

# **Polynomial growth of concept lattices, canonical bases and generators: extremal set theory in Formal Concept Analysis**

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“Although this may seem a paradox, all exact science is dominated by the idea of approximation.”

——— Bertrand Russell



## Preface

We prove that there exist three distinct, comprehensive classes of (formal) contexts with polynomially many concepts. Namely: contexts which are nowhere dense, of bounded breadth or highly convex. Already present in G. Birkhoff's classic monograph [14] is the notion of breadth of a lattice; it equals the number of atoms of a largest boolean suborder. Even though it is natural to define the breadth of a context as being that of its concept lattice, this idea had not been exploited before. We do this and establish many equivalences. Amongst them, it is shown that the breadth of a context equals the size of its largest minimal generator, its largest contranominal-scale subcontext, as well as the Vapnik-Chervonenkis dimension of both its system of extents and of intents.

The polynomiality of the aforementioned classes is proven via upper bounds (also known as majorants) for the number of maximal bipartite cliques in bipartite graphs. These are results obtained by various authors in the last decades. The fact that they yield statements about formal contexts is a reward for investigating how two established fields interact, specifically Formal Concept Analysis (FCA) and graph theory.

We improve considerably the breadth bound. Such improvement is twofold: besides giving a much tighter expression, we prove that it limits the number of minimal generators. This is strictly more general than upper bounding the quantity of concepts. Indeed, it automatically implies a bound on these, as well as on the number of proper premises. A corollary is that this improved result is a bound for the number of implications in the canonical basis too. With respect to the quantity of concepts, this sharper majorant is shown to be best possible. Such fact is established by constructing contexts whose concept lattices exhibit exactly that many elements. These structures are termed, respectively, extremal contexts and extremal lattices. The usual procedure of taking the standard context allows one to work interchangeably with either one of these two extremal structures.

Extremal lattices are equivalently defined as finite lattices which have as many elements as possible, under the condition that they obey two upper limits: one for its number of join-irreducibles, other for its breadth. Subsequently, these structures are characterized in two ways.

Our first characterization is done using the lattice perspective. Initially, we construct extremal lattices by the iterated operation of finding smaller, extremal subsemilattices and duplicating their elements. Then, it is shown that every extremal lattice must be

obtained through a recursive application of this construction principle. A byproduct of this contribution is that extremal lattices are always meet-distributive. Despite the fact that this approach is revealing, the vicinity of its findings contains unanswered combinatorial questions which are relevant. Most notably, the number of meet-irreducibles of extremal lattices escapes from control when this construction is conducted.

Aiming to get a grip on the number of meet-irreducibles, we succeed at proving an alternative characterization of these structures. This second approach is based on implication logic, and exposes an interesting link between number of proper premises, pseudo-extents and concepts. A guiding idea in this scenario is to use implications to construct lattices. It turns out that constructing extremal structures with this method is simpler, in the sense that a recursive application of the construction principle is not needed. Moreover, we obtain with ease a general, explicit formula for the Whitney numbers of extremal lattices. This reveals that they are unimodal, too. Like the first, this second construction method is shown to be characteristic. A particular case of the construction is able to force - with precision - a high number of (in the sense of “exponentially many”) meet-irreducibles.

Such occasional explosion of meet-irreducibles motivates a generalization of the notion of extremal lattices. This is done by means of considering a more refined partition of the class of all finite lattices. In this finer-grained setting, each extremal class consists of lattices with bounded breadth, number of join irreducibles *and meet-irreducibles* as well. The generalized problem of finding the maximum number of concepts reveals itself to be challenging. Instead of attempting to classify these structures completely, we pose questions inspired by Turán’s seminal result in extremal combinatorics. Most prominently: do extremal lattices (in this more general sense) have the maximum permitted breadth?

We show a general statement in this setting: for every choice of limits (breadth, number of join-irreducibles and meet-irreducibles), we produce some extremal lattice with the maximum permitted breadth. The tools which underpin all the intuitions in this scenario are hypergraphs and exact set covers. In a rather unexpected, but interesting turn of events, we obtain for free a simple and interesting theorem about the general existence of “rich” subcontexts. Precisely: every context contains an object/attribute pair which, after removed, results in a context with at least half the original number of concepts.

**Keywords:** Concept lattices, breadth, maximal bicliques, upper bounds, canonical bases, minimal generators, contranominal scales, Vapnik-Chervonenkis dimension, shattered sets, proper premises

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## CHAPTER 1

# Introduction

Formal Concept Analysis was born in an effort to restructure lattice theory. This restructuring endeavor was driven by the perception that the connections of said theory should be reinvigorated. The epicenter of these ideas was a research group based in Darmstadt; temporally, it was the late nineteen seventies/early eighties. At that point in space and time, there was a clear mission to “interpret lattice theory as concretely as possible, thereby promoting better communication between lattice-theorists and potential users of the theory” [59].

Along with the new theory came a methodology - founded in solid principles - which revealed itself able to formally treat data and knowledge. Various application areas rapidly became candidates for the use of this incipient set of methods. They ranged from psychology to civil engineering, but also included computer science and even mathematics itself [32]. There exist notable features of FCA which have allured the attention of researchers from these fields. It is possible to devote a vast space to discuss as to why that is the case. Here, we shall concentrate on two reasons.

First, the visually rich representation of a concept lattice through its labeled diagram is one strong suit of FCA. Such visual aid endows a data analyst with a powerful knowledge representation and processing tool. For instance, it is possible to unfold regularities, discover exceptions and correct inconsistencies in data sets. These extraordinary capabilities play a key role in conceptual information systems, the applications of which have been extensively documented. Indeed, a large body of literature regarding systems of this nature has been consolidated in both international conferences devoted to FCA. Nevertheless, one concern in such systems is that the lattices may be excessively large. This naturally hinders their human readability and complicates the task of displaying such structures in a limited screen interface.

A second remarkable feature of FCA is its mathematically sound foundation. One distinguished goal of the present work is to provide well founded, theoretical justifications for the combinatorial explosion of some lattices. Although this occurrence is handled in a satisfactory way in practice, there exists no deep theoretical understanding as to why this phenomenon occurs.

In FCA, concepts are formal in the sense that they are mathematizations of units of thought. They approximate the philosophical notion which dates back to Aristotle. It is in this context that Bertrand Russell's quote in the beginning of this work should be contemplated. The notion of "approximation" is present in another aspect of this thesis. Specifically, we dedicate efforts to establish what are called upper bounds for interesting quantities in FCA. Many of these values are hard to compute, in a sense made precise by classic computational complexity terms. Not surprisingly, an expressive amount of research deals with discovering restricted classes of data in which such magnitudes can be efficiently computed or estimated.

The number of concepts in a lattice is perhaps an obvious choice, but certainly not the only one. It is equally motivating to investigate how the numbers of minimal generators, proper premises, as well as the number of pseudo-intents behave, just to cite a few examples.

It is our understanding that the interaction of Formal Concept Analysis with other areas of mathematics should be fostered. With this thesis, we intend to establish that extremal set theory has enough contact points and substance to offer. Originally, our investigations were focused not there, but rather on graph theory. The drift from one area to the other was quite natural, corroborating that a vivid interplay between all of these theories is a reality.

### Structure of this thesis and main results

The investigations in Chapter 2 are a natural continuation of the author's master dissertation [2]. In particular, we complement a survey conducted by Albano and do Lago in 2012. Most notably, we present a detailed argumentation with rigorous proofs that there exist three natural, abundant classes of contexts containing polynomially many formal concepts (Theorem 2.6.2).

A summary of combinatorial results regarding implication logic and canonical bases is presented in Chapter 3. In Chapter 4, we propose the use of a famous framework - due to Bollobás - in order to attack combinatorial problems in FCA. This is done by giving details of classic results pertaining to extremal graph theory. A positive effect is that we minimize the impedance between the languages of these two theories and provide an analogy which serves as a motivation for our investigations.

We prove in Chapter 5 that the breadth of a lattice is the maximum size of a contranominal-scale subcontext (Corollary 5.3.2). More generally, we show that contranominal scales, minimal generators and irredundant representations correspond biunivocally to one another (Lemma 5.3.1). Our Theorem 5.2.6 implies an improvement of the order of  $|G|^k \cdot (k-1)!/k$  to the previously best upper bound of [48]. A direct consequence is that we obtain a bound for the size of canonical bases, as exposed in Corollary 5.2.8. In Lemma 5.6.2, we prove that shattered sets correspond to contranominal-scale subcontexts as well, establishing a link with the famous lemma of Sauer and Shelah. As a result, we obtain that this classic result is sharp even for closure systems (Proposition 5.6.5).

A separate investigation is conducted in Chapter 6, where we characterize extremal

lattices through their implication logic. In particular, we obtain an explicit set of implications which generates an extremal lattice with an exponential number of meet-irreducibles (Theorem 6.2.1). We prove in Theorem 6.3.5 that every extremal lattice carries inside its implication logic smaller sets of implications, which are enough to construct a larger extremal lattice. This allows a simple and iterative construction of such structures. Theorems 6.3.6 and 6.4.5 prove that such construction principle is characteristic. We establish the validity of an elegant formula for the Whitney numbers of such lattices in Corollary 6.4.6.

In Chapter 7, we expand the notion of extremal lattices. This means that we consider finite lattices which have (upper) bounded breadth, as well as bounded number of join- and meet-irreducibles. In this more general setting, we prove the existence of extremal lattices with maximum permitted breadth and arbitrary parameters (Claim 7.1.3, the validity of which is proven in Corollary 7.7.8 of Theorem 7.7.6). This also establishes that every context has an object/attribute pair which, when removed, results in a context with at least half of the original number of concepts (Corollary 7.7.9).

The results described in Chapters 2, 6 and 7 were authored and published solely by the author in [5], [4] and [3]. Chapter 5 consists of results obtained in a joint work with Bogdan Chornomaz. These were published in the conference paper [7] and its journal version [6].



## Size of concept lattices

The primitive data model of Formal Concept Analysis is that of a formal context, which is unfolded into a concept lattice for further analysis. There are cases in which the number of concepts is exponentially larger than the number of objects, attributes or incidences which gave rise to the lattice. An example of this phenomenon is the boolean lattice  $B(k)$ , having  $2^k$  elements, the standard context of which is the  $k \times k$  *contranominal scale*  $\mathbb{N}^c(k)$ . For other families of contexts, the associated concept lattices grow much less vigorously (for example, polynomially). In Section 2.2, we make these claims precise and present examples of contexts featuring both types of behavior.

Instead of comparing the number of concepts with three different quantities - that of objects, attributes and incidences - one typically establishes one notion of *size* of a formal context. Very often, this is defined to be  $|G| \cdot |M|$ , i. e. , the product of the numbers of objects and attributes. Besides being more systematic, this approach offers the advantage of delivering assertions which are consistent with computational complexity theory. This matter is elucidated in Section 2.3, alongside with some elementary notation.

Both from the theoretical and the practical point of view, one common desire is to distinguish the formal contexts which have a large number of concepts from the ones which do not. Thus, the endeavor of “estimating” the size of a concept lattice as a function of its context becomes naturally motivated. A first approach, of course, could be that of counting exactly the number of concepts before attempting to determine them. However, the existence of a polynomial-time algorithm which performs this task is unlikely: it was shown by Kuznetsov [42] that such counting problem is #P-complete. In particular, the existence of said polynomial-time algorithm would imply  $P = NP$ . In Section 2.4, we survey known results from different areas which establish upper bounds (that is, majorants) for the number of concepts.

## 2.1. Fundamental terminology and facts

A (formal) context is a triple  $\mathbb{K} = (G, M, I)$  where  $G$  and  $M$  are sets and  $I \subseteq G \times M$  is an incidence relation. The elements of  $G$  are called *objects*, while those of  $M$  are called *attributes*. An element  $(g, m)$  of  $I$  is called an *incidence* and we adopt  $gIm$  to mean  $(g, m) \in I$ . Given a context, one defines the *derivation operators* through

$$\begin{aligned} A' &= \{m \in M \mid gIm \text{ for all } g \in A\} \\ B' &= \{g \in G \mid gIm \text{ for all } m \in B\}, \end{aligned}$$

for arbitrary  $A \subseteq G, B \subseteq M$ . A pair  $(A, B)$  with  $A \subseteq G$  and  $B \subseteq M$  satisfying  $A' = B$  and  $B' = A$  is called a (formal) *concept*. The sets  $A$  and  $B$  are called *extent* and *intent*, respectively. The symbols  $\text{Int}(\mathbb{K})$  and  $\text{Ext}(\mathbb{K})$  are employed, respectively, for the *system of intents* and the *system of extents* of a context  $\mathbb{K}$ .

The hierarchical ordering of concepts is

$$(A_1, B_1) \leq (A_2, B_2) \text{ if and only if } A_1 \subseteq A_2,$$

which is in turn equivalent to  $B_1 \supseteq B_2$ . The derivation of a single object will be denoted  $g'$ . This is a set of attributes, called *object-intent*. Dually, we write  $m'$  and define that this is an *attribute-extent* for any given  $m \in M$ . If  $g' = h'$  implies  $g = h$  for each pair of objects  $g, h$  and, correspondingly, from  $m' = n'$  follows that  $m = n$ , then we say that the context is *clarified*.

It will be sometimes the case that we take intersections of arbitrary families of attributes or objects. In light of this, we make a small but important observation. Suppose that a family of subsets of  $M$  is given, let us call it  $(B_t)_{t \in T}$ . If  $T$  is the empty set, then we *define* that the intersection  $\cap_{t \in T} B_t$  is  $M$ . The following elementary proposition is present in [35] and is common knowledge in the FCA community.

**2.1.1 Proposition** *For any family of sets of objects  $(A_t)_{t \in T}$ , it holds that  $(\cup_{t \in T} A_t)' = \cap_{t \in T} A_t'$ . An analogous statement holds for sets of attributes.*

*Proof* Let  $m$  be an arbitrary attribute. Then:

$$\begin{aligned} m \in (\cup_{t \in T} A_t)' &\Leftrightarrow gIm \text{ for each } g \in \cup_{t \in T} A_t \\ &\Leftrightarrow gIm \text{ for each } g \in A_t, \text{ for each } t \\ &\Leftrightarrow m \in A_t' \text{ for each } t \\ &\Leftrightarrow m \in \cap_{t \in T} A_t'. \end{aligned}$$

A useful consequence of this first proposition is the following, which says that the system of intents is the closure by intersections of all object-intents.

**2.1.2 Proposition**  $\text{Int } \mathbb{K} = \{\cap_{g \in S} g' \mid S \subseteq G\}$ , for any context  $\mathbb{K}$  with object set  $G$ .

An object  $g$  is said to be *reducible* if  $g'$  can be written as the intersection of object-intents associated to other objects. A context is called *object-reduced* if there are no reducible objects. *Full rows*, that is, objects with  $g' = M$  are always reducible. Reducible attributes, attribute-reduced, full columns and full contexts are defined accordingly. We say that a context is *reduced* if it is both object- and attribute-reduced.

The formula given by Proposition 2.1.2 is an explicit description for the system of intents. It will be of great use in the next section, where we survey a small list of notable contexts and their respective quantities of concepts.

## 2.2. Examples of large and moderately-sized concept lattices

In order to achieve objectivity and generality, one does not say that an *individual* context possesses “many” concepts. Instead, this classification is done exclusively to families. Given any definition of size of a context, say,  $s(\mathbb{K})$ , if the number of concepts of a family  $(\mathbb{K}_n)_{n \in \mathbb{N}}$  can be bounded from above by some fixed (i.e., not depending on  $n$ ) polynomial on  $s(\mathbb{K}_n)$ , then we say that such family has *moderately many* concepts. Otherwise, the growth of the number of concepts is superpolynomial (with respect to  $s(\cdot)$ ) and we agree that the family has a *large* or *abundant* number of concepts.

The *nominal scale* is the context  $([n], [n], =)$ , where  $[n] := \{1, 2, \dots, n\}$ . Its extents are all singletons of  $[n]$ , as well as the empty set and  $[n]$  itself, meaning that such context has a total of  $n + 2$  concepts. This quantity is linear in the number of objects (attributes, incidences). Note that it is essentially the square root of  $|G| \cdot |M|$ . In any case, this family has moderately many concepts. For another example, consider the *ordinal scale*  $([n], [n], <)$ . Its system of object-intents is  $\{\emptyset, \{1\}, \{1, 2\}, \dots, [n - 1]\}$ . Thus, its intents are precisely all intervals of the form  $\{1, \dots, a\}$ , as well as the empty set. This makes a total of  $n + 1$  concepts. Even though the numbers of concepts of these two examples are very similar, their number of incidences are very different. Indeed, the average number of attributes possessed by an object in an ordinal scale is roughly  $n/2$ , whereas in the nominal scale this number is precisely one. This shows that the number of incidences in a context may be high without causing a large number of concepts. Yet another example of context having moderately many extents is the *interordinal scale*  $([n], [n], \geq) | ([n], [n], \leq)$ . Its intents are precisely all the intervals of  $[n]$ . Notice that there are exactly  $n(n + 1)/2 + 1$  of them.

The three examples given above feature a moderate amount of concepts. In contrast, the *contranominal scale*  $\mathbb{N}^c(n) := ([n], [n], \neq)$  has as object intents the  $(n - 1)$ -element subsets of  $[n]$ . Clearly, any  $S \subseteq [n]$  may be written as the intersection of some (possibly empty) collection drawn from these subsets. With other words and in lattice-theoretic terms, these object intents form an *infimum-dense* set of the power-set of  $M$ . Therefore, the system of intents (extents) of  $\mathbb{N}^c(n)$  is precisely the power-set of  $[n]$  (for which we employ the symbol  $\mathcal{P}([n])$ ). It is elementary that  $\mathcal{P}([n])$  has  $2^n$  elements.

Another example is the *contra-path*, depicted in Figure 2.1. It possesses  $n - 1$  objects,  $n$

attributes and each object  $i$  has every attribute, except  $i$  and  $i + 1$ . We denote such a context by  $CP(n)$ .

$CP(n)$	1	2	3	...	$n - 1$	$n$
1			×	...	×	×
2	×			...	×	×
⋮				⋮	⋮	⋮
$n - 1$	×	×	×	...		

Figure 2.1.: Contra-path

The concept lattice of  $CP(6)$  is depicted in Figure 2.2. This structure will be revisited later in Section 7.3.

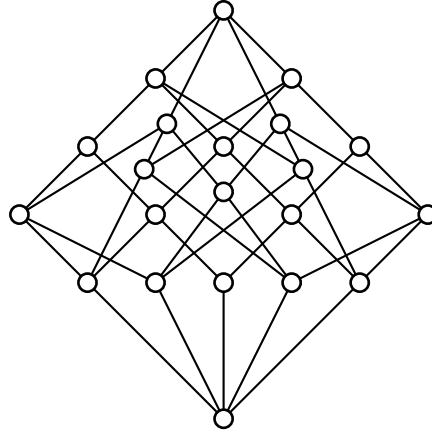


Figure 2.2.: Concept lattice of the contra-path with six attributes.

With a different language and combinatorial model, Austin and Guy showed in [12] that the contra-path has exactly  $\lfloor c\gamma^n \rfloor$  concepts, where  $c \approx 0.722$ ,  $\gamma$  is the real root of  $x^3 - 2x^2 + x - 1$  ( $\gamma \approx 1.754$ ) and the notation  $\lfloor \cdot \rfloor$  refers to the integer which is closest to a given real number. Note that  $CP(6)$  has 21 concepts and  $c \cdot \gamma^6 \approx 21,023$ .

For an example of abundantly many concepts which arises in a more algebraic scenario, consider the following. Let  $K^n$  be a vector space, where  $K = GF(q)$  is a finite field and take the context  $(K^n, K^n, \perp)$ , the incidences of which are given by

$$u \perp v \text{ if and only if } u \text{ is orthogonal to } v.$$

It is not hard to verify that the non-empty extents are the subspaces of  $K^n$ . The number of  $d$ -dimensional subspaces of  $K^n$  is given by  $\binom{n}{d}_q$ , where such symbol denotes the *Gaussian*

*binomial coefficient* and is defined as  $\frac{(q^n-1)(q^n-q)\dots(q^n-q^{n-d+1})}{(q-1)(q^2-1)\dots(q^d-1)}$ . For  $d = \lfloor n/2 \rfloor$ , the number of such subspaces is exponentially larger than  $q^n$ , i.e., the number of objects and attributes.

At this point, we have displayed six examples of families of contexts. Under any reasonable notion of *size* of a context, the first three examples have a polynomial amount of concepts, whereas the last three have a superpolynomial (actually even exponential) number. Observe that each context exemplified until now had the same - or almost the same - number of objects and attributes. If the number of objects and attributes of a context differ drastically, then it becomes vital to specify which notion of size is being used. This is illustrated by the following example: denote by  $\binom{[n]}{r}$  the set of subsets of  $[n]$  with precisely  $r$  elements and let  $n$  be an odd natural number. Take the context  $\left(\binom{[n]}{\lfloor n/2 \rfloor}, [n], \ni\right)$ . Note that it is reduced (since every object intent has the same size, and the same holds for attribute extents). It is easy to establish that its number of concepts is  $2^{n-1} + 1$ . Indeed, for any  $S \subseteq [n]$  with  $|S| \leq \lfloor n/2 \rfloor$ , one may take  $\{T \in \binom{[n]}{\lfloor n/2 \rfloor} \mid T \supseteq S\}$  and it is not hard to verify that the derivation of this collection is precisely  $S$ . In particular,  $S$  is an intent. Adding  $M$  to this counting and using the fact that  $n$  is odd gives the total number of  $2^{n-1} + 1$  intents. Note that this quantity is exponential in  $|M|$  but bounded from above and below by a polynomial in  $|G|$ . As a matter of fact, Stirling's formula yields the asymptotic behavior  $|G| \approx \frac{2^n}{\sqrt{\pi n}}$ . This conundrum is solved by introducing a definition for the size of a context which is indisputable, at least to some extent. This is to be done in the next section.

## 2.3. Notion of size of a formal context

If one seeks a notion of size for some structure and wants it to be compatible with computational complexity theory, then it must be essentially the length of some appropriate<sup>1</sup> representation of said structure [47]. The size of a lattice is defined to be its number of elements. This quantity is sufficient, up to a polynomial factor, to represent the elements of the lattice and its ordering relation. Alternatively, one could represent the *covering relation* instead: given two elements  $x$  and  $y$  of a lattice, we say that  $x$  is *covered by*  $y$  (or  $y$  is an *upper cover* of  $x$ ) if  $x \leq y$  and, for every  $z$  with  $x \leq z \leq y$ , it follows that  $z = x$  or  $z = y$ . The notation  $x < y$  is used to state that  $x$  is covered by  $y$ . The meanings of *lower cover* and  $y$  *covers*  $x$  are the expected. Most of the details about the specific representation are often irrelevant, because one typically compares complexity and algorithmic efficiency through the existence of some polynomial.

We defined the size of a lattice in a completely straightforward way. On the other hand, there are more possibilities to represent a context. Reasonable candidates for the length of such representation include, for instance,  $|G| + |M|$ ,  $|G| \cdot |M|$  and  $|I|$ . For our framework,

<sup>1</sup>An example of *inappropriate* representation is to encode numbers in unary. This yields an unrealistic notion of algorithmic efficiency. For instance, the problem of prime factorization is trivially solvable in polynomial time if one works with this notion. For a detailed discussion, see [47].

these three possibilities are equivalent. This is to be made precise now. With the assumption that both  $G$  and  $M$  are non-empty, the first two definitions are polynomially related, in the following sense: notice that one has

$$\frac{1}{2} \cdot (|G| + |M|) \leq |G| \cdot |M| \leq 2|G| \cdot |M| + |G|^2 + |M|^2 = (|G| + |M|)^2,$$

where the first inequality follows from the fact that  $|G| + |M| \leq 2|G| \cdot |M|$ , which holds because  $G, M \neq \emptyset$  was assumed. Therefore, any polynomial lower (upper) bound in  $|G| \cdot |M|$  induces some polynomial lower (upper) bound in  $|G| + |M|$ , and conversely.

Regarding the notion of size as being given by  $|I|$ , we note that  $|I| \leq |G| \cdot |M|$ , which establishes a one-sided relation with one of the previous notions. However, there is no lower bound for  $|I|$  as a function of  $|G|$  and  $|M|$ . But since we are (initially) interested in the number of concepts of contexts, we may assume that any given context is reduced. In particular, all contexts are clarified and therefore possess at most one empty row and one empty column. This provides us with the lower bound  $\max\{|G|, |M|\} \leq |I| + 1$ , from which follows that  $2 \cdot (|I| + 1) \geq |G| + |M|$  and that the three notions are equivalent. For convenience, we will adopt the convention that the size of a context is the sum of  $|G|$  and  $|M|$ .

## 2.4. Upper bounds present in the literature

Since the beginning of the development of Formal Concept Analysis it was clear that important objects - such as formal concepts - had counterparts in other mathematical areas, such as graph theory. Most of the upper bounds which we are about to see were first presented in graph-theoretical language. It is therefore no surprise that we require definitions before going further. Let  $\mathcal{G} = (V, E)$  be a graph. As usual,  $V$  represents a set of *vertices* and  $E \subseteq \binom{V}{2}$  a set of *edges*. We simply write  $e = uv$  if  $e \in E$ , and call the vertices  $u$  and  $v$  the *endpoints* of  $e$ . In this case,  $u$  and  $v$  are also said to be *adjacent*. A *subgraph* is a pair  $(W, F)$  with  $W \subseteq V$  and  $F \subseteq E \cap \binom{W}{2}$ . To express that a graph  $\mathcal{H}$  is a subgraph of  $\mathcal{G}$ , we use the notation  $\mathcal{H} \leq \mathcal{G}$ . A subset  $S \subseteq V$  is an *independent set* if  $uv \notin E$  for each  $u, v \in S$ . If  $V$  admits a partition  $V_1 \cup V_2$  such that  $V_1$  and  $V_2$  are independent sets, then we call  $\mathcal{G}$  *bipartite*. In this case,  $V_1$  and  $V_2$  are called *vertex classes* or *parts*. Such a graph is termed *complete bipartite* or *biclique* if  $uv \in E$  for each  $u \in V_1, v \in V_2$  and denoted  $K_{n,m}$  where  $n = |V_1|, m = |V_2|$ . These definitions will be valid throughout this work: all our graphs are *undirected* (edges are subsets) and *simple*, meaning that they do not possess multiple edges nor loops.

Maximal biclique subgraphs were rapidly recognized by the FCA community as being precisely formal concepts. An early source is [51]. This fast recognition is in part due to the fact that it is almost immediate to view a context as a bipartite graph. Only one detail requires a tad of attention, namely the disjointness between object and attribute sets. Note

that the definition of context does not require that  $G$  and  $M$  be disjoint. In fact, they may even be the very same set - we saw in Section 2.2 examples in which this occurs. In any case, one can take a copy of  $G$  and of  $M$ , say,  $H$  and  $N$  with  $H \cap N = \emptyset$ . This allows one to work with a bipartite graph  $\mathcal{G} = (H \cup N, E)$ . The sets  $H$  and  $N$  are its vertex classes and  $E$  its edge set. The latter is given naturally after defining  $hn \in E$  if and only if  $(h, n) \in I$ , where  $I$  refers to the incidence relation of the context in question.

Generally, every graph-theoretical result involving maximal bicliques in bipartite graphs lends itself to a translation into the FCA language. In 2012, Albano and do Lago conducted a survey of upper bounds for the number of maximal bicliques in a bipartite graph. Their summary of results is depicted in Figure 2.3. Even though that collection was meant to be non-exhaustive, as of 2017 the authors were not informed of any missing result. In what follows, we make an analysis and introduce the required notions for reading the content of that table.

Name	Parameters	Upper bound	Reference
Smallest class	$ G ,  M $	$ \mathfrak{B}  \leq 2^{\min\{ G ,  M \}}$	trivial
Bipartite Convexity	$ G ,  M , \mathfrak{C}$	$ \mathfrak{B}  \leq 2^{ G  - \mathfrak{C}} \left( \frac{ M ( M +1)}{2} + 1 \right)$	Albano, do Lago [9]
Arboricity	$Y,  G ,  M $	$ \mathfrak{B}  \leq 2^{2Y}( G  +  M ) + 2$	Eppstein [24]
Largest induced crown subgraph	$ G ,  M , \omega$	$ \mathfrak{B}  \leq ( G  M )^\omega + 1$	Prisner [48]
Maximum degree	$\Delta,  G $	$ \mathfrak{B}  \leq  G 2^\Delta + 2$	Alexe, Cramas, Foldes, Hammer, Simeone [10]
Cardinality of $I$	$ I $	$ \mathfrak{B}  \leq \frac{3}{2} \left( 2^{\sqrt{ I +1}} \right) - 1$	Schütt [52]

Figure 2.3.: Bounds present in the literature, as of 2012

Let  $\mathcal{G} = (V, E)$  be a graph. The *neighbors* of a vertex  $v$  are the elements of the set  $N(v) := \{u \in V \mid uv \in E\}$ . Suppose that  $V$  is finite. For any  $v \in V$ , its *degree* is its number of neighbors and denoted  $d(v)$ . The maximum value of  $d(v)$ , with  $v \in V$ , is called *maximum degree* of  $\mathcal{G}$  and for which we employ the symbol  $\Delta(\mathcal{G})$ . The *minimum degree*  $\delta$  is defined in the expected way. If  $\delta = \Delta = r$ , we call the graph *r-regular*. A non-empty sequence of distinct vertices  $(v_1, \dots, v_k)$  such that  $v_i v_{i+1}$  is an edge for  $1 \leq i \leq k-1$  is called a *path* from  $v_1$  to  $v_k$ . If there exists a path from  $u$  to  $v$  for any pair of vertices  $u$  and  $v$ , then we call the graph *connected*. Maximal, connected subgraphs are called *connected components*.

An absolutely trivial upper bound is that of  $2^{\min\{|G|, |M|\}}$ , since it is obvious that an extent determines its associated intent and vice-versa. The *bipartite convexity* of  $\mathcal{G} = (G \cup M, E)$  can be explained as follows. Given a linear order on  $M = \{m_1, \dots, m_{|M|}\}$ , we say that  $g \in G$

has *interval neighborhood* if  $N(g)$  is an interval, i. e., it has the form  $\{m_i, m_{i+1}, \dots, m_{i+k}\}$ . The bipartite convexity of  $\mathcal{G}$  (with respect to  $G$ ) is defined as the maximum number (ranging over all linear orders on  $M$ ) of vertices  $g \in G$  which have interval neighborhood. The symbol employed for bipartite convexity is  $\mathfrak{C}$ . If  $\mathfrak{C}(\mathcal{G}) = |G|$ , then the graph is termed *convex bipartite*. Being convex bipartite is the same as having the “consecutive ones property”, introduced by Fulkerson and Gross in [27]. The dual definition, that of bipartite convexity with respect to  $M$ , is formalized accordingly. Nominal, ordinal and interordinal scales are all convex bipartite with respect to both  $G$  and  $M$ . When such property is fulfilled, the graph may also be called *biconvex*.

The bipartite convexity upper bound can be proved with ease if one is acquainted with FCA. Indeed, the factor  $2^{|G|-\mathfrak{C}}$  is just the aforementioned trivial majorant, while  $\frac{|M|(|M|+1)}{2} + 1$  is the well known upper bound for the number of maximal bicliques in a convex bipartite graph. The fact that the product of both yields an upper bound is not trivial, but follows directly from the well known nested line diagram theorem of [35] (Theorem 7). This is an instance in which Formal Concept Analysis is able to provide sound tools that settle, in a simple and elegant way, questions belonging to other areas of mathematics.

On the other direction, the notion of arboricity of a graph had not been exploited in FCA before Eppstein’s result of [24]. Let  $\mathcal{G} = (V, E)$  be a graph. A *circuit* of  $\mathcal{G}$  is a path  $(v_1, \dots, v_k)$  such that  $k \geq 3$  and  $v_1$  is adjacent to  $v_k$ . We call  $\mathcal{G}$  *acyclic* if it does not have any circuit. Any set of edges  $F \subseteq E$  induces a subgraph, namely the one whose edges are precisely  $F$  and whose vertices is the set of all the endpoints of edges in  $F$ . This is denoted  $\mathcal{G}[F]$  and termed *edge-induced subgraph*. Similarly, a set  $U \subseteq V$  induces a subgraph too (the *vertex-induced subgraph*): its vertex set is  $U$  and edges are all the edges of  $\mathcal{G}$  with both endpoints belonging to  $U$ . The notation is the same:  $\mathcal{G}[U]$ . A subgraph is called *induced* if it is vertex-induced. In FCA, the notion of subcontext corresponds to that of induced subgraph. Formally: given any context  $(G, M, I)$ , a triple of the form  $(H, N, I \cap (H \times N))$  with  $H \subseteq G$  and  $N \subseteq M$  is called a *subcontext*.

We say that  $\mathcal{G}$  is *k-forest-decomposable* if there exists  $E_1, \dots, E_k \subseteq E$  such that  $E = \cup_{i=1}^k E_i$  and each  $\mathcal{G}[E_i]$  is acyclic. Of course, every graph is *k-forest-decomposable* for some  $k$ : an example is  $k = |E|$ , because choosing each edge to be a forest yields a valid decomposition (although a very prodigal one). The least  $k$  such that  $\mathcal{G}$  is *k-forest-decomposable* is the *arboricity* of  $\mathcal{G}$  and denoted  $Y(\mathcal{G})$ .

Arboricity is a measure of hereditary sparsity. This is a consequence of a result independently proven by Tutte in [57] and Nash-Williams in [45]. More specifically, they showed the explicit formula  $Y(\mathcal{G}) = \max_{\mathcal{H} \leq \mathcal{G}} \left\lceil \frac{|E(\mathcal{H})|}{|V(\mathcal{H})|-1} \right\rceil$ , where the maximum ranges over all subgraphs of  $\mathcal{G}$  with at least two vertices. By seeing a context as a graph and combining the formula for  $Y$  with the arboricity majorant, one can say that contexts which are *nowhere dense* have a polynomial number of concepts. Here, *nowhere dense* means not having any subgraph  $\mathcal{H}$  with density  $\frac{|E(\mathcal{H})|}{|V(\mathcal{H})|}$  larger than  $\alpha \log(|V(\mathcal{H})|)$ , with  $\alpha$  constant.

The *crown graph*  $CG(2n)$  is the complete, bipartite graph  $K_{n,n}$  minus a set of edges

constituting a *one-factor*. A one-factor is a special kind of *matching*. A matching, in turn, is a set of edges such that no two edges in the set share an endpoint. We call it a one-factor (or *perfect*) if each vertex from the graph is the endpoint of some edge in the set. Readers more acquainted with graph-theoretical language recognize the crown graph as being the unique  $(n - 1)$ -regular, bipartite graph with  $2n$  vertices. It is also called *cocktail party graph* in [48], although probably by mistake. This graph is important enough for this work to deserve a picture. Observe that it corresponds to a contranominal scale.

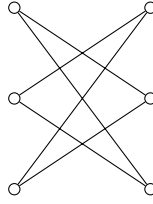


Figure 2.4.: A crown graph with six vertices.

The parameter  $\omega$  featuring in Prisner's upper bound is defined as the maximum size of an induced crown subgraph. Here, *size* means precisely half of the number of vertices. For example, the crown graph depicted above has size three. The term *crown number* also stands for  $\omega$ . Notice that ordinal scales have crown number one, whereas interordinal and nominal scales have crown number two (provided that they have at least two objects and two attributes).

The proof of  $|\mathfrak{B}| \leq (|G||M|)^\omega + 1$  present in [48] requires an ingenious use of a tool called *intersection graph*. Translated into FCA language, the essential idea of the proof is as follows. Given a context  $\mathbb{K}$ , one defines a graph  $\Omega$  whose vertices are concepts of  $\mathbb{K}$  and two vertices  $(A, B), (C, D)$  are joined by an edge if  $(A \cap C) \cup (B \cap D)$  is non-empty. In that case, the existing edge receives the weight  $|A \cap C| + |B \cap D|$ . Prisner then shows that a *maximum spanning tree*<sup>2</sup> of  $\Omega$  provides an encoding which limits the number of concepts. Intersection graphs arose naturally from *interval graphs*, introduced by Hajós in [37].

It is worthwhile to make a remark regarding the complexity of computing the parameters appearing in Figure 2.3. Except for the largest crown subgraph and the bipartite convexity, every parameter present in that table can be computed in polynomial time. This is almost trivial: the only exception which deserves comments is the arboricity. Even though Tutte/Nash-William's formula does not provide a polynomial time algorithm to calculate  $Y$ , it is true that the value of  $Y$  may be calculated using matroid intersection techniques [24] in time  $O(mn \cdot \log n)$  [31]. As usual,  $n$  and  $m$  denote the number of vertices and edges, respectively.

<sup>2</sup>For the definition, we refer the interested reader to [17].

## 2.5. Upper bound subsumption

One of the objectives of this chapter is to reduce (in some sense) Figure 2.3 to three results. This means that three of those upper bounds are sufficient to capture the known polynomial cases of concept lattices.

Our first goal in this section is to show that Eppstein's result captures every polynomial family captured by the maximum degree bound. For that, we need the following. A *k-edge coloring* of a graph  $\mathcal{G} = (V, E)$  is a partition  $E = E_1 \cup \dots \cup E_k$  such that each  $\mathcal{G}[E_i]$  is a matching. If such an object exists, we call  $\mathcal{G}$  *k-edge-colorable*. The intuitive aid here is that each class  $E_i$  represents one color, to be assigned to that subset of edges. The partition describes an assignment of colors such that any two edges sharing an endpoint receive different colors. The minimum  $k$  such that a graph  $\mathcal{G}$  is *k-edge-colorable* is called *chromatic index* and denoted  $\chi'(\mathcal{G})$ . Trivially, every matching induces an acyclic subgraph. This observation guarantees that  $Y \leq \chi'$ , for any graph. Even though there may exist a huge slack between these two values, this approximation is sufficient for our purposes. The famous theorem of Vizing [58], nowadays a classic result in graph theory, is described below. It allows us to ultimately relate the arboricity with the maximum degree of a graph.

**2.5.1 Theorem (Vizing, 1964)** *The chromatic index of any simple, undirected graph is at most its maximum degree plus one.*

The theorem above shows that Eppstein's upper bound subsumes the maximum degree result. As a matter of fact, we have:

**2.5.2 Proposition** *For any family of contexts such that the maximum degree upper bound grows at most polynomially, it holds that the arboricity upper bound grows polynomially as well.*

*Proof* Making use of Vizing's theorem yields  $Y \leq \chi' \leq \Delta + 1$ . As a consequence, the exponential factor of the arboricity majorant may be estimated via  $2^{2Y} \leq 4 \cdot 2^{2\Delta}$ . Then, if  $(\mathbb{K}_n)$  is such that  $|G| \cdot 2^\Delta + 2$  grows polynomially in  $|G| + |M|$ , it holds that  $2^\Delta$  and  $2^{2\Delta}$  grow polynomially as well, forcing the upper bound  $2^{2Y} \cdot (|G| + |M|) + 2$  to exhibit a polynomial limit for its growth too.  $\square$

With the same spirit as above, but relying on a more powerful tool, we show that Eppstein's result subsumes Schütt's bound as well. Recall that, when one views a context  $\mathbb{K}$  as a bipartite graph, its edge set corresponds precisely to the incidence relation of  $\mathbb{K}$ . The theorem below due to Dean et. al. makes it possible to arrive at the subsumption claim. The necessary calculations are elementary: it is sufficient to estimate the factor  $2^{2Y}$  with  $2^{2Y} \leq 2^{2\sqrt{|I|}}$ . Note that the right side of this inequality is essentially the square of the dominating factor present in the cardinality of  $I$  bound.

**2.5.3 Theorem ([18])** *The arboricity of any simple graph with  $e$  edges is at most  $\lceil \sqrt{\frac{e}{2}} \rceil$ , and this bound is best possible.*

The crucial part of the upper bound due to Albano and do Lago is the (one-sided) bipartite convexity  $\mathfrak{C}$ . This parameter depends on the choice of one of the sides of the bipartition: the bound present in Table 2.3 is one of the two possibilities. One obtains the dual version by interchanging  $|G|$  with  $|M|$  and plugging in the bipartite convexity  $\mathfrak{C}$  with respect to the other part. In particular, these two (dual to each other) upper bounds subsume the trivial upper bound  $2^{\min\{|G|, |M|\}}$ .

In light of the considerations above, we arrive at:

**2.5.4 Lemma** *Let  $(\mathbb{K}_n)_{n \in \mathbb{N}}$  be any family of contexts. If some upper bound present in Figure 2.3 grows polynomially with respect to the family, then some upper bound amongst the bipartite convexity, arboricity and crown subgraph grows polynomially.*

## 2.6. Upper bound incomparability

Lemma 2.5.4 reduces the table in Figure 2.3 to three results, the combination of which is capable of capturing all known polynomial cases of concept lattices. Can we push this subsumption effort further? In other words, is it possible to subsume some of these three results using one of the other two? The present section answers this question in the negative. For that, we will show three constructions of families of bipartite graphs.

First, we construct one family for which the crown number majorant performs better than the arboricity and convexity results. Here, “perform better” means to give a polynomial estimate while the other two grow superpolynomially. Then, we proceed to the other two families.

### Crown number performing better than convexity and arboricity

The *subdivision graph*  $\dot{\mathcal{G}} = (V', E')$  of a graph  $\mathcal{G} = (V, E)$  is defined as follows. Its vertex set is  $V' = V \cup \{w^{uv} \mid uv \in E\}$ , meaning that we create one new vertex  $w^{uv}$  for each edge of the original graph. In  $\dot{\mathcal{G}}$ , each original edge  $uv \in E$  is replaced by two edges: one having  $u$  and  $w^{uv}$  as endpoints and the other having  $w^{uv}$  and  $v$ . These are the only elements of  $E'$ . The *claw graph* is the biclique  $K_{1,3}$ . It is depicted on the left of Figure 2.5, together with its subdivision graph (to be seen on the right). This last graph will be called the *subdivided claw* in order to ease the terminology.

We show that the subdivided claw is  $G$ -convex (that is, it has  $G$ -convexity equal to four) but its  $M$ -convexity is two. The first fact is obvious, and can be instantaneously deduced after looking at Figure 2.5, which features linear orderings of both  $G$  and  $M$ . Its  $M$ -convexity is certainly at least two, because of the order on  $G$  shown in that figure. We now argue that it can not be three. Suppose that to  $\{g_1, g_2, g_3, g_4\} = G$  is given an arbitrary linear order (not necessarily the one induced by the indices). Denote by  $<$  the covering relation of such order. If that graph is  $M$ -convex, then the adjacencies of  $m_1$  force that  $g_1 < g_3$  or

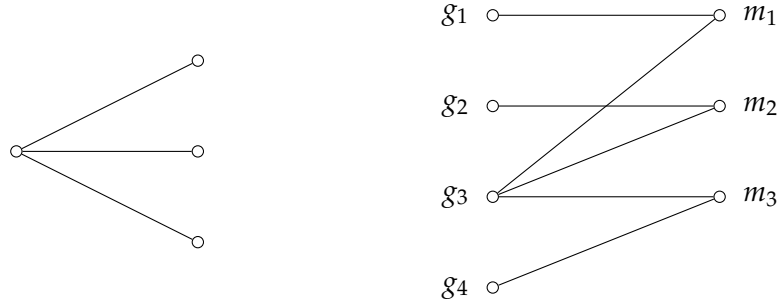


Figure 2.5.: The claw and its subdivision graph

$g_3 < g_1$ . Going further with this argumentation we arrive at the following formula, whose satisfiability is equivalent to the graph's  $M$ -convexity:

$$(g_1 < g_3 \vee g_3 < g_1) \wedge (g_2 < g_3 \vee g_3 < g_2) \wedge (g_3 < g_4 \vee g_4 < g_3).$$

Satisfying the first clause with  $g_1 < g_3$  leads to  $g_1 < g_3 < g_2$ , which in turn results in the unsatisfiability of the third clause. Similarly, if  $g_3 < g_1$  holds, then inevitably  $g_2 < g_3 < g_1$  must occur to satisfy the second, making the third clause false.

Consider the graph family obtained by taking copies of the subdivided claw. By the just conducted argumentation, such graph will have  $M$ -convexity equal to two thirds of  $|M|$ . This makes one version of the convexity result to grow exponentially. But the  $G$ -convexity bound still gives a polynomial quantity (quadratic on  $|M|$ ). To make things irremediably difficult for Albano and do Lago's bound, it suffices to exploit a natural, "mirroring" idea, the result of which is illustrated in Figure 2.6.

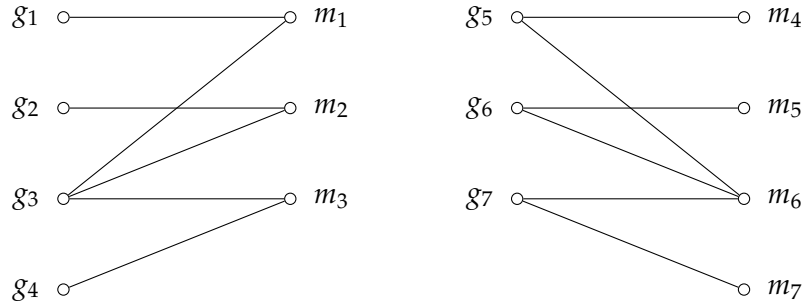


Figure 2.6.: The mirrored, subdivided claw graph

The graph with fourteen vertices present in Figure 2.6 is termed *mirrored, subdivided claw*. If we take copies of this graph, then both versions of the convexity upper bound will show a bad performance. More precisely, for such graph family one has a value of

$|G| - \mathfrak{C} = |M| - \mathfrak{C} = \frac{1}{7}|G|$ , independently of the part to which respect the convexity  $\mathfrak{C}$  is being taken. Thus, the convexity upper bounds grow exponentially in this case.

Making the mentioned majorant to grow superpolynomially (actually, exponentially) is an important step but we are not done yet. There are still two properties that the graph family must have. Namely, first: its arboricity must grow faster than a logarithmic function of the number of vertices (with this,  $2^{2^Y}$  will grow superpolynomially). Second: its crown number must be bounded by a constant value (so that  $(|G||M|)^w$  grows polynomially). Fortunately, it is quite simple to achieve both at the same time.

Denote by  $F(n)$  the graph associated to the (non-reduced) formal context  $([n], [n], \leq)$ . The graph present to the left in Figure 2.7 is  $F(n/4)$ , where  $n$  denotes the number of vertices of an arbitrary member. This means that half of the vertices of the graph is devoted to the subgraph  $F(n/4)$ . The other half builds copies of the mirrored, subdivided claw.

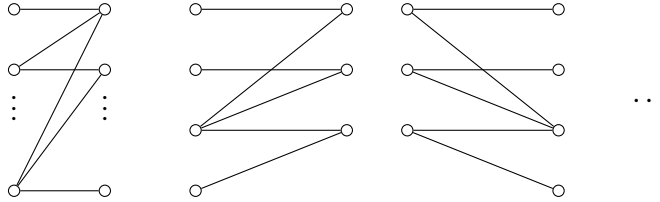


Figure 2.7.: Family with crown number two, arboricity at least  $\frac{n}{16}$  and convexities  $\frac{13}{14}|G|$ .

The family described in Figure 2.7 has, indeed, all the desired properties. Consider an arbitrary member  $\mathcal{G} = \mathcal{G}(n)$  of that family. Clearly, its crown number is two. Its subgraph  $F(n/4)$  has  $1 + 2 + \dots + n/4$  edges, which after simplifying amounts to  $\frac{n^2}{32} + \frac{n}{8}$ . Using this quantity in Tutte/Nash-Williams formula, we obtain the lower bound  $Y(\mathcal{G}) \geq \frac{n}{16}$ . Combining the fact that  $F(n/4)$  is biconvex with the convexity of the mirrored, subdivided claw, we arrive at the convexity value  $\mathfrak{C}(\mathcal{G}) = \frac{13}{14}|G| = \frac{13}{14}|M|$  for both choices of sides of the bipartition.

### Arboricity performing better than convexity and crown number

The claw graph is acyclic. This is certainly not changed by subdividing and/or mirroring. Therefore, the arboricity of all these graphs is precisely one. This fact goes a long way towards constructing the second family.

To make the task difficult for Prisner's upper bound, we need to include some non-bounded, induced crown subgraph in this construction. At the same time, we need to keep the arboricity under control. Thus, we need to somehow estimate the arboricity of the crown graph. Actually, we can use a precise formula. The arboricity of the crown graph was calculated exactly in [2]:

**2.6.1 Lemma** For any positive  $n$ :  $Y(CG(2n)) = \left\lceil \frac{n(n-1)}{2n-1} \right\rceil$ .

Figure 2.8 shows the schematic representation of an arbitrary member of the desired family. The only difference to the first family is that we substituted the graph  $F(n/4)$  by a crown graph of size  $\log n$ . The remaining  $n - 2 \log n$  vertices are, like in the first family, devoted to copies of the mirrored, subdivided claw.

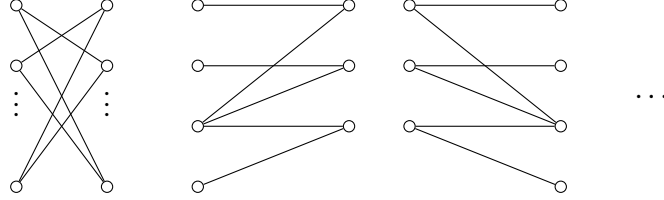


Figure 2.8.: Family with arboricity at most  $\lceil \frac{\log n}{2} \rceil$ , crown number  $\log n$ , and convexities at most  $\frac{13}{14}|G|$ .

### Convexity performing better than arboricity and crown number

The third family arises naturally from the first two. We saw that the arboricity majorant has trouble when it encounters a large subgraph  $F(n)$ . Similarly, Prisner's result does not perform well when an unbounded (by a constant) crown is present as induced subgraph. Combining these two ideas, we arrive at the last family, represented in Figure 2.9.

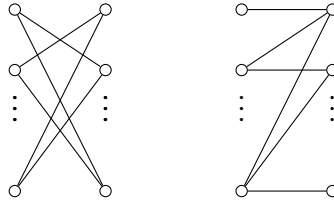


Figure 2.9.: Family with arboricity at least  $\frac{n - \log n}{4}$ , crown number  $\log n$ , and convexities at least  $|G| - \log |G|$ .

The crown subgraph in Figure 2.9 has size  $\log n$ , while the  $F(n - 2 \log n)$  subgraph contains all the other remaining vertices. Once again by the Tutte/Nash-Williams formula, we arrive at the estimate

$$Y(\mathcal{G}(n)) \geq Y(F(n - 2 \log n)) \geq \frac{\left(1 + \frac{n - 2 \log n}{2}\right) \cdot \frac{n - \log n}{2}}{n - 2 \log n} \geq \frac{n - \log n}{4},$$

for the arboricity of an arbitrary member  $\mathcal{G}$ . The assertions regarding crown number and convexities are trivial.

We showed utter disregard to the fact that all graphs constructed in this section are not connected. This leaves an open opportunity for one ad hoc argument, namely, that of applying upper bounds to each connected component of the graph. Fortunately, this style of argumentation is fragile. Any construction can be slightly modified by adding two *universal vertices*, meaning that they are adjacent to every other vertex of the opposing part of the graph. The resulting graph is clearly connected, but has the irritating property of not corresponding to a reduced context. This last, introduced problem may be fixed by creating another pair of vertices such that each one of these is only adjacent to some subset of the original opposite class which is not an extent (intent).

The constructions exposed in this section, combined with Lemma 2.5.4, give rise to the main result of this chapter. In the theorem below, the expression “highly convex” means that the condition  $|G| - \mathfrak{C} \leq \alpha \cdot \log(|G| + |M|)$  (or its dual, with respect to  $M$ -convexity) is fulfilled. Similarly, “bounded” refers to being upper bounded by a constant value, and nowhere dense is as explained before Figure 2.4.

**2.6.2 Theorem** *Families of contexts which are:*

- i) *nowhere dense,*
- ii) *of bounded crown number or*
- iii) *highly convex*

*have associated concept lattices growing polynomially. No one among the three properties above implies any of the other two.*

We will see later in Corollary 5.3.3, Chapter 5, that the crown number coincides with the classic lattice-theoretical notion of breadth.

## 2.7. Hints

A crown subgraph corresponds to an induced matching in the complementary graph. Thus, the problem of calculating the crown number is the same as finding the maximum induced matching (MIM). Stockmeyer and Vazirani introduced in [55] the  $\delta$ -separated-matching problem. The particular case  $\delta = 2$  is MIM, and the authors call it the “risk-free marriage problem”. In that paper, the authors prove that MIM is NP-hard, even when one restricts its instances to graphs of maximum degree four. In terms of approximability, Duckworth, Manlove and Zito proved in [22] that, for every  $r \geq 3$ , the decision version of MIM restricted to  $r$ -regular graphs is a problem that belongs to APX but is also APX-complete. This implies that, assuming  $P \neq NP$ , there is no PTAS (Polynomial Time Approximation Scheme) to compute the crown number.

Besides the maximum induced matching, the crown number equals the *irredundancy number* of a graph as well. This is a parameter which belongs to the theory of dominating sets; a comprehensive treatment is given in [39].

On the other hand, if one is able to solve the independent set problem in a satisfactory way, the computation of  $\omega$  (or some approximation) is conceivable. This is due to the following folklore fact: Given a bipartite graph  $\mathcal{G}$ , the independent sets of  $L^2(\mathcal{G})$  (that is, the square of its line graph) are exactly the induced matchings of  $\mathcal{G}$ .

The problem of computing the bipartite convexity of a graph may be polynomial-time reduced from the NP-hard  $\{0, 1\}$ -travelling salesman problem. This was presented in [2] and in [8]. Apart from this, we are not aware of any approximation or stronger hardness result for this parameter.

Regarding the formula to compute the arboricity, it is common to find in the literature the citation to the 1964 result [46] of Nash-Williams. However, a couple of three years younger theorems of [57] and [45] already implied such formula, albeit in a not so obvious way.

## Size of canonical bases: results in the literature

In applications one often has a predefined, typically small, set of attributes but a very large (or inaccessible) set of objects. By answering queries regarding how the attributes can be meaningfully combined, it is possible to infer the concept lattice from this (attribute) logic. The widely studied - and still very fruitful - techniques of *attribute exploration* gravitate around this approach.

The attribute logic of a formal context is summarized by a set of implications. These constitute a central notion in Formal Concept Analysis, just like the formal concepts studied in the second chapter. The implications of a context reveal dependencies in data and are informative enough to reconstruct its concept lattice. However, the total number of implications is, in general, enormous. Hence, it is common that one restricts attention to a small set of implications, from which all the valid implications can be deduced. Of paramount importance in this scenario are the so-called pseudo-intents. They are the building blocks of the canonical basis of implications. This basis has attracted much attention from researchers in the last decades. One notable reason for this is because it features, amongst all the sound and complete bases, the minimum possible number of implications.

In this chapter we first set the terminology for implications and show central, classic results of the area. After that we exhibit examples appearing in the literature which illustrate how the number of pseudo-intents may behave with respect to the number of attributes and objects, as well as with respect to the number of concepts.

Formal concepts correspond to maximal bicliques; they also enjoy a definition which is very intuitive even for newcomers at the field. This is hardly the case for pseudo-intents. It is therefore not surprising that many questions regarding these have revealed themselves to be quite challenging. As an example, we note that it is known for decades in the FCA community that intents can be enumerated (meaning: to be “listed”) in polynomial delay [33]. It remains elusive whether the same can be done to pseudo-intents. As it was with intents, it is  $\#P$ -hard to count pseudo-intents [43], but it is  $\text{coNP}$ -complete to decide whether a given set of attributes is a pseudo-intent [13]. Moreover, we are not aware of any

non-trivial, contextual upper bound for the number of pseudo-intents.

### 3.1. Basic notions

We will employ notation and a language for implications which is essentially the same as the one in [35] and [34]. Let  $M$  be any set. An implication  $A \rightarrow B$  is a pair of subsets of  $M$ . A family of such pairs of subsets will be called *implication set* or *set of implications* (over  $M$ ). The set  $A$  is called *premise* and  $B$  is called its *conclusion*. A subset  $T \subseteq M$  *respects an implication*  $A \rightarrow B$  if  $A \not\subseteq T$  or  $B \subseteq T$ . Such a subset *respects an implication set*  $\mathcal{L}$  if it respects every implication in  $\mathcal{L}$ . Similarly, we say that an implication  $A \rightarrow B$  *holds* in a family of subsets  $\{T_1, T_2, \dots\}$  if each  $T_i$  respects  $A \rightarrow B$ . If  $A \rightarrow B$  holds in the intents of some context  $\mathbb{K}$ , then we say that  $A \rightarrow B$  is an *implication of  $\mathbb{K}$*  or that  $A \rightarrow B$  *holds* in  $\mathbb{K}$ .

The family of all subsets of  $M$  respecting an implication set  $\mathcal{L}$  is denoted by  $\mathfrak{H}(\mathcal{L})$ . It is well known that  $\mathfrak{H}(\mathcal{L})$  is a closure system. Moreover, we can recover a closure system of intents (and, effectively, the structure of a concept lattice) by making use of the set of all its implications:

**3.1.1 Proposition** *If  $\mathcal{L}$  is an implication set over  $G$ , then  $\mathfrak{H}(\mathcal{L})$  is a closure system. If  $\mathcal{L}$  is the set of all implications of  $\mathbb{K}$ , then  $\mathfrak{H}(\mathcal{L})$  is the system of intents of  $\mathbb{K}$ .*

Now we introduce definitions regarding the semantic inference of implications. An implication  $A \rightarrow B$  *follows semantically* (or just “follows”) from a set of implications  $\mathcal{L}$  if every set which respects  $\mathcal{L}$  also respects  $A \rightarrow B$ . This will be the case if and only if  $\mathfrak{H}(\mathcal{L}) = \mathfrak{H}(\mathcal{L} \cup \{A \rightarrow B\})$ . An implication set  $\mathcal{L}$  is called *closed* if every implication which follows from  $\mathcal{L}$  already belongs to  $\mathcal{L}$ .

As already mentioned in the beginning of this chapter, working with the set of all implications is impracticable. For instance, if we had a full context  $([n], [n], [n] \times [n])$ , the total number of implications would be  $2^{2^n}$  (see proposition below). But this approach is of course extremely pessimistic. All of these implications follow from  $\emptyset \rightarrow [n]$ .

**3.1.2 Proposition** *An implication  $A \rightarrow B$  holds in a context if and only if  $B \subseteq A''$ .*

A tremendously useful feature of implications is that the closed sets of implications admit a syntactic characterization. This is explained in Proposition 3.1.3. Notice that the set of all implications of a context is closed, since it fulfills each of the conditions listed in that proposition. Moreover, Proposition 3.1.3 can also be seen as providing syntactic inference rules which produces *every* semantic consequence of a set of implications.

**3.1.3 Proposition** *An implication set  $\mathcal{L}$  over  $M$  is closed if and only if the following conditions hold, for every  $W, X, Y, Z \subseteq M$  and for an arbitrary index set  $T$ , for all  $Y_t, t \in T$ :*

- $X \rightarrow X \in \mathcal{L}$ ,

- If  $X \rightarrow Y \in \mathcal{L}$ , then  $X \cup Z \rightarrow Y \in \mathcal{L}$ ,
- If  $X \rightarrow Y \in \mathcal{L}$  and  $Y \cup Z \rightarrow W \in \mathcal{L}$ , then  $X \cup Z \rightarrow W \in \mathcal{L}$ ,
- If  $X \rightarrow Y_t \in \mathcal{L}$  for all  $t \in T$ , then  $X \rightarrow \cup_{t \in T} Y_t \in \mathcal{L}$ .

As a remark, we note that the *Armstrong rules of inference* [11] were originally established for functional dependencies. These are intimately related to implications, however, they do have different semantics.

Suppose that  $\mathcal{L}$  is an implication set over the attribute set of a context  $\mathbb{K}$ . If we can generate every implication of  $\mathbb{K}$  by means of the application of the rules present in Proposition 3.1.3, then we say that  $\mathcal{L}$  is *complete* for  $\mathbb{K}$ . It is well known that a set  $\mathcal{L}$  is complete for  $\mathbb{K}$  if and only if  $\mathfrak{H}(\mathcal{L})$  is the system of all intents of  $\mathbb{K}$ . Conversely, if every implication generated in that same way is an implication of  $\mathbb{K}$ , then we say that  $\mathcal{L}$  is *sound*. Evidently, sets which are sound and complete represent faithfully the attribute logic of a context.

An example of implication set which is sound and complete is given by the so-called *proper premises*. Define

$$A^\bullet := A'' \setminus (A \cup \bigcup_{m \in A} (A \setminus \{m\})'').$$

If  $A^\bullet$  is non-empty, then we call  $A$  a *proper premise*. We denote by  $pp(\mathbb{K})$  the set of all proper premises of  $\mathbb{K}$ . Given a context  $\mathbb{K}$ , the set of implications  $\{A \rightarrow A^\bullet \mid A \in pp(\mathbb{K})\}$  is trivially sound because of  $A^\bullet \subseteq A''$  and Proposition 3.1.2. Moreover, the following fact can be found in [35]:

**3.1.4 Proposition** *The set of all implications of the form*

$$A \rightarrow A^\bullet, \text{ with } A \text{ a proper premise}$$

*of a context with a finite attribute set is complete.*

In general, the implication set given by Proposition 3.1.4 does not have the minimum possible number of implications amongst all the sound and complete sets. Duquenne and Guigues showed in [36] that there exists a natural entity which composes a sound and complete set of implications, called *canonical basis*. This basis has the minimum number of implications possible amongst all sound and complete sets.

Let  $(G, M, I)$  be a context. A finite set  $P \subseteq M$  is called a *pseudo-intent* if  $P \neq P''$  and  $Q'' \subseteq P$  for every proper subset  $Q \subsetneq P$  which is a pseudo-intent. This definition is not circular: the subset  $Q$  is proper and therefore has fewer elements than  $P$ . In particular,  $\emptyset$  is a pseudo-intent if and only if  $\emptyset'' \neq \emptyset$ . Notice that, if a set is not closed, but every one of its maximal proper subsets is closed, then it must be a pseudo-intent. The *canonical basis* of a context  $\mathbb{K}$  is defined as the set of implications

$$cb(\mathbb{K}) := \{P \rightarrow P'' \setminus P \mid P \text{ a pseudo-intent}\}.$$

### 3.2. Examples and extremal results

It is not hard to come up with an example of a context which has exponentially more pseudo-intents than attributes: for instance, the contexts  $\mathbb{K}_n := \left( \left( \binom{[n]}{[n/2]} \right), [n], \ni \right)$  for  $n \geq 3$  deliver this. To see that this is indeed the case, note that every subset of  $M$  having exactly  $[n/2]$  elements is an intent (actually, an object-intent). Moreover, any set with precisely  $[n/2] + 1$  attributes can *not* be an intent and, therefore, must be a pseudo-intent. Notice, however, that the number of objects is exponential in  $|M|$ . A natural question to ask is if the number of pseudo-intents can be exponential in  $|G| + |M|$ . The answer to this question is yes, as it was shown by Kautz et. al. in [41] and also by Kuznetsov [43]. The latter result consists of a construction of a context with  $3n$  objects,  $2n + 1$  attributes and exactly  $2^n$  pseudo-intents (see Figure 3.1). In that depiction, each one of the three  $\neq$  symbols is to be understood as a contranominal scale, meaning that the only non-incidences are present along the main diagonal.

	$m_0$	$m_1 \ m_2 \ \dots \ m_n$	$m_{n+1} \ m_{n+2} \ \dots \ m_{2n}$
$g_1$		$\neq$	$\neq$
$\vdots$			
$g_n$			
$g_{n+1}$	$\times$	$\neq$	
$\vdots$	$\vdots$		
$\vdots$	$\vdots$		
$\vdots$	$\vdots$		
$g_{3n}$	$\times$		

Figure 3.1.: Kuznetsov's context family, featuring  $2^n$  pseudo-intents.

Taking just a contranominal scale alone does not serve as an alternative example. For, in that case, every set of attributes is closed, causing no pseudo-intents to exist whatsoever. Nevertheless, the fact that the construction above features three contranominal scales in it (and “large” ones) is quite remarkable.

Kuznetsov's construction may be seen as a case where a context provides a much more succinct representation of data than its canonical basis. The opposite occasionally happens. For a simple justification, it suffices to take a contranominal scale (as we just discussed, it has no pseudo-intents). In order to achieve less degenerate examples, there are constructions, for instance, in [34] where the authors explain that this is the case for partition lattices.

On a different, but related direction, it is worthwhile to mention the following. The size of a reduced context may be about the same as its canonical basis, but its number of intents much higher than both. To see this, take any fixed  $i \geq 2$  and consider  $\mathbb{K}_n := \left( \left( \binom{[n]}{[n-i]} \right), [n], \ni \right)$

for  $n \geq i$ . Note that each  $\mathbb{K}_n$  has size  $n + \binom{n}{n-i} = O(n^i)$ . Every subset of  $[n]$  with at most  $n - i$  attributes must be closed. This implies that there exist at most  $\sum_{j=1}^{i-1} \binom{n}{j}$  pseudo-intents and at least  $2^n - \sum_{j=1}^{i-1} \binom{n}{j}$  intents. Clearly, a constant value of  $i$  causes an exponential gap to exist between these two values. The case  $i = 2$  pertaining to this example is an observation made by Felix Distel in [21]. This value of  $i$  features even a precise counting: there are exactly  $n$  pseudo-intents and precisely  $2^n - n$  intents. The reader may have noticed that, had we allowed  $i = 1$ , then we would have included contranominal scales. We note that this example motivates the investigation of pseudo-intents, in particular, algorithms that enumerate them without the necessity of computing all the intents.

Daniel Borchmann pointed to us the following result, which he derived from the work of Dechter and Pearl [19].

**3.2.1 Theorem** *Let  $\mathbb{K} = (G, M, I)$  be a formal context with finite  $M$ . Then, the implication set*

$$\{A \rightarrow A'' \mid A \notin \text{Int } \mathbb{K} \text{ and } A \setminus \{m\} \in \text{Int } \mathbb{K} \text{ for some } m \in A\} \cup \{\emptyset \rightarrow \emptyset''\}$$

*is complete and sound. In particular,  $|cb(\mathbb{K})| \leq |M| \cdot |\text{Int } \mathbb{K}|$ .*

*Proof* Let  $\mathcal{L}$  denote the implication set above. Its soundness is trivial, and to show its completeness, we prove that every set respecting  $\mathcal{L}$  is an intent. Suppose that  $A$  respects  $\mathcal{L}$  and is not an intent. Then  $A$  contains  $\emptyset''$ , and we may consider an intent  $B$  which is maximal in  $A$ . Since  $A$  is not an intent, the containment must be proper and  $A \setminus B$  is not empty. For  $m \in A \setminus B$  we find that  $B \cup \{m\}$  cannot be an intent (because that would contradict the maximality of  $B$ ), so that  $B \cup \{m\} \rightarrow (B \cup \{m\})''$  is in  $\mathcal{L}$  and therefore is respected by  $A$ . As a consequence,  $(B \cup \{m\})'' \subseteq A$ , which contradicts the maximality of  $B$ .

We have the following consequences:

**3.2.2 Corollary** *Any upper bound for the number of formal concepts induces an upper bound for the number of implications in the canonical basis, namely the original times the number of attributes.*

Combining the corollary above with Theorem 2.6.2, we arrive at:

**3.2.3 Corollary** *Families of contexts which are:*

- i) *nowhere dense,*
- ii) *of bounded crown number or*
- iii) *highly convex*

*have associated canonical bases growing polynomially.*



## A theoretical framework for extremal concept analysis

Extremal combinatorics is a field of discrete mathematics which witnessed huge growth in the past decades. Its questions typically regard the maximum value of an invariant that a combinatorial object may possess, provided that it does not violate some prescribed property. The majorants exposed in Section 2.4 may already be seen as results in this direction. Nevertheless, they do not exhaust the relevant questions that may be posed in such situation. An important goal of this chapter is to present convincing arguments that the characterization of extremal structures is of theoretical interest.

Aiming to provide some degree of chronological and scientific contextualization, we first exhibit in Section 4.1 a result that belongs to the prehistory of the field. Namely, we enunciate and explain Mantel's theorem. Then, the central result due to Turán is exposed and proved in Section 4.2. This result is the milestone of extremal combinatorics, and will serve as an archetypal objective for our developments in Chapters 5, 6 and 7.

The first two sections of this chapter should be understood as a slight but rewarding digression. They serve an illustrative purpose for the third section, where we expose *Bollobás' framework*. It is a description of an abstract setting in which any problem featuring the extremal combinatorics flavor can be stated.

### 4.1. Mantel's theorem

Occasionally, seminal results are only perceived as such after some time has been given for their consequences to mature. The birth of extremal graph theory is considered by many to have occurred in 1941, when Turán's theorem was proved. That result is, however, a generalization of an older result - proved in [44] and presented below - which shared much of that spirit. For this purpose, we define that a *triangle* is a set of three vertices which are pairwise adjacent.

**4.1.1 Theorem (Mantel, 1906)** *If an  $n$ -vertex graph has more than  $\frac{n^2}{4}$  edges, then it must contain a triangle.*

The contraposition of the result above is the following statement: “every triangle-free graph has at most  $n^2/4$  edges”. Thus, Mantel’s theorem may be seen as an upper bound for the number of edges in triangle-free graphs.

Mantel knew that the graph  $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$  provided a limit for how far a result like the above could go. This graph, for even values of  $n$ , is a witness that more than  $n^2/4$  edges are necessary to force some triangle in an *arbitrary* graph. The tools used in the original proof of Theorem 4.1.1 are the handshake lemma and the Cauchy-Schwarz inequality [1]. The latter gives an inequality for the number of edges. Such inequality holds with equality if and only if the graph is precisely  $K_{n/2, n/2}$  for even  $n$ . This fact, most likely overlooked at that time and only implicitly present in that paper from 1906, features the same flavor of statements which would become standard some decades later.

## 4.2. Turán’s theorem

We begin with some definitions. The complementary notion of an independent set is that of a *clique*: we define that to be a set  $K \subseteq V$  with  $uv \in E$  for each  $u, v \in K$  with  $u \neq v$ . Of course, one may see  $K$  as a subgraph too, by considering  $G[K]$ . If there are no cliques of size  $k$  in a graph, we call it  $K_k$ -free.

A triangle can be seen as a clique  $K_3$  or as a circuit. Each one of these possibilities induces a very natural generalization of Theorem 4.1.1. Turán has successfully attacked the first possibility in [56]. He considered  $K_k$ -free graphs with arbitrary  $k$  and established what we will present here as Theorem 4.2.3, although with a different proof. By the way, the second generalization is much harder; we give more details at the end of this chapter.

A graph will be called  $k$ -partite if its vertex set can be partitioned into  $k$  independent sets. Note that, by the pigeonhole principle, every set of  $k + 1$  vertices of a  $k$ -partite graph must contain two vertices belonging to the same class. This implies that  $k$ -partite graphs are always  $K_{k+1}$ -free. We say that a graph is  $(n, k)$ -extremal if it has  $n$  vertices, is  $K_k$ -free and has maximum number of edges (amongst  $K_k$ -free graphs with exactly  $n$  vertices). The values  $n$  and  $k$  are assumed always positive, and if they do not play an important role, we may employ the expression “extremal graph” alone.

The proof of Turán’s theorem which we present in this section is not the original one. Instead, we reproduce the elegant argument due to Alon and Spencer [1], which relies essentially on the following lemma. Observe that the relation used in the result below is, for any graph, reflexive and symmetric. But it is not, in general, transitive.

**4.2.1 Lemma** *If a graph  $(V, E)$  is extremal, then the binary relation on  $V$*

$$u \sim v :\Leftrightarrow u \text{ and } v \text{ are not adjacent}$$

*is an equivalence relation.*

*Proof* Let  $G$  denote such  $(n, k)$ -extremal graph. Reflexivity and symmetry clearly hold. Regarding transitivity, suppose, by contradiction, that  $uw \notin E$ ,  $wv \notin E$  but  $uv \in E$ . We divide in two cases. Suppose that  $d(w) < d(u)$  or  $d(w) < d(v)$ . Without loss of generality, we have  $d(w) < d(u)$ . Call  $G'$  the graph obtained by removing  $w$  and adding a vertex  $u'$  which is a *twin* of  $u$ : that is, they have the same neighbors. In particular,  $u$  and  $u'$  are non-adjacent. Now, every clique of the form  $K \cup \{u'\}$  in  $G'$  does not contain  $u$  and corresponds to a clique  $K \cup \{u\}$  in  $G$ . Therefore,  $G'$  is  $K_k$ -free. Moreover,  $G'$  has  $|E| - d(w) + d(u) > |E|$  edges, a contradiction. For the other case, we have  $d(w) \geq d(u)$  and  $d(w) \geq d(v)$ . Similarly as before, remove  $u$  and  $v$  from the graph and add two vertices, say  $w'$  and  $w''$ , each of them being twin to  $w$ . This obtained graph,  $G'$ , has  $|E| - (d(u) + d(v) - 1) + 2d(w)$  edges, which is strictly larger than  $|E|$ . Similarly as before,  $G'$  is  $K_k$  free and we arrive at a contradiction once again.  $\square$

Since every equivalence relation induces a partition of the underlying set into equivalence classes, the lemma above tells us that every extremal graph must be *complete  $k$ -partite* for some  $k$ , that is,  $k$ -partite and every pair of vertices belonging to different classes must be adjacent. An example of 3-partite graph is given in Figure 4.1, where different styles for the edges were employed exclusively for visual aid. The vertex classes are  $V_1, V_2, V_3$  with  $|V_1| = 4, |V_2| = |V_3| = 2$  and the graph has  $\frac{1}{2} \left( \sum_{i=1}^3 |V_i| \cdot \sum_{j \neq i} |V_j| \right) = 20$  edges.

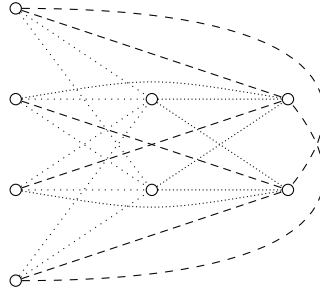


Figure 4.1.: Is this graph  $(8, 4)$ -extremal?

Intuition strongly suggests that complete,  $k$ -partite graphs having partition  $V_1 \cup \dots \cup V_k$  will have more edges if the classes  $V_i$  are as equally-sized as possible. This leads to the following. A partition  $V = V_1 \cup \dots \cup V_k$  is called *balanced* if  $|V_i| - |V_j| \in \{-1, 0, 1\}$  for each  $i, j$ . An  $n$ -vertex graph which is balanced, complete  $k$ -partite is also called a *Turán graph*. It is denoted by  $T(n, k)$  and is unique, up to isomorphism<sup>1</sup>, for each pair of parameters. Observe that  $T(n, n)$  is a clique with  $n$  vertices. For convenience, we define  $T(n, k) = T(n, n)$  whenever  $k > n$ .

<sup>1</sup>The definition of graph isomorphism is what one would expect: a bijection between vertex sets which preserves adjacencies and non-adjacencies.

The reader may verify that, by moving one vertex from  $V_1$  in Figure 4.1 to some other class and adjusting adjacencies accordingly, one obtains  $T(8, 3)$ , which has 21 edges. The general formula follows:

**4.2.2 Proposition** *The Turán graph  $T(n, k)$  has*

$$\frac{1}{2} \left( n^2 - (n \bmod k) \left\lceil \frac{n}{k} \right\rceil^2 - (k - (n \bmod k)) \left\lfloor \frac{n}{k} \right\rfloor^2 \right) \leq \left( 1 - \frac{1}{k} \right) \cdot \frac{n^2}{2}$$

edges.

*Proof* If  $k \geq n + 1$ , the expression above amounts to  $\frac{1}{2}(n^2 - n) = \binom{n}{2}$  and agrees with the number of edges of the clique  $K_n$ . Otherwise, we have  $k \leq n$ . Being balanced and  $k$ -partite gives that  $T(n, k)$  has precisely  $a := n \bmod k$  classes with  $\lceil \frac{n}{k} \rceil$  vertices and  $b := k - a$  classes with  $\lfloor \frac{n}{k} \rfloor$  vertices. The number of ordered pairs of vertices is trivially  $n^2$ , which is of course more than the number of adjacencies. To arrive at the edge count from this quantity, one needs to discount precisely symmetries and adjacencies between vertices belonging to the same class (including loops). Such reasoning yields  $\frac{1}{2} \left( n^2 - a \left\lceil \frac{n}{k} \right\rceil^2 - b \left\lfloor \frac{n}{k} \right\rfloor^2 \right)$ . For the estimate, set  $q := \lfloor \frac{n}{k} \rfloor$ , so that  $n = qk + a$  and  $n^2 = q^2k^2 + 2qka + a^2$ . The following elementary manipulation finishes the proof:

$$1 - \frac{a(q+1)^2 + (k-a)q^2}{n^2} = 1 - \frac{q^2k^2 + 2qka + ak}{kn^2} \leq 1 - \frac{n^2}{kn^2} = 1 - \frac{1}{k}.$$

Now we are in position to state the result which serves as a model for our further developments:

**4.2.3 Theorem (Turán, 1941)** *For any  $n$  and  $k$ , the unique  $(n, k+1)$ -extremal graph is  $T(n, k)$ .*

*Proof* Let  $G$  be  $(n, k+1)$ -extremal with vertex set  $V$ . If  $k \geq n$ , the claim holds trivially. Thus, we assume  $k \leq n - 1$ . By Lemma 4.2.1,  $G$  must be a complete  $l$ -partite graph for some  $l$ . Clearly,  $l \leq k$  because otherwise the graph would contain a clique of size  $k+1$ . Note that  $n \geq l+1$ . Write the partition of  $V$  into  $l$  classes as  $V = V_1 \cup \dots \cup V_l$  and define  $V_i = \emptyset$  for  $l+1 \leq i \leq k$ . Notice that  $n \geq l+1$  forces some class to contain at least two vertices. We will prove that  $l = k$  and that said partition is balanced. To prove both claims, it suffices to show that there does not exist a pair  $i, j$  such that  $|V_i| \geq |V_j| + 2$ . By contradiction, suppose that such a pair exists. Now, remove an arbitrary vertex from  $V_i$  and add a new one to  $V_j$ , say,  $v$ . By adding edges from  $v$  to each other vertex not in  $V_j$ , we arrive at a  $K_{k+1}$ -free graph with  $|E| - |V_j| + (|V_i| - 1)$  edges, contradicting the extremality of  $G$ .  $\square$

We analyze Theorem 4.2.3 by breaking down its consequences into three parts. First, the result implies that every  $K_{k+1}$ -free graph with  $n$  vertices has at most  $\frac{1}{2} \left( n^2 - a \left\lceil \frac{n}{k} \right\rceil^2 - b \left\lfloor \frac{n}{k} \right\rfloor^2 \right)$

edges, where  $a$  and  $b$  depend only on  $n$  and  $k$  and are defined as in the proof of Proposition 4.2.2. Further, it is implicit that this bound is best possible (i.e. that it is *sharp*): indeed, the graph  $T(n, k)$  is defined for every  $n$  and  $k$  and exhibits exactly that number of edges. Lastly, the theorem guarantees that the *only graphs* which attain that maximum number are the Turán graphs.

In general, extremal combinatorialists strive towards results displaying this tripartite nature. According to [25], it is often the case that the precise determination of the extremal value is elusive, and a majorant which is not sharp constitutes already a good enough result. Normally when that happens, the characterization of the extremal structures escapes the grasp.

Turán's theorem describes how some quantity (the number of edges) interacts with a property (being  $K_k$ -free). Evidently, one may ask a myriad of questions regarding other quantities and properties which are relevant for graph theorists. Similarly, the same can be done to hypergraphs (set families) or other combinatorial objects. This realization gave birth to extremal combinatorics and, in particular, extremal graph theory and extremal (finite) set theory. These are nowadays well established fields of research.

### 4.3. Bollobás' framework

Bollobás describes in [16] a scenario which depicts a typical problem/solution pair of extremal graph theory. This setting is flexible enough for us to adapt it in the interests of Formal Concept Analysis. The description reads:

**Scenario:** Given a property  $\mathcal{P}$  and an invariant  $\mu$  for a class  $\mathcal{H}$  of graphs, we wish to determine the least value  $e$  for which every graph  $G$  in  $\mathcal{H}$  with  $\mu(G) > e$  has property  $\mathcal{P}$ . The graphs  $G$  in  $\mathcal{H}$  without property  $\mathcal{P}$  and  $\mu(G) = e$  are called the *extremal graphs* for the problem.

For the just exposed edge/clique problem, we have the following. The number of edges corresponds to  $\mu$ ,  $\mathcal{P}$  is the property of having a  $K_{(k+1)}$ -clique and  $\mathcal{H}$  comprises all graphs with  $n$  vertices. Lastly, the extremal graphs are the members of the two-parameter family  $T(n, k)$ , for which the number of edges  $e$  we know.

#### Relevant questions in FCA

We may use this framework to formalize relevant questions for Formal Concept Analysis. In particular, the ones involving quantity of concepts, as exposed in Chapter 2. In this case,  $\mu$  is  $|\mathfrak{B}(\cdot)|$ . To choose  $\mathcal{P}$ , we need some property which apparently causes a large number of concepts.

There are two facts that suggest that a good candidate for  $\mathcal{P}$  is the property of containing a contranominal scale as a subcontext. The first is Proposition 32 from [35]. It guarantees

that the lattice of a subcontext always order-embeds into the lattice of the whole context. In particular, contexts with some  $\mathbb{N}^c(k)$  must have at least  $2^k$  concepts. The second reason is that the examples shown in Section 2.2 with abundantly many concepts, in some way or another, contain large contranominal scales inside them. Notice that this choice of  $\mathcal{P}$  coincides with the crown number  $\omega$ , present in Prisner's result.

This discussion corresponds to attacking questions 4.3.1 and 4.3.2. We can, however, also highlight problems 4.3.3 and 4.3.4. This last problem was solved by C.Zschalig in [61].

**4.3.1 Question** *What is the maximum number of concepts of a context with  $n$  objects and no  $\mathbb{N}^c(k)$  subcontext?*

**4.3.2 Question** *What is the maximum number of concepts of a context with  $n$  objects,  $m$  attributes and no  $\mathbb{N}^c(k)$  subcontext?*

**4.3.3 Question** *What is the maximum number of  $\mathbb{N}^c(k-1)$  subcontexts of a context with  $n$  objects and no  $\mathbb{N}^c(k)$  subcontext?*

**4.3.4 Question** *What is the maximum number of concepts of a context with  $n$  objects,  $m$  attributes and which has a planar lattice?*

The reader probably noticed that the choice of  $\mathcal{H}$  is more delicate than for the other parameters. As we saw in the description of the problems above, there are at least two reasonable choices for it. It can be the class  $\mathcal{H}^n$  of contexts with  $n$  objects, for instance. But it is equally valid to take the class of contexts with  $n$  objects and  $m$  attributes, say  $\mathcal{H}^{n,m}$ . Note that the second option leads to a more refined result, but a potentially more challenging problem to attack.

In Chapter 5, we solve Questions 4.3.1 and 4.3.3 completely. That chapter consists essentially of joint work between the author and Bogdan Chornomaz. The approach exposed in said chapter however, does not yield any evident insight about the number of attributes in extremal contexts. Motivated by that, in Chapter 6, we give an alternative characterization of the extremal structures from the logical point of view. This second solution is somewhat simpler and gives insight about the number of attributes. This is evidently interesting, given that the solution of Question 4.3.2 is a natural endeavor for us.

The current state of affairs is that Question 4.3.2 is still puzzling and largely unsolved. However, we present in Chapter 7 a general result in that setting. Basically, we show a result regarding the ubiquity of contranominal scales in extremal contexts. More specifically, we prove that for each  $n$  and  $m$ , there exists some extremal context with one as-large-as-permitted contranominal scale.

### Why characterize extremal structures?

Each one of the four questions formally stated above are accompanied by the question: What are the corresponding extremal contexts? But, wait! Why should one care about

that? To answer this question, let us consider for instance, Turán's result once again. The extremal graph  $T(n, k+1)$  consists of a pernicious arrangement of  $K_k$  cliques while not causing not even one  $K_{k+1}$  clique. This gives a theoretical indication that, even though the upper bound may be reached, the structures which do reach the bound *must "look" very special*, in the sense that no one would informally classify them as random-like. This is one objective which we want to achieve in our investigations of extremal concept analysis.

### What about the other two majorants? And pseudo-intents?

It is legitimate to question why we seem to have set aside the arboricity and bipartite convexity majorants in the latest discussions. The reason is simple: quite good "semi-sharpness" results for both upper bounds are known since 2012. And not much happened since, maybe the bounds are simply not sharp. We explain the situation through the table in Figure 4.2:

Upper bound	Sharp? Were the extremal structures characterized?
$ \mathfrak{B}  \leq 2^{2Y}( G  +  M ) + 2$	not known, but a family with $\Omega\left(\frac{2^{2Y} \cdot ( G  +  M )}{Y}\right)$ formal concepts in [8]
$ \mathfrak{B}  \leq ( G  M )^\omega + 1$	discussed in Chapter 5
$ \mathfrak{B}  \leq 2^{ G -\epsilon} \left\lceil \frac{ M ( M +1)}{2} + 1 \right\rceil$	not known, but one family with $\Omega(2^{ G -\epsilon} \cdot  M )$ and other with $\Omega(2^{ G -\epsilon} +  M ^2)$ formal concepts in [8]

Figure 4.2.: Partial results pointing to how sharp the pertinent upper bounds are.

Additionally, Prisner did not give in [48] any construction of graphs with the purpose of showing that the quantity of  $(|G||M|)^\omega + 1$  maximal bicliques can be (partially or totally) attained. This suggests that the bound might be improved. The theorems proved in Chapter 5 will confirm this speculation as being spot on.

The considerations made along Chapter 3 should be put into perspective now. In that part, we discussed pseudo-intents. More precisely, we gave attention to the size of the canonical basis of implications. This thesis does not attack the problem of upper bounding the number of pseudo-intents directly. However, in an exciting turn of events, we will see that when contranominal scales larger than a prescribed size are prohibited, the number of proper premises can not grow too much (and therefore, neither can the number of implications in the canonical basis).

### Hints and further contextualization

The content of Theorem 4.2.3 spanned a huge array of forbidden-graph problems. Nowadays, these are termed *Turán-type problems*. For any fixed graph  $H$ , one defines  $ex(n, H)$  as the maximum number of edges of a graph without any isomorphic to  $H$  subgraph. The case  $H = C_k$ , that is, circuits of length  $k$ , is also a generalization of Mantel's theorem but a much harder one. For instance, even the particular value for circuits of length four (i. e.  $ex(n, C_4)$ ) is known only when  $n$  is of the form  $q^2 + q + 1$ , with  $q$  being a prime power [28–30]. A natural, bipartite version of the question answered by Turán is the so-called *Zarankiewicz problem*. In its most general form, it asks information about the value  $z(n, m; s, t)$ , which is defined as the maximum number of edges that a bipartite graph with parts  $[n]$  and  $[m]$  may have without having a  $K_{s,t}$  subgraph. Extensive research has been done in this direction and mostly asymptotic results have been achieved [30].

Another celebrated result in combinatorics is Ramsey's theorem [49], which escapes the scope of this thesis. Proved in 1930, it predates Turán's result (and actually served as a motivation for it). According to Simonovits [53], since the late 1960s there is an acknowledgement that Turán and Ramsey theorems are very much alike. Actually, Turán himself recognized both results as generalizations of the pigeonhole principle. In his opinion, this is one reason why both results have found many applications in diverse areas [53]. Amongst them, we can cite a few: number theory, logic, geometry and information theory. Because of this understanding, it is not uncommon to find the term "Ramsey-Turán theory" in the literature nowadays.

## Extremal lattices with bounded breadth and number of join-irreducibles

The fourth chapter of the present work inspires the pursue of tripartite results in what we called *extremal concept analysis*. The stunning charm of theorems like Turán's gives at least hope to prove beautiful statements in FCA by using this approach. Allow us to recall what such tripartite nature is. It means that one first proves an upper (lower) bound, then proceeds to show that it is best possible through some construction. Lastly, structures achieving the bound are characterized. A distinguished achievement of this chapter is to prove one such result.

Specifically, we investigate how large concept lattices may be, provided that their contexts do not possess contranominal scales larger than a prescribed size. This vein of investigation was inspired by the crown number majorant, which was proven by Prisner and is the first result in this direction. Our results differ from his in that we do not employ graph-theoretical language. Instead, we make use of common FCA terminology.

In Section 5.1, we start these developments by relating boolean suborders and subcontexts which are contranominal scales. The latter are simply termed contranominal subcontexts and shown to be in strong correspondence with boolean suborders. A remarkably useful consequence is the possibility to work with extremal contexts and extremal lattices interchangeably.

Further, we enunciate Prisner's majorant in Section 5.2 and improve it considerably by using minimal generators. One consequence of this contribution is that the improved upper bound limits the number of proper premises as well. Therefore, the number of pseudo-extents is automatically bounded by the same expression.

Introducing minimal generators to this investigation is shown to be fruitful. Specifically, we prove in Section 5.3 (Lemma 5.3.1) an intimate relationship between irredundant representations, contranominal subcontexts and minimal generators. This connects the established lattice-theoretical notion of breadth with our investigations. Amongst other statements, we conclude that the breadth of a concept lattice is the size of the largest

minimal generator of a realizing context.

In Section 5.4, we show that our improved bound for concepts is the best possible by constructing lattices which attain exactly the improved upper bound. These lattices, i.e., the extremal lattices, are characterized in Section 5.5. In particular, they turn out to be meet-distributive and enjoy interesting properties, such as containing smaller extremal meet-subsemilattices.

Lastly, we give in Section 5.6 an interpretation of our results in terms pertaining to *Vapnik-Chervonenkis (VC) theory*. Such area has found many applications in computational learning [15] and its central notion of *shattered sets* is a widely studied topic in extremal set theory [40].

## 5.1. Fundamentals of boolean suborders

The least and greatest elements of a lattice  $L$  will be denoted, respectively, by  $0_L$  and  $1_L$ . We denote by  $J(L)$  and  $M(L)$ , respectively, the set of (completely) join-irreducible and meet-irreducible elements of  $L$ . Notice that, with these definitions,  $0_L$  ( $1_L$ ) is not join-irreducible (meet-irreducible).

A commonly used condition which is weaker than finiteness is that of doubly foundedness. A complete lattice  $L$  is said to be *doubly founded* if for every  $x, y \in L$  with  $x < y$  there exists  $s$  and  $t$  such that  $s$  is minimal with respect to  $s \leq y$ ,  $s \not\leq x$  and  $t$  is maximal with respect to  $t \geq x$ ,  $t \not\geq y$ . The *standard context* of a doubly founded lattice  $L$  is  $(J(L), M(L), \leq)$ , where  $\leq$  is the order of  $L$ . An *atom* is an element covering  $0_L$ , while a *coatom* is an element covered by  $1_L$ . We denote by  $A(L)$  the set of atoms of a lattice  $L$ . Moreover, for  $l \in L$  we write  $A_l := \{y \in A(L) \mid y \leq x\}$ . A *chain of length  $n$*  is a totally ordered set  $\{x_0, x_1, \dots, x_n\}$ . A lattice  $L$  has *finite length* if a largest chain (w.r.t. length) exists in  $L$ . In this case, the *length* of  $L$  is defined to be the length of one largest chain. For  $l \in L$ , we shall write  $\downarrow l := \{x \in L \mid x \leq l\}$  as well as  $\uparrow l := \{x \in L \mid x \geq l\}$ .

The expression  $\mathbb{K}_1 \leq \mathbb{K}$  denotes that  $\mathbb{K}_1$  is a subcontext of  $\mathbb{K}$ . Two contexts  $(G, M, I)$  and  $(H, N, J)$  are said to be *isomorphic* if there are bijections  $\alpha : G \rightarrow H$ ,  $\beta : M \rightarrow N$  with  $gIm \Leftrightarrow \alpha(g)J\beta(m)$  for each  $g \in G$  and  $m \in M$ . An *order-embedding* between two ordered sets  $(P, \leq_P)$  and  $(Q, \leq_Q)$  is a function  $\alpha : P \rightarrow Q$  such that  $x \leq_P y \Leftrightarrow \alpha(x) \leq_Q \alpha(y)$  for each  $x, y \in P$ . It is automatically injective. If it is actually a bijection, we call it an *order-isomorphism* and say that  $P$  and  $Q$  are order-isomorphic; in this case, the symbol  $P \cong Q$  is employed to express this.

We remind our reader that the boolean lattice with  $k$  atoms, i.e.,  $\mathfrak{B}(\mathbb{N}^c(k))$ , is denoted  $B(k)$ . As anticipated in the beginning of this chapter, we use the term *contranominal subcontext* to refer to a subcontext which is isomorphic to a contranominal scale. For a context to be  $\mathbb{N}^c(k)$ -free means that there does not exist a contranominal subcontext with  $k$  objects (and  $k$  attributes). Similarly, we say that a lattice  $L$  is  $B(k)$ -free whenever  $B(k)$  does not (order-)embed into  $L$ . Using Proposition 32 from [35] one has that  $\mathbb{K}$  is  $\mathbb{N}^c(k)$ -free

whenever  $\underline{\mathfrak{B}}(\mathbb{K})$  is  $B(k)$ -free. The next lemma shows that this is in fact an equivalence.

**5.1.1 Lemma** *A context  $\mathbb{K}$  is  $\mathbb{N}^c(k)$ -free if and only if  $\underline{\mathfrak{B}}(\mathbb{K})$  is  $B(k)$ -free.*

*Proof* Using Proposition 32 from [35] one has that  $B(k)$  embeds into  $\underline{\mathfrak{B}}(\mathbb{K})$  whenever  $\mathbb{N}^c(k) \leq \mathbb{K}$ . To prove the converse, let  $(A_1, B_1), \dots, (A_k, B_k)$  be the atoms of  $B(k)$  in  $\underline{\mathfrak{B}}(\mathbb{K})$ . Similarly, denote its coatoms by  $(C_1, D_1), \dots, (C_k, D_k)$  in such a way that one has  $(A_i, B_i) \leq (C_j, D_j) \Leftrightarrow i \neq j$  for each  $i, j$ . Note that the sets  $A_i$ , as well as the sets  $D_i$ , are non-empty. Let  $i \in [k]$ . Since  $(A_i, B_i) \not\leq (C_i, D_i)$ , we may take an object/attribute pair  $g_i \in A_i, m_i \in D_i$  with  $g_i \not I m_i$ . For every chosen object  $g_i \in A_i$ , one has that  $g_i I m_j$  for every  $j \in [k]$  with  $j \neq i$ , because of  $(A_i, B_i) \leq (C_j, D_j)$ , which implies  $B_i \supseteq D_j$ . Consequently,  $k$  distinct objects  $g_i$  (as well as  $k$  distinct attributes  $m_i$ ) were chosen. Combining both relations results in  $g_i I m_j \Leftrightarrow i \neq j$  for each  $i \in [k]$ , that is, the objects and attributes  $g_i, m_i$  form a contranominal subcontext of  $\mathbb{K}$ .  $\square$

Given a lattice  $L$ , one may apply Lemma 5.1.1 to any context which *realizes*  $L$ : that is, any  $\mathbb{K}$  such that  $\underline{\mathfrak{B}}(\mathbb{K}) \cong L$ . An important particular case for us consists of taking the standard context:

**5.1.2 Corollary** *A doubly founded lattice is  $B(k)$ -free if and only if its standard context is  $\mathbb{N}^c(k)$ -free.*

Even though  $B(k)$  always order-embeds into the lattice of any context with some  $\mathbb{N}^c(k)$  subcontext, it does not hold in general that one can find a lattice embedding (i. e. an injective morphism w.r.t. binary meet and join). For instance, the lattice on Figure 5.1 admits an order embedding - but no lattice embedding - of  $B(4)$ . This is caused by the duplicated element, to be seen on the left of the diagram. The one nearest to the greatest element is join-irreducible, while its unique lower cover is meet-irreducible. Any sublattice of that lattice must have one of the two, but a boolean sublattice can not have such an element.

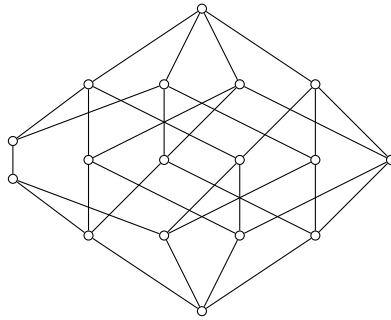


Figure 5.1.: This lattice contains a  $B(4)$ -suborder, but the duplicated element to the left rules out the existence of a  $B(4)$ -lattice-embedding.

An example of a context which has  $\mathbb{N}^c(3)$  as a subcontext along with its concept lattice is depicted in Figure 5.2. One may observe that the context is  $\mathbb{N}^c(4)$ -free, because its lattice has ten concepts (and would have at least sixteen otherwise).

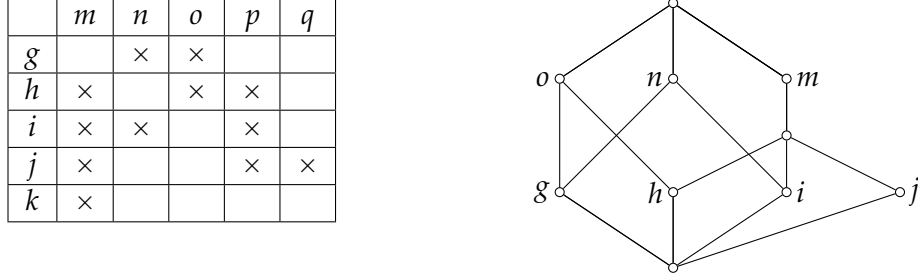


Figure 5.2.: A context  $\mathbb{K}$  with  $\mathbb{N}^c(3) \leq \mathbb{K}$  and its concept lattice.

## 5.2. Improving the bound by connecting it to minimal generators

Prisner gave the first upper bound regarding contranominal-scale free contexts. As already mentioned in Section 2.4, this result was first cast using a graph-theoretical language. More precisely, one which underpins intersection graphs. Reformulated into the FCA terminology, it reads as follows:

**5.2.1 Theorem (Prisner [48])** *Let  $\mathbb{K} = (G, M, I)$  be any  $\mathbb{N}^c(k)$ -free context. Then, it follows that  $|\mathfrak{B}(\mathbb{K})| \leq (|G||M|)^{k-1} + 1$ .*

In this section we will show an improvement of the bound present in Theorem 5.2.1. For that, we will relate minimal generators with contranominal scales. The first step towards this is the equivalence shown in Proposition 5.2.3. Before that, we require a central definition and an important observation.

**5.2.2 Definition** Let  $(G, M, I)$  be a formal context. A set  $S \subseteq G$  is said to be a **minimal generator** (of the extent  $S''$ ) if  $T'' \neq S''$  for every proper subset  $T \subsetneq S$ . The set of all minimal generators of a context  $\mathbb{K}$  will be denoted by  $\text{MinGen}(\mathbb{K})$ .  $\diamond$

As an example, observe that in Figure 5.2 the extent  $\{h, i, j\}$  has three minimal generators, namely each of its two-element subsets. In contrast, the extent  $\{g, i\}$  is its own and unique minimal generator.

*Observation:* In contexts with finitely many objects, every extent has at least one minimal generator. Clearly, two different extents cannot share one same minimal generator. Thus, the upper bound  $|\mathfrak{B}(\mathbb{K})| \leq |\text{MinGen}(\mathbb{K})|$  holds for contexts with finite object sets.

Regarding the next proposition, observe that since derivation operators are always antitone, the  $\neq$  symbol may be substituted by  $\supsetneq$ .

**5.2.3 Proposition** *Let  $(G, M, I)$  be a formal context. A set  $S \subseteq G$  is a minimal generator if and only if for every  $g \in S$ , it holds that  $(S \setminus \{g\})' \neq S'$ .*

*Proof* We will show the two equivalent contrapositions. If  $(S \setminus \{g\})' = S'$ , then, of course,  $(S \setminus \{g\})'' = S''$ , and  $S$  is not a minimal generator. For the converse, suppose that  $S$  is not a minimal generator, and take a proper subset  $T$  of  $S$  with  $T'' = S''$ . Note that  $T'' = S''$  implies  $T' = S'$ . Let  $g \in S \setminus T$ . On one hand,  $(S \setminus \{g\}) \subseteq S$  implies  $(S \setminus \{g\})' \supseteq S'$ . On the other hand,  $(S \setminus \{g\}) \supseteq T$  implies  $(S \setminus \{g\})' \subseteq T' = S'$ . Combining both yields  $(S \setminus \{g\})' = S'$ .  $\square$

The next lemma shows an intimate relationship between minimal generators and contranominal scales. It states that every minimal generator must be the object set of some contranominal subcontext, and conversely.

**5.2.4 Lemma** *Let  $\mathbb{K} = (G, M, I)$  be a context and  $A \subseteq G$ . There exists a contranominal subcontext of  $\mathbb{K}$  having  $A$  as its object set if and only if  $A$  is a minimal generator.*

*Proof* Let  $\mathbb{K} = (G, M, I)$  be a context,  $A \subseteq G$  a minimal generator and let  $g \in A$ . By Proposition 5.2.3, one has that  $(A \setminus \{g\})' \supsetneq A'$ . Hence, there exists an attribute  $m$  with  $g \not\mathcal{X} m$  and  $h \mathcal{I} m$  for every  $h \in A \setminus \{g\}$ . Clearly, two different objects  $g_1, g_2 \in A$  cannot give rise to the same attribute  $m$ , since the two pairs of conditions  $g \not\mathcal{X} m$  and  $h \mathcal{I} m$  for every  $h \in A \setminus \{g_i\}$  cannot be satisfied simultaneously ( $i = 1, 2$ ). Thus, there exists an injection  $\iota : A \rightarrow M$  with  $g \not\mathcal{X} \iota(g)$ ,  $h \mathcal{I} \iota(g)$  for each  $g \in A$  and each  $h \in A \setminus \{g\}$ . By setting  $N = \iota(A)$ , one has that  $(A, N, I \cap (A \times N))$  is a contranominal scale. For the converse, let  $\mathbb{K}_1 = (A, A, \neq) \leq \mathbb{K}$  be a contranominal scale and let  $g \in A$ . Clearly,  $g \notin A'$ . Moreover,  $g \in (A \setminus \{g\})'$ . This amounts to  $(A \setminus \{g\})' \supsetneq A'$  for each  $g \in A$ . By Proposition 5.2.3, the set  $A$  is a minimal generator.  $\square$

**5.2.5 Corollary**  $\mathbb{K}$  is  $\mathbb{N}^c(k)$ -free if and only if  $|A| < k$  for every minimal generator  $A$ .

A consequence of Corollary 5.2.5 is a bound which improves by at least  $(k-1)! \cdot |M|^{k-1}/k$  the original one introduced in the beginning of this section.

**5.2.6 Theorem** *Let  $\mathbb{K} = (G, M, I)$  be any  $\mathbb{N}^c(k)$ -free formal context with finite  $G$ . Then, it follows that  $|\mathfrak{B}(\mathbb{K})| \leq |\text{MINGEN}(\mathbb{K})| \leq \sum_{i=0}^{k-1} \binom{|G|}{i}$ . In the particular case that  $k \leq \frac{|G|}{2}$  holds, one has  $|\mathfrak{B}(\mathbb{K})| \leq k \cdot \frac{|G|^{k-1}}{(k-1)!}$ .*

*Proof* Corollary 5.2.5 guarantees that  $\mathbb{K}$  does not have any minimal generator with  $k$  or more elements. The sum above is the number of subsets of  $G$  having at most  $k-1$  elements.  $\square$

However simple, Lemma 5.2.7 has a powerful consequence: the expression  $\sum_{i=0}^{k-1} \binom{|G|}{i}$  bounds not only the number of concepts, but also the number of pseudo-extents of any formal context. This is more precisely stated in Corollary 5.2.8.

**5.2.7 Lemma** *Every proper premise is a minimal generator.*

*Proof* If  $A$  is a proper premise, then the set  $g \in A'' \setminus (\cup_{B \subseteq A, B \neq A} B'')$  is non-empty. In particular, the closure of every proper subset  $B \subsetneq A$  must be different than  $A''$ . According to Definition 5.2.2, this means that  $A$  is a minimal generator.  $\square$

**5.2.8 Corollary** *Let  $\mathbb{K} = (G, M, I)$  be any  $\mathbb{N}^c(k)$ -free formal context with finite  $G$ . Then, it follows that  $|cb(\mathbb{K})| \leq |pp(\mathbb{K})| \leq |\text{MinGen}(\mathbb{K})| \leq \sum_{i=0}^{k-1} \binom{|G|}{i}$ , where  $cb(\mathbb{K})$  and  $pp(\mathbb{K})$  refer to the object-sided implication logic of the formal context.*

From now on, we denote by  $f(n, k)$  the majorant in Theorem 5.2.6:  $f(n, k) := \sum_{i=0}^{k-1} \binom{n}{i}$ .

Theorem 5.2.6 may be applied indirectly to doubly founded lattices: one just needs to perform the usual operation of taking the standard context beforehand. To put it in another way:

**5.2.9 Corollary**  $|L| \leq f(n, k)$  for any doubly founded,  $B(k)$ -free lattice  $L$ , where  $n = |J(L)|$ .

*Proof* It suffices to take the standard context of  $L$ , apply Corollary 5.1.2 and then Theorem 5.2.6.  $\square$

The upper bound in Theorem 5.2.6 for  $|\mathfrak{B}(\mathbb{K})|$  with  $k \leq \frac{|G|}{2}$  gets worse as  $k$  gets close to  $\frac{|G|}{2}$ . Tighter upper bounds for the sum of binomial coefficients may be found in [60].

Later we will need the following identity involving  $f(n, k)$ .

**5.2.10 Proposition** *The function  $f(n, k)$  satisfies  $f(n, k) = f(n-1, k-1) + f(n-1, k)$ .*

*Proof* This follows from a standard binomial identity:  $f(n-1, k) + f(n-1, k-1) = \sum_{i=0}^{k-1} \binom{n-1}{i} + \sum_{j=0}^{k-2} \binom{n-1}{j} = 1 + \sum_{i=1}^{k-1} \binom{n-1}{i-1} + \sum_{j=1}^{k-1} \binom{n-1}{j} = 1 + \sum_{i=1}^{k-1} \binom{n}{i} = f(n, k)$ .  $\square$

### 5.3. The breadth of a context

We introduce in this section the notion of breadth for lattices and contexts, which we then link to minimal generators. Since we connected minimal generators to contranominal scales and boolean suborders, the consequence is an interplay between all these notions. This is explained by Lemma 5.3.1.

The breadth of a finite lattice was already present in G.Birkhoff's classic [14] and is closely related to *irredundant representations*, whose definitions we explain now. Given an element  $x$  of a lattice  $L$ , we set the notation  $J_x := \{y \in J(L) \mid y \leq x\}$ . Now, suppose that  $L$  is doubly founded. In particular,  $J(L)$  is join-dense and consequently, every element  $x \in L$  is the supremum of some subset of  $J(L)$ ; for example,  $x = \bigvee J_x$ . We call such a subset a *representation of  $x$  through join-irreducible elements* (for brevity, we may say a *representation through irreducibles of  $x$*  or even only a *representation of  $x$* ). A representation  $S \subseteq J(L)$  of  $x$  is

called *irredundant* if  $\bigvee (S \setminus \{y\}) \neq x$  for every  $y \in S$ . The fact that the supremum is monotone easily implies the equivalence  $\bigvee (S \setminus \{y\}) \neq x$  for every  $y \in S \Leftrightarrow \bigvee T \neq x$  for every  $T \subsetneq S$ . We remark that irredundant representations of a given element are not, in general, unique.

The *breadth* of a finite lattice is defined to be the maximum number of elements of an irredundant representation. This notion is self-dual. Which means that if one replaces “join” with “meet” in the necessary definitions, then the same natural number will be obtained, independently of the finite lattice in question. We will not prove this directly, but this fact will become transparent in light of the equivalences established in Lemma 5.3.1. We define that the *breadth of a finite context* is that of its concept lattice.

**5.3.1 Lemma** *Let  $L$  be a doubly founded lattice with standard context  $\mathbb{K}$  and  $A \subseteq J(L)$ . The following are equivalent:*

- i)  *$A$  is the object set of some contranominal subcontext of  $\mathbb{K}$ ;*
- ii)  *$A$  is an irredundant representation (of  $\bigvee A$ );*
- iii)  *$A$  is a minimal generator (of the closure of  $A$  with respect to  $\mathbb{K}$ ).*

*Proof* The equivalence between i) and iii) was achieved by Lemma 5.2.4. For the equivalence between ii) and iii): it suffices to notice that the closure of object sets - with respect to the standard context of  $L$  - is the same operation as the supremum of join-irreducible elements of  $L$  (according to the Basic Theorem on concept lattices [35]).  $\square$

In terms of breadth, we have the following consequence of the preceding lemma:

**5.3.2 Corollary** *The breadth of a finite lattice equals the maximum number of objects of a contranominal subcontext of its standard context.*

Recalling that the maximum number of objects of a contranominal subcontext is precisely the crown number of the graph associated to the whole context, we arrive at the following conclusion:

**5.3.3 Corollary** *Let  $\mathbb{K}$  be a finite context and  $\mathcal{G}$  be its associated bipartite graph. Then, the breadth of  $\mathbb{K}$  equals the crown number of  $\mathcal{G}$ .*

## 5.4. Sharpness of the improved upper bound

To show that the bound present in Theorem 5.2.6 is sharp, one proves that there exists, for each positive  $n$  and  $k$ , a formal context  $\mathbb{K} = \mathbb{K}(n, k)$  such that  $\mathbb{K}$  is  $\mathbb{N}^c(k)$ -free, has  $n$  objects and precisely  $f(n, k)$  concepts. Any context satisfying such conditions is termed an  $(n, k)$ -*extremal context*. It is sometimes more convenient to work with their associated lattices. We therefore, introduce the following:

**5.4.1 Definition** For positive integers  $n$  and  $k$  (called *parameters*), we say that a lattice is  $(n, k)$ -extremal if it has at most  $n$  join-irreducible elements, is  $B(k)$ -free, and has exactly  $f(n, k)$  elements.  $\diamond$

Notice that the concept lattice of an  $(n, k)$ -extremal context is an  $(n, k)$ -extremal lattice. Conversely, every  $(n, k)$ -extremal lattice is realizable by some  $(n, k)$ -extremal context. To exemplify such structures, observe that every  $(n, 1)$ -extremal lattice is trivial, i.e., the lattice with one element. It comes with little effort that every  $(n, 2)$ -extremal lattice must be an  $n + 1$ -element chain, since  $B(2)$  can not be found as a suborder and  $n + 1 = f(n, 2)$ . As a last example, note that if  $n < k$  holds, then the  $B(k)$ -freeness restriction is void, making the extremal lattice to be simply a boolean lattice with  $n$  atoms. Whenever the parameters do not play an important rôle, we may just employ the term *extremal lattice*.

This section is further divided into two parts: first, we prove properties of extremal lattices, even though their existence - except for  $k = 1, k = 2$  and  $n < k$  - is not yet guaranteed. The second part introduces the operation used to construct larger extremal lattices from smaller ones. That will establish the general existence of extremal lattices.

### Meet-distributivity and other properties of extremal lattices

Lemma 5.3.1 has produced important links between vital notions in the last section. We begin this section by showing one last immediate consequence of that lemma, namely Corollary 5.4.2.

It is allowed for an  $(n, k)$ -extremal lattice to possess strictly less than  $n$  join-irreducibles: indeed, the trivial lattice ( $k = 1$ ) is one example. This idiosyncrasy is technical: by working with such definition, one has the advantage that  $(n, 1)$ -extremal lattices exist for any  $n$ . Fortunately, such choice does not incur any nuisance, since for  $k \geq 2$ , such definition is equivalent as requiring precisely  $n$  join-irreducibles:

**5.4.2 Corollary** Every  $(n, k)$ -extremal lattice with  $k \geq 2$  has precisely  $n$  join-irreducible elements.

*Proof* Let  $L$  denote one such lattice. By definition, we have that  $|J(L)| \leq n$  as well as that  $L$  is  $B(k)$ -free. By Corollary 5.1.2 (or Proposition 32 from [35]), it follows that its standard context is  $\mathbb{N}^c(k)$ -free. Hence, Lemma 5.3.1 implies that  $L = \{\bigvee S \mid S \subseteq J(L), |S| \leq k - 1\}$ . Again from the definition of extremal lattices, one has  $|L| = f(n, k)$ . Because  $f(n, k)$  is also the number of subsets of an  $n$ -element set with at most  $k - 1$  elements, this forces  $|J(L)| \geq n$ , since the function  $m \mapsto \binom{m}{k-1}$  is monotone increasing, for any fixed  $k \geq 2$ .  $\square$

As we will see, one important feature of extremal lattices is that they are always meet-distributive. The definition of meet-distributivity used in this work follows that of [23]: a lattice  $L$  is *meet-distributive* if for each  $y \in L$ , the interval  $[y_*, y]$  is a boolean lattice, where  $y_*$  denotes the meet of all elements covered by  $y$ .

Consider a doubly founded lattice  $L$ . For an element  $x \in L$ , there may exist elements in  $J_x$  which belong to every representation of  $x$ : the so-called *extremal points*. An element  $z \in J_x$  is an *extremal point* of  $x$  if there exists a lower cover  $y$  of  $x$  such that  $J_y = J_x \setminus \{z\}$ . Every representation of  $x$  must contain every extremal point  $z$  of  $x$  because, in this case, the supremum  $\bigvee (J_x \setminus \{z\})$  is strictly smaller than  $x$  (and is actually covered by  $x$ ). The next lemma will provide us with a property which is characteristic for meet-distributivity. However, we need to require the lattice to be of finite length.

**5.4.3 Lemma** *Let  $L$  be a lattice of finite length. The following assertions are equivalent:*

- i)  $L$  is meet-distributive;
- ii) Every element  $x \in L$  is the supremum of its extremal points;
- iii) Every element has a unique irredundant representation;
- iv) For every  $x, y \in L$  with  $x < y$ , it holds that  $|J_y \setminus J_x| = 1$ .

*Proof* The equivalence between i), ii) and iii) may be found in Theorem 44 of [35]. Let  $x \in L$  and define  $E_x = \{z \in J_x \mid z \text{ is an extremal point of } x\}$ . We now show that ii) implies iv). Let  $y \in L$  with  $y < x$ . This implies  $J_y \subsetneq J_x$ . The set  $J_y$  does not contain  $E_x$ , because this would force  $y \geq x$ . Therefore,  $y = \bigvee J_y$  is upper bounded by some element in the set  $U = \{\bigvee (J_x \setminus \{z\}) \mid z \in E_x\}$  (note that  $x \notin U$ ). Hence, every lower cover of  $x$  has a representation of the form  $(J_x \setminus \{z\})$  with  $z \in E_x$ . Now we show that iv) implies ii). Define  $y = \bigvee E_x$  and suppose by contradiction that  $y < x$ . Then, there exists an element  $z$  such that  $y \leq z < x$  and  $J_z \supseteq E_x$ . But then,  $z < x$  implies  $J_x \setminus J_z = \{w\}$  for some  $w \in J(L)$ , which means that  $w$  is an extremal point of  $x$ . This contradicts the fact that  $E_x$  contains all extremal points of  $x$ .  $\square$

**5.4.4 Corollary** *Suppose that  $L$  is a meet-distributive, finite length lattice. Then, for every  $x \in L$  it follows that set  $E_x$  of extremal points of  $x$  is the unique irredundant representation of  $x$ .*

In the next subsection, it will be vital to produce lattices which are  $B(k)$ -free. This shall be done by establishing that every element of such lattice possesses *some* irredundant representation through no more than  $k$  join-irreducibles. The next lemma shows in particular that, provided that the lattice is meet-distributive, this condition guarantees  $B(k)$ -freeness.

**5.4.5 Corollary** *If a lattice with finite length is  $B(k+1)$ -free, then every element has some representation of size at most  $k$ . The converse holds if the lattice is meet-distributive.*

*Proof* Suppose that  $L$  is  $B(k+1)$ -free. Given  $x \in L$ , one takes some irredundant representation of  $x$ . By Lemma 5.3.1, such irredundant representation must have at most  $k$  elements. For the converse, suppose that each  $x \in L$  has some representation with at most  $k$  elements. Then, clearly each  $x \in L$  has some irredundant representation, say  $S_x$ , with  $|S_x| \leq k$ . Further, suppose that  $L$  is meet-distributive. Then, by Lemma 5.4.3, the family  $\{S_x\}_{x \in L}$  comprises all irredundant representations and  $L$  is  $B(k+1)$ -free.  $\square$

A complete lattice  $L$  is called *atomistic* if  $x = \bigvee A_x$  holds for every  $x \in L$ . In this case,  $A(L) = J(L)$ . We have now sufficient facts to show that extremal lattices are meet-distributive. This will be done in Lemma 5.4.6. Besides, recall that chains are exactly the  $(n, k)$ -extremal lattices with  $k = 2$ . Said lemma proves that those are the only cases of non-atomistic extremal lattices.

**5.4.6 Lemma** *Any  $(n, k)$ -extremal lattice is meet-distributive. If  $k \geq 2$ , then each of its maximal chains has length  $n$ . In case  $k \geq 3$ , it holds that such lattice is atomistic.*

*Proof* Let  $L$  be an  $(n, k)$ -extremal lattice. Observe that Lemma 5.3.1 and the  $B(k)$ -freeness of  $L$  imply that  $L$  has at most  $f(n, k)$  irredundant representations. Since  $f(n, k)$  is also the number of elements of  $L$ , it holds that each element has exactly one irredundant representation. Meet-distributivity follows from item *i*) of Lemma 5.4.3. Suppose that  $k \geq 2$ . Corollary 5.4.2 yields  $|J(L)| = n$ . Using item *iv*) of Lemma 5.4.3, we have that  $|J_y \setminus J_x| = 1$  for each  $x, y$  with  $x < y$ . Therefore, every maximal chain  $x_0 < x_1 < \dots < x_k$  must satisfy  $|J_{x_i} \setminus J_{x_{i-1}}| = 1$  for  $2 \leq i \leq k$  as well as  $x_0 = 0_L$  and  $x_k = 1_L$ . This forces  $k = n$  because  $J_{x_k} = J(L)$ . Regarding the last claim, suppose that  $k \geq 3$ . If  $L$  were not atomistic, there would exist distinct  $x, y \in J(L)$  with  $x \vee y = x$ , contradicting the fact that each subset of  $J(L)$  of size at most  $k - 1$  is a representation of a different element.  $\square$

### Construction of extremal lattices

At this point, the existence of extremal lattices is clear only for the cases  $k = 2, k = 1$  and  $n < k$ . In order to construct  $(n, k)$ -extremal lattices with arbitrary  $k$ , we will use an operation which we call *doubling* and is defined as follows:

**5.4.7 Definition** Let  $L$  be an ordered set and  $K \subseteq L$ . The **doubling of  $K$  in  $L$**  is defined to be  $L[K] = L \cup \dot{K}$ , where  $\dot{K}$  is a disjoint copy of  $K$ . The order in  $(L[K], \leq')$  is defined as follows:

$$\leq' = \leq \cup \{(x, \dot{y}) \in L \times \dot{K} \mid x \leq y\} \cup \{(\dot{x}, \dot{y}) \in \dot{K} \times \dot{K} \mid x \leq y\}.$$

We will employ the notation  $\dot{x}$  to denote the image under doubling of an element  $x \in K$ . Note that  $x < \dot{x}$  for every  $x \in K$ , and that  $\dot{x}$  is the only upper cover of  $x$  in  $\dot{K}$ . When  $L$  is a family of subsets  $\mathcal{C} \subseteq \mathcal{P}(G)$ , then the diagram of  $L[K]$  can be easily depicted: the doubling  $\mathcal{C}[\mathcal{D}]$  (with  $\mathcal{D} \subseteq \mathcal{C}$ ) corresponds to the family of subsets  $\mathcal{C} \cup \{D \cup \{g\} \mid D \in \mathcal{D}\}$ , where  $g \notin G$  is a new element. As usual, a family of subsets  $\mathcal{C} \subseteq \mathcal{P}(G)$  is called a *closure system* if  $G \in \mathcal{C}$  and  $\mathcal{C}$  is closed under arbitrary intersections. Figure 5.3 illustrates three doubling operations. The first one is the doubling of the chain  $\{\emptyset, \{2\}, \{1, 2\}\}$  inside the closure system  $\mathcal{C}_1 = \mathcal{P}([2])$ , resulting in  $\mathcal{C}_2$ . The (*a fortiori*) closure systems  $\mathcal{C}_3$  and  $\mathcal{C}_4$  are obtained by doubling, respectively, the chains  $\{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$  and  $\{\emptyset, \{2\}, \{2, 3\}, \{1, 2, 3\}\}$  inside  $\mathcal{C}_2$ .

Doubling elements in a concept lattice has a corresponding contextual operation, which we introduce now. Let  $\mathbb{K} = (G, M, I)$  be a formal context, let  $\bullet \notin G$  and  $\ast \notin M$  be “new”

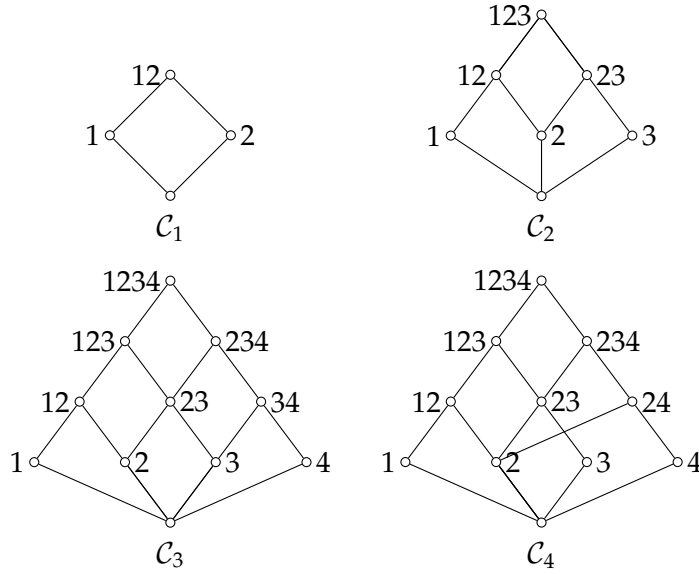


Figure 5.3.: Doubling chains inside closure systems

elements and let  $\mathcal{E} \subseteq \text{Ext}(G, M, I)$  be some set of extents of  $\mathbb{K}$ . Assume  $\mathcal{E} \cap M = \emptyset$  and define a formal context  $\mathbb{K}_{\mathcal{E}} := (G \cup \{\bullet\}, M \cup \{*\} \cup \mathcal{E}, I_{\mathcal{E}})$ , where:

$$I_{\mathcal{E}} := I \cup (G \times \{*\}) \cup (\{\bullet\} \times \mathcal{E}) \cup \{(g, E) \mid g \in G, E \in \mathcal{E}, g \in E\},$$

see Figure 5.4 for a depiction.

	$M$			$*$	$\mathcal{E}$		
$G$	$I$			$\times$	$\in$		
				$\vdots$			
$\bullet$	$\emptyset$	$\dots$	$\emptyset$	$\times$	$\times$	$\dots$	$\times$

Figure 5.4.: Doubling the extents in  $\mathcal{E} \subseteq \text{Ext}(G, M, I)$ 

In the concept lattice  $\underline{\mathfrak{B}}(\mathbb{K}_{\mathcal{E}})$ , the formal concepts  $(*', *')$  and  $(\bullet'', \bullet')$  are such that every element of  $\underline{\mathfrak{B}}(\mathbb{K}_{\mathcal{E}})$  is either below  $(*', *')$  or above  $(\bullet'', \bullet')$ . Moreover,  $\downarrow (*', *')$  is isomorphic to  $\underline{\mathfrak{B}}(G, M, I)$  (in fact, the extents are the same). Similarly,  $\uparrow (\bullet'', \bullet')$  is isomorphic to  $\underline{\mathfrak{B}}(G, \mathcal{E}, \in)$ , which is in turn isomorphic to the closure system generated

by  $\mathcal{E}$ . In particular, if  $\mathcal{E}$  was a finite closure system in the first place, the resulting lattice  $\mathfrak{B}(\mathbb{K}_{\mathcal{E}})$  has  $|\mathfrak{B}(\mathbb{K})| + |\mathcal{E}|$  concepts.

Since we are interested in constructing lattices, it is important to guarantee that the doubling operation produces a lattice. By a *meet-subsemilattice* of a lattice  $L$  is meant a subset  $K$  of  $L$ , endowed with the inherited order, such that  $x \wedge y \in K$  holds for every  $x, y \in K$ . It is called *topped* if  $1_L \in K$ .

**5.4.8 Proposition** *If  $K$  is a topped meet-subsemilattice of a lattice  $L$ , then  $L[K]$  is a lattice.*

*Proof* Let  $x, y \in L[K]$ . If both  $x$  and  $y$  belong to  $L$ , then clearly  $x \wedge y$  and  $x \vee y$  belong to  $L \subseteq L[K]$ . Suppose that only one of  $x$  and  $y$ , say  $x$ , belongs to  $L$ . Then  $y = \dot{z}$  for some  $z \in K$ . First, we establish that  $x \wedge y$  exists. Let  $u \in L[K]$  be arbitrary with  $u \leq x$  and  $u \leq y$ . By the definition of doubling, it follows that  $u \leq z$ . Hence, any lower bound of  $x$  and  $\dot{z}$  must also be a lower bound of  $x$  and  $z$ . Observe that the converse holds trivially. Therefore, in particular, the greatest lower bound of  $x$  and  $z$  (which exists since  $L$  is a lattice) is the meet  $x \wedge y$  as well. For the supremum, set  $S = \{w \in K \mid w \geq x, w \geq z\}$  and  $u = \bigwedge S$ . Note that the fact that  $K$  is topped causes  $S \neq \emptyset$ . Since  $K$  is a meet-subsemilattice, we have that  $u \in K$ . It is clear that  $u$  is the least upper bound of  $x$  and  $z$  which belongs to  $K$ . Thus,  $\dot{u}$  is the least upper bound of  $x$  and  $y$ , because of  $0_K \not\leq u$  and  $y = z \vee 0_K$ . The remaining case is  $x, y \in \dot{K}$  for which, clearly  $x \wedge y$  exists. Moreover, writing  $x = \dot{t}, y = \dot{z}$  with  $t, z \in K$  and setting  $S = \{w \in K \mid w \geq t, w \geq z\}$  as well as  $u = \bigwedge S$  makes clear that  $\dot{u} = x \vee y$ .  $\square$

When considered extrinsically, topped meet-subsemilattices are lattices. Therefore, the meaning of an  $(n, k)$ -*extremal meet-subsemilattice* is well defined, provided that such substructure is topped. Also note that this is compatible with the proof of Proposition 5.4.8, where the supremum and infimum of two elements in  $\dot{K}$  may be easily verified to belong to  $\dot{K}$ : that is,  $\dot{K}$  is actually a sublattice of  $L[K]$ .

Now, we turn our attention to sufficient conditions for  $L[K]$  to be not only a lattice, but a meet-distributive one as well. For that purpose, we define that a suborder  $K$  of an ordered set  $L$  is *cover-preserving* if  $x <_K y$  implies  $x <_L y$  for every  $x, y \in K$ .

**5.4.9 Proposition** *Let  $L$  be a meet-distributive lattice and let  $K$  be a cover-preserving, topped meet-subsemilattice of  $L$ . Then,  $L[K]$  is a meet-distributive lattice.*

*Proof* The fact that  $L[K]$  is a lattice comes from Proposition 5.4.8. Every element  $\dot{x} \in \dot{K}$  has one lower cover in  $K$ , namely,  $x$ . Thus, the total number of lower covers of  $\dot{x}$  is one only if  $x$  does not cover any element in  $K$ , that is,  $x = 0_K$ . Therefore,  $0_K$  is the only join-irreducible of  $L[K]$  which is not a join-irreducible of  $L$ . Let  $x, y \in L[K]$  with  $x <_{L[K]} y$ . We use  $J'_{(\cdot)}$  to denote our  $J$ -notation in  $L[K]$  and  $J_{(\cdot)}$  in  $L$ . To prove that  $L[K]$  is meet-distributive, we show that  $|J'_y \setminus J'_x| = 1$ . Such condition is equivalent to meet-distributivity according to Lemma 5.4.3. If  $x, y \in L$ , then clearly  $J'_y = J_y$  and  $J'_x = J_x$ , which results in

$|J'_y \setminus J'_x| = |J_y \setminus J_x| = 1$ . If  $x, y \notin L$ , then  $x = \dot{z}$  and  $y = \dot{w}$  with  $z, w \in K$  and  $z <_K w$ . From the fact that  $K$  is cover-preserving, we conclude that  $z <_L w$ . Because  $L$  is meet-distributive, it follows that  $|J_w \setminus J_z| = 1$ . Clearly one has  $J'_x = J_z \cup \{\dot{0}_K\}$  and  $J'_y = J_w \cup \{\dot{0}_K\}$ , which yields  $|J'_y \setminus J'_x| = 1$ . For the remaining case, one has necessarily  $x \in L$  and  $y \notin L$ . In these conditions,  $x < y$  results in  $y = \dot{x}$  and, therefore,  $J'_y = J'_x \cup \{\dot{0}_K\}$ , implying  $|J'_y \setminus J'_x| = 1$ .  $\square$

The conditions on  $K$  present in Proposition 5.4.9 suggest how the construction of extremal lattices shall work. Essentially, we apply the doubling of some cover-preserving, topped, extremal meet-subsemilattice in an extremal lattice; the result will be an extremal lattice (with appropriate parameters). One particular case needs no further results and can be depicted now: consider an  $(n, 2)$ -extremal lattice, that is, a chain of length  $n$ . Such lattice may be seen as the doubling of the trivial meet-subsemilattice  $\uparrow 1$  inside an  $(n - 1, 2)$ -extremal lattice. An important objective of this section is to generalize this operation. More precisely, our construction principle is to double cover-preserving, topped,  $(n - 1, k - 1)$ -extremal meet-subsemilattices inside  $(n - 1, k)$ -extremal lattices, yielding  $(n, k)$ -extremal lattices for  $k \geq 3$ . In order to reach this level of generality, we need to prove properties of extremal meet-subsemilattices inside extremal lattices. The first step towards this is done by Proposition 5.4.10, which shows that non-trivial, extremal meet-subsemilattices are always cover-preserving and topped.

**5.4.10 Proposition** *Let  $L$  be an  $(n, k)$ -extremal lattice with  $k \geq 3$  and  $K$  an  $(n, k - 1)$ -extremal meet-subsemilattice of  $L$ . Then,  $K$  is cover-preserving and topped. If  $k \geq 4$ , then  $K$  and  $L$  are atomistic with  $A(K) = A(L)$ .*

*Proof* If  $k = 3$ , then  $K$  is an  $n + 1$ -element chain. By Lemma 5.4.6, we have that the length of any maximal chain of  $L$  is  $n$ . Hence,  $K$  is a maximal chain in  $L$  and  $1_K = 1_L$ . The maximality of  $K$  guarantees that  $K$  is cover-preserving. Now, suppose that  $k \geq 4$ . Corollary 5.4.2 yields that  $L$  and  $K$  have precisely  $n$  join-irreducible elements. Actually, those elements are atoms because Lemma 5.4.6 implies that both  $L$  and  $K$  are atomistic. The same lemma guarantees that each maximal chain of  $K$  - as well as each maximal chain of  $L$  - has length  $n$  and, therefore, every maximal chain of  $K$  is a maximal chain of  $L$  as well. Hence, in particular, one has that  $1_L = 1_K$ ,  $A(K) = A(L)$  and that  $K$  is cover-preserving.  $\square$

As explained after Proposition 5.4.9, the general principle of our construction is to double an extremal meet-subsemilattice of an extremal lattice. The next theorem shows that the lattice produced by this operation is indeed extremal.

**5.4.11 Theorem** *Let  $L$  be an  $(n - 1, k)$ -extremal lattice with  $n \geq 2$ ,  $k \geq 3$  and suppose that  $K$  is an  $(n - 1, k - 1)$ -extremal meet-subsemilattice of  $L$ . Then,  $L[K]$  is an  $(n, k)$ -extremal lattice.*

*Proof* From Lemma 5.4.6 it follows that  $L$  is atomistic and meet-distributive, while Proposition 5.4.10 guarantees that  $K$  is cover-preserving and topped, so that, in particular,  $L[K]$

is a meet-distributive lattice, as a consequence of Proposition 5.4.9. Note that  $J(L[K]) = A(L) \cup \dot{0}_K$ , on account of  $L$  being atomistic. In particular,  $L[K]$  has precisely  $n$  atoms. Because of Proposition 5.2.10, one clearly has that  $|L[K]| = |L| + |K| = f(n, k)$ . Thus, to prove that  $L[K]$  is  $(n, k)$ -extremal it only remains to show that it is  $B(k)$ -free. By Corollary 5.4.5, this can be done by showing that every element of  $L[K]$  has a representation of size at most  $k - 1$ . For that purpose, let  $x \in L[K]$ . If  $x \in L$ , then we can take its irredundant representation in  $L$  (which has size at most  $k - 1$ ) as a representation in  $L[K]$ . Otherwise, one has that  $x \in \dot{K}$  and we represent it as  $x = \dot{y} = y \vee \dot{0}_K$  for some  $y \in K$ . Now, consider two cases. If  $k = 3$  then, by Lemma 5.4.6,  $K$  is a maximal chain in  $L$ . If  $y \in A(L)$ , we are done. Otherwise, take  $z, w \in A(L)$  such that  $z \vee w = y$  and thus  $\dot{y} = x = z \vee w \vee \dot{0}_K$ . Let  $u$  be the only element of  $K$  covered by  $y$ . Then,  $z \not\leq u$  or  $w \not\leq u$ . Without loss of generality, suppose that  $z \not\leq u$ . We show that  $z \vee \dot{0}_K$  is a representation of  $x$ . On the one hand, clearly  $z \vee \dot{0}_K \leq \dot{y} = x$ . On the other hand, note first that a direct consequence of the definition of doubling is the validity of the implication  $\alpha \not\leq \beta \Rightarrow \alpha \not\leq \dot{\beta}$ , for any  $\alpha \in L, \beta \in K$ . In particular, we have that  $z \not\leq \dot{u}$ , on account of  $z \not\leq u$ . Consequently, it holds that  $z \vee \dot{0}_K \not\leq \dot{u}$ . Because both  $z \vee \dot{0}_K$  and  $\dot{u}$  belong to the chain  $\dot{K}$ , we have  $z \vee \dot{0}_K > \dot{u}$ . Clearly,  $\dot{u}$  has only  $\dot{y} = x$  as its upper cover, resulting in  $z \vee \dot{0}_K \geq x$  and, altogether,  $z \vee \dot{0}_K = x$ . For the second case, we have  $k \geq 4$ . Then, by Proposition 5.4.10,  $K$  is an atomistic lattice with  $A(L) = A(K)$ . We use the fact that  $K$  is  $B(k - 1)$ -free, together with Corollary 5.4.5, to write  $y = \bigvee_K S$  for some  $S \subseteq A(K) \subseteq J(L[K])$  with  $|S| \leq k - 2$ . Clearly, in  $L[K]$ , one has  $\dot{y} = \bigvee \dot{S} = \dot{0}_K \vee \bigvee S$ , where the last equality follows from the fact that  $\dot{z} = z \vee \dot{0}_K$  for every  $z \in S$ . Thus, we have a representation of  $\dot{y} = x$  through no more than  $k - 1$  join-irreducible elements of  $L[K]$ .  $\square$

Theorem 5.4.11 sustains the core principle of our construction. Notice that said result requires, as input, some substructure inside an extremal lattice: only then it is able to yield a larger extremal lattice. Since our goal is to construct extremal lattices having arbitrarily given parameters  $n$  and  $k$ , we wish to apply that construction principle indefinitely. This will be possible by making use of Proposition 5.4.12, which aids us in the task of keeping track of extremal meet-subsemilattices inside extremal lattices. For that, the following notion is necessary. A *complete meet-embedding* is a meet-embedding which preserves arbitrary meets, including  $\bigwedge \emptyset$ . As a consequence, the greatest element of one lattice gets mapped to the greatest element of the other. Images of complete meet-embeddings are topped meet-subsemilattices. In Proposition 5.4.12, the symbol  $K[J]$  (for instance) means actually the doubling of the image of  $J$  under the corresponding embedding.

**5.4.12 Proposition** *Suppose that  $J, K$  and  $L$  are lattices with complete meet-embeddings  $\mathcal{E}_1 : J \rightarrow K$  and  $\mathcal{E}_2 : K \rightarrow L$ . Then, there exists a complete meet-embedding from  $K[J]$  into  $L[K]$ .*

*Proof* The fact that  $K[J]$  and  $L[K]$  are lattices comes from Proposition 5.4.8. Of course, there is an induced embedding from  $\dot{J}$  into  $\dot{K}$ , but for which we will use the same symbol  $\mathcal{E}_1$ . The mapping  $\mathcal{E}_3 : K[J] \rightarrow L[K]$  defined by  $\mathcal{E}_3(x) = \mathcal{E}_1(x)$  for  $x \in \dot{J}$  and  $\mathcal{E}_3(x) = \mathcal{E}_2(x)$  for  $x \in K$  may be checked as being a complete meet-embedding.  $\square$

At this point, we find ourselves in the position to establish the existence of extremal lattices for each choice of parameters  $n$  and  $k$ .

**5.4.13 Corollary** *For every  $n$  and  $k$ , there exists at least one  $(n, k)$ -extremal lattice.*

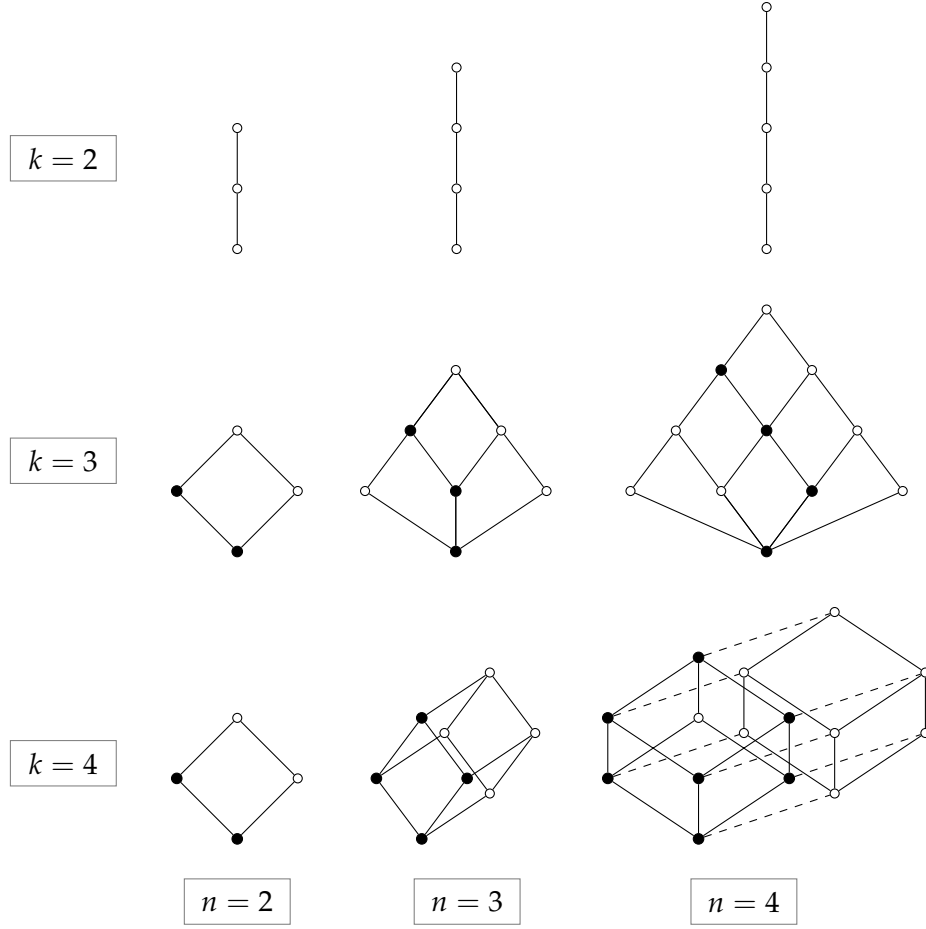
*Proof* Define a partial function  $\Phi$  satisfying

$\Phi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{Lat}$

$$(n, k) \mapsto \begin{cases} ([n], \subseteq), & \text{if } k = 1. \\ (\{\emptyset, \{1\}\}, \subseteq), & \text{if } k \geq 2, n = 1. \\ \Phi(n-1, k)[\mathcal{E}(\Phi(n-1, k-1))], & \text{if } n, k \geq 2 \text{ and there exists a} \\ & \text{complete meet-embedding} \\ & \mathcal{E} : \Phi(n-1, k-1) \rightarrow \\ & \Phi(n-1, k). \end{cases}$$

where  $\mathbf{Lat}$  is the class of all lattices. We prove by induction on  $n$  that  $\Phi(n, k)$  is a total function. The cases  $n = 1$  and  $n = 2$  are trivial. Let  $n \in \mathbb{N}$  with  $n \geq 3$  and suppose that  $\Phi(n-1, k)$  is defined for every  $k \in \mathbb{N}$ . Let  $k \in \mathbb{N}, k \geq 2$ . By the induction hypothesis, the values  $\Phi(n-1, k)$  and  $\Phi(n-1, k-1)$  are defined. If  $k = 2$ , then  $\Phi(n-1, k-1)$  is a trivial lattice and the existence of a complete meet-embedding into  $\Phi(n-1, k)$  is clear and, thereby,  $\Phi(n, k)$  is defined. We therefore assume  $k \geq 3$ . By the definition of  $\Phi$ , one has that  $\Phi(n-1, k) = \Phi(n-2, k)[\mathcal{E}(\Phi(n-2, k-1))]$  and that  $\Phi(n-1, k-1) = \Phi(n-2, k-1)[\mathcal{F}(\Phi(n-2, k-2))]$  for some pair of complete meet-embeddings  $\mathcal{E}$  and  $\mathcal{F}$ . Applying Proposition 5.4.12 with  $\Phi(n-2, k-2)$ ,  $\Phi(n-2, k-1)$  and  $\Phi(n-2, k)$  results in the existence of a complete meet-embedding  $\mathcal{G} : \Phi(n-1, k-1) \rightarrow \Phi(n-1, k)$ , which yields that  $\Phi(n, k)$  is defined. Since  $k$  is arbitrary, every  $\Phi(n, k)$  is defined. The  $(n, k)$ -extremality of each lattice can be proved by induction on  $n$  as well and by invoking Theorem 5.4.11.  $\square$

Figure 5.5 depicts the diagrams of nine  $(n, k)$ -extremal lattices which are constructible by Corollary 5.4.13, where elements shaded in black represent the doubled  $(n-1, k-1)$ -extremal lattices. It is true that, in general,  $(n, k)$ -extremal lattices are not unique up to isomorphism: note that the  $(3, 3)$  and  $(4, 3)$ -extremal lattices in Figure 5.5 are also present in Figure 5.3 as the lattices  $\mathcal{C}_2$  and  $\mathcal{C}_3$ . The lattice  $\mathcal{C}_4$ , depicted in that same figure, is a  $(4, 3)$ -extremal lattice which is not isomorphic to  $\mathcal{C}_3$ . We shall, however, show in the next section that every extremal lattice arises from the construction described in Corollary 5.4.13.

Figure 5.5.: Some  $(n, k)$ -extremal lattices,  $2 \leq n, k \leq 4$ .

## 5.5. Characterization of extremal lattices

In the last section, we constructed lattices whose sizes are exactly the upper bound present in Theorem 5.2.6. In this section, we will show that every lattice meeting those requirements must be obtained from our construction.

**5.5.1 Lemma** *Let  $L$  be an atomistic lattice,  $a$  an atom and  $c$  a coatom with  $A_c = A(L) \setminus \{a\}$ . Then, the mapping  $x \xrightarrow{\mathcal{E}} c \wedge x$  is a complete meet-embedding of  $\uparrow a$  into  $\downarrow c$  such that  $\mathcal{E}(x) < x$  for every  $x \in \uparrow a$ .*

*Proof* The fact that  $\mathcal{E}$  preserves non-empty meets is clear, because  $c$  is a fixed element. Also,  $1_L$  is mapped to  $c = 1_{\downarrow c}$ , so that  $\mathcal{E}$  preserves arbitrary meets. Note that

$$A_{\mathcal{E}(x)} = A_{c \wedge x} = A_x \cap A_c = A_x \setminus \{a\}.$$

Hence,  $\mathcal{E}(x) < x$  as well as  $\mathcal{E}(x) \vee a = x$ . The latter implies injectivity.  $\square$

The next theorem shows that every extremal lattice is constructible by the process described in Corollary 5.4.13, and can be seen as a converse of that result.

**5.5.2 Theorem** *Let  $L$  be an  $(n, k)$ -extremal lattice with  $k \geq 3$ . Then,  $L = J \cup K$  where  $J$  is an  $(n - 1, k)$ -extremal lattice and  $K$  is an  $(n - 1, k - 1)$ -extremal lattice. Moreover, there exists a complete meet-embedding  $\mathcal{E} : K \rightarrow J$  such that  $\mathcal{E}(x) < x$  for every  $x \in K$ . In particular,  $L \cong J[\mathcal{E}(K)]$ .*

*Proof* Let  $c$  be any coatom of  $L$  and let  $1$  denote its greatest element. From Lemma 5.4.6, one has that  $L$  is meet-distributive and atomistic. Using these two properties, as well as Lemma 5.4.3, we arrive at the fact that  $J_1 \setminus J_c = A(L) \setminus A_c$  is a singleton. With that, we have  $A_c = A(L) \setminus \{a\}$ , where  $a$  is the only element of  $A(L) \setminus A_c$ . Consider the lattices  $J = \downarrow c$  and  $K = \uparrow a$ . Observe that  $L = J \cup K$  and let  $\mathcal{E} : K \rightarrow J$  be a complete meet-embedding provided by Lemma 5.5.1. Clearly,  $J$  has  $n - 1$  atoms and is  $B(k)$ -free, therefore,  $|J| \leq f(n - 1, k)$ . Moreover,  $K$  must be  $B(k - 1)$ -free: indeed, if there existed  $B \cong B(k - 1)$  inside  $K$ , then  $B \cup \mathcal{E}(B)$  would be a boolean lattice with  $k$  atoms inside  $L$ , which is impossible. The lattice  $K$  has at most  $n - 1$  atoms, and consequently  $|K| \leq f(n - 1, k - 1)$ , since the function  $n \mapsto \sum_{i=0}^{k-1} \binom{n}{i}$  is monotonic increasing. Now, we have that  $|J| + |K| = |L| = f(n, k) = f(n - 1, k) + f(n - 1, k - 1)$ , where the last equality follows from Proposition 5.2.10. Because of  $|J| \leq f(n - 1, k)$  and  $|K| \leq f(n - 1, k - 1)$ , those two inequalities must hold with equality. Therefore,  $J$  and  $K$  are, respectively,  $(n - 1, k)$  and  $(n - 1, k - 1)$ -extremal.  $\square$

## 5.6. Extremality and shattering

The interplay between contranominal subcontexts and minimal generators was of fundamental importance to establish the upper bound present in Theorem 5.2.6. Making use of minimal generators, however, is not necessary to arrive at such result. In this section, we show an alternative proof which uses the notion of *shattered sets*. These form a widely studied topic in extremal combinatorics and, in particular, extremal set theory. The related, famous lemma of Sauer and Shelah is an important result in at least logic, set theory and probability (see, for example [26, 38, 40, 50, 54]). For the purposes of this section, we set some terminology. Let  $\mathcal{C} \subseteq \mathcal{P}(G)$  be a family of subsets. If  $H \subseteq G$ , then we denote by  $\mathcal{C}|_H$  the *trace of  $\mathcal{C}$  on  $H$* :  $\mathcal{C}|_H := \{A \cap H \mid A \in \mathcal{C}\}$ . A set  $H \subseteq G$  is *shattered* if  $\mathcal{C}|_H = \mathcal{P}(H)$ . The *Vapnik-Chervonenkis dimension* of  $\mathcal{C}$  (also called *VC-dimension*) is the maximum cardinality of a shattered set. A set having precisely  $k$  elements will be called a *k-set*.

**5.6.1 Lemma (Sauer-Shelah)** *If  $\mathcal{A}$  is a family of subsets of  $[n]$  and  $|\mathcal{A}| > f(n, k)$ , then  $\mathcal{A}$  shatters some  $k$ -set.*

*Proof* See Theorem 10.1 of [40]. □

The next lemma provides the connection between shattered sets and contranominal scales. As it was with minimal generators, the relationship here is direct: objects sets of contranominal scales are precisely the collection of sets shattered by the system of extents.

**5.6.2 Lemma** *Let  $\mathbb{K} = (G, M, I)$  be a formal context and  $A \subseteq G$ . There exists a contranominal subcontext of  $\mathbb{K}$  having  $A$  as its object set if and only if  $\text{Ext } \mathbb{K}$  shatters  $A$ . In particular,  $\mathbb{K}$  is  $\mathbb{N}^c(k)$ -free if and only if  $\text{Ext } \mathbb{K}$  does not shatter any  $k$ -set of  $G$ .*

*Proof* Let  $\mathbb{K}_1 \leq \mathbb{K}$  be a contranominal scale with object set  $A$ . Then, it holds that  $\text{Ext } \mathbb{K}_1 = \mathcal{P}(A)$ . By Proposition 32 of [35], we have that the mapping  $S \mapsto S''$  is an order embedding of  $\text{Ext } \mathbb{K}_1$  into  $\text{Ext } \mathbb{K}$ . From the fact that the closure operator is extensive, it follows that  $\text{Ext } \mathbb{K}|_A = \text{Ext } \mathbb{K}_1 = \mathcal{P}(A)$ , that is,  $\text{Ext } \mathbb{K}$  shatters  $A$ . For the converse, suppose that  $\text{Ext } \mathbb{K}$  shatters  $A$ . We may suppose  $A \neq \emptyset$ . The fact that  $\text{Ext } \mathbb{K}$  shatters  $A$  allows us to take one extent  $A_g$  for each  $g \in A$  such that  $A_g \cap A = A \setminus \{g\}$ . In particular, for every  $g \in A$  there exists an attribute  $m_g \in A'_g$  such that  $m'_g \cap A = A \setminus \{g\}$ . Let  $B = \{m_g \mid g \in A\}$ . Then,  $(A, B, I \cap (A \times B))$  is a subcontext of  $\mathbb{K}$  which is a contranominal scale. □

**5.6.3 Corollary** *A  $\mathbb{N}^c(k)$ -free context  $\mathbb{K}$  with  $n$  objects has at most  $f(n, k)$  concepts.*

*Proof* A direct application of Lemma 5.6.2 followed by the contraposition of Lemma 5.6.1. □

Coming up with a family of subsets with  $f(n, k)$  elements and VC-dimension at most  $k$  requires much less effort than going through the doubling construction exposed here; it however does not yield a closure system. More precisely, the following easy observation is present in [40].

**5.6.4 Proposition** *The family of all subsets of  $[n]$  with less than  $k$  elements has  $f(n, k)$  elements and does not shatter any  $k$ -set of  $[n]$ .*

If we restate Corollary 5.4.13 in these terms we get

**5.6.5 Proposition** *There exists a closure system over  $[n]$  that has  $f(n, k)$  elements and does not shatter any  $k$ -set of  $[n]$ .*

*Proof* It suffices to take the system of extents of an  $(n, k)$ -extremal lattice produced by Corollary 5.4.13. □

## The implication logic of extremal lattices

The improved breadth upper bound was shown to be sharp in Chapter 5. This was done by means of a bottom-up, lattice-oriented construction, the principle of which was to make copies of smaller subsemilattices. The resulting structures have a controlled number of join-irreducibles, but it remains elusive how one could estimate their number of meet-irreducibles. This is clearly relevant, since this is the minimum number of attributes of a realizing context.

An approach based on the opposite idea consists of building these structures through some top-down principle. This can be conceived after pondering the following facts. It is well known that the sets which respect a collection of implications form a closure system, and therefore a lattice. Trivially, if the collection is empty, then the associated respecting sets are just one large boolean lattice. If we somehow add implications to this collection, then the associated closure system clearly shrinks. Hence, at some point, one will obtain a lattice which has no boolean suborder larger than the prescribed limit. This chapter establishes that, by carefully choosing implication sets, one obtains a  $B(k)$ -free lattice with the correct number of elements and join-irreducibles: that is, an extremal lattice.

In Section 6.1, we introduce the notion of  $(n, k)$ -extremal sets of implications. Further, in Section 6.2 we explicitly describe  $(n, k)$ -extremal lattices with precisely  $\binom{n}{k-1} + k - 2$  meet-irreducibles. Section 6.3 deals with the construction (Subsection 6.3.1) and characterization (Subsection 6.3.2) of extremal sets of implications. In the fourth and last section, it is established that the introduced notion encompasses the canonical bases of extremal lattices, thereby providing a characterization of these structures through their implication logic.

The construction exposed in this chapter confirms that  $(n, k)$ -extremal lattices sometimes possess a great quantity of meet-irreducibles. This corroborates that the pursuit of a breadth-based, attribute-sensitive majorant is an interesting problem. Such problem acts as the motivation for the first developments here and is more formally explained in Question 6.1.1.

## 6.1. Motivation and fundamental results

An important motivation to understand how many meet-irreducibles the  $(n, k)$ -extremal lattices possess is the following problem:

**6.1.1 Question** *Is it possible to prove an upper bound which is sharper than  $\sum_{i=0}^{k-1} \binom{n}{i}$  by exploiting the number of attributes of  $\mathbb{K}$ ? More generally, what are  $(n, m, k)$ -extremal lattices, where  $m$  stands for the maximum number of meet-irreducibles?*

With other terms, the question above asks for the solution of the problem with  $\mathcal{H} = \mathcal{H}^{n,m}$ , in the sense described in Chapter 4. We begin this investigation by proving some elementary results in this first section. Most of them are very intuitive and some are already well known facts in the area.

In contrast to Chapter 3, our implications here are between objects. Therefore, an implication  $A \rightarrow B$  is a pair of subsets of  $G$ . If  $\mathcal{L}$  is a set of implications and every premise and every conclusion is a subset of  $G$ , then we say that  $G$  is a *set of implications over  $G$* . The *ground set of  $\mathcal{L}$*  is the union of all premises and conclusions and denoted  $\Gamma(\mathcal{L})$ .

The closure operator associated with a set of implications can be described as follows [35]. For  $S \subseteq G$ , define  $S^{\mathcal{L}} = S \cup \bigcup \{B \mid A \rightarrow B \in \mathcal{L}, S \supseteq A\}$ . The application of  $(\cdot)^{\mathcal{L}}$  onto  $S$  may be seen as the one-step *modus ponens* deduction of  $S$ . The set  $S^{\mathcal{L}}$  is not, in general, closed (i. e. it does not respect every implication in  $\mathcal{L}$ ). Instead, the closed sets are precisely the fixed points of this operator. The closure of an arbitrary  $S \subseteq G$  will be denoted by  $S^{\mathcal{L} \dots \mathcal{L}}$ .

As usual, the closure system  $\mathfrak{H}(\mathcal{L})$  may also be seen as a complete lattice: for an arbitrary family  $(T_i)_i$  of sets respecting  $\mathcal{L}$ , its meet is given by intersection of all members and the supremum is the intersection of all sets which contain each  $T_i$ . This lattice has at most  $|G|$  join irreducible elements, as the next well known fact shows:

**6.1.2 Proposition** *Let  $\mathcal{L}$  be a set of implications over  $G$ . Then, every join-irreducible of  $\mathfrak{H}(\mathcal{L})$  is the closure of a singleton  $\{g\} \subseteq G$ .*

*Proof* Contraposition: let  $T$  be a set respecting  $\mathcal{L}$  which is not the closure of a singleton. If  $T$  is the closure of the empty set, then certainly it is the least element of  $\mathfrak{H}(\mathcal{L})$ , therefore not join-irreducible and we are done. Otherwise,  $T$  is the closure of some non-empty set and  $T$  is therefore non-empty. Every set which respects  $\mathcal{L}$  and contains some element, say,  $t \in G$ , must also contain each element in its closure  $\{t\}^{\mathcal{L} \dots \mathcal{L}}$ . Therefore, it is clear that  $T = \cup_{t \in T} (\{t\}^{\mathcal{L} \dots \mathcal{L}}) = \bigvee_{t \in T} \{t\}^{\mathcal{L} \dots \mathcal{L}}$ , which shows that  $T$  is not join-irreducible because of  $T \neq \{t\}^{\mathcal{L} \dots \mathcal{L}}$  for each  $t \in T$ .  $\square$

From now on a lighter notation regarding braces will be adopted:  $g^{\mathcal{L}}$  denotes  $\{g\}^{\mathcal{L}}$ . Occasionally, an implication like  $\{g_1, g_2, \dots, g_k\} \rightarrow \{h_1, h_2, \dots, h_l\}$  will be simply written as  $g_1 g_2 \dots g_k \rightarrow h_1 h_2 \dots h_l$ .

Consider an arbitrary implication  $P \rightarrow Q \in \mathcal{L}$ . Obviously,  $P^{\mathcal{L} \mathcal{L}} \supseteq P^{\mathcal{L}} \supseteq P \cup Q$ . If the containment  $P^{\mathcal{L} \mathcal{L}} \supseteq P \cup Q$  holds with equality, then the second containment collapses as

well and forces  $P^{\mathcal{L}\mathcal{L}} = P^{\mathcal{L}}$ . That is, in this case,  $P^{\mathcal{L}}$  is a fixed point of the operator  $(\cdot)^{\mathcal{L}}$ , causing  $P^{\mathcal{L}} = P \cup Q$  to be the closure of  $P$ . This will be of great utility because it allows an easy determination of the closures of all premises. Hence, we define that a set of implications  $\mathcal{L}$  is *straight* if  $P^{\mathcal{L}\mathcal{L}} = P \cup Q$  for each  $P \rightarrow Q \in \mathcal{L}$ . Being straight is not very restrictive, in the following sense: it will be demonstrated later in Lemma 6.4.2 that the canonical basis of a finite lattice is always straight.

**6.1.3 Lemma** *A set of implications  $\mathcal{L}$  is straight if and only if*

$$P \cup Q \supseteq R \Rightarrow P \cup Q \supseteq S \quad (\text{"condition for straightness"})$$

*holds for every  $P \rightarrow Q, R \rightarrow S \in \mathcal{L}$ .*

*Proof* Let  $P \rightarrow Q \in \mathcal{L}$ . Observe that  $P^{\mathcal{L}} \supseteq P \cup Q$ . Directly from the definition,

$$P^{\mathcal{L}\mathcal{L}} = P^{\mathcal{L}} \cup \bigcup \{S \mid R \rightarrow S, P^{\mathcal{L}} \supseteq R\}. \quad (6.1)$$

For one direction, suppose that  $\mathcal{L}$  is straight. Then,  $P^{\mathcal{L}\mathcal{L}} = P \cup Q$ . Equation 6.1 forces that  $P^{\mathcal{L}}$ , as well as each set  $S$  inside the arbitrary union, to be contained in  $P \cup Q$ . Hence, we have  $P^{\mathcal{L}} = P \cup Q$  and  $S \subseteq P \cup Q$  for every implication  $R \rightarrow S$  with  $P \cup Q = P^{\mathcal{L}} \supseteq R$ , i. e., the condition for straightness. Conversely,  $P^{\mathcal{L}} = P \cup \bigcup \{S \mid R \rightarrow S, P \supseteq R\}$ . Of course,  $P \supseteq R$  implies  $P \cup Q \supseteq R$ , and the condition for straightness guarantees that each set  $S$  appearing in the arbitrary union must satisfy  $S \subseteq P \cup Q$ . Thus,  $P^{\mathcal{L}} = P \cup Q$ . Using Equation 6.1 and making use of the condition again gives  $P^{\mathcal{L}\mathcal{L}} = P \cup Q$ .  $\square$

An implication set is called *injective* if each pair of distinct premises has distinct closures. Consider a formal context  $\mathbb{K} = (G, M, I)$ . We recall that a set  $S \subseteq G$  is called a *minimal generator with respect to the context  $\mathbb{K}$*  if  $T'' \neq S''$  for every proper subset  $T \subsetneq S$ . What follows is a weaker but consistent definition of a minimal generator from the logical perspective. Let  $\mathcal{L}$  be an implication set. We call a set of objects  $S \subseteq G$  a *minimal generator with respect to  $\mathcal{L}$*  if for each  $P \rightarrow Q \in \mathcal{L}$ , the implication  $S \supseteq P \Rightarrow S \cap (Q \setminus P) = \emptyset$  holds. Observe that, in both notions, any subset of a minimal generator is once again a minimal generator.

Soon enough, we will link the absence of a minimal generator with respect to an implication set to the  $B(k)$ -freeness of its associated lattice. The main work in such direction is conducted by the following proposition:

**6.1.4 Proposition** *Let  $\mathcal{L}$  be a set of implications over  $G$ . Further, let  $S \subseteq G$  and suppose that  $\mathfrak{H}(\mathcal{L})$  is doubly-founded. It holds that if  $S$  is a minimal generator with respect to the standard context of  $\mathfrak{H}(\mathcal{L})$ , then  $S$  is a minimal generator with respect to  $\mathcal{L}$ .*

*Proof* Set  $L := \mathfrak{H}(\mathcal{L})$  and let  $\mathbb{K}$  denote the standard context of  $L$ . Denote by  $''$  the closure operator of object sets of  $\mathbb{K}$ . We show the contraposition. Suppose that  $S$  is not a minimal generator with respect to  $\mathcal{L}$ . We may assume  $S \subseteq J(L)$ , since otherwise the claim holds

trivially. Because  $S$  is not a minimal generator with respect to  $\mathcal{L}$ , one can find  $P \rightarrow Q \in \mathcal{L}$  with  $S \supseteq P$  and  $S \cap (Q \setminus P) \neq \emptyset$ . Let  $s \in S \cap (Q \setminus P)$ . Note that  $S \setminus \{s\} \supseteq P$ . Any set  $T$  which respects  $\mathcal{L}$  (equivalently, any extent  $T$  of  $\mathbb{K}$ ) and contains  $S \setminus \{s\}$  must have  $s$  as well, causing  $(S \setminus \{s\})'' = S''$  and implying that  $S$  is not a minimal generator with respect to  $\mathbb{K}$ .  $\square$

An implication set is said to be *r-regular* if every premise has exactly  $r$  elements. For an  $r$ -regular set  $\mathcal{L}$ , we say that  $\mathcal{L}$  is *saturated* if no  $(r + 1)$ -element subset of its ground set is a minimal generator.

To illustrate the properties “straight”, “injective” and “saturated”, we give examples below of regular sets possessing each two of the properties but not the third.

**Examples:** Consider the following sets  $\mathcal{L} = \{3 \rightarrow 21\}$ ,  $\mathcal{M} = \{3 \rightarrow 21, 1 \rightarrow 2, 2 \rightarrow 1\}$  and  $\mathcal{N} = \{3 \rightarrow 2, 2 \rightarrow 1\}$ . The reader should have little or no trouble verifying that  $\mathcal{L}$  is straight and injective but not saturated (“21” is a minimal generator with more than  $r$  elements),  $\mathcal{M}$  is straight and saturated but not injective (we have  $1^{\mathcal{M}} = 2^{\mathcal{M}}$  and this implies  $1^{\mathcal{M} \dots \mathcal{M}} = 2^{\mathcal{M} \dots \mathcal{M}}$ ), and  $\mathcal{N}$  is injective and saturated but not straight.

Whenever  $\mathcal{L}$  is  $r$ -regular and saturated, a natural upper bound is imposed over the numbers of elements of the closure system  $\mathfrak{H}(\mathcal{L})$ , as the next proposition shows.

**6.1.5 Proposition** *Let  $\mathcal{L}$  be an  $r$ -regular, saturated set of implications over a finite set  $G$ . Then, the closure of any  $S \subseteq G$  with  $|S| \geq r + 1$  equals the closure of some premise  $P$  of  $\mathcal{L}$  with  $P \subseteq S$ .*

*Proof* Because  $S$  can not be a minimal generator, there exists an implication  $P \rightarrow Q \in \mathcal{L}$  with  $S \supseteq P$  and an element  $s \in S \cap (Q \setminus P)$ . Now, one clearly has that  $S \setminus \{s\} \supseteq P$  which implies that  $S$  and  $S \setminus \{s\} =: T$  have the same closure. If  $T$  has  $r$  elements, then  $T \supseteq P$  together with  $|P| = r$  (regularity of  $\mathcal{L}$ ) force  $P = T$  and we are done. If  $T$  has more than  $r$  elements, then one repeats this argument a necessary number of times, obtaining at each step another proper subset of  $S$  with the same closure.  $\square$

We summarize the assertion present in Proposition 6.1.5 and a few other facts below:

**6.1.6 Proposition** *The sets which respect an  $r$ -regular, saturated set of implications over a finite  $G$  are precisely the subsets of  $G$  with at most  $r - 1$  elements, together with its  $r$ -element subsets which are not premises and the closures of each premise (which are given by  $P \cup Q$  for each  $P \rightarrow Q$ , in case straightness is satisfied).*

*Proof* An  $r$ -element set which is not a premise must respect the implication set, since regularity forces it not to contain any premise. The other parts of the claim are trivial.  $\square$

Supposing injectivity of the implication set, Proposition 6.1.6 helps us to establish a sufficient set of properties which yield an extremal lattice. We prove the converse of this result in Theorem 6.4.5, characterizing the canonical bases of extremal lattices.

**6.1.7 Proposition** *The sets which respect an injective,  $r$ -regular, saturated set of implications over an  $n$ -element set form an  $(n, r + 1)$ -extremal lattice.*

*Proof* Let  $\mathcal{L}$  denote such an implication set. Being saturated means that  $\mathcal{L}$  does not have any minimal generator with  $r + 1$  elements. It was shown in Lemmas 5.1.1 and 5.2.4 that any formal context without minimal generators having  $r + 1$  elements has a  $B(r + 1)$ -free lattice. Together with Proposition 6.1.4, this implies that the lattice  $\mathfrak{H}(\mathcal{L})$  is  $B(r + 1)$ -free. Regarding the number of join-irreducibles, we use Proposition 6.1.2 which upper bounds the number of join-irreducibles of  $\mathfrak{H}(\mathcal{L})$  by  $n$ . Lastly, Proposition 6.1.6 plus injectivity implies that  $\mathfrak{H}(\mathcal{L})$  has  $\sum_{i=0}^r \binom{n}{i}$  elements, i. e. the correct number of elements for it to be  $(n, r + 1)$ -extremal.  $\square$

It would be natural to define that an implication set is extremal if it is injective, regular and saturated. We will, however, also require that it is straight: these mathematical objects are easier to characterize (as will be discussed after Theorem 6.2.1, making use of straightness helps to prove injectivity). Reaching this characterization is sufficient for the goals of this chapter. Thus, we define that an implication set over an  $n$ -element set is  $(n, k)$ -extremal if it is injective,  $(k - 1)$ -regular, saturated and straight. Like we did with lattices, we may omit  $(n, k)$  and just write “extremal implication set” or even only “extremal set”.

We now illustrate the just introduced notion for small values of  $k$ . For  $k = 1$  and arbitrary  $n$ , it is clear that  $\{\emptyset \rightarrow [n]\}$  is injective, 0-regular, saturated and straight. Therefore, its associated closure system is an  $(n, 1)$ -extremal lattice (that is, it has only one element). On the left side of Figure 6.1, the  $(3, 2)$ -extremal implication set  $\{3 \rightarrow 21, 2 \rightarrow 1\}$  is displayed, together with its respecting sets (i. e. the associated lattice). Similarly, one can see on the right the same representation idea applied for the  $(3, 3)$ -extremal set  $\{13 \rightarrow 2\}$ . If an element has as label an implication  $P \rightarrow Q$ , it is to be understood that the element is the respecting set  $P \cup Q$ .

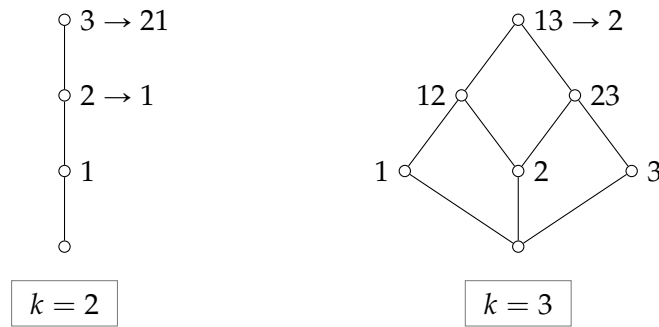


Figure 6.1.: Two small extremal lattices in a depiction which blends  $\mathcal{L}$  and  $\mathfrak{H}(\mathcal{L})$ .

## 6.2. An explicit description of an extremal lattice with many meet-irreducibles

One interesting task reveals itself: is it possible to explicitly describe extremal implication sets for arbitrary  $n, k \geq 2$ ? This question can be promptly answered in the positive, as Theorem 6.2.1 shows. Nevertheless, it is still unclear (but will be revealed after Lemma 6.3.3) how one comes up with that set of implications. The proof of Theorem 6.2.1 gives a hint as to why we require that extremal sets be straight: such property makes the task of proving injectivity much simpler.

**6.2.1 Theorem** *For any  $n$ -element set  $\{g_1, g_2, \dots, g_n\}$  and  $k \geq 2$ ,*

$$\bigcup_{i=1}^{n-k+1} \left\{ P \cup \{g_{i+1}\} \rightarrow \{g_1, \dots, g_i\} \mid P \in \binom{\{g_{i+2}, \dots, g_n\}}{k-2} \right\}$$

*is  $(n, k)$ -extremal.*

*Proof* Denote by  $\mathcal{L}_i$  each set appearing in the union above. Regularity is trivial. Regarding being saturated, consider a set  $T \subseteq G := \{g_1, \dots, g_n\}$  with  $k$  elements. By the pigeonhole principle, the intersection between  $T$  and  $\{g_1, \dots, g_{n-k+2}\}$  must contain at least two elements. Let  $g_i$  and  $g_j$  denote, respectively, the elements in said intersection with smallest and second to smallest indices. Then,  $P := T \setminus \{g_i, g_j\}$  is a subset of  $\{g_{j+1}, \dots, g_n\}$  with  $k-2$  elements and, without effort, one sees that  $P \cup \{g_j\} \rightarrow \{g_1, \dots, g_{j-1}\}$  has  $g_i$  in its conclusion and belongs to  $\mathcal{L}_{j-1}$ , which shows that  $T$  is not a minimal generator with respect to  $\mathcal{L}$ . Regarding straightness, let  $P \rightarrow Q \in \mathcal{L}_i, R \rightarrow S \in \mathcal{L}_j$  for some  $i$  and  $j$ . If  $i \geq j$ , then  $Q \supseteq S$ , which implies  $P \cup Q \supseteq S$  and the condition is satisfied. If  $i < j$ , then it holds that  $P \not\supseteq R$  (both have exactly  $k-1$  elements and are distinct, since the first contains  $g_{i+1}$ , whereas the latter does not). Moreover, observe that  $Q \cap R = \emptyset$  and that this, together with  $P \not\supseteq R$ , yields  $P \cup Q \not\supseteq R$  and the condition for straightness is satisfied. Considering injectivity, let  $P$  be a premise of  $\mathcal{L}_i$  and  $Q$  be a premise of  $\mathcal{L}_j$  with  $P \neq Q$ . Because of straightness,  $P^{\mathcal{L} \dots \mathcal{L}} = P \cup \{g_1, \dots, g_i\}$  and  $Q^{\mathcal{L} \dots \mathcal{L}} = Q \cup \{g_1, \dots, g_j\}$ . If  $i = j$ , then by definition both  $P$  and  $Q$  have empty intersection with  $\{g_1, \dots, g_i\}$ . Thus,  $P \neq Q$  implies  $P^{\mathcal{L} \dots \mathcal{L}} \neq Q^{\mathcal{L} \dots \mathcal{L}}$ . If  $i \neq j$ , and without loss of generality,  $i < j$ , then  $Q \cap \{g_1, \dots, g_i\} = \emptyset$ , which implies  $P^{\mathcal{L} \dots \mathcal{L}} = P \cup \{g_1, \dots, g_i\} \not\supseteq Q$  and, on account of  $Q^{\mathcal{L} \dots \mathcal{L}} \supseteq Q$ , the inequality  $P^{\mathcal{L} \dots \mathcal{L}} \neq Q^{\mathcal{L} \dots \mathcal{L}}$  follows.  $\square$

**6.2.2 Corollary** *Extremal sets of implications, and therefore lattices, exist for every pair of parameters.*

Applying Theorem 6.2.1 with 1234 as ground set (with the natural order) and  $k = 3$ , one obtains the set of implications  $\mathcal{L} = \{32 \rightarrow 1, 42 \rightarrow 1\} \cup \{43 \rightarrow 12\}$ . The associated lattice is

depicted in Figure 6.2. Observe that  $\mathfrak{H}(\mathcal{L})$  has four join-irreducibles, no  $B(3)$  as a suborder and has precisely  $1 + 4 + 6 = 11$  elements: in other words, it is an  $(4, 3)$ -extremal lattice. Even though this particular lattice has 7 meet-irreducible elements (including those which are *doubly-irreducible*: i.e. both join and meet-irreducible), there exist  $(4, 3)$ -extremal lattices with fewer meet-irreducibles: an easy example is an interordinal scale. Such example will be revisited inside our setting later, more precisely, after Theorem 6.3.2.

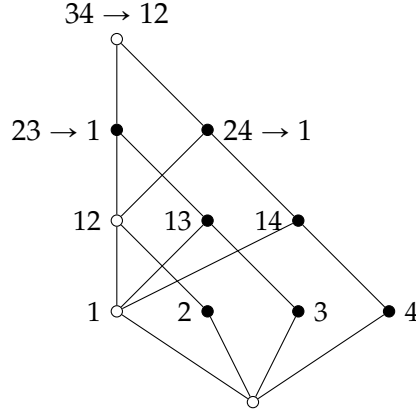


Figure 6.2.: A  $(4, 3)$ -extremal lattice. Black circles represent meet (including doubly)-irreducibles.

In the following lemma, the ordered set  $\{g_1, \dots, g_i\}$  is the same as the one mentioned in Theorem 6.2.1.

**6.2.3 Lemma** *Every set with at least  $k - 1$  elements which respects the implications present in Theorem 6.2.1 has the form  $Q \cup \{g_1, \dots, g_i\}$  with  $|Q| = k - 2$ .*

*Proof* Let  $\mathcal{L}$  denote said set of implications and  $T$  be a set which respects  $\mathcal{L}$ . If  $|T| \geq k$ , then from Proposition 6.1.5 follows that  $T$  must be the closure of a premise. In  $\mathcal{L}$ , such a premise has the form  $P \cup \{g_{i+1}\}$  with corresponding conclusion  $\{g_1, \dots, g_i\}$ . For the other case, we prove that each  $T \subseteq \{g_2, \dots, g_n\}$  with  $|T| = k - 1$  does not respect  $\mathcal{L}$ . Certainly  $T$  is non-empty, so we take  $g_j \in T$  with the smallest index. Then,  $T \setminus \{g_j\}$  belongs to  $\binom{\{g_{j+1}, \dots, g_n\}}{k-2}$ , which implies that  $T$  is a premise of  $\mathcal{L}_{j-1}$ , where  $\mathcal{L}_{j-1}$  is the implication set with  $i = j - 1$  in the description of  $\mathcal{L}$ . Therefore, every respecting set containing  $T$  must contain  $\{g_1, \dots, g_j\}$ . In particular,  $T$  does not respect  $\mathcal{L}$ .  $\square$

The following proposition is a well known necessary and sufficient condition for meet-irreducibility in concept lattices. Together with the lemma above, it will be possible to control the number of meet-irreducibles of the lattice associated to the extremal set described in Theorem 6.2.1.

**6.2.4 Proposition** *Let  $\mathcal{L}$  be a set of implications over a finite set  $G$ . Then, a respecting set  $T$  is meet-irreducible if and only if there exists  $g \in G$  such that  $T$  is maximal amongst the respecting sets which do not have  $g$ .*

*Proof* If  $T$  is meet-irreducible with unique upper cover  $U$ , then  $T$  is maximal amongst the respecting sets which do not have an arbitrary  $g \in U \setminus T$ . For the other direction, suppose that  $T$  is maximal amongst the respecting sets which do not have  $g \in G$ . Then,  $T$  is not the greatest element and therefore has at least one upper cover. Now, suppose by contradiction, that  $T$  has at least two upper covers, say  $U$  and  $V$ . Because of the property on  $T$ , we have that both  $U$  and  $V$  have the element  $g$ . Therefore,  $U \cap V$  is a set which respects  $\mathcal{L}$ , properly contains  $T$  and  $U \cap V \subsetneq U$ , contradicting the fact that  $U$  is an upper cover of  $T$ .  $\square$

Now we are able to determine exactly the number of meet-irreducibles of the lattice associated to the extremal set described in Theorem 6.2.1:

**6.2.5 Theorem** *For every pair of parameters  $n$  and  $k \leq n + 1$ , there exists at least one  $(n, k)$ -extremal lattice with precisely  $\binom{n}{k-1} + k - 2$  meet-irreducibles.*

*Proof* Let  $\mathcal{L}$  denote the  $(n, k)$ -extremal set present in Theorem 6.2.1 and set  $L = \mathfrak{H}(\mathcal{L})$ ,  $G = \{g_1, \dots, g_n\}$ . If  $k = 2$ , then clearly  $L$  is an  $n + 1$  element chain with  $n = \binom{n}{1} + 2 - 2$  meet-irreducibles. Thus, it is assumed that  $k \geq 3$  and the hypothesis forces  $n \geq 2$ . We establish that no  $A \in L$  with  $|A| \leq k - 3$  is meet-irreducible. Indeed, since every subset of  $G$  with at most  $k - 2$  elements respects  $\mathcal{L}$ , it follows that every subset of  $G$  with at most  $k - 3$  elements belongs to  $L$  and has at least two upper covers. Let  $A \in L$  with  $k - 2 \leq |A| \leq n - 2$  and set  $j = |A| - (k - 2)$ . Lemma 6.2.3 gives  $A = Q \cup \{g_1, \dots, g_j\}$ , with  $j$  possibly zero. We show for  $j = 0, \dots, n - k$  that  $A$  is meet-irreducible if and only if  $g_{j+1} \notin A$ . For the converse: suppose that  $g_{j+1} \notin A$ . Lemma 6.2.3 implies that every  $B \in L$  with  $k - 1 + j$  elements must contain  $g_{j+1}$ , so it follows from Proposition 6.2.4 that  $A$  is meet-irreducible. For the other direction, suppose that  $g_{j+1} \in A$ . Because  $\{g_1, \dots, g_j\} \subseteq A$  and  $A$  has at most  $n - 2$  elements, we may take two distinct elements  $h, l \in G \setminus A$  with  $h, l \notin \{g_1, \dots, g_j\}$ . Thus, both sets  $Q_1 := (Q \setminus \{g_{j+1}\}) \cup \{h\}$  and  $Q_2 := (Q \setminus \{g_{j+1}\}) \cup \{l\}$  have exactly  $k - 2$  elements and empty intersection with  $\{g_1, \dots, g_{j+1}\}$ . Therefore,  $Q_1 \cup \{g_1, \dots, g_{j+1}\}$  and  $Q_2 \cup \{g_1, \dots, g_{j+1}\}$  belong to  $L$  and are upper covers of  $A$ . In particular,  $A$  is not meet-irreducible. The number of meet-irreducibles  $A$  with  $k - 2 \leq |A| \leq n - 2$  is, therefore,  $\sum_{i=1}^{n-k+1} \binom{n-i}{k-2} = \binom{n}{k-1} - 1$ . Every element  $A \in L$  with  $|A| = n - 1$  is meet-irreducible (it is a coatom) and, therefore, the total number follows.  $\square$

Even though Theorem 6.2.1 produces extremal sets with arbitrarily given parameters, it is not true that every extremal set is producible by that result. This will be made clear after Theorem 6.3.2, to be presented in the next section.

### 6.3. A characteristic construction of extremal sets of implications

In this section, we completely describe extremal implication sets. First, we show how one may construct an  $(n + 1, k + 1)$ -extremal set supposing that  $(n, k), (n - 1, k), \dots, (k, k)$ -extremal sets are given and that they satisfy some condition of compatibility. Then, we will show that obtaining such (compatible) smaller extremal sets can be done quite easily: it turns out that it is possible to construct an  $(n + 1, k + 1)$ -extremal set after being given solely one  $(n, k)$ -extremal one. Lastly, we proceed to prove that every extremal set must be built through the procedure described in the beginning of the section.

#### 6.3.1. Construction of a larger extremal set through smaller ones

In order to construct extremal implications sets with increasing parameters, one requires some operation which increases the regularity level of an implication set. This service will be performed by the following operation, which is in fact suggested by Theorem 6.2.1. For an implication set  $\mathcal{L}$  over  $G$  and an element  $g$  not in  $G$ , we define the *lift* of  $\mathcal{L}$  to be  $\mathcal{L}^g := \{P \cup \{g\} \rightarrow Q \mid P \rightarrow Q \in \mathcal{L}\}$ . The contrary work is performed by the *drop*, defined as  $\mathcal{L}^{-g} := \{P \setminus \{g\} \rightarrow Q \mid P \rightarrow Q \in \mathcal{L}\}$ .

The content of the next lemma is almost predictable. It will, however, be indispensable for further argumentation.

**6.3.1 Lemma** *Let  $\mathcal{L}$  be a set of implications over  $G \setminus \{g\}$ . Then,*

- (i)  $\mathcal{L}$  is straight if and only if its lift  $\mathcal{L}^g$  is.
- (ii)  $\mathcal{L}$  is injective if and only if its lift  $\mathcal{L}^g$  is.
- (iii) A set  $S \subseteq G \setminus \{g\}$  is a minimal generator with respect to  $\mathcal{L}$  if and only if  $S \cup \{g\}$  is a minimal generator with respect to  $\mathcal{L}^g$ .

*Proof* For item (i), let  $P \rightarrow Q, R \rightarrow S \in \mathcal{L}$ . Because of  $g \notin P \cup Q \cup R \cup S$ , we have

$$\begin{aligned} (P \cup Q \supseteq R \Rightarrow P \cup Q \supseteq S) &\Leftrightarrow \\ (P \cup \{g\} \cup Q \supseteq R \Rightarrow P \cup \{g\} \cup Q \supseteq S), \end{aligned}$$

and the implication on the right-hand side is the condition for straightness in  $\mathcal{L}^g$ . Regarding item (ii), consider an arbitrary implication  $P \rightarrow Q \in \mathcal{L}$ . We will apply the operators  $(\cdot)^{\mathcal{L}}$  to  $P$  and  $(\cdot)^{\mathcal{L}^g}$  to the premise associated to  $P$  in  $\mathcal{L}^g$ , that is,  $P \cup \{g\}$ . It is clear that

$$\{B \mid A \rightarrow B \in \mathcal{L}, P \supseteq A\} = \{B \mid A \rightarrow B \in \mathcal{L}^g, P \cup \{g\} \supseteq A\},$$

which implies  $(P \cup \{g\})^{\mathcal{L}^g} = P^{\mathcal{L}} \cup \{g\}$ . The same argument may be reapplied:

$$\{B \mid A \rightarrow B \in \mathcal{L}, P^{\mathcal{L}} \supseteq A\} = \{B \mid A \rightarrow B \in \mathcal{L}^g, P^{\mathcal{L}} \cup \{g\} \supseteq A\},$$

yielding  $(P \cup \{g\})^{\mathcal{L}^g \mathcal{L}^g} = P^{\mathcal{L} \mathcal{L}} \cup \{g\}$  and so on. Thus, injectivity of  $(\cdot)^{\mathcal{L}^g \mathcal{L}^g \dots \mathcal{L}^g}$  is equivalent to injectivity of  $(\cdot)^{\mathcal{L} \mathcal{L} \dots \mathcal{L}}$ . For item (iii): a set  $S \subseteq G \setminus \{g\}$  is a minimal generator with respect to  $\mathcal{L}$  if and only if  $S \cap (Q \setminus P) = \emptyset$  for each  $P \rightarrow Q \in \mathcal{L}$  with  $S \supseteq P$ , and that holds if and only if  $(S \cup \{g\}) \cap [Q \setminus (P \cup \{g\})] = \emptyset$  is true for each  $P \cup \{g\} \rightarrow Q$  with  $S \cup \{g\} \supseteq P \cup \{g\}$ .  $\square$

In general, the union of straight sets is not straight (take, for instance  $\mathcal{L} = \{1 \rightarrow 2\}$  and  $\mathcal{M} = \{2 \rightarrow 3\}$ ). Of course, straightness is a very desirable property to be maintained during our construction. We therefore define that a family of implication sets  $(\mathcal{L}_i)_{i \in I}$  is *compatible* if  $\cup_i \mathcal{L}_i$  is straight.

Consider a family of implication sets  $(\mathcal{L}_i \mid i \in [a])$ . We say that  $g$  *separates*  $\mathcal{L}_i$  from  $\mathcal{L}_{i-1}$  if  $\Gamma(\mathcal{L}_{i-1}) \setminus \Gamma(\mathcal{L}_i) = \{g\}$ . A family  $(\mathcal{L}_i \mid i \in [a])$  is said to be *cascade* if there exists  $g_i$  which separates  $\mathcal{L}_i$  from  $\mathcal{L}_{i-1}$  for each  $2 \leq i \leq a$ . In this case, the sequence  $g_2, \dots, g_a$  is called the *separating elements* of the family. If  $\mathcal{L} = (\mathcal{L}_i \mid i \in [a])$  is a cascade family and  $g_1$  is any element not in  $\Gamma(\mathcal{L}_1)$ , then we define the *multi-lift* of  $\mathcal{L}$  to be  $\hat{\mathcal{L}} := (\mathcal{L}_i^{g_i} \mid i \in [a])$ , where  $g_2, \dots, g_a$  are the separating elements of  $\mathcal{L}$ .

Theorem 6.3.2 shows how one constructs an  $(n+1, k+1)$ -extremal set, provided that extremal sets with parameters  $(n, k), (n-1, k), \dots, (k, k)$  are available and its multi-lift is compatible. A converse to this result will be shown in Corollary 6.3.7, where we show that every extremal set has this structure.

**6.3.2 Theorem** *If  $\mathcal{L}_1, \dots, \mathcal{L}_{n-k+1}$  is a cascade family such that each  $\mathcal{L}_i$  is  $(n-i+1, k)$ -extremal and  $(\hat{\mathcal{L}}_i)$  is compatible, then  $\cup_i \hat{\mathcal{L}}_i$  is  $(n+1, k+1)$ -extremal.*

*Proof* Set  $\mathcal{M} := \cup_i \hat{\mathcal{L}}_i$ . Denote by  $g_2, \dots, g_{n-k+1}$  the separating elements of  $(\mathcal{L}_i)$ . Let  $g_1$  be the element with  $\Gamma(\mathcal{M}) = \Gamma(\mathcal{L}_1) \cup \{g_1\}$ . Because  $\mathcal{M}$  is straight, one has:

$$\begin{aligned} \{P^{\mathcal{M} \dots \mathcal{M}} \mid P \rightarrow Q \in \mathcal{M}\} &= \{P \cup Q \mid P \rightarrow Q \in \mathcal{M}\} \\ &= \bigcup_{i=1}^{n-k+1} \{P \cup Q \mid P \rightarrow Q \in \mathcal{L}_i^{g_i}\}. \end{aligned}$$

The union above is disjoint: take  $i < j$  and implications  $P \rightarrow Q, R \rightarrow S$  belonging, respectively, to  $\mathcal{L}^{g_i}$  and  $\mathcal{L}^{g_j}$ . Then, it is clear that  $g_i \in P \cup Q$  and, because the ground set of  $\mathcal{L}_j^{g_j}$  does not contain  $g_i$ , it holds that  $g_i \notin R \cup S$ . We develop further:

$$\bigcup_{i=1}^{n-k+1} \{P \cup Q \mid P \rightarrow Q \in \mathcal{L}_i^{g_i}\} = \bigcup_{i=1}^{n-k+1} \{P^{\mathcal{L}_i^{g_i} \dots \mathcal{L}_i^{g_i}} \mid P \rightarrow Q \in \mathcal{L}_i^{g_i}\},$$

where the equality above holds because each  $\mathcal{L}_i^{g_i}$  is straight, according to Lemma 6.3.1. The same lemma gives that every  $\mathcal{L}_i^{g_i}$  is injective and, therefore, so is  $\mathcal{M}$ . To establish that  $\mathcal{M}$  is saturated, let  $T$  be a subset of  $\Gamma(\mathcal{M})$  containing  $k+1$  elements. Observe that we

have  $|\Gamma(\mathcal{M})| = n + 1$ . By the pigeonhole principle,  $T$  contains some element amongst  $\{g_1, \dots, g_{n-k+1}\}$ . Let  $g_i \in T$  be such element with the minimum index. Therefore,  $T \setminus \{g_i\}$  belongs to the ground set of  $\mathcal{L}_i$  and is not a minimal generator, because  $\mathcal{L}_i$  is saturated. Using item (iii) of Lemma 6.3.1 we conclude that  $T$  is not a minimal generator with respect to  $\mathcal{L}_i^{g_i} \subseteq \mathcal{M}$ .  $\square$

Theorem 6.3.2 captures the existence of extremal sets which are not constructible by Theorem 6.2.1. For instance, consider  $\mathcal{L}_1 = \{3 \rightarrow 21, 2 \rightarrow 1\}$  and  $\mathcal{L}_2 = \{1 \rightarrow 2\}$  which are, respectively,  $(3, 2)$  and  $(2, 2)$ -extremal. Then, by calling “4” the new element implicit in the multi-lift, one has that  $\mathcal{M} := \hat{\mathcal{L}}_1 \cup \hat{\mathcal{L}}_2$  equals  $\{34 \rightarrow 21, 24 \rightarrow 1, 13 \rightarrow 2\}$  and is straight. By the theorem above,  $\mathcal{M}$  is  $(4, 3)$ -extremal as well. The sets which respect  $\mathcal{M}$  are depicted in Figure 6.3. Note that the sets 14 and 23 respect  $\mathcal{M}$  and can not be written in the form described by Lemma 6.2.3. Thus, Theorem 6.2.1 does not produce  $\mathfrak{H}(\mathcal{M})$ .

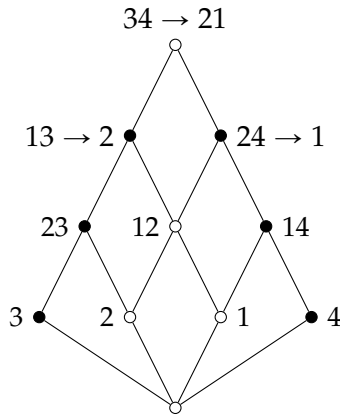


Figure 6.3.: An interordinal scale, which is a particular case of an extremal lattice for  $k = 3$ .

To apply Theorem 6.3.2, one needs a family of implication sets whose multi-lift is compatible. The next lemma shows, in particular, that it suffices to find a family which is *itself* compatible, since compatibility is preserved by the multi-lift operation.

**6.3.3 Lemma** *The multi-lift of a compatible and cascade family is compatible and cascade.*

*Proof* Let  $(\mathcal{L}_i)$  be a cascade family of implications,  $g_1$  an element not in  $\Gamma(\mathcal{L}_1)$  and let  $g_2, \dots, g_a$  be the separating elements. Contraposition: let  $i \neq j$  and take two implications  $P \cup \{g_i\} \rightarrow Q \in \hat{\mathcal{L}}_i$  and  $R \cup \{g_j\} \rightarrow S \in \hat{\mathcal{L}}_j$  with  $P \cup \{g_i\} \cup Q \supseteq R \cup \{g_j\}$  and  $P \cup \{g_i\} \cup Q \not\supseteq S$ . Notice that the mentioned containment forces  $i < j$ , because otherwise  $g_j$  would not belong to the ground set of  $\hat{\mathcal{L}}_i$ . On account of  $i < j$ , we have that  $g_i \notin R$ . Therefore, we also have  $P \cup Q \supseteq R \cup \{g_j\} \supseteq R$  and  $P \cup Q \not\supseteq S$ , which is precisely the violation of straightness for  $\cup_i \mathcal{L}_i$ , i.e.  $(\mathcal{L}_i)$  is not compatible. Being again cascade follows trivially.  $\square$

Theorem 6.3.2, together with Lemma 6.3.3, explains how one comes up with the extremal set shown in Theorem 6.2.1. Let  $n \geq 1, k \geq 2$  and take the family  $(\mathcal{L}_i \mid i = 1, 2, \dots, n-1)$  with  $\mathcal{L}_i = \{j \rightarrow [j-1] \mid j = 2, \dots, n-i+1\}$ . It comes with almost no effort that each  $\mathcal{L}_i$  is  $(i+1, 2)$ -extremal. Of course,  $(\mathcal{L}_i)$  is cascade. The family  $(\mathcal{L}_i)$  is also itself a descending chain, i.e.,  $\mathcal{L}_1 \supseteq \dots \supseteq \mathcal{L}_{n-1}$ . This implies that the union of all of its members is just  $\mathcal{L}_1$  and the family is, therefore, compatible. Lemma 6.3.3 delivers that  $(\hat{\mathcal{L}}_i)$  is compatible and Theorem 6.3.2 may be applied to each subfamily  $(\mathcal{L}_i \mid i = 1, \dots, k)$  for  $k = 1, \dots, n-1$ . This yields the family  $\hat{\mathcal{L}}_1, \hat{\mathcal{L}}_1 \cup \hat{\mathcal{L}}_2, \dots, \cup_{i=1}^{n-1} \hat{\mathcal{L}}_i$  which is readily seen as a chain of  $(i+2, 3)$ -extremal sets. By reapplying Theorem 6.3.2, one obtains  $(i+3, 4)$ -extremal sets and so on. An exercise shows that these extremal sets are obtained by the explicit description present in Theorem 6.2.1.

### 6.3.2. Being the union of a multi-lift is characteristic

In this subsection, we show that every extremal set is the union of the multi-lift of extremal sets with smaller parameters as described in Theorem 6.3.2. We begin with one definition: for an implication set  $\mathcal{L}$  over  $G$  and  $g \in G$ , we set  ${}^g\mathcal{L} := \{P \rightarrow Q \in \mathcal{L} \mid g \in P\}$ .

The claim present in the next lemma appears to be more technical than it is: it can be translated back to the notion of extremal points in extremal lattices.

**6.3.4 Lemma** *Let  $\mathcal{L}$  be  $(n, k)$ -extremal with ground set  $G$  and suppose that  $P \rightarrow Q, R \rightarrow S \in \mathcal{L}$  satisfy  $R \cup S \subseteq P \cup Q$ . Then, for every  $g \in P$  it holds that  $g \in R$  or  $g \notin R \cup S$ . In particular, if  $P \cup Q = G$ , then, for each  $g \in P$  it holds that  $\mathcal{L} \setminus {}^g\mathcal{L}$  is  $(n-1, k)$ -extremal with ground set  $G \setminus \{g\}$ .*

*Proof* Suppose, by contradiction, that  $g \notin R$  and  $g \in R \cup S$ . In particular,  $R \neq P$  and  $k \geq 2$ . Set  $X = (P \setminus \{g\}) \cup R$ . Note that  $g \notin X$  and  $P \cup Q \supseteq X$ . Because  $\mathcal{L}$  is straight, we have that  $P \cup Q$  respects  $\mathcal{L}$  and, therefore, the closure of  $X$  is contained in  $P \cup Q$ . On the other hand, because of  $R \rightarrow S$ ,  $X \supseteq R$  and  $g \in S$ , we have that  $g$  belongs to  $X^{\mathcal{L}}$ . Hence,  $X^{\mathcal{L}\mathcal{L}} \supseteq P \cup Q$ . Combining both, we have that the closure of  $X$  is precisely  $P \cup Q$ , i.e., the closure of  $P$ . Now, if  $X$  is precisely  $R$ , then the contradiction with the injectivity of  $\mathcal{L}$  is clear, since  $R$  is a premise different than  $P$  but with the same closure. Otherwise,  $X$  contains  $R$  properly and, therefore, has at least  $k$  elements. Proposition 6.1.5 gives us a premise  $Y$  of  $\mathcal{L}$ , the closure of which is the same as the closure of  $X$  and  $Y \subseteq X$ . Observe that  $Y \neq P$  because of  $g \notin Y$ . The premises  $P$  and  $Y$  contradict the injectivity of  $\mathcal{L}$ .  $\square$

The hard work regarding the converse has been done in Lemma 6.3.4. We now collect the reward of nice assertions about the structure of extremal sets (and lattices). The following theorem shows that the construction described in Theorem 6.3.2 can be easily bootstrapped: indeed,  $\mathcal{L}$  carries inside itself a compatible family, making the hypothesis of that theorem easy to be satisfied. In particular, it is possible to construct an  $(n+1, k+1)$ -extremal set by having only one  $(n, k)$ -extremal set as initial information.

**6.3.5 Theorem** *Given any  $(n, k)$ -extremal set  $\mathcal{L}$  with  $k \geq 2$ , there exists a chain  $\mathcal{L} = \mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \dots \supseteq \mathcal{L}_{n-k}$  such that each  $\mathcal{L}_i$  is  $(n-i, k)$ -extremal. In particular, the families  $(\mathcal{L}_i)$  and  $(\hat{\mathcal{L}}_i)$  are compatible.*

*Proof* Set  $G = \Gamma(\mathcal{L})$ . If  $n \leq k$ , the claim holds trivially. Thus, assume  $n > k$ . Because  $G$  respects  $\mathcal{L}$  and  $\mathcal{L}$  is  $k$ -regular and saturated, Proposition 6.1.5 gives us an implication  $P \rightarrow Q \in \mathcal{L}$  with  $P \cup Q = G$ . Let  $g \in P$ . By applying the last claim in Lemma 6.3.4, one obtains that the subset  $\mathcal{L}_1 := \mathcal{L} \setminus^g \mathcal{L}$  is an  $(n-1, k)$ -extremal set. If  $n-1 = k$ , we are done. Otherwise, we make use of Lemma 6.3.4 again and obtain  $\mathcal{L}_2 := \mathcal{L}_1 \setminus^h \mathcal{L}_1$ , where  $h$  is some element of the premise of the ground set of  $\mathcal{L}_1$ , that is,  $G \setminus \{g\}$  and so on.  $\square$

Theorem 6.3.6 is the main result of this subsection and its immediate consequence is Corollary 6.3.7, which is the converse of Theorem 6.3.2. In the result below, there is a mention to the premise of the ground set of  $\mathcal{L}$ : to make that expression clear, observe that the ground set of  $\mathcal{L}$  obviously respects  $\mathcal{L}$  and, whenever  $\mathcal{L}$  is  $(n, k)$ -extremal, there exists precisely one implication  $P \rightarrow Q \in \mathcal{L}$  with  $P \cup Q = \Gamma(\mathcal{L})$ . We also employ the term *g-lift* to explicitly refer to the new element required for the lift operation.

**6.3.6 Theorem** *Let  $\mathcal{L}$  be an  $(n, k)$ -extremal set and  $g$  an element belonging to the premise of its ground set. Then, it holds that  ${}^g\mathcal{L}$  is the  $g$ -lift of an  $(n-1, k-1)$ -extremal set and  $\mathcal{L} \setminus^g \mathcal{L}$  is  $(n-1, k)$ -extremal.*

*Proof* Since  $\mathcal{L}$  is straight and straightness is a hereditary property (see Lemma 6.1.3), it follows that  ${}^g\mathcal{L}$  is straight. Moreover, since  $\mathcal{L}$  is injective, it follows with help of straightness of  $\mathcal{L}$  and  ${}^g\mathcal{L}$  that the latter is injective as well. Items (i) and (ii) from Lemma 6.3.1 assure that the drop  ${}^g\mathcal{L}^{-g}$  is straight and injective. It is also clearly  $(k-2)$ -regular. Lemma 6.3.4 guarantees that the ground set of  ${}^g\mathcal{L}^{-g}$  is  $\Gamma(\mathcal{L}) \setminus \{g\}$  and item (iii) from Lemma 6.3.1 shows that it is saturated. The assertion regarding  $\mathcal{L} \setminus^g \mathcal{L}$  follows from the final claim in Lemma 6.3.4.  $\square$

To exemplify Theorem 6.3.6, consider once again  $\mathcal{L} = \{34 \rightarrow 21, 24 \rightarrow 1, 13 \rightarrow 2\}$ , which is  $(4, 3)$ -extremal. Figure 6.4 depicts the decomposition of  $\mathcal{L}$  through the use of Theorem 6.3.6 with the choice  $g = 4$  (the only other option is  $g = 3$ ). The result yields  $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ , where  $\mathcal{L}_1 := {}^4\mathcal{L}$  and  $\mathcal{L}_2 := \mathcal{L} \setminus^4 \mathcal{L}$  is  $(3, 3)$ -extremal. Besides, Theorem 6.3.6 says that  $\mathcal{L}_1$  has the form  $\mathcal{L}_3^4$  with  $\mathcal{L}_3$  being the  $(3, 2)$ -extremal set  $\{3 \rightarrow 21, 2 \rightarrow 1\}$ . The lattice associated to  $\mathcal{L}_2$  is shown inside the dashed region in that figure. The dotted region draws attention to the chain  $\uparrow 4$ , which is a  $(3, 2)$ -extremal sublattice.

**6.3.7 Corollary** *For every  $(n+1, k+1)$ -extremal set  $\mathcal{L}$  there exists a cascade family  $(\mathcal{L}_i)$  whose multi-lift is compatible and  $\mathcal{L} = \cup_{i=1}^{n-k+1} \hat{\mathcal{L}}_i$ , where each  $\mathcal{L}_i$  is  $(n-i+1, k)$ -extremal.*

*Proof* One applies Theorem 6.3.6 repeatedly, by choosing an element  $h$  of the premise of the ground set of  $\mathcal{L} \setminus^g \mathcal{L}$ .  $\square$

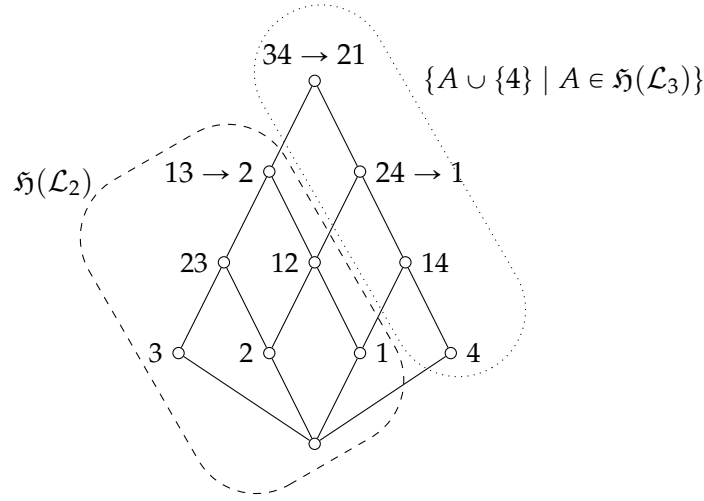


Figure 6.4.: A  $(4, 3)$ -extremal set as the union of one  $(3, 3)$ -extremal set and the lift of one  $(3, 2)$ -extremal one.

## 6.4. Canonical bases of extremal lattices

Extremal sets were characterized in the last section. Now it is time to establish a strong link between that introduced notion and extremal lattices. Said connection is done through the canonical basis of implications: we show that the canonical basis of every extremal lattice is an extremal set of implications (with the same parameters). Since it was already established by Proposition 6.1.7 that the lattice associated to an extremal set is an extremal lattice, the end result is that the notion of extremal implication sets introduced here is consistent with the notion of extremal lattices. Moreover, as it was just shown in Subsections 6.3.1 and 6.3.2, one will have a characteristic construction of such lattices.

### 6.4.1. Basic definitions and results

The scenario here is still the object-sided implication logic of a formal context and its lattice of extents. Consider a finite formal context  $(G, M, I)$ . A set  $P \subseteq G$  is called a *pseudo-extent* if  $P \neq P''$  and  $Q'' \subsetneq P$  for every pseudo-extent  $Q \subsetneq P$ . In particular, note that if  $P \subseteq G$  is not closed but each maximal proper subset of  $P$  is closed, then  $P$  must be a pseudo-extent. If that is the case, then  $P$  is also a proper premise. For a finite lattice  $L$ , its *canonical basis* is  $\{S \rightarrow (\bigvee S) \setminus S \mid S \subseteq J(L), S \text{ a pseudo-extent}\}$ . In this definition, it is implicit that being a pseudo-extent refers to the standard context of  $L$ . This will also be the case whenever we say that a set  $S \subseteq J(L)$  is closed: it means  $\bigvee S = S$ . We recall from Corollary 5.4.5 that any element of a  $B(k)$ -free lattice has an irredundant representation of size at most  $k - 1$ .

Before we proceed to look into the structure of pseudo-extents in extremal lattices, we need to establish some easy facts regarding straightness. Given an implication set  $\mathcal{L}$ , we define that its *expansion* is  $\mathcal{L}^* := \{P \rightarrow P \cup Q \mid P \rightarrow Q \in \mathcal{L}\}$ . Of course, taking the expansion does not change the associated closure system.

**6.4.1 Proposition** *A set of implications is straight if and only if its expansion is.*

*Proof* Let  $P \rightarrow Q, R \rightarrow S \in \mathcal{L}$ . Define  $Q^* = P \cup Q$  and  $S^* = R \cup S$  and consider the pair of implications  $P \rightarrow Q^*, R \rightarrow S^*$  in  $\mathcal{L}^*$ . Then, trivially  $P \cup Q = P \cup Q^*$ , which yields  $P \cup Q \supseteq R$  if and only if  $P \cup Q^* \supseteq R$  and  $P \cup Q \supseteq S$  if and only if  $P \cup Q^* \supseteq S$ .  $\square$

The proposition above helps to prove the following:

**6.4.2 Lemma** *The canonical basis of a finite lattice and its expansion are always straight.*

*Proof* Let  $L$  be a finite lattice,  $\mathcal{L}$  be its canonical basis and set  $\mathcal{M} := \mathcal{L}^*$ . Suppose, by contradiction, that  $\mathcal{M}$  violates the condition for straightness. Then, we take  $P \rightarrow Q \in \mathcal{M}$  (note that  $Q = \bigvee P$ ) and  $R \rightarrow S \in \mathcal{M}$  with  $P \cup Q = Q \supseteq R$  and  $P \cup Q = Q \not\supseteq S$ . Therefore,  $Q$  does not respect  $R \rightarrow S$  and, in particular, does not respect  $\mathcal{M}$ . With symbols, one has that  $Q = \bigvee P \notin \mathfrak{H}(\mathcal{M})$ . This contradicts  $\mathfrak{H}(\mathcal{M}) = \mathfrak{H}(\mathcal{L}) = \{\bigvee X \mid X \subseteq J(L)\}$ . The canonical basis  $\mathcal{L}$  is straight as well, on account of Proposition 6.4.1.  $\square$

#### 6.4.2. The structure of canonical bases of extremal lattices

The necessary logical foundations have now been laid and we begin to look more closely to pseudo-extents of extremal lattices. The following fact is in its essence, item *iii*) of Lemma 5.4.3.

**6.4.3 Proposition** *Suppose that  $L$  is an  $(n, k)$ -extremal lattice. Then, for every  $S, T \subseteq J(L)$  with  $|S|, |T| \leq k - 1$ :*

$$\bigvee S = \bigvee T \Rightarrow S = T.$$

*In particular, every  $S \subseteq J(L)$  with  $|S| = k - 2$  is closed.*

*Proof* The first part was proved in the aforementioned lemma. For the corollary, take a set  $S$  of  $k - 2$  join-irreducibles. Then, if  $S$  were not closed, we would be able to take some  $x \in (\bigvee S) \setminus S$  and we would have  $\bigvee S = \bigvee (S \cup \{x\})$ , contradicting the first part.  $\square$

As a consequence of the proposition above, we have Proposition 6.4.4, which shows the pseudo-extents of an extremal lattice are the sets of exactly  $k - 1$  irreducibles which are not closed. This means that pseudo-extents of extremal lattices are actually proper premises.

**6.4.4 Proposition** *Let  $L$  be an  $(n, k)$ -extremal lattice. Then, every  $S \subseteq J(L)$  with precisely  $k - 1$  elements is either closed or a pseudo-extent. Moreover, every pseudo-extent of  $L$  has exactly  $k - 1$  elements.*

*Proof* Suppose that  $S$  has  $k - 1$  elements and is not closed. On account of Proposition 6.4.3, it follows that every maximal proper subset of  $S$  is closed. Thus, by definition,  $S$  is a pseudo-extent. For the second statement, let  $S$  be a pseudo-extent of  $L$ . Then,  $S$  is not closed and Proposition 6.4.3 forces  $|S| \geq k - 1$ . Suppose by contradiction that equality does not hold. Because  $L$  is  $B(k)$ -free, we may take  $T \subseteq S$  with  $|T| = k - 1$  and  $\bigvee T = \bigvee S$ . Combining the just proven statement with the fact that  $T$  is not closed, we have that  $T$  is a pseudo-extent. This contradicts the fact that  $S$  is a pseudo-extent, because of  $T \subsetneq S$  and  $\bigvee T \supseteq S$ .  $\square$

The next result wraps up the developed theory and introduced notions.

**6.4.5 Theorem** *A lattice is  $(n, k)$ -extremal if and only if its canonical basis is an injective,  $(k - 1)$ -regular, saturated and straight set of implications over an  $n$ -element set; in other words: if and only if its canonical basis is an  $(n, k)$ -extremal implication set.*

*Proof* The “if” direction was already established by Proposition 6.1.7. For the other direction, let  $L$  be an  $(n, k)$ -extremal lattice and  $\mathcal{L}$  be its canonical basis. Set  $G = J(L)$ . Lemma 6.4.2 gives that  $\mathcal{L}$  is straight. Now, Proposition 6.4.4 implies that  $\mathcal{L}$  is  $(k - 1)$ -regular. We now prove that  $\mathcal{L}$  is saturated. The fact that  $L$  is  $B(k)$ -free guarantees the existence of an irredundant representation of size at most  $k - 1$  for each element of  $L$ . In particular, for every  $S \subseteq G$  with  $|S| = k$ , there exists  $T \subseteq S$  with  $|T| = k - 1$  and  $\bigvee T = \bigvee S$ . Note that  $\emptyset \neq S \setminus T \subseteq (\bigvee T) \setminus T$ . Proposition 6.4.4 yields that  $T$  is a pseudo-extent and therefore one has that  $T \rightarrow (\bigvee T) \setminus T$  belongs to  $\mathcal{L}$ . Such implication attests that  $S$  is not a minimal generator, since  $S \supseteq T$  and  $S \cap [(\bigvee T) \setminus T] \neq \emptyset$ . According to Proposition 6.1.6 and because - as established here -  $\mathcal{L}$  is  $(k - 1)$ -regular and saturated,  $|L| = |\mathfrak{H}(\mathcal{L})|$  must be equal to  $\sum_{i=0}^{k-2} \binom{|G|}{i} + |\{P^{\mathcal{L} \dots \mathcal{L}} \mid P \subseteq G, |P| = k - 1\}|$ . This sum must be exactly  $\sum_{i=0}^{k-1} \binom{|G|}{i}$ , implying that  $\mathcal{L}$  is injective.  $\square$

Extremal lattices are meet-distributive. This was first proven in Lemma 5.4.6, but can be deduced from the injectivity and saturated notions introduced in this chapter. Since these lattices have finite length, we obtain for free the fact that they are graded posets, with  $x \mapsto |J_x|$  being one valid rank function. The number of elements having rank  $i$  is called the  $i$ -th *Whitney number* and denoted  $w_i$ . The *Whitney numbers* are the sequence  $(w_0, \dots, w_l)$ , where  $l$  is the length of the lattice. For instance, in Figures 6.2 and 6.3 one can recognize the sequence  $(1, 4, 3, 2, 1)$  as being the Whitney numbers of those  $(4, 3)$ -extremal lattices. It is likewise elementary to determine  $(1, 1, 1, 1)$  as the Whitney numbers of the chain present in Figure 6.1. Both sequences can be obtained using a general formula, described in Corollary 6.4.6. Such formula also implies that extremal lattices are always unimodal, and is particularly easy to read from Pascal’s triangle.

**6.4.6 Corollary** *The Whitney numbers of any  $(n, k)$ -extremal lattice with  $k \geq 2$  are*

$$\left( \binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{k-2}, \binom{n-1}{k-2}, \dots, \binom{k-2}{k-2} \right).$$

*Proof* For any fixed  $n$  and  $k$ , the repeated application of the multi-lift operation yields the same number (and size) of premises and conclusions. Therefore, each  $(n, k)$ -extremal lattice must have the same Whitney numbers. In particular, they must be the same as those of the lattice given by Theorem 6.2.1.  $\square$

**6.4.7 Corollary** *Every  $(n, k)$ -extremal lattice has precisely  $\binom{n-1}{k-1}$  pseudo-extents.*

*Proof* Each element of rank  $r \geq k$  corresponds to a pseudo-extent. Thus, the total number of pseudo-extents can be calculated using the formula given in Corollary 6.4.6, which is  $\sum_{i=0}^{n-k} \binom{k-2+i}{k-2} = \binom{n-1}{k-1}$ .  $\square$



## Generalized extremal lattices with maximum breadth

In this chapter, we turn our attention to the more refined version of the problem of determining extremal lattices and contexts. As it was explained in Section 4.3, we may model our combinatorial problem with classes of contexts  $\mathcal{H}^{n,m}$ , meaning that we impose a limit on the numbers of objects *and* attributes. When contemplated from the lattice perspective, this is equivalent to imposing an upper limit for the number of join- and meet-irreducibles.

Turán's graph is a highly "neatly arranged" structure in the sense that it contains many  $K_{k-1}$  cliques without allowing the presence of not even one  $K_k$  clique. The same phenomenon happens to  $(n, k)$ -extremal lattices: each object set of size at most  $k - 1$  corresponds biunivocally to a contranominal subcontext and, consequently, to a boolean suborder. This follows from the unique irredundant representation through join-irreducibles.

A natural desire is to put all the blame of the exponential growth of concept lattices into contranominal scales. Inspired by this, we prove the main result of this chapter. Namely: in every extremal class of contexts, there exists one which has a contranominal scale which is as large as permitted. More formally, we establish that for any triple  $k \leq n \leq m$  there is one  $(n, m, k + 1)$ -extremal lattice which has breadth  $k$  (that is, a boolean suborder  $B(k)$ ).

### 7.1. A conjecture regarding maximum breadth

Even though this definition was somewhat anticipated, we had not formally introduced it yet:

**7.1.1 Definition** A finite lattice is called  $(n, m, k)$ -*extremal* if it has at most  $n$  join-irreducibles,  $m$  meet-irreducibles, is  $B(k)$ -free and, amongst all lattices satisfying these properties, it has the maximum number of elements.  $\diamond$

We introduce the following conjecture:

**7.1.2 Conjecture** *Given any triple  $k \leq n \leq m$ , every  $(n, m, k + 1)$ -extremal lattice contains a boolean suborder  $B(k)$ .*

The main result of this chapter is the proof of the following statement:

**7.1.3 Claim** *Given any triple  $k \leq n \leq m$ , some  $(n, m, k + 1)$ -extremal lattice contains a boolean suborder  $B(k)$ .*

In Chapter 5, the main tool for constructing our extremal objects was the doubling operation. Although efficient, we can not control very well the number of meet-irreducibles of the resulting extremal lattices. With the multi-lift operation exposed in Chapter 6, at least some control was gained, but we still do not know how exactly the interaction between extremality and number of meet-irreducibles occur. An alternative idea is to perform some *contextual operation*: that is, work directly over the formal contexts, without changing the number of objects or attributes. This idea will guide the present chapter.

## 7.2. Basic notions

We introduce a contextual operation which can be better understood after considering the notion of *direct sum* of contexts. Let  $\mathbb{K}_i = (G_i, M_i, I_i)$ ,  $i = 1, 2$  be two contexts. Their direct sum, denoted  $\mathbb{K}_1 + \mathbb{K}_2$ , is defined as

$$(G_1 \cup G_2, M_1 \cup M_2, I_1 \cup I_2 \cup (G_1 \times M_2) \cup (G_2 \times M_1)).$$

Here, the intersection between object (attribute) sets is assumed to be empty. As usual, disjoint copies may be taken, thereby making this restriction no nuisance at all. Schematically, we have the following:

$$\begin{array}{|c|c|} \hline & M_1 \\ \hline G_1 & I_1 \\ \hline \end{array} + \begin{array}{|c|c|} \hline & M_2 \\ \hline G_2 & I_2 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline & M_1 & M_2 \\ \hline G_1 & I_1 & \times \\ \hline G_2 & \times & I_2 \\ \hline \end{array}$$

Figure 7.1.: Direct sum of two contexts

The lattice of any context sum is isomorphic to the direct product of the lattices of the summands [35]. In particular,  $\mathbb{K} + \mathbb{N}^c(1)$  always has exactly twice as many concepts as  $\mathbb{K}$ .

**7.2.1 Definition (Widening)** Let  $\mathbb{K} = (G, M, I)$  be a context and  $g, m$  be a non-incident object/attribute pair. The *widening of  $\mathbb{K}$  with respect to  $g$  and  $m$*  is  $\mathbb{K}^{g \leftrightarrow m} := (G, M, J)$ , where

$$J = I \cup \{(g, n) \mid n \in M, n \neq m\} \cup \{(h, m) \mid h \in G, h \neq g\}.$$

Figure 7.2 depicts the widening operation.

Observe that  $\mathbb{K}^{g \leftrightarrow m} = \mathbb{K}_{-g, -m} + (\{g\}, \{m\}, \emptyset)$ , where  $\mathbb{K}_{-g, -m}$  denotes the context obtained after deletion of  $g$  and  $m$ . The first indication that the widening of a context is connected to boolean suborders is the following fact:

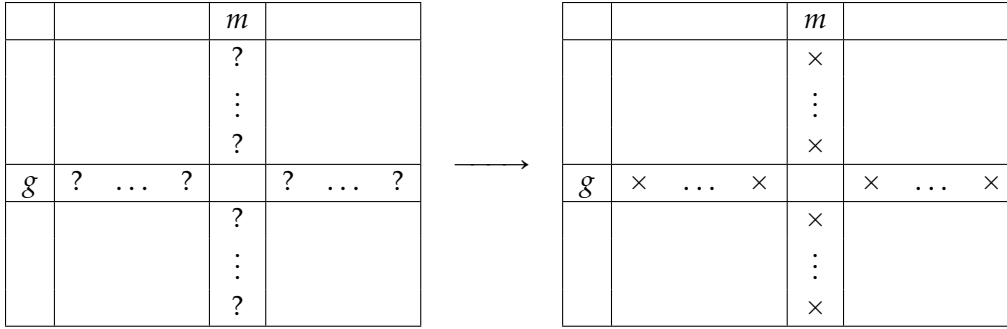


Figure 7.2.: Widening of a formal context

**7.2.2 Proposition** *Widening a finite context does not decrease its breadth.*

*Proof* Let  $g, m$  be a non-incident object/attribute pair in  $\mathbb{K}$ . Denote by  $b$  the breadth of  $\mathbb{K}$  and let  $\mathbb{K}_1$  be a contranominal subcontext with  $b$  objects (and attributes). We divide in two cases. If  $\mathbb{K}_1$  does not contain  $g$  or does not contain  $m$ , then there exists a contranominal subcontext  $\mathbb{K}_2 \leq \mathbb{K}_1$  with at least  $b - 1$  objects that does not contain neither  $g$  or  $m$ . Otherwise,  $\mathbb{K}_1$  contains both  $g$  and  $m$ , and the non-incident between  $g$  and  $m$  guarantees that  $\mathbb{K}_1$  has the form  $\mathbb{K}_2 + (\{g\}, \{m\}, \emptyset)$ . In both cases,  $\mathbb{K}_2$  is a contranominal subcontext with at least  $b - 1$  objects (attributes), none of which is  $g$  (is  $m$ ). It follows immediately that  $\mathbb{K}_2 + (\{g\}, \{m\}, \emptyset)$  is a contranominal subcontext of  $\mathbb{K}^{g \leftrightarrow m}$  with at least  $b$  objects.  $\square$

On the other direction, it is easy to see that widening a context increases its breadth by at most one. Indeed, every subcontext of  $\mathbb{K}^{g \leftrightarrow m}$  induces a subcontext of  $\mathbb{K}_{-g, -m}$  by restricting it to  $G \setminus \{g\}, M \setminus \{m\}$ , where  $G$  and  $M$  are the object and attribute sets of  $\mathbb{K}$ , respectively.

To prove Claim 7.1.3, we will prove the stronger Claim 7.2.3 instead. The fact that the latter is indeed stronger requires a proof; this will be shown in Proposition 7.2.4. Claim 7.2.3 shows, in particular, that a non-full context may always be widened in such a way that its number of concepts do not decrease.

**7.2.3 Claim** *In any context  $\mathbb{K}$ , every non-full column  $m$  has a non-incident with an object  $g$  such that  $\mathbb{K}^{g \leftrightarrow m}$  has at least as many concepts as the original context.*

**7.2.4 Proposition** *The validity of Claim 7.2.3 implies that of Claim 7.1.3.*

*Proof* Let  $k \leq n \leq m$  be natural numbers. We will proceed using the contextual perspective. Amongst all  $(n, m, k + 1)$ -extremal contexts, let  $\mathbb{K}$  be one with the maximum number of incidences. We will prove that  $\mathfrak{B}(\mathbb{K})$  is not  $B(k)$ -free. If  $k = n$ , then forcibly  $|\mathfrak{B}(\mathbb{K})| = 2^n$  is true, in which case the claim holds trivially. Suppose, therefore,  $k < n$ . Now, we split in two cases: suppose that  $\mathbb{K}$  contains some full row, say,  $g$ . We may clearly produce another context with the same number of objects and attributes, call it  $\mathbb{L}$ , by changing the

object-intent  $g'$  to some subset of the attribute set which is not an object-intent. Because  $\mathbb{L}$  has strictly more concepts than  $\mathbb{K}$ , it must be the case that  $\mathfrak{B}(\mathbb{L})$  is not  $B(k+1)$ -free. Thus, it holds that  $\mathfrak{B}(\mathbb{K})$  is not  $B(k)$ -free. For the remaining case, assume that  $\mathbb{K}$  does not have any full row. Thus, each object of  $\mathbb{K}$  has at least one non-incidence. Because of  $k < n$ , we can conclude that there exists a non-incident object/attribute pair  $g, m$  with  $\mathbb{K} \neq \mathbb{K}_{-g, -m} + (\{g\}, \{m\}, \emptyset)$ . Claim 7.2.3 provides an object  $h$  (possibly  $h = g$ ) such that  $\mathbb{L} := \mathbb{K}^{h \leftrightarrow m}$  has at least as many concepts as  $\mathbb{K}$ . The choices of  $g, h$  and  $m$  clearly make  $\mathbb{L}$  have more incidences than  $\mathbb{K}$ . Hence,  $\mathfrak{B}(\mathbb{L})$  can not be  $B(k+1)$ -free, making the lattice  $\mathfrak{B}(\mathbb{K})$  not  $B(k)$ -free.  $\square$

There is an equivalent way of thinking about widening a context while not decreasing its number of concepts. Suppose that  $g$  and  $m$  is a non-incident object/attribute pair. The equality  $\mathbb{K}^{g \leftrightarrow m} = \mathbb{K}_{-g, -m} + (\{g\}, \{m\}, \emptyset)$  yields immediately that  $\mathbb{K}^{g \leftrightarrow m}$  has at least as many concepts as  $\mathbb{K}$  if and only if  $\mathbb{K}_{-g, -m}$  has at least half the original number of concepts. Using well-known facts in the Formal Concept Analysis literature, it is easy to prove that the removal of an object and an attribute yields a context with at least 25% of the original number of concepts. This percentage is, however, distant from 50%.

Removing an object and a non-incident attribute can lead to a loss of more than 50% of the concepts. For example, consider the standard context of the three element chain. It is clear that the removal of its object which has no attributes, along with the removal of the empty column results in a one-by-one full context, which has only one concept. For a less trivial example which results even in reduced subcontexts, consider the formal context present in Figure 7.3. Its lattice has 15 elements and, when object  $g$  and attribute  $m$  both are removed, a (reduced) subcontext with only 7 concepts remains. In contrast, removing  $h$  and  $m$  results in a subcontext with 9 concepts (which is reduced as well).

$\mathbb{K}$	$m$	$n$	$o$	$p$	$q$
$g$		$\times$	$\times$		$\times$
$h$		$\times$		$\times$	$\times$
$i$	$\times$	$\times$		$\times$	
$j$	$\times$		$\times$	$\times$	
$k$	$\times$				$\times$

Figure 7.3.: A formal context

### 7.3. Widening viewed as contracting a star in a hypergraph

A *hypergraph* is a pair  $(V, E)$  where  $V$  is some set of elements called *vertices* and  $E \subseteq \mathcal{P}(V)$  is a set of *hyperedges*. If every hyperedge has precisely  $k$  elements, then the hypergraph is termed *k-uniform*. The case  $k = 2$  corresponds to the notion of graph which we defined in

Section 2.4 and hence hypergraphs generalize graphs. This is not the only way to define graphs and hypergraphs; for instance, in this work we do not allow multiple edges or hyperedges. Given a vertex  $v \in V$ , its *star* is defined as  $\{e \in E \mid \{v\} \subseteq e\}$ .

With these definitions, a hypergraph is essentially a family of subsets. In particular, we can see the object-intents of a clarified formal context  $(G, M, I)$  as a hypergraph having  $M$  as its vertex set. In exactly the same way, we may see the collection of all *complements* of object-intents as a hypergraph. The latter approach provides at times a very good deal of intuition to solve problems, so we will exemplify this. For this purpose we use the term *co-intent* to denote a complement of an intent and the term *co-object-intent* to refer to the complement of an object-intent.

The context we choose to give such example is the contra-path with six attributes, which was already depicted in Figures 2.1 and 2.2 in Section 2.2. Instead of calling its objects 1, 2, 3, 4 and 5, we make use of the letters  $g, h, i, j$  and  $k$ . The hypergraph of its co-object-intents - which is actually a graph since it is 2-uniform - is depicted below:

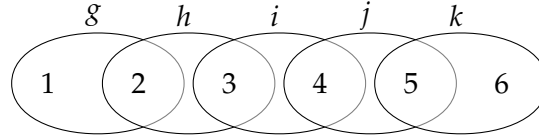


Figure 7.4.: Contra-path with six attributes, represented by the hypergraph of its co-object-intents.

It is well known and was explained in Proposition 2.1.2 that the closure by intersections of all object-intents is the system of intents. Trivially, for any context  $\mathbb{K}$ , the set  $\{\bar{A} \mid A \in \text{Int } \mathbb{K}\}$  is in bijection with  $\text{Int } \mathbb{K}$ . An elementary use of De Morgan's law delivers

$$\{\bar{A} \mid A \in \text{Int } \mathbb{K}\} = \{\overline{\cap_{g \in S} g'} \mid S \subseteq G\} = \{\cup_{g \in S} \bar{g}' \mid S \subseteq G\},$$

and in particular, the closure by unions of all co-object-intents is in bijection with the system of intents.

To exemplify the intuitional aid provided by the co-object-intents hypergraph, we count the number of intents of this particular context by counting the number of unions of co-object-intents. Set  $M = \{1, 2, 3, 4, 5, 6\}$  and let us represent subsets of  $M$  using their characteristic vectors (with respect to the natural order). For instance, the subset  $\{1, 3\}$  is represented by 101000 and 001110 represents the subset  $\{3, 4, 5\}$ . Using the equation above and looking to the hypergraph in Figure 7.4, we conclude that a subset of  $M$  is a co-intent if and only if its characteristic vector does not have any "isolated ones" (i. e. does not begin with 10, does not end with 01 and there is no 010 "factor" in the vector). These arguments clearly work for any contra-path; not only for the one with six vertices. It was this combinatorial model that was considered by Austin and Guy in [12]. They gave a

recurrence which counts the number of binary vectors without isolated ones, which when transported to FCA gives the explicit formula  $|\mathfrak{B}(CP(n))| = [c\gamma^n]$  (for the values of  $c, \gamma$ , see discussion around Figure 2.2).

We give an example in Figure 7.5 of how the hypergraph of co-object-intents gets modified by the widening operation. Specifically, each hyperedge in the star of the widened attribute loses some elements. More precisely, the hyperedge corresponding to the widening object loses every element except  $m$ , whereas the other hyperedges in the star lose only the element  $m$ .

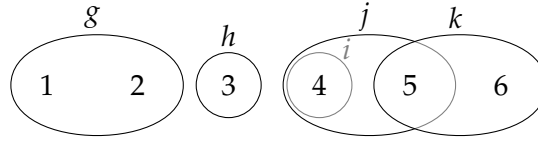


Figure 7.5.:  $CP(6)^{h \leftrightarrow 3}$ : contra-path with six attributes after widening with respect to  $h, 3$ .

## 7.4. Mixed generators

We will frequently deal with at least two incidence relations. Therefore, from now on we denote derivation by writing the incidence relation in a superscript. Generally, the letter  $I$  will denote the incidence relation of some “original” context  $\mathbb{K}$ , whereas  $J$  will mean the incidence relation of the widening  $\mathbb{K}^{g \leftrightarrow m}$ .

The main mathematical object which we use to prove Claim 7.2.3 is a notion which has two aspects having opposite characteristics. Informally speaking: on the one hand, the notion which we introduce resembles an extent because it will be maximal with respect to some subset. On the other hand, it will be like a minimal generator too, in that it will be minimal regarding some other subset. Now, we progressively make these thoughts more concrete. Consider some arbitrary set of objects  $S$  of some formal context. The definition of minimal generators, together with monotonicity of the closure operator yields:

$$S \text{ is a minimal generator} \Leftrightarrow (S \setminus \{g\})^I \neq S^I \text{ for each object } g \text{ in } S. \quad (7.1)$$

In an antipodal manner, the following equivalence dictates whether  $S$  is an extent:

$$S \text{ is an extent} \Leftrightarrow (S \cup \{g\})^I \neq S^I \text{ for each object } g \text{ not in } S. \quad (7.2)$$

One can see the equivalence in 7.2 as stating that an extent is a “maximal generator” (actually, it is the unique) of a closure.

If we previously fix some arbitrary subset  $R \subseteq G$  and require condition in (7.1) to be valid only for objects  $g \in R$  and the second condition (in (7.2)) to hold only for objects  $g \notin R$ , then we will arrive at a notion which is halfway between both. More concisely:

**7.4.1 Definition (mixed generator)** Let  $R \subseteq G$  be fixed. A set  $S \subseteq G$  is called a  $R$ -mixed generator (of the extent  $S^{II}$ ) if, for every  $g \in G$ , both implications below hold:

$$i) g \in (S \cap R) \Rightarrow (S \setminus \{g\})^{II} \neq S^{II}$$

$$ii) g \notin (S \cup R) \Rightarrow (S \cup \{g\})^{II} \neq S^{II}.$$

◇

One valid question is whether every extent has a mixed generator. Under the condition that the object set is finite, Proposition 7.4.2 will answer this in the affirmative.

Note that  $G$ -mixed generators are minimal generators and that  $\emptyset$ -mixed generators are extents. We occasionally refer to a  $R$ -mixed generator simply by *mixed generator* or by *mixgen* if there is no possibility for ambiguity.

We are particularly interested in the case when  $R$  is the set of objects not having some fixed attribute, that is, when  $R = G \setminus m^I$ . For this reason, we set the notation  $m^{\textcircled{I}} = G \setminus m^I$ . Note that, in this case, the set  $R$  is precisely the set of objects whose derivations are changed by the widening operation.

To visualize examples of mixed generators, consider Figure 7.6, which represents the context of Figure 7.3 using the same principle as Figure 7.4 (i. e. representing a clarified context through the hypergraph of its co-object-intents). Further, set  $R = m^{\textcircled{I}} = \{g, h\}$ . In Figure 7.6, the two objects belonging to  $R$  are (represented by) ellipses whereas the other three objects (the ones not in  $R$ ) are closed polygonal curves with rounded corners. The set  $S = \{h, i, j, k\}$  may be verified as being a mixed generator, since the removal of any element belonging to  $S \cap R$  causes its closure (equivalently, its derivation) to change and there is no element in  $G \setminus R$  which can be added to  $S$  without changing its closure. Note that  $S$  is not an extent. Similarly,  $\{g, h, i, j, k\}$  is an extent but not a mixed generator. Lastly,  $\{i, j\}$  is both an extent and a mixed generator, whereas the set  $\{h, j\}$  is neither an extent nor a mixed generator.

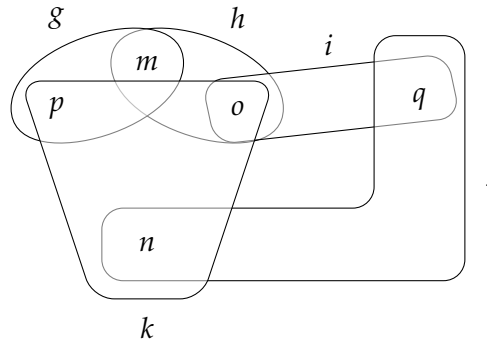


Figure 7.6.: Hypergraph of co-object-intents of the context in Figure 7.3.

In the next subsection, we prove elementary but necessary properties of mixed generators which will allow us to effectively work with them.

### 7.4.1. Elementary properties of mixed generators

We begin this subsection by showing the general existence of mixed generators, provided that the object set is finite. This is a particular case of the statement below, because an extent  $S$  always satisfies condition *ii*) of mixgens. In other words, it satisfies the hypothesis of the proposition below.

**7.4.2 Proposition** *If  $S$  is a finite set satisfying condition *ii*) in the definition of mixed generators, then  $S$  contains a mixed generator of  $S^{II}$ .*

*Proof* Condition *ii*) for  $S$  is equivalent to  $g^I \not\supseteq S^I$  for each  $g \in G \setminus (S \cup R)$ . If  $S$  also satisfies condition *i*), the claim follows trivially. Otherwise, pick some object  $g \in S \cap R$  such that  $(S \setminus \{g\})^I = S^I$  and define  $T := S \setminus \{g\}$ . Note that  $T \cup R$  and  $S \cup R$  are the same set, implying  $G \setminus (T \cup R) = G \setminus (S \cup R)$ . Since  $T^I = S^I$ , we have that condition *ii*) for mixed generators still holds for  $T$ . We repeat the removal of such an object until condition *i*) is fulfilled.  $\square$

Any mixed generator  $S$  has a dual aspect by definition, in that  $S$  has an extensional part and a minimal part. We mean the first to be  $S \cap R$ , whereas the latter is  $S \setminus R$ . The next proposition shows an intuitive fact, namely, that the extensional parts of two mixed generators of the same extent must coincide.

**7.4.3 Proposition** *Let  $S$  and  $T$  be mixed generators with  $S^{II} = T^{II}$ . Then,  $S \setminus R = T \setminus R$ .*

*Proof* Let  $g \in S \setminus R$ . Then,  $g^I \supseteq S^I = T^I$ , implying  $(T \cup \{g\})^{II} = T^{II}$ . Using the contraposition of condition *ii*) in the definition of mixgens gives that  $g \in T \cup R$ . Because of  $g \notin R$ , it follows that  $g \in T$ . The dual containment, i. e.  $T \setminus R \subseteq S \setminus R$ , is established in the same way.  $\square$

Proposition 7.4.4 describes which mixed generators are extents as well.

**7.4.4 Proposition** *Let  $(G, M, I)$  be a context and  $S$  be a  $R$ -mixed generator. Then,  $S$  is an extent if and only if  $(S^{II} \setminus S) \cap R = \emptyset$ .*

*Proof* The direct implication is clear since  $S^{II} \setminus S = \emptyset$  whenever  $S$  is an extent. For the converse, we prove the contraposition. Suppose that  $S$  is not an extent and take  $g \in S^{II} \setminus S$ . Note that  $g \in S^{II}$  implies  $(S \cup \{g\})^{II} = S^{II}$ . Condition *ii*) of the definition of mixgens forces, therefore, that  $g \in S \cup R$ . Because of  $g \notin S$ , it holds that  $g \in R$ .  $\square$

The following proposition is technical, but its usefulness shall become more clear later in the context of Proposition 7.5.3.

**7.4.5 Proposition** *Let  $(G, M, I)$  be a context,  $R = m^{(1)}$  for some attribute  $m$  and  $S \subseteq G$  be a set with  $S \cap R = \emptyset$ . Then,  $S$  is a  $R$ -mixed generator if and only if  $S$  is an extent.*

*Proof* Suppose that  $S$  is a mixed generator. Therefore,  $(S \cup \{g\}) \neq S^{II}$  for every  $g \in G \setminus (S \cup R)$ . Moreover,  $S \cap R = \emptyset$  and  $R = m^{\textcircled{I}}$  clearly imply  $(S \cup \{g\})^{II} \neq S^{II}$  for every  $g \in R$ . Combining both yields  $(S \cup \{g\})^{II} \neq S^{II}$  for every  $g \in G \setminus S$ , i.e.,  $S$  is an extent. For the converse, suppose that  $S$  is an extent. Since  $S \cap R = \emptyset$ , the set  $S$  fulfills trivially condition *i*) of mixgens. Condition *ii*) is likewise fulfilled by  $S$  because of  $(S \cup \{g\})^{II} \neq S^{II}$  for every  $g \in G \setminus S \supseteq G \setminus (S \cup R)$ .  $\square$

An easy but handy fact is the following:

**7.4.6 Proposition** *A set which is a mixed generator and an extent is always the unique mixed generator of itself.*

*Proof* Let  $S, T \subseteq G$  be  $R$ -mixed generators with  $T^{II} = S^{II} = S$ . Proposition 7.4.3 assures that  $S \setminus R = T \setminus R$ . Moreover, one has  $T \cap R \subseteq T^{II} \cap R = S \cap R$ , implying  $T \subseteq S$ . We now show that  $S \subseteq T$ . Suppose, by contradiction, that there exists  $g \in S \setminus T$ . Because of  $S \setminus R = T \setminus R$ , forcibly  $g \in R$  holds. Moreover,  $(S \setminus \{g\}) \supseteq T$ , causing  $(S \setminus \{g\})^{II} \supseteq T^{II} = S$  and ultimately  $(S \setminus \{g\})^{II} = S^{II} = S$ , contradicting that  $S$  is a  $R$ -mixed generator.

The next proposition shows that, for contexts  $\mathbb{K}$  of the form  $\mathbb{K}_1 + (\{g\}, \{m\}, \emptyset)$ , every mixed generator  $S$  without  $g$  comes accompanied by the mixed generator  $S \cup \{g\}$ .

**7.4.7 Proposition** *Let  $\mathbb{K} = (G, M, I)$  be a context such that  $\mathbb{K} = \mathbb{K}_1 + (\{g\}, \{m\}, \emptyset)$  for some object/attribute pair  $g, m$ . Let  $R \subseteq G$  be fixed and suppose that  $S$  is a  $R$ -mixed generator in  $\mathbb{K}$ . Then,  $S \cup \{g\}$  is a  $R$ -mixed generator as well.*

*Proof* Let  $S$  be as above. We may suppose that  $g \notin S$ , since otherwise the claim follows trivially. Note that  $m \in S^I$ . On account of  $g^I = M \setminus \{m\}$  and  $m^I = G \setminus \{g\}$ , it follows that  $(S \cup \{g\})^{II} = S^{II} \cup \{g\}$ , where the last union is disjoint ( $g \notin S^{II}$ ). Let  $h \in G \setminus (S \cup \{g\} \cup R)$ . Because  $S$  is a  $R$ -mixed generator and  $h \notin R$ , one has  $h \notin S^{II}$  (otherwise  $(S \cup \{h\})^{II} = S^{II}$  would hold). Then,  $h \in (S \cup \{g, h\})^{II}$  and  $h \notin S^{II} \cup \{g\} = (S \cup \{g\})^{II}$ , in particular,  $(S \cup \{g, h\})^{II} \neq (S \cup \{g\})^{II}$ . That is, the set  $S \cup \{g\}$  fulfills the second defining condition for mixed generators, since  $h$  was arbitrary. For the first condition, let  $h \in (S \cup \{g\}) \cap R$ . If  $h = g$ , then it is clear that  $S^{II} \neq (S \cup \{g\})^{II}$  and the condition is satisfied. For the other case, suppose that  $h \neq g$ . Then,  $(S \setminus \{h\})^I \neq S^I$ , because  $S$  is a mixed generator. Since  $\mathbb{K} = \mathbb{K}_1 + (\{g\}, \{m\}, \emptyset)$ , it follows that  $[(S \cup \{g\}) \setminus \{h\}]^I = (S \setminus \{h\})^I \setminus \{m\}$  and  $(S \cup \{g\})^I = S^I \setminus \{m\}$ . Combining these last two equations with  $m \in S^I \cap (S \setminus \{h\})^I$  and  $(S \setminus \{h\})^I \neq S^I$ , we arrive at  $[(S \cup \{g\}) \setminus \{h\}]^I \neq (S \cup \{g\})^I$  and that the first defining condition for mixed generators is satisfied too.  $\square$

## 7.5. Describing intents of the widening $\mathbb{K}^{g \leftrightarrow m}$

The objective of this section is to arrive at a sufficient condition for  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})|$ . Mixed generators will play an important role and, more specifically, we will work with

systems of mixed generators which are complete and representative, meaning that we pick exactly one mixed generator for each extent. The next definition sets this rigorously. Remind that, when speaking about mixed generators, it is implicit that some  $R \subseteq G$  is fixed. Such subset defines which are the minimal and which are the extensional parts of the generator.

**7.5.1 Definition (representative systems and complete systems)** Let  $\mathbb{K} = (G, M, I)$  be a formal context. A *representative system of mixed generators* is a family of subsets  $\mathcal{S} \subseteq \mathcal{P}(G)$  such that each  $S \in \mathcal{S}$  is a mixed generator and  $S \mapsto S^{II}$  is an injection from  $\mathcal{S}$  into  $\text{Ext } \mathbb{K}$ . If this (closure) mapping is surjective as well, we call  $\mathcal{S}$  a *complete representative system of mixed generators*. For brevity we shall write only *complete system of mixed generators*.  $\diamond$

**Example:** Setting  $R = m^{(I)}$  in the context of the Figures 7.3 and 7.6 one has that (omitting unnecessary braces and commas)

$$\mathcal{S} = \{\emptyset, g, gh, gi, gj, gk, h, hi, hij, hijk, i, ij, ijk, j, k\} \quad (7.3)$$

is a complete system of mixed generators.

Our strategy to reach this section's objective will be: given a non-incident object/attribute pair  $g, m$ , we take a complete system of  $m^{(I)}$ -mixgens  $\mathcal{S}$  of  $\mathbb{K}$  and proceed to lower bound

$$\{S^J \mid S \in \mathcal{S}\} \subseteq \text{Int}(\mathbb{K}^{g \leftrightarrow m}),$$

where  $J$  is the incidence relation of the widening  $\mathbb{K}^{g \leftrightarrow m}$ . In the family of subsets above, there is no guarantee that each  $S \in \mathcal{S}$  gives rise to a different  $S^J \in \text{Int}(\mathbb{K}^{g \leftrightarrow m})$ ; indeed this will not be the case in general. Nevertheless, we slowly but steadily prove results which lower bound the number of intents in  $\{S^J \mid S \in \mathcal{S}\}$ .

In the next three propositions, we relate mixed generators of an original context with those of its widening. For Propositions 7.5.2, 7.5.3 and 7.5.4, an arbitrary context  $\mathbb{K} = (G, M, I)$  and a non-incident object/attribute pair  $g, m$  is to be considered.

The next proposition shows that a mixed generator  $S$  in  $\mathbb{K}$  is halfway from being a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$ :  $S$  always fulfills condition *ii*) of the definition with respect to the latter context.

**7.5.2 Proposition** Let  $R = m^{(I)}$  and let  $S$  be a  $R$ -mixed generator in  $\mathbb{K}$ . Then, for every  $h \in G$  with  $h \notin S \cup R$  it holds that  $(S \cup \{h\})^{JJ} \neq S^{JJ}$ , where  $J$  denotes derivation in  $\mathbb{K}^{g \leftrightarrow m}$ . In particular, if  $S \cap R = \emptyset$ , then  $S$  is a  $R$ -mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$ .

*Proof* Let  $h \in G \setminus (S \cup R)$ . The fact that  $S$  is a mixgen in  $\mathbb{K}$  implies  $(S \cup \{h\})^{II} \neq S^{II}$ . Hence, one has that  $(S \cup \{h\})^I \neq S^I$ , which is equivalent to  $h^{(I)} \cap S^I \neq \emptyset$ . Now,  $h \notin R$  implies  $h^{(I)} = h^{(I)}$  and this, together with the fact that  $S^J \supseteq S^I$ , yields  $h^{(I)} \cap S^J \neq \emptyset$ . That is equivalent to  $(S \cup \{h\})^{JJ} \neq S^{JJ}$ . The particular case in the conclusion holds because  $S \cap R$  being empty causes condition *i*) of mixgens to be void.  $\square$

Minimal generators form a downset: the removal of any element of a minimal generator yields another minimal generator. For mixed generators this is clearly not the case but, as expected, the same phenomenon happens with respect to the minimal part of mixed generators. Indeed, we have the proposition below.

**7.5.3 Proposition** *For every  $R$ -mixed generator  $S$  and every  $T \subseteq S \cap R$ , it holds that  $S \setminus T$  is a  $R$ -mixed generator. In particular if  $R = m^{(I)}$ , then  $S \setminus R$  is a mixed generator in both contexts,  $\mathbb{K}$  and  $\mathbb{K}^{g \leftrightarrow m}$ . Moreover,  $S \setminus R$  is an extent in  $\mathbb{K}$  and, therefore, its unique mixed generator in that context.*

*Proof* Set  $U = S \setminus T$  so that  $S = U \cup T$  and  $S^I = U^I \cap T^I$ . First we prove that, if the condition *i*) of mixed generators were not valid for  $U$ , then it would also not be valid for the set  $S$ . Suppose, therefore, that there exists  $h \in U \cap R \subseteq S \cap R$  with  $(U \setminus \{h\})^I = U^I$ . Therefore, we have that  $(S \setminus \{h\})^I = [T \cup (U \setminus \{h\})]^I = T^I \cap (U \setminus \{h\})^I = T^I \cap U^I = S^I$ . Now, regarding condition *ii*) of mixed generators, take  $h \in G$  with  $h \notin (U \cup R)$ . Observe that  $U \cap R \subseteq S \cap R$  together with  $S \setminus R = U \setminus R$ ,  $h \notin R$  and  $h \notin U$  imply  $h \notin S$ . Since  $S$  is a mixed generator, we have that  $(S \cup \{h\})^I \neq S^I$ , which means that there exists an attribute  $n \in h^{(I)}$  with  $n \in S^I \subseteq U^I$ . Hence,  $(U \cup \{h\})^I \neq U^I$ . The three final claims (which require  $R = m^{(I)}$ ) come from Propositions 7.5.2, 7.4.5 and 7.4.6.  $\square$

From now on, we will employ the term “mixed generator” to refer to an  $m^{(I)}$ -mixed generator. In such situation, it is to be implicitly understood that some attribute  $m$  is given and that the set  $R$  is precisely  $m^{(I)}$ .

To avoid multiple counting of intents of  $\mathbb{K}^{g \leftrightarrow m}$ , we wish to control which mixed generators (of distinct extents) in  $\mathbb{K}$  have the same derivation with respect to  $\mathbb{K}^{g \leftrightarrow m}$  (and therefore give rise to the same intent in the latter context). It will be a consequence of Proposition 7.5.4 that we can describe these sets by looking to mixed generators in  $\mathbb{K}$  which are *not* mixed generators in  $\mathbb{K}^{g \leftrightarrow m}$ . In other words, mixgens which are not preserved by widening. Therefore, we will establish in Proposition 7.5.4 what is necessary and sufficient for a mixed generator in  $\mathbb{K}$  *not* to be a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$ . Before proving this statement, we give an example to illustrate this situation of a non-preserved mixed generator.

**Example:** Let  $\mathbb{K}$  denote the context of Figures 7.3 and 7.6 and consider the context  $\mathbb{K}^{g \leftrightarrow m}$ , which is depicted in Figure 7.7. Observe that  $\{h, i\}^I = \{i\}^I$  and  $h \in R$ , which implies that  $\{h, i\}$  is not<sup>1</sup> a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$ . Moreover, notice that  $\{h, i\}$  has the following three properties: first, it does not contain  $g$ . Second, its intersection with  $R$  is a singleton. Lastly, the derivation in  $\mathbb{K}$  (and in  $\mathbb{K}^{g \leftrightarrow m}$ ) of  $\{h, i\} \setminus R = \{h, i\} \setminus \{g, h\} = \{i\}$  equals  $\{h, i\}^I \cup \{m\}$ . Proposition 7.5.4 shows that these properties are characteristic.

<sup>1</sup>Notice that the set  $R$  is not, in any sense, “updated”.

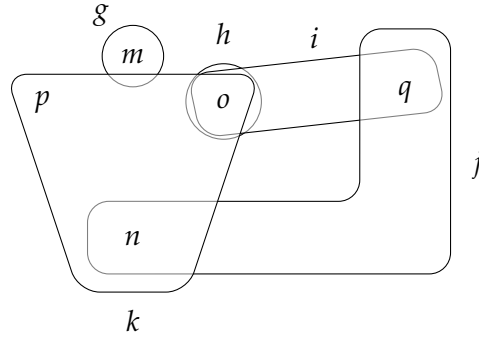


Figure 7.7.: Context  $\mathbb{K}^{g \leftrightarrow m}$ , in which the set  $\{h, i\}$  is not a mixed generator.

**7.5.4 Proposition** Let  $R = m^{\textcircled{1}}$  and let  $S$  be a mixed generator in  $\mathbb{K}$ . Then,  $S$  is not a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$  if and only if  $S \cap R$  is a singleton  $\{h\}$  with  $h \neq g$  and  $(S \setminus \{h\})^J = S^I \cup \{m\}$ .

*Proof* Let  $S$  be a mixed generator in  $\mathbb{K}$ . Suppose that  $S \cap R = \{h\}$  with  $h \neq g$  and that  $(S \setminus \{h\})^J = S^I \cup \{m\}$ . Since  $g \notin S$ , it follows from the definition of widening that  $S^J = S^I \cup \{m\}$ . By transitivity, it holds that  $(S \setminus \{h\})^J = S^J$  and  $S$  is not a mixgen in  $\mathbb{K}^{g \leftrightarrow m}$ . For the converse, by Proposition 7.5.3, we have that  $S \cap R \neq \emptyset$  and, by Proposition 7.5.2 it follows that  $S$  is not a mixgen in  $\mathbb{K}^{g \leftrightarrow m}$  because it fails to fulfill condition *i*) of mixgens. That means that there exists  $h \in S \cap R$  with  $(S \setminus \{h\})^J = S^J$ . This shows  $S \cap R \supseteq \{h\}$ . Note that, since  $g$  is the only object without the attribute  $m$  in  $\mathbb{K}^{g \leftrightarrow m}$ , we have that  $h \neq g$  and that  $h^J = h^I \cup \{m\}$ . From  $h \in (S \setminus \{h\})^J$  follows  $(S \setminus \{h\})^J \subseteq h^J = h^I \cup \{m\}$ . Moreover, note that  $(S \setminus \{h\})^I \subseteq (S \setminus \{h\})^J$  and, by transitivity,  $(S \setminus \{h\})^I \subseteq h^I \cup \{m\}$ . We now argue that  $S \cap R \subseteq \{h\}$ . Suppose, by contradiction, that  $i \in S \cap R$  with  $i \neq h$ . Then,  $i \in (S \setminus \{h\})^J$  which implies  $m \in (S \setminus \{h\})^I$ . Therefore,  $(S \setminus \{h\})^I \subseteq h^I$  which yields  $(S \setminus \{h\})^{II} = S^{II}$ , contradicting the fact that  $S$  is a mixed generator in  $\mathbb{K}$ . Since  $S \cap R = \{h\}$  and  $h \neq g$ , we have that  $S^J = S^I \cup \{m\}$ . By transitivity,  $S^I \cup \{m\} = (S \setminus \{h\})^J$ .  $\square$

The next introduced notion allows us to gain control over the derivation with respect to  $\mathbb{K}^{g \leftrightarrow m}$  of mixed generators.

**7.5.5 Definition (Decomposition of the first kind)** Suppose that a context  $\mathbb{K}$  is given, together with a non-incident object/attribute pair  $g, m$ . If  $\mathcal{S}$  is a system of mixed generators in  $\mathbb{K}$ , its *decomposition of the first kind* is  $\mathcal{S} = \mathcal{N} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ , where:

$$\begin{aligned}
 \mathcal{N} &= \{S \in \mathcal{S} \mid S \text{ is not a mixgen in } \mathbb{K}^{g \leftrightarrow m}\}, \\
 \mathcal{A} &= \{S \in \mathcal{S} \setminus \mathcal{N} \mid S^J = S^I\}, \\
 \mathcal{B} &= \{S \in \mathcal{S} \setminus \mathcal{N} \mid S^J \neq S^I, (S \cup \{g\})^J = S^I\} \text{ and} \\
 \mathcal{C} &= \{S \in \mathcal{S} \setminus \mathcal{N} \mid S^J \neq S^I, (S \cup \{g\})^J \neq S^I\},
 \end{aligned} \tag{7.4}$$

and  $J$  denotes the incidence relation of  $\mathbb{K}^{g \leftrightarrow m}$ . ◇

Notice that whenever  $S \in \mathcal{B}$ , it holds that  $g \notin S$ . In contrast, a mixed generator  $S \in \mathcal{C}$  always contains  $g$ . To see this, it suffices to realize that a subset  $T \subseteq G \setminus \{g\}$  always satisfies (exactly) one of the equalities  $T^J = T^I$  and  $(T \cup \{g\})^J = T^I$ . To develop intuition regarding these four classes of mixgens, we start with an easy lemma about mixed generators in  $\mathcal{N} \cup \mathcal{A}$ .

**7.5.6 Lemma** *For any  $S \in \mathcal{N} \cup \mathcal{A}$ , it holds that  $S \cap R$  contains at most one element. Moreover,  $(S \setminus R)^I \subseteq S^I \cup \{m\}$ .*

*Proof* Let  $S \in \mathcal{N} \cup \mathcal{A}$ . If  $S \in \mathcal{N}$ , then both claims follow from Proposition 7.5.4 (observe that  $(S \setminus \{h\})^J = (S \setminus \{h\})^I$ , where  $\{h\} = S \cap R$ ). Suppose, therefore, that  $S \in \mathcal{A}$ . Then,  $S^J = S^I$ . If  $g \notin S$ , then the equality  $S^J = S^I$  forces  $m \in S^I$  and, as a consequence,  $S \cap R = \emptyset$  and both claims follow trivially. Thus, assume  $g \in S$ . The definition of widening gives

$$\begin{aligned} S^J &= S^I \cup \{n \in M \mid n \in g^{\textcircled{I}}, n \in (S \setminus \{g\})^I, n \neq m\} \\ &= S^I \cup \{n \in M \mid n \in (g^{\textcircled{I}} \setminus \{m\}) \cap (S \setminus \{g\})^I\}. \end{aligned}$$

Hence,  $S^J = S^I$  makes the set on the right be empty, causing  $(S \setminus \{g\})^I \cap g^{\textcircled{I}} \subseteq \{m\}$  and, consequently,  $(S \setminus \{g\})^I \subseteq S^I \cup \{m\}$ . To complete the proof of both claims, we proceed by contradiction. The assumption will be that there exists  $h \in S \cap R$  with  $h \neq g$ . With this,  $h \in S \setminus \{g\}$  assures  $m \notin (S \setminus \{g\})^I$ , making the just proven containment not hold with equality. Since  $S^I \subseteq (S \setminus \{g\})^I$ , it necessarily follows  $(S \setminus \{g\})^I = S^I$ , contradicting the fact that  $S$  is a mixed generator. □

Mixgens in  $\mathcal{A}$  and in  $\mathcal{B}$  are manageable, in the sense that we have control of their  $J$ -derivation (meaning: derivation in  $\mathbb{K}^{g \leftrightarrow m}$ ). This is also the case for mixgens in  $\mathcal{N}$ , but unfortunately their  $J$ -derivations coincide to some of  $\mathcal{A}$ . Thus, in order to avoid repeated counting, we initially count only on the mixgens in  $\mathcal{A} \cup \mathcal{B}$  and *hope* that they form a representative system in  $\mathbb{K}^{g \leftrightarrow m}$ . This hope will not be unfounded: this fact will be proved later in Theorem 7.5.14. The  $J$ -derivations of mixgens in  $\mathcal{N} \cup \mathcal{C}$  will be more challenging to be recovered, but we will satisfactorily make use of them. More precisely, we will “recover” all the mixed generators of  $\mathcal{N}$  and some of  $\mathcal{C}$ . This will be done by applying the *restriction mapping*  $res : S \mapsto S \setminus R$  to inject them into  $\mathcal{S}$  itself. The first evidence that this can be done is the next proposition, which in particular shows that  $res$  maps both  $\mathcal{N}$  and  $\mathcal{C}$  to  $\mathcal{A}$ , provided that  $\mathcal{S}$  is a complete system of mixgens.

The next five propositions shall allow us to recover all mixgens in  $\mathcal{N}$  and some in  $\mathcal{C}$ .

**7.5.7 Proposition** *If  $\mathcal{S}$  is a complete system of mixed generators, then  $S \setminus R \in \mathcal{A} \subseteq \mathcal{S}$  for each mixed generator  $S \in \mathcal{S}$ .*

*Proof* Suppose that  $S \in \mathcal{S}$ . Proposition 7.5.3 assures that  $S \setminus R$  is a mixgen as well. Moreover, the same proposition implies that the set  $S \setminus R$  is an extent and the unique mixed generator of itself, which forces  $S \setminus R \in \mathcal{S}$  whenever  $\mathcal{S}$  is a complete system of mixgens. Also, the equality  $(S \setminus R)^I = (S \setminus R)^J$  is obvious; forcing actually  $S \setminus R \in \mathcal{A}$ .  $\square$

Widening a context changes the derivation of singleton objects in  $R = m^{\textcircled{I}}$ . Therefore, it makes sense to devote special attention to mixgens which have non-empty intersection with  $R$ . In contrast, we need a condition over elements  $h \in R$  that is stronger than  $h \notin S$  to help us identify mixgens  $S$  which are *largely unaffected* by the widening operation. This is introduced now. In the next definition, the attribute  $m$  is considered to be fixed, as if we were about to perform a widening with respect to some object in  $m^{\textcircled{I}}$ .

**7.5.8 Definition (strong avoidance)** Let  $S \subseteq G$  be a set of objects and  $h \in G$ . We say that  $S$  *strongly avoids*  $h$  if  $S^I \cap (h^{\textcircled{I}} \setminus \{m\}) \neq \emptyset$ .  $\diamond$

The condition above means that  $h$  does not belong to  $(S^I \setminus \{m\})^I$  and, in particular, neither to  $S^{II}$  or to  $S$ . Further, given an attribute  $m$  we define:

$$\chi(S) := \{g \in m^{\textcircled{I}} \mid S \text{ strongly avoids } g\}.$$

Even though we defined  $\chi$  for arbitrary object sets, we shall primarily calculate it for mixed generators. One intuition regarding  $\chi(S)$  is that the “closer” this set is to  $R$ , the less the widening operation affects the closure of  $S$ .

The following claim follows directly from the antitone property of the derivation operator:

**7.5.9 Proposition** Let  $S, T \subseteq G$  with  $S \subseteq T$ . Then,  $\chi(T) \subseteq \chi(S)$ .

In order to discern which sets in  $\mathcal{C}$  may be reused to produce intents in  $\mathbb{K}^{g \leftrightarrow m}$ , we define:

$$\mathcal{C}^R = \{S \in \mathcal{C} \mid R \subseteq S\} \text{ and } \mathcal{C}^{-R} = \{S \in \mathcal{C} \mid R \not\subseteq S\}. \quad (7.5)$$

Each set in  $\mathcal{C}^R$  shall be reused, in the sense explained before Proposition 7.5.7. For that purpose, we depend on the function  $\chi$ . This function has a remarkable usefulness which will gradually become apparent. The first manifestation of this is that  $\chi(\cdot)$  is able to distinguish the image sets  $\text{res}(\mathcal{N})$  and  $\text{res}(\mathcal{C}^R)$ , as the following two propositions show.

**7.5.10 Proposition** Let  $S \in \mathcal{C}^R$ . Then,  $\chi(S \setminus R) = R$ .

*Proof* Set  $T = S \setminus R$  and suppose that  $|R| \geq 2$ . Let  $h \in R$ . Note that  $m \notin (S \setminus \{h\})^I$ . Because  $S$  is a mixed generator and  $h \in R \subseteq S$ , it follows that  $(S \setminus \{h\})^I \neq S^I$ . Therefore, there exists  $n \in (S \setminus \{h\})^I$  such that  $n \in h^{\textcircled{I}}$  and  $n \neq m$ . In particular,  $(S \setminus \{h\})^I \cap (h^{\textcircled{I}} \setminus \{m\}) \neq \emptyset$ , which is to say that the set  $S \setminus \{h\}$  strongly avoids  $h$ . That is,  $h \in \chi(S \setminus \{h\}) \subseteq \chi(S \setminus R)$ , where the containment follows from Proposition 7.5.9. Since the object  $h$  was arbitrary, we have

that  $\chi(S \setminus R) = R$ . For the remaining case, necessarily  $R = \{g\}$ . The condition  $S^J \neq S^I$  allows us to take an attribute  $n \in S^J \setminus S^I$ . Clearly  $n \in g^{\textcircled{I}}$ , because  $g$  is the only object whose derivation with respect to  $J$  and  $I$  differ. Similarly,  $n \in h^I$  for every  $h \in S \setminus \{g\}$  because of  $n \in S^J$ . Note that  $S \in \mathcal{C}^R$  forces  $g \in S$  which in turn implies  $m \notin S^J$  and, as a consequence,  $n \neq m$ . We arrive, in particular, at  $(S \setminus \{g\})^I \cap (g^{\textcircled{I}} \setminus \{m\}) \neq \emptyset$ , that is,  $g \in \chi(S \setminus \{g\})$ .  $\square$

**7.5.11 Proposition** *Let  $S \in \mathcal{N}$ . Then,  $\chi(S \setminus R) \neq R$ .*

*Proof* By Proposition 7.5.4, it follows that  $S \cap R = \{h\}$  with  $h \neq g$  and  $(S \setminus \{h\})^J = S^J \cup \{m\}$ . Of course,  $(S \setminus \{h\})^I = (S \setminus \{h\})^J$  and by transitivity,  $(S \setminus \{h\})^I = S^I \cup \{m\}$ . Thus, the only attribute  $n$  satisfying  $n \in h^{\textcircled{I}}$  and  $n \in (S \setminus \{h\})^I$  is  $n = m$ . Consequently, the intersection between  $h^{\textcircled{I}} \setminus \{m\}$  and  $(S \setminus \{h\})^I$  is empty, that is, the set  $S \setminus \{h\} = S \setminus R$  does not strongly avoid  $h$ .  $\square$

If  $\mathcal{S}$  is a complete system of mixgens, then the restriction  $S \mapsto S \setminus R$  is clearly an injection from  $\mathcal{C}^R$  into  $\mathcal{A}$ : this is a trivial consequence of the definition in (7.5) and Proposition 7.5.7. Fortunately, the restriction mapping is injective when applied to  $\mathcal{N}$  as well - but this requires a proof. More specifically, the reader has probably noticed from the characterization of sets in  $\mathcal{N}$  (Proposition 7.5.4) that  $(S \setminus R)^J$  is not much different from  $S^I$  whenever  $S \in \mathcal{N}$ . Indeed, these sets differ only by the presence of the attribute  $m$ . As a result, we have injectivity:

**7.5.12 Proposition** *Suppose that  $\mathcal{S}$  is a complete system of mixgens. Then, the restriction mapping  $\text{res} : \mathcal{N} \rightarrow \mathcal{A}$  given by  $S \mapsto S \setminus R$  is an injection.*

*Proof* The fact that  $\text{res}$  is indeed a mapping to  $\mathcal{A} \subseteq \mathcal{S}$  is guaranteed by Proposition 7.5.7. Let  $S, T \in \mathcal{N}$  and suppose that  $S \setminus R = T \setminus R$ . Applying Proposition 7.5.4, one concludes that  $|S \cap R| = |T \cap R| = 1$  and, in particular,  $m \notin S^I \cup T^I$ . Moreover, Proposition 7.5.4 implies  $(S \setminus R)^J = S^I \cup \{m\}$  as well as  $(T \setminus R)^J = T^I \cup \{m\}$ . From  $S \setminus R = T \setminus R$  follows  $S^I \cup \{m\} = T^I \cup \{m\}$  and  $m \notin S^I \cup T^I$  yields  $S^I = T^I$ . Since  $\mathcal{N} \subseteq \mathcal{S}$  is a representative system of mixgens, this forces  $S = T$ .  $\square$

Thanks to the last three propositions, we now know that  $\mathcal{N}$  and  $\mathcal{C}^R$  can be injected into  $\mathcal{A}$  via the restriction  $S \mapsto S \setminus R$ . Moreover, the third and second to last results guarantee that the image sets of these restrictions is disjoint. This motivates the introduction of a decomposition which is finer than the one of the first kind, described in Definition 7.5.5. With this new decomposition, we will organize into which subset of  $\mathcal{A}$  the referred restriction mappings maps to. Notice that, unlike the first kind of decomposition, the definition below is valid only for complete systems.

**7.5.13 Definition (Decomposition of the second kind)** *Suppose that a context  $\mathbb{K}$  is given, together with a non-incident object/attribute pair  $g, m$ . If  $\mathcal{S}$  is a complete system of mixed generators in  $\mathbb{K}$ , its decomposition of the second kind is*

$$\mathcal{S} = \mathcal{N} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}^{\chi=R} \cup \mathcal{B} \cup \mathcal{C}^R \cup \mathcal{C}^{-R},$$

where:

$$\begin{aligned}\mathcal{A}_1 &= \{S \in \mathcal{A} \mid \chi(S) \neq R, S \notin \text{res}(\mathcal{N})\}, \\ \mathcal{A}^{\chi=R} &= \{S \in \mathcal{A} \mid \chi(S) = R\}, \\ \mathcal{A}_2 &= \text{res}(\mathcal{N})\end{aligned}$$

and  $\mathcal{N}, \mathcal{A}, \mathcal{B}$  are as in Definition 7.5.5 and  $\mathcal{C}^R, \mathcal{C}^{-R}$  are as defined in (7.5).  $\diamond$

**Examples:** Let  $\mathbb{K}$  denote the context depicted in Figures 7.3 and 7.6, together with the complete system of mixed generators  $\mathcal{S}$  given in (7.3). A decomposition of the second kind of  $\mathcal{S}$  with respect to  $\mathbb{K}^{g \leftrightarrow m}$  is given in Figure 7.8. We recall that  $R = m^{\textcircled{1}} = \{g, h\}$  for this particular context. Thus, for its attribute  $m$ , there are two different decompositions of the second kind: one for each choice of object to perform the widening ( $g$  or  $h$ ).

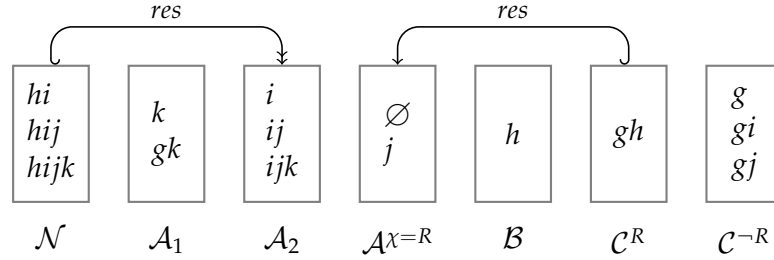


Figure 7.8.: Decomposition of the complete system of mixed generators given in (7.3) with respect to  $\mathbb{K}^{g \leftrightarrow m}$ .

Regarding the same complete system of mixgens but choosing the object  $h$  to perform the widening, we have the decomposition of the second kind depicted in Figure 7.9:

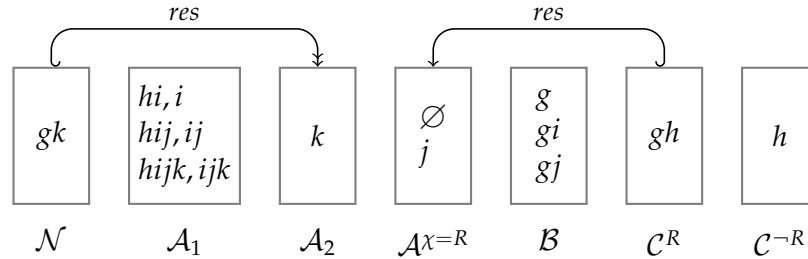


Figure 7.9.: Decomposition of the complete system of mixed generators given in (7.3) with respect to  $\mathbb{K}^{h \leftrightarrow m}$ .

Consider some context  $\mathbb{K}$  and a non-incident object/attribute pair  $g, m$ . Suppose that  $S$  is a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$ . Proposition 7.4.7 guarantees that  $S \cup \{g\}$  is a mixed generator in  $\mathbb{K}^{g \leftrightarrow m}$  as well. Consequently, the collection of all sets of the form  $S$  or  $S \cup \{g\}$  with

$S \in \mathcal{S} \setminus (\mathcal{N} \cup \mathcal{C}) = \mathcal{A} \cup \mathcal{B}$  is a candidate to be a representative system of mixgens in  $\mathbb{K}^{g \leftrightarrow m}$ . Note: we excluded mixgens belonging to  $\mathcal{C}$  because we do not have enough control over their derivation in  $\mathbb{K}^{g \leftrightarrow m}$ . Of course, the mentioned pairs  $S, S \cup \{g\}$  may collapse, i.e., it could be that  $g \in S$ . Therefore, to assure that we count intents only once, we will actually consider a mixgen  $S$  in this pairwise way only when it is certain that  $S$  does not contain  $g$ : that is, when  $S \in \mathcal{A}_2 \cup \mathcal{A}^{\chi=R} \cup \mathcal{B}$ . This construction of a candidate for a representative system is explained in Figure 7.10, where one can also observe an abstract decomposition of the second kind, together with all mappings which are relevant for this development.

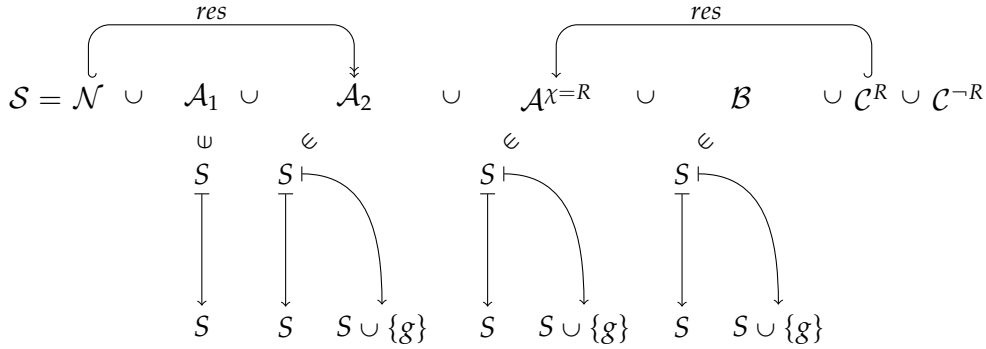


Figure 7.10.: A candidate for a representative system in  $\mathbb{K}^{g \leftrightarrow m}$  is the collection of all subsets having the seven forms described in the lower part of this figure.

Recall that the main objective of this section is to establish a sufficient condition for  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})|$ . This will be achieved by Theorem 7.5.14. That theorem will show that the collection of all sets having one of the seven forms  $S$  and  $S \cup \{g\}$  depicted on the lower part of Figure 7.10 form a representative system of mixed generators in  $\mathbb{K}^{g \leftrightarrow m}$ . Consider for a moment this statement as already proved. Let us count a balance of the number of arrows in Figure 7.10. The class  $\mathcal{N}$  is not mapped directly to the set of all mixed generators of  $\mathbb{K}^{g \leftrightarrow m}$ , but the fact that there are two mappings having  $\mathcal{A}_2$  as domain compensates this absence of a direct mapping. The same argument follows for  $\mathcal{C}^R$ , which is compensated by the two mappings from  $\mathcal{A}^{\chi=R}$ . Hence,  $\mathcal{C}^{-R}$  is the only class which is not mapped and not compensated. This reasoning produces a lower bound relating the number of intents in  $\mathbb{K}^{g \leftrightarrow m}$  and those of  $\mathbb{K}$ : it holds that (supposing finite  $|\mathfrak{B}(\mathbb{K})|$ ):

$$|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})| - |\mathcal{C}^{-R}| + |\mathcal{B}|.$$

Of course, for this inequality to hold, the claim stating that we indeed have a representative system in the widening must be proved first. This shall be done now. Below, we use  $\mathcal{A}$  as in Definition 7.5.5 (i.e. it means  $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}^{\chi=R}$ ). Moreover, the symbol  $|_{\mathcal{A}}$  (for instance) refers to restricting the original mapping's domain by intersecting it with  $\mathcal{A}$ . The term "second kind decomposition" means the same as "decomposition of the second kind".

**7.5.14 Theorem** Let  $\mathcal{S} = \mathcal{N} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}^{\chi=R} \cup \mathcal{B} \cup \mathcal{C}^R \cup \mathcal{C}^{-R}$  be the second kind decomposition of a complete system of mixed generators with respect to  $\mathbb{K}^{g \leftrightarrow m}$ . Then, the mappings  $S \xrightarrow{\alpha} S$  and  $S \xrightarrow{\beta} S \cup \{g\}$  are injections, respectively, from  $\mathcal{A} \cup \mathcal{B}$  and  $\mathcal{A}_2 \cup \mathcal{A}^{\chi=R} \cup \mathcal{B}$  into the set of all mixed generators of  $\mathbb{K}^{g \leftrightarrow m}$ . The images of  $\alpha$  and  $\beta$  are disjoint and their union is a representative system in  $\mathbb{K}^{g \leftrightarrow m}$ . The corresponding intents are given by

$$\begin{aligned} [\alpha|_{\mathcal{A}}(S)]^I &= [\beta|_{\mathcal{B}}(S)]^I = S^I, \\ [\alpha|_{\mathcal{B}}(S)]^I &= S^I \cup \{m\} \text{ and} \\ [\beta|_{\mathcal{A}}(S)]^I &= S^I \setminus \{m\}. \end{aligned}$$

Intents which already were intents of the original context are characterized through the conditions  $\alpha(S)^I \in \text{Int } \mathbb{K} \Leftrightarrow S \in \mathcal{A}$  and  $\beta(S)^I \in \text{Int } \mathbb{K} \Leftrightarrow S \notin \mathcal{A}^{\chi=R}$ . In particular, under the condition that  $\mathcal{S}$  is finite, it holds that  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})| + |\mathcal{B}| - |\mathcal{C}^{-R}|$ .

*Proof* Every  $S \in \mathcal{A} \cup \mathcal{B}$  is a mixed generator in  $\mathbb{L} := \mathbb{K}^{g \leftrightarrow m}$  because of the definition of  $\mathcal{N}$  and  $\mathcal{A} \cap \mathcal{N} = \mathcal{B} \cap \mathcal{N} = \emptyset$ . Similarly, the set  $\beta(S) = S \cup \{g\}$  must be a mixed generator in  $\mathbb{L}$  because of Proposition 7.4.7. We show the formulas for the intents. For both cases where the domain restriction is  $\mathcal{A}$ , let  $S \in \mathcal{A}$  be arbitrary. Note that  $S^I = S^J$ . Therefore,  $\alpha(S)^I = S^I = S^J$ , which shows the formula for  $\alpha|_{\mathcal{A}}$ . Now, for  $\beta|_{\mathcal{A}}$ , suppose that  $S \in \mathcal{A}_2 \cup \mathcal{A}^{\chi=R} \subseteq \mathcal{A}$ . Then,  $\beta(S)^I = (S \cup \{g\})^I = S^I \setminus \{m\} = S^J \setminus \{m\}$ . Let  $S \in \mathcal{B}$  in order to show the formulas for  $\alpha|_{\mathcal{B}}$  and  $\beta|_{\mathcal{B}}$ . Observe that  $S^I \neq S^J$  and  $(S \cup \{g\})^I = S^I$  imply  $S^J = S^I \cup \{m\}$ , with a disjoint union holding. Thus,  $S^J \setminus \{m\} = S^I$ . The first equality gives  $S^I \cup \{m\} = S^J = \alpha(S)^I$ , while the second yields  $\beta(S)^I = (S \cup \{g\})^I = S^I \cap g^I = S^I \setminus \{m\} = S^I$ . To prove that the union of the image sets of these mappings is a representative system, we first show the two mentioned equivalences.

Consider the first equivalence. One direction is quite easy: if  $S \in \mathcal{A}$ , then  $S^I = S^J = \alpha(S)^I$  is an intent of  $\mathbb{K}$ . For the other direction, we prove the contraposition. Thus, suppose that  $S \notin \mathcal{A}$  (and, consequently,  $S \in \mathcal{B}$ ). We show that  $S^I$  is not in  $\text{Int } \mathbb{K}$ . Note that  $S \in \mathcal{B}$  implies  $S^J \neq S^I$  and, therefore,  $S \cap R \neq \emptyset$ . Suppose, by contradiction, that there exists  $T \subseteq G$  with  $T^I = S^I = S^I \cup \{m\}$ . In particular,  $T^I \supseteq S^I$  from which follows  $T^{II} \subseteq S^{II}$ , which in turn implies  $(S \cup \{h\})^{II} = S^{II}$  for each  $h \in T$ . Since  $m \in T^I$ , it holds that  $T \cap R = \emptyset$ . Note that this implies  $T^J = T^I$ . Combining  $(S \cup \{h\})^{II} = S^{II}$  for each  $h \in T$  with the fact that  $S$  is a mixed generator in  $\mathbb{K}$  and  $T \cap R = \emptyset$ , we arrive at  $T \subseteq S$ . Now, on the one hand, we have by the antitone property that  $(S \setminus R)^J \supseteq S^J$ . On the other hand,  $T \cap R = \emptyset$  and  $T \subseteq S$  imply  $S \setminus R \supseteq T$  and, again by antitonicity,  $(S \setminus R)^J \subseteq T^J = T^I = S^I$ . We arrive at  $(S \setminus R)^{JJ} = S^{II}$ , which contradicts the fact that  $S$  is a mixed generator in  $\mathbb{L}$ .

For the second equivalence, we first establish the converse. Suppose that  $S \notin \mathcal{A}^{\chi=R}$  and, consequently,  $S \in \mathcal{A}_2 \cup \mathcal{B}$ . Suppose that  $S \in \mathcal{B}$  holds. Then,  $\beta(S)^I = (S \cup \{g\})^I = S^I$ . In particular,  $\beta(S)^I$  is an intent of  $\mathbb{K}$ . Consider the other possibility, i. e.  $S \in \mathcal{A}_2$ . From the definition of  $\mathcal{A}_2$  follows that  $S \cap R = \emptyset$ , which implies  $m \in S^I$  and of course  $S^I = S^J$ . Note that  $\beta(S)^I = (S \cup \{g\})^I = S^I \cap g^I = S^I \setminus \{m\} = S^J \setminus \{m\}$ . Because of  $\chi(S) \neq R$ , we

may take  $h \in R$  with  $(S^I \setminus \{m\}) \cap h^{\textcircled{I}} = \emptyset$ . Hence,  $S^I \cap h^{\textcircled{I}} \subseteq \{m\}$ . Actually, equality must hold because of  $m \in S^I$  (certainly  $m \in h^{\textcircled{I}}$ , since  $h \in R$ ). Consequently,  $(S \cup \{h\})^I = S^I \cap h^I = S^I \setminus \{m\} = \beta(S)^I$ , where the last equality follows from transitivity. For the other direction, we prove the contraposition. Let  $S \in \mathcal{A}^{\chi=R}$ . We show that  $\beta(S)^I =: B$  is not an intent of  $\mathbb{K}$ . Observe that  $B = S^I \cap g^I = S^I \setminus \{m\}$ . Since  $\chi(S) = R$ , it follows that the set  $(S^I \setminus \{m\}) \cap h^{\textcircled{I}} = B \cap h^{\textcircled{I}}$  is non-empty for each  $h \in R$ . By contradiction, suppose that  $T \subseteq G$  is such that  $T^I = B$ . But then,  $m \notin B$  requires that  $m \notin T^I$ , which clearly forces  $h \in T$  for some  $h \in R$ , causing  $T^I \cap h^{\textcircled{I}} = \emptyset$  and contradicting the non-emptiness of  $B \cap h^{\textcircled{I}}$  for arbitrary  $h \in R$ .

Denote by  $(\cdot)^I$  the derivation operator in  $\mathbb{L}$  and consider the (composite) mappings  $S \mapsto \alpha|_{\mathcal{A}}(S)^I$ ,  $S \mapsto \beta|_{\mathcal{B}}(S)^I$ ,  $S \mapsto \alpha|_{\mathcal{B}}(S)^I$  and  $S \mapsto \beta|_{\mathcal{A}}(S)^I$ . We now establish that each one of these is injective. Recall that  $\mathcal{S}$  is a representative system. Thus, the injectivity of  $\alpha|_{\mathcal{A}} \circ (\cdot)^I$  and  $\beta|_{\mathcal{B}} \circ (\cdot)^I$ , i. e. the first and second mappings, follows trivially from  $S^I = S^I$  (whenever  $S \in \mathcal{A}$ , by definition) and  $(S \cup \{g\})^I = S^I$  (whenever  $S \in \mathcal{B}$ , likewise). For the third, i. e.  $\alpha|_{\mathcal{B}} \circ (\cdot)^I$ , it suffices to use the shown formula  $S^I = S^I \cup \{m\}$  and notice that  $m \notin S^I$  for each  $S \in \mathcal{B}$ . Similarly,  $m \in S^I$  for each  $S \in \mathcal{A}_2 \cup \mathcal{A}^{\chi=R}$ . Thus, since in this case one has  $(S \cup \{g\})^I = S^I \setminus \{m\}$ , injectivity of the fourth follows. We now argue that these four mappings have disjoint image sets. Three pairs of mappings, namely the ones with  $\alpha|_{\mathcal{B}}$ ,  $\beta|_{\mathcal{A}}$  as well as  $\alpha|_{\mathcal{A}}$ ,  $\beta|_{\mathcal{B}}$  and  $\alpha|_{\mathcal{B}}$ ,  $\beta|_{\mathcal{B}}$  need not be checked: the intent formulas (and representativity of  $\mathcal{S}$ ) easily imply the pairwise disjointness of these images. For the remaining, we begin with  $\alpha|_{\mathcal{A}}$  and  $\alpha|_{\mathcal{B}}$ : let  $S \in \mathcal{A}$ ,  $T \in \mathcal{B}$ . By the first equivalence,  $\alpha(S)^I$  is an intent of  $\mathbb{K}$  whereas  $\alpha(T)^I$  is not; consequently  $\alpha(S)^I \neq \alpha(T)^I$ . Now, for  $\beta|_{\mathcal{A}}$  and  $\beta|_{\mathcal{B}}$ : let  $S \in \mathcal{A}_2 \cup \mathcal{A}^{\chi=R}$  and  $T \in \mathcal{B}$ . The intent formula and both equivalences yield that  $\beta(S)^I \cup \{m\}$  is an intent of  $\mathbb{K}$ , whereas  $\beta(T)^I \cup \{m\} = \alpha(T)^I$  is not. Thus,  $\beta(S)^I \neq \beta(T)^I$ . Regarding  $\alpha|_{\mathcal{A}}$  and  $\beta|_{\mathcal{A}}$ , let  $S \in \mathcal{A}$  and  $T \in \mathcal{A}_2 \cup \mathcal{A}^{\chi=R}$ . If  $T \in \mathcal{A}^{\chi=R}$ , then  $\beta(T)^I \notin \text{Int } \mathbb{K}$ , whereas  $\alpha(S)^I \in \text{Int } \mathbb{K}$ ; in particular,  $\alpha(S)^I \neq \beta(T)^I$ . Otherwise, one has  $T \in \mathcal{A}_2$ . The definition of  $\mathcal{A}_2$  implies that a superset of  $T$  belongs to  $\mathcal{N}$ . Using Proposition 7.5.4, we have that such superset must have the form  $T \cup \{h\}$ , with  $h \in R \setminus \{g\}$ . Clearly,  $T \cup \{h\} \neq S$  since the first belongs to  $\mathcal{N}$  and the latter to  $\mathcal{A}$ . Moreover,  $(T \cup \{h\})^I = T^I \setminus \{m\} = T^I \setminus \{m\} = (T \cup \{g\})^I$ . Thus, an equality between  $\beta(T)^I = (T \cup \{g\})^I$  and  $\alpha(S)^I = S^I$  is not possible, since this would imply  $S^I = (T \cup \{h\})^I$  by transitivity, thereby violating the representativity of  $\mathcal{S}$ .

The main objective of this section has been accomplished by the last result. Instead of proceeding directly to the next section, we use the opportunity to develop one last notion which goes a long way to comprehend more easily Section 7.6.

The second kind decomposition is defined only for complete systems because we map  $\mathcal{N}$  and  $\mathcal{C}^R$  “internally” into the very own system of mixgens. This helps understanding our construction and develops intuition - it is however not necessary. Actually, it is easier to develop further the remaining results - alongside with intuition - if we introduce a third

kind of decomposition. This shall be explained now.

Consider the second kind decomposition of some complete system of mixed generators. We can map both  $\mathcal{N}$  and  $\mathcal{C}^R$  directly into the set of intents of  $\mathbb{K}^{g \leftrightarrow m}$  via  $\text{res} \circ \beta \circ J$ , where  $J$  denotes derivation in  $\mathbb{K}^{g \leftrightarrow m}$  and  $\beta$  adds the object  $g$ , that is,  $\beta$  is as defined in Theorem 7.5.14. The corresponding intents can be calculated as follows. The characterization of sets which are mixed generators in  $\mathbb{K}$  but not in  $\mathbb{K}^{g \leftrightarrow m}$  (Proposition 7.5.4) gives us a formula for the intent:  $(\text{res}(S) \cup \{g\})^J = (S^I \cup \{m\}) \cap g^J = S^I$  whenever  $S \in \mathcal{N}$ . Similarly, it is clear that  $(\text{res}(S) \cup \{g\})^J = (S \setminus R)^I \setminus \{m\}$  for every  $S \in \mathcal{C}^R$ . In this setting, without internal mappings, it is not required anymore that  $S$  be complete. Moreover, the need for the symbol  $\mathcal{A}_2$  disappears and we partition  $\mathcal{A}$  in two, instead of three: we set  $\mathcal{A}^{\chi \neq R} = \mathcal{A}_1 \cup \mathcal{A}_2$ . This induces the following:

**7.5.15 Definition (Decomposition of the third kind)** Suppose that a context  $\mathbb{K}$  is given, together with a non-incident object/attribute pair  $g, m$ . If  $\mathcal{S}$  is a system of mixed generators in  $\mathbb{K}$ , its *decomposition of the third kind* is

$$\mathcal{S} = \mathcal{N} \cup \mathcal{A}^{\chi \neq R} \cup \mathcal{A}^{\chi = R} \cup \mathcal{B} \cup \mathcal{C}^R \cup \mathcal{C}^{-R},$$

where:

$$\begin{aligned} \mathcal{A}^{\chi = R} &= \{S \in \mathcal{A} \mid \chi(S) = R\} \\ \mathcal{A}^{\chi \neq R} &= \{S \in \mathcal{A} \mid \chi(S) \neq R\}, \end{aligned}$$

and  $\mathcal{N}, \mathcal{A}, \mathcal{B}$  are as in Definition 7.5.5 and  $\mathcal{C}^R, \mathcal{C}^{-R}$  are as defined in (7.5).  $\diamond$

Of course, another difference between second and third kind decompositions is that one may with the latter decompose systems of mixgens which are not necessarily complete. A depiction of a decomposition of the third kind, along with the relevant mappings, is present in Figure 7.11. Note that this figure synthetises the intent formulas proved in Theorem 7.5.14, and the following fact can be appreciated: such depiction unfolds quite well the statement that the reused mixed generators form a representative system. Indeed, this becomes evident from the rectangle containing four times  $S^I$  and the other containing two intents, which clearly can not collide.

## 7.6. Stability

In the last section, we saw that  $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$  is sufficient for  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})|$ . Before we start conducting investigations of a more combinatorial nature, we need to prove that decompositions of the second (and third) kind enjoy at least some degree of stability.

By “stability”, we mean the following. Consider some context  $\mathbb{K}$ , a fixed attribute  $m$  and take a representative system of mixed generators  $\mathcal{S}$ . Any of the introduced decompositions of  $\mathcal{S}$  depends on choosing a widening  $\mathbb{K}^{g \leftrightarrow m}$  with  $g \in m^{\textcircled{1}}$ . Therefore, it depends on the

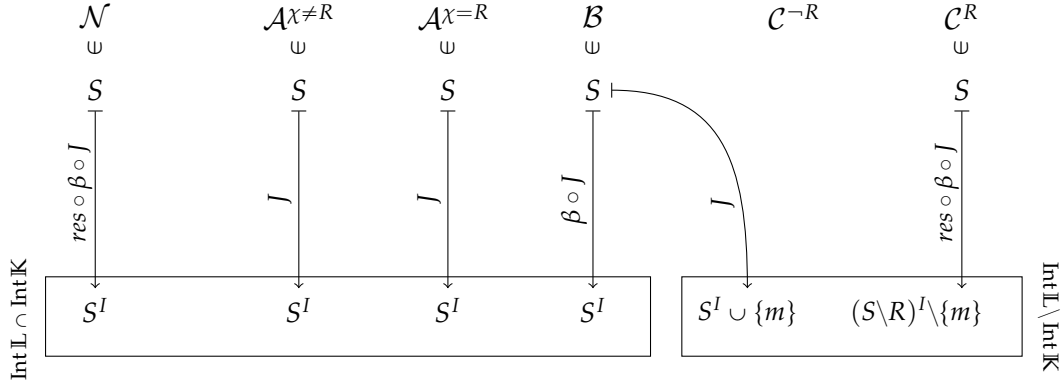


Figure 7.11.: Decomposition of the third kind of a system of mixed generators, together with their intents in  $\mathbb{L} = \mathbb{K}^{g \leftrightarrow m}$ .

choice of some object  $g \in m^{\textcircled{I}}$ . The stability investigation performed in this section answers the following: how do the decomposition classes change for different choices of  $g$  (and fixed  $m$ )? The two concrete examples in Figures 7.8 and 7.9 show that the classes may, indeed, change.

Note that, for any  $S \in \mathcal{S}$ , the value of  $\chi(S)$  depends on  $m$  but does not depend on the object  $g$  chosen to perform a widening  $\mathbb{K}^{g \leftrightarrow m}$ . The function  $\chi$  is the main agent responsible of Theorem 7.6.1.

**7.6.1 Theorem (stability)** *For any context  $\mathbb{K}$  with some fixed attribute  $m$  and a representative system  $\mathcal{S}$  of mixgens, the classes  $\mathcal{A}^{\chi=R}$  and  $\mathcal{C}^R$  do not depend on the choice of the object  $g$  to perform the widening  $\mathbb{K}^{g \leftrightarrow m}$ . Moreover, let  $S \in \mathcal{S}$ . Then,*

$$\begin{aligned} S \in \mathcal{N} \cup \mathcal{A} &\Leftrightarrow \chi(S) = \chi(S \setminus R) \\ S \in \mathcal{B} \cup \mathcal{C} &\Leftrightarrow \chi(S) \subsetneq \chi(S \setminus R). \end{aligned}$$

*In particular,  $\mathcal{B} \cup \mathcal{C}^{-R}$  does not depend on the choice of  $g$  either.*

*Proof* For the claim regarding  $\mathcal{A}^{\chi=R}$ , let  $S$  be a mixed generator in  $\mathbb{K}$  with  $S \in \mathcal{A}^{\chi=R}$  for some choice of  $g$ . Then,  $\chi(S) = R$  implies  $S \cap R = \emptyset$ , therefore  $S^I$  is always  $S^I$ , independently of  $g$ . This implies that  $S$  always belongs to  $\mathcal{A}$ . The fact that  $\chi(\cdot)$  does not depend on  $g$  yields that actually  $S \in \mathcal{A}^{\chi=R}$ . We now assume  $|R| \geq 2$ . Recall that  $R = m^{\textcircled{I}}$ . Let  $S \in \mathcal{C}^R$  for some choice of  $g$ . Further, let  $h \in R$  with  $h \neq g$  and denote by  $J_h$  the incidence relation of  $\mathbb{K}^{h \leftrightarrow m}$ . Observe that  $S \in \mathcal{C}^R$  implies  $R \subseteq S$  and  $h \in S$ . Because  $S$  is a mixgen in  $\mathbb{K}$ , it follows that  $(S \setminus \{h\})^I \neq S^I$ , which means that there exists  $n \in h^{\textcircled{I}}$  with  $n \in (S \setminus \{h\})^I$ . Note that  $n \notin S^I$ . Since  $g \in S \setminus \{h\}$ ,  $n \in (S \setminus \{h\})^I$  and  $g \nmid m$ , we have  $n \neq m$ . Because of  $h^{J_h} = M \setminus \{m\}$  and  $(S \setminus \{h\})^I \subseteq (S \setminus \{h\})^{J_h}$ , it follows that  $n \in S^{J_h}$ . Hence, in particular,  $S^{J_h} \neq S^I$  with  $h \in S$ , which causes  $S \in \mathcal{C}^R$  when choosing  $h$  for the widening.

For the first equivalence, let  $S \in \mathcal{N} \cup \mathcal{A}$ . Further, let  $h \in R$  be arbitrary and observe that  $S^I \cap (h^{\mathbb{I}} \setminus \{m\})$  and  $(S^I \cup \{m\}) \cap (h^{\mathbb{I}} \setminus \{m\})$  are actually the very same set. Moreover, Lemma 7.5.6 yields  $(S \setminus R)^I \subseteq S^I \cup \{m\}$ , which justifies the only one-sided implication in:

$$\begin{aligned} S \setminus R \text{ strongly avoids } h &\Leftrightarrow (S \setminus R)^I \cap (h^{\mathbb{I}} \setminus \{m\}) \neq \emptyset \\ &\Rightarrow (S^I \cup \{m\}) \cap (h^{\mathbb{I}} \setminus \{m\}) \neq \emptyset \\ &\Leftrightarrow S^I \cap (h^{\mathbb{I}} \setminus \{m\}) \neq \emptyset \\ &\Leftrightarrow S \text{ strongly avoids } h. \end{aligned}$$

Conversely, antitonicity of the derivation operator assures the validity of the implication

$$S^I \cap (h^{\mathbb{I}} \setminus \{m\}) \neq \emptyset \Rightarrow (S \setminus R)^I \cap (h^{\mathbb{I}} \setminus \{m\}) \neq \emptyset.$$

Thus, if  $S$  strongly avoids  $h$ , then  $S \setminus R$  does as well. Hence,  $\chi(S) = \chi(S \setminus R)$ . For the converse of the statement, suppose that  $S \notin \mathcal{N} \cup \mathcal{A}$ , that is,  $S \in \mathcal{B} \cup \mathcal{C}$ . Set  $T = S \cap R$ . Clearly,  $T \neq \emptyset$ . We divide in cases. First, suppose that  $|T| \geq 2$  and take  $h \in T$ . Trivially,  $S$  does not strongly avoid  $h$  (on account of  $h \in S$ ). Since  $S$  is a mixgen, it follows that  $(S \setminus \{h\})^I \neq S^I$ , which is equivalent to  $(S \setminus \{h\})^I \cap h^{\mathbb{I}} \neq \emptyset$ . From  $|T| \geq 2$  follows that  $m \notin (S \setminus \{h\})^I$ , which in turn implies that  $S \setminus \{h\}$  strongly avoids  $h$ . Proposition 7.5.9 implies that  $S \setminus R$  strongly avoids  $h$  as well, thus  $\chi(S \setminus R) \neq \chi(S)$ . For the remaining case  $|T| = 1$ , we divide in two subcases. First, suppose that  $S \in \mathcal{B}$ . Then,  $T = \{h\}$  with  $h \neq g$ , since no set in  $\mathcal{B}$  may contain  $g$ . We claim that  $(S \setminus \{h\})^I$  contains properly  $S^I \cup \{m\}$ . The containment is clear, we have to discard equality: if  $(S \setminus \{h\})^I = S^I \cup \{m\}$  were the case, then  $T = S \cap R = \{h\}$  would force  $(S \setminus \{h\})^I = (S \setminus \{h\})^I = S^I \cup \{m\}$  and, by Proposition 7.5.4, the set  $S$  would belong to  $\mathcal{N}$ , and not to  $\mathcal{B}$ . Therefore, there exists  $n \in (S \setminus \{h\})^I$  with  $n \neq m, n \in h^{\mathbb{I}}$ . That is,  $S \setminus \{h\}$  strongly avoids  $h$  and, again by the antitonicity of  $\chi$  (Proposition 7.5.9), it follows that  $\chi(S \setminus R) \neq \chi(S)$ . For the last subcase, suppose that  $S \in \mathcal{C}$ . Then, the inequalities  $S^I \neq S^I$  and  $(S \cup \{g\})^I \neq S^I$  imply  $T = \{g\}$ . Then, it holds that  $m \notin S^I$  and we take some arbitrary element  $n \in S^I \setminus S^I$ . Of course,  $n \neq m$ . Since the only object  $h \in S$  with  $h^I \neq h^I$  is  $g$ , it holds that  $n \in (S \setminus \{g\})^I$  and  $n \in g^{\mathbb{I}}$ . Hence, the set  $(S \setminus \{g\})^I \cap (g^{\mathbb{I}} \setminus \{m\})$  is non-empty, meaning that  $S \setminus \{g\}$  strongly avoids  $g$ . The second equivalence follows from the first and Proposition 7.5.9.  $\square$

Figure 7.12 illustrates the properties stated in Theorems 7.5.14 and 7.6.1. It is a more comprehensive depiction than Figure 7.11.

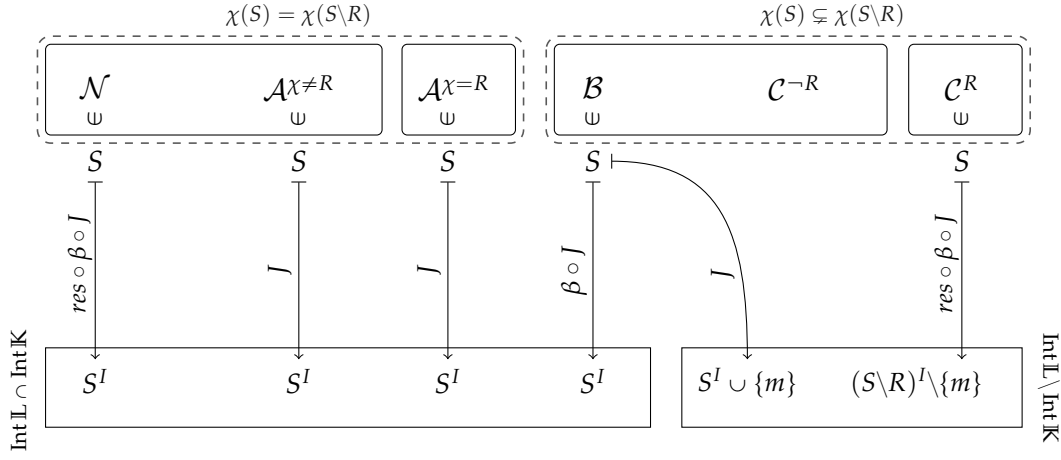


Figure 7.12.: Decomposition of the third kind of a system of mixed generators, their corresponding derivations in  $\mathbb{L} = \mathbb{K}^{g \leftrightarrow m}$  and stability “superclasses”.

Combining the above discussion with the results of Theorems 7.6.1 and 7.5.14, we summarize:

**7.6.2 Corollary** *Let  $\mathbb{K} = (G, M, I)$  be a context,  $m \in M$  and suppose that  $\mathcal{S} \subseteq \mathcal{P}(G)$  is a system of mixed generators. Each object  $g \in m^{\textcircled{I}}$  gives rise to a third kind decomposition*

$$\mathcal{S} = \mathcal{N} \cup \mathcal{A}^{\chi \neq R} \cup \mathcal{A}^{\chi = R} \cup \mathcal{B} \cup \mathcal{C}^{-R} \cup \mathcal{C}^R.$$

*For fixed  $m$ , the classes  $\mathcal{A}^{\chi = R}$  and  $\mathcal{C}^R$  are the same, independently of the choice of  $g$ . The same holds for the unions of classes  $\mathcal{N} \cup \mathcal{A}^{\chi \neq R}$  and  $\mathcal{B} \cup \mathcal{C}^{-R}$ . Moreover, if  $\mathcal{S}$  is complete and finite, then  $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$  is a sufficient condition for  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})|$ .*

## 7.7. Combinatorial proof that an object such that $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$ exists

In this section, we will see that there always exist an object which leads to a decomposition with  $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$ . Before we are able to expose these developments, we require some classic, auxiliary statements. They are presented here as Theorem 7.7.1 and Propositions 7.7.2 and 7.7.3. In Theorem 7.7.1, the symbol  $N_{\mathcal{G}}(S)$  is defined as being the set of vertices adjacent to some vertex in  $S$ . Moreover, *to cover*  $X$  means for a matching that its set of endpoints contains  $X$ . The proof of Theorem 7.7.1 can be found in virtually any basic reference for graph theory, in particular [20] features three different proofs.

**7.7.1 Theorem (Hall)** *Let  $\mathcal{G}$  be a bipartite graph with classes  $X$  and  $Y$ . If the “marriage condition”  $|N_{\mathcal{G}}(S)| \geq |S|$  is fulfilled for each  $S \subseteq X$ , then  $\mathcal{G}$  has a matching which covers  $X$ .*

**7.7.2 Proposition** *Every regular bipartite graph has a perfect matching. Actually, every  $k$ -regular bipartite graph has  $k$  pairwise disjoint perfect matchings.*

*Proof (idea)* The proof usually goes by induction on  $k$  and checking that Hall's condition is fulfilled.  $\square$

**7.7.3 Proposition** *Let  $R$  be a finite set with  $n$  elements and let  $0 \leq k \leq \lfloor n/2 \rfloor$ . There exists a bijection between its  $k$ -element sets and  $(n - k)$ -element sets that associates supersets to each set:  $S \mapsto T$  with  $S \subseteq T$ .*

*Proof* Consider the bipartite graph having all  $k$ -element subsets of  $R$  in one class and all  $(n - k)$ -element subsets in the other class. Adjacencies in this graph are given by set inclusion  $\subseteq$ . Now, every  $k$ -element subset is contained in  $\binom{n-k}{n-2k}$   $(n - k)$ -element subsets and every  $(n - k)$ -element subset contains  $\binom{n-k}{k}$   $k$ -element subsets. On account of  $\binom{n-k}{n-2k} = \binom{n-k}{n-k-k} = \binom{n-k}{k}$ , the graph is regular and Proposition 7.7.2 guarantees the existence of a perfect matching.  $\square$

The next proposition is not really classic but it is very intuitive. It describes a property which, however simple, will be of great use for our combinatorial arguments.

Given any total order on  $G$ , its power-set enjoys at least one important, induced order: the lexicographic order. More precisely, this ordering on  $\mathcal{P}(G)$  is given by  $A < B$  if and only if  $\min\{A \triangle B\} \in B$ , that is, if the smallest element of  $G$  for which the sets  $A$  and  $B$  differ belongs to  $B$ .

**7.7.4 Proposition** *Suppose that  $\mathcal{F} \subseteq \mathcal{P}(G)$  is a lexicographically ordered family of subsets and let  $A, B, C \subseteq G$ . If  $A < B$  and  $B \cap C = \emptyset$ , then  $A \cup C < B \cup C$ .*

*Proof* Let  $A, B$  and  $C$  be as above and set  $i := \min\{A \triangle B\}$ . Since  $A < B$ , we have  $i \in B$ . Thus, it follows easily from  $B \cap C = \emptyset$  that  $i \notin C$ . We argue that  $i$  is also the minimum element of  $(A \cup C) \triangle (B \cup C)$ . Indeed, a simple calculation shows:

$$\begin{aligned} (A \cup C) \triangle (B \cup C) &= [(A \cup C) \setminus (B \cup C)] \cup [(B \cup C) \setminus (A \cup C)] \\ &= [(A \setminus B) \setminus C] \cup [(B \setminus A) \setminus C] \\ &= [(A \setminus B) \cup (B \setminus A)] \setminus C = (A \triangle B) \setminus C. \end{aligned}$$

Hence,  $i \in (A \triangle B) \setminus C$  and, because  $(A \triangle B) \setminus C$  is a subset of  $A \triangle B$ , the element  $i$  must be the minimum amongst the elements in such subset too. In particular,  $A \cup C < B \cup C$ .  $\square$

We may now switch back our mindset to mixed generators. The statement that an object such that  $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$  exists will require two conditions to be satisfied: first, we need the original context to have finitely many objects. Second, we must have some complete system of mixgens satisfying a restricted downset property, which we define now. We say that a

family of  $R$ -mixed generators  $\mathcal{S} \subseteq \mathcal{P}(G)$  has the *semi-downset property* if the implication  $S \in \mathcal{S}, T \subseteq R \Rightarrow S \setminus T \in \mathcal{S}$  holds. Note that this definition is general and does not demand  $R = m^{\textcircled{I}}$ ; this is also the case for Lemma 7.7.5. The next lemma will be an instrument of great importance by allowing us to establish inequalities in Theorem 7.7.6.

**7.7.5 Lemma** *Suppose that a context with finite object set  $G$  is given and let  $R \subseteq G$  be arbitrary. Then, there exists a complete system of  $R$ -mixed generators which has the semi-downset property. In fact, by giving to the power-set of  $G$  an arbitrary lexicographic order, the system obtained by choosing the minimum mixed generator for each extent has this property.*

*Proof* Proposition 7.4.2 yields that each extent has some mixed generator. Suppose that  $<$  is an arbitrary lexicographic order over  $\mathcal{P}(G)$  and for every extent  $A \in \text{Ext } \mathbb{K}$ , define  $S(A) := \min\{S \subseteq G \mid S \text{ is a mixed generator of } A\}$  and  $\mathcal{S} := \{S(A) \mid A \in \text{Ext } \mathbb{K}\}$ . We show that, for every  $S \in \mathcal{S}$  and every  $T \subseteq S \cap R$ , the set  $S \setminus T =: U$  belongs as well to  $\mathcal{S}$ . Note that  $S \setminus R = U \setminus R$ . By Proposition 7.5.3, we have that  $U$  is a mixed generator. Suppose, by contradiction, that  $U \notin \mathcal{S}$ . Then, there exists a mixed generator  $V \in \mathcal{S}$  with  $V < U$  and  $U^{II} = V^{II}$  (therefore  $U^I = V^I$ ). By Proposition 7.4.3, it follows that  $V \setminus R = U \setminus R$  and, by transitivity,  $V \setminus R = S \setminus R$ . The first step to arrive at a contradiction is to prove that  $V \cup T$  is smaller than  $S$  and has the same closure. To that purpose, we apply Proposition 7.7.4, since we have  $V < U$  and  $U \cap T = \emptyset$ ; this allows us to conclude  $V \cup T < U \cup T = S$ . Moreover, clearly  $(V \cup T)^I = V^I \cap T^I = U^I \cap T^I = (U \cup T)^I = S^I$ , implying  $(V \cup T)^{II} = S^{II}$ . The second and final step of this proof is to show that  $V \cup T$  contains some mixed generator with the same closure. Since  $V \setminus R = S \setminus R$  and  $T \subseteq R$ , we trivially have that  $(V \cup T) \setminus R = S \setminus R$ . We argue that  $V \cup T$  satisfies condition *ii*) in the definition of mixed generators. Let  $g \in G \setminus (V \cup T \cup R)$ . Note that  $g \notin R$ , together with  $(V \cup T) \setminus R = S \setminus R$  and  $g \notin V \cup T$  imply that  $g \notin S$ . Now,  $S$  being a mixed generator justifies the only “not equal” symbol below:

$$(V \cup T)^I = S^I \neq (S \cup \{g\})^I = S^I \cap g^I = (V \cup T)^I \cap g^I = (V \cup T \cup \{g\})^I,$$

meaning that  $V \cup T$  fulfills condition *ii*). By Proposition 7.4.2,  $V \cup T$  contains a mixed generator of  $(V \cup T)^{II} = S^{II}$ , and such mixgen is of course not larger (w.r.t.  $<$ ) than  $V \cup T$ . Hence, said mixed generator is strictly smaller than  $S$ , contradicting its minimality.  $\square$

We are now ready for the main result of this section. The statement below remains valid when one substitutes the word “third” by “second”, adjusts the classes accordingly and considers only complete systems of mixed generators.

**7.7.6 Theorem** *Suppose that  $\mathbb{K} = (G, M, I)$  is a context with finite  $G$  and let  $m \in M$  be an attribute not possessed by some object (i.e. not a full column). Further, let  $\mathcal{S}$  be any system of mixed generators such that  $\mathcal{S}$  has the semi-downset property. Then, there exists an object  $g \in m^{\textcircled{I}}$  that gives rise to a third kind decomposition*

$$\mathcal{S} = \mathcal{N} \cup \mathcal{A}^{\chi \neq R} \cup \mathcal{A}^{\chi = R} \cup \mathcal{B} \cup \mathcal{C}^{\neg R} \cup \mathcal{C}^R$$

with  $|\mathcal{B}| \geq |\mathcal{C}^{-R}|$ .

*Proof* Let  $\mathcal{S}$  be as above and consider fixed the attribute  $m$  with  $m^{(1)} \neq \emptyset$ . Set  $R = m^{(1)}$  and  $n = |R|$ . We may of course suppose  $n \geq 2$ : otherwise  $\mathcal{C}^{-R} = \emptyset$  and there is nothing to prove. Stability (Theorem 7.6.1) allows us to define, without choosing an object,  $\mathcal{D}^T := \{S \in \mathcal{B} \cup \mathcal{C}^{-R} \mid S \cap R = T\}$  for each  $T \subseteq R$ . Notice that  $\mathcal{D}^\emptyset = \mathcal{D}^R = \emptyset$ . For each  $g \in R$  we further define

$$\mathcal{D}^g = \bigcup_{\{g\} \subseteq T \subsetneq R} \mathcal{D}^T \quad \text{and} \quad \mathcal{D}^{-g} = \bigcup_{\emptyset \subsetneq T \subseteq R \setminus \{g\}} \mathcal{D}^T.$$

Let  $g \in R$ . The definitions of  $\mathcal{D}^g$  and  $\mathcal{D}^{-g}$ , allied with the fact that  $\mathcal{D}^\emptyset = \mathcal{D}^R = \emptyset$ , make it clear that  $\mathcal{B} \cup \mathcal{C}^{-R} = \mathcal{D}^g \cup \mathcal{D}^{-g}$ . Since no set in  $\mathcal{B}$  contains  $g$ , whereas every set in  $\mathcal{C}^{-R}$  does, actually more holds:  $\mathcal{B} = \mathcal{D}^{-g}$  and  $\mathcal{C}^{-R} = \mathcal{D}^g$ . Now, suppose by contradiction that  $|\mathcal{D}^{-g}| < |\mathcal{D}^g|$  for every  $g \in R$ .

$$|\mathcal{D}^{-g}| < |\mathcal{D}^g| \Leftrightarrow \sum_{\emptyset \subsetneq T \subseteq R \setminus \{g\}} |\mathcal{D}^T| < \sum_{\{g\} \subseteq T \subsetneq R} |\mathcal{D}^T|.$$

Summing for all  $g$ :

$$\sum_{g \in R} \sum_{\emptyset \subsetneq T \subseteq R \setminus \{g\}} |\mathcal{D}^T| < \sum_{g \in R} \sum_{\{g\} \subseteq T \subsetneq R} |\mathcal{D}^T|. \quad (7.6)$$

Consider a proper subset  $T$  of  $R$  and set  $k = |T|$ . For each element of  $R \setminus T$ , there exists a summand  $|\mathcal{D}^T|$  on the left-hand side of the inequality (7.6) ( $n - k$  summands). On the other hand, for each element of  $T$ , there exists a summand  $|\mathcal{D}^T|$  on the right-hand side ( $k$  summands). We calculate a balance: if  $k \leq \lfloor n/2 \rfloor$ , the contribution of  $|\mathcal{D}^T|$  will be only on the left-hand side and multiplied by  $(n - k) - k = (n - 2k)$ . Analogously, if  $k \geq \lfloor n/2 \rfloor + 1$ , then the contribution of  $|\mathcal{D}^T|$  will be only on the right-hand side and multiplied by  $k - (n - k) = (2k - n)$ . We have, therefore

$$\sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k) \cdot \sum_{\substack{T \subseteq R \\ |T|=k}} |\mathcal{D}^T| < \sum_{k=\lfloor n/2 \rfloor + 1}^{n-1} (2k - n) \cdot \sum_{\substack{T \subseteq R \\ |T|=k}} |\mathcal{D}^T| = \sum_{k=\lfloor n/2 \rfloor}^{n-1} (2k - n) \cdot \sum_{\substack{T \subseteq R \\ |T|=k}} |\mathcal{D}^T|. \quad (7.7)$$

(the equality above is indeed valid: for odd  $n$ , it is obvious. For even  $n$ , notice that  $2k - n$  equals zero for  $k = n/2$ ).

We now argue on the other direction, aiming to prove a contradicting inequality. Let us denote the  $k$ -element subsets of  $R$  by  $\mathcal{R}^k := \{T \subseteq R \mid |T| = k\}$  for each  $1 \leq k \leq n - 1$ . The semi-downset property of  $\mathcal{S}$  assures that whenever  $T \supseteq U$ , the restriction  $S \mapsto S \setminus (T \setminus U)$

is an injection from  $\mathcal{D}^T$  into  $\mathcal{D}^U$ . Now, for each  $i$  with  $1 \leq i \leq \lfloor n/2 \rfloor$ , take a bijection  $\varphi^i : \mathcal{R}^{n-i} \rightarrow \mathcal{R}^i$  such that  $\varphi^i(T) \subseteq T$  for every  $T \in \mathcal{R}^{n-i}$  (such bijections exist by Proposition 7.7.3). Now, let  $k \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ , so that  $n - 2k \geq 0$ . Then:

$$\begin{aligned}
\sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^T| &\leq \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^{\varphi^k(T)}| \Leftrightarrow (n - 2k) \cdot \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^T| \leq (n - 2k) \cdot \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^{\varphi^k(T)}| \\
&\Rightarrow \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k) \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^T| \leq \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k) \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^{\varphi^k(T)}| \\
&\Leftrightarrow \sum_{k=n-\lfloor n/2 \rfloor}^{n-1} (2k - n) \sum_{T \in \mathcal{R}^k} |\mathcal{D}^T| \leq \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k) \sum_{T \in \mathcal{R}^{n-k}} |\mathcal{D}^{\varphi^k(T)}| \\
&\Leftrightarrow \sum_{k=\lfloor n/2 \rfloor}^{n-1} (2k - n) \sum_{\substack{T \subseteq R \\ |T|=k}} |\mathcal{D}^T| \leq \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k) \sum_{\substack{T \subseteq R \\ |T|=k}} |\mathcal{D}^T|,
\end{aligned}$$

which contradicts (7.7).  $\square$

As a consequence, the claim which implies our objective gets answered in the affirmative:

**7.7.7 Corollary** Let  $\mathbb{K} = (G, M, I)$  be a context with finite  $G$  and let  $m \in M$  be an attribute which does not correspond to a full column. Then, there exists an object  $g \in m^{(\mathbb{I})}$  such that  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| \geq |\mathfrak{B}(\mathbb{K})|$ . In other words, Claim 7.2.3 is true.

*Proof* One takes a complete system of mixed generators with the semi-downset property provided by Lemma 7.7.5 and applies Theorem 7.7.6 in order to fulfill the sufficient condition guaranteed by Theorem 7.5.14 (or Corollary 7.6.2).  $\square$

In particular, Theorem 7.7.6 shows that Claim 7.1.3 is true.

**7.7.8 Corollary** Given any triple  $k \leq n \leq m$ , some  $(n, m, k + 1)$ -extremal lattice contains a boolean suborder  $\mathcal{B}(k)$ . That is, Claim 7.1.3 holds.

*Proof* Follows from the validity of 7.2.3 as established by Proposition 7.2.4.  $\square$

**7.7.9 Corollary** Let  $\mathbb{K} = (G, M, I)$  be a context with finite  $G$  or finite  $M$ . Then, there exists an object/attribute pair  $g, m$  such that  $|\mathfrak{B}(\mathbb{K}_{-g, -m})| \geq \frac{1}{2} |\mathfrak{B}(\mathbb{K})|$ .

*Proof* If some full column exists, traditional FCA arguments prove the claim. Otherwise, we apply Corollary 7.7.7 and remove the given object/attribute pair.  $\square$

A natural way of attacking this the full conjecture (Conjecture 7.1.2) is by trying to prove that a “smart” choice of  $g$  done by Corollary 7.7.7 actually achieves a strict inequality, i. e.  $|\mathfrak{B}(\mathbb{K}^{g \leftrightarrow m})| > |\mathfrak{B}(\mathbb{K})|$ . The context depicted in Figure 7.13, however, is a counterexample to such approach. For that context, any choice of  $m$  and  $g$  yields a widening  $\mathbb{K}^{g \leftrightarrow m}$  with exactly 22 concepts, which is precisely the number of concepts of  $\mathbb{K}$  as well.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>
1			×	×	×	×
2	×	×			×	×
3	×	×	×	×		
4		×		×		×
5	×		×		×	

Figure 7.13.: A context for which no widening strictly increases the number of concepts.

## 7.8. What happens to extents after widening?

The reader has maybe wondered why we did not give formulas for the extents in Theorem 7.5.14. The reason is that we wish to keep that part of the exposition as simple as possible. We however, find it to be an interesting question and proceed to attack it now. We turn our attention to the closure operator in  $\mathbb{K}^{g \leftrightarrow m}$ . More specifically, we shall relate it to the closure operator in  $\mathbb{K}$  and to the function  $\chi$ . For that, we define

$$\bar{\chi}(S) = R \setminus \chi(S).$$

The functions  $\chi$  (and therefore  $\bar{\chi}$ ) are to be calculated always regarding the incidence relation of  $\mathbb{K}$ .

**7.8.1 Lemma** *Let  $S \in \mathcal{N} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}^R$ . Then:*

$$\begin{aligned} S \in \mathcal{C}^R &\Rightarrow S = S^{II} = S^{JJ}, \\ S \in \mathcal{A} &\Rightarrow S \subseteq S^{II} \subseteq S^{JJ}, \\ S \in \mathcal{N} \cup \mathcal{B} &\Rightarrow S \subseteq S^{II} \setminus \{g\} \subseteq S^{JJ}. \end{aligned}$$

Furthermore,  $S^{JJ} \setminus S^{II} \subseteq \bar{\chi}(S)$  and  $S^{II} \setminus S \subseteq \bar{\chi}(S)$ .

*Proof* Whenever  $S \in \mathcal{C}^R$ , the set  $S$  is a mixgen in  $\mathbb{K}$  and  $\mathbb{K}^{g \leftrightarrow m}$  such that  $R \subseteq S$ , and Proposition 7.4.4 assures that  $S$  is an extent in both contexts. If  $S \in \mathcal{A}$  then, clearly,  $S^J = S^I$  as well as  $S^{JJ} = S^{IJ} \supseteq S^{II}$ . Now, suppose that  $S \in \mathcal{N} \cup \mathcal{B}$ . Proposition 7.5.4 implies that we have  $S^J = S^I \cup \{m\}$  in case that  $S \in \mathcal{N}$ . The same equality follows easily from the definition of  $\mathcal{B}$ , so that that relationship is valid in either case. Then, it follows that  $S^{JJ} = (S^I \cup \{m\})^J = S^{IJ} \cap m^J = S^{IJ} \setminus \{g\} \supseteq S^{II} \setminus \{g\}$ .

Regarding the last two assertions, we assume  $S \notin \mathcal{C}^R$  since otherwise both follow trivially from the already established fact that  $S$  is an extent in both contexts. For the second claim: take an object  $h \in S^{II} \setminus S$ . Since  $S$  is a mixed generator in  $\mathbb{K}$ , we have that  $h \in R$  and it is clear that  $h^{(1)} \cap S^I = \emptyset$  and, consequently,  $S$  does not strongly avoid  $h$ . Now, let  $h \in S^{JJ} \setminus S^{II}$ .

We may suppose  $h \neq g$ : indeed, in  $\mathbb{K}^{S \leftrightarrow m}$ , the object  $g$  is an extremal point of every extent containing it, that is,  $g \in S^{JJ}$  implies  $g \in S \subseteq S^{II}$ . Of course,  $h \notin S^{II}$  is equivalent to the condition of  $h^{\textcircled{I}} \cap S^I$  being non-empty, whereas  $h \in S^{JJ}$  if and only if  $h^{\textcircled{I}}$  and  $S^I$  are disjoint. Since  $S^J \supseteq S^I$  and  $h^{\textcircled{I}} = h^{\textcircled{I}}$  or  $h^{\textcircled{I}} = h^{\textcircled{I}} \setminus \{m\}$ , only the second equality may and must hold, which implies  $h \in R$ , as well as  $h^{\textcircled{I}} \cap S^I = \{m\}$ . In particular, the set  $S$  does not strongly avoid  $h$ .  $\square$

The last lemma accomplishes a great part of the endeavor of describing the extents, as is evidenced by the following proof:

**7.8.2 Theorem** *The extents of the corresponding intents described in Theorem 7.5.14 are given by*

$$\begin{aligned} [\alpha(S)]^{JJ} &= S \cup (\bar{\chi}(S) \setminus \{g\}) \text{ and} \\ [\beta(S)]^{JJ} &= S \cup \bar{\chi}(S) \cup \{g\}. \end{aligned}$$

*Proof* We first prove that  $\bar{\chi}(S) \setminus \{g\} \subseteq S^{JJ}$ . Indeed, for an object  $h \in \bar{\chi}(S) \setminus \{g\}$  one has that  $S$  does not strongly avoid  $h$ , that is,  $h^{\textcircled{I}} \cap S^I \subseteq \{m\}$ . Now, since  $h$  is distinct from  $g$ , it follows that  $h^{\textcircled{I}} = h^{\textcircled{I}} \setminus \{m\}$ . Consequently,  $\emptyset = h^{\textcircled{I}} \cap S^I$  and, in any case, the calculated intent  $S^J$  is a subset of  $S^I \cup \{m\}$ , which causes  $h^{\textcircled{I}} \cap S^J = \emptyset$ , that is,  $h \in S^{JJ}$ . Combining this fact with the following two properties yields the formula for  $\alpha(S)^{JJ}$ : first, for each  $S \in \mathcal{S}$ , one has that  $g \in S^{JJ}$  implies  $g \in S$ , and the converse is obvious. Second, Lemma 7.8.1 guarantees that, in general (except when  $S \in \mathcal{C}^{-R}$ ),  $S \setminus \{g\} \subseteq S^{II} \setminus \{g\} \subseteq S^{JJ} \setminus \{g\}$  as well as  $S^{JJ} \setminus S^{II} \subseteq \bar{\chi}(S)$  and  $S^{II} \setminus S \subseteq \bar{\chi}(S)$ . The formula for  $\beta(S)^{JJ}$  follows from the one for  $\alpha(S)^{JJ}$  and from the fact that  $g$  is an extremal point of every extent containing  $g$ .  $\square$

We are now in position to describe even better the effects of a widening operation seen through the perspective of a decomposition of the third kind. This is done in Figure 7.14.

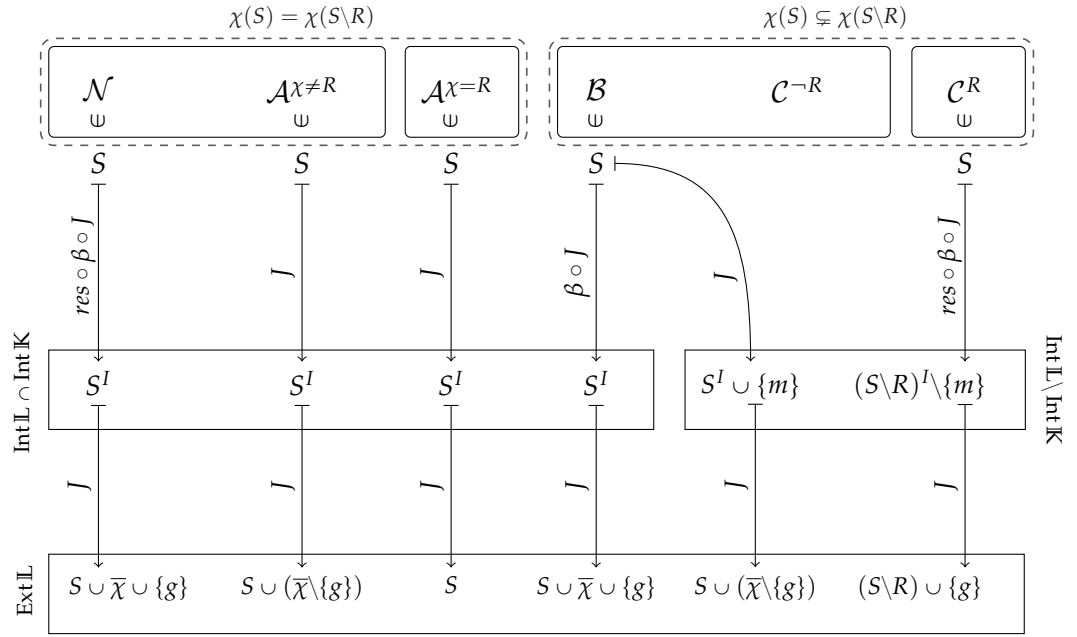


Figure 7.14.: Third kind decomposition with stability superclasses, corresponding derivations and closures in  $\mathbb{L} = \mathbb{K}^{g \leftrightarrow m}$ .

## Conclusion and future developments

This work contributes to the intersection between Formal Concept Analysis and extremal set theory. In particular, we summarized a broad spectrum of combinatorial results regarding the size of concept lattices and canonical bases. Constructions were provided which subsume and separate the known classes of contexts having polynomially many formal concepts. These classes arise from theorems (upper bounds) which explore formal concepts as maximal bicliques in bipartite graphs. Even though such bounds were known before this thesis, we added to the previously known results. This is evidenced - for instance - by the substantial improvement of the breadth majorant, described in Chapter 5.

With respect to breadth, we intimately linked such notion - already present in G.Birkhoff's classic monograph [14] - with at least other four important ones: contranominal subcontexts, minimal generators, shattered sets and boolean suborders.

The famous lemma of Sauer and Shelah in extremal set theory is another contact point between the two theories. The extremal lattices of Chapters 5 and 6 correspond to extremal closure systems with given ground set and bounded Vapnik-Chervonenkis dimension. The latter is an important notion in logic, set theory and computational learning theory [15, 40, 50].

We suspect that the number of meet-irreducibles attained by our explicit construction in Section 6.2 is the maximum, we however do not have and neither are aware of a proof yet. On the other direction, the approach of constructing extremal lattices through their implication logic seems to be the simplest way to obtain the minimum number of meet-irreducibles. It is plausible that this is achievable by selecting some appropriate ordering of subsets having at least  $k - 1$  elements, in a similar fashion as in Theorem 6.2.1.

At times, viewing a formal context as a bipartite graph gives a great deal of intuition to prove relevant theorems. For instance, this is the case for Eppstein's arboricity upper bound. In other situations, viewing complements of object-intents as a hypergraph provides the adequate intuitional aid - this happened with the results we established in Chapter 7. Had not been the visualization of a context as a hypergraph, we would not have come up with the very own definition of strong avoidance (i. e. the function  $\chi$ ), which played a quite

technical but central role in that chapter.

Originally, our approach using mixed generators had as goal the proof of the full conjecture (Conjecture 7.1.2). The intended tool was to prove the general existence of an object/attribute pair such that the widening *strictly* increases the size of the associated concept lattice. The existence of the formal context depicted in Figure 7.13 was, therefore, a negative discovery. However, we note that this does not provide a counterexample for the conjecture and, moreover, it could be that a further exploration of mixed generator classes (as in decompositions of the third kind) yields more insight and ultimately leads to the proof of the full conjecture.

## CHAPTER A

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## Affirmation

- i. Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor in any other country.
- ii. The present thesis has been written since April 2013 at the Institute of Algebra, Department of Mathematics, Faculty of Science, TU Dresden under the supervision of Prof. Bernhard Ganter.
- iii. There have been no prior attempts to obtain a PhD at any university.
- iv. I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued February 23, 2011 with the changes in effect since June 15, 2011.

## Versicherung

- i. Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
- ii. Die vorliegende Dissertation wurde seit April 2013 am Institut für Algebra, Fachrichtung Mathematik, Fakultät Mathematik und Naturwissenschaften, Technische Universität Dresden unter der Betreuung von Prof. Bernhard Ganter angefertigt.
- iii. Es wurden zuvor keine Promotionsvorhaben unternommen.
- iv. Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 23. Februar 2011, in der geänderten Fassung mit Gültigkeit vom 15. Juni 2011 an.

Dresden, July 15, 2017