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PREPRINT

On the relationship of maximal C-clones and maximal clones

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A restricted version of the Galois connection between polymorphisms and invariants, called $\mathrm{Pol}_D - C\,\mathrm{Inv}_D$, is studied, where the invariant relations are restricted to so-called clausal relations. In this context, the relationship of maximal C-clones and maximal clones is investigated. It is shown that, with the exception of one special case occurring for $|D| = 2$, maximal C-clones are never maximal clones.

1 Introduction

In this paper we continue the investigations from [BV10] and [Var10] concerning a special set $C\,\mathrm{R}_D$ of relations on a finite set D called *clausal relations*. A clausal relation is the set of all tuples over D satisfying disjunctions of inequalities of the form $x \geq d$ and $x \leq d$, where $x, d \in D = \{0, 1, \ldots, n-1\}$.

Clones are sets of operations on a fixed domain that are closed under composition and contain all projections. The clones on a finite set D are the Galois closed sets of operations [BKKR69] with respect to the well-known Galois *connection* $\mathrm{Pol}_D - \mathrm{Inv}_D$ induced by the relation "an operation f *preserves* a relation ϱ" (see also [Pös79, Pös80]). In other words, every clone F on D can be described by $F = \mathrm{Pol}_D\, Q$ for some set Q of relations.

We are interested in describing the structure of clones that are determined by sets of clausal relations, so-called *C-clones*. The aim of this paper is to investigate the relationship of the co-atoms in the clone lattice, known as *maximal clones*, and the co-atoms in the lattice of C-clones, the *maximal C-clones*, which have been characterised in the doctoral thesis [Var11] of the second author. During the defence of the latter also the question arose that is answered in this paper.

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2 Preliminaries

Throughout the text, D will denote the finite non-empty set $\{0, \ldots, n-1\}$ $(n > 0)$, $\mathbb{N}$ the set of all natural numbers including zero, and $\mathbb{N}_+$ the set of positive natural numbers. The symbol $\mathfrak{P}(X)$ will stand for the power set of a set X.

Let m be a positive integer. An *m-ary relation* ϱ on D is a subset of the m-fold Cartesian product D^m. By $\mathrm{R}_D^{(m)} := \mathfrak{P}(D^m)$ we denote the *set of all m-ary relations* on D and by $\mathrm{R}_D := \bigcup_{\ell \in \mathbb{N}_+} \mathrm{R}_D^{(\ell)}$ the *set of all finitary relations* on D. For an equivalence relation θ on $\{0, \ldots, m-1\}$, we define an m-ary relation d_θ on D by $d_\theta := \{ (x_0, \ldots, x_{m-1}) \in D^m \mid \forall (i,j) \in \theta \colon x_i = x_j \}$ and call it *diagonal relation*. The special case $d_\theta = D^m$ is also called *trivial relation*. The set of all diagonal relations together with the empty relation $\emptyset$ is denoted by Diag_D.

In this section clones that are determined by sets of clausal relations will be discussed. Next, these relations are defined.

Definition 1. Let $p, q \in \mathbb{N}_+$. For given parameters $\mathbf{a} = (a_1, \ldots, a_p) \in D^p$ and $\mathbf{b} = (b_1, \ldots, b_q) \in D^q$, the *clausal relation* $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ *of arity* $p+q$ is the set of all tuples $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$ satisfying

$$(x_1 \geq a_1) \vee \cdots \vee (x_p \geq a_p) \vee (y_1 \leq b_1) \vee \cdots \vee (y_q \leq b_q). \tag{2.1}$$

In this expression $\leq$ is interpreted as the canonical linear order $\leq_D$ on D and $\geq$ as its dual. ◊

Note that whenever $a_i = 0$ for some $i \in \{1, \ldots, p\}$ or $b_j = n-1$ for some index $j \in \{1, \ldots, q\}$, then the relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ is the full Cartesian power of D, i.e. $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}} = D^{p+q}$, because (2.1) is satisfied for any $(x_1, \ldots, x_p, y_1, \ldots, y_q) \in D^{p+q}$.

Let $p, q \in \mathbb{N}_+$. We use $\mathcal{R}_q^p := \{ \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in D^p, \mathbf{b} \in D^q \}$ to denote the set of all *clausal relations of arity* $p+q$ and $C\,\mathrm{R}_D := \bigcup_{(p,q) \in \mathbb{N}_+^2} \mathcal{R}_q^p$ for the set of all *finitary clausal relations* on D.

The following lemma states that the trivial clausal relations are those we noticed after Definition 1, and that the non-trivial ones can be easily identified by their parameters $\mathbf{a}$ and $\mathbf{b}$.

Lemma 2 ([Var10]). *The set $C\,\mathrm{R}_D$ can be partitioned as*

$$C\,\mathrm{R}_D = \left\{ D^{p+q} \;\middle|\; p, q \in \mathbb{N}_+ \right\} \,\dot\cup\, C\,\mathrm{R}_D^*,$$

where $\{ D^{p+q} \mid p, q \in \mathbb{N}_+ \} = C\,\mathrm{R}_D \cap \mathrm{Diag}_D$ are the trivial clausal relations and $C\,\mathrm{R}_D^* = \{ \mathrm{R}_{\mathbf{b}}^{\mathbf{a}} \mid \mathbf{a} \in (D \setminus \{0\})^p, \mathbf{b} \in (D \setminus \{n-1\})^q;\ p, q \in \mathbb{N}_+ \}$ are the non-trivial clausal relations.

For a positive natural number $k \in \mathbb{N}_+$ we denote by $\mathrm{O}_D^{(k)} := \left\{ f \;\middle|\; f \colon D^k \longrightarrow D \right\}$ the *set of all k-ary operations on D* and by $\mathrm{O}_D := \bigcup_{\ell \in \mathbb{N}_+} \mathrm{O}_D^{(\ell)}$ the *set of all finitary operations on D*.

Next, we will consider a Galois connection between sets of operations and relations that is based on the so-called *preservation relation*. It is the most important tool for our investigations.

Definition 3. Let $m, k \in \mathbb{N}_+$. We say that a k-ary operation $f \in \mathrm{O}_D^{(k)}$ *preserves* an m-ary relation $\varrho \in \mathrm{R}_D^{(m)}$, denoted by $f \rhd \varrho$, if whenever

$$r_1 = (a_{11}, \ldots, a_{m1}) \in \varrho, \ldots, r_k = (a_{1k}, \ldots, a_{mk}) \in \varrho,$$

it follows that also f applied to these tuples belongs to ϱ, i.e.

$$f \circ (r_1, \ldots, r_k) := (f(a_{11}, \ldots, a_{1k}), \ldots, f(a_{m1}, \ldots, a_{mk})) \in \varrho. \qquad \Diamond$$

By definition, to show that a k-ary function $f \in \mathrm{O}_D^{(k)}$ does not preserve a relation $\varrho \in \mathrm{R}_D^{(m)}$, it suffices to exhibit tuples $r_1, \ldots, r_k \in \varrho$ such that

$$f \circ (r_1, \ldots, r_k) =: (b_1, \ldots, b_m) \notin \varrho.$$

In this case we will say that the equation $f \circ (r_1, \ldots, r_k) = (b_1, \ldots, b_m)$ *witnesses* the fact $f \not\rhd \varrho$.

For a set of operations $F \subseteq \mathrm{O}_D$, we denote by $\mathrm{Inv}_D F$ the set of all relations that are *invariant* for all operations $f \in F$:

$$\mathrm{Inv}_D F := \{ \varrho \in \mathrm{R}_D \mid \forall f \in F\colon f \rhd \varrho \}.$$

Similarly, for a set $Q \subseteq \mathrm{R}_D$ of relations, we denote by

$$\mathrm{Pol}_D Q := \{ f \in F \mid \forall \varrho \in Q\colon f \rhd \varrho \}$$

the *set of polymorphisms of* Q. Occasionally, we will write $\mathrm{Pol}_D \varrho$ for $\mathrm{Pol}_D \{\varrho\}$, $\varrho \in \mathrm{R}_D$, and $\mathrm{Inv}_D f$ for $\mathrm{Inv}_D \{f\}$, $f \in \mathrm{O}_D$.

The operators Pol_D and Inv_D define the Galois connection $\mathrm{Pol}_D - \mathrm{Inv}_D$. Below we present a restriction of this connection where the relations are clausal relations.

For $F \subseteq \mathrm{O}_D$ we define $C\,\mathrm{Inv}_D F := C\,\mathrm{R}_D \cap \mathrm{Inv}_D F$. The operators

$$\begin{aligned} C\,\mathrm{Inv}_D\colon \mathfrak{P}(\mathrm{O}_D) \longrightarrow \mathfrak{P}(C\,\mathrm{R}_D) \quad &: \qquad F \mapsto C\,\mathrm{Inv}_D F,\\ \mathrm{Pol}_D\colon \mathfrak{P}(C\,\mathrm{R}_D) \longrightarrow \mathfrak{P}(\mathrm{O}_D) \quad &: \qquad Q \mapsto \mathrm{Pol}_D Q \end{aligned}$$

define a Galois *connection* $\mathrm{Pol}_D - C\,\mathrm{Inv}_D$ between operations and clausal relations.

Definition 4. A set $F \subseteq \mathrm{O}_D$ of operations is called a *C-clone* if $F = \mathrm{Pol}_D Q$ for some set $Q \subseteq C\,\mathrm{R}_D$ of clausal relations, and a set $Q \subseteq C\,\mathrm{R}_D$ is called *relational C-clone* if $Q = C\,\mathrm{Inv}_D F$ for a set F of operations. $\Diamond$

Clearly, every C-clone is a clone as it is a set of polymorphisms of some set of finitary relations.

Every Galois connection gives rise to a pair of closure operators. For the Galois connection $\mathrm{Pol}_D - C\,\mathrm{Inv}_D$, we introduce the following notation.

For any $F \subseteq \mathrm{O}_D$ and any $Q \subseteq C\,\mathrm{R}_D$ we set

$$\langle F \rangle_{\mathcal{C}} := \mathrm{Pol}_D C\,\mathrm{Inv}_D F, \qquad \text{and} \qquad [Q]_{\mathcal{C}} := C\,\mathrm{Inv}_D \mathrm{Pol}_D Q.$$

A C-clone is *maximal* if it is a co-atom in the lattice of all C-clones. Theorem 5 below states that a C-clone is maximal if it can be written as a polymorphism set of only one non-trivial clausal relation of arity two.

Theorem 5 ([Var11]). *Let $M \subseteq \mathrm{O}_D$ be a C-clone. M is maximal if and only if there are elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$ such that $M = \mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$.*

This triggered the question after the relationship of maximal C-clones and maximal clones in general. In the next section we will prove the following main theorem.

Theorem 6. *If $D = \{0, 1\}$, then the only maximal C-clone $\mathrm{Pol}_D \mathrm{R}^{(1)}_{(0)}$ on this set is the maximal clone $\mathrm{Pol}_D \leq_2$ of monotone functions w.r.t. the linear order $0 \leq_2 1$.*

Any other maximal C-clone (that is on any finite domain D with $|D| > 2$) fails to be a maximal clone, hence it is properly contained in some maximal clone.

Another way to state this result is: the maximal clone of monotone Boolean functions is a C-clone, but no other maximal clone on this and larger finite domains is one.

3 Proof of the main theorem

3.1 Principle of proof

Since for a clone F the generated C-clone $\langle F \rangle_{\mathcal{C}}$ is again a clone, if F is maximal we have two possibilities; either $\langle F \rangle_{\mathcal{C}} = \mathrm{O}_D$ is the full clone, or $\langle F \rangle_{\mathcal{C}} \subset \mathrm{O}_D$ is a maximal clone and a maximal C-clone at the same time. This is so because $F \subseteq \langle F \rangle_{\mathcal{C}} \subset \mathrm{O}_D$ by maximality of F yields $F = \langle F \rangle_{\mathcal{C}}$, so F will be a C-clone as well. It has to be a maximal C-clone, because every other non-full C-clone $G \supseteq F$ would be a non-full clone above F and so coincide with F by maximality of F as a usual clone.

We are going to show that, apart from the case of the clone of monotone functions defined on the two-element domain, none of the maximal clones will be a maximal C-clone, that is all maximal clones F will be mapped to the full clone $\langle F \rangle_{\mathcal{C}} = \mathrm{O}_D$ by the C-clone closure. Conversely, this means that almost always all of the maximal C-clones will lie properly below some maximal clone, because every clone on a finite set D either equals O_D or is contained in some maximal clone (see e.g. [PK79, Hauptsatz 3.1.5, p. 80; Vollständigkeitskriterium 5.1.6, p. 123] or [Sze86, Proposition 1.15, p. 27]).

To deduce our main result we will adhere to the following strategy. For every maximal clone F we will try to prove that $\mathrm{Inv}_D F$ does not contain any non-trivial clausal relations, i.e. $C\,\mathrm{Inv}_D F = C\,\mathrm{R}_D \cap \mathrm{Inv}_D F \overset{\text{Lem. 2}}{=} \{D^{p+q} \mid p, q \in \mathbb{N}_+\}$, whence we get $\langle F \rangle_{\mathcal{C}} = \mathrm{Pol}_D C\,\mathrm{Inv}_D F = \mathrm{O}_D$.

Besides the mentioned special case we will always succeed in doing so. We will achieve our goal by exhibiting for each non-trivial clausal relation $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ a function $f \in F$ that does not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. It turns out that we can always find a function of arity at most three, and in some cases one can even find one function $f \in F$ that does not preserve any non-trivial clausal relation $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, i.e. that does not depend on

$\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. In this respect we may always assume $|D| \geq 2$, i.e. $n-1>0$, in the proofs as there only exist trivial clausal relations on a singleton domain D.

It is easy to see that every maximal clone can be written in the form $\mathrm{Pol}_D\, \varrho$ for some non-trivial relation $\varrho \in \mathrm{R}_D$. We are going to benefit from the known description of all maximal clones by Ivo G. Rosenberg ([Ros65, Ros70]) in terms of the occurring relations ϱ that lists six different types of relations. In what follows we are going to develop a different proof for each type of relations occurring in Rosenberg's theorem presented below. To state this theorem, we need some auxiliary definitions:

For $s \in \mathrm{Sym}(D)$ we denote by $\operatorname{graph} s := \{(x, s(x)) \mid x \in D\}$ the *graph* of the permutation s. A permutation s is *prime* if it has only cycles of the same length p, for some prime p. Note that, in particular, such an s cannot have cycles of length one, so it has no fixed points.

For a prime p a group $\mathbf{G} = \langle G; +, -, o\rangle$ is called an *elementary* ABEL*ian* p*-group*, if $\mathbf{G}$ is a commutative group and satisfies the law $x + \cdots + x \approx o$ where the variable symbol x occurs p times in the sum. The latter means that every element in $G \setminus \{o\}$ has order p. If G is finite, then, by the fundamental theorem of finitely generated ABELian groups, $\mathbf{G}$ must be isomorphic to a finite direct power of the cyclic group of order p, so in particular the cardinality of G must be a power of p.

For such a group $\mathbf{G}$, the *affine relation* $\varrho_{\mathbf{G}}$ is defined as

$$\varrho_{\mathbf{G}} := \left\{ (x, y, u, v) \in G^4 \;\middle|\; x + y = u + v \right\}.$$

For $m \in \mathbb{N}_+$, an m-ary relation ϱ is *totally symmetric* if, for every permutation $\pi \in \mathrm{Sym}(m)$, it contains with any tuple $\mathbf{a} = (a_1, \ldots, a_m) \in D^m$ also the permuted tuple $\mathbf{a} \circ \pi = \left(a_{\pi(1)}, \ldots, a_{\pi(m)}\right)$. It is *totally reflexive* if we have $(a_1, \ldots, a_m) \in \varrho$ for all $a_1, \ldots a_m \in D$ satisfying $|\{a_1, \ldots, a_m\}| \leq m-1$.

An element $c \in D$ is a *central element* of ϱ if the relation ϱ contains any tuple $(a_1, \ldots, a_m) \in D^m$ where $c \in \{a_1, \ldots, a_m\}$.

A relation ϱ is *central* if it is totally reflexive, totally symmetric, contains a central element and is not a diagonal relation.

For $h \in \mathbb{N}_{\geq 3}$ let $\iota_h := \left\{ (a_1, \ldots, a_h) \in \{0, \ldots, h-1\}^h \;\middle|\; h > |\{a_1, \ldots, a_h\}| \right\}$. An h-ary relation $\varrho \in \mathrm{R}_D^{(h)}$ is *h-regular*, if there exists an $m \geq 1$ and a surjection $\varphi\colon D \longrightarrow \{0, \ldots, h-1\}^m$ such that

$$\varrho = \left\{ (a_1, \ldots, a_h) \in D^h \;\middle|\; \forall j \in \{1, \ldots, m\} : \left((\varphi(a_1))_j, \ldots, (\varphi(a_h))_j\right) \in \iota_h \right\}.$$

Theorem 7 ([Ros65, Ros70]). *A clone $F \subseteq \mathrm{O}_D$ is maximal if and only if it is of the form $\mathrm{Pol}_D\, \varrho$, where ϱ is a relation belonging to one of the following classes:*

1. *The set of all partial orders with least and greatest element.*

2. *The set of all graphs of prime permutations.*

3. *The set of all non-trivial*[1] *equivalence relations.*

4. *The set of all affine relations w.r.t. some elementary* ABEL*ian p-group on D for some prime p.*

5. *The set of all central relations of arity h $(1 \leq h < |D|)$.*

6. *The set of all h-regular relations $(3 \leq h \leq |D|)$.*

Note that case 4 only occurs if the cardinality of D is a power of a prime p, because the carrier set of the defining elementary ABELian p-group has to be D.

The proof of the main theorem 6 will be spread over several subsections containing separate results for the cases listed in Theorem 7.

In the proofs we will see that in some of the cases we will not even need all the special properties of the relation ϱ given in Theorem 7. For instance, we will not need the primality of the permutations, the non-triviality of the equivalence relations and the full concept of h-regularity of an h-regular relation.

3.2 Bounded orders

Functions in $\mathrm{Pol}_D \preceq$, preserving an order relation $\preceq$ on D, are commonly called monotone functions (w.r.t. $\preceq$). This is a special case of the slightly more general concept of an order preserving function between two arbitrary posets, sometimes called *order homomorphism*, where the function is not necessarily between the finite power of one poset and itself. For notational reasons, it will be easier to formulate the following lemma in this more general setting, even though we do not need the full generality of the statement for the proof of Proposition 9.

Order relations with least and greatest element, the first category of relations mentioned in Theorem 7, are also called *bounded*.

Lemma 8. *If $(P; \leq)$ and $(Q; \sqsubseteq)$ are posets, $p_1, p_2 \in P$ are incomparable elements and $(Q; \sqsubseteq)$ is bounded with top element $\top$ and bottom element $\bot$, then for all values $q_1, q_2 \in Q$ the partial definition $f(p_i) := q_i$ $(i \in \{1, 2\})$ can be extended to an order-homomorphism.*

Proof: We define the homomorphism as follows:

$$\begin{array}{rccc} f\colon & P & \longrightarrow & Q \\ & x & \longmapsto & \begin{cases} \bot & \text{if } x < p_1 \text{ or } x < p_2, \\ q_i & \text{if } x = p_i \ (i \in \{1,2\}), \\ \top & \text{else.} \end{cases} \end{array}$$

To show that f is indeed a homomorphism, we consider any pair $x \leq y$. If $y \not\leq p_1$ and $y \not\leq p_2$, then $f(x) \sqsubseteq \top = f(y)$ and we are done. If $y = p_i$ for some $i \in \{1, 2\}$,

[1]Here *non-trivial* means indeed $\mathrm{Eq}(D) \setminus \{\Delta_D, \nabla_D\}$, in contrast to what was agreed at the beginning of Section 2.

then $f(x) \in \{\bot, q_i\}$ and thus $f(x) \sqsubseteq q_i = f(y)$. The remaining case is $x \leq y < p_i$ for some $i \in \{1,2\}$, whence $f(x) = \bot = f(y)$. $\square$

We will apply this lemma in the proof of the following result to the special case when the order $(P; \leq)$ is a direct power of a bounded order $(Q; \sqsubseteq)$.

Proposition 9. *Let $(D; \preceq)$ be a bounded order on D with greatest element $\top$ and least element $\bot$. If $|D| \neq 2$, then there are no non-trivial clausal relations in the clone $\mathrm{Inv}_D\,\mathrm{Pol}_D \preceq$. If $|D| = 2$, then the only non-trivial clausal relation contained in this clone is $\mathrm{R}^{(1)}_{(0)} = \geq_D$.*

Proof: We only consider the case $|D| \geq 2$. First we note that always one of the conditions $0 \preceq n-1$ or $0 \succeq n-1$ will fail as otherwise we had $0 \preceq n-1 \preceq 0$, implying $0 = n-1$ or $|D| = 1$. It is clear that $\mathrm{Pol}_D \preceq = \mathrm{Pol}_D \succeq$, and the dual order $(D; \succeq)$ is again a bounded order on D. As we wish to prove a property of the clone $\mathrm{Inv}_D\,\mathrm{Pol}_D \preceq = \mathrm{Inv}_D\,\mathrm{Pol}_D \succeq$, we can consider the order $\preceq$ or its dual as we please. So w.l.o.g. we will consider the case that $n-1 \not\preceq 0$. We will now consider an arbitrary non-trivial clausal relation $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ with $\mathbf{a} \in (D \setminus \{0\})^p$ and $\mathbf{b} \in (D \setminus \{n-1\})^q$. Using a case distinction (not all cases will be disjoint), we will show that we can always find a function $f \in \mathrm{Pol}_D \preceq$ of arity $\ell \leq 2$ that does not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. Hence, we will prove $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \notin \mathrm{Inv}_D\,\mathrm{Pol}_D \preceq$.

$\exists i \in \{1, \ldots, p\} : a_i \neq \top$ By Lemma 8 the definition

$$f(n-1, a_i) := 0 \qquad\qquad f(0, \top) := n-1$$

can be extended to an order preserving function w.r.t. $\preceq$ since $n-1 \not\preceq 0$ and $\top \not\preceq a_i$ as $\top \neq a_i$. The function f will not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ as witnessed by the equation $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$ where the tuples r_1 and r_2 are given as $r_1 := (n-1, \ldots, n-1, 0, \ldots, 0)$ and $r_2 := (a_i, \ldots, a_i, \top, \ldots, \top)$.

$\exists j \in \{1, \ldots, q\} : b_j \neq \bot$ Dually to the preceding case, the definition

$$f(n-1, \bot) := 0 \qquad\qquad f(0, b_j) := n-1$$

can be extended to an order preserving function w.r.t. $\preceq$ since $n-1 \not\preceq 0$ and $b_j \not\preceq \bot$ as $\bot \neq b_j$. Again the function f will not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ as witnessed by $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$ for $r_1 := (n-1, \ldots, n-1, 0, \ldots, 0)$ and $r_2 := (\bot, \ldots, \bot, b_j, \ldots, b_j)$.

$n-1 \neq \top$ By Lemma 8 the definition

$$f(n-1, n-1) := 0 \qquad\qquad f(0, \top) := n-1$$

can be extended to an order preserving function w.r.t. $\preceq$ since $n-1 \not\preceq 0$ and $\top \not\preceq n-1$ as $\top \neq n-1$. Putting $r_1 := (n-1, \ldots, n-1, 0, \ldots, 0)$ and $r_2 := (n-1, \ldots, n-1, \top, \ldots, \top)$, the function f cannot preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ since $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$.

$\underline{0 \neq \perp}$ Dually as before the definition

$$f(n-1, \perp) := 0 \qquad\qquad f(0,0) := n-1$$

can be extended to an order preserving function w.r.t. $\preceq$ since $n-1 \not\preceq 0$ and $0 \not\preceq \perp$ as $\perp \neq 0$. Again the function f will not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, witnessed by $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$ for $r_1 := (n-1, \ldots, n-1, 0, \ldots, 0)$ and $r_2 := (\perp, \ldots, \perp, 0, \ldots, 0)$.

If none of the presented cases occurs, then we must have $a_i = \top = n-1$ for all $i \in \{1, \ldots, p\}$ and $b_j = \perp = 0$ for all $j \in \{1, \ldots, q\}$.

If $|D| > 2$ then there exists another element $c \in D \setminus \{0, n-1\}$. It will satisfy $b_j = 0 < c < n-1 = a_i$ for all $i \in \{1, \ldots, p\}$ and all $j \in \{1, \ldots, q\}$. Then we can use the unary constant function f with value c, that will trivially preserve $\preceq$ to show that $f \not\triangleright \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. Namely, we have $f \circ (n-1, \ldots, n-1, 0, \ldots, 0) = (c, \ldots, c) \notin \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ although the argument tuple belongs to the relation $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$.

Otherwise, we have $n = 2$, i.e. $D = \{0, 1\}$. Because of $\perp = 0$ and $\top = n-1 = 1$, the order $\preceq$ coincides with $\leq_D$. We will consider three more cases:

$\underline{p > 1}$ We define $f \in \mathrm{O}^{(2)}_D$ by $f(\mathbf{x}) := 1$ if $\mathbf{x} = (1,1)$, and $f(\mathbf{x}) := 0$ otherwise. Clearly, $f \in \mathrm{Pol}_D \leq_D = \mathrm{Pol}_D \preceq$, but f does not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. Indeed, the equation $f \circ (r_1, r_2) = (0, 0, \ldots, 0, 1, \ldots, 1)$, where $r_1 := (1, 0, \ldots, 0, 1, \ldots, 1)$ and $r_2 := (0, 1, \ldots, 1, 1, \ldots, 1)$, witnesses $f \not\triangleright \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$.

$\underline{q > 1}$ Dually to the preceding case we define $f \in \mathrm{O}^{(2)}_D$ via $f(\mathbf{x}) := 0$ if $\mathbf{x} = (0,0)$, and $f(\mathbf{x}) := 1$ otherwise. Again, the function f preserves $\preceq = \leq_D$, but f does not preserve $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, as witnessed by $f(r_1, r_2) = (0, \ldots, 0, 1, 1, \ldots, 1)$ for $r_1 := (0, \ldots, 0, 0, 1, \ldots, 1)$ and $r_2 := (0, \ldots, 0, 1, 0, \ldots, 0)$.

$\underline{p = q = 1}$ The only non-trivial binary clausal relation on $D = \{0, 1\}$ is

$$\mathrm{R}^{(1)}_{(0)} = \{(1,1), (1,0), (0,0)\} = \geq_D = \succeq = (\preceq)^{-1} \in \mathrm{Inv}_D \, \mathrm{Pol}_D \preceq. \qquad \square$$

3.3 Non-trivial congruences

If $\theta \in \mathrm{Eq}(D)$ is an equivalence relation on D and $f \in \mathrm{Pol}_D \, \theta$, then θ is called a *congruence relation* of the algebra $\langle D; f \rangle$. This motivates the title of this subsection.

The following proposition shows that, to prove our theorem, we do not need the assumption that the equivalence relation is non-trivial as stated in Theorem 7.

Proposition 10. *For every equivalence relation $\theta \in \mathrm{Eq}(D)$, the clone $\mathrm{Inv}_D \, \mathrm{Pol}_D \, \theta$ does not contain any non-trivial clausal relations.*

Proof: We are going to exhibit a function $f \in \mathrm{Pol}_D^{(1)}\theta$ that does not preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. For this we can assume $n-1>0$. The definition of f will depend on whether $(0,n-1)\in\theta$ or not. In both cases f will satisfy $f(0)=n-1$ and $f(n-1)=0$, which ensures that f cannot preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$.

If $(0,n-1)\notin\theta$, then we define $f\in\mathrm{O}_D^{(1)}$ by $f(x):=n-1$ if $x\in[0]_\theta$, and we put $f(x):=0$ otherwise. Such a function f will preserve θ, because it is constant on all blocks of θ. More explicitly, for every tuple $(x,y)\in\theta$, we have $[x]_\theta=[y]_\theta$. If their common θ-class equals $[0]_\theta$, then $f\circ(x,y)=(n-1,n-1)\in\theta$, otherwise, $f\circ(x,y)=(0,0)\in\theta$.

If, otherwise $(0,n-1)\in\theta$, then we define $f\in\mathrm{O}_D^{(1)}$ as $f(x):=n-1$ if $x=0$, and $f(x):=0$ otherwise. To see that $f\rhd\theta$, we consider any tuple $(x,y)\in\theta$. If $x=y$, also $f(x)=f(y)$, and we are done by reflexivity. If $0\notin\{x,y\}$, then $f(x)=f(y)=0$, and $(f(x),f(y))\in\theta$ for the same reason. The case that remains is that $0\in\{x,y\}$ and $x\neq y$. By definition of f we obtain $(f(x),f(y))$ belongs to $\{(0,n-1),(n-1,0)\}$ being a subset of θ. □

3.4 Selfdual functions

Every function $f\in\mathrm{Pol}_D\,\mathrm{graph}\,s$, where $\mathrm{graph}\,s$ is the graph of a permutation $s\in\mathrm{Sym}(D)$ (cf. Subsection 3.1), is called an *s-selfdual* function. We will give a simple characterisation of such functions in Lemma 11 below, and at the same time we will provide a construction for s-selfdual functions f. The crucial point in both, the characterisation and the construction, is how values of f propagate along the orbits of the canonical action of the cyclic permutation group $\langle s\rangle_{\mathrm{Sym}(D)}$ on $D^{\mathrm{ar}(f)}$. We will briefly recall the involved notions:

For any set I, every permutation group $U\leq\mathrm{Sym}(D)$ naturally acts on powers D^I by composition

$$\begin{array}{rccc}\circ\colon & U\times D^I & \longrightarrow & D^I\\ & (s,\mathbf{x}) & \longmapsto & s\circ\mathbf{x}=(s(x(i)))_{i\in I}\,.\end{array}$$

Especially, this holds for finite powers, i.e. we can apply permutations to tuples. As usual the orbit of an element $\mathbf{x}\in D^I$ is defined as $[\mathbf{x}]_U:=\{s\circ\mathbf{x}\mid s\in U\}$, and the set of all orbits $\left\{[\mathbf{x}]_U\;\middle|\;\mathbf{x}\in D^I\right\}$ partitions D^I.

Lemma 11. *Let $s\in\mathrm{Sym}(D)$ and $l\in\mathbb{N}_+$. For a function $f\in\mathrm{O}_D^{(l)}$ we have*

$$\begin{aligned}f\in\mathrm{Pol}_D^{(l)}\,\mathrm{graph}\,s &\iff \forall\mathbf{x}\in D^l\colon s(f(\mathbf{x}))=f(s\circ\mathbf{x}).\\ &\iff \forall k\in\mathbb{N}\;\forall\mathbf{x}\in D^l\colon f\left(s^k\circ\mathbf{x}\right)=s^k\left(f\left(\mathbf{x}\right)\right).\end{aligned}$$

If $f\in\mathrm{Pol}_D^{(l)}\,\mathrm{graph}\,s$, then for all tuples $\mathbf{x}\in D^l$ the value $f(\mathbf{x})$ determines the values $\left\{f(s^k\circ\mathbf{x})\;\middle|\;k\in\mathbb{N}\right\}$, i.e. the values of f on the whole orbit $[\mathbf{x}]_{\langle s\rangle_{\mathrm{Sym}(D)}}$ of the canonical action of $\langle s\rangle_{\mathrm{Sym}(D)}$ on D^l.

Conversely, let $T \subseteq D^l$ be a transversal, i.e. a system of representatives of the orbits of $\langle s\rangle_{\mathrm{Sym}(D)}$ on D^l, and let $f_0\colon T \longrightarrow D$ be any function. Then we define

$$f(\mathbf{x}) := f_0(\mathbf{x}), \qquad\qquad f\left(s^{k+1}\circ\mathbf{x}\right) := s^{k+1}\left(f\left(\mathbf{x}\right)\right) \ \textit{for } k \in \mathbb{N}$$

and all $\mathbf{x} \in T$. In this way a function $f\colon D^l \longrightarrow D$ is completely (well)-defined, and furthermore, $f \in \mathrm{Pol}_D^{(l)}\,\mathrm{graph}\,s$.

Proof: The first mentioned condition is necessary for $f \rhd \mathrm{graph}\,s$, because for any $\mathbf{x} \in D^l$, one has $(x_i, s(x_i)) \in \mathrm{graph}\,s$ for all $1 \leq i \leq l$. Thus, from $f \rhd \mathrm{graph}\,s$ we get $(f(\mathbf{x}), f(s\circ\mathbf{x})) \in \mathrm{graph}\,s$, or $s(f(\mathbf{x})) = f(s\circ\mathbf{x})$. Conversely, if such an equality holds, and $(x_i, y_i) \in \mathrm{graph}\,s$ for $1 \leq i \leq l$, then $y_i = s(x_i)$, and so one obtains $(y_1, \ldots, y_l) = s \circ (x_1, \ldots, x_l)$. Hence

$$\left(f\left(x_1,\ldots,x_l\right), f\left(y_1,\ldots,y_l\right)\right) = \left(f\left(x_1,\ldots,x_l\right), s\left(f\left(x_1,\ldots,x_l\right)\right)\right) \in \mathrm{graph}\,s.$$

The second equivalent condition follows from the one just shown by induction on $k \in \mathbb{N}$. It is equivalent since the first condition is exactly the special case $k = 1$.

For the second part, the function f is well-defined since the orbits of the canonical action of $\langle s\rangle_{\mathrm{Sym}(D)}$ on D^l form a partition of D^l. So every tuple $\mathbf{y} \in D^l$ occurs in $[\mathbf{x}]_{\langle s\rangle_{\mathrm{Sym}(D)}}$ for exactly one $\mathbf{x} \in T$ as $\mathbf{y} = s^k \circ \mathbf{x}$. By definition we have $s\left(f\left(\mathbf{y}\right)\right) = s\left(f\left(s^k\circ\mathbf{x}\right)\right) = s\left(s^k\left(f\left(\mathbf{x}\right)\right)\right) = s^{k+1}\left(f\left(\mathbf{x}\right)\right) = f\left(s^{k+1}\circ\mathbf{x}\right) = f(s\circ\mathbf{y})$, whence f is s-selfdual. $\square$

Proposition 12. *For any permutation $s \in \mathrm{Sym}(D)$ the clone $\mathrm{Inv}_D\,\mathrm{Pol}_D\,\mathrm{graph}\,s$ corresponding to the clone of s-selfdual functions does not contain any non-trivial clausal relations.*

Proof: We are going to exhibit a function $f \in \mathrm{Pol}_D^{(3)}\,\mathrm{graph}\,s$ that does not preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. For this we can assume that $0 < n-1$. By Lemma 11 the partial definition

$$f(n-1, 0, n-1) := 0, \qquad\qquad f(0,0,n-1) := n-1$$

can be extended to a function in $\mathrm{Pol}_D^{(3)}\,\mathrm{graph}\,s$, because the tuples $(n-1, 0, n-1)$ and $(0, 0, n-1)$ are in different orbits of the canonical action of $\langle s\rangle_{\mathrm{Sym}(D)}$ on D^l: because $0 \neq n-1$, the first two entries of the first tuple differ, whereas the first two entries of the second tuple coincide. Thus, the first tuple can never be obtained by applying a power of s to the second one. Consequently, we can choose a transversal of the orbits containing these two tuples, and then apply Lemma 11 to construct an s-selfdual function $f \in \mathrm{O}_D^{(3)}$. Obviously, f will not preserve any non-trivial $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ because $f \circ (r_1, r_2, r_3) = (0, \ldots, 0, n-1, \ldots, n-1) \notin \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ while the arguments $r_1 := (n-1, \ldots, n-1, 0, \ldots, 0)$, $r_2 := (0, \ldots, 0, 0, \ldots, 0)$ and $r_3 := (n-1, \ldots, n-1, n-1, \ldots, n-1)$ belong to $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. $\square$

3.5 Quasilinear functions

Functions $f \in \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$ preserving an affine relation $\varrho_{\mathbf{G}}$ w.r.t. an ABELian group $\mathbf{G} = \langle D; +, -, o\rangle$ (cf. Subsection 3.1) are called *quasilinear* (w.r.t. $\mathbf{G}$). It is easy to see that

$$\mathrm{Pol}_D\, \varrho_{\mathbf{G}} = \bigcup_{k \in \mathbb{N}_+} \left\{ f\colon D^k \longrightarrow D \;\middle|\; \forall \mathbf{x}, \mathbf{y} \in D^k\colon f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) - f(\mathbf{o}) \right\},$$

where $\mathbf{o}$ stands for the tuple $(o, \ldots, o) \in D^k$.

The following simple observation directly follows from the definition of an elementary ABELian p-group.

Observation 13. *Any elementary* ABEL*ian p-group $\mathbf{G} = \langle G; +, -, o\rangle$ (where p is a prime) can be turned into a vector space over the field $\mathsf{GF}(p) = \mathbb{Z}_p$ by defining the scalar multiplication as $\cdot\colon \mathbb{Z}_p \times G \longrightarrow G$ via $(a, g) \mapsto a \cdot g := \sum_{i=1}^{a} g$.*

The following lemma is an auxiliary to be used in the subsequent Proposition 15.

Lemma 14. *Let $\mathbf{G}$ and $\mathbf{H}$ be elementary* ABEL*ian p-groups and $v_1, \ldots, v_t \in G$ with $t \in \mathbb{N}_+$ be linearly independent elements in the associated vector space over $\mathbb{Z}_p$. For any choice of $h_1, \ldots, h_t \in H$, there is a mapping $f\colon G \longrightarrow H$ satisfying*

$$\begin{aligned} f(x+y) &= f(x) + f(y) - f(o_{\mathbf{G}}) && \text{for all } x, y \in G,\\ f(v_i) &= h_i && \text{for all } 1 \le i \le t. \end{aligned}$$

Proof: Since $v_1, \ldots, v_t$ are linearly independent, this list can be completed to a basis B of $\mathbf{G}$. Hence there exists a unique linear extension of the partial definition

$$\begin{aligned} f(v_i) &= h_i && \text{for } 1 \le i \le t,\\ f(b) &= o_{\mathbf{H}} && \text{for } b \in B \setminus \{v_1, \ldots, v_t\}. \end{aligned}$$

By linearity we have $f(o_{\mathbf{G}}) = o_{\mathbf{H}}$ and $f(x+y) = f(x) + f(y) = f(x) + f(y) - o_{\mathbf{H}}$ being equal to $f(x) + f(y) - f(o_{\mathbf{G}})$. □

Proposition 15. *For any prime number p and any elementary* ABEL*ian p-group $\mathbf{G} = \langle D; +, -, o\rangle$, the clone $\mathrm{Inv}_D\, \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$ does not contain any non-trivial clausal relations.*

Proof: We are going to find a function $f \in \mathrm{Pol}_D^{(2)}\, \varrho_{\mathbf{G}}$ that does not preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. As usual we suppose $n - 1 > 0$. By Lemma 14 it will suffice to partially define f on linearly independent vectors of the $\mathbb{Z}_p$-vector space associated with $\mathbf{G}^2$ to ensure $f \in \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$. We will distinguish two cases w.r.t. the relation of o and 0.

$\underline{o = 0}$ We define

$$f(n-1, n-1) := 0, \qquad\qquad f(o, n-1) := n-1.$$

Obviously, the tuples $(n-1, n-1)$ and $(o, n-1)$ are linearly independent since $n-1 \neq 0 = o$. This is enough information to define a function as desired. It will not preserve any non-trivial $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, witnessed by the equation $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$ for $r_1 := (n-1, \ldots, n-1, o, \ldots, o)$ and $r_2 := (n-1, \ldots, n-1, n-1, \ldots, n-1)$.

$\underline{o \neq 0}$ In this case we let

$$f(0, o) := 0, \qquad\qquad f(0, 0) := n-1.$$

Again the tuples $(o, 0)$ and $(0, 0)$ are linearly independent since $0 \neq o$. This suffices to define a function $f \in \mathrm{Pol}_D\, \varrho_{\mathbf{G}}$ as desired. It will not preserve any non-trivial $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, witnessed by $f \circ (r_1, r_2) = (0, \ldots, 0, n-1, \ldots, n-1)$ where $r_1 := (0, \ldots, 0, 0, \ldots, 0)$ and $r_2 := (o, \ldots, o, 0, \ldots, 0)$. □

3.6 Functions preserving central and h-regular relations

The remaining categories of relations listed in Theorem 7 are central and h-regular relations. They share the common property of total symmetry and total reflexivity, and this is, in fact, almost all we need to achieve our goal. The existence of a central element for central relations is only used for binary relations.

Lemma 16. *For $m \in \mathbb{N}_{\geq 3}$ and a totally reflexive m-ary relation $\varrho \in \mathrm{R}^{(m)}_D$, the clone $\mathrm{Inv}_D\, \mathrm{Pol}_D\, \varrho$ does not contain any non-trivial clausal relations.*

Proof: We are going to exhibit a function $f \in \mathrm{Pol}^{(1)}_D\, \varrho$ that does not preserve any non-trivial clausal relation $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. Let us define $f \in \mathrm{O}^{(1)}_D$ by $f(x) := n-1$ if $x = 0$ and $f(x) = 0$ else. Obviously, we have $f \not\triangleright \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ since $(n-1, \ldots, n-1, 0, \ldots, 0) \in \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$, but $f \circ (n-1, \ldots, n-1, 0, \ldots, 0) = (0, \ldots, 0, n-1, \ldots, n-1) \notin \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$ for $n-1 > 0$, which may be assumed without any loss of generality.

Besides, one can show $f \triangleright \varrho$; consider an arbitrary tuple $\mathbf{x} = (x_1, \ldots, x_m) \in \varrho$ and let $\mathbf{y} := f \circ \mathbf{x}$. By definition of f we have $\mathrm{im}\, f = \{0, n-1\}$, so the same holds for $\mathrm{im}\, \mathbf{y} = \mathrm{im}\, f \circ \mathbf{x} \subseteq \mathrm{im}\, f = \{0, n-1\}$, i.e. $\mathbf{y} \in \{0, n-1\}^m$. Since $m \geq 3$, the tuple $\mathbf{y}$ has at least two identical entries, so $\mathbf{y} \in \varrho$ because ϱ is totally reflexive. □

Lemma 17. *For every unary relation $\varrho \in \mathrm{R}^{(1)}_D$, the clone $\mathrm{Inv}_D\, \mathrm{Pol}_D\, \varrho$ does not contain any non-trivial clausal relations.*

Proof: We are going to construct a function $f \in \mathrm{Pol}_D^{(3)} \varrho$ that will not preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$. Assuming $n-1>0$ we define

$$\begin{array}{rccc} f\colon & D^3 & \longrightarrow & D \\ & \mathbf{x}=(x_1,x_2,x_3) & \longmapsto & \begin{cases} n-1 & \text{if } \mathbf{x}=(0,0,n-1), \\ 0 & \text{if } \mathbf{x}=(n-1,0,n-1), \\ x_1 & \text{otherwise.} \end{cases} \end{array}$$

It fails to preserve any non-trivial $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$ as the tuples $r_1 := (n-1,\dots,n-1,0,\dots,0)$, $r_2 := (0,\dots,0,0,\dots,0)$ and $r_3 := (n-1,\dots,n-1,n-1,\dots,n-1)$ in the equation $f \circ (r_1,r_2,r_3) = (0,\dots,0,n-1,\dots,n-1)$ belong to $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$, which is not true for the resulting tuple. Since f is conservative, i.e. its values are always among the input arguments, it preserves any unary relation ϱ. $\square$

Lemma 18. *For every binary central relation $\varrho \in \mathrm{R}_D^{(2)}$, the clone $\mathrm{Inv}_D \mathrm{Pol}_D \varrho$ does not contain any non-trivial clausal relations.*

Proof: A binary central relation ϱ is a reflexive, symmetric relation on D, having a central element $c \in D$ and being different from Δ_D and $\nabla_D = D \times D$. This requires D to have at least three elements.

The function $f \in \mathrm{O}_D^{(1)}$ defined by $f(x) := n-1$ if $x=0$, $f(x) := 0$ if $x=n-1$, and $f(x) := c$ otherwise, does not preserve any non-trivial clausal relation $\mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$, witnessed by $f \circ (n-1,\dots,n-1,0,\dots,0) = (0,\dots,0,n-1,\dots,n-1) \notin \mathrm{R}_{\mathbf{b}}^{\mathbf{a}}$.

It remains to show that f preserves ϱ. If $(x,y) \in \varrho$ and $x=y$, then we have $f(x)=f(y)$, and $(f(x),f(y)) \in \varrho$ by reflexivity. Now let us consider that case that $x \neq y$. If $\{x,y\} \setminus \{0,n-1\} \neq \emptyset$, then $c \in \{f(x),f(y)\}$, so $(f(x),f(y)) \in \varrho$ since c is a central element for ϱ. Otherwise, we have $(x,y) \in \{(0,n-1),(n-1,0)\}$, and, since $(x,y) \in \varrho$, symmetry of ϱ implies $\{(0,n-1),(n-1,0)\} \subseteq \varrho$. Therefore, $(f(x),f(y)) \in \{(n-1,0),(0,n-1)\} \subseteq \varrho$, finishing the argument for $f \rhd \varrho$ and hence completing the proof. $\square$

Corollary 19. *For every central relation $\varrho \in \mathrm{R}_D^{(m)}$ of arity $m \in \mathbb{N}_+$ and every h-regular relation $\sigma \in \mathrm{R}_D^{(h)}$ ($h \geq 3$), the clones $\mathrm{Inv}_D \mathrm{Pol}_D \varrho$ and $\mathrm{Inv}_D \mathrm{Pol}_D \sigma$ do not contain any non-trivial clausal relations.*

Proof: By definition, every central relation is totally reflexive and totally symmetric. It is not hard to show that these properties also follow from the definition of an h-regular relation. So for at least ternary relations the claim will follow from Lemma 16. For unary ones, it is a consequence of Lemma 17, and for binary central relations it is contained in Lemma 18. $\square$

4 Concluding remarks

In this paper we investigated the relationship of maximal C-clones and maximal clones. We benefited from the known description of all maximal clones by Ivo

G. Rosenberg, and we showed that, apart from the case of the clone of monotone functions defined on the two-element domain, none of the maximal clones is a maximal C-clone.

This means that, opposed to conjectures coming from the BOOLEan case, the connection between maximal clones and maximal C-clones is indeed rather loose. As a consequence of Theorem 6, with the one mentioned exception, every C-clone is properly contained in some maximal clone. A natural question for further research would be to characterise which C-clones lie below which types of maximal clones.

With regard to this task we present the following simple lemmas, which settle the cases of s-selfdual and quasilinear functions, and at least ternary central and h-regular relations.

Lemma 20. *If $s \in \mathrm{Sym}(D)$ is a permutation without fixed points, then for all $p, q \in \mathbb{N}_+$ and $\mathrm{R}^{\mathbf{a}}_{\mathbf{b}} \in \mathcal{R}^p_q$, we have $c_{a_1} \in (\mathrm{Pol}_D \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}) \setminus (\mathrm{Pol}_D \operatorname{graph} s)$ where c_{a_1} is the unary constant with value a_1. Thus, in particular no maximal C-clone is a subset of $\mathrm{Pol}_D \operatorname{graph} s$.*

Since prime permutations cannot have fixed points, this result applies to maximal clones in the second case of Theorem 7.

Proof: It is clear that $c_{a_1} \in \mathrm{Pol}_D \mathrm{R}^{\mathbf{a}}_{\mathbf{b}}$. Moreover, it is evident from the definition of preservation that a function $f \in \mathrm{O}^{(\ell)}_D$ belongs to $\mathrm{Pol}_D \operatorname{graph} s$ if and only if $s(f(\mathbf{x})) = f(s \circ \mathbf{x})$ holds for all $\mathbf{x} \in D^\ell$ ($\ell \in \mathbb{N}_+$). For the unary operation c_{a_1} this condition becomes $s(a_1) = s(c_{a_1}(x)) = c_{a_1}(s(x)) = a_1$ for $x \in D$, i.e. that a_1 is fixed by s. This is false by assumption. $\square$

Lemma 21. *For a finite set D, a prime p, an elementary ABELian p-group $\mathbf{G}$ on D and any pair of elements $a \in D \setminus \{0\}$ and $b \in D \setminus \{n-1\}$, the inclusion $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \varrho_{\mathbf{G}}$ always fails.*

Proof: If $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)} \subseteq \mathrm{Pol}_D \varrho_{\mathbf{G}}$ for some elementary ABELian p-group $\mathbf{G}$ with neutral element o, then for all $k \in \mathbb{N}_+$ and every $f \in \mathrm{Pol}^{(k)}_D \mathrm{R}^{(a)}_{(b)}$, the k-ary function $f - f(o, \dots, o)$ were a linear function w.r.t. the $\mathsf{GF}(p)$-vector spaces on D^k and D associated with $\mathbf{G}$. This implied that the set

$$\mathrm{Ker}(f - f(o, \dots, o)) = \left\{ \mathbf{x} \in D^k \,\middle|\, f(\mathbf{x}) - f(o, \dots, o) = o \right\} = f^{-1}[\{f(o, \dots, o)\}]$$

were a subspace of D^k and hence isomorphic to $(\mathsf{GF}(p))^t$ for some $0 \le t \le d \cdot k$ where $n = p^d$. Thus the cardinality of this kernel would necessarily be a power of p. Using a case distinction we will exhibit below functions in $\mathrm{Pol}_D \mathrm{R}^{(a)}_{(b)}$, where this fails to be the case.

If $a < n-1$, we put $v_0 := a$ and $v_1 := a+1$; for $a = n-1$ and $b > 0$, we define $v_0 := b$ and $v_1 := b-1$. In both cases, we have $n \ge 3$ due to $0 < a < n-1$ or $0 < b < n-1$. If $p > 2$, let $c := 2$, otherwise for $p = 2$ we put $c := 3$. Since

$n \geq 3$, we may choose a subset $A \subseteq D$ such that $|A| = c$ and $o \in A$. Define $f\colon D \longrightarrow D$ by $f(x) := v_0$ if $x \in A$ and $f(x) := v_1$ otherwise. We observe that $\operatorname{im} f = \{a, a+1\} \subseteq \{a, \ldots, n-1\}$ or $\operatorname{im} f = \{b, b-1\} \subseteq \{0, \ldots, b\}$, thus $f \rhd \mathrm{R}^{(a)}_{(b)}$. Moreover, we have $f^{-1}\left[\{f\left(o\right)\}\right] = A$, and $|A| = c$, which fails to be a power of p.

The remaining case is $a = n-1$ and $b = 0$, i.e. the relation $\mathrm{R}^{(n-1)}_{(0)}$, being equal to $(\{n-1\} \times D) \cup (D \times \{0\})$. For $2 < p \leq n$ we put $c := 2$, for $p = 2 < n$ we have $n \geq p^2 = 4$ and put $c := 3$. If $o \neq 0$, we may choose $A \subseteq D \setminus \{0\}$ such that $|A| = c$ and $\{o, n-1\} \subseteq A$. We define $f \in \mathrm{O}^{(1)}_D$ by $f\left(x\right) := n-1$ if $x \in A$ and $f\left(x\right) = 0$ else. Since $n-1 \in A$ and $0 \notin A$, we have $f\left(0\right) = 0$ and $f\left(n-1\right) = n-1$, and so $f \in \operatorname{Pol}_D \mathrm{R}^{(n-1)}_{(0)}$. Otherwise, if $o = 0$, we may choose $A \subseteq D \setminus \{n-1\}$ such that $|A| = c$ and $o \in A$. We define $f \in \mathrm{O}^{(1)}_A$ by $f\left(x\right) := 0$ if $x \in A$ and $f\left(x\right) := n-1$ else. Due to $0 \in A$ and $n-1 \notin A$, we can infer $f\left(0\right) = 0$ and $f\left(n-1\right) = n-1$, so $f \rhd \mathrm{R}^{(n-1)}_{(0)}$. In both cases it is $f^{-1}\left[\{f\left(o\right)\}\right] = A$, and $|A| = c$ is not a power of p.

The only remaining case is $p = 2 = n = |D|$ and the relation $\mathrm{R}^{(1)}_{(0)} = \geq_D$. On a two-element domain the only maximal C-clone is the clone $\operatorname{Pol}_D \mathrm{R}^{(1)}_{(0)}$ of monotone Boolean functions (see [Var10, Theorem 2.14]), which does not lie below that of quasilinear Boolean operations, L. This is witnessed, for instance, by $\{\min, \max\} \subseteq \left(\operatorname{Pol}_D \mathrm{R}^{(1)}_{(0)}\right) \setminus L$. □

Lemma 22. *For $m \in \mathbb{N}_{\geq 3}$ and a totally reflexive non-trivial relation $\varrho \in \mathrm{R}^{(m)}_D$, we have $\vee_D, \wedge_D \notin \operatorname{Pol}_D \varrho$, where $\vee_D, \wedge_D$ denote the binary maximum and minimum w.r.t. $\leq_D$, respectively.*

Proof: Since ϱ is non-trivial, we have $\varrho \subsetneq D^m$, and hence there is some tuple $\mathbf{x} := (x_1, \ldots, x_m) \in D^m \setminus \varrho$. By total reflexivity, the entries $x_1, \ldots, x_m$ are pairwise distinct. Choose the unique $i \in \{1, \ldots, m\}$ such that x_i is the least element among $x_1, \ldots, x_m$ w.r.t. $\leq_D$ and pick $j, \ell \in \{1, \ldots, m\}$ such that $|\{i, j, \ell\}| = 3$. This is possible due to $m \geq 3$. Define $\mathbf{y}, \mathbf{z} \in D^m$ by $y_k := x_i$ for $k = j$ and $y_k := x_k$ else; $z_k := x_k$ for $k = j$ and $z_k := x_i$ else. It follows $y_k \vee_D z_k = x_i \vee_D x_k = x_k$ for all $1 \leq k \leq m$, so $\vee_D \circ (\mathbf{y}, \mathbf{z}) = \mathbf{x} \notin \varrho$. This proves $\vee_D \not\rhd \varrho$ due to $y_j = x_i = y_i$, $z_i = x_i = z_\ell$ and total reflexivity of ϱ.

For $\wedge_D$ one chooses $1 \leq i \leq m$ such that x_i is largest among $x_1, \ldots, x_m$. □

Lemma 23. *For all $a, b \in D$ we have $\{\vee_D, \wedge_D\} \subseteq \operatorname{Pol}_D \mathrm{R}^{(a)}_{(b)}$.*

Proof: Let $\left(x_1, y_1\right), \left(x_2, y_2\right) \in \mathrm{R}^{(a)}_{(b)}$. If $x_1 \vee_D x_2 \geq a$ we are done. Otherwise, we have $x_1, x_2 \leq x_1 \vee_D x_2 < a$, so $y_1, y_2 \leq b$, whence $y_1 \vee_D y_2 \leq b$. Dually, we either have $y_1 \wedge_D y_2 \leq b$ or $y_1, y_2 \geq y_1 \wedge_D y_2 > b$, i.e. $x_1, x_2 \geq a$ and hence $x_1 \wedge_D x_2 \geq a$. □

Corollary 24. *If $h \in \mathbb{N}_{\geq 3}$ and $\varrho \subsetneq D^h$ is a central or an h-regular relation, then the clone $\operatorname{Pol}_D \varrho$ does not contain any maximal C-clone.*

Proof: By Theorem 5 maximal C-clones have the form $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}$ with $a, b \in D$. By definition, central relations are totally reflexive, and it is not hard to see that the same property also holds for h-regular relations. Using Lemmas 22 and 23, it is $\vee_D \in \left(\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)}\right) \setminus (\mathrm{Pol}_D \varrho)$, so $\mathrm{Pol}_D \mathrm{R}_{(b)}^{(a)} \nsubseteq \mathrm{Pol}_D \varrho$. $\square$

We leave as an open problem to describe the relationship between maximal clones and maximal C-clones more precisely, and in particular to examine the cases belonging to the other maximal clones mentioned in Theorem 7.

The authors think that solving such problems may be helpful in determining the exact cardinality κ of the lattice of all C-clones on finite sets D of cardinality at least three, which has been shown to satisfy $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$ in [Var10, Proposition 3.1].

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