}$, contradicting $x_{[i-r,j-r)} \neq y_{[i-r,j-r)}$. □

Corollary 7.17. *Injectivity is invariant under conjugacy on all tori for one-dimensional cellular automata.*

Remark 7.18. A cellular automaton f for which there is a sequence of tori $(T_n)_{n\in\mathbb{N}}$ such that f_{T_n} is injective must not necessarily be injective. We saw an example of this phenomenon in the form of the elementary cellular automaton w_{150}. $(w_{150})_{T_n}$ is injective for $T_n := \mathbb{Z}/(6n+1)\mathbb{Z}$ but of course every point has exactly 4 preimages with respect to w_{150}. See Theorem 4.4 by Rhodes in [Rho88] for a non-linear cellular automaton showing this behavior.

This is no longer true in higher dimensions. Kari in [Kar92] (see also [Kar05]) constructed a two-dimensional cellular automaton that is not injective, whose restriction to spatially periodic points, however, is injective. Hence it is natural to ask the following.

Question 7.19. *Is injectivity invariant under conjugacy on all tori?*

It is highly likely that one can answer this question negatively by a modification of the construction mentioned above.

A long standing open problem on the dynamics of cellular automata is the conjecture that surjectivity of cellular automata implies denseness of (jointly) periodic points. By compactness one sees that denseness of periodic points implies surjectivity. The distribution of periodic points for surjective cellular automata on tori, on the other hand, has for example been experimentally investigated in the paper [BL07] by Boyle and Lee. If the conjecture is true, Corollary 7.14 would imply that denseness of periodic points is invariant under conjugacy on all tori. The following might therefore be a more accessible weakening of the conjecture.

Conjecture 7.20. *Denseness of (jointly) periodic points is invariant under conjugacy on all tori.*

Chapter 8.

Conclusion and Open Problems

Our main goal in this thesis was to show that the restrictions of cellular automata on two-sided sofic subshifts to their p-periodic points are conjugate if and only if a natural homeomorphism condition for their periodic points is fulfilled. Let us recapitulate the major new results we obtained along the way.

We showed that although derivative algebras have a significantly simpler axiom system than topological Boolean algebras, these algebraic structures are in fact equivalent. We used derivative algebras to simplify Pierce's work on the classification of closed subsets of the Cantor space for which these algebras are finite.

By using the order on the reals, we showed that topological conjugacy coincides with topological orbit equivalence for periodic dynamical systems on closed subsets of the Cantor space.

For periodic dynamical systems whose periodic point algebra is finite we showed that topological conjugacy can further be characterized by the existence of a homeomorphism mapping p-periodic points onto p-periodic points.

Extending work of Head, we showed that the derivative algebra generated by a finite number of intersecting two-sided sofic subshifts is finite. Combined with the abstract results on periodic dynamical systems with a finite periodic point algebras we thus reached the goal mentioned above. We were also able to construct an algorithm to decide the existence of a topological conjugacy between such systems.

Next we showed that the soficness assumption is crucial. We constructed two-sided subshifts which are not homeomorphic but which become homeomorphic when they are doubled. Based on this, we built a cellular automaton on each of these two subshifts for which every point has minimal period two such that these two cellular automata were not topologically orbit equivalent.

We classified the elementary cellular automata up to topological conjugacy and showed that for this notion of isomorphism there are 83 equivalence classes. Special emphasis was put on the left- and right-permuting cellular automata w_{90} and w_{150}. We showed that these cellular automata are conjugate, but every conjugacy has to use infinitely many local rules, putting them especially far away from the class of strongly conjugate cellular automata pairs.

Finally, we investigated the systems induced by cellular automata on spatially periodic points of a fixed period, or equivalently, on tori. Introducing the notion of being conjugate on all tori, we showed that surjectivity is invariant under this notion of isomorphism.

The diagram in Figure 8.1 gives an overview of most of the various equivalence relations between dynamical systems which we discussed.

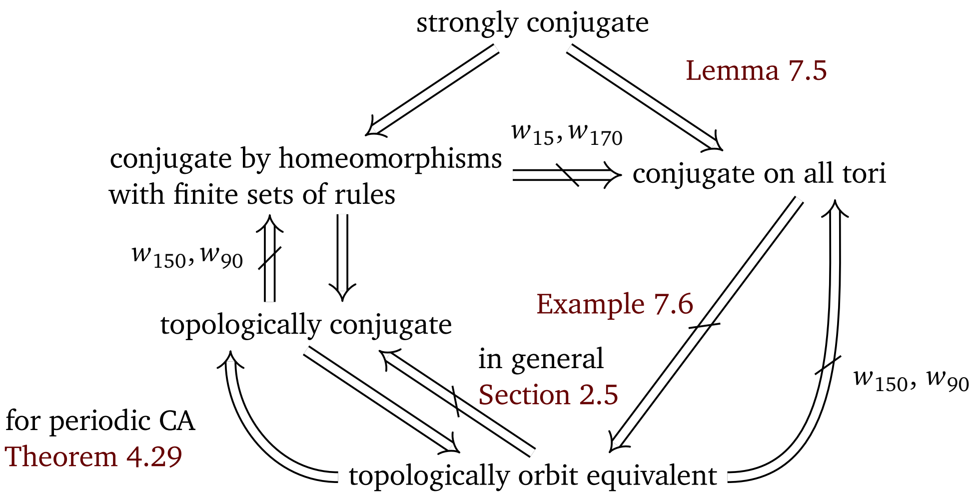


Figure 8.1.: Equivalence relations between cellular automata on two-sided subshifts.

During the research for this thesis naturally many questions came up. Some of them as possible generalizations of results we proved, some of them just like small flowers at the side of the road. We already mentioned many of them in context but some interesting ones, which did not really fit in any other place, we collect here.

We mainly studied topological conjugacy. Many results could have counterparts for strong topological conjugacy.

Question 8.1. *Is there a criterion analogous to Theorem 4.34 for strong conjugacy of periodic cellular automata?*

Question 8.2. *Is there a pair of topologically conjugate cellular automata $f, g : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that f and g are conjugate on all tori but not strongly conjugate?*

The structure of periodic points is another source of many questions.

Question 8.3. *Which sets arise as the periods of a cellular automaton? In other words, can one characterize the family of sets*

$$\left\{ \{k \in \mathbb{N} \mid \widetilde{Per}_k(f) \neq \emptyset\} \;\middle|\; f \text{ is a cellular automaton on } A^{\Lambda} \right\}?$$

For continuous self-maps of the interval, Sharkovskii's theorem [Sha64] answers this question in an astonishing way. Namely, there is a partial order $\trianglelefteq$ on the natural numbers such that

$$\left\{ \{k \in \mathbb{N} \mid \widetilde{\operatorname{Per}}_k(f) \neq \emptyset\} \;\middle|\; f : [0,1] \to [0,1] \text{ is continuous} \right\} = \{\, \{k \in \mathbb{N} \mid k \trianglelefteq a\} \mid a \in \mathbb{N}\,\}.$$

For cellular automata we can show the following. For all $n \in \mathbb{N}$ and $M \subseteq \{1, 2, \ldots, n\}$ there is an alphabet A and a cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that

$$\left\{ k \in \{1, \ldots, n\} \;\middle|\; \widetilde{\operatorname{Per}}_k(f) \neq \emptyset \right\} = M.$$

We can even show the following stronger result.

Theorem 8.4. *For every function $\omega : \{1, \ldots, n\} \to \mathbb{N} \cup \{|\mathbb{N}|, |\mathbb{R}|\}$ there is an alphabet A and a cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ such that $|\widetilde{\operatorname{Per}}_k(f)| = k \cdot \omega(k)$ for all $k \in \{1, \ldots, n\}$.*

Proof. For $k \in \{1, \ldots, n\}$ let $X_k \subseteq \{0,1\}^{\mathbb{Z}}$ be a subshift of finite type with $\omega(k)$ elements, in particular, X_k might be empty. There is $r \in \mathbb{N}$ such that all of these subshifts of finite type are defined by sets of forbidden words of length at most r. Let V_k be all words of length r not appearing in X_k. Define $A := \{0,1\} \times \{1, \ldots, n\} \times \{1, \ldots, n\} \times \{1, \ldots, n+2\}$. For $k \leq \ell$ define a permutation $\tau_{k,\ell} : \{1, \ldots, \ell\} \to \{1, \ldots, \ell\}$ by $\tau_{k,\ell} = (1, \ldots, k)(k+1, \ldots, \ell)$, written here as the product of two cycles. Finally, define $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$f((x_j, y_j, z_j, u_j)_{j \in \mathbb{Z}})_i = \begin{cases} (x_i, y_i, \tau_{y_i,n}(z_i), \tau_{0,n+2}(u_i)) & \text{if } y_i \neq y_{i+1} \text{ or } x_{[i,i+r)} \in V_{y_i} \text{ or } z_i > y_i \\ (x_i, y_i, \tau_{y_i,n}(z_i), \tau_{1,n+2}(u_i)) & \text{otherwise} \end{cases}.$$

If $(x, y, z, u) \in A^{\mathbb{Z}}$ is periodic with respect to f with period at most n, then $u = {}^{\infty}1^{\infty}$, $y = {}^{\infty}k^{\infty}$ for some $k \in \{1, \ldots, n\}$ and $x \in X_k$. Hence

$$\begin{aligned} \widetilde{\operatorname{Per}}_k(f) &= X_k \times \{k\} \times \{1, \ldots, k\} \times \{1\}, \\ |\widetilde{\operatorname{Per}}_k(f)| &= k \cdot \omega(k). \end{aligned}$$

□

Jeandel and Vanier in [JV15a] characterize the sets of periods that can arise for multidimensional subshifts, thus giving recursion theoretic restrictions on the sets of periods that can arise for cellular automata.

Furthermore, most of the questions asked in this thesis can also be asked for factor maps instead of topological conjugacies. We say that a dynamical system $f : X \to X$ factors onto a dynamical system $g : Y \to Y$ if and only if there is a surjective continuous map $\varphi : X \to Y$ with $\varphi \circ f = g \circ \varphi$.

Question 8.5. *Can one characterize when a cellular automaton factors onto another?*

Is this question decidable? The answer seems to be non obvious already for periodic or preperiodic cellular automata.

We conclude with the question that seems to me to be the most attackable.

Question 8.6. *Can one characterize when two preperiodic cellular automata are conjugate to each other?*

That this question might be hard already in the case of idempotent cellular automata is illustrated by the considerations for w_{200} and w_{76} in Section 6.2.

Appendix A.

Data for the Elementary Cellular Automata

In this appendix we present five tables containing various cardinalities calculated for the classification of the 256 elementary cellular automata with radius 1 on the two-sided full shift over the alphabet $\{0, 1\}$. The first table shows for every elementary cellular automaton w_c the Wolfram codes of all those cellular automata which are obtained by conjugating w_c with τ, υ and $\tau \circ \upsilon$, see Section 6.1. Each of these equivalence classes is in the following referred to by its minimal element.

CA	conjugate CA
0	0, 255
1	1, 127
2	16, 191, 247
3	17, 63, 119
4	4, 223
5	5, 95
6	20, 159, 215
7	21, 31, 87
8	64, 239, 253
9	65, 111, 125
10	80, 175, 245
11	47, 81, 117
12	68, 207, 221
13	69, 79, 93
14	84, 143, 213
15	15, 85
16	2, 191, 247
17	3, 63, 119
18	18, 183
19	19, 55
20	6, 159, 215
21	7, 31, 87
22	22, 151
23	23
24	66, 189, 231
25	61, 67, 103
26	82, 167, 181
27	39, 53, 83
28	70, 157, 199
29	29, 71
30	86, 135, 149
31	7, 21, 87
32	32, 251
33	33, 123
34	48, 187, 243
35	49, 59, 115
36	36, 219
37	37, 91
38	52, 155, 211
39	27, 53, 83
40	96, 235, 249
41	97, 107, 121
42	112, 171, 241
43	43, 113
44	100, 203, 217
45	75, 89, 101
46	116, 139, 209
47	11, 81, 117
48	34, 187, 243
49	35, 59, 115
50	50, 179
51	51
52	38, 155, 211
53	27, 39, 83
54	54, 147
55	19, 55
56	98, 185, 227
57	57, 99
58	114, 163, 177
59	35, 49, 115
60	102, 153, 195
61	25, 67, 103
62	118, 131, 145
63	3, 17, 119
64	8, 239, 253
65	9, 111, 125
66	24, 189, 231
67	25, 61, 103
68	12, 207, 221
69	13, 79, 93
70	28, 157, 199
71	29, 71
72	72, 237
73	73, 109
74	88, 173, 229
75	45, 89, 101
76	76, 205
77	77
78	92, 141, 197
79	13, 69, 93
80	10, 175, 245
81	11, 47, 117
82	26, 167, 181
83	27, 39, 53
84	14, 143, 213
85	15, 85
86	30, 135, 149
87	7, 21, 31
88	74, 173, 229
89	45, 75, 101
90	90, 165
91	37, 91
92	78, 141, 197
93	13, 69, 79
94	94, 133
95	5, 95
96	40, 235, 249
97	41, 107, 121
98	56, 185, 227
99	57, 99
100	44, 203, 217
101	45, 75, 89
102	60, 153, 195
103	25, 61, 67
104	104, 233
105	105
106	120, 169, 225
107	41, 97, 121
108	108, 201
109	73, 109
110	124, 137, 193
111	9, 65, 125
112	42, 171, 241
113	43, 113
114	58, 163, 177
115	35, 49, 59
116	46, 139, 209
117	11, 47, 81
118	62, 131, 145
119	3, 17, 63
120	106, 169, 225
121	41, 97, 107
122	122, 161
123	33, 123
124	110, 137, 193
125	9, 65, 111
126	126, 129
127	1, 127
128	128, 254
129	126, 129
130	144, 190, 246
131	62, 118, 145
132	132, 222
133	94, 133
134	148, 158, 214
135	30, 86, 149
136	192, 238, 252
137	110, 124, 193
138	174, 208, 244
139	46, 116, 209
140	196, 206, 220
141	78, 92, 197
142	142, 212
143	14, 84, 213
144	130, 190, 246
145	62, 118, 131
146	146, 182
147	54, 147
148	134, 158, 214
149	30, 86, 135
150	150
151	22, 151
152	188, 194, 230
153	60, 102, 195
154	166, 180, 210
155	38, 52, 211
156	156, 198
157	28, 70, 199
158	134, 148, 214
159	6, 20, 215
160	160, 250
161	122, 161
162	176, 186, 242
163	58, 114, 177
164	164, 218
165	90, 165
166	154, 180, 210
167	26, 82, 181
168	224, 234, 248
169	106, 120, 225
170	170, 240
171	42, 112, 241
172	202, 216, 228
173	74, 88, 229
174	138, 208, 244
175	10, 80, 245
176	162, 186, 242
177	58, 114, 163
178	178
179	50, 179
180	154, 166, 210
181	26, 82, 167
182	146, 182
183	18, 183
184	184, 226
185	56, 98, 227
186	162, 176, 242
187	34, 48, 243
188	152, 194, 230
189	24, 66, 231
190	130, 144, 246
191	2, 16, 247
192	136, 238, 252
193	110, 124, 137
194	152, 188, 230
195	60, 102, 153
196	140, 206, 220
197	78, 92, 141
198	156, 198
199	28, 70, 157
200	200, 236
201	108, 201
202	172, 216, 228
203	44, 100, 217
204	204
205	76, 205
206	140, 196, 220
207	12, 68, 221
208	138, 174, 244
209	46, 116, 139
210	154, 166, 180
211	38, 52, 155
212	142, 212
213	14, 84, 143
214	134, 148, 158
215	6, 20, 159
216	172, 202, 228
217	44, 100, 203
218	164, 218
219	36, 219
220	140, 196, 206
221	12, 68, 207
222	132, 222
223	4, 223
224	168, 234, 248
225	106, 120, 169
226	184, 226
227	56, 98, 185
228	172, 202, 216
229	74, 88, 173

CA	conjugate CA	CA	conjugate CA	CA	conjugate CA	CA	conjugate CA	CA	conjugate CA
230	152, 188, 194	234	168, 224, 248	238	136, 192, 252	242	162, 176, 186	246	130, 144, 190
231	24, 66, 189	235	40, 96, 249	239	8, 64, 253	243	34, 48, 187	247	2, 16, 191
232	232	236	200, 236	240	170, 240	244	138, 174, 208	248	168, 224, 234
233	104, 233	237	72, 237	241	42, 112, 171	245	10, 80, 175	249	40, 96, 235

The following three tables show the cardinalities of $\widetilde{\mathrm{Per}}_{q,p}(w_c)$, $\mathrm{Pre}_{q,p}(w_c)^*$ and $\mathrm{Pre}_{q,p}(w_c)\setminus \mathrm{Pre}_{q,p}(w_c)^*$, respectively. The entry in the first column in each of these tables is the representative c of the equivalence classes mentioned above and the following columns are indexed by the pairs q, p with $q + p \leq 6$. The cardinalities $|\mathbb{N}|$ and $|\mathbb{R}|$ are represented by "N" and "R".

A.1. Cardinality of $\widetilde{\mathbf{Per}}_{q,p}(w_c)$

CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
0	1	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	3	4	5	6	R	0	0	R	R	0	0	0	0	0	0	0	0	0	0
3	0	2	3	0	5	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0
4	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	R	R	0	0	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	0	0
6	N	0	0	N	N	N	R	0	0	R	R	R	0	0	R	R	0	0	R	0	R
7	2	N	0	0	5	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0
8	1	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0
9	0	N	9	0	R	0	0	R	0	0	R	0	R	0	0	0	R	0	0	R	0
10	1	0	3	8	10	12	N	0	R	N	R	0	0	0	0	0	0	0	0	0	0
11	0	2	N	4	10	N	0	N	R	R	R	0	0	R	R	0	0	R	0	0	0
12	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	R	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R
14	N	0	0	4	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N
15	2	2	6	12	30	54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	R	0	R	0	R	R	R	0	R	0	R	R	0	R	R	R	0	R	R	R
19	0	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0
22	3	4	0	R	R	R	R	0	0	R	R	R	0	0	R	R	0	0	R	0	R
23	2	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0
24	1	0	3	4	5	6	N	0	R	R	R	N	0	0	0	0	0	0	0	0	0
25	0	2	3	N	0	0	0	N	N	N	0	0	N	0	N	0	N	0	0	R	0
26	1	6	6	24	N	126	N	6	N	R	R	4	R	R	N	N	24	N	N	48	R
27	0	2	6	0	5	6	0	N	0	0	0	0	0	0	0	0	0	0	0	0	0
28	N	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R
29	2	R	0	0	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	0	0
30	3	0	12	28	45	84	1	0	0	14	5	3	0	0	21	3	0	0	6	0	6
32	1	2	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
33	0	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0
34	1	2	3	4	10	12	N	0	R	R	R	0	0	0	0	0	0	0	0	0	0
35	0	N	3	0	N	0	0	N	R	0	R	0	N	R	0	0	N	R	0	N	0
36	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0
37	3	R	0	0	0	18	3	R	0	0	0	0	R	0	0	0	R	0	0	R	0
38	1	0	6	8	10	18	N	0	0	R	R	N	0	0	0	0	0	0	0	0	0
40	1	2	3	0	5	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
41	0	N	N	0	5	30	0	R	R	0	0	0	R	0	0	0	R	0	0	R	0
42	1	2	6	8	20	30	N	0	0	R	R	0	0	0	0	0	0	0	0	0	0
43	0	N	N	4	N	N	0	N	N	R	N	0	N	N	R	0	N	N	0	N	0
44	R	0	3	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
45	3	2	3	8	55	0	0	2	0	0	0	0	4	0	0	0	4	0	0	4	0
46	1	0	3	4	5	6	N	0	R	R	R	N	0	0	0	0	0	0	0	0	0
50	1	R	0	0	0	0	1	R	0	0	0	0	R	0	0	0	R	0	0	R	0
51	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
54	1	4	0	R	0	24	1	0	0	R	0	R	0	0	R	R	0	0	R	0	R
56	1	2	N	4	N	N	1	R	R	R	R	0	R	R	0	0	R	R	0	R	0
57	0	4	N	0	N	0	0	R	N	0	R	0	R	N	0	0	R	N	0	R	0
58	1	2	3	0	N	0	1	R	R	0	R	0	R	R	0	0	R	R	0	R	0
60	1	0	3	0	15	12	1	0	3	0	15	2	0	6	0	4	0	12	8	0	16
62	1	0	R	0	5	0	1	0	R	0	N	R	0	R	0	0	0	R	0	0	0
72	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0
73	R	R	R	0	R	R	R	R	R	0	R	R	R	R	0	R	R	R	0	R	0
74	N	0	N	N	N	N	N	0	R	N	R	N	0	R	N	N	0	R	N	0	N
76	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0
77	R	2	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
78	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
90	4	12	60	240	1020	4020	12	36	180	720	3060	48	144	720	2880	192	576	2880	768	2304	3072
94	R	R	R	24	0	R	R	R	R	R	0	R	R	R	N	R	R	R	R	R	R
104	R	6	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
105	4	12	60	240	1020	4020	12	36	180	720	3060	48	144	720	2880	192	576	2880	768	2304	3072
106	1	6	12	4	10	12	1	4	0	0	0	3	8	0	0	6	8	0	6	8	6
108	R	R	0	0	0	0	R	R	0	0	0	R	R	0	0	0	0	0	0	0	0
110	1	N	N	0	R	0	1	N	R	0	R	R	R	R	0	R	R	R	N	R	R
122	1	R	0	R	0	R	1	R	0	R	0	R	R	0	R	R	R	0	R	R	R
126	1	R	0	R	0	R	1	R	0	R	0	R	R	0	R	R	R	0	R	R	R
128	2	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
130	2	0	3	4	5	6	R	0	0	R	R	0	0	0	R	0	0	0	0	0	0
132	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
134	N	0	0	N	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N
136	N	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
138	2	0	3	8	15	24	N	0	R	R	R	0	0	0	0	0	0	0	0	0	0
140	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
142	N	0	0	4	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N

Continued on next page

CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
146	2	R	0	R	0	R	R	R	0	R	0	R	R	0	R	R	R	0	R	R	R
150	4	12	60	240	1020	4020	12	36	180	720	3060	48	144	720	2880	192	576	2880	768	2304	3072
152	N	0	N	N	N	N	N	0	N	R	R	N	0	N	R	N	0	N	N	0	N
154	2	6	6	32	30	162	2	6	0	24	0	4	12	0	48	8	24	0	16	48	32
156	N	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R
160	2	2	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R
162	2	2	3	4	10	12	0	R	R	R	R	0	R	0	0	0	R	0	0	R	0
164	R	6	0	24	0	120	R	6	0	24	0	R	12	0	48	R	24	0	R	48	R
168	N	2	3	4	10	12	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R
170	2	2	6	12	30	54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
172	R	0	R	R	R	R	R	0	R	R	R	R	0	R	R	R	0	R	R	0	R
178	2	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0
184	N	2	N	N	N	N	N	R	N	N	R	N	R	N	N	N	R	N	N	R	N
200	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0
204	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
232	R	2	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R

A.2. Cardinality of $\mathbf{Pre}_{q,p}(w_c)^*$

CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
0	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
1	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
2	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
3	0	0	0	0	0	0	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
4	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
5	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
6	3	3	3	11	23	27	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
7	0	4	0	4	0	4	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
8	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
9	0	6	0	6	R	6	0	R	0	R	R	0	R	0	R	0	R	0	0	R	0
10	0	0	0	0	0	0	N	N	R	N	R	N	N	R	N	N	N	R	N	N	N
11	0	0	9	0	0	21	0	3	R	R	R	0	3	R	R	0	3	R	0	3	0
12	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
13	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
14	3	3	3	3	18	27	N	N	N	R	R	N	N	N	R	N	N	N	N	N	N
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	R	0	R	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
19	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
22	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
23	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
24	0	0	0	0	0	0	4	4	R	R	R	N	N	R	R	N	N	R	N	N	N
25	0	0	0	14	0	0	0	3	6	N	0	0	8	6	N	0	N	6	0	R	0
26	0	0	0	0	25	0	3	3	15	R	R	3	R	R	R	15	R	R	27	R	R
27	0	0	0	0	0	0	0	3	0	3	0	0	3	0	3	0	3	0	0	3	0
28	3	R	3	R	3	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
29	0	R	0	R	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
33	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
34	0	0	0	0	0	0	2	2	R	R	R	2	2	R	R	2	2	R	2	2	2
35	0	4	0	4	10	4	0	4	R	4	R	0	4	R	4	0	4	R	0	4	0
36	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
37	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
38	0	0	0	0	0	0	2	2	2	R	R	N	N	N	R	N	N	N	N	N	N
40	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
41	0	6	12	6	0	18	0	R	R	R	0	0	R	R	R	0	R	R	0	R	0
42	0	0	0	0	0	0	2	2	2	R	R	2	2	2	R	2	2	2	2	2	2
43	0	4	12	4	20	40	0	4	12	R	20	0	4	12	R	0	4	12	0	4	0
44	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
45	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	2	2	R	R	R	N	N	R	R	N	N	R	N	N	N
50	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
51	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
54	0	0	0	R	0	0	0	0	0	R	0	R	R	R	R	R	R	R	R	R	R
56	0	0	6	0	10	12	0	R	R	R	R	0	R	R	R	0	R	R	0	R	0
57	0	0	6	0	10	6	0	R	6	R	R	0	R	6	R	0	R	6	0	R	0
58	0	0	0	0	10	0	0	R	R	R	R	0	R	R	R	0	R	R	0	R	0
60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
62	0	0	R	0	0	R	0	0	R	0	10	R	R	R	R	R	R	R	R	R	R
72	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R

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CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
73	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
74	4	4	7	8	9	25	N	N	R	N	R	N	N	R	N	N	N	R	N	N	N
76	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
77	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
78	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
90	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
94	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
104	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
105	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
106	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
108	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
110	0	5	10	5	R	14	0	N	R	N	R	R	R	R	R	R	R	R	R	R	R
122	0	R	0	R	0	R	0	R	0	R	0	R	R	R	R	R	R	R	R	R	R
126	0	R	0	R	0	R	0	R	0	R	0	R	R	R	R	R	R	R	R	R	R
128	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
130	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
132	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
134	3	3	3	11	23	27	N	N	N	R	R	N	N	N	R	N	N	N	N	N	N
136	2	2	2	2	2	2	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
138	0	0	0	0	0	0	3	3	R	R	R	3	3	R	R	3	3	R	3	3	3
140	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
142	4	4	4	4	24	40	4	4	4	R	R	4	4	4	R	4	4	4	4	4	4
146	0	R	0	R	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
150	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
152	2	2	5	6	7	11	N	N	N	R	R	N	N	N	R	N	N	N	N	N	N
154	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
156	N	R	N	R	N	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
160	0	0	0	0	0	0	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
162	0	0	0	0	0	0	0	R	R	R	R	0	R	R	R	0	R	R	0	R	0
164	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
168	2	2	2	2	2	2	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
170	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
172	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
178	0	R	0	R	0	R	0	R	0	R	0	0	R	0	R	0	R	0	0	R	0
184	2	2	8	10	22	32	2	R	8	R	R	2	R	8	R	2	R	8	2	R	2
200	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
204	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R
232	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R	R

A.3. Cardinality of $\mathbf{Pre}_{q,p}(w_c) \setminus \mathbf{Pre}_{q,p}(w_c)^*$

CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	4	5	6	10	0	0	3	0	0	0	0	3	0	0	0	3	0	0	0
3	0	2	3	2	5	5	0	1	3	1	5	0	1	3	1	0	1	3	0	1	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	2	N	2	N	7	N	2	N	2	N	7	2	0	2	0	2	0	2	2	0	2
8	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	N	9	N	0	N	0	0	9	0	0	0	0	9	0	0	0	9	0	0	0
10	1	1	4	9	11	16	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	0	2	N	6	10	N	0	N	0	N	0	0	N	0	N	0	N	0	0	N	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	2	4	8	16	32	64	2	4	8	16	32	2	4	8	16	2	4	8	2	4	2
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	3	7	3	7	3	31	3	7	3	7	3	3	7	3	7	3	7	3	3	7	3
23	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
24	1	1	4	5	6	10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
25	0	2	3	N	0	5	0	N	N	N	0	0	N	N	N	0	N	N	0	N	0
26	1	7	7	31	N	139	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
27	0	2	6	2	5	14	0	N	6	N	5	0	N	6	N	0	N	6	0	N	0
28	N	0	N	0	N	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
29	2	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
30	3	3	15	31	48	99	4	4	16	46	54	7	7	19	70	10	10	22	16	16	22
32	1	3	1	3	1	3	0	2	0	2	0	0	2	0	2	0	2	0	0	2	0
33	0	2	0	2	0	2	0	2	0	2	0	0	2	0	2	0	2	0	0	2	0

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CA	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1
34	1	3	4	7	11	18	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
35	0	N	3	N	N	N	0	N	0	N	0	0	N	0	N	0	N	0	0	N	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	3	3	3	3	3	21	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
38	1	1	7	9	11	25	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
40	1	3	4	3	6	6	0	2	3	2	5	0	2	3	2	0	2	3	0	2	0
41	0	N	N	N	5	N	0	N	0	N	5	0	N	0	N	0	N	0	0	N	0
42	1	3	7	11	21	39	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
43	0	N	N	N	N	N	0	N	N	N	N	0	N	N	N	0	N	N	0	N	0
44	0	0	3	0	0	3	0	0	3	0	0	0	0	3	0	0	0	3	0	0	0
45	3	5	6	13	58	8	3	7	6	15	58	3	11	6	19	3	15	6	3	19	3
46	1	1	4	5	6	10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
50	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
51	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
54	1	5	1	5	1	29	2	6	2	6	2	2	6	2	6	2	6	2	2	6	2
56	1	3	N	7	N	N	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
57	0	4	N	4	N	N	0	2	N	2	0	0	2	N	2	0	2	N	0	2	0
58	1	3	4	3	N	6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
60	1	1	4	1	16	16	2	2	8	2	32	4	4	16	4	8	8	32	16	16	32
62	1	1	1	1	6	1	2	2	2	2	N	2	2	2	2	2	2	2	2	2	2
72	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
73	0	2	0	2	0	26	0	2	0	2	0	0	2	0	2	0	2	0	0	2	0
74	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
76	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
77	0	2	0	2	0	2	0	2	0	2	0	0	2	0	2	0	2	0	0	2	0
78	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
90	4	16	64	256	1024	4096	16	64	256	1024	4096	64	256	1024	4096	256	1024	4096	1024	4096	4096
94	1	1	1	25	1	121	2	2	2	26	2	2	2	2	N	10	10	10	26	26	42
104	0	6	0	6	0	6	0	6	0	6	0	0	6	0	6	0	6	0	0	6	0
105	4	16	64	256	1024	4096	16	64	256	1024	4096	64	256	1024	4096	256	1024	4096	1024	4096	4096
106	1	7	13	11	11	31	2	12	14	16	12	5	23	17	27	11	37	23	17	51	23
108	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
110	1	N	N	N	0	N	2	N	0	N	0	2	N	0	N	2	N	0	N	N	N
122	1	3	1	3	1	3	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
126	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
128	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
130	2	2	5	6	7	11	1	1	4	1	1	1	1	4	1	1	1	4	1	1	1
132	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
134	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
136	N	N	N	N	N	N	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
138	2	2	5	10	17	29	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
140	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
142	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
146	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
150	4	16	64	256	1024	4096	16	64	256	1024	4096	64	256	1024	4096	256	1024	4096	1024	4096	4096
152	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
154	2	8	8	40	32	176	4	16	10	72	34	8	32	14	136	16	64	22	32	128	64
156	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
160	2	4	2	4	2	4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
162	2	4	5	8	12	19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
164	1	7	1	31	1	127	3	15	3	63	3	7	31	7	127	15	63	15	31	127	63
168	N	N	N	N	N	N	0	2	3	6	10	0	2	3	6	0	2	3	0	2	0
170	2	4	8	16	32	64	2	4	8	16	32	2	4	8	16	2	4	8	2	4	2
172	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
178	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
184	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
200	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
204	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
232	0	2	0	2	0	2	0	2	0	2	0	0	2	0	2	0	2	0	0	2	0

A.4. Classification

The last table groups all those equivalence classes together which can not be differentiated by the data in the previous three tables. For space reasons we do not repeat all columns but show only a subset sufficient for the classification. The first column shows the equivalence classes which were grouped together. The next 21 columns show $|\widetilde{\mathrm{Per}}_{q,p}(w_c)|$ for $q \in \{0,\dots,5\}, p \in \{1,\dots,6\}, p+q \leq 6$. The final 12 columns labeled Dk and Ck for $k \in \{1,\dots,6\}$ show $|\mathrm{Pre}_{0,k}(w_c)^*| = |\mathrm{Per}_k(w_c)^*|$ and $|\mathrm{Per}_k(w_c) \setminus \mathrm{Per}_k(w_c)^*|$.

CAs	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1	D1	C1	D2	C2	D3	C3	D4	C4	D5	C5	D6	C6
0	1	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1
1	0	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	R	0	0	0	R	0	0	0	R	0
2	1	0	3	4	5	6	R	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	4	0	5	0	6	0	10
3	0	2	3	0	5	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	3	0	2	0	5	0	5
5	R	R	0	0	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	0	0	R	0	R	0	R	0	R	0	R	0	R	0
6	N	0	0	N	N	N	R	0	0	R	R	R	0	0	R	R	0	0	R	0	R	3	N	3	N	3	N	11	N	23	N	27	N
7	2	N	0	0	5	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	0	2	4	N	0	2	4	N	0	7	4	N
8	1	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1
9	0	N	9	0	R	0	0	R	0	0	R	0	R	0	0	0	R	0	0	R	0	0	0	6	N	0	9	6	N	R	0	6	N
10	1	0	3	8	10	12	N	0	R	N	R	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	4	0	9	0	11	0	16
11	0	2	N	4	10	N	0	N	R	R	R	0	0	R	R	0	0	R	0	0	0	0	0	0	2	9	N	0	6	0	10	21	N
13	R	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R	R	0	R	0	R	0	R	0	R	0	R	0
14	N	0	0	4	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N	3	N	3	N	3	N	3	N	18	N	27	N
18	1	R	0	R	0	R	R	R	0	R	0	R	R	0	R	R	R	0	R	R	R	0	1	R	1	0	1	R	1	0	1	R	1
19	0	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	R	0	0	0	R	0	0	0	R	0
22	3	4	0	R	R	R	R	0	0	R	R	R	0	0	R	R	0	0	R	0	R	0	3	0	7	0	3	R	7	R	3	R	31
25	0	2	3	N	0	0	0	N	N	N	0	0	N	0	N	0	N	0	0	R	0	0	0	0	2	0	3	14	N	0	0	0	5
26	1	6	6	24	N	126	N	6	N	R	R	4	R	R	N	N	24	N	N	48	R	0	1	0	7	0	7	0	31	25	N	0	139
27	0	2	6	0	5	6	0	N	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	6	0	2	0	5	0	14
28	N	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R	3	N	R	0	3	N	R	0	3	N	R	0
29	2	R	0	0	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	R	0	0	2	R	0	0	2	R	0
30	3	0	12	28	45	84	1	0	0	14	5	3	0	0	21	3	0	0	6	0	6	0	3	0	3	0	15	0	31	0	48	0	99
32	1	2	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	0	1	0	3	0	1	0	3	0	1	0	3
33	0	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	0	0	R	2	0	0	R	2	0	0	R	2
34	1	2	3	4	10	12	N	0	R	R	R	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	7	0	11	0	18
35	0	N	3	0	N	0	0	N	R	0	R	0	N	R	0	0	N	R	0	N	0	0	0	4	N	0	3	4	N	10	N	4	N
37	3	R	0	0	0	18	3	R	0	0	0	0	R	0	0	0	R	0	0	R	0	0	3	R	3	0	3	R	3	0	3	R	21
38	1	0	6	8	10	18	N	0	0	R	R	N	0	0	0	0	0	0	0	0	0	0	1	0	1	0	7	0	9	0	11	0	25
40	1	2	3	0	5	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	0	1	0	3	0	4	0	3	0	6	0	6
41	0	N	N	0	5	30	0	R	R	0	0	0	R	0	0	0	R	0	0	R	0	0	0	6	N	12	N	6	N	0	5	18	N
42	1	2	6	8	20	30	N	0	0	R	R	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	7	0	11	0	21	0	39
43	0	N	N	4	N	N	0	N	N	R	N	0	N	N	R	0	N	N	0	N	0	0	0	4	N	12	N	4	N	20	N	40	N
44	R	0	3	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	R	0	R	0	R	3	R	0	R	0	R	3
45	3	2	3	8	55	0	0	2	0	0	0	0	4	0	0	0	4	0	0	4	0	0	3	0	5	0	6	0	13	0	58	0	8
50	1	R	0	0	0	0	1	R	0	0	0	0	R	0	0	0	R	0	0	R	0	0	1	R	1	0	1	R	1	0	1	R	1
51	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	R	0	0	0	R	0	0	0	R	0
54	1	4	0	R	0	24	1	0	0	R	0	R	0	0	R	R	0	0	R	0	R	0	1	0	5	0	1	R	5	0	1	0	29
56	1	2	N	4	N	N	1	R	R	R	R	0	R	R	0	0	R	R	0	R	0	0	1	0	3	6	N	0	7	10	N	12	N
57	0	4	N	0	N	0	0	R	N	0	R	0	R	N	0	0	R	N	0	R	0	0	0	0	4	6	N	0	4	10	N	6	N
58	1	2	3	0	N	0	1	R	R	0	R	0	R	R	0	0	R	R	0	R	0	0	1	0	3	0	4	0	3	10	N	0	6
60	1	0	3	0	15	12	1	0	3	0	15	2	0	6	0	4	0	12	8	0	16	0	1	0	1	0	4	0	1	0	16	0	16
62	1	0	R	0	5	0	1	0	R	0	N	R	0	R	0	0	0	R	0	0	0	0	1	0	1	R	1	0	1	0	6	R	1
73	R	R	R	0	R	R	R	R	R	0	R	R	R	R	0	R	R	R	0	R	0	R	0	R	2	R	0	R	2	R	0	R	26
74	N	0	N	N	N	N	N	0	R	N	R	N	0	R	N	N	0	R	N	0	N	4	N	4	N	7	N	8	N	9	N	25	N
94	R	R	R	24	0	R	R	R	R	R	0	R	R	R	N	R	R	R	R	R	R	R	1	R	1	R	1	R	25	R	1	R	121
104	R	6	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	R	0	R	6	R	0	R	6	R	0	R	6
106	1	6	12	4	10	12	1	4	0	0	0	3	8	0	0	6	8	0	6	8	6	0	1	0	7	0	13	0	11	0	11	0	31

Continued on next page

CAs	0,1	0,2	0,3	0,4	0,5	0,6	1,1	1,2	1,3	1,4	1,5	2,1	2,2	2,3	2,4	3,1	3,2	3,3	4,1	4,2	5,1	D1	C1	D2	C2	D3	C3	D4	C4	D5	C5	D6	C6
108	R	R	0	0	0	0	R	R	0	0	0	R	R	0	0	0	0	0	0	0	0	R	0	R	0	R	0	R	0	R	0	R	0
110	1	N	N	0	R	0	1	N	R	0	R	R	R	R	0	R	R	R	N	R	R	0	1	5	N	10	N	5	N	R	0	14	N
122	1	R	0	R	0	R	1	R	0	R	0	R	R	0	R	R	R	0	R	R	R	0	1	R	3	0	1	R	3	0	1	R	3
126	1	R	0	R	0	R	1	R	0	R	0	R	R	0	R	R	R	0	R	R	R	0	1	R	1	0	1	R	1	0	1	R	1
128	2	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	0	2	0	2	0	2	0	2	0	2	0	2
130	2	0	3	4	5	6	R	0	0	R	R	0	0	0	R	0	0	0	0	0	0	0	2	0	2	0	5	0	6	0	7	0	11
132	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	R	1	R	1	R	1	R	1	R	1	R	1
134	N	0	0	N	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N	3	N	3	N	3	N	11	N	23	N	27	N
136	N	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	2	N	2	N	2	N	2	N	2	N	2	N
138	2	0	3	8	15	24	N	0	R	R	R	0	0	0	0	0	0	0	0	0	0	0	2	0	2	0	5	0	10	0	17	0	29
142	N	0	0	4	N	N	N	0	0	R	R	N	0	0	R	N	0	0	N	0	N	4	N	4	N	4	N	4	N	24	N	40	N
146	2	R	0	R	0	R	R	R	0	R	0	R	R	0	R	R	R	0	R	R	R	0	2	R	2	0	2	R	2	0	2	R	2
152	N	0	N	N	N	N	N	0	N	R	R	N	0	N	R	N	0	N	N	0	N	2	N	2	N	5	N	6	N	7	N	11	N
154	2	6	6	32	30	162	2	6	0	24	0	4	12	0	48	8	24	0	16	48	32	0	2	0	8	0	8	0	40	0	32	0	176
156	N	R	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R	N	N	R	N	N	N	R	N	N	N	R	N
160	2	2	0	0	0	0	R	R	0	0	0	R	R	0	0	R	R	0	R	R	R	0	2	0	4	0	2	0	4	0	2	0	4
162	2	2	3	4	10	12	0	R	R	R	R	0	R	0	0	0	R	0	0	R	0	0	2	0	4	0	5	0	8	0	12	0	19
164	R	6	0	24	0	120	R	6	0	24	0	R	12	0	48	R	24	0	R	48	R	R	1	R	7	R	1	R	31	R	1	R	127
168	N	2	3	4	10	12	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	2	N	2	N	2	N	2	N	2	N	2	N
172	R	0	R	R	R	R	R	0	R	R	R	R	0	R	R	R	0	R	R	0	R	R	0	R	0	R	0	R	0	R	0	R	0
184	N	2	N	N	N	N	N	R	N	N	R	N	R	N	N	N	R	N	N	R	N	2	N	2	N	8	N	10	N	22	N	32	N
204	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	R	0	R	0	R	0	R	0	R	0	R	0
15, 170	2	2	6	12	30	54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	4	0	8	0	16	0	32	0	64
23, 178	2	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	0	2	R	2	0	2	R	2	0	2	R	2
24, 46	1	0	3	4	5	6	N	0	R	R	R	N	0	0	0	0	0	0	0	0	0	0	1	0	1	0	4	0	5	0	6	0	10
36, 72	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	0	0	0	0	0	0	R	0	R	0	R	0	R	0	R	0	R	0
77, 232	R	2	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	R	0	R	2	R	0	R	2	R	0	R	2
78, 140	R	0	0	0	0	0	R	0	0	0	0	R	0	0	0	R	0	0	R	0	R	R	0	R	0	R	0	R	0	R	0	R	0
90, 105, 150	4	12	60	240	1020	4020	12	36	180	720	3060	48	144	720	2880	192	576	2880	768	2304	3072	0	4	0	16	0	64	0	256	0	1024	0	4096
4, 12, 76, 200	R	0	0	0	0	0	R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	R	0	R	0	R	0	R	0	R	0	R	0

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List of Symbols

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