Optimal Control Problems with Singularly Perturbed Differential Equations as Side Constraints: Analysis and Numerics

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Preface

A wide variety of physical processes exhibits the phenomenon that the physical quantities of interest change drastically in a small region. Such phenomena are called layers for the quantities of interest. For example, air flows form boundary layers along wings of planes (cf. [GEL04, SG03]). These layers are most important in the understanding of the capability of planes to fly. Other examples include the skin effect used for surface hardening of steel (cf. [FH99]) and the depletion layers of the p-n junctions of semiconductor devices (cf. [Sel84, PHS87]).

Providing and modifying of physical prototypes for tests is often very expensive. Sometimes it is even impossible to place measuring instruments properly. Therefore, one is interested in mathematical models for such processes. These models are singularly perturbed differential equations. Their solutions are difficult to obtain. Usually, it is not known or very difficult how to derive analytic solutions. On the other hand the application of standard techniques, such as *finite element method* (FEM) on a uniform mesh, to solve differential equations using a computer does not produce satisfying results. This is mainly due to the existence of the layers mentioned above. In the past many approaches were developed to overcome these difficulties. Some of them are presented in the comprehensive books [RST08, Lin10]. Nevertheless, this field is far from being understood conclusively.

In most cases, the models derived to describe such physical processes are very complex. Hence, it is an open problem how to analyze these models and solution algorithms mathematically. Therefore, one considers as a first step simpler differential equation problems. Those problems are obtained by some mathematical simplifications like linearization or usage of plausible assumptions. Additionally, one considers only problems with simple geometries or time independent solutions. Of course, this is motivated by the hope that the simpler problems still have similar properties compared to the original ones. A mathematical model problem frequently used in this context is the convection-diffusion problem given by

$$-\varepsilon\Delta u + b_1 u_x + cu = f \quad \text{in} \quad \Omega \subseteq \mathbb{R}^n, \qquad u\big|_{\partial\Omega} = g \tag{1}$$

for a very small parameter $0 < \varepsilon \ll 1$ and some functions b_1, c, f, g that are independent of ε. Since ε is very small it seems reasonable to assume some similarity in the behavior of the solution to the excluded case $\varepsilon = 0$, but especially in the layer regions this is not true. Hence, this problem is called singularly perturbed.

In the last years one observes a widely spread interest in the analysis of optimal control problems with singularly perturbed differential equations as side constraints (cf. [CH02, BV07, HYZ09, HL10, LH12]). These problems arise naturally (cf. [Tr10]) when one is interested in optimizing physical processes that lead to mathematical models which are singular perturbed. However, the results known for singularly perturbed differential equations are so far not adopted to the optimization problems in their full extend.

As indicated above, the basic analysis (application of the Céa-Lemma) of the FEM we consider here, using some uniform mesh leads for problem (1) to an error estimate of the numerical solution u^N compared with the exact solution u of the form

$$\|u - u^N\|_\varepsilon \le C\left(\frac{1}{N} + \frac{1}{N^2\sqrt{\varepsilon}}\right)\|u\|_{2,2}$$

where N is a number indicating the amount of work and memory invested into the numeric solution, $\|\cdot\|_\varepsilon$ is a measurement of the error, arising naturally from the mathematical formulation of problem (1), and $\|\cdot\|_{2,2}$ is some measure for the quality of a function. Obviously, this estimate becomes meaningless for very small values of ε and in fact one observes that the numerical error of the standard FEM increases as ε gets smaller. These drawbacks also apply to the usual way to analyze optimal control problems (cf. [Tr10]) with singularly perturbed differential equations as side constraints.

Using appropriate techniques one can construct algorithms that admit improved error bounds, cf. [CH02, BV07, HYZ09, HL10]. The presented bounds are of the form

$$\|u - u^N\|_\varepsilon + \ldots \le \frac{C}{N}\|u\|_{2,2}.$$

for various algorithms. This seems to be the solution to the problems described above. On a first glance these bounds look very strict even for small values of ε. But in general the value of $\|u\|_{2,2}$ is proportional to $\varepsilon^{-3/2}$. Hence, it tends to infinity for small values of ε and the estimates become meaningless again. Nevertheless this is a big improvement, especially since similar algorithms for singularly perturbed differential equations without optimization admit localization of this estimates to subdomains. Thus, it is reasonable to expect that also in the case of optimal control problems one may be able to prove localized estimates. This would ensure that the algorithms produce good results away from layer regions. But as was already observed in [HL10] the method analyzed therein does not produce as good results as one would expect away from layers. This indicates that not all proposed algorithms admit convenient localized estimates for the optimal control problem, although such estimates are proved for related algorithms for single differential equation problems.

In this thesis we apply additional techniques known for singularly perturbed differential equations to optimal control problems. We strive to develop an algorithm with proved bounds of the form

$$\|u - u^N\|_\varepsilon \le C\frac{\ln(N)}{N},$$

where the right hand side constant is independent of ε. Still such an algorithm is not the salvation from all problems since even then the computed solution may show some peculiar oscillations that are not present in the exact solution. However, the results presented here allow a better understanding of the structure of the solution, because we derive detailed estimates for the solution and its derivatives. Future research may use this achievements to develop even better algorithms.

During the work that lead to this thesis it has been observed, that almost all results in the field of singularly perturbed differential equations so far use data b_1, c, f and g in problem (1) that are smooth. In optimal control we have to work with a nonlinear term that is not very smooth. Thus, it became necessary to establish results similar to already known estimates only using minimal smoothness prerequisites. Such results are also present here.

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Contents

Notation

Throughout this text C (sometimes superscripted) denotes a generic constant that is independent of the perturbation parameter ε and the mesh size N. Its value may vary in different equations, but to clarify the used mathematics we add or modify superscripts in one equation to signal the change of the constant.

We write $|\cdot|$ without any subscripts for the absolute value.

For real vectors we denote the euclidean norm by $\|\cdot\|$ and the inner product by $\langle\cdot,\cdot\rangle$.

By $\mathcal{L}^p(D)$ we denote the Lebesgue space of measurable functions on D with a bounded p-norm

$$\|f\|_{p,D} := \left(\int_D f^p \,\mathrm{d}\lambda\right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1,\infty) \quad \text{and} \quad \|f\|_{\infty,D} := \operatorname*{ess\,sup}_D |f|.$$

The $\mathcal{L}^2(D)$ inner product will be denoted by $\langle\cdot,\cdot\rangle_D$.

For a subset $D \subseteq \mathbb{R}^n$ we denote the boundary by ∂D. Ihe Lebesgue measure of D will be referred to by $\lambda(D)$.

By $W^{k,p}(D)$ we denote the Sobolev spaces of measurable functions on D with all weak derivatives of order up to k in $\mathcal{L}^p(D)$. The corresponding norms are denoted by $\|\cdot\|_{k,p,D}$. The half norms of $W^{k,p}(D)$ considering only the k-th order derivatives are denoted as $|\cdot|_{k,p,D}$. Also, we will use an ε-weighted $W^{1,2}$-norm $\|\cdot\|_{\varepsilon,D} := \|\cdot\|_{2,D} + \sqrt{\varepsilon}|\cdot|_{1,2,D}$, referred to as ε-norm. When D coincides with the domain of the arguments we omit it as a subscript to the inner product or the norms.

The completion of the set of smooth continuous functions with compact support $\mathcal{C}_0^\infty(D)$ in $W^{1,2}(D)$ (i.e. the functions in $W^{1,2}(D)$ that vanish on ∂D) is denoted by $H_0^1(D) \subset W^{1,2}(D)$.

At some time we need the spaces $\mathcal{C}^k$, $\mathcal{C}^{k,\alpha}$ of functions with continuou, Hölder-continuous derivatives of order up to k, respectively.

Later on we will split functions into a smooth part and layer parts located at $x = 0$, $x = 1$, the sum of the layers at $y \in \{0,1\}$, the sum of the corner layers located at $(x,y) \in 0 \times \{0,1\}$, $(x,y) \in 1 \times \{0,1\}$ and a nonsmooth but small part. For such a function f we will denote the parts by

$$f^S,\ f^{x0},\ f^{x1},\ f^y,\ f^{c0},\ f^{c1},\ f^n, \quad \text{respectively.}$$

If we need several components contributing to such a part we will append an index $k \in I$ for some index set $I \subseteq \mathbb{N}$ in the form

$$f^{\cdots} = \sum_{k\in I} f^{\cdots,k}.$$

For the boundary and corner layer terms we will derive pointwise bounds, where the functions

$$\mathcal{E}_0^x(x) := e^{-\frac{\beta x}{\varepsilon}}, \quad \mathcal{E}_1^x(x) := e^{-\frac{\beta(1-x)}{\varepsilon}}, \quad \mathcal{E}^y(y) := e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}$$

play a key role.

Furthermore, we use continuation operators $\mathfrak{C}_D^\Delta$ from D to $\Delta \supseteq D$ with

$$\left\|\mathfrak{C}_D^\Delta(f)\right\|_{k,p,\Delta} \leq C \left\|f\right\|_{k,p,D},$$

for any $k \in \mathbb{N}$, $p \in [1,\infty]$ (cf. [Ro04]).

Also, we denote by $B_\tau(x,y) := \left\{(\xi,\eta) \in \mathbb{R}^2 \;\middle|\; (x-\xi)^2 + (y-\eta)^2 < \tau^2\right\}$ the ball with radius τ and center (x,y).

We use the Landau big $\mathcal{O}$ notation with respect to the limit to 0, i.e.

$$\mathcal{O}\big(f(x)\big) := \left\{g : \mathbb{R} \to \mathbb{R} \middle| \exists x_0 > 0, C > 0 \forall x \in (0,x_0) : |g(x)| \leq Cf(x)\right\}$$

the class of all functions that are asymptotically not larger than f.

To distinguish it from the regularly used index i we denote the imaginary unit by $\hat{\imath}$.

Since we work with functions in Sobolev spaces all pointwise estimates are to be considered λ-almost-everywhere, with λ the Lebesgue measure. But for shortening of notation we omit this in the following text.

Introduction

In this thesis we are mainly interested in the problem

$$\min_{u,q\in\mathcal{L}^2}\left(\frac{1}{2}\|u-u_d\|_2^2+\frac{\mu}{2}\|q\|_2^2\right) \tag{0.1a}$$

subject to the singularly perturbed convection-diffusion equation

$$Lu := -\varepsilon\Delta u + bu_x + cu = f + q \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0 \tag{0.1b}$$

and the box constraints

$$q \in Q_{\mathrm{ad}} := \left\{w \in \mathcal{L}^2 \middle| q_a \le w \le q_b\right\} \tag{0.1c}$$

for the control, where the lower bound q_a or the upper bound q_b might be absent. As domain we consider $\Omega = (0,1)^n \subset \mathbb{R}^n$ for the one-dimensional ($n = 1$) and two-dimensional ($n = 2$) case.

As stated in the Preface the standard Finite Element Method and optimization methods fail in general to compute good solutions for small values of ε. The main problem is that the solution has boundary layers and for this reason the global L^2-norms of the second order derivatives have very large values. Although problem (0.1) was discussed in the literature (cf. [CH02, BV07, HYZ09, HL10, LH12]) there is not much known about the layer structure of its solution. With exception of [LH12] the proofs published so far are not very restrictive in the case of boundary layers since they contain norms on the right hand side of the error estimates that scale like $\varepsilon^{-3/2}$. The proof presented in [LH12] uses a solution to the reduced system attained by setting $\varepsilon = 0$ in the differential equation and neglecting some boundary conditions. For the optimization problem without control constraints $-q_a = q_b = \infty$ this works well. But this approach can not be generalized to the case with active control constraints since the solution of the reduced problem may not be in $W^{2,2}(\Omega)$ but the corresponding norm would be required. Therefore, we strive to prove detailed bounds for the solution of the optimization problem (0.1) as a first step.

In optimal control theory it is well known that the solution (u, q) of problem (0.1) together with the adjoint state v satisfy the optimality system

$$Lu = -\varepsilon\Delta u + bu_x + \qquad cu = f + q \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \tag{0.2a}$$

$$L^*v = -\varepsilon\Delta v - bv_x + (c - b')v = u - u_d \text{ in } \Omega, \quad v\big|_{\partial\Omega} = 0, \tag{0.2b}$$

$$\langle v + \mu q,\, w - q\rangle = 0 \text{ for all } w \in Q_{\mathrm{ad}}. \tag{0.2c}$$

For the case $Q_{\mathrm{ad}} = \mathcal{L}^2$ this system simplifies to a linear system of two coupled differential equations. This system has the interesting property that the sign in front of the convection coefficient b is different in the second equation compared to the first one. There is some analysis done for systems of singularly perturbed differential equations in the literature, e.g. [Cen05, Lin07]. The proofs presented there only apply to the case, where both convection terms have the same sign. Linß claims in [Lin07] to show bounds for the case with different signs, but applies the results of [KT78] incorrectly for the derivatives of order k with $k \geq 2$. Thus, it is necessary to establish rigor proofs for the properties of the solution of (0.2).

We start by considering a linear system of singularly perturbed differential equations in 1D that is a slight generalization to (0.2) for $Q_{\text{ad}} = \mathcal{L}^2$. We construct an asymptotic expansion for the solution and demonstrate sharp bounds for its derivatives up to the second order. This technique can easily be extended to prove estimates for arbitrary derivatives, given the data are smooth enough. Subsequently, we use the attained information on the solution to prove convergence results for standard FEM on some layer adapted Shishkin meshes.

In Chapter 2 we attend to the optimization problem (0.1) for the one-dimensional case. It is nontrivial to adopt the method of asymptotic expansion to this case since the optimality system is not linear for $Q_{\text{ad}} \neq \mathcal{L}^2$. Hence, we adopt the techniques presented in [KT78] to derive bounds for the solution similar to the results from Chapter 1. By using the optimality properties of the solution we are able to derive these bounds with less requirements than we used for the asymptotic expansion. Due to the nonlinearity of the optimality system these proofs can not be iterated to acquire bounds for arbitrary derivatives.

We use the attained bounds to provide convergence results for various discretization schemes motivated in Chapter 1 and the discussion of semi-discretization (cf. [HYZ09]) and full-discretization (cf. [BV07]). The proofs presented in this section carry over to the multidimensional case provided we are able to prove sufficient convergence results for the primal and adjoint equation neglecting the optimization.

In Chapter 3 we present estimates for the solution u and its derivatives of the problem

$$Lu = -\varepsilon\Delta u + \beta u_x + cu = f \text{ in } \Omega := (0,1)^2, \quad u\big|_{\partial\Omega} = 0 \tag{0.3}$$

under the mild regularity assumption $f \in W^{1,\infty}$. We do so to adopt the convergence results of Chapter 2 to the two-dimensional case. Surprisingly there is not much known for such a non-smooth right hand side f. Most of the known techniques can not be generalize to this case, because the convenient tool of differentiation is used excessively, and one requires something like $f \in \mathcal{C}^{2,\alpha}$ for some $\alpha \in (0,1)$ and compatibility conditions (cf. [KS05, LS01, OS08, Ro02]). This seems to be unnecessary since we have $u \in W^{2,2}$ without any of these additional requirements. To conceive sharp bounds on u that can be used to prove ε-uniform convergence we rely heavily on the Green's function of problem (0.3). We adopt the estimates presented in [FK12] to serve in our case. Especially to get bounds for the layer corrections this is very technical. Unfortunately, so far there is one term we can not prove a sufficiently sharp bound for, although numeric calculations indicate this term behaves nicely – even better than required for the convergence proofs.

One may ask why we assume $f \in W^{1,\infty}$ even though we have $u \in W^{2,2}$ for $f \in \mathcal{L}^2$ (cf. [LU68]). To answer this we consider the simple example $u = x(1-x)\sin(\varepsilon^{-1/2}\pi y)$ for $\varepsilon^{-1/2} \in \mathbb{N}$. It is easy to check we have $\|Lu\|_2 \leq C$. However in the preponderant part of the domain we have $u_{yy} \sim \varepsilon^{-1}$. Thus, we can not expect that there is a piecewise bilinear function on a relatively coarse mesh that approximates u_y with the order of $\varepsilon^{-1/2}$. But if there is no such function we will not be able to construct an algorithm that computes a good solution approximation with respect to the ε-norm. This shows that the assumption $\|f\|_2 \leq C$ is too weak to expect appropriate convergence results.

Finally, we state in Chapter 4 some facts for the optimization problem (0.1) in 2D. The analysis for this problem is not complete. Especially the characteristic layers induce problems that are not understood completely so far. In literature there is only the article [Sh00] that tackles a similar problem. But the bounds derived there in are very weak. Further research is required to get useful results concerning this problem.

1. A System of Weakly Coupled Singularly Perturbed Convection-Diffusion Equations in 1D

This chapter is primarily a replication of [RR11], the initial work that led to the results presented in the following chapters.

We consider the system

$$-\varepsilon u_1'' + b_1 u_1' + c_{11} u_1 + c_{12} u_2 = f_1, \qquad u_1(0) = u_1(1) = 0, \tag{1.1a}$$
$$-\varepsilon u_2'' - b_2 u_2' - c_{21} u_1 + c_{22} u_2 = f_2, \qquad u_2(0) = u_2(1) = 0, \tag{1.1b}$$

under the assumptions

$$0 < \varepsilon \ll \beta, \tag{1.2a}$$
$$b_1,\, b_2 \geq \beta > 0, \tag{1.2b}$$
$$c_{11},\, c_{22} \geq 0, \tag{1.2c}$$
$$c_{12}c_{21} > 0, \quad |c_{12}|,\, |c_{21}| \geq \gamma > 0. \tag{1.2d}$$

We assume that the functions b_i, c_{ij}, f_i, $(i, j \in \{1, 2\})$ are sufficiently smooth.

For simplification in writing we assume furthermore

$$c_{12},\ c_{21} \geq \gamma > 0$$

but the results can easily be generalized to include the case $c_{12},\ c_{21} \leq -\gamma < 0$.

The system (1.1) has the interesting property that the convection in the second differential equation opposes the convection in the first equation. These special assumptions about the sign properties of b_i and c_{ij} are due to the application of the mathematics derived in this chapter. In Chapter 2 we will see that the optimality systems of some optimal control problems have the form we consider here – hence, the special sign structure of system (1.1).

In Section 1.1 we construct a solution decomposition and derive various estimates for its terms. Some technical details of the construction are presented in Section 1.2. Subsequently, we use the discovered solution properties to analyze a special FEM for solving convection-diffusion problems of the form (1.1). Finally we present some computational results for our method to confirm our theoretical results.

1.1. Properties of the Exact Solution

First we formulate sufficient conditions for the existence of a weak solution of the system (1.1).

Theorem 1.1
If the assumptions

$$2c_{11}c_{21} - (b_1c_{21} + \varepsilon c_{21}')' \geq 0 \tag{1.3a}$$
$$2c_{22}c_{12} + (b_2c_{12} - \varepsilon c_{12}')' \geq 0 \tag{1.3b}$$

hold, the system (1.1) *has a unique weak solution* $u \in V := \left(H_0^1(0,1)\right)^2$.

Proof
First we multiply the first and second equation of the system (1.1) by c_{21} and c_{12}, respectively. This leads to an equivalent system with corresponding bilinear form

$$\begin{aligned}\tilde{a}(u,v) := &\int_0^1 \varepsilon c_{21}u_1'v_1' + (b_1c_{21} + \varepsilon c_{21}')u_1'v_1 + (c_{11}c_{21}u_1 + c_{12}c_{21}u_2)v_1 \,\mathrm{d}\lambda \\ &+ \int_0^1 \varepsilon c_{12}u_2'v_2' - (b_2c_{12} - \varepsilon c_{12}')u_2'v_2 + (c_{22}c_{12}u_2 - c_{21}c_{12}u_1)v_2 \,\mathrm{d}\lambda. \end{aligned}\tag{1.4}$$

We use the well known relation

$$\int_0^1 \tilde{b}u_iu_i' \,\mathrm{d}\lambda = \tilde{b}u_i^2\big|_{x=0}^1 - \int_0^1 \tilde{b}'u_i^2 + \tilde{b}u_iu_i' \,\mathrm{d}\lambda \quad \Leftrightarrow \quad 2\int_0^1 \tilde{b}u_iu_i' \,\mathrm{d}\lambda = -\int_0^1 \tilde{b}'u_i^2 \,\mathrm{d}\lambda$$

for $u_i \in H_0^2(0,1)$ and the fact, that the terms $c_{12}c_{21}u_2v_1$ and $-c_{21}c_{12}u_1v_2$ cancel each other for $u_i = v_i$, to acquire

$$\begin{aligned}\tilde{a}(u,u) := &\int_0^1 \varepsilon c_{21}\left(u_1'\right)^2 + \left(-\frac{(b_1c_{21} + \varepsilon c_{21}')'}{2} + c_{11}c_{21}\right)u_1^2 \,\mathrm{d}\lambda \\ &+ \int_0^1 \varepsilon c_{12}\left(u_2'\right)^2 + \left(\frac{(b_2c_{12} - \varepsilon c_{12}')'}{2} + c_{22}c_{12}\right)u_2^2 \,\mathrm{d}\lambda.\end{aligned}$$

The V-ellipticity of this bilinear form is obviously ensured by condition (1.3). Thus, the Lax-Milgram lemma establishes the unique solvability. □

Easily we get from the V-ellipticity of the bilinear form $\tilde{a}(\cdot,\cdot)$ the a priori estimate

$$\varepsilon\left(|u_1|_1^2 + |u_2|_1^2\right) \leq C\left(\|f_1\|_0^2 + \|f_2\|_0^2\right). \tag{1.5}$$

As a next step we want to prove precise bounds for the derivatives of u_1 and u_2. While in [Lin07] the inverse monotony of the matrix

$$\Gamma := \begin{pmatrix} 1 & -\left\|\frac{c_{12}}{c_{11}}\right\|_\infty \\ -\left\|\frac{c_{21}}{c_{22}}\right\|_\infty & 1 \end{pmatrix} \tag{1.6}$$

to split the system into two single equations with known bounds for the right hand side is used to prove these bounds, we use an asymptotic expansion. To do so we need an existence theorem for the solution of the reduced system.

Let us consider the reduced problem

$$b_1v_1' + c_{11}v_1 + c_{12}v_2 = \tilde{f}_1, \qquad u_{1,l}(0) = \nu_1, \tag{1.7a}$$
$$-b_2v_2' + c_{22}v_2 - c_{21}v_1 = \tilde{f}_2, \qquad u_{2,l}(1) = \nu_2. \tag{1.7b}$$

Theorem 1.2
Assume

$$c_{11}c_{22} + c_{12}c_{21} - b_2 c_{12}\left(\frac{c_{11}}{c_{12}}\right)' \geq 0 \quad or \tag{1.8a}$$

$$c_{11}c_{22} + c_{12}c_{21} + b_1 c_{21}\left(\frac{c_{22}}{c_{21}}\right)' \geq 0. \tag{1.8b}$$

Then the reduced problem (1.7) has a unique solution u. Its smoothness depends on the smoothness of $\tilde{f}_1$ and $\tilde{f}_2$.

Proof
The system (1.7) can be transformed to a second order boundary value problem in two ways. This leads to

$$\begin{aligned} b_1 b_2 v_1'' + &\left(b_2 c_{12}\left(\frac{b_1}{c_{12}}\right)' - b_1 c_{22} + b_2 c_{11}\right) v_1' - \left(c_{11}c_{22} + c_{12}c_{21} - b_2 c_{12}\left(\frac{c_{11}}{c_{12}}\right)'\right) v_1 \\ &= c_{12}\tilde{f}_2 - c_{22}\tilde{f}_1 + b_2 c_{12}\left(\frac{\tilde{f}_1}{c_{12}}\right)' \quad \text{and} \\ -b_1 b_2 v_2'' - &\left(b_1 c_{21}\left(\frac{b_2}{c_{21}}\right)' - b_1 c_{22} + b_2 c_{11}\right) v_2' + \left(c_{11}c_{22} + c_{12}c_{21} + b_1 c_{21}\left(\frac{c_{22}}{c_{21}}\right)'\right) v_2 \\ &= c_{21}\tilde{f}_1 + c_{11}\tilde{f}_2 + b_1 c_{21}\left(\frac{\tilde{f}_2}{c_{21}}\right)'. \end{aligned}$$

Conditions (1.8) ensure the applicability of maximum principle to one of the resulting boundary value problems. Thus, the method of continuity (cf. [GT01]) gives existence of a solution. □

For sufficiently smooth coefficients this lemma implies the relation

$$\|v_1\|_{k+2,2} + \|v_2\|_{k+2,2} \leq C\left(\|\tilde{f}_1\|_{k+1,2} + \|\tilde{f}_2\|_{k+1,2}\right). \tag{1.9}$$

Remark 1.3
In case of constant coefficients c_{11}, c_{12}, c_{21} and c_{22} the prerequisites (1.2) imply the requirement (1.8). If additionally b_1 and b_2 are constant the prerequisites also imply condition (1.3).

Remark 1.4
The system

$$-\varepsilon u_1'' + b_1 u_1' + c_{11}u_1 + c_{12}u_2 = f_1, \qquad u_1(0) = u_1(1) = 0, \tag{1.10a}$$

$$-\varepsilon u_2'' + b_2 u_2' + c_{22}u_2 + c_{21}u_1 = f_2, \qquad u_2(0) = u_2(1) = 0, \tag{1.10b}$$

with $b_1, b_2 > 0$ is significantly different from our system (1.1). First, the reduced problem leads to an initial value problem which always has a unique solution. Second, the transformation $u_i = e^{\vartheta x} v_i$, $i \in \{1, 2\}$ leads to a system where we can choose ϑ in such a way that Γ defined in (1.6) is inverse monotone for all sufficiently small values of ε. Therefore, the coefficients c_{11}, c_{12}, c_{21}, c_{22} have little influence on the behavior of the solution of system (1.10). These are significant differences to the system (1.1) we study here.

Next we construct an asymptotic expansion for u_1, u_2 and introduce the local variables $\xi := x/\varepsilon$, $\eta := (1-x)/\varepsilon$:

$$u_1 = \sum_{l=0}^{n} \varepsilon^l u_1^{S,l} + \sum_{l=1}^{n} \varepsilon^l u_1^{x0,l}(\xi) + \sum_{l=0}^{n} \varepsilon^l u_1^{x1,l}(\eta) + R_{1,n}, \tag{1.11a}$$

$$u_2 = \sum_{l=0}^{n} \varepsilon^l u_2^{S,l} + \sum_{l=1}^{n} \varepsilon^l u_2^{x0,l}(\xi) + \sum_{l=0}^{n} \varepsilon^l u_2^{x1,l}(\eta) + R_{2,n}. \tag{1.11b}$$

For details see Section 1.2.

Combining the results for the asymptotic expansion we get:

Theorem 1.5

If the data are sufficiently smooth and the assumptions (1.3) *and* (1.8) *hold, the solution of system* (1.1) *can be decomposed as*

$$u_1 = u_1^S + u_1^{x0} + u_1^{x1}, \quad u_2 = u_2^S + u_2^{x0} + u_2^{x1} \tag{1.12a}$$

with

$$\|u_1^S\|_{2,2} + \|u_2^S\|_{2,2} \le C, \tag{1.12b}$$

$$|u_1^{x0^{(k)}}(x)| \le C\varepsilon^{1-k}\mathcal{E}_0^x(x), \tag{1.12c}$$

$$|u_1^{x1^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_1^x(x), \tag{1.12d}$$

$$|u_2^{x0^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_0^x(x), \tag{1.12e}$$

$$|u_2^{x1^{(k)}}(x)| \le C\varepsilon^{1-k}\mathcal{E}_1^x(x) \tag{1.12f}$$

for $k \le 2$. Recall that the constant C is independent of ε.

Proof

To show the result, we consider the asymptotic expansion (1.11) for $n = 1$. Thus, estimate (1.20) gives

$$\|R_{i,1}\|_{2,2} \le C \text{ for } i \in \{1,2\}.$$

Furthermore, we can use estimate (1.9) to get

$$\|u_i^{S,0}\|_{2,2} + \|u_i^{S,1}\|_{2,2} \le C.$$

Combining this results we get for

$$u_i^S := \sum_{l=0}^{1} \varepsilon^l u_i^{S,l} + R_{i,1}$$

the estimate

$$\|u_i^S\|_{2,2} \le \|u_i^{S,0}\|_{2,2} + \varepsilon\|u_i^{S,1}\|_{2,2} + \|R_{i,1}\|_{2,2} \le C.$$

We define the layer correction terms via

$$u_1^{x0} := \sum_{l=1}^{n} \varepsilon^l u_1^{x0,l}, \quad u_1^{x1} := \sum_{l=0}^{n} \varepsilon^l u_1^{x1,l}, \quad u_2^{x0} := \sum_{l=0}^{n} \varepsilon^l u_2^{x0,l} \quad \text{and} \quad u_2^{x1} := \sum_{l=1}^{n} \varepsilon^l u_2^{x1,l}.$$

From Theorem 1.7 we know that the boundary term u_1^{x1} of the asymptotic expansion (1.11) has the form $\sum_{l=0}^{1} \varepsilon^l \mathbb{P}_l(x/\varepsilon)\exp(-\hat{b}x/\varepsilon)$. Differentiation proves the estimates. Analogously, one can prove the bounds for the other layer terms u_1^{x0}, u_2^{x0} and u_2^{x1}. □

This result also yields estimates for $\|u\|_\infty$ and $\|u'\|_\infty$. To receive them one has to apply Sobolev imbedding theorems to the smooth part. For the other components we have pointwise estimates already.

1.2. Asymptotic Expansion

In the following we construct an asymptotic expansion

$$u_1 = \sum_{l=0}^{n} \varepsilon^l u_1^{S,l} + \sum_{l=1}^{n} \varepsilon^l u_1^{x0,l}(\xi) + \sum_{l=0}^{n} \varepsilon^l u_1^{x1,l}(\eta) + R_{1,n}, \tag{1.13a}$$

$$u_2 = \sum_{l=0}^{n} \varepsilon^l u_2^{S,l} + \sum_{l=1}^{n} \varepsilon^l u_2^{x0,l}(\xi) + \sum_{l=0}^{n} \varepsilon^l u_2^{x1,l}(\eta) + R_{2,n}. \tag{1.13b}$$

of the solution of system (1.1) using the local variables $\xi := x/\varepsilon$, $\eta := (1-x)/\varepsilon$. The construction can mainly be done the same way it is done for a single differential equation (cf. [RST08, Section 1.1.1]), but the coupling of the two solutions u_1 and u_2 requires the consideration of a boundary layer on either side of the domain. We desire that the differential equations of system (1.1) are fulfilled for $\sum_{l=0}^{n} \varepsilon^l u_1^{S,l}$ and $\sum_{l=0}^{n} \varepsilon^l u_2^{S,l}$. Here we skip the boundary condition at the right and left side in the first and second line of the system, respectively. Furthermore, we demand that the corresponding homogeneous differential equations of system (1.1) are satisfied for the boundary terms. Transformation of the resulting system to the local variables ξ and η leads to the following equations.

For the first component of the smooth parts we attain the reduced system

$$b_1 u_1^{S,0\prime} + c_{11} u_1^{S,0} + c_{12} u_2^{S,0} = f_1, \qquad u_1^{S,0}(0) = 0, \tag{1.14a}$$

$$-b_2 u_2^{S,0\prime} + c_{22} u_2^{S,0} - c_{21} u_1^{S,0} = f_2, \qquad u_2^{S,0}(1) = 0. \tag{1.14b}$$

For the subsequent components ($l \geq 1$) of the smooth parts we have to correct the error introduced by $u^{S,l-1}$ and boundary errors from layer components via

$$b_1 u_1^{S,l\prime} + c_{11} u_1^{S,l} + c_{12} u_2^{S,l} = u_1^{S,l-1\prime\prime}, \qquad u_1^{S,l}(0) = -u_1^{x0,l}(0), \tag{1.15a}$$

$$-b_2 u_2^{S,l\prime} + c_{22} u_2^{S,l} - c_{21} u_1^{S,l} = u_2^{S,l-1\prime\prime}, \qquad u_2^{S,l}(1) = -u_2^{x1,l}(1). \tag{1.15b}$$

The correction at $x = 0$ has to compensate the neglected boundary condition of $u_2^{S,l}$, thus we have

$$-u_1^{x0,l\prime\prime} + \tilde{b}_{1,0} u_1^{x0,l\prime} = -\sum_{j=1}^{l} \left(\tilde{b}_{1,j} u_1^{x0,l-j\prime} + \tilde{c}_{11,j-1} u_1^{x0,l-j} + \tilde{c}_{12,j-1} u_2^{x0,l-j} \right), \tag{1.16a}$$

$$\lim_{\xi\to\infty} u_1^{x0,l}(\xi) = 0, \tag{1.16b}$$

$$-u_2^{x0,l\prime\prime} - \tilde{b}_{1,0} u_2^{x0,l\prime} = -\sum_{j=1}^{l} \left(-\tilde{b}_{2,j} u_2^{x0,l-j\prime} + \tilde{c}_{22,j-1} u_2^{x0,l-j} - \tilde{c}_{21,j-1} u_1^{x0,l-j} \right), \tag{1.16c}$$

$$\lim_{\xi\to\infty} u_2^{x0,l}(\xi) = 0, \quad u_2^{x0,l}(0) = -u_2^{S,l}(0) \tag{1.16d}$$

for $l \geq 0$. Analogously, the correction at $x = 1$ has to compensate the neglected boundary condition of $u_1^{S,l}$, and we have

$$-u_1^{x1,l''} - \hat{b}_{1,0} u_1^{x1,l'} = -\sum_{j=1}^{l} \left(-\hat{b}_{1,j} u_1^{x1,l-j'} + \hat{c}_{11,j-1} u_1^{x1,l-j} + \hat{c}_{12,j-1} u_2^{x1,l-j} \right), \tag{1.17a}$$

$$\lim_{\eta \to \infty} u_1^{x1,l}(\eta) = 0, \quad u_1^{x1,l}(0) = -u_1^{S,l}(1), \tag{1.17b}$$

$$-u_2^{x1,l''} + \hat{b}_{2,0} u_2^{x1,l'} = -\sum_{j=1}^{l} \left(\hat{b}_{2,j} u_2^{x1,l-j'} + \hat{c}_{22,j-1} u_2^{x1,l-j} - \hat{c}_{21,j-1} u_1^{x1,l-j} \right), \tag{1.17c}$$

$$\lim_{\eta \to \infty} u_2^{x1,l}(\eta) = 0 \tag{1.17d}$$

for $l \geq 0$. Here $\tilde{z}_i$ and $\hat{z}_i$ denotes the i-th coefficient of the Taylor expansion of $z(\varepsilon\xi)$ and $z(1-\varepsilon\eta)$ at $\xi = 0$ and $\eta = 0$, respectively. Note the difference $u_1^{S,l}(0) = -u_1^{x0,l}(0)$, $u_2^{S,l}(1) = -u_2^{x1,l}(1)$ from the standard expansion. This modification is necessary because the limitary condition for $\xi, \eta \to \infty$ determines $u_1^{x0,l}(0)$ and $u_2^{x1,l}(1)$ completely. To prove this we need the following theorems.

Theorem 1.6

The first terms of the boundary layer correction have the form

$$u_1^{x0,0}(\xi) = 0, \qquad u_1^{x1,0}(\eta) = -u_1^{S,0}(1)\, e^{-b_1(1)\eta}, \tag{1.18a}$$

$$u_2^{x0,0}(\xi) = -u_2^{S,0}(0)\, e^{-b_2(0)\xi}, \qquad u_2^{x1,0}(\eta) = 0. \tag{1.18b}$$

Therefore, the solutions u_1 and u_2 have a strong boundary layer only at the right and left boundary, respectively.

Proof

By solving the explicitly given boundary value problems (1.16) and (1.17). □

Theorem 1.7

The terms of boundary layer correction have the form

$$u_1^{x0,l}(\xi) \in \mathbb{P}_{l-1}(\xi)\, e^{-b_2(0)\xi}, \qquad u_1^{x1,l}(\eta) \in \mathbb{P}_{l}(\eta)\, e^{-b_1(1)\eta}, \tag{1.19a}$$

$$u_2^{x0,l}(\xi) \in \mathbb{P}_{l}(\xi)\, e^{-b_2(0)\xi}, \qquad u_2^{x1,l}(\eta) \in \mathbb{P}_{l-1}(\eta)\, e^{-b_1(1)\eta}, \tag{1.19b}$$

where $\mathbb{P}_n(x)$ denotes the set of polynomials in the unknown x of degree at most n.

Proof

By inductive solution of the ordinary boundary value problems for $u_1^{x0,l}$, $u_1^{x1,l}$, $u_2^{x0,l}$ and $u_2^{x1,l}$. □

Combining the previous results we get

$$R_{i,n}(0) \in \mathcal{O}(\varepsilon^{n+1}), \quad R_{i,n}(1) \in \mathcal{O}\left(\varepsilon^{n+1}\right),$$
$$\|L(R_{1,n}, R_{2,n})\|_2 \leq C\, \|\varepsilon^{n+1} + \varepsilon^n e^{-\beta\xi} + \varepsilon^n e^{-\beta\eta}\|_2 \in \mathcal{O}\left(\varepsilon^{n+\frac{1}{2}}\right).$$

Thus, we get by our a priori estimate (1.5) the information

$$\|R_{i,n}\|_{1,2} \in \mathcal{O}(\varepsilon^n)$$

for $i \in \{1, 2\}$. For the $W^{2,2}$-norm we get

$$\|R_{i,n}\|_{2,2} \leq \varepsilon^{-1} C(\|L(R_{1,n}, R_{2,n})\|_2 + \|R_{i,1}\|_{1,2}) \in \mathcal{O}\left(\varepsilon^{n-1}\right). \tag{1.20}$$

1.3. Error Estimates for Linear FEM on Shishkin Meshes

Next we want to use the previous estimates to prove an a priori estimate for the error of a finite element method. We are using meshes that are adapted to the layer properties of the solution.

In this section we discretize the system (1.1)

$$\begin{aligned} -\varepsilon u_1'' + b_1 u_1' + c_{11} u_1 + c_{12} u_2 &= f_1, & u_1(0) = u_1(1) = 0,\\ -\varepsilon u_2'' - b_2 u_2' - c_{21} u_1 + c_{22} u_2 &= f_2, & u_2(0) = u_2(1) = 0, \end{aligned}$$

with linear finite elements. We start from the weak formulation

$$\begin{aligned} \tilde{a}(u,v) &= \langle f,\, v\rangle \text{ for all } v \in V = \left(H_0^1(0,1)\right)^2 \quad \text{with}\\ \tilde{a}(u,v) &:= \int_0^1 \varepsilon b_{21} u_1' v_1' + (a_1 b_{21} + \varepsilon b_{21}') u_1' v_1 + (b_{11} b_{21} u_1 + b_{12} b_{21} u_2) v_1 \,\mathrm{d}\lambda\\ &\quad + \int_0^1 \varepsilon b_{12} u_2' v_2' - (a_2 b_{12} - \varepsilon b_{12}') u_2' v_2 + (b_{22} b_{12} u_2 - b_{21} b_{12} u_1) v_2 \,\mathrm{d}\lambda. \end{aligned}$$

Recall we denote the $\mathcal{L}^2$ inner product by $\langle f,\, v\rangle = \int_0^1 f_1 v_1 + f_2 v_2 \,\mathrm{d}\lambda$. Denoting our finite element space by $V^N \subset V$, the finite element method reads:

$$\text{Find } u^N \in V^N \text{ such that } \tilde{a}(u^N, v) = \langle f,\, v\rangle \text{ for all } v \in V^N.$$

Based on the information from Theorem 1.5 concerning the layer structure we use a Shishkin mesh for the discretization. Because u_1 has a strong layer at $x = 1$ and u_2 at $x = 0$, we use different meshes for the two solution components. We neglect the weak layers for the construction of the mesh. A Shishkin mesh is a piecewise equidistant mesh. To cope with a boundary layer at $x = 0$ one chooses the transition point $\sigma_0 := \min\{1/2,\, 2\varepsilon \ln(N)/\beta\}$ and uses for the two subdomains $\Omega_f := [0, \sigma_0]$, $\Omega_c := [\sigma_0, 1]$ an equidistant mesh with $N/2$ nodes each. Analogously, one chooses the transition point $\sigma_1 := \max\{1/2,\, 1 - 2\varepsilon \ln(N)/\beta\}$ to take account for a boundary layer at $x = 1$. This leads to meshes of a form shown in Figure 1.1.

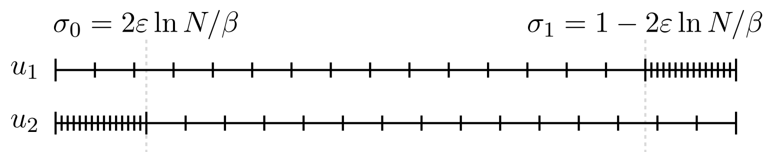

Figure 1.1.: Used Shishkin meshes

Now we will prove a priori estimates for our Galerkin method in the ε-norm. This norm is inherent to the problem (1.1), because it is equivalent to the energy norm induced by $\|u\|_{\tilde{a}} := \tilde{a}(u,u)$ (cf. (1.5)). Denoting the nodal linear interpolant of u by u^I, we first bound the interpolation error $\|u - u^I\|_\varepsilon$ by using the inequalities of formula (1.12).

For some terms we use on the fine part Ω_f of the mesh a different estimation than on the coarse part Ω_c. We denote the locally constant mesh size by h_{Ω_f} and $h = h_{\Omega_c}$.

Theorem 1.8
Provided the solution $u = (u_1, u_2)$ *has a decomposition that satisfies the bounds of formula* (1.12) *for* $k \leq 2$*, the interpolation error satisfies*

$$\|u - u^I\|_\varepsilon \leq CN^{-1}\ln(N) \quad \text{and} \quad \|u - u^I\|_2 \leq CN^{-\frac{3}{2}}. \tag{1.21}$$

Proof
By standard interpolation results we estimate

$$\|u_1^S - u_1^{S^I}\|_2 \leq \tilde{C}h^2 |u_1^S|_{2,2} \leq CN^{-2}, \tag{1.22a}$$
$$\|u_1^{x1} - u_1^{x1^I}\|_{2,\Omega_f} \leq \tilde{C}h_{\Omega_f}^2 |u_1^{x1}|_{2,2,\Omega_f} \leq C\sqrt{\varepsilon}\, N^{-2}\ln^2(N), \tag{1.22b}$$
$$|u_1^S - u_1^{S^I}|_{1,2} \leq \tilde{C}h|u_1^S|_{2,2} \leq CN^{-1}, \tag{1.22c}$$
$$|u_1^{x1} - u_1^{x1^I}|_{1,2,\Omega_f} \leq \tilde{C}h_{\Omega_f}|u_1^{x1}|_{2,2,\Omega_f} \leq C\varepsilon^{-\frac{1}{2}}N^{-1}\ln(N). \tag{1.22d}$$

Using the decaying of the boundary terms we derive furthermore

$$\|u_1^{x1} - u_1^{x1^I}\|_{2,\Omega_c} \leq \|u_1^{x1}\|_{2,\Omega_c} + \|u_1^{x1^I}\|_{2,\Omega_c} \leq 2\|u_1^{x1}\|_{\infty,\Omega_c} \leq CN^{-2}, \tag{1.22e}$$
$$\begin{aligned} |u_1^{x1} - u_1^{x1^I}|_{1,2,\Omega_c} &\leq |u_1^{x1}|_{1,2,\Omega_c} + |u_1^{x1^I}|_{1,2,\Omega_c} \\ &\leq \|u_1^{x1'}\|_{2,\Omega_c} + h_{\Omega_c}^{-1}\|u_1^{x1}\|_{\infty,\Omega_c} \leq C\left(\varepsilon^{-\frac{1}{2}}N^{-2} + N^{-1}\right). \end{aligned} \tag{1.22f}$$

These estimates are all attained by well-known techniques used e.g. in [Lin10]. The interpolation error of the weak boundary layer can be bounded by

$$\begin{aligned} |u_1^{x0} - u_1^{x0^I}|_{1,2} &\leq \tilde{C}h|u_1^{x0}|_{2,2} \leq C\varepsilon^{-\frac{1}{2}}N^{-1}, \\ \|u_1^{x0} - u_1^{x0^I}\|_2 &\leq \tilde{C}h|u_1^{x0}|_{1,2} \leq C\varepsilon^{\frac{1}{2}}N^{-1} \quad \text{and} \\ \|u_1^{x0} - u_1^{x0^I}\|_2 &\leq \tilde{C}h^2|u_1^{x0}|_{2,2} \leq C\varepsilon^{-\frac{1}{2}}N^{-2} \end{aligned} \tag{1.22g}$$

which follow from the usual interpolation error estimates. This implies

$$\|u_1^{x0} - u_1^{x0^I}\|_2 \leq C\min\left\{\varepsilon^{\frac{1}{2}}N^{-1}, \varepsilon^{-\frac{1}{2}}N^{-2}\right\} \leq CN^{-\frac{3}{2}}. \tag{1.22h}$$

The bounds of the terms u_2^S, u_2^{x0} and u_2^{x1} can be proved similarly. Combining all these results proves the theorem. □

From the previously attained interpolation error estimates we can deduce an error estimate for $\|u - u^N\|_\varepsilon$:

Theorem 1.9
Provided the solution u *of the system* (1.1) *has a decomposition that satisfies the estimates* (1.12) *for* $k \leq 2$ *the finite element error on a mesh as depicted in Figure 1.1 satisfies*

$$\|u - u^N\|_\varepsilon \leq CN^{-1}\ln(N). \tag{1.23}$$

Proof
In the following we use the abbreviations $\chi := u^I - u^N$ and $\psi := u^I - u$. The coercivity of $\tilde{a}(\cdot,\cdot)$ and the Galerkin orthogonality of our method provide

$$\begin{aligned} \gamma\|\chi\|_\varepsilon^2 &\leq \tilde{a}(\chi,\chi) = \tilde{a}(\psi,\chi) \\ &\leq \varepsilon C|\chi|_{1,2}|\psi|_{1,2} + C\|\chi\|_2\,\|\psi\|_2 + C\left|\int_\Omega \chi_1\psi_1'\,\mathrm{d}\lambda\right| + C\left|\int_\Omega \chi_2\psi_2'\,\mathrm{d}\lambda\right|. \end{aligned} \tag{1.24}$$

To estimate the remaining integral terms we split ψ as we did in (1.22). This way we obtain for the smooth part u_1^S of the solution u using estimate (1.22a)

$$\left|\int_\Omega \chi_1 \left(u_1^S - u_1^{SI}\right)' \, \mathrm{d}\lambda\right| \le \|\chi_1\|_2 \left\|\left(u_1^S - u_1^{SI}\right)'\right\|_2 \le CN^{-1}\|\chi_1\|_2 \le CN^{-1}\|\chi_1\|_\varepsilon. \tag{1.25a}$$

For the estimation of the boundary layer terms we transform the integral via integration by parts

$$\left|\int_\Omega \chi_1 \left(u_1^x - u_1^{xI}\right)' \, \mathrm{d}\lambda\right| = \left|\int_\Omega -\chi_1' \left(u_1^x - u_1^{xI}\right) \, \mathrm{d}\lambda\right| \le \|\chi_1'\|_2 \, \|u_1^x - u_1^{xI}\|_2.$$

Using this transformation we can estimate by (1.22h) the weak layer term via

$$\begin{aligned}\left|\int_\Omega \chi_1 \left(u_1^{x0} - u_1^{x0I}\right)' \, \mathrm{d}\lambda\right| &\le \|\chi_1'\|_2 \, \|u_1^{x0} - u_1^{x0I}\|_2 \\ &\le \sqrt{\varepsilon}CN^{-1}|\chi_1|_{1,2} \le CN^{-1}\|\chi_1\|_\varepsilon. \end{aligned} \tag{1.25b}$$

For the strong boundary layer term we split the integral at the mesh transition point and use an inverse inequality on the coarse part of the mesh domain. This leads to the following formulas

$$\left|\int_{\Omega_f} \chi_1 \left(u_1^{x1} - u_1^{x1I}\right)' \, \mathrm{d}\lambda\right| \le C\sqrt{\varepsilon}N^{-2}\ln^2(N)\|\chi_1'\|_{2,\Omega_f} \le CN^{-1}\|\chi_1\|_{\varepsilon,\Omega_f}, \tag{1.25c}$$

$$\left|\int_{\Omega_c} \chi_1 \left(u_1^{x1} - u_1^{x1I}\right)' \, \mathrm{d}\lambda\right| \le \tilde{C}N^{-2}\|\chi_1'\|_{2,\Omega_c} \le CN^{-1}\|\chi_1\|_{\varepsilon,\Omega_c}. \tag{1.25d}$$

The integral containing χ_2 can be estimated analogously. Combining (1.21), (1.24), (1.25) and their equivalents for χ_2 we get the result

$$\|u^I - u^N\|_\varepsilon = \|\chi\|_\varepsilon \le \tilde{C}\left(\|\psi\|_\varepsilon + N^{-1}\right) \le CN^{-1}\ln N.$$

A triangle inequality completes the proof. □

For a single convection-diffusion equation it is well-known that one has supercloseness of the type

$$\|u^I - u^N\|_\varepsilon \le C\left(N^{-1}\ln(N)\right)^2. \tag{1.26}$$

This leads to the optimal estimate

$$\|u - u^N\|_2 \le C\left(N^{-1}\ln(N)\right)^2. \tag{1.27}$$

Note that in the singularly perturbed case it is not possible to use the Aubin-Nitsche-trick to attain optimal $\mathcal{L}^2$-estimates that are independent of ε.

For our system the interpolation error estimate (1.22h) indicates that we do not have optimal $\mathcal{L}^2$-error bounds if we ignore the weak layers for the mesh construction (cf. numerical experiments in Section 1.4).

If we, however, use equal meshes for u_1 and u_2 with a refinement at each side of the domain (cf. Figure 1.2) we can adopt the proofs for a single equation (cf. [Lin10]). The estimates of the weak boundary layer do no longer pose a problem; due to the refinement of the grid they can be handled the same way the strong layers are. Consequently we obtain

$$\|u^I - u^N\|_\varepsilon \le C\left(N^{-1}\ln(N)\right)^2 \quad \text{and} \quad \|u - u^N\|_2 \le C\left(N^{-1}\ln(N)\right)^2.$$

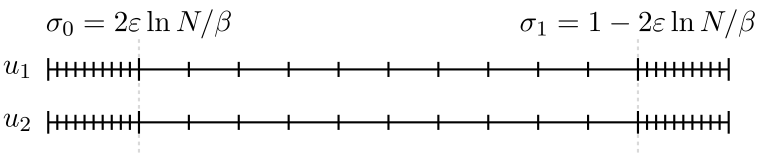

Figure 1.2.: Two-sided Shishkin mesh

1.4. Computational Results

In the following we solve the test problem

$$-\varepsilon u_1'' + \sqrt{2}u_1' + u_2 = 2, \qquad u_1(0) = u_1(1) = 0, \qquad (1.28a)$$
$$-\varepsilon u_2'' - \sqrt{2}u_2' - u_1 = 1, \qquad u_2(0) = u_2(1) = 0 \qquad (1.28b)$$

numerically. Obviously, our theory from the previous chapters applies, because this problem has constant coefficients.

An explicit solution of the system (1.28) is given by

$$u_1 = -1 + \sum_{i=1}^{4} \bar{u}_i\, e^{\lambda_i x}, \qquad u_2 = 2 + \sum_{i=1}^{4} \bar{u}_i\, p_i\, e^{\lambda_i x}$$

with

$$\lambda_i := \pm\frac{\sqrt{1 \pm \sqrt{1-\varepsilon^2}}}{\varepsilon}, \qquad p_i := (\varepsilon\lambda_i - \sqrt{2})\lambda_i.$$

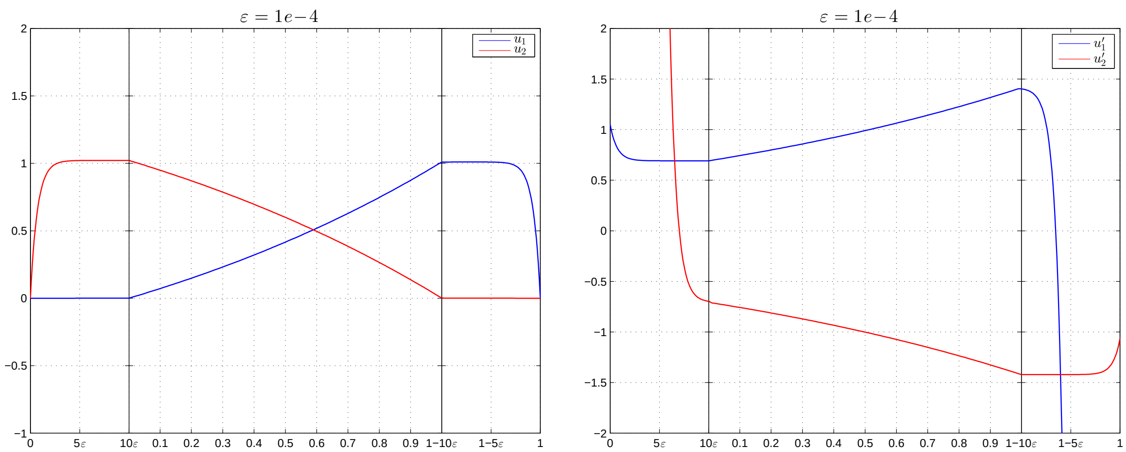

Figure 1.3.: Solution to model problem (1.28)

Here $\bar{u} \in \mathbb{R}^4$ is the solution of the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} & e^{\lambda_4} \\ p_1 & p_2 & p_3 & p_4 \\ p_1 e^{\lambda_1} & p_2 e^{\lambda_2} & p_3 e^{\lambda_3} & p_4 e^{\lambda_4} \end{pmatrix} \bar{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

derived from the boundary conditions of the problem (1.28). A plot of the solution is given in Figure 1.3. The regions near the boundary are stretched for a better visualization of the layer behavior. The wedge in the plot of the solution at the points 10ε and $1-10\varepsilon$ is induced by this stretching. Having this exact solution, we can compute the discrepancy of the numerical to the explicit solution in various norms.

As in the previous analysis of Theorem 1.9, we first use a Shishkin mesh which only accounts for the strong boundary layers for the computations. Using $N+1$ mesh intervals we have N degrees of freedom for u_1 and u_2 each. From these computations we attained the results shown in Figure 1.4.

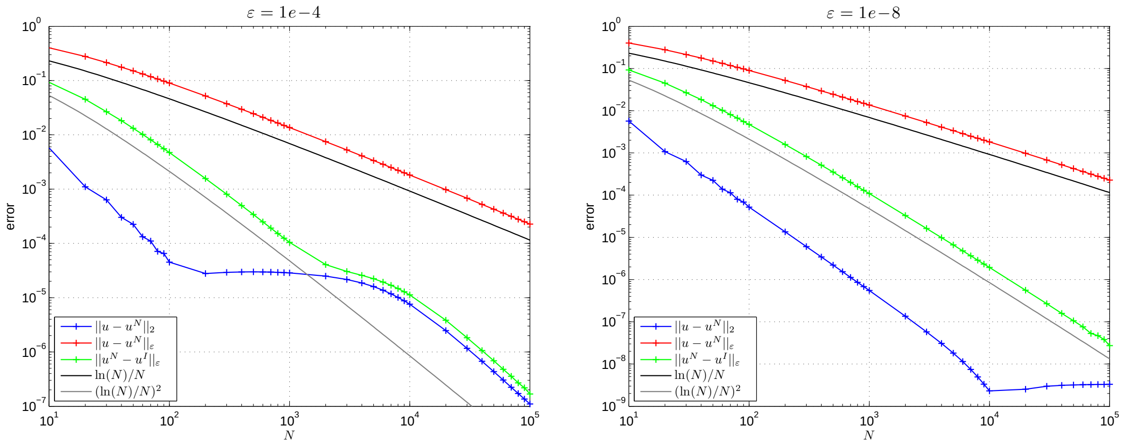

Figure 1.4.: Error of the linear FEM on one-sided Shishkin meshes (cf. Figure 1.1)

These numerical results confirm the theoretical result of an ε-independent convergence in the ε-norm. However, they do not show the almost second order convergence measured in the $\mathcal{L}^2$-norm one could expect knowing the superconvergence results for a single equation. The $\mathcal{L}^2$-error rather exhibits a range of stagnating convergence in the order of magnitude of the perturbation parameter.

Next we compare the results from the previous calculations with the error attained using a two-sided version of the Shishkin mesh where we refine in the region of the weak boundary layers as well as in the region of the strong ones.

The results of this computations are presented in Figure 1.5. As predicted we now get almost second order convergence in the $\mathcal{L}^2$-norm. Furthermore, the range of stagnating convergence does not exist. But the absolute error measured in the ε-norm is larger compared to the previous calculations. This is not surprising because in the first calculations we have more nodes of the grid to resolve the strong layer and the smooth region of the solution.

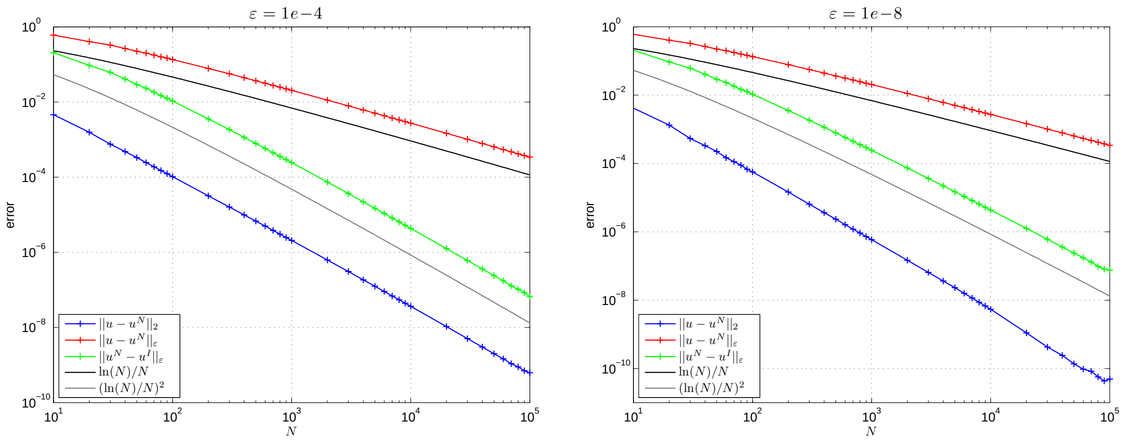

Figure 1.5.: Error of the linear FEM on a two-sided Shishkin mesh (cf. Figure 1.2)

2. Optimal Control with Singularly Perturbed Convection-Diffusion Equations in 1D

In the following we want to analyze the optimal control problem

$$\min_{u,q\in\mathcal{L}^2} J(u,q) := \min_{u,q} \left(\frac{1}{2}\|u-u_d\|_2^2 + \frac{\mu}{2}\|q\|_2^2\right) \tag{2.1a}$$

subject to the singularly perturbed convection-diffusion equation

$$Lu := -\varepsilon u'' + bu' + cu = f + q \text{ in } (0,1), \quad u(0) = u(1) = 0 \tag{2.1b}$$

and the box constraints

$$q \in Q_{\mathrm{ad}} \;:\Longleftrightarrow\; -\infty \le q_a \le q \le q_b \le \infty \text{ in } (0,1) \tag{2.1c}$$

for the control q.

It is well known from optimal control theory (cf. [Tr10]) that (u,q) is a solution for the problem (2.1) if and only if there is an adjoint state v such that the system

$$Lu = f + q \;, \quad u(0) = u(1) = 0, \tag{2.2a}$$

$$L^*v = u - u_d, \quad v(0) = v(1) = 0, \tag{2.2b}$$

$$\langle v + \mu q,\, w - q\rangle = 0 \text{ for all } w \in Q_{\mathrm{ad}} \tag{2.2c}$$

is fulfilled, where L^* is the adjoint operator to L. One can derive easily that the last equation (2.2c) is equivalent to

$$q = \max\left\{q_a, \min\left\{q_b, -\mu^{-1}v\right\}\right\} =: \Pi_{[q_a,q_b]}\left(-\mu^{-1}v\right) \text{ almost everywhere.} \tag{2.3}$$

For simplification of notation we define

$$v_a := -\mu q_b \quad \text{and} \quad v_b := -\mu q_a.$$

Using these definitions we have

$$q = \Pi_{[q_a,q_b]}\left(-\mu^{-1}v\right) = -\mu^{-1}\Pi_{[v_a,v_b]}(v).$$

Thus, the problem above can be written as

$$Lu = -\varepsilon u'' + bu' + cu = f - \mu^{-1}\Pi_{[v_a,v_b]}(v), \qquad u(0) = u(1) = 0, \tag{2.4a}$$

$$L^*v = -\varepsilon v'' - bv'' + (c - b')v = -u_d + u, \qquad v(0) = v(1) = 0. \tag{2.4b}$$

In the following we assume

$$
\begin{align}
0 < \varepsilon \ll \beta, \tag{2.5a}\\
b, c \in \mathcal{C}^2, \tag{2.5b}\\
b \geq \tilde{\beta} > \beta > 0, \tag{2.5c}\\
c \geq 0, \tag{2.5d}\\
2c - b' \geq 2\gamma > 0, \tag{2.5e}\\
\mu \geq 0, \tag{2.5f}\\
|f^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k}\mathcal{E}_0^x(x) + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_1^x(x)\right), \tag{2.5g}\\
|u_d^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_0^x(x) + \varepsilon^{-k}\mathcal{E}_1^x(x)\right), \tag{2.5h}
\end{align}
$$

for $k \in \{0, 1\}$. For the lower constraint q_a we either assume

$$
|q_a^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k}\mathcal{E}_0^x(x) + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_1^x(x)\right) \text{ for } k \in \{0,1\} \quad \text{or} \quad q_a = -\infty. \tag{2.5i}
$$

Analogously, we assume that q_b meets either

$$
|q_b^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k}\mathcal{E}_0^x(x) + \varepsilon^{-k-\frac{1}{2}}\mathcal{E}_1^x(x)\right) \text{ for } k \in \{0,1\} \quad \text{or} \quad q_b = \infty. \tag{2.5j}
$$

The requirement (2.5e) seems not really restrictive since it can be assured via a variable transform (cf. [RST08, Remark 1.6]) for sufficient small $\varepsilon \leq \varepsilon_0$. However, this transform leads to a change of the norms one has to consider to attain an equivalent optimization problem. In that case the optimality system (2.2) has an other structure and our theory can not be applied.

The coefficient $\varepsilon^{-k-\frac{1}{2}}$ in prerequisite (2.5g), (2.5h), (2.5i) and (2.5j) seems a bit unusual. Knowing [KT78] or the fact $|L\mathcal{E}_1^x| \leq C\varepsilon^{-1}\mathcal{E}_1^x$ one might expect the coefficient ε^{-k-1}. However, we also require the $\mathcal{L}^2$-norm of the respective functions to be bounded ε-uniformly. Since we have $\|\mathcal{E}_1^x\|_2 = C\sqrt{\varepsilon}$ we can allow coefficients down to the order $\varepsilon^{-k-\frac{1}{2}}$ for our proofs to work.

Remark 2.1
For $q_a = -\infty$ and $q_b = \infty$ the system (2.2) simplifies to a system of the form (1.1). Since the corresponding coefficients $c_{12} = \mu^{-1}$ and $c_{21} = 1$ are constant the assumptions (1.3) and (1.8) from the previous chapter read

$$
2c - b' \geq 0, \quad c(c - b') + \mu^{-1} - bc' \geq 0, \quad c(c - b') + \mu^{-1} + b(c - b')' \geq 0.
$$

Obviously, they are satisfied by constant coefficients b and c.

In the following we will use the special structure of the optimization problem to omit some requirements. Additionally, the following proofs are applicable to the case of box constraints for the control q.

We start by constructing a solution decomposition and derive detailed estimates for its terms. Some technical details are deferred to Section 2.2. Subsequently, we consider different ways of discretizing the optimal control problem (2.1). We modify two proofs from the literature to provide ε-uniform convergence rates. Also, we present a new proof that does not need the requirement that we have to use the same mesh for the discretization of the state u and the adjoint state v. In Section 2.4 we present computational results to illustrate the theoretical results.

2.1. Analytic Properties of the Solution

As a first step we establish the solvability of the optimization problem (2.1) and establish some preliminary regularity estimates. From our prerequisites (2.5) we get the coercitivity of the bilinear form a associated with L (cf. proof of Theorem 1.1), i.e.

$$a(u,u) = \int_0^1 \varepsilon u'^2 + \left(c - \frac{b'}{2}\right) u^2 \,\mathrm{d}\lambda \geq \varepsilon |u|_{1,2} + \gamma \|u\|_2.$$

Thus, we have the estimate

$$C\|u\|_\varepsilon^2 \leq a(u,u) = \langle f+q,\, u\rangle \leq \|f+q\|_2 \|u\|_2 \quad \Rightarrow \quad \|u\|_2 \leq \|u\|_\varepsilon \leq C\|f+q\|_2 \tag{2.6}$$

and standard optimization theory (cf. [Tr10]) assures the existence of a unique solution $(u,q) \in \mathcal{L}^2 \times \mathcal{L}^2$. Hence, we can apply the theory for differential equations (cf. [GT01, LU68]) to acquire $u \in W^{2,2} \cap H_0^1$ and $v \in W^{2,2} \cap H_0^1$. From properties of the max- and min-operators (cf. [GT01]) we get

$$|q^{(k)}| \leq |q_a^{(k)}| + \mu^{-1}|v^{(k)}| + |q_b^{(k)}| \quad \text{for} \quad k \in \{0,1\}. \tag{2.7}$$

Hence, we have $q \in W^{1,2}$ and $\|q\|_{1,2} \leq \|q_a\|_{1,2} + \mu^{-1}\|v\|_{1,2} + \|q_b\|_{1,2}$. In fact, we will see later on bounds in $W^{2,\infty}$ for u and v.

Next we analyze the solution properties in more detail.

Lemma 2.2
The optimal control satisfies

$$\|q\|_2 \leq C. \tag{2.8}$$

Proof
We define

$$q_0 := \Pi_{[q_a,q_b]}(0) \in U_{\mathrm{ad}}.$$

Since (2.5i) and (2.5j) hold we have

$$\|q_0\|_2 \leq \|q_a\|_2 + \|q_b\|_2 \leq C.$$

Let u_0 be the solution of the primary equation (2.2a) for the inhomogeneity $f + q_0$. Thus, the standard estimates for convection-diffusion problems and the bound (2.5g) give by (2.6) the estimate

$$\|u_0\|_2 \leq \tilde{C}\big(\|f\|_2 + \|q_0\|_2\big) \leq C.$$

Using (2.5h) we conclude

$$\frac{\mu}{2}\|q\|_2^2 \leq J(u,q) \leq J(u_0,q_0) = \frac{1}{2}\|u_0 - u_d\|_2^2 + \frac{\mu}{2}\|q_0\|_2^2 \leq C. \qquad \square$$

Using estimates for some single differential equation problems like (2.2a) or (2.2b) we derive in the next section, we are able to prove explicit bounds for the solution and its derivatives.

Theorem 2.3
If the prerequisites (2.5) *hold, the solution* (u,q,v) *of problem* (2.2) *satisfies almost everywhere*

$$|u^{(k)}(x)| \le C\left(1+\varepsilon^{1-k}\mathcal{E}_0^x(x)+\varepsilon^{-k}\mathcal{E}_1^x(x)\right), \tag{2.9a}$$

$$|v^{(k)}(x)| \le C\left(1+\varepsilon^{-k}\mathcal{E}_0^x(x)+\varepsilon^{1-k}\mathcal{E}_1^x(x)\right) \tag{2.9b}$$

for $k\in\{0,1,2\}$ *and*

$$|q^{(k)}(x)| \le C\left(1+\varepsilon^{-k}\mathcal{E}_0^x(x)+\varepsilon^{1-k}\mathcal{E}_1^x(x)\right) \tag{2.9c}$$

for $k\in\{0,1\}$.

Proof
By Lemma 2.2 and the bound (2.5g) for f, we have $\|Lu\|_2 \le C$. Hence, Lemma 2.5 gives

$$u = u^S + u^{x1}, \quad \|u^S\|_{1,2}\le C, \quad |u^{x1^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_1^x(x)$$

for $k\in\{0,1\}$. Using this estimate and the bound (2.5h) of u_d we can invoke the Lemma 2.5 for the adjoint problem and obtain

$$v = v^S + v^{x0}, \quad \|v^S\|_{1,2}\le C, \quad |v^{x0^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_0^x(x)$$

for $k\in\{0,1\}$. Thus, the Sobolev imbedding theorem ensures

$$\|u^S(x)\|_\infty + \|v^S(x)\|_\infty \le C$$

and the bounds (2.9a) and (2.9b) for $k=0$ are established. These estimates and the projection property (2.7) imply the stated bound for $|q|$.

Hence, we can conclude

$$\begin{aligned}
|-\varepsilon u''+bu'| &= \quad\;\; |-cu+f+\;\; q| \le C\left(1+\varepsilon^{-1}\mathcal{E}_1^x(x)\right) \quad \text{and}\\
|-\varepsilon v''-bv'| &= |-(c-b)v+u-u_d| \le C\left(1+\varepsilon^{-1}\mathcal{E}_0^x(x)\right).
\end{aligned}$$

Using Lemma 2.6 with $i=1$ provides the stated estimates (2.9) ($k=1$) for u' and v'. Invoking (2.7) again we can conclude $|q'| \le |q_a'| + \mu^{-1}|v'| + |q_b'|$. Therefore, we have established the bound (2.9c) ($k=1$) for q'.

By differentiation of the primal and adjoint equation (2.2a, 2.2b) we get

$$\begin{aligned}
|-\varepsilon(u')''+b(u')'| &= |-b'u'-c'u-cu'+f'+q'|\\
&\le C\left(1+\varepsilon^{-1}\mathcal{E}_0^x(x)+\varepsilon^{-2}E_1(x)\right)\\
|-\varepsilon(v')''-b(v')'| &= |b'v-(c-b')'v-(c-b')v'+u'-u_d'|\\
&\le C\left(1+\varepsilon^{-2}\mathcal{E}_0^x(x)+\varepsilon^{-1}E_1(x)\right)
\end{aligned}$$

and Lemma 2.6 ($i=2$) gives the stated bounds for u'' and v''. □

The estimates of Theorem 2.3 for u and v yield decompositions

$$u = u^S + u^{x0} + u^{x1}, \quad v = v^S + v^{x0} + v^{x1}, \tag{2.10a}$$

$$\|u^S\|_{2,\infty}\le C, \qquad |u^{x0^{(k)}}(x)| \le C\varepsilon^{1-k}\mathcal{E}_0^x(x), \qquad |u^{x1^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_1^x(x), \tag{2.10b}$$

$$\|v^S\|_{2,\infty}\le C, \qquad |v^{x0^{(k)}}(x)| \le C\varepsilon^{-k}\mathcal{E}_0^x(x), \qquad |v^{x1^{(k)}}(x)| \le C\varepsilon^{1-k}\mathcal{E}_1^x(x) \tag{2.10c}$$

for $k\in\{0,1,2\}$. This is proved the way devised by Linß in [Lin00].

2.2. Auxiliary Estimates for some Single Differential Equation Problems

First we consider the so called reduced problem.

Lemma 2.4
The solution w of

$$bw' + cw = g, \quad w(0) = w_0 \tag{2.11}$$

satisfies under the assumptions

$$\|g\|_p \leq C, \quad |w_0| \leq C, \quad b, c \in \mathcal{L}^\infty, \quad b \geq \tilde{\beta} \tag{2.12}$$

for $p \in [1, \infty]$ the estimate

$$\|w\|_\infty + \|w\|_{1,p} \leq C. \tag{2.13}$$

Proof
It is easy to check that the solution of problem (2.11) is given by

$$w(x) := \int_0^x \frac{g(\xi)}{b(\xi)}\, \eta(\xi, x)\, \mathrm{d}\xi + K\eta(0, x) \quad \text{with}$$
$$\eta(r, s) := \exp\left(-\int_r^s \frac{c(t)}{b(t)}\, \mathrm{d}t\right)$$

for $x \in (0, 1)$. The boundary condition gives

$$w_0 = w(0) = K.$$

Since $b \geq \tilde{\beta} > 0$ and $c \geq 0$ we have

$$0 < \eta(r, s) \leq 1$$

for $r \leq s$. This yields by application of the Hölder inequality

$$|w(x)| \leq \int_0^x \frac{|g(\xi)|}{\tilde{\beta}}\, \mathrm{d}\xi + |w_0| \leq \tilde{C}\big(\|g\|_1 + 1\big) \leq C.$$

Furthermore, we have

$$w'(x) = \int_0^x -\frac{g(\xi)}{b(\xi)}\, \eta(\xi, x) \frac{c(x)}{b(x)}\, \mathrm{d}\xi + \frac{g(x)}{b(x)} - K\eta(0, x) \frac{c(x)}{b(x)}.$$

Thus, we obtain

$$\|w'\|_p \leq \|g\|_p \left(\frac{\|c\|_\infty}{\tilde{\beta}^2} + \frac{1}{\tilde{\beta}}\right) + w_0 \frac{\|c\|_\infty}{\tilde{\beta}} \leq C.$$

□

Ideas motivated by asymptotic expansions of solutions of singularly perturbed problems (cf. [RST08]) lead to the following

Lemma 2.5
If the assumption

$$\|g\|_2 \le C \tag{2.14}$$

holds, then the solution of the problem

$$Lw = -\varepsilon w'' + bw' + cw = g, \quad w(0) = w(1) = 0 \tag{2.15}$$

has a decomposition in a smooth and a boundary layer part satisfying

$$w = w^S + w^{x1}, \tag{2.16a}$$

$$\|w^S\|_{1,2} \le C, \tag{2.16b}$$

$$|w^{x1(k)}(x)| \le C\varepsilon^{-k}\mathcal{E}_1^x(x) \tag{2.16c}$$

for $k \in \{0, 1\}$.

Proof
First, we look at the reduced problem (2.11). Let us denote the solution of this reduced problem by w^r. Application of Lemma 2.4 establishes $\|w^r\|_\infty + \|w^r\|_{1,2} \le C$. Furthermore, we define the boundary layer correction

$$w^{x1}(x) := -w^r(1)e^{-\frac{b(1)\,(1-x)}{\varepsilon}}.$$

Since w^r is ε-uniformly bounded, w^{x1} satisfies the bounds (2.16c).

From

$$\begin{aligned}|Lw^{x1}(x)| &= \left|-\frac{b(1)^2}{\varepsilon}w^{x1}(x) + b(x)\frac{b(1)}{\varepsilon}w^{x1}(x) + c(x)w^{x1}(x)\right| \\ &= \left|-\frac{b(1)}{\varepsilon}\int_x^1 b'(t)\,\mathrm{d}t + c(x)\right| |w^{x1}(x)| \\ &\le \left(\frac{1-x}{\varepsilon}\|b\|_\infty\|b'\|_\infty + \|c(x)\|_\infty\right)|w^{x1}(x)|\end{aligned}$$

we conclude

$$\|Lw^{x1}\|_2^2 \le \tilde{C}\int_0^1 \left(\frac{1-x}{\varepsilon}\mathcal{E}_1^x(x) + \mathcal{E}_1^x(x)\right)^2 \mathrm{d}x \le C\varepsilon. \tag{2.17}$$

Defining $R := w - w^r - w^{x1}$ we get

$$|R(0)| = |w^r(1)|e^{-\frac{b(1)}{\varepsilon}} \le C\varepsilon, \quad R(1) = 0.$$

Thus, $\tilde{R} := R - (1-x)R(0)$ satisfies homogeneous boundary conditions. Let $a(\cdot,\cdot)$ denote the bilinear form of the weak formulation associated with L, i.e.

$$a(w, v) := \int_0^1 \varepsilon w'v' + bw'v + cwv\,\mathrm{d}\lambda.$$

Using (2.17) and Lemma 2.4 we conclude

$$\begin{aligned}\|\tilde{R}\|_\varepsilon^2 &\le \tilde{C}\big|a(\tilde{R},\tilde{R})\big| = \tilde{C}\left|a(w,\tilde{R}) - a(w^r,\tilde{R}) - a(w^{x1},\tilde{R}) - a\big((1-x)R(0),\tilde{R}\big)\right| \\ &= \tilde{C}\left|-\varepsilon\int_0^1 w^{r\prime}\tilde{R}'\,\mathrm{d}\lambda + \int_0^1 (Lw^{x1})\tilde{R}\,\mathrm{d}\lambda - R(0)\int_0^1 L(1-x)\tilde{R}\,\mathrm{d}\lambda\right| \\ &\le \hat{C}\left(\varepsilon|w^r|_{1,2}|\tilde{R}|_{1,2} + \|Lw^{x1}\|_2\|\tilde{R}\|_2 + |R(0)|\|\tilde{R}\|_2\right) \\ &\le C\sqrt{\varepsilon}\|\tilde{R}\|_\varepsilon.\end{aligned}$$

From this we derive
$$\|R\|_{1,2} \le \|\tilde{R}\|_{1,2} + 2|R(0)| \le C.$$
Setting $w^S := w^r + R$ we have proved the lemma. □

Finally, we use pointwise estimates for a related differential equation problem.

Lemma 2.6
Let us assume the bounds
$$|g(x)| \le C\left(1 + \varepsilon^{1-i}\mathcal{E}_0^x(x) + \varepsilon^{-i}\mathcal{E}_1^x(x)\right), \tag{2.18a}$$
$$|w_0| + |w_1| \le C\varepsilon^{1-i} \tag{2.18b}$$
hold for $i \ge 1$. Then the solution w of the differential equation problem
$$-\varepsilon w'' + bv' = g, \quad w(0) = w_0, \quad w(1) = w_1 \tag{2.19}$$
with $b \ge \tilde{\beta} > \beta$ satisfies
$$|w'(x)| \le C\left(1 + \varepsilon^{1-i}\mathcal{E}_0^x(x) + \varepsilon^{-i}\mathcal{E}_1^x(x)\right). \tag{2.20}$$

The proof of this Lemma follows ideas presented in [KT78].

Proof
One can check that the solution of problem (2.19) is given by
$$w(x) := \int_x^1 -\hat{w}(\xi)\,\mathrm{d}\xi + K_1\big(1 - \eta(x,1)\big) + K_2 \quad \text{with}$$
$$\hat{w}(\xi) := \int_\xi^1 \frac{g(\zeta)}{\varepsilon}\eta(\xi,\zeta)\,\mathrm{d}\zeta, \quad \eta(r,s) := \exp\left(-\int_r^s \frac{b(t)}{\varepsilon}\,\mathrm{d}t\right)$$
for $x \in (0,1)$. Using the upper and lower bounds of b we get
$$0 < e^{-\frac{\|b\|_\infty(s-r)}{\varepsilon}} \le \eta(r,s) \le e^{-\frac{\tilde{\beta}(s-r)}{\varepsilon}} \le e^{-\frac{\beta(s-r)}{\varepsilon}} \tag{2.21}$$
for $r \le s$. These estimates yield
$$\begin{aligned}|\hat{w}(\xi)| &\le C\int_\xi^1 \varepsilon^{-1}e^{-\frac{\beta(\zeta-\xi)}{\varepsilon}} + \varepsilon^{-i}e^{-\frac{\beta(2\zeta-\xi)}{\varepsilon}} + \varepsilon^{-i-1}e^{-\frac{\beta-\tilde{\beta}\xi+(\tilde{\beta}-\beta)\zeta}{\varepsilon}}\,\mathrm{d}\zeta\\ &\le C\left(\beta^{-1} + \frac{\varepsilon^{1-i}}{2\beta}e^{-\frac{\beta\xi}{\varepsilon}} + \frac{\varepsilon^{-i}}{\tilde{\beta}-\beta}e^{-\frac{\beta(1-\xi)}{\varepsilon}}\right).\end{aligned} \tag{2.22}$$
We apply this result to get
$$\begin{aligned}\left|-\int_x^1 \hat{w}(\xi)\,\mathrm{d}\xi\right| &\le \tilde{C}\int_x^1 1 + \varepsilon^{1-i}\mathcal{E}_0^x(\xi) + \varepsilon^{-i}\mathcal{E}_1^x(\xi)\,\mathrm{d}\xi\\ &\le \tilde{C}\left(1 + \frac{\varepsilon^{2-i}}{\beta}\mathcal{E}_0^x(x) + \frac{\varepsilon^{1-i}}{\beta}\right) \le C\varepsilon^{1-i}\end{aligned}$$
since we assumed $i \ge 1$. Thus, we observe
$$|K_2| = |w(1)| \le C\varepsilon^{1-i}$$

and acquire the estimate

$$|K_1| = \left| \frac{w_0 + \int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi - K_2}{1 - \eta(0,1)} \right| \leq C\varepsilon^{1-i}. \tag{2.23}$$

Obviously, $w'(x) = \hat{w}(x) - K_1 \frac{b(x)}{\varepsilon}\eta(x,1)$ holds, thus we derive from (2.21), (2.22) and (2.23) the final estimate

$$|w'(x)| \leq C\Big(1 + \varepsilon^{1-i}\mathcal{E}_0^x(x) + \varepsilon^{-i}\mathcal{E}_1^x(x)\Big) + C\varepsilon^{-i}\mathcal{E}_1^x(x) \qquad \square$$

The results above can by applied to the corresponding problems of (2.11), (2.15), (2.19) with b substituted by $-b$ by a change of the variable $\tilde{x} = 1 - x$. Note that this leads to an exchange of the boundary points and the respective layer terms.

2.3. Numerical Analysis

In this section we will use the decomposition of the solution (2.10) derived above to prove convergence rates for some finite element methods.

We choose piecewise linear ansatz functions on Shishkin meshes. As in the previous analysis of the system of differential equations (cf. Chapter 1), we may resolve the strong layers only – or we choose to resolve also the weak ones. This leads to meshes shown in Figure 1.1 or Figure 1.2. By the previous analysis we can apply Theorem 1.8 to acquire

$$\|u - u^I\|_\varepsilon + \|v - v^I\|_\varepsilon \leq CN^{-1}\ln N \tag{2.24}$$

for the nodal interpolants u^I, v^I to u, v, respectively.

In the literature one can find different approaches to prove convergence of a numerical method for solving the optimality system (2.2). In [CH02, HL10, LH12] the linearity of the problem for $Q_{\mathrm{ad}} = \mathcal{L}^2$ is used. Therefore, we can not adopt their approach to the case $Q_{\mathrm{ad}} \neq \mathcal{L}^2$. Other approaches ([BV07, HYZ09]) rely on the fact that the discrete problem is also an optimization problem. This is not the case when we use different meshes to discretize u and v as we have in the Chapter 1.

Remark 2.7
In the case $Q_{\mathrm{ad}} = \mathcal{L}^2$ we can use the approach of Theorem 1.9. Thus, we do not need any further requirements like (2.5e) (assuring the coercitivity of the primal and dual equation) to prove convergence, provided the solution decomposition (2.10) and the corresponding estimates are valid. Hence, we have

$$\|u - u^N\|_\varepsilon + \|v - v^N\|_\varepsilon + \|q - q^N\|_\varepsilon \leq CN^{-1}\ln N.$$

In the following we derive estimates for $Q_{\mathrm{ad}} \neq \mathcal{L}^2$. Due to the projection Π the resulting optimality system is not linear, thus the analysis is more involved than in Chapter 1. For the analysis in the following sections we use the linear solution operator $\mathcal{S}$ of the primal problem (2.1b) and its counterpart $\mathcal{S}^*$ for the adjoint problem (2.2b). Also, we will denote the corresponding operators for the discrete problems by $\mathcal{S}^N$ and $(\mathcal{S}^*)^N$, respectively.

We present three discretizations with different limitations and prove their convergence. The so called full-discrete problem presented in Section 2.3.2 has the big advantage of being easy to implement. However, the convergence proof is pretty long and more technical than for the other two algorithms.

Note that the convergence proofs presented here are easily adopted to the case of a multidimensional domain for the considered functions in the optimal control problem. The basis for the following proofs is an ε-uniform convergence estimate for single differential equation problems. The additions to that only rely on techniques also available for multidimensional domains.

2.3.1. Symmetric Discretization Using the Continuous Projection

The discrete problem we present here is based on the ideas of Hinze (e.g. [HYZ09]) to discretize only the differential equations and to leave the projection unchanged. The mathematics to analyze such a semi-discretization are very neat. The implementation on the other hand is more difficult, especially for arbitrary functions q_a and q_b. One has to evaluate integrals of the form $\int \Pi(v)w \,\mathrm{d}\lambda$ for $v, w \in V^N$. Since $\Pi(v)'$ may have jumps not only at the nodes of the mesh, one has to pay special attention to this term. We discuss the details of the implementation later.

The convergence proof presented in [HYZ09] for this semi-discretization is based on $\left(\mathcal{S}^N\right)^* = (\mathcal{S}^*)^N$. We modify this way of thought to derive ε-independent convergence rates. To do so, we need a symmetric discretization of the primal and adjoint equation. This implies that we have to use the same mesh to discretize u and v. Consequently, we attain a discrete problem that is an optimality system of a discrete optimization problem.

In Section 2.3.3 we present an other approach where we use something like V-ellipticity of the nonlinear (semi-discrete) systems. But this leads to a lower bound for the regularization parameter μ, which may be undesirable.

As stated above we use the same mesh to discretize the primal and adjoint equation. Hence, we refine on both ends of the interval (cf. Figure 1.2) and use the same discrete space $V^N \subset H_0^1$ of piecewise linear ansatz functions to discretize u, v and the test functions. We do not discretize the projection $\Pi_{[v_a,v_b]}(v)$. We attain the semi-discrete problem

$$u^N \in V^N : a^N(u^N, w) = \left\langle f - \mu^{-1}\Pi_{[v_a,v_b]}(v^N),\, w\right\rangle \text{ for all } w \in V^N, \tag{2.25a}$$

$$v^N \in V^N : a^N(w, v^N) = \left\langle u^N - u_d,\, w\right\rangle \quad \text{for all } w \in V^N, \tag{2.25b}$$

where $a^N(\cdot,\cdot)$ denotes the bilinear form associated with L. For simplification in writing we define $q^N := -\mu^{-1}\Pi_{[v_a,v_b]}(v^N)$. Thus, we have

$$a^N(u^N, w) = \left\langle f + q^N,\, w\right\rangle \text{ for all } w \in V^N \quad \text{and} \tag{2.25c}$$

$$\left\langle v^N + \mu q^N,\, w - q^N\right\rangle \geq 0 \text{ for all } w \in Q_{\mathrm{ad}}. \tag{2.25d}$$

We start the analysis of the semi-discretization by recalling some convergence results for singularly perturbed differential equation problems.

Lemma 2.8
For the numeric solution $\tilde{u}^N(q) := \mathcal{S}^N(f+q)$ of the primal equation using the continuous optimal control q on the right hand side we have

$$\|u - \tilde{u}^N(q)\|_\varepsilon \leq CN^{-1}\ln N. \tag{2.26}$$

Analogously, we have for the numeric solution $\tilde{v}^N(u) := (\mathcal{S}^*)^N\,(u - u_d)$ of the adjoint equation using the continuous optimal state u on the right hand side the estimate

$$\|v - \tilde{v}^N(u)\|_\varepsilon \leq CN^{-1}\ln N. \tag{2.27}$$

Proof
The proof works the same way as the proof of Theorem 1.9. □

Lemma 2.9
For the solution (u^N, v^N, q^N) *of the semi-discretization* (2.25) *we have the estimate*

$$\|u - u^N\|_2 + \|q - q^N\|_2 \le CN^{-1} \ln N. \tag{2.28}$$

Proof
As indicated above this proof is done with methods from [HYZ09].

First we note, that due to the discretization with equal meshes for the primal and adjoint equation we have

$$\left(\mathcal{S}^N\right)^* = (\mathcal{S}^*)^N. \tag{2.29}$$

We test the projection property (2.25d) with q and add the continuous optimality condition (2.2c) tested by $q^N \in Q_{\text{ad}}$. Hence, we attain

$$\begin{aligned} 0 &\le \left\langle v^N + \mu q^N,\, q - q^N \right\rangle + \left\langle v + \mu q,\, q^N - q \right\rangle \\ &= \left\langle v - v^N + \mu(q - q^N),\, q^N - q \right\rangle = \left\langle v - v^N,\, q^N - q \right\rangle - \mu \|q - q^N\|_2^2. \end{aligned}$$

This and an application of Young's inequality leads to the relation

$$\begin{aligned} \mu\|q - q^N\|_2^2 &\le \left\langle v - v^N,\, q^N - q \right\rangle = \left\langle v - \tilde{v}^N(u),\, q^N - q \right\rangle + \left\langle \tilde{v}^N(u) - v^N,\, q^N - q \right\rangle \\ &\le \|v - \tilde{v}^N(u)\|_2 \|q^N - q\|_2 + \left\langle \tilde{v}^N(u) - v^N,\, q^N - q \right\rangle \\ &\le \frac{1}{2\mu}\|v - v^N(u)\|_2^2 + \frac{\mu}{2}\|q^N - q\|_2^2 + \left\langle \tilde{v}^N(u) - v^N,\, q^N - q \right\rangle \end{aligned}$$

where $\tilde{v}^N(u) = (\mathcal{S}^*)^N (u - u_d)$ denotes the numerical solution of the adjoint equation starting from the analytic solution u on the right hand side. Using the weak formulation of the discrete primal equation (2.25b) we acquire

$$\left\langle \tilde{v}^N(u) - v^N,\, q^N - q \right\rangle = a^N\big(u^N - \tilde{u}^N(q), \tilde{v}^N(u) - v^N\big)$$

where $\tilde{u}^N(q) = \mathcal{S}^N(f + q)$ denotes the numerical solution of the primal equation starting from the analytic solution q on the right hand side. Since we have $u^N - \tilde{u}^N(q) \in V^N$ and the relation (2.29) we can furthermore derive, using the weak formulation of the discrete adjoint equation (2.25b), that

$$\begin{aligned} a^N\big(u^N - \tilde{u}^N(q), \tilde{v}^N(u) - v^N\big) &= \left\langle u - u^N,\, u^N - \tilde{u}^N(q) \right\rangle \\ &= \left\langle u - u^N,\, u - \tilde{u}^N(q) \right\rangle - \|u - u^N\|_2^2 \end{aligned}$$

holds. Combining these estimates we get

$$\begin{aligned} \mu\|q - q^N\|_2^2 &\le \frac{1}{2\mu}\|v - \tilde{v}^N(u)\|_2^2 + \frac{\mu}{2}\|q^N - q\|_2^2 + \left\langle u - u^N,\, u - \tilde{u}^N(q) \right\rangle - \|u - u^N\|_2^2 \\ &\le \frac{1}{2\mu}\|v - \tilde{v}^N(u)\|_2^2 + \frac{\mu}{2}\|q - q^N\|_2^2 + \frac{1}{2}\|u - \tilde{u}^N(q)\|_2^2 - \frac{1}{2}\|u - u^N\|_2^2 \end{aligned}$$

where we used again Young's inequality. By Lemma 2.8 we conclude

$$\mu\|q - q^N\|_2^2 + \|u - u^N\|_2^2 \le \frac{1}{\mu}\|v - \tilde{v}^N(u)\|_2^2 + \|u - \tilde{u}^N(q)\|_2^2 \le C\left(N^{-1}\ln N\right)^2.$$ □

Theorem 2.10
Assume that

$$\|q - q^N\|_2 \le CN^{-1} \ln N. \tag{2.30}$$

Than we have

$$\|u - u^N\|_\varepsilon + \|v - v^N\|_\varepsilon \le CN^{-1} \ln N. \tag{2.31}$$

Proof
By standard stability estimates we get

$$\begin{aligned}\|u^N - \tilde{u}^N(q)\|_\varepsilon^2 &\le a^N\left(u^N - \tilde{u}^N(q), u^N - \tilde{u}^N(q)\right) = \langle q^N - q,\, u^N - \tilde{u}^N(q)\rangle \\ &\le \|q^N - q\|_2 \|u^N - \tilde{u}^N(q)\|_2 \le C\|u^N - \tilde{u}^N(q)\|_\varepsilon N^{-1} \ln N.\end{aligned}$$

Thus, the result of Lemma 2.8 and a triangle inequality gives

$$\|u^N - u\|_\varepsilon \le \|u^N - \tilde{u}^N(q)\|_\varepsilon + \|\tilde{u}^N(q) - u\|_\varepsilon \le CN^{-1} \ln N.$$

Analogue argumentation gives the bound for $\|v - v^N\|_\varepsilon$. □

Combining Lemma 2.9 and Theorem 2.10 we proved

Corollary 2.11
The semi-discrete problem (2.25) *using a symmetric mesh as depicted in Figure 1.2 satisfies*

$$\|u - u^N\|_\varepsilon + \|v - v^N\|_\varepsilon \le CN^{-1} \ln N. \tag{2.32}$$

2.3.2. Symmetric Discretization with Discretized Projection

In this section we discretize not only the primal and adjoint equation, but also the projection. The convergence proof presented in this section resembles the ideas presented in [BV07]. As in the previous section we use $\left(\mathcal{S}^N\right)^* = (\mathcal{S}^*)^N$ for the solution operators S^N and $(\mathcal{S}^*)^N$ of the discrete primal equation (2.33a) and adjoint equation (2.33b), respectively. To have this property we use again piecewise linear ansatz functions on a Shishkin mesh that accounts for both boundary layer terms for both the state u and the adjoint v (cf. Figure 1.2). We discretize the box constraints of Q_{ad} by enforcing them only in the grid points. This gives the method

$$u^N \in V^N : a^N(u^N, w) = \langle f + q^N,\, w\rangle \text{ for all } w \in V^N, \tag{2.33a}$$
$$v^N \in V^N : a^N(w, v^N) = \langle u^N - u_d,\, w\rangle \text{ for all } w \in V^N, \tag{2.33b}$$
$$q^N \in V^N : q^N(x_i) = \min\left\{\max\left\{-\mu^{-1} v^N(x_i), q_a(x_i)\right\}, q_b(x_i)\right\}. \tag{2.33c}$$

We call this the full-discrete problem.

A solution of this problem can easily be found by applying an active set algorithm. Thus, the implementation is much easier than for the semi-discretization presented in Section 2.3.1.

Our convergence proof presented below uses some additional assumptions: The lower bound q_a satisfies either

$$\left|q_a^{(k)}(x)\right| \le C\left(1 + \varepsilon^{-k}\mathcal{E}_0^x(x) + \varepsilon^{-k}\mathcal{E}_1^x(x)\right) \text{ for } k \in \{0,1\} \quad \text{or} \quad q_a = -\infty. \tag{2.34a}$$

Analogously, we assume that q_b meets either

$$\left|q_b^{(k)}(x)\right| \le C\left(1+\varepsilon^{-k}\mathcal{E}_0^x(x)+\varepsilon^{-k}\mathcal{E}_1^x(x)\right) \text{ for } k\in\{0,1\} \quad \text{or} \quad q_b=\infty \tag{2.34b}$$

and we have an ε-independent constant $\delta_q \in \mathbb{R}$ with

$$q_b(x)-q_a(x)\ge \delta_q \quad \text{for all} \quad x\in(0,1). \tag{2.34c}$$

Obviously, the optimal control q is not necessarily in $Q_{\text{ad}}^N = \left\{w\in V^N \middle| q_a^I \le w \le q_b^I\right\}$. Thus, we first construct an approximation $\tilde{q}\in Q_{\text{ad}}^N$ of q by linear interpolation of

$$\tilde{q}(x_i) := \begin{cases} q_a(x_i), & \exists\, x\in(x_{i-1},x_{i+1}):\ q(x) > -\mu^{-1}v(x),\\ q_b(x_i), & \exists\, x\in(x_{i-1},x_{i+1}):\ q(x) < -\mu^{-1}v(x),\\ q(x_i), & \text{otherwise.}\end{cases}$$

In other words we set the interpolant to the upper or lower bound in x_i if the upper or lower bound is reached in the vicinity of the grid point. Otherwise we set the interpolant to the solution q.

Obviously, this interpolation is well defined in case $\max\{-q_a,q_b\}=\infty$, i.e. there is no upper or no lower bound for q. Otherwise the assumptions (2.34) enable us to show that this interpolant is well-defined for $N\ge N_0$ by

Lemma 2.12
There is an N_0 independent of ε such that

$$\delta_{q,i} := \operatorname*{ess\,max}_{(x_{i-1},x_i)} \hat{q} - \operatorname*{ess\,min}_{(x_{i-1},x_i)} \hat{q} \le \frac{\delta_q}{3} \quad \textit{for} \quad i\in\{1,\dots,N\} \tag{2.35}$$

holds for the function $\hat{q} := q-q_a$ and all $N\ge N_0$.

Proof
By the previous requirements (2.34) and Theorem 2.3 we can split $\hat{q}=\hat{q}^S+\hat{q}^x$ with

$$|\hat{q}^{S(k)}(x)|\le C, \quad |\hat{q}^{x(k)}(x)| \le C\left(\varepsilon^{-k}\mathcal{E}_0^x(x)+\varepsilon^{-k}\mathcal{E}_1^x(x)\right)$$

for $k\in\{0,1\}$.

From the Bramble-Hilbert lemma we know

$$\operatorname*{ess\,max}_{(x_{i-1},x_i)} w - \operatorname*{ess\,min}_{(x_{i-1},x_i)} w \le Ch\|w'\|_{\infty,(x_{i-1},x_i)}.$$

Hence, we have in the coarse part of the grid $\Omega_c := \left(\frac{2\varepsilon}{\beta}\ln(N), 1-\frac{2\varepsilon}{\beta}\ln(N)\right)$ the relation

$$\delta_{q,i} \le \tilde{C}\left(2\|\mathcal{E}_0^x\|_{\infty,\Omega_c}+2\|\mathcal{E}_1^x\|_{\infty,\Omega_c}+h\|\hat{q}^{S\prime}\|_{\infty,\Omega_c}\right) \le \tilde{C}\left(\frac{4}{N^2}+\frac{3}{N}\|\hat{q}^{S\prime}\|_\infty\right)\le CN^{-1}.$$

In the fine part of the grid we have $h=C\varepsilon N^{-1}\ln N$ and conclude

$$\delta_{q,i}\le \tilde{C}h\|\hat{q}'\|_\infty \le CN^{-1}\ln N.$$

Thus, the term $\delta_{q,i}$ diminishes ε-uniformly in both parts of the grid and we can find an N_0 with $3\delta_{q,i}\le\delta_q$ for all $i\in\{1,\dots,N\}$. □

From the proof above and the usual interpolation estimate (cf. Theorem 1.8) we can derive

$$\|\tilde{q} - q\|_2 \leq \|\tilde{q} - q^I\|_2 + \|q^I - q\|_2 \leq CN^{-1} \ln N. \tag{2.36}$$

Furthermore, we have the useful approximation property

Lemma 2.13
The estimate

$$\|\tilde{q} - q^N\|_2 \leq CN^{-1} \ln N \tag{2.37}$$

holds.

Proof
By $\left(\mathcal{S}^N\right)^* = (\mathcal{S}^*)^N$ we have

$$\begin{aligned}
\mu\|\tilde{q} - q^N\|_2^2 &\leq \left\langle \mathcal{S}^N(\tilde{q} - q^N), \, \mathcal{S}^N(\tilde{q} - q^N)\right\rangle + \mu \left\langle \tilde{q} - q^N, \, \tilde{q} - q^N\right\rangle \\
&\leq \left\langle (\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} + f) - u_d\right) + \mu\tilde{q} - (\mathcal{S}^*)^N \left(\mathcal{S}^N(q^N + f) - u_d\right) - \mu q^N, \, \tilde{q} - q^N\right\rangle .
\end{aligned}$$

The projection property (2.33c) ensures

$$\left\langle (\mathcal{S}^*)^N \left(\mathcal{S}^N(q^N + f) - u_d\right) + \mu q^N, \, \tilde{q} - q^N\right\rangle = \left\langle v^N + \mu q^N, \, \tilde{q} - q^N\right\rangle \geq 0$$

for $\tilde{q} \in Q_{\mathrm{ad}}^N$.

By construction of $\tilde{q}$ we have

$$\begin{aligned}
v + \mu q < 0 \quad &\Leftrightarrow \quad -v/\mu > q = q_b \quad \Rightarrow \quad \tilde{q} = q_b^I \quad \Rightarrow \quad \tilde{q} \geq q^N, \\
v + \mu q > 0 \quad &\Leftrightarrow \quad -v/\mu < q = q_a \quad \Rightarrow \quad \tilde{q} = q_a^I \quad \Rightarrow \quad \tilde{q} \leq q^N.
\end{aligned}$$

and can conclude

$$\begin{aligned}
\left\langle \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right) + \mu q, \, \tilde{q} - q^N\right\rangle &= \left\langle v + \mu q, \, \tilde{q} - q^N\right\rangle \\
&= \left\langle v + \mu q, \, \tilde{q} - q^N\right\rangle_{v+\mu q<0} + \left\langle v + \mu q, \, \tilde{q} - q^N\right\rangle_{v+\mu q>0} \leq 0.
\end{aligned}$$

Combining these estimates we get

$$\begin{aligned}
&\mu\|\tilde{q} - q^N\|_2^2 \\
&\leq \left\langle (\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} + f) - u_d\right) + \mu\tilde{q} - \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right) - \mu q, \, \tilde{q} - q^N\right\rangle \\
&\leq \left(\left\|(\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} + f) - u_d\right) - \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right)\right\|_2 + \mu\|\tilde{q} - q\|_2\right) \|\tilde{q} - q^N\|_2 \\
&\leq \frac{1}{2\mu} \left(\left\|(\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} + f) - u_d\right) - \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right)\right\|_2 + \mu\|\tilde{q} - q\|_2\right)^2 + \frac{\mu}{2}\|\tilde{q} - q^N\|_2^2
\end{aligned}$$

where we used Young's inequality. This yields

$$\begin{aligned}
\mu\|\tilde{q} - q^N\|_2 &\leq \left\|(\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} + f) - u_d\right) - \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right)\right\|_2 + \mu\|\tilde{q} - q\|_2 \\
&\leq \left\|(\mathcal{S}^*)^N \left(\mathcal{S}^N(q + f) - u_d\right) - \mathcal{S}^* \left(\mathcal{S}(q + f) - u_d\right)\right\|_2 \\
&\qquad + \left\|(\mathcal{S}^*)^N \left(\mathcal{S}^N(\tilde{q} - q)\right)\right\|_2 + \mu\|\tilde{q} - q\|_2.
\end{aligned}$$

By the usual stability result

$$\gamma \left\|(\mathcal{S}^*)^N(u_1)\right\|_2^2 \le a^N\Big((\mathcal{S}^*)^N(u_1), (\mathcal{S}^*)^N(u_1)\Big)$$
$$= \Big\langle u_1, (\mathcal{S}^*)^N(u_1)\Big\rangle \le \|u_1\|_2 \left\|(\mathcal{S}^*)^N(u_1)\right\|_2$$

for the adjoint equation and its analogue for the primal equation we have

$$\begin{aligned}
&\left\|(\mathcal{S}^*)^N\left(\mathcal{S}^N(q+f)-u_d\right)-\mathcal{S}^*\left(\mathcal{S}(q+f)-u_d\right)\right\|_2+\left\|(\mathcal{S}^*)^N\left(\mathcal{S}^N(\tilde{q}-q)\right)\right\|_2\\
&\le \left\|(\mathcal{S}^*)^N\left(\mathcal{S}^N(q+f)-\mathcal{S}(q+f)\right)\right\|_2+\left\|\tilde{v}^N(u)-v\right\|_2+\gamma^{-1}\left\|\mathcal{S}^N(\tilde{q}-q)\right\|_2\\
&\le \gamma^{-1}\left\|\tilde{u}^N(q)-u\right\|_2+\left\|\tilde{v}^N(u)-v\right\|_2+\gamma^{-2}\left\|\tilde{q}-q\right\|_2.
\end{aligned}$$

Combining these estimates, the approximation error (2.36) and Lemma 2.8 leads to

$$\mu\|\tilde{q}-q^N\|_2 \le CN^{-1}\ln N. \qquad \square$$

A triangle inequality gives $\|q^N - q\| \le CN^{-1}\ln N$ and we can apply Theorem 2.10 to acquire

Corollary 2.14
Under the additional assumptions (2.34) *the full-discrete problem* (2.33) *using a symmetric mesh as depicted in Figure 1.2 satisfies*

$$\|u-u^N\|_\varepsilon + \|v-v^N\|_\varepsilon \le CN^{-1}\ln N. \tag{2.38}$$

2.3.3. Non-Symmetric Discretization Using the Continuous Projection

In the following we derive estimates for the semi-discrete problem (2.25) that allow to chose a different discretization of the primal and the adjoint equation. Thus, we are able to refine only in the part of the domain, where the boundary layers impair the interpolation estimates. As in Section 2.3.1 the implementation of the algorithm presented in this section has to deal with the problem of evaluating integrals for non-smooth functions. We assume that at least one of the bounds is finite (cf. Remark 2.7). For simplification we assume furthermore that we have $q_b < \infty$. But all the proofs can be modified to serve for the case $-\infty < q_a$, $q_b = \infty$. As an additional requirement for our proofs to work we need

$$c - \frac{b'}{2} \ge \gamma \ge \frac{1}{\sqrt{\mu}}. \tag{2.39}$$

As a first step we reformulate the optimality system (2.4) as follows

$$Lu = f - \mu^{-1}v_a - \mu^{-1}\Pi_{[0,v_b-v_a]}(v-v_a), \qquad u(0)=u(1)=0, \tag{2.40a}$$
$$L^*v = \qquad -u_d + u, \qquad v(0)=v(1)=0. \tag{2.40b}$$

To simplify the notation in this section we define

$$\tilde{\Pi} := v \mapsto \Pi_{[0,v_b-v_a]}(v-v_a).$$

As a first ingredient we use the pointwise estimate

Lemma 2.15
For $x \in (0,1)$ we have

$$\left|\tilde{\Pi}(v_1)(x) - \tilde{\Pi}(v_2)(x) - \big(v_1(x) - v_2(x)\big)\right| \leq |v_1(x) - v_2(x)|. \tag{2.41}$$

Proof
For brevity we omit the argument x of the functions in this proof. Nevertheless everything is meant pointwise for any $x \in (0,1)$.

Obviously, the statement is true for $\tilde{\Pi}(v_1) = \tilde{\Pi}(v_2)$.

Thus, we assume without loss of generality $\tilde{\Pi}(v_1) < \tilde{\Pi}(v_2)$ – otherwise we could switch v_1 and v_2. This relation is only possible for $v_1 < v_2$ and $0 \leq \tilde{\Pi}(v_1) < \tilde{\Pi}(v_2) \leq v_b - v_a$. Hence, we conclude

$$\begin{aligned}\tilde{\Pi}(v_1) &= \max\{0, v_1 - v_a\} \geq v_1 - v_a \quad \text{and}\\ \tilde{\Pi}(v_2) &= \min\{v_2 - v_a, v_b - v_a\} \leq v_2 - v_a.\end{aligned}$$

Using these estimates we acquire

$$\begin{aligned}\left|\tilde{\Pi}(v_1) - \tilde{\Pi}(v_2) - v_1 + v_2\right| &= \Big(\tilde{\Pi}(v_1) - (v_1 - v_a)\Big) + \Big((v_2 - v_a) - \tilde{\Pi}(v_2)\Big)\\ &= (v_2 - v_1) + \Big(\tilde{\Pi}(v_1) - \tilde{\Pi}(v_2)\Big)\\ &\leq v_2 - v_1 = |v_1 - v_2|.\end{aligned}$$

□

Next we multiply the primal equation with μ and consider the associated weak formulation of (2.40):

$$a(u, v; \varphi, \psi) = \langle(\mu f - v_a),\, \varphi\rangle - \langle u_d,\, \psi\rangle \text{ for all } \varphi, \psi \in H_0^1(0,1) \tag{2.42a}$$

with

$$\begin{aligned}a(u, v; \varphi, \psi) := \int_0^1 &\varepsilon\mu u'\varphi' + b\mu u'\varphi + c\mu u\varphi + \tilde{\Pi}(v)\varphi\\ &+ \varepsilon v'\psi' - bv'\psi + (c - b')v\psi - u\psi \,\mathrm{d}\lambda.\end{aligned} \tag{2.42b}$$

Lemma 2.16
Under the assumption (2.39) *the form $a(\cdot,\cdot;\cdot,\cdot)$ is uniformly monotone (cf. [Zei90]), i.e.*

$$\begin{aligned}a(u_1, v_1; u_1 - u_2, v_1 - v_2) &- a(u_2, v_2; u_1 - u_2, v_1 - v_2)\\ &\geq C\left(\|u_1 - u_2\|_\varepsilon^2 + \|v_1 - v_2\|_\varepsilon^2\right).\end{aligned} \tag{2.43}$$

Proof
Using Lemma 2.15 and Young's inequality we conceive

$$\begin{aligned}&\int_0^1 \Big(\tilde{\Pi}(v_1) - \tilde{\Pi}(v_2) - v_1 + v_2\Big)(u_1 - u_2)\,\mathrm{d}\lambda\\ &\geq -\left\|\tilde{\Pi}(v_1) - \tilde{\Pi}(v_2) - v_1 + v_2\right\|_2 \|u_1 - u_2\|_2\\ &\geq -\|v_1 - v_2\|_2\|u_1 - u_2\|_2\\ &\geq -\frac{1}{2\sqrt{\mu}}\|v_1 - v_2\|_2^2 - \frac{\sqrt{\mu}}{2}\|u_1 - u_2\|_2^2.\end{aligned}$$

By integration by parts, we have the well-known result

$$\int_0^1 bw'w \,\mathrm{d}\lambda = -\int_0^1 b'w^2 \,\mathrm{d}\lambda - \int_0^1 bw'w \,\mathrm{d}\lambda$$

for $w \in H_0^1(0,1)$. Thus, we can deduce the estimate

$$\begin{aligned}
&a(u_1, v_1; u_1 - u_2, v_1 - v_2) - a(u_2, v_2; u_1 - u_2, v_1 - v_2) \\
&\quad = \int_0^1 \varepsilon\mu(u_1 - u_2)'^2 + \left(c - \tfrac{b'}{2}\right)\mu(u_1 - u_2)^2 + \varepsilon(v_1 - v_2)'^2 + \left(c - \tfrac{b'}{2}\right)(v_1 - v_2)^2 \\
&\qquad\quad + \left(\tilde{\Pi}(v_1) - \tilde{\Pi}(v_2) - (v_1 - v_2)\right)(u_1 - u_2)\,\mathrm{d}\lambda \\
&\quad \geq \varepsilon\mu|u_1 - u_2|_{1,2}^2 + \gamma\mu\|u_1 - u_2\|_2^2 + \varepsilon|v_1 - v_2|_{1,2}^2 + \gamma\|v_1 - v_2\|_2^2 \\
&\qquad\quad - \tfrac{\sqrt{\mu}}{2}\|u_1 - u_2\|_2^2 - \tfrac{1}{2\sqrt{\mu}}\|v_1 - v_2\|_2^2.
\end{aligned}$$

Using the assumption (2.39) we get

$$\begin{aligned}
&a(u_1, v_1; u_1 - u_2, v_1 - v_2) - a(u_2, v_2; u_1 - u_2, v_1 - v_2) \\
&\qquad \geq C\left(\|u_1 - u_2\|_\varepsilon^2 + \|v_1 - v_2\|_\varepsilon^2\right).
\end{aligned}$$ □

Theorem 2.17
Under the assumption (2.39) *the problem* (2.40) *and its discrete counterparts have a unique solution in the weak sense. The error of the semi-discrete problem corresponding to* (2.42) *– that is equivalent to the semi-discretization* (2.25) *– using linear finite elements on meshes that only account for the strong boundary layers (cf. Figure 1.1) is bounded by*

$$\|u - u^N\|_\varepsilon + \|v - v^N\|_\varepsilon \leq CN^{-1}\ln N. \tag{2.44}$$

Proof
From Lemma 2.16 we know that $a(\cdot,\cdot;\cdot,\cdot)$ is uniformly monotone. Together with the continuity of $a(\cdot,\cdot;\cdot,\cdot)$ this implies that $a(\cdot,\cdot;\cdot,\cdot)$ is hemicontinuous. Therefore, problem (2.42) and its discrete counterparts are uniquely solvable (cf. [Zei90]).

Obviously, we have $\tilde{\Pi}(0) = 0$ and $a(0,0;\varphi,\psi) = 0$. Thus, we get from Lemma 2.16 the estimate

$$a(u,v;u,v) = a(u,v;u,v) - a(0,0;u,v) \geq C\left(\|u\|_\varepsilon^2 + \|v\|_\varepsilon^2\right).$$

The new inhomogeneity $\tilde{f} := f - \mu^{-1}v_a$ of the primal equation satisfies the same bounds as f. Thus, we have a solution decomposition of the form (2.10) and can apply the techniques of the proof of Theorem 1.9 to derive the desired bounds for $u^N - u^I$ and $v^N - v^I$. □

2.4. Computational Results

We illustrate our theoretical results from the previous sections with some examples.

2.4.1. Example with a Known Good Approximation to the Solution

As a first example we consider the test problem

$$\min_{u,q\in\mathcal{L}^2} J(u,q) := \min_{u,q}\left(\frac{1}{2}\|u-1\|_2^2 + \frac{1}{20}\|q\|_2^2\right) \tag{2.45a}$$

subject to

$$-\varepsilon u'' + \sqrt{2}u' + 4u = e^x - 2x + q \text{ in } (0,1), \quad u(0) = u(1) = 0 \quad \text{and} \tag{2.45b}$$

$$q \in Q_{\text{ad}} := \left\{w \in \mathcal{L}^2(0,1) \middle| q_a \le w\right\} \tag{2.45c}$$

with

$$q_a := -\tfrac{17}{100}. \tag{2.45d}$$

The problem has constant coefficients and we have $c^2 = 16 > 10 = \mu^{-1}$. Thus, the theory from the previous sections can be applied.

The solution of this problem is given by

$$u(x) = -\varepsilon v''(x) + \sqrt{2}v'(x) + 4v(x), \qquad v(x) = \begin{cases} v_1(x), & x \le x_1, \\ v_2(x), & x_1 \le x \le x_2, \\ v_3(x), & x_2 \le x \end{cases}$$

with

$$\begin{aligned}
v_1(x) &= \tfrac{e^x}{8+(4-\varepsilon)^2} - \tfrac{x}{13} + D_1 e^{\kappa_1 x} + D_2 e^{\kappa_1(x_1-x)} + D_3 e^{\kappa_2 x} + D_4 e^{\kappa_2(x_1-x)}, \\
v_2(x) &= \tfrac{e^x}{-2+(4-\varepsilon)^2} - \tfrac{x+\frac{17}{200}}{8} + D_5 e^{\kappa_3(x-x_1)} + D_6 e^{\kappa_3(x_2-x)} + D_7 e^{\kappa_4(x-x_1)} + D_8 e^{\kappa_4(x_2-x)}, \\
v_3(x) &= \tfrac{e^x}{8+(4-\varepsilon)^2} - \tfrac{x}{13} + D_9 e^{\kappa_1(x-x_2)} + D_{10} e^{\kappa_1(1-x)} + D_{11} e^{\kappa_2(x-x_2)} + D_{12} e^{\kappa_2(1-x)},
\end{aligned}$$

$$\kappa_1 = -\varepsilon^{-1}\sqrt{4\varepsilon+1-\sqrt{8\varepsilon+1-10\varepsilon^2}}, \qquad \kappa_2 = -\varepsilon^{-1}\sqrt{4\varepsilon+1+\sqrt{8\varepsilon+1-10\varepsilon^2}},$$
$$\kappa_3 = -\varepsilon^{-1}\sqrt{4\varepsilon+1-\sqrt{8\varepsilon+1}}, \qquad \kappa_4 = -\varepsilon^{-1}\sqrt{4\varepsilon+1+\sqrt{8\varepsilon+1}}$$

for some unknown parameters x_i and D_i. The solution must furthermore satisfy

$$v \in \mathcal{C}^4, \quad v(0) = u(0) = 0, \quad v(1) = u(1) = 0, \quad v\big|_{[x_1,x_2]} \ge \tfrac{17}{1000}, \quad v\big|_{[0,x_1]\cup[x_2,1]} \le \tfrac{17}{1000}.$$

Thus, the solution satisfies

$$v(0) = 0 \qquad u(0) = 0 \qquad v(1) = 0 \qquad u(1) = 0, \tag{2.46a}$$
$$v_1(x_1) = \tfrac{17}{1000} \qquad v_2(x_1) = \tfrac{17}{1000} \qquad v_2(x_2) = \tfrac{17}{1000} \qquad v_3(x_2) = \tfrac{17}{1000}, \tag{2.46b}$$
$$v_1'(x_1) = v_2'(x_1), \quad v_1''(x_1) = v_2''(x_1), \quad v_1'''(x_1) = v_2'''(x_1), \tag{2.46c}$$
$$v_2'(x_2) = v_3'(x_2), \quad v_2''(x_2) = v_3''(x_2), \quad v_2'''(x_2) = v_3'''(x_2). \tag{2.46d}$$

The relations $v_1^{(4)}(x_1) = v_2^{(4)}(x_1)$ and $v_2^{(4)}(x_2) = v_3^{(4)}(x_2)$ follow from $v \in \mathcal{C}^3$ and the differential equations. However, the requirements $v\big|_{[x_1,x_2]} \ge \frac{17}{1000}$ and $v\big|_{[0,x_1]\cup[x_2,1]} \le \frac{17}{1000}$ are neglected in the equations (2.46), but it shows later on that these conditions are satisfied for the values of ε we use.

The nonlinear system (2.46) is difficult to solve analytically. Thus, we decided to use Newton's method to acquire the solution of system (2.46). Unfortunately, the required

Jacobians have a high condition number (about 10^{19} for $\varepsilon = 10^{-8}$), at least for small ε. We circumvented this problem by using MATLAB's *variable-precision arithmetic* to do all calculations with about 64 decimal digits. As a stopping criterion for the Newton iterations we used

$$\|F(D_1^i, \dots, D_{12}^i, x_1^i, x_2^i)\| + \|(D_1^i - D_1^{i+1}, \dots, D_{12}^i - D_{12}^{i+1}, x_1^i - x_1^{i+1}, x_2^i - x_2^{i+1})\| \le 10^{-32}.$$

This gives a very good approximation to the analytic solution of our test problem. A plot of the solution is given in Figure 2.1. For better visualization we stretched the region of

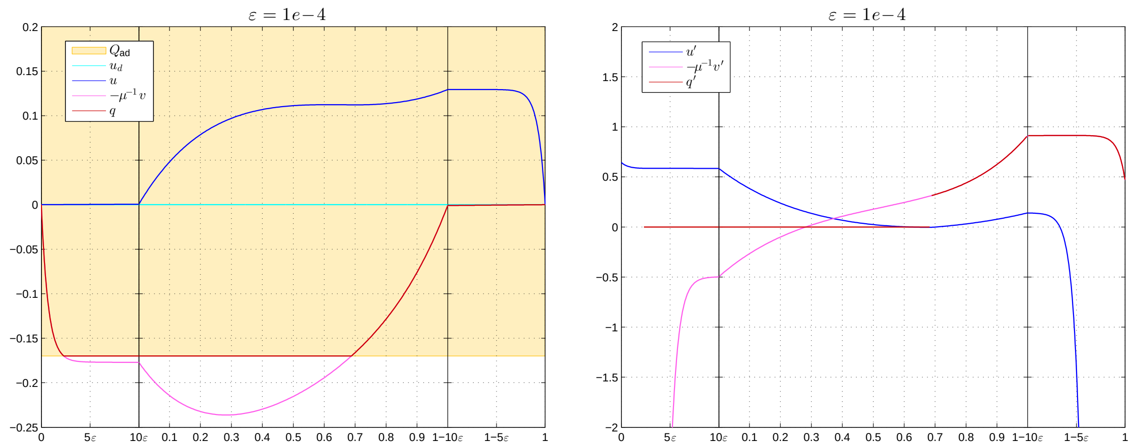

Figure 2.1.: Solution to model problem (2.45)

the boundary layers in the plots. Note, that the wedges in the plots of the solution at 10ε and $1 - 10\varepsilon$ are a result of this stretching. As in Chapter 1 we perceive the strong layers of u and v at $x = 1$ and $x = 0$, respectively. Also we notice the weak layers at the opposing side of the domain in u and v. Finally, we observe that there is no layer in the vicinity of the boundary of the active set at $x_1 \approx 2.3\varepsilon$ and $x_2 \approx 0.69$ in u or v.

First we use Shishkin meshes which only account for the strong boundary layers. The number of degrees of freedom is denoted by N which gives us $N + 1$ mesh intervals. We solve the problem

$$\begin{aligned} u^N \in V^N &: a^N(u^N, w) = \langle f + q^N, w\rangle \text{ for all } w \in V^N, \\ v^N \in W^N &: a^N(w, v^N) = \langle u^N - u_d, w\rangle \text{ for all } w \in W^N, \\ q^N \in W^N &: q^N(x_i) = \min\left\{\max\left\{-\mu^{-1}v^N(x_i), q_a(x_i)\right\}, q_b(x_i)\right\} \end{aligned}$$

that corresponds to full-discrete scheme (2.33) for $W^N = V^N$ where we enforce the upper and lower bound of the control only in the mesh points. We solve this problem by using an active set algorithm (cf. [Tr10]). From these computations we attained the results shown in Figure 2.2.

Next we use a semismooth Newton's method (cf. [HPUU09]) to compute a numeric solution of problem (2.25) which fulfills the projection everywhere in Ω.

This algorithm has to evaluate integrals with integrands of very low regularity, i.e. integrands with wedges (jumps in the first derivative) or even jumps. To accomplish this, we use an adaptive Simpson rule algorithm. In case of a wedge the convergence order drops to one, so it is slow but works. In case of a jump this algorithm may not converge

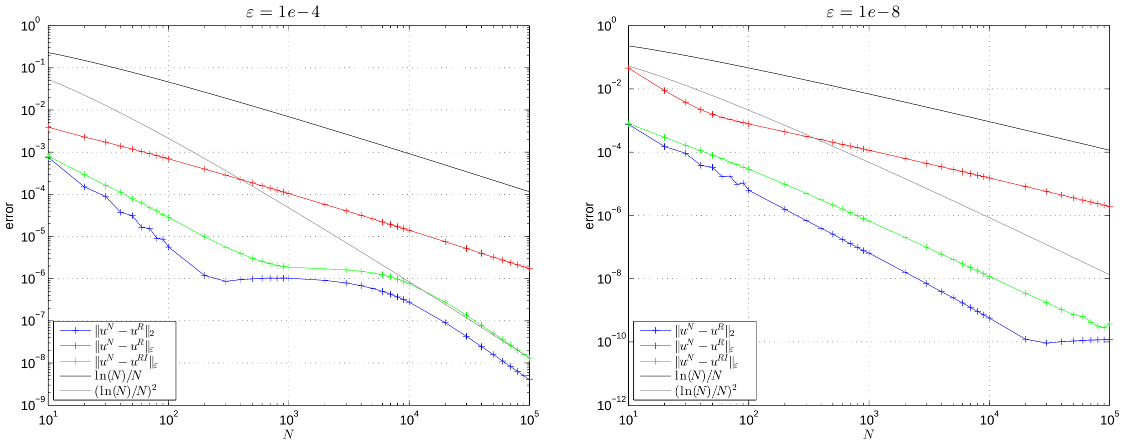

Figure 2.2.: Error of the linear FEM satisfying the control constraints in the grid points, on one-sided Shishkin meshes (cf. Figure 1.1)

at all. To circumvent this problem we add a jump detection to the algorithm, for details see Section A.3.

The results are shown in Figure 2.3.

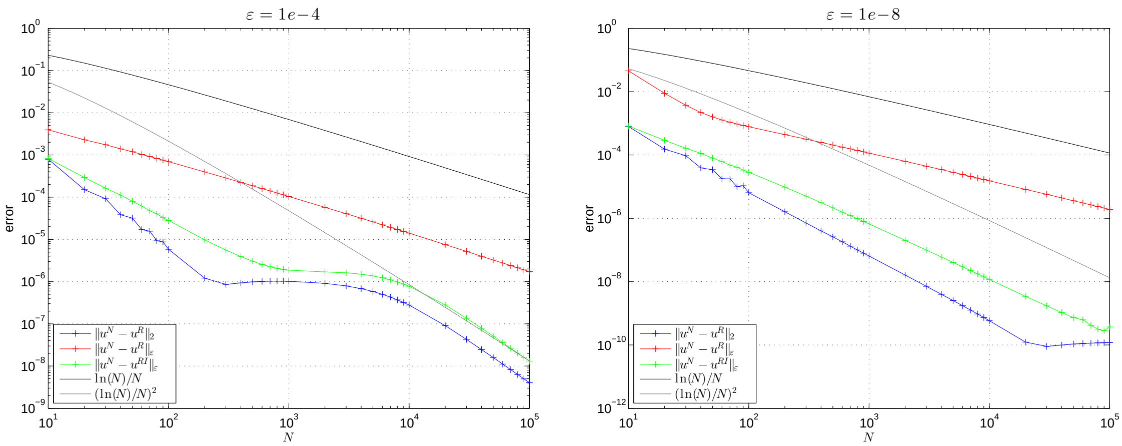

Figure 2.3.: Error of the linear FEM satisfying the control constraints everywhere, on one-sided Shishkin meshes (cf. Figure 1.1)

These numerical results confirm the theoretical findings of an ε-independent convergence in the ε-norm for the semi-discrete algorithm. We also note that even the first algorithm shows comparable convergence, although we were not able to prove this. Furthermore, we observe a behavior of the $\mathcal{L}^2$-error $\|u^N - u^R\|_2$ and the superconvergence error $\|u^N - u^{R^I}\|_\varepsilon$ that are very similar to the convergence of the corresponding errors in the previous chapter. We observe a range of stagnating convergence when the error has the order of magnitude of $10^{-2}\varepsilon$.

Next we consider a Shishkin mesh where we refine in the region of the weak boundary layers as well as in the region of the strong ones. The results of this computations are presented in Figures 2.4 and 2.5.

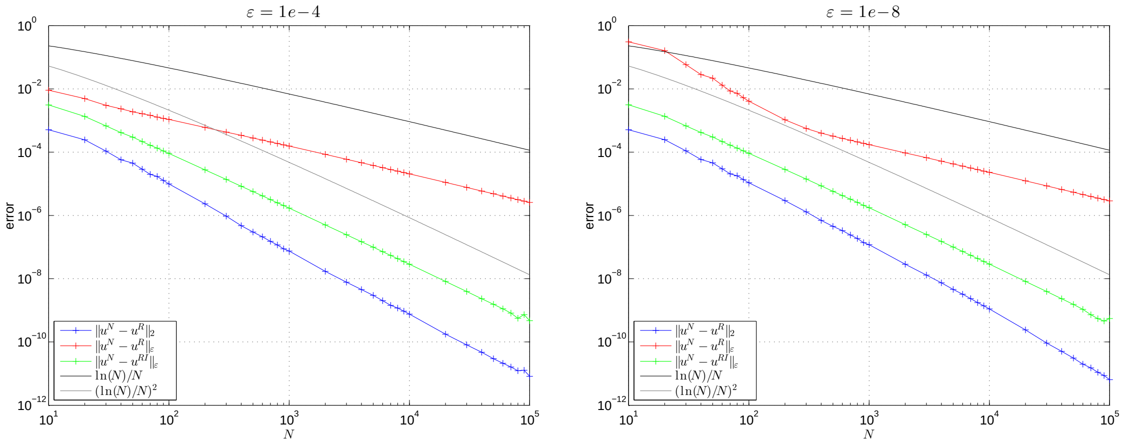

Figure 2.4.: Error of the linear FEM satisfying the control constraints in the grid points, on a two-sided Shishkin mesh (cf. Figure 1.2)

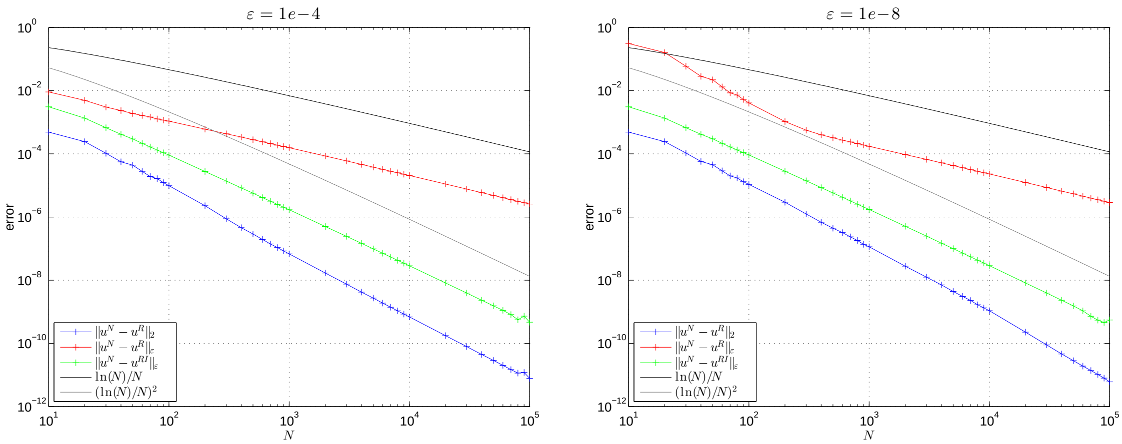

Figure 2.5.: Error of the linear FEM satisfying the control constraints everywhere, on a two-sided Shishkin mesh (cf. Figure 1.2)

As in the previous chapter we now get almost second order convergence in the $\mathcal{L}^2$-norm and the range of stagnating convergence does not exist. The same is true for the superconvergence error $\|u^N - u^{RI}\|_\varepsilon$.

These results indicate that for first order FEM it is not necessary to use the continuous projection. This very costly method improves the quality of the results only marginally, but from a theoretical point of view, this technique is rather helpful.

2.4.2. A more Complex Example

Now we consider a second example with a different admissable set and corresponding projection

$$\min_{u,q\in\mathcal{L}^2} J(u,q) := \min_{u,q}\left(\frac{1}{2}\|u-1\|_2^2 + \frac{1}{20}\|q\|_2^2\right) \tag{2.47a}$$

subject to

$$-\varepsilon u'' + \sqrt{2}u' + 4u = 5 + q \text{ in } (0,1), \quad u(0) = u(1) = 0 \quad \text{and} \tag{2.47b}$$
$$q \in Q_{\text{ad}} := \left\{ w \in \mathcal{L}^2(0,1) \middle| q_a \le q \le q_b \right\} \tag{2.47c}$$

with

$$q_a := x - e^{-4(1-x)} - \tfrac{1}{2}, \tag{2.47d}$$
$$q_b := \tfrac{1}{2} - \tfrac{3}{8}\sin(4\pi x) + \max\left(\tfrac{1}{4}\cos(16\pi x), 0\right). \tag{2.47e}$$

As in the first example we are able to apply our theory. The upper bound q_b is chosen in a fashion that it is active in its convex and its concave regime – thus, it seems unlikely to get positive side effects from $Q^N_{\text{ad}} \subseteq Q_{\text{ad}}$ or $q^N \in Q_{\text{ad}}$ in the full-discrete scheme (2.33).

Due to the complex structure – especially the nonlinearity from the projection – we do not know an exact solution to the problem (2.47). To overcome this problem we use a reference solution on a relatively fine grid (two-sided Shishkin mesh, $N = 10^7$) to compute the numerical errors in various norms. Because this grid is very fine we use the double-double-precision number class from the *QD-library* [HLB08] to avoid the pollution of the reference solution by round-off errors. Such a reference solution is depicted in Figure 2.6.

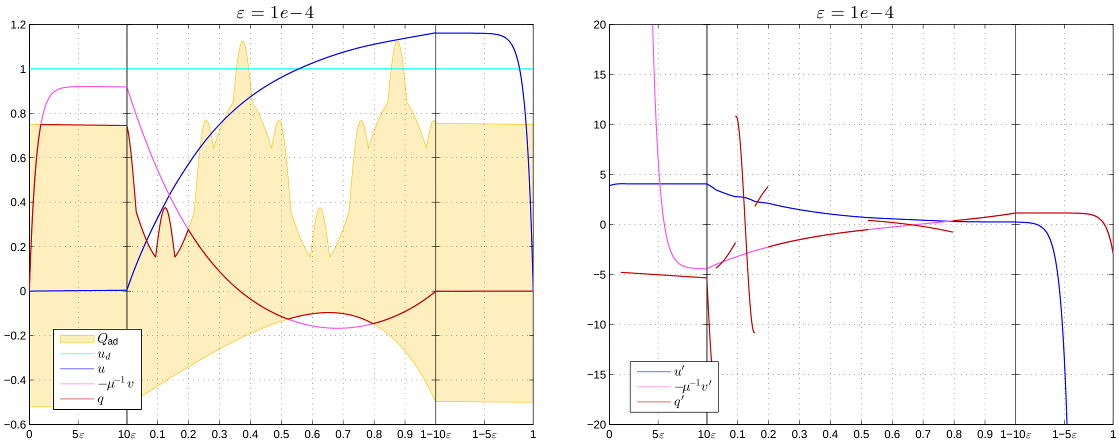

Figure 2.6.: Reference solution to model problem (2.47)

As in the first example we apply all four versions of our algorithm. The attained results are shown in Figures 2.7 to 2.10.

These numerical results correspond to the results we attained for the first example (2.45). Again, we see the ε-independent convergence in the ε-norm and the range of stagnating convergence in the $\mathcal{L}^2$-error for the one-sided Shishkin meshes. For the two-sided Shishkin mesh, we even see second order convergence in the $\mathcal{L}^2$-error.

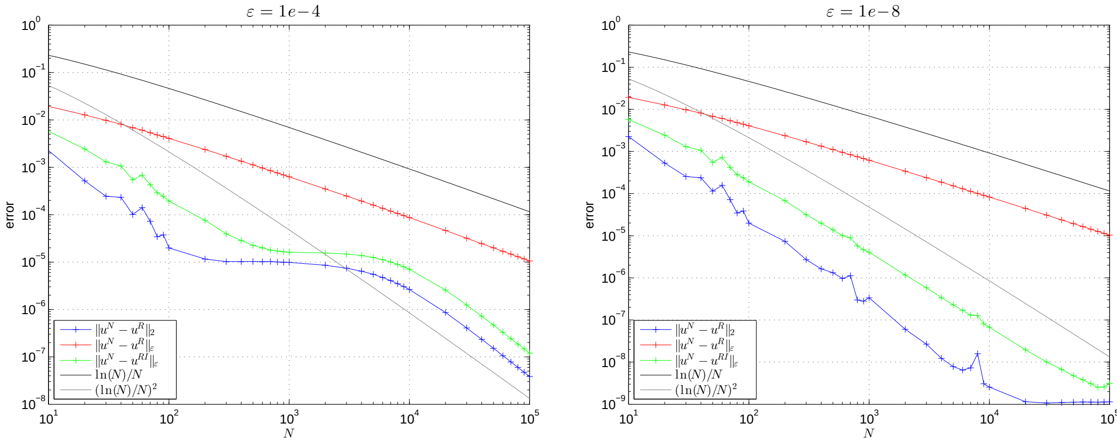

Figure 2.7.: Error of the linear FEM satisfying the control constraints in the grid points, on one-sided Shishkin meshes (cf. Figure 1.1)

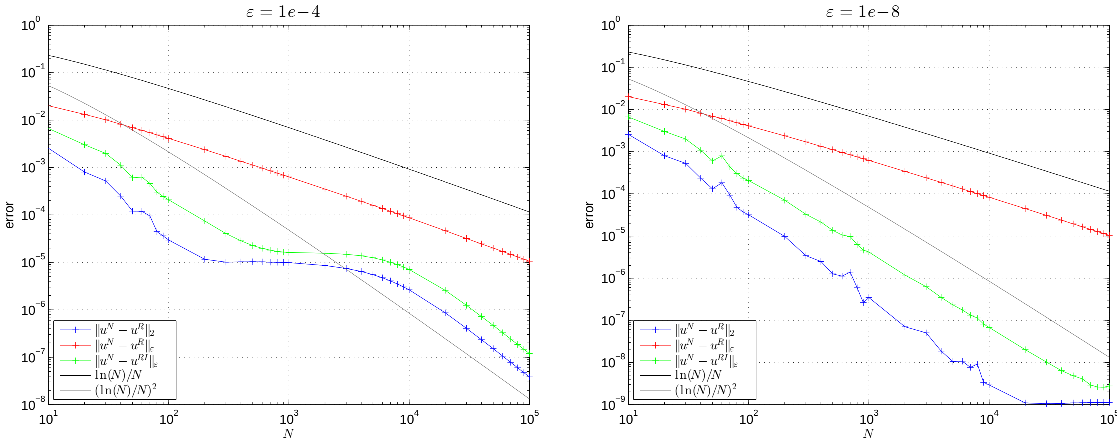

Figure 2.8.: Error of the linear FEM satisfying the control constraints everywhere, on one-sided Shishkin meshes (cf. Figure 1.1)

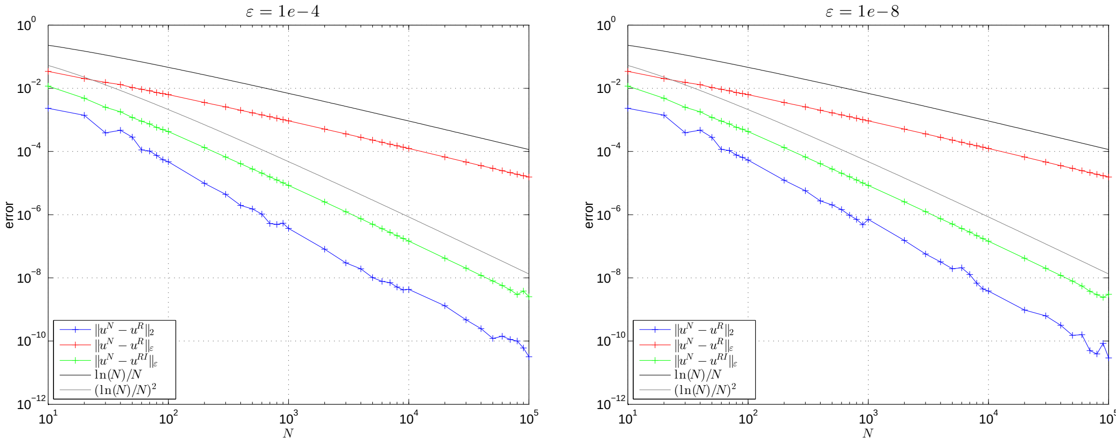

Figure 2.9.: Error of the linear FEM satisfying the control constraints in the grid points, on a two-sided Shishkin mesh (cf. Figure 1.2)

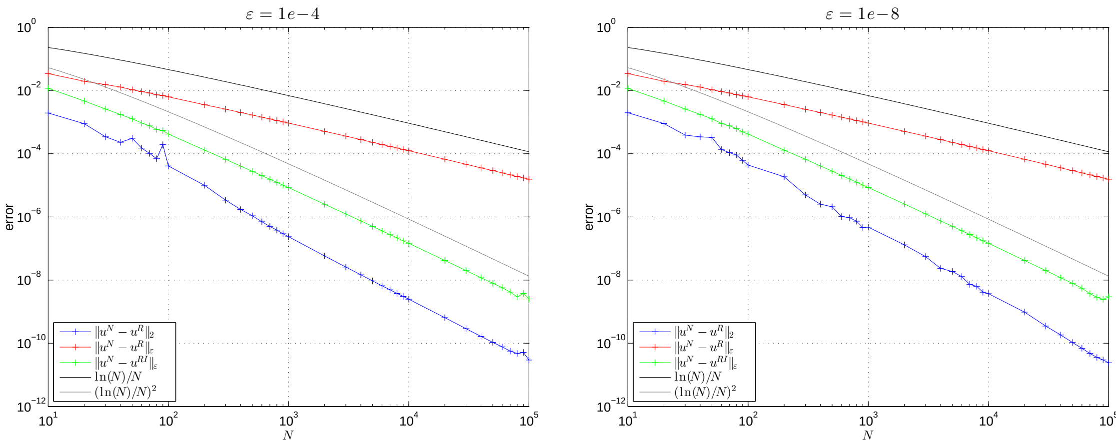

Figure 2.10.: Error of the linear FEM satisfying the control constraints everywhere, on a two-sided Shishkin mesh (cf. Figure 1.2)

3. A Singularly Perturbed Convection-Diffusion Equation with Low Regularity of the Inhomogeneity in 2D

In this chapter we consider the problem

$$Lu = -\varepsilon\Delta u + \beta u_x + cu = f \text{ in } \Omega := (0,1)^2, \quad u\big|_{\partial\Omega} = 0. \tag{3.1}$$

We analyze the properties of its solution u under the relatively weak assumption

$$f \in W^{1,\infty}(\Omega). \tag{3.2}$$

From the standard regularity theory for elliptic equations (cf. [GT01, LU68]) we immediately get $u \in H_0^1(\Omega) \cap W^{2,2}(\Omega)$, although the corresponding norms may not be bounded ε-uniformly. But a proper decomposition of u with estimates of its terms will be sufficient to prove satisfactory convergence results for the problem on a layer adapted mesh.

As we have seen in Chapter 2 the optimal state of a special optimal control problem satisfies an equation similar to (3.1). Its right hand side f includes the projection of the adjoint state. This projection entails a low regularity of f and may impair the properties of the optimal state. Therefore, we are interested in the properties of the solution of problem (3.1).

We assume for the data

$$\varepsilon \in (0,\beta] \cap (0,\tfrac{1}{2}], \tag{3.3a}$$
$$\beta \in (0,\infty), \tag{3.3b}$$
$$c \in \mathcal{C}^2\big((0,\infty)\times\mathbb{R}\big), \tag{3.3c}$$
$$c \geq \gamma > 0, \tag{3.3d}$$
$$\|f\|_{1,\infty} \leq C. \tag{3.3e}$$

Note we require the function c to be defined and meet the requirements on $(0,\infty)\times\mathbb{R}$. The prerequisite $\varepsilon \leq \frac{1}{2}$ is only for simplifying the notation for we can use the fact $|\ln(\varepsilon)| \geq |\ln(\frac{1}{2})| > \frac{1}{2}$.

The assumption that the coefficient of the convective term has to be constant is required because we use the explicitly known Green's function of the differential equation

$$\hat{L}u := -\varepsilon\Delta u + \beta u_x = f$$

to derive various estimates. It may be possible to extend this results to the Green's function of the problem with a non-constant coefficient as was done for related estimates in [FK12].

In the following we construct a decomposition of the solution u. Unfortunately, we are not able to prove sharp bounds for all derivatives of one of the used terms. However, we present a numeric example that motivates that the term might behave nicely. The proofs presented use very technical estimates of norms of Green's functions. These estimates are presented and used in in Sections 3.1.1 to 3.1.3. Some of the integrals used in this sections are derived in Section A.1. Subsequently, we use this decomposition to prove almost linear convergence of the standard FEM on a layer adapted Shishkin mesh. Finally, we present computational results to confirm our theoretical results.

3.1. Analytic Properties of the Solution

Theorem 3.1
The solution u of problem (3.1) *can be decomposed into four parts* $u = u^S\big|_\Omega + u^{x1}\big|_\Omega + u^y\big|_\Omega + u^c1\big|_\Omega$. *The smooth part u^S satisfies*

$$\|u^S\|_{1,\infty,\mathcal{H}} + \sqrt{\varepsilon}\|u^S_{xx}\|_{\infty,\mathcal{S}} + \sqrt{\varepsilon}\|u^S_{yy}\|_{\infty,\mathcal{H}} \le C, \tag{3.4a}$$

$$\|u^S_{xx}\|_{2,\Omega} + \|u^S_{xy}\|_{\infty,\mathcal{H}} \le C|\ln(\varepsilon)| \tag{3.4b}$$

with $\mathcal{H} := (0,\infty) \times \mathbb{R}$ *and* $\mathcal{S} := (0,1) \times \mathbb{R}$. *The outflow layer part u^{x1} meets*

$$|u^{x1}(x,y)| + |u^{x1}_y(x,y)| + \sqrt{\varepsilon}|u^{x1}_{yy}(x,y)| \le C\mathcal{E}^x_1(x), \tag{3.4c}$$

$$\varepsilon|u^{x1}_x(x,y)| + \varepsilon^2|u^{x1}_{xx}(x,y)| \le C\mathcal{E}^x_1(x), \quad \|u^{x1}_{xy}(x,\cdot)\|_{2,(0,1)} \le \frac{C}{\varepsilon}\mathcal{E}^x_1(x). \tag{3.4d}$$

The characteristic layer part u^y satisfies

$$|u^y(x,y)| + |u^y_x(x,y)| + \sqrt{\varepsilon}|u^y_y(x,y)| + \varepsilon|u^y_{yy}(x,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big) \tag{3.4e}$$

and the corner layer part u^{c1} satisfies

$$|u^{c1}(x,y)| + \varepsilon|u^{c1}_x(x,y)| + \sqrt{\varepsilon}|u^{c1}_y(x,y)| \le C\mathcal{E}^x_1(x)\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big), \tag{3.4f}$$

$$\|u^{c1}_{xx}\|_{2,\Omega} \le C\varepsilon^{-\frac{5}{4}}, \quad \|u^{c1}_{xy}\|_{2,\Omega} \le C\varepsilon^{-\frac{3}{4}} \quad \text{and} \quad \|u^{c1}_{yy}\|_{2,\Omega} \le C\varepsilon^{-\frac{1}{4}}. \tag{3.4g}$$

Proof
The line of argumentation is very lengthy and technical. Therefore, we give at this point only an overview. The details are presented in the following subsections.

We define the smooth part u^S by a continuation of the problem (3.1) to the half plane $\mathcal{H} = (0,\infty) \times \mathbb{R}$ via

$$Lu^S = -\varepsilon\Delta u^S + \beta u^S_x + cu^S = \omega_B \mathfrak{E}^{\mathcal{H}}_\Omega f =: f^u \text{ in } \mathcal{H}, \tag{3.5a}$$

$$u^S(0,\cdot) = 0, \quad \lim_{\|(x,y)\|\to\infty} u^S(x,y) = 0, \tag{3.5b}$$

where $\omega_B \in \mathcal{C}^\infty(\mathcal{H})$ is a suitable cut-off function with $\omega_B\big|_\Omega = 1$ and $\omega_B\big|_{\mathcal{H}\setminus B_2(0,0)} = 0$. Note we have $\|f^u\|_{1,\infty,\mathcal{H}} \le C\|f\|_{1,\infty,\Omega}$ and $\operatorname{supp}(f^u) \subseteq B_2(0,0)$. The bounds stated in the theorem are derived in Lemma 3.20.

Next, we define a layer correction term u^{x1} to compensate the neglected boundary condition at $x = 1$. To this end we consider the problem

$$Lu^{x1} = 0, \quad u^{x1}(0,\cdot) = 0, \quad u^{x1}(1,\cdot) = -\omega_I(\cdot)u^S(1,\cdot) \tag{3.6}$$

on the stripe $\mathcal{S} = (0,1) \times \mathbb{R}$, where $\omega_I \in \mathcal{C}^\infty(\mathbb{R})$ is a cut-off function with $\operatorname{supp}\omega_I \subseteq [-2,3]$, $\omega_I\big|_{[0,1]} = 1$ and $|\omega_I| \leq 1$. By Lemma 3.21 u^{x1} satisfies the stated bounds.

The correction term u^y that accounts for the so far ignored Dirichlet boundary conditions at $y \in \{0,1\}$ is defined via

$$Lu^y = 0, \quad u^y(0,\cdot) = 0, \quad u^y(\cdot,0) = -u^S(\cdot,0)\omega_I(\cdot), \quad u^y(\cdot,1) = -u^S(\cdot,1)\omega_I(\cdot). \tag{3.7}$$

Since we have $u^S(0,\cdot) = 0$ the imposed boundary conditions of u^y are continuous. The estimates are provided in Lemma 3.22.

As a last step we define a corner layer by

$$Lu^{c1} = 0, \quad u^{c1}(0,\cdot)\big|_{x=0} = 0, \quad u^{c1}\big|_{y\in\{0,1\}} = -u^{x1}, \quad u^{c1}\big|_{x=1} = -u^y. \tag{3.8}$$

Note that the boundary conditions posed on u^{c1} are continuous because the constructions of u^{x1} and u^y yield $-u^{x1}(1,0) = u^S(1,0) = -u^y(1,0)$ and $-u^{x1}(1,1) = u^S(1,1) = -u^y(1,1)$. The stated bounds are derived in Lemma 3.23 $\square$

Remark 3.2
The bound for u^S_{yy} seems to be to loose, but consider a case where the solution can be written as the product of two functions $u^S(x,y) = \varphi(x)\psi(y)$. This induces a behavior in y-direction that is similar to the solution of

$$-\varepsilon\psi_{yy} + c\psi = g, \quad \psi(0) = \psi(1) = 0.$$

The solution to this problem for $c = 4$ and $g(y) = 4 - 4\,|1-2y|$ is given by

$$\psi(y) = \begin{cases} 2y - \sqrt{\varepsilon}\dfrac{e^{\frac{2y-1}{\sqrt{\varepsilon}}} - e^{-\frac{2y+1}{\sqrt{\varepsilon}}}}{1+e^{-\frac{2}{\sqrt{\varepsilon}}}}, & y \leq \frac{1}{2}, \\[2ex] 2 - 2y + \sqrt{\varepsilon}\dfrac{e^{\frac{2y-3}{\sqrt{\varepsilon}}} - e^{-\frac{2y-1}{\sqrt{\varepsilon}}}}{1+e^{-\frac{2}{\sqrt{\varepsilon}}}}, & y \leq \frac{1}{2}. \end{cases}$$

The second derivative of ψ is

$$\psi''(y) = \begin{cases} \dfrac{4}{\sqrt{\varepsilon}}\dfrac{e^{\frac{2y-1}{\sqrt{\varepsilon}}} - e^{-\frac{2y+1}{\sqrt{\varepsilon}}}}{1+e^{-\frac{2}{\sqrt{\varepsilon}}}}, & y \leq \frac{1}{2}, \\[2ex] \dfrac{4}{\sqrt{\varepsilon}}\dfrac{e^{\frac{2y-3}{\sqrt{\varepsilon}}} - e^{-\frac{2y-1}{\sqrt{\varepsilon}}}}{1+e^{-\frac{2}{\sqrt{\varepsilon}}}}, & y \leq \frac{1}{2}. \end{cases}$$

Obviously, this is of order $\varepsilon^{-1/2}$ in the vicinity of the wedge in the inhomogeneity g at $y = \frac{1}{2}$. This indicates that u^S_{yy} is of order $\varepsilon^{-1/2}$ near wedges in the inhomogeneity that are parallel to the convection. Thus, the $\mathcal{L}^\infty$-bound of u^S_{yy} may be sharp.

Conjecture 3.3
We assume to have

$$\|u^y_{xx}\|_{2,\Omega} \leq C|\ln(\varepsilon)|\varepsilon^{\frac{1}{4}}. \tag{3.9}$$

Remark 3.4
Knowing the usual pointwise results (e.g. [KS05, Lin10, RST08]) for layer terms of the structure above it is reasonable to assume something like

$$u^y_{xx}(x,y) \sim u^y_{xx}(x,0)\mathcal{E}^y_0(y) + u^y_{xx}(x,1)\mathcal{E}^y_1(y)$$

and we would get the assumed bound from the previous estimates.

A further indication that the assumption may be valid is given by the numerical example that follows. We consider the problem

$$-\varepsilon\Delta u + u_x + u = 0 \quad \text{in} \quad (0,10)\times(0,1),$$
$$u(x,0) = \nu(x), \quad u(x,1) = 0, \quad u(0,y) = u(10,y) = 0,$$
$$\nu(x) := \begin{cases} \nu^1(x), & x \in [0,2\varepsilon], \\ \nu^2(x-2\varepsilon), & x \in (2\varepsilon, 1+2\varepsilon], \\ \nu^3(x-1-2\varepsilon), & x \in (1+2\varepsilon, 2+2\varepsilon], \\ 0, & x \in (2+2\varepsilon, 10] \end{cases}$$

with the functions

$$\nu^1(x) := x + \frac{x^2\sin\left(1+\frac{x}{\varepsilon}\right)}{\sqrt{\varepsilon}},$$
$$\tilde{\nu}^2(\xi) := \sin(\xi) + \varepsilon^2\sin\left(\frac{\pi\xi}{\varepsilon}\right) - \ln(\varepsilon)(\xi+2\varepsilon)^2\ln(\xi+2\varepsilon) + \frac{|\sin(10\pi\xi)|\sin(10\pi\xi)}{10},$$
$$\nu^2(\xi) := \tilde{\nu}^2(\xi) - \tilde{\nu}^2(0) + \nu^1(2\varepsilon) + (1-e^{-\xi})\left(\nu^1_x(2\varepsilon) - \tilde{\nu}^2_\xi(0)\right),$$
$$\nu^3(\xi) := \nu^2(1)(\xi-1)^2(2\xi+1) + \nu^2_\xi(1)\xi(\xi-1)^2.$$

The function ν defined above meets the bounds we have for $u^S(x,y)\omega_I(x)$ and it violates further smoothness in several ways. The second derivative ν_{xx} is discontinuous at $\left\{2\varepsilon + \frac{k}{10} \middle| k \in \{0,1,\cdots,10\}\right\} \cup \{2\}$ and in the remaining domain the third derivative ν_{xxx} is of order $\varepsilon^{-3/2}$.

We solve this problem numerically using the finite element method on a special tensor grid. As basis functions we use the products of third order Hermite base in x-direction and second order Lagrangian base in y-direction, thus we get an element with the degrees of freedom depicted in Figure 3.1. As a consequence the solution u^{yN}

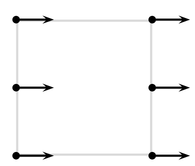

Figure 3.1.: Element used for the numeric example for u^y

and its first order x-derivative are continuous. Therefore, the second derivative in x-direction is in $\mathcal{L}^\infty \subseteq \mathcal{L}^2$ and we can evaluate its $\mathcal{L}^2$-norm by elementwise integration.

For the computations we use a Shishkin type mesh. To be on the safe side we refine at $y=0$ and $y=1$ to account for possible layer terms of the form $e^{-y/\varepsilon}$ and the form $e^{-y/\sqrt{\varepsilon}}$. In x-direction we use nested Shishkin grids at $x=0$ and $x=10$ also

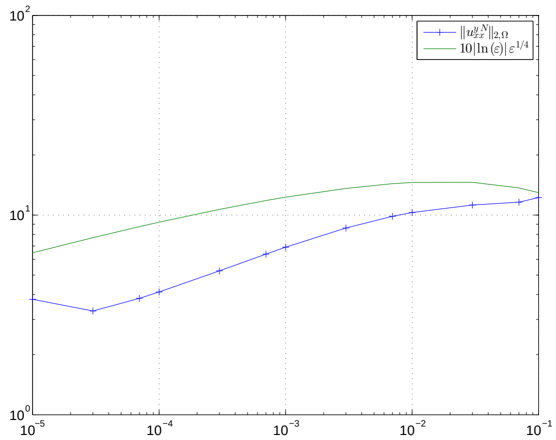

Figure 3.2.: $\mathcal{L}^2$-norm of u^{yN}_{xx} for the numerical example for u^y

for terms of the form $e^{-x/\varepsilon}$ and $e^{-x/\sqrt{\varepsilon}}$ and additionally we use the locations of the discontinuities of ν_{xx} as mesh points. The results are depicted in Figure 3.2.

These results support our conjecture. Unfortunately, we can only compute u^{yN}_{xx} for relatively large $\varepsilon \geq CN_x^{-1}$. For smaller ε the solution shows strange oscillations in the derivatives. The attempt to resolve these oscillations better by refining the grid locally leads to vanishing of the oscillations on one hand, but induces oscillations at the rim of the refinement. Therefore, we believe this is a problem due to numerical issues.

It remains to show a simple consequence of our conjecture for the norm of u^y_{xx} and the estimate we have for u^y_{yy}:

Lemma 3.5
If we assume Conjecture 3.3 holds we have

$$\|u^y_{xy}\|_{2,\Omega} \leq C|\ln(\varepsilon)|\varepsilon^{-\frac{1}{4}}. \tag{3.10}$$

Proof
We consider the solution of problem (3.7) in the domain $\Omega = (0,1)^2$. We can split it as $u^y = \tilde{u} + \hat{u}$ with

$$\hat{u}(x,y) := u^y(1,y)x - \frac{\omega(x)\Big(u^S(x,0)\Big(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}(2-y)}\Big) + u^S(x,1)\Big(e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1+y)}\Big)\Big)}{1 - e^{-2\sqrt{\frac{\gamma}{\varepsilon}}}}.$$

Using the variable transform $\tilde{x} := x$, $\tilde{y} := \frac{y}{\sqrt{\varepsilon}}$, $\tilde{\Omega} := (0,1) \times (0, \frac{1}{\sqrt{\varepsilon}})$ we get

$$\|\tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} = \varepsilon^{\frac{3}{4}}\|\tilde{u}_{yy}\|_{2,\Omega}, \quad \|\tilde{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}} = \varepsilon^{-\frac{1}{4}}\|\tilde{u}_{xx}\|_{2,\Omega} \quad \text{and} \quad \tilde{u}\big|_{\partial\Omega} = 0.$$

Thus, the estimates above and a usual norm estimate for the second order derivatives of the solution to the Laplace equation (cf. [LU68]) give

$$\|\tilde{u}_{xy}\|_{2,\Omega} = \varepsilon^{-\frac{1}{4}}\|\tilde{u}_{\tilde{x}\tilde{y}}\|_{2,\tilde{\Omega}} \leq \varepsilon^{-\frac{1}{4}}\left(\|\tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} + \|\tilde{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}}\right) \leq C|\ln(\varepsilon)|\varepsilon^{-\frac{1}{4}}.$$

A triangle inequality completes the proof. □

Remark 3.6
The bounds above raise the question: What happend to the corner layers from the probably violated first compatibility condition $f(0,0) = f(0,1) = 0$? Numerical experiments and the publication [Vo65] suggest that only the mixed derivative u_{xy} tends to infinity for $(x,y) \to (0,0)$ and $(x,y) \to (0,1)$. Nevertheless, the $\mathcal{L}^2$-norm of u_{xy} is bounded (probably not ε-uniform).

Subsequently, we present the details of the proof of Theorem 3.1.

3.1.1. Estimates for Some Half Plane Problems

In the definition of the smooth part (3.5) we use a half plane problem of the form

$$Lu = -\varepsilon\Delta u + \beta u_x + cu = f \text{ in } \mathcal{H}, \quad u(0,\cdot) = \nu, \quad \lim_{\|(x,y)\|\to\infty} u(x,y) = 0, \tag{3.11}$$

where we have $\lim_{|x|\to\infty} \nu(x) = 0$. Therefore, we establish some properties for this problem in the following.

Lemma 3.7
The solution u of problem (3.11) *satisfies*

$$\|u\|_\infty \le \max\left\{\frac{1}{\gamma}\|f\|_\infty, \|\nu\|_\infty\right\}. \tag{3.12}$$

Proof
Since we have $\lim_{|x|\to\infty} \nu(x) = 0$ and $\lim_{\|(x,y)\|\to\infty} u(x,y) = 0$, there is a ball $B_\tau(0,0)$ with

$$\|u\|_{\infty,\mathcal{H}\setminus B_\tau(0,0)} \le \max\left\{\frac{1}{\gamma}\|f\|_\infty, \|\nu\|_\infty\right\}.$$

Thus, we can apply a maximum principle (cf. [GT01]) on $B_\tau(0,0)$ with the comparison function $w^c = \gamma^{-1}$ to get

$$\|u\|_{B_\tau(0,0)} \le \max\left\{\frac{1}{\gamma}\|f\|_\infty, \|\nu\|_\infty\right\}$$

and we have proved the lemma. □

Lemma 3.8
For $\operatorname{supp}(f) \subseteq B_\tau(0,0)$*,* $|\nu| \le C$ *and* $\operatorname{supp}(\nu) \subseteq (-\tau,\tau)$ *the solution u of problem* (3.11) *satisfies*

$$|u(x,y)| \le Ce^{-\alpha\varrho}, \quad \varrho := \sqrt{x^2+y^2}, \quad \alpha := -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\gamma}{\beta}} > 0. \tag{3.13}$$

Proof
We consider the domain $\mathcal{H} \setminus \overline{B_\tau(0,0)}$. In this domain u satisfies the homogeneous differential equation with inhomogeneous but bounded (cf. Lemma 3.7) boundary conditions on $\mathcal{H} \cap \partial B_\tau(0,0)$. For the comparison function $w^c = e^{-\alpha\varrho} > 0$ we have

$$Lw^c = \left(\tfrac{\varepsilon\alpha}{\varrho} - \varepsilon\alpha^2 - \tfrac{\beta\alpha x}{\varrho} + c\right) e^{-\alpha\varrho} \ge \left(-\beta\alpha^2 - \beta\alpha + \gamma\right) e^{-\alpha\varrho} = 0.$$

In combination with Lemma 3.7 we have established

$$|u| \le \max\left\{\frac{1}{\gamma}\|f\|_\infty, \|\nu\|_\infty\right\} e^{\alpha\tau} w^c = Ce^{-\alpha\varrho}.$$

□

An important tool in our analysis is a Green's function g of

$$\hat{L}u := -\varepsilon\Delta u + \beta u_x = f \text{ in } \mathcal{H}, \quad u(0,\cdot) = \nu, \quad \lim_{\|(x,y)\|\to\infty} u(x,y) = 0. \tag{3.14}$$

that is given by (cf. [FK12, RST08])

$$\begin{aligned} g(x,y;\xi,\eta) &:= \frac{1}{2\pi\varepsilon} e^{q\frac{\varphi}{\varepsilon}} \left[K_0\left(q\frac{r^{[x]}}{\varepsilon}\right) - K_0\left(q\frac{r^{[-x]}}{\varepsilon}\right)\right] \quad \text{with} \\ q := \tfrac{\beta}{2}, \quad r^{[s]} &:= \sqrt{(s-\xi)^2+\psi^2}, \quad \varphi := x-\xi, \quad \psi := y-\eta, \end{aligned} \tag{3.15}$$

where K_i denotes the *modified Bessel functions of the second kind* of order i. Thus, we have

$$u(x,y) = \int_{\mathcal{H}} g(x,y;\xi,\eta) f(\xi,\eta)\,\mathrm{d}\lambda(\xi,\eta) + \int_{\mathbb{R}} g_\xi(x,y;0,\eta)\nu(\eta)\,\mathrm{d}\lambda(\eta). \tag{3.16}$$

Differentiation gives

$$\begin{aligned} g_x &= \frac{q}{2\pi\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left[K_0\left(q\frac{r^{[x]}}{\varepsilon}\right) - K_0\left(q\frac{r^{[-x]}}{\varepsilon}\right) - \frac{\varphi}{r^{[x]}} K_1\left(q\frac{r^{[x]}}{\varepsilon}\right) + \frac{x+\xi}{r^{[-x]}} K_1\left(q\frac{r^{[-x]}}{\varepsilon}\right)\right], \\ g_y &= -\frac{q}{2\pi\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left[\frac{\psi}{r^{[x]}} K_1\left(q\frac{r^{[x]}}{\varepsilon}\right) - \frac{\psi}{r^{[-x]}} K_1\left(q\frac{r^{[-x]}}{\varepsilon}\right)\right]. \end{aligned}$$

In order to use these relations to get estimates for u and its derivatives we need some properties of the modified Bessel functions.

Remark 3.9
From [AS84] we get that K_0 and K_1 are monotonically decreasing and that the following holds

$$\forall s > 0 : 0 < K_0(s) \le K_1(s), \quad \forall s \in (0, \tfrac{1}{2}) : K_0(s) \le -C\ln(s), \tag{3.17a}$$

$$\forall s > 0 : K_1(s) \le \frac{C}{s} \quad \text{and} \quad \forall s \ge C > 0 : K_0(s) \le K_1(s) \le \frac{\tilde{C}}{\sqrt{s}} e^{-s}. \tag{3.17b}$$

Easily we deduce from this properties that

$$\forall s > 0 : K_0(s) \le K_1(s) \le \frac{C}{s} e^{-\frac{s}{2}} \tag{3.17c}$$

holds. As shown in detail in Section A.2 we have

$$\left| K_0\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) - \frac{s}{\mathfrak{r}} K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right)\right| \le \left(\frac{\psi^2}{\mathfrak{r}(\mathfrak{r}+s)} + \frac{\varepsilon}{2q\mathfrak{r}}\right) K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) \tag{3.18}$$

for $\mathfrak{r} := \sqrt{s^2+\psi^2}$.

For simplification in writing we omit the superscript of r for $s = x$ and define

$$G^1(x,y;\xi,\eta) := \frac{1}{\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) \quad \text{and} \tag{3.19}$$

$$G^x(x,y;\xi,\eta) := \frac{1}{\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left(\frac{\psi^2}{r(r+\varphi)} + \frac{\varepsilon}{r}\right) K_1\left(q\tfrac{r}{\varepsilon}\right). \tag{3.20}$$

Obviously, we have

$$|g_y(x,y;\xi,\eta)| \le \frac{q|\psi|}{2\pi r} G^1(x,y;\xi,\eta) \quad \text{and} \quad |g_x(x,y;\xi,\eta)| \le \frac{3q}{2\pi} G^1(x,y;\xi,\eta). \tag{3.21a}$$

Using (3.18) we also get

$$\begin{aligned}
|g_x(x,y;\xi,\eta)| &\le \frac{q}{2\pi\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left[\left| K_0\left(q\tfrac{r}{\varepsilon}\right) - \frac{\varphi}{r} K_1\left(q\tfrac{r}{\varepsilon}\right)\right| + \left| K_0\left(q\tfrac{r^{[-x]}}{\varepsilon}\right) - \frac{x+\xi}{r^{[-x]}} K_1\left(q\tfrac{r^{[-x]}}{\varepsilon}\right)\right|\right] \\
&\le \frac{q}{2\pi\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left[\left(\frac{\psi^2}{r(r+\varphi)} + \frac{\varepsilon}{2qr}\right) K_1\left(q\tfrac{r}{\varepsilon}\right) \right. \\
&\qquad \left. + \left(\frac{\psi^2}{r^{[-x]}(r^{[-x]}+x+\xi)} + \frac{\varepsilon}{2qr^{[-x]}}\right) K_1\left(q\tfrac{r^{[-x]}}{\varepsilon}\right)\right] \\
&\le \frac{q}{\pi\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} \left(\frac{\psi^2}{r(r+\varphi)} + \frac{\varepsilon}{2qr}\right) K_1\left(q\tfrac{r}{\varepsilon}\right) \\
&\le \max\left\{\tfrac{q}{\pi}, \tfrac{1}{2\pi}\right\} G^x(x,y;\xi,\eta).
\end{aligned} \tag{3.21b}$$

Subsequently, we derive several bounds for (weighted) norms of the Green's function. They will be used in Section 3.1.3 to acquire estimates for derivatives of the solution of problems of the form (3.14).

Lemma 3.10
For $x \in [0,\varepsilon]$ we have

$$\|g_\zeta(x,y;\cdot,\cdot)\|_{1,\mathcal{H}} \le C \tag{3.22a}$$

for $\zeta \in \{x,y\}$. For larger $x \in (\varepsilon, 1]$ we have

$$\|g_x(x,y;\cdot,\cdot)\|_{1,\mathcal{H}} \le C|\ln(\varepsilon)|, \quad \|g_y(x,y;\cdot,\cdot)\|_{1,\mathcal{H}} \le \frac{C}{\sqrt{\varepsilon}}. \tag{3.22b}$$

For $x \ge 0$ and $w^e(\xi,\eta) := e^{-\alpha\sqrt{\xi^2+\eta^2}}$ we have

$$\|g_x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\mathcal{H}} \le \frac{C|\ln(\varepsilon)|}{\varrho}, \quad \|g_y(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\mathcal{H}} \le \frac{C}{\sqrt{\varepsilon\varrho}} \tag{3.22c}$$

with $\varrho := \sqrt{x^2+y^2}$ for $\zeta \in \{x,y\}$ and any $\alpha > 0$. Finally, we have

$$\|g_y(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\mathcal{S}} \le \frac{C}{\sqrt{\varepsilon}} \mathcal{E}_1^x(x) \tag{3.22d}$$

for $\mathcal{S} := (0,1)\times\mathbb{R}$ and $w^{\mathcal{E}^x}(\xi,\eta) := \mathcal{E}_1^x(\xi)$ ad $x \in (0,1)$.

Proof
By (3.21) it suffices to prove the desired bounds for g_y only for $\frac{|\psi|}{r}G^1 \le G^1$. Analogously, it suffices to show the bounds desired for g_x for G^1 or G^x.

The rest of the proof is a modification of the proofs in [FK12]. In analogy to this reference we split the domain of integration $\mathcal{H}$ into two subdomains Ω_1, Ω_2 (cf. Figure 3.3).

More explicitly we define $\Omega_1 := \left\{(\xi,\eta)\in\mathbb{R}^2 \middle| \varphi < \max\{\varepsilon, \frac{|\psi|}{4}\}\right\}$ where we used the definitions $\varphi = x-\xi$ and $\psi = y-\eta$ from above. Via the transformation to polar coordinates (r,ϑ) and the relations (cf. Remark 3.9)

$$\varphi < \varepsilon + \tfrac{|\psi|}{4} \le \varepsilon + \tfrac{r}{4} \quad \text{and} \quad K_1(s) \le Cs^{-1}e^{-\frac{s}{2}}$$

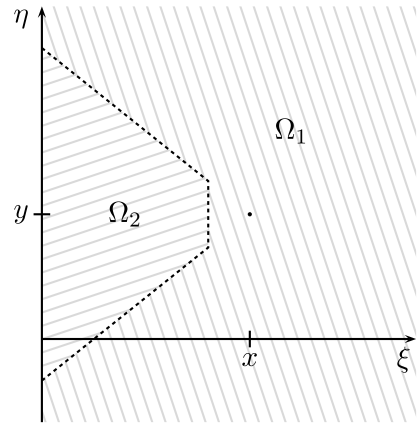

Figure 3.3.: Splitting of the domain

we get

$$0 \le e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) \le \tilde{C}\frac{\varepsilon}{qr} e^{q\frac{\varphi}{\varepsilon}-\frac{qr}{2\varepsilon}} \le \tilde{C}\frac{\varepsilon}{qr} e^{q(1+\frac{r}{4\varepsilon}-\frac{r}{2\varepsilon})} \le C\frac{\varepsilon}{r} e^{-\frac{qr}{4\varepsilon}}.$$

Thus, we conclude

$$\begin{aligned}\|G^1(x,y;\cdot,\cdot)\|_{1,\Omega_1} &\le \int_0^\infty \int_0^{2\pi} \frac{1}{\varepsilon^2} e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) r \,\mathrm{d}\vartheta \,\mathrm{d}r \\ &\le \int_0^\infty \int_0^{2\pi} \frac{\tilde{C}}{\varepsilon} e^{-\frac{qr}{4\varepsilon}} \,\mathrm{d}\vartheta \,\mathrm{d}r \le C.\end{aligned}$$

The first bound (3.22a) of the lemma follows since we have $\Omega_2 = \varnothing$ for $x \le \varepsilon$. Using the techniques above and the triangular inequality $\sqrt{\xi^2+\eta^2} \ge \sqrt{x^2+y^2} - r = \varrho - r$ we get the bound

$$\begin{aligned}\|G^1(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_1} &\le \int_0^\infty \frac{\tilde{C}}{\varepsilon} e^{-\frac{qr}{4\varepsilon}} \int_0^{2\pi} w^e \,\mathrm{d}\vartheta \,\mathrm{d}r \\ &\le \frac{2\pi\tilde{C}}{\varepsilon}\left(\int_0^\varrho e^{-\frac{qr}{4\varepsilon}} e^{-\alpha(\varrho-r)} \,\mathrm{d}r + \int_\varrho^\infty e^{-\frac{qr}{4\varepsilon}} \,\mathrm{d}r\right) \\ &\le \hat{C}\left(e^{-\alpha\varrho} + e^{-\frac{q\varrho}{4\varepsilon}}\right) \le C e^{-\alpha\varrho}.\end{aligned}$$

We start the estimate of the norm including the weight $w^{\mathcal{E}^x}$ by noting

$$\|G^1(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\Omega_1\cap\mathcal{S}} = \frac{1}{\varepsilon^2} e^{-\frac{\beta(1-x)}{\varepsilon}} \int_{\Omega_1\cap\mathcal{S}} e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) e^{-\frac{2q\varphi}{\varepsilon}} \,\mathrm{d}\lambda(\xi,\eta).$$

To obtain the desired bounds we split $\Omega_1 \cap \mathcal{S}$ again into subdomains Ω_{1a}, Ω_{1b} (cf. Figure 3.4).

In $\Omega_{1a} := \left\{(\xi,\eta) \in \mathbb{R}^2 \middle| |\varphi| < \max\{\varepsilon, \frac{|\psi|}{4}\}\right\}$ we have

$$|\varphi| < \varepsilon + \tfrac{|\psi|}{4} \le \varepsilon + \tfrac{r}{4}$$

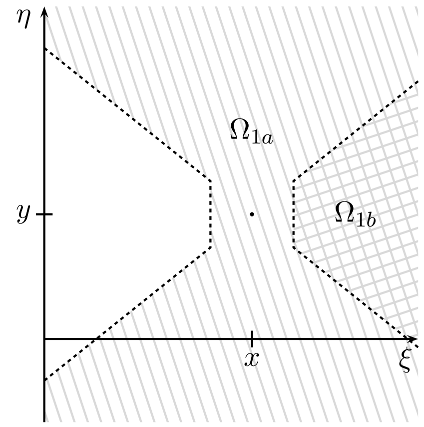

Figure 3.4.: Splitting of Ω_1

and receive similarly to the estimates above

$$\begin{aligned}\|G^1(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\Omega_{1a}} &\leq \frac{\tilde{C}}{\varepsilon}e^{-\frac{\beta(1-x)}{\varepsilon}}\int_0^\infty e^{-q\frac{\varphi}{\varepsilon}-\frac{qr}{2\varepsilon}}\,\mathrm{d}r\\ &\leq \frac{\tilde{C}}{\varepsilon}e^{-\frac{\beta(1-x)}{\varepsilon}}\int_0^\infty e^{1-\frac{qr}{4\varepsilon}}\,\mathrm{d}r \leq Ce^{-\frac{\beta(1-x)}{\varepsilon}}.\end{aligned}$$

In $\Omega_{1b} := \left\{(\xi,\eta)\in\mathbb{R}^2 \middle| -1\leq\varphi\leq -\max\{\varepsilon,\frac{|\psi|}{4}\}\right\}$ we use the variable transform $\tilde{\varphi}=-\varphi$ to get

$$\begin{aligned}\|\tfrac{|\psi|}{r}G^1(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\Omega_{1b}} &\leq \int_{-1}^{-\varepsilon}\int_0^\infty \frac{2\psi}{\varepsilon^2 r}e^{q\frac{\varphi-2+2\xi}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right)\,\mathrm{d}\psi\,\mathrm{d}\varphi\\ &\leq 2e^{-\frac{\beta(1-x)}{\varepsilon}}\int_{-1}^{-\varepsilon}\int_0^\infty \frac{\psi}{\varepsilon^2|\varphi|}e^{q\frac{-\varphi}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right)\,\mathrm{d}\psi\,\mathrm{d}\varphi\\ &\leq 2e^{-\frac{\beta(1-x)}{\varepsilon}}\int_{\varepsilon}^{1}\int_0^\infty \frac{\psi}{\varepsilon^2\tilde{\varphi}}e^{q\frac{\tilde{\varphi}}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right)\,\mathrm{d}\psi\,\mathrm{d}\tilde{\varphi}.\end{aligned}$$

The remaining integral coincides with an intermediate result in the norm estimates on the domain Ω_2. Using the results presented below in (3.23) for Ω_2 we receive the estimate

$$\|\tfrac{|\psi|}{r}G^1(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\Omega_{1b}} \leq \frac{C}{\sqrt{\varepsilon}}e^{-\frac{\beta(1-x)}{\varepsilon}}.$$

Next we analyze the integrals on $\Omega_2 := \left\{(\xi,\eta)\in\mathbb{R}^2 \middle| \max\{\varepsilon,\frac{|\psi|}{4}\}\leq\varphi<x\right\}$. As in [FK12] we estimate by using the relations

$$\begin{gathered}\frac{qr}{\varepsilon}\geq\frac{q\varphi}{\varepsilon}\geq q \quad\Rightarrow\quad K_1\left(q\tfrac{r}{\varepsilon}\right)\leq C\sqrt{\frac{\varepsilon}{qr}}e^{-q\frac{r}{\varepsilon}},\\ \varphi\leq r=\sqrt{\varphi^2+\psi^2}\leq\sqrt{17}\varphi,\\ 0>\varphi-r=\frac{\varphi^2-r^2}{\varphi+r}\begin{cases}\leq-\frac{\psi^2}{d\varphi},\\ \geq-\frac{\psi^2}{\varphi},\end{cases}\qquad d:=1+\sqrt{17}.\end{gathered}$$

Thus, we have

$$0 \le e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) \le C\sqrt{\frac{\varepsilon}{qr}} e^{q\frac{\varphi - r}{\varepsilon}} \le C\sqrt{\frac{\varepsilon}{q\varphi}} e^{-q\frac{\psi^2}{d\varepsilon\varphi}}.$$

We use this to estimate

$$\begin{aligned}
\|\tfrac{|\psi|}{r} G^1(x,y;\cdot,\cdot)\|_{1,\Omega_2} &\le \tilde{C} \int_\varepsilon^1 \int_0^\infty \frac{\psi}{\varepsilon^2\varphi} e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) \,\mathrm{d}\psi \,\mathrm{d}\varphi \\
&\le \hat{C} \int_\varepsilon^1 \int_0^\infty \frac{\psi}{\varepsilon^{3/2}\varphi^{3/2}} e^{-q\frac{\psi^2}{d\varepsilon\varphi}} \,\mathrm{d}\psi \,\mathrm{d}\varphi \\
&= \hat{C} \int_\varepsilon^1 \frac{d}{2q\sqrt{\varepsilon\varphi}} \,\mathrm{d}\varphi \le \frac{C}{\sqrt{\varepsilon}} \qquad (3.23)
\end{aligned}$$

for $x \in [0,1]$. This result implies immediately

$$\|\tfrac{|\psi|}{r} G^1(x,y;\cdot,\cdot) w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\Omega_2} \le \|\tfrac{|\psi|}{r} G^1(x,y;\cdot,\cdot)\|_{1,\Omega_2} \|w^{\mathcal{E}^x}(\cdot,\cdot)\|_{\infty,\Omega_2} \le \frac{C}{\sqrt{\varepsilon}} e^{-\beta\frac{1-x}{\varepsilon}}.$$

Similarly, we obtain

$$\begin{aligned}
\|G^x(x,y;\cdot,\cdot)\|_{1,\Omega_2} &\le \tilde{C} \int_\varepsilon^1 \int_0^\infty \left(\frac{\psi^2}{\varepsilon^{3/2}\varphi^{5/2}} + \frac{1}{\sqrt{\varepsilon}\varphi^{3/2}}\right) e^{-q\frac{\psi^2}{d\varepsilon\varphi}} \,\mathrm{d}\psi \,\mathrm{d}\varphi \\
&\le C \int_\varepsilon^1 \frac{1}{\varphi} \,\mathrm{d}\varphi = C|\ln(\varepsilon)|. \qquad (3.24)
\end{aligned}$$

From $|y| - |\eta| \le |y - \eta| \le 4(x - \xi) \le 4x$ in Ω_2 we conceive $|\eta| \ge |y| - 4x$. For $|y| \ge 5x$ we have $|\eta| \ge x$ and deduce

$$\sqrt{\xi^2 + \eta^2} \ge |\eta| \ge |y| - 4x \ge \frac{|y|}{5} = \frac{1}{\sqrt{26}}\sqrt{\tfrac{y^2}{25} + y^2} \ge \frac{1}{\sqrt{26}}\varrho.$$

Hence, we have $\|w^e\|_{\infty,\Omega_2} \le e^{-\frac{\alpha}{\sqrt{26}}\varrho}$ and conclude

$$\begin{aligned}
\|g_y(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\le \|g_y(x,y;\cdot,\cdot)\|_{1,\Omega_2} \|w^e\|_{\infty,\Omega_2} \le \frac{C}{\sqrt{\varepsilon}} e^{-\frac{\alpha}{\sqrt{26}}\varrho}, \\
\|g_x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\le \|g_x(x,y;\cdot,\cdot)\|_{1,\Omega_2} \|w^e\|_{\infty,\Omega_2} \le C|\ln(\varepsilon)| e^{-\frac{\alpha}{\sqrt{26}}\varrho}.
\end{aligned}$$

So we are left with the case $|y| < 5x$. Using the first part of (3.24) we receive

$$\begin{aligned}
\|G^x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\le \tilde{C} \int_\varepsilon^x \int_0^\infty \left(\frac{\psi^2}{\varepsilon^{3/2}\varphi^{5/2}} + \frac{1}{\sqrt{\varepsilon}\varphi^{3/2}}\right) e^{-q\frac{\psi^2}{d\varepsilon\varphi}} e^{-\alpha\xi} \,\mathrm{d}\psi \,\mathrm{d}\varphi \\
&\le \hat{C} e^{-\alpha x} \int_\varepsilon^x \frac{1}{\varphi} e^{\alpha\varphi} \,\mathrm{d}\varphi = C e^{-\alpha x}\Big[\mathrm{Ei}(\alpha x) - \mathrm{Ei}(\alpha\varepsilon)\Big]
\end{aligned}$$

where Ei denotes the *exponential integral* (cf. [AS84]). Furthermore, we know

$$\begin{aligned}
\frac{1}{x} e^{\alpha x} = \sum_{n=0}^\infty \frac{\alpha^n x^{n-1}}{n!} &= \frac{1}{x} + \alpha + \alpha \sum_{n=1}^\infty \frac{\alpha^n x^n}{(n+1)!} \ge \alpha \sum_{n=1}^\infty \frac{\alpha^n x^n}{2n\, n!} \\
\Rightarrow \quad e^{-\alpha x} \sum_{n=1}^\infty \frac{\alpha^n x^n}{n\, n!} &\le \frac{2}{\alpha x}. \qquad (3.25)
\end{aligned}$$

Therefore, we conceive from the series expansion of Ei the estimate

$$\begin{aligned}\|G^x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\leq \tilde{C}e^{-\alpha x}\left(\ln(x)-\ln(\varepsilon)+\sum_{n=1}^{\infty}\frac{\alpha^n x^n}{n\,n!}\right)\\ &\leq C\frac{|\ln(\varepsilon)|}{x}\leq C\sqrt{26}\frac{|\ln(\varepsilon)|}{\varrho}.\end{aligned}$$

Analogously, we can estimate using the first part of (3.23) as follows

$$\begin{aligned}\|\tfrac{|\psi|}{r}G^x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\leq C\int_\varepsilon^x\int_0^\infty \frac{\psi}{\varepsilon^{3/2}\varphi^{3/2}}e^{-q\frac{\psi^2}{d\varepsilon\varphi}}e^{-\alpha\xi}\,\mathrm{d}\psi\,\mathrm{d}\varphi\\ &= Ce^{-\alpha x}\int_\varepsilon^x \frac{d}{2q\sqrt{\varepsilon\varphi}}e^{\alpha\varphi}\,\mathrm{d}\varphi\\ &= -\hat{\imath}\frac{Cd}{2q}\sqrt{\frac{\pi}{\alpha\varepsilon}}e^{-\alpha x}\left[\operatorname{erf}(\hat{\imath}\sqrt{\alpha x})-\operatorname{erf}(\hat{\imath}\sqrt{\alpha\varepsilon})\right]\end{aligned}$$

where erf denotes the *error function*. Using the series expansion of the error function (cf. [AS84]) we get

$$\begin{aligned}\|\tfrac{|\psi|}{r}G^x(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\Omega_2} &\leq \frac{\tilde{C}}{\sqrt{\varepsilon}}e^{-\alpha x}\left[\frac{1}{\sqrt{x}}\sum_{n=0}^{\infty}\frac{(\alpha x)^{n+1}}{n!(2n+1)}-\frac{1}{\sqrt{\varepsilon}}\sum_{n=0}^{\infty}\frac{(\alpha\varepsilon)^{n+1}}{n!(2n+1)}\right]\\ &\leq \frac{\tilde{C}}{\sqrt{\varepsilon x}}e^{-\alpha x}\sum_{n=0}^{\infty}\frac{(\alpha x)^{n+1}}{(n+1)!}\leq\frac{C}{\sqrt{\varepsilon x}}\leq\frac{C\sqrt[4]{26}}{\sqrt{\varepsilon\varrho}}.\end{aligned}$$

Combining the results on the subdomains, the lemma is proved. □

Lemma 3.11
For $x\in[2\varepsilon,1]$ and $w^l(\xi,\eta):=\max\bigl(-\ln(\xi),1\bigr)$ we have

$$\|g_x(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\mathcal{H}}\leq C|\ln(\varepsilon)|\Bigl(1+|\ln(x)|\Bigr) \tag{3.26a}$$

and for $x\in(0,2\varepsilon)$ we have

$$\|g_x(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\mathcal{H}}\leq C|\ln(x)| \tag{3.26b}$$

Proof
As in the previous proof it suffices by (3.21) to prove the desired bounds for G^1 or G^x.

First we consider the case $x\in[2\varepsilon,1]$. We will again split the domain of integration $\mathcal{H}$, but this time into three pieces $\hat{\Omega}_i$ (cf. Figure 3.5) with

$$\begin{aligned}\hat{\Omega}_1 &:= \Bigl\{(\xi,\eta)\in(x-\varepsilon,\infty)\times\mathbb{R}\Bigr\}\subseteq\Omega_1,\\ \hat{\Omega}_2 &:= \Omega_2=\Bigl\{(\xi,\eta)\in\mathbb{R}^2\Bigm|\max\{\varepsilon,\tfrac{|\psi|}{4}\}\leq\varphi<x\Bigr\}\quad\text{and}\\ \hat{\Omega}_3 &:= \Bigl\{(\xi,\eta)\in\mathbb{R}^2\Bigm|\varepsilon<\varphi\leq x\wedge 4\varphi<|\psi|\Bigr\}\subseteq\Omega_1.\end{aligned}$$

To establish the following bounds we use the calculations and estimates we obtained in the proof of Lemma 3.10.

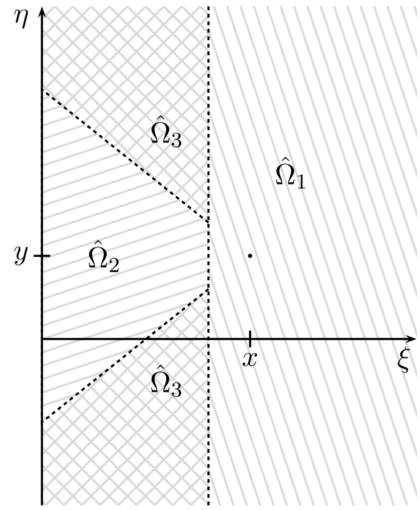

Figure 3.5.: Splitting of the domain

We start by estimating

$$\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\hat\Omega_1} \le \|G^1(x,y;\cdot,\cdot)\|_{1,\Omega_1}\|w^l\|_{\infty,\hat\Omega_1} \le C|\ln(\varepsilon)|.$$

Furthermore, we get similarly to (3.24) the bound

$$\begin{aligned}\|G^x(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\hat\Omega_2} &\le C\int_\varepsilon^x \frac{1}{\varphi}\big(1-\ln(\xi)\big)\,\mathrm{d}\varphi = C\int_\varepsilon^x \frac{1}{\varphi} - \frac{\ln(x-\varphi)}{\varphi}\,\mathrm{d}\varphi\\ &= C\Big(\ln(x)-\ln(\varepsilon)+\ln(x)\ln\left(\tfrac{\varepsilon}{x}\right)+\mathrm{dilog}(0)-\mathrm{dilog}\left(\tfrac{x-\varepsilon}{x}\right)\Big)\end{aligned}$$

where dilog denotes the *dilogarithm* (cf. [AS84]). From

$$|\mathrm{dilog}(s)| \le \sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6} \quad \text{for} \quad s\in[0,2] \tag{3.27}$$

we get

$$\begin{aligned}\|G^x(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\hat\Omega_2} &\le \tilde C\left(|\ln(\varepsilon)|+\ln(x)\Big(\ln(\varepsilon)-\ln(x)\Big)+\frac{\pi^2}{3}\right)\\ &\le C|\ln(\varepsilon)|\Big(1+|\ln(x)|\Big).\end{aligned}$$

On $\hat\Omega_3$ we use

$$4\varphi < |\psi| \quad \Rightarrow \quad r=\sqrt{\varphi^2+\psi^2} > \sqrt{\varphi^2+16\varphi^2} = \sqrt{17}\varphi$$

and $r\ge\varepsilon$ to estimate

$$0 < e^{q\frac{\varphi}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right) \le e^{q\frac{\varphi}{\varepsilon}}\sqrt{\frac{\varepsilon}{qr}}e^{-q\frac{r}{\varepsilon}} \le \sqrt{\frac{\varepsilon}{qr}}e^{q\frac{r}{\varepsilon}\left(\frac{1}{\sqrt{17}}-1\right)} = \sqrt{\frac{\varepsilon}{qr}}e^{-\sqrt{2}\tilde q\frac{r}{\varepsilon}}$$

with $\tilde{q} := \frac{\sqrt{17}-1}{\sqrt{34}} > 0$. Thus, we get from $r \geq \frac{1}{\sqrt{2}}\big(|\varphi| + |\psi|\big)$ the bound

$$\begin{aligned}
\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\hat{\Omega}_3} &\leq \int_\varepsilon^x \int_0^\infty \frac{C}{\varepsilon^{3/2}\sqrt{r}} e^{-\tilde{q}\frac{|\varphi|+|\psi|}{\varepsilon}} \Big(1 - \ln(x-\varphi)\Big)\,\mathrm{d}\psi\,\mathrm{d}\varphi \\
&\leq \int_0^{x-\varepsilon} \int_0^\infty \frac{C}{\varepsilon^2} e^{-\tilde{q}\frac{x-\xi+|\psi|}{\varepsilon}} \Big(1 - \ln(\xi)\Big)\,\mathrm{d}\psi\,\mathrm{d}\xi \\
&= \int_0^{x-\varepsilon} \frac{C}{\tilde{q}\varepsilon} e^{-\tilde{q}\frac{x-\xi}{\varepsilon}} \Big(1 - \ln(\xi)\Big)\,\mathrm{d}\xi \\
&= \frac{C}{\tilde{q}^2} e^{-\tilde{q}\frac{x}{\varepsilon}} \Bigg(e^{\tilde{q}\frac{x-\varepsilon}{\varepsilon}} - 1 + \mathrm{Ei}\left(\tfrac{\tilde{q}}{\varepsilon}(x-\varepsilon)\right) - \gamma^e \\
&\qquad - \ln\left(\tfrac{\tilde{q}}{\varepsilon}\right) - e^{\frac{\tilde{q}}{\varepsilon}(x-\varepsilon)} \ln(x-\varepsilon) \Bigg)
\end{aligned}$$

where $\gamma^e \approx 0.58$ denotes Euler's constant. In (3.25) we established

$$e^{-\frac{\tilde{q}}{\varepsilon}x} \sum_{n=1}^\infty \frac{\tilde{q}^n x^n}{\varepsilon^n n\, n!} \leq \frac{2\varepsilon}{\tilde{q}x}$$

and from [AS84] we know

$$\mathrm{Ei}(s) = \gamma^e + \ln(s) + \sum_{n=1}^\infty \frac{s^n}{n\, n!}.$$

Therefore, we conclude using the monotonicity of Ei that we have

$$\begin{aligned}
\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\hat{\Omega}_3} &\leq \tilde{C} + \tilde{C} e^{-\tilde{q}\frac{x}{\varepsilon}} \left(\mathrm{Ei}\left(\tfrac{\tilde{q}}{\varepsilon}x\right) - \gamma^e - \ln\left(\tfrac{\tilde{q}}{\varepsilon}\right) - e^{\frac{\tilde{q}}{\varepsilon}(x-\varepsilon)} \ln(x-\varepsilon) \right) \\
&\leq \tilde{C} + \tilde{C} e^{-\tilde{q}\frac{x}{\varepsilon}} \left(\ln(x) + \sum_{n=1}^\infty \frac{\tilde{q}^n x^n}{\varepsilon^n n\, n!} - e^{\frac{\tilde{q}}{\varepsilon}(x-\varepsilon)} \ln(x-\varepsilon) \right) \\
&\leq \tilde{C} + \tilde{C} \left(\frac{2\varepsilon}{\tilde{q}x} - e^{-\tilde{q}} \ln(x-\varepsilon) \right) \leq C|\ln(\varepsilon)|.
\end{aligned}$$

Next we analyze the case $x \in [0, 2\varepsilon]$. In this case we split the domain of integration into

$$\tilde{\Omega}_1 := \Big\{(\xi,\eta) \in \big(\tfrac{x}{2}, \infty\big) \times \mathbb{R}\Big\} \subseteq \Omega_1 \quad \text{and} \quad \tilde{\Omega}_2 := \Big\{(\xi,\eta) \in \big(0, \tfrac{x}{2}\big) \times \mathbb{R}\Big\}.$$

Easily we get

$$\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\tilde{\Omega}_1} \leq \|G^1(x,y;\cdot,\cdot)\|_{1,\Omega_1} \|w^l\|_{\infty,\tilde{\Omega}_1} \leq C|\ln(x)|.$$

On $\tilde{\Omega}_2$ we use

$$0 < \xi < \frac{x}{2} \leq \varepsilon, \quad |K_1(s)| \leq \frac{1}{s} e^{-\frac{s}{2}} \quad \text{and} \quad (x-\xi) + |\psi| \leq \sqrt{2}r$$

to get

$$\begin{aligned}
\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\tilde{\Omega}_2} &\leq \frac{C}{\varepsilon} \int_0^\infty \int_0^{\frac{x}{2}} \frac{-\ln(\xi)}{(x-\xi)+\psi} e^{-q\frac{\psi}{2\varepsilon}}\,\mathrm{d}\xi\,\mathrm{d}\psi \\
&= \frac{C}{\varepsilon} \int_0^\infty \int_{\frac{x}{2}+\psi}^{x+\psi} \frac{-\ln(x+\psi-s)}{s} e^{-q\frac{\psi}{2\varepsilon}}\,\mathrm{d}s\,\mathrm{d}\psi \\
&= \frac{C}{\varepsilon} \int_0^\infty \left[\mathrm{dilog}(0) + \ln(x+\psi) \ln\left(\tfrac{\frac{x}{2}+\psi}{x+\psi}\right) - \mathrm{dilog}\left(1 - \tfrac{\frac{x}{2}+\psi}{x+\psi}\right)\right] e^{-q\frac{\psi}{2\varepsilon}}\,\mathrm{d}\psi.
\end{aligned}$$

For $x \leq \psi$ we have

$$0 > \ln\left(\frac{\frac{x}{2}+\psi}{x+\psi}\right) \geq \ln\left(\frac{\psi}{2\psi}\right) = \ln\left(\frac{1}{2}\right)$$

and for $x > \psi$ we can estimate

$$0 > \ln\left(\frac{\frac{x}{2}+\psi}{x+\psi}\right) > \ln\left(\frac{\frac{x}{2}}{2x}\right) = \ln\left(\frac{1}{4}\right).$$

Using the bound (3.27) of the dilogarithm, we conclude

$$\begin{aligned}\|G^1(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\tilde{\Omega}_2} &\leq \frac{\tilde{C}}{\varepsilon}\int_0^\infty \left(\frac{\pi^2}{3} + \ln(x+\psi)\ln\left(\frac{\frac{x}{2}+\psi}{x+\psi}\right)\right) e^{-q\frac{\psi}{2\varepsilon}}\,\mathrm{d}\psi \\ &\leq \frac{\tilde{C}}{\varepsilon}\left(\frac{\pi^2}{3} + \ln(x)\ln\left(\frac{1}{4}\right)\right)\int_0^\infty e^{-q\frac{\psi}{2\varepsilon}}\,\mathrm{d}\psi \leq C|\ln(x)|.\end{aligned}$$

Combining the results for the subdomains we have proved the lemma. □

Additionally, we will use an approximation of the Green's function for the problem

$$\hat{L}u = -\varepsilon\Delta u + \beta u_x = f \text{ in } \mathcal{S} = (0,1)\times\mathbb{R}, \quad u(0,\cdot) = 0, \quad u(1,\cdot) = 0. \tag{3.28}$$

Defining the approximation

$$\begin{aligned}\tilde{g}(x,y;\xi,\eta) := g(x,y;\xi,\eta) + \frac{1}{2\pi\varepsilon}e^{q\frac{\varphi}{\varepsilon}}\Bigg[&K_0\left(q\frac{r^{[-2+x]}}{\varepsilon}\right) - K_0\left(q\frac{r^{[2-x]}}{\varepsilon}\right) \\ &+ K_0\left(q\frac{r^{[2+x]}}{\varepsilon}\right) - K_0\left(q\frac{r^{[-2-x]}}{\varepsilon}\right)\Bigg]\end{aligned}$$

with $\tilde{g}(x,y;0,\eta) = 0$ and $\tilde{g}(x,y;1,\eta) \neq 0$ of the Green's function of problem (3.28) we have the representation

$$u(x,y) = \int_{\mathcal{S}} \tilde{g}(x,y;\xi,\eta)f(\xi,\eta)\,\mathrm{d}\lambda(\xi,\eta) - \int_{\mathbb{R}} \tilde{g}(x,y;1,\eta)u_x(1,\eta)\,\mathrm{d}\lambda(\eta) \tag{3.29}$$

for the solution u of problem (3.28).

Corollary 3.12
Obviously, we have

$$|\tilde{g}_y(x,y;\xi,\eta)| \leq \frac{3q|\psi|}{2\pi r}G^1(x,y:\xi,\eta) \tag{3.30}$$

for $(x,y) \in \mathcal{S}$.

Since all estimates for g_y in Lemma 3.10 were deduced using $|g_y| \leq C\frac{|\psi|}{r}G^1$, they are also valid for $\tilde{g}_y$ on $\mathcal{S} \subseteq \mathcal{H}$:

$$\|\tilde{g}_y(x,y;\cdot,\cdot)\|_{1,\mathcal{S}} \leq C \quad \textit{for} \quad x \in [0,\varepsilon], \tag{3.31a}$$

$$\|\tilde{g}_y(x,y;\cdot,\cdot)\|_{1,\mathcal{S}} \leq \frac{C}{\sqrt{\varepsilon}}, \tag{3.31b}$$

$$\|\tilde{g}_y(x,y;\cdot,\cdot)w^e(\cdot,\cdot)\|_{1,\mathcal{S}} \leq \frac{C}{\sqrt{\varepsilon\varrho}} \quad \textit{and} \tag{3.31c}$$

$$\|\tilde{g}_y(x,y;\cdot,\cdot)w^{\mathcal{E}^x}(\cdot,\cdot)\|_{1,\mathcal{S}} \leq \frac{C}{\sqrt{\varepsilon}}\mathcal{E}_1^x(x). \tag{3.31d}$$

Lemma 3.13
We have

$$\|\tilde{g}_y(x,y;1,\cdot)\|_{1,\mathbb{R}} \leq C\frac{e^{-\frac{\beta}{2\varepsilon}}}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}} \tag{3.32}$$

for $(x,y) \in \mathcal{S}$.

Proof
Differentiation gives

$$\begin{aligned}|\tilde{g}_y(x,y;1,\eta)| &= \frac{q}{2\pi\varepsilon^2}e^{-q\frac{1-x}{\varepsilon}}\left|\frac{\psi}{\sqrt{(3+x)^2+\psi^2}}K_1\left(q\frac{\sqrt{(3+x)^2+\psi^2}}{\varepsilon}\right)\right.\\ &\qquad\left. -\frac{\psi}{\sqrt{(3-x)^2+\psi^2}}K_1\left(q\frac{\sqrt{(3-x)^2+\psi^2}}{\varepsilon}\right)\right|\\ &\leq \frac{q}{2\pi\varepsilon^2}e^{-q\frac{1-x}{\varepsilon}}K_1\left(q\frac{\sqrt{4+\psi^2}}{\varepsilon}\right)\end{aligned}$$

for $x \in [0,1]$. Using Remark 3.9 we can estimate

$$|\tilde{g}_y(x,y;1,\eta)| \leq \frac{q}{2\pi\varepsilon^2}e^{-q\frac{1-x}{\varepsilon}}K_1\left(q\frac{\sqrt{4+\psi^2}}{\varepsilon}\right) \leq \frac{C}{\varepsilon^2}e^{-q\frac{1-x}{\varepsilon}}e^{-q\frac{\sqrt{4+\psi^2}}{\varepsilon}}.$$

Integration leads to

$$\begin{aligned}\|\tilde{g}_y(x,y;1,\cdot)\|_{1,\mathbb{R}} &\leq \frac{\tilde{C}}{\varepsilon^2}e^{-q\frac{1-x}{\varepsilon}}\left[\int_0^1 e^{-\frac{2q}{\varepsilon}}\,\mathrm{d}\psi + \int_1^\infty e^{-q\frac{\sqrt{4+\psi}}{\varepsilon}}\,\mathrm{d}\psi\right]\\ &\leq \frac{C}{\varepsilon}e^{-q\frac{1-x}{\varepsilon}}\left(e^{-\frac{2q}{\varepsilon}}+e^{-\frac{\sqrt{5}q}{\varepsilon}}\right) \leq \frac{2C}{\varepsilon}e^{-q\frac{3-x}{\varepsilon}} \leq 2C\frac{e^{-\frac{\beta}{2\varepsilon}}}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}}. \quad \square\end{aligned}$$

Later on we will use information of the solution of

$$-\Delta\tilde{u} = 0 \text{ in } \mathcal{H}, \quad \tilde{u}(0,y) = \nu(y), \quad \lim_{\|(x,y)\|\to\infty}\tilde{u}(x,y) = 0. \tag{3.33}$$

The advantage of this problem is the simple structure of the associated Green's function. It allows us to get a sharper estimate for the norms of the derivatives.

Lemma 3.14
If $\|\nu\|_{1,\infty,\mathbb{R}} \leq C$ *holds we have for the solution* $\tilde{u}$ *of* (3.33) *the estimates*

$$|\tilde{u}(x,y)| \leq \pi\|\nu\|_{0,\infty,\mathbb{R}} \quad \textit{and} \quad |\tilde{u}_x(x,y)| \leq \begin{cases} C|\ln(x)|, & x \in (0,\frac{1}{2}),\\ Cx^{-1}, & x \geq \frac{1}{2}.\end{cases} \tag{3.34}$$

Proof
By using the Green's function

$$\bar{g}^{\Delta}(x,y;\xi,\eta) := \frac{1}{2\pi}\ln\left(r^{[x]}\right), \quad g^{\Delta}(x,y;\xi,\eta) := \bar{g}^{\Delta}(x,y;\xi,\eta) - \bar{g}^{\Delta}(-x,y;\xi,\eta),$$

we have the representation (cf. [GT01])

$$\tilde{u}(x,y) = \int_{-\infty}^{\infty} g_\xi^{\Delta}(x,y;0,\eta)\nu(\eta)\,\mathrm{d}\eta = -\int_{-\infty}^{\infty}\frac{x}{\pi r^2}\nu(y-\psi)\,\mathrm{d}\psi.$$

Consequently, we can estimate

$$|\tilde{u}(x,y)| \le \|\nu\|_{0,\infty,\mathbb{R}} \int_{-\infty}^{\infty} \frac{x}{\pi r^2}\,\mathrm{d}\psi = \|\nu\|_{0,\infty,\mathbb{R}} \left[\arctan\left(\tfrac{\psi}{x}\right)\right]_{\psi=-\infty}^{\infty} = \pi\|\nu\|_{0,\infty,\mathbb{R}}$$

for $x > 0$.

Also, we have

$$\tilde{u}_x(x,y) = \int_{-\infty}^{\infty} g^{\Delta}_{\xi x}(x,0;0,\psi)\nu(y-\psi)\,\mathrm{d}\psi = \int_{-\infty}^{\infty} \frac{x^2-\psi^2}{\pi r^4}\nu(y-\psi)\,\mathrm{d}\psi$$

and conclude for $x \in (0, \frac{1}{2})$

$$\begin{aligned}
|\tilde{u}_x(x,y)| &\le \left| \int_{-1}^{1} \frac{x^2-\psi^2}{\pi r^4}\left(\nu(y) - \int_0^{\psi} \nu'(y-s)\,\mathrm{d}s\right)\mathrm{d}\psi \right| \\
&\quad + \int_1^{\infty} \frac{\psi^2-x^2}{\pi r^4}\,|\nu(y-\psi)+\nu(y+\psi)|\,\mathrm{d}\psi \\
&\le \tilde{C}\left[\left|\int_{-1}^{1} \frac{x^2-\psi^2}{r^4}\,\mathrm{d}\psi\right| + \int_0^x \frac{x^2-\psi^2}{r^4}\psi\,\mathrm{d}\psi \right.\\
&\quad \left. + \int_x^1 \frac{\psi^2-x^2}{r^4}\psi\,\mathrm{d}\psi + \int_1^{\infty} \frac{\psi^2-x^2}{r^4}\,\mathrm{d}\psi\right] \\
&\le \tilde{C}\Big(4 + |\ln(x)|\Big) \le C|\ln(x)|.
\end{aligned}$$

For $x \ge \frac{1}{2}$ we have

$$|\tilde{u}_x(x,y)| \le C\int_0^{\infty} \left|\frac{x^2-\psi^2}{r^4}\right|\mathrm{d}\psi = \frac{C}{x}.$$

□

Remark 3.15

The logarithmic bound in Lemma 3.14 is not caused by improper estimates, but a sharp bound. This leads to the result, that the first derivative in x-direction is not bounded near the edge $x = 0$. In fact for $\nu(y) = |y-1| + |y+1| - 2|y| \in W^{1,\infty}$ we have

$$\begin{aligned}
\tilde{u}(x,0) &= -\int_{-1}^{1} \frac{x}{\pi(x^2+\eta^2)}\Big(|\eta-1| + |\eta+1| - 2|\eta|\Big)\,\mathrm{d}\eta \\
&= -\frac{4}{\pi}\int_0^1 \frac{x}{x^2+\eta^2}(1-\eta)\,\mathrm{d}\eta = \frac{2}{\pi}\left(x\ln(x^2+1) - 2x\ln(x) + 2\arctan\left(\frac{1}{x}\right)\right).
\end{aligned}$$

Differentiation gives

$$\begin{aligned}
\tilde{u}_x(x,0) &= \frac{2}{\pi}\left(\ln(x^2+1) + \frac{2x^2}{x^2+1} - 2\ln(x) - 2 - \frac{2}{(1+\frac{1}{x^2})x^2}\right) \\
&= \frac{2}{\pi}\left(\ln(x^2+1) - \frac{4}{x^2+1} - 2\ln(x)\right)
\end{aligned}$$

and we can see, that this has a logarithmic singularity at $x = 0$.

3.1.2. Estimates for a One-Dimensional Auxiliary Problem

Later on we use some one-dimensional estimates to acquire sharp bounds for x-derivatives of solutions. To that end we consider the auxiliary problem

$$L^{1D}w := -\varepsilon w_{xx} + \beta w_x = g, \quad w(0) = w_0, \quad w(1) = w_1. \tag{3.35}$$

The techniques we use to prove the following lemmata are modifications of the proofs presented in [KT78] that we also used in Chapter 2.

Lemma 3.16
For $\|g\|_\infty \le C$ and $|w_0| + |w_1| \le C$ we have for the solution w of (3.35) the estimates

$$|w_x(x)| \le C\left(1 + \varepsilon^{-1}\mathcal{E}_1^x(x)\right) \quad \textit{and} \quad |w_{xx}(x)| \le C\left(\varepsilon^{-1} + \varepsilon^{-2}\mathcal{E}_1^x(x)\right). \tag{3.36a}$$

If additionally $\|g\|_{1,\infty} \le C$ holds, then we have

$$|w_{xx}(x)| \le C\left(1 + \varepsilon^{-2}\mathcal{E}_1^x(x)\right). \tag{3.36b}$$

Proof
It is easy to check that the solution w is given by

$$w(x) = -\int_x^1 \hat{w}(\xi)\,\mathrm{d}\xi + K_1 + K_2\left(e^{-\beta\frac{1-x}{\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}\right) \quad \text{with}$$
$$\hat{w}(\xi) = \int_\xi^1 \frac{g(\zeta)e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta.$$

First we note

$$w_0 = w(0) = -\int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi + K_1.$$

Using the prerequisites on g we deduce

$$|K_1| \le |w_0| + \int_0^1\int_\xi^1 \frac{\|g\|_\infty e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta\,\mathrm{d}\xi = |w_0| + \int_0^1 \frac{\|g\|_\infty}{\beta}\left(1 - e^{-\beta\frac{1-\xi}{\varepsilon}}\right)\mathrm{d}\xi \le C.$$

Thus, we derive

$$|K_2| = \frac{|w_1 - K_1|}{1 - e^{-\frac{\beta}{\varepsilon}}} < |w_1| + |K_1| \le C.$$

For the first derivative we estimate

$$\begin{aligned}|w_x(x)| &= \left|\int_x^1 \frac{g(\eta)}{\varepsilon}e^{-\beta\frac{\eta-x}{\varepsilon}}\,\mathrm{d}\eta + K_2\frac{\beta}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}}\right| \\ &\le \frac{\|g\|_\infty}{\beta}\left(1 - e^{-\beta\frac{1-x}{\varepsilon}}\right) + |K_2|\frac{\beta}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}} \le C\left(1 + \varepsilon^{-1}e^{-\beta\frac{1-x}{\varepsilon}}\right).\end{aligned}$$

For the second derivative we have

$$\begin{aligned}|w_{xx}(x)| &= \left|-\frac{g(x)}{\varepsilon} + \int_x^1 \frac{\beta g(\eta)}{\varepsilon^2}e^{-\beta\frac{\eta-x}{\varepsilon}}\,\mathrm{d}\eta + K_2\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}}\right| \\ &\le \frac{\|g\|_\infty}{\varepsilon} + \frac{\|g\|_\infty}{\varepsilon}\left(1 - e^{-\beta\frac{1-x}{\varepsilon}}\right) + |K_2|\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}} \le C\left(\varepsilon^{-1} + \varepsilon^{-2}e^{-\beta\frac{1-x}{\varepsilon}}\right).\end{aligned}$$

Using the derivative of g we can sharpen this bound as follows

$$\begin{aligned}
|w_{xx}(x)| &= \left| -\frac{g(x)}{\varepsilon} + \int_x^1 \frac{\beta}{\varepsilon^2}\left(g(x) + \int_x^\eta g'(\xi)\,\mathrm{d}\xi\right) e^{-\beta\frac{\eta-x}{\varepsilon}}\,\mathrm{d}\eta + K_2\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}} \right| \\
&\le \left| -\frac{g(x)}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}} + \int_x^1 \frac{\beta\|g'\|_\infty}{\varepsilon^2}(\eta - x)e^{-\beta\frac{\eta-x}{\varepsilon}}\,\mathrm{d}\eta + K_2\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}} \right| \\
&\le \frac{\varepsilon\|g\|_\infty + \beta^2|K_2|}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}} + \frac{\beta\|g'\|_\infty}{\varepsilon^2}\int_0^{1-x} se^{-\beta\frac{s}{\varepsilon}}\,\mathrm{d}s \\
&= \frac{\varepsilon\|g\|_\infty + \beta^2|K_2|}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}} + \|g'\|_\infty\left(\frac{1}{\beta} - \frac{1-x}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}} - \frac{1}{\beta}e^{-\beta\frac{1-x}{\varepsilon}}\right) \\
&\le C\left(1 + \varepsilon^{-2}e^{-\beta\frac{1-x}{\varepsilon}}\right). \qquad \square
\end{aligned}$$

Lemma 3.17
For $|g(x)| \le C\left(e^{-\beta\frac{x}{\varepsilon}} + \varepsilon^l\right)$ with $l \ge 0$ and $w_0 = w_1 = 0$ we have

$$|w^{(k)}(x)| \le C\varepsilon^{1-k}\left(e^{-\beta\frac{x}{\varepsilon}} + \varepsilon^l + \varepsilon^{\min\{l-1,0\}}e^{-\beta\frac{1-x}{\varepsilon}}\right) \tag{3.37}$$

for $k \in \{1, 2\}$.

Proof
Similarly to the previous proof we consider the solution representation

$$w(x) = -\int_x^1 \hat{w}(\xi)\,\mathrm{d}\xi + K\left(1 - e^{-\beta\frac{1-x}{\varepsilon}}\right) \quad \text{where} \quad \hat{w}(\xi) = \int_\xi^1 \frac{g(\zeta)e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta.$$

From the prerequisites on g we deduce

$$\begin{aligned}
|\hat{w}(\xi)| &\le \int_\xi^1 \frac{\tilde{C}\left(e^{-\beta\frac{\zeta}{\varepsilon}} + \varepsilon^l\right)e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta \le C\left(e^{-\beta\frac{\xi}{\varepsilon}} + \varepsilon^l\right) \quad \text{and} \\
|\hat{w}'(\xi)| &= \left|\frac{\beta}{\varepsilon}\hat{w}(\xi) - \frac{g(\xi)}{\varepsilon}\right| \le C\left(\frac{1}{\varepsilon}e^{-\beta\frac{\xi}{\varepsilon}} + \varepsilon^{l-1}\right).
\end{aligned}$$

The boundary condition at $x = 0$ implies

$$|K| = \left|\frac{\int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi}{1 - e^{-\beta\frac{1}{\varepsilon}}}\right| \le \frac{C\varepsilon^{\min\{l,1\}}}{1 - e^{-\frac{\beta}{\varepsilon}}} \le C\varepsilon^{\min\{l,1\}}.$$

Using this estimates we get

$$\begin{aligned}
|w_x(x)| &= \left|\hat{w}(x) - K\frac{\beta}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}}\right| \le C\left(e^{-\beta\frac{x}{\varepsilon}} + \varepsilon^l + \varepsilon^{\min\{l-1,0\}}e^{-\beta\frac{1-x}{\varepsilon}}\right) \quad \text{and} \\
|w_{xx}(x)| &= \left|\hat{w}'(x) - K\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}}\right| \le C\left(\frac{1}{\varepsilon}e^{-\beta\frac{x}{\varepsilon}} + \varepsilon^{l-1} + \varepsilon^{\min\{l-1,0\}-1}e^{-\beta\frac{1-x}{\varepsilon}}\right). \quad \square
\end{aligned}$$

Lemma 3.18
The solution w of the special case $g(x) = 1 - \ln(1-x)$, $w_0 = 1$, $w_1 = 0$ of (3.35) *satisfies*

$$0 \le w(x) \le C. \tag{3.38}$$

Proof
By applying maximum principle we conclude from $g > 0$, $w_1 > 0$ that $w(x) \geq 0$ holds.
Again, we use the solution representation

$$w(x) = -\int_x^1 \hat{w}(\xi)\,\mathrm{d}\xi + K\left(1 - e^{-\beta\frac{1-x}{\varepsilon}}\right) \quad \text{where} \quad \hat{w}(\xi) = \int_\xi^1 \frac{g(\zeta)e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta.$$

From the boundary conditions and

$$\begin{aligned} 0 \leq \int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi &= \int_0^1 \Big(1 - \ln(1-\zeta)\Big) \int_0^\zeta \frac{e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\xi\,\mathrm{d}\zeta \\ &= \int_0^1 \Big(1 - \ln(1-\zeta)\Big)\frac{1 - e^{-\beta\frac{\zeta}{\varepsilon}}}{\beta}\,\mathrm{d}\zeta \leq \frac{2}{\beta} \end{aligned}$$

we conclude

$$|K| = \left|\frac{\int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi + w_0}{1 - e^{-\frac{\beta}{\varepsilon}}}\right| \leq \frac{\frac{2}{\beta} + 1}{1 - e^{-\frac{\beta}{\varepsilon}}} \leq C.$$

Since we know $\hat{w}(\xi) \geq 0$ we conceive

$$0 \leq \int_x^1 \hat{w}(\xi)\,\mathrm{d}\xi \leq \int_0^1 \hat{w}(\xi)\,\mathrm{d}\xi \leq \frac{2}{\beta}.$$

Thus, we conclude $w(x) \leq C$. □

Lemma 3.19
For $|g(x)| \leq Ce^{-\beta\frac{1-x}{\varepsilon}}$ *and* $|w_0| + |w_1| \leq C$ *we have*

$$|w^{(k)}(x)| \leq C\varepsilon^{-k}e^{-\beta\frac{1-x}{\varepsilon}} \tag{3.39}$$

for $k \in \{1, 2\}$.

Proof
The solution w is given by

$$w(x) = -\int_0^x \tilde{w}(\xi)\,\mathrm{d}\xi + K_1\left(e^{-\beta\frac{1-x}{\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}\right) + K_2 \quad \text{with}$$
$$\tilde{w}(\xi) = \int_0^\xi \frac{g(\zeta)e^{-\beta\frac{\zeta-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta.$$

Using the prerequisites on g we deduce

$$\begin{aligned} |\tilde{w}(\xi)| &\leq \int_0^\xi \frac{Ce^{-\beta\frac{1-\xi}{\varepsilon}}}{\varepsilon}\,\mathrm{d}\zeta = \frac{C}{\varepsilon}\xi e^{-\beta\frac{1-\xi}{\varepsilon}} \quad \text{and} \\ |\tilde{w}'(\xi)| &= \left|\frac{\beta}{\varepsilon}\tilde{w}(\xi) + \frac{g(\xi)}{\varepsilon}\right| \leq C\frac{\xi + \varepsilon}{\varepsilon^2}e^{-\beta\frac{1-\xi}{\varepsilon}}. \end{aligned}$$

The boundary condition at $x = 0$ implies

$$|K_2| = |w(0)| = |w_0| \leq C.$$

From the estimate of $\tilde{w}$ from above we get

$$\left|\int_0^1 \tilde{w}(\xi)\,\mathrm{d}\xi\right| \leq \int_0^1 \frac{C}{\varepsilon} e^{-\beta\frac{1-\xi}{\varepsilon}}\,\mathrm{d}\xi \leq \frac{C}{\beta}$$

and we obtain, using the boundary condition at $x = 1$,

$$|K_1| = \left|\frac{-\int_0^1 \tilde{w}(\xi)\,\mathrm{d}\xi - K_2 + w_1}{1 - e^{-\frac{\beta}{\varepsilon}}}\right| \leq \frac{\tilde{C} + |K_2| + |w_1|}{1 - e^{-\frac{\beta}{\varepsilon}}} \leq C.$$

Combining this estimates we get

$$\begin{aligned} |w_x(x)| &= \left|-\tilde{w}(x) + K_1\frac{\beta}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}}\right| \leq \frac{C}{\varepsilon}e^{-\beta\frac{1-x}{\varepsilon}} \quad \text{and} \\ |w_{xx}(x)| &= \left|-\tilde{w}'(x) + K_1\frac{\beta^2}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}}\right| \leq \frac{C}{\varepsilon^2}e^{-\beta\frac{1-x}{\varepsilon}}. \end{aligned}$$ □

3.1.3. Estimation Details for the Solution Decomposition

We recall that the smooth part u^S is defined via problem (3.5), a prolongation of the differential equation to the half plane $\mathcal{H} = (0, \infty) \times \mathbb{R}$:

$$\begin{aligned} Lu^S = -\varepsilon\Delta u^S + \beta u^S_x + cu^S = \omega_B \mathfrak{E}^{\mathcal{H}}_{\Omega} f = f^u \text{ in } \mathcal{H}, \\ u^S(0, \cdot) = 0, \qquad \lim_{\|(x,y)\|\to\infty} u^S(x, y) = 0. \end{aligned}$$

Lemma 3.20
We have

$$\|u^S\|_{1,\infty,\mathcal{H}} + \sqrt{\varepsilon}\|u^S_{xx}\|_{\infty,\mathcal{S}} + \sqrt{\varepsilon}\|u^S_{yy}\|_{\infty,\mathcal{H}} \leq C \quad \textit{and} \tag{3.40a}$$

$$\|u^S_{xx}\|_{2,\Omega} + \|u^S_{xy}\|_{\infty,\mathcal{H}} \leq C|\ln(\varepsilon)|. \tag{3.40b}$$

Proof
Note, we have $\operatorname{supp} f^u \subseteq B_2(0, 0)$. Applying Lemma 3.8 to the differential equation problem (3.5) we get $|u^S| \leq Ce^{-\alpha\varrho}$. Reordering the differential equation leads to

$$-\varepsilon\Delta u^S + \beta u^S_x = f^u - cu^S =: f^{u*}.$$

From $|f^{u*}| \leq Ce^{-\alpha\varrho}$ and the estimates for the Green's function of this problem provided in Lemma 3.10 we conclude

$$\begin{aligned} \|u^S_\zeta\|_{\infty,(0,\varepsilon)\times\mathbb{R}} &\leq \|f^{u*}\|_\infty \|g_\zeta(x, y; \cdot, \cdot)\|_1 \leq C \quad \text{and} \\ |u^S_\zeta(x, y)| &\leq \tilde{C}\|g_\zeta(x, y; \cdot, \cdot)w^e(\cdot, \cdot)\|_1 \leq \frac{C}{\sqrt{\varepsilon\varrho}} \end{aligned}$$

for $\zeta \in \{x, y\}$. Therefore, we can differentiate the equation (3.5) and receive

$$Lu^S_\zeta = -\varepsilon\Delta(u^S_\zeta) + \beta(u^S_\zeta)_x + cu^S_\zeta = f^u_\zeta - c_\zeta u^S, \tag{3.41a}$$

$$|u^S_\zeta(0, \cdot)| \leq C, \qquad \lim_{\|(x,y)\|\to\infty} u^S(x, y) = 0 \tag{3.41b}$$

for $\zeta \in \{x, y\}$. Hence, we can apply Lemma 3.7 and receive

$$\|u_x^S\|_\infty + \|u_y^S\|_\infty \leq C.$$

Reordering the terms of (3.41) gives

$$-\varepsilon\Delta(u_y^S) + \beta(u_y^S)_x = f_y^u - c_y u^S - c u_y^S =: f^{u*,y} \text{ in } \mathcal{H},$$
$$u_y^S(0,\cdot) = 0, \qquad \lim_{\|(x,y)\|\to\infty} u_y^S(x,y) = 0.$$

Since we know already $\|f^{u*,y}\|_\infty \leq C$ we can apply Lemma 3.10 to acquire

$$|u_{yx}^S(x,y)| \leq \|f^{u*,y}\|_\infty \|g_x(x,y;\cdot,\cdot)\|_1 \leq \begin{cases} C, & x \in [0,\varepsilon], \\ C|\ln(\varepsilon)|, & x \in (\varepsilon, 1), \end{cases}$$
$$|u_{yy}^S(x,y)| \leq \|f^{u*,y}\|_\infty \|g_y(x,y;\cdot,\cdot)\|_1 \leq \frac{C}{\sqrt{\varepsilon}} \text{ for } x \in [0,1].$$

Due to the nonzero boundary condition for the differentiated problem for u_x^S we can not apply this technique to get bounds for u_{xx}^S. We circumvent this problem by the splitting

$$u^S(x,\cdot) = \tilde{u}(x,\cdot) + \hat{u}(x,\cdot) \quad \text{with} \quad \begin{aligned} L^{1\mathrm{D}}\tilde{u} &= f^{u*}, \quad \tilde{u}(0,\cdot) = 0, \quad \tilde{u}(1,\cdot) = u^S(1,\cdot), \\ L^{1\mathrm{D}}\hat{u} &= \varepsilon u_{yy}^S, \quad \hat{u}(0,\cdot) = 0, \quad \hat{u}(1,\cdot) = 0. \end{aligned}$$

From the previous calculations we know $\|f^{u*}\|_{1,\infty,\Omega} \leq C$ and $\|\varepsilon u_{yy}^S\|_{\infty,\Omega} \leq C\sqrt{\varepsilon}$. Thus, we can apply Lemma 3.16 to acquire

$$|\tilde{u}_{xx}| \leq C\Big(1 + \varepsilon^{-2}\mathcal{E}_1^x(x)\Big) \quad \text{and} \quad |\hat{u}_{xx}| \leq C\left(\varepsilon^{-\frac{1}{2}} + \varepsilon^{-\frac{3}{2}}\mathcal{E}_1^x(x)\right),$$

which gives the preliminary bound

$$|u_{xx}^S(x,y)| \leq C\left(\varepsilon^{-\frac{1}{2}} + \varepsilon^{-2}\mathcal{E}_1^x(x)\right).$$

For $x \in (2\varepsilon, 1)$ we estimate by considering the problem

$$-\varepsilon\Delta(u_x^S) + \beta(u_x^S)_x = f_x^u - c_x u^S - c u_x^S =: f^{u*,x} \text{ in } \mathcal{H},$$
$$u_x^S(0,\cdot) = \nu^x, \qquad \lim_{\|(x,y)\|\to\infty} u_x^S(x,y) = 0$$

which we receive from reordering the terms of (3.41). Note that we have by the estimates above

$$\|\nu^x\|_{1,\infty,\mathbb{R}} = \|u_x^S(0,\cdot)\|_{\infty,\mathbb{R}} + \|u_{xy}^S(0,\cdot)\|_{\infty,\mathbb{R}} \leq C, \quad \|f^{u*,x}\|_\infty \leq C.$$

Now we split $u_x^S = \tilde{u} + \hat{u}$ in two parts satisfying

$$\begin{aligned} -\Delta\tilde{u} &= 0, & \tilde{u}(0,y) &= \nu^x(y), & \lim_{\|(x,y)\|\to\infty} \tilde{u}(x,y) &= 0, \\ -\varepsilon\Delta\hat{u} + \beta\hat{u}_x &= f^{u*,x} - \beta\tilde{u}_x, & \hat{u}(0,y) &= 0, & \lim_{\|(x,y)\|\to\infty} \hat{u}(x,y) &= 0. \end{aligned}$$

From Lemma 3.14 we know $|-\varepsilon\Delta\tilde{u} + \beta\tilde{u}_x| = \beta|\tilde{u}_x| \leq C\max\bigl(-\ln(x), 1\bigr)$. Thus, the Green's function representation gives with Lemma 3.11 the estimate

$$|\hat{u}_x(x,y)| \leq \tilde{C}\|g_x(x,y;\cdot,\cdot)w^l(\cdot,\cdot)\|_{1,\mathcal{H}} \leq C|\ln(\varepsilon)|(1+|\ln(x)|).$$

Combining these results we obtain

$$|u^S_{xx}| \leq \begin{cases} \frac{C}{\sqrt{\varepsilon}}, & x \leq 2\varepsilon, \\ C|\ln(\varepsilon)|(1+|\ln(x)|), & x > 2\varepsilon. \end{cases}$$

By integration we receive

$$\|u^S_{xx}\|_{2,\Omega} \leq C|\ln(\varepsilon)|.$$

□

Next we recall the definition of the layer correction term u^{x1} given in (3.6)

$$Lu^{x1} = 0, \quad u^{x1}(0,\cdot) = 0, \quad u^{x1}(1,\cdot) = -\omega_I(\cdot)u^S(1,\cdot).$$

Lemma 3.21
For u^{x1} we have the estimates

$$|u^{x1}(x,y)| + |u^{x1}_y(x,y)| + \sqrt{\varepsilon}|u^{x1}_{yy}(x,y)| \leq C\mathcal{E}^x_1(x), \tag{3.42a}$$

$$\varepsilon|u^{x1}_x(x,y)| + \varepsilon^2|u^{x1}_{xx}(x,y)| \leq C\mathcal{E}^x_1(x) \quad \textit{and} \quad \|u^{x1}_{xy}(x,\cdot)\|_{2,(0,1)} \leq \frac{C}{\varepsilon}\mathcal{E}^x_1(x). \tag{3.42b}$$

Proof
By application of maximum principle with comparison function $w^c = e^{-\beta\frac{1-x}{\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}$ and successive differentiation in y-direction we get

$$|\partial^k_y u^{x1}(x,y)| \leq \|\partial^k_y u^S\|_{\infty,\mathbb{R}}\left(e^{-\beta\frac{1-x}{\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}\right)$$

for $k \in \{0,1,2\}$.

Reordering the differential equation leads to

$$L^{1\mathrm{D}}u^{x1} = -\varepsilon u^{x1}_{xx} + \beta u^{x1}_x = \varepsilon u^{x1}_{yy} - cu^{x1}, \quad u^{x1}(0,\cdot) = 0, \quad u^{x1}(1,\cdot) = \nu_1(\cdot).$$

Since $|\nu_1(y)| \leq \|u^{x1}\|_\infty \leq C$ we can apply Lemma 3.19 to acquire

$$|\partial^k_x u^{x1}(x,y)| \leq C\varepsilon^{-k}\mathcal{E}^x_1(x)$$

for $k \in \{1,2\}$.

To obtain bounds for the mixed derivative, we split $u^{x1} = \tilde{u} + \hat{u}$ with

$$\begin{aligned} &\hat{L}\tilde{u} = 0, && \tilde{u}(0,\cdot) = 0, && \tilde{u}(1,\cdot) = -\omega_I u^S \quad \text{and} \\ &\hat{L}\hat{u} = -cu^{x1}, && \hat{u}(0,\cdot) = 0, && \hat{u}(1,\cdot) = 0. \end{aligned}$$

We estimate $\hat{u}$ by differentiation of the differential equation and receive

$$\hat{L}\hat{u}_y = -c_y u^{x1} - cu^{x1}_y =: \hat{f}, \qquad \hat{u}_y(0,\cdot) = 0, \qquad \hat{u}_y(1,\cdot) = 0, \tag{3.43a}$$

$$\hat{L}\hat{u}_{yy} = \hat{f}_y, \qquad \hat{u}_{yy}(0,\cdot) = 0, \qquad \hat{u}_{yy}(1,\cdot) = 0. \tag{3.43b}$$

From the previous estimates we know $|\hat{f}(x,\cdot)| + \sqrt{\varepsilon}|\hat{f}_y(x,\cdot)| \leq C\mathcal{E}_1^x(x)$. Applying a maximum principle with comparison function $w^c = (1-x)\left(e^{-\beta\frac{1-x}{\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}\right)$ to each problem of (3.43) we get

$$|\hat{u}_y(x,\cdot)| + \sqrt{\varepsilon}|\hat{u}_{yy}(x,\cdot)| \leq C\mathcal{E}_1^x(x) \quad \text{and} \quad |\hat{u}_{xyy}(1,\cdot)| \leq \frac{C}{\sqrt{\varepsilon}}.$$

Next we apply the Green's function representation formula (3.29) to the differentiated problem (3.43b)

$$\hat{u}_{yyy}(x,y) = \int_{\mathbb{S}} \tilde{g}_y(x,y;\xi,\eta)\hat{f}_y(\xi,\eta)\,\mathrm{d}\lambda(\xi,\eta) - \int_{\mathbb{R}} \tilde{g}_y(x,y;1,\eta)\hat{u}_{xyy}(1,\eta)\,\mathrm{d}\lambda(\eta)$$

and deduce from Corollary 3.12 and Lemma 3.13 the estimate

$$|\hat{u}_{yyy}(x,y)| \leq \frac{\tilde{C}}{\sqrt{\varepsilon}}\left(\|\tilde{g}_y(x,y;\cdot,\cdot)w^{\mathcal{E}^x}\|_{1,\mathbb{S}} + \|\tilde{g}_y(x,y;1,\cdot)\|_{1,\mathbb{R}}\right) \leq \frac{C}{\varepsilon}\mathcal{E}_1^x(x).$$

Hence, we have

$$L^{1\mathrm{D}}\hat{u}_y := -\varepsilon\hat{u}_{xxy} + \beta\hat{u}_{xy} = \hat{f} + \varepsilon\hat{u}_{yyy}, \quad \hat{u}_y(0,\cdot) = 0, \quad \hat{u}_y(1,\cdot) = 0$$

and conclude using Lemma 3.16 and Lemma 3.19 that

$$|\hat{u}_{xy}(x,y)| \leq \tilde{C}\left(\frac{1}{\varepsilon}\mathcal{E}_1^x(x) + \frac{1}{\sqrt{\varepsilon}}\mathcal{E}_1^x(x)\right) \leq \frac{C}{\varepsilon}\mathcal{E}_1^x(x)$$

holds.

To get bounds for $\tilde{u}$, we use techniques similar to the proof of [NKS09, Lemma 3.1]. We define the Fourier transform in y-direction via

$$\mathcal{F}\tilde{u}(x,\eta) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iy\eta}\tilde{u}(x,y)\,\mathrm{d}y.$$

Then, we get

$$0 = \mathcal{F}(\hat{L}\tilde{u}) = -\varepsilon\mathcal{F}(\tilde{u})_{xx} + \beta\mathcal{F}(\tilde{u})_x + \varepsilon\eta^2\mathcal{F}(\tilde{u}),$$
$$\mathcal{F}(\tilde{u})(0,\cdot) = 0, \quad \mathcal{F}(\tilde{u})(1,\cdot) = \mathcal{F}(-\omega_I u^S)(1,\cdot) =: \tilde{\nu}(\cdot).$$

Consequently, we have

$$\mathcal{F}(\tilde{u})(x,\eta) = \tilde{\nu}(\eta)\frac{e^{-\frac{\left(\beta+\sqrt{\beta^2+4\varepsilon^2\eta^2}\right)(1-x)}{2\varepsilon}} - e^{-\frac{\beta}{\varepsilon}}e^{-\frac{\left(-\beta+\sqrt{\beta^2+4\varepsilon^2\eta^2}\right)(1+x)}{2\varepsilon}}}{1-e^{-\frac{\sqrt{\beta^2+4\varepsilon^2\eta^2}}{\varepsilon}}}$$

and

$$|\mathcal{F}(\tilde{u}_{xy})(x,\eta)| = |\eta\mathcal{F}(\tilde{u})_x(x,\eta)| \leq \frac{C}{\varepsilon}|\eta\tilde{\nu}(\eta)|\mathcal{E}_1^x(x) = \frac{C}{\varepsilon}\left|\mathcal{F}\left((\omega_I u^S)_y\right)(x,\eta)\right|\mathcal{E}_1^x(x).$$

From this formula we get by Plancherel's theorem immediately

$$\|\tilde{u}_{xy}(x,\cdot)\|_{2,\mathbb{R}} = \|\mathcal{F}(\tilde{u}_{xy})(x,\cdot)\|_{2,\mathbb{R}} \leq \frac{\tilde{C}}{\varepsilon}\|\omega_I' u^S + \omega_I u_y^S\|_{2,\mathbb{R}}\mathcal{E}_1^x(x)$$
$$\leq \frac{\hat{C}}{\varepsilon}\|u^S(1,\cdot)\|_{1,\infty,(-1,2)}\mathcal{E}_1^x(x) \leq \frac{C}{\varepsilon}\mathcal{E}_1^x(x). \qquad \square$$

Recall the definition (3.7) of the characteristic layer correction at $y \in \{0,1\}$

$$Lu^y = 0, \quad u^y(0,\cdot) = 0, \quad u^y(\cdot,0) = -\omega_I(\cdot)u^S(\cdot,0), \quad u^y(\cdot,1) = -u^S(\cdot,1)\omega_I(\cdot). \tag{3.44}$$

Note that the boundary conditions are continuous because we have $u^S(0,\cdot) = 0$.

Lemma 3.22
We have

$$|u^y(x,y)| + |u^y_x(x,y)| + \sqrt{\varepsilon}|u^y_y(x,y)| + \varepsilon|u^y_{yy}(x,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big) \tag{3.45}$$

for $(x,y) \in \Omega$.

Proof
Since we know $|u^S| + |u^S_x| \le C$ and $u^S(0,\cdot) = 0$ we apply a maximum principle with $w^c = \mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)$ to acquire

$$|u^y(\cdot,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big).$$

Additionally, we can apply on Ω a maximum principle with $w^c = x\big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\big)$ to get

$$|u^y_x(0,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big).$$

Since we know $u^y_x\big|_{y\in\{0,1\}} = -\omega_I u^S_x\big|_{y\in\{0,1\}} - \omega_I' u^S\big|_{y\in\{0,1\}}$ we can use a maximum principle on the differentiated equation to prove

$$|u^y_x(x,y)| \le C(1+x)\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big).$$

To acquire bounds for u^y_y we split $u^y = \tilde{u} - \hat{u}$ with

$$\hat{u}(x,y) := u^S(x,0)\omega_I(x)\frac{e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}}{1 - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}} + u^S(x,1)\omega_I(x)\frac{e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}}{1 - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}}. \tag{3.46}$$

Thus, we have

$$|L\tilde{u}| = |L\hat{u}| \le C\left(\varepsilon\|u^S_{xx}\|_\infty + \|u^S_x\|_\infty + \|u^S\|_\infty\right)\left(\mathcal{E}^y_0 + \mathcal{E}^y_1\right) \le C\big(\mathcal{E}^y_0 + \mathcal{E}^y_1\big)$$

and

$$\tilde{u}(0,\cdot) = 0, \quad \tilde{u}(\cdot,0) = \tilde{u}(\cdot,1) = 0.$$

Hence, we can apply a maximum principle with comparison function

$$w^{c0} := \frac{C}{\sqrt{\varepsilon}}\left(ye^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + (1-y)e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}\frac{e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}y} + e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}(1-y)}}{1 + e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}}}\right)$$

to deduce

$$|\tilde{u}(x,y)| \le w^{c0}$$

and conclude

$$\|u^y_y\|_{\infty,(0,\infty)\times\{0,1\}} \le \|\tilde{u}_y\|_{\infty,(0,\infty)\times\{0,1\}} + \|\hat{u}_y\|_{\infty,(0,\infty)\times\{0,1\}} \le \frac{C}{\sqrt{\varepsilon}}.$$

We use $u_y^y(0,\cdot)=0$ and a maximum principle on the differentiated equation and get

$$|u_y^y(x,y)| \le \frac{C}{\sqrt{\varepsilon}}(1+x)\Big(\mathcal{E}_0^y(y)+\mathcal{E}_1^y(y)\Big).$$

For the second order y-derivative we have by $\partial_x^k u^y\big|_{(0,\infty)\times\{0,1\}} = -\partial_x^k(\omega_I u^S)$ for $k\in\{0,1,2\}$ and the differential equation

$$\begin{aligned}\varepsilon\|u_{yy}^y\|_{\infty,(0,\infty)\times\{0,1\}} &= \|-\varepsilon u_{xx}^y + \beta u_x^y + cu^y\|_{\infty,(0,\infty)\times\{0,1\}}\\ &\le \tilde{C}\left(\varepsilon\|u_{xx}^S\|_{\infty,(0,\infty)\times\{0,1\}} + \|u^S\|_{1,\infty,(0,\infty)\times\{0,1\}}\right) \le C.\end{aligned}$$

Thus, we have

$$|Lu_{yy}^y| \le \frac{C}{\sqrt{\varepsilon}}\big(\mathcal{E}_0^y(y)+\mathcal{E}_1^y(y)\big),$$
$$u_{yy}^y(0,\cdot)=0, \quad \|u_{yy}^y\|_{\infty,(0,\infty)\times\{0,1\}} = \frac{C}{\varepsilon}$$

and conclude using a maximum principle

$$|u_{yy}^y(x,y)| \le \frac{C}{\varepsilon}(1+x)\big(\mathcal{E}_0^y(y)+\mathcal{E}_1^y(y)\big).$$

□

As a last step we recall the definition (3.8) of the corner layer correction

$$Lu^{c1}=0, \quad u^{c1}\big|_{x=0}=0, \quad u^{c1}\big|_{y\in\{0,1\}} = -u^{x1}, \quad u^{c1}\big|_{x=1} = -u^y.$$

Note that the boundary conditions posed on u^{c1} are continuous since we have $-u^{x1}(1,0) = u^S(1,0) = -u^y(1,0)$ and $-u^{x1}(1,1) = u^S(1,1) = -u^y(1,1)$.

Lemma 3.23
We have

$$|u^{c1}(x,y)| + \varepsilon|u_x^{c1}(x,y)| + \sqrt{\varepsilon}|u_y^{c1}(x,y)| \le C\mathcal{E}_1^x(x)\Big(\mathcal{E}_0^y(y)+\mathcal{E}_1^y(y)\Big), \tag{3.47a}$$

$$\|u_{xx}^{c1}\|_{2,\Omega} \le C\varepsilon^{-\frac{5}{4}}, \quad \|u_{xy}^{c1}\|_{2,\Omega} \le C\varepsilon^{-\frac{3}{4}} \quad \textit{and} \quad \|u_{yy}^{c1}\|_{2,\Omega} \le C\varepsilon^{-\frac{1}{4}}. \tag{3.47b}$$

Proof
From a maximum principle we get

$$|u^{c1}(x,y)| \le Ce^{-\frac{\beta}{\varepsilon}(1-x)}\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right).$$

We split $u^{c1} = \tilde{u} - \hat{u}$ with

$$\hat{u}(x,y) := \Big(u^y(1,y) - Y\big(u^{x1}(1,0), u^{x1}(1,1), y\big)\Big)X(x) + Y\big(u^{x1}(x,0), u^{x1}(x,1), y\big),$$

$$X(x) := \frac{e^{-\frac{\beta}{\varepsilon}(1-x)} - e^{-\frac{\beta}{\varepsilon}}}{1-e^{-\frac{\beta}{\varepsilon}}}, \quad Y(r,s,y) := \frac{r\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}\right) + s\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}\right)}{1-e^{-\sqrt{\frac{\gamma}{\varepsilon}}}}.$$

Using $|-\varepsilon u_{xx}^{x1} + \beta u_x^{x1}| = |\varepsilon u_{yy}^{x1} - cu^{x1}| \le C\mathcal{E}_1^x$ we conclude

$$\begin{aligned}|L\hat{u}| &\le |-\varepsilon u_{yy}^y + cu^y|X + CY(1,1,\cdot)X + |-\varepsilon u_{xx}^{x1} + \beta u_x^{x1}|Y(1,1,\cdot) + |u^{x1}|Y(1,1,\cdot)\\ &\le C\mathcal{E}_1^x\big(\mathcal{E}_0^y + \mathcal{E}_1^y\big).\end{aligned}$$

Hence, $\tilde{u}$ satisfies

$$|L\tilde{u}| = |L\hat{u}| \le C\mathcal{E}_1^x(x)\Big(\mathcal{E}_0^y(y) + \mathcal{E}_1^y(y)\Big), \quad \tilde{u}\big|_{\partial\Omega} = 0. \tag{3.48}$$

We apply a maximum principle with comparison function

$$w^c = (1-x)\left(e^{-\frac{\beta}{\varepsilon}(1-x)} - e^{-\frac{\beta}{\varepsilon}}\right)\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right)$$

to (3.48) and acquire

$$|\tilde{u}_x\big|_{x=0}| \le C\frac{e^{-\frac{\beta}{\varepsilon}}}{\varepsilon}\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right) \quad \text{and}$$
$$|\tilde{u}_x\big|_{x=1}| \le C\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right).$$

Similarly, we use a maximum principle with comparison function

$$w^c = \frac{e^{-\frac{\beta}{\varepsilon}(1-x)}}{\sqrt{\varepsilon}}\left(ye^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + (1-y)e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)} - e^{-\sqrt{\frac{\gamma}{\varepsilon}}}\frac{e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}y} + e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}(1-y)}}{1+e^{-\sqrt{\frac{\|c\|_\infty}{\varepsilon}}}}\right)$$

to obtain

$$|\tilde{u}_y\big|_{y\in\{0,1\}}| \le \frac{C}{\sqrt{\varepsilon}}e^{-\frac{\beta}{\varepsilon}(1-x)}.$$

Thus, we have

$$|Lu_x^{c1}| = |c_x u^{c1}| \le C\mathcal{E}_1^x\big(\mathcal{E}_0^y + \mathcal{E}_1^y\big), \qquad |u_x^{c1}\big|_{\partial\Omega}| \le \frac{C}{\varepsilon}\mathcal{E}_1^x\big(\mathcal{E}_0^y + \mathcal{E}_1^y\big) \quad \text{and}$$
$$|Lu_y^{c1}| = |c_y u^{c1}| \le C\mathcal{E}_1^x\big(\mathcal{E}_0^y + \mathcal{E}_1^y\big), \qquad |u_y^{c1}\big|_{\partial\Omega}| \le \frac{C}{\sqrt{\varepsilon}}\mathcal{E}_1^x\big(\mathcal{E}_0^y + \mathcal{E}_1^y\big).$$

Application of a maximum principle gives

$$|u_x^{c1}(x,y)| \le C\frac{2-x}{\varepsilon}e^{-\frac{\beta}{\varepsilon}(1-x)}\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right) \quad \text{and}$$
$$|u_y^{c1}(x,y)| \le C\frac{2-x}{\sqrt{\varepsilon}}e^{-\frac{\beta}{\varepsilon}(1-x)}\left(e^{-\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right).$$

For the second order derivatives we use the splitting $u^{c1} = \tilde{u} - \hat{u}$ again. Thus, we have

$$\Delta\tilde{u} = \Delta u^{c1} + \Delta\hat{u} = \varepsilon^{-1}(\beta u_x^{c1} + cu^{c1}) + \hat{u}_{xx} + \hat{u}_{yy}.$$

We use the variable transform $\tilde{x} := x, \tilde{y} := \sqrt{\varepsilon}y$, $\tilde{\Omega} := (0,1)\times(0,\sqrt{\varepsilon})$ and get

$$\begin{aligned}\|\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} &\le \beta\varepsilon^{-\frac{3}{4}}\|u_x^{c1}\|_{2,\Omega} + \|c\|_\infty\varepsilon^{-\frac{3}{4}}\|u^{c1}\|_{2,\Omega} + \varepsilon^{\frac{1}{4}}\|\hat{u}_{xx}\|_{2,\Omega} + \varepsilon^{-\frac{3}{4}}\|\hat{u}_{yy}\|_{2,\Omega}\\ &\le C\varepsilon^{-1}.\end{aligned}$$

Thus, the usual norm estimates for the second order derivatives of the solution to the Laplace equation (cf. [LU68]) provides

$$\|\tilde{u}_{xx}\|_{2,\Omega} = \varepsilon^{-\frac{1}{4}}\|\tilde{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}} \le C\varepsilon^{-\frac{5}{4}}, \quad \|\tilde{u}_{xy}\|_{2,\Omega} = \varepsilon^{\frac{1}{4}}\|\tilde{u}_{\tilde{x}\tilde{y}}\|_{2,\tilde{\Omega}} \le C\varepsilon^{-\frac{3}{4}} \quad \text{and}$$
$$\|\tilde{u}_{yy}\|_{2,\Omega} = \varepsilon^{\frac{3}{4}}\|\tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} \le C\varepsilon^{-\frac{1}{4}}.$$

A triangle inequality completes the proof. □

3.2. Error Estimates for Bilinear FEM on a Shishkin Mesh

We use a Shishkin mesh of the form depicted in Figure 3.6, where we split the domain in a course part and several finer subparts to compensate the boundary layers. In every part of the splitting we refine with an equidistant tensor grid using $N/2$ and $N/3$ intervals in x- and y-direction, respectively.

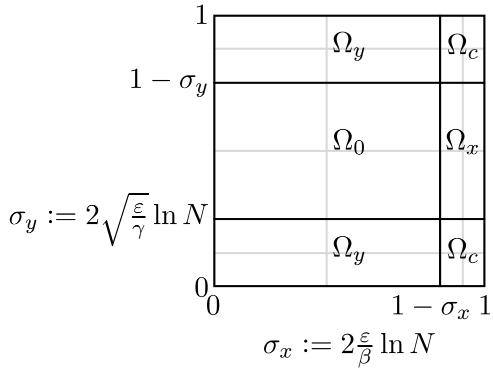

Figure 3.6.: Used Shishkin Mesh

We denote the mesh width in the coarse subdomain by $H \in \mathcal{O}(N^{-1})$. The fine mesh width in x-, y-direction in Ω_x, Ω_y we denote by $h_x \in \mathcal{O}(\varepsilon N^{-1}\ln N)$, $h_y \in \mathcal{O}(\sqrt{\varepsilon}N^{-1}\ln N)$, respectively. Additionall, we define $\Omega_{xc} := \Omega_x \cup \Omega_c$, $\Omega_{0y} := \Omega_0 \cup \Omega_y$, $\Omega_{yc} := \Omega_y \cup \Omega_c$ and $\Omega_{0x} := \Omega_0 \cup \Omega_x$, the subdomains with equal subdivision in either x- or y-direction. Furthermore, we define $\Omega_{0xy} := \Omega \setminus \Omega_c$, the subdomain without the corner sections.

Theorem 3.24
Provided the solution u of problem (3.1) *has a decomposition as in Theorem 3.1 and Conjecture 3.3 we have*

$$\|u - u^I\|_\varepsilon \le CN^{-1}\ln(N). \tag{3.49}$$

Proof
By standard anisotropic interpolation results (cf. [Ape99]) we can estimate

$$\begin{aligned}
\|u^S - u^{S^I}\|_2 &\le \tilde{C}H|u^S|_{1,2} \le CN^{-1},\\
\|u^{x1} - u^{x1^I}\|_{2,\Omega_{xc}} &\le \tilde{C}\left(h_x\|u_x^{x1}\|_{2,\Omega_{xc}} + H\|u_y^{x1}\|_{2,\Omega_{xc}}\right) \le C\varepsilon^{\frac{1}{2}}N^{-1}\ln(N),\\
\|u^y - u^{y^I}\|_{2,\Omega_{yc}} &\le \tilde{C}\left(H^2\|u_{xx}^y\|_{2,\Omega_{yc}} + Hh_y\|u_{xy}^y\|_{2,\Omega_{yc}} + h_y^2\|u_{yy}^y\|_{2,\Omega_{yc}}\right)\\
&\le C\varepsilon^{\frac{1}{4}}\left(|\ln(\varepsilon)|N^{-2}\ln(N) + N^{-2}\ln^2(N)\right),\\
\|u^{c1} - u^{c1^I}\|_{2,\Omega_c} &\le \tilde{C}\left(h_x^2\|u_{xx}^{c1}\|_{2,\Omega_c} + h_xh_y\|u_{xy}^{c1}\|_{2,\Omega_c} + h_y^2\|u_{yy}^{c1}\|_{2,\Omega_c}\right)\\
&\le C\varepsilon^{\frac{3}{4}}N^{-2}\ln^2(N),\\
|u^S - u^{S^I}|_{1,2} &\le \tilde{C}H|u^S|_{2,2} \le C\varepsilon^{-\frac{1}{2}}N^{-1},\\
|u^{x1} - u^{x1^I}|_{1,2,\Omega_{xc}} &\le \tilde{C}\left(h_x\|u_{xx}^{x1}\|_{2,\Omega_{xc}} + H\|u_{xy}^{x1}\|_{2,\Omega_{xc}} + H\|u_{yy}^{x1}\|_{2,\Omega_{xc}}\right)\\
&\le C\varepsilon^{-\frac{1}{2}}N^{-1}\ln(N),
\end{aligned}$$

$$
\begin{aligned}
|u^y - u^{yI}|_{1,2,\Omega_{yc}} &\le \tilde{C}\left(H\|u^y_{xx}\|_{2,\Omega_{yc}} + H\|u^y_{xy}\|_{2,\Omega_{yc}} + h_y\|u^y_{yy}\|_{2,\Omega_{yc}}\right)\\
&\le C\varepsilon^{-\frac{1}{4}}|\ln(\varepsilon)|N^{-1}\ln(N),\\
|u^{c1} - u^{c1I}|_{1,2,\Omega_c} &\le \tilde{C}\left(h_x\|u^{c1}_{xx}\|_{2,\Omega_c} + h_y\|u^{c1}_{xy}\|_{2,\Omega_c} + h_y\|u^{c1}_{yy}\|_{2,\Omega_c}\right)\\
&\le C\varepsilon^{-\frac{1}{4}}N^{-1}\ln(N).
\end{aligned}
$$

Using the decay of the boundary terms and inverse estimates we furthermore derive

$$
\begin{aligned}
\|u^{x1} - u^{x1I}\|_{2,\Omega_{0y}} &\le 2\|u^{x1}\|_{\infty,\Omega_{0y}} \le CN^{-2},\\
|u^{x1} - u^{x1I}|_{1,2,\Omega_{0y}} &\le |u^{x1}|_{1,2,\Omega_{0y}} + |u^{x1I}|_{1,2,\Omega_{0y}}\\
&\le \|u^{x1}_x\|_{2,\Omega_{0y}} + \|u^{x1}_y\|_{2,\Omega_{0y}} + 2h_y^{-1}\|u^{x1}\|_{\infty,\Omega_{0y}} \le C\varepsilon^{-\frac{1}{2}}N^{-1},\\
\|u^y - u^{yI}\|_{2,\Omega_{0x}} &\le 2\|u^y\|_{\infty,\Omega_{0x}} \le CN^{-2},\\
\|(u^y - u^{yI})_x\|_{2,\Omega_{0x}} &\le \tilde{C}H|u^y_x|_{1,2,\Omega_{0x}} \le C\varepsilon^{-\frac{1}{4}}|\ln(\varepsilon)|N^{-1},\\
\|(u^y - u^{yI})_y\|_{2,\Omega_{0x}} &\le \|u^y_y\|_{2,\Omega_{0x}} + \|u^{yI}_y\|_{2,\Omega_{0x}}\\
&\le \|u^y_y\|_{2,\Omega_{0x}} + 2H^{-1}\|u^y\|_{\infty,\Omega_{0x}} \le C\varepsilon^{-\frac{1}{4}}N^{-2},\\
\|u^{c1} - u^{c1I}\|_{2,\Omega_{0xy}} &\le 2\|u^{c1}\|_{\infty,\Omega_{0xy}} \le CN^{-2},\\
\|(u^{c1} - u^{c1I})_y\|_{2,\Omega_{0xy}} &\le \|u^{c1}_y\|_{2,\Omega_{0xy}} + 2h_y^{-1}\|u^{c1}\|_{\infty,\Omega_{0xy}} \le C\varepsilon^{-\frac{1}{2}}N^{-1},\\
\|(u^{c1} - u^{c1I})_x\|_{2,\Omega_{0y}} &\le \|u^{c1}_x\|_{2,\Omega_{0y}} + 2H^{-1}\|u^{c1}\|_{\infty,\Omega_{0y}} \le C\varepsilon^{-\frac{1}{2}}N^{-1},\\
\|(u^{c1} - u^{c1I})_x\|_{2,\Omega_x} &\le \|u^{c1}_x\|_{2,\Omega_x} + \|u^{c1I}_x\|_{2,\Omega_x}\\
&\le \|u^{c1}_x\|_{2,\Omega_x} + 2\sqrt{\lambda(\Omega_x)}h_x^{-1}\|u^{c1}\|_{\infty,\Omega_x} \le C\varepsilon^{-\frac{1}{2}}N^{-1}.
\end{aligned}
$$

All this estimates are attained using well-known techniques used e.g. in [Lin10]. □

Theorem 3.25
Assume the solution u of problem (3.1) has a decomposition as in Theorem 3.1 and Conjecture 3.3. Then the solution u^N of the first order FEM discretization of (3.1) on a Shishkin mesh according to Figure 3.6 has a numerical error satisfying

$$\|u^N - u^I\|_\varepsilon \le CN^{-1}\big(\ln(N) + |\ln(\varepsilon)|\big). \tag{3.50}$$

Proof
By the V-ellipticity of $a(\cdot,\cdot)$ and the Galerkin orthogonality we have

$$
\begin{aligned}
\tilde{C}\|u^I - u^N\|_\varepsilon^2 &\le a(u^I - u^N, u^I - u^N) = a(u^I - u, u^I - u^N)\\
&\le C\|u^I - u\|_\varepsilon\|u^I - u^N\|_\varepsilon + \beta\left|\left\langle (u^I - u)_x,\, u^I - u^N\right\rangle\right|.
\end{aligned}
$$

To estimate the remaining integral we use integration by parts in x-direction

$$\left\langle (u^I - u)_x,\, u^I - u^N\right\rangle_{(0,1)\times Y} = -\left\langle u^I - u,\, (u^I - u^N)_x\right\rangle_{(0,1)\times Y}$$

for subdomains $Y \subseteq (0,1)$. Now we split $u^I - u$ according to our solution decomposition and can estimate for the smooth part

$$
\begin{aligned}
\left|\left\langle (u^{SI} - u^S)_x,\, u^I - u^N\right\rangle\right| &\le \|u^{SI}_x - u^S_x\|_2\|u^I - u^N\|_2\\
&\le \tilde{C}H\left(\|u^S_{xx}\|_2 + \|u^S_{xy}\|_2\right)\|u^I - u^N\|_2\\
&\le CN^{-1}|\ln(\varepsilon)|\|u^I - u^N\|_\varepsilon.
\end{aligned}
$$

For the boundary layer term u^{x1} we get by an inverse inequality

$$\begin{aligned}
&\left|\left\langle (u^{x1^I} - u^{x1})_x,\, u^I - u^N \right\rangle\right| \\
&\quad = \left| \left\langle u^{x1^I} - u^{x1},\, (u^I - u^N)_x \right\rangle_{\Omega_{xc}} + \left\langle u^{x1^I} - u^{x1},\, (u^I - u^N)_x \right\rangle_{\Omega_{0y}} \right| \\
&\quad \le \tilde{C}\left(h_x |u^{x1}_x|_{2,\Omega_{xc}} + H |u^{x1}_y|_{2,\Omega_{xc}} \right) \frac{1}{\sqrt{\varepsilon}} \|u^I - u^N\|_{\varepsilon,\Omega_{xc}} \\
&\qquad + 2\|u^{x1}\|_{\infty,\Omega_{0y}} \frac{1}{H} \|u^I - u^N\|_{2,\Omega_{0y}} \\
&\quad \le C N^{-1} \ln(N) \|u^I - u^N\|_\varepsilon.
\end{aligned}$$

For the characteristic layer term u^y we can estimate similarly

$$\begin{aligned}
&\left|\left\langle (u^{y^I} - u^y)_x,\, u^I - u^N \right\rangle\right| \\
&\quad \le \left| \left\langle (u^{y^I} - u^y)_x,\, u^I - u^N \right\rangle_{\Omega_{yc}} \right| + \left| \left\langle u^{y^I} - u^y,\, (u^I - u^N)_x \right\rangle_{\Omega_{0x}} \right| \\
&\quad \le \|(u^{y^I} - u^y)_x\|_{2,\Omega_{yc}} \|u^I - u^N\|_{2,\Omega_{yc}} \\
&\qquad + \left| \left\langle u^{y^I} - u^y,\, (u^I - u^N)_x \right\rangle_{\Omega_0} \right| + \left| \left\langle u^{y^I} - u^y,\, (u^I - u^N)_x \right\rangle_{\Omega_x} \right| \\
&\quad \le \tilde{C}\left(H |u^y_{xx}|_{2,\Omega_{yc}} + h_y |u^y_{xy}|_{2,\Omega_{yc}} \right) \|u^I - u^N\|_{2,\Omega_{yc}} \\
&\qquad + 2\|u^y\|_{\infty,\Omega_0} \frac{1}{H} \|u^I - u^N\|_{2,\Omega_0} + 2\sqrt{\lambda(\Omega_x)} \|u^y\|_{\infty,\Omega_x} \frac{1}{\sqrt{\varepsilon}} \|u^I - u^N\|_{\varepsilon,\Omega_0} \\
&\quad \le C N^{-1} \ln(N) \|u^I - u^N\|_\varepsilon.
\end{aligned}$$

Finally, we estimate

$$\begin{aligned}
&\left|\left\langle (u^{c1^I} - u^{c1})_x,\, u^I - u^N \right\rangle\right| \\
&\quad \le \left| \left\langle u^{c1^I} - u^{c1},\, (u^I - u^N)_x \right\rangle_{\Omega_{0y}} \right| + \left| \left\langle u^{c1^I} - u^{c1},\, (u^I - u^N)_x \right\rangle_{\Omega_{xc}} \right| \\
&\quad \le 2\|u^{c1}\|_{\infty,\Omega_{0y}} \frac{1}{H} \|u^I - u^N\|_{2,\Omega_{0y}} + \|u^{c1^I} - u^{c1}\|_{2,\Omega_{xc}} \frac{1}{\sqrt{\varepsilon}} \|u^I - u^N\|_{\varepsilon,\Omega_{xc}} \\
&\quad \le 2\|u^{c1}\|_{\infty,\Omega_{0y}} \frac{1}{H} \|u^I - u^N\|_{2,\Omega_{0y}} \\
&\qquad + \tilde{C}\left(\sqrt{\lambda(\Omega_x)} \|u^{c1}\|_{\infty,\Omega_x} + h_x |u^{c1}_x|_{2,\Omega_c} + h_y |u^{c1}_y|_{2,\Omega_c} \right) \frac{1}{\sqrt{\varepsilon}} \|u^I - u^N\|_{\varepsilon,\Omega_{xc}} \\
&\quad \le C N^{-1} \ln(N) \|u^I - u^N\|_\varepsilon.
\end{aligned}$$

Combining these results we have

$$\|u^I - u^N\|_\varepsilon^2 \le C\left(\|u^I - u\|_\varepsilon + N^{-1}\ln(N) + N^{-1}|\ln(\varepsilon)| \right) \|u^I - u^N\|_\varepsilon$$

and the assertion follows by Theorem 3.24. □

3.3. Computational Results

In the following we solve the test problem

$$-\varepsilon\Delta u + u_x + u = f := \left|x - \tfrac{1}{2}\right| \left|y - \tfrac{1}{3}\right| + \left|\sin\!\left(6xy - \tfrac{3}{4}\right)\right|, \quad u\big|_{\partial\Omega} = 0. \tag{3.51}$$

Obviously, the first order weak derivatives of f exist and are bounded. Nevertheless, they have discontinuities along the curves $x = \frac{1}{2}$, $y = \frac{1}{3}$, $xy = \frac{1}{8}$ and $xy = \frac{1}{24}(4\pi + 3)$. Thus, f has wedges in Ω with almost any direction, minded the symmetry of the problem in the y-axis.

Unfortunately, we do not know an analytic solution of our test problem (3.51). To handle this difficulty we consider a reference solution u^R on a very fine mesh ($N = 7200$) computed with our method to be almost the exact solution. All errors shown below are calculated using this reference solution u^R instead of the exact solution u.

A plot of u^R for $\varepsilon = 10^{-4}$ is given in Figure 3.7. Note that the layer regions are stretched in the plot. We can see clearly the different boundary layers.

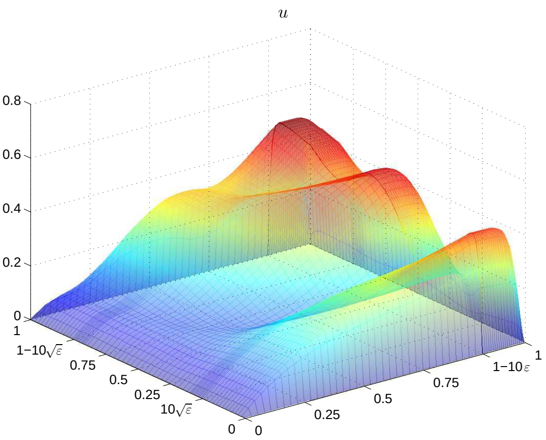

Figure 3.7.: Plot of the reference solution u^R for $\varepsilon = 10^{-4}$

We discretize our test problem (3.51) with bilinear finite elements using a Shishkin mesh as depicted in Figure 3.6 with $N + 1$ intervals in x- and y-direction each. Thus, we have a total of N^2 degrees of freedom, the function values at the inner mesh points. The attained

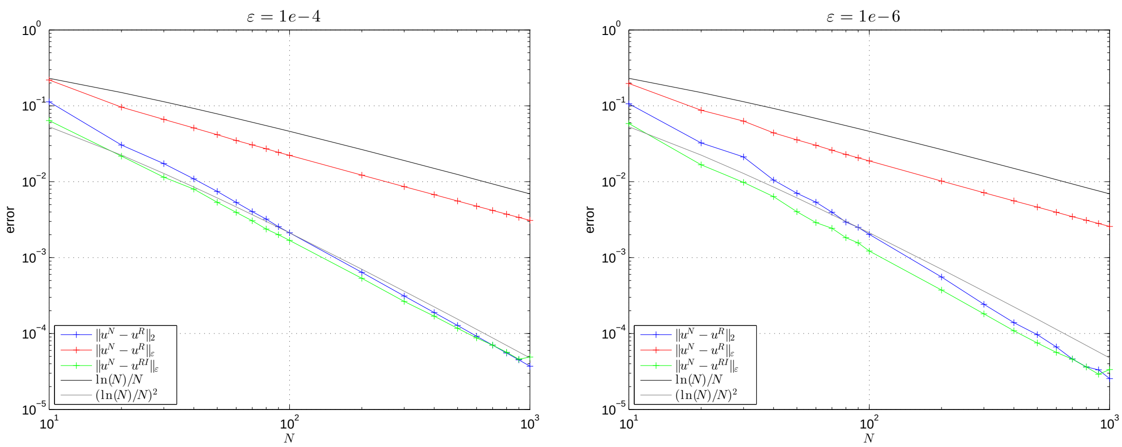

Figure 3.8.: Error of the bilinear FEM on a Shishkin mesh (cf. Figure 3.6)

convergence rates are shown in Figure 3.8.

These results confirm the assertion of an almost ε-independent almost first order convergence from Theorem 3.25. In fact, we are not able to verify the factor $|\ln(\varepsilon)|$ in the error. The attained rates rather suggest that this factor does not occur in practice but is an oddity in the proofs presented here.

Moreover, we observe even an ε-independent almost second order superconvergence of $\|u^N - u^I\|_\varepsilon$, although we were not able to prove this. The convergence break-down one may suspect at the end of the plot most likely results from the error $\|u^R - u\|_\varepsilon$, since the reference solution is not evaluated at its grid points to attain u^{RI} on the considered mesh. This improved convergence may result from the fact that the inhomogeneity f is a very smooth function in the preponderant part of the domain.

Finally, we present the plots of the reference solution along the line $x = 0.6$ in Figure 3.9. Note that the inhomogeneity has wedges on this line at $y_0 = \frac{5}{24} \approx 0.208$ and

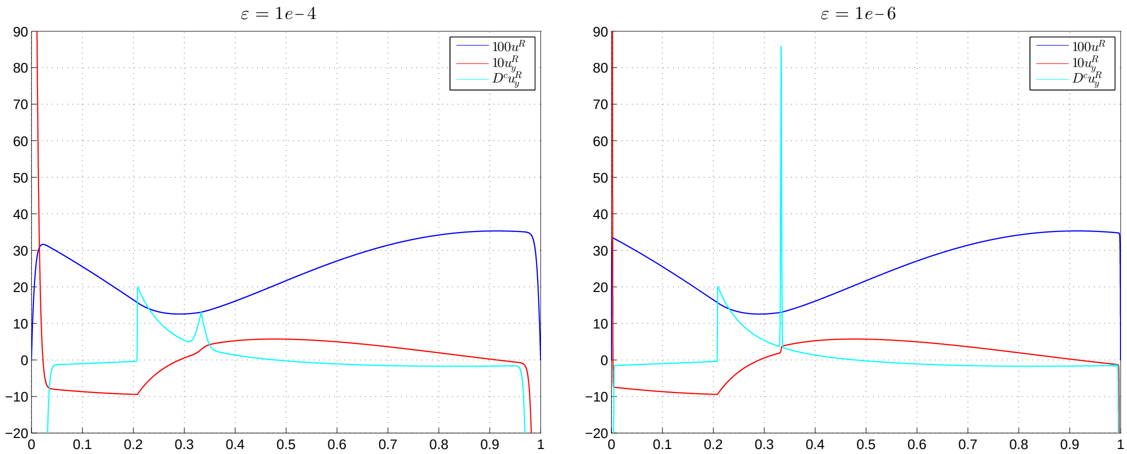

Figure 3.9.: Reference solution u^R plotted along the line $x = 0.6$

$y_1 = \frac{1}{3} \approx 0.333$. Since we only use bilinear elements to compute the reference solution we use central differences to get an approximation $D^c u_y^R$ of the derivative u_{yy}. As discussed in Remark 3.2 we expect u_{yy} to be of order $\varepsilon^{-1/2}$. At least near the wedge that is parallel to the x-axis at y_1 we observe this behavior. In the vicinity of the almost parallel wedge at y_0 this is not the case. Although we have something that looks like a layer near y_0, its magnitude does not change when ε gets smaller.

4. Optimal Control with Singularly Perturbed Convection-Diffusion Equations in 2D

In the following we want to analyze the optimal control problem

$$\min_{u,q\in\mathcal{L}^2} J(u,q) := \min_{u,q\in\mathcal{L}^2}\left(\frac{1}{2}\|u-u_d\|_2^2+\frac{\mu}{2}\|q\|_2^2\right) \tag{4.1a}$$

subject to the singularly perturbed convection-diffusion equation

$$Lu := -\varepsilon\Delta u+\beta u_x+cu = f+q \text{ in } \Omega := (0,1)^2, \quad u\big|_{\partial\Omega}=0. \tag{4.1b}$$

and the box constraints

$$q\in Q_{\text{ad}} :\Longleftrightarrow -\infty\le q_a\le q\le q_b\le\infty \text{ in } \Omega \tag{4.1c}$$

for the control q.

As in Chapter 2 we introduce the adjoint state v and receive the equivalent optimality system

$$Lu = f-\mu^{-1}\Pi_{[v_a,v_b]}(v), \tag{4.2a}$$
$$L^*v = u-u_d, \tag{4.2b}$$

with

$$v_a := -\mu q_b \quad\text{and}\quad v_b := -\mu q_a.$$

We assume

$$\varepsilon\in(0,\beta]\cap(0,\tfrac{1}{2}], \tag{4.3a}$$
$$\beta\ge 0, \tag{4.3b}$$
$$c\in\mathcal{C}^2\left(\mathbb{R}^2\right), \tag{4.3c}$$
$$c\ge\gamma>0, \tag{4.3d}$$
$$\mu>\frac{1}{\gamma^2}>0, \tag{4.3e}$$
$$\|f\|_{1,\infty}+\|u_d\|_{1,\infty}\le C, \tag{4.3f}$$
$$q_a,q_b\in\mathbb{R}\cup\{-\infty,\infty\}. \tag{4.3g}$$

Without loss of generality we assume $q_a\le 0\le q_b$, otherwise one can modify the optimality system as we have done in (2.40).

Note that we require the function c to be defined and meet the requirements on $\mathbb{R}^2$ because some terms of the solution decomposition we construct are defined on half planes containing Ω. The prerequisite $\varepsilon\le\frac{1}{2}$ is only for simplifying the notation for we can use the fact $|\ln(\varepsilon)|\ge|\ln(\frac{1}{2})|\ge\frac{1}{2}$.

Remark 4.1
Note that the following proofs can also be applied to the differential equation system

$$L^1 u := -\varepsilon\Delta u + \beta u_x + c^1 u = f^1 - d^1 v \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{4.4a}$$
$$L^2 v := -\varepsilon\Delta v - \beta v_x + c^2 v = f^2 - d^2 u \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega \tag{4.4b}$$

under the assumptions

$$\begin{aligned} \varepsilon &\in (0,\beta] \cap (0,\tfrac{1}{2}], \\ c^i &\in \mathcal{C}^2\left(\mathbb{R}^2\right), \\ c^i &\geq \gamma > 0, \\ \|f^i\|_{1,\infty} &\leq C, \\ d^i &\in \mathcal{C}_0^1\big(B_2(0,0)\big), \\ \|d^1\|_\infty \|d^2\|_\infty &< \gamma^2, \end{aligned}$$

for $i \in \{1,2\}$.

4.1. Analytic Properties of the Solution

Theorem 4.2
The solution u, v of (4.2) can be decomposed as

$$u = u^S + u^{x1} + u^{x0} + u^y + u^{c1} + u^n + u^r \quad \text{and} \tag{4.5a}$$
$$v = v^S + v^{x0} + v^{x1} + v^y + v^{c0} + v^n + v^r. \tag{4.5b}$$

The smooth parts u^S, v^S satisfy

$$\|u^S\|_{1,\infty} + \|v^S\|_{1,\infty} + \sqrt{\varepsilon}\left(\|u^S_{xx}\|_\infty + \|v^S_{xx}\|_\infty + \|u^S_{yy}\|_\infty + \|v^S_{yy}\|_\infty\right) \leq C, \tag{4.5c}$$
$$\|u^S_{xx}\|_2 + \|v^S_{xx}\|_2 + \|u^S_{xy}\|_\infty + \|v^S_{xy}\|_2 \leq C|\ln(\varepsilon)|. \tag{4.5d}$$

The outflow layer parts u^{x1}, v^{x0} meet

$$|\partial_x^k u^{x1}(x,y)| \leq \frac{C}{\varepsilon^k}\mathcal{E}_1^x(x), \quad |u_y^{x1}(x,y)| \leq C\Big(\mathcal{E}_1^x(x) + \sqrt{\varepsilon}\chi_{(1-\sigma^*,1]}\Big), \tag{4.5e}$$
$$\|u^{x1}_{xy}\|_2 \leq \frac{C|\ln(\varepsilon)|}{\sqrt{\varepsilon}}, \quad |u^{x1}_{yy}(x,y)| \leq \frac{C}{\sqrt{\varepsilon}}\Big(\mathcal{E}_1^x(x) + \chi_{(1-\sigma^*,1]}\Big), \tag{4.5f}$$
$$\varepsilon^k|\partial_x^k v^{x0}(x,y)| + |v_y^{x0}(x,y)| + \sqrt{\varepsilon}|v_{yy}^{x0}(x,y)| \leq C\mathcal{E}_0^x(x), \quad \|v^{x0}_{xy}\|_2 \leq \frac{C}{\sqrt{\varepsilon}} \tag{4.5g}$$

for $k \in \{0,1,2\}$, $\sigma^* := -\frac{\varepsilon}{\beta}\ln(\varepsilon)$ and χ_I the characteristic function of $I \times \mathbb{R}$. The inflow layer parts u^{x0}, v^{x1} satisfy

$$|\partial_x^k u^{x0}(x,y)| \leq C\varepsilon^{\frac{1}{2}-k}\mathcal{E}_0^x(x), \qquad |u_y^{x0}(x,y)| + \sqrt{\varepsilon}|u_{yy}^{x0}(x,y)| \leq C\chi_{[0,\sigma^*)}, \tag{4.5h}$$
$$|\partial_x^k v^{x1}(x,y)| \leq C\varepsilon^{1-k}\mathcal{E}_1^x(x), \qquad |v_y^{x1}(x,y)| + \sqrt{\varepsilon}|v_{yy}^{x1}(x,y)| \leq C\chi_{(1-\sigma^*,1]}, \tag{4.5i}$$
$$\|u^{x0}_{xy}\|_2 \leq \frac{C|\ln(\varepsilon)|}{\sqrt{\varepsilon}}, \qquad \|v^{x1}_{xy}\|_2 \leq C \tag{4.5j}$$

for $k \in \{0,1,2\}$. *The characteristic layer parts* u^y, v^y *satisfy*

$$|u^y(x,y)| + |u^y_x(x,y)| + \sqrt{\varepsilon}|u^y_y(x,y)| + \varepsilon|u^y_{yy}(x,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big), \tag{4.5k}$$

$$|v^y(x,y)| + |v^y_x(x,y)| + \sqrt{\varepsilon}|v^y_y(x,y)| + \varepsilon|v^y_{yy}(x,y)| \le C\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big). \tag{4.5l}$$

The corner layer parts u^{c1}, v^{c0} *meet*

$$|u^{c1}(x,y)| + \varepsilon|u^{c1}_x(x,y)| + \sqrt{\varepsilon}|u^{c1}_y(x,y)| \le C\mathcal{E}^x_1(x)\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big), \tag{4.5m}$$

$$|v^{c0}(x,y)| + \varepsilon|v^{c0}_x(x,y)| + \sqrt{\varepsilon}|v^{c0}_y(x,y)| \le C\mathcal{E}^x_0(x)\Big(\mathcal{E}^y_0(y) + \mathcal{E}^y_1(y)\Big), \tag{4.5n}$$

$$\|u^{c1}_{xx}\|_{2,\Omega} + \|v^{c0}_{xx}\|_{2,\Omega} \le C\varepsilon^{-\frac{5}{4}}, \quad \|u^{c1}_{xy}\|_{2,\Omega} + \|v^{c0}_{xy}\|_{2,\Omega} \le C\varepsilon^{-\frac{3}{4}} \quad \text{and} \tag{4.5o}$$

$$\|u^{c1}_{yy}\|_{2,\Omega} + \|v^{c0}_{yy}\|_{2,\Omega} \le C\varepsilon^{-\frac{1}{4}}. \tag{4.5p}$$

The small but non-smooth parts u^n, v^n *fulfill*

$$\|u^n\|_2 + \|v^n\|_2 \le C\varepsilon, \quad |u^n|_{1,2} \le C\sqrt{\varepsilon}|\ln(\varepsilon)|, \quad |v^n|_{1,2} \le C\sqrt{\varepsilon} \quad \text{and} \tag{4.5q}$$

$$|u^n|_{2,2} + |v^n|_{2,2} \le \frac{C}{\sqrt{\varepsilon}} \tag{4.5r}$$

and the remaining parts u^r, v^r *that contain some characteristic layer components and corner layer components satisfy*

$$|u^r| + |v^r| \le C\left(e^{-2\sqrt{\frac{\gamma}{\varepsilon}}y} + e^{-2\sqrt{\frac{\gamma}{\varepsilon}}(1-y)}\right). \tag{4.5s}$$

Proof

Note that some of the different parts of the decomposition itself are a sum of terms that arise from different subproblems. In this way we have

$$\begin{aligned} u^S &= u^{S,1}, & u^{x1} &= u^{x1,1} + u^{x1,2}, & u^n &= u^{n,2} + u^{n,3},\\ v^S &= v^{S,1} + v^{S,2}, & v^{x0} &= v^{x0,1} + v^{x0,2}, & v^n &= v^{n,3}. \end{aligned}$$

We start by defining a smooth approximation by

$$Lu^{S,1} = -\varepsilon\Delta u^{S,1} + \beta u^{S,1}_x + cu^{S,1} = f^u - \mu^{-1}P^+_{[v_a,v_b]}v^{S,1} \text{ in } \mathcal{H}^+, \tag{4.6a}$$

$$u^{S,1}(0,\cdot) = 0, \quad \lim_{\|(x,y)\|\to\infty} u^{S,1}(x,y) = 0, \tag{4.6b}$$

$$L^*v^{S,1} = -\varepsilon\Delta v^{S,1} - \beta v^{S,1}_x + cv^{S,1} = f^v + P^- u^{S,1} \text{ in } \mathcal{H}^-, \tag{4.6c}$$

$$v^{S,1}(1,\cdot) = 0, \quad \lim_{\|(x,y)\|\to\infty} v^{S,1}(x,y) = 0 \tag{4.6d}$$

where we define $\mathcal{H}^+ := (0,\infty)\times\mathbb{R}$, $\mathcal{H}^- := (-\infty,1)\times\mathbb{R}$ and

$$\left(P^+_{[w_a,w_b]}v\right)(x,y) = \begin{cases} \omega_B \Pi_{[w_a,w_b]}(v)(x,y), & x \ge 0,\\ 0, & x < 0, \end{cases} \tag{4.6e}$$

$$\left(P^- v\right)(x,y) = \begin{cases} \omega_B v(x,y), & x \le 1,\\ 0, & x > 1. \end{cases} \tag{4.6f}$$

The inhomogeneities are defined as

$$f^u := \omega_B \mathfrak{C}_\Omega^{\mathcal{H}^+} f \quad \text{and} \quad f^v := -\omega_B \mathfrak{C}_\Omega^{\mathcal{H}^-} u_d$$

with a suitable cut-off function $\omega_B \in \mathcal{C}^\infty(\mathbb{R}^2)$ that satisfies $\omega_B\big|_\Omega = 1$, $0 \le \omega_B \le 1$ and $\operatorname{supp}(\omega_B) \subseteq B_2(0,0)$. Hence, we have

$$\|f^u\|_{1,\infty} \le C\|f\|_{1,\infty}, \quad \|f^v\|_{1,\infty} \le C\|u_d\|_{1,\infty} \quad \text{and}$$
$$\operatorname{supp}(f^u) \cup \operatorname{supp}(f^v) \subseteq B_2(0,0).$$

By the $\mathcal{L}^\infty$-stability of the system (4.6) we are able to prove $W^{1,\infty}$-estimates. Using the techniques from Chapter 3 we conceive the remaining bounds of $u^{S,1}$ and $v^{S,1}$ stated above. The details are presented in Lemma 4.10 and Corollary 4.11.

As next parts we define the layer corrections $u^{x1,1}$, $v^{x0,1}$ at the outflow boundary, u^y, v^y at the characteristic boundaries and the corner layer corrections u^{c1}, v^{c0} in the same way we defined the corresponding terms in Chapter 3. Hence, we conceive similar estimates applying Lemmata 3.21 and 3.22. Mind that the opposed convection direction in the adjoint equation leads to a layer at $x = 0$. The previously presented proofs can be applied using the variable transform $\tilde{x} = 1 - x$.

We proceed by defining a correction for the influence of $u^{x1,1}$ at the right hand side of the adjoint equation (4.2b) via

$$L^*\tilde{v}^{x1} = u^{x1,1}, \quad \tilde{v}^{x1}(0,\cdot) = \tilde{v}^{x1}(1,\cdot) = 0. \tag{4.7}$$

In Lemma 4.12 we prove

$$|\tilde{v}^{x1}| + |\tilde{v}^{x1}_y| + \sqrt{\varepsilon}|\tilde{v}^{x1}_{yy}| \le C\varepsilon, \quad \|\tilde{v}^{x1}_{xy}\|_{2,\Omega} \le C \quad \text{and}$$
$$|\partial_x^k \tilde{v}^{x1}| \le C\varepsilon^{1-k}\big(\varepsilon + \mathcal{E}_0^x(x) + \mathcal{E}_1^x(x)\big), \ k \in \{1,2\}.$$

Subsequently we construct in Lemma 4.13 a splitting $\tilde{v}^{x1} = v^{x0,2} + v^{S,2} + v^{x1}$ satisfying the bounds above.

Due to the projection, we are not able to apply Lemma 4.12 to a similar correction for the term $v^{x0,1}$ defined by

$$L\tilde{u}^{x0} = -\mu^{-1}\Pi_{[v_a^S, v_b^S]}(v^{x0,1}), \quad \tilde{u}^{x0}(0,\cdot) = \tilde{u}^{x0}(1,\cdot) = 0 \tag{4.8}$$

where $v_a^S := v_a - \Pi_{[v_a,v_b]}(v^{S,1})$ and $v_b^S := v_b - \Pi_{[v_a,v_b]}(v^{S,1})$. However, using the Green's function representation formula (3.29) we are able to derive the weaker estimates

$$|\tilde{u}^{x0}| \le C\varepsilon, \quad |\tilde{u}^{x0}_y| \le C \min\left\{1, \sqrt{\tfrac{\varepsilon}{x}}\right\}$$
$$|\tilde{u}^{x0}_{yy}| \le \frac{C}{\sqrt{\varepsilon}}, \quad \|\tilde{u}^{x0}_{xy}\|_{2,\Omega} \le C\varepsilon^{-\frac{3}{4}} \quad \text{and}$$
$$|\partial_x^k \tilde{u}^{x0}| \le C\varepsilon^{\frac{1}{2}-k}\big(\varepsilon + \mathcal{E}_0^x(x) + \mathcal{E}_1^x(x)\big), \ k \in \{1,2\},$$

in Lemma 4.15. In the follow-up we construct a splitting $\tilde{u}^{x0} = u^{x0} + u^{n,2} + u^{x1,2}$ satisfying the bounds above.

The needed correction for the influence of $\tilde{u}^{x0}$ and $\tilde{v}^{x1}$ on the right hand side of the optimality system we define by

$$Lu^{n,3} = -\mu^{-1}\Pi_{[v_a^{x0}, v_b^{x0}]}(\tilde{v}^{x1} + v^{n,3}), \qquad u^{n,3}\big|_{\partial\Omega} = 0, \tag{4.9a}$$
$$L^*v^{n,3} = \tilde{u}^{x0} + u^{n,3}, \qquad v^{n,3}\big|_{\partial\Omega} = 0 \tag{4.9b}$$

with $v_a^{x0} := v_a - \Pi_{[v_a,v_b]}(v^{S,1} - v^{x0,1})$ and $v_b^{x0} := v_b - \Pi_{[v_a,v_b]}(v^{S,1} - v^{x0,1})$. Obviously, we have a right hand side of this system of order ε. Thus, $\|u^{n,3}\|_2 + \|v^{n,3}\|_2 \le C\varepsilon$ follows and the simple estimates for singularly perturbed problems (cf. [RST08, LU68]) give

$$\|u^{n,3}\|_\varepsilon + \|v^{n,3}\|_\varepsilon \le C\varepsilon \quad \text{and} \quad |u^{n,3}|_{2,2} + |v^{n,3}|_{2,2} \le \frac{C}{\sqrt{\varepsilon}}.$$

It remains to construct corrections for the influence of u^y, v^y, u^{c1} and v^{c0} on the right hand side of system (4.2). We may do so by

$$Lu^r = -\mu^{-1} P^+_{[v_a^n, v_b^n]}(v^y + v^{c0} + v^r), \qquad u^r\big|_{\partial\Omega} = 0, \tag{4.10a}$$

$$L^* v^r = P^-(u^y + u^{c1} + u^r), \qquad v^r\big|_{\partial\Omega} = 0 \tag{4.10b}$$

with $v_a^n := v_a - \Pi_{[v_a,v_b]}(v^{S,1} - v^{x0,1} - \tilde{v}^{x1} - v^n)$ and $v_b^n := v_b + (v_a^n - v_a)$. Note, we define the remaining layer compensation as system of differential equations since we do not have the property $\|L^{-1}\mathcal{E}_0^y\|_\infty \le C\varepsilon^\delta$ for some $\delta > 0$, which we used for the layer correction terms in x-direction. Unfortunately, we are not able to prove useful bounds for this remaining terms or a splitting considering the corrections for (u^y, v^y) and (u^{c1}, v^{c0}) separately. We only conceive the stated bound for u^r and v^r by some maximum principle arguments in Lemma 4.17. □

In analogy to Conjecture 3.3 and Lemma 3.5 it is reasonable to assume similar properties for the second order derivatives of the characteristic layer parts:

Conjecture 4.3
We assume we have

$$\|u_{xx}^{y,1}\|_{2,\Omega} + \|v_{xx}^{y,1}\|_{2,\Omega} \le C|\ln(\varepsilon)|\varepsilon^{\frac{1}{4}} \quad \text{and} \tag{4.11a}$$

$$\|u_{xy}^{y,1}\|_{2,\Omega} + \|v_{xy}^{y,1}\|_{2,\Omega} \le C|\ln(\varepsilon)|\varepsilon^{-\frac{1}{4}}. \tag{4.11b}$$

Subsequently, we discuss some problems concerning the estimates we attained in Theorem 4.2.

Remark 4.4
The estimates for $\tilde{u}^{x0}$ may not be sharp, but the definition we used here introduces the layer $\tilde{u}^{x0}$ that is slightly stronger than $\tilde{v}^{x1}$. This is due to the fact that $\partial_y \Pi(v^{x0,1})$ may not be exponentially small away from $x = 0$. Possibly, the definition of the correction is not adequate and could be improved.

Remark 4.5
The estimates we have for u^{x0} do not suffice to get first order convergence of the interpolant u^{x0^I} on a one-sided Shishkin mesh refined only at $x = 1$. Instead we have to refine also at $x = 0$. In the latter case we can adapt the estimates we used in Theorem 3.24 easily to the terms u^{x0}, u^{x1}, v^{x0} and v^{x1}.

Surprisingly, this situation improves in the case of system (4.4) where only the right hand side is not smooth. In this case we can apply Lemmata 4.12 and 4.13 to the correction term $\tilde{u}^{x0}$ and attain similar bounds as for $\tilde{v}^{x1} = v^{x0} + v^{S,2} + v^{x1,2}$. Then it suffices to refine at $x = 1$ for the discretization of u.

Similarly, we can derive improved bounds for $\tilde{u}^{x0}$ when we assume the bounds q_a and q_b to be constant. In this case we know

$$|\partial_y \Pi(v^{x0,1})| \le |v_y^{x0,1}|$$

and regain the exponential decay of the derivatives of the right hand side in the definition (4.8) of $\tilde{u}^{x0}$.

Remark 4.6
There is very little about differential equation systems with parabolic layers in the literature. Essentially we have [Sh00]. There one can find some results for a problem similar to (4.4) on a stripe $\mathcal{S} := \mathbb{R} \times (0,1)$. For the layer correction V_1 of this system, it is only shown

$$|\partial_x^k \partial_y^l V_1| \leq C\varepsilon^{-\frac{l}{2}},$$

which would correspond to our expectations neglecting the exponential decline away from the boundary. But instead of low regularity they used a smooth inhomogeneity and a nonzero coefficient b_1 for V_{1y} in $\mathcal{S}$, satisfying only $b_1\big|_{\mathcal{S}} = 0$.

Remark 4.7
The bounds we conceived for the strong layer part u^{x1} seem to be too weak to acquire sharp interpolation error estimates similar to the results of Theorem 3.24. However, we have

$$\sigma^* = -\frac{\varepsilon}{\beta}\ln(\varepsilon) \leq 2\frac{\varepsilon}{\beta}\ln(N) = \sigma_x$$

for the mesh transition point σ_x defined in Section 3.2 for the most interesting case $N < \varepsilon^{-1}$. Therefore, we still have

$$\|u_y^{x1}\|_{\infty,\Omega_{0y}} \leq CN^{-2}$$

on the coarse part Ω_{0y} of the mesh in x-direction. In the second order y-derivative we loose a logarithmic factor in the $\mathcal{L}^2$-norm, receiving only

$$\|u_{yy}^{x1}\|_2 \leq C\sqrt{\frac{|\ln(\varepsilon)|}{\varepsilon}}.$$

This loss carries over to the interpolation and error estimates for this term.

Remark 4.8
The newly introduced part u^n poses no problem in the interpolation and error estimates. Recall that it is only necessary to estimate the convective term to attain an estimate for the numerical error (cf. proof of Theorem 3.25). Straight forward computations show

$$\begin{aligned}
\|u^n - u^{nI}\|_{2,\Omega} &\leq \tilde{C}H\|u^n\|_{1,2,\Omega} \leq C\sqrt{\varepsilon}N^{-1},\\
|u^n - u^{nI}|_{1,2,\Omega} &\leq \tilde{C}H\|u^n\|_{2,2,\Omega} \leq C\varepsilon^{-\frac{1}{2}}N^{-1},\\
\left|\left\langle (u^{nI} - u^n)_x,\, u^I - u^N \right\rangle\right| &= \left|\left\langle u^{nI} - u^n,\, (u^I - u^N)_x \right\rangle\right|\\
&\leq \|u^n - u^{nI}\|_2 \frac{1}{\sqrt{\varepsilon}}\|u^I - u^N\|_\varepsilon \leq CN^{-1}\|u^I - u^N\|_\varepsilon.
\end{aligned}$$

4.2. Details of the Estimates

We start by some preliminary considerations for the prolongations P^+ and P^-. We omit the subscript of P^+ in the case $P^+_{[v_a,v_b]}$ in the following.

Easily, we realize that for any lower and upper bound with $w_a \leq 0 \leq w_b$ we have the properties

$$|P^+_{[w_a,w_b]}(w)| \leq |w| \quad \text{and} \quad |P^-(w)| \leq |w|. \tag{4.12a}$$

We also have the bound

$$\left|\partial_\zeta P^\pm(w)(x,y)\right| \leq |\omega(x,y)| \left(\|w_\zeta\|_\infty + \|w_{a\zeta}\|_\infty + \|w_{b\zeta}\|_\infty\right) + |\omega_\zeta(x,y)| \, \|w\|_\infty \tag{4.12b}$$

for $\zeta \in \{x, y\}$ that can be derived from

$$\begin{aligned} \left|\partial_\zeta \Pi_{[w_a,w_b]}(w)(x,y)\right| &= \begin{cases} |w_\zeta(x,y)|, & w_a(x,y) \leq w(x,y) \leq w_b(x,y), \\ |w_{a\zeta}(x,y)|, & w(x,y) < w_a(x,y), \\ |w_{b\zeta}(x,y)|, & w_b(x,a) < w(x,y) \end{cases} \\ &\leq \|w_\zeta\|_\infty + \|w_{a\zeta}\|_\infty + \|w_{b\zeta}\|_\infty. \end{aligned} \tag{4.13}$$

Now we examine the smooth parts $u^{S,1}$ and $v^{S,1}$ defined by system (4.6). Note that we can apply the results from Chapter 3 to the first and second equation independently provided we have sufficient information on the right hand side. For the adjoint equation we only have to use the variable transformation $\tilde{x} = 1 - x$ and then it corresponds to a differential equation with an operator of the form L on $\mathcal{H}^+$. Hence, it suffices to combine this estimates with the technique from [Lin07] to get estimates for the system:

Lemma 4.9
For the solution $(u^{S,1}, v^{S,1})$ of (4.6) we have

$$|u^{S,1}| + |v^{S,1}| \leq Ce^{-\alpha\varrho}. \tag{4.14}$$

Proof
We apply Lemma 3.7 to each line of the system and receive

$$\begin{aligned} \|u^{S,1}\|_\infty &\leq \frac{1}{\gamma}\|f^u - \mu^{-1}P^+v^s\|_\infty \leq C\|f\|_\infty + \frac{\mu^{-1}}{\gamma}\|v^{S,1}\|_\infty, \\ \|v^{S,1}\|_\infty &\leq \frac{1}{\gamma}\|f^v + P^-u^{S,1}\|_\infty \leq C\|u_d\|_\infty + \frac{1}{\gamma}\|u^{S,1}\|_\infty. \end{aligned}$$

Thus, we have

$$\Gamma \begin{pmatrix} \|u^{S,1}\|_\infty \\ \|v^{S,1}\|_\infty \end{pmatrix} \leq C \begin{pmatrix} \|f\|_\infty \\ \|u_d\|_\infty \end{pmatrix} \quad \text{with} \quad \Gamma := \begin{pmatrix} 1 & -\mu^{-1}/\gamma \\ -1/\gamma & 1 \end{pmatrix}. \tag{4.15}$$

Obviously, we get

$$\Gamma^{-1} = \frac{\gamma^2}{\gamma^2 - \mu^{-1}} \begin{pmatrix} 1 & \mu^{-1}/\gamma \\ 1/\gamma & 1 \end{pmatrix}.$$

Thus, Γ is inverse monotone (i.e. $\Gamma^{-1} \geq 0$) for $\mu^{-1} < \gamma^2 \;\Leftrightarrow\; \mu > \gamma^{-2}$ (cf. (4.3e)) and we conceive

$$\|u^{S,1}\|_\infty + \|v^{S,1}\|_\infty \leq C.$$

This and the fact $\operatorname{supp}(f^u) \cup \operatorname{supp}(P^+v^{S,1}) \cup \operatorname{supp}(f^v) \cup \operatorname{supp}(P^-u^{S,1}) \subseteq B_2(0,0)$ enables us to apply Lemma 3.8 to each differential equation of (4.6) to conclude

$$|u^{S,1}(x,y)| + |v^{S,1}(x,y)| \leq Ce^{-\alpha\varrho}. \qquad \square$$

Lemma 4.10
For the solution $(u^{S,1}, v^{S,1})$ *of* (4.6) *we have*

$$\|u^{S,1}\|_{1,\infty} + \|v^{S,1}\|_{1,\infty} \le C. \tag{4.16}$$

Proof
From Lemma 4.9 we have for the reordered differential equation system

$$\hat{L}^+ u^{S,1} := -\varepsilon \Delta u^{S,1} + \beta u_x^{S,1} = f^u - \mu^{-1} P^+ v^{S,1} - c u^{S,1} =: f^{u*} \text{ in } \mathcal{H}, \tag{4.17a}$$
$$u^{S,1}(0,\cdot) = 0, \qquad \lim_{\|(x,y)\|\to\infty} u^{S,1}(x,y) = 0, \tag{4.17b}$$
$$\hat{L}^- v^{S,1} := -\varepsilon \Delta v^{S,1} - \beta v_x^{S,1} = f^v + P^- u^{S,1} - c v^{S,1} =: f^{v*} \text{ in } \tilde{\mathcal{H}}, \tag{4.17c}$$
$$v^{S,1}(1,\cdot) = 0, \qquad \lim_{\|(x,y)\|\to\infty} v^{S,1}(x,y) = 0 \tag{4.17d}$$

the estimate

$$|f^{u*}| + |f^{v*}| \le C e^{-\alpha \varrho}.$$

By application of Lemma 3.10 we acquire

$$\begin{aligned}
\|u_\zeta^{S,1}\|_{\infty,(0,\varepsilon)\times\mathbb{R}} &\le \|f^{u*}\|_\infty \|g_\zeta(x,y;\cdot,\cdot)\|_1 \le C,\\
|u_\zeta^{S,1}(x,y)| &\le \tilde{C}\|g_\zeta(x,y;\cdot,\cdot) w^e(\cdot,\cdot)\|_1 \le \frac{C}{\sqrt{\varepsilon\varrho}},\\
\|v_\zeta^{S,1}\|_{\infty,(0,\varepsilon)\times\mathbb{R}} &\le \|f^{v*}\|_\infty \|g_\zeta(x,y;\cdot,\cdot)\|_1 \le C \quad \text{and}\\
|v_\zeta^{S,1}(x,y)| &\le \tilde{C}\|g_\zeta(x,y;\cdot,\cdot) w^e(\cdot,\cdot)\|_1 \le \frac{C}{\sqrt{\varepsilon\varrho}}
\end{aligned}$$

for $\zeta \in \{x, y\}$.
Differentiating and reordering of (4.6) gives

$$L(u_\zeta^{S,1}) = f_\zeta^u - c_\zeta u^{S,1} - \mu^{-1} \partial_\zeta P^+ v^{S,1} =: f^{u,\zeta} \text{ in } \mathcal{H}^+, \tag{4.18a}$$
$$u_\zeta^{S,1}(0,\cdot) = \nu^{u^{S,1}\zeta}, \qquad \lim_{\|(x,y)\|\to\infty} u_\zeta^{S,1}(x,y) = 0, \tag{4.18b}$$
$$L^*(v_\zeta^{S,1}) = f_\zeta^v - c_\zeta v^{S,1} + \partial_\zeta P^- u^{S,1} =: f^{v,\zeta} \text{ in } \mathcal{H}^-, \tag{4.18c}$$
$$v_\zeta^{S,1}(1,\cdot) = \nu^{v^{S,1}\zeta}, \qquad \lim_{\|(x,y)\|\to\infty} v_\zeta^{S,1}(x,y) = 0. \tag{4.18d}$$

We use (4.12) to get the relations

$$\begin{aligned}
\|f_\zeta^u - c_\zeta u^{S,1} - \mu^{-1}\partial_\zeta P^+ v^{S,1}\|_\infty &\le \|f_\zeta^u - c_\zeta u^{S,1}\|_\infty + \mu^{-1}\|v_\zeta^{S,1}\|_\infty\\
&\quad + C\mu^{-1}\left(\|v^{S,1}\|_\infty + \|q_a\|_{1,\infty} + \|q_b\|_{1,\infty}\right) \quad \text{and}\\
\|f_\zeta^v - c_\zeta v^{S,1} + \partial_\zeta P^- u^{S,1}\|_\infty &\le \|f_\zeta^v - c_\zeta v^{S,1}\|_\infty + C\|u^{S,1}\|_\infty + \|u_\zeta^{S,1}\|_\infty
\end{aligned}$$

with $\zeta \in \{x, y\}$. Using Lemma 3.7 we conceive

$$\begin{aligned}
\|u_\zeta^{S,1}\|_\infty &\le \|\nu^{u^{S,1}\zeta}\|_\infty + \frac{1}{\gamma}\|f_\zeta^u - c_\zeta u^{S,1} - \mu^{-1}\partial_\zeta P^+ v^{S,1}\|_\infty \le C + \frac{1}{\gamma\mu}\|v_\zeta^{S,1}\|_\infty,\\
\|v_\zeta^{S,1}\|_\infty &\le \|\nu^{v^{S,1}\zeta}\|_\infty + \frac{1}{\gamma}\|f_\zeta^v - c_\zeta v^{S,1} + \partial_\zeta P^- u^{S,1}\|_\infty \le C + \frac{1}{\gamma}\|u_\zeta^{S,1}\|_\infty
\end{aligned}$$

which is equivalent to

$$\Gamma \begin{pmatrix} \|u_\zeta^{S,1}\|_\infty \\ \|v_\zeta^{S,1}\|_\infty \end{pmatrix} \leq C$$

where Γ is the inverse monotone matrix defined in (4.15). Consequently, we have

$$\|u_\zeta^{S,1}\|_\infty + \|u_\zeta^{S,1}\|_\infty \leq C$$

for $\zeta \in \{x, y\}$. □

Recalling the reordered system (4.18), we have established $\|f^{u,\zeta}\|_\infty + \|f^{v,\zeta}\|_\infty \leq C$. Hence, we can apply the techniques from Lemma 3.20 to each equation separately to receive:

Corollary 4.11
The second order derivatives of the smooth components $u^{S,1}$ and $v^{S,1}$ satisfy

$$\|u_{xx}^{S,1}\|_{\infty,\mathcal{H}^+} + \|u_{yy}^{S,1}\|_{\infty,\mathcal{H}^+} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \|u_{xy}^{S,1}\|_{\infty,\mathcal{H}^+} + \|u_{xx}^{S,1}\|_{2,\Omega} \leq C|\ln(\varepsilon)|, \tag{4.19a}$$

$$\|v_{xx}^{S,1}\|_{\infty,\mathcal{H}^-} + \|v_{yy}^{S,1}\|_{\infty,\mathcal{H}^-} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \|v_{xy}^{S,1}\|_{\infty,\mathcal{H}^-} + \|v_{xx}^{S,1}\|_{2,\Omega} \leq C|\ln(\varepsilon)|. \tag{4.19b}$$

Next we recall the definition (4.7) of the compensation for the contribution of $u^{x1,1}$ at the right hand side

$$L^*\tilde{v}^{x1} = u^{x1,1}, \quad \tilde{v}^{x1}(0,\cdot) = \tilde{v}^{x1}(1,\cdot) = 0.$$

Lemma 4.12
We have the estimates

$$|\tilde{v}^{x1}| + |\tilde{v}_y^{x1}| + \sqrt{\varepsilon}|\tilde{v}_{yy}^{x1}| \leq C\varepsilon, \quad \|\tilde{v}_{xy}^{x1}\|_{2,\Omega} \leq C \quad \textit{and} \tag{4.20a}$$

$$|\partial_x^k \tilde{v}^{x1}| \leq C\varepsilon^{1-k}\Big(\varepsilon + \mathcal{E}_0^x(x) + \mathcal{E}_1^x(x)\Big), \; k \in \{1,2\}. \tag{4.20b}$$

Proof
For $w^c = 1 + e^{-\frac{\beta}{\varepsilon}} - \mathcal{E}_0^x(x) - \mathcal{E}_1^x(x)$ we have $\hat{L}^- w^c = \frac{2\beta^2}{\varepsilon}\mathcal{E}_1^x(x)$ (cf. (4.17)) and

$$w_x^c = \frac{\beta}{\varepsilon}\Big(\mathcal{E}_0^x(x) - \mathcal{E}_1^x(x)\Big) \begin{cases} > 0, & x \in [0, \frac{1}{2}), \\ < 0, & x \in (\frac{1}{2}, 1]. \end{cases}$$

In combination with $w^c(0,\cdot) = w^c(1,\cdot) = 0$ this leads to $w^c \geq 0$. Thus, using w^c as a comparison function to the (differentiated) equation (4.7) we conclude

$$|\tilde{v}^{x1}| \leq C\varepsilon w^c, \quad |\tilde{v}_y^{x1}| \leq C\varepsilon w^c \quad \text{and} \quad |\tilde{v}_{yy}^{x1}| \leq C\sqrt{\varepsilon} w^c.$$

By applying the one-dimensional estimates of Lemma 3.17 to the reordered differential equation we get

$$|\partial_x^k \tilde{v}^{x1}| \leq C\varepsilon^{1-k}\Big(\varepsilon + \mathcal{E}_0^x(x) + \mathcal{E}_1^x(x)\Big)$$

for $k \in \{1, 2\}$.

We consider the solution $\tilde{v}^{x1}$ in the domain $\Omega = (0,1)^2$. We can split it as $\tilde{v}^{x1} = \bar{v} + \hat{v}$ with

$$\hat{v}(x,y) := \tilde{v}^{x1}(x,0)(1-y) + \tilde{v}^{x1}(x,1)y.$$

Using the variable transform $\tilde{x} := x$, $\tilde{y} := \sqrt{\varepsilon}y$, $\tilde{\Omega} := (0,1)\times(0,\sqrt{\varepsilon})$ we get

$$\|\bar{v}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} = \varepsilon^{-\frac{3}{4}}\|\bar{v}_{yy}\|_{2,\Omega}, \quad \|\bar{v}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}} = \varepsilon^{\frac{1}{4}}\|\bar{v}_{xx}\|_{2,\Omega} \quad \text{and} \quad \bar{v}\big|_{\partial\Omega} = 0.$$

Thus, the estimates above and a usual norm estimate for the second order derivatives of the solution to the Laplace equation (cf. [LU68]) give

$$\|\bar{v}_{xy}\|_{2,\Omega} = \varepsilon^{\frac{1}{4}}\|\bar{v}_{\tilde{x}\tilde{y}}\|_{2,\tilde{\Omega}} \le \varepsilon^{\frac{1}{4}}\left(\|\bar{v}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} + \|\bar{v}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}}\right) \le C.$$

A triangle inequality completes the proof. □

Now we construct a splitting of $\tilde{v}^{x1}$ into three parts. Two of them can be added to already existing terms and do not deteriorate the corresponding estimates significantly.

Lemma 4.13
We can split $\tilde{v}^{x1} = v^{x0,2} + v^{S,2} + v^{x1}$ *satisfying*

$$|\partial_x^k v^{x0,2}| \le C\varepsilon^{1-k}\mathcal{E}_0^x(x), \quad |v_y^{x0,2}| + \sqrt{\varepsilon}|v_{yy}^{x0,2}| \le C\mathcal{E}_0^x(x), \quad \|v_{xy}^{x0,2}\|_{2,\Omega} \le C, \tag{4.21a}$$

$$\|v^{S,2}\|_{1,\infty} + \|v_{xx}^{S,2}\|_\infty + \varepsilon\|v_{xy}^{S,2}\|_{2,\Omega} + \sqrt{\varepsilon}\|v_{yy}^{S,2}\|_\infty \le C\varepsilon, \tag{4.21b}$$

$$|\partial_x^k v^{x1}| \le C\varepsilon^{1-k}\mathcal{E}_1^x(x), \quad |v_y^{x1}| + \sqrt{\varepsilon}|v_{yy}^{x1}| \le C\varepsilon\chi_{[1-\sigma^*,1]}, \quad \|v_{xy}^{x1}\|_{2,\Omega} \le C \tag{4.21c}$$

for $k \in \{0,1,2\}$ *and the characteristic function* $\chi_{[1-\sigma^*,1]}(x,y)$ *of the set* $[1-\sigma^*,1]\times\mathbb{R}$ *with* $\sigma^* := -\frac{\varepsilon}{\beta}\ln(\varepsilon)$.

Proof
As was devised in [Lin00] we set $\sigma^* := -\frac{\varepsilon}{\beta}\ln(\varepsilon)$ and define

$$v^{S,2} := \mathfrak{C}_{(\sigma^*,1-\sigma^*)\times\mathbb{R}}^{(0,1)\times\mathbb{R}}\tilde{v}^{x1},$$

$$v^{x1} := \begin{cases} 0, & x \le 1-\sigma^*, \\ \tilde{v}^{x1} - v^{S,2}, & x > 1-\sigma^* \end{cases} \quad \text{and}$$

$$v^{x0,2} := \begin{cases} \tilde{v}^{x1} - v^{S,2}, & x < \sigma^*, \\ 0, & x \ge \sigma^*. \end{cases}$$

Thus, we have

$$\|v^{S,2}\|_{1,\infty} + \|v_{xx}^{S,2}\|_\infty + \varepsilon\|v_{xy}^{S,2}\|_{2,\Omega} + \sqrt{\varepsilon}\|v_{yy}^{S,2}\|_\infty \le C\varepsilon.$$

Also, we get

$$|\partial_x^k v^{x1}| \le C\varepsilon^{1-k}\mathcal{E}_1^x(x),\ k \in \{1,2\}, \quad |v_y^{x1}| + \sqrt{\varepsilon}|v_{yy}^{x1}| \le C\varepsilon, \quad \|v_{xy}^{x1}\|_{2,\Omega} \le C$$

and via integrating v_x^{x1} (cf. [Lin00]) we conceive

$$|v^{x1}| \le C\varepsilon\mathcal{E}_1^x(x).$$

Since we know $v^{x1}(x,y) = 0$ for $x < 1-\sigma^*$ and have $\mathcal{E}_1^x(1-\sigma^*) = \varepsilon$, we conclude

$$|v_y^{x1}| + \sqrt{\varepsilon}|v_{yy}^{x1}| \le C\mathcal{E}_1^x(x).$$

Analogously, we deduce the bounds for $v^{x0,2}$. □

Recall the definition (4.8)

$$L\tilde{u}^{x0} = -\mu^{-1}\Pi_{[v_a^S, v_b^S]}(v^{x0,1}), \quad \tilde{u}^{x0}(0,\cdot) = \tilde{u}^{x0}(1,\cdot) = 0.$$

Note, we have $v_a^S \le 0 \le v_b^S$. To prove sufficient bounds for $\tilde{u}^{x0}$ we use some estimates for the Green's function:

Lemma 4.14
Using the weight function $w^{\mathcal{E}_0^x}(\xi,\eta) = \mathcal{E}_0^x(\xi)$ we have for the approximation $\tilde{g}$ of the Green's function on the stripe $\mathcal{S}$ (cf. (3.29)) the estimate

$$\|\tilde{g}_y(x,y;\cdot,\cdot)w^{\mathcal{E}_0^x}(\cdot,\cdot)\|_{1,(0,1)\times\mathbb{R}} \le C\mathcal{E}_0^x(x) \tag{4.22a}$$

for $x \in (0,\varepsilon)$. For larger x we have

$$\|\tilde{g}_y(x,y;\cdot,\cdot)w^{\mathcal{E}_0^x}(\cdot,\cdot)\|_{1,(0,1)\times\mathbb{R}} \le C\sqrt{\frac{\varepsilon}{x}}. \tag{4.22b}$$

Proof
By Corollary 3.12 it suffices to prove the desired bounds for $\frac{|\psi|}{r}G^1$.

In analogy to the proofs of Lemma 3.10, 3.11 and [FK12] we split the domain of integration $\mathcal{S} \subseteq \mathcal{H}^+$ into subdomains Ω_1, Ω_2 (cf. Figure 3.3), but use a different splitting angle.

More explicitly we define $\Omega_1 := \left\{(\xi,\eta) \in \mathbb{R}^2 \middle| \varphi < \max\{\varepsilon, \frac{|\psi|}{8}\}\right\}$. Via the transformation to polar coordinates (r,ϑ) and the relations (cf. Remark 3.9)

$$\varphi < \varepsilon + \tfrac{|\psi|}{8} \le \varepsilon + \tfrac{r}{8} \quad \text{and} \quad K_1(s) \le Cs^{-1}e^{-\frac{s}{2}}$$

we get

$$0 \le e^{\beta\frac{\varphi}{\varepsilon}}e^{q\frac{\varphi}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right) \le \tilde{C}\frac{\varepsilon}{qr}e^{q\frac{3\varphi}{\varepsilon}-\frac{qr}{2\varepsilon}} \le \tilde{C}\frac{\varepsilon}{qr}e^{q(3+\frac{3r}{8\varepsilon}-\frac{r}{2\varepsilon})} \le C\frac{\varepsilon}{r}e^{-\frac{qr}{8\varepsilon}}$$

and conclude

$$\begin{aligned}\|G^1(x,y;\cdot,\cdot)w^{\mathcal{E}_0^x}(\cdot,\cdot)\|_{1,\Omega_1} &\le e^{-\beta\frac{x}{\varepsilon}}\int_0^\infty\int_0^{2\pi}\frac{1}{\varepsilon^2}e^{\beta\frac{\varphi}{\varepsilon}}e^{q\frac{\varphi}{\varepsilon}}K_1\left(q\tfrac{r}{\varepsilon}\right)r\,\mathrm{d}\vartheta\,\mathrm{d}r\\ &\le \tilde{C}e^{-\beta\frac{x}{\varepsilon}}\int_0^\infty\int_0^{2\pi}\frac{1}{\varepsilon}e^{-\frac{qr}{8\varepsilon}}\,\mathrm{d}\vartheta\,\mathrm{d}r \le Ce^{-\beta\frac{x}{\varepsilon}}.\end{aligned}$$

Since $\Omega_2 = \varnothing$ for $x \le \varepsilon$ the first assertion of the Lemma is established.

Next, we analyze the integral on $\Omega_2 := \left\{(\xi,\eta) \in \mathbb{R}^2 \middle| \max\{\varepsilon, \frac{|\psi|}{8}\} \le \varphi < x\right\}$. We estimate by using the relations

$$\begin{gathered}\frac{qr}{\varepsilon} \ge \frac{q\varphi}{\varepsilon} \ge q \quad \Rightarrow \quad K_1(s) \le Cs^{-\frac{1}{2}}e^{-s},\\ \varphi \le r = \sqrt{\varphi^2+\psi^2} \le \sqrt{65}\varphi,\\ 0 > \varphi - r = \frac{\varphi^2 - r^2}{\varphi + r}\begin{cases}\le -\frac{\psi^2}{d\varphi},\\ \ge -\frac{\psi^2}{\varphi},\end{cases} \qquad d := 1+\sqrt{65}.\end{gathered}$$

Thus, we have

$$0 \le e^{q\frac{\varphi}{\varepsilon}} K_1\left(q\tfrac{r}{\varepsilon}\right) \le C\sqrt{\frac{\varepsilon}{qr}} e^{q\frac{\varphi-r}{\varepsilon}} \le C\sqrt{\frac{\varepsilon}{q\varphi}} e^{-q\frac{\psi^2}{d\varepsilon\varphi}}.$$

This can be used to estimate

$$\begin{aligned}\|\tfrac{|\psi|}{r} G^1(x,y;\cdot,\cdot) w^{\mathcal{E}_0^x}(\cdot,\cdot)\|_{1,\Omega_2} &\le \tilde{C} e^{-\beta\frac{x}{\varepsilon}} \int_\varepsilon^x \int_0^\infty \frac{\psi}{\varepsilon^{3/2}\varphi^{3/2}} e^{-q\frac{\psi^2}{d\varepsilon\varphi}} e^{2q\frac{\varphi}{\varepsilon}} \,\mathrm{d}\psi\,\mathrm{d}\varphi \\ &= \tilde{C} e^{-\beta\frac{x}{\varepsilon}} \int_\varepsilon^x \frac{d}{2q\sqrt{\varepsilon\varphi}} e^{2q\frac{\varphi}{\varepsilon}} \,\mathrm{d}\varphi \\ &= C e^{-\beta\frac{x}{\varepsilon}} \hat{\imath}\left[\operatorname{erf}\left(\hat{\imath}\sqrt{\beta}\right) - \operatorname{erf}\left(\hat{\imath}\sqrt{\tfrac{\beta x}{\varepsilon}}\right)\right].\end{aligned}$$

Using the series expansion of the error function (cf. [AS84]) we conceive

$$\begin{aligned}\hat{\imath}\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\left(\hat{\imath}\sqrt{\beta}\right) - \operatorname{erf}\left(\hat{\imath}\sqrt{\tfrac{\beta x}{\varepsilon}}\right)\right] &= \hat{\imath}\sum_{n=0}^\infty \frac{\hat{\imath}\beta^{n+\frac{1}{2}}}{n!(2n+1)} - \hat{\imath}\sum_{n=0}^\infty \frac{\hat{\imath}\left(\frac{\beta x}{\varepsilon}\right)^{n+\frac{1}{2}}}{n!(2n+1)} \\ &\le \sqrt{\frac{\varepsilon}{\beta x}} \sum_{n=0}^\infty \frac{\left(\frac{\beta x}{\varepsilon}\right)^{n+1}}{(n+1)!} \le \sqrt{\frac{\varepsilon}{\beta x}} e^{\frac{\beta x}{\varepsilon}}.\end{aligned}$$

Thus, we can conclude

$$\|\tfrac{|\psi|}{r} G^1(x,y;\cdot,\cdot) w^{\mathcal{E}_0^x}(\cdot,\cdot)\|_{1,\Omega_2} \le C\sqrt{\frac{\varepsilon}{x}}$$

and the Lemma is proved. □

Lemma 4.15
We have the estimates

$$|\tilde{u}^{x0}| \le C\varepsilon, \quad |\tilde{u}_y^{x0}| \le C\min\left(1, \sqrt{\tfrac{\varepsilon}{x}}\right) \tag{4.23a}$$

$$|\tilde{u}_{yy}^{x0}| \le \frac{C}{\sqrt{\varepsilon}}, \quad \|\tilde{u}_{xy}^{x0}\|_{2,\Omega} \le C\varepsilon^{-\frac{3}{4}} \quad \textit{and} \tag{4.23b}$$

$$|\partial_x^k \tilde{u}^{x0}| \le C\varepsilon^{\frac{1}{2}-k}\left(\varepsilon + \sqrt{\varepsilon}\mathcal{E}_0^x(x) + \mathcal{E}_1^x(x)\right), \; k \in \{1,2\}. \tag{4.23c}$$

Proof
As in the proof of Lemma 4.12 we use the comparison function $w^c = 1 + e^{-\frac{\beta}{\varepsilon}} - \mathcal{E}_0^x(x) - \mathcal{E}_1^x(x)$ with $\hat{L}^+ w^c = \frac{2\beta^2}{\varepsilon}\mathcal{E}_0^x(x)$. Applying a maximum principle we get

$$|\tilde{u}^{x0}| \le C\varepsilon w^c.$$

We proceed by considering the representation formula (3.29) for the reordered equation

$$\hat{L}^+ \tilde{u}^{x0} = -c\tilde{u}^{x0} - \mu^{-1}\Pi_{[v_a^S, v_b^S]}(v^{x0,1}), \quad \tilde{u}^{x0}(0,\cdot) = \tilde{u}^{x0}(1,\cdot) = 0.$$

Applying Corollary 3.12 and Lemmata 3.13 and 4.14 we get

$$|\tilde{u}_y^{x0}| \le \begin{cases} C\mathcal{E}_0^x, & x \le \varepsilon, \\ \sqrt{\frac{\varepsilon}{x}}, & x > \varepsilon. \end{cases}$$

Differentiation in y-direction leads to

$$\hat{L}^+\tilde{u}^{x0}_y = -c_y\tilde{u}^{x0} - c\tilde{u}^{x0}_y - \mu^{-1}\partial_y\Pi_{[v_a^S, v_b^S]}(v^{x0,1}), \quad \bar{u}_y(0,\cdot) = 0.$$

From the bound on the projection (4.13) we get $|\hat{L}^+\tilde{u}^{x0}_y| \le C$. Hence, we can apply Corollary 3.12 and Lemma 3.13 again to obtain

$$|\tilde{u}^{x0}_{yy}| \le \frac{C}{\sqrt{\varepsilon}}.$$

By applying the one-dimensional estimates of Lemma 3.17 to the reordered differential equation we get

$$|\partial_x^k\tilde{u}^{x0}| \le C\varepsilon^{1-k}\left(\sqrt{\varepsilon} + \mathcal{E}_0^x(x) + \tfrac{1}{\sqrt{\varepsilon}}\mathcal{E}_1^x(x)\right)$$

for $k \in \{1,2\}$.

As a last step we consider the solution $\tilde{u}^{x0}$ in the domain $\Omega = (0,1)^2$. We can split it as $\tilde{u}^{x0} = \bar{u} + \hat{u}$ with

$$\hat{u}(x,y) := \tilde{u}^{x0}(x,0)(1-y) + \tilde{u}^{x0}(x,1)y.$$

Using the variable transform $\tilde{x} := x$, $\tilde{y} := \varepsilon^{1/4}y$, $\tilde{\Omega} := (0,1)\times(0,\varepsilon^{1/4})$ we get

$$\|\bar{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} = \varepsilon^{-\frac{3}{8}}\|\bar{u}_{yy}\|_{2,\Omega}, \quad \|\bar{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}} = \varepsilon^{\frac{1}{8}}\|\bar{u}_{xx}\|_{2,\Omega} \quad \text{and} \quad \bar{u}\big|_{\partial\Omega} = 0.$$

Thus, the estimates above and a usual norm estimate for the second order derivatives of the solution to the Laplace equation (cf. [LU68]) give

$$\|\bar{u}_{xy}\|_{2,\Omega} = \varepsilon^{\frac{1}{8}}\|\bar{u}_{\tilde{x}\tilde{y}}\|_{2,\tilde{\Omega}} \le \varepsilon^{\frac{1}{8}}\left(\|\bar{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} + \|\bar{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}}\right) \le C\varepsilon^{-\frac{3}{4}}.$$

A triangle inequality completes the proof. □

As for $\tilde{v}^{x1}$ we now construct a splitting of $\tilde{u}^{x0}$ into three parts. One of them can be added to an already existing one without deteriorating the corresponding bounds decisively.

Lemma 4.16
We can split $\tilde{u}^{x0} = u^{x0} + u^{n,2} + u^{x1,2}$ where the separate parts satisfy

$$|\partial_x^k u^{x0}| \le C\varepsilon^{\frac{1}{2}-k}\mathcal{E}_0^x(x), \quad |u^{x0}_y| + \sqrt{\varepsilon}|u^{x0}_{yy}| \le C\chi_{(0,\sigma^*)}, \tag{4.24a}$$

$$\|u^{n,2}\|_2 \le C\varepsilon, \quad |u^{n,2}|_{1,2} \le C\sqrt{\varepsilon}|\ln(\varepsilon)|, \quad |u^{n,2}|_{2,2} \le \frac{C}{\sqrt{\varepsilon}}, \tag{4.24b}$$

$$|\partial_x^k u^{x1,2}| \le C\varepsilon^{\frac{1}{2}-k}\mathcal{E}_1^x(x) \quad \text{and} \quad |u^{x1,2}_y| + \varepsilon|u^{x1,2}_{yy}| \le C\sqrt{\varepsilon}\chi_{(1-\sigma^*,1)} \tag{4.24c}$$

where $k \in \{0,1,2\}$ and $\chi_I(x,y)$ is the characteristic function of $I \times \mathbb{R}$. Furthermore, we have

$$\|u^{x0}_{xy}\|_2 + \|u^{x1,2}_{xy}\|_2 \le \frac{C|\ln(\varepsilon)|}{\sqrt{\varepsilon}}. \tag{4.24d}$$

Proof
We define the separate terms via

$$\begin{aligned} u^{n,2} &= \mathfrak{C}^{(0,1)\times\mathbb{R}}_{(\sigma^*,1-\sigma^*)\times\mathbb{R}}\tilde{u}^{x0}, \\ u^{x0} &:= \begin{cases} \tilde{u}^{x0} - u^{n,2}, & x < \sigma^*, \\ 0, & x \geq \sigma^*, \end{cases} \quad \text{and} \\ u^{x1,2} &:= \begin{cases} 0, & x \leq 1-\sigma^*, \\ \tilde{u}^{x0} - u^{n,2}, & x > 1-\sigma^* \end{cases} \end{aligned}$$

with $\sigma^* := -\frac{\varepsilon}{\beta}\ln(\varepsilon)$. Hence, we receive

$$\|u^{n,2}\|_2 \leq C\varepsilon, \quad |u^{n,2}|_{1,2} \leq C\sqrt{\varepsilon}|\ln(\varepsilon)| \quad \text{and} \quad \|u^{n,2}_{xx}\|_2 + \|u^{n,2}_{yy}\|_2 \leq \frac{C}{\sqrt{\varepsilon}}.$$

Furthermore, we can use the norm estimate for Laplace equation with $\bar{u} = u^{n,2} - \hat{u}$ in $\Omega' := (\sigma^*, 1-\sigma^*) \times (0,1)$ with

$$\begin{aligned} \hat{u}(x,y) := {} & u^{n,2}(x,0)(1-y) + u^{n,2}(x,1)y \\ & + \Big(u^{n,2}(\sigma^*,y) - u^{n,2}(\sigma^*,0)(1-y) - u^{n,2}(\sigma^*,1)y\Big)\frac{1-\sigma^*-x}{1-2\sigma^*} \\ & + \Big(u^{n,2}(1-\sigma^*,y) - u^{n,2}(1-\sigma^*,0)(1-y) - u^{n,2}(1-\sigma^*,1)y\Big)\frac{x-\sigma^*}{1-2\sigma^*} \end{aligned}$$

and obtain

$$\|\bar{u}_{xy}\|_{2,\Omega'} \leq C\Big(\|u^{n,2}_{xx}\|_{2,\Omega'} + \|\hat{u}_{xx}\|_{2,\Omega'} + \|u^{n,2}_{yy}\|_{2,\Omega'} + \|\hat{u}_{yy}\|_{2,\Omega'} + \|\hat{u}_{xx}\|_{2,\Omega'}\Big) \leq \frac{C}{\sqrt{\varepsilon}}.$$

A triangle inequality yields $\|u^{n,2}_{xy}\|_{2,\Omega} \leq \tilde{C}\|u^{n,2}_{xy}\|_{2,\Omega'} \leq \frac{C}{\sqrt{\varepsilon}}$.

In analogy to the proof of Lemma 4.13 we get

$$\begin{aligned} |\partial^k_x u^{x0}| &\leq C\varepsilon^{\frac{1}{2}-k}\mathcal{E}^x_0(x), \quad |u^{x0}_y| \leq C\chi_{(0,\sigma^*)}, \quad |u^{x0}_{yy}| \leq \frac{C}{\sqrt{\varepsilon}}\chi_{(0,\sigma^*)}, \\ |\partial^k_x u^{x1,2}| &\leq C\varepsilon^{\frac{1}{2}-k}\mathcal{E}^x_1(x), \quad |u^{x1,2}_y| \leq C\sqrt{\varepsilon}\chi_{(1-\sigma^*,1)} \quad \text{and} \quad |u^{x1,2}_{yy}| \leq \frac{C}{\sqrt{\varepsilon}}\chi_{(1-\sigma^*,1)} \end{aligned}$$

for $k \in \{0,1,2\}$. Note that the $\mathcal{L}^2$-norm of the y- and yy-derivative improves over the $\mathcal{L}^\infty$-norm, by using the facts $u^{x0}(x,y) = 0$ for $x > \sigma^*$ and $u^{x1,2}(x,y) = 0$ for $x < 1-\sigma^*$.

By definition we have $u^{x0}_{xy}(x,y) = 0$ for $x \geq \sigma^*$. Thus, we only consider the domain $\hat{\Omega} := (0,\sigma^*) \times (0,1)$ to acquire bounds for the mixed second order derivative. We split $u^{x0}_{xy}\big|_{\hat{\Omega}} = \tilde{u} + \hat{u}$ with

$$\begin{aligned} \hat{u}(x,y) = {} & (1-y)u^{x0}(x,0) + yu^{x0}(x,1) \\ & + \Big(u^{x0}(0,y) - (1-y)u^{x0}(0,0) - yu^{x0}(0,1)\Big)\frac{e^{-\frac{x}{\varepsilon}} - e^{-\frac{\sigma^*}{\varepsilon}}}{1-e^{-\frac{\sigma^*}{\varepsilon}}}. \end{aligned}$$

This construction provides

$$\|\tilde{u}_{xx}\|_{2,\hat{\Omega}} \leq \|u^{x0}_{xx}\|_{2,\hat{\Omega}} + \|\hat{u}_{xx}\|_{2,\hat{\Omega}} \leq \frac{C}{\varepsilon},$$
$$\|\tilde{u}_{yy}\|_{2,\hat{\Omega}} \leq \|u^{x0}_{yy}\|_{2,\hat{\Omega}} + \|\hat{u}_{yy}\|_{2,\hat{\Omega}} \leq C|\ln(\varepsilon)|, \quad \text{and} \quad \tilde{u}\big|_{\partial\hat{\Omega}} = 0.$$

Using the variable transform $\tilde{x} := x$, $\tilde{y} := \sqrt{\varepsilon}y$, $\tilde{\Omega} := (0,\sigma^*) \times (0,\sqrt{\varepsilon})$ we get

$$\|\tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} = \varepsilon^{-\frac{3}{4}}\|\tilde{u}_{yy}\|_{2,\hat{\Omega}}, \quad \|\tilde{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}} = \varepsilon^{\frac{1}{4}}\|\tilde{u}_{xx}\|_{2,\hat{\Omega}}.$$

Thus, the estimates above and a usual norm estimate for the second order derivatives of the solution to the Laplace equation (cf. [LU68]) give

$$\|\tilde{u}_{xy}\|_{2,\Omega} = \|\tilde{u}_{xy}\|_{2,\hat{\Omega}} = \varepsilon^{\frac{1}{4}}\|\tilde{u}_{\tilde{x}\tilde{y}}\|_{2,\tilde{\Omega}} \leq \varepsilon^{\frac{1}{4}}\left(\|\tilde{u}_{\tilde{y}\tilde{y}}\|_{2,\tilde{\Omega}} + \|\tilde{u}_{\tilde{x}\tilde{x}}\|_{2,\tilde{\Omega}}\right) \leq \frac{C|\ln(\varepsilon)|}{\sqrt{\varepsilon}}.$$

A triangle inequality completes the proof.
The mixed derivative of $u^{x1,2}$ can be estimated analogously. □

As a last step we recall the definition of the remainder

$$L u^r = -\mu^{-1}P^+_{[v^n_a,v^n_b]}(v^y + v^{c0} + v^r), \qquad u^r\big|_{\partial\Omega} = 0,$$
$$L^* v^r = P^-(u^y + u^{c1} + u^r), \qquad v^r\big|_{\partial\Omega} = 0$$

given in (4.10). So far it is an open problem to prove sufficient estimates for the remainder. At least we have the estimate

Lemma 4.17
For $\mu > \frac{4}{\gamma^2}$ we have

$$|u^r| + |v^r| \leq C\left(e^{-\sqrt{\frac{\gamma}{2\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{2\varepsilon}}(1-y)}\right). \tag{4.25}$$

Proof
Let us define

$$\omega := e^{-\sqrt{\frac{\gamma}{2\varepsilon}}y} + e^{-\sqrt{\frac{\gamma}{2\varepsilon}}(1-y)}, \quad \|f\|_\omega := \left\|\tfrac{f}{\omega}\right\|_{\infty,\Omega}.$$

A maximum principle yields

$$\|u^r\|_\omega \leq \frac{2}{\gamma}\|Lu^r\|_\omega \leq C\|v^y + v^{c0}\|_\omega + \frac{2}{\mu\gamma}\|v^r\|_\omega \quad \text{and}$$
$$\|v^r\|_\omega \leq \frac{2}{\gamma}\|Lv^r\|_\omega \leq C\|u^y + u^{c1}\|_\omega + \frac{2}{\gamma}\|u^r\|_\omega.$$

Hence, we have

$$\Gamma\begin{pmatrix}\|u^r\|_\omega\\ \|v^r\|_\omega\end{pmatrix} \leq C\begin{pmatrix}\|v^y + v^{c0}\|_\omega\\ \|u^y + u^{c1}\|_\omega\end{pmatrix} \quad \text{with} \quad \Gamma := \begin{pmatrix}1 & -2/(\mu\gamma)\\ -2/\gamma & 1\end{pmatrix}.$$

The condition assumed for μ assures the inverse monotonicity of Γ and we conclude

$$\|u^r\|_\omega + \|v^r\|_\omega \leq C. \qquad \square$$

4.3. Computational Results

Although our analysis of the solution properties is not conclusive, we got some insights in the solution structure. As stated before it is easy to adapt the convergence proofs of Chapter 2 to the higher dimensional case, provided one has good estimates for the primal and adjoint equation errors. At the moment we are not able to establish rigor proofs of this convergence. In the following we illustrate the results and assumptions we have derived in the previous section with some numerical computations.

We consider the example

$$\min_{u,q}\left(\frac{1}{2}\|u-u_d\|_2^2+\frac{3}{20}\|q\|_2^2\right),\quad q_a\le q\le q_b, \tag{4.26a}$$

$$Lu := -\varepsilon\Delta u + u_x + 2u = f + q \text{ in } \Omega=(0,1)^2,\quad u\big|_{\partial\Omega}=0 \tag{4.26b}$$

with

$$f = x^3 - x - y^2 + e^{xy}, \tag{4.26c}$$

$$u_d = \cos\big(\pi(x-y^2)\big), \tag{4.26d}$$

$$q_a = -\big(2+3\cos(5y+2\sqrt{x})+5\sin(10x)+5x(1-x)\big)/10, \tag{4.26e}$$

$$q_b = \big(7-4\sin(6x)\big)/10. \tag{4.26f}$$

Note we have $\mu^{-1}=\frac{10}{3}\le 4=\gamma^2$ and hence our previously presented theory applies. We emphasize that our example only uses very regular data f, u_d, q_a and q_b. The irregularities involved are only induced by the projection $\Pi_{[q_a,q_b]}(q)$. However, this problem is hard to solve. Hence, we do not know an analytic solution. We circumvent the related difficulties by using a reference solution u^R, v^R computed on a fine mesh ($N=3500$). All errors presented below are computed with respect to this reference solution instead of the exact solution.

In Figure 4.1 we present a plot of the solution. For better visualization we stretched the layer parts. The upper two images show the state u^R and the adjoint v^R. Clearly, we see the strong layers. The third image depicts the optimal control q^R. To get a better idea where the control bounds are active, we changed the coloring in those regions to a blueish shade. In the remaining domain, we applied a reddish shade.

In Chapter 2 we discussed several options of discretizing the optimal control problem. There we noted that the results do not show big discrepancies of the various solutions. The semi-discrete schemes were difficult to implement since integrals over nonsmooth functions had to be evaluated. It is obvious that this difficulties increase in the higher-dimensional case. Thus, in this chapter we only present results for the full-discrete scheme although we have no convergence prove for the use of different meshes for u and v.

As we outlined in Remark 4.5, our analysis is not able to prove ε-independent convergence if the mesh is not refined in the region of the weak layers. Thus, we start using the mesh depicted in Figure 4.2. Recall that our incomplete estimates for the terms u^r and v^r suggest the existence of characteristic layers of the form $e^{-\sqrt{\gamma/(2\varepsilon)}\,y}$ and $e^{-\sqrt{\gamma/(2\varepsilon)}\,(1-y)}$. Therefore, we use, in contrast to Chapter 3, the mesh transition point $\sigma_y=2\sqrt{\frac{2\varepsilon}{\gamma}}\ln N$. The numeric results are presented in Figure 4.3.

As motivated by our analysis and the results attained in Chapter 2, we observe almost first order convergence. Even the logarithmic factor $|\ln(\varepsilon)|$ in our analysis seems to be an artifact of the proofs. Moreover we even get almost second order super convergence.

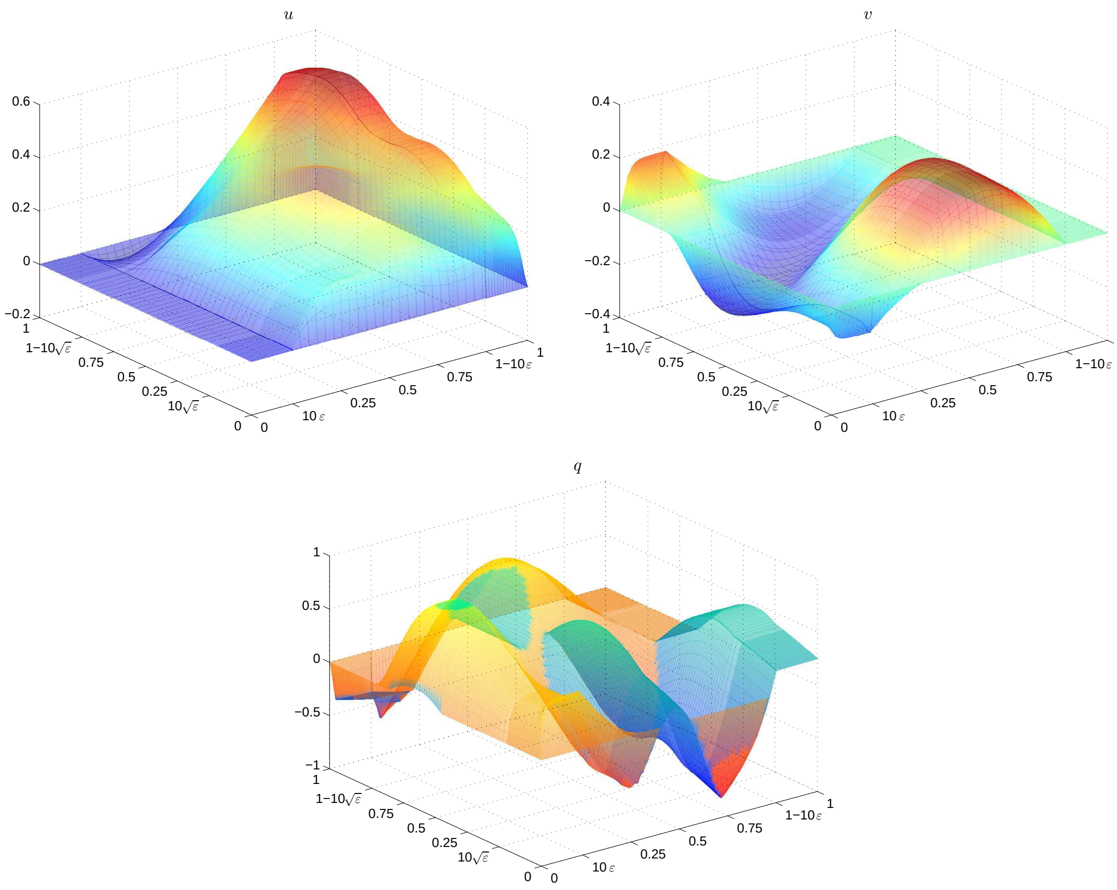

Figure 4.1.: Plot of the reference solution u^R, v^R and q^R for $\varepsilon = 10^{-4}$

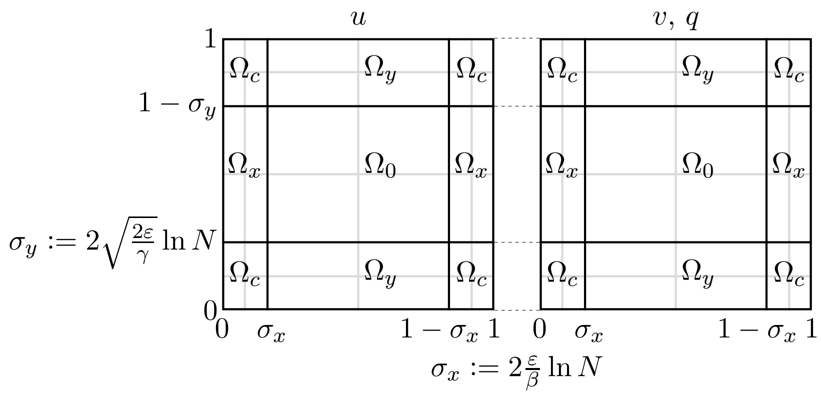

Figure 4.2.: Two-sided Shishkin mesh

Now we test our assumptions for the characteristic layers hidden in u^r and v^r. To this end we use the mesh transition point $\tilde{\sigma}_y := 2\sqrt{\frac{\varepsilon}{\gamma}}\ln N = \sigma_y/\sqrt{2}$ instead of σ_y. If the analysis is sharp the superconvergence observed in the first calculations should be reduced this time (for we have only $e^{-\sqrt{\gamma/(2\varepsilon)}\,\tilde{\sigma}_y} = N^{-\sqrt{2}}$). The results of this second computation are given in Figure 4.4. As we can see, we attain an ε-independent second

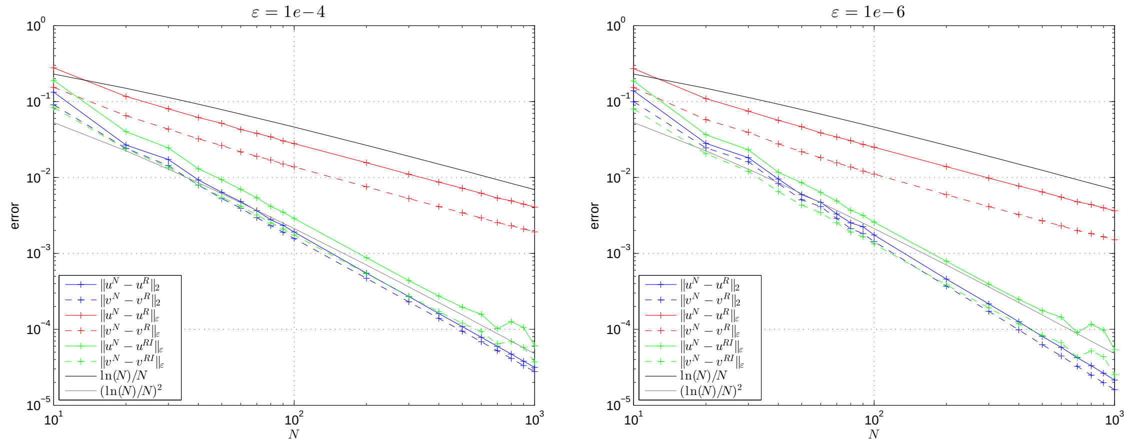

Figure 4.3.: Error of the bilinear FEM on a two-sided Shishkin mesh (cf. Figure 4.2)

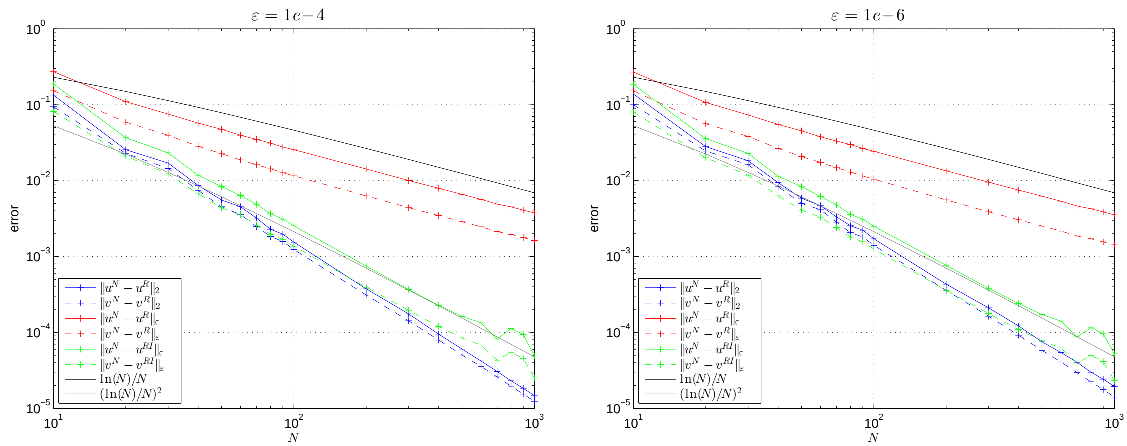

Figure 4.4.: Error of the bilinear FEM on a two-sided Shishkin mesh (cf. Figure 4.2) using $\tilde{\sigma}_y = 2\sqrt{\frac{\varepsilon}{\gamma}} \ln N$ instead of $\sigma_y = 2\sqrt{\frac{2\varepsilon}{\gamma}} \ln N$

order superconvergence again. This indicates that the terms u^r and v^r admit a much sharper bound. However, we also see that there is almost no difference in the errors from the first and second computation. A direct comparison of the errors yields that the computation using $\tilde{\sigma}_y$ is better by about 3%. Thus, an overestimate of the layer width appears to be relatively harmless.

In Chapter 2 we also discussed the case of only refining the strong layer part. Although our analysis is too weak to give satisfying results (cf. Remark 4.5) we present tests for this case. We use one-sided Shishkin meshes as depicted in Figure 4.5 for the discretization. The attained numerical errors are given in Figure 4.6.

Obviously, we attain an ε-independent almost first order convergence, thus our decomposition of the solution and the corresponding bounds seem not to be sharp enough. In contrast to the expectations based on the results of Chapter 2 it seems as if we had second order superconvergence. But at the lower end of the plot for $\varepsilon = 10^{-4}$ we observe some convergence break-down. To investigate if this is merely a numerical error in context of

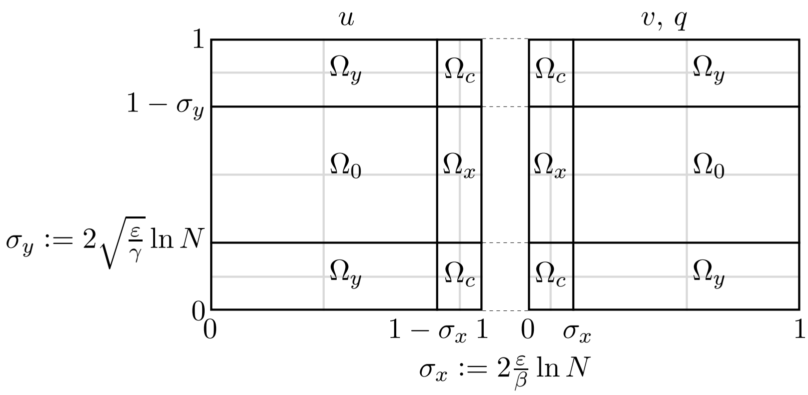

Figure 4.5.: One-sided Shishkin meshes

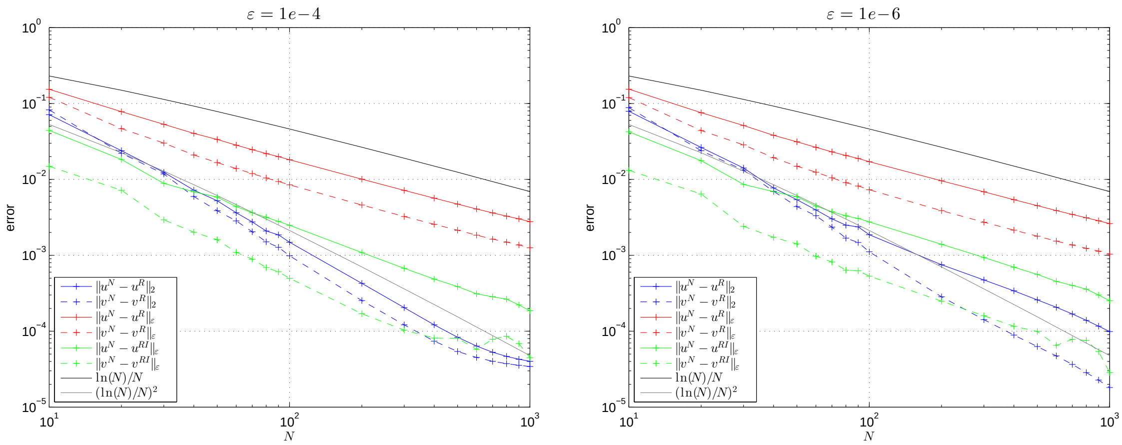

Figure 4.6.: Error of the bilinear FEM on one-sided Shishkin meshes (cf. Figure 4.5)

using a reference solution and not the exact solution for computing the errors we present additional results in Figure 4.7.

The situation is not as obvious as in Chapter 2, but we see a range where the convergence of the numerical solution u^N measured in the $\mathcal{L}^2$-norm is diminished. This range starts when the error is in the order of magnitude of ε.

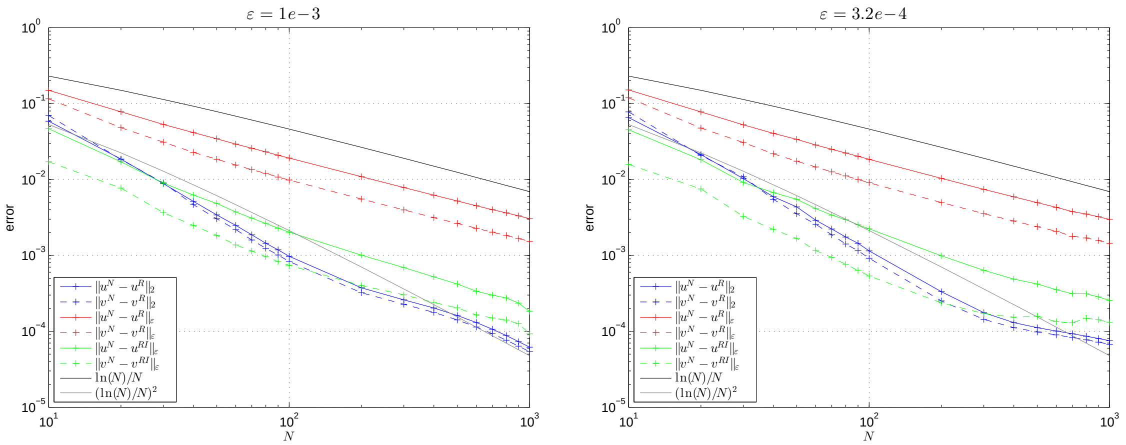

Figure 4.7.: Error of the bilinear FEM on one-sided Shishkin meshes (cf. Figure 4.5)

Conclusions and Outlook

Recalling the central results of Theorem 2.3, Theorem 3.1 and Theorem 4.2 we have seen, that the solutions of optimal control problems with singularly perturbed differential equations as side constraints have a special structure. They exhibit not only an outflow boundary layer and characteristic layers as one would expect knowing the results for singularly perturbed differential equations but also a weak boundary layer at the inflow boundary. Furthermore, we learned that the solutions lose regularity away from the boundary layers, especially the derivative orthogonal to the convection direction. This is induced by the projection to the admissible set.

Of course, there are open problems. For example, we have no sharp bounds for the characteristic layer terms – neither for a single singularly perturbed differential equation with low regularity inhomogeneity nor for the optimal control problem. Further research is needed to get rigor bounds for these terms.

Nevertheless, we constructed adapted algorithms that show an ε-independent convergence rate for the considered problems. These algorithms have a very simple structure, i.e. we only adapted the mesh according to ideas devised by Shishkin. In further investigations one should use the information we attained on the solution to construct even better algorithms. Especially some stabilized methods could be considered.

Because the solution of the optimal control problem with box constraints for the control is not very regular, it is not easy to construct higher order solving methods. In this context it may be helpful to get more information on the behavior of the solution near the boundaries of the active set. Such additional estimates could be useful for the construction of adapted meshes near this boundary.

A. Appendix

A.1. Integrals

In the following we derive for some of the more complex integrands the indefinite integrals.

Via substitution we derive

$$\begin{aligned}
\int e^{-A\sqrt{B+x}}\,\mathrm{d}x &= \int 2\xi e^{-A\xi}\,\mathrm{d}\xi = -\frac{2}{A^2}e^{-A\sqrt{B+x}}\Big(A\sqrt{B+x}+1\Big),\\
\int e^{-Ax^2}\,\mathrm{d}x &= \int \frac{1}{\sqrt{A}}e^{-\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}}{2\sqrt{A}}\operatorname{erf}\left(\sqrt{A}x\right),\\
\int xe^{-Ax^2}\,\mathrm{d}x &= \int \frac{1}{2}e^{-A\xi}\,\mathrm{d}\xi = -\frac{1}{2A}e^{-Ax^2},\\
\int \frac{1}{\sqrt{x}}e^{Ax}\,\mathrm{d}x &= -\int \frac{2\hat{\imath}}{\sqrt{A}}e^{-\xi^2}\,\mathrm{d}\xi = -\hat{\imath}\sqrt{\frac{\pi}{A}}\operatorname{erf}\left(\hat{\imath}\sqrt{Ax}\right),\\
\int e^{-Ax^2+Bx}\,\mathrm{d}x &= \int \frac{e^{\frac{B^2}{4A}}}{\sqrt{A}}e^{-\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}e^{\frac{B^2}{4A}}}{2\sqrt{A}}\operatorname{erf}\left(\sqrt{A}x-\frac{B}{2\sqrt{A}}\right),\\
\int xe^{-Ax^2+Bx}\,\mathrm{d}x &= \int -\frac{1}{2A}e^{\xi}\,\mathrm{d}\xi + \frac{B}{2A}\int e^{-Ax^2+Bx}\,\mathrm{d}x\\
&= -\frac{1}{2A}e^{-Ax^2+Bx} + \frac{\sqrt{\pi}Be^{\frac{B^2}{4A}}}{4A^{\frac{3}{2}}}\operatorname{erf}\left(\sqrt{A}x-\frac{B}{2\sqrt{A}}\right),\\
\int -\frac{\ln(A-x)}{x}\,\mathrm{d}x &= -\int \frac{\ln(A)+\ln(1-\xi)}{\xi}\,\mathrm{d}\xi = -\ln(A)\ln(\xi) + \int \frac{\ln(\eta)}{1-\eta}\,\mathrm{d}\eta\\
&= -\ln(A)\ln\left(\frac{x}{A}\right) + \operatorname{dilog}\left(1-\frac{x}{A}\right)
\end{aligned}$$

where $\hat{\imath}$ denotes the imaginary unit and A and B are positive real constants and dilog denotes the *dilogarithm* (cf. [AS84]) defined by

$$\operatorname{dilog}(x) = -\int_1^x \frac{\ln(t)}{t-1}\,\mathrm{d}t.$$

Using these integrals and integration by parts we obtain

$$\begin{aligned}
\int x^2e^{-Ax^2}\,\mathrm{d}x &= -\frac{x}{2A}e^{-Ax^2} + \int \frac{1}{2A}e^{-Ax^2}\,\mathrm{d}x = -\frac{x}{2A}e^{-Ax^2} + \frac{\sqrt{\pi}}{4A^{\frac{3}{2}}}\operatorname{erf}\left(\sqrt{A}x\right),\\
\int x^2e^{-Ax^2+Bx}\,\mathrm{d}x &= \int\left(\frac{\xi^2}{A}+\frac{B\xi}{A^{3/2}}+\frac{B^2}{4A^2}\right)\frac{e^{\frac{B^2}{4A}}}{\sqrt{A}}e^{-\xi^2}\,\mathrm{d}\xi\\
&= -\left(\frac{x}{2A}+\frac{B}{4A^2}\right)e^{-Ax^2+Bx} + \frac{\sqrt{\pi}e^{\frac{B^2}{4A}}}{8A^{5/2}}(2A+B^2)\operatorname{erf}\left(\sqrt{A}x-\frac{B}{2\sqrt{A}}\right).
\end{aligned}$$

Next, we recall the definition of the *exponential integral* (cf. [AS84])

$$\operatorname{Ei}(x) = ⨍_{-\infty}^{x}\frac{e^t}{t}\,\mathrm{d}t$$

where $⨍$ denotes the Cauchy principal value of the integral. We derive

$$\int_a^b \frac{e^{Ax}}{x}\,\mathrm{d}x = ⨍_{-\infty}^{Ab} \frac{e^{\xi}}{\xi}\,\mathrm{d}\xi - ⨍_{-\infty}^{Aa} \frac{e^{\xi}}{\xi}\,\mathrm{d}\xi = \Big[\mathrm{Ei}(Ax)\Big]_{x=a}^{b}$$

where we assumed $A, B > 0$, $b > a > 0$.

Last but not least, we have

Lemma A.1
We have

$$\int_0^s -\ln(\xi)e^{A\xi}\,\mathrm{d}\xi = \frac{1}{A}\big(\mathrm{Ei}(As) - \gamma^e - \ln(A) - e^{As}\ln(s)\big) =: F(s) \tag{A.1}$$

where $\gamma^e \approx 0.58$ *denotes Euler's constant.*

Proof
First we note

$$\begin{aligned} F'(s) &= \frac{1}{A}\left(\mathrm{Ei}'(As)A - Ae^{As}\ln(s) - \frac{e^{As}}{s}\right) \\ &= \frac{1}{A}\left(\frac{e^{As}}{s} - Ae^{As}\ln(s) - \frac{e^{As}}{s}\right) = -e^{As}\ln(s). \end{aligned}$$

Also we have by the series expansion of Ei the relation

$$\begin{aligned} A\lim_{s\searrow 0} F(s) &= \lim_{s\searrow 0}\left(\gamma^e + \ln(As) + \sum_{n=1}^{\infty}\frac{A^n s^n}{n\,n!} - \gamma^e - \ln(A) - e^{As}\ln(s)\right) \\ &= \lim_{s\searrow 0}\big(\ln(s) - e^{As}\ln(s)\big) = \lim_{s\searrow 0}\frac{\frac{1}{s}}{(1-e^{As})^{-2}Ae^{As}} = \lim_{s\searrow 0}\frac{(1-e^{As})^2}{Ase^{As}} \\ &= \frac{1}{A}\lim_{s\searrow 0}\frac{-2(1-e^{As})Ae^{As}}{e^{As}+Ase^{As}} = 0. \end{aligned}$$

□

A.2. Properties of Bessel Function

From [AS84] we know

$$K_\nu(s) = \int_0^\infty e^{-s\cosh(t)}\cosh(\nu t)\,\mathrm{d}t.$$

Since the integrand is positive we have $K_\nu(s) > 0$. Using $\cosh(x) \geq 1 = \cosh(0)$ for all $x \geq 0$ we can deduce

$$K_0(s) = \int_0^\infty e^{-s\cosh(t)}\,\mathrm{d}t \leq \int_0^\infty e^{-s\cosh(t)}\cosh(t)\,\mathrm{d}t = K_1(s).$$

Also from [AS84] we know

$$\sqrt{\frac{\pi}{2s}}e^{-s}\left(1-\frac{1}{8s}\right) \leq K_0(s) \quad \text{and} \quad K_1(s) \leq \sqrt{\frac{\pi}{2s}}e^{-s}\left(1+\frac{3}{8s}\right)$$

for $s > 0$. Hence, we have

$$K_1(s)\left(1-\frac{1}{2s}\right) \le \sqrt{\frac{\pi}{2s}}e^{-s}\left(1-\frac{1}{8s}-\frac{3}{16s}\right) \le \sqrt{\frac{\pi}{2s}}e^{-s}\left(1-\frac{1}{8s}\right) \le K_0(s).$$

Thus, we can estimate for $\mathfrak{r} := \sqrt{s^2+\psi^2}$, $q > 0$ and $\varepsilon > 0$ as follows

$$\begin{aligned}\left|K_0\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) - \tfrac{s}{\mathfrak{r}}K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right)\right| &\le \begin{cases} K_0\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) - \tfrac{s}{\mathfrak{r}}K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) & \le \left(1-\tfrac{s}{\mathfrak{r}}\right)K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) \\ \tfrac{s}{\mathfrak{r}}K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) - K_0\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) & \le \tfrac{\varepsilon}{2q\mathfrak{r}}K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right) \end{cases} \\ &\le \left(\frac{\psi^2}{\mathfrak{r}(\mathfrak{r}+s)} + \frac{\varepsilon}{2q\mathfrak{r}}\right)K_1\left(q\tfrac{\mathfrak{r}}{\varepsilon}\right).\end{aligned}$$

A.3. Jump detection for numerical integration

We consider the integral

$$I_{(a,b)} := \int_a^b f(x)\,\mathrm{d}x$$

where f is piecewise smooth but has a finite number of jumps at unknown locations. As a basic integration algorithm we use an adaptive Simpson rule to evaluate I up to a given precision $\bar{\varepsilon}$, where the adaptivity is guided by an $\frac{h}{2}$-strategy (cf. [Her11]). Our goal is to add an automatic detection of jumps of f, thus enabling us to integrate to a given precision. Note that f is bounded in the bounded set (a,b) since it is piecewise smooth.

The main idea to handle a jump is to locate it in an sufficiently small interval $(x_l, x_u) \subseteq (a,b)$ and then proceed using

$$\begin{aligned} I^* &:= \int_{x_l}^{x_u} f(x)\,\mathrm{d}x \approx (x_u - x_l)\frac{f(x_l)+f(x_u)}{2} =: I^{*N}, \\ E^* &:= |I^* - I^{*N}| \approx (x_u - x_l)|f(x_u) - f(x_l)| =: E^{*N}. \end{aligned}$$

For a sufficiently small interval width $h^* := x_u - x_l$ the error E^* will be small since f is bounded. Obviously, the same is true for the estimate E^{*N}.

Next we consider the problem, how to detect a jump. For the adaptive Simpson rule we have to evaluate f in the mesh points $x_i := a + ih \in [a,b]$. We want to use this function values for the jump detection to reduce the overhead of our algorithm. We define

$$\Delta_i := |f(x_i) - f(x_{i+2})| \quad \text{and} \quad \delta_i := |f(x_i) - f(x_{i+1})|.$$

For a smooth integrand and small h it is reasonable to expect

$$\Delta_i \approx 2\delta_i \tag{A.2}$$

since we have $\Delta_i = 2hf'(\xi_{\Delta i})$ and $\delta_i = hf'(\xi_{\delta i})$ for $\xi_{\Delta i} \in (x_i, x_{i+2})$ and $\xi_{\delta i} \in (x_i, x_{i+1})$ and therefore $\xi_{\Delta i} \approx \xi_{\delta i}$. In contrast we would expect

$$\Delta_i \approx \delta_i \approx |\lim_{x \searrow x^*} f(x) - \lim_{x \nearrow x^*} f(x)| \tag{A.3}$$

in the vicinity of a jump at $x^* \in (x_i, x_{i+1})$. Thus, the algorithm decides to expect a jump if we have

$$\Delta_i < e^{\frac{\ln 2 + \ln 1}{2}}\delta_i = \sqrt{2}\delta_i, \tag{A.4}$$

i.e. the quotient is more like 1 than 2.

In case this rule indicates a jump, we refine the corresponding interval and search again for a jump until either the rule (A.4) does not apply anymore or the error estimator ${E^*}^N$ has a sufficiently small value ${E^*}^N \leq \frac{\bar{\varepsilon}}{3}$. For the intervals (a, x_l), (x_u, b) we call recursively the adaptive routine with the error bound $\frac{\bar{\varepsilon}}{3}$.

Note that the improvement to the pure $\frac{h}{2}$-strategy comes from the fact that we allow a much larger error to be accepted in the vicinity of the jump compared to the pure recursive call of the adaptive Simpson rule ($\bar{\varepsilon} \in \mathcal{O}(h)$).

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Dresden, den 28. November 2014

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