Analytical solution of a linear, elliptic, inhomogeneous partial differential equation with inhomogeneous mixed Dirichlet- and Neumann-type boundary conditions for a special rotationally symmetric problem of linear elasticity

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July 29, 2014

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Abstract

The analytical solution of a given inhomogeneous boundary value problem of a linear, elliptic, inhomogeneous partial differential equation and a set of inhomogeneous mixed Dirichlet- and Neumann-type boundary conditions is derived in the present paper. In the context of elasticity theory, the problem arises for a non-conservative symmetric ansatz and an extended constitutive law shown earlier. For convenient user application, the scalar function expressed in cylindrical coordinates is primarily obtained for the general case before being expatiated on a special case of linear boundary conditions.

1 Introduction

1.1 Motivation

As mentioned earlier [1], analytical solutions for more or less general cases of boundary value problem (BVP) consisting of systems of partial differential equations (PDEs) and given boundary conditions (BCs) are of scientific interest especially against the background of the assessment of found numerical solutions, cf. e.g. [2–6] The present paper can be considered an addendum of [1], as an analytical solution of the BVP is derived for a different set of BCs.

1.2 Inhomogeneous boundary value problem

In addition to [1], the following inhomogeneous boundary value problem (iBVP) is considered: The linear, elliptic, inhomogeneous partial differential equation (iPDE)

$$G\,r\,\frac{\partial^2}{\partial r^2}u_z(r,z)+G\,\frac{\partial}{\partial r}u_z(r,z)+(K+\frac{4}{3}\,G)\,r\,\frac{\partial^2}{\partial z^2}u_z(r,z)=2\,G\,\varepsilon_{rz}^{\mathrm{pl}},\qquad \forall(r,z)\in D:=(0,r_{\mathrm{a}})\times(0,L)\quad (1)$$

for the real solution $u_z\colon\overline{D}\to\mathbb{R}^1$ with given real parameters $r_{\mathrm{a}}, L>0$. Furthermore the corresponding set of inhomogeneous boundary conditions (iBCs) is given in form of Dirichlet-type BCs by

$$u_z(r_{\mathrm{a}},z)\overset{!}{=}\Phi_1(z),\qquad \forall z\in(0,L)\quad (2)$$

$$\frac{\partial}{\partial z}u_z(r,0)\overset{!}{=}0,\qquad \forall r\in[0,r_{\mathrm{a}}]\quad (3)$$

$$\frac{\partial}{\partial z}u_z(r,L)\overset{!}{=}0,\qquad \forall r\in[0,r_{\mathrm{a}}]\quad (4)$$

with given real function Φ_1 that is not specified further at this point for generality reasons.
Remark: Due to the same remarks as in [1], the iBVP does not lack a BC for $r=0$ as it appears at first sight.

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2 Results

For abbreviative purposes, the iPDE (1) is written as

$$G\, r\, u_z''(r,z) + G\, u_z'(r,z) + (K + \frac{4}{3} G)\, r\, \ddot{u}_z(r,z) = 2\, G\, \varepsilon_{rz}^{\mathrm{pl}}, \qquad \forall (r,z) \in D \tag{5}$$

with the symbolic abbreviations $(\cdot)' := \frac{\partial}{\partial r}(\cdot)$ and $\dot{(\cdot)} := \frac{\partial}{\partial z}(\cdot)$. Similar to [1], the analytical solution of the iBVP, i.e. iPDE (5) with iBCs (2)–(4), is derived in the following with the well-known Fourier series approach – based on the separability of the form

$$u_z(r,z) \stackrel{!}{=} R(r) \cdot Z(z), \qquad \forall (r,z) \in \overline{D} \tag{6}$$

with real functions $R\colon [0, r_\mathrm{a}] \to \mathbb{R}^1$ and $Z\colon [0, L] \to \mathbb{R}^1$. Thus, the analytical total solution for the iBVP given above splits into the two (real) partial solutions $u_{z,\mathrm{i}}(r,z)$, $\mathrm{i} = \mathrm{I}, \mathrm{II}$ [7]:

1. solution $u_{z,\mathrm{I}}(r,z)$ of the respective homogeneous PDE (hPDE) with iBCs in r, but homogeneous BCs (hBCs) in z
2. solution $u_{z,\mathrm{II}}(r,z)$ of the iPDE (5) with both hBCs in r and z

The solution $u_z(r,z)$ of the given iBVP equals the superposition of these partial solutions:

$$u_z(r,z) = u_{z,\mathrm{I}}(r,z) + u_{z,\mathrm{II}}(r,z), \qquad \forall (r,z) \in \overline{D} \tag{7}$$

2.1 Solution $u_{z,\mathrm{I}}(r,z)$

By analogy with [1], the ansatz (6) converts the hPDE

$$G\, r\, u_z''(r,z) + G\, u_z'(r,z) + (K + \frac{4}{3} G)\, r\, \ddot{u}_z(r,z) = 0, \qquad \forall (r,z) \in D, \tag{8}$$

into the problem

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\,\frac{R'(r)}{R(r)} = -\frac{3K+4G}{3G}\,\frac{\ddot{Z}(z)}{Z(z)} \stackrel{!}{=} l, \qquad \forall (r,z) \in D \tag{9}$$

with $l \in \mathbb{R}^1$, that is separable into the two ODEs

$$-\ddot{Z}(z) - l\,\frac{3G}{3K+4G}\,Z(z) = 0, \qquad \forall z \in (0, L), \tag{10}$$

$$r\,R''(r) + R'(r) - l\,r\,R(r) = 0, \qquad \forall r \in (0, r_\mathrm{a}) \tag{11}$$

for the non-zero functions R and Z, respectively. Moreover, using (6), the hBCs (3) and (4) in z turn into

$$\dot{Z}(0) \stackrel{!}{=} 0 \stackrel{!}{=} \dot{Z}(L). \tag{12}$$

2.1.1 Eigenvalue problem for $Z(z)$

From (10) and the hBCs (12) the eigenvalue problem (EVP)

$$-\ddot{Z}(z) = \underbrace{l\,\frac{3G}{3K+4G}}_{=:\tilde{l} \in \mathbb{R}^1} Z(z), \qquad \forall z \in (0,L), \qquad \dot{Z}(0) = \dot{Z}(L) = 0 \tag{13}$$

evolves. Consideration of the three cases $\tilde{l} > 0$, $\tilde{l} = 0$ and $\tilde{l} < 0$ leads to the two non-trivial solutions and to the respective eigenvalues $\tilde{l}_0 = 0$ and $\tilde{l}_n = (\frac{n\pi}{L})^2$ for $n = 1, 2, \ldots$. The corresponding eigenfunctions are $Z(z) \equiv Z_0 = 1$ and $Z(z) \equiv Z_n(z) = \cos(\frac{n\pi}{L} z)$, respectively. Hence,

$$u_{z,\mathrm{I}}(r,z) = R_0(r) + \sum_{n=1}^{\infty} R_n(r)\, \cos(\frac{n\pi}{L} z), \qquad \forall (r,z) \in \overline{D}. \tag{14}$$

2.1.2 Bessel ODE for $R_n(r)$

Insertion of the two classes of eigenvalues $\tilde{l}_0 = 0$ and $\tilde{l}_n > 0$, calculated in section 2.1.1, into (11) leads – for fixed $n \in \mathbb{N}$ and under consideration of (5) – to the determining equations

$$\text{for } \tilde{l}_0 = 0: \qquad r\,R_0''(r) + R_0'(r) = 0, \qquad \forall r \in (0, r_\mathrm{a}), \tag{15}$$

$$\text{for } l_n \overset{(13)}{=} \frac{3K+4G}{3G}\,(\frac{n\pi}{L})^2 > 0: \quad r\,R_n''(r) + R_n'(r) - l_n\,r\,R_n(r) = 0, \qquad \forall r \in (0, r_\mathrm{a}),\; n \in \mathbb{N}, \tag{16}$$

for the functions $R_0(r)$ and $R_n(r)$ with $n \in \mathbb{N}$, yet unknown in (14). Each being a homogeneous ODE (hODE), (15) is a EULERIAN-type, while (16) belongs to the BESSEL-type [8, 9]. Thus, (for fixed $n \in \mathbb{N}$) the corresponding two classes of solutions are

$$R_0(r) = A_0 + B_0\,\ln(r), \qquad \forall r \in [0, r_\mathrm{a}], \tag{17}$$

$$R_n(r) = A_n\,I_0(\sqrt{l_n}\,r) + B_n\,K_0(\sqrt{l_n}\,r), \qquad \forall r \in [0, r_\mathrm{a}],\; n \in \mathbb{N}, \tag{18}$$

with the unknown coefficients A_0, B_0, A_n and $B_n \in \mathbb{R}^1$ ($n \in \mathbb{N}$). Here, the so-called modified BESSEL functions I_0 and $K_0\colon \mathbb{R}^1 \to \mathbb{R}^1$ are denoted according to [8, 9]. Because of $\lim_{r\to 0+} \ln(r) = -\infty$ as well as $\lim_{r\to 0+} K_0(\sqrt{l_n}\,r) = +\infty$, it follows that $B_0 \overset{!}{=} 0$ as well as $B_n \overset{!}{=} 0$, $n \in \mathbb{N}$. From a physical point of view, this correlates with finite displacements of the material.
Using (17) and (18) in (14) yields

$$u_{z,\mathrm{I}}(r,z) = A_0 + \sum_{n=1}^{\infty} A_n\,I_0(\sqrt{l_n}\,r)\,\cos(\frac{n\pi}{L}z), \qquad \forall (r,z) \in \overline{D} \tag{19}$$

whereas the (yet unknown) real coefficients A_0 and A_n are specified by the iBC in r as follows.

2.1.3 Consideration of the iBCs in r

By analogy with [1], substituting (19) into the iBC (2) in r leads to the linear equation

$$\Phi_1(z) \overset{(2)}{=} u_{z,\mathrm{I}}(r_\mathrm{a}, z) \overset{(19)}{=} A_0 + \sum_{n=1}^{\infty} A_n\,I_0(\sqrt{l_n}\,r_\mathrm{a})\,\cos(\frac{n\pi}{L}z), \qquad \forall z \in (0, L) \tag{20}$$

for the coefficients A_0 and A_n, $n \in \mathbb{N}$. In order to determine the former, equation (20) is integrated according to $\int_{z=0}^{L} (\cdot)\,dz$ [7] which yields

$$A_0 = \frac{1}{L} \int_{z=0}^{L} \Phi_1(z)\,dz. \tag{21}$$

The coefficients A_n are obtained by multiplication of (20) with $\cos(\frac{\kappa\pi}{L}z)$ for $\kappa = 1, 2, ...$ and then integrating analog to the mentioned above. Doing so, (20) turns (for $\kappa = 1, 2, ...$) into

$$\int_{z=0}^{L} \Phi_1(z)\,\cos(\frac{\kappa\pi}{L}z)\,dz \overset{!}{=} A_0 \underbrace{\int_{z=0}^{L} \cos(\frac{\kappa\pi}{L}z)\,dz}_{=0} + \int_{z=0}^{L} \left[\sum_{n=1}^{\infty} A_n\,I_0(\sqrt{l_n}\,r_\mathrm{a})\,\cos(\frac{n\pi}{L}z)\,\cos(\frac{\kappa\pi}{L}z)\right] dz, \tag{22}$$

and is solved for the coefficients A_n, giving

$$A_\kappa = \frac{2}{L}\,\frac{1}{I_0(\sqrt{l_\kappa}\,r_\mathrm{a})} \int_{z=0}^{L} \Phi_1(z)\,\cos(\frac{\kappa\pi}{L}z)\,dz, \qquad \kappa = 1, 2, ... \tag{23}$$

Therefore, using $l_0 = 0$ and $l_n = \frac{3K+4G}{3G}\,(\frac{n\pi}{L})^2 > 0$ from section 2.1.1, the final expression for $u_{z,\mathrm{I}}(r,z)$ can be written as

$$\boxed{u_{z,\mathrm{I}}(r,z) = \frac{1}{L} \int_{z=0}^{L} \Phi_1(z)\,dz + \frac{2}{L} \sum_{n=1}^{\infty} \left[\frac{I_0(\sqrt{\frac{3K+4G}{3G}}\,\frac{n\pi}{L}\,r)}{I_0(\sqrt{\frac{3K+4G}{3G}}\,\frac{n\pi}{L}\,r_\mathrm{a})} \int_{z=0}^{L} \Phi_1(z)\,\cos(\frac{n\pi}{L}z)\,dz\right] \cos(\frac{n\pi}{L}z), \qquad \forall (r,z) \in \overline{D}.} \tag{24}$$

Remark: For further specification of the coefficients A_0 in (21) and A_κ in (23) for their usage in (24), the function $\Phi_1(z)$ in (2) has to be specified itself. By analogy with [1], $\Phi_1(z)$ can be estimated to be linear in z in first-order approximation, i.e. with the measured values $\hat{\Phi}_{1L} \approx \Phi_1(L)$ and $\hat{\Phi}_{10} \approx \Phi_1(0)$,

$$\Phi_1(z) \overset{!}{=} \frac{1}{L}\left[\hat{\Phi}_{1L}\,z + \hat{\Phi}_{10}\,(L - z)\right], \qquad \text{with } \hat{\Phi}_{1L}, \hat{\Phi}_{10} \in \mathbb{R}^1 \tag{25}$$

is used. Thus, the unknown coefficients A_0 and A_κ can be derived from (21) and (23), respectively. Thus,

$$A_0 = \frac{\hat{\Phi}_{1L} + \hat{\Phi}_{10}}{2}, \tag{26}$$

$$A_\kappa = \frac{2}{(\pi\kappa)^2} \frac{(-1)^{\kappa+1}}{I_0(\sqrt{l_\kappa}\, r_\mathrm{a})} \left[\hat{\Phi}_{1L} - \hat{\Phi}_{10}\right], \qquad \kappa = 1, 2, \dots\,, \tag{27}$$

and therefore converting (24) into

$$\boxed{\hat{u}_{z,\mathrm{I}}(r,z) = \frac{\hat{\Phi}_{1L} + \hat{\Phi}_{10}}{2} + \frac{2(\hat{\Phi}_{1L} - \hat{\Phi}_{10})}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \frac{I_0(\sqrt{\frac{3K+4G}{3G}}\,\frac{n\pi}{L}\, r)}{I_0(\sqrt{\frac{3K+4G}{3G}}\,\frac{n\pi}{L}\, r_\mathrm{a})} \cos(\frac{n\pi}{L} z), \qquad \forall (r,z) \in \overline{D}.} \tag{28}$$

2.2 Solution $u_{z,\mathrm{II}}(r,z)$

In [1], a possibility to obtain the solution of the iPDE (5) with a set of hBCs is shown. By analogy, the partial solution $u_{z,\mathrm{II}}(r,z)$ of this iPDE with the corresponding set

$$u_{z,\mathrm{II}}(r_\mathrm{a}, z) \stackrel{!}{=} 0, \qquad \forall z \in (0, L) \tag{29}$$

$$\dot{u}_{z,\mathrm{II}}(r, 0) \stackrel{!}{=} 0, \qquad \forall r \in [0, r_\mathrm{a}] \tag{30}$$

$$\dot{u}_{z,\mathrm{II}}(r, L) \stackrel{!}{=} 0, \qquad \forall r \in [0, r_\mathrm{a}] \tag{31}$$

of hBCs (instead of the iBCs (2)-(4)) is determined – by using the ansatz

$$u_{z,\mathrm{II}}(r,z) = \widehat{C}_0(r) + \sum_{n=1}^{\infty} \widehat{C}_n(r) \cos(\frac{n\pi}{L} z), \qquad \forall (r,z) \in \overline{D} \tag{32}$$

with the coefficients $\widehat{C}_n(r)$ with $n = 0, 1, 2, \dots$, yet to determine.
Motivation of the ansatz above is the fact of the functions $Z_0 = 1$ and $Z_n(z) = \cos(\frac{n\pi}{L} z)$ being the eigenfunctions of the EVP (13). Moreover, (32) ensures the satisfaction of (30) and (31). By analogy with [1],

$$\ddot{u}_{z,\mathrm{II}}(r,z) = -\frac{\pi^2}{L^2} \sum_{n=1}^{\infty} n^2\, \widehat{C}_n(r) \cos(\frac{n\pi}{L} z), \qquad \forall (r,z) \in \overline{D} \tag{33}$$

inserted in (5) yields

$$r\, \widehat{C}_0''(r) + \widehat{C}_0'(r) + \sum_{n=1}^{\infty} \left[r\, \widehat{C}_n''(r) + \widehat{C}_n'(r) - \frac{3K+4G}{3G} \frac{n^2\pi^2}{L^2}\, r\, \widehat{C}_n(r) \right] \cos(\frac{n\pi}{L} z) = 2\, \varepsilon_{rz}^{\mathrm{pl}}, \qquad \forall (r,z) \in D. \tag{34}$$

Thus, the solution of (34) is obtained by solving the following two ODEs:

$$\varepsilon_{rz}^{\mathrm{pl}} \stackrel{!}{=} r\, \widehat{C}_0''(r) + \widehat{C}_0'(r), \qquad \forall r \in (0, r_\mathrm{a}) \tag{35}$$

$$0 \stackrel{!}{=} r\, \widehat{C}_n''(r) + \widehat{C}_n'(r) - \underbrace{\frac{3K+4G}{3G} \frac{n^2\pi^2}{L^2}}_{>0}\, r\, \widehat{C}_n(r), \qquad \forall r \in (0, r_\mathrm{a}) \tag{36}$$

By analogy with (17) and (18) in section 2.1.2, the solutions of the Eulerian-type ODE (35) and the Bessel-type ODE (36) are

$$\widehat{C}_0(r) = \widehat{C}_{0,1} + \widehat{C}_{0,2} \ln(r) + 2\, \varepsilon_{rz}^{\mathrm{pl}}\, r, \qquad \forall r \in [0, r_\mathrm{a}] \tag{37}$$

and

$$\widehat{C}_n(r) = \widehat{C}_{n,1}\, I_0(\sqrt{\frac{3K+4G}{3G}}\, \frac{n\pi}{L}\, r) + \widehat{C}_{n,2}\, K_0(\sqrt{\frac{3K+4G}{3G}}\, \frac{n\pi}{L}\, r), \qquad \forall r \in [0, r_\mathrm{a}],\ n \in \mathbb{N}, \tag{38}$$

respectively. The (yet unknown) coefficients $\widehat{C}_{n,1}$, $\widehat{C}_{n,1} \in \mathbb{R}^1$ $(n \in \mathbb{N}_+)$ are determined as shown below.
Firstly, analog arguments as described in section 2.1.2 lead to $\widehat{C}_{n,2} \stackrel{!}{=} 0$ $(n \in \mathbb{N}_+)$. Secondly, the usage of the hBC (29) gives

$$\widehat{C}_{0,1} = -2\, \varepsilon_{rz}^{\mathrm{pl}}\, r_\mathrm{a} \tag{39}$$

$$\widehat{C}_{n,1} = 0, \qquad \forall n \in \mathbb{N}, \tag{40}$$

and thus $\widehat{C}_n(r) = 0$, $\forall r \in [0, r_\mathrm{a}]$ and $\forall n \in \mathbb{N}$. Hence, (32) yields the final expression of

$$\boxed{u_{z,\mathrm{II}}(r,z) = u_{z,\mathrm{II}}(r) = 2\, \varepsilon_{rz}^{\mathrm{pl}}\, (r - r_\mathrm{a}), \qquad \forall (r,z) \in \overline{D}.} \tag{41}$$

2.3 Total solution $u_z(r, z)$

The partial solutions $u_{z,\mathrm{I}}(r, z)$ and $u_{z,\mathrm{II}}(r)$ – given in (24) and (41), respectively – result according to (7) into the (total) solution $u_z(r, z)$ of the iBVP, given in section 1.2.
Additionally, in case of a linear form of the iBC (2), i.e. (25), $u_{z,\mathrm{I}}(r, z)$ simplifies according to (28) to $\hat{u}_{z,\mathrm{I}}(r, z)$, yielding the corresponding (total) solution $\hat{u}_z(r, z) = \hat{u}_{z,\mathrm{I}}(r, z) + u_{z,\mathrm{II}}(r)$.

3 Conclusion

Studying the solution of the static equilibrium conditions in the context of elasticity theory while using a non-conservative symmetry ansatz and extended constitutive law as shown in [1], the general analytical solution of a special rotationally symmetric iBVP is derived above. Furthermore, the solution is then expatiated on a special case of linear iBCs for user convenience.

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Acknowledgements

Fruitful discussions with R. Vanselow (NM, TUD) are gratefully acknowledged. This work has been financially supported by the European Union and the Free State of Saxonia in the framework of the International Graduate School (IGS) of the European Centre for Emerging Materials and Processes (ECEMP), contract no. 100111842, which is highly appreciated.