Local Convergence of Newton-type Methods for Nonsmooth Constrained Equations and Applications

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Notation

$\mathbb{N}$	set of all natural numbers (including 0)
$\mathbb{R}$	set of all real numbers
$\mathbb{R}_+$	set of all nonnegative real numbers, i.e., $\{x \in \mathbb{R} \mid x \geq 0\}$
$\mathbb{R}_{++}$	set of all positive real numbers, i.e., $\{x \in \mathbb{R} \mid x > 0\}$
$\mathcal{M} \subseteq \mathcal{N}$	every element of the set $\mathcal{M}$ belongs to the set $\mathcal{N}$
$\mathcal{M} \subset \mathcal{N}$	$\mathcal{M} \subseteq \mathcal{N}$ but $\mathcal{M} \neq \mathcal{N}$
$\emptyset$	empty set
$\forall$	universal quantifier
$\exists$	existential quantifier
$\\| \cdot \\|$	Euclidean vector norm or corresponding induced matrix norm
$\\| \cdot \\|_\infty$	maximum norm
$\mathcal{B}_\delta(z)$	closed ball around $z \in \mathbb{R}^n$ with radius $\delta > 0$ concerning the Euclidean norm, i.e., $\{s \in \mathbb{R}^n \mid \\|s - z\\| \leq \delta\}$
$\operatorname{dist}[z, Z]$	Euclidean distance of $z \in \mathbb{R}^n$ to a nonempty set $Z \subseteq \mathbb{R}^n$, i.e., $\inf\{\\|w - z\\| \mid w \in Z\}$
$\operatorname{conv}(Z)$	convex hull of a set $Z \subseteq \mathbb{R}^n$, i.e., the smallest convex set which contains Z
I_n	identity matrix with n rows and n columns
1_n	vector with n components consisting of ones only, i.e., $1_n := (1, \ldots, 1)^\top \in \mathbb{R}^n$
e^i	i-th canonical unit vector in $\mathbb{R}^n$, i.e., $e^i_j := \delta_{ij}$ $(j = 1, \ldots, n)$ where δ_{ij} denotes the Kronecker delta
$A_{i,\cdot}$	i-th row of a matrix A
$A_{\cdot,j}$	j-th column of a matrix A
$\operatorname{rank}(A)$	rank of a matrix A
$\operatorname{diag}(z_i)_{i=1}^n$	diagonal matrix with the numbers $z_1, \ldots, z_n \in \mathbb{R}$ on the main diagonal
$\operatorname{block}(A_i)_{i=1}^n$	block diagonal matrix with the (in general rectangular) matrices $A_1, \ldots, A_n$ on the main diagonal
$v \circ w$	Hadamard product of two vectors $v, w \in \mathbb{R}^n$, i.e., $(v \circ w)_i := v_i w_i$ $(i = 1, \ldots, n)$

$\min\{v,w\}$	minimum of two vectors $v, w \in \mathbb{R}^n$ where the minimum has to be taken componentwise, i.e., $(\min\{v,w\})_i := \min\{v_i, w_i\}$ $(i = 1, \ldots, n)$
$\max\{v,w\}$	maximum of two vectors $v, w \in \mathbb{R}^n$ where the maximum has to be taken componentwise, i.e., $(\max\{v,w\})_i := \max\{v_i, w_i\}$ $(i = 1, \ldots, n)$
$F'(z)$	Jacobian of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ at $z \in \mathbb{R}^n$
$\nabla F(z)$	transposed of the Jacobian of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$ at $z \in \mathbb{R}^n$, i.e., $F'(z)^\top$
$\nabla_x F(x,y)$	transposed of the Jacobian of a differentiable function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^m$ at (x,y) with respect to x only
$\nabla^2 f(z)$	Hessian of a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at $z \in \mathbb{R}^n$

If $v \in \mathbb{R}^m$ is a vector and $\mathcal{J} \subseteq \{1, \ldots, m\}$ is an index set, then we denote by $v_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$ the vector consisting of all components v_j of v whose indices belong to $\mathcal{J}$. Similarly, if $F : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function, then $F_{\mathcal{J}} : \mathbb{R}^n \to \mathbb{R}^{|\mathcal{J}|}$ consists of all components F_j of F whose indices belong to $\mathcal{J}$.

The elements of $\mathbb{R}^n$ are regarded as column vectors. However, we sometimes write (x, y) instead of $(x^\top, y^\top)^\top$ to indicate the elements of $\mathbb{R}^{n_x+n_y}$.

Chapter 1

Introduction

In this thesis the problem of finding a solution of a constrained system of equations

$$F(z) = 0 \quad \text{s.t.} \quad z \in \Omega \tag{1.1}$$

is considered, i.e., a nonlinear system with the additional constraint $z \in \Omega$. The set $\Omega \subseteq \mathbb{R}^n$ is assumed to be nonempty and closed. If Ω equals $\mathbb{R}^n$, then (1.1) reduces to an unconstrained system of equations. The function $F : \Omega \to \mathbb{R}^m$ is supposed to be at least continuous. The main focus of this thesis is on local Newton-type methods for the solution of (1.1) which converge locally quadratically under mild assumptions implying neither differentiability of F nor the local uniqueness of solutions.

In general, nonlinear systems of equations are important for many practical applications. For instance, they may arise after modeling problems from engineering, economics, or sciences. Moreover, nonlinear equations are obtained by the first order optimality conditions of unconstrained optimization problems or least squares problems.

The nonlinear systems we have mainly in mind arise from systems of equations and inequalities which contain complementarity constraints, i.e., which have the form

$$P(z) = 0, \quad Q(z) \geq 0, \quad R(z) \geq 0, \; S(z) \geq 0, \; R(z)^\top S(z) = 0 \tag{1.2}$$

with given functions $P : \mathbb{R}^n \to \mathbb{R}^p$, $Q : \mathbb{R}^n \to \mathbb{R}^q$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$. We call (1.2) *complementarity system*. It is well known that (1.2) can be equivalently reformulated as unconstrained or constrained system of equations (1.1). There are many problems which have the form (1.2), for example linear or nonlinear complementarity problems, and Karush-Kuhn-Tucker (KKT) systems arising from optimization problems, variational inequalities, or generalized Nash equilibrium problems (GNEPs).

In the case $m = n$ and $\Omega = \mathbb{R}^n$, where (1.1) reduces to an unconstrained nonlinear system, Newton's method is probably the most famous locally fast convergent method. In order to recall its subproblems, let us assume that $s \in \mathbb{R}^n$

denotes the current iterate. Then the new iterate has to be determined as solution of the linear system

$$F(s) + F'(s)(z - s) = 0.$$

It is well known that any sequence generated by the Newton method converges with a Q-quadratic rate to a zero z^* of F if the function F is differentiable at z^*, its Jacobian is locally Lipschitz continuous near z^*, $F'(z^*)$ is nonsingular, and the starting point belongs to a sufficiently small neighborhood of z^*, see for example [83, Satz 5.1.2].

So, the Newton method has powerful local convergence properties. However, the convergence assumptions are too strong for many applications, in particular they restrict the applicability of Newton's method for the solution of complementarity systems and related problems. The nonsingularity assumption implies the local uniqueness of z^* as zero of F. Moreover, typical reformulations of (1.2) as nonlinear system have the property that the function F is not differentiable at z^* if strict complementarity is violated. We say that a solution z^* of (1.2) satisfies the *strict complementarity condition* if, for every $j = 1, \ldots, r$, the relation $R_j(z^*)+S_j(z^*) > 0$ is satisfied. Of course, there are also smooth reformulations of (1.2) as nonlinear system of equations. But then the nonsingularity assumption cannot be expected to hold, even if z^* is a locally unique solution.

Particularly, for solutions of KKT systems arising from GNEPs both the local uniqueness of solutions and the strict complementarity condition are often too strong requirements. Therefore, the question concerning further methods arises which converge locally quadratically under milder assumptions.

There are several approaches to extend the classical Newton method to nonsmooth systems of equations. In order to describe the probably most important among them, let $s \in \mathbb{R}^n$ again denote the current iterate. Then a matrix $V \in \partial F(s)$ has to be determined and the linear system of equations

$$F(s) + V(z - s) = 0$$

must be solved. By $\partial F(s)$ Clarke's generalized Jacobian at s is indicated, its definition is recalled in Section 2.2 below. To the best of the author's knowledge, Kojima and Shindo [57] were the first who analyzed a realization of the nonsmooth Newton method described above for the case that F is a piecewise continuously differentiable (PC^1) function. Later on, it was considered in [58, 59, 74, 77] for more general nonsmooth systems. The following famous result on local convergence is proved in [77, Theorem 3.2]. Any sequence generated by the nonsmooth Newton method converges Q-quadratically to a zero z^* of F if the function F is strongly semismooth at z^*, all elements of $\partial F(z^*)$ are nonsingular, and the starting point belongs to a sufficiently small neighborhood of z^*. The definition of strong semismoothness is recalled in Section 2.3 below.

The nonsmooth Newton method and inexact versions of it were applied for the solution of reformulations of linear and nonlinear complementarity problems, see for example [8, 9, 33, 34, 52], and of KKT systems arising from optimization problems, variational inequalities, or quasi-variational inequalities, see for example [20, 24, 32, 75]. The assumptions which guarantee local quadratic convergence of the nonsmooth Newton method do not imply differentiability of F but still the local uniqueness of z^* as zero of F.

The paper [90] of Yamashita and Fukushima is probably a milestone on the way to local superlinear convergent methods for the solution of nonlinear systems of equations without requiring the local uniqueness of solutions. Instead of the nonsingularity of the Jacobian, some local error bound condition is used in [90] for the analysis of local convergence properties of the Levenberg-Marquardt method. Suppose that Z denotes the set of all zeros of F and that z^* is an arbitrary but fixed element of Z. Then we say that F *provides a local error bound* for the distance to Z near z^* if there are $\omega > 0$ and $\delta > 0$ such that

$$\operatorname{dist}[s, Z] \leq \omega \|F(s)\| \tag{1.3}$$

holds for all $s \in \mathcal{B}_\delta(z^*)$. During the last years, it turned out that local error bound conditions are the key for proving local superlinear convergence. It is well known that the nonsingularity of $F'(z^*)$ is sufficient for the above local error bound condition to hold. However, the latter does not imply that z^* is an isolated zero of F. Therefore, the local error bound condition (1.3) is strictly weaker than the nonsingularity of $F'(z^*)$.

Besides [90], there are many further papers where local convergence properties of the Levenberg-Marquardt method are analyzed under the local error bound condition (1.3), see for example [7, 21, 28, 29, 30, 39, 91]. In order to describe the subproblems of this method, let us assume that s denotes the current iterate and that $\mu : \mathbb{R}^n \to \mathbb{R}_{++}$ is a given function. Then the following unconstrained optimization problem with a strongly convex, quadratic objective function must be solved:

$$\|F(s) + F'(s)(z - s)\|^2 + \mu(s)\|z - s\|^2 \to \min_z .$$

It is known that the Levenberg-Marquardt method converges locally quadratically to some zero of F if the function F is sufficiently smooth, there are $\omega > 0$ and $\delta > 0$ such that (1.3) holds for all $s \in \mathcal{B}_\delta(z^*)$, and the function μ is appropriately chosen, see for example [30, Theorem 2.2].

Local error bounds and related conditions were also used for the local convergence analysis of further methods for different problem classes such as generalized equations [36, 64], nonlinear complementarity problems [6, 85, 89], and KKT systems arising from optimization problems or variational inequalities [31, 35, 42, 46, 48, 87, 88].

As mentioned at the beginning of the Introduction, we consider constrained nonlinear systems of equations in this thesis, i.e., we are also interested in extensions of Newton-type methods for the solution of (1.1) with $\Omega \neq \mathbb{R}^n$. There are several reasons why the adding of the constraint $z \in \Omega$ might be reasonable. For instance, the function F might be not defined everywhere on $\mathbb{R}^n$. Furthermore, for some problems it is a priori known that the solutions must belong to some feasible area, for example the variables could be physical quantities for which negative values make no sense. More importantly for us, considering the complementarity system (1.2), we know that the function values of Q, R, and S have to be nonnegative at any solution so that the corresponding inequalities could be put into Ω. It will turn out in this thesis that the introduction of some feasible set Ω can also be reasonable because stronger convergence results of the Newton-type methods which we are going to analyze might be obtained. More precisely, we will see, at least for special problem classes, that milder assumptions are sufficient for local quadratic convergence of our methods if the iterates are required to belong to some prescribed set, although the solution set of the constrained system coincides with the set of all zeros of F, i.e., with the solution set of the unconstrained system.

In [55] an extension of the Levenberg-Marquardt method for the solution of (1.1) is proposed where Ω is supposed to be convex. In order to describe the subproblems of the constrained Levenberg-Marquardt method, let $s \in \Omega$ denote the current iterate and assume that $\mu : \Omega \to \mathbb{R}_{++}$ is a given function. Then the following constrained optimization problem has to be solved:

$$\|F(s) + F'(s)(z - s)\|^2 + \mu(s)\|z - s\|^2 \to \min_z \quad \text{s.t.} \quad z \in \Omega. \tag{1.4}$$

For the local convergence analysis, a local error bound condition is used in [55] which is adapted to the constrained system (1.1). Let Z be the solution set of (1.1) now and let $z^* \in Z$ be arbitrary but fixed. Then we say that F *provides a local error bound* for the distance to Z near z^* *on* Ω if there are $\omega > 0$ and $\delta > 0$ such that (1.3) holds for all $s \in \mathcal{B}_\delta(z^*) \cap \Omega$. Under this local error bound condition, any sequence generated by the constrained Levenberg-Marquardt method converges locally with a Q-quadratic rate to a solution of (1.1) if F is differentiable at z^*, its derivative is locally Lipschitz continuous near z^*, the function μ is appropriately chosen, and the starting point belongs to a sufficiently small neighborhood of z^*. The latter result follows from [55, Theorem 2.11].

The differentiability assumptions on F are actually not required in [55]. Instead, $F'(s)$ is replaced by some matrix $G(s)$ in (1.4), and it is proved that the local quadratic convergence is kept if the differentiability assumptions are replaced by some weaker condition. However, this condition implies that F is differentiable at all points which belong to the interior of Ω and are sufficiently close to z^*, see [84, Lemma 5.3.1] and also Section 3.2 below. Besides that, local quadratic convergence of nonsmooth (unconstrained or constrained) Levenberg-Marquardt methods has, to the best of the author's knowledge, not been proved in the literature under assumptions which allow nonisolated solutions.

In [18] the LP-Newton method is proposed for the solution of (1.1) where, for any given point $s \in \Omega$, a solution of the optimization problem

$$\gamma \to \min_{z,\gamma} \quad \text{s.t.} \quad \begin{array}{l} z \in \Omega, \\ \|F(s) + G(s)(z - s)\|_\infty \leq \gamma \|F(s)\|_\infty^2, \\ \|z - s\|_\infty \leq \gamma \|F(s)\|_\infty, \\ \gamma \geq 0 \end{array}$$

has to be determined where $G(s)$ is a suitable substitute for $F'(s)$ (if the Jacobian at s exists, then $G(s) = F'(s)$ can be taken). The above problem can be regarded as a linear program if Ω is *polyhedral*, i.e., defined by affine inequalities and equations. Four assumptions are used in [18] to prove local quadratic convergence of the LP-Newton method. The first one is very weak and particularly satisfied if F is locally Lipschitz continuous. The second assumption is the local error bound condition on Ω which was recalled above. The further two assumptions relax the differentiability of F. It turns out that all four assumptions together do neither imply the local uniqueness of solutions of (1.1) nor the differentiability of F at a solution.

The paper [18] can be regarded as initial point of this thesis. Our first aim is to show that the assumptions which are used in [18] also lead to local quadratic convergence of a nonsmooth version of the constrained Levenberg-Marquardt method described by (1.4). To this end, we describe a general Newton-type algorithm for the solution of (1.1) and prove local quadratic convergence under the assumptions from [18]. Afterwards, it is shown that both the LP-Newton method and the nonsmooth constrained Levenberg-Marquardt method are special realizations of the general Newton-type algorithm and enjoy the same local convergence properties.

The second aim of this thesis is a detailed analysis of the convergence assumptions from [18], at least for the case that the constrained system (1.1) arises from special problem classes. In particular, the two assumptions which relax the differentiability of F seem quite technical and difficult to check. Therefore, we are interested in sufficient conditions which are still very mild but which seem to be more familiar. First, the case that F is a PC^1-function is considered. We provide conditions which imply the whole set of the convergence assumptions from [18]. In particular, it will turn out that some set of local error bound conditions is sufficient for the whole set of the convergence assumptions to hold if in addition a certain condition on Ω is satisfied. Afterwards, our new conditions are discussed for a reformulation of the complementarity system (1.2) as constrained system of equations by means of the minimum function where the resulting function F is a PC^1-function. Particularly, we provide conditions which guarantee local quadratic convergence of the LP-Newton method as well as the constrained Levenberg-Marquardt method for the solution of KKT systems arising from optimization problems, variational inequalities, or GNEPs; or FJ systems arising from GNEPs.

In the following, the organization of this thesis and the main contributions of the author are described in more detail.

Chapter 2 recalls some basic definitions and results concerning constraint qualifications implying the local error bound condition for systems of inequalities and equations, Clarke's generalized Jacobian, and semismooth functions.

In Chapter 3 we describe a general Newton-type algorithm and prove local quadratic convergence under the same four assumptions and by a similar proof technique that were used in [18] for the local convergence analysis of the LP-Newton method. Moreover, based on [18, Section 3.1], we relate the convergence assumptions to some existing conditions from the literature. Afterwards, it is shown that, under suitable regularity conditions, the classical Newton method as well as nonsmooth and inexact variants of it can be regarded as special realizations of the general Newton-type algorithm. More importantly, we prove that, without requiring any additional conditions, the LP-Newton method, an inexact version of it, and the constrained Levenberg-Marquardt method are special realizations of the general algorithm and therefore enjoy the same local convergence properties. The latter results were obtained in a joint work with Francisco Facchinei and Andreas Fischer in [17]. There, we also described an inexact constrained Levenberg-Marquardt method, see [17, Algorithm 3], and proved local quadratic convergence under the assumptions from [18]. In Section 3.5 of this thesis local quadratic convergence of a more general inexact constrained Levenberg-Marquardt method is shown under the same assumptions. The latter result is exploited to prove, under appropriate conditions, local quadratic convergence of both [17, Algorithm 3] and the inexact method which is considered in [1].

Chapter 4 provides an in-depth discussion of the four convergence assumptions from [18] for the case that F is a PC^1-function. First, for each assumption sufficient and possibly necessary conditions are derived separately. Thereafter, conditions are developed which imply the whole set of the convergence assumptions. Furthermore, we discuss the reformulation of (1.1) by means of slack variables. If Ω is described by nonlinear inequalities and thus nonpolyhedral, the introduction of slack variables is reasonable since otherwise the subproblems of the LP-Newton as well as of the constrained Levenberg-Marquardt method would be difficult to solve. The main result of Chapter 4 is Theorem 4.19. It provides conditions where each of them is sufficient for the whole set of convergence assumptions to hold, not only for the original problem but also for the reformulation with slack variables. The weakest among these conditions requires that some set of local error bound conditions, together with some condition on the set Ω, is satisfied. In the last section of Chapter 4 the complementarity system (1.2) is considered. We present a suitable reformulation as constrained system of equations by means of the minimum function where the resulting function F is a PC^1-function. Afterwards, we provide an adapted formulation of Theorem 4.19 and, in addition, some new constant rank condition which is shown to be sufficient for the whole set of the convergence assumptions to hold. The results of

Chapter 4 will in large part be published together with Andreas Fischer, Alexey Izmailov, and Mikhail Solodov in the technical report [37]. In addition, the constant rank condition for KKT systems arising from GNEPs which will appear in [37] is extended in this thesis to the case of general complementarity systems.

Chapter 5 is devoted to the application of the results of Chapter 4 to special classes of complementarity systems. Based on Theorem 4.19 and the constant rank condition from Chapter 4, we provide, for each particular problem class, adapted conditions. Each of them implies the whole set of the convergence assumptions and is therefore sufficient for the local quadratic convergence of the general Newton-type algorithm and its special realizations. Moreover, we prove relations to some existing conditions from the literature. In Section 5.1 KKT systems arising from optimization problems or variational inequalities are considered. We recover assertions from [18, Theorems 5 and 6]. It is particularly shown that the whole set of our convergence assumptions is implied by a second-order condition which was used for the local convergence analysis of a stabilized SQP method, see for example [31, 48, 87]. New conditions implying the whole set of the convergence assumptions are presented as well, for instance some constant rank condition or some set of local error bound conditions. Section 5.2 deals with KKT systems arising from GNEPs. We will particularly recover and extend a result from [49] by proving that the full row rank of a certain matrix is sufficient for our convergence assumptions to hold. The results of Section 5.2 will also be published in the technical report [37]. In Section 5.3 we discuss the applicability of our Newton-type methods for the solution of Fritz-John (FJ) systems arising from GNEPs. The consideration of FJ systems instead of KKT systems is motivated by an example in [13] which shows that it cannot be expected in general that every solution of a GNEP yields a solution of the corresponding KKT system. This example is stable with respect to small perturbations of the problem functions. In particular, we prove that some full row rank condition implies the whole set of the convergence assumptions. Moreover, we show that generically this full row rank condition is satisfied at all solutions. Thus, new results concerning local quadratic convergence of the general Newton-type algorithm for the solution of FJ systems arising from GNEPs are obtained.

In Chapter 6 an idea for a possible globalization of our Newton-type methods is described, at least for the case that the constrained system arises from a certain smooth reformulation of the KKT system of a GNEP. We present a hybrid method which enjoys, under appropriate conditions, both global convergence and local fast convergence. The local part of the hybrid method is the LP-Newton method whereas the global part is a potential reduction algorithm. The results of Chapter 6 are based on a paper [15] which was published by Axel Dreves, Francisco Facchinei, Andreas Fischer, and the author. Some numerical results can be found in [15], too.

Chapter 7 summarizes the most important results of this thesis and gives an outlook concerning future research.

Chapter 2

Preliminaries

In this chapter some basic definitions and results are recalled which will be needed later on. Section 2.1 is devoted to the notion of the local error bound condition for a system consisting of inequalities and equations. Moreover, relations to some constraint qualifications are recalled. Section 2.2 deals with Clarke's generalized Jacobian and the B-subdifferential of locally Lipschitz continuous functions. In Section 2.3 we recall the notions of semismoothness and strong semismoothness.

2.1 Local Error Bound Condition

In this section we consider a system of inequalities and equations

$$g(z) \leq 0, \quad h(z) = 0 \tag{2.1}$$

with given functions $g : \mathbb{R}^n \to \mathbb{R}^{m_g}$ and $h : \mathbb{R}^n \to \mathbb{R}^{m_h}$ which are assumed to be at least continuous. We will recall the notion of the local error bound condition at a solution of such a system. After its definition, relations to some constraint qualifications are recalled.

Let $\mathcal{Z}$ denote the solution set of (2.1) throughout the rest of this section, i.e.,

$$\mathcal{Z} := \{z \in \mathbb{R}^n \mid g(z) \leq 0,\ h(z) = 0\}.$$

It is assumed that $\mathcal{Z}$ is nonempty and that $z^* \in \mathcal{Z}$ denotes an arbitrary but fixed solution of (2.1).

We say that the system (2.1) satisfies the *local error bound condition* at z^* if there are $\omega_{EB} > 0$ and $\delta_{EB} > 0$ such that

$$\operatorname{dist}[z, \mathcal{Z}] \leq \omega_{EB}(\|\min\{0, -g(z)\}\| + \|h(z)\|)$$

holds for all $s \in \mathcal{B}_{\delta_{EB}}(z^*)$. According to our notation in the Introduction, (2.1) satisfies the local error bound condition at z^* if and only if the function

$$z \mapsto \begin{pmatrix} \min\{0, -g(z)\} \\ h(z) \end{pmatrix}$$

provides a local error bound for the distance to $\mathcal{Z}$ near z^*.

From now on, it is assumed that g and h are continuously differentiable. By $\mathcal{G}_0$ we indicate the index set of all inequality constraints which are active at z^*, i.e.,

$$\mathcal{G}_0 := \{i \in \{1, \dots, m_g\} \mid g_i(z^*) = 0\}.$$

It is well known [43, Proposition 1] that the local error bound condition implies *Abadie's constraint qualification* (ACQ). We say that the ACQ holds at z^* if $\mathcal{T}(z^*)$ equals $\mathcal{L}(z^*)$ where

$$\begin{aligned}\mathcal{T}(z^*) \;\; := \;\; & \{d \in \mathbb{R}^n \mid \exists \{z^k\} \subseteq \mathcal{Z},\, \{t_k\} \subset (0, \infty) : \lim_{k\to\infty} z^k = z^*, \\ & \lim_{k\to\infty} t_k = 0,\; \lim_{k\to\infty} \frac{z^k - z^*}{t_k} = d\}\end{aligned}$$

denotes the *tangent cone* at z^* and

$$\mathcal{L}(z^*) := \{d \in \mathbb{R}^n \mid g'_{\mathcal{G}_0}(z^*)d \le 0,\; h'(z^*)d = 0\}$$

indicates the *linearization cone* at z^*.

We are interested in conditions which imply the validity of the local error bound condition. Let us begin with Theorem 2.1 below which is a consequence of a famous result of Hoffman [45]. It says that the system (2.1) satisfies a global error bound condition if g and h are affine. Then the local error bound condition holds at z^* with arbitrary $\delta_{EB} > 0$.

Theorem 2.1. *(Hoffman) Suppose that g and h are affine. Then there is $\omega_H > 0$ such that*

$$\operatorname{dist}[z, \mathcal{Z}] \le \omega_H(\|\min\{0, -g(z)\}\| + \|h(z)\|)$$

holds for all $z \in \mathbb{R}^n$.

The *Mangasarian-Fromovitz constraint qualification* (MFCQ) is satisfied at z^* if the matrix $h'(z^*)$ has full row rank and there is some vector $d \in \mathbb{R}^n$ such that

$$g'_{\mathcal{G}_0}(z^*)d < 0 \quad \text{and} \quad h'(z^*)d = 0$$

hold. The MFCQ was introduced in [65]. It is well known that the local error bound condition is implied by the MFCQ, see for example [79] and [82, Example 9.44].

The next condition which we want to recall is the *relaxed constant rank constraint qualification* (RCRCQ). The RCRCQ is satisfied at z^* if there is $\delta_{CR} > 0$ such that, for every subset $\mathcal{G} \subseteq \mathcal{G}_0$, the matrices

$$\begin{pmatrix} g'_{\mathcal{G}}(z) \\ h'(z) \end{pmatrix}$$

have the same rank for all $z \in \mathcal{B}_{\delta_{CR}}(z^*)$. To the best of the author's knowledge, the RCRCQ was used in [60] for the first time. The name "relaxed constant rank constraint qualification" was given in [69]. The RCRCQ is sufficient for the local error bound condition to hold. The latter result can be found in [69, Theorem 3] and is also proved in [62, Proposition 3.3] where some gap could be closed in [63].

The RCRCQ is a relaxation of the *constant rank constraint qualification* (CRCQ) which was defined in [51]. If the system (2.1) consists of inequalities only, RCRCQ and CRCQ coincide. However, if (2.1) contains equations, the latter is stronger.

The MFCQ is neither sufficient nor necessary for the RCRCQ to hold. In fact, [51, Example 2.1] provides an example where MFCQ holds but CRCQ is not valid. Since only inequalities appear in this example, RCRCQ is not satisfied as well. Conversely, if both g and h are affine, then RCRCQ is always satisfied whereas MFCQ might be violated. But there are also problems which are described by nonlinear functions and where RCRCQ holds and MFCQ does not hold, see for instance [69, Example 2].

Finally, let us recall the *linear independence constraint qualification* (LICQ) which requires that the matrix

$$\begin{pmatrix} g'(z^*) \\ h'(z^*) \end{pmatrix}$$

has full row rank. It is not difficult to show that the LICQ implies both RCRCQ and MFCQ. In particular, the LICQ is sufficient for the local error bound condition to hold.

Let us summarize the conditions which we recalled in this section and which imply the local error bound condition.

Theorem 2.2. *The local error bound condition is satisfied at z^* if one of the following conditions holds: LICQ, MFCQ, RCRCQ.*

From Theorem 2.2 a well-known result regarding local error bounds for systems of equations can be derived.

Theorem 2.3. *Suppose that* (2.1) *is a system of equations $h(z) = 0$ only. Moreover, assume that $h'(z^*)$ has full (row or column) rank. Then the local error bound condition is satisfied at z^*.*

Proof. Due to the continuity of the function $z \mapsto h'(z)$, there is $\delta_{CR} > 0$ such that $h'(z)$ has still full rank for all $z \in \mathcal{B}_{\delta_{CR}}(z^*)$. In particular, the RCRCQ holds at z^*. Thus, the local error bound condition follows from Theorem 2.2. □

There are further constraint qualifications implying the local error bound condition, for instance the constant positive linear dependence condition (CPLD). The latter was introduced in [78], its sufficiency for the local error bound condition is stated in [69, Theorem 4]. However, the CPLD and further constraint qualifications will not be discussed in this thesis.

2.2 Clarke's Generalized Jacobian

Throughout this section, a locally Lipschitz continuous function $F : \mathbb{R}^n \to \mathbb{R}^m$ is considered, i.e., F is assumed to be Lipschitz continuous on any compact subset of $\mathbb{R}^n$. We will introduce the generalized Jacobian ∂F of F in the sense of Clarke as well as the B-subdifferential $\partial_B F$ of F. Afterwards, some properties of the set-valued mappings $z \mapsto \partial F(z)$ and $z \mapsto \partial_B F(z)$ are recalled.

Note that a locally Lipschitz continuous function is in general not everywhere differentiable. For instance, the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(z) := |z|$ is locally Lipschitz continuous on $\mathbb{R}$ but not differentiable at $z = 0$. Even the directional differentiability is not implied by the local Lipschitz continuity of F, see [34, page 521] for an example, where F is called *directional differentiable* at z if

$$F'(z,d) := \lim_{t \downarrow 0} \frac{F(z+td) - F(z)}{t} \tag{2.2}$$

exists for all $d \in \mathbb{R}^n$. If the limes exists for some $z \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, then $F'(z,d)$ is called *directional derivative* of F at z in the direction d.

However, it is well known by Rademacher's Theorem that locally Lipschitz continuous functions are differentiable almost everywhere. Let us recall this famous theorem. A proof can be found, for instance, in [5, Corollary 4.19].

Theorem 2.4. *(Rademacher) The locally Lipschitz continuous function F is differentiable almost everywhere, i.e., the Lebesgue measure of the set consisting of all points where F is not differentiable is equal to zero.*

In the following, we denote by D_F the set of all points $z \in \mathbb{R}^n$ where F is differentiable. For any point $z \in \mathbb{R}^n$, the *Bouligand-subdifferential* (B-subdifferential for short) of F at z [74] is defined by

$$\partial_B F(z) := \{V \in \mathbb{R}^{m \times n} \mid \exists \{z^k\} \subset D_F : \lim_{k \to \infty} z^k = z, \ \lim_{k \to \infty} F'(z^k) = V\}.$$

Note that Theorem 2.4 guarantees that for any $z \in \mathbb{R}^n$ there is a sequence $\{z^k\} \subset D_F$ converging to z. The convex hull of $\partial_B F(z)$ is denoted by $\partial F(z)$, i.e.,

$$\partial F(z) := \operatorname{conv}(\partial_B F(z)).$$

It was introduced by Clarke [4] and is called *Clarke's generalized Jacobian* of F at z. The following proposition states some properties of the sets $\partial_B F(z)$ and $\partial F(z)$ for any fixed z.

Proposition 2.5. *Let $z \in \mathbb{R}^n$ be a given point. Then the following assertions are satisfied.*

(a) If F is continuously differentiable at z, $\partial_B F(z) = \partial F(z) = \{F'(z)\}$ holds.

(b) The sets $\partial_B F(z)$ and $\partial F(z)$ are nonempty and compact.

Proof. (a) The proof is essentially taken from [86]. Let $\{z^k\} \subset D_F$ be an arbitrary but fixed sequence converging to z. Due to the continuity of F' at z, we have $\lim_{k\to\infty} F'(z^k) = F'(z)$. Since $\{z^k\}$ was arbitrarily chosen, $\partial_B F(z) = \{F'(z)\}$ holds. Obviously, the convex hull of a singleton is still a singleton so that $\partial F(z) = \{F'(z)\}$ follows.

(b) We omit the proof of this item. In [4, Proposition 2.6.2] it is shown that $\partial F(z)$ is nonempty and compact. A detailed proof of the nonemptiness and the compactness of both $\partial F(z)$ and $\partial_B F(z)$ can be found in [53, Satz 4.3]. □

Next, we recall some properties of the point-to-set mappings $z \mapsto \partial_B F(z)$ and $z \mapsto \partial F(z)$. To this end, we need some definitions. Let X and Y be real Banach spaces. A point-to-set mapping $T : X \rightrightarrows Y$ is called *closed* at $x \in X$ if, for any sequences $\{x_k\} \subseteq X$ and $\{y_k\} \subseteq Y$ satisfying $\lim_{k\to\infty} x_k = x$, $y_k \in T(x_k)$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} y_k = y$, the point y belongs to $T(x)$. If T is closed at all points $x \in X$, we say that T is *closed* on X. Moreover, $T : X \rightrightarrows Y$ is called *upper semicontinuous* at $x \in X$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$T(w) \subseteq T(x) + \mathcal{B}_\varepsilon(0)$$

holds for all $w \in \mathcal{B}_\delta(x)$. If T is upper semicontinuous at all points $x \in X$, we say that T is *upper semicontinuous* on X.

In [4, Proposition 2.6.2] the closedness and the upper semicontinuity of the point-to-set mapping $z \mapsto \partial F(z)$ are shown. For a detailed proof of the following proposition, i.e., of both properties for $z \mapsto \partial F(z)$ as well as for $z \mapsto \partial_B F(z)$, we refer to [53].

Proposition 2.6. *The point-to-set mappings $z \mapsto \partial_B F(z)$ and $z \mapsto \partial F(z)$ are both closed and upper semicontinuous on $\mathbb{R}^n$.*

The upper semicontinuity of ∂F yields the following corollary which is [53, Bemerkung 4.6 (b)].

Corollary 2.7. *Let $z \in \mathbb{R}^n$ be a given point and suppose that $\{z^k\} \subset \mathbb{R}^n$ is a sequence converging to z. Let $\{V_k\}$ be a sequence such that $V_k \in \partial F(z^k)$ holds for all $k \in \mathbb{N}$. Then $\{V_k\}$ is bounded.*

Proof. Due to the upper semicontinuity of ∂F at z, there is $\delta > 0$ such that

$$\partial F(s) \subseteq \partial F(z) + \mathcal{B}_1(0)$$

holds for all $s \in \mathcal{B}_\delta(z)$. Taking into account this inclusion and $\lim_{k\to\infty} z^k = z$, there is $k_0 \in \mathbb{N}$ such that

$$\partial F(z^k) \subseteq \partial F(z) + \mathcal{B}_1(0)$$

is satisfied for all $k \geq k_0$. Therefore, we have $V_k \in \partial F(z) + \mathcal{B}_1(0)$ for all $k \geq k_0$. Since $\partial F(z)$ is compact due to Proposition 2.5, and $\mathcal{B}_1(0)$ is compact, the boundedness of the sequence $\{V_k\}$ follows. □

Note that the assertion of Corollary 2.7 stays true if $\partial F(z^k)$ is replaced by $\partial_B F(z^k)$ since the B-subdifferential is always a subset of Clarke's generalized Jacobian.

Next, we are going to state some properties on the B-subdifferential in a neighborhood of points z where all elements of $\partial_B F(z)$ have full column rank. In [74] the notion of strongly BD-regularity was introduced. Let us recall the definition. A locally Lipschitz continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *strongly BD-regular* at z if all $V \in \partial_B F(z)$ are nonsingular. Assuming F is strongly BD-regular at z, it was shown in [74, Lemma 2.6] that, for any s in a sufficiently small neighborhood of z, all matrices belonging to $\partial_B F(s)$ are still nonsingular. Moreover, the norms of the inverses are bounded from above.

Proposition 2.9 below extends both results to the case $m \geq n$. The proof is a straightforward generalization of [74, Lemma 2.6]. At first, let us recall a result on the Moore-Penrose pseudoinverse of a matrix with full column rank. A matrix $A^+ \in \mathbb{R}^{n\times m}$ is called *(Moore-Penrose) pseudoinverse* of a matrix $A \in \mathbb{R}^{m\times n}$ if the identities

$$AA^+A = A \quad \text{and} \quad A^+AA^+ = A^+$$

hold and if both AA^+ and A^+A are symmetric. It can be shown that every matrix has exactly one pseudoinverse [56, Satz 8.1.8]. The following lemma summarizes some properties of the pseudoinverse of matrices with full column rank. The assertions can be easily checked, see also [56, Satz 8.1.9].

Lemma 2.8. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix and suppose that A has full column rank. Then the following assertions are satisfied.*

(a) The pseudoinverse is given by

$$A^+ = (A^\top A)^{-1}A^\top.$$

In particular, $A^+A = I_n$ holds.

(b) If m equals n, then $A^+ = A^{-1}$ is valid.

The proof of the subsequent proposition can be straightforwardly derived from the proof of [74, Lemma 2.6].

Proposition 2.9. *Let $z \in \mathbb{R}^n$ be a given point and assume that all elements of $\partial_B F(z)$ have full column rank. Then there are $\varrho > 0$ and $c > 0$ such that, for any $s \in \mathcal{B}_\varrho(z)$,*

$$\operatorname{rank} V = n \quad \textit{and} \quad \|V^+\| \leq c$$

hold for all $V \in \partial_B F(s)$.

Proof. Let us assume the contrary. Then there are a sequence $\{s^k\} \subset \mathbb{R}^n$ converging to z and a corresponding sequence $\{V_k\} \subset \mathbb{R}^{m\times n}$ satisfying $V_k \in \partial_B F(s^k)$ for all $k \in \mathbb{N}$ such that either $\operatorname{rank} V_k < n$ holds for all $k \in \mathbb{N}$, or the matrices V_k have full column rank for all $k \in \mathbb{N}$ but $\|V_k^+\|$ grows to infinity for $k \to \infty$. Due to Corollary 2.7, the sequence $\{V_k\}$ is bounded. Consequently, there is an infinite subset $\mathcal{K}_1 \subseteq \mathbb{N}$ such that $\{V_k\}_{k\in\mathcal{K}_1}$ converges to some $V \in \mathbb{R}^{m\times n}$. Since the point-to-set mapping $\partial_B F$ is closed by Proposition 2.6, $V \in \partial_B F(z)$ holds. By the assumption of the proposition, $\operatorname{rank} V = n$ follows.

Suppose that $\operatorname{rank} V_k < n$ is valid for all $k \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$, there is some $d^k \in \mathbb{R}^n$ such that $\|d^k\| = 1$ and $V_k d^k = 0$ are satisfied. Obviously, the sequence $\{d^k\}_{k\in\mathcal{K}_1}$ is bounded. Thus, there is an infinite subset $\mathcal{K}_2 \subseteq \mathcal{K}_1$ such that the corresponding subsequence $\{d^k\}_{k\in\mathcal{K}_2}$ is convergent to some $d \in \mathbb{R}^n$ satisfying $\|d\| = 1$. Taking into account $V_k d^k = 0$ for all $k \in \mathcal{K}_2$ and $\lim_{k\to\infty,\, k\in\mathcal{K}_2} V_k = V$, we obtain $Vd = 0$. However, this contradicts $\operatorname{rank} V = n$.

Now let us assume that $\operatorname{rank} V_k = n$ is valid for all $k \in \mathbb{N}$ but $\|V^+\|$ grows to infinity for $k \to \infty$. Due to Lemma 2.8, $V_k^+ = (V_k^\top V_k)^{-1} V_k^\top$ holds for all $k \in \mathbb{N}$. Just as well, $V^+ = (V^\top V)^{-1} V^\top$ is satisfied. The convergence of the sequence $\{V_k\}_{k\in\mathcal{K}_1}$ to V implies

$$\lim_{k\to\infty,\, k\in\mathcal{K}_1} V_k^\top = V^\top \quad \text{and} \quad \lim_{k\to\infty,\, k\in\mathcal{K}_1} V_k^\top V_k = V^\top V.$$

Since the mapping $A \mapsto A^{-1}$ is continuous on the set of all nonsingular (n,n)-matrices,

$$\lim_{k\to\infty,\, k\in\mathcal{K}_1} V_k^+ = \lim_{k\to\infty,\, k\in\mathcal{K}_1} (V_k^\top V_k)^{-1} V_k^\top = (V^\top V)^{-1} V^\top = V^+$$

follows. This contradicts $\|V_k^+\| \to \infty$ for $k \to \infty$. □

Remark 2.1. The assertion of Proposition 2.9 stays true if, in the assumption as well as in the conclusion, the B-subdifferentials are replaced by Clarke's generalized Jacobians.

2.3 Semismooth Functions

The concept of semismoothness was introduced in [68] for real-valued functions and extended in [77] to functions $F : \mathbb{R}^n \to \mathbb{R}^m$. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called *semismooth* at z if F is locally Lipschitz continuous near z and, for any $d \in \mathbb{R}^n$,

$$\lim_{\substack{V \in \partial F(z+td') \\ d' \to d,\, t \downarrow 0}} V d' \tag{2.3}$$

exists. In [77] it is shown that the semismoothness of F at z implies that the directional derivative $F'(z,d)$ exists for all $d \in \mathbb{R}^n$ and equals the limes in (2.3).

For the definition of $F'(z,d)$ we refer to (2.2). We say that F is *semismooth* if F is semismooth at all points $z \in \mathbb{R}^n$. The following characterization of semismooth functions by means of directional derivatives is proved in [77, Theorem 2.3].

Proposition 2.10. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous near z and directional differentiable at z. Then F is semismooth at z if and only if*

$$\lim_{d\to 0} \frac{1}{\|d\|} \sup\{\|Vd - F'(z,d)\| \mid V \in \partial F(z+d)\} = 0$$

holds.

The latter characterization of semismoothness motivates to define a stronger notion as follows. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ which is semismooth at z is called *strongly semismooth* at z if there are $K > 0$ and $\varepsilon > 0$ such that

$$\sup\{\|Vd - F'(z,d)\| \mid V \in \partial F(z+d)\} \leq K\|d\|^2$$

holds for all $d \in \mathcal{B}_\varepsilon(0)$. Note that strong semismoothness coincides with 1-order semismoothness defined in [77]. The following relation between differentiability and semismoothness is not difficult to verify. A proof can be found in [86, Satz 3.4.4].

Proposition 2.11. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable at z. Then F is semismooth at z. If, in addition, F' is locally Lipschitz continuous near z, then F is strongly semismooth at z.*

The next proposition contains an implication of semismoothness and strong semismoothness, respectively, which we will refer to in Section 3.2.

Proposition 2.12. *The following assertions hold.*

(a) If $F : \mathbb{R}^n \to \mathbb{R}^m$ is semismooth at z, then

$$\lim_{d\to 0} \frac{1}{\|d\|} \sup\{\|F(z+d) - F(z) - Vd\| \mid V \in \partial F(z+d)\} = 0$$

is satisfied.

(b) If $F : \mathbb{R}^n \to \mathbb{R}^m$ is strongly semismooth at z, then there are $\tilde{\varepsilon} > 0$ and $\tilde{K} > 0$ such that

$$\sup\{\|F(z+d) - F(z) - Vd\| \mid V \in \partial F(z+d)\} \leq \tilde{K}\|d\|^2$$

is satisfied for all $d \in \mathcal{B}_{\tilde{\varepsilon}}(0)$.

Proof. As stated in [34], the assertions follow from the definitions of semismoothness and strong semismoothness, respectively, and [34, Lemma 2]. The latter coincides with a remark at the end of [77, Section 2]. Besides, a proof of item (a) can be found in [73, Proposition 1], and item (b) is also proven in [22, Proposition 2]. □

Chapter 3

A Family of Newton-type Methods

This chapter is devoted to the description and the analysis of local methods for the solution of the constrained system of equations (1.1), i.e.,

$$F(z) = 0 \quad \text{s.t.} \quad z \in \Omega$$

where $\Omega \subseteq \mathbb{R}^n$ is a nonempty and closed set and $F : \Omega \to \mathbb{R}^m$ is an at least continuous function. Throughout this chapter, it is assumed that the solution set

$$Z := \{z \in \Omega \mid F(z) = 0\}$$

of (1.1) is nonempty. By z^* we denote an arbitrary but fixed element of Z. Moreover, let a function $G : \Omega \to \mathbb{R}^{m \times n}$ be given. Later on, we will set conditions on this matrix-valued function and specify its choice for certain problem classes. For the moment, it is sufficient to think of G as the Jacobian of F if it exists, or a suitable substitute otherwise.

In Section 3.1 local convergence properties of a general Newton-type algorithm for the solution of (1.1) are analyzed. We prove local quadratic convergence under the four assumptions which were used in [18] for the local convergence analysis of the LP-Newton method. Section 3.2 is devoted to a first discussion of the convergence assumptions. Particularly, their relation to some existing regularity conditions from the literature is analyzed. It will turn out that neither differentiability of F nor the local uniqueness of solutions are implied by the whole set of the convergence assumptions. Except Proposition 3.4, Section 3.2 is based on [18, Section 3.1]. The further sections of this chapter deal with special realizations of the general Newton-type algorithm. Unlike the algorithm from Section 3.1 itself, which is rather a general framework for Newton-type methods for the solution of (1.1), these realizations are computationally implementable. First, it is shown in Section 3.3 that the classical Newton method as well as nonsmooth and inexact variants of it can be regarded as special cases of the general Newton-type algorithm if suitable conditions are satisfied. However, these conditions are quite restrictive, in particular they imply that z^* is a locally unique solution of (1.1).

So, although the results from Section 3.3 are interesting from the theoretical point of view, our focus lies on the methods which are described in the last two sections of this chapter. Section 3.4 is devoted to the LP-Newton method introduced in [18] and an inexact version of it. In Section 3.5 the constrained Levenberg-Marquardt method and several inexact extensions are described. Both the LP-Newton method and the constrained Levenberg-Marquardt method are special realizations of the general Newton-type algorithm from Section 3.1 without requiring any additional conditions. Therefore, these methods converge locally with a Q-quadratic rate under the same assumptions as the general algorithm itself. The results of Sections 3.3–3.5 were in large part obtained in a joint work with Francisco Facchinei and Andreas Fischer in [17]. However, we describe a more general inexact constrained Levenberg-Marquardt method in Section 3.5 compared to [17] and show that both [17, Algorithm 3] and the inexact method from [1] are special cases, the latter at least under appropriate conditions.

3.1 A General Framework

In this section a general iterative algorithm for the solution of (1.1) is described. The subproblems consist of finding an element of a certain set. We will show that the algorithm is always well defined. Afterwards, four assumptions are presented which will be used to analyze local convergence properties. The main result of this section is Theorem 3.3 on the local quadratic convergence of the general Newton-type algorithm to a solution of (1.1).

Let us define, for any point $s \in \Omega$ and any number $\gamma \geq 0$, the set

$$\mathcal{F}(s,\gamma) := \{z \in \Omega \mid \|F(s) + G(s)(z-s)\| \leq \gamma \|F(s)\|^2,\ \|z-s\| \leq \gamma \|F(s)\|\}.$$

The following iterative algorithm for the solution of (1.1) is a slight modification of [17, Algorithm 1].

Algorithm 3.1. (General Newton-type Algorithm)

(S.0) Choose $z^0 \in \Omega$, $\Gamma_0 > 0$, $\alpha \geq 1$, and $\beta > 1$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) Compute $\Gamma_{k+1} := \min\{\beta^\ell \Gamma_k \mid \ell = 0,1,2,\ldots\}$ such that $\mathcal{F}(z^k, \Gamma_{k+1})$ is nonempty.

(S.3) Determine $z^{k+1} \in \mathcal{F}(z^k, \alpha\Gamma_{k+1})$.

(S.4) Set $k := k+1$ and go to (S.1).

Algorithm 3.1 should be regarded as a general framework for Newton-type methods for the solution of (1.1). Of course, the question concerning a computational implementation may arise. An answer is given in Sections 3.4 and 3.5 where special realizations of Algorithm 3.1 are described. It will turn out that these realizations are implementable and keep the local convergence properties which we prove for Algorithm 3.1 in the sequel.

Unlike [17, Algorithm 1], the values Γ_k in the above algorithm are updated in each step and therefore depend on the iteration index k. The updating rule in step (S.2) guarantees that Algorithm 3.1 is always well defined. This is shown in the next proposition.

Proposition 3.1. *Algorithm 3.1 is well defined for any $z^0 \in \Omega$, $\Gamma_0 > 0$, $\alpha \geq 1$, and $\beta > 1$.*

Proof. It suffices to prove that steps (S.2) and (S.3) of the algorithm are always well defined. To this end, let us fix $k \in \mathbb{N}$ and let $z^k \in \Omega$ and $\Gamma_k > 0$ be given. It is not difficult to see that $s \in \mathcal{F}(s, \|F(s)\|^{-1})$ is satisfied for all $s \in \Omega \backslash Z$. Thus, since z^k in step (S.2) of Algorithm 3.1 is not yet a solution of (1.1), $\mathcal{F}(z^k, \|F(z^k)\|^{-1})$ is nonempty. Taking into account $\beta > 1$, there is $\bar{\ell} \in \mathbb{N}$ such that

$$\beta^{\bar{\ell}}\Gamma_k \geq \|F(z^k)\|^{-1}$$

holds. This implies that $\mathcal{F}(z^k, \|F(z^k)\|^{-1})$ is a subset of $\mathcal{F}(z^k, \beta^{\bar{\ell}}\Gamma_k)$ so that the latter is nonempty. Of course, there is also a smallest natural number $\ell \leq \bar{\ell}$ with the property that the set $\mathcal{F}(z^k, \beta^{\ell}\Gamma_k)$ is nonempty. Therefore, Γ_{k+1} in step (S.2) is well defined. Due to $\alpha \geq 1$, the set $\mathcal{F}(z^k, \Gamma_{k+1})$ is a subset of $\mathcal{F}(z^k, \alpha\Gamma_{k+1})$. Consequently, taking into account $\mathcal{F}(z^k, \Gamma_{k+1}) \neq \emptyset$, the set $\mathcal{F}(z^k, \alpha\Gamma_{k+1})$ is nonempty. Thus, step (S.3) of Algorithm 3.1 is well defined. □

In order to prove local quadratic convergence of Algorithm 3.1, we use the subsequent assumptions. The same assumptions were used in [18] for the analysis of local convergence properties of the LP-Newton method.

Assumption 1. There are $L > 0$ and $\delta_1 > 0$ such that

$$\|F(s)\| \leq L \operatorname{dist}[s, Z]$$

holds for all $s \in \mathcal{B}_{\delta_1}(z^*) \cap \Omega$.

This assumption is quite weak since it is particularly satisfied if F is locally Lipschitz continuous on Ω. In order to justify this, let $\delta_1 > 0$ be given and $s \in \mathcal{B}_{\delta_1}(z^*) \cap \Omega$ be arbitrarily chosen. Due to the continuity of F, the set Z is closed so that there is $\bar{s} \in Z$ with the property $\operatorname{dist}[s, Z] = \|s - \bar{s}\|$. Particularly,

$$\|\bar{s} - z^*\| \leq \|\bar{s} - s\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\delta_1$$

is valid. Therefore, if $L > 0$ denotes a Lipschitz constant of F on $\mathcal{B}_{2\delta_1}(z^*) \cap \Omega$,

$$\|F(s)\| = \|F(s) - F(\bar{s})\| \leq L\|s - \bar{s}\| = L \operatorname{dist}[s, Z]$$

follows. Hence, Assumption 1 holds.

Assumption 2. There are $\omega > 0$ and $\delta_2 > 0$ such that

$$\operatorname{dist}[s, Z] \leq \omega\|F(s)\|$$

holds for all $s \in \mathcal{B}_{\delta_2}(z^*) \cap \Omega$.

Assumption 2 requires that F provides a local error bound for the distance to the solution set of (1.1) near z^* on Ω. If both Assumption 1 and Assumption 2 are satisfied, the norm of F and the distance to Z are proportional, at least for points in a certain neighborhood of z^* intersected with Ω.

During the last years it turned out that local error bound conditions are the key for proving local superlinear convergence of algorithms, particularly if the problem has nonisolated solutions. If $n = m$ holds, Ω equals $\mathbb{R}^n$, and F is continuously differentiable, then it is well known that Assumption 2 is implied by the nonsingularity of $F'(z^*)$, see also Theorem 2.3. The latter condition is usually required for proving local superlinear convergence of the classical Newton method and variants of it. However, unlike the nonsingularity of the Jacobian at z^*, Assumption 2 does not imply the local uniqueness of z^* as a solution of (1.1).

Local error bound conditions were particularly used for proving local fast convergence of the Levenberg-Marquardt method and variants of it, see for instance [7, 21, 28, 29, 30, 39, 90, 91] for the application to unconstrained systems and [1, 2, 27, 55] for the extension to constrained systems. Assumption 2 and related conditions were also used for the local convergence analysis of further methods for different problem classes such as generalized equations [36, 64], nonlinear complementarity problems [6, 85, 89], KKT systems arising from optimization problems or variational inequalities [31, 35, 42, 46, 48, 87, 88], and KKT systems arising from GNEPs [15, 49].

Assumption 3. There are $\Gamma > 0$ and $\delta_3 > 0$ such that for any $s \in \mathcal{B}_{\delta_3}(z^*) \cap \Omega$ the set $\mathcal{F}(s, \Gamma)$ is nonempty.

In [18] a different formulation of this assumption was used for the analysis of local convergence properties of the LP-Newton method. However, both formulations are equivalent, see Remark 3.4 in Section 3.4. A condition which is, supposed that Assumptions 1 and 2 hold, equivalent to Assumption 3 was already introduced in [84, Condition 5.5.1]. The equivalence follows from [18, Proposition 3]. Assumption 4 below was considered in [84], too. Both conditions were discussed there for several reformulations of the nonlinear complementarity problem.

Assumption 4. There are $\kappa > 0$ and $\delta_4 > 0$ such that

$$z \in \{z \in \Omega \mid \|F(s) + G(s)(z - s)\| \leq \delta^2,\ \|z - s\| \leq \delta\}$$

implies

$$\|F(z)\| \leq \kappa\delta^2$$

for all $s \in (\mathcal{B}_{\delta_4}(z^*) \cap \Omega) \setminus Z$ and all $\delta \in [0, \delta_4]$.

Assumption 4 requires that, for any s in a certain neighborhood of z^*, the affine mapping $z \mapsto F(s) + G(s)(z - s)$ approximates the mapping $z \mapsto F(z)$ in some sense, at least for points z belonging to Ω and being sufficiently close to s. Assumption 3 guarantees, for any $s \in \mathcal{B}_{\delta_3}(z^*) \cap \Omega$, the existence of some $z \in \Omega$ near s for which the expression $\|F(s) + G(s)(z - s)\|$ has the same order as $\|F(s)\|^2$. Therefore, one can say that Assumptions 3 and 4 together imply that, for any $s \in \Omega$ belonging to a sufficiently small neighborhood of z^*, there is a point $z \in \Omega$ whose function value has the same order as $\|F(s)\|^2$.

Assumptions 3 and 4 are quite technical so that a deeper discussion on them is appropriate. In Section 3.2 we will relate them to some existing regularity conditions from the literature. In particular, it will be shown that Assumption 4 holds if F is differentiable and has a locally Lipschitz continuous derivative. We will see that Assumption 3 is also satisfied in that setting if Assumption 2 is additionally valid. Conversely, Assumptions 3 and 4 together do not imply differentiability of F. Chapter 4 provides a detailed analysis of the assumptions for the case that F is a PC^1-function.

The next aim is to prove local quadratic convergence of our general Newton-type algorithm supposed that Assumptions 1–4 hold. The proof which is presented below is similar to the proof given in [18] for the theorem on local quadratic convergence of the LP-Newton method. Before the main result of this section is stated and shown, we need the following lemma.

Lemma 3.2. *Let Assumptions 1 and 4 be satisfied. Then, for every $\gamma > 0$, there are $\epsilon(\gamma) > 0$ and $C(\gamma) > 0$ such that for any $s \in \mathcal{B}_{\epsilon(\gamma)}(z^*) \cap \Omega$*

$$\|F(z)\| \leq C(\gamma)\, \|F(s)\|^2 \leq \frac{1}{2}\|F(s)\|$$

holds for all $z \in \mathcal{F}(s, \gamma)$.

Proof. Let $\gamma > 0$ be arbitrary but fixed and let us define $\tilde{C}(\gamma) := \max\{\gamma, \sqrt{\gamma}\}$ and

$$\epsilon(\gamma) := \min\left\{\delta_1,\ \frac{\delta_4}{L\,\tilde{C}(\gamma)},\ \frac{1}{2\kappa L\,\tilde{C}(\gamma)^2}\right\}. \tag{3.1}$$

Now let $s \in \mathcal{B}_{\epsilon(\gamma)}(z^*) \cap \Omega$ be arbitrarily chosen and suppose that $\mathcal{F}(s, \gamma)$ is nonempty. If $s \in Z$ holds, then $\mathcal{F}(s, \gamma) = \{s\}$ follows so that nothing is to show.

So let us assume $s \notin Z$ and let us set $\delta := \tilde{C}(\gamma)\|F(s)\|$. Due to the definition of $\epsilon(\gamma)$, we have $s \in \mathcal{B}_{\delta_1}(z^*) \cap \Omega$ so that Assumption 1 can be applied and yields, again together with the definition of $\epsilon(\gamma)$,

$$\delta \leq \tilde{C}(\gamma)L\,\mathrm{dist}[s,Z] \leq \tilde{C}(\gamma)L\|s-z^*\| \leq \tilde{C}(\gamma)L\epsilon(\gamma) \leq \delta_4. \tag{3.2}$$

Now let us take any $z \in \mathcal{F}(s,\gamma)$. Using the definitions of $\mathcal{F}(s,\gamma)$, $\tilde{C}(\gamma)$, and δ, we obtain

$$\|F(s)+G(s)(z-s)\| \leq \gamma\|F(s)\|^2 \leq \delta^2 \tag{3.3}$$

and

$$\|z-s\| \leq \gamma\|F(s)\| \leq \delta. \tag{3.4}$$

Taking into account (3.2)–(3.4) and $z \in \Omega$, Assumption 4 implies

$$\|F(z)\| \leq \kappa\delta^2 = \kappa\tilde{C}(\gamma)^2\|F(s)\|^2.$$

This, Assumption 1, and the definition of $\epsilon(\gamma)$ yield

$$\begin{aligned}\|F(z)\| &\leq \kappa\tilde{C}(\gamma)^2\|F(s)\|^2 \\ &\leq \kappa\tilde{C}(\gamma)^2L\,\mathrm{dist}[s,Z]\|F(s)\| \\ &\leq \kappa\tilde{C}(\gamma)^2L\epsilon(\gamma)\|F(s)\| \\ &\leq \frac{1}{2}\|F(s)\|.\end{aligned}$$

Hence, with $C(\gamma) := \kappa\tilde{C}(\gamma)^2$, the lemma is proved. □

Now we are in the position to state the main result of this section.

Theorem 3.3. *Let Assumptions 1–4 be satisfied. Then there is $\rho > 0$ such that the following assertions hold for any infinite sequences $\{z^k\}$ and $\{\Gamma_k\}$ generated by Algorithm 3.1 with starting points $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ and $\Gamma_0 > 0$.*

(a) The sequence $\{\Gamma_k\}$ is bounded.

(b) The sequence $\{z^k\}$ converges Q-quadratically to a solution of (1.1).

Proof. Let us define $\gamma := \alpha\max\{\Gamma_0, \beta\Gamma\}$ and $\epsilon := \min\{\epsilon(\gamma), \delta_3\}$, where $\epsilon(\gamma)$ is given according to (3.1), again with $\tilde{C}(\gamma) := \max\{\gamma, \sqrt{\gamma}\}$. Moreover, we set

$$\rho := \frac{\epsilon}{1+2\gamma L}.$$

First, we show by induction that

$$\alpha\Gamma_k \leq \gamma \tag{3.5}$$

and

$$z^k \in \mathcal{B}_\epsilon(z^*) \cap \Omega \tag{3.6}$$

hold for all $k \in \mathbb{N}$. By the definitions of γ and ρ, both (3.5) and (3.6) are obviously satisfied for $k = 0$. Now assume that (3.5) and (3.6) are valid for all $k = 0, \dots, l$. Let us verify (3.5) for $k = l + 1$. The updating rule in step (S.2) of Algorithm 3.1 implies $\Gamma_{l+1} = \beta^{\ell_{l+1}}\Gamma_l$ where ℓ_{l+1} is the smallest number $\ell \in \mathbb{N}$ for which $\mathcal{F}(z^l, \beta^\ell\Gamma_l)$ is nonempty. By (3.6) for $k = l$, we have $z^l \in \mathcal{B}_\epsilon(z^*) \cap \Omega$ so that, taking into account the definition of ϵ, Assumption 3 yields that $\mathcal{F}(z^l, \Gamma)$ is nonempty. Together with the definitions of ℓ_{l+1} and γ,

$$\Gamma_{l+1} = \beta^{\ell_{l+1}}\Gamma_l \le \beta\Gamma \le \frac{1}{\alpha}\gamma$$

follows. Hence, (3.5) is satisfied for $k = l + 1$.

Next, we have to prove (3.6) for $k = l + 1$. The triangle inequality leads to

$$\|z^{l+1} - z^*\| \le \|z^l - z^*\| + \|z^{l+1} - z^l\| \le \|z^0 - z^*\| + \sum_{i=0}^{l} \|z^{i+1} - z^i\|. \tag{3.7}$$

The rule in step (S.3) of Algorithm 3.1 and (3.5) for $k = 1, \dots, l + 1$ yield

$$z^{i+1} \in \mathcal{F}(z^i, \alpha\Gamma_{i+1}) \subseteq \mathcal{F}(z^i, \gamma) \tag{3.8}$$

is valid for all $i = 0, \dots, l$. Thus, by the definition of the sets $\mathcal{F}(z^i, \gamma)$, we obtain

$$\|z^{i+1} - z^i\| \le \gamma\|F(z^i)\| \tag{3.9}$$

for all $i = 0, \dots, l$. Moreover, taking into account (3.6) for $k = 0, \dots, l$, (3.8), and the definition of ϵ, the repeated application of Lemma 3.2 implies

$$\|F(z^i)\| \le \left(\frac{1}{2}\right)^i \|F(z^0)\| \tag{3.10}$$

for all $i = 0, \dots, l$. By the definition of ϵ, z^0 belongs to $\mathcal{B}_{\delta_1}(z^*) \cap \Omega$ so that Assumption 1 yields

$$\|F(z^0)\| \le L\,\mathrm{dist}[z^0, Z]. \tag{3.11}$$

Combining (3.7)–(3.11), we obtain

$$\|z^{l+1} - z^*\| \le \underbrace{\|z^0 - z^*\|}_{\le\rho} + \gamma L \underbrace{\mathrm{dist}[z^0, Z]}_{\le\rho} \underbrace{\sum_{i=0}^{l} \left(\frac{1}{2}\right)^i}_{\le 2} \le (1 + 2\gamma L)\rho = \epsilon.$$

Hence, z^{l+1} belongs to $\mathcal{B}_\epsilon(z^*)$. Moreover, (3.8) for $i = l$ implies $z^{l+1} \in \Omega$. Therefore, (3.6) is shown for $k = l + 1$. Thus, it is proved that (3.5) and (3.6) hold for all $k \in \mathbb{N}$. In particular, the boundedness of the sequence $\{\Gamma_k\}$ follows from (3.5).

Next, let us prove the convergence properties of the sequence $\{z^k\}$. The rule in step (S.3) of Algorithm 3.1 and (3.5) imply that (3.8) is satisfied for all $i \in \mathbb{N}$. With this, (3.6), and the definition of ϵ, Lemma 3.2 yields the existence of some $C := C(\gamma) > 0$ such that

$$\|F(z^{k+1})\| \leq C\|F(z^k)\|^2 \leq \frac{1}{2}\|F(z^k)\| \tag{3.12}$$

holds for all $k \in \mathbb{N}$. This implies

$$\lim_{k\to\infty} \|F(z^k)\| = 0. \tag{3.13}$$

Now let us take any $j, k \in \mathbb{N}$ with $k > j$. Using Lemma 3.2 again, we obtain

$$\|z^k - z^j\| \leq \sum_{i=j}^{k-1} \|z^{i+1} - z^i\| \leq \gamma\|F(z^j)\| \sum_{i=j}^{k-1} \left(\frac{1}{2}\right)^{i-j} \leq 2\gamma\|F(z^j)\|. \tag{3.14}$$

By (3.13) and (3.14), $\|z^k - z^j\| \to 0$ follows for $j, k \to \infty$, i.e., $\{z^k\}$ is a Cauchy sequence and thus converges to some $\bar{z}$. Since F is continuous, $F(\bar{z}) = 0$ follows from (3.13). Moreover, $\bar{z} \in \Omega$ holds due to the closedness of Ω. Therefore, $\bar{z}$ belongs to Z.

It remains to prove the convergence rate. Using (3.14) for $k+1$ instead of j and $k+j$ instead of k and (3.12), we obtain

$$\|z^{k+j} - z^{k+1}\| \leq 2\gamma\|F(z^{k+1})\| \leq 2C\gamma\|F(z^k)\|^2$$

for all $k, j \in \mathbb{N}$ with $j > 1$. For $j \to \infty$,

$$\|\bar{z} - z^{k+1}\| \leq 2C\gamma\|F(z^k)\|^2 \leq 2C\gamma L^2 \operatorname{dist}[z^k, Z]^2 \leq 2C\gamma L^2\|\bar{z} - z^k\|^2$$

follows where Assumption 1 was used in the second inequality. Therefore, the sequence $\{z^k\}$ converges Q-quadratically to $\bar{z} \in Z$. □

Note that Assumption 2 was not explicitly used in the proof of Theorem 3.3. However, it is implied by Assumptions 1, 3, and 4 together. This implication is proved in Section 3.2, see Proposition 3.4. Therefore, it does not matter if Assumption 2 is explicitly required to hold in Theorem 3.3 or not.

3.2 A First Discussion of the Convergence Assumptions

In the preceding section we presented four assumptions and used them to prove local quadratic convergence of our general Newton-type algorithm. This section deals with some relations between Assumptions 1–4 among each other and to existing regularity conditions from the literature. First, it is shown that Assumption 2 is implied by Assumptions 1, 3, and 4 together. Afterwards, we prove that the whole set of our convergence assumptions is implied by conditions which were used in [55] to prove local fast convergence of the constrained Levenberg-Marquardt method. These conditions allow nonisolated solutions but imply differentiability of F, at least at all points which are sufficiently close to z^* and belong to the interior of Ω. Finally, conditions are recalled which were used to analyze local convergence properties of nonsmooth Newton methods. They allow nondifferentiability of F but imply the local uniqueness of solutions. It is shown that these conditions are sufficient for the whole set of Assumptions 1–4 to hold. The quintessence of this section is that our convergence assumptions imply neither the local uniqueness of solutions nor differentiability of F. A deeper discussion of Assumptions 1–4 for the case that F is a PC^1-function can be found in Chapter 4.

The first result we are going to show is the fact that Assumptions 1, 3, and 4 together are sufficient for Assumption 2 to hold. In order to prove this implication, we use similar arguments as in the first part of the proof of Theorem 3.3.

Proposition 3.4. *Let Assumptions 1, 3, and 4 be satisfied. Then Assumption 2 holds.*

Proof. We define $\epsilon := \min\{\epsilon(\Gamma), \delta_3\}$, where $\epsilon(\Gamma)$ is given by (3.1) with $\gamma := \Gamma$. Moreover, we set

$$\delta_2 := \frac{\epsilon}{1 + 2\Gamma L}.$$

Let us take any $s \in \mathcal{B}_{\delta_2}(z^*) \cap \Omega$. At first, we show by induction that there is a sequence $\{s^k\}$ with $s^0 := s$ such that

$$s^k \in \mathcal{B}_\epsilon(z^*) \cap \Omega \tag{3.15}$$

and

$$s^{k+1} \in \mathcal{F}(s^k, \Gamma) \tag{3.16}$$

hold for all $k \in \mathbb{N}$. Since $s^0 = s$ belongs to $\mathcal{B}_{\delta_2}(z^*) \cap \Omega$ and δ_2 is smaller than ϵ, (3.15) is satisfied for $k = 0$. With $\epsilon \leq \delta_3$, Assumption 3 yields the existence of some $s^1 \in \mathcal{F}(s^0, \Gamma)$ so that (3.16) is valid for $k = 0$. Now let us assume that $s^0, \ldots, s^l, s^{l+1}$ are given such that (3.15) and (3.16) hold for all $k = 0, \ldots, l$. We show that s^{l+1} belongs to $\mathcal{B}_\epsilon(z^*) \cap \Omega$. Using (3.16) for $k = l$ and the definition of $\mathcal{F}(s^l, \Gamma)$, $s^{l+1} \in \Omega$ follows. It remains to prove that s^{l+1} belongs to $\mathcal{B}_\epsilon(z^*)$. The triangle inequality yields

$$\|s^{l+1} - z^*\| \leq \|s^0 - z^*\| + \sum_{i=0}^{l} \|s^{i+1} - s^i\|. \tag{3.17}$$

By (3.16) for $k = 0, \dots, l$ and the definition of the sets $\mathcal{F}(s^k, \Gamma)$,

$$\|s^{i+1} - s^i\| \le \Gamma \|F(s^i)\| \tag{3.18}$$

follows for all $i = 0, \dots, l$. Taking into account (3.15) and (3.16) for $k = 0, \dots, l$ and the definition of ϵ, the repeated application of Lemma 3.2 yields

$$\|F(s^i)\| \le \left(\frac{1}{2}\right)^i \|F(s^0)\| \tag{3.19}$$

for all $i = 0, \dots, l$. By the definition of ϵ, s^0 belongs to $\mathcal{B}_{\delta_1}(z^*) \cap \Omega$ so that Assumption 1 implies

$$\|F(s^0)\| \le L \operatorname{dist}[s^0, Z]. \tag{3.20}$$

Combining (3.17)–(3.20), we have

$$\|s^{l+1} - z^*\| \le \underbrace{\|s^0 - z^*\|}_{\le \delta_2} + \Gamma L \underbrace{\operatorname{dist}[s^0, Z]}_{\le \delta_2} \underbrace{\sum_{i=0}^{l} \left(\frac{1}{2}\right)^i}_{\le 2} \le (1 + 2\Gamma L)\delta_2 = \epsilon.$$

Hence, (3.15) is shown for $k = l + 1$. Since $\epsilon \le \delta_3$ is valid, the existence of some $s^{l+2} \in \mathcal{F}(s^{l+1}, \Gamma)$ follows from Assumption 3. Thus, it is proved that there is a sequence $\{s^k\}$ with starting point $s^0 := s$ such that (3.15) and (3.16) are satisfied for all $k \in \mathbb{N}$.

Taking into account (3.15) and (3.16), Lemma 3.2 yields

$$\|F(s^{k+1})\| \le \frac{1}{2} \|F(s^k)\| \tag{3.21}$$

for all $k \in \mathbb{N}$ and therefore

$$\lim_{k \to \infty} \|F(s^k)\| = 0. \tag{3.22}$$

Let us take any $j, k \in \mathbb{N}$ with $k > j$. The triangle inequality, (3.16), and the definition of the sets $\mathcal{F}(s^k, \Gamma)$ imply

$$\|s^k - s^j\| \le \sum_{i=j}^{k-1} \|s^{i+1} - s^i\| \le \Gamma \sum_{i=j}^{k-1} \|F(s^i)\|.$$

Using this and (3.21),

$$\|s^k - s^j\| \le \Gamma \|F(s^j)\| \sum_{i=j}^{k-1} \left(\frac{1}{2}\right)^{i-j} \le 2\Gamma \|F(s^j)\| \tag{3.23}$$

follows. The latter, together with (3.22), implies $\|s^k - s^j\| \to 0$ for $j, k \to \infty$. Consequently, $\{s^k\}$ is a Cauchy sequence and therefore converges to some $\bar{s}$. Due to (3.22) and the continuity of F, $F(\bar{s}) = 0$ follows. Moreover, $\bar{s}$ belongs to Ω since Ω is closed. Thus, we have $\bar{s} \in Z$. Considering (3.23) for $j = 0$ and using $s^0 = s$, we obtain

$$\|s^k - s\| \leq 2\Gamma\|F(s)\|$$

for all $k \in \mathbb{N}$. For $k \to \infty$,

$$\operatorname{dist}[s, Z] \leq \|\bar{s} - s\| \leq 2\Gamma\|F(s)\|$$

follows. Hence, Assumption 2 holds with $\omega := 2\Gamma$. □

The rest of this section is devoted to the relation of Assumptions 1–4 to existing regularity conditions from the literature. The following condition was used, besides Assumptions 1 and 2, in [55] to prove local quadratic convergence of the constrained Levenberg-Marquardt method.

Condition 1. There are $K_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\|F(s) + G(s)(z - s) - F(z)\| \leq K_1\|z - s\|^2$$

holds for all pairs (z, s) with $z \in \mathcal{B}_{\varepsilon_1}(z^*) \cap \Omega$ and $s \in (\mathcal{B}_{\varepsilon_1}(z^*) \cap \Omega) \setminus Z$.

Condition 1 is particularly satisfied if F is defined and differentiable in an open neighborhood U of z^*, has a locally Lipschitz continuous Jacobian there, and $G(s) = F'(s)$ holds for all $s \in U \cap \Omega$. Conversely, Condition 1 implies the differentiability of F at any interior point of $(\mathcal{B}_{\varepsilon_1}(z^*) \cap \Omega) \setminus Z$. The proof of the latter assertion can be found in [84, Lemma 5.3.1]. The next theorem is [18, Proposition 2] and shows that Assumptions 3 and 4 together are implied by Condition 1 if Assumption 2 is additionally satisfied.

Theorem 3.5. *Let Condition 1 be valid. Then the following assertions are true.*

(a) Assumption 3 holds if Assumption 2 is satisfied.

(b) Assumption 4 holds.

Proof. (a) Let us define $\delta_3 := \min\{\delta_2, \frac{1}{2}\varepsilon_1\}$ and let us take any $s \in \mathcal{B}_{\delta_3}(z^*) \cap \Omega$. If s is an element of Z, then $\mathcal{F}(s, \Gamma) = \{s\}$ is valid for arbitrary $\Gamma > 0$, in particular $\mathcal{F}(s, \Gamma)$ is nonempty in that case. So let us assume $s \notin Z$ and let $\bar{s} \in Z$ be a point such that $\|s - \bar{s}\| = \operatorname{dist}[s, Z]$ holds. Note that a point with this property exists since Z is closed and was supposed to be nonempty. By Assumption 2, we have

$$\|\bar{s} - s\| = \operatorname{dist}[s, Z] \leq \omega\|F(s)\|. \tag{3.24}$$

With

$$\|\bar{s} - z^*\| \leq \|\bar{s} - s\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\delta_3 \leq \varepsilon_1,$$

Condition 1 for $z := \bar{s}$ and (3.24) imply

$$\begin{aligned}\|F(s) + G(s)(\bar{s} - s)\| &= \|F(s) + G(s)(\bar{s} - s) - F(\bar{s})\| \\ &\leq K_1\|\bar{s} - s\|^2 \\ &\leq K_1\omega^2\|F(s)\|^2.\end{aligned}$$

Consequently, $\bar{s} \in \mathcal{F}(s, \Gamma)$ holds for $\Gamma := \max\{\omega, K_1\omega^2\}$. Thus, Assumption 3 is valid.

(b) We set $\delta_4 := \frac{1}{2}\varepsilon_1$. Let $s \in (\mathcal{B}_{\delta_4}(z^*) \cap \Omega) \setminus Z$ and $\delta \in [0, \delta_4]$ be arbitrarily chosen and let $z \in \Omega$ be any point such that

$$\|F(s) + G(s)(z - s)\| \leq \delta^2 \quad \text{and} \quad \|z - s\| \leq \delta \tag{3.25}$$

hold. Taking into account

$$\|z - z^*\| \leq \|z - s\| + \|s - z^*\| \leq \delta + \delta_4 \leq 2\delta_4 = \varepsilon_1,$$

Condition 1, together with the second inequality in (3.25), yields

$$\begin{aligned}\|F(z)\| - \|F(s) + G(s)(z - s)\| &\leq \|F(s) + G(s)(z - s) - F(z)\| \\ &\leq K_1\|z - s\|^2 \\ &\leq K_1\delta^2.\end{aligned}$$

Thus, using the first inequality in (3.25), we obtain

$$\|F(z)\| \leq \|F(s) + G(s)(z - s)\| + K_1\delta^2 \leq (1 + K_1)\delta^2.$$

Hence, Assumption 4 is satisfied with $\kappa := 1 + K_1$.

□

As already mentioned above, Assumptions 1 and 2 were used besides Condition 1 in [55]. Therefore, Assumptions 1–4 together, which guarantee local quadratic convergence of our general Newton-type algorithm, are implied by the whole set of conditions which were used in [55] for proving local quadratic convergence of the constrained Levenberg-Marquardt method.

Since Condition 1 particularly holds if F is sufficiently smooth near z^* and G is suitably chosen, Theorem 3.5 yields items (b) and (c) of the following corollary. Item (a) is true because the smoothness assumptions on F imply the local Lipschitz continuity near z^* and therefore Assumption 1.

Corollary 3.6. *Let F be defined and differentiable in an open neighborhood U of z^* and suppose that F has a locally Lipschitz continuous derivative there. Then the following assertions are true if $G(s) = F'(s)$ is satisfied for all $s \in U \cap \Omega$.*

(a) Assumption 1 holds.

(b) *Assumption 3 holds if Assumption 2 is satisfied.*

(c) *Assumption 4 holds.*

A consequence of Proposition 3.4 and Corollary 3.6 is the following corollary. It says that Assumptions 2 and 3 are equivalent if F is sufficiently smooth and G coincides with the derivative near z^*.

Corollary 3.7. *Let F be defined and differentiable in an open neighborhood U of z^* and suppose that F has a locally Lipschitz continuous derivative there. Moreover, assume that $G(s) = F'(s)$ is valid for all $s \in U \cap \Omega$. Then Assumption 2 is satisfied if and only if Assumption 3 holds.*

Proof. Due to items (a) and (c) of Corollary 3.6, Assumptions 1 and 4 are satisfied. The equivalence of Assumptions 2 and 3 follows from Proposition 3.4 and item (b) of Corollary 3.6. □

In the next corollary it is supposed that $\Omega = \mathbb{R}^n$ holds, i.e., (1.1) is assumed to be an unconstrained system of equations. In that case, under the assumptions of Corollary 3.6, Assumption 2 particularly holds if the matrix $F'(z^*)$ has full (row or column) rank. The latter fact follows from Theorem 2.3. This leads to the following result.

Corollary 3.8. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable with a locally Lipschitz continuous derivative and assume that $\Omega = \mathbb{R}^n$ holds. Moreover, suppose that $F'(z^*)$ has full (row or column) rank and let $G(s) = F'(s)$ be valid for all $s \in \mathbb{R}^n$. Then Assumptions 1–4 are satisfied.*

The latter corollary shows that our convergence assumptions are particularly satisfied under the conditions which guarantee local quadratic convergence of the classical Newton method, i.e., the validity of $n = m$, sufficient smoothness of F, and the nonsingularity of $F'(z^*)$.

Throughout the rest of this section, we make the blanket assumption that F is defined on the whole space $\mathbb{R}^n$ and locally Lipschitz continuous there. Moreover, it is supposed that (1.1) is an unconstrained system, i.e., $\Omega = \mathbb{R}^n$ holds. The next aim is to analyze relations between our convergence assumptions and Condition 2 below. Condition 2 was used, together with further regularity conditions implying the local uniqueness of solutions, in [32] for the analysis of a nonsmooth Newton method for the solution of KKT systems arising from optimization problems, and in [22] to prove local fast convergence of a nonsmooth and inexact Levenberg-Marquardt method for the solution of nonlinear complementarity problems.

Condition 2. There are $K_2 > 0$ and $\varepsilon_2 > 0$ such that

$$\sup\{\|F(s) + V(z^* - s)\| \mid V \in \partial F(s)\} \leq K_2 \|z^* - s\|^2$$

holds for all $s \in \mathcal{B}_{\varepsilon_2}(z^*)$.

Condition 2 particularly holds if F is strongly semismooth at z^*. This follows from Proposition 2.12. We refer to Section 2.3 for the definition of strong semismoothness. The definition of Clarke's generalized Jacobian ∂F can be found in Section 2.2.

Now let us discuss Assumptions 1–4. Since we made the blanket assumption that F is locally Lipschitz continuous, Assumption 1 is satisfied. The following lemma shows some relation between Assumption 3 and Condition 2. We prove Lemma 3.9 by slightly modifying the proof of [18, Proposition 4].

Lemma 3.9. *Let Assumption 2 and Condition 2 be satisfied and assume that z^* is a locally unique solution of* (1.1)*. Then Assumption 3 holds if $G(s) \in \partial F(s)$ is valid for all $s \in \mathbb{R}^n$.*

Proof. Let $\delta_3 \in (0, \min\{\delta_2, \varepsilon_2\}]$ be small enough such that $\operatorname{dist}[s, Z] = \|s - z^*\|$ holds for all $s \in \mathcal{B}_{\delta_3}(z^*)$. Note that there is some δ_3 with this property because z^* is assumed to be a locally unique solution of (1.1). Let $s \in \mathcal{B}_{\delta_3}(z^*)$ be arbitrary but fixed. Assumption 2 implies

$$\|z^* - s\| = \operatorname{dist}[s, Z] \leq \omega \|F(s)\|. \tag{3.26}$$

Using $G(s) \in \partial F(s)$, Condition 2, and (3.26), we obtain

$$\begin{aligned} \|F(s) + G(s)(z^* - s)\| &\leq \sup\{\|F(s) + V(z^* - s)\| \mid V \in \partial F(s)\} \\ &\leq K_2 \|z^* - s\|^2 \\ &\leq K_2 \omega^2 \|F(s)\|^2. \end{aligned}$$

Therefore, defining $\Gamma := \max\{\omega, K_2\omega^2\}$, we have $z^* \in \mathcal{F}(s, \Gamma)$. Hence, Assumption 3 is satisfied. □

For the case $m = n$ it is proved in [73, Proposition 3] that both Assumption 2 and the local uniqueness of z^* as a solution of (1.1) hold if F is semismooth and strongly BD-regular at z^*. We refer to Section 2.2 for the definition of strong BD-regularity. By similar arguments, the following lemma shows the validity of the assumptions of Lemma 3.9 for the case $m \geq n$, assumed that Condition 2 is valid and that the rank of all matrices belonging to $\partial_B F(z^*)$ is equal to n. The proof is essentially based on the proof of [18, Corollary 2].

Lemma 3.10. *Let Condition 2 be satisfied and assume that all matrices $V \in \partial_B F(z^*)$ have full column rank. Then z^* is a locally unique solution of* (1.1) *and Assumption 2 is satisfied.*

Proof. We define

$$\delta_2 := \min\left\{\varepsilon_2, \varrho, \frac{1}{2cK_2}\right\}$$

with ϱ and c from Proposition 2.9. Let us take any $s \in \mathcal{B}_{\delta_2}(z^*)$ and let $V \in \partial_B F(s)$ be arbitrary but fixed. By the definition of δ_2, Proposition 2.9 yields $\operatorname{rank} V = n$ and $\|V^+\| \leq c$. Furthermore, by Lemma 2.8, we obtain $V^+V = I_n$. Using this and Condition 2,

$$\begin{aligned}
\|z^* - s\| &= \|V^+V(z^* - s)\| \\
&= \|V^+(F(s) + V(z^* - s)) - V^+F(s)\| \\
&\leq \|V^+(F(s) + V(z^* - s))\| + \|V^+F(s)\| \\
&\leq cK_2\|z^* - s\|^2 + c\|F(s)\|
\end{aligned}$$

follows. Hence, we have

$$c\|F(s)\| \geq \|z^* - s\|(1 - cK_2\|z^* - s\|). \tag{3.27}$$

With

$$1 - cK_2\|z^* - s\| \geq 1 - cK_2\delta_2 \geq \frac{1}{2}$$

due to the definition of δ_2, (3.27) implies

$$\|z^* - s\| \leq 2c\|F(s)\|. \tag{3.28}$$

Since $s \in \mathcal{B}_{\delta_2}(z^*)$ was arbitrarily chosen, (3.28) shows that z^* is the unique solution of (1.1) within $\mathcal{B}_{\delta_2}(z^*)$. Moreover, with $\omega := 2c$, the validity of Assumption 2 follows from (3.28) due to $\operatorname{dist}[s, Z] \leq \|z^* - s\|$. □

Remark 3.1. It is not difficult to show that the conclusions of Lemma 3.10 stay true if the following weaker requirement is used instead of Condition 2:

$$\lim_{s \to z^*} \frac{1}{\|z^* - s\|} \sup\{\|F(s) + V(z^* - s)\| \mid V \in \partial F(s)\} = 0.$$

Due to Proposition 2.12, the latter condition is particularly satisfied if F is semismooth at z^*. Thus, [73, Proposition 3] is extended to the case $m \geq n$.

The following lemma is [18, Proposition 5] for the case $\Omega = \mathbb{R}^n$. It says that Assumption 4 holds under the assumptions of Lemma 3.10 if, for any $s \in \mathbb{R}^n$, $G(s)$ is chosen as an element of the B-subdifferential of F at s.

Lemma 3.11. *Let Condition 2 be satisfied and assume that all matrices $V \in \partial_B F(z^*)$ have full column rank. Moreover, suppose that $G(s) \in \partial_B F(s)$ holds for all $s \in \mathbb{R}^n$. Then Assumption 4 is satisfied.*

Proof. Let us set

$$\delta_4 := \min\left\{\varepsilon_2, \varrho, \frac{1}{8cK_2}\right\} \tag{3.29}$$

with ϱ and c from Proposition 2.9. Due to the local Lipschitz continuity of F, there is $L_0 > 0$ such that

$$\|F(z) - F(s)\| \le L_0\|z - s\| \tag{3.30}$$

is valid for all $s, z \in \mathcal{B}_{2\delta_4}(z^*)$.

Now let $s \in \mathcal{B}_{\delta_4}(z^*) \setminus Z$ and $\delta \in [0, \delta_4]$ be arbitrarily chosen. Moreover, let $z \in \mathbb{R}^n$ be any point satisfying

$$\|F(s) + G(s)(z - s)\| \le \delta^2 \quad \text{and} \quad \|z - s\| \le \delta. \tag{3.31}$$

The triangle inequality yields

$$\|z - z^*\| \le \|z - s\| + \|s - z^*\| \le 2\delta_4. \tag{3.32}$$

Consequently, (3.30) implies

$$\|F(z)\| = \|F(z) - F(z^*)\| \le L_0\|z - z^*\|. \tag{3.33}$$

By the definition of δ_4 and $G(s) \in \partial_B F(s)$, we have $\operatorname{rank} G(s) = n$, $\|G(s)^+\| \le c$, and $G(s)^+G(s) = I_n$ by Proposition 2.9 and Lemma 2.8. Hence,

$$\begin{aligned} z - z^* &= z - s + s - z^* \\ &= G(s)^+(G(s)(z - s) - G(s)(z^* - s)) \\ &= G(s)^+(F(s) + G(s)(z - s) - F(s) - G(s)(z^* - s)). \end{aligned}$$

holds. Therefore, using Proposition 2.9, the first inequality in (3.31), Condition 2, the triangle inequality, and the second inequality in (3.31),

$$\begin{aligned} \|z - z^*\| &\le \|G(s)^+\|(\|F(s) + G(s)(z - s)\| + \|F(s) + G(s)(z^* - s)\|) \\ &\le c(\delta^2 + K_2\|z^* - s\|^2) \\ &\le c(\delta^2 + K_2\|z - z^*\|^2 + 2K_2\|z - z^*\|\|z - s\| + K_2\|z - s\|^2) \\ &\le c(\delta^2 + K_2\|z - z^*\|^2 + 2K_2\delta\|z - z^*\| + K_2\delta^2) \end{aligned}$$

follows. This yields

$$\|z - z^*\|(1 - cK_2\|z - z^*\| - 2cK_2\delta) \le c(1 + K_2)\delta^2. \tag{3.34}$$

Using (3.32), $\delta \le \delta_4$, and (3.29), we obtain

$$1 - cK_2\|z - z^*\| - 2cK_2\delta \ge 1 - 4cK_2\delta_4 \ge \frac{1}{2}. \tag{3.35}$$

Thus, combining (3.33)–(3.35),

$$\|F(z)\| \le 2cL_0(1 + K_2)\delta^2$$

follows. Hence, Assumption 4 is satisfied with $\kappa := 2cL_0(1 + K_2)$. □

Let us summarize the results of Lemmas 3.9–3.11.

Theorem 3.12. *Let Condition 2 be satisfied and assume that all matrices belonging to* $\partial_B F(z^*)$ *have full column rank. Then* z^* *is a locally unique solution of* (1.1)*. Moreover, Assumptions 1–4 hold if* $G(s) \in \partial_B F(s)$ *is valid for all* $s \in \mathbb{R}^n$*.*

As already mentioned, Condition 2 particularly holds if F is strongly semismooth at z^*. Therefore, we obtain the following corollary.

Corollary 3.13. *Let* F *be strongly semismooth at* z^* *and assume that all matrices belonging to* $\partial_B F(z^*)$ *have full column rank. Then* z^* *is a locally unique solution of* (1.1)*. Moreover, Assumptions 1–4 are satisfied if* $G(s) \in \partial_B F(s)$ *holds for all* $s \in \mathbb{R}^n$*.*

Remark 3.2. It is easy to see that the assertions of Lemmas 3.10 and 3.11 as well as of Theorem 3.12 and Corollary 3.13 stay true if the B-subdifferential is replaced everywhere by Clarke's generalized Jacobian. The reason is that the assertion of Proposition 2.9 is kept, see Remark 2.1.

3.3 Nonsmooth Inexact Newton Method

In this section we recall a nonsmooth and inexact Newton method and show that it can be regarded as a special realization of Algorithm 3.1, at least locally and under suitable regularity conditions. A well-known result on local quadratic convergence of the nonsmooth inexact Newton method is recovered by applying Theorem 3.3.

Throughout this section, it is assumed that $m = n$ and $\Omega = \mathbb{R}^n$ hold and that $F : \mathbb{R}^n \to \mathbb{R}^n$ is at least locally Lipschitz continuous. Note that (1.1) reduces to the unconstrained system of equations $F(z) = 0$ in this setting. As in the previous sections, z^* denotes an arbitrary but fixed solution of this system.

Let us start with some historical overview on the Newton method and variants of it as well as important local convergence results. First, the subproblems of the classical Newton method are recalled. To this end, let us assume for the moment that F is differentiable. Denoting by $s \in \mathbb{R}^n$ the current iterate, a solution of the following linear system of equations has to be determined:

$$F(s) + F'(s)(z - s) = 0. \tag{3.36}$$

It is well known that any sequence generated by the Newton method converges quadratically to z^* if F' is locally Lipschitz continuous near z^*, $F'(z^*)$ is nonsingular, and the starting point belongs to a sufficiently small neighborhood of z^*, see for instance [83, Satz 5.1.2].

Local convergence properties of an inexact version of the classical Newton method are analyzed in [10]. Instead of (3.36), the following linear system of equations is solved for a given point s:

$$F(s) + F'(s)(z - s) = \pi(s). \tag{3.37}$$

The function $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is used to describe the inexactness which may arise from computational inaccuracies or the truncation of iterative solution algorithms. In other words, the inexact Newton method allows to determine an approximate solution of (3.36) only. It is proved in [10] that the local quadratic convergence to z^* is kept if there is some constant $c_\pi > 0$ such that

$$\|\pi(s)\| \leq c_\pi \|F(s)\|^2 \tag{3.38}$$

holds for all s in a certain neighborhood of z^*.

From now on, differentiability of F is not required any longer. There are several approaches to extend the Newton method for nonsmooth systems of equations. Let us describe the probably most important among them. To this end, let s again denote the current iterate. Then an element $G(s)$ of Clarke's generalized Jacobian $\partial F(s)$ has to be computed, and afterwards a solution of the linear system

$$F(s) + G(s)(z - s) = 0 \tag{3.39}$$

must be determined. To the best of the author's knowledge, Kojima and Shindo [57] were the first who analyzed a realization of this method. They considered the case that F is a PC^1-function and proved local quadratic convergence under suitable regularity conditions. The result from [57] was recovered and extended by Kummer [58] where the nonsmooth Newton method described by (3.39) was furthermore considered for the case that F is locally Lipschitz continuous only. In that setting, local quadratic convergence to z^* was proved under the assumption that $\partial F(z^*)$ is a singleton and the unique element is nonsingular. The nonsmooth Newton method was also considered in [59] where, besides conditions being sufficient, necessary conditions for local convergence were presented.

A famous result on local convergence of the nonsmooth Newton method described by (3.39) is proved in [77]. If F is strongly semismooth at z^* and all elements of $\partial F(z^*)$ are nonsingular, then any sequence generated by the nonsmooth Newton method converges quadratically to z^* if the starting point belongs to a sufficiently small neighborhood of z^*, see [77, Theorem 3.2]. The local quadratic convergence is kept if $G(s) \in \partial_B F(s)$ holds for all $s \in \mathbb{R}^n$ and the nonsingularity of all elements of $\partial F(z^*)$ is replaced by the strong BD-regularity of F at z^*, i.e., the nonsingularity of all matrices belonging to $\partial_B F(z^*)$, see [74].

There are further extensions of the classical Newton method to nonsmooth equations. In [71, 72] it is assumed that F is B-differentiable, and, using the B-derivative, a method is described where a nonlinear system of equations must be solved in each iteration. The notion of B-differentiability goes back to Robinson [80]. Assuming that all elements of $\partial F(z^*)$ are nonsingular and $G(s) \in \partial F(s)$ is appropriately chosen for any s, the method from [71, 72] can locally be regarded as a special realization of the nonsmooth Newton method described by (3.39), see [77, Proposition 3.4]. In [59, 81] local convergence properties of Newton-type methods for nonsmooth equations are analyzed where, instead of a linearization of the function, more general point-based approximations are used in each step.

An inexact version of the nonsmooth Newton method characterized by the subproblems (3.39) is considered in [67]. In order to describe its subproblems, assume that, for any $s \in \mathbb{R}^n$, $G(s)$ is an element of $\partial_B F(s)$ and that $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is a given function, again used to describe the inexactness. Similar to the smooth case (3.37), the following linear system is solved for a given point s:

$$F(s) + G(s)(z - s) = \pi(s). \tag{3.40}$$

It is proved in [67, Theorem 3] that the resulting inexact nonsmooth Newton method is locally superlinearly convergent to z^* if F is semismooth and strongly BD-regular at z^* and π satisfies

$$\lim_{s \to z^*} \frac{\|\pi(s)\|}{\|F(s)\|} = 0.$$

In [20, Theorem 3.2] it is shown that the method converges even locally quadratically if in addition F is strongly semismooth at z^* and there is $c_\pi > 0$ such that (3.38) is satisfied for all s in a certain neighborhood of z^*.

There are many papers where the nonsmooth Newton method and inexact versions of it are applied to nondifferentiable systems of equations arising from special problem classes, and the convergence assumptions are discussed in the particular contexts. Examples are reformulations of linear and nonlinear complementarity problems [8, 9, 33, 34, 52] and of KKT systems which arise from optimization problems, variational inequalities, or quasi-variational inequalities [20, 24, 32, 75]. We also refer to [26, Chapter 7], [50, Sections 2.4 and 3.2], and [76] for a survey and further references on nonsmooth Newton methods and applications.

Algorithm 3.2 below is the nonsmooth inexact Newton method from [67] and [20] where subproblems of the form (3.40) are solved in each step. We assume that the functions $\pi : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are given and that $G(s) \in \partial_B F(s)$ is satisfied for all $s \in \mathbb{R}^n$.

Algorithm 3.2. (Nonsmooth Inexact Newton Method)

(S.0) Choose $z^0 \in \mathbb{R}^n$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) Determine a solution z^{k+1} of the linear system (3.40) with $s := z^k$.

(S.3) Set $k := k + 1$ and go to (S.1).

Our aim is to show that, locally and under suitable regularity assumptions, Algorithm 3.2 is a special realization of the general Newton-type algorithm from Section 3.1. Then classical local convergence results can be recovered by Theorem 3.3.

Proposition 3.14. *Let Condition 2 be satisfied and assume that F is strongly BD-regular at z^*. Moreover, suppose that there are $c_\pi > 0$ and $\varepsilon_\pi > 0$ such that (3.38) holds for all $s \in \mathcal{B}_{\varepsilon_\pi}(z^*)$. Then there is $\bar{\rho} > 0$ such that Algorithm 3.2 is well defined for any $z^0 \in \mathcal{B}_{\bar{\rho}}(z^*)$. Let $\{z^k\}$ be an infinite sequence generated by Algorithm 3.2 with starting point $z^0 \in \mathcal{B}_{\bar{\rho}}(z^*)$. Then this sequence can, together with some sequence $\{\Gamma_k\}$, also be obtained by Algorithm 3.1, for suitable $\Gamma_0 > 0$, arbitrary $\alpha \geq 1$, and arbitrary $\beta > 1$.*

Proof. Since all elements of $\partial_B F(z^*)$ are nonsingular and $G(s) \in \partial_B F(s)$ is valid for all $s \in \mathbb{R}^n$, we know from Proposition 2.9, together with item (b) of Lemma 2.8, that there are $\varrho > 0$ and $c > 0$ such that, for any $s \in \mathcal{B}_\varrho(z^*)$, the matrix $G(s)$ is nonsingular and

$$\|G(s)^{-1}\| \leq c \tag{3.41}$$

holds. Due to the local Lipschitz continuity of F, there is $L_0 > 0$ such that

$$\|F(z) - F(s)\| \leq L_0 \|z - s\| \tag{3.42}$$

is satisfied for all $s, z \in \mathcal{B}_\varrho(z^*)$. We set

$$\gamma := \max\{c(1 + c_\pi L_0 \varrho), c_\pi\}. \tag{3.43}$$

By Theorem 3.12, Assumptions 1–4 are valid. Therefore, Lemma 3.2 can be applied and yields the existence of some $\epsilon(\gamma) > 0$ such that, for any $s \in \mathcal{B}_{\epsilon(\gamma)}(z^*)$,

$$\|F(z)\| \leq \frac{1}{2}\|F(s)\| \tag{3.44}$$

holds for all $z \in \mathcal{F}(s, \gamma)$. Let us define $\epsilon := \min\{\varepsilon_\pi, \varrho, \epsilon(\gamma)\}$ and

$$\bar{\rho} := \frac{\epsilon}{1 + 2\gamma L_0}.$$

Now let us take any $z^0 \in \mathcal{B}_{\bar{\rho}}(z^*)$. We are going to show by induction that, for every $k \in \mathbb{N}$,

$$z^k \in \mathcal{B}_\epsilon(z^*) \tag{3.45}$$

holds and the linear system (3.40) with s replaced by z^k has a unique solution z^{k+1} so that step (S.2) of Algorithm 3.2 is well defined. The validity of (3.45) for $k = 0$ is obvious since $\bar{\rho}$ is smaller than ϵ. Therefore, taking into account $\epsilon \leq \varrho$, $G(z^0)$ is nonsingular so that the linear system (3.40) with $s := z^0$ has a unique solution z^1. Thus, step (S.2) of Algorithm 3.2 is well defined for $k = 0$.

Now let us assume that, for every $k = 0, \ldots, l$, problem (3.40) with $s := z^k$ has a unique solution and that z^{k+1} denotes this solution. In particular, the points $z^1, \ldots, z^{l+1}$ are generated by Algorithm 3.2. Moreover, assume that (3.45) is satisfied for all $k = 0, \ldots, l$. In order to prove (3.45) for $k = l + 1$, we need some preliminaries. For every $i = 0, \ldots, l$, the vector z^{i+1} solves (3.40) with s replaced by z^i. This, together with (3.38) and the definition of γ, yields

$$\|F(z^i) + G(z^i)(z^{i+1} - z^i)\| = \|\pi(z^i)\| \leq c_\pi \|F(z^i)\|^2 \leq \gamma \|F(z^i)\|^2 \tag{3.46}$$

for all $i = 0, \ldots, l$. Since the vectors z^i belong to $\mathcal{B}_\epsilon(z^*)$ and $\epsilon \leq \varrho$ holds, the matrices $G(z^i)$ are nonsingular for all $i = 0, \ldots, l$. Therefore, we have

$$z^{i+1} - z^i = G(z^i)^{-1}(\pi(z^i) - F(z^i))$$

for all $i = 0, \ldots, l$. This, (3.38), (3.41), (3.42), and the definition of ϵ imply

$$\begin{aligned}
\|z^{i+1} - z^i\| &\leq \|G(z^i)^{-1}\|(\|\pi(z^i)\| + \|F(z^i)\|) \\
&\leq c(c_\pi \|F(z^i)\| + 1)\|F(z^i)\| \\
&\leq c(c_\pi L_0 \|z^i - z^*\| + 1)\|F(z^i)\| \\
&\leq c(c_\pi L_0 \varrho + 1)\|F(z^i)\|
\end{aligned}$$

for all $i = 0, \ldots, l$. By the definition of γ,

$$\|z^{i+1} - z^i\| \leq \gamma \|F(z^i)\| \tag{3.47}$$

follows for all $i = 0, \ldots, l$. Combining (3.46) and (3.47),

$$z^{i+1} \in \mathcal{F}(z^i, \gamma) \tag{3.48}$$

holds for all $i = 0, \ldots, l$.

Now let us verify (3.45) for $k = l + 1$. To this end, we use the triangle inequality and (3.47) and obtain

$$\|z^{l+1} - z^*\| \leq \|z^0 - z^*\| + \sum_{i=0}^{l} \|z^{i+1} - z^i\| \leq \bar{\rho} + \gamma \sum_{i=0}^{l} \|F(z^i)\|. \tag{3.49}$$

Taking into account (3.45) for $k = 0, \ldots, l$ and (3.48), the repeated application of (3.44) implies

$$\|F(z^i)\| \leq \left(\frac{1}{2}\right)^i \|F(z^0)\| \tag{3.50}$$

for all $i = 0, \ldots, l$. Using (3.49), (3.50), (3.42), and the definition of $\bar{\rho}$,

$$\begin{aligned}
\|z^{l+1} - z^*\| &\leq \bar{\rho} + \gamma \|F(z^0)\| \sum_{i=0}^{l} \left(\frac{1}{2}\right)^i \\
&\leq \bar{\rho} + 2\gamma L_0 \|z^0 - z^*\| \\
&\leq (1 + 2\gamma L_0)\bar{\rho} \\
&= \epsilon
\end{aligned}$$

follows. Thus, (3.45) is shown for $k = l+1$. In particular, z^{l+1} belongs to $\mathcal{B}_\varrho(z^*)$ so that $G(z^{l+1})$ is nonsingular. Therefore, (3.40) with $s := z^{l+1}$ has a unique solution z^{l+2} so that step (S.2) of Algorithm 3.2 is well defined for $k = l+1$. This completes the induction. Particularly, it is shown that Algorithm 3.2 is well defined for any starting point $z^0 \in \mathcal{B}_{\bar{\rho}}(z^*)$. Moreover, we know that all iterates belong to $\mathcal{B}_\epsilon(z^*)$ and that (3.48) is satisfied for all $i \in \mathbb{N}$.

It remains to prove that any sequence generated by Algorithm 3.2 with some starting point within $\mathcal{B}_{\bar{\rho}}(z^*)$ can, together with some sequence $\{\Gamma_k\}$, also be generated by Algorithm 3.1. To this end, let $\{z^k\}$ be a sequence generated by Algorithm 3.2 with starting point $z^0 \in \mathcal{B}_{\bar{\rho}}(z^*)$. We set $\Gamma_0 := \gamma$ where γ is given according to (3.43). Moreover, let $\alpha \geq 1$ and $\beta > 1$ be arbitrary but fixed and let, for every $k \in \mathbb{N}$, Γ_{k+1} be defined by the updating rule in step (S.2) of Algorithm 3.1, i.e., $\Gamma_{k+1} := \min\{\beta^\ell \Gamma_k \mid \ell \in \mathbb{N},\ \mathcal{F}(z^k, \beta^\ell \Gamma_k) \neq \emptyset\}$. Let us show by induction that

$$\Gamma_k = \gamma \tag{3.51}$$

holds for all $k \in \mathbb{N}$. The definition of Γ_0 obviously yields (3.51) for $k = 0$. Now suppose that (3.51) is satisfied for all $k = 0, \ldots, l$. By (3.48) for $i = l$, together with (3.51) for $k = l$, the set $\mathcal{F}(z^l, \Gamma_l)$ is nonempty. Thus, the definition of Γ_{l+1} implies

$$\Gamma_{l+1} = \Gamma_l = \gamma$$

so that (3.51) is proved for $k = l+1$ and the induction is complete.

Using (3.48), (3.51), and $\alpha \geq 1$,

$$z^{k+1} \in \mathcal{F}(z^k, \Gamma_{k+1}) \subseteq \mathcal{F}(z^k, \alpha\Gamma_{k+1})$$

follows for all $k \in \mathbb{N}$. Hence, it is shown that $\{z^k\}$ can also be obtained by Algorithm 3.1. □

As a consequence of the latter proposition, together with Theorems 3.3 and 3.12, we obtain the following local convergence result for Algorithm 3.2.

Theorem 3.15. *Let Condition 2 be satisfied and assume that F is strongly BD-regular at z^*. Moreover, suppose that there are $c_\pi > 0$ and $\varepsilon_\pi > 0$ such that* (3.38) *holds for all $s \in \mathcal{B}_{\varepsilon_\pi}(z^*)$. Then there is $\rho > 0$ such that any infinite sequence $\{z^k\}$ generated by Algorithm 3.2 with starting point $z^0 \in \mathcal{B}_\rho(z^*)$ converges to z^* with a Q-quadratic rate.*

Since Condition 2 is satisfied if F is strongly semismooth at z^*, see Proposition 2.12, the subsequent corollary follows which recovers [20, Theorem 3.2].

Corollary 3.16. *Let F be strongly semismooth and strongly BD-regular at z^*. Moreover, suppose that there are $c_\pi > 0$ and $\varepsilon_\pi > 0$ such that* (3.38) *holds for all $s \in \mathcal{B}_{\varepsilon_\pi}(z^*)$. Then there is $\rho > 0$ such that any infinite sequence $\{z^k\}$ generated by Algorithm 3.2 with starting point $z^0 \in \mathcal{B}_\rho(z^*)$ converges to z^* with a Q-quadratic rate.*

Remark 3.3. The assertions of Proposition 3.14, Theorem 3.15, and Corollary 3.16 stay true if G satisfies $G(s) \in \partial F(s)$ (instead of $G(s) \in \partial_B F(s)$) for all $s \in \mathbb{R}^n$ and the strong BD-regularity of F at z^* is replaced by the nonsingularity of all elements of $\partial F(z^*)$. In order to justify this, note that Proposition 2.9 and Theorem 3.12 can be transferred to that setting, see Remarks 2.1 and 3.2. Thus, [77, Theorem 3.2] is recovered.

3.4 LP-Newton Method

This section deals with the LP-Newton method. This is an iterative method for the solution of (1.1) and was introduced in [18]. In each step a solution of an optimization problem has to be determined which turns out to be a linear program if Ω is a polyhedral set. That is the reason for the name "LP-Newton method". We will show that, without requiring any additional conditions, the LP-Newton method is well defined and can be regarded as a special realization of our general Newton-type algorithm from Section 3.1. Consequently, by Theorem 3.3, the LP-Newton method converges locally with a Q-quadratic rate to a solution of (1.1) if Assumptions 1–4 are satisfied. In the second part of this section we recall the inexact version of the LP-Newton method from [17] which allows to determine approximate solutions of the LP-Newton subproblems only. It will turn out that this inexact version is a special realization of Algorithm 3.1, too, and therefore enjoys the same local convergence properties.

At first, let us describe the subproblems of the LP-Newton method. To this end, let $s \in \Omega$ denote the current iterate. Then the following optimization problem has to be solved:

$$\gamma \to \min_{z,\gamma} \quad \text{s.t.} \quad \begin{aligned} & z \in \Omega, \\ & \|F(s) + G(s)(z-s)\|_\infty \le \gamma \|F(s)\|_\infty^2, \\ & \|z-s\|_\infty \le \gamma \|F(s)\|_\infty, \\ & \gamma \ge 0. \end{aligned} \tag{3.52}$$

Since the maximum norm is used, (3.52) can be regarded as a linear program if Ω is a polyhedral set. In fact, the inequalities $\|F(s) + G(s)(z-s)\|_\infty \le \gamma\|F(s)\|_\infty^2$ and $\|z-s\|_\infty \le \gamma\|F(s)\|_\infty$ can be equivalently rewritten as $2(m+n)$ linear constraints. If Ω is not polyhedral, the solution of (3.52) is in general difficult from the computational point of view. However, if Ω is described by nonlinear inequalities, (1.1) can be equivalently reformulated by means of slack variables such that the feasible set of the new constrained system is polyhedral, see Section 4.2.

Before we formally describe the LP-Newton method, let us prove some properties of (3.52). Particularly, it is shown in the following proposition that this optimization problem has always a solution. Moreover, relations between the set $\mathcal{F}(s,\gamma)$ and the feasible set of (3.52) are discussed. In order to shorten the notation, let us define, for any $s \in \Omega$ and any $\gamma \geq 0$, the set

$$\mathcal{F}_\infty(s,\gamma) := \{z \in \Omega \mid \|F(s)+G(s)(z-s)\|_\infty \leq \gamma\|F(s)\|_\infty^2,\ \|z-s\|_\infty \leq \gamma\|F(s)\|_\infty\}.$$

Obviously, a point z belongs to $\mathcal{F}_\infty(s,\gamma)$ if and only if (z,γ) is feasible for (3.52). Let us further define

$$M := \max\{m, \sqrt{n}\}.$$

Proposition 3.17. *The following assertions are satisfied for any $s \in \Omega$.*

(a) The optimization problem (3.52) *has a solution.*

(b) The optimal value of (3.52) *is equal to zero if and only if s is a solution of* (1.1)*.*

(c) The inclusions $\mathcal{F}_\infty(s,\gamma) \subseteq \mathcal{F}(s,M\gamma)$ and $\mathcal{F}(s,\gamma) \subseteq \mathcal{F}_\infty(s,M\gamma)$ are satisfied for all $\gamma \geq 0$.

Proof. Let $s \in \Omega$ be arbitrary but fixed.

(a) If s belongs to Z, then $(z,\gamma) := (s,0)$ is feasible for (3.52) and, obviously, solves this optimization problem. Now let us assume $s \notin Z$. The feasible set of (3.52) is nonempty because, for instance, the point $(z,\gamma) := (s,\hat{\gamma})$ with

$$\hat{\gamma} := \|F(s)\|_\infty^{-1}$$

is feasible. Assume that (3.52) is modified by adding the constraint $\gamma \leq \hat{\gamma}$. Then the resulting problem has a nonempty, compact feasible set and a continuous objective. Therefore, due to the theorem of Weierstrass, the modified problem is solvable. Obviously, every solution of the modified problem is also a solution of (3.52). Hence, the assertion is shown.

(b) Assume that s is a solution of (1.1). We already stated in the proof of item (a) that the point $(z,\gamma) := (s,0)$ is a solution of (3.52) in that case. Therefore, the optimal value of (3.52) is equal to zero.

Conversely, let us suppose that $(\bar{z},\bar{\gamma})$ is a solution of (3.52) and that $\bar{\gamma} = 0$ is satisfied. Then

$$\|\bar{z} - s\|_\infty \leq \bar{\gamma}\|F(s)\|_\infty = 0$$

follows from the second inequality in (3.52). Consequently, $\bar{z} = s$ holds. Using this and the first inequality in (3.52), we obtain

$$\|F(s)\|_\infty = \|F(s) + G(s)(\bar{z} - s)\|_\infty \leq \bar{\gamma}\|F(s)\|_\infty^2 = 0.$$

Thus, $s \in Z$ follows.

(c) Let $\gamma \geq 0$ be arbitrarily chosen. Suppose that z is an arbitrary but fixed element of $\mathcal{F}_\infty(s,\gamma)$. Then, using relations between the Euclidean norm and the maximum norm,

$$\begin{aligned}\|F(s)+G(s)(z-s)\| &\leq \sqrt{m}\|F(s)+G(s)(z-s)\|_\infty \\ &\leq \sqrt{m}\gamma\|F(s)\|_\infty^2 \\ &\leq \sqrt{m}\gamma\|F(s)\|^2 \qquad (3.53)\end{aligned}$$

and

$$\|z-s\| \leq \sqrt{n}\|z-s\|_\infty \leq \sqrt{n}\gamma\|F(s)\|_\infty \leq \sqrt{n}\gamma\|F(s)\| \qquad (3.54)$$

hold. By (3.53), (3.54), the definition of M, and $z \in \Omega$, we have $z \in \mathcal{F}(s, M\gamma)$. Hence, the inclusion $\mathcal{F}_\infty(s,\gamma) \subseteq \mathcal{F}(s, M\gamma)$ is proved.

Conversely, assume that z belongs to $\mathcal{F}(s,\gamma)$. Then, using relations between the Euclidean norm and the maximum norm again, we obtain

$$\begin{aligned}\|F(s)+G(s)(z-s)\|_\infty &\leq \|F(s)+G(s)(z-s)\| \\ &\leq \gamma\|F(s)\|^2 \\ &\leq m\gamma\|F(s)\|_\infty^2 \qquad (3.55)\end{aligned}$$

and

$$\|z-s\|_\infty \leq \|z-s\| \leq \gamma\|F(s)\| \leq \sqrt{n}\gamma\|F(s)\|_\infty. \qquad (3.56)$$

Combining (3.55), (3.56), $z \in \Omega$, and the definition of M, we have $z \in \mathcal{F}_\infty(s, M\gamma)$. Therefore, the inclusion $\mathcal{F}(s,\gamma) \subseteq \mathcal{F}_\infty(s, M\gamma)$ is satisfied.

□

Remark 3.4. After the introduction of Assumption 3 in Section 3.1 it was already mentioned that our formulation of this assumption differs from [18, Assumption 3]. Now we are in the position to justify the equivalence of both formulations. In [18, Assumption 3] it is required that there are $\tilde{\Gamma} \geq 1$ and $\tilde{\delta}_3 > 0$ such that

$$\gamma(s) \leq \tilde{\Gamma} \qquad (3.57)$$

holds for all $s \in \mathcal{B}_{\tilde{\delta}_3}(z^*) \cap \Omega$ where, for any s, $\gamma(s)$ denotes the optimal value of the optimization problem (3.52). Note that $\gamma(s)$ is well defined for all $s \in \Omega$ due to item (a) of Proposition 3.17.

Let us assume that Assumption 3 is satisfied with some constants $\Gamma > 0$ and $\delta_3 > 0$. Moreover, let $s \in \mathcal{B}_{\delta_3}(z^*) \cap \Omega$ be arbitrarily chosen. Then the set $\mathcal{F}(s,\Gamma)$ is nonempty. Let z be an arbitrary but fixed element of $\mathcal{F}(s,\Gamma)$. Due to item (c) of Proposition 3.17, z belongs to $\mathcal{F}_\infty(s, M\Gamma)$, i.e., $(z, M\Gamma)$ is feasible for (3.52). Thus, $\gamma(s) \leq M\Gamma$ follows. Consequently, defining $\tilde{\Gamma} := \max\{1, M\Gamma\}$ and $\tilde{\delta}_3 := \delta_3$, (3.57) holds for all $s \in \mathcal{B}_{\tilde{\delta}_3}(z^*) \cap \Omega$.

Conversely, suppose that there are $\tilde{\Gamma} \geq 1$ and $\tilde{\delta}_3 > 0$ such that (3.57) is satisfied for all $s \in \mathcal{B}_{\tilde{\delta}_3}(z^*) \cap \Omega$. Moreover, let $s \in \mathcal{B}_{\tilde{\delta}_3}(z^*) \cap \Omega$ be arbitrary but fixed and let $(\bar{z}, \gamma(s))$ be a solution of (3.52). Then $(\bar{z}, \gamma(s))$ is particularly feasible for (3.52). Due to $\gamma(s) \leq \tilde{\Gamma}$, the point $(\bar{z}, \tilde{\Gamma})$ is feasible for (3.52), too. This, together with item (c) of Proposition 3.17, implies that the set $\mathcal{F}(s, M\tilde{\Gamma})$ is nonempty. Hence, Assumption 3 is valid with $\Gamma := M\tilde{\Gamma}$ and $\delta_3 := \tilde{\delta}_3$.

Next, let us formally describe the LP-Newton method. The following algorithm is [18, Algorithm 1].

Algorithm 3.3. (LP-Newton Method)

(S.0) Choose $z^0 \in \Omega$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) Compute a solution (z^{k+1}, γ_{k+1}) of (3.52) with $s := z^k$.

(S.3) Set $k := k + 1$ and go to (S.1).

From item (a) of Proposition 3.17 we can deduce the following result on the well-definedness of Algorithm 3.3.

Proposition 3.18. *Algorithm 3.3 is well defined for any $z^0 \in \Omega$.*

Before we show that Algorithm 3.3 is a realization of Algorithm 3.1, let us make some further comments on the LP-Newton method and its subproblems (3.52).

Remark 3.5. (a) From the computational point of view it might be reasonable to solve the following modification of (3.52) in each step of the LP-Newton method:

$$\tilde{\gamma} \to \min_{z,\gamma} \quad \text{s.t.} \quad \begin{aligned} & z \in \Omega, \\ & \|F(s) + G(s)(z - s)\|_\infty \leq \tilde{\gamma}\|F(s)\|_\infty, \\ & \|z - s\|_\infty \leq \tilde{\gamma}, \\ & \tilde{\gamma} \geq 0. \end{aligned} \tag{3.58}$$

Note that a point (z, γ) is feasible for (3.52) if and only if $(z, \tilde{\gamma})$ with $\tilde{\gamma} := \gamma\|F(s)\|_\infty$ is feasible for (3.58). Therefore, since the objectives of (3.52) and (3.58) are proportional, for any solution of (3.52) there is a solution of (3.58) with the same z-part and vice versa. The motivation to consider (3.58) instead of (3.52) is that the former might be better scaled if $\|F(s)\|_\infty$ is very small, i.e., if s is very close to a solution of (1.1).

(b) If $s \in \Omega$ is not yet a solution of (1.1), the constraint $\gamma \geq 0$ in (3.52) is implied by the other two inequalities. Therefore, in a computational implementation of Algorithm 3.3 this constraint can be omitted since the algorithm stops once a solution of (1.1) is found.

The following proposition says that the LP-Newton method can be regarded as a special realization of our general Newton-type algorithm from Section 3.1, without requiring any additional conditions.

Proposition 3.19. *Let* $\{z^k\}$ *be an infinite sequence generated by Algorithm 3.3 with starting point* $z^0 \in \Omega$. *Then this sequence can, together with some sequence* $\{\Gamma_k\}$, *also be obtained by Algorithm 3.1, for arbitrary* $\Gamma_0 > 0$, $\alpha := M^2$, *and arbitrary* $\beta > 1$.

Proof. For every $k \in \mathbb{N}$, we denote by γ_{k+1} the optimal value of (3.52) with s replaced by z^k. Let $\Gamma_0 > 0$ and $\beta > 1$ be arbitrarily chosen and let, for every $k \in \mathbb{N}$, Γ_{k+1} be defined according to the updating rule in step (S.2) of Algorithm 3.1, i.e., $\Gamma_{k+1} := \min\{\beta^\ell \Gamma_k \mid \ell \in \mathbb{N},\ \mathcal{F}(z^k, \beta^\ell \Gamma_k) \neq \emptyset\}$.

We are going to show that

$$\gamma_{k+1} \leq M\Gamma_{k+1} \tag{3.59}$$

is satisfied for all $k \in \mathbb{N}$. Assume the contrary. Then there is $k \in \mathbb{N}$ such that $\gamma_{k+1} > M\Gamma_{k+1}$ holds. Due to the definition of Γ_{k+1}, the set $\mathcal{F}(z^k, \Gamma_{k+1})$ is nonempty. Using this and item (c) of Proposition 3.17, there is $z \in \Omega$ such that $(z, M\Gamma_{k+1})$ is feasible for (3.52) with $s := z^k$. However, since γ_{k+1} is the optimal value of (3.52) with s replaced by z^k, this is a contradiction to $\gamma_{k+1} > M\Gamma_{k+1}$. Therefore, (3.59) is valid for all $k \in \mathbb{N}$.

Now let us prove that

$$z^{k+1} \in \mathcal{F}(z^k, \alpha\Gamma_{k+1}) \tag{3.60}$$

holds for all $k \in \mathbb{N}$. To this end, let $k \in \mathbb{N}$ be arbitrary but fixed. The point (z^{k+1}, γ_{k+1}) is a solution of (3.52) with $s := z^k$ due to the rule in step (S.2) of Algorithm 3.3. Therefore, (z^{k+1}, γ_{k+1}) is in particular feasible for (3.52) with $s := z^k$. By item (c) of Proposition 3.17, $z^{k+1} \in \mathcal{F}(z^k, M\gamma_{k+1})$ is valid. Using this, (3.59), and the definition of α, we obtain

$$z^{k+1} \in \mathcal{F}(z^k, M\gamma_{k+1}) \subseteq \mathcal{F}(z^k, M^2\Gamma_{k+1}) = \mathcal{F}(z^k, \alpha\Gamma_{k+1}).$$

Thus, it is shown that $\{z^k\}$ can also be obtained by Algorithm 3.1. □

The latter proposition and Theorem 3.3 yield the following local convergence result for the LP-Newton method which recovers [18, Theorem 1].

Theorem 3.20. *Let Assumptions 1–4 be satisfied. Then there is* $\rho > 0$ *such that any infinite sequence* $\{z^k\}$ *generated by Algorithm 3.3 with starting point* $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ *converges Q-quadratically to a solution of* (1.1).

The rest of this section is devoted to a modification of the LP-Newton method where it suffices to determine a suitable feasible point of (3.52) in each step instead of solving this optimization problem exactly. This means that an iterative algorithm for the solution of the LP-Newton subproblem (3.52) can be truncated once a suitable feasible point is found.

Algorithm 3.4 below is the inexact LP-Newton method which was described and analyzed in [17]. Our aim is to show that it is a special realization of our general Newton-type algorithm from Section 3.1 and enjoys the same local convergence properties.

Algorithm 3.4. (Inexact LP-Newton Method)

(S.0) Choose $z^0 \in \Omega$, $\gamma_0 > 0$, and $\bar{\alpha} > 1$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) If $\mathcal{F}_\infty(z^k, \gamma_k) \neq \emptyset$, then determine a feasible point $(z^{k+1}, \tilde{\gamma}_{k+1})$ of (3.52) with $s := z^k$ such that $\tilde{\gamma}_{k+1} \leq \gamma_k$ holds. Set $\gamma_{k+1} := \gamma_k$.
Else, compute a solution $(z^{k+1}, \tilde{\gamma}_{k+1})$ of (3.52) with $s := z^k$ and set $\gamma_{k+1} := \bar{\alpha}\tilde{\gamma}_{k+1}$.

(S.3) Set $k := k + 1$ and go to (S.1).

In a computational implementation of Algorithm 3.4, step (S.2) in the k-th iteration means that an iterative algorithm for the solution of the LP-Newton subproblem (3.52) with s replaced by z^k can be truncated once a feasible point is found whose γ-part is less than or equal to γ_k. If a point with this property is not found before the exact solution is determined, the latter is taken as the new iterate and γ_k is increased.

The subsequent proposition on the well-definedness of Algorithm 3.4 follows from the definition of the sets $\mathcal{F}_\infty(z^k, \gamma_k)$ and from item (a) of Proposition 3.17.

Proposition 3.21. *Algorithm 3.4 is well defined for any $z^0 \in \Omega$, $\gamma_0 > 0$, and $\bar{\alpha} > 1$.*

Next, we show that the inexact LP-Newton method can be regarded as a special realization of the general Newton-type algorithm from Section 3.1.

Proposition 3.22. *Let $\{z^k\}$ and $\{\gamma_k\}$ be infinite sequences generated by Algorithm 3.4 with starting points $z^0 \in \Omega$ and $\gamma_0 > 0$. Then the sequence $\{z^k\}$ can, together with some sequence $\{\Gamma_k\}$, also be obtained by Algorithm 3.1, for $\Gamma_0 := \gamma_0$, $\alpha := \bar{\alpha}M^2$, and arbitrary $\beta > 1$.*

Proof. For every $k \in \mathbb{N}$ some number $\tilde{\gamma}_{k+1}$ is determined in step (S.2) of Algorithm 3.4, regardless of whether $\mathcal{F}_\infty(z^k, \gamma_k)$ is empty or not. This number has the property that $(z^{k+1}, \tilde{\gamma}_{k+1})$ is feasible for the optimization problem (3.52) with s replaced by z^k. Taking into account item (c) of Proposition 3.17, z^{k+1} belongs to $\mathcal{F}(z^k, M\tilde{\gamma}_{k+1})$ for all $k \in \mathbb{N}$. Moreover, $\tilde{\gamma}_{k+1} \leq \gamma_{k+1}$ is obviously valid for all $k \in \mathbb{N}$. This follows from the rule in step (S.2) of Algorithm 3.4 again, regardless of whether $\mathcal{F}_\infty(z^k, \gamma_k)$ is empty or not. Combining the latter observations, we obtain

$$z^{k+1} \in \mathcal{F}(z^k, M\tilde{\gamma}_{k+1}) \subseteq \mathcal{F}(z^k, M\gamma_{k+1}) \tag{3.61}$$

for all $k \in \mathbb{N}$.

Now let $\beta > 1$ be arbitrarily chosen and let, for every k, Γ_{k+1} be defined according to the updating rule in step (S.2) of Algorithm 3.1, i.e., $\Gamma_{k+1} := \min\{\beta^\ell \Gamma_k \mid \ell \in \mathbb{N},\ \mathcal{F}(z^k, \beta^\ell \Gamma_k) \neq \emptyset\}$. We show by induction that

$$\gamma_k \leq \bar{\alpha} M \Gamma_k \tag{3.62}$$

holds for all $k \in \mathbb{N}$. By $\bar{\alpha} > 1$, $M \geq 1$, and the definition of Γ_0, inequality (3.62) is obviously valid for $k = 0$. Now suppose that (3.62) is satisfied for all $k = 0, \ldots, l$. Let us verify the validity for $k = l + 1$. If $\mathcal{F}_\infty(z^l, \gamma_l)$ is nonempty, then $\gamma_{l+1} = \gamma_l$ holds due to the updating rule in step (S.2) of Algorithm 3.4. Using this, (3.62) for $k = l$, and the definition of Γ_{l+1}, we obtain

$$\gamma_{l+1} = \gamma_l \leq \bar{\alpha} M \Gamma_l \leq \bar{\alpha} M \Gamma_{l+1}.$$

Otherwise, if $\mathcal{F}_\infty(z^l, \gamma_l)$ is empty, $\tilde{\gamma}_{l+1}$ is the optimal value of (3.52) with $s := z^l$, and $\gamma_{l+1} = \bar{\alpha}\tilde{\gamma}_{l+1}$ holds. Due to the definition of Γ_{l+1}, the set $\mathcal{F}(z^l, \Gamma_{l+1})$ is nonempty. Thus, with item (c) of Proposition 3.17, there is $z \in \Omega$ such that $(z, M\Gamma_{l+1})$ is feasible for (3.52) with s replaced by z^l. Since $\tilde{\gamma}_{l+1}$ is the optimal value, $\tilde{\gamma}_{l+1} \leq M\Gamma_{l+1}$ follows. Therefore, we obtain

$$\gamma_{l+1} = \bar{\alpha}\tilde{\gamma}_{l+1} \leq \bar{\alpha} M \Gamma_{l+1}.$$

In each case, (3.62) is shown for $k = l + 1$. This completes the induction.

Combining (3.61), (3.62), and the definition of α,

$$z^{k+1} \in \mathcal{F}(z^k, M\gamma_{k+1}) \subseteq \mathcal{F}(z^k, \bar{\alpha} M^2 \Gamma_{k+1}) = \mathcal{F}(z^k, \alpha\Gamma_{k+1})$$

follows for all $k \in \mathbb{N}$. Thus, it is shown that $\{z^k\}$ can also be obtained by Algorithm 3.1. □

Theorem 3.23 below essentially follows from Proposition 3.22 and Theorem 3.3 and says that the inexact LP-Newton method converges locally with a Q-quadratic rate to a solution of (1.1) if Assumptions 1–4 are satisfied. Moreover, it is shown that the exact solution of a subproblem is needed a finite number of times only.

Theorem 3.23. *Let Assumptions 1–4 be satisfied. Then there is $\rho > 0$ such that the following assertions hold for any infinite sequences $\{z^k\}$ and $\{\gamma_k\}$ generated by Algorithm 3.4 with starting points $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ and $\gamma_0 > 0$.*

(a) The sequence $\{\gamma_k\}$ is bounded.

(b) The sequence $\{z^k\}$ converges Q-quadratically to a solution of (1.1).

Moreover, the "else"-branch in step (S.2) of Algorithm 3.4 is taken a finite number of times only.

Proof. Let $\{z^k\}$ and $\{\gamma_k\}$ be infinite sequences which are generated by Algorithm 3.4. Then Proposition 3.22 yields that $\{z^k\}$ can, together with some sequence $\{\Gamma_k\}$, also be obtained by Algorithm 3.1, for $\Gamma_0 := \gamma_0$, $\alpha := \bar{\alpha}$, and arbitrary $\beta > 1$. Assume that z^0 belongs to $\mathcal{B}_\rho(z^*) \cap \Omega$, with $\rho > 0$ from Theorem 3.3. Then we know from Theorem 3.3 that $\{z^k\}$ converges Q-quadratically to a solution of (1.1) and that the sequence $\{\Gamma_k\}$ is bounded. We have shown in the proof of Proposition 3.22 that $\gamma_k \leq \bar{\alpha} M \Gamma_k$ holds for all $k \in \mathbb{N}$. Therefore, the sequence $\{\gamma_k\}$ is bounded, too.

It is not difficult to see that $\gamma_{k+1} \geq \gamma_k$ holds for all $k \in \mathbb{N}$. If the "else"-branch is taken, we even have $\gamma_{k+1} > \bar{\alpha}\gamma_k$. In fact, the set $\mathcal{F}_\infty(z^k, \gamma_k)$ is empty in that case whereas the set $\mathcal{F}_\infty(z^k, \tilde{\gamma}_{k+1})$ is nonempty where $\tilde{\gamma}_{k+1}$ denotes the optimal value of (3.52) with s replaced by z^k. Therefore, $\tilde{\gamma}_{k+1} > \gamma_k$ follows. Thus, by the definition of γ_{k+1},

$$\gamma_{k+1} = \bar{\alpha}\tilde{\gamma}_{k+1} > \bar{\alpha}\gamma_k$$

is satisfied. Using this, $\bar{\alpha} > 1$, and the boundedness of $\{\gamma_k\}$, the "else"-branch cannot be taken an infinite number of times. Thus, all assertions are proved. □

3.5 Constrained Levenberg-Marquardt Method

This section is devoted to the constrained Levenberg-Marquardt method and several inexact versions of it. In each step of the constrained Levenberg-Marquardt method an optimization problem has to be solved which is a strongly convex, quadratic program if Ω is polyhedral. We will describe a quite general algorithm which provides a framework for an inexact constrained Levenberg-Marquardt method, see Algorithm 3.5. Afterwards, we show that every realization of this general Levenberg-Marquardt-type algorithm can also be regarded as a special realization of the general Newton-type algorithm from Section 3.1, without requiring any additional conditions. It will turn out that both the (exact) constrained Levenberg-Marquardt method and the inexact version from [17] are special realizations of Algorithm 3.5. Therefore, both methods converge locally quadratically to a solution of (1.1) if Assumptions 1–4 are satisfied. Furthermore, we will consider the inexact variant of the constrained Levenberg-Marquardt method which is analyzed in [1]. It is proved that, under suitable conditions, this method can be regarded as a special realization of Algorithm 3.5, too, and therefore enjoys the same local convergence properties.

At first, let us give some historical overview on the Levenberg-Marquardt method and important local convergence results. We start with the classical Levenberg-Marquardt method for the solution of unconstrained systems of equations. Let us assume for the moment that Ω equals $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, and F' is locally Lipschitz continuous near z^*. The Levenberg-Marquardt method goes back to Levenberg [61] and Marquardt [66]. In order to describe its subproblems, let $\mu : \mathbb{R}^n \to \mathbb{R}_{++}$ be a given function and let $s \in \mathbb{R}^n$ denote the current iterate. Then a solution of the following linear system of equations has to be determined:

$$F'(s)^\top F(s) + (F'(s)^\top F'(s) + \mu(s) I_n)(z - s) = 0. \tag{3.63}$$

Since $\mu(s)$ is strictly positive, the system matrix in (3.63) is positive definite. Therefore, for any $s \in \mathbb{R}^n$, (3.63) has a unique solution, even if $F'(s)$ is singular or $m \neq n$ holds. Thus, the resulting algorithm is always well defined. This is a great advantage of the Levenberg-Marquardt method compared to the Newton or the Gauss-Newton method. The latter would be obtained by setting $\mu(s) = 0$ for all $s \in \mathbb{R}^n$.

It is not difficult to see that the Levenberg-Marquardt subproblem can be equivalently written as an unconstrained optimization problem. In fact, the solution of (3.63) is also the unique solution of the quadratic program

$$\psi_0(z, s) := \|F(s) + F'(s)(z - s)\|^2 + \mu(s)\|z - s\|^2 \to \min_z \tag{3.64}$$

and vice versa. Let us briefly justify this equivalence. The objective function $\psi_0(\cdot, s)$ of (3.64) is strongly convex. Therefore, (3.64) has a unique solution and the optimality condition $\nabla_z \psi_0(z, s) = 0$ is both necessary and sufficient. The left hand side of the equation (3.63) is equal to $\frac{1}{2}\nabla_z \psi_0(z, s)$. Hence, the equivalence is shown.

It is well known that the full column rank of $F'(z^*)$ leads to local fast convergence of the classical Levenberg-Marquardt method if the function μ is suitably chosen, see for instance [11, Theorem 10.2.6] and [83, Satz 10.2.9].

The Levenberg-Marquardt method has gained in importance since Yamashita and Fukushima [90] observed that the local fast convergence is kept if the full column rank of the Jacobian at z^* is replaced by a local error bound condition which is precisely our Assumption 2 for the case $\Omega = \mathbb{R}^n$. We know from Theorem 2.3 that Assumption 2 is implied by the full column rank of $F'(z^*)$. However, unlike the latter, Assumption 2 allows nonisolated solutions. It is shown in [90, Theorem 2.1] that any sequence $\{z^k\}$ generated by the Levenberg-Marquardt method converges to a solution of (1.1) if Assumption 2 holds, z^0 is sufficiently close to z^*, and $\mu(s)$ is equal to $\|F(s)\|^2$ for all $s \notin Z$. Moreover, it is proved that the sequence $\{\text{dist}[z^k, Z]\}$ of the corresponding distances to the solution set converges quadratically to zero.

More robust versions of the Levenberg-Marquardt method are considered in [30, 36, 91] where "more robust" means that a larger value of $\mu(s)$ is allowed if s is close to the solution set. In particular, it is shown in [30, Theorem 2.2] that any sequence generated by the Levenberg-Marquardt method converges quadratically to a solution of (1.1) if Assumption 2 holds, $\mu(s) = \|F(s)\|$ is satisfied for all $s \notin Z$, and the starting point belongs to a sufficiently small neighborhood of z^*.

An inexact version of the Levenberg-Marquardt method for the solution of unconstrained systems of equations is analyzed in [7]. Let us assume again that $s \in \mathbb{R}^n$ is the current iterate. Then, instead of (3.63), a solution of the linear system

$$F'(s)^\top F(s) + (F'(s)^\top F'(s) + \mu(s)I_n)(z - s) = \pi(s) \tag{3.65}$$

is determined where the function $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is used to describe the inexactness which may arise from computational inaccuracies or the truncation of iterative solution algorithms. Suppose that Assumption 2 is satisfied, z^0 is sufficiently close to z^*, $\mu(s)$ equals $\|F(s)\|^2$ for all $s \notin Z$, and there is some $c_\pi > 0$ such that $\|\pi(s)\| \le c_\pi \|F(s)\|^4$ holds for all s in a certain neighborhood of z^*. Then any sequence generated by the inexact Levenberg-Marquardt method converges to a solution of (1.1), and the sequence of the corresponding distances to the solution set converges quadratically to zero, see [7, Theorem 2.2 and Remark 2.1].

This result is improved in [28, 29, 39] where local quadratic convergence of the generated sequence $\{z^k\}$ itself is shown under Assumption 2. Moreover, larger values of $\mu(s)$ and $\|\pi(s)\|$ are allowed. In particular, it is proved in [39, Theorem 2.10] that the inexact Levenberg-Marquardt method converges locally quadratically to a solution of (1.1) if Assumption 2 holds, $\mu(s) = \|F(s)\|$ is valid for all $s \notin Z$, and π satisfies, for some $c_\pi > 0$, $\|\pi(s)\| \le c_\pi \|F(s)\|^2$ for all s in a certain neighborhood of z^*.

From now on, we drop the assumption $\Omega = \mathbb{R}^n$. Instead, we assume that $\Omega \subseteq \mathbb{R}^n$ is nonempty and closed. However, the differentiability assumptions on F are still kept. Moreover, Ω is supposed to be convex for the moment. In [55] a constrained Levenberg-Marquardt method is proposed where, for any given iterate $s \in \Omega$, the constrained optimization problem

$$\psi_0(z, s) \to \min_z \quad \text{s.t.} \quad z \in \Omega \tag{3.66}$$

has to be solved. The function ψ_0 is the same as in (3.64). Due to the continuity and strong convexity of $\psi_0(\cdot, s)$, together with the assumptions on Ω, (3.66) has a unique solution for any $s \in \Omega$. Therefore, the resulting algorithm is well defined for any starting point. It follows from [55, Theorem 2.11] that any sequence generated by the constrained Levenberg-Marquardt method converges quadratically to a solution of (1.1) if Assumption 2 holds, $\mu(s)$ equals $\|F(s)\|^2$ for all $s \notin Z$, and the starting point belongs to a sufficiently small neighborhood of z^*. Note that the differentiability assumptions on F are actually not required in [55]. Instead, $F'(s)$ is replaced by some matrix $G(s)$ in the definition of ψ_0, and the weaker Condition 1 which we recalled in Section 3.2, together with the local Lipschitz continuity of F and Assumption 2, is used to prove local quadratic convergence of the constrained Levenberg-Marquardt method. However, note that Condition 1 implies that F is differentiable at all points which are sufficiently close to z^* and belong to the interior of Ω, see [84, Lemma 5.3.1].

An inexact version of the constrained Levenberg-Marquardt method is described and analyzed in [1]. In order to recall the subproblems, let us assume that a function $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is given to describe the inexactness and let us define the function ψ_π by

$$\psi_\pi(z, s) := \|F(s) + F'(s)(z - s)\|^2 + \mu(s)\|z - s\|^2 + \pi(s)^\top(z - s).$$

Then, for any given iterate $s \in \Omega$, the following subproblem is solved instead of (3.66):

$$\psi_\pi(z, s) \to \min_z \quad \text{s.t.} \quad z \in \Omega. \tag{3.67}$$

This kind of an inexact method is motivated by the unconstrained case where it can be easily verified that a point z solves (3.65) if and only if it minimizes the function $\psi_\pi(\cdot, s)$. Of course, for any function π and any point s, $\psi_\pi(\cdot, s)$ is a strongly convex function so that (3.67) has a unique solution and the resulting algorithm is well defined. In [1, Theorem 1] it is shown that the local quadratic convergence is kept if $\mu(s) = \|F(s)\|$ is satisfied for all $s \notin Z$ and there is a constant $c_\pi > 0$ such that $\|\pi(s)\| \leq c_\pi\|F(s)\|^2$ holds for all s near z^*.

The projected Levenberg-Marquardt method is a further extension of the classical Levenberg-Marquardt method to constrained systems of equations. For a given iterate s, this method requires to solve the linear system (3.63) and afterwards to project its solution onto Ω. The projected Levenberg-Marquardt method has advantages if the projection onto Ω is computationally cheap, for example if Ω is defined by bound constraints only. However, to guarantee local quadratic convergence, a local error bound condition is needed which is stronger than Assumption 2 and particularly implies that F has no zeros outside of Ω. More precisely, it is required that there are $\omega > 0$ and $\delta_2 > 0$ such that the inequality from Assumption 2 is satisfied for all $s \in \mathcal{B}_{\delta_2}(z^*)$ (not intersected with Ω), see [55, Theorem 3.11]. The projected Levenberg-Marquardt method is also considered in [2] where local linear convergence is proved under the validity of two weaker error bound conditions. Moreover, approximate projections are allowed which might be useful if Ω is not described by bounds only.

Now let us return to the setting we are mainly interested in, i.e., not necessarily differentiable constrained systems. To this end, let $F : \Omega \to \mathbb{R}^m$ be at least continuous and $G : \Omega \to \mathbb{R}^{m\times n}$ be a given function. Moreover, we drop the convexity assumption on Ω, so we just require that Ω is nonempty and closed. For the unconstrained case, nonsmooth Levenberg-Marquardt methods are described and analyzed in [22] and [54]. In both cases, the methods are used for the solution of nonsmooth reformulations of nonlinear complementarity problems. In [22] an inexact method is described. The linear system (3.65) is solved in each step where $F'(s)$ is replaced by $G(s)$ and the latter is chosen from the B-subdifferential $\partial_B F(s)$. The subproblems in [54] consist of solving (3.63) for a given point s where $F'(s)$ is replaced by an element of the C-subdifferential. For its definition, we refer to [54]. Local quadratic convergence of the resulting methods is proved in [22] and [54], respectively, under suitable regularity conditions which particularly imply the local uniqueness of solutions.

To the best of the author's knowledge, local convergence properties of nonsmooth Levenberg-Marquardt methods for systems with nonisolated solutions were not analyzed previous to [17], except [55]. In the latter reference the constrained Levenberg-Marquardt method is analyzed under Assumption 2, which allows nonisolated solutions, together with Condition 1. However, as already mentioned, Condition 1 implies the differentiability of F at all interior points of Ω being sufficiently close to z^*.

The aim of the rest of this section is to analyze local convergence properties of nonsmooth, inexact constrained Levenberg-Marquardt methods for the solution of (1.1) under our Assumptions 1–4. In particular, we do not require that z^* is a local unique solution of (1.1). Moreover, taking into account the results from Section 3.2, we know that Assumptions 1–4 together are weaker than the whole set of conditions being used in [55], see Theorem 3.5 and the discussion after it.

First, we describe an algorithm which generalizes [17, Algorithm 3] and can be regarded as a general framework for inexact constrained Levenberg-Marquardt methods. Let us redefine the function $\psi_0 : \mathbb{R}^n \times \Omega \to \mathbb{R}$ by replacing $F'(s)$ in the old definition by $G(s)$, i.e.,

$$\psi_0(z, s) := \|F(s) + G(s)(z - s)\|^2 + \mu(s)\|z - s\|^2. \tag{3.68}$$

From now on, the function $\mu : \Omega \to \mathbb{R}_{++}$ is defined according to

$$\mu(s) := \begin{cases} \|F(s)\|^2 & \text{if} \quad s \notin Z, \\ 1 & \text{if} \quad s \in Z. \end{cases} \tag{3.69}$$

If Ω is convex, then it is well known that, for any $s \in \Omega$, the optimization problem (3.66) with ψ_0 defined according to (3.68) has a unique solution. Otherwise, at least the existence of a solution is still guaranteed. This is shown in the following proposition.

Proposition 3.24. *The following assertions are true.*

(a) The optimization problem (3.66) *with* ψ_0 *given by* (3.68) *has a solution for any* $s \in \Omega$.

(b) If s *belongs to* Z*, then* $z = s$ *is the unique solution of* (3.66) *and the optimal value is equal to zero.*

Proof. (a) Let $s \in \Omega$ be arbitrarily chosen. The function $\psi_0(\cdot, s)$ is continuous and strongly convex on $\mathbb{R}^n$. Therefore, the level set

$$\mathcal{N}(s) := \{z \in \mathbb{R}^n \mid \psi_0(z, s) \leq \psi_0(s, s)\}$$

is compact. Due to the closedness of Ω, the set $\mathcal{N}(s) \cap \Omega$ is compact as well. Moreover, the latter set is nonempty since s belongs to it. Thus, by the theorem of Weierstrass, the optimization problem

$$\psi_0(z, s) \to \min_z \quad \text{s.t.} \quad z \in \mathcal{N}(s) \cap \Omega \tag{3.70}$$

has a solution. Obviously, the solution sets of (3.70) and (3.66) coincide so that the latter problem is solvable as well.

(b) Let $s \in Z$ be arbitrary but fixed. Then

$$0 = \psi_0(s, s) \leq \psi_0(z, s)$$

is satisfied for all $z \in \Omega$ where actually $\psi_0(s, s) < \psi_0(z, s)$ holds for all $z \in \Omega \setminus \{s\}$. Hence, the assertions are shown.

□

If Ω is a polyhedral set, then (3.66) is a quadratic program. Otherwise, the solution of (3.66) might be difficult from the computational point of view. However, if Ω is described by nonlinear inequalities, (1.1) can be equivalently reformulated by introducing slack variables such that the new constrained system has a polyhedral feasible set, see Section 4.2.

Throughout the rest of this section, we denote, for any $s \in \Omega$, by $\psi_0^{\text{opt}}(s)$ the optimal value of (3.66). Due to Proposition 3.24 this value is always well defined. In order to describe our general framework for inexact constrained Levenberg-Marquardt methods, let us define, for any point $s \in \Omega$ and any number $\gamma \geq 0$, the set

$$\mathcal{E}(s, \gamma) := \{z \in \Omega \mid \psi_0(z, s) \leq \gamma^2 \|F(s)\|^4\}.$$

Moreover, we need, for any $s \in \Omega$, the number $\gamma_0(s)$, defined by

$$\gamma_0(s) := \begin{cases} \sqrt{\psi_0^{\text{opt}}(s)} \|F(s)\|^{-2} & \text{if} \quad s \notin Z, \\ 0 & \text{if} \quad s \in Z. \end{cases}$$

The following lemma will be helpful in the sequel. It states some properties of the sets $\mathcal{E}(s, \gamma)$. In particular, relations to the sets $\mathcal{F}(s, \gamma)$ are discussed.

Lemma 3.25. *The following assertions are satisfied for any $s \in \Omega$.*

(a) The inclusions $\mathcal{E}(s, \gamma) \subseteq \mathcal{F}(s, \gamma)$ and $\mathcal{F}(s, \gamma) \subseteq \mathcal{E}(s, \sqrt{2}\gamma)$ hold for all $\gamma \geq 0$.

(b) Every solution z_0^s of (3.66) belongs to $\mathcal{E}(s, \gamma_0(s))$.

(c) For every $\gamma \geq 0$ it holds that $\mathcal{E}(s, \gamma)$ is nonempty if and only if $\gamma \geq \gamma_0(s)$ is valid.

Proof. Let $s \in \Omega$ be arbitrary but fixed.

(a) If s is a solution of (1.1), then it is not difficult to see that $\mathcal{E}(s,\gamma) = \mathcal{F}(s,\gamma) = \{s\}$ holds for all $\gamma \geq 0$. So let us assume $s \notin Z$ and let $\gamma \geq 0$ be arbitrarily chosen. Suppose that $\mathcal{E}(s,\gamma)$ is nonempty and let us take any $z \in \mathcal{E}(s,\gamma)$. Then z belongs to Ω and

$$\psi_0(z,s) = \|F(s) + G(s)(z-s)\|^2 + \mu(s)\|z-s\|^2 \leq \gamma^2\|F(s)\|^4$$

holds due to the definition of $\mathcal{E}(s,\gamma)$. This, together with (3.69), implies

$$\|F(s) + G(s)(z-s)\| \leq \gamma\|F(s)\|^2 \quad \text{and} \quad \|z-s\| \leq \gamma\|F(s)\|.$$

Hence, $z \in \mathcal{F}(s,\gamma)$ follows.

Conversely, suppose that $\mathcal{F}(s,\gamma)$ is nonempty and let $z \in \mathcal{F}(s,\gamma)$ be arbitrary but fixed. Then we have $z \in \Omega$ and, taking into account (3.69),

$$\begin{aligned} \psi_0(z,s) &= \|F(s) + G(s)(z-s)\|^2 + \mu(s)\|z-s\|^2 \\ &\leq \gamma^2\|F(s)\|^4 + \|F(s)\|^2 \cdot \gamma^2\|F(s)\|^2 \\ &= 2\gamma^2\|F(s)\|^4. \end{aligned}$$

Thus, z belongs to $\mathcal{E}(s,\sqrt{2}\gamma)$.

(b) If $s \in Z$ is valid, then $z_0^s = s$ is the unique solution of (3.66) due to item (b) of Proposition 3.24. Moreover, it is not difficult to see that $\mathcal{E}(s,\gamma_0(s)) = \{s\}$ holds. Thus, the assertion is valid.

Now assume that s does not belong to Z and let z_0^s be an arbitrary but fixed solution of (3.66). Then we obtain

$$\psi_0(z_0^s, s) = \psi_0^{\text{opt}}(s) = \gamma_0(s)^2\|F(s)\|^4$$

by the definition of $\gamma_0(s)$. This, together with $z_0^s \in \Omega$, yields $z_0^s \in \mathcal{E}(s,\gamma_0(s))$.

(c) Let $\gamma \geq 0$ be arbitrarily chosen such that $\mathcal{E}(s,\gamma)$ is nonempty and let $z \in \mathcal{E}(s,\gamma)$ be arbitrary but fixed. Assume that $\gamma < \gamma_0(s)$ holds. This implies $\gamma_0(s) > 0$ so that the definitions of $\mathcal{E}(s,\gamma)$ and $\gamma_0(s)$ yield

$$\psi_0(z,s) \leq \gamma^2\|F(s)\|^4 < \gamma_0(s)^2\|F(s)\|^4 = \psi_0^{\text{opt}}(s).$$

This is a contradiction to the fact that $\psi_0^{\text{opt}}(s)$ is the optimal value of (3.66). Thus, $\gamma \geq \gamma_0(s)$ must be valid.

Conversely, let us take any $\gamma \geq \gamma_0(s)$. Then the inclusion $\mathcal{E}(s,\gamma_0(s)) \subseteq \mathcal{E}(s,\gamma)$ is valid. Hence, the assertion is shown since $\mathcal{E}(s,\gamma_0(s))$ is nonempty due to item (b), together with Proposition 3.24.

□

Now we are in the position to describe Algorithm 3.5 which is a general inexact constrained Levenberg-Marquardt algorithm and generalizes [17, Algorithm 3]. The latter can be regarded as a special instance of Algorithm 3.5, see Remark 3.6. Moreover, it will turn out that the (exact) constrained Levenberg-Marquardt method itself as well as the inexact method described by the subproblems (3.67) are special realizations of Algorithm 3.5, the latter if π satisfies some suitable condition.

Algorithm 3.5. (Inexact Constrained Levenberg-Marquardt Algorithm I)

(S.0) Choose $z^0 \in \Omega$, $\gamma_0 > 0$, $\bar{\alpha} > 1$, and $\hat{\alpha} \geq 1$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) If $\mathcal{E}(z^k, \gamma_k) \neq \emptyset$, then set $\gamma_{k+1} := \gamma_k$.
Else, determine $\psi_0^{\text{opt}}(z^k)$ and set $\gamma_{k+1} := \bar{\alpha}\gamma_0(z^k)$.

(S.3) Determine $z^{k+1} \in \mathcal{E}(z^k, \hat{\alpha}\gamma_{k+1})$.

(S.4) Set $k := k + 1$ and go to (S.1).

Remark 3.6. (a) The (exact) constrained Levenberg-Marquardt method itself is a special realization of Algorithm 3.5, for arbitrary $\bar{\alpha} > 1$ and arbitrary $\hat{\alpha} \geq 1$. In fact, the updating rule in step (S.2), together with $\hat{\alpha} \geq 1$, implies that $\mathcal{E}(z^k, \hat{\alpha}\gamma_{k+1})$ in step (S.3) is always nonempty so that every solution of (3.66) with s replaced by z^k particularly belongs to it. The latter follows from items (b) and (c) of Lemma 3.25. Therefore, the computation of z^{k+1} as an exact solution of (3.66) with $s := z^k$ in each step provides a special realization of Algorithm 3.5.

(b) The inexact constrained Levenberg-Marquardt method from [17] can also be regarded as a special realization of Algorithm 3.5. In fact, it is obtained if $\hat{\alpha}$ equals 1 and if a solution of (3.66) with $s := z^k$ is taken as the new iterate z^{k+1} whenever $\mathcal{E}(z^k, \gamma_k)$ is empty. Based on this observation, let us describe a reasonable realization of steps (S.2) and (S.3) in a computational implementation of Algorithm 3.5. An iterative algorithm for the solution of the subproblem (3.66) with $s := z^k$ can be truncated once a feasible point is found such that the corresponding function value of $\psi_0(\cdot, z^k)$ is less than or equal to $\gamma_k^2\|F(z^k)\|^4$. If a point with this property is not found before an exact solution $z_0^{z^k}$ is determined, the latter is taken as the new iterate.

(c) Later on in this section, it is shown that the inexact constrained Levenberg-Marquardt method described by the subproblems (3.67) can also be regarded as a special realization of Algorithm 3.5 if some suitable condition on π is satisfied.

The following result on the well-definedness of Algorithm 3.5 essentially follows from Proposition 3.24.

Proposition 3.26. *Algorithm 3.5 is well defined for any $z^0 \in \Omega$, $\gamma_0 > 0$, $\bar{\alpha} > 1$, and $\hat{\alpha} \geq 1$.*

Our next aim is to prove that Algorithm 3.5 can be regarded as a special realization of our general Newton-type algorithm from Section 3.1, without requiring any additional conditions. In particular, taking into account Remark 3.6, the exact constrained Levenberg-Marquardt method as well as [17, Algorithm 3] are special realizations of Algorithm 3.1.

Proposition 3.27. *Let $\{z^k\}$ and $\{\gamma_k\}$ be infinite sequences generated by Algorithm 3.5 with starting points $z^0 \in \Omega$ and $\gamma_0 > 0$. Then the sequence $\{z^k\}$ can, together with some sequence $\{\Gamma_k\}$, also be obtained by Algorithm 3.1, for $\Gamma_0 := \gamma_0$, $\alpha := \sqrt{2}\bar{\alpha}\hat{\alpha}$, and arbitrary $\beta > 1$.*

Proof. Let $\beta > 1$ be arbitrarily chosen and let, for every $k \in \mathbb{N}$, Γ_{k+1} be defined according to the updating rule in step (S.2) of Algorithm 3.1, i.e., $\Gamma_{k+1} := \min\{\beta^\ell \Gamma_k \mid \ell \in \mathbb{N},\ \mathcal{F}(z^k, \beta^\ell \Gamma_k) \neq \emptyset\}$. We show by induction that

$$\gamma_k \leq \sqrt{2}\bar{\alpha}\Gamma_k \tag{3.71}$$

holds for all $k \in \mathbb{N}$. By $\bar{\alpha} > 1$ and the definition of Γ_0, (3.71) is obviously valid for $k = 0$. Now let us assume that (3.71) is satisfied for all $k = 0, \ldots, l$. We prove the validity for $k = l + 1$. If $\mathcal{E}(z^l, \gamma_l)$ is nonempty, then γ_{l+1} equals γ_l due to the rule in step (S.2) of Algorithm 3.5. Therefore, using (3.71) for $k = l$ and the definition of Γ_{l+1}, we obtain

$$\gamma_{l+1} = \gamma_l \leq \sqrt{2}\bar{\alpha}\Gamma_l \leq \sqrt{2}\bar{\alpha}\Gamma_{l+1}.$$

Now suppose that $\mathcal{E}(z^l, \gamma_l)$ is empty. Let us assume that (3.71) is not satisfied for $k = l + 1$, i.e., $\gamma_{l+1} > \sqrt{2}\bar{\alpha}\Gamma_{l+1}$ holds. The definition of Γ_{l+1} implies that the set $\mathcal{F}(z^l, \Gamma_{l+1})$ is nonempty. Let $z \in \mathcal{F}(z^l, \Gamma_{l+1})$ be arbitrary but fixed. By item (a) of Lemma 3.25, z belongs to $\mathcal{E}(z^l, \sqrt{2}\Gamma_{l+1})$. Consequently,

$$\psi_0(z, z^l) \leq 2\Gamma_{l+1}^2 \|F(z^l)\|^4 < \frac{1}{\bar{\alpha}^2}\gamma_{l+1}^2 \|F(z^l)\|^4 = \gamma_0(z^l)^2 \|F(z^l)\|^4$$

follows where the last identity holds due to the rule in step (S.2) of Algorithm 3.5. Thus, with the definition of $\gamma_0(z^l)$, we obtain

$$\psi_0(z, z^l) < \psi_0^{\text{opt}}(z^l).$$

This is a contradiction to the fact that $\psi_0^{\text{opt}}(z^l)$ is the optimal value of (3.66) with $s := z^l$. Therefore, (3.71) must be valid for $k = l + 1$. Hence, the induction is complete.

Using the rule in step (S.3) of Algorithm 3.5, (3.71), the definition of α, and item (a) of Lemma 3.25, we have

$$z^{k+1} \in \mathcal{E}(z^k, \hat{\alpha}\gamma_{k+1}) \subseteq \mathcal{E}(z^k, \alpha\Gamma_{k+1}) \subseteq \mathcal{F}(z^k, \alpha\Gamma_{k+1})$$

for all $k \in \mathbb{N}$. Hence, it is shown that $\{z^k\}$ can also be obtained by Algorithm 3.1. □

By Proposition 3.27 and Theorem 3.3, we obtain the following theorem on local quadratic convergence of Algorithm 3.5. Moreover, it is shown that the exact solution of the subproblems (3.66) is needed a finite number of times only.

Theorem 3.28. *Let Assumptions 1–4 be satisfied. Then there is $\rho > 0$ such that the following assertions hold for any infinite sequences $\{z^k\}$ and $\{\gamma_k\}$ generated by Algorithm 3.5 with starting points $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ and $\gamma_0 > 0$.*

(a) The sequence $\{\gamma_k\}$ is bounded.

(b) The sequence $\{z^k\}$ converges Q-quadratically to a solution of (1.1).

Moreover, the "else"-branch in step (S.2) of Algorithm 3.5 is taken a finite number of times only.

Proof. Let $\{z^k\}$ and $\{\gamma_k\}$ be infinite sequences which are generated by Algorithm 3.5. Then Proposition 3.27 yields that $\{z^k\}$ can, together with some sequence $\{\Gamma_k\}$, also be obtained by Algorithm 3.1, for $\Gamma_0 := \gamma_0$, $\alpha := \sqrt{2}\bar{\alpha}\hat{\alpha}$, and arbitrary $\beta > 1$. Assume that z^0 belongs to $\mathcal{B}_\rho(z^*) \cap \Omega$, with $\rho > 0$ from Theorem 3.3. Then we know from Theorem 3.3 that $\{z^k\}$ converges with a Q-quadratic rate to a solution of (1.1) and that $\{\Gamma_k\}$ is bounded. We have shown in the proof of Proposition 3.27 that $\gamma_k \le \sqrt{2}\bar{\alpha}\Gamma_k$ holds for all $k \in \mathbb{N}$. Therefore, the sequence $\{\gamma_k\}$ is bounded, too.

It is not difficult to see that $\gamma_{k+1} \ge \gamma_k$ is valid for all $k \in \mathbb{N}$. If the "else"-branch is taken for some k, we even have $\gamma_{k+1} > \bar{\alpha}\gamma_k$. In fact, the set $\mathcal{E}(z^k, \gamma_k)$ is empty in that case whereas, due to item (b) of Lemma 3.25 together with Proposition 3.24, the set $\mathcal{E}(z^k, \gamma_0(z^k))$ is nonempty. Therefore, $\gamma_0(z^k) > \gamma_k$ holds. Thus, with the definition of γ_{k+1},

$$\gamma_{k+1} = \bar{\alpha}\gamma_0(z^k) > \bar{\alpha}\gamma_k$$

follows. Using this, $\bar{\alpha} > 1$, and the boundedness of $\{\gamma_k\}$, the "else"-branch cannot be taken an infinite number of times. Thus, all assertions are proved. □

By Remark 3.6 and Theorem 3.28, we know that particularly the exact constrained Levenberg-Marquardt method converges locally with a Q-quadratic rate to a solution of (1.1) if Assumptions 1–4 are satisfied. Therefore, the assertions of [55, Theorem 2.11] could be improved since all of Assumptions 1–4 are implied by the whole set of assumptions being used there, see Theorem 3.5 and the discussion after it. Moreover, because [17, Algorithm 3] is a special realization of Algorithm 3.5, Theorem 3.28 recovers [17, Corollary 2].

The rest of this section is devoted to the inexact constrained Levenberg-Marquardt method which is analyzed in [1] and described by the subproblems (3.67). We assume that $\pi : \Omega \to \mathbb{R}^n$ is a given function which is used to describe the inexactness. For our analysis we will suppose that there is some constant $c_\pi > 0$ such that

$$\|\pi(s)\| \leq c_\pi \|F(s)\|^3 \tag{3.72}$$

holds for all $s \in \Omega$. Unlike [1], we consider an extension to nonsmooth constrained systems of equations. To this end, we redefine the function $\psi_\pi : \mathbb{R}^n \times \Omega \to \mathbb{R}$ by replacing $F'(s)$ by $G(s)$, i.e.,

$$\psi_\pi(z, s) := \|F(s) + G(s)(z - s)\|^2 + \mu(s)\|z - s\|^2 + \pi(s)^\top (z - s). \tag{3.73}$$

Still, it is supposed that the function μ is defined according to (3.69). As (3.66), the optimization problem (3.67) with ψ_π defined according to (3.73) has always a solution. This is shown in the following proposition.

Proposition 3.29. *The following assertions are true.*

(a) The optimization problem (3.67) *with ψ_π given by* (3.73) *has a solution for any $s \in \Omega$.*

(b) Assume that there is $c_\pi > 0$ such that (3.72) *is satisfied for all $s \in \Omega$. If s belongs to Z, then $z = s$ is the unique solution of* (3.67) *and the optimal value is equal to zero.*

Proof. (a) The proof of this item is omitted since it is very similar to the proof of item (a) of Proposition 3.24. Note that $\psi_\pi(\cdot, s)$ is still continuous and strongly convex on $\mathbb{R}^n$.

(b) Let $s \in Z$ be arbitrary but fixed. Then $\pi(s) = 0$ follows by (3.72). Using this, we obtain

$$0 = \psi_\pi(s, s) \leq \psi_\pi(z, s)$$

for all $z \in \Omega$ where actually $\psi_\pi(s, s) < \psi_\pi(z, s)$ holds for all $z \in \Omega \setminus \{s\}$. Hence, the assertions are shown.

□

Algorithm 3.6 below formally describes the variant of an inexact constrained Levenberg-Marquardt method we are going to analyze in the sequel.

Algorithm 3.6. (Inexact Constrained Levenberg-Marquardt Algorithm II)

(S.0) Choose $z^0 \in \Omega$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) Determine z^{k+1} as a solution of (3.67) with $s := z^k$.

(S.3) Set $k := k + 1$ and go to (S.1).

Since the subproblems (3.67) have always a solution due to Proposition 3.29, we obtain the following result on the well-definedness of Algorithm 3.6.

Proposition 3.30. *Algorithm 3.6 is well defined for any $z^0 \in \Omega$.*

Our aim is to show that, for μ defined by (3.69) and π chosen such that (3.72) holds for some $c_\pi > 0$ and all $s \in \Omega$, any sequence generated by the above inexact constrained Levenberg-Marquardt method can also be obtained by the more general Algorithm 3.5. At first, we need two lemmas.

Lemma 3.31. *Assume that there is $c_\pi > 0$ such that* (3.72) *is satisfied for all $s \in \Omega$. Then*

$$\psi_\pi(z_\pi^s, s) \leq (1 + c_\pi)\gamma^2 \|F(s)\|^4$$

holds for all pairs $(s, \gamma) \in \Omega \times [1, \infty)$ satisfying that $\mathcal{E}(s, \gamma)$ is nonempty and all solutions z_π^s of (3.67).

Proof. Let $s \in \Omega$ and $\gamma \geq 1$ be arbitrarily chosen such that $\mathcal{E}(s, \gamma)$ is nonempty. Moreover, let z_π^s be an arbitrary but fixed solution of (3.67). If s belongs to Z, then $\psi_\pi(z_\pi^s, s) = 0$ holds due to item (b) of Proposition 3.29. This implies the validity of the assertion.

Now let us assume $s \notin Z$ and let us take any $z \in \mathcal{E}(s, \gamma)$. Then the definition of $\mathcal{E}(s, \gamma)$ yields

$$\psi_0(z, s) \leq \gamma^2 \|F(s)\|^4. \tag{3.74}$$

Moreover, using item (a) of Lemma 3.25, we obtain $z \in \mathcal{F}(s, \gamma)$. The latter particularly implies

$$\|z - s\| \leq \gamma \|F(s)\|. \tag{3.75}$$

Since z is feasible for the optimization problem (3.67), $\psi_\pi(z_\pi^s, s) \leq \psi_\pi(z, s)$ holds. Using this together with (3.74), the Cauchy-Schwarz inequality, (3.72), and (3.75),

$$\begin{aligned}
\psi_\pi(z_\pi^s, s) &\leq \psi_\pi(z, s) \\
&= \psi_0(z, s) + \pi(s)^\top (z - s) \\
&\leq \gamma^2 \|F(s)\|^4 + \|\pi(s)\| \|z - s\| \\
&\leq \gamma^2 \|F(s)\|^4 + c_\pi \|F(s)\|^3 \cdot \gamma \|F(s)\| \\
&\leq (1 + c_\pi)\gamma^2 \|F(s)\|^4
\end{aligned}$$

follows. In the last inequality we have used that $\gamma \leq \gamma^2$ holds due to $\gamma \geq 1$. Thus, the assertion is proved. □

Lemma 3.32. *Assume that there is $c_\pi > 0$ such that (3.72) is satisfied for all $s \in \Omega$. Then there is $\hat{\alpha} > 1$ such that*

$$z_\pi^s \in \mathcal{E}(s, \hat{\alpha}\gamma)$$

holds for all pairs $(s, \gamma) \in \Omega \times [1, \infty)$ satisfying that $\mathcal{E}(s, \gamma)$ is nonempty and all solutions z_π^s of (3.67).

Proof. At first, we show that there is $C > 0$ such that

$$\|z_\pi^s - s\| \le C\gamma \|F(s)\| \tag{3.76}$$

is valid for all pairs $(s, \gamma) \in \Omega \times [1, \infty)$ satisfying that $\mathcal{E}(s, \gamma)$ is nonempty and all solutions z_π^s of (3.67). For every s belonging to Z, item (b) of Proposition 3.29 yields that $z_\pi^s = s$ is the unique solution of (3.67). This implies (3.76) for arbitrary $\gamma \ge 1$ and arbitrary $C > 0$. So it suffices to prove the existence of some $C > 0$ such that (3.76) holds for all pairs $(s, \gamma) \in (\Omega \setminus Z) \times [1, \infty)$ satisfying that $\mathcal{E}(s, \gamma)$ is nonempty and all solutions z_π^s of (3.67).

Let us assume the contrary. Then there are sequences $\{s^k\} \subset \Omega \backslash Z$, $\{z_\pi^{s^k}\} \subset \Omega$ and $\{\gamma_k\} \subset [1, \infty)$ such that, for every $k \in \mathbb{N}$, $z_\pi^{s^k}$ is a solution of (3.67) with $s := s^k$, the set $\mathcal{E}(s^k, \gamma_k)$ is nonempty, and

$$\|z_\pi^{s^k} - s^k\| \ge k\gamma_k \|F(s^k)\| \tag{3.77}$$

holds. Using the definition of ψ_π, Lemma 3.31, the Cauchy-Schwarz inequality, and (3.72),

$$\begin{aligned} \mu(s^k)\|z_\pi^{s^k} - s^k\|^2 &\le \psi_\pi(z_\pi^{s^k}, s^k) - \pi(s^k)^\top (z_\pi^{s^k} - s^k) \\ &\le (1 + c_\pi)\gamma_k^2 \|F(s^k)\|^4 + \|\pi(s^k)\| \|z_\pi^{s^k} - s^k\| \\ &\le (1 + c_\pi)\gamma_k^2 \|F(s^k)\|^4 + c_\pi \|F(s^k)\|^3 \|z_\pi^{s^k} - s^k\| \end{aligned}$$

is satisfied for all $k \in \mathbb{N}$. Dividing this inequality by $\mu(s^k)$ and taking into account $s^k \notin Z$, (3.69), and (3.77), we obtain

$$\begin{aligned} &\|z_\pi^{s^k} - s^k\|^2 \\ &\le (1 + c_\pi)\gamma_k^2 \|F(s^k)\|^2 + c_\pi \|F(s^k)\| \|z_\pi^{s^k} - s^k\| \\ &\le (1 + c_\pi) \cdot \frac{1}{k^2} \|z_\pi^{s^k} - s^k\|^2 + c_\pi \cdot \frac{1}{k\gamma_k} \|z_\pi^{s^k} - s^k\|^2 \end{aligned} \tag{3.78}$$

for all $k \in \mathbb{N}$, $k \ge 1$. Due to $s^k \notin Z$ and (3.77), $z_\pi^{s^k} \ne s^k$ holds for all $k \ge 1$. Therefore, dividing (3.78) by $\|z_\pi^{s^k} - s^k\|^2$ and taking into account $\gamma_k \ge 1$,

$$1 \le (1 + c_\pi) \cdot \frac{1}{k^2} + c_\pi \cdot \frac{1}{k}$$

follows for all $k \geq 1$. However, this inequality cannot hold for sufficiently large $k \geq 1$. Hence, there is some $C > 0$ such that (3.76) is valid for all pairs $(s, \gamma) \in \Omega \times [1, \infty)$ satisfying that $\mathcal{E}(s, \gamma)$ is nonempty and all solutions z_π^s of (3.67).

Now let $s \in \Omega$ and $\gamma \geq 1$ be arbitrarily chosen such that $\mathcal{E}(s, \gamma)$ is nonempty. Moreover, let z_π^s be an arbitrary but fixed solution of (3.67). Using Lemma 3.31, the Cauchy-Schwarz inequality, (3.72), (3.76), and $\gamma \geq 1$, we obtain

$$\begin{aligned} \psi_0(z_\pi^s, s) &= \psi_\pi(z_\pi^s, s) - \pi(s)^\top (z_\pi^s - s) \\ &\leq (1 + c_\pi)\gamma^2 \|F(s)\|^4 + \|\pi(s)\| \|z_\pi^s - s\| \\ &\leq (1 + c_\pi + C c_\pi)\gamma^2 \|F(s)\|^4. \end{aligned}$$

Hence, with $\hat{\alpha} := \sqrt{1 + (1 + C)c_\pi}$, we have $z_\pi^s \in \mathcal{E}(s, \hat{\alpha}\gamma)$. □

Now we are in the position to prove that Algorithm 3.6 is a special realization of Algorithm 3.5 if π satisfies (3.72) for some constant $c_\pi > 0$ and for all $s \in \Omega$.

Proposition 3.33. *Assume that there is $c_\pi > 0$ such that* (3.72) *is satisfied for all $s \in \Omega$. Moreover, let $\{z^k\}$ be an infinite sequence generated by Algorithm 3.6 with starting point $z^0 \in \Omega$. Then $\{z^k\}$ can, together with some sequence $\{\gamma_k\}$, also be generated by Algorithm 3.5, for arbitrary $\gamma_0 \geq 1$, arbitrary $\bar{\alpha} > 1$, and $\hat{\alpha}$ from Lemma 3.32.*

Proof. Let $\gamma_0 \geq 1$ and $\bar{\alpha} > 1$ be arbitrarily chosen and let, for every $k \in \mathbb{N}$, γ_{k+1} be defined according the updating rule in step (S.2) of Algorithm 3.5, i.e., $\gamma_{k+1} := \gamma_k$ if $\mathcal{E}(z^k, \gamma_k)$ is nonempty, and $\gamma_{k+1} := \bar{\alpha}\gamma_0(z^k)$ otherwise. In each case, $\mathcal{E}(z^k, \gamma_{k+1}) \neq \emptyset$ is satisfied. Moreover, $\gamma_{k+1} \geq 1$ holds for all $k \in \mathbb{N}$ since $\gamma_0 \geq 1$ is valid and the sequence $\{\gamma_k\}$ is monotonically nondecreasing. Therefore, Lemma 3.32, together with the rule in step (S.2) of Algorithm 3.6, yields

$$z^{k+1} \in \mathcal{E}(z^k, \hat{\alpha}\gamma_{k+1})$$

for all $k \in \mathbb{N}$. Hence, it is shown that $\{z^k\}$ can be obtained by Algorithm 3.5. □

Theorem 3.28 and the preceding proposition yield the following result on local convergence properties of the inexact constrained Levenberg-Marquardt method described by the subproblems (3.67) with μ defined according to (3.69) and π satisfying (3.72) for some constant $c_\pi > 0$ and for all $s \in \Omega$.

Theorem 3.34. *Let Assumptions 1–4 be satisfied. Moreover, suppose that there is $c_\pi > 0$ such that* (3.72) *is satisfied for all $s \in \Omega$. Then there is $\rho > 0$ such that any infinite sequence $\{z^k\}$ generated by Algorithm 3.5 with starting point $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ converges Q-quadratically to a solution of* (1.1).

If F is differentiable and has a locally Lipschitz continuous derivative, it is known that the inexact constrained Levenberg-Marquardt method described by the subproblems (3.67) converges locally fast for more robust choices of μ and π. More precisely, local quadratic convergence is proved in [1, Theorem 1] if $\mu(s)$ equals $\|F(s)\|$ for all $s \notin Z$ and if there is some $c_\pi > 0$ such that $\|\pi(s)\| \leq c_\pi \|F(s)\|^2$ holds for all s near z^*. It is an open question if more robust choices of μ and π are also possible in our setting, where F is in general not differentiable everywhere, without loosing the local fast convergence.

Chapter 4

Application to PC^1-systems

In the preceding chapter a general Newton-type algorithm for the solution of the constrained system (1.1) was described and local quadratic convergence was proved under the validity of four assumptions, see Theorem 3.3. Moreover, in Sections 3.4 and 3.5 special realizations of Algorithm 3.1 were presented and it was shown that the local quadratic convergence is kept if Assumptions 1–4 are satisfied. Section 3.2 provided a first discussion of our convergence assumptions by relating them to existing regularity conditions from the literature. However, the sufficient conditions given there still imply the local uniqueness of a solution or differentiability of F near a solution, at least in the interior of the feasible set Ω. So up to now, the advantages of our general framework from Section 3.1 and the convergence assumptions which we used might not be clear yet.

This chapter is devoted to an in-depth discussion of Assumptions 1–4 for the case that F is a PC^1-function. The main results are Theorems 4.13 and 4.19 providing conditions which imply the whole set of Assumptions 1–4. These sufficient conditions are still mild but seem to be more familiar, at least compared to the rather technical Assumptions 3 and 4.

First, let us recall the notion of a PC^1-function. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called *piecewise continuously differentiable* (PC^1) if F is continuous and there is a finite number of continuously differentiable functions $F^1, \dots, F^t : \mathbb{R}^n \to \mathbb{R}^m$ such that $F(z) \in \{F^1(z), \dots, F^t(z)\}$ holds for all $z \in \mathbb{R}^n$. In that case the functions $F^1, \dots, F^t$ are called *selection functions*. If all selection functions are affine, then F is called *piecewise affine*. For any point $z \in \mathbb{R}^n$ we say that a selection function F^i is *active* at z if $F(z) = F^i(z)$ holds. The index set of all selection functions being active at z is denoted by $\mathcal{A}(z)$, i.e.,

$$\mathcal{A}(z) := \{i \in \{1, \dots, t\} \mid F(z) = F^i(z)\}.$$

Throughout this chapter, the constrained system (1.1) is considered again, and $F : \mathbb{R}^n \to \mathbb{R}^m$ is assumed to be a PC^1-function. We further suppose that $G : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ satisfies

$$G(z) \in \{(F^i)'(z) \mid i \in \mathcal{A}(z)\} \tag{4.1}$$

for all $z \in \mathbb{R}^n$. For a better understanding of the above definitions, let us present an example for a PC^1-function.

Example 4.1. Let $P : \mathbb{R}^n \to \mathbb{R}^p$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$ be given continuously differentiable functions. Then $F : \mathbb{R}^n \to \mathbb{R}^{p+r}$ defined according to

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R(z), S(z)\} \end{pmatrix}, \tag{4.2}$$

where the minimum has to be taken componentwise, is a PC^1-function with $t = 2^r$ selection functions $F^1, \ldots, F^{2^r} : \mathbb{R}^n \to \mathbb{R}^{p+r}$. In this example the set $\mathcal{A}(z)$ contains $2^{|\mathcal{I}_=(z)|}$ elements for any $z \in \mathbb{R}^n$ where $\mathcal{I}_=(z)$ is defined by

$$\mathcal{I}_=(z) := \{j \in \{1, \ldots, r\} \mid R_j(z) = S_j(z)\}.$$

One possible choice of the function $G : \mathbb{R}^n \to \mathbb{R}^{(p+r)\times n}$ is the following:

$$\begin{array}{lcll} (G(z))_{k,\cdot} & := & P_k'(z) & (k = 1, \ldots, p), \\ (G(z))_{p+j,\cdot} & := & \begin{cases} R_j'(z) & \text{if } R_j(z) \le S_j(z), \\ S_j'(z) & \text{if } R_j(z) > S_j(z) \end{cases} & (j = 1, \ldots, r). \end{array}$$

With this definition it is guaranteed that (4.1) holds for all $z \in \mathbb{R}^n$. □

PC^1-functions which have a similar structure like F in Example 4.1 will play an important role in the sequel. There are many problems which can be equivalently reformulated as the problem of finding a zero of such a function. Examples are nonlinear and linear complementarity problems, KKT systems, and, more generally, complementarity systems. Therefore, our algorithms from Chapter 3 can be used to find solutions of such problems. Complementarity systems are considered in Section 4.3. KKT systems are the matter of Chapter 5 where Section 5.1 is devoted to KKT systems arising from optimization problems or variational inequalities, and Section 5.2 deals with KKT systems arising from GNEPs.

It will turn out that the whole set of Assumptions 1–4 cannot be expected to hold for the unconstrained system $F(z) = 0$, at least if classical (and possibly strong) regularity conditions like strict complementarity or the nonsingularity of certain Jacobians are violated. However, if a suitable set Ω is introduced such that all zeros of F still belong to Ω, there is a chance that the whole set of Assumptions 1–4 holds for the resulting constrained system (1.1) without requiring strong regularity conditions. Thus, the introduction of some set Ω and the consideration of a constrained system instead of the unconstrained one might have the advantage that our algorithms from Chapter 3 converge locally fast under much milder regularity conditions, even if the solution set of (1.1) coincides with the set of all zeros of F. Therefore, a further aim of this chapter is to provide suitable choices of Ω.

Besides the blanket assumptions from above, we suppose for the rest of this chapter that the selection functions $F^1, \dots, F^t$ have locally Lipschitz continuous derivatives. Moreover, as in the preceding chapter, $\Omega \subseteq \mathbb{R}^n$ is assumed to be nonempty and closed, Z indicates the solution set of (1.1), and by $z^* \in Z$ an arbitrary but fixed solution is denoted. For every $i \in \{1, \dots, t\}$, the set of all zeros of the selection function F^i intersected with Ω is denoted by Z_i, i.e.,

$$Z_i := \{z \in \Omega \mid F^i(z) = 0\}.$$

Note that at least for $i \in \mathcal{A}(z^*)$ the set Z_i is nonempty since z^* belongs to it.

The rest of this chapter is organized as follows. In Section 4.1 sufficient conditions for our convergence assumptions from Section 3.1 to hold are presented. In particular, it will turn out that the whole set of Assumptions 1–4 is implied by the validity of some set of local error bound conditions if in addition some condition on Ω is satisfied, see Theorems 4.9 and 4.13. Besides, we present suitable choices of the feasible set Ω for the case that a zero of a function is looked for which has a similar structure like F in Example 4.1, see Examples 4.7 and 4.8. Section 4.2 deals with the reformulation of (1.1) by means of slack variables and a discussion of Assumptions 1–4 for the modified problem. The main result of Section 4.2 is Theorem 4.19. It provides conditions which are slightly stronger than those from Section 4.1 but which are sufficient for Assumptions 1–4 to hold, not only for the original problem (1.1) but also for its reformulation with slack variables. Note that the introduction of slack variables is advisable from the computational point of view if Ω is defined by nonlinear inequalities because in that case the subproblems of the LP-Newton as well as of the constrained Levenberg-Marquardt method would be difficult to solve. In Section 4.3 complementarity systems are considered. We present a reformulation as a constrained system of equations by means of the minimum function. The resulting function F is a PC^1-function so that the results from Sections 4.1 and 4.2 can be applied. The main result of Section 4.3 is Theorem 4.20 which provides conditions implying local quadratic convergence of our Newton-type algorithm from Section 3.1 and its special realizations from Sections 3.4 and 3.5 to a solution of the complementarity system.

The results of this chapter will in large part be published together with Andreas Fischer, Alexey Izmailov, and Mikhail Solodov in the technical report [37]. In addition, the constant rank condition for KKT systems arising from GNEPs which will appear in [37] is extended to the case of general complementarity systems in Section 4.3 below.

4.1 Discussion of the Convergence Assumptions

In this section we present sufficient conditions for Assumptions 1–4 from Section 3.1 to hold for the case that $F : \mathbb{R}^n \to \mathbb{R}^m$ is a PC^1-function, the derivatives of the selection functions are locally Lipschitz continuous, and $G : \mathbb{R}^n \to \mathbb{R}^{m \times n}$

satisfies (4.1) for all $z \in \mathbb{R}^n$. At first, we will verify that Assumption 1 is always valid in this context since PC^1-functions are locally Lipschitz continuous. The latter fact is well known and will be recalled in Subsection 4.1.1. Assumptions 2 and 3 are discussed in Subsection 4.1.2. It is shown that Assumption 2 holds if and only if those selection functions which are active at z^* provide, for certain points $s \in \Omega$, local error bounds for the distance to the solution set of (1.1) near z^*. Moreover, if these selection functions provide local error bounds to their own set of zeros intersected with Ω, then both Assumption 3 and Assumption 2 are satisfied, the latter if in addition a condition on Ω holds which will be introduced as Ω-property. In Subsection 4.1.3 a condition is introduced which turns out to be sufficient for Assumption 4 to hold and which requires that, in a neighborhood of z^*, the norm of F does not increase faster than the norm of those selection functions being active at z^*. Subsection 4.1.4 summarizes the most important results until then and provides conditions which imply the whole set of Assumptions 1–4.

4.1.1 Assumption 1

Since F is a PC^1-function, Assumption 1 is always satisfied in our setting. In fact, any PC^1-function is locally Lipschitz continuous. This follows from [41, Theorem 2.1]. Nevertheless, we want to present a proof here which is a bit more adapted to our notation. However, the proof technique and the main idea are essentially the same as in [41, Theorem 2.1].

Proposition 4.1. *The PC^1-function F is locally Lipschitz continuous. In particular, Assumption 1 is satisfied.*

Proof. It suffices to prove the local Lipschitz continuity of F. Then Assumption 1 holds, see the discussion after the introduction of Assumption 1 in Section 3.1.

Let us take any nonempty and compact set $X \subset \mathbb{R}^n$. Without loss of generality, we assume that X is convex (if necessary, we consider the convex hull of X and prove the Lipschitz continuity of F on $\mathrm{conv}(X)$ which obviously implies the Lipschitz continuity on X). By the local Lipschitz continuity of the selection functions, there is, for every $i \in \{1, \dots, t\}$, some $L_i > 0$ such that

$$\|F^i(z) - F^i(s)\| \leq L_i \|z - s\| \tag{4.3}$$

holds for all $s, z \in X$.

Let $s, z \in X$ with $s \neq z$ be arbitrarily chosen. We denote by W the line with starting point s and end point z, i.e.,

$$W := \{w = s + \lambda(z - s) \mid 0 \leq \lambda \leq 1\}.$$

Since X was supposed to be convex, W is a subset of X. Let $\mathcal{A}(W)$ be defined according to

$$\mathcal{A}(W) := \{i \in \{1, \dots, t\} \mid \exists w \in W : i \in \mathcal{A}(w)\}.$$

For every $i \in \mathcal{A}(W)$ we indicate by $\lambda(i)$ the unique solution of the maximization problem

$$\lambda \to \max_{\lambda} \quad \text{s.t.} \quad F(s+\lambda(z-s)) = F^i(s+\lambda(z-s)),\ 0 \leq \lambda \leq 1. \tag{4.4}$$

The corresponding element $w(i) := s + \lambda(i)(z-s)$ of W is that point which has, among all elements of W where F^i is active, the smallest distance to z. Let us briefly justify that (4.4) actually has a unique solution. Obviously, the objective function of (4.4) is continuous. Since i belongs to $\mathcal{A}(W)$, there is $\lambda \in [0,1]$ such that $F(s+\lambda(z-s)) = F^i(s+\lambda(z-s))$ is valid. Therefore, the feasible set of the maximization problem (4.4) is nonempty. Moreover, it is compact. In fact, the boundedness of the feasible set is obvious, and the closedness follows from the continuity of F and F^i. The theorem of Weierstrass yields the solvability of (4.4). Since the objective function is strictly increasing, the solution is unique.

Now let us define finite sequences $\{\lambda_k\}_{k=0}^l \subset [0,1]$ and $\{w^k\}_{k=0}^l \subset W$ recursively as follows. We set $\lambda_0 := 0$ and $w^0 := s$. Now assume that $\lambda_0, \ldots, \lambda_{k-1}$ as well as $w^0, \ldots, w^{k-1}$ are generated. Then we define

$$\lambda_k := \max\{\lambda(i) \mid i \in \mathcal{A}(w^{k-1})\} \quad \text{and} \quad w^k := s + \lambda_k(z-s).$$

If $\lambda_k = 1$ (and therefore $w^k = z$) holds, then we set $l := k$ and stop the recursion. Otherwise, we continue with $k+1$ instead of k.

Note that it is guaranteed that the recursion stops after a finite number of steps due to the continuity of F and the finiteness of $\mathcal{A}(W)$. Moreover,

$$0 = \lambda_0 < \lambda_1 < \ldots < \lambda_l = 1$$

is valid. This, together with the definition of $w^0, \ldots, w^l$, yields

$$\begin{aligned} \sum_{k=1}^{l} \|w^k - w^{k-1}\| &= \sum_{k=1}^{l} \|(\lambda_k - \lambda_{k-1})(z-s)\| \\ &= \sum_{k=1}^{l} (\lambda_k - \lambda_{k-1})\|z-s\| \\ &= (\lambda_l - \lambda_0)\|z-s\| \\ &= \|z-s\|. \end{aligned} \tag{4.5}$$

For every $k = 1, \ldots, l$ let $i_k \in \mathcal{A}(w)$ denote an index such that $\lambda_k = \lambda(i_k)$ holds. Then for every $k = 1, \ldots, l$ the index i_k belongs to both $\mathcal{A}(w^{k-1})$ and $\mathcal{A}(w^k)$. In particular, $F(s) = F^{i_1}(s) = F^{i_1}(w^0)$ and $F(z) = F^{i_l}(z) = F^{i_l}(w^l)$ are valid. Moreover, $F^{i_k}(w^k) = F^{i_{k-1}}(w^k)$ is satisfied for all $k = 2, \ldots, l$. Using this, the triangle inequality, (4.3), and (4.5), we obtain

$$\begin{aligned}
\|F(z) - F(s)\| &= \|F^{i_l}(w^l) - F^{i_1}(w^0)\| \\
&\leq \sum_{k=1}^{l} \|F^{i_k}(w^k) - F^{i_k}(w^{k-1})\| \\
&\leq \sum_{k=1}^{l} L_{i_k} \|w^k - w^{k-1}\| \\
&\leq \max\{L_i \mid i \in \{1, \dots, t\}\} \|z - s\|.
\end{aligned}$$

Hence, F is Lipschitz continuous on X with the Lipschitz constant $L_0 := \max\{L_i \mid i \in \{1, \dots, t\}\}$. Since X was arbitrarily chosen, F is locally Lipschitz continuous on $\mathbb{R}^n$. □

4.1.2 Assumptions 2 and 3

In this subsection sufficient conditions for the validity of Assumptions 2 and 3 are provided. The main result is Proposition 4.4 where it is shown that both assumptions are satisfied if some set of local error bound conditions, together with some condition on Ω, is valid.

First, we introduce a condition which turns out to be both sufficient and necessary for Assumption 2 to hold. To this end, let us denote, for every $i \in \{1, \dots, t\}$, by Ω_i the set of all points belonging to Ω where the selection function F^i is active, i.e.,

$$\Omega_i := \{z \in \Omega \mid F(z) = F^i(z)\}.$$

Condition 3. There are $K_3 > 0$ and $\varepsilon_3 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\operatorname{dist}[s, Z] \leq K_3 \|F^i(s)\|$$

holds for all $s \in \mathcal{B}_{\varepsilon_3}(z^*) \cap \Omega_i$.

The above condition requires that every selection function F^i which is active at z^* provides a local error bound for the distance to the solution set of problem (1.1) near z^* on Ω_i. For the case that (1.1) arises from a reformulation of a complementarity system by means of the minimum function, a condition which is similar to Condition 3 was already considered in [49, Lemma 1] and the sufficiency for Assumption 2 was proved.

The equivalence of Assumption 2 and Condition 3 is intuitively clear because, for every i, $F^i(s) = F(s)$ is valid for all $s \in \Omega_i$, and $\mathcal{A}(s) \subseteq \mathcal{A}(z^*)$ holds for all s in a certain neighborhood of z^*. The latter follows from the continuity of F. Nevertheless, we present a detailed proof.

Proposition 4.2. *Assumption 2 is satisfied if and only if Condition 3 holds.*

Proof. Suppose that Assumption 2 is valid. Let $i \in \mathcal{A}(z^*)$ be arbitrarily chosen and let us take any $s \in \mathcal{B}_{\delta_2}(z^*) \cap \Omega_i$. Then $F(s) = F^i(s)$ holds and Assumption 2 yields

$$\operatorname{dist}[s, Z] \leq \omega \|F(s)\| = \omega \|F^i(s)\|.$$

Therefore, Condition 3 is satisfied with $\varepsilon_3 := \delta_2$ and $K_3 := \omega$.

Conversely, let us assume that Condition 3 holds. Let $\delta_2 \in (0, \varepsilon_3]$ be small enough such that $\mathcal{A}(s) \subseteq \mathcal{A}(z^*)$ is satisfied for all $s \in \mathcal{B}_{\delta_2}(z^*)$. Note that there is some δ_2 with this property since F is continuous. Now let us choose any $s \in \mathcal{B}_{\delta_2}(z^*) \cap \Omega$ and let i be an element of $\mathcal{A}(s)$. Then we have $F(s) = F^i(s)$ and $s \in \Omega_i$ so that Condition 3 implies

$$\operatorname{dist}[s, Z] \leq K_3 \|F^i(s)\| = K_3 \|F(s)\|.$$

Hence, Assumption 2 is valid with $\omega := K_3$. □

Note that the equivalence of Assumption 2 and Condition 3 stays true if the selection functions are not necessarily differentiable but only continuous. In fact, the differentiability was not needed in the above proof.

Condition 3 does in general not imply Assumption 3, even if F has the structure as in Example 4.1. In order to realize this, let us consider the subsequent example which particularly provides an instance where Assumption 2 is satisfied but Assumption 3 is not.

Example 4.2. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined according to

$$F(z) := F(x, y) := \min\{x^2 - y, -2y\}.$$

Obviously, F is a PC^1-function with the selection functions F^1 and F^2 given by

$$F^1(z) := F^1(x, y) := x^2 - y \quad \text{and} \quad F^2(z) := F^2(x, y) := -2y.$$

We define the set Ω such that those points are excluded for which one of the selection functions is negative, i.e.,

$$\Omega := \{(x, y)^\top \in \mathbb{R}^2 \mid x^2 - y \geq 0,\ -2y \geq 0\} = \{(x, y)^\top \in \mathbb{R}^2 \mid y \leq 0\}.$$

The solution set of the constrained system (1.1) is given by

$$Z = \{(x, y)^\top \in \Omega \mid \min\{x^2 - y, -2y\} = 0\} = \{(x, y)^\top \in \mathbb{R}^2 \mid y = 0\}.$$

Obviously, Z coincides with the set of all zeros of F, i.e., with the solution set of the unconstrained system $F(z) = 0$. However, the introduction of the set Ω is appropriate for our purpose if F is the minimum of two functions. This will become clear later on in this section, see the discussion after the introduction of the Ω-property.

In particular, $z^* := (0,0)^\top$ is a solution of (1.1) and both selection functions are active at z^*, i.e., $\mathcal{A}(z^*) = \{1,2\}$ holds. Obviously, for every $s = (x,y)^\top \in \mathbb{R}^2$, the distance to Z equals $|y|$. With $y \le 0$ for all $s = (x,y)^\top \in \Omega$, we obtain

$$\operatorname{dist}[s, Z] = |y| = -y \le x^2 - y = |F^1(s)| \tag{4.6}$$

and

$$\operatorname{dist}[s, Z] = |y| = -y \le -2y = |F^2(s)| \tag{4.7}$$

for all $s \in \Omega$. In particular, (4.6) holds for all $s \in \Omega_1$, and (4.7) is valid for all $s \in \Omega_2$. Therefore, Condition 3 is satisfied at z^* with arbitrary $\varepsilon_3 > 0$ and $K_3 := 1$.

However, Assumption 3 does not hold at z^*. In order to verify this, let $G : \mathbb{R}^2 \to \mathbb{R}^{1\times 2}$ be any function satisfying (4.1) for all $z \in \mathbb{R}^2$ and let us consider the sequence $\{s^k\}_{k\in\mathbb{N},k\ge 1} \subset \Omega$ defined by $s^k := (\frac{1}{k}, -\frac{2}{k^2})^\top$. Obviously, this sequence converges to z^*. For every $k \ge 1$,

$$F^1(s^k) = \frac{3}{k^2} < \frac{4}{k^2} = F^2(s^k)$$

holds. Therefore, $\mathcal{A}(s^k) = \{1\}$ follows for all $k \ge 1$ so that we have

$$F(s^k) = F^1(s^k) = \frac{3}{k^2} \quad \text{and} \quad G(s^k) = (F^1)'(s^k) = \begin{pmatrix} \frac{2}{k} & -1 \end{pmatrix}.$$

Suppose that Assumption 3 is satisfied at z^* with some $\Gamma > 0$ and some $\delta_3 > 0$. Then $k_0 \ge 1$ exists such that, for every $k \ge k_0$, there is $z^k = (x_k, y_k)^\top \in \mathcal{F}(s^k, \Gamma)$. By $\|z^k - s^k\| \le \Gamma|F(s^k)| = \frac{3}{k^2}\Gamma$, we obtain

$$\left| x_k - \frac{1}{k} \right| \le \frac{3}{k^2}\Gamma$$

for all $k \ge k_0$. Using this, $|F(s^k) + G(s^k)(z^k - s^k)| \le \Gamma|F(s^k)|^2 = \frac{9}{k^4}\Gamma$, and $y_k \le 0$, which is implied by $z^k \in \Omega$,

$$\begin{aligned}
\frac{9}{k^4} &\ge \left| \frac{3}{k^2} + \frac{2}{k}\left(x_k - \frac{1}{k}\right) - \left(y_k + \frac{2}{k^2}\right) \right| \\
&= \left| -y_k + \frac{1}{k^2} + \frac{2}{k}\left(x_k - \frac{1}{k}\right) \right| \\
&\ge \left| -y_k + \frac{1}{k^2} \right| - \frac{2}{k}\left| x_k - \frac{1}{k} \right| \\
&\ge \frac{1}{k^2} - \frac{6}{k^3}\Gamma
\end{aligned}$$

follows for all $k \ge k_0$. However, this inequality cannot hold for all sufficiently large k. This is a contradiction, i.e., Assumption 3 is not valid at z^*. □

Next, we introduce a further condition which turns out to be sufficient for Assumption 3 to hold. Condition 4 below requires that every selection function F^i being active at z^* provides a local error bound for the distance to Z_i near z^* on Ω_i. So unlike Condition 3, the distance to Z_i, i.e., to the set of all zeros of F^i intersected with Ω is considered instead of the distance to Z.

Condition 4. There are $K_4 > 0$ and $\varepsilon_4 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\operatorname{dist}[s, Z_i] \leq K_4 \|F^i(s)\| \tag{4.8}$$

holds for all $s \in \mathcal{B}_{\varepsilon_4}(z^*) \cap \Omega_i$.

Note that Condition 4 is slightly weaker than [18, Condition 4]. The kind of weakening is the following: for every index from $\mathcal{A}(z^*)$ the validity of (4.8) is only required for points where the particular selection function is active, whereas [18, Condition 4] requires that (4.8) holds for all $s \in \Omega$ in a sufficiently small neighborhood of z^*. Later on, we will prove that Assumption 3 is implied by Condition 4. If in addition the following condition, which we call Ω-property, is satisfied, then Assumption 2 also follows from Condition 4.

Ω-property. There is $\varepsilon_\Omega > 0$ such that

$$Z_i \cap \mathcal{B}_{\varepsilon_\Omega}(z^*) \subseteq Z$$

holds for all $i \in \mathcal{A}(z^*)$.

The Ω-property requires that the set Ω excludes all zeros of those selection functions being active at z^* which are not zeros of F, at least in a certain neighborhood of z^*. In other words, it is required that, for each $i \in \mathcal{A}(z^*)$, every zero of F^i which is sufficiently close to z^* and which belongs to Ω is also a zero of F and therefore a solution of (1.1). In the case that F is given by (4.2), the Ω-property is satisfied if it excludes those zeros $\bar{z}$ of the selection functions for which some components of $R(\bar{z})$ or $S(\bar{z})$ are negative. This is in particular guaranteed if Ω is defined according to

$$\Omega := \{z \in \mathbb{R}^n \mid R(z) \geq 0,\ S(z) \geq 0\}. \tag{4.9}$$

Conversely, no zero of F itself is excluded by the latter definition of Ω, i.e., the solution set of the constrained system (1.1) with F given by (4.2) equals the set of all zeros of F. In the sequel the Ω-property will play a crucial role in the discussion of our convergence assumptions.

Now we are in the position to state the next proposition which says that Condition 4, together with the Ω-property, implies Condition 3. Afterwards, a relation between Condition 4 and Assumptions 2 and 3 is shown. The following two propositions will be part of [37].

Proposition 4.3. *Let Condition 4 be valid. Moreover, assume that the Ω-property holds. Then Condition 3 is satisfied.*

Proof. We set $\varepsilon_3 := \min\{\varepsilon_4, \frac{1}{2}\varepsilon_\Omega\}$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let $s \in \mathcal{B}_{\varepsilon_3}(z^*) \cap \Omega_i$ be arbitrarily chosen. The set Z_i is nonempty and closed because z^* belongs to Z_i, Ω is closed, and F^i is continuous. Thus, there is some $\bar{s} \in Z_i$ such that

$$\operatorname{dist}[s, Z_i] = \|s - \bar{s}\| \tag{4.10}$$

holds. Taking into account

$$\|\bar{s} - z^*\| \leq \|\bar{s} - s\| + \|s - z^*\| \leq 2\varepsilon_3 \leq \varepsilon_\Omega,$$

$\bar{s}$ belongs to Z due to the Ω-property. Using this, (4.10), and Condition 4,

$$\operatorname{dist}[s, Z] \leq \|s - \bar{s}\| = \operatorname{dist}[s, Z_i] \leq K_4\|F^i(s)\|$$

follows. Hence, Condition 3 is satisfied with $K_3 := K_4$. □

Proposition 4.4. *Let Condition 4 be satisfied. Then the following assertions are valid.*

(a) Assumption 2 holds if the Ω-property is satisfied.

(b) Assumption 3 holds.

Proof. (a) By Proposition 4.3, Condition 3 is satisfied. Thus, Assumption 2 follows from Proposition 4.2.

(b) Let $\delta_3 \in (0, \varepsilon_4]$ be small enough such that $\mathcal{A}(s) \subseteq \mathcal{A}(z^*)$ holds for all $s \in \mathcal{B}_{\delta_3}(z^*)$. Note that there is some δ_3 with this property due to the continuity of F. Since the derivatives of the selection functions are locally Lipschitz continuous, there is $C > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\|F^i(s) + (F^i)'(s)(z - s) - F^i(z)\| \leq C\|z - s\|^2 \tag{4.11}$$

is satisfied for all $s, z \in \mathcal{B}_{2\delta_3}(z^*)$.

Now let us take any $s \in \mathcal{B}_{\delta_3}(z^*) \cap \Omega$ and let i be an index belonging to $\mathcal{A}(s)$ such that $G(s) = (F^i)'(s)$ holds. Let $\bar{s}$ be an element of Z_i such that

$$\operatorname{dist}[s, Z_i] = \|s - \bar{s}\| \tag{4.12}$$

is valid. Note that there is some $\bar{s}$ with this property since Z_i is nonempty because z^* belongs to it, and Z_i is closed due to the continuity of F^i and the closedness of Ω. Using (4.12), Condition 4, and $s \in \Omega_i$, we obtain

$$\|s - \bar{s}\| = \operatorname{dist}[s, Z_i] \leq K_4\|F^i(s)\| = K_4\|F(s)\|. \tag{4.13}$$

With
$$\|\bar{s} - z^*\| \leq \|s - \bar{s}\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\delta_3,$$
(4.11) implies
$$\|F(s) + G(s)(\bar{s} - s)\| = \|F^i(s) + (F^i)'(s)(\bar{s} - s) - F^i(\bar{s})\| \leq C\|s - \bar{s}\|^2.$$
Using this and (4.13),
$$\|F(s) + G(s)(\bar{s} - s)\| \leq CK_4^2\|F(s)\|^2 \tag{4.14}$$
follows. By (4.13), (4.14), and $\bar{s} \in Z_i \subseteq \Omega$, we obtain $\bar{s} \in \mathcal{F}(s, \Gamma)$ with $\Gamma := \max\{K_4, CK_4^2\}$. Hence, Assumption 3 is satisfied.

□

In [18, Corollary 3] it was already proved that the slightly stronger version of our Condition 4 (where for each $i \in \mathcal{A}(z^*)$ the inequality is required to hold in a neighborhood of z^* intersected with Ω instead of Ω_i) implies Assumption 3. However, our proof from above seems simpler. Furthermore, the relation to Assumption 2 was not analyzed in [18].

In the following example, Condition 4 is satisfied whereas Assumption 2 does not hold. This shows that the Ω-property cannot be omitted as assumption in item (a) of Proposition 4.4. Particularly, Example 4.3 provides an instance where Assumption 3 is valid but Assumption 2 is not.

Example 4.3. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$F(z) := F(x, y) := \min\{y - x^2, -y\}$$
and suppose that $\Omega = \mathbb{R}^2$ holds. Thus, problem (1.1) becomes an unconstrained system of equations and the solution set Z is equal to the set of all zeros of F. Let us compute Z. We have $F(x, y) \leq -y < 0$ for all $(x, y)^\top \in \mathbb{R}^2$ with $y > 0$, and $F(x, y) \leq y - x^2 < 0$ for all $(x, y)^\top \in \mathbb{R}^2$ with $y < 0$. Thus, the y-part of any element of Z must be equal to zero. Moreover, $F(x, 0) = -x^2 < 0$ is valid for all $x \neq 0$. Therefore, we obtain $Z = \{(0, 0)^\top\}$, i.e., Z is a singleton.

Let us verify that Assumption 2 does not hold. To this end, let the sequence $\{s^k\}_{k\in\mathbb{N}, k\geq 1} \subset \mathbb{R}^2$ be defined by $s^k := (\frac{1}{k}, 0)^\top$. Obviously, this sequence converges to $z^* := (0, 0)^\top$. Furthermore, we have
$$F(s^k) = -\frac{1}{k^2} \quad \text{and} \quad \mathrm{dist}[s^k, Z] = \|s^k\| = \frac{1}{k}$$
for all $k \geq 1$. Hence,
$$\frac{\mathrm{dist}[s^k, Z]}{|F(s^k)|} = k \overset{k\to\infty}{\longrightarrow} \infty$$
follows so that Assumption 2 cannot be valid.

However, Condition 4 is satisfied. Note that F is a PC^1-function where the selection functions are given by

$$F^1(x,y) := y - x^2 \quad \text{and} \quad F^2(x,y) := -y.$$

Since Ω equals $\mathbb{R}^2$, the sets Z_1 and Z_2 coincide with the sets of all zeros of F^1 and F^2, respectively, i.e., we have

$$Z_1 = \{(x,y)^\top \in \mathbb{R}^2 \mid y = x^2\} \quad \text{and} \quad Z_2 = \{(x,y)^\top \in \mathbb{R}^2 \mid y = 0\}.$$

Obviously, the Ω-property is not satisfied near z^* since in any neighborhood of z^* there are zeros of the selection functions which are not zeros of F. In order to prove Condition 4, let us take any $s = (x,y)^\top \in \mathbb{R}^2$. Then the distance of s to Z_1 can be estimated according to

$$\text{dist}[s, Z_1] \le \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x \\ x^2 \end{pmatrix} \right\| = |y - x^2| = |F^1(s)|,$$

and the distance of s to Z_2 equals

$$\text{dist}[s, Z_2] = |y| = |F^2(s)|.$$

Hence, Condition 4 is satisfied with arbitrary $\varepsilon_4 > 0$ and $K_4 := 1$. By Proposition 4.4, Assumption 3 is valid as well. □

As an application of Proposition 4.4, we are going to extend a result from [18]. In Proposition 4.5 below it is assumed that z is split according to $z = (x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ with $n_x + n_y = n$. The quintessence of Proposition 4.5 is that Assumption 3 is satisfied if Assumption 2 holds, the x-part of the fixed solution $z^* = (x^*, y^*)$ is locally unique, $F(x^*, \cdot)$ is piecewise affine, and $\Omega_y(x^*)$ defined according to

$$\Omega_y(x^*) := \{y \in \mathbb{R}^{n_y} \mid (x^*, y) \in \Omega\} \tag{4.15}$$

is a polyhedral set.

Proposition 4.5. *Let Assumption 2 be satisfied. Moreover, let z be split according to $z = (x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that $\Omega_y(x^*)$ defined by (4.15) is polyhedral and that, for every $i \in \mathcal{A}(z^*)$, the function $F^i(x^*, \cdot)$ is affine. Furthermore, suppose that there is $\varepsilon > 0$ such that $z = (x,y) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$. Then Condition 4 is valid. In particular, Assumption 3 is satisfied.*

Proof. We show that Condition 4 holds. Then Assumption 3 follows from Proposition 4.4. Let us set $\varepsilon_4 := \min\{\delta_2, \frac{1}{2}\varepsilon\}$. By the local Lipschitz continuity of the selection functions, there is $L_0 > 0$ such that

$$\|F^i(z) - F^i(s)\| \le L_0 \|z - s\| \tag{4.16}$$

is valid for all $i \in \mathcal{A}(z^*)$ and all $s, z \in \mathcal{B}_{\varepsilon_4}(z^*)$. Since $\Omega_y(x^*)$ is a polyhedral set, there are a number $l \in \mathbb{N}$ and an affine function $a(x^*, \cdot) : \mathbb{R}^{n_y} \to \mathbb{R}^l$ such that

$$\Omega_y(x^*) = \{y \in \mathbb{R}^{n_y} \mid a(x^*, y) \geq 0\}$$

holds.

Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed. We define the set $Z_i(x^*)$ by

$$\begin{aligned} Z_i(x^*) &:= \{y \in \Omega_y(x^*) \mid F^i(x^*, y) = 0\} \\ &= \{y \in \mathbb{R}^{n_y} \mid a(x^*, y) \geq 0,\ F^i(x^*, y) = 0\}. \end{aligned}$$

Note that $Z_i(x^*)$ is nonempty because y^* belongs to it. Since $a(x^*, \cdot)$ and $F^i(x^*, \cdot)$ are affine, $Z_i(x^*)$ is a polyhedral set. Thus, Theorem 2.1 yields the existence of some $\omega_H^{(i)} = \omega_H^{(i)}(x^*) > 0$ such that

$$\operatorname{dist}[y, Z_i(x^*)] \leq \omega_H^{(i)}(\|F^i(x^*, y)\| + \|\min\{0, a(x^*, y)\}\|)$$

holds for all $y \in \mathbb{R}^{n_y}$. Particularly, we have

$$\operatorname{dist}[y, Z_i(x^*)] \leq \omega_H^{(i)}\|F^i(x^*, y)\| \tag{4.17}$$

for all $y \in \Omega_y(x^*)$.

Now let us take any $s = (x, y) \in \mathcal{B}_{\varepsilon_4}(z^*) \cap \Omega_i$. Let $\bar{y} \in Z_i(x^*)$ be a point with the property

$$\operatorname{dist}[y, Z_i(x^*)] = \|y - \bar{y}\|. \tag{4.18}$$

Then $F^i(x^*, \bar{y}) = 0$ holds. Moreover, due to $\bar{y} \in \Omega_y(x^*)$, $(x^*, \bar{y})$ belongs to Ω. Therefore, we have $(x^*, \bar{y}) \in Z_i$. Using this, (4.18), and (4.17), we obtain

$$\operatorname{dist}[s, Z_i] \leq \|x - x^*\| + \|y - \bar{y}\| \leq \|x - x^*\| + \omega_H^{(i)}\|F^i(x^*, y)\|. \tag{4.19}$$

We want to estimate the right hand side of this inequality from above. Let us suppose that $s^\perp = (x^\perp, y^\perp) \in Z$ satisfies

$$\operatorname{dist}[s, Z] = \|s - s^\perp\|. \tag{4.20}$$

With the definition of ε_4,

$$\|s^\perp - z^*\| \leq \|s^\perp - s\| + \|s - z^*\| \leq 2\varepsilon_4 \leq \varepsilon$$

follows. This implies $x^\perp = x^*$. Using this, (4.20), Assumption 2, and $s \in \Omega_i$, we have

$$\|x - x^*\| = \|x - x^\perp\| \leq \|s - s^\perp\| = \operatorname{dist}[s, Z] \leq \omega\|F(s)\| = \omega\|F^i(s)\|. \tag{4.21}$$

Furthermore, the triangle inequality and (4.16) yield

$$\|F^i(x^*, y)\| \leq \|F^i(x^*, y) - F^i(x, y)\| + \|F^i(x, y)\| \leq L_0\|x - x^*\| + \|F^i(s)\|.$$

This together with (4.21) implies

$$\|F^i(x^*, y)\| \leq (L_0\omega + 1)\|F^i(s)\|. \tag{4.22}$$

Combining (4.19), (4.21), and (4.22), we obtain

$$\text{dist}[s, Z_i] \leq (\omega + \omega_H^{(i)}(L_0\omega + 1))\|F^i(s)\|.$$

Since $i \in \mathcal{A}(z^*)$ was arbitrarily chosen, Condition 4 is satisfied with

$$K_4 := \omega + \omega_H(L_0\omega + 1)$$

where ω_H is defined according to $\omega_H := \max\{\omega_H^{(i)} \mid i \in \mathcal{A}(z^*)\}$. □

Proposition 4.5 extends [18, Theorem 3]. There, in addition to the assumptions of the above proposition, F is assumed to be split into a differentiable part and a piecewise affine part where the latter is supposed to be independent of x. Moreover, it is supposed that there is a polyhedral set $\Omega_y \subseteq \mathbb{R}^{n_y}$ such that $\Omega = \mathbb{R}^{n_x} \times \Omega_y$ holds. The latter is obviously a stronger requirement than our condition on Ω. Particularly, we allow that the constraints which define Ω are dependent on x.

4.1.3 Assumption 4

At the beginning of this subsection a condition implying Assumption 4 is introduced. Afterwards, some relations between Assumption 4 and the Ω-property are discussed. Furthermore, we provide suitable choices of the set Ω for the case that F is given according to (4.2) such that the validity of Assumption 4 for the resulting constrained system is always guaranteed.

The following condition says that, for every $i \in \mathcal{A}(z^*)$, the norm of F^i provides, locally on Ω, an overestimate for the norm of F. In other words, Condition 5 requires that there is a certain neighborhood of z^* where the norm of F does not increase faster than the norm of each selection function which is active at z^*.

Condition 5. There are $K_5 > 0$ and $\varepsilon_5 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\|F(s)\| \leq K_5\|F^i(s)\|$$

holds for all $s \in \mathcal{B}_{\varepsilon_5}(z^*) \cap \Omega$.

The subsequent proposition shows that Condition 5 is sufficient for Assumption 4 to hold. Note that, unlike Conditions 3 and 4, the inequality in Condition 5 is not only required for points s where the particular selection function is active but for all points which belong to Ω and are sufficiently close to z^*. It will become clear in the proof of the following proposition that this requirement is really necessary. Proposition 4.6 will also be part of [37].

Proposition 4.6. *Let Condition 5 be satisfied. Then Assumption 4 holds.*

Proof. Let $\delta_4 \in (0, \frac{1}{2}\varepsilon_5]$ be small enough such that $\mathcal{A}(s) \subseteq \mathcal{A}(z^*)$ is satisfied for all $s \in \mathcal{B}_{\delta_4}(z^*)$. Note that there is some δ_4 with this property due to the continuity of F. Since the selection functions are differentiable and have locally Lipschitz continuous derivatives, there is $C > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\|F^i(s) + (F^i)'(s)(z - s) - F^i(z)\| \le C\|z - s\|^2 \tag{4.23}$$

holds for all $s, z \in \mathcal{B}_{2\delta_4}(z^*)$. Now let $s \in (\mathcal{B}_{\delta_4}(z^*) \cap \Omega) \setminus Z$ and $\delta \in [0, \delta_4]$ be arbitrarily chosen. Moreover, let i be an element of $\mathcal{A}(s)$ such that $G(s) = (F^i)'(s)$ is valid. Suppose that $z \in \Omega$ is a point for which the inequalities

$$\|F(s) + G(s)(z - s)\| \le \delta^2 \quad \text{and} \quad \|z - s\| \le \delta \tag{4.24}$$

are satisfied. The triangle inequality, together with the second inequality in (4.24), yields

$$\|z - z^*\| \le \|z - s\| + \|s - z^*\| \le \delta + \delta_4 \le 2\delta_4 \le \varepsilon_5.$$

Using Condition 5, $G(s) = (F^i)'(s)$, (4.23), and (4.24), we obtain

$$\begin{aligned}
\|F(z)\| &\le K_5\|F^i(z)\| \\
&\le K_5\left(\|F^i(s) + (F^i)'(s)(z - s) - F^i(z)\| + \|F^i(s) + G(s)(z - s)\|\right) \\
&\le K_5\left(C\|z - s\|^2 + \delta^2\right) \\
&\le K_5(C + 1)\delta^2.
\end{aligned}$$

Hence, Assumption 4 holds with $\kappa := K_5(C + 1)$. □

It is not difficult to see that Condition 5 particularly provides conditions on Ω. In fact, the Ω-property is implied by Condition 5. This is shown in the next proposition.

Proposition 4.7. *Let Condition 5 be satisfied. Then the Ω-property is valid.*

Proof. Let us define $\varepsilon_\Omega := \varepsilon_5$ and let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed. Then, for any $s \in Z_i \cap \mathcal{B}_{\varepsilon_\Omega}(z^*)$, Condition 5 yields

$$\|F(s)\| \le K_5\|F^i(s)\| = 0.$$

The latter implies $F(s) = 0$ and therefore $s \in Z$. Thus, the assertion is shown. □

However, it turns out that the Ω-property is not sufficient for Condition 5 to hold. In fact, not even Assumption 4 is implied by the Ω-property. In order to justify this, let us consider the following example.

Example 4.4. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by $F(z) := F(x,y) := \min\{x,y\}$. Obviously, F is a PC^1-function with the selection functions $F^1(x,y) := x$ and $F^2(x,y) := y$. Let $G : \mathbb{R}^2 \to \mathbb{R}^{1\times 2}$ be any function satisfying (4.1) for all $z \in \mathbb{R}^2$. We define Ω by

$$\Omega := \mathbb{R}^2_+ \cup \{(x,y)^\top \in \mathbb{R}^2 \mid y \geq x^2\}.$$

The set Ω is illustrated in Figure 4.1.

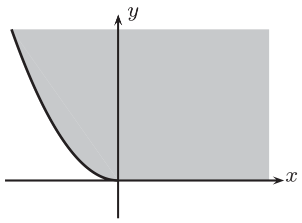

Figure 4.1: The set Ω in Example 4.4.

The points on the negative half axes do not belong to Ω so that all zeros of the selection functions F^1 and F^2 which are not zeros of F are excluded. Therefore, the Ω-property is satisfied with arbitrary $\varepsilon_\Omega > 0$ near any solution of (1.1).

The solution set of the constrained system (1.1) is given by

$$Z := \{(x,y)^\top \in \mathbb{R}^2_+ \mid x = 0 \text{ or } y = 0\}.$$

We are going to show that Assumption 4 does not hold at $z^* := (0,0)^\top$. To this end, let us assume the contrary, i.e., that Assumption 4 is satisfied with some $\delta_4 > 0$ and some $\kappa > 0$. Let the sequences $\{s^k\}_{k\in\mathbb{N},\,k\geq 1} \subset \Omega \setminus Z$, $\{z^k\}_{k\in\mathbb{N},\,k\geq 1} \subset \Omega$, and $\{\delta^{(k)}\}_{k\in\mathbb{N},\,k\geq 1} \subset (0,\infty)$ be defined as follows:

$$s^k := \left(\frac{2}{k^2}, \frac{1}{k^2}\right)^\top, \quad z^k := \left(-\frac{1}{k}, \frac{1}{k^2}\right)^\top, \quad \delta^{(k)} := \frac{3}{k}.$$

Obviously, $\{s^k\}$ converges to z^* and $\{\delta^{(k)}\}$ goes to zero. Consequently, there is $k_0 \geq 1$ such that $s^k \in (\mathcal{B}_{\delta_4}(z^*) \cap \Omega) \setminus Z$ and $\delta^{(k)} \in [0, \delta_4]$ hold for all $k \geq k_0$. Moreover,

$$\|z^k - s^k\| = \left|-\frac{1}{k} - \frac{2}{k^2}\right| \leq \frac{3}{k} = \delta^{(k)} \tag{4.25}$$

is satisfied for all $k \geq 1$. Furthermore, for every $k \geq 1$,

$$F^1(s^k) = \frac{2}{k^2} > \frac{1}{k^2} = F^2(s^k)$$

is valid so that $\mathcal{A}(s^k) = \{2\}$ follows. Therefore, we have

$$F(s^k) = \frac{1}{k^2} \quad \text{and} \quad G(s^k) = (F^2)'(s^k) = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

for all $k \geq 1$. This implies, for every $k \geq 1$,

$$|F(s^k) + G(s^k)(z^k - s^k)| = \left|\frac{1}{k^2} + \frac{1}{k^2} - \frac{1}{k^2}\right| = \frac{1}{k^2} < \frac{9}{k^2} = (\delta^{(k)})^2. \tag{4.26}$$

By $z^k \in \Omega$, (4.25), and (4.26), Assumption 4 yields

$$|F(z^k)| \leq \kappa(\delta^{(k)})^2 = \frac{9}{k^2}\kappa$$

for all $k \geq k_0$. Thus,

$$\frac{9}{k^2}\kappa \geq |F(z^k)| = |\min\{-\frac{1}{k}, \frac{1}{k^2}\}| = \frac{1}{k}$$

follows for all $k \geq k_0$. This is a contradiction, i.e., Assumption 4 cannot hold at z^*. □

The next example shows that, unlike Condition 5, Assumption 4 is in general not sufficient for the Ω-property to hold.

Example 4.5. As in Example 4.4, we define $F : \mathbb{R}^2 \to \mathbb{R}$ according to $F(z) := F(x, y) := \min\{x, y\}$. The selection functions are given by $F^1(x, y) := x$ and $F^2(x, y) := y$. Let $G : \mathbb{R}^2 \to \mathbb{R}^{1\times 2}$ be any function satisfying (4.1) for all $z \in \mathbb{R}^2$. Moreover, let Ω be defined by

$$\Omega := \{(x, y)^\top \in \mathbb{R}^2 \mid x \leq 0,\ y = 0\},$$

i.e., Ω is just the nonpositive part of the x-axis. Note that $z^* := (0, 0)^\top$ is the only solution of the constrained system (1.1), i.e., Z is a singleton. Of course, the Ω-property is not satisfied near z^*. However, Assumption 4 is valid at z^*. In order to verify this, let $s = (x_s, 0)^\top \in \Omega \setminus Z$ and $z = (x_z, 0)^\top \in \Omega$ be arbitrarily chosen such that, for some $\delta \geq 0$,

$$\|z - s\| \leq \delta \quad \text{and} \quad |F(s) + G(s)(z - s)| \leq \delta^2 \tag{4.27}$$

hold. Taking into account $s \notin Z$, we have $x_s < 0$ and therefore $\mathcal{A}(s) = \{1\}$. Thus, $G(s) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ follows so that the second inequality in (4.27) yields

$$\delta^2 \geq |F(s) + G(s)(z - s)| = \left| x_s + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_z - x_s \\ 0 \end{pmatrix} \right| = |x_z|.$$

Since $F(z) = x_z$ is valid due to $x_z \leq 0$, $|F(z)| \leq \delta^2$ follows. Hence, Assumption 4 is satisfied with arbitrary $\delta_4 > 0$ and $\kappa := 1$. □

Examples 4.4 and 4.5 together make clear that generally Assumption 4 neither implies nor is implied by the Ω-property. Since the Ω-property is necessary for Condition 5 to hold, Example 4.5 also shows that Assumption 4 is in general strictly weaker than Condition 5.

Example 4.5 actually provides an instance where all of Assumptions 1–4 are satisfied whereas the Ω-property does not hold. In fact, Assumption 1 holds due to Proposition 4.1. Assumption 2 is valid with arbitrary $\delta_2 > 0$ and $\omega := 1$ because, taking into account $Z = \{(0,0)^\top\}$ and the definition of Ω,

$$\mathrm{dist}[s, Z] = |x_s| = |\min\{x_s, 0\}| = |F(s)|$$

holds for all $s = (x_s, 0)^\top \in \Omega$. Moreover, it is not difficult to show that for any $s = (x_s, 0)^\top \in \mathcal{B}_1(z^*) \cap \Omega$ the point $z := (-x_s^2, 0)^\top$ belongs to $\mathcal{F}(s, 1)$. Therefore, Assumption 3 is satisfied with $\delta_3 := 1$ and $\Gamma := 1$.

But of course, the choice of Ω in Example 4.5 is anything but natural. If F has a similar structure like F in Example 4.1 and a zero of F is looked for, our aim is always to choose Ω such that all zeros of F are included. This is obviously not satisfied in Example 4.5. Moreover, the question concerning a generalization arises, for example if $F : \mathbb{R}^n \to \mathbb{R}$ is given by $F(z) := \min\{R(z), S(z)\}$ with real-valued functions $R : \mathbb{R}^n \to \mathbb{R}$ and $S : \mathbb{R}^n \to \mathbb{R}$. A straightforward generalization for the definition of Ω could be

$$\Omega := \{z \in \mathbb{R}^n \mid R(z) \leq 0,\ S(z) = 0\}.$$

But then it would not be guaranteed that the resulting constrained system (1.1) has a solution. The latter considerations underline that the Ω-property can be regarded as a reasonable condition. In the sequel we will also regard Condition 5 as reasonable since it guarantees both the Ω-property and Assumption 4.

Therefore, we are interested in conditions on Ω which imply Condition 5, at least if the PC^1-function F has a certain structure. To this end, the simple function $F(x, y) := \min\{x, y\}$ is considered again in the next example. We present possible choices of the set Ω which guarantee the validity of Condition 5. Afterwards, in Examples 4.7 and 4.8 we provide suitable choices of Ω in larger settings where F has the structure as in Example 4.1.

Example 4.6. Once again, we consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by $F(z) := F(x, y) := \min\{x, y\}$ which is a PC^1-function with the selection functions $F^1(x, y) := x$ and $F^2(x, y) := y$. For any $\varepsilon > 0$ we define

$$\Omega_\varepsilon := \{(x, y)^\top \in \mathbb{R}^2 \mid |y| \geq \varepsilon|x| \text{ if } x < 0,\ |x| \geq \varepsilon|y| \text{ if } y < 0\}.$$

Figure 4.2 shows illustrations of Ω_ε for two different values of ε. On the left hand side, ε is smaller than 1, whereas the right hand side of Figure 4.2 shows Ω_ε for some $\varepsilon > 1$. Obviously, for any $\varepsilon > 0$, $z^* := (0, 0)^\top$ is a solution of the constrained system (1.1) with $\Omega := \Omega_\varepsilon$. Now let $\varepsilon > 0$ be arbitrary but fixed. We

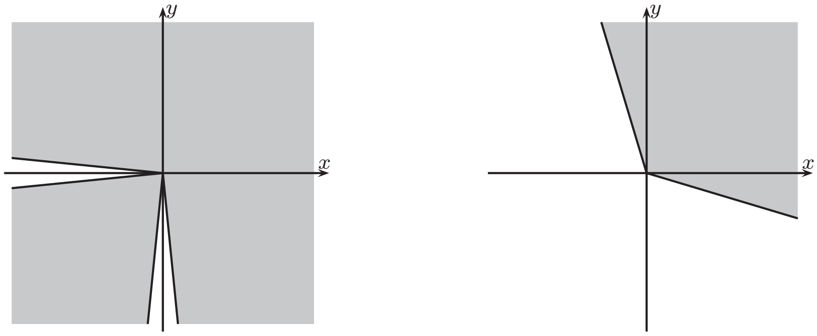

Figure 4.2: The set Ω_ε in Example 4.6 – on the left for $\varepsilon < 1$ and on the right for $\varepsilon > 1$.

are going to show that Condition 5 is satisfied at z^*. To this end, let us take any $s = (x, y)^\top \in \Omega_\varepsilon$. If s belongs to R^2_+,

$$|\min\{x, y\}| \leq |x| \quad \text{and} \quad |\min\{x, y\}| \leq |y|$$

obviously hold. Otherwise, the definition of Ω_ε yields

$$|\min\{x, y\}| \leq \max\left\{\frac{1}{\varepsilon}, 1\right\}|x| \quad \text{and} \quad |\min\{x, y\}| \leq \max\left\{\frac{1}{\varepsilon}, 1\right\}|y|.$$

In each case, we have

$$|F(x, y)| = |\min\{x, y\}| \leq \max\left\{\frac{1}{\varepsilon}, 1\right\}|x| = \max\left\{\frac{1}{\varepsilon}, 1\right\}|F^1(x, y)|$$

and

$$|F(x, y)| = |\min\{x, y\}| \leq \max\left\{\frac{1}{\varepsilon}, 1\right\}|y| = \max\left\{\frac{1}{\varepsilon}, 1\right\}|F^2(x, y)|.$$

Thus, Condition 5 is satisfied at z^* with $K_5 := \max\left\{\frac{1}{\varepsilon}, 1\right\}$ and arbitrary $\varepsilon_5 > 0$. It is obvious that Condition 5 is also valid for any subset of Ω_ε which contains z^*. In particular, it is satisfied for $\Omega = \mathbb{R}^2_+$. □

We have seen in Example 4.4 that the exclusion of all zeros of the selection functions which are not zeros of F is not sufficient for Condition 5 (not even for Assumption 4) to hold. Taking into account the preceding example, it seems that, to guarantee Condition 5, there has to be a certain angle between Ω and the set of the "wrong" zeros of the selection functions.

In the following two examples, we consider PC^1-functions F which have a similar structure like the function in Example 4.1, i.e., which consist of a smooth part and a part which is described by the minimum function. As already discussed after Example 4.1, many problems can be equivalently reformulated into the problem of finding a zero of such a function, for example complementarity problems and KKT systems arising from optimization problems, variational inequalities, or GNEPs, see also Section 4.3 and Chapter 5. The next two examples provide suitable choices of Ω which certainly imply Condition 5 and therefore, due to Propositions 4.6 and 4.7, both Assumption 4 and the Ω-property.

Example 4.7. Consider the function $F : \mathbb{R}^n \to \mathbb{R}^{p+r}$ in (4.2), i.e.,

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R(z), S(z)\} \end{pmatrix}$$

with given continuously differentiable functions $P : \mathbb{R}^n \to \mathbb{R}^p$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$ whose derivatives are assumed to be locally Lipschitz continuous. As already mentioned in Example 4.1, F is a PC^1-function with 2^r selection functions $F^1, \dots, F^{2^r}$.

Let $\Omega \subseteq \mathbb{R}^n$ be defined according to

$$\Omega := \{z \in \mathbb{R}^n \mid R(z) + S(z) \geq 0\} \tag{4.28}$$

and let z^* denote an arbitrary but fixed solution of the constrained system (1.1). Then Condition 5 is satisfied. In order to prove this, let us take any $s \in \Omega$ and let $i \in \mathcal{A}(z^*)$ and $j \in \{1, \dots, r\}$ be arbitrarily chosen. Without loss of generality we assume that $R_j(s) \leq S_j(s)$ is valid. If $R_j(s)$ is nonnegative, then

$$|\min\{R_j(s), S_j(s)\}| = R_j(s) = \min\{|R_j(z)|, |S_j(z)|\}$$

holds. Otherwise, assuming $R_j(s) < 0$, we have $S_j(s) \geq |R_j(s)| > 0$ due to the definition of Ω. Therefore,

$$|\min\{R_j(s), S_j(s)\}| = -R_j(s) = \min\{|R_j(z)|, S_j(z)\} = \min\{|R_j(z)|, |S_j(z)|\}$$

follows. In each case, we obtain

$$|\min\{R_j(s), S_j(s)\}| = \min\{|R_j(z)|, |S_j(z)|\}. \tag{4.29}$$

Since j was arbitrarily chosen, (4.29) is valid for all $j = 1, \dots, r$. Thus, taking into account the structure of F^i, we have

$$\|F(s)\| \leq \|F^i(s)\|.$$

Hence, Condition 5 is satisfied with $K_5 := 1$ and arbitrary $\varepsilon_5 > 0$. Assumption 4 is valid as well due to Proposition 4.6. This recovers [18, Corollary 5]. Note that Condition 5 stays true for any subset of Ω in (4.28), in particular for the set defined according to (4.9), i.e., $\{z \in \mathbb{R}^n \mid R(z) \geq 0,\ S(z) \geq 0\}$. □

Example 4.8. Consider the function $F : \mathbb{R}^n \to \mathbb{R}^{p+r}$ defined by

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R^1(z), \dots, R^l(z)\} \end{pmatrix}$$

with continuously differentiable functions $P : \mathbb{R}^n \to \mathbb{R}^p$ and $R^k : \mathbb{R}^n \to \mathbb{R}^r$ $(k = 1, \dots, l)$ whose derivatives are assumed to be locally Lipschitz continuous. The function F is a PC^1-function with l^r selection functions $F^1, \dots, F^{l^r}$.

Let $\Omega \subseteq \mathbb{R}^n$ be given by

$$\Omega := \{z \in \mathbb{R}^n \mid \forall k \in \{1, \dots, l\} : R^k(z) \geq 0\}.$$

Assume that z^* denotes an arbitrary but fixed solution of (1.1). Moreover, let us take any $s \in \Omega$ and any $i \in \mathcal{A}(z^*)$. The definition of Ω implies

$$|\min\{R_j^1(s), \dots, R_j^l(s)\}| = \min\{R_j^1(s), \dots, R_j^l(s)\} \leq R_j^k(s) = |R_j^k(s)|$$

for all $k = 1, \dots, l$ and all $j = 1, \dots, r$. Thus, taking into account the structure of F^i,

$$\|F(s)\| \leq \|F^i(s)\|$$

follows. Hence, Condition 5 is valid with $K_5 := 1$ and arbitrary $\varepsilon_5 > 0$. Assumption 4 is satisfied as well due to Proposition 4.6. Thus, [18, Theorem 4] is recovered. □

4.1.4 Sufficient Conditions for the Whole Set of Assumptions 1–4

Let us briefly summarize the most important results of the previous subsections. We introduced Condition 4 which requires that every selection function F^i being active at z^* provides a local error bound for the distance to Z_i near z^* on Ω_i, where Z_i is the set of all zeros of F^i belonging to Ω. It was proved that Condition 4 always implies Assumption 3. Moreover, we showed that Assumption 2 is also implied by Condition 4 if in addition the Ω-property is satisfied, i.e., if Ω excludes those zeros of the selection functions which are not zeros of F, at least in a certain neighborhood of z^*. Furthermore, Condition 5 was introduced. This condition requires that, in a sufficiently small neighborhood of z^*, the norm of F can be estimated from above by the norm of each selection function which is active at z^*. It turned out that both Assumption 4 and the Ω-property are implied by Condition 5. For the important case that F has a similar structure as in Example 4.1 we provided suitable choices of the set Ω which always guarantee the validity of Condition 5.

The aim of this subsection is to provide a condition which is sufficient for all convergence assumptions from Section 3.1 to hold if in addition the Ω-property is valid. We will see that this aim is achieved by Condition 6 below. This condition extends Condition 4. As in the latter, each selection function F^i which is active at z^* must provide a local error bound for the distance to Z_i near z^*. However, unlike Condition 4, the corresponding inequality is not only required for points belonging to Ω_i but for all elements of Ω which are sufficiently close to z^*.

Condition 6. There are $K_6 > 0$ and $\varepsilon_6 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\mathrm{dist}[s, Z_i] \leq K_6 \|F^i(s)\|$$

holds for all $s \in \mathcal{B}_{\varepsilon_6}(z^*) \cap \Omega$.

Condition 6 was already introduced in [18]. It was shown in [18, Corollary 3] that Assumption 3 is implied by Condition 6. In Subsection 4.1.2 we extended the latter result by showing that Condition 6 can be replaced by the slightly weaker Condition 4. Moreover, it was proved that Condition 4 is also sufficient for Assumption 2 to hold if the Ω-property is satisfied.

In the sequel, we will show that Condition 6, together with the Ω-property, is actually sufficient for all convergence assumptions from Section 3.1 to hold. To this end, we need the following proposition.

Proposition 4.8. *Let Condition 6 be satisfied. Then the following assertions are valid.*

(a) Condition 4 holds.

(b) Condition 5 holds if the Ω-property is satisfied.

Proof. (a) It is obvious that Condition 4 is valid with $K_4 := K_6$ and $\varepsilon_4 := \varepsilon_6$.

(b) Let us set $\varepsilon_5 := \min\{\varepsilon_6, \frac{1}{2}\varepsilon_\Omega\}$. By Proposition 4.1, F is locally Lipschitz continuous. Therefore, there is $L_0 > 0$ such that

$$\|F(z) - F(s)\| \leq L_0 \|z - s\| \tag{4.30}$$

holds for all $s, z \in \mathcal{B}_{2\varepsilon_5}(z^*)$. Now let $i \in \mathcal{A}(z^*)$ and $s \in \mathcal{B}_{\varepsilon_5}(z^*) \cap \Omega$ be arbitrarily chosen. The set Z_i is closed due to the continuity of F^i. Moreover, Z_i is nonempty because z^* belongs to it. Thus, there is $\bar{s} \in Z_i$ such that

$$\|s - \bar{s}\| = \mathrm{dist}[s, Z_i]$$

is valid. This, together with Condition 6, implies

$$\|s - \bar{s}\| \leq K_6 \|F^i(s)\|. \tag{4.31}$$

With

$$\|\bar{s} - z^*\| \leq \|\bar{s} - s\| + \|s - z^*\| \leq 2\varepsilon_5 \leq \varepsilon_\Omega,$$

the Ω-property yields $\bar{s} \in Z$ and therefore $F(\bar{s}) = 0$. Using this, (4.30), and (4.31), we obtain

$$\|F(s)\| = \|F(s) - F(\bar{s})\| \leq L_0 \|s - \bar{s}\| \leq L_0 K_6 \|F^i(s)\|.$$

Hence, Condition 5 is satisfied with $K_5 := L_0 K_6$.

□

Combining Propositions 4.1, 4.4, 4.6, and 4.8, we obtain the following theorem which will also be part of [37].

Theorem 4.9. *Let Condition 6 and the Ω-property be satisfied. Then Assumptions 1–4 hold.*

Based on the latter theorem we want to provide some further conditions which imply the whole set of Assumptions 1–4. Due to Hoffman's error bound, Condition 6 is in particular satisfied if all selection functions are affine and Ω is polyhedral. This leads to the following proposition.

Proposition 4.10. *Let F be piecewise affine and assume that Ω is a polyhedral set. Then Condition 6 holds. If the Ω-property is additionally valid, then Assumptions 1–4 are satisfied.*

Proof. Since Ω is polyhedral, there are a number $l \in \mathbb{N}$ and an affine function $a : \mathbb{R}^n \to \mathbb{R}^l$ such that

$$\Omega = \{z \in \mathbb{R}^n \mid a(z) \geq 0\}$$

holds. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed. Since F^i is affine, the set

$$Z_i = \{z \in \mathbb{R}^n \mid a(z) \geq 0, \; F^i(z) = 0\}$$

is polyhedral. By Theorem 2.1, there is $\omega_H^{(i)} > 0$ such that

$$\mathrm{dist}[s, Z_i] \leq \omega_H^{(i)} (\|F^i(s)\| + \|\min\{0, a(s)\}\|)$$

holds for all $s \in \mathbb{R}^n$. In particular, we obtain

$$\mathrm{dist}[s, Z_i] \leq \omega_H^{(i)} \|F^i(s)\|$$

for all $s \in \Omega$. Therefore, Condition 6 is satisfied with arbitrary $\varepsilon_6 > 0$ and

$$K_6 := \max\{\omega_H^{(i)} \mid i \in \mathcal{A}(z^*)\}.$$

The second assertion of the proposition follows from Theorem 4.9. □

It is not difficult to show that Condition 6 coincides with Assumption 2 if $\mathcal{A}(z^*)$ is a singleton, i.e., if exactly one of the selection functions is active at z^*. Moreover, the Ω-property is satisfied in that case. Thus, we obtain the following result.

Proposition 4.11. *Let Assumption 2 be valid and assume that $\mathcal{A}(z^*)$ is a singleton. Then both Condition 6 and the Ω-property are satisfied. In particular, Assumptions 1–4 hold.*

Proof. Suppose that the index i_0 is the unique element of $\mathcal{A}(z^*)$. The continuity of F implies the existence of some $\varepsilon > 0$ such that $\mathcal{A}(s) = \{i_0\}$ holds for all $s \in \mathcal{B}_\varepsilon(z^*)$, i.e. F coincides with F^{i_0} on $\mathcal{B}_\varepsilon(z^*)$. In particular, we have

$$Z_{i_0} \cap \mathcal{B}_\varepsilon(z^*) = Z \cap \mathcal{B}_\varepsilon(z^*). \tag{4.32}$$

Therefore, the Ω-property is satisfied with $\varepsilon_\Omega := \varepsilon$. Let us define $\varepsilon_6 := \min\{\frac{1}{2}\varepsilon, \delta_2\}$ and let $s \in \mathcal{B}_{\varepsilon_6}(z^*) \cap \Omega$ be arbitrarily chosen. The definition of ε_6, together with (4.32), implies $\operatorname{dist}[s, Z_{i_0}] = \operatorname{dist}[s, Z]$. Using this, Assumption 2, and $F(s) = F^{i_0}(s)$,

$$\operatorname{dist}[s, Z_{i_0}] = \operatorname{dist}[s, Z] \leq \omega \|F(s)\| = \omega \|F^{i_0}(s)\|$$

follows. Hence, Condition 6 is valid with $K_6 := \omega$. Theorem 4.9 yields the validity of Assumptions 1–4. □

Remark 4.1. If $\mathcal{A}(z^*)$ is a singleton, then F is differentiable at z^* and its derivative is locally Lipschitz continuous near z^*. Therefore, assuming that Assumption 2 is satisfied, the validity of Assumptions 3 and 4 already follows from Corollary 3.6. Nevertheless, we think it was worth to state and prove the latter proposition because it shows that Condition 6 and the Ω-property together are somewhere between Assumption 2 together with the smoothness of F and the whole set of Assumptions 1–4.

In the next proposition the setting of Proposition 4.5 is considered where the vectors z are split into an x-part and a y-part, the x-part of the solution $z^* = (x^*, y^*)$ is locally unique, $F(x^*, \cdot)$ is piecewise affine, and $\Omega_y(x^*)$ defined according to (4.15) is polyhedral. We will show, by similar arguments as in the proof of Proposition 4.5, that both Condition 6 and the Ω-property hold in this setting if in addition Assumption 2 and Condition 5 are valid.

Proposition 4.12. *Let Assumption 2 and Condition 5 be satisfied. Moreover, let z be split according to $z = (x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that $\Omega_y(x^*)$ defined by* (4.15) *is polyhedral and that, for every $i \in \mathcal{A}(z^*)$, the function $F^i(x^*, \cdot)$ is affine. Furthermore, suppose that there is $\varepsilon > 0$ such that $z = (x, y) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$. Then both Condition 6 and the Ω-property are valid. In particular, Assumptions 1–4 hold.*

Proof. Let us set $\varepsilon_6 := \min\{\delta_2, \varepsilon_5, \frac{1}{2}\varepsilon\}$. By the local Lipschitz continuity of the selection functions, there is $L_0 > 0$ such that

$$\|F^i(z) - F^i(s)\| \leq L_0 \|z - s\| \tag{4.33}$$

holds for all $i \in \mathcal{A}(z^*)$ and all $s, z \in \mathcal{B}_{\varepsilon_6}(z^*)$. Since $\Omega_y(x^*)$ is a polyhedral set, there are a number $l \in \mathbb{N}$ and an affine function $a(x^*, \cdot) : \mathbb{R}^{n_y} \to \mathbb{R}^l$ such that

$$\Omega_y(x^*) = \{y \in \mathbb{R}^{n_y} \mid a(x^*, y) \geq 0\}$$

holds.

Let $i \in \mathcal{A}(z^*)$ be arbitrarily chosen. As in the proof of Proposition 4.5, we define the set $Z_i(x^*)$ by

$$\begin{aligned} Z_i(x^*) &:= \{y \in \Omega_y(x^*) \mid F^i(x^*, y) = 0\} \\ &= \{y \in \mathbb{R}^{n_y} \mid a(x^*, y) \geq 0,\ F^i(x^*, y) = 0\}. \end{aligned}$$

Note that $Z_i(x^*)$ is nonempty because y^* belongs to it. Since $F^i(x^*, \cdot)$ is affine, $Z_i(x^*)$ is a polyhedral set so that there is, due to Theorem 2.1, $\omega_H^{(i)} = \omega_H^{(i)}(x^*) > 0$ such that

$$\operatorname{dist}[y, Z_i(x^*)] \leq \omega_H^{(i)}(\|F^i(x^*, y)\| + \|\min\{0, a(x^*, y)\}\|)$$

holds for all $y \in \mathbb{R}^{n_y}$. In particular, we have

$$\operatorname{dist}[y, Z_i(x^*)] \leq \omega_H^{(i)} \|F^i(x^*, y)\| \tag{4.34}$$

for all $y \in \Omega_y(x^*)$.

Now let us take any $s = (x, y) \in \mathcal{B}_{\varepsilon_6}(z^*) \cap \Omega$. Let $\bar{y} \in Z_i(x^*)$ be a point with the property

$$\operatorname{dist}[y, Z_i(x^*)] = \|y - \bar{y}\|. \tag{4.35}$$

Then $F^i(x^*, \bar{y}) = 0$ holds. Moreover, taking into account $\bar{y} \in \Omega_y(x^*)$, $(x^*, \bar{y})$ belongs to Ω. Therefore, we have $(x^*, \bar{y}) \in Z_i$. Using this, (4.35), and (4.34),

$$\operatorname{dist}[s, Z_i] \leq \|x - x^*\| + \|y - \bar{y}\| \leq \|x - x^*\| + \omega_H^{(i)} \|F^i(x^*, y)\| \tag{4.36}$$

follows. Let us estimate the right hand side of this inequality from above. Suppose that $s^\perp = (x^\perp, y^\perp) \in Z$ satisfies

$$\operatorname{dist}[s, Z] = \|s - s^\perp\|.$$

The triangle inequality and the definition of ε_6 yield

$$\|s^\perp - z^*\| \leq \|s^\perp - s\| + \|s - z^*\| \leq 2\varepsilon_6 \leq \varepsilon.$$

This implies $x^\perp = x^*$. Therefore, by Assumption 2 and Condition 5, we have

$$\|x - x^*\| = \|x - x^\perp\| \leq \operatorname{dist}[s, Z] \leq \omega \|F(s)\| \leq K_5 \omega \|F^i(s)\|. \tag{4.37}$$

Furthermore, due to the triangle inequality and (4.33),

$$\|F^i(x^*, y)\| \leq \|F^i(x^*, y) - F^i(x, y)\| + \|F^i(x, y)\| \leq L_0 \|x - x^*\| + \|F^i(s)\|$$

is satisfied. Using this and (4.37), we obtain

$$\|F^i(x^*, y)\| \leq (L_0 K_5 \omega + 1)\|F^i(s)\|. \tag{4.38}$$

Combining (4.36)–(4.38),

$$\operatorname{dist}[s, Z_i] \leq (K_5\omega + \omega_H^{(i)}(L_0 K_5 \omega + 1))\|F^i(s)\|$$

follows. Since $i \in \mathcal{A}(z^*)$ was arbitrarily chosen, Condition 6 holds with

$$K_6 := K_5\omega + \omega_H(L_0 K_5 \omega + 1)$$

where ω_H is defined according to $\omega_H := \max\{\omega_H^{(i)} \mid i \in \mathcal{A}(z^*)\}$. The validity of the Ω-property follows from Proposition 4.7. Thus, Theorem 4.9 yields Assumptions 1–4. □

Remark 4.2. Note that we already obtain the sufficiency of the assumptions of Proposition 4.12 for all of Assumptions 1–4 by combining the assertions from Propositions 4.1, 4.5, and 4.6. Nevertheless, we think that it was worth to state and prove the latter proposition because it shows that Condition 6 and the Ω-property together are somewhere between the assumptions of Proposition 4.12 and the whole set of Assumptions 1–4.

Combining Theorem 4.9 and Propositions 4.10–4.12, we obtain the following theorem which is the main result of this section and provides sufficient conditions for the whole set of Assumptions 1–4 to hold for the case that F is a PC^1-function.

Theorem 4.13. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a PC^1-function and assume that all selection functions have locally Lipschitz continuous derivatives. Let one of the following conditions be true.*

(i) The function F is piecewise affine, and Ω is a polyhedral set. Furthermore, the Ω-property is satisfied at z^*.

(ii) Assumption 2 holds at z^* and $\mathcal{A}(z^*)$ is a singleton.

(iii) Assumption 2 and Condition 5 are satisfied at z^*. Moreover, the vectors z are split according to $z = (x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that $\Omega_y(x^*)$ defined by (4.15) is polyhedral and that, for every $i \in \mathcal{A}(z^*)$, the function $F^i(x^*, \cdot)$ is affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x, y) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$.

(iv) Condition 6 and the Ω-property are satisfied at z^*.

Then Assumptions 1–4 are valid if the function $G : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ satisfies (4.1) for all $z \in \mathbb{R}^n$.

Remark 4.3. By Propositions 4.10–4.12, we know that condition (iv) in Theorem 4.13 is implied by each of conditions (i)–(iii). Nevertheless, we have presented all four sufficient conditions since we will explicitly discuss them in the sequel for different settings.

By the above theorem and our results from Chapter 3, each of conditions (i)–(iv) from Theorem 4.13 is sufficient for local quadratic convergence of our general Newton-type algorithm from Section 3.1 and its special realizations from Sections 3.4 and 3.5 to a solution of the constrained system (1.1) if F is a PC^1-function. Conditions (i)–(iv) from Theorem 4.13 are those which will be further analyzed in the subsequent sections and in Chapter 5. We will discuss them in different settings and present sufficient conditions for them to hold for special problem classes.

We have introduced many conditions in this section. Figure 4.3 gives an overview on them and illustrates the relations which were proved.

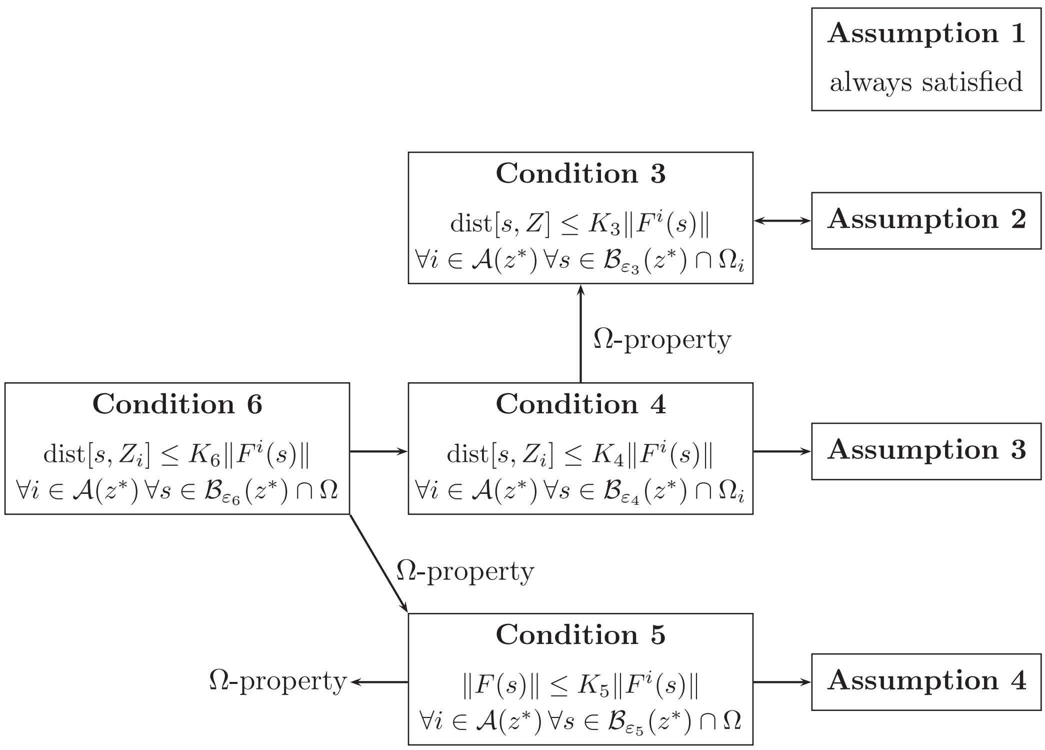

Figure 4.3: Scheme on relations of the conditions and assumptions.

4.2 Reformulation with Slack Variables

In the preceding section sufficient conditions for the convergence assumptions from Section 3.1 were presented. In particular, it turned out that all of Assumptions 1–4 are satisfied whenever one of the conditions (i)–(iv) from Theorem 4.13 holds. In that case we know that Algorithm 3.1 converges locally quadratically to a solution of (1.1). Particularly, by Theorems 3.20 and 3.28, the LP-Newton as well as the constrained Levenberg-Marquardt method converge locally with a Q-quadratic rate.

The set Ω is often described by inequalities. For instance, in Example 4.7 it was given according to $\Omega = \{z \in \mathbb{R}^n \mid R(z) + S(z) \geq 0\}$ with functions $R, S : \mathbb{R}^n \to \mathbb{R}^r$, and in Example 4.8 it was defined by $R^k(z) \geq 0$ $(k = 1, \ldots, l)$ with $R^1, \ldots, R^l : \mathbb{R}^n \to \mathbb{R}^r$. However, if some of the constraint functions are nonlinear, then the feasible set of the optimization problem (3.52) is nonpolyhedral. So the subproblems of the LP-Newton method are nonlinear and therefore difficult to solve from the computational point of view. The same holds for the subproblems (3.66) of the constrained Levenberg-Marquardt method. Their feasible sets are nonpolyhedral in that case, too, so that they are not quadratic programs.

Therefore, it is advisable, from the computational point of view, to reformulate the original problem (1.1) by means of slack variables and to apply the methods to the resulting modified constrained system. In this section we will show how the reformulation is obtained. Afterwards, our main convergence result from the preceding section, Theorem 4.13, is stated for the resulting reformulation. To this end, we formulate Assumption 2, Condition 5, Condition 6, and the Ω-property for the new system. In the second part of this section we establish conditions on the original problem only which imply the whole set of Assumptions 1–4 for both the original constrained system (1.1) and its reformulation with of slack variables, see Theorem 4.19. The latter is the main result of this section.

Throughout this section, we assume that Ω is given by

$$\Omega := \{z \in \mathbb{R}^n \mid a(z) \geq 0,\ b(z) \geq 0\} \tag{4.39}$$

with an affine function $a : \mathbb{R}^n \to \mathbb{R}^{l_a}$ and a differentiable function $b : \mathbb{R}^n \to \mathbb{R}^{l_b}$ which is assumed to have a locally Lipschitz continuous derivative. The function b can be thought of as a nonlinear function for which the introduction of slack variables is advisable. However, the results of this section are also true if some components of b are affine. Sometimes, the introduction of slacks for affine constraints could also make sense in order to obtain bound constraints only in the reformulation.

We consider the following reformulation of (1.1) by means of slack variables $w \in \mathbb{R}^{l_b}$:

$$\hat{F}(z, w) := \begin{pmatrix} F(z) \\ b(z) - w \end{pmatrix} = 0 \quad \text{s.t.} \quad (z, w) \in \hat{\Omega} \tag{4.40}$$

with

$$\hat{\Omega} := \{(z, w) \in \mathbb{R}^n \times \mathbb{R}^{l_b} \mid a(z) \geq 0,\ w \geq 0\}. \tag{4.41}$$

Obviously, a point $\bar{z}$ solves the original problem (1.1) if and only if $(\bar{z}, \bar{w})$, with $\bar{w} := b(\bar{z})$, is a solution of (4.40). We denote by $\hat{Z}$ the solution set of (4.40), i.e.,

$$\hat{Z} := \{(z, w) \in \hat{\Omega} \mid \hat{F}(z, w) = 0\}.$$

Moreover, as in the previous sections, by z^* an arbitrary but fixed solution of (1.1) is denoted. We indicate by (z^*, w^*), with $w^* := b(z^*)$, the corresponding solution of (4.40).

It is still supposed that F is a PC^1-function and that the selection functions $F^1, \ldots, F^t$ are differentiable and have locally Lipschitz continuous Jacobians. Then these properties are transferred to the function from (4.40). In fact, $\hat{F}$ is a PC^1-function with the selection functions $\hat{F}^1, \ldots, \hat{F}^t$ defined by

$$\hat{F}^i(z, w) := \begin{pmatrix} F^i(z) \\ b(z) - w \end{pmatrix} \quad (i = 1, \ldots, t).$$

By our differentiability assumptions on b and $F^1, \ldots, F^t$, the functions $\hat{F}^1, \ldots, \hat{F}^t$ are differentiable and their derivatives are locally Lipschitz continuous. For any point $(z, w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$, the index set of all selection functions $\hat{F}^i$ being active at (z, w) is denoted by $\mathcal{A}(z, w)$. It is not difficult to see that $\mathcal{A}(z, w)$ coincides with $\mathcal{A}(z)$ and is hence independent on w. Therefore, we will still use $\mathcal{A}(z)$ instead of $\mathcal{A}(z, w)$ in the sequel. For every $i \in \{1, \ldots, t\}$, the set of all zeros of $\hat{F}^i$ belonging to $\hat{\Omega}$ is denoted by $\hat{Z}_i$, i.e.,

$$\hat{Z}_i := \{(z, w) \in \hat{\Omega} \mid \hat{F}^i(z, w) = 0\}.$$

Note that at least for $i \in \mathcal{A}(z^*)$ this set is nonempty since (z^*, w^*) belongs to it. The function $\hat{G} : \mathbb{R}^n \times \mathbb{R}^{l_b} \to \mathbb{R}^{(m+l_b)\times(n+l_b)}$ is assumed to satisfy

$$\hat{G}(z, w) \in \{(\hat{F}^i)'(z, w) \mid i \in \mathcal{A}(z)\} \tag{4.42}$$

for all $(z, w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$.

Now let us assume that the general Newton-type algorithm from Section 3.1 or one of its realizations from Sections 3.4 or 3.5 is applied to system (4.40). Our aim is to state Theorem 4.13 for (4.40) and therefore to provide conditions under which local quadratic convergence is guaranteed. So we are interested in a formulation of the conditions (i)–(iv) from Theorem 4.13 for the new constrained system. To this end, let us introduce Assumption $\hat{2}$, Condition $\hat{5}$, and Condition $\hat{6}$ which are the analoga of Assumption 2, Condition 5, and Condition 6, respectively, for (4.40).

Assumption $\hat{\mathbf{2}}$. There are $\hat{\omega} > 0$ and $\hat{\delta}_2 > 0$ such that

$$\operatorname{dist}[(s, w), \hat{Z}] \leq \hat{\omega} \|\hat{F}(s, w)\|$$

holds for all $(s, w) \in \mathcal{B}_{\hat{\delta}_2}(z^*, w^*) \cap \hat{\Omega}$.

Condition $\hat{5}$. There are $\hat{K}_5 > 0$ and $\hat{\varepsilon}_5 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\|\hat{F}(s,w)\| \leq \hat{K}_5 \|\hat{F}^i(s,w)\|$$

holds for all $(s,w) \in \mathcal{B}_{\hat{\varepsilon}_5}(z^*,w^*) \cap \hat{\Omega}$.

Condition $\hat{6}$. There are $\hat{K}_6 > 0$ and $\hat{\varepsilon}_6 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\operatorname{dist}[(s,w), \hat{Z}_i] \leq \hat{K}_6 \|\hat{F}^i(s,w)\|$$

holds for all $(s,w) \in \mathcal{B}_{\hat{\varepsilon}_6}(z^*,w^*) \cap \hat{\Omega}$.

Furthermore, we need the analogon of the Ω-property for the new system (4.40) which we call $\hat{\Omega}$-property and which is introduced below.

$\hat{\Omega}$-property. There is $\hat{\varepsilon}_{\hat{\Omega}} > 0$ such that

$$\hat{Z}_i \cap \mathcal{B}_{\hat{\varepsilon}_{\hat{\Omega}}}(z^*,w^*) \subseteq \hat{Z}$$

holds for all $i \in \mathcal{A}(z^*)$.

Now we are in the position to state Theorem 4.13 for the reformulation (4.40) of the original system (1.1) by means of slack variables. Theorem 4.14 below provides conditions on the system (4.40) where each of them is sufficient for all of Assumptions 1–4 for (4.40) to hold.

Theorem 4.14. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a PC^1-function and assume that all selection functions have locally Lipschitz continuous derivatives. Moreover, let Ω be defined according to* (4.39) *with an affine function $a : \mathbb{R}^n \to \mathbb{R}^{l_a}$ and a function $b : \mathbb{R}^n \to \mathbb{R}^{l_b}$ which is assumed to be differentiable with a locally Lipschitz continuous Jacobian. Furthermore, let $\hat{\Omega}$ be given by* (4.41)*. Let one of the following conditions be true.*

(i) The function $\hat{F}$ is piecewise affine. Furthermore, the $\hat{\Omega}$-property is satisfied at (z^*,w^*).

(ii) Assumption $\hat{2}$ holds and $\mathcal{A}(z^*)$ is a singleton.

(iii) Assumption $\hat{2}$ and Condition $\hat{5}$ are satisfied. Moreover, the vectors z are split according to $z = (x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that, for every $i \in \mathcal{A}(z^*)$, the function $\hat{F}^i(x^*,\cdot,\cdot)$ is affine. Furthermore, there is $\varepsilon > 0$ such that $(z,w) = (x,y,w) \in \hat{Z} \cap \mathcal{B}_\varepsilon(z^*,w^*)$ implies $x = x^*$.

(iv) Condition $\hat{6}$ and the $\hat{\Omega}$-property are satisfied.

Then the analoga of Assumptions 1–4 for (4.40) *hold if the function $\hat{G}$ satisfies* (4.42) *for all $(z,w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$.*

Note that conditions (i) and (iii) from Theorem 4.13 set additional conditions on Ω whose analoga do not explicitly appear in conditions (i) and (iii) from Theorem 4.14. Condition (i) from Theorem 4.13 required that Ω is polyhedral. Since a is an affine function, $\hat{\Omega}$ is polyhedral so that an explicit requirement is not necessary in condition (i) from Theorem 4.14. Moreover, in condition (iii) from Theorem 4.13 the set $\Omega_y(x^*)$ defined by (4.15) was assumed to be polyhedral. The analogon for the new system (4.40) says that

$$\hat{\Omega}_{y,w}(x^*) := \{(y,w) \in \mathbb{R}^{n_y} \times \mathbb{R}^{l_b} \mid (x^*, y, w) \in \hat{\Omega}\}$$

is polyhedral. Since $\hat{\Omega}$ is a polyhedral set, the latter is clear so that an explicit requirement in condition (iii) from Theorem 4.14 is not necessary.

Theorem 4.13 provides conditions on the original system (1.1) where each of them guarantees local quadratic convergence of our general Newton-type algorithm to a solution of (1.1). In Theorem 4.14 conditions on the reformulation (4.40) are presented which yield local quadratic convergence of Algorithm 3.1 and its special realizations from Sections 3.4 and 3.5 if they are applied to the new system (4.40). Our next aim is to find conditions on the original problem only which imply the whole set of Assumptions 1–4 for both the original system (1.1) and its reformulation (4.40). To this end, we look for conditions on the original problem implying Assumption $\hat{2}$, Condition $\hat{5}$, Condition $\hat{6}$, and the $\hat{\Omega}$-property, respectively. Let us start with the latter. It turns out that the $\hat{\Omega}$-property is equivalent to the Ω-property itself.

Proposition 4.15. *The Ω-property is satisfied if and only if the $\hat{\Omega}$-property holds.*

Proof. Assume that the Ω-property holds near z^* with some $\varepsilon_\Omega > 0$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let $(z,w) \in \hat{Z}_i \cap \mathcal{B}_{\varepsilon_\Omega}(z^*, w^*)$ be arbitrarily chosen. Then $z \in Z_i \cap \mathcal{B}_{\varepsilon_\Omega}(z^*)$ holds. By the Ω-property, $z \in Z$ follows so that $F(z) = 0$ is valid. Furthermore, $w = b(z)$ holds due to $(z,w) \in \hat{Z}_i$ and the definition of $\hat{F}^i$. Consequently, $\hat{F}(z,w) = 0$ holds. Hence, the $\hat{\Omega}$-property is valid with $\hat{\varepsilon}_{\hat{\Omega}} := \varepsilon_\Omega$.

Conversely, let the $\hat{\Omega}$-property be satisfied near (z^*, w^*) with some $\hat{\varepsilon}_{\hat{\Omega}} > 0$. Let $L_0 > 0$ denote a Lipschitz constant of b on $\mathcal{B}_{\hat{\varepsilon}_{\hat{\Omega}}}(z^*)$, i.e.,

$$\|b(z) - b(s)\| \le L_0 \|z - s\| \tag{4.43}$$

holds for all $s, z \in \mathcal{B}_{\hat{\varepsilon}_{\hat{\Omega}}}(z^*)$. Now let us define $\varepsilon_\Omega := \frac{\hat{\varepsilon}_{\hat{\Omega}}}{1+L_0}$, let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed, and let us take any $z \in Z_i \cap \mathcal{B}_{\varepsilon_\Omega}(z^*)$. Moreover, let us set $w := b(z)$. Then $(z,w) \in \hat{Z}_i$ holds. By $w^* = b(z^*)$ and (4.43),

$$\|w - w^*\| = \|b(z) - b(z^*)\| \le L_0 \|z - z^*\| \le L_0 \varepsilon_\Omega$$

follows. Using the definition of ε_Ω, we obtain

$$\left\| \begin{pmatrix} z \\ w \end{pmatrix} - \begin{pmatrix} z^* \\ w^* \end{pmatrix} \right\| \le \|z - z^*\| + \|w - w^*\| \le (1 + L_0)\varepsilon_\Omega = \hat{\varepsilon}_{\hat{\Omega}}$$

and therefore $(z,w) \in \hat{Z}_i \cap \mathcal{B}_{\hat{\varepsilon}_{\hat{\Omega}}}(z^*,w^*)$. The $\hat{\Omega}$-property implies $(z,w) \in \hat{Z}$. Particularly, we have $z \in Z$. Hence, the Ω-property near z^* follows. □

Next, we introduce Conditions 7, 8, and 9 which are conditions on the original constrained system (1.1) and extend Assumption 2, Condition 5, and Condition 6, respectively.

Condition 7. *There are $K_7 > 0$ and $\varepsilon_7 > 0$ such that*

$$\text{dist}[s,Z] \leq K_7(\|F(s)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|)$$

holds for all $s \in \mathcal{B}_{\varepsilon_7}(z^*)$.

Condition 8. *There are $K_8 > 0$ and $\varepsilon_8 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,*

$$\|F(s)\| \leq K_8(\|F^i(s)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|)$$

holds for all $s \in \mathcal{B}_{\varepsilon_8}(z^*)$.

Condition 9. *There are $K_9 > 0$ and $\varepsilon_9 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,*

$$\text{dist}[s,Z_i] \leq K_9(\|F^i(s)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|)$$

holds for all $s \in \mathcal{B}_{\varepsilon_9}(z^*)$.

The next aim is to prove relations of Conditions 7, 8, and 9 to former assumptions and conditions. We start with relations between Conditions 6, $\hat{6}$, and 9. It is obvious that Condition 6 is implied by Condition 9. In the subsequent proposition we show that the converse of this implication is also true if the system $a(z) \geq 0$, $b(z) \geq 0$, which characterizes Ω, satisfies the local error bound condition at z^*. The definition of the local error bound condition for a system of inequalities and equations can be found in Section 2.1. Moreover, we prove that Conditions $\hat{6}$ and 9 are equivalent. Afterwards, in Propositions 4.17 and 4.18, similar relations between Assumption 2, Assumption $\hat{2}$, and Condition 7, and between Conditions 5, $\hat{5}$, and 8 are shown. The proof techniques of the subsequent three propositions are very similar.

Proposition 4.16. *The following assertions are true.*

(a) Condition 9 implies Condition 6. The converse is valid if in addition the system

$$a(z) \geq 0, \quad b(z) \geq 0$$

satisfies the local error bound condition at z^*.

(b) Condition 9 holds if and only if Condition $\hat{6}$ is satisfied.

Proof. Throughout this proof, we denote by $L_0 > 0$ a common Lipschitz constant on $\mathcal{B}_1(z^*)$ for b and those selection functions being active at z^*, i.e., the inequalities

$$\begin{array}{rcl} \|b(z) - b(s)\| & \leq & L_0\|z - s\|, \\ \|F^i(z) - F^i(s)\| & \leq & L_0\|z - s\| \end{array} \tag{4.44}$$

hold for all $i \in \mathcal{A}(z^*)$ and all $s, z \in \mathcal{B}_1(z^*)$. Note that such a constant L_0 exists due to the local Lipschitz continuity of b and the selection functions.

(a) It is obvious that Condition 6 is implied by Condition 9 with $K_6 := K_9$ and $\varepsilon_6 := \varepsilon_9$. Conversely, let Condition 6 be valid and assume that the system $a(z) \geq 0$, $b(z) \geq 0$ satisfies the local error bound condition at z^*. The latter implies that there are $\omega_{EB} > 0$ and $\delta_{EB} > 0$ such that

$$\operatorname{dist}[s, \Omega] \leq \omega_{EB}(\|\min\{0, a(s)\}\| + \|\min\{0, b(s)\}\|) \tag{4.45}$$

holds for all $s \in \mathcal{B}_{\delta_{EB}}(z^*)$. Now let us set $\varepsilon_9 := \min\{\delta_{EB}, \frac{1}{2}\varepsilon_6, \frac{1}{2}\}$ and let $i \in \mathcal{A}(z^*)$ and $s \in \mathcal{B}_{\varepsilon_9}(z^*)$ be arbitrarily chosen. We denote by $s^\perp \in \Omega$ a point satisfying

$$\operatorname{dist}[s, \Omega] = \|s - s^\perp\|. \tag{4.46}$$

With

$$\|s^\perp - z^*\| \leq \|s^\perp - s\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\varepsilon_9 \leq \min\{\varepsilon_6, 1\},$$

Condition 6, the triangle inequality, and (4.44) yield

$$\begin{array}{rcl} \operatorname{dist}[s^\perp, Z_i] & \leq & K_6\|F^i(s^\perp)\| \\ & \leq & K_6(\|F^i(s^\perp) - F^i(s)\| + \|F^i(s)\|) \\ & \leq & K_6(L_0\|s - s^\perp\| + \|F^i(s)\|). \end{array} \tag{4.47}$$

Since Z_i is nonempty and closed, there is $\bar{s} \in Z_i$ such that

$$\operatorname{dist}[s^\perp, Z_i] = \|s^\perp - \bar{s}\|$$

holds. Using this, (4.47), (4.46), and (4.45), we obtain

$$\begin{array}{rcl} \operatorname{dist}[s, Z_i] & \leq & \|s - \bar{s}\| \\ & \leq & \|s - s^\perp\| + \|s^\perp - \bar{s}\| \\ & \leq & (K_6 L_0 + 1)\|s - s^\perp\| + K_6\|F^i(s)\| \\ & \leq & (K_6 L_0 + 1)\omega_{EB}(\|\min\{0, a(s)\}\| + \|\min\{0, b(s)\}\|) \\ & & + K_6\|F^i(s)\| \\ & \leq & \max\{(K_6 L_0 + 1)\omega_{EB}, K_6\}(\|F^i(s)\| + \|\min\{0, a(s)\}\| \\ & & + \|\min\{0, b(s)\}\|). \end{array}$$

Hence, Condition 9 is satisfied with $K_9 := \max\{(K_6 L_0 + 1)\omega_{EB}, K_6\}$.

(b) Suppose that Condition 9 is valid. We set $\hat{\varepsilon}_6 := \min\{\varepsilon_9, \frac{1}{2}\}$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let us take any $(s, w) \in \mathcal{B}_{\hat{\varepsilon}_6}(z^*, w^*) \cap \hat{\Omega}$. Then $s \in \mathcal{B}_{\varepsilon_9}(z^*)$ follows so that Condition 9 implies

$$\operatorname{dist}[s, Z_i] \leq K_9(\|F^i(s)\| + \|\min\{0, a(s)\}\| + \|\min\{0, b(s)\}\|). \tag{4.48}$$

Since (s, w) belongs to $\hat{\Omega}$, $a(s) \geq 0$ holds. This leads to

$$\|\min\{0, a(s)\}\| = 0. \tag{4.49}$$

Let us estimate $\|\min\{0, b(s)\}\|$ from above. To this end, let $j \in \{1, \ldots, l_b\}$ be arbitrary but fixed. If $b_j(s)$ is nonnegative, then

$$|\min\{0, b_j(s)\}| = 0 \leq |b_j(s) - w_j|$$

is obviously satisfied. Otherwise, taking into account $w_j \geq 0$, we have

$$|\min\{0, b_j(s)\}| = -b_j(s) \leq -b_j(s) + w_j = |b_j(s) - w_j|.$$

In each case, $|\min\{0, b_j(s)\}| \leq |b_j(s) - w_j|$ holds. Since $j \in \{1, \ldots, l_b\}$ was arbitrarily chosen,

$$\|\min\{0, b(s)\}\| \leq \|b(s) - w\| \tag{4.50}$$

follows. Combining (4.48)–(4.50), we obtain

$$\operatorname{dist}[s, Z_i] \leq K_9(\|F^i(s)\| + \|b(s) - w\|). \tag{4.51}$$

By the nonemptiness and closedness of Z_i, there is $\bar{s} \in Z_i$ such that

$$\operatorname{dist}[s, Z_i] = \|s - \bar{s}\| \tag{4.52}$$

is valid. We set $\bar{w} := b(\bar{s})$. With

$$\|\bar{s} - z^*\| \leq \|\bar{s} - s\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\hat{\varepsilon}_6 \leq 1,$$

(4.44) yields

$$\|w - \bar{w}\| \leq \|w - b(s)\| + \|b(s) - b(\bar{s})\| \leq \|b(s) - w\| + L_0\|s - \bar{s}\|. \tag{4.53}$$

Obviously, $(\bar{s}, \bar{w})$ is an element of $\hat{Z}_i$. Therefore, using (4.53), (4.52), (4.51), and the definition of $\hat{F}^i$, we obtain

$$\begin{aligned}
\operatorname{dist}[(s, w), \hat{Z}_i] &\leq \|s - \bar{s}\| + \|w - \bar{w}\| \\
&\leq (L_0 + 1)\|s - \bar{s}\| + \|b(s) - w\| \\
&\leq K_9(L_0 + 1)(\|F^i(s)\| + \|b(s) - w\|) + \|b(s) - w\| \\
&\leq \max\{K_9(L_0 + 1), 1\}(\|F^i(s)\| + \|b(s) - w\|) \\
&\leq \sqrt{2}\max\{K_9(L_0 + 1), 1\}\|\hat{F}^i(s, w)\|.
\end{aligned}$$

Hence, Condition $\hat{6}$ is satisfied with $\hat{K}_6 := \sqrt{2}\max\{K_9(L_0+1), 1\}$.

Conversely, assume that Condition $\hat{6}$ holds. Since $\hat{\Omega}$ is a polyhedral set, there is some $\omega_H > 0$ such that

$$\operatorname{dist}[(s,w),\hat{\Omega}] \leq \omega_H(\|\min\{0,a(s)\}\| + \|\min\{0,w\}\|)$$

is valid for all $(s,w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$. This follows from Theorem 2.1. In particular, the system $a(z) \geq 0$, $w \geq 0$ satisfies the local error bound condition at z^* (with arbitrary $\delta_{EB} > 0$). Thus, the application of the second assertion of item (a) of the current proposition to the constrained system (4.40) implies the validity of the analogon of Condition 9 for (4.40). The latter means that there are $\hat{K}_9 > 0$ and $\hat{\varepsilon}_9 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\operatorname{dist}[(s,w),\hat{Z}_i] \leq \hat{K}_9(\|\hat{F}^i(s,w)\| + \|\min\{0,a(s)\}\| + \|\min\{0,w\}\|) \quad (4.54)$$

holds for all $(s,w) \in \mathcal{B}_{\hat{\varepsilon}_9}(z^*,w^*)$. Now let us define ε_9 according to $\varepsilon_9 := \min\{\frac{\hat{\varepsilon}_9}{L_0+1}, 1\}$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let $s \in \mathcal{B}_{\varepsilon_9}(z^*)$ be arbitrarily chosen. We set $w := b(s)$. By $w^* = b(z^*)$ and (4.44),

$$\|w - w^*\| = \|b(s) - b(z^*)\| \leq L_0\|s - z^*\|$$

follows. Consequently, we have

$$\left\|\begin{pmatrix} s \\ w \end{pmatrix} - \begin{pmatrix} z^* \\ w^* \end{pmatrix}\right\| \leq \|s - z^*\| + \|w - w^*\| \leq (L_0+1)\varepsilon_9 \leq \hat{\varepsilon}_9.$$

Therefore, (4.54), together with the definitions of w and $\hat{F}^i$, implies

$$\operatorname{dist}[(s,w),\hat{Z}_i] \leq \hat{K}_9(\|F^i(s)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|). \quad (4.55)$$

Since $\hat{Z}_i$ is nonempty and closed, there is $(\bar{s},\bar{w}) \in \hat{Z}_i$ such that

$$\operatorname{dist}[(s,w),\hat{Z}_i] = \left\|\begin{pmatrix} s \\ w \end{pmatrix} - \begin{pmatrix} \bar{s} \\ \bar{w} \end{pmatrix}\right\| \quad (4.56)$$

holds. Obviously, $\bar{s}$ belongs to Z_i. Thus, by (4.56) and (4.55),

$$\begin{aligned} \operatorname{dist}[s,Z_i] &\leq \|s - \bar{s}\| \\ &\leq \operatorname{dist}[(s,w),\hat{Z}_i] \\ &\leq \hat{K}_9(\|F^i(s)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|) \end{aligned}$$

follows. Hence, Condition 9 is satisfied with $K_9 := \hat{K}_9$.

□

Proposition 4.17. *The following assertions are true.*

(a) Condition 7 implies Assumption 2. The converse is valid if in addition the system

$$a(z) \geq 0, \quad b(z) \geq 0$$

satisfies the local error bound condition at z^*.

(b) Condition 7 holds if and only if Assumption $\hat{2}$ *is satisfied.*

Proof. The detailed proof is omitted here because it is almost the same, word for word, as the proof of Proposition 4.16. We have just to replace F^i by F, $\hat{F}^i$ by $\hat{F}$, Z_i by Z, $\hat{Z}_i$ by $\hat{Z}$, Condition 6 by Assumption 2, Condition $\hat{6}$ by Assumption $\hat{2}$, Condition 9 by Condition 7, K_6 by ω, $\hat{K}_6$ by $\hat{\omega}$, K_9 by K_7, $\hat{K}_9$ by $\hat{K}_7$, ε_6 by δ_2, $\hat{\varepsilon}_6$ by $\hat{\delta}_2$, ε_9 by ε_7, and $\hat{\varepsilon}_9$ by $\hat{\varepsilon}_7$. Note that $L_0 > 0$ denotes a common Lipschitz constant on $\mathcal{B}_1(z^*)$ for b and F in that setting. Such a constant exists since b is continuously differentiable, and F is locally Lipschitz continuous due to Proposition 4.1. □

Proposition 4.18. *The following assertions are true.*

(a) Condition 8 implies Condition 5. The converse is valid if in addition the system

$$a(z) \geq 0, \quad b(z) \geq 0$$

satisfies the local error bound condition at z^*.

(b) Condition 8 holds if and only if Condition $\hat{5}$ *is satisfied.*

Proof. Throughout this proof, $L_0 > 0$ denotes a common Lipschitz constant on $\mathcal{B}_1(z^*)$ for b, F, and those selection functions being active at z^*, i.e., the inequalities

$$\begin{aligned} \|b(z) - b(s)\| &\leq L_0\|z - s\|, \\ \|F(z) - F(s)\| &\leq L_0\|z - s\|, \\ \|F^i(z) - F^i(s)\| &\leq L_0\|z - s\| \end{aligned} \tag{4.57}$$

are valid for all $i \in \mathcal{A}(z^*)$ and all $s, z \in \mathcal{B}_1(z^*)$. Note that such a constant L_0 exists due to the local Lipschitz continuity of b and the selection functions and Proposition 4.1.

(a) If Condition 8 holds, then Condition 5 is obviously satisfied with $K_5 := K_8$ and $\varepsilon_5 := \varepsilon_8$. Conversely, suppose that Condition 5 is valid and the system $a(z) \geq 0$, $b(z) \geq 0$ satisfies the local error bound condition at z^*. The latter implies that there are $\omega_{EB} > 0$ and $\delta_{EB} > 0$ such that

$$\operatorname{dist}[s, \Omega] \leq \omega_{EB}(\|\min\{0, a(s)\}\| + \|\min\{0, b(s)\}\|) \tag{4.58}$$

holds for all $s \in \mathcal{B}_{\delta_{EB}}(z^*)$. Now let us set $\varepsilon_8 := \min\{\delta_{EB}, \frac{1}{2}\varepsilon_5, \frac{1}{2}\}$ and let $i \in \mathcal{A}(z^*)$ and $s \in \mathcal{B}_{\varepsilon_8}(z^*)$ be arbitrarily chosen. We denote by $s^\perp \in \Omega$ a point satisfying

$$\operatorname{dist}[s, \Omega] = \|s - s^\perp\|. \tag{4.59}$$

With

$$\|s^\perp - z^*\| \leq \|s^\perp - s\| + \|s - z^*\| \leq 2\|s - z^*\| \leq 2\varepsilon_8 \leq \min\{\varepsilon_5, 1\},$$

(4.57) and Condition 5 yield

$$\begin{aligned} \|F(s)\| &\leq \|F(s) - F(s^\perp)\| + \|F(s^\perp)\| \\ &\leq L_0\|s - s^\perp\| + \|F(s^\perp)\| \\ &\leq L_0\|s - s^\perp\| + K_5\|F^i(s^\perp)\| \\ &\leq L_0\|s - s^\perp\| + K_5(\|F^i(s^\perp) - F^i(s)\| + \|F^i(s)\|) \\ &\leq K_5\|F^i(s)\| + L_0(K_5 + 1)\|s - s^\perp\|. \end{aligned}$$

Using this, (4.59), and (4.58), we obtain

$$\begin{aligned} \|F(s)\| &\leq K_5\|F^i(s)\| + L_0(K_5 + 1)\omega_{EB}(\|\min\{0, a(s)\}\| \\ &\qquad + \|\min\{0, b(s)\}\|) \\ &\leq \max\{K_5, L_0\omega_{EB}(K_5 + 1)\}(\|F^i(s)\| + \|\min\{0, a(s)\}\| \\ &\qquad + \|\min\{0, b(s)\}\|). \end{aligned}$$

Hence, Condition 8 is satisfied with $K_8 := \max\{K_5, L_0\omega_{EB}(K_5 + 1)\}$.

(b) Suppose that Condition 8 is valid. We set $\hat{\varepsilon}_5 := \min\{\varepsilon_8, \frac{1}{2}\}$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let us take any $(s, w) \in \mathcal{B}_{\hat{\varepsilon}_5}(z^*, w^*) \cap \hat{\Omega}$. Then $s \in \mathcal{B}_{\varepsilon_8}(z^*)$ holds so that Condition 8 implies

$$\|F(s)\| \leq K_8(\|F^i(s)\| + \|\min\{0, a(s)\}\| + \|\min\{0, b(s)\}\|). \tag{4.60}$$

Since (s, w) belongs to $\hat{\Omega}$, we have $a(s) \geq 0$. This leads to

$$\|\min\{0, a(s)\}\| = 0. \tag{4.61}$$

Moreover, it was shown in the proof of Proposition 4.16 that

$$\|\min\{0, b(s)\}\| \leq \|b(s) - w\| \tag{4.62}$$

is satisfied. Combining (4.60)–(4.62) and using the definition of $\hat{F}^i$, we obtain

$$\|F(s)\| \leq \sqrt{2}K_8\|\hat{F}^i(s, w)\|.$$

This and the definition of $\hat{F}$ yield

$$\|\hat{F}(s,w)\| \le \|F(s)\| + \|b(s) - w\| \le (\sqrt{2}K_8 + 1)\|\hat{F}^i(s,w)\|.$$

Thus, Condition $\hat{5}$ holds with $\hat{K}_5 := \sqrt{2}K_8 + 1$.

Conversely, let Condition $\hat{5}$ be satisfied. Since $\hat{\Omega}$ is a polyhedral set, there is some $\omega_H > 0$ such that

$$\text{dist}[(s,w),\hat{\Omega}] \le \omega_H(\|\min\{0,a(s)\}\| + \|\min\{0,w\}\|)$$

is valid for all $(s,w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$. This follows from Theorem 2.1. In particular, the system $a(z) \ge 0$, $w \ge 0$ satisfies the local error bound condition at z^* (with arbitrary $\delta_{EB} > 0$). Thus, the application of the second assertion of item (a) of the current proposition to the constrained system (4.40) implies the validity of the analogon of Condition 8 for (4.40). The latter means that there are $\hat{K}_8 > 0$ and $\hat{\varepsilon}_8 > 0$ such that, for every $i \in \mathcal{A}(z^*)$,

$$\|\hat{F}(s,w)\| \le \hat{K}_8(\|\hat{F}^i(s,w)\| + \|\min\{0,a(s)\}\| + \|\min\{0,w\}\|) \qquad (4.63)$$

holds for all $(s,w) \in \mathcal{B}_{\hat{\varepsilon}_8}(z^*,w^*)$. Now let us define ε_8 according to $\varepsilon_8 := \min\{\frac{\hat{\varepsilon}_8}{L_0+1}, 1\}$. Let $i \in \mathcal{A}(z^*)$ be arbitrary but fixed and let us take any $s \in \mathcal{B}_{\varepsilon_8}(z^*)$. We set $w := b(s)$. By $w^* = b(z^*)$ and (4.57),

$$\|w - w^*\| = \|b(s) - b(z^*)\| \le L_0\|s - z^*\|$$

follows. Consequently, we have

$$\left\| \begin{pmatrix} s \\ w \end{pmatrix} - \begin{pmatrix} z^* \\ w^* \end{pmatrix} \right\| \le \|s - z^*\| + \|w - w^*\| \le (L_0 + 1)\varepsilon_8 \le \hat{\varepsilon}_8.$$

Therefore, (4.63), together with the definitions of w, $\hat{F}$, and $\hat{F}^i$, implies

$$\begin{aligned} \|F(s)\| &= \|\hat{F}(s,w)\| \\ &\le \hat{K}_8(\|\hat{F}^i(s,w)\| + \|\min\{0,a(s)\}\| + \|\min\{0,w\}\|) \\ &= \hat{K}_8(\|F^i(s,w)\| + \|\min\{0,a(s)\}\| + \|\min\{0,b(s)\}\|). \end{aligned}$$

Hence, Condition 8 is satisfied with $K_8 := \hat{K}_8$.

□

Now we are in the position to state the main result of this section. The subsequent theorem is obtained by combining the assertions of Theorem 4.13, Theorem 4.14, and Propositions 4.15–4.18. Theorem 4.19 provides conditions on the original problem (1.1) where each of them implies the whole set of Assumptions 1–4, not only for (1.1) but also for the reformulation (4.40) by means of slack variables. Therefore, each of the conditions (i)–(iv) in the following theorem guarantees local quadratic convergence of our general Newton-type algorithm from Section 3.1 and its special realizations from Sections 3.4 and 3.5 if they are applied to the original problem and also if they are applied to (4.40).

Theorem 4.19. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a PC^1-function and assume that all selection functions $F^1, \dots, F^t$ have locally Lipschitz continuous derivatives. Moreover, let Ω be defined according to* (4.39) *with an affine function $a : \mathbb{R}^n \to \mathbb{R}^{l_a}$ and a function $b : \mathbb{R}^n \to \mathbb{R}^{l_b}$ which is assumed to be differentiable with a locally Lipschitz continuous Jacobian. Let one of the following conditions be true.*

(i) The function F is piecewise affine and b is an affine function. Furthermore, the Ω-property is satisfied at z^*.

(ii) Condition 7 holds and $\mathcal{A}(z^*)$ is a singleton.

(iii) Conditions 7 and 8 are satisfied. Moreover, the vectors z are split according to $z = (x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that the function $b(x^*, \cdot)$ is affine and, for every $i \in \mathcal{A}(z^*)$, the function $F^i(x^*, \cdot)$ is affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x, y) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$.

(iv) Condition 9 and the Ω-property are satisfied.

Then the following assertions are valid.

(a) Assumptions 1–4 hold if the function G satisfies (4.1) *for all $z \in \mathbb{R}^n$.*

(b) The analoga of Assumptions 1–4 for (4.40) *are satisfied if the function $\hat{G}$ satisfies* (4.42) *for all $(z, w) \in \mathbb{R}^n \times \mathbb{R}^{l_b}$.*

In Subsection 4.1.3 examples were considered where F had a similar structure like the function in Example 4.1. We presented suitable choices of the set Ω implying Condition 5, see Examples 4.7 and 4.8. In the subsequent examples it is shown that even Condition 8 is satisfied for the resulting constrained systems.

Example 4.9. Let $F : \mathbb{R}^n \to \mathbb{R}^{p+r}$ be defined according to (4.2), i.e.,

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R(z), S(z)\} \end{pmatrix}$$

with given continuously differentiable functions $P : \mathbb{R}^n \to \mathbb{R}^p$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$ whose derivatives are assumed to be locally Lipschitz continuous. The function F is a PC^1-function with 2^r selection functions $F^1, \dots, F^{2^r}$. Moreover, let $\Omega \subseteq \mathbb{R}^n$ be the set from Example 4.7, i.e.,

$$\Omega := \{z \in \mathbb{R}^n \mid R(z) + S(z) \geq 0\},$$

and let z^* be an arbitrary but fixed solution of (1.1). We are going to prove Condition 8. To this end, let us take any $s \in \mathbb{R}^n$ and let $i \in \mathcal{A}(z^*)$ and $j \in \{1, \ldots, r\}$ be arbitrarily chosen. Without loss of generality we assume that $R_j(s) \leq S_j(s)$ holds. If $R_j(s) + S_j(s)$ is nonnegative, we have

$$|\min\{R_j(s), S_j(s)\}| = \min\{|R_j(s)|, |S_j(s)|\}.$$

This was verified in Example 4.7, see (4.29). Now let us suppose that $R_j(s) + S_j(s) < 0$ is satisfied. Then $R_j(s)$ is negative. If $S_j(s)$ is negative as well,

$$|\min\{R_j(s), S_j(s)\}| = -R_j(s) < -R_j(s) - S_j(s) = |\min\{0, R_j(s) + S_j(s)\}|$$

is valid. Otherwise, if $S_j(s) \geq 0$ holds, we obtain $S_j(s) = |S_j(s)| < |R_j(s)|$ due to $R_j(s) + S_j(s) < 0$. This implies

$$\begin{aligned} |\min\{R_j(s), S_j(s)\}| &= -R_j(s) \\ &= -R_j(s) - S_j(s) + S_j(s) \\ &= |\min\{0, R_j(s) + S_j(s)\}| + \min\{|R_j(s)|, |S_j(s)|\}. \end{aligned}$$

In each case, regardless of whether $R_j(s) + S_j(s)$ is negative or nonnegative,

$$|\min\{R_j(s), S_j(s)\}| \leq \min\{|R_j(s)|, |S_j(s)|\} + |\min\{0, R_j(s) + S_j(s)\}| \quad (4.64)$$

is valid. Since j was arbitrarily chosen, (4.64) holds for all $j = 1, \ldots, r$. Thus, taking into account the structure of F^i, we obtain

$$\|F(s)\| \leq \|F^i(s)\| + \|\min\{0, R(s) + S(s)\}\|$$

Hence, Condition 8 is satisfied with $K_8 := 1$ and arbitrary $\varepsilon_8 > 0$. □

Example 4.10. Consider the setting of Example 4.8, i.e., let $F : \mathbb{R}^n \to \mathbb{R}^{p+r}$ be defined according to

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R^1(z), \ldots, R^l(z)\} \end{pmatrix}$$

and let Ω be given by

$$\Omega := \{z \in \mathbb{R}^n \mid \forall k \in \{1, \ldots, l\} : R^k(z) \geq 0\}.$$

The functions $P : \mathbb{R}^n \to \mathbb{R}^p$ and $R^k : \mathbb{R}^n \to \mathbb{R}^r$ $(k = 1, \ldots, l)$ are assumed to be differentiable with locally Lipschitz continuous derivatives. The function F is a PC^1-function with l^r selection functions $F^1, \ldots, F^{l^r}$. Let z^* be an arbitrary but fixed solution of (1.1). In order to verify Condition 8, let us take any $s \in \mathbb{R}^n$ and let $i \in \mathcal{A}(z^*)$ and $j \in \{1, \ldots, r\}$ be arbitrarily chosen. Assume that

$k_0 \in \{1, \dots, l\}$ is an index with the property $\min\{R_j^1(s), \dots, R_j^l(s)\} = R_j^{k_0}(s)$. If $R_j^{k_0}(s)$ is nonnegative, then

$$|\min\{R_j^1(s), \dots, R_j^l(s)\}| = R_j^{k_0}(s) = \min\{|R_j^1(s)|, \dots, |R_j^l(s)|\}$$

is obviously valid. Otherwise, if $R_j^{k_0}(s) < 0$ holds, we obtain

$$|\min\{R_j^1(s), \dots, R_j^l(s)\}| = -R_j^{k_0}(s) = |\min\{0, R_j^{k_0}(s)\}|.$$

In each case, regardless of whether $R_j^{k_0}(s)$ is negative or nonnegative,

$$\begin{aligned} &|\min\{R_j^1(s), \dots, R_j^l(s)\}| \\ \leq\ & \min\{|R_j^1(s)|, \dots, |R_j^l(s)|\} + \sum_{k=1}^{l} |\min\{0, R_j^k(s)\}| \end{aligned} \tag{4.65}$$

is satisfied. Since j was arbitrarily chosen, (4.65) holds for all $j = 1, \dots, r$. Thus, taking into account the structure of F^i,

$$\|F(s)\| \leq \|F^i(s)\| + \sum_{k=1}^{l} \|\min\{0, R^k(s)\}\|$$

follows. Hence, Condition 8 is satisfied with $K_8 := 1$ and arbitrary $\varepsilon_8 > 0$. □

4.3 Application to Complementarity Systems

This section is devoted to the application of Algorithm 3.1 and its special realizations for the solution of the complementarity system (1.2), i.e., the system

$$P(z) = 0, \quad Q(z) \geq 0, \quad R(z) \geq 0,\ S(z) \geq 0,\ R(z)^\top S(z) = 0.$$

The functions $P : \mathbb{R}^n \to \mathbb{R}^p$, $Q : \mathbb{R}^n \to \mathbb{R}^q$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$ are assumed to be differentiable with locally Lipschitz continuous derivatives. As already mentioned in the Introduction, there are many problems which have the form (1.2), for instance nonlinear and linear complementarity problems and KKT systems arising from optimization problems, variational inequalities, or GNEPs.

We will present an equivalent reformulation of (1.2) as a constrained system of equations. Thereafter, conditions (i)–(iv) from Theorem 4.19 are discussed for the resulting system. In particular, it will turn out that Condition 8 and the Ω-property are always satisfied. The main result of this section is Theorem 4.20 which is an adapted formulation of Theorem 4.19 for the constrained system arising from (1.2) by reformulation and therefore provides conditions implying all of Assumptions 1–4. Afterwards, a constant rank condition is presented which turns out to be also sufficient for the whole set of Assumptions 1–4 to hold.

There are many possibilities to reformulate (1.2) as a constrained system of equations. We consider the following nonsmooth one:

$$F(z) := \begin{pmatrix} P(z) \\ \min\{R(z), S(z)\} \end{pmatrix} \quad \text{s.t.} \quad z \in \Omega \tag{4.66}$$

where Ω is given by

$$\Omega := \{z \in \mathbb{R}^n \mid Q(z) \geq 0,\ R(z) \geq 0,\ S(z) \geq 0\}. \tag{4.67}$$

Note that Ω is closed since Q, R, and S are continuous. Moreover, we assume that Ω is nonempty. It is not difficult to see that every solution of (1.2) is also a solution of (4.66) and vice versa. Note that the constraints $R(z) \geq 0$ and $S(z) \geq 0$ are not necessary for the definition of Ω to state the latter equivalence. However, we know from Section 4.1 that local quadratic convergence of Algorithm 3.1 cannot be expected in general without including these or similar inequalities to describe Ω.

Throughout this section, Z denotes the solution set of the complementarity system (1.2) (which coincides with the solution set of the constrained system (4.66). We suppose that Z is nonempty and denote by z^* an arbitrary but fixed element of Z.

The function F in (4.66) is the same function as in Example 4.1. As stated there, F is a PC^1-function with 2^r selection functions $F^1, \ldots, F^{2^r}$. We assume that $G : \mathbb{R}^n \to \mathbb{R}^{(p+r)\times n}$ is a function satisfying (4.1) for all $z \in \mathbb{R}^n$. One possibility for an appropriate definition of G is given in Example 4.1.

Our aim is to formulate Theorem 4.19 in the context of the complementarity system (1.2) and its reformulation (4.66). To this end, let us discuss Conditions 7–9 for (4.66). Taking into account the definitions of F and Ω, the following condition is equivalent to Condition 7 for problem (4.66) with Ω given according to (4.67).

Condition 7a. There are $K_{7a} > 0$ and $\varepsilon_{7a} > 0$ such that

$$\begin{aligned} \operatorname{dist}[s, Z] \quad \leq \quad & K_{7a}(\|P(s)\| + \|\min\{R(s), S(s)\}\| + \|\min\{0, Q(s)\}\| \\ & + \|\min\{0, R(s)\}\| + \|\min\{0, S(s)\}\|) \end{aligned}$$

holds for all $s \in \mathcal{B}_{\varepsilon_{7a}}(z^*) \cap \Omega$.

Next, let us consider Condition 8 for (4.66). It follows from Example 4.10 that, for every $i \in \mathcal{A}(z^*)$,

$$\|F(s)\| \leq \|F^i(s)\| + \|\min\{0, R(s)\}\| + \|\min\{0, S(s)\}\|$$

holds for all $s \in \mathbb{R}^n$. In particular,

$$\|F(s)\| \leq \|F^i(s)\| + \|\min\{0, Q(s)\}\| + \|\min\{0, R(s)\}\| + \|\min\{0, S(s)\}\|$$

is valid for all $i \in \mathcal{A}(z^*)$ and all $s \in \mathbb{R}^n$. The latter shows that Condition 8 is always satisfied for (4.66), with arbitrary $\varepsilon_8 > 0$. Moreover, due to Propositions 4.18 and 4.7, the Ω-property always holds for the constrained system (4.66) with Ω given by (4.67).

In order to discuss Condition 9, let us consider the structure of the selection functions being active at z^*. To this end, the following index sets are introduced:

$$\begin{aligned}
\mathcal{I}_R &:= \{j \in \{1, \dots, r\} \mid 0 = R_j(z^*) < S_j(z^*)\},\\
\mathcal{I}_S &:= \{j \in \{1, \dots, r\} \mid 0 = S_j(z^*) < R_j(z^*)\},\\
\mathcal{I}_= &:= \{j \in \{1, \dots, r\} \mid 0 = R_j(z^*) = S_j(z^*)\}.
\end{aligned}$$

Obviously, these index sets partition the set $\{1, \dots, r\}$, i.e., they are pairwise disjoint and their union is the set $\{1, \dots, r\}$ itself. It is not difficult to see that a selection function F^i is active at z^* (i.e., $i \in \mathcal{A}(z^*)$) if and only if there is a partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$ such that F^i is, after some row permutations if necessary, equal to the function $F^{\mathcal{I}_1,\mathcal{I}_2}$ defined according to

$$F^{\mathcal{I}_1,\mathcal{I}_2}(z) := \begin{pmatrix} P(z) \\ R_{\mathcal{I}_R \cup \mathcal{I}_1}(z) \\ S_{\mathcal{I}_S \cup \mathcal{I}_2}(z) \end{pmatrix}. \tag{4.68}$$

The set of all zeros of $F^{\mathcal{I}_1,\mathcal{I}_2}$ belonging to Ω is denoted by $Z_{\mathcal{I}_1,\mathcal{I}_2}$, i.e.,

$$\begin{aligned}
Z_{\mathcal{I}_1,\mathcal{I}_2} &:= \{z \in \mathbb{R}^n \mid F^{\mathcal{I}_1,\mathcal{I}_2}(z) = 0,\ Q(z) \geq 0,\ R(z) \geq 0,\ S(z) \geq 0\}\\
&= \{z \in \mathbb{R}^n \mid P(z) = 0,\ R_{\mathcal{I}_R \cup \mathcal{I}_1}(z) = 0,\ S_{\mathcal{I}_S \cup \mathcal{I}_2}(z) = 0,\ Q(z) \geq 0,\\
&\qquad R_{\mathcal{I}_S \cup \mathcal{I}_2}(z) \geq 0,\ S_{\mathcal{I}_R \cup \mathcal{I}_1}(z) \geq 0\}.
\end{aligned}$$

The strict complementarity condition is satisfied at z^* if $\mathcal{I}_=$ is empty. In that case, exactly one of the selection functions is active at z^*, i.e., $\mathcal{A}(z^*)$ is a singleton.

Now we are in the position to formulate Condition 9a which is equivalent to Condition 9 for (4.66) with Ω defined according to (4.67).

Condition 9a. There are $K_{9a} > 0$ and $\varepsilon_{9a} > 0$ such that, for every partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$,

$$\begin{aligned}
\operatorname{dist}[s, Z_{\mathcal{I}_1,\mathcal{I}_2}] \leq\ & K_{9a}(\|P(s)\| + \|R_{\mathcal{I}_R \cup \mathcal{I}_1}(s)\| + \|S_{\mathcal{I}_S \cup \mathcal{I}_2}(s)\| + \|\min\{0, Q(s)\}\|\\
&+ \|\min\{0, R_{\mathcal{I}_S \cup \mathcal{I}_2}(s)\}\| + \|\min\{0, S_{\mathcal{I}_R \cup \mathcal{I}_1}(s)\}\|)
\end{aligned}$$

holds for all $s \in \mathcal{B}_{\varepsilon_{9a}}(z^*)$.

In other words, Condition 9a requires that, for every partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$, the following system of equations and inequalities satisfies the local error bound condition at z^*:

$$\begin{gathered}
P(z) = 0,\ Q(z) \geq 0,\ R_{\mathcal{I}_R \cup \mathcal{I}_1}(z) = 0,\ S_{\mathcal{I}_S \cup \mathcal{I}_2}(z) = 0,\\
R_{\mathcal{I}_S \cup \mathcal{I}_2}(z) \geq 0,\ S_{\mathcal{I}_R \cup \mathcal{I}_1}(z) \geq 0.
\end{gathered} \tag{4.69}$$

For the definition of the local error bound condition we refer to Section 2.1. Sufficient conditions for it to hold can also be found there. In particular, it follows from Theorem 2.2 that Condition 9a holds if, for every partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$, the system (4.69) satisfies MFCQ or RCRCQ at z^*.

The subsequent theorem is the main result of this section. It provides conditions on the complementarity system (1.2) where each of them implies the whole set of Assumptions 1–4 for the constrained system (4.66). Furthermore, each of the conditions (i)–(iv) in Theorem 4.20 is sufficient for the analoga of Assumptions 1–4 for suitable reformulations of (4.66) by means of slack variables, see Remark 4.4 below. The weakest among the conditions (i)–(iv) in the following theorem is condition (iv), i.e., Condition 9a, see Subsection 4.1.4 for details.

Theorem 4.20. *Let $P : \mathbb{R}^n \to \mathbb{R}^p$, $Q : \mathbb{R}^n \to \mathbb{R}^q$, $R : \mathbb{R}^n \to \mathbb{R}^r$, and $S : \mathbb{R}^n \to \mathbb{R}^r$ be differentiable functions with locally Lipschitz continuous derivatives. Let one of the following conditions be true.*

(i) The functions P, Q, R, and S are affine.

(ii) Condition 7a holds and the set $\mathcal{I}_=$ is empty, i.e., the strict complementarity condition is satisfied at z^*.

(iii) Condition 7a is satisfied. Moreover, the vectors z are split according to $z = (x, y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$ such that the functions $P(x^*, \cdot)$, $Q(x^*, \cdot)$, $R(x^*, \cdot)$, and $S(x^*, \cdot)$ are affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x, y) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$.

(iv) Condition 9a is satisfied.

Then Assumptions 1–4 for (4.66) *hold if the function G satisfies* (4.1) *for all $z \in \mathbb{R}^n$.*

Proof. The assertions follow from Theorem 4.19, together with the considerations of the current section up to now. In particular, we have verified that Condition 8 and the Ω-property are satisfied for (4.66) with Ω given according to (4.67). Therefore, an explicit requirement in the conditions (i)–(iv) above is not necessary. □

Remark 4.4. (a) The constrained system

$$\hat{F}(z, w^1, w^2, w^3) = 0 \quad \text{s.t.} \quad (z, w^1, w^2, w^3) \in \hat{\Omega} \tag{4.70}$$

with

$$\hat{F}(z, w^1, w^2, w^3) := \begin{pmatrix} P(z) \\ \min\{R(z), S(z)\} \\ Q(z) - w^1 \\ R(z) - w^2 \\ S(z) - w^3 \end{pmatrix}$$

and

$$\hat{\Omega} := \mathbb{R}^n \times \mathbb{R}^q_+ \times \mathbb{R}^r_+ \times \mathbb{R}^r_+$$

is a suitable reformulation of (4.66) by means of slack variables. Remember that such a reformulation is advisable from the computational point of view. The analoga of Assumptions 1–4 for (4.70) are implied by each of the conditions (i)–(iv) from Theorem 4.20 (if the corresponding matrix-valued function $\hat{G}$ is suitably chosen). This follows from item (b) of Theorem 4.19, together with the considerations of the current section up to now. If some components of the functions Q, R, or S are affine, then less than $q + 2r$ slack variables are necessary.

(b) The solution set of (4.70) does not change if the function $\hat{F}$ is replaced by $\tilde{F}$ defined according to

$$\tilde{F}(z, w^1, w^2, w^3) := \begin{pmatrix} P(z) \\ \min\{w^2, w^3\} \\ Q(z) - w^1 \\ R(z) - w^2 \\ S(z) - w^3 \end{pmatrix}.$$

It is not difficult to prove that the analoga of Assumptions 1–4 for the resulting constrained system are still implied by each of the conditions (i)–(iv) from Theorem 4.20 (if the corresponding matrix-valued function $\tilde{G}$ is suitably chosen).

(c) The assertions of Theorem 4.20 stay true if Ω in (4.66) is defined by

$$\Omega := \{z \in \mathbb{R}^n \mid Q(z) \geq 0,\ R(z) + S(z) \geq 0\}. \tag{4.71}$$

In fact, we know from Example 4.9 that Condition 8 is satisfied for the resulting constrained system. Therefore, the Ω-property holds as well. Furthermore, Condition 7a is satisfied if and only if its analogon for (4.66) with Ω from (4.71) is valid. This essentially follows from the fact that

$$\begin{aligned} &|\min\{0, R_j(z)\}| + |\min\{0, S_j(z)\}| \\ &\quad\leq\ |\min\{R_j(z), S_j(z)\}| + |\min\{0, R_j(z) + S_j(z)\}| \end{aligned}$$

and

$$|\min\{0, R_j(z) + S_j(z)\}| \leq |\min\{0, R_j(z)\}| + |\min\{0, S_j(z)\}|$$

hold for all $j = 1, \ldots, r$ and all $z \in \mathbb{R}^n$. The latter inequalities are not difficult to verify. By similar arguments, it can be justified that Condition 9a is satisfied if and only if its analogon for (4.66) with Ω defined by (4.71) holds.

Similarly, the assertions of item (a) of this remark stay true if $\hat{\Omega}$ is given according to

$$\hat{\Omega} := \{(z, w^1, w^2, w^3) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^r \times \mathbb{R}^r \mid w^1 \geq 0,\ w^2 + w^3 \geq 0\}.$$

At the end of this section we present a sufficient condition for Condition 9a to hold. More precisely, it is proved in the following proposition that a certain constant rank condition implies Condition 9a. In particular, using Theorem 4.20, this constant rank condition yields the whole set of Assumptions 1–4 for (4.66). We need the index set $\mathcal{I}_Q$ defined by

$$\mathcal{I}_Q := \{j \in \{1, \ldots, q\} \mid Q_j(z^*) = 0\}.$$

Proposition 4.21. *Assume that there is some $\varepsilon > 0$ such that, for every triple $(\mathcal{K}_Q, \mathcal{K}_R, \mathcal{K}_S)$ of index sets $\mathcal{K}_Q \subseteq \mathcal{I}_Q$, $\mathcal{K}_R \subseteq \mathcal{I}_=$, and $\mathcal{K}_S \subseteq \mathcal{I}_=$, the matrices*

$$\begin{pmatrix} P'(s) \\ Q'_{\mathcal{K}_Q}(s) \\ R'_{\mathcal{I}_R \cup \mathcal{K}_R}(s) \\ S'_{\mathcal{I}_S \cup \mathcal{K}_S}(s) \end{pmatrix} \tag{4.72}$$

have the same rank for all $s \in \mathcal{B}_\varepsilon(z^*)$. Then Condition 9a is valid. In particular, Assumptions 1–4 for (4.66) *hold if G satisfies* (4.1) *for all $z \in \mathbb{R}^n$.*

Proof. It suffices to show that Condition 9a is satisfied. Then the second assertion follows from Theorem 4.20. Let $(\mathcal{I}_1, \mathcal{I}_2)$ be an arbitrary but fixed partition of $\mathcal{I}_=$. We are going to show that the system (4.69) satisfies the RCRCQ at z^*. To this end, let us take any index sets $\mathcal{M}_Q \subseteq \mathcal{I}_Q$, $\mathcal{M}_R \subseteq \mathcal{I}_2$, and $\mathcal{M}_S \subseteq \mathcal{I}_1$. Note that the inequalities $Q_{\mathcal{I}_Q}(z) \geq 0$, $R_{\mathcal{I}_2}(z) \geq 0$, and $S_{\mathcal{I}_1}(z) \geq 0$ are precisely those inequalities in (4.69) which are active at z^*. Using the assumption of the proposition for $\mathcal{K}_Q := \mathcal{M}_Q$, $\mathcal{K}_R := \mathcal{I}_1 \cup \mathcal{M}_R$, and $\mathcal{K}_S := \mathcal{I}_2 \cup \mathcal{M}_S$, we obtain that the matrices

$$\begin{pmatrix} P'(s) \\ Q'_{\mathcal{M}_Q}(s) \\ R'_{\mathcal{I}_R \cup \mathcal{I}_1 \cup \mathcal{M}_R}(s) \\ S'_{\mathcal{I}_S \cup \mathcal{I}_2 \cup \mathcal{M}_S}(s) \end{pmatrix}$$

have the same rank for all $s \in \mathcal{B}_\varepsilon(z^*)$. Since $\mathcal{M}_Q$, $\mathcal{M}_R$, and $\mathcal{M}_S$ were arbitrarily chosen, RCRCQ is satisfied for (4.69) at z^*. By Theorem 2.2, the local error bound condition for (4.69) at z^* follows. Thus, since $(\mathcal{I}_1, \mathcal{I}_2)$ was an arbitrarily chosen partition of $\mathcal{I}_=$, Condition 9a holds. □

We want to emphasize that the rank of the matrices in (4.72) is allowed to be dependent on the triple $(\mathcal{K}_Q, \mathcal{K}_R, \mathcal{K}_S)$ in the above proposition. It is only required that the rank does not depend on s for all points s in some neighborhood of z^*.

Chapter 5

Application to Special Problem Classes

In Chapter 3 a general Newton-type algorithm was described and local quadratic convergence was shown under very mild assumptions. Moreover, we proved that the LP-Newton method and the constrained Levenberg-Marquardt method are special realizations of the general algorithm and therefore enjoy the same local convergence properties. Chapter 4 was devoted to conditions being sufficient for the convergence assumptions to hold in the case that F is a PC^1-function. Particularly, we considered reformulations of complementarity systems as constrained systems of equations and discussed Assumptions 1–4 in that setting, see Section 4.3. The main result concerning complementarity systems was Theorem 4.20 where four conditions (i)–(iv) were provided. It was shown that each of these conditions implies the whole set of Assumptions 1–4, regardless of whether slack variables are introduced or not.

This chapter deals with special problem classes which have the form (1.2), i.e., which are complementarity systems. For each problem class we will present a suitable reformulation as constrained system of equations and formulate Conditions 7a and 9a for the particular instances. The latter conditions played a crucial role in Section 4.3. Then adapted versions of Theorem 4.20 are stated which provide conditions where each of them implies all of Assumptions 1–4 and therefore local quadratic convergence of Algorithm 3.1 and its special realizations to a solution of the particular problem. Moreover, we relate our conditions to some existing ones from the literature implying local fast convergence of several algorithms.

Section 5.1 is devoted to KKT systems arising from optimization problems or variational inequalities. We recover assertions from [18, Theorems 5 and 6]. In particular, it will turn out that the whole set of Assumptions 1–4 is implied by a second-order condition (SOC) which was used in [31, 48, 87] for the local convergence analysis of a stabilized SQP method. Moreover, we provide new conditions which are sufficient for all of Assumptions 1–4 to hold, for instance a certain constant rank condition or some set of local error bound conditions. Unlike the SOC, our new conditions do not imply the local uniqueness of the primal part of a solution. In Section 5.2 our convergence assumptions are discussed in the context of KKT systems arising from GNEPs. We will see that Assumptions 1–4 together are weaker than conditions which were used in [21] to prove local fast convergence of the (unconstrained) Levenberg-Marquardt method to a solution of the KKT system of a GNEP. Particularly, Assumptions 1–4 do not imply strict complementarity. Moreover, we extend a result from [49] by proving that the full row rank of a certain matrix is sufficient for the whole set of Assumptions 1–4 to hold. The results of Section 5.2 will in large part be published together with Andreas Fischer, Alexey Izmailov, and Mikhail Solodov in the technical report [37]. Section 5.3 deals with FJ systems arising from GNEPs. The consideration of FJ systems instead of KKT systems is motivated by an example in [13] which shows that it cannot be expected that every solution of a GNEP yields a solution of the corresponding KKT system. This example is stable with respect to small perturbations of the problem functions. We use results from [13] to prove that generically some full rank condition is satisfied at any solution of the FJ system of a GNEP. This full rank condition implies the whole set of Assumptions 1–4 if the functions which define the GNEP are sufficiently smooth.

5.1 KKT Systems of Optimization Problems or Variational Inequalities

In this section we consider the problem of finding a solution of the Karush-Kuhn-Tucker (KKT) system

$$\begin{aligned} &H(x) + \nabla g(x)u + \nabla h(x)v = 0,\ h(x) = 0, \\ &g(x) \le 0,\ u \ge 0,\ u^\top g(x) = 0. \end{aligned} \tag{5.1}$$

Throughout, we assume that $H : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is differentiable and has a locally Lipschitz continuous Jacobian. Moreover, $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ and $h : \mathbb{R}^{n_x} \to \mathbb{R}^{m_h}$ are supposed to be twice differentiable with locally Lipschitz continuous second-order derivatives.

It is obvious that (5.1) is a complementarity system, i.e., it has the form (1.2). Therefore, the results of Section 4.3 can be applied. We will present a suitable reformulation of (5.1) as a constrained system of equations and discuss Assumptions 1–4 for the resulting system. The main result of this section is Theorem 5.3 which is an adapted formulation of Theorem 4.20. The second part of this section is devoted to further conditions implying the whole set of Assumptions 1–4. In particular, we recall a second-order condition (SOC) which was frequently used in the past to prove local fast convergence of the stabilized SQP method and further methods for the solution of KKT systems, see for example [31, 46, 48, 87, 88].

In Proposition 5.4 it is proved that the SOC implies all of Assumptions 1–4. Moreover, exploiting a result from Section 4.3, we show that a certain constant rank condition is sufficient for all of our convergence assumptions to hold. At the end of this section examples are presented where both the SOC and strict complementarity are violated but Assumptions 1–4 are valid.

First, let us recall some well-known relations between the KKT system (5.1) and the corresponding optimization problem or the corresponding variational inequality, respectively. The KKT system (5.1) may arise from an optimization problem

$$f(x) \to \min \quad \text{s.t.} \quad g(x) \leq 0,\ h(x) = 0 \tag{5.2}$$

with a twice differentiable function $f : \mathbb{R}^{n_x} \to \mathbb{R}$ whose second-order derivative is assumed to be locally Lipschitz continuous. In that case, H coincides with ∇f. A proof of the following proposition can be found in [40, Satz 2.36 and Satz 2.46].

Proposition 5.1. *(a) Let x^* denote a local solution of the optimization problem* (5.2) *and assume that the ACQ is satisfied at x^*. Then there are $u^* \in \mathbb{R}^{m_g}$ and $v^* \in \mathbb{R}^{m_h}$ such that (x^*, u^*, v^*) solves* (5.1) *with $H := \nabla f$.*

(b) Suppose that the functions $f, g_1, \dots, g_{m_g}$ are convex and that h is affine. Let (x^*, u^*, v^*) be a solution of (5.1) *with $H := \nabla f$. Then x^* is a (global) solution of the optimization problem* (5.2).

In item (a) of the latter proposition the ACQ is needed. Its definition and sufficient conditions for it to hold can be found in Section 2.1. Furthermore, the term "local solution" occurs in item (a) of Proposition 5.1. We call a point x^* *local solution* of the optimization problem (5.2) if $f(x^*) \leq f(x)$ holds for all x which are feasible for (5.2) and which belong to a sufficiently small neighborhood of x^*.

The KKT system (5.1) may also arise from a variational inequality

$$H(x)^\top (y - x) \geq 0 \quad \text{for all } y \in \mathbb{R}^{n_x} \text{ satisfying } g(y) \leq 0 \text{ and } h(y) = 0. \tag{5.3}$$

Proposition 5.2 below recalls relations between (5.3) and (5.1). A proof can be found in [25, Proposition 1.3.4].

Proposition 5.2. *(a) Let x^* be a solution of the variational inequality* (5.3) *and suppose that the ACQ is satisfied at x^*. Then there are $u^* \in \mathbb{R}^{m_g}$ and $v^* \in \mathbb{R}^{m_h}$ such that (x^*, u^*, v^*) solves* (5.1).

(b) Assume that the functions $g_1, \dots, g_{m_g}$ are convex and that h is affine. Let (x^*, u^*, v^*) be a solution of (5.1). *Then x^* solves the variational inequality* (5.3).

In order to shorten the notation, let us introduce the function $\Psi^{5.1} : \mathbb{R}^{n_x} \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h} \to \mathbb{R}^{n_x}$ given by

$$\Psi^{5.1}(x, u, v) := H(x) + \nabla g(x)u + \nabla h(x)v.$$

Moreover, we define the matrix-valued function $\Psi_x^{5.1} : \mathbb{R}^{n_x} \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h} \to \mathbb{R}^{n_x \times n_x}$ according to

$$\begin{aligned}\Psi_x^{5.1}(x,u,v) &:= \nabla_x \Psi^{5.1}(x,u,v)\\ &= H'(x) + \sum_{j=1}^{m_g} u_j \nabla^2 g_j(x) + \sum_{k=1}^{m_h} v_k \nabla^2 h_k(x).\end{aligned}$$

The upper index "5.1" indicates the current section. In Sections 5.2 and 5.3 we will introduce similar functions with the upper indices "5.2" and "5.3", respectively.

It is easy to see that the KKT system (5.1) is a complementarity system, i.e., it has the form (1.2) with $n := n_x + m_g + m_h$, $p := n_x + m_h$, $q := 0$, $r := m_g$, $z := (x,u,v)$, $R(z) := -g(x)$, $S(z) := u$, and

$$P(z) := \begin{pmatrix} \Psi^{5.1}(x,u,v) \\ h(x) \end{pmatrix}.$$

In this section we denote by Z the solution set of (5.1) and by $z^* = (x^*, u^*, v^*) \in Z$ an arbitrary but fixed solution. A suitable reformulation of (5.1) as a constrained system of equations is

$$F(z) := \begin{pmatrix} \Psi^{5.1}(x,u,v) \\ h(x) \\ \min\{-g(x), u\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega \tag{5.4}$$

with

$$\Omega := \{z = (x,u,v) \in \mathbb{R}^{n_x} \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h} \mid g(x) \le 0,\ u \ge 0\}.$$

Every solution of (5.1) is also a solution of (5.4) and vice versa. Obviously, F is a PC^1-function with 2^{m_g} selection functions $F^1, \ldots, F^{2^{m_g}}$.

Our next aim is to formulate Conditions 7a and 9a in the context of KKT systems and then to state an adapted version of Theorem 4.20 for (5.1). It is not difficult to see that the following condition is equivalent to Condition 7a for (5.1) and (5.4), respectively.

Condition 7b. There are $K_{7b} > 0$ and $\varepsilon_{7b} > 0$ such that

$$\begin{aligned}\operatorname{dist}[s,Z] \le\ & K_{7b}(\|\Psi^{5.1}(x,u,v)\| + \|h(x)\| + \|\min\{-g(x),u\}\| \\ & + \|\min\{0,-g(x)\}\| + \|\min\{0,u\}\|)\end{aligned}$$

holds for all $s = (x,u,v) \in \mathcal{B}_{\varepsilon_{7b}}(z^*)$.

Now let us consider Condition 9a in the context of KKT systems arising from optimization problems or variational inequalities. First, we define the following index sets which depend on the fixed solution $z^* = (x^*, u^*, v^*)$:

$$\begin{aligned}\mathcal{I}_g &:= \{j \in \{1,\ldots,m_g\} \mid g_j(x^*) = 0 < u_j^*\},\\ \mathcal{I}_u &:= \{j \in \{1,\ldots,m_g\} \mid g_j(x^*) < 0 = u_j^*\},\\ \mathcal{I}_= &:= \{j \in \{1,\ldots,m_g\} \mid g_j(x^*) = u_j^* = 0\}.\end{aligned}$$

The sets $\mathcal{I}_g$ and $\mathcal{I}_=$ partition the set

$$\mathcal{G}_0 := \{j \in \{1, \ldots, m_g\} \mid g_j(x^*) = 0\}$$

consisting of the indices of those constraints being active at x^* where the elements of $\mathcal{I}_=$ correspond to those inequalities for which strict complementarity is violated. The set $\mathcal{I}_u$ coincides with the index set of the inactive constraints. Obviously, $\mathcal{I}_g$ and $\mathcal{I}_u$ are the analoga of $\mathcal{I}_R$ and $\mathcal{I}_S$, respectively, from Section 4.3.

We denote, for any partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$, by $Z_{\mathcal{I}_1,\mathcal{I}_2}$ the solution set of the following system of equations and inequalities:

$$\begin{gathered} \Psi^{5.1}(x,u,v) = 0,\ h(x) = 0,\ g_{\mathcal{I}_g \cup \mathcal{I}_1}(x) = 0,\ u_{\mathcal{I}_u \cup \mathcal{I}_2} = 0, \\ g_{\mathcal{I}_u \cup \mathcal{I}_2}(x) \le 0,\ u_{\mathcal{I}_g \cup \mathcal{I}_1} \ge 0. \end{gathered} \tag{5.5}$$

Condition 9b below requires that, for every partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$, the system (5.5) satisfies the local error bound condition at $z^* = (x^*, u^*, v^*)$. It is easy to see that Condition 9b is equivalent to Condition 9a for the KKT system (5.1) and its reformulation (5.4), respectively.

Condition 9b. There are $K_{9b} > 0$ and $\varepsilon_{9b} > 0$ such that, for every partition $(\mathcal{I}_1, \mathcal{I}_2)$ of $\mathcal{I}_=$,

$$\begin{aligned} \operatorname{dist}[s, Z_{\mathcal{I}_1,\mathcal{I}_2}] \ \le\ & K_{9b}(\|\Psi^{5.1}(x,u,v)\| + \|h(x)\| + \|g_{\mathcal{I}_g \cup \mathcal{I}_1}(x)\| + \|u_{\mathcal{I}_u \cup \mathcal{I}_2}\| \\ & + \|\min\{0, -g_{\mathcal{I}_u \cup \mathcal{I}_2}(x)\}\| + \|\min\{0, u_{\mathcal{I}_g \cup \mathcal{I}_1}\}\|) \end{aligned}$$

holds for all $s = (x,u,v) \in \mathcal{B}_{\varepsilon_{9b}}(z^*)$.

Now we are in the position to state Theorem 4.20 in the context of KKT systems arising from optimization problems or variational inequalities. Theorem 5.3 below provides conditions where each of them is sufficient for the whole set of Assumptions 1–4 for (5.4) to hold. In particular, it turns out that our convergence assumptions are satisfied if Condition 7b holds and in addition one of the following two conditions is valid at z^*: strict complementarity or the local uniqueness of the x-part of all solutions near z^*. It already follows from [18, Theorem 5] that each of the conditions (i)–(iii) in the subsequent theorem implies the whole set of Assumptions 1–4. However, condition (iv), i.e. Condition 9b, is a new sufficient condition. We know from Subsection 4.1.4 that the latter is the weakest among the conditions (i)–(iv).

Theorem 5.3. *Let $H : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ be differentiable with a locally Lipschitz continuous Jacobian. Moreover, assume that $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ and $h : \mathbb{R}^{n_x} \to \mathbb{R}^{m_h}$ are twice differentiable and have locally Lipschitz continuous second-order derivatives. Let one of the following conditions be true.*

(i) The functions H, g, and h are affine.

(ii) *Condition 7b holds and the set* $\mathcal{I}_=$ *is empty, i.e., strict complementarity is valid at* z^*.

(iii) *Condition 7b is satisfied. Moreover, there is* $\varepsilon > 0$ *such that* $z = (x, u, v) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ *implies* $x = x^*$.

(iv) *Condition 9b holds.*

Then Assumptions 1–4 for (5.4) *are valid if the function* G *satisfies* (4.1) *for all* $z = (x, u, v) \in \mathbb{R}^n$.

Proof. The assertions follow from Theorem 4.20, together with the considerations of the current section up to now. Concerning condition (iii) note that we have $R(x^*, u, v) = -g(x^*)$, $S(x^*, u, v) = u$, and

$$P(x^*, u, v) = \begin{pmatrix} H(x^*) + \nabla g(x^*)u + \nabla h(x^*)v \\ h(x^*) \end{pmatrix}$$

in the setting of the current section. Therefore, $P(x^*, \cdot, \cdot)$, $R(x^*, \cdot, \cdot)$, and $S(x^*, \cdot, \cdot)$ are affine. Thus, condition (iii) above implies condition (iii) from Theorem 4.20. □

Remark 5.1. (a) The assertions of Theorem 5.3 stay true if condition (iii) is replaced by the following weaker condition (iii)'.

(iii)' Condition 7b is satisfied. Moreover, the vectors x are split according to $x = (x_a, x_b) \in \mathbb{R}^{n_{x_a}} \times \mathbb{R}^{n_{x_b}}$ such that the functions $H(x_a^*, \cdot)$, $g(x_a^*, \cdot)$, and $h(x_a^*, \cdot)$ are affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x_a, x_b, u, v) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x_a = x_a^*$.

Condition (iii)' allows a nonisolated primal part of the solution but requires that the problem functions H, g, and h are affine regarding the vector x_b of nonisolated components.

(b) If the corresponding matrix-valued function $\hat{G}$ is suitably chosen, then each of the conditions (i)–(iv) from Theorem 5.3 implies, besides Assumptions 1–4 for the constrained system (5.4), also the whole set of Assumptions 1–4 for the following reformulation of (5.4) by means of slack variables w for the inequality constraints:

$$\hat{F}(z, w) := \begin{pmatrix} \Psi^{5.1}(x, u, v) \\ h(x) \\ \min\{-g(x), u\} \\ g(x) + w \end{pmatrix} = 0 \quad \text{s.t.} \quad (z, w) \in \hat{\Omega}$$

with

$$\hat{\Omega} := \{(z, w) = (x, u, v, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{m_g} \times \mathbb{R}^{m_h} \times \mathbb{R}^{m_g} \mid u \geq 0,\ w \geq 0\},$$

see Section 4.2 and item (a) of Remark 4.4 for details. Note that slack variables for the constraints $u \geq 0$ are not needed since these constraints are already affine (even bound constraints only).

We are interested in further sufficient conditions for Assumptions 1–4 for the constrained system (5.4) to hold. Let us begin with a second-order condition (SOC) which is used in [31] for proving local fast convergence of an extension of the stabilized SQP method for the solution of KKT systems arising from variational inequalities. In order to define the SOC, we introduce the cone

$$\mathcal{C}(x^*) := \{d \in \mathbb{R}^{n_x} \mid H(x^*)^\top d = 0,\ g'_{\mathcal{G}_0}(x^*)d \leq 0\}.$$

We say that the SOC is satisfied at $z^* = (x^*, u^*, v^*)$ if

$$d^\top \Psi_x^{5.1}(x^*, u^*, v^*)d \neq 0$$

holds for all $d \in \mathcal{C}(x^*) \setminus \{0\}$. In the case of optimization problems this condition is the classical second-order sufficiency condition (where "$\neq$" is replaced by "$>$"). The latter is used, for example, in [48, 87] for proving local quadratic convergence of the stabilized SQP method and in [46, 88] for the local convergence analysis of further methods for the solution of KKT systems arising from optimization problems.

Proposition 5.4. *Suppose that the SOC holds at z^*. Then Assumptions 1–4 for* (5.4) *are valid if the function G satisfies* (4.1) *for all $z = (x, u, v) \in \mathbb{R}^n$.*

Proof. There are $\varepsilon > 0$ and $C > 0$ such that $(\bar{x}, \bar{u}, \bar{v}) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $\bar{x} = x^*$ and

$$\operatorname{dist}[s, Z] \leq C(\|\Psi^{5.1}(x, u, v)\| + \|h(x)\| + \|\min\{-g(x), u\}\|)$$

holds for all $s = (x, u, v) \in \mathcal{B}_\varepsilon(z^*)$. For the case of inequality constraints only, this follows from [31, Theorem 4]. By slightly extending the proof of that theorem, it turns out that the result stays true if equality constraints occur. Therefore, condition (iii) from Theorem 5.3 holds so that the assertion follows from Theorem 5.3. □

The SOC allows nonisolated Lagrange multipliers but implies the local uniqueness of the primal part of z^* as solution of (5.1). Example 5.1 below, which is [19, Example 8], provides an instance where Assumptions 1–4 are satisfied although the x-part of z^* is nonunique. This shows that Assumptions 1–4 together are strictly weaker than the SOC. Besides, strict complementarity is violated at all solutions in the following example.

Example 5.1. Consider the optimization problem

$$f(x) := x_1^2 \to \min \quad \text{s.t.} \quad g_1(x) := -x_1 \leq 0,\ g_2(x) := -x_2 \leq 0.$$

The corresponding KKT system is given by

$$\begin{aligned} \nabla f(x)+\nabla g(x)u &= \begin{pmatrix} 2x_1-u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ g_1(x) &= -x_1 \le 0,\ u_1\ge 0,\ u_1x_1=0, \\ g_2(x) &= -x_2 \le 0,\ u_2\ge 0,\ u_2x_2=0. \end{aligned} \tag{5.6}$$

The elements of the solution set Z of (5.6) have the same u-part but nonunique primal parts:

$$Z=\{(x_1,x_2,u_1,u_2)^\top=(0,\tau,0,0)^\top \mid \tau\in\mathbb{R}_+\}.$$

Of course, the SOC cannot be satisfied at any solution of (5.6). Moreover, strict complementarity is violated at all solutions. However, since ∇f, g_1, and g_2 are affine, condition (i) from Theorem 5.3 holds. Consequently, Assumptions 1–4 for the constrained system (5.4) are valid at any solution if G satisfies (4.1) for all $z=(x,u)\in\mathbb{R}^4$. □

At the end of this section examples will be presented where SOC and strict complementarity are still violated but where not all of the problem functions are affine, see Examples 5.2 and 5.3. Before, let us provide a further condition implying the whole set of Assumptions 1–4 for the constrained system (5.4). Exploiting a result from Section 4.3, we obtain Proposition 5.6 below which says that some constant rank condition is sufficient for Condition 9b and therefore for all of Assumptions 1–4 to hold (the latter if G is suitably chosen). In order to prove Proposition 5.6, we need the following lemma.

Lemma 5.5. *Let $A_1\in\mathbb{R}^{n_1\times n_2}$ be some matrix whose rank is equal to l. Moreover, let $a\in\mathbb{R}^{n_1}$ be some vector and $a_0\neq 0$ be some number. Then the rank of the matrix*

$$A_2 := \left(\begin{array}{c|c} A_1 & a \\ \hline 0 & a_0 \end{array}\right) \in \mathbb{R}^{(n_1+1)\times(n_2+1)}$$

equals $l+1$.

Proof. Obviously, the last column of A_2 is linearly independent on the other columns. Therefore, the maximal number of linear independent columns is, compared to A_1, increased by one. Thus, the rank of A_2 is also increased by one compared to the rank of A_1. Hence, the lemma is proved. □

Proposition 5.6. *Assume that there is some $\varepsilon>0$ such that, for every pair $(\mathcal{J}_1,\mathcal{J}_2)$ of index sets $\mathcal{J}_2\subseteq\mathcal{J}_1\subseteq\mathcal{I}_=$, the matrices*

$$\left(\begin{array}{c|c|c} \Psi_x^{5.1}(x,u,v) & \nabla g_{\mathcal{I}_g\cup\mathcal{J}_2}(x) & \nabla h(x) \\ \hline h'(x) & 0 & 0 \\ \hline g'_{\mathcal{I}_g\cup\mathcal{J}_1}(x) & 0 & 0 \end{array}\right) \tag{5.7}$$

have the same rank for all $s = (x, u, v) \in \mathcal{B}_\varepsilon(z^*)$. Then Condition 9b is satisfied. In particular, Assumptions 1–4 for (5.4) *hold if the function G satisfies* (4.1) *for all $z = (x, u, v) \in \mathbb{R}^n$.*

Proof. We show that Condition 9b is satisfied. Then the second assertion follows from Theorem 5.3. We want to exploit Proposition 4.21. Note that we have $s = (x, u, v)$,

$$P(s) = \begin{pmatrix} \Psi^{5.1}(x, u, v) \\ h(x) \end{pmatrix} = \begin{pmatrix} H(x) + \nabla g(x)u + \nabla h(x)v \\ h(x) \end{pmatrix},$$

$R(s) = -g(x)$, and $S(s) = u$ in the setting of the current section. The function Q does not occur. Moreover, the index sets $\mathcal{I}_R$ and $\mathcal{I}_S$ coincide with $\mathcal{I}_g$ and $\mathcal{I}_u$, respectively. Using the latter observations, it follows from Proposition 4.21 that Condition 9b, which is the analogon of Condition 9a, holds if there is $\varepsilon > 0$ such that, for every pair $(\mathcal{K}_g, \mathcal{K}_u)$ of index sets $\mathcal{K}_g \subseteq \mathcal{I}_=$ and $\mathcal{K}_u \subseteq \mathcal{I}_=$, the matrices

$$\left(\begin{array}{c|c|c|c} \Psi^{5.1}_x(x, u, v) & \nabla g_{\mathcal{I}_g \cup (\mathcal{K}_g \setminus \mathcal{K}_u)}(x) & \nabla g_{\mathcal{I}_u \cup \mathcal{K}_u}(x) & \nabla h(x) \\ \hline h'(x) & 0 & 0 & 0 \\ \hline -g'_{\mathcal{I}_g \cup \mathcal{K}_g}(x) & 0 & 0 & 0 \\ \hline 0 & 0 & I_{|\mathcal{I}_u \cup \mathcal{K}_u|} & 0 \end{array}\right) \tag{5.8}$$

have the same rank for all $s = (x, u, v) \in \mathcal{B}_\varepsilon(z^*)$. Note that the four column blocks in the above matrix contain the derivatives of P, $R_{\mathcal{I}_g \cup \mathcal{K}_g}$, and $S_{\mathcal{I}_u \cup \mathcal{K}_u}$ with respect to the variables in the following order:

$$x,\ u_{\mathcal{I}_g \cup (\mathcal{K}_g \setminus \mathcal{K}_u)},\ u_{\mathcal{I}_u \cup \mathcal{K}_u},\ v.$$

Using Lemma 5.5, the rank of the matrix in (5.8) equals, for every $s = (x, u, v) \in \mathbb{R}^n$, the rank of

$$\left(\begin{array}{c|c|c} \Psi^{5.1}_x(x, u, v) & \nabla g_{\mathcal{I}_g \cup (\mathcal{K}_g \setminus \mathcal{K}_u)}(x) & \nabla h(x) \\ \hline h'(x) & 0 & 0 \\ \hline -g'_{\mathcal{I}_g \cup \mathcal{K}_g}(x) & 0 & 0 \end{array}\right) \tag{5.9}$$

plus $|\mathcal{I}_u \cup \mathcal{K}_u|$. In particular, the matrices in (5.8) have the same rank for all $s = (x, u, v) \in \mathcal{B}_\varepsilon(z^*)$ if and only if the matrices in (5.9) have this property. The matrices in (5.9) actually have the same rank for all $s = (x, u, v) \in \mathcal{B}_\varepsilon(z^*)$ due to the assumption of the proposition with $\mathcal{J}_1 := \mathcal{K}_g$ and $\mathcal{J}_2 := \mathcal{K}_g \setminus \mathcal{K}_u$. □

Similar to our comment after Proposition 4.21, we want to emphasize that the rank of the matrices in (5.7) may depend on the pair $(\mathcal{J}_1, \mathcal{J}_2)$ of index sets. The rank is only required to be independent on s.

The next example is a slight modification of Example 5.1 where ∇f is not longer affine so that condition (i) from Theorem 5.3 does not hold. Moreover, neither SOC nor strict complementarity are satisfied. However, we will show that the constant rank condition from Proposition 5.6 is satisfied at every solution.

Example 5.2. Let us consider the optimization problem

$$f(x) := \mathrm{e}^{x_1^2} \to \min \quad \text{s.t.} \quad g_1(x) := -x_1 \leq 0,\ g_2(x) := -x_2 \leq 0$$

and the corresponding KKT system

$$\begin{aligned} &\nabla f(x) + \nabla g(x) u = \begin{pmatrix} 2x_1 \mathrm{e}^{x_1^2} - u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &g_1(x) = -x_1 \leq 0,\ u_1 \geq 0,\ u_1 x_1 = 0, \\ &g_2(x) = -x_2 \leq 0,\ u_2 \geq 0,\ u_2 x_2 = 0. \end{aligned} \tag{5.10}$$

The solution set of this system is the same as in Example 5.1:

$$Z = \{(x_1, x_2, u_1, u_2)^\top = (0, \tau, 0, 0)^\top \mid \tau \in \mathbb{R}_+\}.$$

We are going to show that the constant rank assumption from Proposition 5.6 holds at $z^* := (0, 0, 0, 0)^\top$. Note that strict complementarity is violated for both inequality constraints at z^* so that $\mathcal{I}_= = \{1, 2\}$ is valid. Let us consider the pair $(\mathcal{J}_1, \mathcal{J}_2)$ with $\mathcal{J}_1 := \mathcal{J}_2 := \{1, 2\}$. We have to prove that the matrices

$$M(x, u) := \begin{pmatrix} \nabla^2 f(x) + u_1 \nabla^2 g_1(x) + u_2 \nabla^2 g_2(x) & \nabla g_1(x) & \nabla g_2(x) \\ g_1'(x) & 0 & 0 \\ g_2'(x) & 0 & 0 \end{pmatrix}$$

have the same rank for all points (x, u) in a certain neighborhood of z^*. We obtain

$$M(x, u) = \begin{pmatrix} (2 + 4x_1^2)\,\mathrm{e}^{x_1^2} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to see that $\operatorname{rank} M(x, u)$ does not depend on the particular point (x, u) and equals 4 for all $(x, u) \in \mathbb{R}^4$. Thus, the assumption of Proposition 5.6 is shown for $\mathcal{J}_1 = \mathcal{J}_2 = \{1, 2\}$. Furthermore, it can be easily verified that the rank of any submatrix of $M(x, u)$ is not dependent on (x, u) as well. Therefore, the constant rank assumption of Proposition 5.6 is satisfied for all pairs $(\mathcal{J}_1, \mathcal{J}_2)$ of index sets $\mathcal{J}_2 \subseteq \mathcal{J}_1 \subseteq \mathcal{I}_=$. Using Proposition 5.6, we know that Condition 9b is valid so that Assumptions 1–4 for (5.4) hold at z^* if G satisfies (4.1) for all $z = (x, u) \in \mathbb{R}^4$.

It is easy to verify that the constant rank assumption of Proposition 5.6 stays true at any further solution of (5.10). □

The following example shows that Condition 9b is strictly weaker than the constant rank condition in Proposition 5.6. Example 5.3 is a slight modification of [47, Example 2.2] and was also considered in [19, Example 9].

Example 5.3. Let us consider the optimization problem

$$f(x) := x_1^2 - x_2^2 + x_3^2 \to \min \quad \text{s.t.} \quad g_1(x) := x_1^2 + x_2^2 - x_3^2 \le 0,\; g_2(x) := x_1x_3 \le 0$$

and the corresponding KKT system

$$\begin{aligned} &\nabla f(x) + \nabla g(x)u = \begin{pmatrix} 2x_1 + 2u_1x_1 + u_2x_3 \\ -2x_2 + 2u_1x_2 \\ 2x_3 - 2u_1x_3 + u_2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ &g_1(x) = x_1^2 + x_2^2 - x_3^2 \le 0,\; u_1 \ge 0,\; u_1(x_1^2 + x_2^2 - x_3^2) = 0, \\ &g_2(x) = x_1x_3 \le 0,\; u_2 \ge 0,\; u_2x_1x_3 = 0. \end{aligned} \tag{5.11}$$

The solution set of (5.11) is given by

$$\begin{aligned} Z \;=\; &\{(x_1, x_2, x_3, u_1, u_2)^\top = (0, 0, 0, \tau_1, \tau_2)^\top \mid \tau_1, \tau_2 \in \mathbb{R}_+\} \\ &\cup \{(x_1, x_2, x_3, u_1, u_2)^\top = (0, \tau, \tau, 1, 0)^\top \mid \tau \in \mathbb{R}\} \\ &\cup \{(x_1, x_2, x_3, u_1, u_2)^\top = (0, \tau, -\tau, 1, 0)^\top \mid \tau \in \mathbb{R}\}. \end{aligned}$$

We set $z^* := (x^*, u^*) := (0, 1, 1, 1, 0)^\top$. The u-part of this solution is locally unique whereas the x-part is not. Thus, the SOC is not satisfied at z^*. Obviously, both inequalities are active at x^* where the second inequality violates strict complementarity. Therefore, we have

$$\mathcal{I}_g = \{1\}, \quad \mathcal{I}_u = \emptyset, \quad \mathcal{I}_= = \{2\}.$$

Let us prove that the assumption of Proposition 5.6 is not satisfied at z^*. To this end, we define, for any point $s = (x, u)$, the matrix

$$\begin{aligned} M(x, u) \;&:=\; \left(\begin{array}{c|c} \nabla^2 f(x) + u_1\nabla^2 g_1(x) + u_2\nabla^2 g_2(x) & \nabla g_1(x) \\ \hline g_1'(x) & 0 \end{array}\right) \\ &= \begin{pmatrix} 2 + 2u_1 & 0 & u_2 & 2x_1 \\ 0 & 2u_1 - 2 & 0 & 2x_2 \\ u_2 & 0 & 2 - 2u_1 & -2x_3 \\ 2x_1 & 2x_2 & -2x_3 & 0 \end{pmatrix} \end{aligned}$$

and show that for any $\varepsilon > 0$ there is some $(x, u) \in \mathcal{B}_\varepsilon(z^*)$ such that the rank of $M(x, u)$ is greater than the rank of $M(x^*, u^*)$. Then it is proved that the assumption of Proposition 5.6 is violated. Note that the matrices $M(x, u)$ are precisely the matrices (5.7) from Proposition 5.6 for $\mathcal{J}_1 := \mathcal{J}_2 := \emptyset$.

First, let us compute the matrix $M(x^*, u^*)$ and its rank. We obtain

$$M(x^*, u^*) = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \end{pmatrix}.$$

The rank of this matrix is equal to 3. Now let us consider the sequence $\{(x^k, u^k)\}_{k\in\mathbb{N},\, k\geq 1}$ defined according to

$$(x_1^k, x_2^k, x_3^k, u_1^k, u_2^k) := (0, 1 + \frac{1}{k}, 1, 1 + \frac{1}{k}, 0).$$

Obviously, $\{(x^k, u^k)\}$ converges to (x^*, u^*). Furthermore, we have

$$M(x^k, u^k) = \begin{pmatrix} 4+\frac{2}{k} & 0 & 0 & 0 \\ 0 & \frac{2}{k} & 0 & 2+\frac{2}{k} \\ 0 & 0 & -\frac{2}{k} & -2 \\ 0 & 2+\frac{2}{k} & -2 & 0 \end{pmatrix}$$

for all $k \geq 1$. The determinant of this matrix equals

$$\det M(x^k, u^k) = 16(2 + \frac{1}{k})(\frac{1}{k^3} + \frac{2}{k^2}) \neq 0$$

so that

$$\operatorname{rank} M(x^k, u^k) = 4$$

follows for all $k \geq 1$. Hence, it is shown that for any $\varepsilon > 0$ there is some $(x, u) \in \mathcal{B}_\varepsilon(z^*)$ such that the ranks of $M(x, u)$ and of $M(x^*, u^*)$ do not coincide.

Now let us verify Condition 9b. First, we consider the partition $(\{2\}, \emptyset)$ of the set $\mathcal{I}_=$, i.e., we set $\mathcal{I}_1 := \{2\}$ and $\mathcal{I}_2 := \emptyset$. Then system (5.5) is given by

$$\begin{aligned} \nabla f(x) + \nabla g(x) u = \begin{pmatrix} 2x_1 + 2u_1x_1 + u_2x_3 \\ -2x_2 + 2u_1x_2 \\ 2x_3 - 2u_1x_3 + u_2x_1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ g_1(x) = x_1^2 + x_2^2 - x_3^2 = 0,\ u_1 \geq 0,& \\ g_2(x) = x_1x_3 = 0,\ u_2 \geq 0.& \end{aligned} \tag{5.12}$$

It is not difficult to prove that the solution set $Z_{\{2\},\emptyset}$ of (5.12) coincides with Z. We are going to show that the inequality in Condition 9b belonging to the system (5.12) holds for some $K_{9b} > 0$ and all points $s \in \mathcal{B}_{\frac{1}{2}}(z^*)$. To this end, let us take any $s = (x_1, x_2, x_3, u_1, u_2)^\top \in \mathcal{B}_{\frac{1}{2}}(z^*)$. Then we have

$$\begin{gathered} x_1 \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right],\ x_2 \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right],\ x_3 \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right], \\ u_1 \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right],\ u_2 \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]. \end{gathered} \tag{5.13}$$

Obviously, the point

$$\bar{s} := (0, \frac{1}{2}(x_2 + x_3), \frac{1}{2}(x_2 + x_3), 1, 0)^\top \tag{5.14}$$

belongs to $Z_{\{2\},\emptyset}$. Therefore,

$$\text{dist}[s, Z_{\{2\},\emptyset}] \leq \|s - \bar{s}\| \leq |x_1| + |x_2 - x_3| + |u_1 - 1| + |u_2| \tag{5.15}$$

holds. Let us estimate each summand of the right-hand side from above. To this end, the triangle inequality and (5.13) are used:

$$\begin{aligned}
|x_1| &= \frac{1}{x_3}|x_1 x_3| \\
&\leq 2|x_1 x_3|, \\
|x_2 - x_3| &= \frac{1}{x_2 + x_3}|x_2^2 - x_3^2| \\
&\leq |x_2^2 - x_3^2| \\
&\leq |x_1^2 + x_2^2 - x_3^2| + x_1^2 \\
&\leq |x_1^2 + x_2^2 - x_3^2| + \frac{1}{2}|x_1| \\
&\leq |x_1^2 + x_2^2 - x_3^2| + |x_1 x_3|, \\
|u_1 - 1| &= \frac{1}{x_2}| - x_2 + u_1 x_2| \\
&\leq | - 2x_2 + 2u_1 x_2|, \\
|u_2| &= \frac{1}{x_3}|u_2 x_3| \\
&\leq 2|2x_1 + 2u_1 x_1 + u_2 x_3| + 2|2x_1 + 2u_1 x_1| \\
&= 2|2x_1 + 2u_1 x_1 + u_2 x_3| + 4(1 + u_1)|x_1| \\
&\leq 2|2x_1 + 2u_1 x_1 + u_2 x_3| + 10|x_1| \\
&\leq 2|2x_1 + 2u_1 x_1 + u_2 x_3| + 20|x_1 x_3|.
\end{aligned}$$

Combining these estimates and taking into account (5.15),

$$\begin{aligned}
\text{dist}[s, Z_{\{2\},\emptyset}] &\leq 2|2x_1 + 2u_1 x_1 + u_2 x_3| + | - 2x_2 + 2u_1 x_2| \\
&\quad + |x_1^2 + x_2^2 - x_3^2| + 23|x_1 x_3| \\
&\leq 2\sqrt{2}\|\nabla f(x) + \nabla g(x) u\| + 23\sqrt{2}\|g(x)\| + \|\min\{0, u\}\|
\end{aligned}$$

follows so that the inequality in Condition 9b belonging to the system (5.12) is valid for $K_{9b} := 23\sqrt{2}$.

Next, the partition $(\emptyset, \{2\})$ of the set $\mathcal{I}_=$ is considered, i.e., we set $\mathcal{I}_1 := \emptyset$ and $\mathcal{I}_2 := \{2\}$. In that case, system (5.5) is given according to

$$\begin{aligned} &\nabla f(x) + \nabla g(x)u = \begin{pmatrix} 2x_1 + 2u_1x_1 + u_2x_3 \\ -2x_2 + 2u_1x_2 \\ 2x_3 - 2u_1x_3 + u_2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \\ &g_1(x) = x_1^2 + x_2^2 - x_3^2 = 0, \ u_1 \geq 0, \\ &g_2(x) = x_1x_3 \leq 0, \ u_2 = 0. \end{aligned} \tag{5.16}$$

The solution set $Z_{\emptyset,\{2\}}$ of (5.16) is given by

$$\begin{aligned} Z_{\emptyset,\{2\}} \ = \ & \{(x_1, x_2, x_3, u_1, u_2)^\top = (0, 0, 0, \tau, 0)^\top \mid \tau \in \mathbb{R}_+\} \\ & \cup \{(x_1, x_2, x_3, u_1, u_2)^\top = (0, \tau, \tau, 1, 0)^\top \mid \tau \in \mathbb{R}\} \\ & \cup \{(x_1, x_2, x_3, u_1, u_2)^\top = (0, \tau, -\tau, 1, 0)^\top \mid \tau \in \mathbb{R}\}. \end{aligned}$$

Again, let us take any $s = (x_1, x_2, x_3, u_1, u_2)^\top \in \mathcal{B}_{\frac{1}{2}}(z^*)$ and let $\bar{s}$ be defined by (5.14). Obviously, $\bar{s}$ belongs to $Z_{\emptyset,\{2\}}$ so that

$$\text{dist}[s, Z_{\emptyset,\{2\}}] \leq \|s - \bar{s}\| \leq |x_1| + |x_2 - x_3| + |u_1 - 1| + |u_2| \tag{5.17}$$

follows. The estimation of $|u_2|$ is not necessary. The third summand $|u_1 - 1|$ of the right-hand side in (5.17) can be estimated as in the case $\mathcal{I}_1 = \{2\}$, $\mathcal{I}_2 = \emptyset$ above. For $|x_1|$ we need another estimation. We use the triangle inequality and (5.13):

$$\begin{aligned} |x_1| \ &= \ \frac{1}{2 + 2u_1}|2x_1 + 2u_1x_1| \\ &\leq \ \frac{1}{3}|2x_1 + 2u_1x_1 + u_2x_3| + \frac{1}{3}|u_2x_3| \\ &\leq \ \frac{1}{3}|2x_1 + 2u_1x_1 + u_2x_3| + \frac{1}{2}|u_2|. \end{aligned}$$

The expression $|x_2 - x_3|$ in (5.17) is estimated as in the case $\mathcal{I}_1 = \{2\}$, $\mathcal{I}_2 = \emptyset$ above, however, the last inequality is replaced by the following one:

$$|x_1^2 + x_2^2 - x_3^2| + \frac{1}{2}|x_1| \leq |x_1^2 + x_2^2 - x_3^2| + \frac{1}{6}|2x_1 + 2u_1x_1 + u_2x_3| + \frac{1}{4}|u_2|.$$

Combining the estimations for the summands of the right-hand side in (5.17), we obtain

$$\begin{aligned} \text{dist}[s, Z_{\emptyset,\{2\}}] \ &\leq \ \frac{1}{2}|2x_1 + 2u_1x_1 + u_2x_3| + |-2x_2 + 2u_1x_2| \\ &\qquad + |x_1^2 + x_2^2 - x_3^2| + \frac{7}{4}|u_2| \\ &\leq \ \sqrt{2}\|\nabla f(x) + \nabla g(x)u\| + |g_1(x)| + \frac{7}{4}|u_2| \\ &\qquad + |\min\{0, -g_2(x)\}| + |\min\{0, u_1\}|. \end{aligned}$$

Hence, the inequality in Condition 9b belonging to the system (5.16) is proved for $K_{9b} := \frac{7}{4}$.

After all, it is proved that Condition 9b is satisfied with $\varepsilon_{9b} := \frac{1}{2}$ and $K_{9b} := 23\sqrt{2}$. In particular, due to Theorem 5.3, Assumptions 1–4 for (5.4) hold if G satisfies (4.1) for all $z = (x, u) \in \mathbb{R}^5$. □

5.2 KKT Systems of GNEPs

In this section the KKT system of a *generalized Nash equilibrium problem* (GNEP) with N players $\nu = 1, \dots, N$ is considered. After recalling some basic notation and definitions, we present a suitable reformulation of the KKT system as a constrained system of equations. Theorem 5.8, which is an adapted formulation of Theorem 4.20, is the main result of this section. It provides conditions where each of them implies all of Assumptions 1–4 and therefore local quadratic convergence of Algorithm 3.1 and its special realizations to a solution of the KKT system. We will particularly see that Assumptions 1–4 together are weaker than conditions which are used in [21] for proving local quadratic convergence of the (unconstrained) Levenberg-Marquardt method. In the second part of this section further conditions being sufficient for our convergence assumptions to hold are provided. In particular, exploiting Proposition 4.21, it will turn out that a certain constant rank condition implies the whole set of Assumptions 1–4, see Proposition 5.9. This implication is used to prove that both the full row rank condition considered in [49] and the nonsingularity condition from [15] are sufficient for Assumptions 1–4 to hold.

First, let us recall some basic notation and results. Further results on GNEPs and methods for their solution can be found in the survey papers [23, 38] and references therein. We denote, for every $\nu \in \{1, \dots, N\}$, by n_ν the number of the decision variables controlled by the ν-th player. The strategy vectors of player ν are indicated by

$$x^\nu := (x_1^\nu, \dots, x_{n_\nu}^\nu)^\top \in \mathbb{R}^{n_\nu}.$$

The vectors consisting of the variables of all players are denoted by x, i.e.,

$$x := \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix} \in \mathbb{R}^{n_x}$$

where n_x is defined according to $n_x := \sum_{\nu=1}^{N} n_\nu$. In order to emphasize the ν-th player's variables, we often use the notation $(x^\nu, x^{-\nu})$ instead of x where $x^{-\nu}$ includes the variables of the rival players, i.e., $x^{-\nu} := (x^\iota)_{\iota=1,\iota\neq\nu}^{N}$.

We assume that the strategy spaces of the players are defined by inequality constraints. Moreover, for the sake of simplicity, a GNEP with shared constraints only is considered. Therefore, if the strategies $x^{-\nu}$ of the rival players are fixed, the aim of player ν is to solve the optimization problem

$$\theta_\nu(x^\nu, x^{-\nu}) \to \min_{x^\nu} \quad \text{s.t.} \quad g(x^\nu, x^{-\nu}) \leq 0 \tag{5.18}$$

where $\theta_\nu : \mathbb{R}^{n_x} \to \mathbb{R}$ denotes the objective function of the ν-th player and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ is used to describe the shared constraints. Throughout this section, the functions $\theta_1, \ldots, \theta_N$, and g are supposed to be twice differentiable with locally Lipschitz continuous second-order derivatives.

A vector $x^* \in \mathbb{R}^{n_x}$ is called *generalized Nash equilibrium* or simply *solution* of the GNEP if, for every $\nu \in \{1, \ldots, N\}$, the strategy vector $x^{*,\nu}$ is a solution of (5.18) with $x^{-\nu}$ replaced by $x^{*,-\nu}$.

Let us assume for the moment that $\nu \in \{1, \ldots, N\}$ and $x^{-\nu} \in \mathbb{R}^{n_x - n_\nu}$ are fixed. Then the corresponding KKT system of the optimization problem (5.18) is given by

$$\begin{aligned}
&\nabla_{x^\nu}\theta_\nu(x^\nu, x^{-\nu}) + \nabla_{x^\nu} g(x^\nu, x^{-\nu})u^\nu = 0,\\
&g(x^\nu, x^{-\nu}) \leq 0,\ u^\nu \geq 0,\ (u^\nu)^\top g(x^\nu, x^{-\nu}) = 0
\end{aligned}$$

with some multiplier vector $u^\nu \in \mathbb{R}^{m_g}$. We obtain the KKT system of the GNEP by concatenating the KKT systems of all players:

$$\begin{aligned}
&\Theta^{5.2}(x) + B(x)u = 0,\\
&g(x) \leq 0,\ u \geq 0,\\
&(u^\nu)^\top g(x) = 0 \quad \text{for all } \nu = 1, \ldots, N
\end{aligned} \tag{5.19}$$

with

$$u := \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}, \quad \Theta^{5.2}(x) := \begin{pmatrix} \nabla_{x^1}\theta_1(x) \\ \vdots \\ \nabla_{x^N}\theta_N(x) \end{pmatrix}, \quad B(x) := \text{block}(\nabla_{x^\nu} g(x))_{\nu=1}^N.$$

The upper index of the function $\Theta^{5.2}$ indicates the current section. In Section 5.3 a similar function is introduced which has the upper index “5.3”.

Let us recall some well-known relations between the GNEP and its corresponding KKT system. The following proposition essentially follows from Proposition 5.1, see also [23, Theorem 4.6] or [38, Theorem 7].

Proposition 5.7. *(a) Let x^* be a solution of the GNEP and assume that, for every $\nu \in \{1, \ldots, N\}$, the ACQ is satisfied at $x^{*,\nu}$ regarding the system of inequalities $g(x^\nu, x^{*,-\nu}) \leq 0$. Then there is $u^* \in \mathbb{R}^{Nm_g}$ such that (x^*, u^*) solves* (5.19).

(b) Suppose that, for every $\nu \in \{1, \ldots, N\}$, the functions $\theta_\nu(\cdot, x^{-\nu})$ and $g(\cdot, x^{-\nu})$ are convex for all $x^{-\nu} \in \mathbb{R}^{n_x - n_\nu}$. Moreover, assume that (x^*, u^*) solves (5.19). *Then x^* is a solution of the GNEP.*

Let us define the functions $\Psi^{5.2}: \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \to \mathbb{R}^{n_x}$ and $\Psi^{5.2}_x: \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \to \mathbb{R}^{n_x \times n_x}$ by

$$\Psi^{5.2}(x,u) := \Theta^{5.2}(x) + B(x)u \tag{5.20}$$

and

$$\Psi^{5.2}_x(x,u) := \nabla_x \Psi^{5.2}(x,u), \tag{5.21}$$

respectively. It is easy to see that (5.19) is a complementarity system, i.e., it has the form (1.2) with $n := n_x + Nm_g$, $p := n_x$, $q := 0$, $r := Nm_g$, $z := (x,u)$, $P(z) := \Psi^{5.2}(x,u)$, $R(z) := (-g(x))_{\nu=1}^N$, and $S(z) := u = (u^\nu)_{\nu=1}^N$. Throughout this section, Z denotes the solution set of (5.19) and by $z^* = (x^*, u^*) \in Z$ an arbitrary but fixed solution is indicated. A suitable reformulation of (5.19) as a constrained system of equations is

$$F(z) := \begin{pmatrix} \Psi^{5.2}(x,u) \\ \min\{-g(x), u^1\} \\ \vdots \\ \min\{-g(x), u^N\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega \tag{5.22}$$

with

$$\Omega := \{z = (x,u) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \mid g(x) \leq 0,\ u \geq 0\}.$$

Every solution of (5.19) is also a solution of (5.22) and vice versa. Of course, F is a PC^1-function with 2^{Nm_g} selection functions $F^1, \ldots, F^{2^{Nm_g}}$.

Our next aim is to formulate Conditions 7a and 9a in the context of KKT systems arising from GNEPs and then to state an adapted formulation of Theorem 4.20. It is not difficult to see that the following condition is equivalent to Condition 7a for (5.19) and (5.22), respectively.

Condition 7c. There are $K_{7c} > 0$ and $\varepsilon_{7c} > 0$ such that

$$\begin{aligned} \operatorname{dist}[s, Z] \quad \leq \quad & K_{7c}(\|\Psi^{5.2}(x,u)\| + \sum_{\nu=1}^{N} \|\min\{-g(x), u^\nu\}\| \\ & + \|\min\{0, -g(x)\}\| + \|\min\{0, u\}\|) \end{aligned}$$

holds for all $s = (x,u) \in \mathcal{B}_{\varepsilon_{7c}}(z^*)$.

In order to formulate Condition 9a for the KKT system (5.19) and its reformulation (5.22), respectively, let us introduce some index sets which depend on the fixed solution $z^* = (x^*, u^*)$. We define, for every $\nu \in \{1, \ldots, N\}$,

$$\begin{aligned} \mathcal{I}^\nu_g &:= \{j \in \{1, \ldots, m_g\} \mid g_j(x^*) = 0 < u_j^{*,\nu}\}, \\ \mathcal{I}^\nu_= &:= \{j \in \{1, \ldots, m_g\} \mid g_j(x^*) = u_j^{*,\nu} = 0\}. \end{aligned} \tag{5.23}$$

For every $\nu \in \{1, \ldots, N\}$, the sets $\mathcal{I}^\nu_g$ and $\mathcal{I}^\nu_=$ partition the index set

$$\mathcal{G}_0 := \{j \in \{1, \ldots, m_g\} \mid g_j(x^*) = 0\} \tag{5.24}$$

consisting of the indices of all constraints which are active at x^*. Moreover, we introduce

$$\mathcal{I}_u := \{j \in \{1, \dots, m_g\} \mid \forall \nu = 1, \dots, N: \ g_j(x^*) < 0 = u_j^{*,\nu}\}, \tag{5.25}$$

i.e., $\mathcal{I}_u$ coincides with the index set of all constraints being inactive at x^*. Note that for each inactive constraint the corresponding multipliers of all players must be equal to zero so that $\mathcal{I}_u$ is independent on ν. The index sets $\mathcal{I}_g^\nu$, $\mathcal{I}_u$, and $\mathcal{I}_=^\nu$ correspond to the index sets $\mathcal{I}_R$, $\mathcal{I}_S$, and $\mathcal{I}_=$, respectively, from Section 4.3.

Let us further introduce the sets

$$\begin{array}{rcl} \mathcal{I}_g^\cup & := & \bigcup_{\nu=1}^N \mathcal{I}_g^\nu = \{j \in \mathcal{G}_0 \mid \exists \nu \in \{1, \dots, N\}: \ u_j^{*,\nu} > 0\}, \\ \mathcal{I}_=^\cap & := & \bigcap_{\nu=1}^N \mathcal{I}_=^\nu = \{j \in \mathcal{G}_0 \mid \forall \nu \in \{1, \dots, N\}: \ u_j^{*,\nu} = 0\} \end{array} \tag{5.26}$$

which are a partition of $\mathcal{G}_0$. The set $\mathcal{I}_g^\cup$ contains the indices of all constraints being active at x^* for which the corresponding multiplier of at least one player is strictly positive whereas $\mathcal{I}_=^\cap$ consists of the indices of all active constraints where the corresponding multipliers of all players are equal to zero.

For any family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$ $(\nu = 1, \dots, N)$, we denote by $Z_{\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N}$ the solution set of the following system of equations and inequalities:

$$\begin{array}{c} \Psi^{5.2}(x, u) = 0, \ g_{\mathcal{I}_g^\cup \cup \mathcal{I}_1}(x) = 0, \ g_{\mathcal{I}_u \cup \mathcal{I}_2}(x) \le 0, \\ u^\nu_{\mathcal{I}_u \cup \mathcal{I}_2^\nu} = 0, \ u^\nu_{\mathcal{I}_g^\nu \cup \mathcal{I}_1^\nu} \ge 0 \quad (\nu = 1, \dots, N) \end{array} \tag{5.27}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined by

$$\mathcal{I}_1 := \bigcup_{\nu=1}^N \mathcal{I}_1^\nu \quad \text{and} \quad \mathcal{I}_2 := \left(\bigcup_{\nu=1}^N \mathcal{I}_2^\nu \right) \setminus \mathcal{I}_1. \tag{5.28}$$

It is not difficult to show that the following condition is equivalent to Condition 9a for the KKT system (5.19) and its reformulation (5.22), respectively.

Condition 9c. There are $K_{9c} > 0$ and $\varepsilon_{9c} > 0$ such that, for every family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$,

$$\begin{aligned} \operatorname{dist}[s, Z_{\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N}] \ \le \ & K_{9c}(\|\Psi^{5.2}(x, u)\| + \|g_{\mathcal{I}_g^\cup \cup \mathcal{I}_1}(x)\| + \|\min\{0, -g_{\mathcal{I}_u \cup \mathcal{I}_2}(x)\}\| \\ & + \sum_{\nu=1}^N (\|u^\nu_{\mathcal{I}_u \cup \mathcal{I}_2^\nu}\| + \|\min\{0, u^\nu_{\mathcal{I}_g^\nu \cup \mathcal{I}_1^\nu}\}\|)) \end{aligned}$$

holds for all $s = (x, u) \in \mathcal{B}_{\varepsilon_{9c}}(z^*)$ where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined according to (5.28).

In other words, Condition 9c requires that, for every family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$, the system (5.27) satisfies the local error bound condition at $z^* = (x^*, u^*)$. Now we are in the position to state Theorem 4.20 in the context of KKT systems arising from GNEPs.

Theorem 5.8. *Let the functions $\theta_1, \ldots, \theta_N : \mathbb{R}^{n_x} \to \mathbb{R}$, and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ be twice differentiable with locally Lipschitz continuous second-order derivatives. Let one of the following conditions be true.*

(i) The functions $\theta_1, \ldots, \theta_N$ are quadratic and the function g is affine.

(ii) Condition 7c holds and the sets $\mathcal{I}^\nu_=$ are empty for all $\nu = 1, \ldots, N$, i.e., strict complementarity is valid at z^*.

(iii) Condition 7c is satisfied. Moreover, there is $\varepsilon > 0$ such that $z = (x, u) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$.

(iv) Condition 9c holds.

Then Assumptions 1–4 for (5.22) *are valid if the function G satisfies* (4.1) *for all $z = (x, u) \in \mathbb{R}^n$.*

Proof. The assertions follow from Theorem 4.20, together with the considerations of the current section up to now. Regarding condition (i) note that the function $\Theta^{5.2}(\cdot)$ is affine since the functions $\theta_1, \ldots, \theta_N$ are quadratic. Moreover, $B(\cdot)$ is independent on x because g is affine. Therefore, taking into account $P(z) = \Theta^{5.2}(x) + B(x)u$, $R(z) = (-g(x))_{\nu=1}^N$, and $S(z) = u$, the functions P, R, and S are affine.

Concerning condition (iii) note that we have $P(x^*, u) = \Theta^{5.2}(x^*) + \nabla g(x^*)u$, $R(x^*, u) = (-g(x^*))_{\nu=1}^N$, and $S(x^*, u) = u$. Consequently, $P(x^*, \cdot)$, $R(x^*, \cdot)$, and $S(x^*, \cdot)$ are affine. Thus, condition (iii) above implies condition (iii) from Theorem 4.20. □

The latter theorem shows that Assumptions 1–4 together are weaker than conditions which are used in [21] for proving local quadratic convergence of the (unconstrained) Levenberg-Marquardt method. In fact, it is proved in [21, Theorem 5] that the Levenberg-Marquardt method converges locally with a Q-quadratic rate to a solution of (5.19) if condition (ii) from the above theorem is satisfied. Furthermore, it is shown in [21, Theorem 8] that condition (i) of the above theorem, together with strict complementarity at z^*, leads to local quadratic convergence.

Remark 5.2. (a) The assertions of Theorem 5.8 stay true if condition (iii) is replaced by the following weaker condition (iii)'.

(iii)' Condition 7c is satisfied. Moreover, the vectors x are split according to $x = (x_a, x_b) \in \mathbb{R}^{n_{x_a}} \times \mathbb{R}^{n_{x_b}}$ such that the functions $\theta_1(x_a^*, \cdot), \ldots, \theta_N(x_a^*, \cdot)$ are quadratic and the function $g(x_a^*, \cdot)$ is affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x_a, x_b, u) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x_a = x_a^*$.

Condition (iii)' allows a nonisolated primal part of the solution z^* but requires that the objective functions $\theta_1, \ldots, \theta_N$ are quadratic regarding the vector x_b of nonisolated components and that the function g is affine regarding x_b.

(b) Assuming that the corresponding matrix-valued function $\hat{G}$ is suitably defined, each of the conditions (i)–(iv) from Theorem 5.8 implies, besides Assumptions 1–4 for (5.22), also the whole set of Assumptions 1–4 for the following constrained system arising from (5.22) by introducing slack variables w for the inequality constraints:

$$\hat{F}(z,w) := \begin{pmatrix} \Psi^{5.2}(x,u) \\ \min\{-g(x), u^1\} \\ \vdots \\ \min\{-g(x), u^N\} \\ g(x) + w \end{pmatrix} = 0 \quad \text{s.t.} \quad (z,w) \in \hat{\Omega} \tag{5.29}$$

with

$$\hat{\Omega} := \{(z,w) = (x,u,w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \times \mathbb{R}^{m_g} \mid u \geq 0,\ w \geq 0\},$$

see Section 4.2 and item (a) of Remark 4.4 for details. Of course, slack variables for the constraints $u \geq 0$ are not necessary since they are already affine (even bound constraints only).

Exploiting Proposition 4.21, we obtain the following result which says that a certain constant rank condition is sufficient for Condition 9c to hold. Proposition 5.9 will appear in [37].

Proposition 5.9. *Assume that there is some $\varepsilon > 0$ such that, for every $(N+1)$-tuple $(\mathcal{J}, \mathcal{J}^1, \ldots, \mathcal{J}^N)$ of index sets $\mathcal{J} \subseteq \mathcal{I}_=^{\cap}$ and $\mathcal{J}^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \ldots, N)$, the matrices*

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x,u) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup \mathcal{J}^\nu}(x)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^{\cup} \cup \mathcal{J}}(x) & 0 \end{array}\right)$$

have the same rank for all $s = (x,u) \in \mathcal{B}_\varepsilon(z^*)$. Then Condition 9c is valid. In particular, Assumptions 1–4 for (5.22) *hold if G satisfies* (4.1) *for all $z = (x,u) \in \mathbb{R}^n$.*

Proof. It suffices to show that Condition 9c is satisfied. Then the second assertion follows from Theorem 5.8. We want to exploit Proposition 4.21. Note that we have $s = (x,u)$, $P(s) = \Psi^{5.2}(x,u) = \Theta^{5.2}(x) + B(x)u$,

$$R(s) = (-g(x))_{\nu=1}^N = \begin{pmatrix} -g(x) \\ \vdots \\ -g(x) \end{pmatrix}, \quad \text{and} \quad S(s) = u = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}$$

in the setting of the current section. The function Q does not occur. Moreover, the index sets $\mathcal{I}_R$, $\mathcal{I}_S$, and $\mathcal{I}_=$ correspond to $\mathcal{I}_g^\nu$, $\mathcal{I}_u$, and $\mathcal{I}_=^\nu$, respectively. Using the latter observations, it follows from Proposition 4.21 that Condition 9c, which is the analogon of Condition 9a, holds if there is $\varepsilon > 0$ such that, for every family $\{(\mathcal{K}_g^\nu, \mathcal{K}_u^\nu)\}_{\nu=1}^N$ of index sets $\mathcal{K}_g^\nu \subseteq \mathcal{I}_=^\nu$ and $\mathcal{K}_u^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \ldots, N)$, the matrices

$$\left(\begin{array}{c|c|c} \Psi_x^{5.2}(x,u) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)}(x)\right)_{\nu=1}^N & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_u \cup \mathcal{K}_u^\nu}(x)\right)_{\nu=1}^N \\ \hline -g'_{\mathcal{I}_g^1 \cup \mathcal{K}_g^1}(x) & & \\ \vdots & 0 & 0 \\ -g'_{\mathcal{I}_g^N \cup \mathcal{K}_g^N}(x) & & \\ \hline 0 & 0 & \text{block}\left(I_{|\mathcal{I}_u \cup \mathcal{K}_u^\nu|}\right)_{\nu=1}^N \end{array}\right) \tag{5.30}$$

have the same rank for all $s = (x, u) \in \mathcal{B}_\varepsilon(z^*)$. Note that the three column blocks in the above matrix contain the derivatives of P, $R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}$, and $S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}$ with respect to the variables in the following order:

$$x,\ (u^\nu_{\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)})_{\nu=1}^N,\ (u^\nu_{\mathcal{I}_u \cup \mathcal{K}_u^\nu})_{\nu=1}^N.$$

By $R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}$ and $S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}$ we mean the functions given according to

$$R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}(s) := \begin{pmatrix} \vdots \\ -g_{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu}(x) \\ \vdots \end{pmatrix}_{\nu=1}^N$$

and

$$S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}(s) := \begin{pmatrix} \vdots \\ u^\nu_{\mathcal{I}_u \cup \mathcal{K}_u^\nu} \\ \vdots \end{pmatrix}_{\nu=1}^N,$$

respectively. The rank of the matrix in (5.30) does not change if the second row block is multiplied by -1 and repeated rows are deleted. Therefore, taking into account the definition of the index set $\mathcal{I}_g^\cup$, the matrix in (5.30) has, for any $s = (x, u) \in \mathbb{R}^n$, the same rank as the matrix

$$\left(\begin{array}{c|c|c} \Psi_x^{5.2}(x,u) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)}(x)\right)_{\nu=1}^N & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_u \cup \mathcal{K}_u^\nu}(x)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{K}_g}(x) & 0 & 0 \\ \hline 0 & 0 & \text{block}\left(I_{|\mathcal{I}_u \cup \mathcal{K}_u^\nu|}\right)_{\nu=1}^N \end{array}\right) \tag{5.31}$$

where we set $\mathcal{K}_g := \bigcap_{\nu=1}^N \mathcal{K}_g^\nu$. The latter set is a subset of $\mathcal{I}_=^\cap$. Using Lemma 5.5, the rank of the matrix in (5.31) equals, for any $s = (x, u) \in \mathbb{R}^n$, the rank of the matrix

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x,u) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)}(x)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{K}_g}(x) & 0 \end{array}\right) \tag{5.32}$$

plus $\sum_{\nu=1}^{N} |\mathcal{I}_u \cup \mathcal{K}_u^\nu|$. After all, we obtain that the matrices in (5.30) have the same rank for all $s = (x, u) \in \mathcal{B}_\varepsilon(z^*)$ if and only if the matrices in (5.32) have this property. The matrices in (5.32) actually have the same rank for all $s = (x, u) \in \mathcal{B}_\varepsilon(z^*)$ due to the assumption of the proposition with $\mathcal{J} := \mathcal{K}_g$ and $\mathcal{J}^\nu := \mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu$ $(\nu = 1, \dots, N)$. □

The following corollary is a consequence of Proposition 5.9. It says that Condition 9c is satisfied if a certain matrix has full row rank. This implication is already shown in [49, Theorem 1], at least for the case of two players. Corollary 5.10 will also be part of [37].

Corollary 5.10. *Suppose that the matrix*

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{G}_0}(x^*) & 0 \end{array}\right)$$

has full row rank. Then Condition 9c is valid. In particular, Assumptions 1–4 for (5.22) *hold if* G *satisfies* (4.1) *for all* $z = (x, u) \in \mathbb{R}^n$.

Proof. We are going to show that the assumption of Proposition 5.9 holds. To this end, let us take any $(N+1)$-tuple $(\mathcal{J}, \mathcal{J}^1, \dots, \mathcal{J}^N)$ of index sets $\mathcal{J} \subseteq \mathcal{I}_=^\cap$ and $\mathcal{J}^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \dots, N)$. Since $\mathcal{G}_0$ equals $\mathcal{I}_g^\cup \cup \mathcal{I}_=^\cap$, we have $\mathcal{I}_g^\cup \cup \mathcal{J} \subseteq \mathcal{G}_0$. Therefore, by the assumption of the corollary, the matrix

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g \cup \mathcal{J}}(x^*) & 0 \end{array}\right)$$

has full row rank. The rows of the matrix

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup \mathcal{J}^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{J}}(x^*) & 0 \end{array}\right)$$

are still linearly independent since at most the number of columns has increased. Due to the continuity of all functions which are involved in this matrix, the rows stay linearly independent for all $s = (x, u)$ in a sufficiently small neighborhood of z^*. Since the index sets $\mathcal{J}, \mathcal{J}^1, \dots, \mathcal{J}^N$ were arbitrarily chosen, the assumption of Proposition 5.9 is satisfied. Thus, the assertions follow from Proposition 5.9. □

In [15] the nonsingularity of a certain matrix, together with the condition $\mathcal{I}_=^\cap = \emptyset$, is used to analyze convergence properties of a hybrid method for the solution of KKT systems arising from GNEPs. The following corollary shows that these conditions together imply the full row rank of the matrix in Corollary 5.10. In particular, Condition 9c is implied. Note that the emptiness of $\mathcal{I}_=^\cap$ is a weaker condition than strict complementarity. It only requires that for any active constraint the corresponding multiplier of at least one player is strictly positive.

Corollary 5.11. [1] *Assume that $\mathcal{I}_=^\cap$ is empty. Moreover, suppose that for any $j \in \mathcal{G}_0$ an index $\nu(j)$ exists such that $j \in \mathcal{I}_g^{\nu(j)}$ holds and the matrix*

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{J}^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{G}_0}(x^*) & 0 \end{array}\right) \tag{5.33}$$

is nonsingular where, for any $\nu \in \{1, \ldots, N\}$, the index set $\mathcal{J}^\nu$ is defined according to $\mathcal{J}^\nu := \{j \in \mathcal{G}_0 \mid \nu = \nu(j)\}$. Then Condition 9c is valid. In particular, Assumptions 1–4 for (5.22) *hold if G satisfies* (4.1) *for all $z = (x, u) \in \mathbb{R}^n$.*

Proof. Obviously, for any ν, $\mathcal{J}^\nu \subseteq \mathcal{I}_g^\nu$ holds so that the matrix in (5.33) is a submatrix of the matrix from Corollary 5.10 where the latter has the same number of rows but more columns. Therefore, the matrix from Corollary 5.10 has full row rank due to the nonsingularity of the matrix in (5.33). Thus, the assertions follow from Corollary 5.10. □

Finally, let us present an example where classical regularity assumptions like local uniqueness of the x-part of the solution or strict complementarity are violated whereas the assumption of Corollary 5.11 is satisfied. So in particular, Condition 9c and therefore Assumptions 1–4 are valid (the latter if G is suitably chosen). Example 5.4 is [23, Example 1.1] and was also considered in [15, Example 1] and in [49, Example 1].

Example 5.4. Let us consider a GNEP with two players where each player controls one variable only and one shared constraint occurs. The optimization problems of the players are given by

$$\theta_1(x^1, x^2) := (x^1 - 1)^2 \to \min \quad \text{s.t.} \quad g(x^1, x^2) := x^1 + x^2 - 1 \leq 0$$

and

$$\theta_2(x^1, x^2) := (x^2 - \frac{1}{2})^2 \to \min \quad \text{s.t.} \quad g(x^1, x^2) \leq 0,$$

respectively. The corresponding KKT system looks as follows:

$$\begin{aligned} &\Theta^{5.2}(x) + B(x)u = \begin{pmatrix} 2x^1 - 2 + u^1 \\ 2x^2 - 1 + u^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ &g(x) = x^1 + x^2 - 1 \leq 0, \; u^1 \geq 0, \; u^2 \geq 0, \\ &u^1(x^1 + x^2 - 1) = u^2(x^1 + x^2 - 1) = 0. \end{aligned}$$

Its solution set is given by

$$Z := \{(x^1, x^2, u^1, u^2)^\top = (\tau, 1 - \tau, 2 - 2\tau, 2\tau - 1)^\top \mid \frac{1}{2} \leq \tau \leq 1\}.$$

[1] The condition that $j \in \mathcal{I}_g^{\nu(j)}$ (as already used in [15, Assumption 1]) was added after the defense. It guarantees that the matrix in (5.33) is indeed a submatrix of the matrix from Corollary 5.10.

The function value of g is equal to zero at all solutions. Let us verify that classical regularity conditions are not satisfied. Obviously, neither the x-parts nor the u-parts of the solutions are locally unique. Furthermore, strict complementarity is violated for the solutions corresponding to $\tau = \frac{1}{2}$ and $\tau = 1$.

However, the conditions of Corollary 5.11 hold at any solution $z^* = (x^*, u^*)$. In fact, for every $\tau \in [\frac{1}{2}, 1]$, at least one of the values $u^{*,1} = 2 - 2\tau$ and $u^{*,2} = 2\tau - 1$ is strictly positive so that $\mathcal{I}_=^\cap = \emptyset$ is valid. Moreover, both matrices

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \nabla_{x^1} g(x^*) \\ \hline g'(x^*) & 0 \end{array}\right) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*, u^*) & \nabla_{x^2} g(x^*) \\ \hline g'(x^*) & 0 \end{array}\right) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

are nonsingular. Note that the former matrix coincides with the matrix in (5.33) for $\nu(1) = 1$, and the latter matrix coincides with the matrix in (5.33) for $\nu(1) = 2$. In particular, the nonsingularity assumption of Corollary 5.11 holds. Therefore, Condition 9c holds by Corollary 5.11. Particularly, Assumptions 1–4 are valid if G satisfies (4.1) for all $z = (x, u) \in \mathbb{R}^4$. □

5.3 FJ Systems of GNEPs

We consider the setting of Section 5.2 again, i.e., a GNEP with N players where the optimization problem of the ν-th player is given by (5.18). Again, the inequalities $g(x) \leq 0$ describe the shared constraints. The functions $\theta_1, \ldots, \theta_N : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ are assumed to be twice differentiable with locally Lipschitz continuous second-order derivatives. As in Section 5.2, n_x denotes the number of the variables of all players. In this section, we consider the corresponding Fritz-John (FJ) system of the GNEP. It is well known that every solution of the GNEP yields a solution of the FJ system. In [13] an example is described which shows that it cannot be expected in general that every solution of a GNEP provides a solution of the KKT system, see [13, Example 2]. The example is stable with respect to small perturbations of the problem functions g and θ_ν. This motivates to consider the FJ system instead of the KKT system.

First, we will justify that the FJ system of a GNEP is a complementarity system, i.e., it has the form (1.2). Afterwards, an adapted formulation of Theorem 4.20 is provided, see Theorem 5.13. Moreover, similar to the preceding section, we present some constant rank condition as well as some full row rank condition, and it will be proved that each of these conditions implies the whole set of Assumptions 1–4, see Proposition 5.14 and Corollary 5.15, respectively. In the second part of this section, we show that generically the full row rank condition is satisfied at any solution of the FJ system of a GNEP. To this end, we recall the notion of a generically satisfied condition and use results from [13]. At the end of this section, we relate our convergence assumptions to those which were used in [13] for the analysis of local convergence properties of a nonsmooth projection method.

We begin with the description of the FJ system of a GNEP. For the moment, let us assume that $\nu \in \{1, \ldots, N\}$ and $x^{-\nu} \in \mathbb{R}^{n_x - n_\nu}$ are fixed. The corresponding FJ system of the ν-th players' optimization problem (5.18) is given by

$$\begin{aligned} & u_0^\nu \nabla_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) + \nabla_{x^\nu} g(x^\nu, x^{-\nu}) u^\nu = 0, \\ & g(x^\nu, x^{-\nu}) \le 0, \; u^\nu \ge 0, \; (u^\nu)^\top g(x^\nu, x^{-\nu}) = 0, \\ & u_0^\nu \ge 0, \; u_0^\nu + 1_{m_g}^\top u^\nu - 1 = 0 \end{aligned} \tag{5.34}$$

with some multiplier vector $u^\nu \in \mathbb{R}^{m_g}$ and some number $u_0^\nu \in \mathbb{R}$. By 1_{m_g} the vector with m_g components consisting of ones only is denoted. Concatenating the FJ systems of all players, we obtain the FJ system of the GNEP:

$$\begin{aligned} & \Theta^{5.3}(x) u_0 + B(x) u = 0, \\ & g(x) \le 0, \; u \ge 0, \; u_0 \ge 0, \\ & (u^\nu)^\top g(x) = 0 \quad \text{for all } \nu = 1, \ldots, N, \\ & u_0 + Eu - 1_N = 0 \end{aligned} \tag{5.35}$$

with

$$u := \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}, \quad \Theta^{5.3}(x) := \operatorname{block}\left(\nabla_{x^\nu} \theta_\nu(x)\right)_{\nu=1}^N, \quad B(x) := \operatorname{block}\left(\nabla_{x^\nu} g(x)\right)_{\nu=1}^N,$$

$$u_0 := (u_0^1, \ldots, u_0^N)^\top, \quad E := \operatorname{block}\left(1_{m_g}^\top\right)_{\nu=1}^N.$$

In the following proposition it is recalled that every solution of the GNEP yields a solution of the corresponding FJ system (5.35).

Proposition 5.12. *Let x^* be a solution of the GNEP. Then there are $u_0^* \in \mathbb{R}^N$ and $u^* \in \mathbb{R}^{N m_g}$ such that (x^*, u_0^*, u^*) solves the FJ system* (5.35).

Proof. For every $\nu = 1, \ldots, N$ the vector $x^{*,\nu}$ solves the optimization problem (5.18) with $x^{-\nu} := x^{*,-\nu}$ since x^* is a solution of the GNEP. Therefore, for every ν, there are $u_0^{*,\nu} \in \mathbb{R}$ and $u^{*,\nu} \in \mathbb{R}^{m_g}$ such that $(x^{*,\nu}, u_0^{*,\nu}, u^{*,\nu})$ solves (5.34) with $x^{-\nu}$ replaced by $x^{*,-\nu}$. This follows from a well-known relation between an optimization problem and the corresponding FJ system, see for example [40, Satz 2.53]. Then it is not difficult to see that (x^*, u_0^*, u^*), with

$$u_0^* := \begin{pmatrix} u_0^{*,1} \\ \vdots \\ u_0^{*,N} \end{pmatrix} \quad \text{and} \quad u^* := \begin{pmatrix} u^{*,1} \\ \vdots \\ u^{*,N} \end{pmatrix},$$

is a solution of (5.35). □

Let us define the functions $\Psi^{5.3} : \mathbb{R}^{n_x} \times \mathbb{R}^N \times \mathbb{R}^{Nm_g} \to \mathbb{R}^{n_x}$ and $\Psi_x^{5.3} : \mathbb{R}^{n_x} \times \mathbb{R}^N \times \mathbb{R}^{Nm_g} \to \mathbb{R}^{n_x \times n_x}$ by

$$\Psi^{5.3}(x, u_0, u) := \Theta^{5.3}(x)u_0 + B(x)u$$

and

$$\Psi_x^{5.3}(x, u_0, u) := \nabla_x \Psi^{5.3}(x, u_0, u),$$

respectively. Obviously, (5.35) is a complementarity system, i.e., it has the form (1.2) with $n := n_x + N + Nm_g$, $p := n_x + N$, $q := N$, $r := Nm_g$, $z := (x, u_0, u)$, $Q(z) := u_0$, $R(z) := (-g(x))_{\nu=1}^N$, $S(z) := u = (u^\nu)_{\nu=1}^N$, and

$$P(z) := \begin{pmatrix} \Psi^{5.3}(x, u_0, u) \\ u_0 + Eu - 1_N \end{pmatrix}.$$

In this section we denote by Z the solution set of (5.35) and by $z^* = (x^*, u_0^*, u^*) \in Z$ an arbitrary but fixed solution. A suitable reformulation of (5.35) as a constrained system of equations is

$$F(z) := \begin{pmatrix} \Psi^{5.3}(x, u_0, u) \\ u_0 + Eu - 1_N \\ \min\{-g(x), u^1\} \\ \vdots \\ \min\{-g(x), u^N\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega \tag{5.36}$$

with

$$\Omega := \{z = (x, u_0, u) \in \mathbb{R}^{n_x} \times \mathbb{R}^N \times \mathbb{R}^{Nm_g} \mid g(x) \leq 0,\ u_0 \geq 0,\ u \geq 0\}.$$

Every solution of (5.35) is also a solution of (5.36) and vice versa. The function F is a PC^1-function with 2^{Nm_g} selection functions $F^1, \ldots, F^{2^{Nm_g}}$.

Our next aim is to formulate Conditions 7a and 9a in the context of FJ systems arising from GNEPs and then to present an adapted version of Theorem 4.20. It is not difficult to see that the following condition is equivalent to Condition 7a for (5.35) and (5.36), respectively.

Condition 7d. There are $K_{7d} > 0$ and $\varepsilon_{7d} > 0$ such that

$$\begin{aligned} \operatorname{dist}[s, Z] \quad \leq \quad & K_{7d}(\|\Psi^{5.3}(x, u_0, u)\| + \|u_0 + Eu - 1_N\| + \sum_{\nu=1}^N \|\min\{-g(x), u^\nu\}\| \\ & + \|\min\{0, -g(x)\}\| + \|\min\{0, u_0\}\| + \|\min\{0, u\}\|) \end{aligned}$$

holds for all $s = (x, u_0, u) \in \mathcal{B}_{\varepsilon_{7d}}(z^*)$.

In order to formulate Condition 9a for the FJ system (5.35) and its reformulation (5.36), respectively, we use the index sets $\mathcal{I}_g^\nu$, $\mathcal{I}_=^\nu$, $\mathcal{G}_0$, $\mathcal{I}_u$, $\mathcal{I}_g^\cup$, and $\mathcal{I}_=^\cap$ (the former two ones for $\nu = 1, \ldots, N$) which are defined according to (5.23)– (5.26). Moreover, let us introduce, for any family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$ $(\nu = 1, \ldots, N)$, the set $Z_{\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N}$ denoting the solution set of the following system of equations and inequalities:

$$\begin{array}{c} \Psi^{5.3}(x, u_0, u) = 0,\ u_0 + Eu - 1_N = 0,\ g_{\mathcal{I}_g^\cup \cup \mathcal{I}_1}(x) = 0,\ g_{\mathcal{I}_u \cup \mathcal{I}_2}(x) \leq 0, \\ u_0 \geq 0,\ u^\nu_{\mathcal{I}_u \cup \mathcal{I}_2^\nu} = 0,\ u^\nu_{\mathcal{I}_g^\nu \cup \mathcal{I}_1^\nu} \geq 0 \quad (\nu = 1, \ldots, N). \end{array} \tag{5.37}$$

The index sets $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined as in Section 5.2, see (5.28). It is not difficult to verify that the following condition is equivalent to Condition 9a for (5.35) and (5.36), respectively.

Condition 9d. There are $K_{9d} > 0$ and $\varepsilon_{9d} > 0$ such that, for every family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$,

$$\begin{aligned} \operatorname{dist}[s, Z_{\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N}] \ \leq\ & K_{9d}(\|\Psi^{5.3}(x, u_0, u)\| + \|u_0 + Eu - 1_N\| + \|g_{\mathcal{I}_g^\cup \cup \mathcal{I}_1}(x)\| \\ & + \|\min\{0, -g_{\mathcal{I}_u \cup \mathcal{I}_2}(x)\}\| + \|\min\{0, u_0\}\| \\ & + \sum_{\nu=1}^N (\|u^\nu_{\mathcal{I}_u \cup \mathcal{I}_2^\nu}\| + \|\min\{0, u^\nu_{\mathcal{I}_g^\nu \cup \mathcal{I}_1^\nu}\}\|)) \end{aligned}$$

holds for all $s = (x, u_0, u) \in \mathcal{B}_{\varepsilon_{9d}}(z^*)$ where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined according to (5.28).

In other words, Condition 9d requires that, for every family $\{(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)\}_{\nu=1}^N$ of partitions $(\mathcal{I}_1^\nu, \mathcal{I}_2^\nu)$ of $\mathcal{I}_=^\nu$, the system (5.37) satisfies the local error bound condition at $z^* = (x^*, u_0^*, u^*)$. Now we are in the position to state an adapted formulation of Theorem 4.20 for FJ systems arising from GNEPs.

Theorem 5.13. *Let the functions $\theta_1, \ldots, \theta_N : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ be twice differentiable with locally Lipschitz continuous second-order derivatives. Let one of the following conditions be true.*

(i) The functions $\theta_1, \ldots, \theta_N$, and g are affine.

(ii) Condition 7d holds and the sets $\mathcal{I}_=^\nu$ are empty for all $\nu = 1, \ldots, N$, i.e., strict complementarity is valid at z^*.

(iii) Condition 7d is satisfied. Moreover, there is $\varepsilon > 0$ such that $z = (x, u_0, u) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x = x^*$.

(iv) Condition 9d holds.

Then Assumptions 1–4 for (5.36) are valid if the function G satisfies (4.1) for all $z = (x, u_0, u) \in \mathbb{R}^n$.

Proof. The assertions follow from Theorem 4.20, together with the considerations of the current section up to now. Regarding condition (i) note that the function $\Theta^{5.3}(\cdot)$ is independent on x since the functions $\theta_1, \ldots, \theta_N$ are affine. Moreover, $B(\cdot)$ is independent on x because g is affine. Therefore, taking into account

$$P(z) = \begin{pmatrix} \Theta^{5.3}(x)u_0 + B(x)u \\ u_0 + Eu - 1_N \end{pmatrix},$$

$Q(z) = u_0$, $R(z) = (-g(x))_{\nu=1}^N$, and $S(z) = u$, the functions P, Q, R, and S are affine.

Concerning condition (iii) note that we have

$$P(x^*, u_0, u) = \begin{pmatrix} \Theta^{5.3}(x^*)u_0 + B(x^*)u \\ u_0 + Eu - 1_N \end{pmatrix},$$

$Q(x^*, u_0, u) = u_0$, $R(x^*, u_0, u) = (-g(x^*))_{\nu=1}^N$, and $S(x^*, u_0, u) = u$. Consequently, $P(x^*, \cdot, \cdot)$, $Q(x^*, \cdot, \cdot)$, $R(x^*, \cdot, \cdot)$, and $S(x^*, \cdot, \cdot)$ are affine. Thus, condition (iii) above implies condition (iii) from Theorem 4.20. □

Remark 5.3. (a) The assertions of Theorem 5.13 stay true if condition (iii) is replaced by the following weaker condition (iii)'.

(iii)' Condition 7d is satisfied. Moreover, the vectors x are split according to $x = (x_a, x_b) \in \mathbb{R}^{n_{x_a}} \times \mathbb{R}^{n_{x_b}}$ such that the functions $\theta_1(x_a^*, \cdot), \ldots, \theta_N(x_a^*, \cdot)$, and $g(x_a^*, \cdot)$ are affine. Furthermore, there is $\varepsilon > 0$ such that $z = (x_a, x_b, u_0, u) \in Z \cap \mathcal{B}_\varepsilon(z^*)$ implies $x_a = x_a^*$.

Condition (iii)' allows a nonisolated primal part of the solution z^* but requires that the functions $\theta_1, \ldots, \theta_N$, and g are affine regarding the vector x_b of nonisolated components.

(b) Assuming that the corresponding matrix-valued function $\hat{G}$ is suitably defined, each of the conditions (i)–(iv) from Theorem 5.13 implies, besides Assumptions 1–4 for (5.36), also the whole set of Assumptions 1–4 for the following constrained system arising from (5.36) by introducing slack variables w for the inequality constraints:

$$\hat{F}(z, w) := \begin{pmatrix} \Psi^{5.3}(x, u_0, u) \\ u_0 + Eu - 1_N \\ \min\{-g(x), u^1\} \\ \vdots \\ \min\{-g(x), u^N\} \\ g(x) + w \end{pmatrix} = 0 \quad \text{s.t.} \quad (z, w) \in \hat{\Omega}$$

with

$$\begin{aligned}\hat{\Omega} \;\; &:= \;\; \{z = (x, u_0, u, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^N \times \mathbb{R}^{Nm_g} \times \mathbb{R}^{m_g} \mid \\ & \qquad u_0 \geq 0,\ u \geq 0,\ w \geq 0\},\end{aligned}$$

see Section 4.2 and item (a) of Remark 4.4 for details. Of course, slack variables for the constraints $u_0 \geq 0$ and $u \geq 0$ are not necessary since they are already affine (even bound constraints only).

Similar to the previous sections, the next aim is to provide sufficient conditions for Condition 9d to hold. We begin with a condition based on the constant rank of certain matrices. To this end, we need the following index sets which depend on the u_0-part of the fixed solution $z^* = (x^*, u_0^*, u^*)$:

$$\begin{aligned}\mathcal{I}_{u_0} \;\; &:= \;\; \{\nu \in \{1, \dots, N\} \mid u_0^{*,\nu} = 0\}, \\ \overline{\mathcal{I}}_{u_0} \;\; &:= \;\; \{\nu \in \{1, \dots, N\} \mid u_0^{*,\nu} > 0\}.\end{aligned}$$

Obviously, $\mathcal{I}_{u_0}$ and $\overline{\mathcal{I}}_{u_0}$ partition the index set $\{1, \dots, N\}$. Moreover, let us define, for any $\nu \in \{1, \dots, N\}$ and any $x \in \mathbb{R}^{n_x}$, the vector $d^\nu(x) \in \mathbb{R}^{n_x}$ according to

$$d^\nu(x) := \begin{pmatrix} v^1(x) \\ \vdots \\ v^N(x) \end{pmatrix}$$

where, for every $\iota = 1, \dots, N$, the vector $v^\iota(x) \in \mathbb{R}^{n_\iota}$ is given by

$$v^\iota(x) := \begin{cases} \nabla_{x^\nu} \theta_\nu(x) & \text{if} \quad \iota = \nu, \\ 0 & \text{if} \quad \iota \neq \nu. \end{cases}$$

Furthermore, let us denote, for any $\nu = 1, \dots, N$, by $e^\nu \in \mathbb{R}^N$ the ν-th canonical unit vector, i.e.,

$$e^\nu_\iota := \begin{cases} 1 & \text{if} \quad \iota = \nu, \\ 0 & \text{if} \quad \iota \neq \nu \end{cases} \qquad (\iota = 1, \dots, N).$$

For any index set $\mathcal{M} \subseteq \{1, \dots, N\}$ we define the matrices

$$M^e_{\mathcal{M}} := (\cdots\ e^\nu\ \cdots)_{\nu \in \mathcal{M}} \in \mathbb{R}^{N \times |\mathcal{M}|}$$

and, for any vector $x \in \mathbb{R}^{n_x}$,

$$M^d_{\mathcal{M}}(x) := (\cdots\ d^\nu(x)\ \cdots)_{\nu \in \mathcal{M}} \in \mathbb{R}^{n_x \times |\mathcal{M}|}.$$

Now we are in the position to state and prove Proposition 5.14 which provides a certain constant rank condition being sufficient for Condition 9d to hold.

Proposition 5.14. *Assume that there is some $\varepsilon > 0$ such that, for every $(N+2)$-tuple $(\mathcal{J}, \mathcal{J}_{u_0}, \mathcal{J}^1, \dots, \mathcal{J}^N)$ of index sets $\mathcal{J} \subseteq \mathcal{I}_=^{\cap}$, $\mathcal{J}_{u_0} \subseteq \mathcal{I}_{u_0}$, and $\mathcal{J}^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \dots, N)$, the matrices*

$$\left(\begin{array}{c|c|c} \Psi_x^{5.3}(x, u_0, u) & M^d_{\overline{\mathcal{I}}_{u_0} \cup \mathcal{J}_{u_0}}(x) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup \mathcal{J}^\nu}(x)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0} \cup \mathcal{J}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu \cup \mathcal{J}^\nu|}\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{J}}(x) & 0 & 0 \end{array}\right)$$

have the same rank for all $s = (x, u_0, u) \in \mathcal{B}_\varepsilon(z^*)$. Then Condition 9d is valid. In particular, Assumptions 1–4 for (5.36) *hold if the function G satisfies* (4.1) *for all $z = (x, u_0, u) \in \mathbb{R}^n$.*

Proof. We show that Condition 9d is satisfied. Then the second assertion follows from Theorem 5.13. We want to exploit Proposition 4.21. Note that we have $s = (x, u_0, u)$,

$$P(s) = \begin{pmatrix} \Psi^{5.3}(x, u_0, u) \\ u_0 + Eu - 1_N \end{pmatrix} = \begin{pmatrix} \Theta^{5.3}(x)u_0 + B(x)u \\ u_0 + Eu - 1_N \end{pmatrix}, \quad Q(s) = u_0,$$

$$R(s) = (-g(x))_{\nu=1}^N = \begin{pmatrix} -g(x) \\ \vdots \\ -g(x) \end{pmatrix}, \quad \text{and} \quad S(s) = u = \begin{pmatrix} u^1 \\ \vdots \\ u^N \end{pmatrix}$$

in the setting of the current section. Moreover, the index sets $\mathcal{I}_Q$, $\mathcal{I}_R$, $\mathcal{I}_S$, and $\mathcal{I}_=$ correspond to $\mathcal{I}_{u_0}$, $\mathcal{I}_g^\nu$, $\mathcal{I}_u$, and $\mathcal{I}_=^\nu$, respectively. Using the latter observations, it follows from Proposition 4.21 that Condition 9d, which is the analogon of Condition 9a, holds if there is $\varepsilon > 0$ such that, for every index set $\mathcal{K}_{u_0} \subseteq \mathcal{I}_{u_0}$ and every family $\{(\mathcal{K}_g^\nu, \mathcal{K}_u^\nu)\}_{\nu=1}^N$ of index sets $\mathcal{K}_g^\nu \subseteq \mathcal{I}_=^\nu$ and $\mathcal{K}_u^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \dots, N)$, the matrices

$$\left(\begin{array}{c|c|c|c|c} \Psi_x^{5.3}(x, u_0, u) & M^d_{\mathcal{M}^0}(x) & M^d_{\mathcal{K}_{u_0}}(x) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{M}^\nu}(x)\right)_{\nu=1}^N & \operatorname{block}\left(\nabla_{x^\nu} g_{\overline{\mathcal{M}}^\nu}(x)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\mathcal{M}^0} & M^e_{\mathcal{K}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{M}^\nu|}\right)_{\nu=1}^N & \operatorname{block}\left(1^\top_{|\overline{\mathcal{M}}^\nu|}\right)_{\nu=1}^N \\ \hline 0 & 0 & M^e_{\mathcal{K}_{u_0}} & 0 & 0 \\ \hline -g'_{\mathcal{I}_g^1 \cup \mathcal{K}_g^1}(x) & & & & \\ \vdots & 0 & 0 & 0 & 0 \\ -g'_{\mathcal{I}_g^N \cup \mathcal{K}_g^N}(x) & & & & \\ \hline 0 & 0 & 0 & 0 & \operatorname{block}\left(I_{|\overline{\mathcal{M}}^\nu|}\right)_{\nu=1}^N \end{array}\right) \tag{5.38}$$

have the same rank for all $s = (x, u_0, u) \in \mathcal{B}_\varepsilon(z^*)$ where we set

$$\mathcal{M}^0 := \overline{\mathcal{I}}_{u_0} \cup (\mathcal{I}_{u_0} \setminus \mathcal{K}_{u_0})$$

and, for every $\nu = 1, \dots, N$,

$$\begin{aligned} \mathcal{M}^\nu &:= \mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu), \\ \overline{\mathcal{M}}^\nu &:= \mathcal{I}_u \cup \mathcal{K}_u^\nu. \end{aligned}$$

Note that the five column blocks in the matrix (5.38) contain the derivatives of P, $Q_{\mathcal{K}_{u_0}}$, $R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}$, and $S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}$ with respect to the variables in the following order:

$$x,\ [u_0]_{\mathcal{M}^0},\ [u_0]_{\mathcal{K}_{u_0}},\ (u^\nu_{\mathcal{M}^\nu})_{\nu=1}^N,\ (u^\nu_{\overline{\mathcal{M}}^\nu})_{\nu=1}^N.$$

As in the proof of Proposition 5.9, we mean by $R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}$ and $S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}$ the functions given according to

$$R_{\{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu\}_{\nu=1}^N}(s) := \begin{pmatrix} \vdots \\ -g_{\mathcal{I}_g^\nu \cup \mathcal{K}_g^\nu}(x) \\ \vdots \end{pmatrix}_{\nu=1}^N$$

and

$$S_{\{\mathcal{I}_u \cup \mathcal{K}_u^\nu\}_{\nu=1}^N}(s) := \begin{pmatrix} \vdots \\ u^\nu_{\mathcal{I}_u \cup \mathcal{K}_u^\nu} \\ \vdots \end{pmatrix}_{\nu=1}^N,$$

respectively. The rank of the matrix in (5.38) does not change if the fourth row block is multiplied by -1 and repeated rows are deleted. Therefore, taking into account the definition of the index set $\mathcal{I}_g^\cup$, the matrix in (5.38) has, for any $s = (x, u_0, u) \in \mathbb{R}^n$, the same rank as the matrix

$$\left(\begin{array}{c|c|c|c|c} \Psi_x^{5.3}(x,u_0,u) & M^d_{\mathcal{M}^0}(x) & M^d_{\mathcal{K}_{u_0}}(x) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{M}^\nu}(x)\right)_{\nu=1}^N & \operatorname{block}\left(\nabla_{x^\nu} g_{\overline{\mathcal{M}}^\nu}(x)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\mathcal{M}^0} & M^e_{\mathcal{K}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{M}^\nu|}\right)_{\nu=1}^N & \operatorname{block}\left(1^\top_{|\overline{\mathcal{M}}^\nu|}\right)_{\nu=1}^N \\ \hline 0 & 0 & M^e_{\mathcal{K}_{u_0}} & 0 & 0 \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{K}_g}(x) & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \operatorname{block}\left(I_{|\overline{\mathcal{M}}^\nu|}\right)_{\nu=1}^N \end{array}\right) \quad (5.39)$$

where we set $\mathcal{K}_g := \bigcap_{\nu=1}^N \mathcal{K}_g^\nu$. The latter set is a subset of $\mathcal{I}_=^\cap$. Using Lemma 5.5, the rank of the matrix in (5.39) equals, for any $s = (x, u_0, u) \in \mathbb{R}^n$, the rank of the matrix

$$\left(\begin{array}{c|c|c} \Psi_x^{5.3}(x,u_0,u) & M^d_{\overline{\mathcal{I}}_{u_0} \cup (\mathcal{I}_{u_0} \setminus \mathcal{K}_{u_0})}(x) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)}(x)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0} \cup (\mathcal{I}_{u_0} \setminus \mathcal{K}_{u_0})} & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu \cup (\mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu)|}\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{K}_g}(x) & 0 & 0 \end{array}\right) \quad (5.40)$$

plus $|\mathcal{K}_{u_0}| + \sum_{\nu=1}^N |\mathcal{I}_u \cup \mathcal{K}_u^\nu|$. After all, we obtain that the matrices in (5.38) have the same rank for all $s = (x, u_0, u) \in \mathcal{B}_\varepsilon(z^*)$ if and only if the matrices in (5.40) have this property. The matrices in (5.40) actually have the same rank for all $s = (x, u_0, u) \in \mathcal{B}_\varepsilon(z^*)$ due to the assumption of the proposition with $\mathcal{J} := \mathcal{K}_g$, $\mathcal{J}_{u_0} := \mathcal{I}_{u_0} \setminus \mathcal{K}_{u_0}$, and $\mathcal{J}^\nu := \mathcal{I}_=^\nu \setminus \mathcal{K}_u^\nu$ ($\nu = 1, \ldots, N$). □

Corollary 5.15 below is a consequence of Proposition 5.14 and says that the full row rank of a certain matrix is sufficient for Condition 9d. The proof is very similar to the proof of Corollary 5.10.

Corollary 5.15. *Suppose that the matrix*

$$\left(\begin{array}{c|c|c} \Psi_x^{5.3}(x^*,u_0^*,u^*) & M^d_{\overline{\mathcal{I}}_{u_0}}(x^*) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu|}\right)_{\nu=1}^N \\ \hline g'_{\mathcal{G}_0}(x^*) & 0 & 0 \end{array}\right) \tag{5.41}$$

has full row rank. Then Condition 9d is satisfied. In particular, Assumptions 1–4 for (5.36) *hold if the function G satisfies* (4.1) *for all $z = (x, u_0, u) \in \mathbb{R}^n$.*

Proof. We are going to show that the assumption of Proposition 5.14 holds. To this end, let us take any $(N+2)$-tuple $(\mathcal{J}, \mathcal{J}_{u_0}, \mathcal{J}^1, \dots, \mathcal{J}^N)$ of index sets $\mathcal{J} \subseteq \mathcal{I}^\cap_=$, $\mathcal{J}_{u_0} \subseteq \mathcal{I}_{u_0}$, and $\mathcal{J}^\nu \subseteq \mathcal{I}^\nu_=$ $(\nu = 1, \dots, N)$. Since $\mathcal{G}_0$ equals $\mathcal{I}_g^\cup \cup \mathcal{I}^\cap_=$, we have $\mathcal{I}_g^\cup \cup \mathcal{J} \subseteq \mathcal{G}_0$. Therefore, by the assumption of the corollary, the matrix

$$\left(\begin{array}{c|c|c} \Psi_x^{5.3}(x^*,u_0^*,u^*) & M^d_{\overline{\mathcal{I}}_{u_0}}(x^*) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu|}\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{J}}(x^*) & 0 & 0 \end{array}\right)$$

has full row rank. The rows of the matrix

$$\left(\begin{array}{c|c|c} \Psi_x^{5.3}(x^*,u_0^*,u^*) & M^d_{\overline{\mathcal{I}}_{u_0} \cup \mathcal{J}_{u_0}}(x^*) & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup \mathcal{J}^\nu}(x^*)\right)_{\nu=1}^N \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0} \cup \mathcal{J}_{u_0}} & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu \cup \mathcal{J}^\nu|}\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup \cup \mathcal{J}}(x^*) & 0 & 0 \end{array}\right)$$

are still linearly independent since at most the number of columns has increased. Due to the continuity of all functions which are involved in this matrix, the rows stay linearly independent for all $s = (x, u_0, u)$ in a sufficiently small neighborhood of z^*. Since the index sets $\mathcal{J}, \mathcal{J}_{u_0}, \mathcal{J}^1, \dots, \mathcal{J}^N$ were arbitrarily chosen, the assumption of Proposition 5.14 is satisfied. Thus, the assertions follow from Proposition 5.14. □

The aim of the second part of this section is to prove that generically the full row rank condition from Corollary 5.15 is satisfied at any solution of the FJ system of a GNEP. From now on, we drop the assumption that the second-order derivatives of the functions $\theta_1, \dots, \theta_N$, and g are locally Lipschitz continuous. Instead, these functions are required to be twice continuously differentiable only.

First, let us briefly explain what is meant by "generically satisfied condition". We restrict ourselves to the explanation in [38, Section 4.1], details can be found in [12, 13] and references therein. Our GNEP is characterized by its problem functions $\theta_1, \ldots, \theta_N$, and g. Assume that the space $C^2(\mathbb{R}^n)$ of all real-valued, twice continuously differentiable functions is endowed with the *Whitney topology*. Furthermore, let the product space of GNEP defining functions

$$\mathcal{D} := \prod_{\nu=1}^{N} \underbrace{C^2(\mathbb{R}^n)}_{\theta_\nu} \times \prod_{j=1}^{m_g} \underbrace{C^2(\mathbb{R}^n)}_{g_j}$$

be endowed with the product Whitney topology. We refer to [12, 13] and references therein for the definition of the Whitney topology. We say that a condition (on a GNEP or the corresponding FJ system) is *generically satisfied* if there is, regarding the Whitney topology, an open and dense subset $\widehat{\mathcal{D}} \subseteq \mathcal{D}$ such that the condition holds for any GNEP which is characterized by an instance $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_{m_g}) \in \widehat{\mathcal{D}}$.

Let us consider the following nonsmooth system of equations, which is also considered in [13], there even for the more general case that individual constraints may occur:

$$T(x, \xi_0, \xi) := \begin{pmatrix} \Theta^{5.3}(x) \max\{0, \xi_0\} + B(x) \max\{0, \xi\} \\ g(x) + \left(\xi^1, \ldots, \xi^N\right)^* \\ \max\{0, \xi_0\} + E \max\{0, \xi\} - 1_N \end{pmatrix} = 0 \tag{5.42}$$

with

$$\xi_0 = \begin{pmatrix} \xi_0^1 \\ \vdots \\ \xi_0^N \end{pmatrix} \in \mathbb{R}^N, \quad \xi^\nu = \begin{pmatrix} \xi_1^\nu \\ \vdots \\ \xi_{m_g}^\nu \end{pmatrix} \in \mathbb{R}^{m_g} \ (\nu = 1, \ldots, N), \quad \xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^N \end{pmatrix} \in \mathbb{R}^{N m_g}.$$

The maximum has to be taken componentwise. Moreover, $(\xi^1, \ldots, \xi^N)^* \in \mathbb{R}^{m_g}$ is a vector whose j-th component ($j = 1, \ldots, m_g$) is defined according to

$$\begin{aligned} \left[\left(\xi^1, \ldots, \xi^N\right)^*\right]_j \quad &:= \quad \begin{cases} \prod_{\nu=1}^{N} |\xi_j^\nu| & \text{if} \quad \xi_j^\nu < 0 \ \forall \nu = 1, \ldots, N, \\ 0 & \text{else} \end{cases} \\ &= \quad \left|\min\{0, \xi_j^1\} \cdot \ldots \cdot \min\{0, \xi_j^N\}\right| \\ &= \quad (-1)^N \min\{0, \xi_j^1\} \cdot \ldots \cdot \min\{0, \xi_j^N\}. \end{aligned}$$

Using the Hadamard product notation, the function T in (5.42) can be written as follows:

$$T(x, \xi_0, \xi) = \begin{pmatrix} \Theta^{5.3}(x) \max\{0, \xi_0\} + B(x) \max\{0, \xi\} \\ g(x) + (-1)^N \cdot \min\{0, \xi^1\} \circ \ldots \circ \min\{0, \xi^N\} \\ \max\{0, \xi_0\} + E \max\{0, \xi\} - 1_N \end{pmatrix}.$$

The following lemma, which follows from [13, Lemma 2.1], states relations between the solution sets of (5.35) and (5.42).

Lemma 5.16. *If* (x^*, ξ_0^*, ξ^*) *is a solution of the nonlinear system* (5.42)*, then there are* $u_0^* \in \mathbb{R}^N$ *and* $u^* \in \mathbb{R}^{Nm_g}$ *such that* (x^*, u_0^*, u^*) *solves the FJ system* (5.35)*. Conversely, if* (x^*, u_0^*, u^*) *is a solution of* (5.35)*, then there are* $\xi_0^* \in \mathbb{R}^N$ *and* $\xi^* \in \mathbb{R}^{Nm_g}$ *such that* (x^*, ξ_0^*, ξ^*) *solves* (5.42)*.*

Proof. Let (x^*, ξ_0^*, ξ^*) be a solution of (5.42). Then it is not difficult to show that (x^*, u_0^*, u^*) with $u_0^* := \max\{0, \xi_0^*\}$ and $u^* := \max\{0, \xi^*\}$ solves the FJ system (5.35).

Conversely, assume that (x^*, u_0^*, u^*) is a solution of (5.35). Then the point (x^*, ξ_0^*, ξ^*) with $\xi_0^* := u_0^*$ and

$$\xi_j^{*,\nu} := \begin{cases} u_j^{*,\nu} & \text{if} \quad j \in \mathcal{G}_0, \\ -\,|g_j(x^*)|^{\frac{1}{N}} & \text{if} \quad j \in \mathcal{I}_u \end{cases} \qquad (\nu = 1, \ldots, N,\ j = 1, \ldots, m_g) \tag{5.43}$$

solves (5.42). The latter fact can be easily verified. □

Obviously, the function T from (5.42) is not everywhere differentiable. However, T is locally Lipschitz continuous so that, for any $(x, \xi_0, \xi) \in \mathbb{R}^{n_x} \times \mathbb{R}^N \times \mathbb{R}^{Nm_g}$, Clarke's generalized Jacobian $\partial T(x, \xi_0, \xi)$ at (x, ξ_0, ξ) is nonempty, see Proposition 2.5. For the definition of Clarke's generalized Jacobian we refer to Section 2.2. In the following, we denote again by $z^* = (x^*, u_0^*, u^*)$ an arbitrary but fixed solution of (5.35) and by (x^*, ξ_0^*, ξ^*) the corresponding solution of (5.42) defined as in the proof of Lemma 5.16, i.e., ξ_0^* equals u_0^*, and ξ^* is given by (5.43).

Proposition 5.17 below shows that the full row rank of all matrices belonging to $\partial T(x^*, \xi_0^*, \xi^*)$ is sufficient for the full row rank of the matrix from Corollary 5.15 to hold if there are at least two players. The latter implication stays true in the case $N = 1$ if in addition the strict complementarity condition is satisfied at z^*. Note that the latter condition is equivalent to $\mathcal{I}_=^\cap = \emptyset$ if there is only one player.

Proposition 5.17. *Suppose that all matrices belonging to* $T(x^*, \xi_0^*, \xi^*)$ *have full row rank. Moreover, assume that one of the following conditions is valid.*

(i) $N \geq 2$ *holds.*

(ii) $N = 1$ *holds and* $\mathcal{I}_=^\cap$ *is empty, i.e., strict complementarity is satisfied.*

Then the matrix in (5.41) *has full row rank.*

Proof. It is not difficult to see that the function T is not differentiable at (x^*, ξ_0^*, ξ^*) with respect to the variables ξ_0^ν ($\nu \in \mathcal{I}_{u_0}$) and ξ_j^ν ($\nu \in \{1, \ldots, N\}$, $j \in \mathcal{I}_=^\nu$) because $\xi_0^{*,\nu} = 0$ holds for all $\nu \in \mathcal{I}_{u_0}$, and $\xi_j^{*,\nu} = 0$ is valid for all $\nu = 1, \ldots, N$ and all

$j \in \mathcal{I}_{=}^{\nu}$. Let $\{(x^k, \xi_0^k, \xi^k)\}_{k\in\mathbb{N}}$ be a sequence converging to (x^*, ξ_0^*, ξ^*) such that the following relations hold for all $k \in \mathbb{N}$:

$$\xi_0^{k,\nu} > 0 \qquad (\nu \in \overline{\mathcal{I}}_{u_0}), \tag{5.44}$$

$$\xi_0^{k,\nu} < 0 \qquad (\nu \in \mathcal{I}_{u_0}), \tag{5.45}$$

$$\xi_j^{k,\nu} > 0 \qquad (\nu \in \{1,\dots,N\},\ j \in \mathcal{I}_g^{\nu}), \tag{5.46}$$

$$\xi_j^{k,\nu} < 0 \qquad (\nu \in \{1,\dots,N\},\ j \in \mathcal{I}_{=}^{\nu}), \tag{5.47}$$

$$\xi_j^{k,\nu} < 0 \qquad (\nu \in \{1,\dots,N\},\ j \in \mathcal{I}_u). \tag{5.48}$$

Taking into account the definitions of ξ_0^* and ξ^*, there is actually a sequence $\{(x^k, \xi_0^k, \xi^k)\}$ with these properties. Let us fix k for the moment. The relations (5.44)–(5.48) are still true for all points (x, ξ_0, ξ) in a sufficiently small neighborhood of (x^k, ξ_0^k, ξ^k). Using this, it is not difficult to see that for all points (x, ξ_0, ξ) being sufficiently close to (x^k, ξ_0^k, ξ^k) the corresponding function value $T(x, \xi_0, \xi)$ is given by

$$T(x,\xi_0,\xi) = \begin{pmatrix} \left(\xi_0^{\nu}\nabla_{x^\nu}\theta_\nu(x) + \sum_{j\in\mathcal{I}_g^{\nu}} \xi_j^{\nu}\nabla_{x^\nu} g_j(x)\right)_{\nu\in\overline{\mathcal{I}}_{u_0}} \\ \left(\sum_{j\in\mathcal{I}_g^{\nu}} \xi_j^{\nu}\nabla_{x^\nu} g_j(x)\right)_{\nu\in\mathcal{I}_{u_0}} \\ (g_j(x))_{j\in\mathcal{I}_g^{\cup}} \\ \left(g_j(x) + (-1)^N \xi_j^1 \cdot \ldots \cdot \xi_j^N\right)_{j\in\mathcal{I}_{=}^{\cap}\cup\mathcal{I}_u} \\ \left(\xi_0^{\nu} + \sum_{j\in\mathcal{I}_g^{\nu}} \xi_j^{\nu} - 1\right)_{\nu\in\overline{\mathcal{I}}_{u_0}} \\ \left(\sum_{j\in\mathcal{I}_g^{\nu}} \xi_j^{\nu} - 1\right)_{\nu\in\mathcal{I}_{u_0}} \end{pmatrix}.$$

In particular, it follows that T is differentiable at (x^k, ξ_0^k, ξ^k) and the Jacobian coincides, if necessary after some row and column permutations, with the matrix

$$T'(x^k, \xi_0^k, \xi^k) =$$

$$\left(\begin{array}{c|c|c|c|c|c} \Psi_x^{5.3}(x^k, \max\{0,\xi_0^k\}, \max\{0,\xi^k\}) & M^d_{\overline{\mathcal{I}}_{u_0}}(x^k) & 0 & 0 & 0 & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^{\nu}}(x^k)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^{\cup}}(x^k) & 0 & 0 & 0 & 0 & 0 \\ \hline g'_{\mathcal{I}_{=}^{\cap}}(x^k) & 0 & 0 & 0 & M_2(\xi^k) & 0 \\ \hline g'_{\mathcal{I}_u}(x^k) & 0 & 0 & M_1(\xi^k) & 0 & 0 \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0}} & 0 & 0 & 0 & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^{\nu}|}\right)_{\nu=1}^N \end{array}\right).$$

The six column blocks of the matrix $T'(x^k, \xi_0^k, \xi^k)$ contain the derivatives of T at the point (x^k, ξ_0^k, ξ^k) with respect to the variables in the following order:

$$x,\ [\xi_0]_{\overline{\mathcal{I}}_{u_0}},\ [\xi_0]_{\mathcal{I}_{u_0}},\ (\xi^{\nu}_{\mathcal{I}_u})_{\nu=1}^N,\ (\xi^{\nu}_{\mathcal{I}_{=}^{\nu}})_{\nu=1}^N,\ (\xi^{\nu}_{\mathcal{I}_g^{\nu}})_{\nu=1}^N.$$

The matrices $M_1(\xi^k)$ and $M_2(\xi^k)$ appear in $T'(x^k, \xi_0^k, \xi^k)$. For any vector $\xi \in \mathbb{R}^{Nm_g}$, the matrix $M_1(\xi)$ is given by

$$M_1(\xi) := \left(M_1^1(\xi), \dots, M_1^N(\xi)\right) \in \mathbb{R}^{|\mathcal{I}_u|\times(N\cdot|\mathcal{I}_u|)}$$

with

$$[M_1^\nu(\xi)]_{jl} := \begin{cases} (-1)^N \prod\limits_{\iota \neq \nu} \xi_j^\iota & \text{if} \quad j = l, \\ 0 & \text{if} \quad j \neq l \end{cases} \qquad (\nu \in \{1, \dots, N\},\, j, l \in \mathcal{I}_u),$$

and the matrix $M_2(\xi)$ is defined according to

$$M_2(\xi) := \left(M_2^1(\xi), \dots, M_2^N(\xi)\right) \in \mathbb{R}^{|\mathcal{I}_=^\cap| \times \sum_{\nu=1}^N |\mathcal{I}_=^\nu|}$$

with

$$[M_2^\nu(\xi)]_{jl} := \begin{cases} (-1)^N \prod\limits_{\iota \neq \nu} \xi_j^\iota & \text{if} \quad j = l, \\ 0 & \text{else} \end{cases} \qquad (\nu \in \{1, \dots, N\},\, j \in \mathcal{I}_=^\cap,\, l \in \mathcal{I}_=^\nu).$$

Note that, for every $\nu = 1, \dots, N$, the matrix $M_2^\nu(\xi)$ has at least as many columns as rows and may contain columns consisting of zeros only. For $k \to \infty$ the sequence (x^k, ξ_0^k, ξ^k) converges to (x^*, ξ_0^*, ξ^*). Therefore, the matrices $M_1(\xi^k)$ converge to $M_1(\xi^*)$, and the matrices $M_2(\xi^k)$ converge to $M_2(\xi^*)$.

Let us assume that condition (i) of the proposition is satisfied, i.e., there are at least two players. Then the matrix $M_2(\xi^*)$ consists of zeros only since $\xi_j^{*,\nu} = 0$ holds for all $\nu \in \{1, \dots, N\}$ and all $j \in \mathcal{I}_=^\cap$. Therefore, the Jacobians $T'(x^k, \xi_0^k, \xi^k)$ converge to a matrix J which is defined by

$$J := \left(\begin{array}{c|c|c|c|c|c} \Psi_x^{5.3}(x^*, \max\{0, \xi_0^*\}, \max\{0, \xi^*\}) & M^d_{\overline{\mathcal{I}}_{u_0}}(x^*) & 0 & 0 & 0 & \operatorname{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{I}_g^\cup}(x^*) & 0 & 0 & 0 & 0 & 0 \\ \hline g'_{\mathcal{I}_=^\cap}(x^*) & 0 & 0 & 0 & 0 & 0 \\ \hline g'_{\mathcal{I}_u}(x^*) & 0 & 0 & M_1(\xi^*) & 0 & 0 \\ \hline 0 & M^e_{\overline{\mathcal{I}}_{u_0}} & 0 & 0 & 0 & \operatorname{block}\left(1^\top_{|\mathcal{I}_g^\nu|}\right)_{\nu=1}^N \end{array}\right).$$

By the definition of Clarke's generalized Jacobian, this matrix J is an element of $\partial T(x^*, \xi_0^*, \xi^*)$. The assumption of the proposition yields that J has full row rank. Let us denote by J° the matrix which arises from J by deleting the row block where $g'_{\mathcal{I}_u}(x^*)$ and $M_1(\xi^*)$ are included and those columns of the resulting matrix consisting of zeros only. The matrix J° has still full row rank. Taking into account $u_0^* = \xi_0^* = \max\{0, \xi_0^*\}$, $u^* = \max\{0, \xi^*\}$, and $\mathcal{G}_0 = \mathcal{I}_g^\cup \cup \mathcal{I}_=^\cap$, J° actually coincides with the matrix in (5.41) so that the assertion is proved.

Now suppose that condition (ii) of the proposition is satisfied, i.e., there is only one player and strict complementarity holds. Then the relations $\mathcal{I}_=^\cap = \mathcal{I}_=^1 = \emptyset$ and $\mathcal{I}_g^\cup = \mathcal{I}_g^1 = \mathcal{G}_0$ are valid. Moreover, the variables ξ_0 are real numbers only, and exactly one of the sets $\mathcal{I}_{u_0}$ and $\overline{\mathcal{I}}_{u_0}$ is empty while the other one is a singleton.

The Jacobians $T'(x^k, \xi_0^k, \xi^k)$ converge to a matrix $\tilde{J}$ which is defined by

$$\tilde{J} := \left(\begin{array}{c|c|c|c} \Psi_x^{5.3}(x^*, \max\{0, \xi_0^*\}, \max\{0, \xi^*\}) & \tilde{M}^d_{\overline{\mathcal{I}}_{u_0}}(x^*) & 0 & \nabla g_{\mathcal{G}_0}(x^*) \\ \hline g'_{\mathcal{G}_0}(x^*) & 0 & 0 & 0 \\ \hline g'_{\mathcal{I}_u}(x^*) & 0 & M_1(\xi^*) & 0 \\ \hline 0 & \tilde{M}^e_{\overline{\mathcal{I}}_{u_0}} & 0 & 1^\top_{|\mathcal{G}_0^\nu|} \end{array}\right)$$

where we set

$$\tilde{M}^d_{\overline{\mathcal{I}}_{u_0}}(x^*) := \begin{cases} \nabla\theta_1(x^*) & \text{if } \xi_0^* > 0, \\ 0 & \text{if } \xi_0^* = 0 \end{cases} \in \mathbb{R}^{n_x}$$

and

$$\tilde{M}^e_{\overline{\mathcal{I}}_{u_0}} := \begin{cases} 1 & \text{if } \xi_0^* > 0, \\ 0 & \text{if } \xi_0^* = 0 \end{cases} \in \mathbb{R}.$$

Note that $\tilde{M}^d_{\overline{\mathcal{I}}_{u_0}}(x^*)$ and $\tilde{M}^e_{\overline{\mathcal{I}}_{u_0}}$, respectively, coincide with $M^d_{\overline{\mathcal{I}}_{u_0}}(x^*)$ and $M^e_{\overline{\mathcal{I}}_{u_0}}$, respectively, if $\overline{\mathcal{I}}_{u_0}$ is a singleton and $\mathcal{I}_{u_0}$ is empty.

The matrix $\tilde{J}$ belongs to $\partial T(x^*, \xi_0^*, \xi^*)$. Consequently, $\tilde{J}$ has full row rank due to the assumption of the proposition. It follows that ξ_0^* is positive since otherwise the second column block of $\tilde{J}$ would consist of zeros only. The latter would imply that $\tilde{J}$ has at most $n_x + m_g$ columns with nonzero components. That would contradict the full row rank because $\tilde{J}$ has $n_x + m_g + 1$ rows. Therefore, taking into account $u_0^* = \xi_0^* = \max\{0, \xi_0^*\}$ and $u^* = \max\{0, \xi^*\}$, the matrix in (5.41) arises from $\tilde{J}$ by deleting the row block where $g'_{\mathcal{I}_u}(x^*)$ and $M_1(\xi^*)$ are included and those columns of the resulting matrix consisting of zeros only. Thus, the matrix in (5.41) has still full row rank. □

The condition that, for any solution $(\bar{x}, \bar{\xi}_0, \bar{\xi})$ of (5.42), all elements of $\partial T(\bar{x}, \bar{\xi}_0, \bar{\xi})$ have full row rank, is generically satisfied. Moreover, if there is only one player, the condition that strict complementarity is valid at all solutions $(\bar{x}, \bar{\xi}_0, \bar{\xi})$ of (5.42) is generically satisfied. The latter assertions follow from [13, Theorem 2.2 and Remark 2]. Thus, Proposition 5.17 yields the following corollary.

Corollary 5.18. *There is an open and dense subset $\widehat{\mathcal{D}} \subseteq \mathcal{D}$ such that, for any tuple $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_{m_g}) \in \widehat{\mathcal{D}}$ of GNEP defining functions and any solution $z^* = (x^*, u_0^*, u^*)$ of the FJ system* (5.35) *of the corresponding GNEP, the matrix in* (5.41) *has full row rank.*

By Corollary 5.15 we know that the full row rank of the matrix in (5.41) implies the whole set of Assumptions 1–4 for (5.36) supposed that the second-order derivatives of the problem functions $\theta_1, \ldots, \theta_N, g_1, \ldots, g_{m_g}$ are locally Lipschitz continuous. Unfortunately, there might be tuples $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_{m_g})$ belonging to the set $\widehat{\mathcal{D}}$ from Corollary 5.18 such that the second-order derivative of at least one of the functions is not locally Lipschitz continuous. Therefore, local quadratic convergence of our Newton-type algorithms from Chapter 3 applied to (5.36) cannot be expected for all instances of $\widehat{\mathcal{D}}$.

However, it is known that the space $C^3(\mathbb{R}^n)$ of all real-valued, three times continuously differentiable functions is dense in $C^2(\mathbb{R}^n)$ regarding the Whitney topology, see for example [44, Theorem 2.6]. Consequently, the space of all real-valued, twice differentiable functions with locally Lipschitz continuous second-order derivatives is, regarding the Whitney topology, dense in $C^2(\mathbb{R}^n)$ as well. It follows that for any tuple $(\theta_1, \dots, \theta_N, g_1, \dots, g_{m_g}) \in \mathcal{D}$ and any open neighborhood $\mathcal{O} \subseteq \mathcal{D}$ of this tuple there is a tuple $(\tilde{\theta}_1, \dots, \tilde{\theta}_N, \tilde{g}_1, \dots, \tilde{g}_{m_g}) \in \mathcal{O}$ such that the second-order derivatives of the functions $\tilde{\theta}_1, \dots, \tilde{\theta}_N, \tilde{g}_1, \dots, \tilde{g}_{m_g}$ are locally Lipschitz continuous. Moreover, we know that for any tuple $(\theta_1, \dots, \theta_N, g_1, \dots, g_{m_g}) \in \widehat{\mathcal{D}}$ there is an open neighborhood of this tuple which is included in $\widehat{\mathcal{D}}$ since the latter set is open.

In [13] a Newton-type method for the solution of (5.42) is proposed where in each step the pseudoinverse of an element of Clarke's generalized Jacobian is used. This method is called nonsmooth projection method in [13]. It follows from [13, Theorem 3.1] that, for each tuple $(\theta_1, \dots, \theta_N, g_1, \dots, g_{m_g}) \in \widehat{\mathcal{D}}$, a step of the method is locally, near any solution, well defined. Moreover, it is claimed in [13] that the nonsmooth projection method converges locally quadratically to a solution of (5.42) if strict complementarity is satisfied at some fixed solution, see [13, Theorem 3.1] and the discussion in [13, Section 3.1]. For the proof of the local quadratic convergence the reader is referred to [3]. However, in the latter reference only linear convergence of the method is proved. Therefore, it is at least not clear if the claim regarding local quadratic convergence in [13, Theorem 3.1] is true.

Regardless of whether that claim is true or not, our Newton-type methods from Chapter 3 are locally quadratically convergent for the instances $(\theta_1, \dots, \theta_N, g_1, \dots, g_{m_g})$ belonging to $\widehat{\mathcal{D}}$ even if strict complementarity is violated. The only condition which remains to guarantee is the local Lipschitz continuity of the second-order derivatives of the functions $\theta_1, \dots, \theta_N, g_1, \dots, g_{m_g}$.

Chapter 6

A Hybrid Method for KKT Systems of GNEPs

The previous chapters were devoted to an in-depth analysis of local convergence properties of a general Newton-type algorithm for the solution of constrained systems of equations. We proved in Chapter 3 that Algorithm 3.1 converges locally with a Q-quadratic rate to a solution of (1.1) if Assumptions 1–4 are satisfied at some fixed solution z^*. Moreover, it was shown that the LP-Newton method as well as the constrained Levenberg-Marquardt method are special realizations of Algorithm 3.1 and therefore enjoy the same local convergence properties. Chapter 4 provided a detailed discussion of our convergence assumptions for the case that F is a PC^1-function. The results of Chapters 3 and 4 were applied in Chapter 5 where we discussed the application of the general Newton-type algorithm and its special realizations for the solution of special problem classes such as KKT systems arising from optimization problems, variational inequalities, or GNEPs. Conditions were provided which are sufficient for all of Assumptions 1–4 to hold and therefore guarantee local quadratic convergence of our algorithms.

Of course, the question concerning a reasonable globalization of Algorithm 3.1, in particular of the LP-Newton method and the constrained Levenberg-Marquardt method, may arise. In this chapter we want to describe at least one possible globalization for the case that the constrained system arises from a certain smooth reformulation of the KKT system of a GNEP. More precisely, a hybrid method is presented where the local part is the LP-Newton method from Section 3.4 and the global part is a potential reduction algorithm.

In Section 6.1 we present a smooth reformulation of a KKT system arising from a GNEP. Afterwards, the potential reduction algorithm for the solution of the resulting constrained system is described where we consider a slight modification of [16, Algorithm 4.1]. Finally, the hybrid method is presented. The main result of Section 6.1 is Theorem 6.4 which provides conditions under which the hybrid method converges globally and locally fast to a solution of the KKT system. Section 6.1 is based on results which were published by Axel Dreves, Francisco Facchinei, Andreas Fischer, and the author in [15]. Numerical results can be found in the latter reference, too.

Concerning the local part it turns out that Assumption 2 for the smooth constrained system is sufficient for implying local quadratic convergence of the hybrid method. In Section 6.2 we show that Assumption 2 for the smooth system is satisfied if and only if Assumption 2 for the constrained system (5.29) from Remark 5.2 holds, supposed that in addition the following weakened form of strict complementarity is valid at the considered solution: for each constraint which is active the corresponding multiplier of at least one player is strictly positive, see Proposition 6.5. The latter result will be published together with Andreas Fischer, Alexey Izmailov, and Mikhail Solodov in the technical report [37].

6.1 Description and a Convergence Result

In this section a smooth reformulation of the KKT system (5.19) arising from a GNEP is considered. We will briefly discuss our convergence assumptions from Section 3.1 for the resulting constrained system. It will turn out that Assumption 2 already implies the whole set of Assumptions 1–4. Afterwards, we describe a potential reduction algorithm which is a slight modification of [16, Algorithm 4.1]. Finally, a hybrid method is presented which combines the LP-Newton method and the potential reduction algorithm. Theorem 6.4 provides conditions implying global and local fast convergence of the hybrid method.

As in Section 5.2, a GNEP with N players $\nu = 1, \ldots, N$ is considered and it is assumed that the players have shared constraints only which are described by inequalities. So, if the variables $x^{-\nu}$ of the rival players are fixed, player ν aims to solve the optimization problem

$$\theta_\nu(x^\nu, x^{-\nu}) \to \min_{x^\nu} \quad \text{s.t.} \quad g(x) \le 0.$$

The functions $\theta_1, \ldots, \theta_N : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_g}$ are assumed to be twice differentiable with locally Lipschitz continuous second-order derivatives. As in Sections 5.2 and 5.3, n_x denotes the number of the variables of all players.

The corresponding KKT system of the GNEP is given by (5.19). Unlike Section 5.2, we consider the following reformulation of (5.19) as a smooth constrained system of equations in this chapter:

$$F(z) := F(x, u, w) := \begin{pmatrix} \Psi^{5.2}(x, u) \\ g(x) + w \\ u^1 \circ w \\ \vdots \\ u^N \circ w \end{pmatrix} \quad \text{s.t.} \quad z \in \Omega \tag{6.1}$$

where the function $\Psi^{5.2}$ is defined according to (5.20), Ω is given by

$$\Omega := \{z = (x, u, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \times \mathbb{R}^{m_g} \mid u \ge 0,\ w \ge 0\}, \tag{6.2}$$

and, for every $\nu = 1, \ldots, N$, $u^\nu \circ w$ denotes the Hadamard product of the vectors u^ν and w. It is not difficult to see that a point (x^*, u^*) solves the KKT system (5.19) if and only if (x^*, u^*, w^*), with $w^* := -g(x^*)$, is a solution of the constrained system (6.1). In this chapter we denote by Z the solution set of (6.1) and assume that Z is nonempty. By $z^* = (x^*, u^*, w^*) \in Z$ an arbitrary but fixed solution is indicated. The number of all variables is denoted by n, i.e., $n := n_x + (N+1)m_g$.

The aim of this section is to describe and analyze a hybrid method for the solution of (6.1). First, we analyze local convergence properties of the LP-Newton method if it is applied to (6.1). The latter will be the local part of the hybrid method. So let us discuss Assumptions 1–4 for (6.1). Due to our differentiability assumptions on $\theta_1, \ldots, \theta_N$, and g, the function F is differentiable and its Jacobian is locally Lipschitz continuous. Therefore, by Corollary 3.6, we obtain the following proposition.

Proposition 6.1. *Assume that Assumption 2 for* (6.1) *is satisfied and* $G(z) = F'(z)$ *is valid for all* $z = (x, u, w) \in \mathbb{R}^n$. *Then all of Assumptions 1–4 hold.*

It follows from Proposition 6.1, together with Theorem 3.20, that the LP-Newton method converges locally with a Q-quadratic rate to a solution of (6.1) if Assumption 2 holds. Since we use the LP-Newton method later on in this chapter, we want to state this local convergence result explicitly.

Corollary 6.2. *Assume that Assumption 2 for* (6.1) *is satisfied and* $G(z) = F'(z)$ *is valid for all* $z = (x, u, w) \in \mathbb{R}^n$. *Then there is* $\rho > 0$ *such that any infinite sequence* $\{z^k\} = \{(x^k, u^k, w^k)\}$ *generated by Algorithm 3.3 with starting point* $z^0 = (x^0, u^0, w^0) \in \mathcal{B}_\rho(z^*) \cap \Omega$ *converges locally with a Q-quadratic rate to a solution of* (6.1).

The global part of the hybrid method will be a potential reduction algorithm. This is an interior point method which was proposed in [70] for the solution of general constrained systems and is based on the minimization of a potential function. In [16] it is used for the solution of KKT systems arising from GNEPs. It is not supposed in [16] that the players have shared constraints only. Moreover, for each constraint of each player an own slack variable is introduced, regardless of whether the constraint is shared by all players or not. Consequently, in our setting where the players have shared constraints only, the following reformulation of (5.19) is considered in [16]:

$$\tilde{F}(x, u, \tilde{w}) := \begin{pmatrix} \Psi^{5.2}(x, u) \\ g(x) + \tilde{w}^1 \\ \vdots \\ g(x) + \tilde{w}^N \\ u^1 \circ \tilde{w}^1 \\ \vdots \\ u^N \circ \tilde{w}^N \end{pmatrix} = 0 \quad \text{s.t.} \quad (x, u, \tilde{w}) \in \tilde{\Omega} \tag{6.3}$$

with

$$\tilde{w} := \begin{pmatrix} \tilde{w}^1 \\ \vdots \\ \tilde{w}^N \end{pmatrix} \in \mathbb{R}^{Nm_g}$$

and

$$\tilde{\Omega} := \{(x, u, \tilde{w}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g} \times \mathbb{R}^{Nm_g} \mid u \geq 0,\ \tilde{w} \geq 0\}.$$

We will not consider the constrained system (6.3) in the sequel. Instead, we describe a slight modification of [16, Algorithm 4.1] which is adapted to the constrained system (6.1) with Ω given by (6.2).

Let us define the set $\Omega_I \subseteq \Omega$ consisting of all points which belong to the interior of Ω and for which the last $(N+1)m_g$ components of the function F are positive, i.e.,

$$\Omega_I := \{z = (x, u, w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g}_{++} \times \mathbb{R}^{m_g}_{++} \mid g(x) + w > 0\}.$$

Moreover, let the vector $a \in \mathbb{R}^{n_x+(N+1)m_g}$ be given by

$$a := (\underbrace{0, \dots, 0}_{n_x}, \underbrace{1, \dots, 1}_{(N+1)m_g})^\top$$

throughout the rest of this section. Furthermore, we introduce, for a given real number $\zeta > \frac{1}{2}(N+1)m_g$, the function $\Phi : \mathbb{R}^{n_x} \times \mathbb{R}^{(N+1)m_g}_{++} \to \mathbb{R}$ according to

$$\Phi(\check{v}, \hat{v}) := \zeta \ln\left(\|\check{v}\|^2 + \|\hat{v}\|^2\right) - \sum_{j=1}^{(N+1)m_g} \ln(\hat{v}_j).$$

The potential function $\phi : \Omega_I \to \mathbb{R}$ is defined by

$$\phi(z) := \Phi(F(z)).$$

Now we are in the position to describe a slight modification of the potential reduction algorithm from [16] which is adapted to our constrained system (6.1). Particularly, Algorithm 6.1 below differs from [16, Algorithm 4.1] in the choice of ζ and the definition of the vector a. Besides, the linear system in step (S.2) of the following algorithm is allowed to be solved inexactly in [16, Algorithm 4.1].

Algorithm 6.1. (Potential Reduction Algorithm for (6.1))

(S.0) Choose $z^0 = (x^0, u^0, w^0) \in \Omega_I$ and parameters $\beta, \eta \in (0, 1)$, $\zeta > \frac{1}{2}(N+1)m_g$. Set $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.

(S.2) Choose $\sigma_k \in [0, 1)$ and compute a solution d^k of the linear system

$$F'(z^k)d = -F(z^k) + \sigma_k \frac{a^\top F(z^k)}{\|a\|^2} a.$$

(S.3) Compute a step size $t_k := \max\{\beta^\ell \mid \ell = 0, 1, 2, \ldots\}$ such that

$$z^k + t_k d^k \in \Omega_I$$

and

$$\phi(z^k + t_k d^k) \leq \phi(z^k) + \eta t_k \nabla\phi(z^k)^\top d^k.$$

(S.4) Set $z^{k+1} := z^k + t_k d^k$, $k := k + 1$, and go to (S.1).

Let us state a convergence result for Algorithm 6.1. The proof is omitted here. It can be obtained by slightly modifying the proof of the convergence result for [16, Algorithm 4.1], see [16, Remark 4.2 and Theorem 4.3].

Proposition 6.3. *Assume that the Jacobians $F'(z)$ are nonsingular for all $z \in \Omega_I$ and that the sequence $\{\sigma_k\}$ satisfies the condition $\limsup_{k\to\infty} \sigma_k < 1$. Then Algorithm 6.1 is well defined. Moreover, the following assertions hold for any infinite sequence $\{z^k\} = \{(x^k, u^k, w^k)\}$ generated by Algorithm 6.1.*

(a) The sequence $\{F(z^k)\}$ is bounded.

(b) Any accumulation point of $\{z^k\}$ is a solution of (6.1).

Next, we describe a hybrid method for the solution of (6.1) which enjoys, under suitable assumptions, both global convergence and local quadratic convergence. Algorithm 6.2 below combines the potential reduction algorithm (Algorithm 6.1) with the LP-Newton method (Algorithm 3.3). Our hybrid method is essentially the same as [15, Algorithm 3]. The only difference is that [15, Algorithm 3] is a method for the constrained system (6.3) where for each constraint of each player an own slack variable is introduced.

Algorithm 6.2. (Hybrid Method for (6.1))

(S.0) Choose $z^0 = (x^0, u^0, w^0) \in \Omega_I$ and parameters $\beta, \eta, \vartheta \in (0, 1)$, $\zeta > \frac{1}{2}(N + 1)m_g$, $0 < \tau_{\min} \leq \tau_{\max}, \tau_0 \in [\tau_{\min}, \tau_{\max}]$. Set $\hat{z} := z^0$ and $k := 0$.

(S.1) If $F(z^k) = 0$: STOP.
If $\|F(z^k)\| \leq \tau_k$, then set $\sigma_k := 0$ and go to (S.4), else go to (S.2).

(S.2) Choose $\sigma_k \in [0, 1)$ and compute a solution d^k of the linear system

$$F'(z^k)d = -F(z^k) + \sigma_k \frac{a^\top F(z^k)}{\|a\|^2} a.$$

(S.3) Compute a step size $t_k := \max\{\beta^\ell \mid \ell = 0, 1, 2, \dots\}$ such that

$$z^k + t_k d^k \in \Omega_I$$

and

$$\phi(z^k + t_k d^k) \leq \phi(z^k) + \eta t_k \nabla\phi(z^k)^\top d^k.$$

Set $z^{k+1} := z^k + t_k d^k$, $\hat{z} := z^{k+1}$, $\tau_{k+1} := \tau_k$, $k := k + 1$, and go to (S.1).

(S.4) Compute a solution $(\tilde{z}^{k+1}, \gamma_{k+1})$ of the optimization problem

$$\gamma \to \min_{z,\gamma} \quad \text{s.t.} \quad \begin{aligned} & z \in \Omega, \\ & \|F(z^k) + F'(z^k)(z - z^k)\|_\infty \leq \gamma \|F(z^k)\|_\infty^2, \\ & \|z - z^k\|_\infty \leq \gamma \|F(z^k)\|_\infty, \\ & \gamma \geq 0. \end{aligned}$$

If $\|F(\tilde{z}^{k+1})\| \leq \vartheta \|F(z^k)\|$ holds, then set $z^{k+1} := \tilde{z}^{k+1}$, $\tau_{k+1} := \tau_k$, $k := k+1$, and go to (S.1).
Else, set $z^{k+1} := \hat{z}$, $k := k + 1$, choose $\tau_{k+1} \in [\tau_{\min}, \tau_{\max}]$, and go to (S.2).

Algorithm 6.2 coincides with the potential reduction algorithm until the norm of $F(z^k)$ is sufficiently small in a certain sense. Then a step of the LP-Newton method is performed. If the solution of the LP-Newton subproblem leads to a sufficient decrease of the norm of F, then its z-part is taken as the new iterate. Otherwise, the algorithm switches back to the potential reduction algorithm with $\hat{z}$ as starting point which denotes the last iterate before the switch to the LP-Newton method. Note that the z-part of the solution of the LP-Newton subproblem cannot be used in general as starting point for the next step of the potential reduction algorithm because it might not belong to Ω_I.

The subsequent theorem is the main result of this section and states assertions on global and local convergence properties of Algorithm 6.2. The assertions of Theorem 6.4 below essentially coincide with the assertions of [15, Theorem 4] but, of course, we consider the constrained system (6.1) whereas in [15] the system (6.3) is considered. However, the proof of [15, Theorem 4] can be transferred to the following theorem. Therefore, we omit the proof of Theorem 6.4 here.

Theorem 6.4. *Assume that the Jacobians $F'(z)$ are nonsingular for all $z \in \Omega_I$ and that the sequence $\{\sigma_k\}$ satisfies the condition $\limsup_{k\to\infty} \sigma_k < 1$. Then Algorithm 6.2 is well defined. Moreover, the following assertions hold for any infinite sequence $\{z^k\} = \{(x^k, u^k, w^k)\}$ generated by Algorithm 6.2.*

(a) The sequence $\{F(z^k)\}$ is bounded.

(b) Any accumulation point of $\{z^k\}$ is a solution of (6.1).

(c) Suppose that $\{z^k\}$ has an accumulation point z^* and that Assumption 2 for (6.1) is satisfied at z^*. Then the whole sequence $\{z^k\}$ converges to z^* with a Q-quadratic rate.

Remark 6.1. (a) The assertions of Theorem 6.4 stay true if the constrained Levenberg-Marquardt method or any other realization of Algorithm 3.1 is used instead of the LP-Newton method as local part in Algorithm 6.2.

(b) The critical assumption regarding global convergence in Proposition 6.3 and Theorem 6.4 is the nonsingularity of $F'(z)$ for all $z \in \Omega_I$. Note that this assumption does not require the nonsingularity of the Jacobian at any solution of (6.1) since $\Omega_I \cap Z = \emptyset$ holds. In particular, this nonsingularity assumption does not imply local uniqueness of solutions of (6.1).

(c) In [16, Theorems 4.6–4.8] sufficient conditions for the nonsingularity of $\tilde{F}'(x,u,\tilde{w})$ for some $(x,u,\tilde{w}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g}_{++} \times \mathbb{R}^{Nm_g}_{++}$ are provided. From these results, we can also obtain conditions implying the nonsingularity of $F'(z)$ for some $z = (x,u,w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g}_{++} \times \mathbb{R}^{m_g}_{++}$. In fact, let $(x,u,w) \in \mathbb{R}^{n_x} \times \mathbb{R}^{Nm_g}_{++} \times \mathbb{R}^{m_g}_{++}$ be arbitrary but fixed and let us set

$$\tilde{w} := \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} \in \mathbb{R}^{Nm_g}_{++}.$$

We are going to show that the nonsingularity of $\tilde{F}'(x,u,\tilde{w})$ implies the nonsingularity of $F'(x,u,w)$. So suppose that

$$\tilde{F}'(x,u,\tilde{w}) = \left(\begin{array}{c|c|ccc} \Psi_x^{5.2}(x,u) & \text{block}\,(\nabla_{x^\nu} g(x))_{\nu=1}^N & & 0 & \\ \hline g'(x) & & I_{m_g} & & \\ \vdots & 0 & & \ddots & \\ g'(x) & & & & I_{m_g} \\ \hline & & \text{diag}(u_j^1)_{j=1}^{m_g} & & \\ 0 & \text{block}\left(\text{diag}(w_j)_{j=1}^{m_g}\right)_{\nu=1}^N & & \ddots & \\ & & & & \text{diag}(u_j^N)_{j=1}^{m_g} \end{array}\right)$$

is nonsingular where $\Psi_x^{5.2}$ is defined according to (5.21). The matrix stays nonsingular if we add, for each $j = 1,\dots,m_g$ and each $\nu = 2,\dots,N$, the $(n_x + Nm_g + (\nu-1)m_g + j)$-th column to the $(n_x + Nm_g + j)$-th column, i.e., the resulting matrix

$$\tilde{M}(x,u,\tilde{w}) :=$$

$$\left(\begin{array}{c|c|cccc}
\Psi_x^{5.2}(x,u) & \operatorname{block}\left(\nabla_{x^\nu} g(x)\right)_{\nu=1}^N & & 0 & & \\
\hline
g'(x) & & I_{m_g} & & & \\
g'(x) & & I_{m_g} & I_{m_g} & & \\
\vdots & 0 & \vdots & & \ddots & \\
g'(x) & & I_{m_g} & & & I_{m_g} \\
\hline
 & & \operatorname{diag}(u_j^1)_{j=1}^{m_g} & & & \\
 & & \operatorname{diag}(u_j^2)_{j=1}^{m_g} & \operatorname{diag}(u_j^2)_{j=1}^{m_g} & & \\
0 & \operatorname{block}\left(\operatorname{diag}(w_j)_{j=1}^{m_g}\right)_{\nu=1}^N & \vdots & & \ddots & \\
 & & \operatorname{diag}(u_j^N)_{j=1}^{m_g} & & & \operatorname{diag}(u_j^N)_{j=1}^{m_g}
\end{array}\right)$$

is still nonsingular. Now it is not difficult to see that

$$F'(x,u,w) =$$

$$\left(\begin{array}{c|c|c}
\Psi_x^{5.2}(x,u) & \operatorname{block}\left(\nabla_{x^\nu} g(x)\right)_{\nu=1}^N & 0 \\
\hline
g'(x) & 0 & I_{m_g} \\
\hline
 & & \operatorname{diag}(u_j^1)_{j=1}^{m_g} \\
0 & \operatorname{block}\left(\operatorname{diag}(w_j)_{j=1}^{m_g}\right)_{\nu=1}^N & \vdots \\
 & & \operatorname{diag}(u_j^N)_{j=1}^{m_g}
\end{array}\right)$$

is nonsingular, too. In fact, if $F'(x,u,w)v = 0$ holds for some vector

$$v = \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} \in \mathbb{R}^{n_x + N m_g + m_g},$$

then $\tilde{M}(x,u,w)\bar{v} = 0$ follows, with

$$\bar{v} := \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ 0 \end{pmatrix} \in \mathbb{R}^{n_x + N m_g + m_g + (N-1) m_g}.$$

Due to the nonsingularity of $\tilde{M}(x,u,w)$, the latter implies $v = 0$.

6.2 Discussion of Local Convergence

In this section we are interested in conditions which imply Assumption 2 for the constrained system (6.1). The latter turned out to be the only one which remains to guarantee in order to obtain local quadratic convergence of the LP-Newton method, see Corollary 6.2, and of the hybrid method, see Theorem 6.4. We show in Proposition 6.5 that, supposed that a certain relaxation of the strict complementarity condition holds, Assumption 2 for (6.1) is satisfied if and only if Assumption 2 for (5.29) is valid. The latter reformulation was considered in Remark 5.2.

Again, we denote by $z^* = (x^*, u^*, w^*)$ an arbitrary but fixed solution of (6.1). Moreover, we will use the index sets $\mathcal{I}_g^\nu$, $\mathcal{I}_=^\nu$, $\mathcal{G}_0$, $\mathcal{I}_u$, $\mathcal{I}_g^\cup$, and $\mathcal{I}_=^\cap$ (the former two ones for $\nu = 1, \ldots, N$) which are defined according to (5.23)–(5.26).

In the following, we consider, besides (6.1), the constrained system

$$F_{\min}(z) := F_{\min}(x, u, w) := \begin{pmatrix} \Psi^{5.2}(x, u) \\ \min\{-g(x), u^1\} \\ \vdots \\ \min\{-g(x), u^N\} \\ g(x) + w \end{pmatrix} \quad \text{s.t.} \quad z \in \Omega \tag{6.4}$$

with Ω defined according to (6.2). Note that (6.4) is precisely the reformulation of the KKT system (5.19) which was considered in item (b) of Remark 5.2. Obviously, the solution sets of (6.1) and (6.4) coincide.

In Proposition 6.5 below we show that Assumption 2 for (6.1) is both necessary and sufficient for Assumption 2 for (6.4) to hold if in addition the set $\mathcal{I}_=^\cap$ is empty. The latter condition means that for each constraint which is active at z^* the corresponding multiplier of at least one player is strictly positive. Note that this condition is weaker than strict complementarity.

Proposition 6.5. *Suppose that $\mathcal{I}_=^\cap$ is empty. Then the following assertions are equivalent.*

(a) There are $\omega > 0$ and $\delta_2 > 0$ such that

$$\operatorname{dist}[s, Z] \le \omega \|F(s)\| \tag{6.5}$$

is satisfied for all $s = (x, u, w) \in \mathcal{B}_{\delta_2}(z^*) \cap \Omega$.

(b) There are $\omega_{\min} > 0$ and $\delta_{2,\min} > 0$ such that

$$\operatorname{dist}[s, Z] \le \omega_{\min} \|F_{\min}(s)\| \tag{6.6}$$

holds for all $s = (x, u, w) \in \mathcal{B}_{\delta_{2,\min}}(z^*) \cap \Omega$.

Proof. Since the left-hand sides of (6.5) and (6.6) coincide, and since all norms in a finite dimensional space are equivalent, it suffices to show that there are constants $\varepsilon > 0$, $c_1 > 0$, and $c_2 > 0$ such that

$$c_1 \|F(s)\|_\infty \le \|F_{\min}(s)\|_\infty \le c_2 \|F(s)\|_\infty \tag{6.7}$$

holds for all $s = (x, u, w) \in \mathcal{B}_\varepsilon(z^*) \cap \Omega$. Let us define numbers $\sigma > 0$ and $\tau > 0$ according to

$$\begin{aligned}
\sigma &:= \frac{1}{2}\min\{\, \{w_j^* \mid j \in \mathcal{I}_u\} \cup \{u_j^{*,\nu} \mid j \in \mathcal{I}_g^\nu,\, \nu \in \{1, \ldots, N\}\}\,\},\\
\tau &:= 2\max\{\, \{w_j^* \mid j \in \mathcal{I}_u\} \cup \{u_j^{*,\nu} \mid j \in \mathcal{I}_g^\nu,\, \nu \in \{1, \ldots, N\}\}\,\}.
\end{aligned}$$

Note that these numbers are well defined since there is at least one $\nu \in \{1, \ldots, N\}$ such that $\mathcal{I}_g^\nu$ is nonempty because of $\mathcal{I}_=^\cap = \emptyset$. Moreover, both σ and τ are positive due to the definition of the sets $\mathcal{I}_u$ and $\mathcal{I}_g^\nu$. Let $\varepsilon > 0$ be small enough such that the following relations are satisfied for all $s = (x, u, w) \in \mathcal{B}_\varepsilon(z^*) \cap \Omega$:

$$\forall j \in \mathcal{I}_u : \quad \tau \geq w_j \geq \sigma > 0, \tag{6.8}$$
$$\forall j \in \mathcal{G}_0 : \quad 0 \leq w_j \leq \sigma, \tag{6.9}$$
$$\forall \nu \in \{1, \ldots, N\}\, \forall j \in \mathcal{I}_g^\nu : \quad \tau \geq u_j^\nu \geq \sigma > 0, \tag{6.10}$$
$$\forall \nu \in \{1, \ldots, N\}\, \forall j \in \mathcal{I}_=^\nu \cup \mathcal{I}_u : \quad 0 \leq u_j^\nu \leq \sigma. \tag{6.11}$$

Now let us verify, for some suitable $c_1 > 0$, the left inequality in (6.7). To this end, let $s = (x, u, w) \in \mathcal{B}_\varepsilon(z^*) \cap \Omega$ be arbitrarily chosen. Of course, due to the definition of $F_{\min}$,

$$|[\Psi^{5.2}(x, u)]_k| \leq \|F_{\min}(s)\|_\infty \tag{6.12}$$

holds for all $k = 1, \ldots, n_x$, and

$$|g_j(x) + w_j| \leq \|F_{\min}(s)\|_\infty \tag{6.13}$$

is valid for all $j = 1, \ldots, m_g$. Let us take any $\nu \in \{1, \ldots, N\}$ and any $j \in \{1, \ldots, m_g\}$. Using (6.8)–(6.11), $\tau \geq \sigma$, the triangle inequality, and the Lipschitz continuity of the function $\min\{\cdot, u_j^\nu\}$ with Lipschitz constant 1, we have

$$\begin{aligned}
|w_j u_j^\nu| &\leq \tau |\min\{w_j, u_j^\nu\}|\\
&\leq \tau(|\min\{-g_j(x), u_j^\nu\}| + |\min\{w_j, u_j^\nu\} - \min\{-g_j(x), u_j^\nu\}|)\\
&\leq \tau(|\min\{-g_j(x), u_j^\nu\}| + |g_j(x) + w_j|)\\
&\leq 2\tau\|F_{\min}(s)\|_\infty.
\end{aligned} \tag{6.14}$$

Combining (6.12)–(6.14) and taking into account the definition of F, we obtain

$$\|F(s)\|_\infty \leq 2\tau\|F_{\min}(s)\|_\infty.$$

Hence, for $c_1 := \frac{1}{2\tau}$, the left inequality in (6.7) is proved.

Next, we are going to show the right inequality in (6.7) for some $c_2 > 0$. Again, let $s = (x, u, w) \in \mathcal{B}_\varepsilon(z^*) \cap \Omega$ be arbitrarily chosen. It is easy to see that

$$|[\Psi^{5.2}(x, u)]_k| \leq \|F(s)\|_\infty \tag{6.15}$$

holds for all $k = 1, \dots, n_x$, and

$$|g_j(x) + w_j| \leq \|F(s)\|_\infty \tag{6.16}$$

is valid for all $j = 1, \dots, m_g$. Now let us take any $\nu \in \{1, \dots, N\}$ and any $j \in \{1, \dots, m_g\}$. The triangle inequality and the Lipschitz continuity of the function $\min\{\cdot, u_j^\nu\}$ with Lipschitz constant 1 yield

$$\begin{aligned} &|\min\{-g_j(x), u_j^\nu\}| \\ &\quad\leq\ |\min\{-g_j(x), u_j^\nu\} - \min\{w_j, u_j^\nu\}| + |\min\{w_j, u_j^\nu\}| \\ &\quad\leq\ |g_j(x) + w_j| + |\min\{w_j, u_j^\nu\}|. \end{aligned} \tag{6.17}$$

Let us estimate the second summand from above. If j belongs to $\mathcal{I}_u$, then we have $\min\{w_j, u_j^\nu\} = u_j^\nu$ due to (6.8) and (6.11). Hence, using (6.8) again,

$$|\min\{w_j, u_j^\nu\}| = |u_j^\nu| = \frac{1}{w_j}|w_j u_j^\nu| \leq \frac{1}{\sigma}\|F(s)\|_\infty \tag{6.18}$$

follows. Otherwise, if j is an element of $\mathcal{G}_0$, then $j \in \mathcal{I}_g^\cup$ holds since $\mathcal{I}_=^\cap$ is empty. Thus, there is some $\nu(j) \in \{1, \dots, N\}$ such that $u_j^{\nu(j)} > 0$ is valid. The latter implies $j \in \mathcal{I}_g^{\nu(j)}$. By (6.10), we even have $u_j^{\nu(j)} \geq \sigma > 0$. This, together with $w_j \geq 0$ and $u_j^\nu \geq 0$, yields

$$|\min\{w_j, u_j^\nu\}| = \min\{w_j, u_j^\nu\} \leq w_j = \frac{1}{u_j^{\nu(j)}}|w_j u_j^{\nu(j)}| \leq \frac{1}{\sigma}\|F(s)\|_\infty. \tag{6.19}$$

Using (6.17)–(6.19),

$$|\min\{-g_j(x), u_j^\nu\}| \leq |g_j(x) + w_j| + \frac{1}{\sigma}\|F(s)\|_\infty \leq \left(1 + \frac{1}{\sigma}\right)\|F(s)\|_\infty \tag{6.20}$$

follows, regardless of whether j belongs to $\mathcal{I}_u$ or to $\mathcal{G}_0$. Combining (6.15), (6.16), and (6.20), we obtain

$$\|F_{\min}(s)\|_\infty \leq \left(1 + \frac{1}{\sigma}\right)\|F(s)\|_\infty.$$

Hence, for $c_2 := 1 + \frac{1}{\sigma}$, the right inequality in (6.7) is proved. □

The benefit of Proposition 6.5 is the following. Each condition which implies Assumption 2 for (6.4) is also sufficient for Assumption 2 for (6.1) to hold if in addition $\mathcal{I}_=^\cap$ is empty. In Section 5.2 we actually provided conditions implying all of Assumptions 1–4 for (6.4). For instance, each of the conditions (i)–(iv) from Theorem 5.8 is sufficient for the whole set of Assumptions 1–4 for (6.4) to hold, see item (b) of Remark 5.2. Therefore, each of the conditions (i)–(iv) from Theorem 5.8 does also imply Assumption 2 for the smooth constrained system (6.1) if in addition $\mathcal{I}_=^\cap = \emptyset$ holds. Moreover, using the latter observations, together with Proposition 5.9 and Corollary 5.10, respectively, and taking into account $\mathcal{I}_g^\cup = \mathcal{G}_0$ in the case $\mathcal{I}_=^\cap = \emptyset$, we obtain the following corollaries.

Corollary 6.6. *Suppose that* $\mathcal{I}_=^\cap$ *is empty. Moreover, assume that there is some* $\varepsilon > 0$ *such that, for every* N*-tuple* $(\mathcal{J}^1, \ldots, \mathcal{J}^N)$ *of index sets* $\mathcal{J}^\nu \subseteq \mathcal{I}_=^\nu$ $(\nu = 1, \ldots, N)$*, the matrices*

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x,u) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu \cup \mathcal{J}^\nu}(x)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{G}_0}(x) & 0 \end{array}\right)$$

have the same rank for all $(x,u) \in \mathcal{B}_\varepsilon(x^*, u^*)$*. Then Assumption 2 for* (6.1) *is valid.*

Corollary 6.7. *Suppose that* $\mathcal{I}_=^\cap$ *is empty. Moreover, assume that the matrix*

$$\left(\begin{array}{c|c} \Psi_x^{5.2}(x^*,u^*) & \text{block}\left(\nabla_{x^\nu} g_{\mathcal{I}_g^\nu}(x^*)\right)_{\nu=1}^N \\ \hline g'_{\mathcal{G}_0}(x^*) & 0 \end{array}\right)$$

has full row rank. Then Assumption 2 for (6.1) *is valid.*

A result which is similar to Corollary 6.7 is proved in [14] where even the more general case is considered that individual constraints may appear. Assuming that the players have shared constraints only, it follows from [14, Theorem 2] that Assumption 2 for the constrained system (6.3) holds if the assumptions of Corollary 6.7 are satisfied. However, further sufficient conditions for Assumption 2 are not considered in [14]. We obtained more general and weaker conditions implying Assumption 2 for (6.1), at least for the case that the players have shared constraints only.

At the end of this section, let us summarize the results of this chapter and make some further comments. We described a hybrid method for the solution of KKT systems arising from GNEPs which enjoys, under suitable assumptions, both global convergence and local fast convergence. In [15, Section 3.4] numerical results can be found which show that the hybrid method works well. It turned out in the current chapter that Assumption 2 for the smooth system (6.1) is crucial for local quadratic convergence of the hybrid method. We showed that this assumption is satisfied if and only if Assumption 2 for the nonsmooth system (6.4) holds, supposed that in addition for each active constraint the corresponding multiplier of at least one player is strictly positive. The latter condition seems mild. In particular, it is a weaker requirement than strict complementarity.

However, if the problem reduces to a KKT system arising from an optimization problem or a variational inequality, the condition $\mathcal{I}_=^\cap = \emptyset$ coincides with strict complementarity. So the hybrid method for the smooth system could be transferred to such KKT systems but then Assumption 2 cannot be expected to hold if strict complementarity is violated. This motivates to look for further globalization techniques.

In [14, Algorithm 2] a modification of the hybrid method was considered where the steps of the potential reduction algorithm are performed for the smooth system whereas the steps of the LP-Newton method are applied to the reformulation with the minimum function. Numerical results for the modified hybrid method can be found in [14] as well. This way of globalization could also be transferred to KKT systems arising from optimization problems or variational inequalities without requiring strict complementarity to guarantee local fast convergence.

Finally, it is to say that reasonable approaches to globalize the Newton-type methods from Chapter 3 are still rare and topics of future research.

Chapter 7

Conclusions and Outlook

In this thesis Newton-type methods for the solution of the constrained system of equations (1.1) were considered. We analyzed local convergence properties under very mild assumptions. Moreover, the convergence assumptions were discussed, at least for important problem classes. In this final chapter the most important results are summarized. In the second part we give some outlook concerning future research.

The initial point of this thesis was the paper [18] where the LP-Newton method was described and local quadratic convergence was shown under four assumptions implying neither local uniqueness of solutions nor differentiability of F at solutions. In Chapter 3 we used the same assumptions (Assumptions 1–4) to prove local quadratic convergence of a general Newton-type algorithm. Moreover, we showed that, besides the LP-Newton method, the constrained Levenberg-Marquardt method is a special realization of the general Newton-type algorithm and therefore enjoys the same local convergence properties. To the best of the author's knowledge, this is the first time that local fast convergence of a non-smooth Levenberg-Marquardt method is proved without requiring assumptions implying the local uniqueness of solutions.

In Chapter 4 we discussed Assumptions 1–4 in detail for the case that the function F is piecewise continuously differentiable. Furthermore, the reformulation of (1.1) by means of slack variables was discussed. We developed conditions which imply the whole set of Assumptions 1–4, not only for the original constrained system but also for the reformulation with slack variables, see Theorem 4.19. The weakest among these sufficient conditions is condition (iv) in Theorem 4.19. It requires that some set of local error bound conditions is satisfied and that the set Ω excludes those zeros of the selection functions which are not zeros of F, at least in a certain neighborhood of some fixed solution. The latter condition on Ω was called Ω-property. The results were applied to constrained systems arising from reformulations of complementarity systems. We presented suitable reformulations where the Ω-property is always satisfied so that some set of local error bound conditions is actually sufficient for the whole set of Assumptions 1–4 to hold. Besides, further conditions implying all of Assumptions 1–4 were provided, in particular some new constant rank condition.

The results of Chapter 4 were applied to special classes of complementarity systems in Chapter 5. More precisely, we considered KKT systems arising from optimization problems, variational inequalities, or GNEPs and FJ systems arising from GNEPs. We provided adapted formulations of the conditions from Chapter 4. Thus, we obtained for each problem class conditions which imply local quadratic convergence of our general Newton-type algorithm and its special realizations to a solution of the particular problem. Some of these conditions are, to the best of the author's knowledge, new in the field of KKT systems and FJ systems, in particular some constant rank condition or, again the weakest among the sufficient conditions, some set of local error bound conditions. Moreover, we proved for FJ systems arising from GNEPs that generically some full row rank condition is satisfied at any solution. The latter condition implies the whole set of Assumptions 1–4 at any solution of the FJ system if the second-order derivatives of all functions which characterize the GNEP are locally Lipschitz continuous.

In Chapter 6 a smooth reformulation arising from the KKT system of a GNEP was considered. We described a hybrid algorithm whose local part is the LP-Newton method. The hybrid algorithm turned out to be, under suitable conditions, both globally convergent and locally fast convergent. Moreover, we presented sufficient conditions for the local quadratic convergence.

Finally, let us discuss some possible directions for future research. It turned out in this thesis that the general Newton-type algorithm and its special realizations have strong local convergence properties. However, the question concerning a reasonable globalization is not completely answered yet. In Chapter 6 we just described an idea for the case that the constrained system is a certain smooth reformulation of a KKT system arising from a GNEP. However, we have seen in this thesis that nonsmooth reformulations of KKT systems and related problems may lead to local quadratic convergence of our general Newton-type algorithm under milder assumptions. So the question concerning a reasonable globalization for nonsmooth constrained systems arises. A first step in this direction is done in [14], at least for the case of KKT systems of GNEPs.

A further topic might be the search for more conditions implying the whole set of Assumptions 1–4. For the case of complementarity systems we provided some set of local error bound conditions which seems to be still very mild and which is sufficient for all of Assumptions 1–4 to hold. Using this and the sufficiency of the RCRCQ for the local error bound condition, we obtained some constant rank condition which implies Assumptions 1–4. There are further constraint qualifications which are sufficient for the local error bound condition to hold, for example CPLD. The latter is neither sufficient nor necessary for the RCRCQ to hold. Maybe, the CPLD can also be used to derive some new condition implying the whole set of Assumptions 1–4.

Moreover, numerical tests for large examples are a topic of future work. Related to this, the question concerning reasonable solvers for the LP-Newton subproblems and the Levenberg-Marquardt subproblems, respectively, arise. It seems that both kinds of subproblems are ill-conditioned in a certain sense near a solution of (1.1). Thus, solvers for linear and quadratic programs which provide a high accuracy might be necessary.

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Die vorliegende Dissertation habe ich an der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. Andreas Fischer angefertigt.

Es wurden zuvor keine Promotionsvorhaben unternommen.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 23.02.2011, in der geänderten Fassung vom 18.06.2014, an.

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