Relational Structure Theory
A Localisation Theory for Algebraic Structures

DISSERTATION

zur Erlangung des akademischen Grades
Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Dipl.-Math. Mike Behrisch

geboren am 14. Juni 1984 in Dresden

Eingereicht am 26. April 2013

Die Dissertation wurde in der Zeit von Dezember 2010 bis
April 2013 am Institut für Algebra angefertigt.

Gutachter:	Prof. Dr. rer. nat. habil. Reinhard Pöschel (TU Dresden)
	Prof. Dr. László Zádori (Szegedi Tudományegyetem)
Tag der Disputation:	17. Juli 2013

Abstract

This thesis extends a localisation theory for non-indexed algebras, originally introduced in [Kea01] and further studied in [Beh09], from the case of finite structures to infinite ones. Based on a general GALOIS theory connecting finitary operations and relations, the presented *Relational Structure Theory* (RST) allows an exploration of general algebraic structures up to local term equivalence, i.e. up to equality of the associated clone of invariant relations.

With the development of RST, we address the open problem to derive an analogy of so-called *Tame Congruence Theory* (TCT, see e.g. [HM88]) that does not only consider the clone of (local) polynomial operations of an algebra but that of (local) term operations.

Within this thesis suitable concepts and conditions are identified that allow a generalisation of most results from preliminary work [Kea01, Beh09] to at least all those algebras generating locally finite varieties.

In this connection we address three principal aspects: first, determining appropriate subsets for localisation, second, the actual localisation procedure, in particular how given knowledge about a global structure can be transferred to its localisations, and third, the globalisation process, showing how information obtained by local reasoning can be combined to derive insights about the original structure.

With respect to the first task, we prove that, as in the finite case, images of idempotent unary operations belonging to the least locally closed clone containing all operations of a given 1-locally finite algebra are exactly the right choice, provided one wants to ensure compatibility of the canonical restriction process for relations w.r.t. the structure of the relational clone of invariant relations. Such images are called *neighbourhoods* of an algebra.

Concerning the second issue, via the underlying GALOIS connection, every algebra $\mathbf{A}$ is associated with a canonical relational structure $\underset{\sim}{\mathbf{A}}$, carrying precisely all invariant relations of the algebra. Exploiting the compatibility of the restriction process for relations established by the choice of neighbourhoods, the localisation of $\mathbf{A}$ to U is chosen as the operational counterpart of the canonically induced relational substructure $\underset{\sim}{\mathbf{A}}\restriction_U$. In this way a precise analogy to the localisation process known from [HM88] is obtained.

Regarding the third aspect, we adopt the notion of a *cover* from [Kea01, Beh09] as a collection of neighbourhoods that collectively have the same strength w.r.t. separation of pairs of invariant relations as the full algebra. We introduce the concepts of *jointly finite local retract* and *local retract* in order to characterise covers of arbitrary algebras by local decomposition equations, jointly finite local retracts and a product-local-retract construction. This constitutes the first main theorem of the thesis and generalises results known for the finite case that used products and retractions. In this way, globalisation can be achieved in a very strong sense: namely it is possible to completely reconstruct the original algebra up to local term equivalence from restricted algebras corresponding to neighbourhoods in a cover. It is also shown that for locally finite algebras generating a 1-locally finite variety, including in particular all algebras in locally finite varieties, one can get a nicer product-retract construction as in the finite case.

Furthermore, connections of RST to matrix products and applications of the theory regarding categorical equivalence of varieties are discussed. Especially, it is argued that under mild assumptions, e.g. for algebras in locally finite varieties, the globalisation construction via product and retract can be turned into an equivalence of varieties as categories sending an algebra to the matrix product corresponding to a fixed cover.

Moreover, we define an optimality concept called *q-non-refinability* for collections of neighbourhoods, where q is a quasiorder on the set of neighbourhoods of an algebra. Likewise, in this thesis many other notions from [Kea01, Beh09] are parametrised via quasiorders q. Further, we introduce *q-cover prebases* and *q-cover bases* and we show how to construct irredundant q-cover bases from them. The second main theorem of the thesis demonstrates that the latter are q-non-refinable covers and that q-non-refinable covers are unique up to q-isomorphism, subject to the existence of q-cover bases. Combining this with a constructive intrinsic characterisation of $\lessapprox$-cover bases for poly-Artinian algebras fulfilling the finite iteration property, e.g. poly-Artinian algebras in 1-locally finite varieties, we obtain a concrete description of non-refinable covers, which solves an open problem from [Beh09]. What is more, in this way we considerably extend the existence and uniqueness result concerning non-refinable covers proven in [Beh09] for finite algebras to all poly-Artinian algebras in 1-locally finite varieties.

Contents

1 Introduction

Divide et impera, Latin for "divide and rule", is a maxim describing a strategy how to stay in power by ruling in such a way that competing forces are kept small to prevent them from attacking the one implementing the strategy, usually the sovereign. Often attributed to the ancient Romans, notably Julius Cæsar, there is no actually documented evidence that the proverb has ever been used by him, although the strategy is indeed recognisable within his political action.

In this thesis we are going to pursue this motto in a figurative sense, much as it is used in computer science for the design of algorithms. There, a complicated problem is usually separated into a number of smaller, more manageable subtasks, which are then solved, often in parallel, and whose individual solutions are afterwards combined into an answer to the original question.

Here, we apply this approach to algebraic structures, i.e. pairs consisting of a set, together with a set of functions on it. Such structures are also called algebras, for brevity. For them we want to develop a localisation theory, called *Relational Structure Theory* (RST), comprising the following central components: the first question that needs to be answered is to determine the "right division into subtasks", that is for algebras, to find suitable local structures. Second, one needs to specify how the actual localisation process should work, that is, how a global problem can be partitioned into smaller subproblems, and how global preconditions can be transferred to the local level. Finally, the third ingredient is to understand how knowledge obtained from local reasoning can be combined into information about the global structure. This process is called globalisation.

The method suggested in this thesis profits much from using a relational view on algebraic structures. Namely, based on a natural compatibility notion between functions and relations, we can translate structures with functions into relational structures, i.e. sets together with a collection of relations. This translation process is realised by a so-called GALOIS connection, more details about this are presented in Chapter 2.

Application of this translation paradigm implies that algebras that are transformed into the same relational structure cannot be distinguished by our theory and therefore need to be considered equivalent. The precise term for this relationship is *local term equivalence* to be defined in the following chapter. Consequently, globalisation can only enable us to infer properties about algebras that are invariant w.r.t. this sort of equivalence.

Relaxing the equivalence notion further from local term equivalence to so-called *categorical equivalence*, exhibits a special feature of Relational Structure Theory. Starting with an appropriately chosen collection of local structures, called *cover*, globalisation can be achieved by a complete reconstruction of the original structure, up to categorical equivalence. Under a few mild assumptions, any algebra is categorically equivalent to some sort of product structure (called *matrix product*) of local structures belonging to some cover of the algebra. This means that by using our localisation approach, absolutely no information is lost that concerns the so-called *variety* generated by an algebra and can be expressed in terms of categories.

In the following paragraphs we try to give a non-technical overview of the historic origin of the particular approach we pursue and about the content of the thesis. Experts looking for a more detailed introduction are kindly referred to the introduction of Chapter 3 where the material covered in this thesis is presented in a mathematically more precise way, including references to the relevant sources.

Two important developments in universal algebra, both linked to the same person, have influenced the origination of Relational Structure Theory. During the 1980s mainly out of a combination of results by P. P. Pálfy, P. Pudlák and R. McKenzie a powerful structure theory for algebraic structures emerged. It is known under the name *Tame Congruence Theory* (TCT) and has appeared in a condensed form in the foundational monograph "The Structure of Finite Algebras" by R. McKenzie and D. Hobby in 1988. Since then, TCT has proven useful to solve many problems in universal algebra by studying algebras via examining their *congruences*. These are special binary relations, that can be seen as generalisations of the notion of normal subgroup or ideal to the world of general algebraic structures.

Via the compatibility notion mentioned above, this way of looking at algebras entails another equivalence notion in a natural way: TCT provides methods to study finite algebras and algebras lying in so-called *locally finite varieties* up to *polynomial equivalence*. The latter allows a less precise classification than term equivalence mentioned earlier. Motivated by the successful applicability of TCT, the universal algebraic research community has uttered the wish for an analogy of TCT that would not only provide insight up to polynomial equivalence, but up to the more specific term equivalence. The development of TCT, and in particular this open question related to it, constitutes the first significant influence on Relational Structure Theory.

The second one is again due to Ralph McKenzie. In the early 1990s he presented a characterisation of categorical equivalence of varieties, and thus especially for single algebras, using two concepts: one of them are finite matrix powers, the model for matrix products appearing in RST; the other is restriction of algebras to images of idempotent unary term operations, which is closely related to the localisation process of RST.

Finally, special credit belongs to Keith Kearnes and Ágnes Szendrei for compiling the previously mentioned mathematical ideas into a decomposition theory for algebras. Lecture notes of their talks given at a workshop on Tame Congruence Theory in 2001, although sketchy, can be considered as the foundation of RST. It was Kearnes and Szendrei's idea to replace the role of congruences in TCT by the so-called full *relational clone* of all *invariant* relations of an algebra and that of so-called *polynomials* by *terms*. In this way they base their refinement of TCT on the GALOIS theory between functions and relations mentioned earlier. They show also that many basic concepts from TCT still work in this more general context, setting in this way the localisation part of RST. Furthermore, they point out to use matrix products and McKenzie's characterisation of categorical equivalence for globalisation. What is more, they maintain that among the many ways to decompose a finite algebra into local pieces, there are some which are in a certain well-defined sense optimal. Such decompositions are given by so-called *non-refinable covers*. The lecture notes from the workshop state as one of the main theorems that decompositions of finite algebras via non-refinable covers exist and are essentially unique, more precisely, unique up to a canonically defined notion of isomorphism.

It was Reinhard Pöschel, who, interested by the use of relational clones at the basis of Kearnes and Szendrei's theory, brought these ideas to Dresden. There, the author of this thesis in 2009 undertook a first attempt to collect and fundamentally elaborate the basics of the new variant of TCT for term operations. The resulting diploma thesis, for simplicity, focussed mainly on finite algebras. Beginning towards the end of 2010, the theory has been systematically studied within a research project funded by Deutsche Forschungsgemeinschaft [German Research Foundation] with the aim to extend it both in depth and in applicability from finite to infinite structures. For a better recognisability, the author of this thesis and co-authors have invented the name "Relational Structure Theory" for it, which is motivated by the fundamental use of invariant relations in RST.

The present thesis forms part of the achievements produced within the mentioned project. It extends the scope of a huge fraction of the results found in the diploma thesis from 2009 from finite to infinite algebras, at least to those generating so-called locally finite varieties and often further than that. This applies in particular to the main characterisation of the cover property, describing how many localisations are sufficient for globalisation, and to the existence and uniqueness result for non-refinable covers. Furthermore, the text contains a precise constructive description of such covers, which has remained open in 2009, as well as algorithms helping to find them. As a special advance of the presentation of the theory, we wish to point out the invention of several new properties of algebras capturing behaviour of finite algebras that may also take place within infinite ones. Examples are the so-called finite iteration property (FIP), ARTINianness and (pre)cover bases.

However, this work needs to be conceived as just a segment of a more comprehensive picture that is now available and does not fully fit into the scope of this thesis. This concerns for one thing the description of non-refinable covers for certain sorts of algebras, that is only slightly broached in Example 3.6.1. Second, this statement holds even more for possible applications of RST re characterisation of categorical equivalence of finite algebras. At the end of Section 3.4 we hint at a theorem for which we claim to have a proof, but which has been omitted in order not to strain the limits of this thesis further. Moreover, we are convinced that the way how existence and uniqueness of non-refinable covers is proven collectively suggests that RST can be extended to a more general axiomatic localisation theory that is not limited to algebras and relational structures. We collect these and a few other aspects in the chapter on open problems.

Besides, in parallel to the writing of this thesis, a different generalisation of the original theory by Kearnes and Szendrei has been effected. In 2012 Friedrich Martin Schneider has presented an extension to the level of topological algebras, based on a GALOIS theory between continuous functions and closed relations. In this respect, one can rightfully argue that some of the results presented here follow as special cases in his theory when the algebra is equipped with the discrete topology. Yet, we basically have two arguments why it nevertheless makes sense to read this thesis, too.

Due to the more specific setting, many of our proofs become much simpler using entirely algebraic arguments, and we consider it overkill to develop them via topological reasoning. Moreover, some of our results also provide more detailed characterisations, partly because these details do not (or not easily) generalise to the topological setting, partly because some details have not been considered in Schneider's localisation theory. Second, and more importantly, the scope of applications is different. It is well-known that there are several issues with the definition of topological varieties re homomorphic images and factors, respectively. This is why, w.r.t. reconstruction, Schneider only considers the *topological quasivariety* generated by a topological algebra. Since a (generated) *topological* quasivariety is only closed w.r.t. the formation of *closed* substructures, even in the discrete case, it is generally different from the generated quasivariety, which needs to contain *all* substructures, let alone the generated variety. Thus, in view of applications w.r.t. categorical equivalence of algebras, our theory clearly is the method of choice.

Moreover, nothing seems to be wrong with considering a topological algebra after all simply as an algebra and to apply results from this thesis to obtain knowledge about it. The additional information about the topology, which is discarded in this way, can still be used to ensure, for instance, that the assumptions of our theorems are fulfilled. (To be fair the converse is also possible by considering an arbitrary algebra as topological algebra equipped with the discrete topology and applying Schneider's localisation theory to it.)

We finish the introduction with a few organisational remarks. It has been our intention to keep the presentation of RST almost self-contained. Thus, we have tried to manage the balancing act between introducing most of the universal algebraic notions that are required to understand the principal chapter on Relational Structure Theory, and the need to prevent the chapter dedicated to preliminaries from going out of hand. Whether we have been successful with this attempt, the gentle reader may estimate by himself.

Certainly, reading Chapter 2 cannot replace an introductory course in universal algebra, clone theory and a basic knowledge of set theory, order theory and category theory, which is necessary for Chapter 3. The preliminaries in Chapter 2 are written in such a way that readers who are generally used to the listed topics become acquainted with the notation we use and are reminded of most of the relevant notions and their interrelationships. For those who are utterly new to the mentioned areas and nonetheless endeavour to immerse themselves in Relational Structure Theory, we have given references to standard monographs for the respective fields, which should also serve for looking up unfamiliar terms or theorems.

The main part of the thesis is Chapter 3, which aims at an audience having a sufficiently strong background in general algebra measured by the fact that the content of Chapter 2 should mostly be easy to follow. The principal chapter is subdivided into eight sections whose content and order of reading are addressed in a separate introduction to Chapter 3.

Finally, the text finishes with a list of open questions and problems for further research in Chapter 4.

2 Preliminaries and Notation

In this chapter we will make the reader familiar with some notations and conventions used throughout the following text. In particular, we will regard functions and relations, especially finitary operations and finitary relations on one fixed set. We then combine them to algebras in the sense of universal algebra and relational structures, and we also look at clones of operations and relations.

We wish to point out that none of the results or definitions in this chapter is original. The main goal is to fix our notation and to recollect some knowledge from the folklore of general algebra that we want to build our theory onto.

We begin with a few basic notational remarks mainly related to set theory. Throughout the text, the symbol $\emptyset$ denotes the *empty set*, and for a set A we write $\mathfrak{P}(A)$ for the *powerset of* A, i.e. the set containing all subsets of A. Further, we denote *inclusion* of sets A and B by $A \subseteq B$ and *proper set inclusion*, i.e. $A \subseteq B$ and $A \neq B$, by $A \subset B$. Moreover, we write $\mathbb{N}$ for the set of all natural numbers, including zero. The set $\mathbb{N} \setminus \{0\}$ of positive natural numbers is denoted by $\mathbb{N}_+$. We implicitly use the standard set-theoretic representation of natural numbers (as finite ordinals) due to John von Neumann: every natural number $n \in \mathbb{N}$ is the set of all its predecessors: $n = \{i \in \mathbb{N} \mid i < n\}$, i.e. $n = \{0, \ldots, n-1\}$. Thereby, every natural number n is in particular a set, and if it occurs in some expression that has been defined for sets, then the meaning induced by viewing the number as an index set with the entries $0, \ldots, n-1$ is the precise meaning we intend to use.

Furthermore, for a set A, by $|A|$ we refer to its cardinality. We use $\aleph_0$ to denote the smallest infinite cardinal number, i.e. the cardinality of $\mathbb{N}$.

Finally, we comment on the use of category theoretic notions in the text. They will only occur very sparsely, and their understanding is not really central to the topic we want to present. Besides, we shall only use the very basics such as the notions of category, morphism, isomorphism, isomorphic objects, retraction, retract, to mention a few, and probably this list is almost exhaustive. Occasionally, the word “functor” has crept into our language. Most of the time, the notions have been defined in concrete terms, too, such that an understanding of the category theoretic background is not needed, and has more of an illustrative character. There are a few places where we cannot avoid to rely on category theoretic language, because it is implicit in the problem we are discussing, e.g. when speaking about applications of our theory w.r.t. categorical equivalence of varieties. However, these are side remarks that are addressed to experts who know the terminology anyway. Therefore, in the following paragraph we will just give a very rough idea what a category and functors are. For more information we refer to excellent introductory monographs such as [AHS06, Awo10].

A category can be understood as a structural abstraction of sets together with functions and their composition, much in the same way as groups are an abstraction of permutation groups on one specific set.

A *category* consists of a class of so-called *objects* (playing the role of the sets), together with a class of so-called *morphisms* (playing the role of functions between sets) such that each morphism has precisely one object as a definite starting point and one object as a target. Moreover, there needs to be a distinguished *identity* morphism id_A for each object A, which starts and ends there. The last piece of data needed for a category is a *composition law*, associating for every three objects A, B, C in the class of objects and any two morphisms, f from A to B and g from B to C, a morphism $g \circ f$ starting in A and ending in C. This partial composition is required to be associative, and the identity morphisms to be neutral, subject to the compositions being defined, respectively. A prototypical example of a category is $\mathcal{S}et$, the category of sets, where the objects are all sets, the morphisms are the functions between them, and the composition is the normal composition of functions as described in Section 2.1.

Now, a functor is the same to categories what a group homomorphism is to groups as abstractions of permutation groups. Functors are structure preserving “morphisms” between categories and consist of two parts, usually denoted with the same symbol: one assignment between the object classes of the two involved categories and one between the morphism classes. They need to be compatible with the structure of the categories in the following sense: let F be a functor from $\mathcal{C}$ to $\mathcal{D}$, then it needs to satisfy three properties by definition. For every object A in $\mathcal{C}$ we must have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, moreover, if a morphism f in $\mathcal{C}$ goes from A to B, then $F(f)$ needs to go from $F(A)$ to $F(B)$. Third, if morphisms f and g can be composed in $\mathcal{C}$, then by the above, $F(f)$ and $F(g)$ can be composed in $\mathcal{D}$, and we require $F(f \circ g) = F(f) \circ F(g)$.

With these two definitions we leave the abstract world of categories and turn back to very concrete objects, constituting the crucial ingredients for the body of theory we want to develop in this thesis.

2.1 Functions, operations and relations

Functions and relations, especially finitary operations and finitary relations defined on one fixed set, will be the basic objects of discourse in the following chapter. Hence, we cannot avoid to address some notational issues associated with these concepts in the paragraphs below. These will include images and preimages of functions, composition of several functions, certain actions, restriction of functions and relations, and the definition of some special functions and relations.

For sets A and B, a set $f \subseteq A \times B$ of pairs (in the sense of Kuratowski) is the graph of a *function* (A, f, B) if it is left-total and right-unique. Putting this explicitly, for every $a \in A$ there has to exist exactly one $b \in B$, denoted $f(a)$, such that $(a, b) \in f$. We use the names *function*, *map* and *mapping* as synonyms, and we denote a function between A and B by $f\colon A \longrightarrow B$ and the *set of all* such *functions* by B^A. If $U \subseteq A$ and $V \subseteq B$ are subsets, then we denote by

$$f[U] := \{ f(u) \mid u \in U\}$$

the *image of U under f* and by

$$f^{-1}[V] := \{a \in A \mid f(a) \in V\}$$

the *preimage of V under f*. Especially, we write $\operatorname{im} f$ for $f[A]$ and call this the *image (range)* of f.

If A, B and C are sets, and $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ are functions, then we write

$$\begin{array}{rccc} g \circ f\colon & A & \longrightarrow & C \\ & a & \longmapsto & g(f(a)) \end{array}$$

for the *composition g after f*.

Let I be an index set and B_i for $i \in I$ sets. Then their product $\prod_{i\in I} B_i$ is defined as $\left\{ b \in \left(\bigcup_{i\in I} B_i\right)^I \;\middle|\; \forall\, i \in I\colon b(i) \in B_i \right\}$. We also write the members $b \in \prod_{i\in I}$ as *tuples* (or sequences) $(b(i))_{i\in I}$ or as $(b_i)_{i\in I}$. If for every $i \in I$ the set B_i is equal to some fixed set B, then the product $\prod_{i\in I} B_i$ becomes the set B^I of all mappings from I to B, called *I-th power of B*. Especially, if $I = n$ is a natural number, then via the interpretation of n as a set, we obtain B^n as the set of all n-tuples with entries from B, which we can thus view as functions from $\{0, \ldots, n-1\}$ to B. Nevertheless, we allow ourselves to write also $(b_1, \ldots, b_n)$ for tuples in B^n instead of $(b_0, \ldots, b_{n-1})$, whenever this is convenient.

If A is another set and for $i \in I$ there are mappings $f_i\colon A \longrightarrow B_i$, we are going to denote the *tupling* of the mappings $(f_i)_{i\in I}$ (in category theoretic language, the universal morphism into the product) by

$$\begin{array}{rccc} (f_i)_{i\in I}\colon & A & \longrightarrow & \prod_{i\in I} B_i \\ & a & \longmapsto & (f_i(a))_{i\in I}\,. \end{array}$$

In this way for $g\colon \prod_{i\in I} B_i \longrightarrow C$ the composition $g \circ (f_i)_{i\in I}$ is well-defined.

Next, we introduce some more notational abbreviations. For a set A and an integer $n \in \mathbb{N}$ we will denote the *set of n-ary operations on A*, defined as A^{A^n}, by $\mathrm{O}_A^{(n)}$. For a function $f \in \mathrm{O}_A^{(n)}$ the integer n is said to be the *arity* of the operation f, sometimes also denoted by $\operatorname{ar}(f)$. Furthermore, the *set of all finitary operations on A* is defined as as

$$\mathrm{O}_A := \bigcup \left\{ \mathrm{O}_A^{(n)} \;\middle|\; n \in \mathbb{N} \right\}.$$

For any subset $F \subseteq \mathrm{O}_A$ we will write $F^{(n)} := F \cap \mathrm{O}_A^{(n)}$ for its n-ary part, $n \in \mathbb{N}$. Note that this is consistent with the notation for the set of all n-ary operations. Moreover, observe that, by the above, for $f \in \mathrm{O}_A^{(n)}$ and $(g_0, \ldots, g_{n-1}) \in \left(\mathrm{O}_A^{(m)}\right)^n$ the *general composition* $f \circ (g_0, \ldots, g_{n-1}) \in \mathrm{O}_A^{(m)}$ is defined.

We now turn to the second important concept of this thesis: relations. If I is an index set and for $i \in I$ we have sets A_i, then a *relation ϱ between the sets $(A_i)_{i\in I}$* is simply any subset $\varrho \subseteq \prod_{i\in I} A_i$ of the product of the sets A_i, $i \in I$. A *relation on A* is any subset of the power A^I. If I is a natural number $m \in \mathbb{N}$, such a relation is called *finitary* and m is said to be the arity of ϱ, written as $\operatorname{ar}(\varrho)$. Analogously, as for operations, we write $\mathrm{R}_A^{(m)}$ for the set $\mathfrak{P}(A^m)$ of all m-ary relations on a set A, $m \in \mathbb{N}$. Similarly, we abbreviate by R_A the set $\bigcup \left\{ \mathrm{R}_A^{(m)} \;\middle|\; m \in \mathbb{N} \right\}$ of all *finitary relations on A*, and for any set $Q \subseteq \mathrm{R}_A$ we define $Q^{(m)} := Q \cap \mathrm{R}_A^{(m)}$ to be the *m-ary part* of Q. Again, this notation is consistent.

As m-tuples $a \in A^m$ are, by the definition we use, functions from the set m to A, we can do everything with them, that we can do with functions. For instance, we can take the image $\operatorname{im} a = \operatorname{im}(a_0, \ldots, a_{m-1}) = \{a_i \mid 0 \leq i < m\}$, or we can compose them with functions. If $m, n \in \mathbb{N}$, $f \in \mathrm{O}_A^{(n)}$ and $(r_0, \ldots, r_{n-1}) \in (A^m)^n$, then from the above, $f \circ (r_0, \ldots, r_{n-1}) \in A^m$ is again a well-defined m-tuple with the entries

$$f \circ (r_0, \ldots, r_{n-1}) = \begin{pmatrix} f(& r_0(0), & \ldots, & r_{n-1}(0) &) \\ & & \vdots & & \\ f(& r_0(m-1), & \ldots, & r_{n-1}(m-1) &) \end{pmatrix}.$$

In particular, if $f \in \mathrm{O}_A^{(1)}$ is unary and $\mathbf{a} := (a_0, \ldots, a_{m-1}) \in A^m$ is an m-tuple, then $f \circ \mathbf{a} = (f(a_0), \ldots, f(a_{m-1})) \in A^m$. This defines a left-action of unary operations on tuples, and of course this can be extended to sets of tuples in a standard way. By taking images, we obtain for $f \in \mathrm{O}_A^{(n)}$ and $S \in \mathrm{R}_A^{(m)}$

$$f \circ [S^n] = \{ f \circ (r_0, \ldots, r_{n-1}) \mid (r_0, \ldots, r_{n-1}) \in S^n \} \in \mathrm{R}_A^{(m)},$$

and in case of $n = 1$, one gets

$$f \circ [S] = \{ (f(a_0), \ldots, f(a_{m-1})) \mid (a_0, \ldots, a_{m-1}) \in S \},$$

i.e. an action on $\mathrm{R}_A^{(m)}$, which we refer to as the canonical action of unary operations on m-ary relations. We use the same notation also for relations of arbitrary arity, i.e. for a set I, $S \subseteq A^I$ and $f \in \mathrm{O}_A^{(1)}$, we put $f \circ [S] := \{ f \circ r \mid r \in S\}$.

Furthermore, we are going to need *restrictions* of functions and relations. If $S \in \mathrm{R}_A^{(m)}$ is an m-ary relation, and $U \subseteq A$ is a subset, we will write $S{\restriction}_U$ for the intersection $S \cap U^m$. If $f\colon A \longrightarrow B$ is a function, and $U \subseteq A$ and $V \subseteq B$ are subsets of domain and codomain of f such that $f[U] \subseteq V$, we can restrict the mapping as follows

$$\begin{aligned} f|_U^V\colon \quad U &\longrightarrow V \\ x &\longmapsto f(x). \end{aligned}$$

We will use the convention that in expressions using functions, composition and restrictions the restriction symbol $|$ binds stronger to the function symbol than the composition. That is, $f \circ g|_U := f \circ (g|_U)$. A special case of restricting functions occurs with n-ary operations. If $n \in \mathbb{N}$ is the arity of $f \in \mathrm{O}_A^{(n)}$ and $U \subseteq A$ is a subset such that[1] $f[U^n] \subseteq U$, then we abbreviate $f|_U := f|_{U^{\operatorname{ar} f}}^{U}$.

We continue this section with the introduction of notations for some special operations and relations on a set A. We start with some finitary operations that are canonically given by the finite powers of A. For every $n \in \mathbb{N} \setminus \{0\}$ and $1 \leq i \leq n$ we denote by

$$\begin{array}{rccc} e_i^{(n)}\colon & A^n & \longrightarrow & A \\ & (x_1, \ldots, x_n) & \longmapsto & x_i \end{array}$$

the *n-ary projection on the i-th coordinate*. The *set of all* such *projections* is then defined as $\mathrm{J}_A := \bigcup_{n \in \mathbb{N}_+} \left\{ e_i^{(n)} \;\middle|\; 1 \leq i \leq n \right\}$. We remark that there are no nullary projections, because the empty product has not got any factors.

Projection operations on A are trivial in the sense that they preserve (see Section 2.3) every finitary relation on A. Somewhat dually, we shall introduce names for two relations, which will always be preserved by every function on A (see again Section 2.3)

$$\begin{aligned} \Delta_A &:= \{ (x,x) \mid x \in A \} \\ \nabla_A &:= \{ (x,y) \mid x, y \in A \} = A^2. \end{aligned}$$

They are usually called *binary diagonal relations*, the first is the identity relation and the second the full Cartesian product of A with itself.

Moreover, certain collections of unary relations on a set A, i.e. subsets of $\mathfrak{P}(A)$, will be important to us. A collection $\mathcal{C} \subseteq \mathfrak{P}(A)$ is called a *closure system on A* if it is closed w.r.t. arbitrary intersections, including the empty intersection yielding A as a result. Putting this explicitly, $\mathcal{C} \subseteq \mathfrak{P}(A)$ is a closure system if for every subcollection $\mathcal{S} \subseteq \mathcal{C}$ the intersection $\bigcap \mathcal{S}$ is already a unary relation in $\mathcal{C}$.

Closure systems on A are connected to so-called *closure operators on A*. These are unary operations $\mathfrak{c}\colon \mathfrak{P}(A) \longrightarrow \mathfrak{P}(A)$ on $\mathfrak{P}(A)$ satisfying the following three axioms: for every $X \subseteq A$ we have $X \subseteq \mathfrak{c}(X)$ (*extensivity*), for all $X_1 \subseteq X_2 \subseteq A$ it is $\mathfrak{c}(X_1) \subseteq \mathfrak{c}(X_2)$ (*monotonicity*) and $\mathfrak{c} \circ \mathfrak{c} = \mathfrak{c}$ (idempotency).

The relationship between the two concepts is as follows: if $\mathcal{C} \subseteq \mathfrak{P}(A)$ is a closure system on A, one can define a closure operator $\langle\,\rangle_{\mathcal{C}}$ on A, called *the induced* or *corresponding* closure operator, by letting $\langle X \rangle_{\mathcal{C}} := \bigcap \{ C \in \mathcal{C} \mid X \subseteq C \}$ for $X \subseteq A$. It associates to each subset $X \subseteq A$ the least (w.r.t. set inclusion) member of $\mathcal{C}$ containing X. Conversely, if a closure operator $\mathfrak{c} \in \mathrm{O}_{\mathfrak{P}(A)}^{(1)}$ is given, then its image $\operatorname{im} \mathfrak{c}$ is a subset of $\mathfrak{P}(A)$ forming a closure system. One can easily check that these two constructions actually invert each other such that it is the same to give a closure operator on A or to describe a closure system on this set.

Finally, we recall three types of binary relations that are not only important as objects of the theory we want to educe in the following chapter, but also as auxiliaries on the meta level to develop the theory. The first of them are *quasiorders*

[1] We will later say that the function f preserves the subset U.

on a set A. These are binary relations that are reflexive, i.e. contain Δ_A, and transitive, i.e. with every two pairs (x, y) and (y, z) they also need to contain the pair (x, z). Let us denote by $\operatorname{Quord} A$ the set containing all quasiorders on A.

There are two more frequently occurring specialisations of quasiorders. One is obtained by adding the axiom of symmetry, i.e. whenever they contain a pair (x, y), they also need to contain the reversed pair (y, x). Such relations are called *equivalences* on A, and we collect all of them in the set $\operatorname{Eq} A$. They can also be understood as those binary relations on A arising as *kernels of mappings* $f: A \longrightarrow B$ into some other set B, i.e. as $\ker(f) = \{(x, y) \in A^2 \mid f(x) = f(y)\}$.

The other specialisation we have in mind are *order relations* (sometimes called *partial orders*). In addition to the axioms of a quasiorder, they are, by definition, antisymmetric, which means that they can only contain pairs (x, y) and (y, x) if $x = y$.

We call a pair (A, q) an *ordered set* (alternatively a *partially ordered set*, or *poset*, for short) if and only if $q \subseteq A^2$ is an order relation on A. In the same way the notion of *quasiordered set* is defined by q being a quasiorder on A. We mention that for binary relations $q \subseteq A^2$, especially for orders and quasiorders, we sometimes use the more suggestive notation $x\ q\ y$ instead the formal $(x, y) \in q$.

The following fundamental relationships between quasiorders, orders and equivalences will play an important role quite often in the next chapter. Every quasiorder q contains a symmetric core, $q \cap q^{-1}$, which is the largest equivalence relation contained in q. Thus, it partitions the set A into equivalence classes. These can be ordered via $[x]_{q \cap q^{-1}} \leq [y]_{q \cap q^{-1}}$ if and only if $(x, y) \in q$. This is well-defined, since two equivalence classes X and Y can be shown to be in relation $\leq$ if and only if every element $x \in X$ satisfies $(x, y) \in q$ for any $y \in Y$. Moreover, we have thus defined indeed an order relation on the factor set $A/(q \cap q^{-1})$. Usually, we will denote the order $\leq$ by $q/(q \cap q^{-1})$, and call the pair $(A/(q \cap q^{-1}), q/(q \cap q^{-1}))$ the *canonically* or *naturally associated factor poset* of q. So every quasiorder on A gives rise to a *corresponding equivalence relation* $q \cap q^{-1}$ and a canonically associated factor poset on the equivalence classes. Conversely, whenever one has got a pair consisting of an equivalence relation θ on a set A and an order relation on the partition A/θ then these two define a quasiorder on A by putting $(x, y) \in q$ if and only if $[x]_\theta$ lies below $[y]_\theta$ in the order relation. It is not hard to see that the two presented constructions are mutual inverses of each other. However, most of the time, we shall only need the first direction, i.e. that every quasiorder can be understood as a partition, whose blocks are ordered.

Before we finish this section, let us quickly introduce a few additional bits of notation concerning quasiordered sets. For a quasiordered set (A, q), a subset $U \subseteq A$ is called an *upset* (sometimes also *upper set* or *upward closed set*) if it has the property that with every element $x \in U$ it also contains any other $y \in A$ such that $(x, y) \in q$. One easily checks that the collection of all upsets w.r.t. (A, q) forms a closure system on A. Therefore, we can associate with every subset $V \subseteq A$ the least upset containing V, which we call the *upset generated by* V and denote by $\uparrow_{(A,q)} V$ or more briefly $\uparrow_q V$. If the quasiorder is thought to be clear from the context, we may also omit any index of the upwards directed arrow. It follows immediately from the definition that $\uparrow_{(A,q)} V = \{y \in A \mid \exists\, x \in V \colon (x,y) \in q\}$. *Principal upsets* are such generated by a singleton subset $V \subseteq A$. For them we sometimes prefer to write $\uparrow_{(A,q)} x$ instead of $\uparrow_{(A,q)} \{x\}$.

The dual notion of an upset is that of a *downset* (also *downward closed set*). Formally, $U \subseteq A$ is a downset in a quasiordered set (A,q) if and only if it is an upset in (A, q^{-1}). We express *generated downsets* as $\downarrow_{(A,q)} V$ and *principal downsets* as $\downarrow_{(A,q)} x$ for $V \subseteq A$ and $x \in A$. Moreover, we use similar abbreviations for downsets as those defined for upsets.

Besides, it will be important to speak about *minimal* and *maximal* elements of quasiordered sets. An element $x \in A$ of a quasiordered set (A,q) is said to be minimal if any other element $y \in A$ cannot be below x without being above x at the same time. Maximal elements are defined dually. Formally we put

$$\begin{aligned} \operatorname{Min}(A,q) &:= \{x \in A \mid \forall\, y \in A \colon (y,x) \in q \implies (x,y) \in q\} \\ \operatorname{Max}(A,q) &:= \{x \in A \mid \forall\, y \in A \colon (x,y) \in q \implies (y,x) \in q\} \end{aligned}$$

as the sets of all minimal and maximal element of the quasiordered set, respectively.

A few times we shall also encounter ordered sets in the form of *(semi)lattices*. We simply mention that a poset is a *(join) semilattice* if any two of its elements have got a least common upper bound, usually called *join* or *supremum*. A poset is a *(meet) semilattice* if the poset with the reversed order is a join semilattice (the dual join is called *infimum* or *meet*), and it is a *lattice* if it is both, a meet and a join semilattice. Lattices and semilattices are called *complete*, if the existence of least common upper bounds (and largest common lower bounds, respectively) is not only required for two-element subsets, but for any subset of elements.

A *chain* in a poset is any subset, in which any two elements are comparable (w.r.t. the order relation). Contrarily, an *antichain* in a quasiordered set is any subset, in which only pairs of identical elements are comparable.

Now having defined the main pieces of notation for relations and operations, we can bundle some finitary relations (or operations) on a fixed set together. The result will be a so-called relational structure (or an algebra in the case of operations). These two concepts will be discussed in the following section.

2.2 Algebras and relational structures

In the following chapter we are going to study *algebras* in their operational form, denoted by boldface letters $\mathbf{A}$, and also *relational structures*, denoted by $\underset{\sim}{\mathbf{A}}$. Both kinds of structures exist in two variants: typed structures indexed by a so-called signature, and untyped structures, where the operations, or relations, respectively, are simply collected in a set. In this section we are going to provide mainly notation, brief definitions and a few relationships concerning the following concepts arising in connection with the mentioned structures: signatures, homomorphisms, isomorphisms, homomorphic images, congruences, substructures, direct products, varieties, terms, term operations and identities for algebras; signatures, homomorphisms, isomorphisms, substructures, image substructures, induced substructures, embeddings and retractions, retracts and products for relational structures. However, we assume later in the text that the reader is more familiar with these fundamental concepts from universal algebra than he can just get from reading this section, at least as far as algebras are concerned. For further information about algebraic structures the reader is referred to an introductory textbook such as [BS81, MMT87, Coh65].

We first discuss the untyped versions of algebras and relational structures, then we address signatures and concepts derived from this.

An *(untyped) algebra* is simply a pair $\langle A; F\rangle$ where A is a set and $F \subseteq \mathrm{O}_A$ is any set of finitary operations. Similarly, an *(untyped) relational structure* is a pair $\langle A; Q\rangle$ consisting of a set A and a collection $Q \subseteq \mathrm{R}_A$ of finitary relations. In both cases the set A is called *carrier* or *underlying set*, and the operations in F, or the relations in Q, respectively, are said to be *fundamental*. In this text, algebras are generally denoted by boldface letters, such as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ etc., and relational structures are symbolised by wavy underlined boldface letters, such as $\underset{\sim}{\mathbf{A}}$, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{C}}$ etc. If nothing else is said the carrier set of such structures is denoted by the same non-boldface letter as the structure. If we want to be explicit, we will write $\mathbf{A} = \langle A; F\rangle$ for an algebra having F as set of fundamental operations, and $\underset{\sim}{\mathbf{A}} = \langle A; Q\rangle$ for a relational structure with fundamental relations belonging to Q. If the sets F or Q are finite, one often just lists their elements without writing the parentheses delimiting the set F or Q, respectively.

Next, we outline the second variant, which are typed structures. For this we introduce *signatures*. They are important to define concepts such as products, homomorphisms, terms, identities and varieties.

Formally, a signature is a triple $\Sigma = (S_{\mathrm{op}}, S_{\mathrm{rel}}, \mathrm{ar})$ consisting of two disjoint sets S_{op} and S_{rel} (interpreted as *operation symbols* and *relation symbols*) together with a function $\mathrm{ar}\colon S_{\mathrm{op}} \cup S_{\mathrm{rel}} \longrightarrow \mathbb{N}$ assigning to each symbol a natural number, called *arity*. The signature is called *algebraic* if $S_{\mathrm{rel}} = \emptyset$, and *relational* if $S_{\mathrm{op}} = \emptyset$.

A structure corresponding to such a signature Σ is a pair $\left(A, {}^{\mathbb{A}}\right)$ consisting of a carrier set A and a mapping ${}^{\mathbb{A}}\colon S_{\mathrm{op}} \cup S_{\mathrm{rel}} \longrightarrow \mathrm{O}_A \cup \mathrm{R}_A$ such that $f^{\mathbb{A}} \in \mathrm{O}_A^{(\mathrm{ar}(f))}$ for every $f \in S_{\mathrm{op}}$ and $T^{\mathbb{A}} \in \mathrm{R}_A^{(\mathrm{ar}(T))}$ for every $T \in S_{\mathrm{rel}}$. If the signature Σ is algebraic, we thus obtain a *(typed) algebra* of signature Σ, and if Σ is relational, we get a *(typed) relational structure* of signature Σ. Corresponding with the previously introduced notation for untyped algebras, we usually write $f^{\mathbf{A}}$ instead of $f^{\mathbb{A}}$ for $f \in S_{\mathrm{op}}$ and algebraic signatures, and, similarly, $T^{\underset{\sim}{\mathbf{A}}}$ instead of $T^{\mathbb{A}}$ for $T \in S_{\mathrm{rel}}$ and relational signatures. If $\mathrm{ar}\colon S_{\mathrm{op}} \longrightarrow \mathbb{N}$ describes an algebraic signature, then it is often more convenient to collect all symbols of a certain fixed arity in one set, i.e. to define $\Omega^{(n)} := \mathrm{ar}^{-1}\left[\{n\}\right]$ for $n \in \mathbb{N}$. Then an algebraic signature can equivalently be given by a sequence of sets $\Omega := \left(\Omega^{(n)}\right)_{n\in\mathbb{N}}$, where the n-th entry contains all n-ary symbols. Following this approach, typed algebras are often also described by listing their carrier set, together with a sequence of operations, indexed in a similar manner: $\mathbf{A} = \left\langle A; \left(\left(\omega^{\mathbf{A}}\right)_{\omega\in\Omega^{(n)}}\right)_{n\in\mathbb{N}}\right\rangle$. The analogous remark applies of course also to relational signatures and relational structures.

We wish to point out explicitly that the carrier sets of our structures for typed and untyped, algebraic and relational structures do not need to fulfil any further conditions: neither do we require finiteness, nor do we exclude empty carrier sets as it is sometimes done in the literature. Likewise, the definitions above include the possibility to have nullary operations or relations. This does not pose problems, but contrarily unifies the presentation. For a few selected aspects, we will even discuss the relevance of nullary operations or relations for our theory, explicitly.

Next, we briefly address the relationship of typed and untyped structures. Certainly, we may regard every typed algebraic structure as an untyped algebra by simply collecting all fundamental operations in a set $F := \left\{ f^{\mathbf{A}} \,\middle|\, f \in S_{\mathrm{op}} \right\} \subseteq \mathrm{O}_A$ and forming the pair $\langle A; F\rangle$. Similarly, we may collect the interpretations of relation symbols $\left\{ \underset{\sim}{T}^{\mathbf{A}} \,\middle|\, T \in S_{\mathrm{rel}} \right\}$ in a set Q of fundamental relations and construct $\langle A; Q\rangle$ for a typed relational structure. This way of "forgetting the indexing" becomes important for the theory we want to develop. Most of its aspects only depend on the set of fundamental operations of an algebra, even less specific, only on the least locally closed clone generated by this set (see below). Therefore, we formulate our theory mainly for untyped algebras. However, in some remarks we shall speak of course about connections with other universal algebraic concepts, such as e.g. satisfaction of identities, which requires typed algebras in order to be well-defined. Then "forgetting about the signature" is exactly the way how we want our theory to be applied, even though indexed and non-indexed algebras are formally different things.

Conversely, if a non-indexed structure is given, then one can interpret it also as a typed structure w.r.t. a *canonically induced signature*. Namely, one simply uses the given fundamental operations (or relations, respectively) as symbols and their inherent arities as the arities of the symbols. This naive way of indexing is not always very helpful, especially in connection with other structures. Therefore, we sometimes try to avoid it, see e.g. the discussion in Remark 3.3.5 and the paragraph following Lemma 3.4.39.

With regard to applications the following extension operation on (typed or untyped) algebras plays a role. Even though, we do not work with it explicitly, it is mentioned several times in the rest of the text. If $\mathbf{A} = \langle A; F\rangle$ is an algebra, then $\mathbf{A}_A$ denotes the *polynomial expansion* derived from $\mathbf{A}$. It is the algebra on the same carrier set A, having the fundamental operations of $\mathbf{A}$ plus one nullary constant operation $c_a^{(0)}$ for every element $a \in A$, i.e. $\mathbf{A}_A = \left\langle A; F \cup \left\{ c_a^{(0)} \,\middle|\, a \in A \right\}\right\rangle$. If $\mathbf{A}$ has got a signature, then this signature is extended via a disjoint union of the operation symbols with the carrier set of A, and each new symbol $a \in A$ is interpreted as $a^{\mathbf{A}_A} := c_a^{(0)}$, whereas the old symbols are interpreted as in $\mathbf{A}$.

As already mentioned, typed structures are necessary to define concepts as homomorphism, product, variety, and some more. We first introduce the important ones concerning algebraic structures, then we turn to relational ones.

If $\mathbf{A}$ and $\mathbf{B}$ are algebras of the same signature, then a mapping $h\colon A \longrightarrow B$ between the carrier sets of the structures is called a *homomorphism*, written as $h\colon \mathbf{A} \longrightarrow \mathbf{B}$, if $f^{\mathbf{B}}(h \circ x) = h\left(f^{\mathbf{A}}(x)\right)$ holds for all $f \in S_{\mathrm{op}}$ and every $x \in A^{\operatorname{ar} f}$. A homomorphism from $h\colon \mathbf{A} \longrightarrow \mathbf{B}$ is called an isomorphism if there exists (a necessarily unique) inverse homomorphism $h'\colon \mathbf{B} \longrightarrow \mathbf{A}$ such that $h' \circ h = \mathrm{id}_A$ and $h \circ h' = \mathrm{id}_B$. This is equivalent to being a bijective homomorphism. Two algebras $\mathbf{A}$ and $\mathbf{B}$ are called isomorphic, in symbols $\mathbf{A} \cong \mathbf{B}$, if there exists an isomorphism between them.

An algebra $\mathbf{U}$ is a *subalgebra* of an algebra $\mathbf{A}$ if $U \subseteq A$ and the restriction $\mathrm{id}_A|_U^A\colon U \longrightarrow A$ of the identity map on A to U constitutes a homomorphism $\mathrm{id}_A|_U^A\colon \mathbf{U} \longrightarrow \mathbf{A}$. We sometimes denote this condition by $\mathbf{U} \leq \mathbf{A}$. A subset $U \subseteq A$ of the carrier of $\mathbf{A}$ is called a *subuniverse of* $\mathbf{A}$ if there exists a subalgebra $\mathbf{U} \leq \mathbf{A}$ having U as carrier set (such an algebra is necessarily unique). Again, we denote the fact that $U \subseteq A$ forms a subuniverse of $\mathbf{A}$ with a similar notation, namely $U \leq \mathbf{A}$. Since the truth of $U \leq \mathbf{A}$ does not depend on the indexing of the operations in $\mathbf{A}$, but just on the set of fundamental operations, we can also use this notation for non-indexed algebras. Then it expresses that the subset U, interpreted as a unary relation, is invariant (see also Section 2.3 below), i.e. $f\left[U^{\operatorname{ar}(f)}\right] \subseteq U$ for all fundamental operations f of $\mathbf{A}$.

Moreover, we write $\operatorname{Sub}\mathbf{A}$ for the set $\{U \subseteq A \mid U \leq \mathbf{A}\}$ of all subuniverses of an algebra $\mathbf{A}$. It is easy to see that this collection forms a closure system on A. The associated closure operator is denoted by $\langle\ \rangle_{\mathbf{A}}$ and maps any subset $V \subseteq A$ to the least subuniverse w.r.t. set inclusion containing the set V, which is usually called *subuniverse generated by V in* $\mathbf{A}$.

If I is an index set, and we have algebras $\mathbf{A}_i$ of the same type for every $i \in I$, then we can define an algebra $\mathbf{P} := \prod_{i\in I}\mathbf{A}_i$, called *direct product*, on the set $P := \prod_{i\in I} A_i$ as follows. For $f \in S_{\mathrm{op}}$ of arity n and $\left((a_{i,0})_{i\in I}, \dots, (a_{i,n-1})_{i\in I}\right) \in P^n$ we let $f^{\mathbf{P}}\left((a_{i,0})_{i\in I}, \dots, (a_{i,n-1})_{i\in I}\right) := \left(f^{\mathbf{A}_i}(a_{i,0}, \dots, a_{i,n-1})\right)_{i\in I}$. In particular if $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$ and some algebra $\mathbf{A}$, we have in this way defined the power algebra $\mathbf{A}^I$.

To mention a few basic facts, whenever $h\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism, then the image $h[A]$ is a subuniverse of $\mathbf{B}$. The corresponding unique subalgebra of $\mathbf{B}$ is denoted by $h[\mathbf{A}]$ and is called *homomorphic image* of $\mathbf{A}$ under h. Up to isomorphism, all homomorphic images of an algebra can be described by so-called *congruences*. Namely, if $h\colon \mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism, then its kernel is not only an equivalence relation on A, but a congruence, that is, an equivalence $\theta \subseteq A^2$ satisfying $\theta \leq \mathbf{A} \times \mathbf{A}$. This property ensures that an algebra $\mathbf{A}/\theta$ on the equivalence classes A/θ is well-defined by application of the fundamental operations of $\mathbf{A}$ to the representatives of the equivalence classes. It is the content of the well-known *homomorphism theorem* that we always have $\mathbf{A}/\ker(h) \cong h[\mathbf{A}]$. Thus, knowing all congruences of an algebra, one can describe all of its homomorphic images. This underlines the importance of congruences as particularly interesting examples of relations within the theory to be developed later on. We abbreviate the set of all congruences of an algebra $\mathbf{A}$ by $\operatorname{Con}\mathbf{A}$. As a slight generalisation of this, we also define the set $\operatorname{Quord}\mathbf{A}$ of all compatible quasiorders of $\mathbf{A}$ to be $\{q \subseteq A^2 \mid q \leq \mathbf{A}\times\mathbf{A} \text{ and } q \text{ quasiorder on } A\}$.

Classes of algebras that are closed under forming arbitrary direct products, taking subalgebras and homomorphic images of their members are called *varieties* in universal algebra. Varieties can be *generated* by just closing a given class $\mathcal{K}$ of algebras of the same signature against these operations. The result is the smallest variety containing $\mathcal{K}$ as a subclass and is here denoted by $\operatorname{Var}\mathcal{K}$. In case $\mathcal{K} = \{\mathbf{A}\}$, we also write $\operatorname{Var}\mathbf{A}$ instead of $\operatorname{Var}\{\mathbf{A}\}$. Due to a well-known theorem by Birkhoff varieties are exactly those classes of algebras that can be axiomatised using identities. To understand this, we need to speak about terms in the following paragraphs.

The notion of *term* depends on an algebraic signature and on some set X thought to contain *variable symbols*. In most cases one uses a countably infinite set $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ or a finite subset of it. The set of all terms w.r.t. a given signature and a set X of variable symbols is the smallest set containing X as a subset and being closed against forming new terms $ft_0, \ldots, t_{\operatorname{ar}(f)-1}$ from terms $t_0, \ldots, t_{\operatorname{ar}(f)-1}$ and operation symbols $f \in S_{\mathrm{op}}$. Abstractly speaking, any member of any form of the absolutely free algebra over the generating set X in the class of all algebras of a given signature can be understood to be a term over that signature and variable symbols from X, but we fear that this abstraction rather spoils the intuition one should have about terms than being helpful.

By an explicit construction of all terms over a given signature, one can prove that, even though the set X of variable symbols may be infinite, every term t contains only finitely many members of X. This is caused by the fact that the arities of all operation symbols in our signatures are natural numbers. Let us suppose that the variables occurring in t are a subset of $\{x_0, \ldots, x_{n-1}\}$. We say that the term is *n-ary* in this case. If $\mathbf{A}$ is an algebra over the same signature as t and we assign values $a_0, \ldots, a_{n-1} \in A$ to the variable symbols $x_0, \ldots, x_{n-1}$, then we can *evaluate* the term t over the algebra $\mathbf{A}$, by substituting the values for the variables and the fundamental operations of $\mathbf{A}$ for the operation symbols from the signature. Calculating the result of this expression yields a value in A, denoted by $t^{\mathbf{A}}(a_0, \ldots, a_{n-1})$. Now letting the evaluation tuple $(a_0, \ldots, a_{n-1}) \in A^n$ vary, we have defined an n-ary operation $t^{\mathbf{A}} \colon A^n \longrightarrow A$ on $\mathbf{A}$, a so-called *n-ary term function* (or *term operation*) of $\mathbf{A}$. In the set $\operatorname{Term}^{(n)}(\mathbf{A})$ we collect all n-ary term operations of $\mathbf{A}$, that is all operations in $\mathrm{O}_A^{(n)}$ that arise from some n-ary term over the signature of $\mathbf{A}$. A *term operation* of $\mathbf{A}$ is any finitary operation in $\operatorname{Term}(\mathbf{A}) := \bigcup_{n\in\mathbb{N}} \operatorname{Term}^{(n)}(\mathbf{A})$. As a side remark we mention that term operations of $\mathbf{A}_A$ are called *polynomial operations* of $\mathbf{A}$.

Terms and term operations have a number of useful applications in universal algebra. We first use them to understand the connection between identities and varieties. Again, we consider a fixed algebraic signature as before. An *identity* in this signature is just any pair of terms (s,t), usually written as $s \approx t$. Since both terms contain only finitely many variable symbols, we may consider them as n-ary for some $n \in \mathbb{N}$. We say that the identity $s \approx t$ *holds* in an algebra corresponding to the signature of s and t (or that the identity is *satisfied* in $\mathbf{A}$, or that $\mathbf{A}$ is a *model of the identity*) if the corresponding n-ary term operations agree: $s^{\mathbf{A}} = t^{\mathbf{A}}$. If this condition is fulfilled, we also denote this by $\mathbf{A} \models s \approx t$.

Now the connection to varieties is as follows: if we start with a set of identities in some signature, e.g. the axioms describing a monoid, and collect all algebras of this signature, satisfying all given identities, then the resulting class of algebras is a variety. In the example, we mentioned, this is the variety of all monoids. Furthermore, every variety can be described in this way, namely, it can be axiomatised by using simply all identities holding in all members of the given variety as axioms. This is the content of Birkhoff's theorem we mentioned above. Moreover, it follows, that the variety generated by a class $\mathcal{K}$ of algebras can also be obtained by taking all models of all identities satisfied by the members of $\mathcal{K}$. This implies that any equality between term operations of an algebra $\mathbf{A}$ automatically turns into an identity holding everywhere in the variety $\mathbf{A}$ generates. We mention this here because it helps to understand the consequences of one of our main theorems. In the case of finite algebras, it links a relational property (that of being a so-called cover) to an equality between term operations of the algebra (see Corollary 3.4.36(k)).

As a second application of term operations, we want to present how they can be used to describe generated subuniverses in power algebras. This becomes important in the proof of Lemma 3.4.29. As the statement, like everything presented in this section, belongs to the folklore of universal algebra, we just give a very brief reason for it. Suppose that I is a set and $\mathbf{A}$ is an algebra. Then for any subset $G \subseteq A^I$ the subpower generated by G can be expressed as

$$\langle G \rangle_{\mathbf{A}^I} = \bigcup_{n \in \mathbb{N}} \left\{ h \circ (g_0, \ldots, g_{n-1}) \;\middle|\; h \in \mathrm{Term}^{(n)}(\mathbf{A}) \wedge (g_0, \ldots, g_{n-1}) \in G^n \right\},$$

where we use the composition notion introduced in the previous section. It is clear that all the sets making up the big union on the right-hand side of the equality need to be subsets of $\langle G \rangle_{\mathbf{A}^I}$ since the elements $g_0, \ldots, g_{n-1}$ belong to $G \subseteq \langle G \rangle_{\mathbf{A}^I}$, and the latter is closed against application of fundamental operations of $\mathbf{A}$ due to being a subuniverse (and thus by induction also against application of term operations). Conversely, the set on the right-hand side contains G since, for instance the identity is a unary term operation. Moreover, one can check that the union is indeed a subuniverse by appropriately combining terms and fundamental operations to new terms. This demonstrate the inclusion of $\langle G \rangle_{\mathbf{A}^I}$ in the set on the right-hand side since $\langle G \rangle_{\mathbf{A}^I}$ is the least subuniverse of $\mathbf{A}^I$ containing G.

With a similar argument one can prove the equality

$$\langle G \rangle_{\mathbf{A}^I} = \left\{ h \circ (g_0, \ldots, g_{n-1}) \;\middle|\; h \in \mathrm{Term}^{(n)}(\mathbf{A}) \right\}$$

in case $G = \{g_0, \ldots, g_{n-1}\}$. This is used in the proof of Corollary 3.4.36. As a special case we obtain that the set of n-ary term operations is a subpower generated by the n projection operations:

$$\left\langle \left\{ e_i^{(n)} \;\middle|\; 1 \leq i \leq n \right\} \right\rangle_{\mathbf{A}^{A^n}} = \left\{ h \circ \left(e_1^{(n)}, \ldots, e_n^{(n)} \right) \;\middle|\; h \in \mathrm{Term}^{(n)}(\mathbf{A}) \right\}$$
$$= \mathrm{Term}^{(n)}(\mathbf{A}).$$

This fact is exploited in the proof of Corollary 3.5.13.

Having dealt with the fundamental concepts of algebraic structures, we now shift our focus to relational ones. Again, we discuss concepts such as homomorphism, product, substructure.

For this we consider again a fixed signature, this time for relational structures. If $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ are structures of this signature, then a mapping $h\colon A \longrightarrow B$ is said to be *relation preserving* w.r.t. $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ if $h \circ x \in T^{\underset{\sim}{\mathbf{B}}}$ for every $T \in S_{\mathrm{rel}}$ and all $x \in T^{\underset{\sim}{\mathbf{A}}}$. This simply says that the natural operation of h on tuples of A maps elements of the fundamental relations from $\underset{\sim}{\mathbf{A}}$ to the corresponding relation from $\underset{\sim}{\mathbf{B}}$. The function h is called *relation reflecting* w.r.t. $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ if for any $T \in S_{\mathrm{rel}}$ and all $x \in A^{\mathrm{ar}\,T}$ the condition $h \circ x \in T^{\underset{\sim}{\mathbf{B}}}$ implies $x \in T^{\underset{\sim}{\mathbf{A}}}$. Employing the short notation from the previous section, we can state the condition of preserving relations as $h \circ \left[T^{\underset{\sim}{\mathbf{A}}}\right] \subseteq T^{\underset{\sim}{\mathbf{B}}}$ for $T \in S_{\mathrm{rel}}$, and reflection of relations by the preimage inclusion $(h\circ)^{-1}\left[T^{\underset{\sim}{\mathbf{B}}}\right] \subseteq T^{\underset{\sim}{\mathbf{A}}}$ for all $T \in S_{\mathrm{rel}}$.

A homomorphism from $\underset{\sim}{\mathbf{A}}$ to $\underset{\sim}{\mathbf{B}}$, written as $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$, is by definition just any relation preserving map w.r.t. $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$. We write $\mathrm{Hom}\left(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}\right)$ for the set $\left\{ h \mid h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}} \right\}$ of all homomorphisms from $\underset{\sim}{\mathbf{A}}$ to $\underset{\sim}{\mathbf{B}}$. Furthermore, we call a homomorphism from a relational structure $\underset{\sim}{\mathbf{A}}$ into itself an *endomorphism* and we put $\mathrm{End}\,\underset{\sim}{\mathbf{A}} := \mathrm{Hom}\left(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{A}}\right)$ as the set of all endomorphisms of $\underset{\sim}{\mathbf{A}}$.

A structure $\underset{\sim}{\mathbf{U}}$ of the same signature as $\underset{\sim}{\mathbf{A}}$ is called a *substructure* of $\underset{\sim}{\mathbf{A}}$ if $U \subseteq A$ and the restriction of the identity mapping $\mathrm{id}_A\,|_U^A\colon U \longrightarrow A$ forms a homomorphism between $\underset{\sim}{\mathbf{U}}$ and $\underset{\sim}{\mathbf{A}}$. This expresses simply the condition that for every $T \in S_{\mathrm{rel}}$ the relation $T^{\underset{\sim}{\mathbf{U}}}$ is a subrelation, that is a subset, of $T^{\underset{\sim}{\mathbf{A}}}$. In contrast to algebras, substructures are not completely determined by their carrier sets. This is only true for *full* (also called *induced*) substructures, where $T^{\underset{\sim}{\mathbf{U}}} = T^{\underset{\sim}{\mathbf{A}}}\restriction_U$ holds for all $T \in S_{\mathrm{rel}}$. For them however the existence does not, as for algebras, depend on the subset $U \subseteq A$. Every set $U \in \mathfrak{P}(A)$ can be the carrier set of an induced substructure, namely of $\underset{\sim}{\mathbf{A}}\restriction_U := \left\langle U; \left(T^{\underset{\sim}{\mathbf{A}}}\restriction_U\right)_{T \in S_{\mathrm{rel}}} \right\rangle$.

So for relational structures substructures are not very interesting. Instead one studies so-called retracts. Before we look at retracts in a little bit more detail let us introduce some additional terminology about homomorphisms. As for algebras a homomorphism $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ is called an isomorphism if there exists another homomorphism $h'\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ such that $h' \circ h = \mathrm{id}_A$ and $h \circ h' = \mathrm{id}_B$. This can be characterised by requiring that h is a bijective *relation preserving* and *relation reflecting* map, equivalently a bijection satisfying $h \circ \left[T^{\underset{\sim}{\mathbf{A}}}\right] = T^{\underset{\sim}{\mathbf{B}}}$ for all $T \in S_{\mathrm{rel}}$. Moreover, we say that two structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ of identical signature are isomorphic, written $\underset{\sim}{\mathbf{A}} \cong \underset{\sim}{\mathbf{B}}$, if there exists an isomorphism between them.

Every homomorphism $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ naturally comes with an *image* substructure $h[\underset{\sim}{\mathbf{A}}] := \left\langle h[A]\,; \left(h \circ \left[T^{\underset{\sim}{\mathbf{A}}}\right]\right)_{T\in S_{\mathrm{rel}}}\right\rangle$, which is of course a substructure of $\underset{\sim}{\mathbf{B}}$, even of $\underset{\sim}{\mathbf{B}}\restriction_{h[A]}$, but not necessarily a full one.

Furthermore, a homomorphism $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ is called an *embedding* if it is injective and relation reflecting. This means that embeddings are precisely those injections that are relation preserving and relation reflecting. We say that $\underset{\sim}{\mathbf{A}}$ is *embedded* into $\underset{\sim}{\mathbf{B}}$, and denote this fact by $\underset{\sim}{\mathbf{A}} \lesssim \underset{\sim}{\mathbf{B}}$, if there exists an embedding of $\underset{\sim}{\mathbf{A}}$ into $\underset{\sim}{\mathbf{B}}$.

In case that $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ is an embedding, the restriction $h|_A^{h[A]}\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}\restriction_{h[A]}$ to the induced substructure of $\underset{\sim}{\mathbf{B}}$ on the image of h is a bijective relation preserving and relation reflecting map. Thus, by the above it is an isomorphism between $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}\restriction_{h[A]}$. Hence, we have verified that $\underset{\sim}{\mathbf{A}} \lesssim \underset{\sim}{\mathbf{B}}$ entails the existence of a subset $U \subseteq B$ such that $\underset{\sim}{\mathbf{A}} \cong \underset{\sim}{\mathbf{B}}\restriction_U$. It is not hard to see that this condition is also sufficient for $\underset{\sim}{\mathbf{A}}$ to be embedded into $\underset{\sim}{\mathbf{B}}$. Namely, one simply extends an isomorphism between $\underset{\sim}{\mathbf{A}}$ and an induced substructure of $\underset{\sim}{\mathbf{B}}$ to an injective relation reflecting homomorphism between $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$, which consequently is an embedding.

Now we can turn our focus to retracts. A structure $\underset{\sim}{\mathbf{B}}$ of the same type as $\underset{\sim}{\mathbf{A}}$ is said to be a *retract* of $\underset{\sim}{\mathbf{A}}$ if there exists a retraction $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$. The latter is by definition any homomorphism $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ such that there exists a second homomorphism $h'\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$, a so-called *co-retraction*, fulfilling $h \circ h' = \mathrm{id}_B$. This equality has a number of consequences: first, it implies that h is surjective, while h' must be injective. Furthermore, we obtain the equality $h \circ \left[T^{\underset{\sim}{\mathbf{A}}}\right] = T^{\underset{\sim}{\mathbf{B}}}$ for all $T \in S_{\mathrm{rel}}$. This means that the image substructure $h[\underset{\sim}{\mathbf{A}}]$ is equal to the induced substructure of $\underset{\sim}{\mathbf{B}}$ on $h[A]$, i.e. to $\underset{\sim}{\mathbf{B}}$ (this fact is also discussed in more detail in Lemma 3.4.22(a)). Moreover, since h is relation preserving and inverts h', we get that h' is relation reflecting. In other words, h' embeds $\underset{\sim}{\mathbf{B}}$ into $\underset{\sim}{\mathbf{A}}$ whence the retract $\underset{\sim}{\mathbf{B}}$ is isomorphic to the substructure $\underset{\sim}{\mathbf{A}}\restriction_{\mathrm{im}\, h'}$ that $\underset{\sim}{\mathbf{A}}$ induces on the subset $h'[B]$. Finally, it follows that the composition $h' \circ h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}$ is an idempotent[2] endomorphism of $\underset{\sim}{\mathbf{A}}$ having the same image as h'. This means, up to isomorphism, the retract $\underset{\sim}{\mathbf{B}}$ is an induced substructure of $\underset{\sim}{\mathbf{A}}$ on the image of an idempotent endomorphism of $\underset{\sim}{\mathbf{A}}$.

Conversely, if $e\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}$ is an idempotent endomorphism of $\underset{\sim}{\mathbf{A}}$ and $U := \mathrm{im}\, e$ is its image, then $e(u) = u$ for $u \in U$ (see Lemma 3.1.3 below). Therefore, $\underset{\sim}{\mathbf{A}}\restriction_U$ is a retract of $\underset{\sim}{\mathbf{A}}$ via $e|_A^U\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_U$ as a retraction and $\mathrm{id}_A|_U^A\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{A}}$ as a co-retraction.

Consequently, a concrete description of all retracts of a relational structure $\underset{\sim}{\mathbf{A}}$ up to isomorphism is given by all induced substructures of $\underset{\sim}{\mathbf{A}}$ on images of idempotent endomorphisms of $\underset{\sim}{\mathbf{A}}$. This underlines the importance of images of idempotent endomorphisms, which will play a central role in the theory we present in the following chapter. We will sometimes call these special retracts $\underset{\sim}{\mathbf{A}}\restriction_U$ coming from idempotent endomorphisms of $\underset{\sim}{\mathbf{A}}$ *idempotent retracts* of $\underset{\sim}{\mathbf{A}}$ to distinguish them from

[2] An operation $e\colon A \longrightarrow A$ is called idempotent if $e \circ e = e$.

general retracts which may live on other sets that are not necessarily subsets of A.

Finally, as for algebras we provide the notion of direct product of relational structures. The main aspect of forming products of relational structures is to define a relation on the direct product from individual relations on the factors. To this end, let I and J be sets and suppose that for $i \in I$ we are given a set A_i and a relation $\varrho_i \subseteq A_i^J$. We now define a relation on $P := \prod_{i\in I} A_i$ in a canonical way. This relation has the same arity as the individual relations and can be regarded as a product of the relations $(\varrho_i)_{i\in I}$. However, as it is not the usual Cartesian product of the sets ϱ_i, $i \in I$, but involves a lot of re-indexing, we introduce a new symbol for it:

$$\prod\!\!\!\prod_{i\in I} \varrho_i := \left\{ \left((a_{j,i})_{i\in I}\right)_{j\in J} \in P^J \,\middle|\, \forall\, i \in I\colon\ (a_{j,i})_{j\in J} \in \varrho_i \right\}.$$

With the help of this notation, we can easily define the *product* of relational structures $\underset{\sim}{\mathbf{A}}_i$, $i \in I$, of identical signature:

$$\prod_{i\in I} \underset{\sim}{\mathbf{A}}_i := \left\langle \prod_{i\in I} A_i; \left(\prod\!\!\!\prod_{i\in I} T^{\underset{\sim}{\mathbf{A}}_i} \right)_{T\in S_{\mathrm{rel}}} \right\rangle.$$

This is clearly a structure of the same type as all its factors and it is easily seen to fulfil the universal property of a product w.r.t. the projection homomorphisms onto the factors and arbitrary homomorphisms from one structure into the factors $\underset{\sim}{\mathbf{A}}_i$, $i \in I$.

As a last remark about relational structures, we mention that, in analogy to the case of algebras, the notion of *relation variety* has been defined. The latter describes classes of structures of common signature that are closed under the formation of arbitrary products, as just defined, and retracts. Relation varieties have been studied, for instance, in [Zád97b]. For more information about them we refer the interested reader to this article.

Up to now, we have considered functions and relations, or alternatively algebras and relational structures separately. Yet, what we need in the following chapter is a machinery that allows us to change our point of view from either of the two to the other. This will be developed in the next section on clones.

2.3 Clones

Clones are special collections of finitary operations or also of finitary relations on a fixed set. If this set is finite, both notions of clone are in a one-to-one correspondence via the Galois connection of so-called polymorphisms and invariant relations.

We first clarify the two concepts of clones, then we discuss their relationship to the mentioned Galois connection. In this respect we roughly follow the exposition chosen in [Beh11].

2.3.1 Definition. A *clone* on a set A is any subset $F \subseteq \mathrm{O}_A$ subject to the following two conditions: first, it contains all projection operations, i.e. $\mathrm{J}_A \subseteq F$, and second, it is closed against general composition of finitary operations. In a little bit more detail, this means that whenever $f \in F^{(n)}$ and $(g_0, \ldots, g_{n-1}) \in \left(F^{(m)}\right)^n$, then it is required that also the composition $f \circ (g_0, \ldots, g_{n-1})$ belongs to the set F. ◊

We mention that the set $F \subseteq \mathrm{O}_A$ in Definition 2.3.1 is allowed to contain nullary operations. This is not quite standard in clone theory. For historical reasons the majority of publications on clones, including the standard monographs such as [PK79, Sze86, Ros70, Lau06], defines the notion in the same way as above but exclusively for sets $F \subseteq \mathrm{O}_A \setminus \mathrm{O}_A^{(0)}$ of non-nullary operations. There are however a few sources, e.g. [MMT87, Coh65], acknowledging the usefulness of nullary operations for clones as we are doing this here. For a more detailed discussion of the advantages of this more general approach see also Section 1 of [Beh11]. The latter also provides a detailed analysis of the relationship between the traditional notions of clone, the corresponding closure operators and its Galois theory to the counterparts in the setting allowing nullary operations. For the classical case without nullary operations the main results we are going to need can be read, for instance, from [Pös79, Pös80] or [Sze86]. A comprehensive treatment of clones, especially focussing on finite carrier sets, can be found in [PK79] (in German) or in [Lau06] (in English).

The results presented in this section belong to the well-established background knowledge of clone theory. We shall give references to [Beh11] in a few cases, because the proofs we omit here can all be found there. The results proven in the latter report are in fact nothing but slight technical improvements of the theorems stated in [Pös80]. We deliberately do not bother with tracking all of these statements back to their original sources for we fear attributing them to the wrong authors due to insufficient knowledge. A detailed account of the sometimes quite intricate constellation of traces to the historic origins of these and related theorems can also be found in [Pös80].

As already announced there is a second notion of clone for finitary relations. The definition uses a construction which looks quite technical, but this is only at first glance. The idea is actually simple: one takes tuples from relations, glues them together on a big tuple that can be chosen as large as necessary, and then one projects to a finite number of coordinates. All the tuples arising in this way are collected in a new relation, which is then required to be in the relational clone, too. What makes this construction powerful is the vast choice in the parameters defining exactly how the gluing and the projection is done.

We first define this general composition operation in a bit more generality than we need it for relational clones. To the author's best knowledge, this construction first appears under the name *general superposition* in 3.2 Definitions(R4) of [Pös80, p. 27] for relations of finite arity.

2.3.2 Definition. Let A be a fixed carrier set of relations, I an (index) set and m, κ, and m_i for $i \in I$, be sets (thought to play the roles of arities). Moreover, let

$\alpha_i\colon m_i \longrightarrow \kappa$ for $i \in I$ and $\beta\colon m \longrightarrow \kappa$ be mappings and $\varrho_i \subseteq A^{m_i}$ for $i \in I$ be sets of mappings (interpreted as relations). Then

$$\prod_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} := \{a \circ \beta \mid a \in A^\kappa \wedge \forall\, i \in I\colon a \circ \alpha_i \in \varrho_i\}$$

defines a relation contained in A^m, which is called *general composition* (or *general superposition*) of the relations $(\varrho_i)_{i\in I}$. ◊

Note that the carrier set A of the relations is an implicit parameter of the operator $\prod$, which we do not include in the notation. Besides, we have used here again the double role played by tuples in relations as actually being functions. Thereby, we can compose them with other functions, this time on the side of indices and not on the side of values as in Section 2.1.

With the operators just defined at hand it is now easy to describe what relational clones are.

2.3.3 Definition. A *relational clone* (or *clone of relations*) on a set A is any subset $Q \subseteq \mathrm{R}_A$ of finitary relations that is closed w.r.t. arbitrary general compositions that result again in finitary relations.

Formally, we require that for any sets I and κ, natural numbers m and any sequence $(m_i)_{i\in I} \in \mathbb{N}^I$, mappings $\beta\colon m \longrightarrow \kappa$ and $(\alpha_i)_{i\in I} \in \prod_{i\in I} \kappa^{m_i}$ the following implication holds: whenever $(\varrho_i)_{i\in I} \in \prod_{i\in I} Q^{(m_i)}$, then also $\prod_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} \in Q$. ◊

We mention that often in the literature relational clones are only studied on finite carrier sets A. Then much simpler definitions can be used. In fact for finite A, one only needs to consider the closure w.r.t. a finite number of partial operations on finitary relations. Details about this can be found, for instance, in [PK79, 1.1.2, 1.1.3 and 1.1.8] or [Pös80, 2.3]. In the general case at least a small simplification can be made: depending on A there exist cardinal bounds on the sets I and κ needed in Definition 2.3.3.

With the two definitions of clone we have established the structure of a partial algebra on the sets of all finitary operations and relations, respectively. In more detail, the projection operations can be seen as nullary operations on O_A, and composition of n-ary operations with operations of other, yet common, arity can be seen as an $(n+1)$-ary partial[3] operation on O_A. This interpretation yields a function $\circ_n\colon \mathrm{O}_A^{(n)} \times \bigcup_{k\in\mathbb{N}} \left(\mathrm{O}_A^{(k)}\right)^n \longrightarrow \mathrm{O}_A$ for each $n \in \mathbb{N}$. Definition 2.3.1 simply states that a subset $F \subseteq \mathrm{O}_A$ is a clone on A precisely if it is a subuniverse[4] of this partial algebra. One can easily extend each of the operations $\circ_n$, $n \in \mathbb{N}$, in a conservative way to a full $(n+1)$-ary operation on O_A, namely by projecting

[3] A partial operation is simply a function that is defined on a specific subset of its domain.

[4] Subuniverse is to be understood in a similar way as described above for algebras of total operations, namely as a subset containing all results of partial fundamental operations applied to argument tuples from the subset lying in the domain of the respective operation.

to the first argument if the arities of the argument functions do not fit into the composition scheme. Similarly, the operations $\prod^{\beta}_{(\alpha_i)_{i\in I}} : \prod_{i\in I} \mathrm{R}_A^{(m_i)} \longrightarrow \mathrm{R}_A$ used in Definition 2.3.3 are infinitary partial operations on R_A. Again these can be extended to total operations between ${\mathrm{R}_A}^I$ and R_A by conservatively projecting to one specific coordinate in I in case the arities of the argument relations do not match the $(m_i)_{i\in I}$. As for clones of operations, a subset $Q \subseteq \mathrm{R}_A$ is a relational clone on A if and only if it forms a subuniverse of the described partial algebra[5] of infinitary operations on R_A (or equivalently of the extended algebra of infinitary operations).

Employing the interpretation of clones as algebras of partial operations, we may also use the notion of *clone homomorphism*. The latter is any mapping between two clones that does not change the arities of the arguments and commutes with the partial operations of the clones wherever they are defined. If the partial operations are extended via projection to total ones as outlined above, then being a clone homomorphism is certainly equivalent to being an arity-preserving homomorphism between the algebras of total (albeit infinitary for relational clones) operations belonging to the clones.

It is easy to see from the definitions of clones of operations and relations (or from our interpretation as subuniverses of certain algebraic structures) that the collections of all clones form closure systems on O_A and R_A, respectively. The description of the corresponding closure operator $[\,]_{\mathrm{R}_A}$ on R_A is technical and not needed for our work. On finite carrier sets A the *relational clone generated by a set* $Q \subseteq \mathrm{R}_A$ of finitary relations can be described as closure against all interpretations of primitive positive first-order formulæ using atomic relations from Q and the equality predicate. Replacing an arbitrary such formula by its equivalent prenex normal form, it suffices to close against interpretations of formulæ that can be written as a finite prefix of existential quantifiers followed by a finite conjunction of variable substitutions in the basic relations and equalities of variables. For infinite sets A one needs to use stronger logics allowing for instance at least infinite conjunctions and infinite sequences of existential quantifiers.

However, for sets $F \subseteq \mathrm{O}_A$ of finitary operations we can describe the *clone* $\langle F\rangle_{\mathrm{O}_A}$ *generated* by the operations in F in a very elegant way. Regarding $\mathbf{A} = \langle A; F\rangle$ as an indexed algebra via its canonically induced signature, one can show that the least clone on A containing F equals the set $\mathrm{Term}\,(\mathbf{A})$ of all term operations of $\mathbf{A}$. On the other hand, if $\mathbf{A}$ is an algebra indexed by a signature, then $\mathrm{Term}\,(\mathbf{A})$ is a clone on A, namely the clone generated by the set $\left\{ f^{\mathbf{A}} \;\middle|\; f \in S_{\mathrm{op}} \right\}$ of fundamental operations.

This relationship allows two interpretations. First, generating clones via term operations can be understood as a syntactic description of the closure as, at least

[5]To turn it into an algebra, we need to ensure that the operations considered for the closure actually form a set and not a proper class. Since A is fixed, this can be achieved by using the cardinality bounds on I and κ mentioned above and only considering ordinals below these bounds instead of arbitrary sets I and κ.

in theory, one first produces all terms and then evaluates them to term operations. Second, describing the set of term operations as the least clone containing the fundamental operations of an algebra yields a presentation independent way of obtaining the clone of term operations. Operation symbols and terms are actually not required, just the fundamental operations and closure against projections and composition (or more abstractly intersection of all clones containing the given operations).

Next, we link both types of clones via "[t]he most basic Galois connection in algebra" [MMT87, p. 147, l. 20 et seq.]. Before we describe this in detail, let us recall what a Galois connection is in general.

If X and Y are sets, then, formally, a Galois *connection* between X and Y is any pair (φ, ψ) of mappings $\varphi\colon \mathfrak{P}(X) \longrightarrow \mathfrak{P}(Y)$ and $\psi\colon \mathfrak{P}(Y) \longrightarrow \mathfrak{P}(X)$ such that they are both *antitone*, i.e. φ is a relation preserving map from $(\mathfrak{P}(X), \subseteq)$ to $(\mathfrak{P}(Y), \supseteq)$ and similarly for ψ, and their compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are extensive (see Section 2.1 above). A straightforward calculation shows that then $\varphi \circ \psi \circ \varphi = \varphi$ and $\psi \circ \varphi \circ \psi = \psi$, and that the compositions $\psi \circ \varphi$ and $\varphi \circ \psi$ are closure operators on X and Y, respectively. Furthermore, it is well-known that whenever $R \subseteq X \times Y$ is a binary relation, then one can define a Galois connection via $\varphi(A) := \{y \in Y \mid \forall x \in A\colon (x, y) \in R\}$ for every subset $A \subseteq X$ and $\psi(B) := \{x \in X \mid \forall y \in B\colon (x, y) \in R\}$ for $B \subseteq Y$. In fact, any abstract Galois connection (φ, ψ) between X and Y can be given in this way by using the binary relation $R := \{(x, y) \in X \times Y \mid \{x\} \subseteq \psi(\{y\})\}$. What is more, the described correspondence between Galois connections and binary relations is indeed one-to-one, such that Galois connections in the concrete sense above are in fact really nothing else but binary relations.

How does now the binary relation belonging to "[t]he most basic Galois connection in algebra" look like? As it is supposed to link clones of operations, which are subsets of O_A, to relational clones, which are subsets of R_A, it needs to be a relation between the sets $X = \mathrm{O}_A$ and $Y = \mathrm{R}_A$. It is called *preservation relation* (or *compatibility relation* or *invariance relation*) and can be defined as

$$\rhd := \left\{ (f, S) \in \mathrm{O}_A \times \mathrm{R}_A \;\middle|\; f \circ \left[S^{\operatorname{ar} f}\right] \subseteq S \right\}.$$

One can verify in a quiet moment of calculation that for $m, n \in \mathbb{N}$, $f \in \mathrm{O}_A^{(n)}$ and $S \in \mathrm{R}_A^{(m)}$ the preservation property can be characterised as follows:

$$\begin{aligned} f \rhd S \iff & S \leq \langle A; f\rangle^m \\ \iff & f \in \operatorname{Hom}\left(\langle A; S\rangle^n, \langle A; S\rangle\right). \end{aligned}$$

This explains the following naming conventions. If $f \rhd S$, read as f *preserves* S, S *is compatible with* f or S *is invariant under* f, then S is a subuniverse of the m-th direct power of the algebra $\langle A; f\rangle$. Thus, it is sometimes called a *subpower*. Using the second characterisation, f is a (homo)morphism from a finite direct power of $\langle A; S\rangle$ into this relational structure. Therefore, f is also said to be a *polymorphism* of S in this case.

Using the preservation relation we can now define a Galois connection (Pol, Inv) in the canonical way, the Galois connection of polymorphisms and invariant relations. Explicitly, for sets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ we have

$$\mathrm{Pol}_A\, Q := \{ f \in \mathrm{O}_A \mid \forall S \in Q\colon f \rhd S \}$$

as the set of all polymorphisms of Q, and

$$\mathrm{Inv}_A\, F := \{ S \in \mathrm{R}_A \mid \forall f \in F\colon f \rhd S \}$$

as the set of all invariant relations of F. We extend these operators to work also with algebras and relational structures in the obvious way: for $\mathbf{A} = \langle A; F \rangle$ we put $\mathrm{Inv}\,\mathbf{A} := \mathrm{Inv}_A\, F$ and $\mathrm{Pol}\,\underset{\approx}{\mathbf{A}} := \mathrm{Pol}_A\, Q$ for $\underset{\approx}{\mathbf{A}} = \langle A; Q \rangle$. Using these abbreviations, we can concisely formulate the following two simple characterisations that follow from the characterisation of the preservation condition above:

$$\mathrm{Pol}\,\underset{\approx}{\mathbf{A}} = \bigcup_{n \in \mathbb{N}} \mathrm{Hom}\,(\underset{\approx}{\mathbf{A}}^n, \underset{\approx}{\mathbf{A}}) \qquad\qquad \mathrm{Inv}\,\mathbf{A} = \bigcup_{m \in \mathbb{N}} \mathrm{Sub}\,(\mathbf{A}^m)\,.$$

Moreover, for $n \in \mathbb{N}$ we abbreviate $\mathrm{Pol}^{(n)}\underset{\approx}{\mathbf{A}} := \mathrm{Pol}_A^{(n)} Q := \mathrm{O}_A^{(n)} \cap \mathrm{Pol}_A\, Q$ and likewise $\mathrm{Inv}^{(n)}\mathbf{A} := \mathrm{Inv}_A^{(n)} F := \mathrm{R}_A^{(n)} \cap \mathrm{Inv}_A\, F$. Thus, it is $\mathrm{Pol}^{(n)}\underset{\approx}{\mathbf{A}} = \mathrm{Hom}\,(\underset{\approx}{\mathbf{A}}^n, \underset{\approx}{\mathbf{A}})$ and $\mathrm{Inv}^{(n)}\mathbf{A} = \mathrm{Sub}\,(\mathbf{A}^n)$. In particular, for $n = 1$ it follows that $\mathrm{Pol}^{(1)}\underset{\approx}{\mathbf{A}} = \mathrm{End}\,(\underset{\approx}{\mathbf{A}})$.

With the following displayed formula we stress the definition

$$\mathrm{Clo}\,(\mathbf{A}) := \mathrm{Pol}_A\, \mathrm{Inv}\,\mathbf{A}$$

for any algebra $\mathbf{A}$, in contrast to many other authors who use the symbol $\mathrm{Clo}\,(\mathbf{A})$ to mean $\mathrm{Term}\,(\mathbf{A})$. We refer to the defined set as *the locally closed clone associated with* $\mathbf{A}$, or simply *clone of* $\mathbf{A}$ for short, as opposed to the clone of term operations of $\mathbf{A}$. Throughout the text we call the functions in $\mathrm{Clo}\,(\mathbf{A})$ *clone operations of* $\mathbf{A}$ to distinguish them from term operations. The reasons for the difference will be comprehensible quite soon. First, however, we introduce the same sort of abbreviation for Clo as for Inv and Pol above: $\mathrm{Clo}^{(n)}\,(\mathbf{A}) := \mathrm{O}_A^{(n)} \cap \mathrm{Clo}\,(\mathbf{A})$ denotes the n-ary part of the clone of $\mathbf{A}$, i.e. $\mathrm{Pol}_A^{(n)}\, \mathrm{Inv}\,\mathbf{A}$.

Understanding why $\mathrm{Clo}\,(\mathbf{A})$ is called (locally closed) clone associated with $\mathbf{A}$ requires a more detailed investigation of the operators $\mathrm{Pol}_A\,\mathrm{Inv}_A \colon \mathfrak{P}\,(\mathrm{O}_A) \longrightarrow \mathfrak{P}\,(\mathrm{O}_A)$ and $\mathrm{Inv}_A\,\mathrm{Pol}_A \colon \mathfrak{P}\,(\mathrm{R}_A) \longrightarrow \mathfrak{P}\,(\mathrm{R}_A)$ coming with the Galois connection (Pol, Inv). This is, of course, inextricably linked with the study of the corresponding closure systems $\mathrm{im}\,\mathrm{Pol}_A$ and $\mathrm{im}\,\mathrm{Inv}_A$.

The easy part of the answer is the following result, which determines the closures to be certain clones and occurs as part of Lemma 2.6 in [Beh11]. For clones without nullary operations, one may also consider Lemma 2.5 and Proposition 3.8 of [Pös80]. We point out again that these references are not meant to reflect the exact historic origin of the statement.

2.3.4 Lemma. *For any carrier set A and all sets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ the polymorphism set $\mathrm{Pol}_A\, Q$ is a clone of operations on A and $\mathrm{Inv}_A\, F$ is a relational clone on A. This implies the inclusions*

$$\langle F \rangle_{\mathrm{O}_A} \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F \qquad \textit{and} \qquad [Q]_{\mathrm{R}_A} \subseteq \mathrm{Inv}_A \mathrm{Pol}_A Q.$$

This lemma states that for any algebra $\mathbf{A} = \langle A; F \rangle$ we have the inclusion

$$\mathrm{Term}\,(\mathbf{A}) = \langle F \rangle_{\mathrm{O}_A} \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F = \mathrm{Clo}\,(\mathbf{A})\,,$$

i.e., $\mathrm{Clo}\,(\mathbf{A})$ is a clone of operations containing the clone of term operations generated by the fundamental operations of $\mathbf{A}$. However, in general (for infinite carrier sets A) this inclusion, as both inclusions in Lemma 2.3.4, can be proper. So, generally, the two clone closures are not strong enough to describe the closure operators belonging to $(\mathrm{Pol}, \mathrm{Inv})$. To compensate this deficiency one needs additional closure operators on O_A and R_A, respectively. These are called local closures and are in fact topological closures arising naturally if one interprets A as a discrete topological space and constructs the corresponding product space.

2.3.5 Definition. For any set A and subsets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$ we define

$$\mathrm{Loc}_A F := \bigcup_{n \in \mathbb{N}} \left\{ f \in \mathrm{O}_A^{(n)} \;\middle|\; \forall B \subseteq A^n, 0 \leq |B| < \aleph_0 \, \exists\, g \in F^{(n)} \colon g|_B^A = f|_B^A \right\}$$

$$\mathrm{LOC}_A Q := \bigcup_{m \in \mathbb{N}} \left\{ S \in \mathrm{R}_A^{(m)} \;\middle|\; \forall B \subseteq S, 0 \leq |B| < \aleph_0 \, \exists\, T \in Q^{(m)} \colon B \subseteq T \subseteq S \right\}$$

and call these sets *local closures* of F and Q, respectively. ◊

It is not difficult to check that the operations $\mathrm{Loc}_A \in \mathrm{O}_{\mathrm{O}_A}^{(1)}$ and $\mathrm{LOC}_A \in \mathrm{O}_{\mathrm{R}_A}^{(1)}$ defined in this way are indeed closure operators. We call a set $F \subseteq \mathrm{O}_A$ or $Q \subseteq \mathrm{R}_A$ *locally closed* if it belongs to the corresponding closure system, i.e. if $\mathrm{Loc}_A F = F$ or $\mathrm{LOC}_A Q = Q$, respectively. Furthermore, as above, we agree on the short notations $\mathrm{Loc}_A^{(n)} F := \mathrm{O}_A^{(n)} \cap \mathrm{Loc}_A F$ and $\mathrm{LOC}_A^{(n)} Q := \mathrm{R}_A^{(n)} \cap \mathrm{LOC}_A Q$ for $n \in \mathbb{N}$. Using Definition 2.3.5, it is not hard to see (and a proof can be found as part of [Beh11, Lemma 2.8]) that the local closure operators work arity wise, i.e. that we have

$$\mathrm{Loc}_A^{(n)} F = \mathrm{Loc}_A \left(F^{(n)} \right) \qquad \text{and} \qquad \mathrm{LOC}_A^{(n)} Q = \mathrm{LOC}_A \left(Q^{(n)} \right)$$

for $F \subseteq \mathrm{O}_A$, $Q \subseteq \mathrm{R}_A$ and $n \in \mathbb{N}$.

Besides, it is also proven in Lemma 2.8 of [Beh11] that

$$\mathrm{Loc}_A \mathrm{Pol}_A Q = \mathrm{Pol}_A Q \qquad \text{and} \qquad \mathrm{LOC}_A \mathrm{Inv}_A F = \mathrm{Inv}_A F$$

hold for all $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$. This means that polymorphism sets are not only clones, but moreover, locally closed clones of operations, and similarly for sets of invariant relations and relational clones. Thus, for an algebra $\mathbf{A}$, the set $\mathrm{Clo}\,(\mathbf{A})$

is *a* locally closed clone containing all fundamental operations of $\mathbf{A}$. Why is it *the* locally closed clone associated with $\mathbf{A}$?

Combining the two inclusions stated in Lemma 2.3.4 with the previous observation that all Galois closures w.r.t. $(\mathrm{Pol}, \mathrm{Inv})$ are locally closed, we obtain

$$\mathrm{Loc}_A \langle F \rangle_{\mathrm{O}_A} \subseteq \mathrm{Pol}_A \mathrm{Inv}_A F \qquad \text{and} \qquad \mathrm{LOC}_A [Q]_{\mathrm{R}_A} \subseteq \mathrm{Inv}_A \mathrm{Pol}_A Q$$

for $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$. The following theorem states that these inclusions are actually equalities, and therefore it characterises the Galois closures w.r.t. $(\mathrm{Pol}, \mathrm{Inv})$ as precisely those (relational) clones on A that are locally closed. Consequently, $\mathrm{Clo}(\mathbf{A})$ is the least locally closed clone on A w.r.t. set inclusion, containing the fundamental operations of $\mathbf{A}$ (or the set $\mathrm{Term}(\mathbf{A})$).

2.3.6 Theorem. *For any carrier set A and all subsets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$, we have*

$$\mathrm{Loc}_A \langle F \rangle_{\mathrm{O}_A} = \mathrm{Pol}_A \mathrm{Inv}_A F \qquad \text{and} \qquad \mathrm{LOC}_A [Q]_{\mathrm{R}_A} = \mathrm{Inv}_A \mathrm{Pol}_A Q.$$

A proof of this result can be found in [Beh11, Theorem 3.23], or in [Pös80, Theorems 4.1(a),4.2(a)] for clones without nullary operations. There, one can also find an extensive list of references to precursors and related results.

A simplification of this result can be obtained in case that the carrier set A is finite. Then for every $n \in \mathbb{N}$ also the power A^n is a finite set, and so is any of its subsets, i.e. any n-ary relation on A. Thus, one can always choose the largest possible set B in the definition of the local closure operators (see Definition 2.3.5) to show that $\mathrm{Loc}_A F \subseteq F$ and $\mathrm{LOC}_A Q \subseteq Q$ holds for all $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$. In other words, on a finite carrier set A all sets of operations or relations are automatically locally closed, which results in the following corollary and demonstrates that examples exhibiting proper inclusions in Lemma 2.3.4 need to live on infinite carrier sets.

2.3.7 Corollary. *For any finite carrier set A and all subsets $F \subseteq \mathrm{O}_A$ and $Q \subseteq \mathrm{R}_A$, we have*

$$\langle F \rangle_{\mathrm{O}_A} = \mathrm{Pol}_A \mathrm{Inv}_A F \qquad \text{and} \qquad [Q]_{\mathrm{R}_A} = \mathrm{Inv}_A \mathrm{Pol}_A Q.$$

This result was certainly known before Theorem 2.3.6 was first published. In the literature it is often attributed to [Gei68, BKKR69], and we simply copy this reference here.

Focussing once more on algebras $\mathbf{A}$ explicitly, Theorem 2.3.6 states that generally $\mathrm{Clo}(\mathbf{A}) = \mathrm{Loc}_A \mathrm{Term}(\mathbf{A})$. If the carrier set of $\mathbf{A}$ is finite, then we can omit the local closure operator and there is no difference between term operations and clone operations of $\mathbf{A}$.

In the next chapter we want to build a decomposition theory based on the described Galois theory between finitary operations and relations. Since we do not restrict our focus to finite structures, by Theorem 2.3.6, the natural sort of clones to look at will be locally closed ones. More specifically, with every algebra $\mathbf{A} = \langle A; F\rangle$ we will associate a relational structure on the same carrier set, carrying all invariant relations of $\mathbf{A}$ as fundamental relations. It is inherent to our approach that we will not be able to discern algebras on the same carrier set having identical relational clones of invariant relations. This is due to the fact that our main definitions simply will only depend on $\operatorname{Inv} \mathbf{A}$ or alternatively on $\operatorname{Clo}(\mathbf{A})$, not on the actual fundamental operations of $\mathbf{A}$. Thus, our theory allows an analysis of algebras up to the following sort of equivalence that, for lack of a better word, has been called *algebraic equivalence* in [Beh09].

Here we want to coin a more intuitive name: in the following we shall call two algebras $\mathbf{A}$ and $\mathbf{B}$ *locally term equivalent*[6] if $A = B$ and $\operatorname{Inv} \mathbf{A} = \operatorname{Inv} \mathbf{B}$. By virtue of our Galois connection, this condition is equivalent to equality of the carrier sets, $A = B$, and equality of the associated locally closed clones $\operatorname{Clo}(\mathbf{A}) = \operatorname{Clo}(\mathbf{B})$.

Of course $\operatorname{Clo}(\mathbf{A}) \subseteq \operatorname{Clo}(\mathbf{B})$ implies $F \subseteq \operatorname{Clo}(\mathbf{B}) = \operatorname{Loc}_A \operatorname{Term}(\mathbf{B})$ for the fundamental operations F of $\mathbf{A}$. In other words, every fundamental operation of $\mathbf{A}$ is locally approximated by term operations of $\mathbf{B}$. This is also sufficient for the inclusion $\operatorname{Clo}(\mathbf{A}) \subseteq \operatorname{Clo}(\mathbf{B})$. Namely, by monotonicity of the clone closure and the fact that $\operatorname{Clo}(\mathbf{B})$ is a clone we obtain $\langle F\rangle_{\mathrm{O}_A} \subseteq \operatorname{Clo}(\mathbf{B})$, and hence

$$\operatorname{Clo}(\mathbf{A}) = \operatorname{Loc}_A \operatorname{Term}(\mathbf{A}) = \operatorname{Loc}_A \langle F\rangle_{\mathrm{O}_A} \subseteq \operatorname{Loc}_A \operatorname{Clo}(\mathbf{B}) = \operatorname{Clo}(\mathbf{B})$$

by monotonicity of the local closure and the fact that $\operatorname{Clo}(\mathbf{B})$ is locally closed.

Thus, local term equivalence of $\mathbf{A}$ and $\mathbf{B}$ expresses that the operations of $\mathbf{A}$ are locally, that is on finite subsets of their domain, term operations (or clone operations) of $\mathbf{B}$ and vice versa. This interpretation explains why we have called the equivalence local term equivalence.

Furthermore, local term equivalence is an evident relaxation of the notion of term equivalence, that is well-recognised throughout the literature. Two *finite* algebras $\mathbf{A}$ and $\mathbf{B}$ are locally term equivalent if and only if their carriers and their clones of term operations coincide. This is precisely the defining condition for *term equivalence* of algebras.

Thus, the theory to be presented in the following chapter allows to study algebras up to local term equivalence, and moreover finite algebras up to standard term equivalence.

[6] The definition would also suggest to call such algebras *relationally equivalent*.

3 Relational Structure Theory

In Chapter 1 we have already presented details on the historical development of Relational Structure Theory (abbreviated RST). Here we recollect the following central facts.

Tame Congruence Theory (TCT), see e.g. [McK83, HM88, Kis97, CV02], is a deep structure theory for mainly finite universal algebras and such lying in so-called locally finite varieties. As its name suggests it provides understanding of algebraic structures via properties of their congruence lattices in relationship to their clones of polynomial operations.

Having been applied successfully over the last 25 years, e.g. to settle several instances of the so-called RS-conjecture [KKV99, KK03] or the study of complexity of constraint satisfaction problems (CSP), there has been the wish of the research community for a generalisation of TCT to work with term operations instead of polynomials, too. This is reflected, for instance, in [KSKM01, Problem 9.3] and allows a more fine grained understanding of algebras, because any two term equivalent algebras also have coinciding clones of polynomial operations[1], but not conversely.

Attacking this problem, Keith Kearnes and Ágnes Szendrei in July, 2001 presented ideas on how main notions and constructions from TCT could also work for term operations. Lecture notes of the talks given during the workshop titled "A Course in Tame Congruence Theory" in Budapest can be found in [Kea01]. Based on the Galois theory of polymorphisms and invariant relations, they replaced the role of polynomial operations by term operations and that of congruences by arbitrary invariant relations. Thereby they laid the foundations for a localisation theory for finite algebras and such generating locally finite varieties. Combining this body of theory with the ground-breaking characterisation of categorical equivalence of varieties by Ralph McKenzie, [McK96], in terms of matrix powers and restriction to images of idempotents, they explained that their theory—in contrast to standard TCT—allowed a complete reconstruction of a finite algebra from sufficiently many local pieces, up to term equivalence.

The first detailed written account of these relationships, including proofs for the main theorems, was given in 2009 with [Beh09], which focussed chiefly on finite algebras. From then on the author of this thesis and collaborators have denominated the body of theory that came into being with Kearnes and Szendrei's work "Relational Structure Theory", abbreviated "RST". This acknowledges its descent from TCT, which is a structure theory for algebras, and the fundamental

[1] They are also called *polynomially equivalent*.

use of invariant relations within the theory.

Subsequent publications closely related to RST concerned the following topics: [Beh12] provides a simpler proof for the characterisation of so-called irreducible subalgebra primal algebras given in [Beh09], the article [KL10] contains a characterisation of a special kind of Maľcev condition (congruence 3-permutability plus existence of a near unanimity term) for finitely generated varieties via intrinsic properties of RST, and [Iza13] characterises categorical equivalence of finite algebras by so-called weak isomorphism of matrix products belonging to non-refinable covers. Finally, as of 2012, there is a generalisation of the whole theory to the more comprehensive setting of topological algebras available, see [Sch12].

In this chapter we extend the presentation of RST for finite algebras that was given in [Beh09] to algebras generating locally finite varieties and often to infinite algebras beyond that limit. Thereby, we solve the fifth open problem stated in [Beh09, p. 148]. We elaborate the main topics of [Beh09] not only on this more general level, thereby suitably adapting definitions and results, as well as introducing new concepts, but also provide a much more detailed analysis of RST-notions and their interrelationships.

In this respect, we shall in particular replace flat finiteness reasoning from [Beh09] by conceptual arguments giving a better insight into the causal structure of the theory. For a better understanding we give a short example: it follows from finiteness of an algebra, that its monoid of unary clone operations only contains a finite number of functions. Therefore, each of them can be iterated a finite number of times until we obtain a power of the function that is idempotent. We will introduce the name *finite iteration property* (FIP) for this behaviour of unary clone operations of an algebra. Certainly, this is a weaker, i.e. more general property, than having a finite monoid of unary clone operations or than having a finite carrier set.

Other concepts modelled from the finite case in a similar way include neighbourhood self-embedding simplicity, different degrees of ARTINianness or the existence of precover and cover bases of algebras. We mention that even in the infinite case the way how we present the results, the techniques we use to prove them and the notions we define are still heavily inspired by the arguments used for finite algebras. This is done on purpose, so it is a feature of our theory, contributing to an easier applicability, rather than a bug.

For instance, in some places, we could probably have replaced the assumption of the FIP by a local version, only requiring a similar property w.r.t. approximation on finite subsets. This would have allowed us to get stronger results, but at the price of losing the spirit of finiteness. The more local approximation we use—and for some of the main results we do, the more we implicitly apply topological reasoning (hidden by considering our algebras as discrete topological algebras) instead of algebraic arguments. This however, can be done taking a much more general perspective as in [Sch12], where the work of [Kea01] is extended to (HAUSDORFF) topological algebras. Quite a few results there are similar to ours and imply parts of the statements here via simply considering the special case of the underlying topology being discrete. In this regard, we have tried to give precise references to the corresponding results. Yet, in most cases our characterisations are more detailed, and towards the end of the chapter, especially in Sections 3.5, 3.6 and 3.7, we focus on different aspects than done in [Sch12]. For instance, in Section 3.7 we present a solution to the third open problem from [Beh09, p. 147 et seq.] asking for a constructive description of so-called non-refinable covers.

We now give an overview of the content of the individual sections in this chapter, mention the main results and comment on their importance for RST. Generally a localisation theory should at least address the following three aspects: it needs to provide information about which structures are appropriate to localise to. In case of RST this means to identify suitable subsets of an algebra. Second, it should clarify how localisation is done, and how local structures inherit given information from the global structure. Third, one wants to understand how knowledge obtained by local reasoning can be combined to yield knowledge about the original structure. For RST, at least for algebras in locally finite varieties, more than just that is possible. From so-called *covers*, we can reconstruct our original algebra up to (local) term equivalence. Combining the results applicable to algebras generating locally finite varieties in Section 3.4 with the theorems about categorical equivalence of varieties proven in Section 3.5 of [Beh09], we can achieve more. We can infer any information about the global algebra that can be encoded inside $\operatorname{Var}\mathbf{A}$ in terms of category theory also from the local structures corresponding to a cover.

The aim of this chapter is a systematic and detailed presentation of RST. In this respect, we follow the exposition in [Kea01] and [Beh09], at least for the first few sections. In the course of presentation of the theory, certain concepts or results remind of similar ones in classical TCT. We will give references to [HM88] for further reading, where appropriate.

Section 3.1 deals with the first question about identifying suitable subsets for localisation. Its main result is Proposition 3.1.12 characterising the "good" subsets, called *neighbourhoods* in Section 3.2. There we examine especially the relational structure of neighbourhoods with the main results being Propositions 3.2.8 and 3.2.10, characterising isomorphism and embedding of neighbourhoods. Subsequently, we consider the second main aspect of localisation, namely how to localise, i.e. to compute local structures and to transfer information to them. This is done in Section 3.3, its main achievement probably being the description of the local algebraic structure in Lemma 3.3.4 and the examples explaining how to transfer properties from the global to the local level.

Section 3.4 is dedicated to globalisation, i.e. to provide reconstructibility via covers. The principal contributions of this section are the characterisation of the cover implication in Lemma 3.4.17, the definition of ARTIN*ianness* in 3.4.18 and that of *crucial pair* in 3.4.19, and finally the arity-wise characterisation of covers in Theorem 3.4.31, as well as its Corollaries 3.4.32–3.4.37. Finally, this section finishes with connections of covers to categorical equivalence of algebras, especially of finite ones as studied in [Beh09]. Since, compared to Section 3.5 of [Beh09], we have nothing essentially new to add to the reconstruction process via categorical equivalence, we just sketch it and refer to the mentioned source for details. A concretely elaborated extension of the essential RST concepts, such as the distribution of neighbourhoods, irreducibility, covering etc. from one algebra to its generated (assumed to be locally finite) variety would of course be desirable, yet breaking the mould. We have therefore moved this project into the chapter on open problems. Moreover, we also comment on possible applications of RST in connection with characterising categorical equivalence of finite algebras in the spirit of [Iza13]. Likewise, we do not give any details in order not to go beyond the constraints of this thesis.

The next three sections essentially study existence and uniqueness of decompositions allowing a reconstruction as described in Section 3.4. Generally, existence of decompositions is no problem, however, many are trivial or extremely redundant. So, to get to uniqueness, we need to restrict the pool of admissible decompositions to such that are in a certain sense "optimal". This optimality condition for covers is termed *non-refinablility*. Furthermore, by re-introducing parametric versions of notions from [Kea01, Beh09] and defining new ones with parameters being quasiorder relations on the set of all neighbourhoods of an algebra, we prepare a possible generalisation of this localisation theory ranging further than just algebras or relational structures. Details about this can be found in open problem (12).

At least with the way how we educe existence and uniqueness of "optimal" decompositions in these sections we intend to exhibit what such a result really depends on. We are convinced that the concepts we develop in the course of the demonstration reveal that, somewhat surprisingly, existence and uniqueness of non-refinable covers is a global property of an algebra and only marginally related to our localisation[2]. This is a conclusion that is hard to be seen from the proofs of [Beh09] whose exposition is in this respect clearly inferior to the current one.

In detail, Section 3.5 introduces a *refinement* quasiorder on collections of neighbourhoods and studies and characterises non-refinability. The principal goal of this section is to develop algorithms producing—at least theoretically—non-refinable covers. They are primarily intended for application to finite algebras, but usually formulated using more general assumptions. As a by-product the notion of *irreducibility* pops up, which describes algebras, or alternatively neighbourhoods, that do not possess further non-trivial decompositions. The principal result of this section is Algorithm 4 together with Lemma 3.5.32 proving its correctness and giving conditions for termination.

Section 3.6 studies irreducibility in more detail, and presents a list of familiar irreducible structures, see Example 3.6.1. Moreover, an irreducibility notion for neighbourhoods is introduced which has a pair of invariant relations as parameter (so-called (S,T)-*irreducibility*). This is then used to characterise irreducibility of neighbourhoods in Proposition 3.6.15 and that of finite neighbourhoods in Propos-

[2]Certainly, iterative localisation of structures can help in determining such covers, but it is not essential. Just examining the properties of the localising subsets of the global structure is in general sufficient.

ition 3.6.16. Furthermore, under suitable assumptions it is shown that properties of (S, T)-irreducible neighbourhoods for special pairs of relations, e.g. crucial pairs from Section 3.4, are easier to control than those of arbitrary irreducible neighbourhoods, see Corollary 3.6.19, Proposition 3.6.20 and its corollary, and Proposition 3.6.22.

In Section 3.7 the previously established properties are cast into abstract concepts, called *q-cover base* and *q-cover prebase* where q is a quasiorder on the set of neighbourhoods. Proposition 3.7.5 then describes a way how to turn arbitrary q-cover bases, whose existence is ensured by results in the previous section, into irredundant q-cover bases. After developing an appropriate notion of isomorphism, Theorem 3.7.14 finally shows that these are q-non-refinable covers and, moreover, that they are unique up to q-isomorphism. This result, together with the following corollaries, providing different levels of more concrete instances, constitutes a substantial generalisation of the existence and uniqueness result [Kea01, Theorem 5.3] and [Beh09, Theorem 3.8.1], proven for finite algebras.

Finally, Section 3.8 presents in detail how non-refinable covers can be constructed for an example algebra on four elements using the techniques developed in the other sections. Readers in need of an example while consuming the previous sections are encouraged to consult this section in parallel as far as the necessary terminology has already been introduced. Concepts demonstrated in Section 3.8 include: computation of unary term operations and idempotents, the set of all neighbourhoods, isomorphism and embedding of neighbourhoods, proving the covering condition via finding a decomposition equation, demonstrating irreducibility via (S, T)-irreducibility, checking irredundancy of covers, and constructing non-refinable covers from isomorphism types of $\precapprox$-maximal strictly irreducible neighbourhoods. As a side-application, this example is also used to disprove a characterisation of non-refinable covers stated in in [KL10]. It demonstrates furthermore, that the collection of strictly irreducible neighbourhoods introduced in Section 3.6 is in general distinct from that of irreducible ones.

Before we begin we make a last remark on the order in which the individual sections should be read. Generally, it is recommended to study Sections 3.1–3.7 in the linear order given by the ascending numeration. Technically, Section 3.1 is not necessarily to be read completely for the rest of the text as it basically contains the motivation for the choice of the subsets called *neighbourhoods* in Section 3.2. If one is willing to accept this, one can start there, but a few later results rely on technical lemmas from the first section. In Section 3.3 the motivational part about transfer of relational properties (including Lemma 3.3.3) and of identities to restricted algebras (Remark 3.3.5), as well as Lemma 3.3.6 can be skipped if desired. Likewise, if one wants to go strictly for the technical part, one can start reading Section 3.4 with Definition 3.4.2, exclude weak and strong embedding of sets of neighbourhoods (Definition 3.4.8 and Lemma 3.4.9) and finally everything following Corollary 3.4.37 if one is not interested in connections of RST with categorical equivalence of algebras. Section 3.5 should at least be read until Corollary 3.5.23, the remainder concerns mainly algorithmic questions that are not of crucial importance for the subsequent sections. Again, if one wishes to ignore examples and interpretation of irreducibility, one can begin the lecture of Section 3.6 with Definition 3.6.6. Furthermore, the criteria and characterisations established between Lemma 3.6.13 and Lemma 3.6.17 are of minor relevance for the description of non-refinable covers in the following section. However, the results of Lemma 3.6.18 through Proposition 3.6.22 are crucial. Finally, one can stop reading Section 3.7 with Corollary 3.7.18 if one is satisfied with this result. For an improved version, one has to continue the lecture until Corollaries 3.7.22 and 3.7.23. Concerning the final section, containing an example, the esteemed reader is encouraged to read it in parallel to the previous sections or at any convenient time.

3.1 Finding suitable subsets for localisation

Localisation in Relational Structure Theory pays special attention to the relational aspect of an algebra, i.e. to its clone of invariant relations. The first task in this respect is to identify subsets of an algebra that are suitable for localisation. We start with a well-known fact about localisation of algebras in their operational form (almost literally copied from Section 3.1 of [Beh09]) and then use this to motivate an analogous result concerning relational structures.

3.1.1 Lemma. *Let* $\mathbf{A}$ *be an algebra,* $F := \operatorname{Clo}(\mathbf{A})$ *its locally closed clone of operations, and let* $U \subseteq A$ *be a subset. Then the following holds*

(a) $F \cap \operatorname{Pol}_A \{U\}$ *is a clone on* A*,*

(b) $\begin{array}{rccc} |_U\colon & F \cap \operatorname{Pol}_A\{U\} & \longrightarrow & \mathrm{O}_U \\ & f & \longmapsto & f|_U = f|^U_{U^{\operatorname{ar} f}} \end{array}$ *is a clone homomorphism, and*

(c) $[F \cap \operatorname{Pol}_A \{U\}]|_U := \{ f|_U \mid f \in F \wedge f \rhd U\}$ *is a clone on the set* U*.*

Proof: (a) Since the clone lattice is a complete lattice w.r.t. set inclusion, the intersection $F \cap \operatorname{Pol}_A \{U\}$ is again a clone.

(b) This item follows by a straightforward calculation: for $n \in \mathbb{N}$ and $1 \leq i \leq n$ the i-th n-ary projection on A restricts to the i-th projection on U:

$$\left.\left(\begin{array}{rccc} e_i^{(n)}\colon & A^n & \longrightarrow & A \\ & (x_1,\ldots,x_n) & \longmapsto & x_i \end{array}\right)\right|_U = \begin{array}{rccc} e_i^{(n)}\colon & U^n & \longrightarrow & U \\ & (x_1,\ldots,x_n) & \longmapsto & x_i. \end{array}$$

Moreover, for $m, n \in \mathbb{N}$ and any $f \in \mathrm{O}_A^{(n)}$ and $(g_1,\ldots,g_n) \in \left(\mathrm{O}_A^{(m)}\right)^n$ such that $f, g_1, \ldots, g_n \in F \cap \operatorname{Pol}_A \{U\}$, composition and restriction to U commute:

$$\left.\left(\begin{array}{rccc} f\circ(g_1,\ldots,g_n)\colon & A^m & \longrightarrow & A \\ & x & \longmapsto & f(g_1(x),\ldots,g_n(x)) \end{array}\right)\right|_U =$$
$$\begin{array}{rccc} f|_U\circ(g_1|_U,\ldots,g_n|_U)\colon & U^m & \longrightarrow & U \\ & u & \longmapsto & f(g_1(u),\ldots,g_n(u)). \end{array}$$

(c) This is a consequence of (b) as images of homomorphisms are subuniverses. One can find finitely many finitary operations on O_A (see [PK79, 1.1.2 and 1.1.3]), in other words an algebra $\mathrm{O}_{\mathbf{A}}$, such that its subuniverses are exactly the clones on A. All fundamental operations of $\mathrm{O}_{\mathbf{A}}$ can be expressed as term operations using composition $\circ$ and projections. Therefore, $|_U$ is a homomorphism w.r.t. those fundamental operations, as well, and its image is a subuniverse, i.e. a clone on U. $\square$

3.1.2 Corollary. *Let* $\mathbf{A}$ *be an algebra,* $\mathrm{Clo}(\mathbf{A})$ *its clone of operations, and let* $U \subseteq A$ *be a subset. Then*

$$f|_U \colon \mathrm{Clo}(\mathbf{A}) \longrightarrow \mathrm{O}_U \text{ is a well-defined clone homomorphism} \iff U \leq \mathbf{A}.$$

Proof: The conditions $\mathrm{Clo}(\mathbf{A}) \cap \mathrm{Pol}_A \{U\} = \mathrm{Clo}(\mathbf{A})$, $\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Pol}_A \{U\}$ and U being a subuniverse of $\mathbf{A}$ are certainly all equivalent. $\square$

One might ask why in case that $\mathbf{A}$ contains nullary operations, the condition $U \leq \mathbf{A}$ in the corollary forbids the empty set $U = \emptyset$ for restriction. What goes wrong here is, that there do not exist any nullary operations on the empty set, thus a restriction of the nullary operations of $\mathbf{A}$ is impossible.

From the point of view of clones of operations, the appropriate subsets for localisation are subuniverses of an algebra. Looking at algebras in this way leads to the study of the lattice of subuniverses, which are precisely the unary invariant relations. Following this path, one is lead to study also the clone of invariant relations of an algebra if one considers the algebra inside its generated (quasi-)variety. This way to look at things is certainly valuable and well-known. Here, we are not going to continue along this line.

Instead we intend to work with the invariant relations of an algebra directly. To this end we are going to present a result similar to Corollary 3.1.2 concerning algebras in their relational form. However, before, we make an easy observation about idempotent unary operations, which is very useful throughout the whole text. Let us recall that a unary operation $e \in \mathrm{O}_A^{(1)}$ on a set A is called *idempotent* if it is an idempotent element in the full transformation semigroup $\left(\mathrm{O}_A^{(1)}, \circ\right)$, that is, if $e \circ e = e$. We denote the set of all unary idempotent operations on a set A by $\mathrm{Idem}\, A$.

The content of the following lemma can be considered folklore. Parts of it can also be found in Lemma 3.1.3 of [Beh09].

3.1.3 Lemma. *Let A be a set, $e \in \mathrm{O}_A^{(1)}$ a unary operation and $e[A]$ its image, denoted U. Then e is idempotent, i.e. $e^2 = e$, if and only if it operates identically on its image U, i.e. if $e(u) = u$ for every $u \in U$.*

Furthermore, for an idempotent $e \in \mathrm{Idem}\, A$ and any finitary operation $f \in \mathrm{O}_A$ the condition $\mathrm{im}\, f \subseteq \mathrm{im}\, e$ can be equivalently expressed by $e \circ f = f$. In particular two idempotents $e, f \in \mathrm{Idem}\, A$ have the same image if and only if $e \circ f = f$ and $f \circ e = e$.

Proof: We start by showing that being the identity on their image characterises idempotent operations. After that we turn to the second part of the lemma.

"⇒" Let $e \in \mathrm{O}_A^{(1)}$ be idempotent and $u \in U = e[A]$. Then $u = e(a)$ for some $a \in A$, so $e(u) = e(e(a)) = e^2(a) = e(a) = u$.

"⇐" Conversely, assume that the condition from above holds for all $u \in U$. In this case for every $a \in A$, one can set $u := e(a)$ and obtains

$$e^2(a) = e(e(a)) = e(u) = u = e(a).$$

Hence, $e^2 = e$.

For the other statement let still $e \in \operatorname{Idem} A$ and consider $f \in \mathrm{O}_A^{(n)}$ where $n \in \mathbb{N}$. If $\operatorname{im} f \subseteq \operatorname{im} e$ then for every $\mathbf{x} \in A^n$ we have $f(\mathbf{x}) \in \operatorname{im} f \subseteq \operatorname{im} e$. Since the idempotent operation e is the identity on its image, we get $e(f(\mathbf{x})) = f(\mathbf{x})$, hence $e \circ f = f$.

Conversely, if this holds, then every element $f(\mathbf{x})$ in the image of f can be written as $f(\mathbf{x}) = e(f(\mathbf{x}))$, which obviously belongs to $\operatorname{im} e$. Therefore, we get $\operatorname{im} f \subseteq \operatorname{im} e$.

Now for two idempotents $e, f \in \operatorname{Idem} A$, writing the condition $\operatorname{im} f = \operatorname{im} e$ as a conjunction of two inclusions and applying the characterisation we just proved, yields the last fact of the lemma. □

We will apply this result a number of times on our way to the promised analogy (see Proposition 3.1.12) of Corollary 3.1.2, that identifies the appropriate subsets for localisation of relational structures. The announced result says that for finite carrier sets (more generally, for invariant relations of 1-locally finite algebras) restriction of relations to a subset is a homomorphism between relational clones, if and only if the subset is the image of an idempotent endomorphism of the relations. Such a mapping is often also known as a *retraction* (even though this is technically incorrect[3]), and the restricted relational structure is called *(idempotent) retract*.

When dealing with algebras $\mathbf{A}$, the relational structure we have in mind w.r.t. Proposition 3.1.12 is the one living on the same carrier set as $\mathbf{A}$ and bearing all its invariant relations as fundamental relations (even though, actually, a generating set of this clone would suffice). In this setting the appropriate subsets for localisation are then the unary idempotent endomorphisms of the invariants of $\mathbf{A}$, which are exactly the unary idempotent operations in $\operatorname{Clo}(\mathbf{A})$. In the next section we shall call the corresponding images, which we will use for localisation, *neighbourhoods*.

In order to simplify the proof of Proposition 3.1.12, we first collect further properties of idempotent unary operations and idempotent endomorphisms of relational structures, in particular. Restriction of relations preserved by an idempotent endomorphism e to the image of the endomorphism can be expressed by applying e component-wise to the tuples in the relation. In other words, for preserved relations, restriction to the image of an idempotent endomorphism is the same as the result of the canonical action of the endomorphism on the relation. This is

[3]However, the restriction to the induced substructure on the image actually *is* a retraction.

even true for relations of arbitrary infinite arity (for finitary relations, see also Lemma 3.2.4 in [Beh09]).

3.1.4 Lemma. *Let $e \in IdemA$ be an idempotent unary operation on a set A. For any set I and each subset $S \subseteq A^I$ of mappings from I to A, we have the inclusion $\{f \in S \mid \operatorname{im} f \subseteq \operatorname{im} e\} \subseteq e \circ [S]$. If S is closed under the action of e, i.e. $e \circ [S] = \{e \circ f \mid f \in S\} \subseteq S$, or equivalently, if S is a subuniverse of the power algebra $\langle A; e\rangle^I$, then $\{f \in S \mid \operatorname{im} f \subseteq \operatorname{im} e\} = e \circ [S]$.*

In particular for every $S \in \operatorname{Inv}_A \{e\}$, we have $S\restriction_{\operatorname{im} e} = e \circ [S]$.

Proof: Let us abbreviate $U := \operatorname{im} e$, fix a set I and consider an arbitrary $S \subseteq A^I$. Now if f is in S such that $\operatorname{im} f \subseteq U$, then Lemma 3.1.3 implies $f = e \circ f \in e \circ [S]$. Conversely, if we assume $e \circ [S] \subseteq S$, then for any $f \in S$ we have the inclusion $\operatorname{im}(e \circ f) \subseteq \operatorname{im} e = U$ and, furthermore, $e \circ f \in e \circ [S] \subseteq S$.

Finally, consider the special case that $I = m \in \mathbb{N}$ and $S \in \operatorname{Inv}_A^{(m)} \{e\}$. The condition $S \in \operatorname{Inv}_A^{(m)} \{e\}$ is equivalent to $e \circ [S] \subseteq S$, so we only have to prove that $\{x \in S \mid \operatorname{im} x \subseteq U\} = S\restriction_U$. This is true since it is $\operatorname{im} x = \{x_1, \ldots, x_m\}$ for any tuple $x = (x_1, \ldots, x_m)$. $\square$

The following observation about polymorphisms of relational structures is a simple application of the previous result.

3.1.5 Corollary. *For every relational structure $\underset{\sim}{\mathbf{A}}$ and any idempotent endomorphism $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$ with image $U := \operatorname{im} e$ we have*

$$\left\{ f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \;\middle|\; \operatorname{im} f \subseteq U \right\} = \left\{ e \circ f \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\}$$

for every arity $n \in \mathbb{N}$.

Proof: Set $I := A^n$ and $S := \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}}$ in Lemma 3.1.4. The assumption that S is closed under the action of e is fulfilled since the composition of an endomorphism of $\underset{\sim}{\mathbf{A}}$ with an n-ary polymorphism of this structure yields again a polymorphism of this arity. $\square$

The mentioned analogy of Corollary 3.1.2 for relational structures will be part of a more substantial characterisation. We shall now prepare its proof with a number of small lemmas.

3.1.6 Lemma. *If $\underset{\sim}{\mathbf{A}}$ is a relational structure and $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$ is an idempotent endomorphism having image $U := \operatorname{im} e$, then*

$$\begin{aligned}\left\{ (e \circ f)\,|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\} &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\} \\ &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \right\}\end{aligned}$$

holds for every arity $n \in \mathbb{N}$.

Proof: The operations occurring in Corollary 3.1.5 can be restricted to the image U of the endomorphism e. Restricting every operation in the set

$$\left\{ f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \;\middle|\; \operatorname{im} f \subseteq U \right\} = \left\{ e \circ f \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\}$$

yields

$$\left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\} = \left\{ (e \circ f)\,|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\},$$

which is the first stated equality. Since every function having image in U must preserve U, the inclusion

$$\left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\} \subseteq \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \right\}$$

is also clear. For the converse inclusion let us consider any $f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}}$ that preserves U. This implies, in combination with Lemma 3.1.3, that $(e \circ f)\,|_U = f|_U$, so

$$\left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \right\} \subseteq \left\{ (e \circ f)\,|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\},$$

which was equal to $\left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\}$. □

It is interesting to see that the condition exhibited in the previous lemma does not only follow from U being the image of an idempotent endomorphism of a relational structure. It also necessitates that U must be of this form:

3.1.7 Lemma. *For a relational structure $\underset{\sim}{\mathbf{A}}$ and a subset $U \subseteq A$ the following conditions are equivalent:*

(a) $U = \operatorname{im} e$ *for some idempotent* $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$.

(b) $\{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \} = \{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \}$.

(c) The equalities

$$\begin{aligned} \left\{ (e \circ f)\,|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\} &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\} \\ &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \right\} \end{aligned}$$

hold for every arity $n \in \mathbb{N}$.

Proof: We will show that (a) ⇒ (c) ⇒ (b) ⇒ (a).

"(a) ⇒ (c)" This implication is a consequence of Lemma 3.1.6.

"(c) ⇒ (b)" The condition on endomorphisms is a special case ($n = 1$) of the assumption on polymorphisms, so the claim follows by specialisation.

"(b) $\Rightarrow$ (a)" Certainly, the identity is an endomorphism of $\underset{\sim}{\mathbf{A}}$ which preserves U, and furthermore, it restricts to the identity mapping on U. So by assumption, we get

$$\mathrm{id}_U \in \{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \} = \{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \}\,.$$

This means there must exist some endomorphism $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$ whose image is a subset of U, and which satisfies $e|_U = \mathrm{id}_U$, i.e.

$$U = \mathrm{id}_U\,[U] = e|_U\,[U] \subseteq e\,[A] = \operatorname{im} e \subseteq U.$$

Hence, $\operatorname{im} e = U$ and e is the identity on this set. Now Lemma 3.1.3 implies that e is idempotent. Consequently, U is the image of an idempotent endomorphism of $\underset{\sim}{\mathbf{A}}$. □

Another pleasant property of idempotent endomorphisms is that restriction to their images always produces clone homomorphisms between relational clones. As we shall see later (see Proposition 3.1.12), in the finite case (moreover, for 1-locally finite algebras) this fact is even equivalent to the localising subset being of this form.

We shall first prove that the natural action of endomorphisms satisfies a homomorphism property w.r.t. the general composition of invariant relations of arbitrary high arity, i.e. subpowers of arbitrary high exponent, not only finite ones. From this, one gets the result for relational clones as an easy consequence.

3.1.8 Lemma. *Let $e \in \operatorname{Idem} A$ be an idempotent operation on a set A, and let I, κ, m and m_i for $i \in I$ be (index) sets, $\alpha_i \colon m_i \longrightarrow \kappa$ for $i \in I$ and $\beta \colon m \longrightarrow \kappa$ be mappings, and suppose that ϱ_i is a subalgebra of the power $\langle A;\, e\rangle^{m_i}$ for every $i \in I$. Then the following equality holds*

$$e \circ \left[\prod_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} \right] = \prod_{(\alpha_i)_{i\in I}}^{\beta} (e \circ [\varrho_i])_{i\in I}\,,$$

where $\prod_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I}$ is defined as $\{ a \circ \beta \mid a \in A^\kappa \wedge \forall i \in I \colon a \circ \alpha_i \in \varrho_i \}$ w.r.t. the domain A, and the other operator in a similar way w.r.t. the domain $U := \operatorname{im} e$.

Proof: Writing out the left-hand side as $\{ e \circ a \circ \beta \mid a \in A^\kappa \wedge \forall i \in I \colon a \circ \alpha_i \in \varrho_i \}$ it is easy to see that it is contained in the right-hand side, which can be spelt out as $\{ u \circ \beta \mid u \in U^\kappa \wedge \forall i \in I \colon u \circ \alpha_i \in e \circ [\varrho_i] \}$. For the converse inclusion we note that $e \circ [\varrho_i] \subseteq \varrho_i$ since ϱ_i is a subpower. Therefore, every $u \in U^\kappa \subseteq A^\kappa$ satisfying $u \circ \alpha_i \in e \circ [\varrho_i]$ for all $i \in I$ also fulfils $u \circ \alpha_i \in \varrho_i$ for any $i \in I$. Applying Lemma 3.1.3, every $u \in U^\kappa$ can be written as $u = e \circ u$, and thus $u \circ \beta = e \circ u \circ \beta$. Hence, the right-hand side is a subset of the left-hand side, whence we are done. □

3.1.9 Corollary. *If $\underset{\sim}{\mathbf{A}}$ is a relational structure, $Q \subseteq \operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}}$ is a relational clone, $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$ is an idempotent endomorphism and $U := \operatorname{im} e$ its image, then restriction to U*

$$\begin{array}{rccc} \restriction_U\colon & Q & \longrightarrow & \mathrm{R}_U \\ & \varrho & \longmapsto & \varrho\restriction_U = \varrho \cap U^{\operatorname{ar}\varrho} \end{array}$$

is a homomorphism between relational clones.

Proof: From the assumption $Q \subseteq \operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}}$ one obtains that

$$e \in \operatorname{End} \underset{\sim}{\mathbf{A}} = \operatorname{Pol}_A^{(1)} \operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}} \subseteq \operatorname{Pol}_A^{(1)} Q.$$

To show that restriction to U is a homomorphism between relational clones, we have to show that it commutes with the general composition of relations. For any set I, an ordinal κ and natural numbers $m \in \mathbb{N}$, $(m_i)_{i\in I} \in \mathbb{N}^I$, mappings $(\alpha_i\colon m_i \longrightarrow \kappa)_{i\in I}$ and $\beta\colon m \longrightarrow \kappa$ we need to prove that for every choice of relations $(\varrho_i)_{i\in I} \in \prod_{i\in I} Q^{(m_i)}$ the following equality holds

$$\left(\bigsqcap_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I} \right)\restriction_U = \bigsqcap_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i\restriction_U)_{i\in I}\,.$$

Since Q is a clone, the relation $\varrho := \bigsqcap_{(\alpha_i)_{i\in I}}^{\beta} (\varrho_i)_{i\in I}$ belongs to Q. Since $\varrho, \varrho_i \in Q$ $(i \in I)$ and $e \in \operatorname{Pol}_A^{(1)} Q$, we have $e \in \operatorname{Pol}_A^{(1)} (\{\varrho\} \cup \{\, \varrho_i \mid i \in I\})$ or equivalently that $\{\varrho\} \cup \{\, \varrho_i \mid i \in I\} \subseteq \operatorname{Inv}_A \{e\}$. So these relations are subuniverses of powers of $\langle A; e\rangle$. Applying Lemma 3.1.4, we can restate the previous displayed equality equivalently as

$$e \circ [\varrho] = \bigsqcap_{(\alpha_i)_{i\in I}}^{\beta} (e \circ [\varrho_i])_{i\in I}\,,$$

which has been proven in Lemma 3.1.8 for a more general setting. □

As a further corollary we get that using images of idempotent endomorphisms for restriction of relations in a relational clone, we obtain again a relational clone.

3.1.10 Corollary (cp. Corollary 3.1.5 in [Beh09]). *For a relational structure $\underset{\sim}{\mathbf{A}}$, a relational clone Q contained in $\operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}}$, and an idempotent endomorphism $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$, the restriction*

$$[Q]\restriction_U = \{\, \varrho\restriction_U \mid \varrho \in Q\}$$

to the image $U := \operatorname{im} e$ is again a relational clone on the set U.

Proof: As in the proof of Lemma 3.1.1(c) we are going to use the fact that images of homomorphisms are subuniverses. For any set I, any ordinal κ, natural numbers $m, m_i \in \mathbb{N}$ $(i \in I)$ and mappings $(\alpha_i\colon m_i \longrightarrow \kappa)_{i\in I}$ and $\beta\colon m \longrightarrow \kappa$ and m_i-ary relations $\sigma_i \in [Q]\restriction_U$ we need to show that the general composition $\prod^{\beta}_{(\alpha_i)_{i\in I}} (\sigma_i)_{i\in I}$ belongs to $[Q]\restriction_U$. First, for every $i \in I$, by choice of σ_i we can find some $\varrho_i \in Q^{(m_i)}$ such that $\sigma_i = \varrho_i\restriction_U$ holds. Then we simply use the homomorphism property of restriction to U (see Corollary 3.1.9) to get that

$$\prod^{\beta}_{(\alpha_i)_{i\in I}} (\sigma_i)_{i\in I} = \prod^{\beta}_{(\alpha_i)_{i\in I}} (\varrho_i\restriction_U)_{i\in I} = \left(\prod^{\beta}_{(\alpha_i)_{i\in I}} (\varrho_i)_{i\in I}\right)\restriction_U,$$

which belongs to $[Q]\restriction_U$ for $\prod^{\beta}_{(\alpha_i)_{i\in I}} (\varrho_i)_{i\in I}$ is in Q as the latter is a relational clone and thus closed under general composition. □

3.1.11 Remark. Let us note that the second equality in Lemma 3.1.6 can also be seen as a consequence of the compatibility condition in Lemma 3.1.8. Indeed, projection of subpowers onto a subset of coordinates can be understood as a special instance of generalised composition, namely for $I = \{1\}$, $m_1 = \kappa$, $\alpha_1 = \mathrm{id}_\kappa$, m being a set and $\beta\colon m \hookrightarrow \kappa$ being an injective mapping. Then it is $\prod^{\beta}_{\mathrm{id}_\kappa} = \mathrm{pr}^{\kappa}_{m}$, i.e. we obtain projection of κ-th subpowers to m-indices singled out by β. To get the equality

$$\left\{ f|_U \;\middle|\; f \in \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U\right\} = \left\{ f|_U \;\middle|\; f \in \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} \wedge f \rhd U\right\}$$

in Lemma 3.1.6, we consider the set $\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}$, which is a subpower of $\langle A; e\rangle^{A^n}$ since e is an endomorphism of $\underset{\sim}{\mathbf{A}}$. The homomorphism condition from Lemma 3.1.8, specialised to the case of projections, now yields

$$e \circ \left[\mathrm{pr}^{A^n}_{U^n} \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right] = \mathrm{pr}^{A^n}_{U^n}\, e \circ \left[\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right].$$

Using Lemma 3.1.4, this equation can be rewritten as

$$\left(\mathrm{pr}^{A^n}_{U^n} \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right)\restriction_U = \mathrm{pr}^{A^n}_{U^n} \left(\left(\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right)\restriction_U\right).$$

The set $\mathrm{pr}^{A^n}_{U^n} \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}$ contains all domain restrictions of n-ary polymorphisms of $\underset{\sim}{\mathbf{A}}$ to U^n, its restriction to U only those whose image of U^n is contained in U, i.e. which preserve U. Thus, we get $\left(\mathrm{pr}^{A^n}_{U^n} \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right)\restriction_U = \left\{ f|_U \;\middle|\; f \in \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} \wedge f \rhd U\right\}$. On the other hand we have $\left(\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right)\restriction_U = \left\{ f \in \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} \;\middle|\; \operatorname{im} f \subseteq U\right\}$, and projecting this set onto U^n yields $\mathrm{pr}^{A^n}_{U^n}\left(\left(\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}}\right)\restriction_U\right) = \left\{ f|_U \;\middle|\; f \in \mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U\right\}$.

Combining these observations with the condition obtained from Lemma 3.1.8, we get the desired equality. ♢

The next proposition will characterise, which subsets are preferable for restriction from a relational point of view. When focussing on finite structures, the argument just given can be used to prove the most difficult of its implications, namely that item (b) implies item (e). Here the finiteness requirement is needed to ensure that the subpowers considered in the argument are actually of finite power such that the strong condition from Lemma 3.1.8 can be replaced by item (b).

In the following proposition however, we want to weaken the assumption of finiteness a little bit to allow application to 1-locally finite algebras instead of just finite ones. Therefore, our arguments need to be slightly more careful.

Let us recall that for $n \in \mathbb{N}$ an algebra is called *n-locally finite* if all its subuniverses generated by n elements are finite. Certainly, every finite algebra is *locally finite*, meaning n-locally finite for all $n \in \mathbb{N}$.

Furthermore, we note that for finite algebras the equivalence of items (b) and (a) of the subsequent proposition is already formulated in [Kea01, Theorem 2.3.] and also in [Beh09, Lemma 3.1.4.]. Lemma 5.2.11 of [Sch12] contains a more general formulation of this equivalence for weakly 1-operationally compact HAUSDORFF topological algebras, see [Sch12] for the corresponding definition. If we equip a 1-locally finite algebra with the discrete topology, then Remark 5.2.9 of [Sch12] shows that the assumption of n-local finiteness is equivalent to weak n-operational compactness. Hence, our result for 1-locally finite algebras follows as a special case of Lemma 5.2.11 in [Sch12]. Nevertheless, we give a direct proof here because we wish to avoid the detour via general topological algebras.

3.1.12 Proposition. *Let $\underset{\sim}{\mathbf{A}}$ be a relational structure, such that $\langle A; \operatorname{End} \underset{\sim}{\mathbf{A}}\rangle$ is 1-locally finite, and let $U \subseteq A$ be any subset. Then the following facts are equivalent:*

(a) $U = \operatorname{im} e$ *for some idempotent endomorphism* $e \in \operatorname{End} \underset{\sim}{\mathbf{A}}$.

(b) $\begin{array}{rccc} \restriction_U\colon & \operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}} & \longrightarrow & \mathrm{R}_U \\ & \varrho & \longmapsto & \varrho\restriction_U = \varrho \cap U^{\operatorname{ar} \varrho} \end{array}$ *is a homomorphism between relational clones.*

(c) $\begin{array}{rccc} \restriction_U\colon & \operatorname{Inv}_A \operatorname{Pol} \underset{\sim}{\mathbf{A}} & \longrightarrow & \mathrm{R}_U \\ & \varrho & \longmapsto & \varrho\restriction_U = \varrho \cap U^{\operatorname{ar} \varrho} \end{array}$ *is a homomorphism w.r.t. all coordinate projection operations* pr^k_m, $m, k \in \mathbb{N}$, $m \leq k$, *as described on page 43.*

(d) $\{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge f \rhd U\} = \{ f|_U \mid f \in \operatorname{End} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U\}$.

(e) The equalities

$$\begin{aligned} \left\{ (e \circ f)\,|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \right\} &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge \operatorname{im} f \subseteq U \right\} \\ &= \left\{ f|_U \;\middle|\; f \in \operatorname{Pol}^{(n)} \underset{\sim}{\mathbf{A}} \wedge f \rhd U \right\} \end{aligned}$$

hold for every arity $n \in \mathbb{N}$.

Proof: By Lemma 3.1.7 items (a), (e) and (d) are equivalent. So it remains to show that (a), (b) and (c) are equivalent. Clearly, the implication "(a) $\Rightarrow$ (b)" follows from Corollary 3.1.9, and item (b) entails item (c). So we are left with the converse implication "(c) $\Rightarrow$ (a)".

Assume that $\restriction_U$ is a homomorphism w.r.t. finite coordinate projections. We consider the set $\operatorname{End}\underset{\sim}{\mathbf{A}}$ as an invariant relation of (possibly infinite) arity. In other words, $\operatorname{End}\underset{\sim}{\mathbf{A}}$ is a subuniverse of $\langle A; \operatorname{Pol}\underset{\sim}{\mathbf{A}}\rangle^A$ since substitution of endomorphisms into a polymorphism yields again endomorphisms. Moreover, we take an arbitrary finite subset $X \subseteq A$, then $\varrho_X := \operatorname{pr}^A_X \operatorname{End}\underset{\sim}{\mathbf{A}}$ is subuniverse of $\langle A; \operatorname{Pol}\underset{\sim}{\mathbf{A}}\rangle^X$ because coordinate projections are homomorphisms between powers of the same algebra. As the exponent X is finite, ϱ_X corresponds to some invariant relation in $\operatorname{Inv}_A \operatorname{Pol}\underset{\sim}{\mathbf{A}}$ via some bijection between X and its cardinality, which fixes some indexing. Thus, we can use the homomorphism property w.r.t. finite coordinate projections, in particular for $\operatorname{pr}^X_{U\cap X}$. Exploiting the double role of ϱ_X as a relation and a set of unary operations, we note that application of $\operatorname{pr}^X_{U\cap X}$ is the same as restricting the domain of all operations in the relation to the subset $X \cap U$, and restriction $\restriction_U$ to U is realised by just taking those operations whose image lies in U.

Hence, we get

$$\begin{aligned}
\operatorname{pr}^X_{U\cap X}\varrho_X &= \operatorname{pr}^X_{U\cap X}\operatorname{pr}^A_X \operatorname{End}\underset{\sim}{\mathbf{A}} = \operatorname{pr}^A_{U\cap X}\operatorname{End}\underset{\sim}{\mathbf{A}} = \left\{ f|^A_{U\cap X} \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}}\right\},\\
\left(\operatorname{pr}^X_{U\cap X}\varrho_X\right)\restriction_U &= \left\{ f|^A_{U\cap X} \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \wedge f[X\cap U] \subseteq U\right\},\\
\varrho_X\restriction_U &= \left(\operatorname{pr}^A_X \operatorname{End}\underset{\sim}{\mathbf{A}}\right)\restriction_U = \left\{ f|^A_X \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \wedge f[X] \subseteq U\right\}
\end{aligned}$$

and

$$\operatorname{pr}^X_{U\cap X}(\varrho_X\restriction_U) = \left\{ f|^A_{U\cap X} \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \wedge f[X] \subseteq U\right\}.$$

So, from our assumption, we can infer

$$\begin{aligned}
\operatorname{id}_A|^A_{U\cap X} \in \left\{ f|^A_{U\cap X} \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \wedge f[X\cap U] \subseteq U\right\} &= \left(\operatorname{pr}^X_{U\cap X}\varrho_X\right)\restriction_U\\
&= \operatorname{pr}^X_{U\cap X}(\varrho_X\restriction_U) = \left\{ f|^A_{U\cap X} \;\middle|\; f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \wedge f[X] \subseteq U\right\},
\end{aligned}$$

in other words, the set

$$G_X := \left\{ f \in \operatorname{End}\underset{\sim}{\mathbf{A}} \;\middle|\; f[X] \subseteq U \wedge f|^A_{U\cap X} = \operatorname{id}_A|^A_{U\cap X}\right\}$$

is non-empty. Furthermore, we have $\operatorname{Loc}_A G_X = G_X$. This holds because every function $g \in \operatorname{Loc}_A G_X \subseteq \operatorname{Loc}_A \operatorname{End}\underset{\sim}{\mathbf{A}} = \operatorname{End}\underset{\sim}{\mathbf{A}}$ is interpolated by some $f \in G_X$ on every finite subset of its domain, especially on X. So, it follows $g[X] = f[X] \subseteq U$ and $g|^A_{U\cap X} = f|^A_{U\cap X} = \operatorname{id}_A|^A_{U\cap X}$, which proves $g \in G_X$. Moreover, for a finite number $\ell \geq 1$ of finite subsets $X_1, X_2, \ldots, X_\ell \subseteq A$, it is easy to see that

$$\emptyset \neq G_{X_1\cup X_2\cup\cdots\cup X_\ell} \subseteq G_{X_1} \cap G_{X_2} \cap \cdots \cap G_{X_\ell}.$$

Thus, $\mathcal{G} := \{G_X \mid X \subseteq A \text{ finite}\}$ is a collection of closed subsets of the topological space A^A equipped with the product topology when A is understood as a discrete space. Moreover, we have seen that $\mathcal{G}$ satisfies the finite intersection property, that is, intersections of arbitrary non-empty finite subcollections are non-empty. Now we shall use the assumption of 1-local finiteness. By Remark 5.2.9 of [Sch12] this condition is equivalent to the fact that $\operatorname{End}\underset{\sim}{\mathbf{A}}$ forms a compact subspace of A^A. It is a basic observation from topology that compactness can equivalently be characterised by the property that collections of closed subsets having finite intersection property have non-empty intersection at all. Thus, $\bigcap\mathcal{G}$ is non-void, which means, there must exist some $e \in \operatorname{End}\underset{\sim}{\mathbf{A}}$ such that $e[X] \subseteq U$ and $e|_{U\cap X}^{A} = \mathrm{id}_A|_{U\cap X}^{A}$ hold for every finite subset $X \subseteq A$. Choosing $X = \{a\}$ for any $a \in A$ we obtain $e(a) \in U$, that is, $\operatorname{im} e \subseteq U$, and similarly using $X = \{u\}$ for $u \in U$ yields $e(u) = \mathrm{id}_A|_{\{u\}}^{A}(u) = u$. Thus, $\operatorname{im} e = U$ and e acts identically on this set, which by Lemma 3.1.3 is equivalent to idempotency of e. This proves the truth of item (a). □

3.1.13 Remark. We wish to point out one advantage of our choice to include nullary invariant relations in our framework. Without this such a neat formulation of Proposition 3.1.12 would have been impossible. If only invariant relations of positive arity were allowed, one would have had to add the condition $U \neq \emptyset$ to items (b) and (c). This is so because on a non-empty set A images of idempotent mappings always contain at least one element, whereas restriction of relations to the empty set U is a homomorphism between $\operatorname{Inv}_A \operatorname{Pol}\underset{\sim}{\mathbf{A}}$ and the relational clone on the empty set (provided one considers only relations of positive arity). By allowing nullary invariant relations this discrepancy is automatically removed: for $U = \emptyset$ restriction of invariant relations is not compatible with projection to nullary relations.

Take for ϱ a non-empty invariant relation of positive arity, for instance the unary relation A and suppose that $U = \emptyset$. Then projecting the non-empty relation ϱ onto the empty coordinate set gives the non-empty nullary invariant relation containing the empty tuple. It does not change when restricted to U. On the other hand restricting the relation ϱ to $U = \emptyset$ yields the empty relation because the arity of ϱ was non-zero. Projecting an empty relation keeps the relation being empty, so we get $\mathrm{pr}_{\emptyset}^{\operatorname{ar}\varrho}(\varrho\restriction_U) = \emptyset \neq \left(\mathrm{pr}_{\emptyset}^{\operatorname{ar}\varrho}\,\varrho\right)\restriction_U$. This inequality shows that restriction to the empty set violates the homomorphism condition, which keeps our result tidy. ◇

The crucial equivalence of Proposition 3.1.12 is that between items (a) and (b). Corollary 3.1.9 shows that "(a) ⇒ (b)" is generally true without further assumptions. For the converse implication we used the additional assumption of 1-local finiteness in the proof above. Remark 3.1.11 shows that extending our framework to allow clones of invariant relations of infinite arity at least $|A|$ would also make the equivalence true without further finiteness conditions. In principle a wider theory including such general relations is possible, however, it is not compatible with the choice that is commonly made for relational clones. It can be argued that infinitary relations are rather non-algebraic objects, at least topological (continuous) methods seem to be more appropriate for them.

In any case, these arguments, Proposition 3.1.12 and also Corollary 3.1.10 support the point that, generally, images of idempotent endomorphisms are a suitable choice for restriction of structures from a relational point of view.

Since many aspects and definitions of the localisation theory we want to present here have been influenced by Tame Congruence Theory, it seems natural also to discuss our choice of local subsets from the perspective of TCT. Indeed, looking at [HM88, Lemma 2.3], our result identifying the images of idempotent endomorphisms as suitable subsets for restriction is not very surprising. There, the images U of idempotent polynomials of a finite algebra $\mathbf{A}$ were identified by the fact that $\restriction_U$ is a surjective lattice homomorphism from $\operatorname{Con}\mathbf{A}$ onto $\operatorname{Con}\mathbf{A}|_U$ where the restricted algebra $\mathbf{A}|_U$ defined in [HM88] is nearly the same as the one we shall use below. In fact, if the algebra $\mathbf{A}$ contains all constant operations in its clone (or as fundamental operations), then our choice of subsets U does not differ from the choice made in [HM88].

As a last point before the end of this section, we want to strengthen Corollary 3.1.10 a little bit by proving that restriction to images of idempotent endomorphisms does not only map relational clones to relational clones but moreover maps locally closed relational clones to locally closed relational clones. This is a desirable property since in the infinite case only the locally closed clones of relations are closures w.r.t. the Galois connection $\operatorname{Pol}-\operatorname{Inv}$. In the following sections we want to employ this Galois connection to translate between algebras and relational structures. Hence, it is preferable that the restriction process we use works well together with this Galois connection.

In fact, we shall prove that restriction to any subset U commutes with the local closure operator on finitary relations provided the set of relations we apply it to is closed under arbitrary non-empty intersections. This will clearly be the case for relational clones.

As a first step, we note that the following inclusion is generally true.

3.1.14 Lemma. *For $U \subseteq A$ and any set $Q \subseteq \mathrm{R}_A$ of finitary relations, we have*

$$[\mathrm{LOC}_A\, Q]\restriction_U \subseteq \mathrm{LOC}_U\, [Q]\restriction_U.$$

Proof: For every $\varrho \in \mathrm{LOC}_A\, Q$ we need to show that $\varrho\restriction_U$ belongs to $\mathrm{LOC}_U\, [Q]\restriction_U$, i.e. that it can be approximated on finite sets by restricted relations. Hence, consider any finite subset $B \subseteq \varrho\restriction_U \subseteq \varrho$. Since $\varrho \in \mathrm{LOC}_A\, Q$ one can find some relation $\sigma \in Q$ such that $B \subseteq \sigma \subseteq \varrho$. Restricting this chain of inclusions to U yields $B\restriction_U \subseteq \sigma\restriction_U \subseteq \varrho\restriction_U$. As $B \subseteq \varrho\restriction_U \subseteq U^{\operatorname{ar}\varrho}$, we have $B = B\restriction_U$, and so we obtain $B = B\restriction_U \subseteq \sigma\restriction_U \subseteq \varrho\restriction_U$. This proves that $\varrho\restriction_U$ is interpolated on B by the relation $\sigma\restriction_U \in [Q]\restriction_U$. In other words, as this holds for arbitrary finite $B \subseteq \varrho\restriction_U$, we have verified $\varrho\restriction_U \in \mathrm{LOC}_U\, [Q]\restriction_U$. □

Assuming closure under arbitrary non-empty intersection of relations of the same arity, we can achieve equality in the previous lemma:

3.1.15 Lemma. *Every set $Q \subseteq \mathrm{R}_A$ of finitary relations that is stable under arbitrary non-empty intersections of relations of the same arity satisfies*

$$\mathrm{LOC}_U\, [Q]\restriction_U = [\mathrm{LOC}_A\, Q]\restriction_U$$

for any $U \subseteq A$.

Proof: From Lemma 3.1.14 we can infer $\mathrm{LOC}_U\,[Q]\restriction_U \supseteq [\mathrm{LOC}_A\,Q]\restriction_U$, so we only need to deal with the converse inclusion.

Let $\varrho \in \mathrm{LOC}_U\,[Q]\restriction_U$, that is, for every finite subset $B \subseteq \varrho$ there exists a relation $\sigma \in Q$ of the same arity as ϱ such that $B \subseteq \sigma\restriction_U \subseteq \varrho$. In other words, the set

$$\Sigma_B := \left\{ \sigma \in Q^{(\operatorname{ar}\varrho)} \;\middle|\; B \subseteq \sigma\restriction_U \subseteq \varrho \right\} \subseteq Q^{(\operatorname{ar}\varrho)}$$

is non-empty, so by assumption on Q the relation $\varrho_B := \bigcap \Sigma_B$ belongs to $Q^{(\operatorname{ar}\varrho)}$. Using that restriction to U is defined via intersection with some power of U according to the arity of the relation, we get

$$\begin{aligned}\varrho_B\restriction_U = \varrho_B \cap U^{\operatorname{ar}\varrho} = \bigcap \Sigma_B \cap U^{\operatorname{ar}\varrho} &= \bigcap_{\sigma\in\Sigma_B} \sigma \cap U^{\operatorname{ar}\varrho} = \bigcap_{\sigma\in\Sigma_B} \sigma\restriction_U \\ &= \bigcap \left\{ \tilde{\sigma} \in \left[Q^{(\operatorname{ar}\varrho)}\right]\restriction_U \;\middle|\; B \subseteq \tilde{\sigma} \subseteq \varrho \right\}.\end{aligned}$$

Since each of the relations we intersect contains B, also $\varrho_B\restriction_U$ contains B, furthermore, Σ_B is non-empty and each of the intersecting relations is contained in ϱ, so $\varrho_B\restriction_U \subseteq \varrho$. Hence, we have shown that ϱ_B is the least element (w.r.t. inclusion) of Σ_B. The set $Q' := \{\, \varrho_B \mid B \subseteq \varrho \text{ finite}\}$ is a non-empty directed subcollection of $Q^{(\operatorname{ar}\varrho)}$. Indeed, if $B_1 \subseteq B_2 \subseteq \varrho$ are finite subsets, then $B_1 \subseteq B_2 \subseteq \varrho_{B_2}\restriction_U \subseteq \varrho$ since $\varrho_{B_2} \in \Sigma_{B_2}$, so $\varrho_{B_2} \in \Sigma_{B_1}$. Hence, we obtain the inclusion $\varrho_{B_1} = \bigcap \Sigma_{B_1} \subseteq \varrho_{B_2}$. It follows $\varrho_{B_1} \cup \varrho_{B_2} \subseteq \varrho_{B_1\cup B_2}$ for all finite subsets $B_1, B_2 \subseteq \varrho$. As $B_1 \cup B_2$ is again a finite subset of ϱ, the collection Q' of relations of arity $\operatorname{ar}\varrho$ is upwards directed and non-empty.

It is known (see e.g. [Pös80, Proposition 1.13]) that for sets of relations that are intersection stable as Q in the assumption of the lemma, their local closure $\mathrm{LOC}_A\,Q$ can be computed by collecting all directed unions of non-empty subsets consisting of relations of the same arity. Thus, in particular, we get $\sigma := \bigcup Q' \in \mathrm{LOC}_A\,Q$. Moreover, its restriction to U is $\sigma\restriction_U = \bigcup \{\, \varrho_B\restriction_U \mid B \subseteq \varrho \text{ finite}\}$ since intersection distributes over arbitrary unions. As ϱ_B belongs to Σ_B, we have $\varrho_B\restriction_U \subseteq \varrho$ for every finite $B \subseteq \varrho$. So each of the relations in the union describing $\sigma\restriction_U$ is contained in ϱ, implying that $\sigma\restriction_U \subseteq \varrho$. On the other hand, take any tuple $x \in \varrho$ and set $B := \{x\}$. Then $x \in B \subseteq \varrho_B\restriction_U \subseteq \sigma\restriction_U$, i.e., $\varrho \subseteq \sigma\restriction_U$. Consequently, $\varrho = \sigma\restriction_U \in [\mathrm{LOC}_A\,Q]\restriction_U$.□

3.1.16 Corollary. *For $U \subseteq A$ and any relational clone $Q \subseteq \mathrm{R}_A$ we have*

$$\mathrm{LOC}_U\,[Q]\restriction_U = [\mathrm{LOC}_A\,Q]\restriction_U.$$

If Q is a locally closed clone of relations (a Galois *closure w.r.t.* $\mathrm{Pol} - \mathrm{Inv}$*) and U is the image of an idempotent endomorphism of $\langle A; Q\rangle$, then the restricted relational clone $[Q]\restriction_U = \mathrm{LOC}_U\,[Q]\restriction_U$ is again locally closed, i.e. a* Galois *closure.*

Proof: It readily follows from the definition of relational clone (see e.g. the paragraph following Definition 2.2 in [Beh11, p. 9]) that these are closed w.r.t. arbitrary (non-empty) intersections of relations of the same arity. Hence, Lemma 3.1.15 applies here.

Assuming local closedness of the clone Q, we can substitute $\mathrm{LOC}_A\, Q$ by Q in the derived equality $\mathrm{LOC}_U\, [Q]{\restriction_U} = [\mathrm{LOC}_A\, Q]{\restriction_U}$, and this proves that $[Q]{\restriction_U}$ is again locally closed. It remains a clone of relations by the additional assumptions on U and Corollary 3.1.10. □

3.2 Neighbourhoods

Based on the results of the previous section we will here define *neighbourhoods* of an algebra, the subsets that are suitable for our localisation theory. Then we define a canonical relational structure on them, which gives rise to a natural notion of embedding and isomorphism. Furthermore, we characterise these concepts using unary clone operations. After that we will show that isomorphic neighbourhoods distinguish exactly the same invariant relations, which demonstrates that the chosen isomorphism concept is reasonable from a relational point of view.

It is our aim to establish a structure theory which studies algebras via relational structures. To this end we associate with every algebra $\mathbf{A}$ the relational structure $\underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle$ living on the same base set as $\mathbf{A}$ and carrying all invariant relations of $\mathbf{A}$. We call $\underset{\sim}{\mathbf{A}}$ the *relational counterpart* of $\mathbf{A}$. A priori, $\underset{\sim}{\mathbf{A}}$ is a non-indexed relational structure, but of course, we can view it as an indexed structure via its canonically associated signature. The result is then the indexed structure $\underset{\sim}{\mathbf{A}}{\restriction_A}$ presented in Definition 3.2.2 below. We often do not distinguish sharply between $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{A}}{\restriction_A}$.

We want to localise algebras $\mathbf{A}$ to subsets $U \subseteq A$ via restricting the relations in the relational counterpart of $\mathbf{A}$. Hence, we should consider a subset $U \subseteq A$ suitable for localisation, if the restriction process is compatible with the clone structure on the relational counterpart $\underset{\sim}{\mathbf{A}}$. The previous section contains an extensive discussion which subsets are suitable in this sense. Its main results, particularly Proposition 3.1.12 and the arguments on page 46, suggest to use images of idempotent endomorphisms of $\underset{\sim}{\mathbf{A}}$, i.e. images of idempotent mappings in $\mathrm{End}\,\underset{\sim}{\mathbf{A}} = \mathrm{Pol}_A^{(1)}\,\mathrm{Inv}\,\mathbf{A} = \mathrm{Clo}^{(1)}\,(\mathbf{A})$. Moreover, according to Lemma 3.1.4, for such subsets U restricting the relational counterpart to U is the same as taking the retract of $\underset{\sim}{\mathbf{A}}$ w.r.t. the corresponding idempotent endomorphism, an operation which is very natural in the study of relation varieties, too (cp. e.g. [Zád97b, p. 562 et seq.]). Motivated by these arguments we will use images of idempotent operations in $\mathrm{Clo}^{(1)}\,(\mathbf{A})$ for localisation, and we will call these images *neighbourhoods*.

Speaking in terms of relational structures, the neighbourhoods of an algebra $\mathbf{A}$ are precisely the carrier sets of the idempotent retracts of its relational counterpart $\underset{\sim}{\mathbf{A}}$.

In the context of the word "neighbourhood" we should point out that the connection with the usual topological notion is very vague. The main commonality seems to lie in that both concepts refer to a local piece of a space or of an algebra, respectively. So, maybe, "localhood" would also be good terminology, which would not interfere with the established notion from topology. However, we stay with the term “neighbourhood” for we think that no confusion is to be expected as we are not dealing with topological algebras. This choice is mainly motivated by the wish to ensure compatibility with the language introduced in some works on Tame Congruence Theory for the TCT analogy of our neighbourhoods, see, for example, [Kis97, Sect. 2, p. 2], [CV02, Sect. 2, p. 5], and [Kea01, Beh09] for use in the context of Relational Structure Theory.

In order to fix some notation we give the following formal definition of neighbourhood and related concepts (see also Definition 2.4 in [Kea01], Definition 3.2.1 in [Beh09] and Definition 2.1 in [KL10]).

3.2.1 Definition. Let $\mathbf{A}$ be an algebra.

(i) We denote by
$$\operatorname{Idem}\mathbf{A} := \left\{ e \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; e \circ e = e \right\}$$
the *set of all idempotent unary operations in the clone of* $\mathbf{A}$.

(ii) A subset $U \subseteq A$ will be called a *neighbourhood* of $\mathbf{A}$ if it is the image of one such function $e \in \operatorname{Idem}\mathbf{A}$. We naturally call a neighbourhood U a *subneighbourhood* of some neighbourhood V of $\mathbf{A}$ if $U \subseteq V$.

(iii) The *set of all neighbourhoods* will be denoted by
$$\operatorname{Neigh}\mathbf{A} := \{ \operatorname{im} e \mid e \in \operatorname{Idem}\mathbf{A} \}. \qquad \Diamond$$

With this definition we have decided to use the narrower notion of neighbourhoods, which are given as images of idempotent unary clone operations. One can obtain a more fine-grained theory by studying as neighbourhoods all subsets for which the restriction operation as in Corollary 3.1.9 is a clone homomorphism between relational clones. This approach has been taken in [Sch12]. There, what we call neighbourhood is called *regular neighbourhood* of a discrete topological algebra. However, according to Proposition 3.1.12, for 1-locally finite algebras, both concepts coincide. So for the objects of our main interest, finite algebras and algebras in locally finite varieties, there is no difference between neighbourhoods and regular neighbourhoods.

Tame Congruence Theory is interested in congruence relations of (locally) finite algebras, and these are reflexive. That explains why in TCT polynomial operations are generally preferred over term operations (or clone operations). Correspondingly, TCT uses images of idempotent unary polynomial operations as neighbourhoods, see e.g. [HM88, Definition 2.1], [Kis97, Sect. 2, p. 2] and [CV02, Sect. 2, p. 5]. To avoid confusion we have not kept the notation $\operatorname{E}(\mathbf{A})$, which is commonly defined in TCT as $\operatorname{E}(\mathbf{A}) := \left\{ e \in \operatorname{Term}^{(1)}(\mathbf{A}_A) \;\middle|\; e^2 = e \right\}$, but introduced the notation $\operatorname{Idem}\mathbf{A}$. However, if $\operatorname{Clo}^{(1)}(\mathbf{A})$ contains all the unary constants and $\operatorname{Term}^{(1)}(\mathbf{A})$ is locally closed, then both sets coincide: $\operatorname{E}(\mathbf{A}) = \operatorname{Idem}\mathbf{A}$. This is, for instance, the case for polynomial expansions $\mathbf{A}_A$ of finite algebras, i.e. structures obtained by enriching algebras $\mathbf{A}$ with all constant operations. Apart from the difference between clone functions and polynomial functions our choice of idempotents and neighbourhoods is the same as in [HM88].

Next, we add some (relational) structure to neighbourhoods, in fact to arbitrary subsets of an algebra.

3.2.2 Definition. For an algebra $\mathbf{A}$ and a subset $U \subseteq A$ of its carrier set, we define the *relational structure that* $\mathbf{A}$ *induces on* U (or the *restriction*[4] *of the relational counterpart of* $\mathbf{A}$ *to* U) to be the structure

$$\underset{\sim}{\mathbf{A}}\restriction_U := \left\langle U;\ \left((S\restriction_U)_{S\in\mathrm{Inv}^{(m)}\mathbf{A}}\right)_{m\in\mathbb{N}}\right\rangle$$

of type $\left(\mathrm{Inv}^{(m)}\mathbf{A}\right)_{m\in\mathbb{N}}$. ◇

As mentioned before, for neighbourhoods we can view these restricted structures also as idempotent retracts.

3.2.3 Remark. Suppose that $\mathbf{A}$ is an algebra and $e \in \operatorname{Idem}\mathbf{A}$, then according to Lemma 3.1.4, the relational structure that $\mathbf{A}$ induces on the neighbourhood $\operatorname{im} e$ is the retract of $\underset{\sim}{\mathbf{A}}$ under the idempotent endomorphism $e \in \operatorname{End}\underset{\sim}{\mathbf{A}}$:

$$\begin{aligned}\underset{\sim}{\mathbf{A}}\restriction_{\operatorname{im} e} &= \left\langle \operatorname{im} e;\ \left((S\restriction_{\operatorname{im} e})_{S\in\mathrm{Inv}^{(m)}\mathbf{A}}\right)_{m\in\mathbb{N}}\right\rangle \\ &= \left\langle \operatorname{im} e;\ \left((e\circ[S])_{S\in\mathrm{Inv}^{(m)}\mathbf{A}}\right)_{m\in\mathbb{N}}\right\rangle = e\left[\underset{\sim}{\mathbf{A}}\right].\end{aligned}$$ ◇

The structure exhibited in Definition 3.2.2 yields a natural way of defining a homomorphism concept between neighbourhoods of an algebra (see Lemma 3.2.7 to see that our notion coincides with the definition of *neighbourhood morphism* chosen in [KL10, Definition 2.5]).

3.2.4 Definition. Let $\mathbf{A}$ be an algebra and $U, V \in \operatorname{Neigh}\mathbf{A}$ neighbourhoods. We call a mapping $h\colon U \longrightarrow V$ a *homomorphism* between these neighbourhoods if it is a homomorphism between the induced relational structures $\underset{\sim}{\mathbf{A}}\restriction_U$ and $\underset{\sim}{\mathbf{A}}\restriction_V$, i.e. a relation preserving map. Furthermore, we set

$$\operatorname{Hom}(U,V) := \operatorname{Hom}\left(\underset{\sim}{\mathbf{A}}\restriction_U, \underset{\sim}{\mathbf{A}}\restriction_V\right)$$

as the set of all such homomorphisms. ◇

3.2.5 Remark. Defining homomorphisms between neighbourhoods of an algebra $\mathbf{A}$ in the sense of Definition 3.2.4 means that we have in fact established a small concrete category on the object set $\operatorname{Neigh}\mathbf{A}$, which is isomorphic to the full subcategory of all relational structures of type $\left(\mathrm{Inv}^{(m)}\mathbf{A}\right)_{m\in\mathbb{N}}$ given by the object set

[4] In point of fact, $\underset{\sim}{\mathbf{A}}\restriction_U$ is exactly the induced substructure of the relational counterpart $\underset{\sim}{\mathbf{A}}$ (viewed as an indexed structure with the natural indexing) on the subset U.

$\{\underset{\approx}{\mathbf{A}}\restriction_U \mid U \in \operatorname{Neigh}\mathbf{A}\}$. This category isomorphism allows us to translate all category theoretic concepts from the mentioned subcategory to neighbourhoods. For instance, this is the case for isomorphism of neighbourhoods. Furthermore, the isomorphism functor between the two categories is identity on morphisms. So more concrete concepts, e.g. embedding, that are related to morphisms in categories of relational structures can be transferred, too. ♢

To make the examples in the previous remark more explicit, we state the following definition of isomorphism and embedding. The same notion of isomorphism between neighbourhoods has been stated in Definitions 2.5 of [Kea01] and 3.2.1(iv) of [Beh09]. However, our concept of neighbourhood embedding is new in this wording. We shall see in Proposition 3.2.10 below that it induces the same quasiorder on neighbourhoods of an algebra as defined in [KL10] after Definition 2.8.

3.2.6 Definition. Let $\mathbf{A}$ be an algebra and $U, V \in \operatorname{Neigh}\mathbf{A}$ be neighbourhoods.

(i) A mapping $h \in \operatorname{Hom}(U, V)$ is called *isomorphism* between the neighbourhoods U and V if it is an isomorphism between $\underset{\approx}{\mathbf{A}}\restriction_U$ and $\underset{\approx}{\mathbf{A}}\restriction_V$, i.e., if there is a homomorphism $h' \in \operatorname{Hom}(V, U)$ such that $h' \circ h = \mathrm{id}_U$ and $h \circ h' = \mathrm{id}_V$.

The neighbourhoods U and V are called *isomorphic (within* $\mathbf{A}$*)*, written as $U \cong V$ if there is an isomorphism between them, i.e. if the induced relational structures $\underset{\approx}{\mathbf{A}}\restriction_U$ and $\underset{\approx}{\mathbf{A}}\restriction_V$ are isomorphic as structures of type $\left(\operatorname{Inv}^{(m)}\mathbf{A}\right)_{m\in\mathbb{N}}$.

(ii) A mapping $h \in \operatorname{Hom}(U, V)$ is called *neighbourhood embedding of* U *in* V if it is an embedding as a homomorphism between $\underset{\approx}{\mathbf{A}}\restriction_U$ and $\underset{\approx}{\mathbf{A}}\restriction_V$, i.e. an injective relation preserving and relation reflecting map, and its image $h[U]$ can be obtained as the image of an idempotent endomorphism of $\underset{\approx}{\mathbf{A}}\restriction_V$.

We say that the neighbourhood U is *embedded* in V, symbolically $U \underset{\approx}{\lesssim} V$, if there exists a neighbourhood embedding between them. ♢

The abstract approach of transferring known concepts via a category isomorphism ensures that the defined notions are reasonable and useful. However, we certainly wish to see more concrete characterisations for them, to simplify the work with neighbourhoods. The following lemma points out that unary clone operations of the underlying algebra should play a role in such characterisations.

3.2.7 Lemma (cf. Definition 2.5 of [KL10]). *For an algebra* $\mathbf{A}$*, idempotents* $e_U, e_V \in \operatorname{Idem}\mathbf{A}$ *and corresponding neighbourhoods* $U := \operatorname{im} e_U$ *and* $V := \operatorname{im} e_V$*, we have*

$$\begin{aligned}\operatorname{Hom}(U, V) &= \left\{ f|_U^V \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge f[U] \subseteq V \right\}\\ &= \left\{ f|_U^V \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge \operatorname{im} f \subseteq V \right\}.\end{aligned}$$

Proof: We will prove the inclusions

$$\begin{aligned}\operatorname{Hom}(U,V) &\overset{(1)}{\subseteq} \left\{ f|_U^V \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge f[U] \subseteq V \right\} \\ &\overset{(2)}{\subseteq} \left\{ f|_U^V \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge \operatorname{im} f \subseteq V \right\} \overset{(3)}{\subseteq} \operatorname{Hom}(U,V).\end{aligned}$$

(1) For a given homomorphism $\varphi \in \operatorname{Hom}(U,V) = \operatorname{Hom}(\underset{\sim}{\mathbf{A}}{\restriction_U}, \underset{\sim}{\mathbf{A}}{\restriction_V})$, let us define $f := \operatorname{id}_A|_V^A \circ \varphi \circ e_U|_A^U$. Since e_U is in Idem $\mathbf{A}$, it belongs to

$$\operatorname{Clo}^{(1)}(\mathbf{A}) = \operatorname{Pol}_A^{(1)} \operatorname{Inv} \mathbf{A} = \operatorname{Pol}^{(1)} \underset{\sim}{\mathbf{A}} = \operatorname{End} \underset{\sim}{\mathbf{A}}.$$

Therefore, the restriction to its image $e[\underset{\sim}{\mathbf{A}}] = \underset{\sim}{\mathbf{A}}{\restriction_U}$, the map $e_U|_A^U \colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_U}$, is a homomorphism, too. The same is correct for the inclusion morphism $\operatorname{id}_A|_V^A \colon \underset{\sim}{\mathbf{A}}{\restriction_V} \longrightarrow \underset{\sim}{\mathbf{A}}$, whence f is a homomorphism between the relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{A}}$. In other words, f is an endomorphism of $\underset{\sim}{\mathbf{A}}$, so it belongs to $\operatorname{Clo}^{(1)}(\mathbf{A})$. Furthermore,

$$f[U] = \operatorname{id}_A[\varphi[e_U[U]]] = \varphi[e_U[U]] = \varphi[U] \subseteq V,$$

and for every $u \in U$ we have $f(u) = \operatorname{id}_A(\varphi(e_U(u))) = \varphi(e_U(u)) = \varphi(u)$, such that $f|_U^V = \varphi$.

(2) Consider a unary operation $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ satisfying $f[U] \subseteq V$. Let us define $g := e_V \circ f$, which again belongs to $\operatorname{Clo}^{(1)}(\mathbf{A})$. Certainly, its image is contained in $\operatorname{im} e_V = V$, and for $u \in U$ we have $g(u) = e_V(f(u)) = f(u)$ since the latter element belongs to V by assumption on f. Hence, $g|_U^V = f|_U^V$.

(3) Here we need to check that the restriction of any $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ with $\operatorname{im} f \subseteq V$ yields a homomorphism $f|_U^V \colon \underset{\sim}{\mathbf{A}}{\restriction_U} \longrightarrow \underset{\sim}{\mathbf{A}}{\restriction_V}$. For this consider any $m \in \mathbb{N}$, $S \in \operatorname{Inv}^{(m)} \mathbf{A}$ and an arbitrary tuple $x \in S{\restriction_U}$. Then we have $f|_U^V \circ x \in V^m$ since $\operatorname{im} f \subseteq V$. Furthermore, $f|_U^V \circ x = f \circ x \in S$ as f needs to preserve S for it belongs to $\operatorname{Clo}^{(1)}(\mathbf{A}) = \operatorname{Pol}_A^{(1)} \operatorname{Inv} \mathbf{A}$. Putting this together we obtain $f|_U^V \circ x \in S{\restriction_V}$ as desired. □

This result enables us to characterise the isomorphism relation between neighbourhoods. The equivalence of items (a) and (b) of the subsequent proposition has already been noted in Lemma 2.6 of [Kea01]. Furthermore, with the following we provide a more detailed version of Lemma 3.2.2 from [Beh09].

3.2.8 Proposition. *For an algebra $\mathbf{A}$ and idempotent operations $e_U, e_V \in \operatorname{Idem} \mathbf{A}$ such that $U := e_U[A]$ and $V := e_V[A]$ are two neighbourhoods of $\mathbf{A}$, the following statements are equivalent*

(a) $U \cong V$.

(b) *There exist* $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ *such that* $f[U] \subseteq V$, $g[V] \subseteq U$ *and*

$$\begin{aligned} \forall\, u \in U\colon &\quad g(f(u)) = u \\ \forall\, v \in V\colon &\quad f(g(v)) = v. \end{aligned}$$

(c) *There exist* $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ *such that* $\operatorname{im} f = f[U] = V$, $\operatorname{im} g = g[V] = U$ *and*

$$\begin{aligned} \forall\, u \in U\colon &\quad g(f(u)) = u \\ \forall\, v \in V\colon &\quad f(g(v)) = v. \end{aligned}$$

(d) *There exist* $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ *such that*

$$\begin{aligned} e_V \circ f \circ e_U &= f \circ e_U & \qquad e_U \circ g \circ e_V &= g \circ e_V \\ g \circ f \circ e_U &= e_U & \qquad f \circ g \circ e_V &= e_V. \end{aligned}$$

(e) *There exist* $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ *such that*

$$\begin{aligned} e_V \circ f &= f & \qquad e_U \circ g &= g \\ g \circ f \circ e_U &= e_U & \qquad f \circ g \circ e_V &= e_V. \end{aligned}$$

Item (b) is a useful sufficient condition for neighbourhood isomorphism, whereas item (c) should be used as a necessary condition. The characterisations (d) and (e) can be interpreted as semigroup theoretic reformulations of items (b) and (c) in $\left(\mathrm{Clo}^{(1)}(\mathbf{A}), \circ\right)$, respectively. We note, furthermore, that, apart from the small difference between clone and polynomial operations, the characterisation of neighbourhood isomorphism in (c) is the same as the definition of polynomial isomorphism between subsets in Definition 2.7 of [HM88].

Proof: We will show that (a) $\Rightarrow$ (c) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a).

"(a) $\Rightarrow$ (c)" Suppose that $U \cong V$. According to Definition 3.2.6 there exist mutual inverse isomorphism between these neighbourhoods. Using Lemma 3.2.7 we can obtain these homomorphisms as restrictions of unary clone operations. In more detail, there exist $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ such that $\operatorname{im} f \subseteq V$, $\operatorname{im} g \subseteq U$, $(g \circ f)|_U = g|_V^U \circ f|_U^V = \mathrm{id}_U$ and $(f \circ g)|_V = f|_U^V \circ g|_V^U = \mathrm{id}_V$. The latter can be written element-wise as $g(f(u)) = u$ for $u \in U$ and $f(g(v)) = v$ for $v \in V$. Therefore, for $u \in U$ we have $u = g(f(u)) \in g[V]$ because $f(u) \in \operatorname{im} f \subseteq V$. Thus, $U \subseteq g[V] \subseteq \operatorname{im} g \subseteq U$, i.e., $U = g[V] = \operatorname{im} g$. Analogously, we can infer $V = f[U] = \operatorname{im} f$.

"(c) $\Rightarrow$ (e)" Suppose the existence of unary clone operations $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ with the assumptions of item (c). They imply $\operatorname{im} f \subseteq V$ and $\operatorname{im} g \subseteq U$, which by Lemma 3.1.3 can be rewritten as $e_V \circ f = f$ and $e_U \circ g = g$, respectively. Furthermore, for $a \in A$ we have $e_U(a) \in U$ such that $g(f(e_U(a))) = e_U(a)$ follows from the assumptions. That is, $g \circ f \circ e_U = e_U$, and similarly, we obtain $f \circ g \circ e_V = e_V$.

"(e) ⇒ (d)" If we are given unary clone operations $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ obeying the stated equalities, then these automatically satisfy the equalities of item (d).

"(d) ⇒ (b)" Assume unary clone operations $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ satisfy the equalities listed in item (d). By Lemma 3.1.3 the condition $e_V \circ f \circ e_U = f \circ e_U$ is equivalent to $V \supseteq \mathrm{im}(f \circ e_U) = f[e_U[A]] = f[U]$. From the second assumed equality we obtain $U \supseteq g[V]$. The remaining two equalities are an obvious consequence of idempotency of e_U and e_V and the other two assumed conditions, respectively.

"(b) ⇒ (a)" Having clone functions $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ fulfilling the conditions of item (b), the operations $\varphi := f|_U^V$ and $\psi := g|_V^U$ are well-defined. By assumption, we have $\psi \circ \varphi = \mathrm{id}_U$ and $\varphi \circ \psi = \mathrm{id}_V$. It only remains to show that these maps are homomorphisms. This follows since they are restrictions of clone operations of $\mathbf{A}$. Namely, for $m \in \mathbb{N}$ and $S \in \mathrm{Inv}^{(m)}\mathbf{A}$ we have

$$\varphi \circ [S\restriction_U] = f \circ [S\restriction_U] \subseteq f \circ [S] \subseteq S$$

since $f \in \mathrm{Pol}_A^{(1)} \mathrm{Inv}\,\mathbf{A}$ preserves the invariant relation S. Furthermore, since φ has its image in V, we get $\varphi \circ [S\restriction_U] \subseteq V^m$. Putting this together we obtain $\varphi \circ [S\restriction_U] \subseteq S\restriction_V$, and this means that φ is a relation preserving map. In the same way one can see that ψ is relation preserving, i.e. a homomorphism. □

We want to give a similar characterisation of the neighbourhood embedding relation. We begin with a lemma describing a necessary condition.

3.2.9 Lemma. *Let $\mathbf{A}$ be an algebra, $U, V \in \mathrm{Neigh}\,\mathbf{A}$ and $f \in \mathrm{Hom}(U, V)$ be a neighbourhood embedding. Then $V' := f[U] \subseteq V$ is a neighbourhood of $\mathbf{A}$ and $U \cong V'$.*

Proof: It is part of the definition that $V' = f[U]$ is the image of some idempotent endomorphism $e' \in \mathrm{End}\,\underset{\sim}{\mathbf{A}}\restriction_V$. Furthermore, let $e_V \in \mathrm{Idem}\,\mathbf{A}$ such that $V = \mathrm{im}\,e_V$. Clearly, the restriction $e_V|_A^V$ is a member of $\mathrm{Hom}(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{A}}\restriction_V)$, and also $\mathrm{id}_A|_V^A$ belongs to $\mathrm{Hom}(\underset{\sim}{\mathbf{A}}\restriction_V, \underset{\sim}{\mathbf{A}})$. So the composition $e := \mathrm{id}_A|_V^A \circ e' \circ e_V|_A^V$ is a homomorphism, too, an endomorphism of $\underset{\sim}{\mathbf{A}}$. Since $\mathrm{im}\,e' = V' \subseteq V$, Lemma 3.1.3 implies $e_V|_A^V \circ \mathrm{id}_A|_V^A \circ e' = e_V|_V \circ e' = e'$. Hence we get

$$\begin{aligned} e \circ e &= \mathrm{id}_A|_V^A \circ e' \circ e_V|_A^V \circ \mathrm{id}_A|_V^A \circ e' \circ e_V|_A^V = \mathrm{id}_A|_V^A \circ e' \circ e' \circ e_V|_A^V \\ &= \mathrm{id}_A|_V^A \circ e' \circ e_V|_A^V = e. \end{aligned}$$

Therefore, $e \in \mathrm{Idem}\,A \cap \mathrm{End}\,\underset{\sim}{\mathbf{A}} = \mathrm{Idem}\,\mathbf{A}$, and its image is

$$\mathrm{im}\,e = e'[e_V[A]] = e'[V] = V' = f[U].$$

So $V' \subseteq V$ is a subneighbourhood of V.

Next, we prove that U and V' are isomorphic. Since f is a neighbourhood embedding, the restriction $f|_U^{V'}\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_{V'}$ is a relation preserving bijection. Let us denote the inverse mapping by $g\colon V' \longrightarrow U$. It is also relation preserving because for every $m \in \mathbb{N}$, $S \in \operatorname{Inv}^{(m)}\mathbf{A}$ and $y \in S\restriction_{V'}$ we have $y \in V'^m$, so $x := g \circ y \in U^m$. This tuple satisfies $f \circ x = f \circ g \circ y = y \in S\restriction_{V'} \subseteq S\restriction_V$. Since f reflects relations, we obtain that $g \circ y = x \in S\restriction_U$. This shows that g belongs to $\operatorname{Hom}(\underset{\sim}{\mathbf{A}}\restriction_{V'}, \underset{\sim}{\mathbf{A}}\restriction_U)$, so f and g are inverse isomorphisms, which demonstrate $U \cong V'$. $\square$

The following result shows that the condition of being isomorphic to some subneighbourhood, which was exhibited in the previous lemma, is indeed characteristic for neighbourhood embedding.

3.2.10 Proposition. *For an algebra $\mathbf{A}$, idempotents $e_U, e_V \in \operatorname{Idem}\mathbf{A}$ with corresponding neighbourhoods $U := \operatorname{im} e_U$ and $V := \operatorname{im} e_V$ the following facts are equivalent:*

(a) $U \precapprox V$.

(b) U is isomorphic to some subneighbourhood of V, i.e. there exists a neighbourhood $V' \in \operatorname{Neigh}\mathbf{A}$ such that $U \cong V'$ and $V' \subseteq V$. (Abstractly speaking, this means that the pair (U, V) belongs to the relation product $\cong \circ \subseteq$.)

(c) There exist $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that $f[U] \subseteq V$ and

$$\forall\, u \in U\colon \quad g(f(u)) = u.$$

(d) There exist $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that $\operatorname{im} f = f[U] \subseteq V$, $\operatorname{im} g = g[V] = U$ and

$$\forall\, u \in U\colon \quad g(f(u)) = u.$$

(e) There exist $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that

$$\begin{aligned} e_V \circ f \circ e_U &= f \circ e_U \\ g \circ f \circ e_U &= e_U. \end{aligned}$$

(f) There exist $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that

$$\begin{aligned} e_V \circ f &= f \\ e_U \circ g &= g \\ g \circ f \circ e_U &= e_U. \end{aligned}$$

Again item (c) is intended as a means sufficient to prove embedding of neighbourhoods, and item (d) is to be exploited as a necessary condition. The conditions in (e) and in (f) can again be understood as semigroup theoretic reformulations of (c) and (d). Characterisation (b) can be seen as the neighbourhood analogy of embedding in the model theoretic sense: being isomorphic to an induced submodel. It is also the definition of neighbourhood embedding that is used in [KL10].

Proof: We will show that (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (c) $\Rightarrow$ (a).

"(a) $\Rightarrow$ (b)" This implication immediately follows from Lemma 3.2.9.

"(b) $\Rightarrow$ (d)" We assume that there is some neighbourhood $V' \in \operatorname{Neigh} \mathbf{A}$ such that $V' \subseteq V$ and $U \cong V'$. The characterisation of the isomorphism relation in Proposition 3.2.8(c) yields the existence of $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that

$$\begin{aligned} \operatorname{im} f = f[U] = V' \subseteq V, \\ g[V'] \subseteq g[V] \subseteq g[A] = \operatorname{im} g = g[V'] = U \end{aligned}$$

and the equality $g(f(u)) = u$ holds for all $u \in U$.

"(d) $\Rightarrow$ (f)" By assumption we have $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that $\operatorname{im} f \subseteq V$, $\operatorname{im} g \subseteq U$ and $g(f(u)) = u$ for $u \in U$. Using Lemma 3.1.3, the inclusions can be expressed equivalently as $e_V \circ f = f$ and $e_U \circ g = g$, respectively. As $e_U(a) \in U$ for every $a \in A$, we obtain $g(f(e_U(a))) = e_U(a)$ for any $a \in A$, that is to say, $g \circ f \circ e_U = e_U$.

"(f) $\Rightarrow$ (e)" This implication is an immediate consequence.

"(e) $\Rightarrow$ (c)" Due to Lemma 3.1.3, the first assumed equality for $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ is equivalent to the inclusion $V \supseteq \operatorname{im}(f \circ e_U) = f[e_U[A]] = f[U]$. As we have $e_U(u) = u$ for $u \in U$, the second equality yields $g(f(u)) = u$ for all $u \in U$.

"(c) $\Rightarrow$ (a)" If we are given $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ satisfying $f[U] \subseteq V$ and $g(f(u)) = u$ for all $u \in U$, then clearly the restriction $\varphi := f|_U^V$ is a well-defined mapping. Since g inverts f on U, the mapping φ is injective. It is also a homomorphism from $\underset{\sim}{\mathbf{A}}\restriction_U$ to $\underset{\sim}{\mathbf{A}}\restriction_V$ because it preserves relations $S\restriction_U$ for $S \in \operatorname{Inv} \mathbf{A}$: indeed, we have $\varphi \circ [S\restriction_U] = f \circ [S\restriction_U] \subseteq f \circ [S] \subseteq S$ because $f \in \operatorname{Pol}_A \operatorname{Inv} \mathbf{A}$ preserves the invariant S. Furthermore, the image of φ lies in V such that $\varphi \circ [S\restriction_U] \subseteq V^{\operatorname{ar} S}$, yielding, together with the previous, that $\varphi \circ [S\restriction_U] \subseteq S\restriction_V$.

It remains to show that the injective homomorphism $\varphi\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_V$ also reflects relations. For this consider again some $S \in \operatorname{Inv}^{(m)} \mathbf{A}$ of arity $m \in \mathbb{N}$. For any $u \in U^m$ such that $y := \varphi \circ u \in S\restriction_V$, the assumed invertibility of f on U yields $g \circ y = g \circ \varphi \circ u = g \circ f \circ u = u$. Moreover, as $g \in \operatorname{Clo}(\mathbf{A})$, it must preserve S, hence $u = g \circ y \in S$. For u is in U^m, it belongs to $S\restriction_U$, and this proves that φ is a relation reflecting homomorphism, and hence an embedding.

We still need to show that $V' := \operatorname{im} \varphi = f[U]$ is the image of an idempotent endomorphism of $\underset{\sim}{\mathbf{A}}\restriction_V$. Let us consider the unary clone operation $e := f \circ e_U \circ g$. As above the invertibility condition $g(f(u)) = u$ for $u \in U$ is equivalent to $g \circ f \circ e_U = e_U$. This yields that e is idempotent:

$$e \circ e = f \circ e_U \circ g \circ f \circ e_U \circ g = f \circ e_U \circ e_U \circ g = f \circ e_U \circ g = e.$$

Since $f[U] \subseteq V$, we have $u = g(f(u)) \in g[f[U]] \subseteq g[V]$ for $u \in U$. This means that $U \subseteq g[V]$. Hence, $U = e_U[U] \subseteq e_U[g[V]]$, and we obtain

$$V' = f[U] \subseteq f[e_U[g[V]]] = e[V] \subseteq f[e_U[A]] = f[U] = V'.$$

Thus, $e[V] = V' \subseteq V$, in particular e preserves V. Therefore, it can be restricted to V, and the restriction $e|_V$ has image V' and is still idempotent. For e belongs to $\mathrm{Clo}^{(1)}(\mathbf{A})$ it is an endomorphism of $\underset{\approx}{\mathbf{A}}$. Consequently, the restriction $e|_V$ is an endomorphism of $\underset{\approx}{\mathbf{A}}\restriction_V$ having image V'. $\square$

The following corollary shows that given two unary clone operations, one inverting the other on a neighbourhood U, the image of U under the embedding automatically is a neighbourhood, too, which is isomorphic to U.

3.2.11 Corollary. *Assume* $\mathbf{A}$ *is an algebra,* $U \in \mathrm{Neigh}\,\mathbf{A}$ *and* $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ *satisfy* $g(f(u)) = u$ *for all* $u \in U$. *Then for any subneighbourhood* $U' \in \mathrm{Neigh}\,\mathbf{A}$, $U' \subseteq U$, *the images* $V' := f[U']$ *and* $V := f[U]$ *are neighbourhoods of* $\mathbf{A}$ *and satisfy* $U' \cong V'$, $U \cong V$ *and the subneighbourhood relation* $V' \subseteq V$.

If U *is finite and* $U' \subsetneq U$ *is a proper subneighbourhood, then so is* $V' \subsetneq V$.

Proof: The given assumption is sufficient for item (c) of the previous proposition to show $U' \precapprox A$. Reading again the proof of the implication "(c) $\Rightarrow$ (a)" of the previous proposition, we see that $f|_U^A \in \mathrm{Hom}(U', A)$ is an embedding. Combining this observation with Lemma 3.2.9, gives us that the image of the embedding $V' = f[U']$ is a neighbourhood of $\mathbf{A}$, which is isomorphic to U'. Certainly, this argument is also true for the fullsubneighbourhood $U' = U$, such that $V \in \mathrm{Neigh}\,\mathbf{A}$ and $U \cong V$. The subneighbourhood relationship $V' \subseteq V$ holds by definition.

At last we show that proper subneighbourhoods are mapped to proper subneighbourhoods if U is finite. In fact, we demonstrate the contrapositive, i.e. we show that $V' = V$ implies $U = U'$. Taking cardinalities from the condition $U \cong V = V' \cong U' \subseteq U$ yields $|U| = |V| = |V'| = |U'| \leq |U|$, i.e. $|U'| = |U|$. Now by finiteness of U, the neighbourhood U' cannot be a proper subset of U. $\square$

Moreover, from Proposition 3.2.10 we obviously see that neighbourhood embedding is a common generalisation of containment and the isomorphism relation.

3.2.12 Corollary. *For an algebra* $\mathbf{A}$ *and neighbourhoods* $U, V \in \mathrm{Neigh}\,\mathbf{A}$ *the following implications hold:*

(a) $U \subseteq V \implies U \precapprox V$.

(b) $U \cong V \implies U \precapprox V$.

Proof: This is a direct consequence of item (b) of Proposition 3.2.10 and reflexivity of set inclusion and the isomorphism relation. $\square$

The third corollary mentions that neighbourhood embedding is a quasiorder and states that, for finite algebras, the associated equivalence relation is the isomorphism relation. This fact also occurs in [KL10].

3.2.13 Corollary. *For any algebra* $\mathbf{A}$ *the structure* $(\mathrm{Neigh}\,\mathbf{A}; \precapprox)$ *is a quasiordered set. Moreover, neighbourhood embedding is even the least quasiorder on* $\mathrm{Neigh}\,\mathbf{A}$ *extending set inclusion and the isomorphism relation, i.e. it is their least common generalisation in the quasiorder lattice on* $\mathrm{Neigh}\,\mathbf{A}$*:* $\precapprox \; = \; \cong \vee \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$.

Any two mutually embeddable neighbourhoods, one of which is finite, must be isomorphic. Thus, for finite algebras $\mathbf{A}$ *the equivalence relation* $\precapprox \cap \succapprox$*, the mutual embedding relation, is the isomorphism relation on neighbourhoods of* $\mathbf{A}$.

Proof: Reflexivity of the embedding relation follows from reflexivity of set inclusion and Corollary 3.2.12(a). Transitivity can easily be verified using the characterisation in Proposition 3.2.10(c). Thus $\precapprox$ is a quasiorder on neighbourhoods.

To emphasise their role as quasiorders on the set $\mathrm{Neigh}\,\mathbf{A}$, let us denote for a moment the subset relation on $\mathrm{Neigh}\,\mathbf{A}$ by ψ and neighbourhood isomorphism by θ. Proposition 3.2.10(b) says that $\precapprox \; = \theta \circ \psi$, so, by Corollary 3.2.12, we have $\theta \cup \psi \subseteq \; \precapprox \; = \theta \circ \psi \subseteq \theta \vee \psi$. We just saw that $\precapprox$ is a quasiorder relation on $\mathrm{Neigh}\,\mathbf{A}$. Since it contains θ and ψ, it must also contain their join $\theta \vee \psi$. Hence, we get $\precapprox \; = \theta \vee \psi = \; \cong \vee \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$.

Now suppose that $U \precapprox V \precapprox U$ and U is finite. According to the statement in item (b) of Proposition 3.2.10, we can find neighbourhoods $U', V' \in \mathrm{Neigh}\,\mathbf{A}$ such that

$$U \cong U' \subseteq V \cong V' \subseteq U.$$

Translating this into a statement about cardinalities yields

$$|U| = |U'| \leq |V| = |V'| \leq |U|,$$

which means $|U| = |U'| = |V| = |V'|$. Now finiteness of U ensures that V' cannot be a proper subset of U, hence $V \cong V' = U$. □

The last statement of Corollary 3.2.13 is not only true for finite algebras. In the following definition we are going to introduce two conceptual properties of algebras ensuring, among other things, that the mutual embedding relation coincides with the isomorphism relation. Using these conditions allows us to transfer techniques familiar from finite algebras to certain classes of infinite algebras. This will become important, especially in Section 3.6. In Corollary 3.5.14, we shall also see that all algebras from 1-locally finite varieties, hence not only finite ones, exhibit the following two qualities:

3.2.14 Definition. (i) An algebra $\mathbf{A}$ is said to have the *finite iteration property* (abbreviated *FIP*) if for every unary operation $f \in \mathrm{Clo}^{(1)}(\mathbf{A})$ there exists a finite exponent $n \in \mathbb{N}_+$ such that the power $f^n \in \mathrm{Idem}\,\mathbf{A}$.

(ii) We say that an algebra $\mathbf{A}$ is *neighbourhood self-embedding simple* if for all $U, V \in \operatorname{Neigh} \mathbf{A}$ the conditions $V \subseteq U$ and $U \precapprox V$ imply $U = V$. $\Diamond$

In our next steps, we will first observe a quite obvious, however sometimes useful, equivalent formulation of the FIP. Second, we shall give a semigroup theoretic characterisation of the FIP, then understand that this property implies neighbourhood self-embedding simplicity, and after that verify that the latter indeed ensures that mutually embeddable neighbourhoods are isomorphic.

3.2.15 Remark. An algebra $\mathbf{A}$ has the FIP if and only if for every finite subset $F \subseteq \operatorname{Clo}^{(1)}(\mathbf{A})$ there exists a finite exponent $n \in \mathbb{N}_+$ such that $f^n \in \operatorname{Idem} \mathbf{A}$ for all $f \in F$. $\Diamond$

Proof: Clearly, the FIP follows from the stated condition by letting $F \subseteq \operatorname{Clo}^{(1)}(\mathbf{A})$ be a singleton set. For the converse we first note that, by induction on $k \in \mathbb{N}_+$, every idempotent $e \in \operatorname{Idem} \mathbf{A}$ satisfies $e^k = e \in \operatorname{Idem} \mathbf{A}$. Therefore, if $n, N \in \mathbb{N}_+$ are such that $n \mid N$, and $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ satisfies $f^n \in \operatorname{Idem} \mathbf{A}$, then we also have $f^N = (f^n)^{N/n} = f^n \in \operatorname{Idem} \mathbf{A}$. Now, for a finite set $F \subseteq \operatorname{Clo}^{(1)}(\mathbf{A})$, using the FIP, we obtain an integer $n_f \in \mathbb{N}_+$ for each $f \in F$ such that $f^{n_f} \in \operatorname{Idem} \mathbf{A}$. Letting $N := \operatorname{lcm}\{n_f \mid f \in F\}$, we obtain a common multiple of all these integers, which is greater than zero provided F is non-empty. Consequently, we can then infer $f^N = f^{n_f} \in \operatorname{Idem} \mathbf{A}$ for all $f \in F$. For $F = \emptyset$ the claim is trivially true choosing $N = 1$. $\square$

The following lemma shows that the FIP is connected to *periodic* semigroups, i.e. those which are by definition 1-locally finite (see page 44).

3.2.16 Lemma. *An algebra $\mathbf{A}$ has the FIP if and only if $\left\langle \operatorname{Clo}^{(1)}(\mathbf{A}), \circ \right\rangle$ is a periodic transformation semigroup.*

Proof: First, assume that $\mathbf{A}$ has the FIP. We need to show that the semigroup $\mathbf{S} := \left\langle \operatorname{Clo}^{(1)}(\mathbf{A}), \circ \right\rangle$ is 1-locally finite. For every $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ the monogenic subsemigroup $\langle\{f\}\rangle_{\mathbf{S}} = \left\{ f^k \;\middle|\; k \in \mathbb{N}_+ \right\}$ has to be checked for finiteness. Applying the FIP, we find some $n \in \mathbb{N}_+$ such that f^n is idempotent, i.e. $f^{2n} = f^n$. From this we get by induction that $f^{n+k} = f^{n+(k \bmod n)}$ for all $k \in \mathbb{N}$. Thus $\langle\{f\}\rangle_{\mathbf{S}}$ equals $\left\{ f^k \;\middle|\; 1 \leq k < 2n \right\}$, which is a finite set.

For the converse, we assume that all monogenic subsemigroups of $\mathbf{S}$ are finite. So for every $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ the set $\langle\{f\}\rangle_{\mathbf{S}} = \left\{ f^k \;\middle|\; k \in \mathbb{N}_+ \right\}$ has finite cardinality, whence there is a least exponent $k \in \mathbb{N}_+$ such that $f^k \in \{f^m \mid 1 \leq m < k\}$. So we can find some $m \in \{1, \dots, k-1\}$ such that $f^k = f^m$. For any $\ell \geq m$ we can now show by induction on $r \in \mathbb{N}$ that $f^{\ell + r(k-m)} = f^\ell$. Now choose ℓ to be $m(k-m) \geq m$, and let $r = m$. It follows $f^\ell = f^{\ell + m(k-m)} = f^{\ell+\ell} = f^{2\ell} = \left(f^\ell\right)^2$, such that f^ℓ is an idempotent power of f. $\square$

Particularly, every *idempotent semigroup* (also called *band*), where all elements are idempotent, is periodic. A special subclass of bands consists of the commutative ones, namely semi-lattices. Lemma 3.2.16 says we may take any periodic transformation semigroup on A that is locally closed and consider the unary algebra having these transformations as fundamental operations. The resulting structure will automatically have the FIP.

A simple consequence of the previous characterisation is the following:

3.2.17 Corollary. *If for an algebra* $\mathbf{A}$ *the set* $\mathrm{Clo}^{(1)}(\mathbf{A})$ *is finite, then* $\mathbf{A}$ *has the FIP. In particular, finite algebras have the FIP.*

Proof: It is clear that every finite algebra is 1-locally finite. Using this fact for semigroups and applying Lemma 3.2.16 tells us that finiteness of $\mathrm{Clo}^{(1)}(\mathbf{A})$ implies the FIP for $\mathbf{A}$. □

Next, we shall see that the FIP is at least as strong as neighbourhood self-embedding simplicity.

3.2.18 Lemma. *(a) An algebra* $\mathbf{A}$ *is neighbourhood self-embedding simple if and only if for all* $U, V \in \operatorname{Neigh}\mathbf{A}$ *the conditions* $V \subseteq U$ *and* $U \cong V$ *imply* $U = V$.

(b) Every algebra $\mathbf{A}$ *having the FIP is neighbourhood self-embedding simple.*

Proof: (a) First, if the algebra $\mathbf{A}$ is neighbourhood self-embedding simple, and $U, V \in \operatorname{Neigh}\mathbf{A}$ satisfy $V \subseteq U$ and $U \cong V$, then we have $U \precapprox V$ by Corollary 3.2.12(b). Thus, our assumption yields $U = V$. Conversely, if the implication stated in (a) holds and we are given $U, W \in \operatorname{Neigh}\mathbf{A}$ such that $W \subseteq U$ and $U \precapprox W$, then Proposition 3.2.10(b) yields that $U \cong V \subseteq W \subseteq U$ for some neighbourhood $V \in \operatorname{Neigh}\mathbf{A}$. By the assumed condition, we get $U = V$, and hence $U = V \subseteq W \subseteq U$, i.e. $U = W$. This shows that $\mathbf{A}$ is neighbourhood self-embedding simple.

(b) To prove that the FIP implies neighbourhood self-embedding simplicity, we show that it implies the condition occurring in (a). For this let us consider idempotents $e_U, e_V \in \operatorname{Idem}\mathbf{A}$ where $U \cong V \subseteq U$ holds for the corresponding neighbourhoods $U := \operatorname{im} e_U$ and $V := \operatorname{im} e_V$. According to Proposition 3.2.8(e), we can find $f, g \in \mathrm{Clo}^{(1)}(\mathbf{A})$ satisfying

$$e_V \circ f = f \tag{3.1}$$

$$e_U \circ g = g \tag{3.2}$$

$$g \circ f \circ e_U = e_U. \tag{3.3}$$

Furthermore, we have $e_U \circ e_V = e_V$ since $V \subseteq U$ and $e_U \in \operatorname{Idem}\mathbf{A}$. Thus, we get

$$e_U \circ f \overset{(3.1)}{=} e_U \circ e_V \circ f = e_V \circ f \overset{(3.1)}{=} f. \tag{3.4}$$

We mention that equations (3.1), (3.2) and (3.4) express $\operatorname{im} f \subseteq V$, $\operatorname{im} g \subseteq U$ and $\operatorname{im} f \subseteq U$. Therefore, we shall also use the numbers of those equations, when referring to these inclusions.

Next, we show by induction on k that $g^k[A] = g^k[U] = U$ holds for all $k \in \mathbb{N}_+$. The base step $k = 1$ works as follows:

$$U = e_U[A] \overset{(3.3)}{=} g \circ f \circ e_U[A] = g \circ f[U] \subseteq g \circ f[A] \overset{(3.4)}{\subseteq} g[U] \subseteq g[A] \overset{(3.2)}{\subseteq} U.$$

So we have established $U = g[U] = g[A]$. For the inductive step we suppose $g^k[A] = g^k[U] = U$ for some $k \in \mathbb{N}_+$. Then, using the inductive assumption in the second and the fifth equality, we can infer

$$U \overset{k=1}{=} g[U] = g\left[g^k[U]\right] = g^{k+1}[U] \subseteq g^{k+1}[A] = g\left[g^k[A]\right] = g[U] \overset{k=1}{=} U,$$

whence we get $U = g^{k+1}[U] = g^{k+1}[A]$.

Now, exploiting the FIP, we may choose a finite number $n \in \mathbb{N}_+$ such that $g^n \in \operatorname{Idem} \mathbf{A}$. Since $\operatorname{im} g^n = U$, we have $g \circ h = h$ for every $h \in \mathrm{O}_A^{(1)}$ having its image in U. Especially, for $h \in \{f, e_U\}$ this implies

$$g^n \circ f = f \tag{3.5}$$
$$g^n \circ e_U = e_U. \tag{3.6}$$

Thus, it is

$$\begin{aligned} f \circ g \circ e_U &\overset{(3.5)}{=} g^n \circ f \circ g \circ e_U = g^{n-1} \circ g \circ f \circ g \circ e_U \\ &\overset{(3.2)}{=} g^{n-1} \circ g \circ f \circ e_U \circ g \circ e_U \overset{(3.3)}{=} g^{n-1} \circ e_U \circ g \circ e_U \overset{(3.2)}{=} g^{n-1} \circ g \circ e_U \\ &= g^n \circ e_U \overset{(3.6)}{=} e_U, \end{aligned} \tag{3.7}$$

and so $e_V \circ e_U \overset{(3.7)}{=} e_V \circ f \circ g \circ e_U \overset{(3.1)}{=} f \circ g \circ e_U \overset{(3.7)}{=} e_U$. Therefore, we obtain $U = \operatorname{im} e_U = \operatorname{im}(e_V \circ e_U) \subseteq \operatorname{im} e_V = V \subseteq U$, which proves $U = V$. □

The following result now generalises the situation shown to be true for finite algebras in Corollary 3.2.13.

3.2.19 Lemma. *For every neighbourhood self-embedding simple algebra* $\mathbf{A}$ *we have* $\precsim \cap \succsim = \cong$. *Especially, this equality is true for all algebras having the FIP.*

Proof: Since by Corollary 3.2.12(b) neighbourhood isomorphism implies embedding, and $\cong$ is a symmetric relation, we have the inclusion $\cong \subseteq \precsim \cap \succsim$. For the converse let $U, V \in \operatorname{Neigh} \mathbf{A}$ such that $U \precsim V \precsim U$. By Proposition 3.2.10(b) there exists some $V' \in \operatorname{Neigh} \mathbf{A}$ fulfilling $U \precsim V \cong V' \subseteq U$. Again from Corollary 3.2.12(b) we can infer $U \precsim V \precsim V'$, and by transitivity (see Corollary 3.2.13) $U \precsim V' \subseteq U$.

Now, the assumption of $\mathbf{A}$ being neighbourhood self-embedding simple immediately concludes the proof by $U = V' \cong V$.

By Lemma 3.2.18(b), algebras with FIP satisfy the assumption of this lemma, and hence for them mutual embeddability of neighbourhoods and isomorphism coincide. □

With this we finish studying relationships between neighbourhoods and instead focus a bit on their interplay with invariant relations. The notions and results obtained in this connection prepare the part of the theory we will encounter in Section 3.4.

It is a major topic of Tame Congruence Theory to distinguish pairs of congruence relations, on the one hand by unary polynomial operations, and on the other hand by restriction to subsets. Since our theory is inspired by TCT, we are interested to do the same for pairs of invariant relations and unary clone operations or neighbourhoods, respectively.

3.2.20 Definition. For an algebra $\mathbf{A}$, we say that a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ *distinguishes* or *separates* a pair of invariant relations $S, T \in \operatorname{Inv}\mathbf{A}$ if $S{\restriction_U} \neq T{\restriction_U}$.

We define the *set of separated pairs of invariant relations of common arity*, or *separation set of* U for short, as

$$\operatorname{Sep}_{\mathbf{A}}(U) := \left\{ (S,T) \in (\operatorname{Inv}\mathbf{A})^2 \;\middle|\; \operatorname{ar}(S) = \operatorname{ar}(T) \wedge S{\restriction_U} \neq T{\restriction_U} \right\}$$

for $U \in \operatorname{Neigh}\mathbf{A}$ and

$$\begin{aligned}\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) &:= \bigcup \{ \operatorname{Sep}_{\mathbf{A}}(U) \mid U \in \mathcal{U} \} \\ &= \left\{ (S,T) \in (\operatorname{Inv}\mathbf{A})^2 \;\middle|\; \operatorname{ar}(S) = \operatorname{ar}(T) \wedge \exists U \in \mathcal{U} \colon S{\restriction_U} \neq T{\restriction_U} \right\}\end{aligned}$$

for collections $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$. ◇

3.2.21 Remark. Clearly, the two defined notions of separation sets are compatible with each other, that is, the equality $\operatorname{Sep}_{\mathbf{A}}(\{U\}) = \operatorname{Sep}_{\mathbf{A}}(U)$ holds for every neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ of any algebra $\mathbf{A}$. ◇

One might ask why in the definition of separation set we make a restriction to pairs of invariant relations of common arity. First, these are the only ones we will be interested in distinguishing later on. Second, apart from pathological cases where both restrictions are empty, every neighbourhood automatically separates every pair of relations of different arity. So, characteristic differences of distinct neighbourhoods will only appear among pairs of relations of identical arity. Third, in case $\mathbf{A}$ contains at least one nullary operation and $U = \{u\}$ is a singleton neighbourhood, then it is induced by a constant unary clone operation with range $\{u\}$. Hence, the nullary operation with the same value is also a clone operation of $\mathbf{A}$, and so every invariant relation contains the tuple $(u, \ldots, u)$. Therefore, any two invariant relations of identical arity restrict to the same local relation $\{(u,\dots,u)\}$, which means $\operatorname{Sep}_{\mathbf{A}}(U)=\emptyset$. Evidently, by definition, the separation set of an empty collection of neighbourhoods is $\operatorname{Sep}_{\mathbf{A}}(\emptyset)=\emptyset$, too, so a singleton neighbourhood is not more powerful w.r.t. separation of invariant relations than the empty collection. If we had not restricted to invariants of common arity, then this would not be true. Thus, the choice we made in Definition 3.2.20 reduces the importance of singleton neighbourhoods if $\mathbf{A}$ contains nullary constants, a fact becoming again relevant in Section 3.4 when dealing with covers. Neglecting singleton neighbourhoods is of particular interest when studying polynomial expansions $\mathbf{A}_A$ of algebras as these are full of such neighbourhoods which do not carry much structural information.

At the end of this section we will examine how embedding and isomorphism of neighbourhoods relates to their power w.r.t. separating invariant relations. The first result is the following.

3.2.22 Lemma. *For an algebra $\mathbf{A}$ and neighbourhoods $U,V\in\operatorname{Neigh}\mathbf{A}$ such that $\underset{\approx}{\mathbf{A}}\restriction_U$ is embedded[5] in $\underset{\approx}{\mathbf{A}}\restriction_V$, then $\operatorname{Sep}_{\mathbf{A}}(U)\subseteq\operatorname{Sep}_{\mathbf{A}}(V)$, i.e. V distinguishes every pair of invariant relations of $\mathbf{A}$ that U does. Explicitly, for $S,T\in\operatorname{Inv}\mathbf{A}$, it is $S\restriction_V\neq T\restriction_V$ whenever $S\restriction_U\neq T\restriction_U$ holds.*

Proof: Suppose that $f\colon\underset{\approx}{\mathbf{A}}\restriction_U\longrightarrow\underset{\approx}{\mathbf{A}}\restriction_V$ is an embedding of relational structures, that is, an injective, relation preserving and relation reflecting map. Consider invariant relations $S,T\in\operatorname{Inv}\mathbf{A}$ such that $S\restriction_U\neq T\restriction_U$, i.e. there exists some tuple $x\in((S\setminus T)\cup(T\setminus S))\restriction_U$. Without loss of generality we can assume that x belongs to $T\restriction_U$ but not to $S\restriction_U$. As f preserves $T\restriction_U$, we have $f\circ x\in T\restriction_V$. Furthermore, this tuple cannot belong to $S\restriction_V$ because if it did, the fact that f reflects relations would imply $x\in S\restriction_U$, which was excluded above. Thus, $f\circ x\in T\restriction_U\setminus S\restriction_U$, implying $T\restriction_U\neq S\restriction_U$. □

As a corollary we get the fact that mutually embeddable neighbourhoods, and, in particular, isomorphic ones distinguish exactly the same invariant relations.[6] So they have the same power w.r.t. separation of invariant relations. This is not too surprising as according to Corollary 3.2.13, in the case of a finite algebra the concept of mutual embeddability even coincides with the isomorphism relation.

3.2.23 Corollary. *For an algebra $\mathbf{A}$ and neighbourhoods $U,V\in\operatorname{Neigh}\mathbf{A}$ the following assertions are true.*

(a) The condition $U\lessapprox V$, especially $U\subseteq V$, implies $\operatorname{Sep}_{\mathbf{A}}(U)\subseteq\operatorname{Sep}_{\mathbf{A}}(V)$, i.e. V distinguishes every pair of invariant relations of $\mathbf{A}$ that U does.

(b) Mutually embeddable neighbourhoods separate the same pairs of invariant relations: if $U\lessapprox V$ and $V\lessapprox U$, then $\operatorname{Sep}_{\mathbf{A}}(U)=\operatorname{Sep}_{\mathbf{A}}(V)$, i.e., for every pair of invariants $S,T\in\operatorname{Inv}\mathbf{A}$ we have

$$S\restriction_U\neq T\restriction_U\iff S\restriction_V\neq T\restriction_V.$$

[5] Note that a simple embedding of relational structures is enough, we do not require that its image is again a neighbourhood.

[6] In Section 3.4 we shall call such neighbourhoods *covering equivalent* (cp. Corollary 3.4.6(b)).

(c) Isomorphic neighbourhoods separate the same pairs of invariant relations, i.e.

$$U \cong V \implies \mathrm{Sep}_{\mathbf{A}}(U) = \mathrm{Sep}_{\mathbf{A}}(V)\,.$$

We remark that the statement of item (c) is already proven in a more direct fashion in Lemma 3.2.3 of [Beh09].

Proof: (a) The definition of neighbourhood embedding contains that the corresponding restricted relational structures are embedded, so the claim follows from Lemma 3.2.22.

(b) This is a direct inference from item (a).

(c) This claim follows as a combination of the symmetry of the isomorphism relation, Corollary 3.2.12(b) and item (b). □

3.3 The restricted algebra $\mathbf{A}|_U$

Still motivated by Proposition 3.1.12, where the neighbourhoods of an algebra were identified as the appropriate subsets for localisation, we now add an operational structure to them. Our general programme to build a localisation theory upon a Galois theory between functions and relations clearly dictates how this should be done. We simply translate the restriction of the relational counterpart of an algebra to a neighbourhood via the polymorphism operator.

The localisation process is best expressed in the following schema where we have forgotten about the indexing of the relational structures:

$$\begin{tikzcd}
\mathbf{A} \arrow[dr, "\mathrm{Inv}"] & \\
\langle A; \mathrm{Clo}(\mathbf{A})\rangle \arrow[r, shift left, "\mathrm{Inv}_A"] & \underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle \arrow[l, shift left, "\mathrm{Pol}_A"] \arrow[d, "\restriction_U"] \\
\mathbf{A}|_U := \langle U; \mathrm{Pol}_U([\mathrm{Inv}\,\mathbf{A}]\restriction_U)\rangle \arrow[r, shift left, "\mathrm{Inv}_U"] & \underset{\sim}{\mathbf{A}}\restriction_U = \langle U; [\mathrm{Inv}\,\mathbf{A}]\restriction_U\rangle \arrow[l, shift left, "\mathrm{Pol}_U"]
\end{tikzcd}$$

We formally record this in the following definition originally coming from [Kea01, Definition 2.7] and here almost literally quoted from [Beh09, Definition 3.3.1]:

3.3.1 Definition. Let $\mathbf{A}$ be an algebra and $e \in \mathrm{Idem}\,\mathbf{A}$ an idempotent endomorphism having image $U := e[A]$. Then *the algebra that* $\mathbf{A}$ *induces on* U, written $\mathbf{A}|_U$ or $e(\mathbf{A})$, is

$$\begin{aligned}\mathbf{A}|_U &:= \langle U;\ \mathrm{Pol}_U([\mathrm{Inv}\,\mathbf{A}]\restriction_U)\rangle \\ &= \langle U;\ \mathrm{Pol}_U\{\varrho\restriction_U \mid \varrho \in \mathrm{Inv}\,\mathbf{A}\}\rangle\,.\end{aligned}$$

We call $\mathbf{A}|_A = \langle A; \mathrm{Clo}(\mathbf{A})\rangle$ the *saturated algebra* belonging to $\mathbf{A}$. ◇

3.3.2 Remark. As indicated in the restriction schema above, for a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ of an algebra $\mathbf{A}$, the structures $\mathbf{A}|_U$ and $\underset{\sim}{\mathbf{A}}\restriction_U$ are in one-to-one correspondence via the GALOIS connection $\operatorname{Pol}_U - \operatorname{Inv}_U$. This is true because the neighbourhood U is, by definition, an image of an idempotent endomorphism of the structure $\underset{\sim}{\mathbf{A}}$ carrying the locally closed relational clone $\operatorname{Inv} \mathbf{A}$. Hence, by Corollary 3.1.16, the restriction $[\operatorname{Inv} \mathbf{A}]\restriction_U$ is again a GALOIS closed clone of relations, i.e. $\operatorname{Inv}_U \operatorname{Pol}_U ([\operatorname{Inv} \mathbf{A}]\restriction_U) = [\operatorname{Inv} \mathbf{A}]\restriction_U$. This means that

$$\begin{aligned} \operatorname{Inv} \mathbf{A}|_U &= [\operatorname{Inv} \mathbf{A}]\restriction_U, \\ \operatorname{Pol} \underset{\sim}{\mathbf{A}}\restriction_U &= \operatorname{Clo}(\mathbf{A}|_U) \end{aligned}$$

hold, i.e. $\mathbf{A}|_U$ is a saturated algebra. ♢

Both, the choice of neighbourhoods for localisation, and the way how we defined the local algebraic structure on them, are tailored to profit from one central point: the fact that restriction to a neighbourhood is a surjective homomorphism between relational clones. In the following we shall separately discuss the value that lies in both aspects, being a clone homomorphism, and being surjective.

Having a clone homomorphism allows us to translate structure from $\mathbf{A}$ (or $\underset{\sim}{\mathbf{A}}$) to its localisations. What is meant here by "structure"? The answer is, of course, in terms of invariant relations. We can pass on properties of invariant relations expressed in the language of relational clones, i.e. using the general composition operation (see Definition 2.3.2) to their restrictions to a neighbourhood. This encompasses all relations that are definable from given relations and equality using primitive positive formulæ of first order logic, i.e. finite conjunctions of variable substitutions in given (atomic) relations and variable identifications, surrounded by a finite number of existential quantifications. Transportable properties are, for instance, equalities between relations definable in such a way.

A simple example to illustrate this is the condition that two binary invariant relations S, T commute w.r.t. relation product: $S \circ T = T \circ S$. Each side of the equality can be seen as the evaluation of a binary term in clone theoretic language at the pair (S, T). Depending on which definition one chooses for relational clone, the operation $\circ$ is either a fundamental operation (then $S \circ T$ is the interpretation of the term $x \circ y$ at (S, T)), or it is more difficult to express. In our definition, it is the interpretation of the term $\prod^{\beta}_{(\alpha_1,\alpha_2)}(x, y)$, where $\alpha_1, \alpha_2, \beta\colon 2 \longrightarrow 3$ are given by $\alpha_1(\nu) := \nu$, $\alpha_2(\nu) := \nu + 1$ and $\beta(\nu) := 2\nu$ for $\nu \in 2 = \{0, 1\}$. Indeed,

$$\begin{aligned} \prod^{\beta}_{(\alpha_1,\alpha_2)}(S,T) &= \left\{ (a_0,a_1,a_2)\circ\beta \;\middle|\; \begin{array}{r} (a_0,a_1,a_2) \in A^3 \wedge (a_0,a_1,a_2)\circ\alpha_1 \in S \\ \wedge\ (a_0,a_1,a_2)\circ\alpha_2 \in T \end{array} \right\} \\ &= \left\{ (a_0,a_2) \;\middle|\; (a_0,a_1,a_2) \in A^3 \wedge (a_0,a_1) \in S \wedge (a_1,a_2) \in T \right\} \\ &= S \circ T. \end{aligned}$$

Alternatively, using definability by first order formulæ we can express $S \circ T$ as

$$\left\{ (x,y) \in A^2 \;\middle|\; (A;\, S,T) \models \varphi \right\}$$

where $\varphi := \exists z\,(\varrho_0(x,z) \wedge \varrho_1(z,y))$ and ϱ_0 is a symbol to be substituted by the first relation and ϱ_1 by the second.

Now the homomorphism property of restriction to a neighbourhood U yields

$$(S \circ T){\restriction_U} = \left(\prod^{\beta}_{(\alpha_1,\alpha_2)} (S,T) \right){\restriction_U} = \prod^{\beta}_{(\alpha_1,\alpha_2)} (S{\restriction_U}, T{\restriction_U}) = S{\restriction_U} \circ T{\restriction_U}.$$

Combining this with the commutativity condition $S \circ T = T \circ S$, we infer that the restricted relations have the same property:

$$S{\restriction_U} \circ T{\restriction_U} = (S \circ T){\restriction_U} = (T \circ S){\restriction_U} = T{\restriction_U} \circ S{\restriction_U}.$$

So the restricted invariants inherit commutativity from S and T. Clearly, more complicated properties than just commutativity can be transported in this way. In fact, the presented idea works completely the same for any property of relations which can be written as an evaluation of an identity $s \approx t$, where s and t are (possibly infinitary) terms over the (possibly infinitary) fundamental operations of a relational clone, yielding relations of a common arity. If $s \approx t$ holds for certain invariant relations of $\mathbf{A}$, then it also holds for their restrictions to the neighbourhood U.

We mention here that with our definition of relational clone, no generality is lost in simply considering terms s and t containing at most one operational symbol, that is variable projections or variable substitutions in one of the fundamental operations of a relational clone. This is so because substitution of relations obtained by general compositions into a general composition can be expressed again as another general composition (see the discussion below equation (1) on page 10 of [Beh11] for some details). Note, however, that, in general, variable projections cannot be expressed as an application of a fundamental operation of a relational clone: for a relational clone on a non-empty set including the empty relation[7], any operation of the form $\prod^{\beta}_{(\alpha_i)_{i\in I}}$ will depend on all its positions (use the empty invariant relation and non-empty full powers of the carrier set to see this), whereas a projection only depends on the coordinate, it projects to. The exception to this imbalance is, of course, the identity (unary projection, $I = \{i_0\}$), which is expressible via $\alpha_{i_0} = \beta = \mathrm{id}_{\mathrm{ar}(x_{i_0})}$.

This means no generality is lost in saying that properties of relations that are

[7] Yet, for clones not containing the empty relation, variable projections can be written as general compositions, whence the fundamental operations acting on the finitary relations of the clone, under which it is closed form some “clone-like” structure. It is not a clone in the sense of Definition 2.3.1, but a multi-sorted clone of possibly infinitary operations.

expressible as the truth of evaluations of identities of the form[8]

$$\prod_{(\alpha_i)_{i\in I}}^{\beta} (x_i)_{i\in I} \approx \prod_{(\gamma_j)_{j\in J}}^{\delta} (y_j)_{j\in J}\,, \tag{3.8}$$

where β and δ have identical domains, for certain invariant relations $(S_i)_{i\in I}$, $(T_j)_{j\in J}$ in $\operatorname{Inv}\mathbf{A}$ remain true for the restrictions $(S_i{\restriction_U})_{i\in I}$, $(T_j{\restriction_U})_{j\in J}$ belonging to $\underset{\sim}{\mathbf{A}}{\restriction_U}$.

So far, we have been discussing the usefulness of having a homomorphism between relational clones, but where does surjectivity come into play? There are at least two answers to this question.

First, passing on properties of relations via restriction to a neighbourhood U can be understood as exerting some kind of control over a part of the relations of $\underset{\sim}{\mathbf{A}}{\restriction_U}$, namely those which are given as restrictions of relations in $\operatorname{Inv}\mathbf{A}$. Now surjectivity means that this control is total as every relation of the structure $\underset{\sim}{\mathbf{A}}{\restriction_U}$ arises in this way. So whenever we are presented with relations of $\underset{\sim}{\mathbf{A}}{\restriction_U}$ for which we have to prove some fact, we know that these can be written as restrictions of certain invariant relations in $\operatorname{Inv}\mathbf{A}$. For them possible manipulations are usually known, and assumptions of the given algebra $\mathbf{A}$ can be exploited. Using the clone homomorphism, we can then transport these arguments back to $\underset{\sim}{\mathbf{A}}{\restriction_U}$ and thus turn them into properties of $\mathbf{A}|_U$.

The second answer is similar, but has a different focus. We recall that, as explained above, we could convert properties of specific invariant relations of $\mathbf{A}$ into properties of their restrictions. Yet, often properties of an algebra translate into properties of the relational counterpart that are universally quantified. These do not only concern a few selected members of $\operatorname{Inv}\mathbf{A}$, but all of them (at least all of certain arities). For sure we can turn these all-quantified properties into properties of the restricted invariant relations via the clone homomorphism. However, it is surjectivity which guarantees that in this way we obtain again a universally quantified property of $\underset{\sim}{\mathbf{A}}{\restriction_U}$, which can be transferred into a statement about $\mathbf{A}|_U$ resembling the original property of $\mathbf{A}$.

A familiar example in this sense would be the following. Suppose $\mathbf{A}$ is congruence permutable, that is, $\theta\circ\psi=\psi\circ\theta$ holds for all $\theta,\psi\in\operatorname{Con}\mathbf{A}$. Using the clone homomorphism we can transform this into $\theta{\restriction_U}\circ\psi{\restriction_U}=\psi{\restriction_U}\circ\theta{\restriction_U}$ for all $\theta,\psi\in\operatorname{Con}\mathbf{A}$. This is a property which holds for all restrictions of congruences of $\mathbf{A}$. Surjectivity (see Lemma 3.3.3 below) then entails that not only a part of the congruences of $\mathbf{A}|_U$ permute but that the whole restricted algebra is again congruence permutable.

Looking a bit more general at what we explained above, we can conclude that one advantage of a surjective relational clone homomorphism is the following. Any

[8] Actually, even evaluations of equalities of the form $\prod^{\beta}_{(\alpha_i)_{i\in I}} (x_i)_{i\in I} \approx \prod^{\delta}_{(\gamma_i)_{i\in I}} (x_i)_{i\in I}$ do suffice since in any particular instantiation of the more general expression above, the resulting relations on both sides are either equally empty or non-empty. So with some minor modifications of the index mappings, one can also use an evaluation of terms of the form $\prod^{\tilde\beta}_{\left((\tilde\alpha_i)_{i\in I},(\tilde\gamma_j)_{j\in J}\right)} \left((x_i)_{i\in I},(y_j)_{j\in J}\right) \approx \prod^{\tilde\beta}_{\left((\tilde\alpha_i)_{i\in I},(\tilde\gamma_j)_{j\in J}\right)} \left((x_i)_{i\in I},(y_j)_{j\in J}\right)$.

universally quantified identity, using terms including the fundamental operations of relational clones, that is fulfilled by the relational clone $\operatorname{Inv}\mathbf{A}$ also holds in the relational clone $[\operatorname{Inv}\mathbf{A}]\restriction_U$. In other words, if the relational clone of $\mathbf{A}$ belongs to some "variety" of relational clones described by satisfaction of a set of identities as above, then its homomorphic image, the relational clone belonging to $\mathbf{A}|_U$, is a member of the same "variety". This is an analogy to the familiar fact that ordinary varieties of algebras are closed under homomorphic images.

On the other hand, the mentioned benefits obtained from having a surjective clone homomorphism at hand come at a price. It is immediate from the definition of the localisation process that algebras $\mathbf{A}$ and $\mathbf{A}'$ sharing the same carrier set A and the same saturated algebra $\mathbf{A}|_A = \mathbf{A}'|_A$, i.e. the same locally closed clone of operations, have identical local structure within our theory. In other words, algebras on the same base set having the same clone of invariant relations cannot be discerned in Relational Structure Theory and hence must be considered equivalent. So RST only allows investigation of algebras up to this sort of equivalence relation, that we called local term equivalence. For instance, we cannot expect RST to yield assertions about properties of algebras, which are not invariant under local term equivalence. However, fortunately, many properties that are of interest in universal algebra *are* invariant under local term equivalence (or at least under term equivalence, which coincides with local term equivalence in the case of finite algebras). With this statement we finish the general discussion of the localisation process and turn to a special sort of invariants.

In universal algebra invariant relations that are of particular importance are congruence relations, i.e. binary invariants that are equivalence relations. In the example mentioned in the discussion above, we have already referred to a result about them. Our clone homomorphism obediently restricts tocongruences yielding a complete lattice homomorphism, which still commutes with all the other operations of relational clones, e.g. relation product. Under reasonable assumptions, which are evidently fulfilled for polynomial expansions of algebras, this lattice homomorphism becomes surjective. So we can again exhibit another similarity of our notions with those known from classical Tame Congruence Theory. There neighbourhoods were distinguished from other subsets by precisely the fact that restriction to a neighbourhood U constitutes a surjective lattice homomorphism from $\operatorname{Con}\mathbf{A}$ onto $\operatorname{Con}\mathbf{A}|_U$. However, even if restriction $\restriction_U$ to a neighbourhood U is not always a surjective mapping in our case, it still remains a lattice homomorphism from $\operatorname{Con}\mathbf{A}$ to $\operatorname{Con}\mathbf{A}|_U$. If it is a surjection, then as in the example above, it allows us to transfer the truth of congruence identities in $\operatorname{Con}\mathbf{A}$ to local algebras, where the terms on both sides of the identity can be composed from relational clone operations preserving equivalences and arbitrary joins and meets in the congruence lattice.

We mention that, for finite algebras, the statement about congruences in item (d) of the following result is already contained as Exercise (1) in [Kea01] and as Lemma 3.3.2 in [Beh09].

3.3.3 Lemma. *For an algebra* $\mathbf{A}$ *and a neighbourhood* $U \in \operatorname{Neigh}\mathbf{A}$*, the following statements are true.*

(a) Restriction to U *has the homomorphism property w.r.t. arbitary unions of relations of common arity that are again invariant. In detail, for every* $m \in \mathbb{N}$ *and every subset* $Q \subseteq \operatorname{Inv}^{(m)}\mathbf{A}$ *where* $\bigcup Q$ *is again an invariant relation of* $\mathbf{A}$*, we have* $(\bigcup Q){\restriction_U} = \bigcup [Q]{\restriction_U}$.

(b) Restriction to U *is a homomorphism w.r.t. non-empty directed unions of invariant relations of common arity. That is, for every* $m \in \mathbb{N}$ *and any upwards directed*[9] *subset* $\emptyset \neq Q \subseteq \operatorname{Inv}^{(m)}\mathbf{A}$*, it is* $\bigcup Q \in \operatorname{Inv}^{(m)}\mathbf{A}$ *and* $(\bigcup Q){\restriction_U} = \bigcup [Q]{\restriction_U}$.

(c) Restriction to U *is a homomorphism w.r.t. transitive closure of reflexive binary invariant relations, i.e.* $\langle S\rangle^{\text{trans}} \in \operatorname{Inv}^{(2)}\mathbf{A}$*, and* $\langle S\rangle^{\text{trans}}{\restriction_U} = \langle S{\restriction_U}\rangle^{\text{trans}}$ *holds for every* $S \in \operatorname{Inv}^{(2)}\mathbf{A}$ *containing* Δ_A.

(d) The restriction mapping

$$h_c\colon \quad \begin{array}{ccc} \operatorname{Con}\mathbf{A} & \longrightarrow & \operatorname{Con}\mathbf{A}|_U \\ \theta & \longmapsto & \theta{\restriction_U} \end{array}$$

is a well-defined complete lattice homomorphism. If $\langle U\rangle_{\mathbf{A}} = A$*, then* h_c *is surjective.*

Although less important, the same facts hold, more generally, for restriction of compatible quasiorders. The operation

$$h_q\colon \quad \begin{array}{ccc} \operatorname{Quord}\mathbf{A} & \longrightarrow & \operatorname{Quord}\mathbf{A}|_U \\ \theta & \longmapsto & \theta{\restriction_U} \end{array}$$

is a complete lattice homomorphism, and it is surjective whenever the set U *generates* $\mathbf{A}$.

Proof: (a) The first fact about arbitrary unions is just a reformulation of distributivity of intersection over union: if $m \in \mathbb{N}$ and $Q \subseteq \operatorname{Inv}^{(m)}\mathbf{A}$ is such that $\bigcup Q \in \operatorname{Inv}^{(m)}\mathbf{A}$, then

$$\begin{aligned}\Big(\bigcup Q\Big){\restriction_U} &= \Big(\bigcup Q\Big) \cap U^m = \bigcup \{\, S \cap U^m \mid S \in Q\} = \bigcup \{\, S{\restriction_U} \mid S \in Q\} \\ &= \bigcup [Q]{\restriction_U}.\end{aligned}$$

(b) Invariant relations of common arity $m \in \mathbb{N}$ are closed under arbitrary intersections. For such sets of relations it is well-known that non-empty upwards directed unions belong to their local closure (see also [Pös80, Proposition 1.13]). However, invariant relations form a locally closed clone of relations, so $\bigcup Q$ belongs to $\operatorname{Inv}\mathbf{A}$, for any non-empty upwards directed $Q \subseteq \operatorname{Inv}^{(m)}\mathbf{A}$. Now, the statement follows from item (a).

[9] Upwards directed is meant here w.r.t. set inclusion, i.e. for every $R, S \in Q$ there has to exist a relation $T \in Q$ such that $R \cup S \subseteq T$.

(c) Suppose that $S \in \operatorname{Inv}^{(2)}\mathbf{A}$ is reflexive. It is well-known that relational clones are closed under relation product (see also the discussion on page 66), especially under powers. Furthermore, one can express the transitive closure of a binary relation as

$$\langle S \rangle^{\text{trans}} = \bigcup_{n \in \mathbb{N}_+} S^n,$$

where the powers are taken w.r.t. relation product. It is easy to see that for $n \in \mathbb{N}_+$ we can write the n-th power of S as

$$S^n = \left\{ (x_0, x_n) \in A^2 \;\middle|\; \begin{array}{l} \exists x_1, \dots, x_{n-1} \in A\colon \\ (x_0, x_1), (x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n) \in S \end{array} \right\}.$$

Now, for positive integers $m \leq n$, we have $S^m \subseteq S^n$ because S is reflexive. Thus, the set of all finite non-zero powers $\{ S^n \mid n \in \mathbb{N}_+ \}$ is non-empty, upwards directed and contains only binary invariant relations of $\mathbf{A}$. Upon application of item (b), we can infer that the directed union $\langle S \rangle^{\text{trans}}$ belongs to $\operatorname{Inv}^{(2)}\mathbf{A}$ and satisfies

$$\langle S \rangle^{\text{trans}}\restriction_U = \left(\bigcup_{n \in \mathbb{N}_+} S^n \right)\restriction_U = \bigcup_{n \in \mathbb{N}_+} S^n\restriction_U = \bigcup_{n \in \mathbb{N}_+} (S\restriction_U)^n = \langle S\restriction_U \rangle^{\text{trans}},$$

where we have also used the homomorphism property of restriction w.r.t. relation product.

(d) Remember that congruences are precisely the (binary) invariant equivalences. As explained in Remark 3.3.2, for a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ binary invariants of $\mathbf{A}$ restrict to binary invariants of $\mathbf{A}|_U$. Since, obviously, the restriction of an equivalence relation on A is again an equivalence relation (on U), h_c is a well-defined mapping. Similarly, compatible quasiorders are, by definition, (binary) invariant quasiorders, and so they restrict to invariant quasiorders of $\mathbf{A}|_U$, making h_q well-defined.

Next, we demonstrate that h_c and h_q are complete lattice homomorphisms. They obviously are complete meet homomorphisms, since the infimum operation in $\operatorname{Con}\mathbf{A}$ and $\operatorname{Quord}\mathbf{A}$ coincides with $\bigcap$, and restriction to U is intersection with U^2.

To prove that h_c and h_q are also homomorphisms w.r.t. $\bigvee$, we need a little trick. Let us define $P := \operatorname{Clo}^{(1)}(\mathbf{A}_A)$ containing all unary polynomial operations of $\mathbf{A}$, and the unary algebra $\mathbf{A}' := \langle A; P \rangle$. It is easy to see that every reflexive invariant of $\mathbf{A}$ is preserved by all constant operations on A and is hence an invariant of $\mathbf{A}'$. Furthermore, it is well-known that every invariant quasiorder of $\mathbf{A}'$ is an invariant of $\mathbf{A}$ because P contains all unary polynomial operations of $\mathbf{A}$. This is to say that

$$\begin{aligned} \operatorname{Quord}\mathbf{A} &= \operatorname{Quord} A \cap \operatorname{Inv}\mathbf{A} = \operatorname{Quord} A \cap \operatorname{Inv}\mathbf{A}' = \operatorname{Quord}\mathbf{A}', \\ \operatorname{Con}\mathbf{A} &= \operatorname{Eq} A \cap \operatorname{Inv}\mathbf{A} = \operatorname{Eq} A \cap \operatorname{Inv}\mathbf{A}' = \operatorname{Con}\mathbf{A}'. \end{aligned}$$

Moreover, assume that U is given as the image of an idempotent operation e in $\mathrm{Clo}^{(1)}(\mathbf{A}) \subseteq P \subseteq \mathrm{Clo}^{(1)}(\mathbf{A}')$. Then e also belongs to $\mathrm{Idem}\,\mathbf{A}'$, so the set U is a neighbourhood of $\mathbf{A}'$, too. Hence, restriction to U also constitutes a homomorphism of the relational clone of invariant relations of $\mathbf{A}'$.

Compared to $\mathrm{Inv}\,\mathbf{A}$, the clone $\mathrm{Inv}\,\mathbf{A}'$ has the advantage that it is closed under arbitrary unions of relations of the same arity, since $\mathbf{A}'$ contains only unary operations. So, according to item (a) applied to $\mathbf{A}'$, restriction to U has the homomorphism property w.r.t. arbitrary unions of intermediate invariant relations of $\mathbf{A}'$ (that are not necessarily invariant for $\mathbf{A}$). We can use this in order to deal with joins of congruences and, more generally, compatible quasiorders $Q \subseteq \mathrm{Quord}\,\mathbf{A} = \mathrm{Quord}\,\mathbf{A}'$. Joins are given as transitive closure of the union: $\bigvee Q = \langle \Delta_A \cup \bigcup Q \rangle^{\mathrm{trans}}$. The inner relation $S := \Delta_A \cup \bigcup Q \in \mathrm{Inv}^{(2)}\mathbf{A}'$ is reflexive, so by item (c) we obtain

$$\left(\bigvee Q\right){\restriction_U} = \langle S \rangle^{\mathrm{trans}}{\restriction_U} = \langle S{\restriction_U} \rangle^{\mathrm{trans}}.$$

Further exploiting the homomorphism property w.r.t. $\bigcup$, we see

$$S{\restriction_U} = \left(\Delta_A \cup \bigcup Q\right){\restriction_U} = \Delta_A{\restriction_U} \cup \bigcup [Q]{\restriction_U} = \Delta_U \cup \bigcup [Q]{\restriction_U}.$$

Putting these observations together yields

$$\left(\bigvee Q\right){\restriction_U} = \langle S{\restriction_U} \rangle^{\mathrm{trans}} = \left\langle \Delta_U \cup \bigcup [Q]{\restriction_U} \right\rangle^{\mathrm{trans}} = \bigvee [Q]{\restriction_U}.$$

Last, but not least, we are going to prove surjectivity of h_q and h_c if U is a generating set for $\mathbf{A}$. As, by the definition of neighbourhood, the map ${\restriction_U}\colon \mathrm{Inv}\,\mathbf{A} \longrightarrow \mathrm{Inv}\,\mathbf{A}|_U$ is a surjective relational clone homomorphism, every $\theta \in \mathrm{Quord}\,\mathbf{A}|_U$ is a restriction of some binary invariant $\varrho \in \mathrm{Inv}^{(2)}\mathbf{A}$. From this relation we will now construct an invariant quasiorder relation ϱ'' having the same property as ϱ. This will prove that h_q is surjective. We shall see on the way, that we can ensure symmetry of ϱ'' if θ is symmetric, i.e. a congruence, which will demonstrate that h_c is surjective, too.

Indeed, for symmetric θ, let us define $\varrho' := \varrho \cap \varrho^{-1} \in \mathrm{Inv}^{(2)}\mathbf{A}$. Then ϱ' is symmetric, and $\theta = \varrho{\restriction_U}$ contains $(\varrho \cap \varrho^{-1}){\restriction_U} = \varrho'{\restriction_U}$. However, since θ is symmetric, with every pair $(u, v) \in \theta$ also $(v, u) \in \theta \subseteq \varrho$, so $(u, v) \in \varrho \cap \varrho^{-1}$. Hence, $(u, v) \in \varrho'{\restriction_U}$, thus $\theta = \varrho'{\restriction_U}$. Otherwise, if θ is just a quasiorder, i.e. in the proof for h_q, we simply put $\varrho' := \varrho$.

As θ is reflexive, we have $\Delta_U \subseteq \theta = \varrho'{\restriction_U} \subseteq \varrho'$, so $\langle \Delta_U \rangle_{\mathbf{A}^2} \subseteq \varrho'$. We are going to show that $\langle \Delta_U \rangle_{\mathbf{A}^2} = \Delta_A$. Obviously, $\langle \Delta_U \rangle_{\mathbf{A}^2} \subseteq \Delta_A$ because the diagonal $\Delta_A \in \mathrm{Inv}^{(2)}\mathbf{A}$ is an invariant containing Δ_U. Conversely, since U generates $\mathbf{A}$, for every $a \in A$ there exists some $n \in \mathbb{N}$, some n-ary term operation $f \in \mathrm{Clo}^{(n)}(\mathbf{A})$ and a tuple $(u_1, \dots, u_n) \in U^n$ such that $a = f(u_1, \dots, u_n)$. Therefore,

$$(a, a) = (f(u_1, \dots, u_n), f(u_1, \dots, u_n)) \in \langle \Delta_U \rangle_{\mathbf{A}^2},$$

i.e. $\Delta_A \subseteq \langle \Delta_U \rangle_{\mathbf{A}^2}$. Consequently, $\Delta_A = \langle \Delta_U \rangle_{\mathbf{A}^2} \subseteq \varrho'$, and ϱ' is a reflexive invariant of $\mathbf{A}$ restricting to θ. If θ was symmetric, then so is ϱ'.

Now let $\varrho'' := \langle \varrho' \rangle^{\mathrm{trans}}$. This relation is reflexive and transitive, i.e. a quasiorder. It is symmetric, i.e. an equivalence, depending on θ being one. As ϱ' was reflexive, item (c) ensures that ϱ'' belongs to $\mathrm{Inv}^{(2)}\mathbf{A}$, whence it is a compatible quasiorder or even a congruence of $\mathbf{A}$ if θ was symmetric. Using the homomorphism property claimed in item (c), we can infer

$$\varrho''\restriction_U = \langle \varrho' \rangle^{\mathrm{trans}}\restriction_U = \langle \varrho'\restriction_U \rangle^{\mathrm{trans}} = \langle \theta \rangle^{\mathrm{trans}} = \theta$$

from $\varrho'\restriction_U = \theta$. However, this shows $h_c(\varrho'') = \theta$ or $h_q(\varrho'') = \theta$ where $\varrho'' \in \mathrm{Con}\,\mathbf{A}$ or $\varrho'' \in \mathrm{Quord}\,\mathbf{A}$ for $\theta \in \mathrm{Con}\,\mathbf{A}|_U$ or $\theta \in \mathrm{Quord}\,\mathbf{A}|_U$, respectively. Thus h_q and h_c are both surjective. $\square$

In the subsequent lemma we are going to expatiate upon the direct way from $\mathbf{A}$ to $\mathbf{A}|_U$ without computing the relational clone of invariant relations, restricting these and calculating their polymorphisms. This result will also give a motivation for the notation $e(\mathbf{A})$ for $\mathbf{A}|_U$, which was introduced in Definition 3.3.1, and shows an analogy to the retract notation $e[\underset{\sim}{\mathbf{A}}]$ for the restricted relational counterpart $\underset{\sim}{\mathbf{A}}\restriction_U$. The lemma originates from Lemma 2.8 in [Kea01] and largely coincides with Lemma 3.3.3 from [Beh09].

3.3.4 Lemma. *Let* $\mathbf{A}$ *be an algebra and* $e \in \mathrm{Idem}\,\mathbf{A}$ *an idempotent defining a neighbourhood* $U := e[A] \in \mathrm{Neigh}\,\mathbf{A}$*. Furthermore, let* $Q_0 \subseteq \mathrm{R}_A$ *be a set of relations such that* $\mathrm{Clo}(\mathbf{A}) = \mathrm{Pol}_A Q_0$*, then*

$$\mathrm{Pol}_U \{ \varrho\restriction_U \mid \varrho \in Q_0 \} = e(\mathrm{Clo}(\mathbf{A}))$$

where

$$\begin{aligned} e(\mathrm{Clo}(\mathbf{A})) := \{ (e \circ f)|_U \mid f \in \mathrm{Clo}(\mathbf{A}) \} &= \{ f|_U \mid f \in \mathrm{Clo}(\mathbf{A}) \wedge f \rhd U \} \\ &= \{ f|_U \mid f \in \mathrm{Clo}(\mathbf{A}) \wedge \mathrm{im}\, f \subseteq U \}. \end{aligned}$$

Especially, the algebra induced by $\mathbf{A}$ *on* U *is*

$$\mathbf{A}|_U = \langle U; e(\mathrm{Clo}(\mathbf{A})) \rangle,$$

and, in more detail, the fundamental operations of $\mathbf{A}|_U$ *are*

$$e(\mathrm{Clo}(\mathbf{A})) = \mathrm{Pol}_U([\mathrm{Inv}\,\mathbf{A}]\restriction_U) = \mathrm{Pol}\,\underset{\sim}{\mathbf{A}}\restriction_U = \mathrm{Clo}(\mathbf{A}|_U).$$

Proof: The two different characterisations listed after the definition of $e(\mathrm{Clo}(\mathbf{A}))$ follow arity-wise by applying Lemma 3.1.6 to the structure $\underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A} \rangle$. In this setting, the equality $\mathrm{Pol}^{(n)}\underset{\sim}{\mathbf{A}} = \mathrm{Pol}_A^{(n)}\,\mathrm{Inv}\,\mathbf{A} = \mathrm{Clo}^{(n)}(\mathbf{A})$ holds for every $n \in \mathbb{N}$, especially $\mathrm{End}\,\underset{\sim}{\mathbf{A}} = \mathrm{Pol}^{(1)}\underset{\sim}{\mathbf{A}} = \mathrm{Clo}^{(1)}(\mathbf{A})$. So Lemma 3.1.6 indeed contains the desired result.

The final characterisation of $\mathbf{A}|_U$ follows by choosing $Q_0 := \operatorname{Inv}\mathbf{A}$ in the main claim of this lemma. Then the assumption $\operatorname{Pol}_A Q_0 = \operatorname{Pol}_A \operatorname{Inv}\mathbf{A} = \operatorname{Clo}(\mathbf{A})$ is trivially fulfilled and the result above is applicable. For $Q_0 = \operatorname{Inv}\mathbf{A}$, it says explicitly, $\operatorname{Pol}_U([\operatorname{Inv}\mathbf{A}]\restriction_U) = e(\operatorname{Clo}(\mathbf{A}))$, and, by definition of the relational counterpart, we have $\operatorname{Pol}_U([\operatorname{Inv}\mathbf{A}]\restriction_U) \overset{3.2.2}{=} \operatorname{Pol}\underset{\sim}{\mathbf{A}}\restriction_U \overset{3.3.2}{=} \operatorname{Clo}(\mathbf{A}|_U)$.

It remains to be shown that for $Q_0 \subseteq \mathrm{R}_A$ satisfying $\operatorname{Pol}_A Q_0 = \operatorname{Clo}(\mathbf{A})$, it is

$$\operatorname{Pol}_U \{\, \varrho\restriction_U \mid \varrho \in Q_0 \} = e(\operatorname{Clo}(\mathbf{A}))\,.$$

Before we begin the proof, we note that our assumption $\operatorname{Clo}(\mathbf{A}) = \operatorname{Pol}_A Q_0$ entails the inclusion $Q_0 \subseteq \operatorname{Inv}_A \operatorname{Pol}_A Q_0 = \operatorname{Inv}_A \operatorname{Clo}(\mathbf{A}) = \operatorname{Inv}\mathbf{A}$. We now demonstrate the remaining equality by discussing both set inclusions separately.

"$\supseteq$" Let $n \in \mathbb{N}$, $f \in \operatorname{Clo}^{(n)}(\mathbf{A})$ with $f \rhd U$; set $g := f|_U$. It has to be shown that $g \rhd \varrho\restriction_U$ for every $\varrho \in Q_0$. So we consider some arbitrarily chosen n-tuple of tuples $(x_1, \ldots, x_n) \in (\varrho\restriction_U)^n = (\varrho \cap U^m)^n$ for some m-ary relation $\varrho \in {Q_0}^{(m)}$ ($m \in \mathbb{N}$). Hence, ϱ also belongs to $\operatorname{Inv}^{(m)}\mathbf{A}$. Clearly, $g \circ (x_1, \ldots, x_n) \in U^m$ as $g \in \mathrm{O}_U$. Furthermore, due to $g = f|_U$, we have

$$g \circ (x_1, \ldots, x_n) = f \circ (x_1, \ldots, x_n) \in \varrho$$

because $f \rhd \varrho$. So $g \circ (x_1, \ldots, x_n) \in \varrho\restriction_U$, and $g \rhd \varrho\restriction_U$.

"$\subseteq$" Conversely, take $n \in \mathbb{N}$ and some $g \in \operatorname{Pol}_U^{(n)} \{\, \varrho\restriction_U \mid \varrho \in Q_0 \}$. Define the operation $f := \mathrm{id}_A|_U^A \circ g \circ \left(e|_A^U \circ e_1^{(n)}, \ldots, e|_A^U \circ e_n^{(n)}\right) \in \mathrm{O}_A^{(n)}$, that is, we have set $f(a) = g(e \circ a)$ for all $a \in A^n$. Note that for the special case $n = 0$, the expression $\left(e|_A^U \circ e_1^{(n)}, \ldots, e|_A^U \circ e_n^{(n)}\right)$ denotes the empty tupling, that is, the unique mapping $A^0 \longrightarrow U^0$ into the terminal object U^0. We have to show that $f \in \operatorname{Clo}(\mathbf{A})$, f preserves U and $g = f|_U$. Indeed, for all $(u_1, \ldots, u_n) \in U^n$ we have

$$f(u_1, \ldots, u_n) = g(e \circ (u_1, \ldots, u_n)) \overset{3.1.3}{=} g(u_1, \ldots, u_n) \in U$$

as $g \in \mathrm{O}_U$. This shows that $f \rhd U$ and $g = f|_U$. It is left to demonstrate $f \in \operatorname{Clo}(\mathbf{A}) = \operatorname{Pol}_A Q_0$. To this end, we take some arbitrary $m \in \mathbb{N}$ and $\varrho \in {Q_0}^{(m)}$, and show $f \rhd \varrho$. For all $n \in \mathbb{N}_+$ and $(x_1, \ldots, x_n) \in \varrho^n$, it follows

$$\begin{aligned}
f \circ (x_1, \ldots, x_n) &= \mathrm{id}_A|_U^A \circ g \circ \left(e|_A^U \circ e_1^{(n)}, \ldots, e|_A^U \circ e_n^{(n)}\right) \circ (x_1, \ldots, x_n)\\
&= g \circ \left(e|_A^U \circ e_1^{(n)}, \ldots, e|_A^U \circ e_n^{(n)}\right) \circ (x_1, \ldots, x_n)\\
&= g \circ \left(e|_A^U \circ e_1^{(n)} \circ (x_1, \ldots, x_n), \ldots, e|_A^U \circ e_n^{(n)} \circ (x_1, \ldots, x_n)\right)\\
&= g \circ \left(e|_A^U \circ x_1, \ldots, e|_A^U \circ x_n\right)\\
&= g \circ (e \circ x_1, \ldots, e \circ x_n)
\end{aligned}$$

which belongs to $\varrho\restriction_U \subseteq \varrho$ as $e \circ x_1, \ldots, e \circ x_n \in \varrho\restriction_U$ by Lemma 3.1.4 and g preserves $\varrho\restriction_U$. For the same reason we have

$$(f(\emptyset), \ldots, f(\emptyset)) = (g(\emptyset), \ldots, g(\emptyset)) \in \varrho\restriction_U \subseteq \varrho$$

in case $n = 0$. Hence, f always preserves ϱ, i.e. $f \in \mathrm{Clo}(\mathbf{A})$. □

The previous Lemma 3.3.4 shows that, up to the difference between clone and polynomial operations, our restriction process yields the same restriction as defined in [HM88, Definition 2.2]. Again, if the algebra $\mathbf{A}$ contains all nullary constants in $\mathrm{Clo}^{(0)}(\mathbf{A})$ and $\mathrm{Term}(\mathbf{A})$ is locally closed, then the restriction $\mathbf{A}|_U$ defined in [HM88] is identical with the one in Definition 3.3.1.

Looking at Definition 3.3.1 it is natural to consider restricted algebras $\mathbf{A}|_U$ as non-indexed structures. However, Lemma 3.3.4 indicates a canonical way to interpret them as an algebra with a signature. This is especially useful, if the original algebra $\mathbf{A}$ was given as an indexed structure, and allows us to speak about identities. In this way we can not only transfer relational properties to local algebras, but also such that are given by operations.

3.3.5 Remark. If $e \in \mathrm{Idem}\,\mathbf{A}$ is an idempotent unary clone operation of an algebra $\mathbf{A}$ and $U := \mathrm{im}\, e$ is the corresponding neighbourhood, then the previous lemma suggests to understand the restricted algebra $\mathbf{A}|_U$ as an indexed structure in the following way: the operation symbols are the operations in $\mathrm{Clo}(\mathbf{A})$, keeping the arities they already have as functions. Interpreting a symbol $f \in \mathrm{Clo}^{(n)}(\mathbf{A})$ as an operation $f^{\mathbf{U}} := (e \circ f)|_U$ yields a well-defined n-ary operation of $\mathbf{A}|_U$. In this way the algebra $\mathbf{U} = \left\langle U; \left(f^{\mathbf{U}}\right)_{f \in \mathrm{Clo}(\mathbf{A})}\right\rangle$ has got the same carrier set and the same *set* of fundamental operations as $\mathbf{A}|_U$ (cp. Lemma 3.3.4).

In order to discuss now the transfer of identities from $\mathbf{A}$ to $\mathbf{A}|_U$ (more precisely to $\mathbf{U}$), we further assume that $\mathbf{A}$ itself is an algebra having a certain signature. If $n \in \mathbb{N}$ and s and t are n-variable terms in the signature of $\mathbf{A}$, then the truth of the identity $s \approx t$ in $\mathbf{A}$ entails that the identity $s^{\mathbf{A}} x_0 \cdots x_{n-1} \approx t^{\mathbf{A}} x_0 \cdots x_{n-1}$ holds in $\mathbf{U}$, where $s^{\mathbf{A}}$ and $t^{\mathbf{A}}$ denote the n-ary term operations of $\mathbf{A}$ belonging to s and t, respectively.

This is true, because, by definition, the term operation of $\mathbf{U}$ belonging to the term $s^{\mathbf{A}} x_0 \cdots x_{n-1}$ is $\left(s^{\mathbf{A}} x_0 \cdots x_{n-1}\right)^{\mathbf{U}} = \left(e \circ s^{\mathbf{A}}\right)|_U$, and similarly for t. The truth of the identity $s \approx t$ in $\mathbf{A}$ is equivalent to $s^{\mathbf{A}} = t^{\mathbf{A}}$, whence

$$\left(s^{\mathbf{A}} x_0 \cdots x_{n-1}\right)^{\mathbf{U}} = \left(s^{\mathbf{A}}\right)^{\mathbf{U}} = \left(e \circ s^{\mathbf{A}}\right)|_U = \left(e \circ t^{\mathbf{A}}\right)|_U = \left(t^{\mathbf{A}}\right)^{\mathbf{U}} = \left(t^{\mathbf{A}} x_0 \cdots x_{n-1}\right)^{\mathbf{U}},$$

i.e. $\mathbf{U} \models s^{\mathbf{A}} x_0 \cdots x_{n-1} \approx t^{\mathbf{A}} x_0 \cdots x_{n-1}$ follows.

So every identity of the global algebra $\mathbf{A}$ translates into a linear identity[10] for each local algebra $\mathbf{A}|_U$, $U \in \mathrm{Neigh}\,\mathbf{A}$. This is of course caused by the change of signature coming with the localisation process.

[10] An identity is called *linear* if in both terms there occurs at most one operation symbol.

We give a warning example to show that not too much should be expected from this translation property. Assume that the original algebra $\mathbf{A}$ is a semigroup (using a signature with just one binary operation symbol $*$). Then the result stated above should not be misunderstood to say that all restricted algebras have again semigroup operations.

Of course, the derived algebra $\mathbf{U}$ has again binary operations, in particular the operation $*^{\mathbf{U}} = \left(e \circ *^{\mathbf{A}}\right)|_U$ derived from the fundamental operation $*^{\mathbf{A}}$ of $\mathbf{A}$. However, the method sketched above does not allow us to infer directly that this or some other derived binary operation is again associative. Indeed, associativity of $*^{\mathbf{U}}$ would mean that for all $u, v, w \in U$ the equality

$$e\left(u *^{\mathbf{A}} e\left(v *^{\mathbf{A}} w\right)\right) = u *^{\mathbf{U}} \left(v *^{\mathbf{U}} w\right) = \left(u *^{\mathbf{U}} v\right) *^{\mathbf{U}} w = e\left(e\left(u *^{\mathbf{A}} v\right) *^{\mathbf{A}} w\right)$$

were true, which is not necessarily the case.

We also see from this equation that associativity of $*^{\mathbf{U}}$ would follow in case that the operation $e\colon A \longrightarrow A$ were a homomorphism w.r.t. to the fundamental operations of $\mathbf{A}$. However, this is a very strong property with many consequences that should be studied on its own.

It is also obvious that the relationship from above does not enable us to turn the associativity law of $\mathbf{A}$ into an associativity law for $\mathbf{A}|_U$. Namely, the identity $x * (y * z) \approx (x * y) * z$, which holds in $\mathbf{A}$, results in a linear identity between two ternary fundamental operations of $\mathbf{A}|_U$, namely in

$$(x * (y * z))^{\mathbf{A}} x_0 x_1 x_2 \approx ((x * y) * z)^{\mathbf{A}} x_0 x_1 x_2$$

which is true because of $e \circ f = e \circ g$ for $f, g \in \operatorname{Term}^{(3)}(\mathbf{A}) \subseteq \operatorname{Clo}^{(3)}(\mathbf{A})$ given by $f(a, b, c) := a *^{\mathbf{A}} \left(b *^{\mathbf{A}} c\right)$ and $g(a, b, c) := \left(a *^{\mathbf{A}} b\right) *^{\mathbf{A}} c$ for $a, b, c \in A$.

Now, after discussing a disappointing example, we also want to show a situation where the method from above is useful. Let us assume that $\mathbf{A}$ has got an n-ary term operation (more generally an n-ary clone operation) satisfying a linear identity in $\mathbf{A}$. This means, we have some $f \in \operatorname{Clo}^{(n)}(\mathbf{A})$ satisfying an equality of the form $f \circ \left(e_{i_1}^{(m)}, \dots, e_{i_n}^{(m)}\right) = f \circ \left(e_{j_1}^{(m)}, \dots, e_{j_n}^{(m)}\right)$ or $f \circ \left(e_{i_1}^{(m)}, \dots, e_{i_n}^{(m)}\right) = e_j^{(m)}$ for some $m \in \mathbb{N}$ and $i_1, \dots, i_n, j_1, \dots, j_n, j \in \{1, \dots, m\}$. Then, by what was shown above, the restricted algebra $\mathbf{A}|_U$ has a fundamental n-ary operation, namely $f^{\mathbf{U}} = (e \circ f)|_U$ satisfying a similarly shaped identity (on U). In detail, composing both sides of the given identities with e and then restricting to U yields

$$\begin{aligned}\left(e \circ f \circ \left(e_{i_1}^{(m)}, \dots, e_{i_n}^{(m)}\right)\right)|_U &= (e \circ f)|_U \circ \left(e_{i_1}^{(m)}|_U, \dots, e_{i_n}^{(m)}|_U\right) \\ &= f^{\mathbf{U}} \circ \left(e_{i_1}^{(m)}|_U, \dots, e_{i_n}^{(m)}|_U\right),\end{aligned}$$

similarly for the other side, and

$$\left(e \circ e_j^{(m)}\right)|_U = e|_U \circ e_j^{(m)}|_U = e_j^{(m)}|_U,$$

because e is the identity on U. Combining this with the originally given identities, we obtain $f^{\mathbf{U}} \circ \left(e^{(m)}_{i_1}|_U, \dots, e^{(m)}_{i_n}|_U\right) = f^{\mathbf{U}} \circ \left(e^{(m)}_{j_1}|_U, \dots, e^{(m)}_{j_n}|_U\right)$ in the first case, or $f^{\mathbf{U}} \circ \left(e^{(m)}_{i_1}|_U, \dots, e^{(m)}_{i_n}|_U\right) = e^{(m)}_{j}|_U$, in the second.

Certainly, these arguments work for algebras $\mathbf{A}$ having a minority, a MAL'CEV ([Mal'54]), a PIXLEY ([Pix63]), a cyclic (idempotent) operation ([BKM+09, BK12]), a [weak] near unanimity, a TAYLOR ([Tay77]), a SIGGERS ([Sig10]), an edge, a cube ([BIM+10]) or a parallelogram operation ([KS12]) in $\operatorname{Clo}(\mathbf{A})$. The same is true for DAY operations ([Day69]), GUMM operations ([Gum81]) and JÓNSSON operations ([Jón67]).

This shows that local structures inherit different sorts of Mal'cev conditions from the global algebra, a fact, which is not extremely surprising in view of the discussion earlier on how relational properties can be passed on from the global to the local level. ♢

We mention in connection with the previous remark that [KL10] proves an interesting characterisation of Mal'cev conditions for finitely generated varieties purely in terms of RST. Namely, a finitely generated variety is congruence 3-permutable (see the paragraph preceding Example 3.4.1) and contains a near unanimity term if and only if the size of irreducible (see Definition 3.5.16 below) neighbourhoods in polynomial expansions of all finite algebras of the variety is bounded.

The next lemma shows what can be said about $\mathbf{A}|_U$ if, by coincidence, a neighbourhood U happens to be a subuniverse[11] of $\mathbf{A}$. For the case of finite algebras, one can find this result as Remark 3.3.4 in [Beh09].

3.3.6 Lemma. *For any algebra $\mathbf{A} = \langle A; F\rangle$ having a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ that forms a subuniverse $U \leq \mathbf{A}$, the induced algebra on U coincides with the saturated algebra belonging to the subalgebra $\mathbf{U} = \langle U;\ \{ f|_U \mid f \in F\}\rangle$:*

$$\mathbf{A}|_U = \mathbf{U}|_U.$$

Proof: By assumption, we have $U \in \operatorname{Inv}_A F = \operatorname{Inv}_A \operatorname{Pol}_A \operatorname{Inv}_A F$, so all operations in the clone $\operatorname{Clo}(\mathbf{A}) = \operatorname{Pol}_A \operatorname{Inv}_A F$ can be restricted to U. The proof proceeds in three steps.

(1) First we show that

$$\operatorname{Inv}_U \{ f|_U \mid f \in F\} \subseteq \operatorname{Inv}_A F.$$

Clearly, if $m \in \mathbb{N}$ and $S \in \operatorname{Inv}^{(m)}_U \{ f|_U \mid f \in F\}$, then $S \subseteq U^m \subseteq A^m$, so S is a finitary relation on A, too. Furthermore, for $n \in \mathbb{N}$, every $f \in F^{(n)}$ and all $(x_1, \dots, x_n) \in S^n$, we have

$$f \circ (x_1, \dots, x_n) = f|_U \circ (x_1, \dots, x_n) \in S,$$

[11] This happens, for instance, for finite ABELian groups or neighbourhoods of polynomial expansions of distributive lattices.

because $S \in \operatorname{Inv}_U \{ f|_U \mid f \in F \}$ and $(x_1, \dots, x_n) \in S^n$. Writing this out for $n = 0$ explicitly, we get $(f(\emptyset), \dots, f(\emptyset)) = (f|_U(\emptyset), \dots, f|_U(\emptyset)) \in S$. Consequently, all fundamental operations of $\mathbf{A}$ preserve the relations that are invariant for $\{ f|_U \mid f \in F \}$, and that was to be shown.

(2) Next we infer that

$$\operatorname{Inv}_U \{ f|_U \mid f \in F \} = \operatorname{Inv}_U \{ f|_U \mid f \in \operatorname{Pol}_A \operatorname{Inv}_A F \} = [\operatorname{Inv} \mathbf{A}]\restriction_U .$$

From the previous item we get that

$$\operatorname{Inv}_U \{ f|_U \mid f \in F \} \subseteq \{ S\restriction_U \mid S \in \operatorname{Inv}_A F \} ,$$

because due to (1) every $S \in \operatorname{Inv}_U \{ f|_U \mid f \in F \}$ satisfies $S\restriction_U = S \in \operatorname{Inv}_A F$. Clearly, by Lemmas 3.3.2 and 3.3.4, we have

$$\begin{aligned} &\{ S\restriction_U \mid S \in \operatorname{Inv}_A F \} = [\operatorname{Inv} \mathbf{A}]\restriction_U \overset{3.3.2}{=} \operatorname{Inv} \mathbf{A}|_U \overset{3.3.4}{=} \operatorname{Inv}_U e\,(\operatorname{Clo}(\mathbf{A})) \\ &\overset{3.3.4}{=} \operatorname{Inv}_U \{ f|_U \mid f \in \operatorname{Pol}_A \operatorname{Inv}_A F \wedge f \rhd U \} = \operatorname{Inv}_U \{ f|_U \mid f \in \operatorname{Clo}(\mathbf{A}) \} . \end{aligned}$$

So putting the previous displayed equalities together, the inclusion

$$\operatorname{Inv}_U \{ f|_U \mid f \in F \} \subseteq \operatorname{Inv}_U \{ f|_U \mid f \in \operatorname{Pol}_A \operatorname{Inv}_A F \} = [\operatorname{Inv} \mathbf{A}]\restriction_U$$

follows. Finally, we know $F \subseteq \operatorname{Clo}(\mathbf{A})$, so anti-monotonicity of Inv_U implies $\operatorname{Inv}_U \{ f|_U \mid f \in F \} \supseteq \operatorname{Inv}_U \{ f|_U \mid f \in \operatorname{Pol}_A \operatorname{Inv}_A F \}$, which proves the desired equality.

(3) At last, applying the operator Pol_U to the equality just proven, we obtain

$$\operatorname{Pol}_U \operatorname{Inv}_U \{ f|_U \mid f \in F \} = \operatorname{Pol}_U [\operatorname{Inv} \mathbf{A}]\restriction_U .$$

According to Definition 3.3.1, the set on the left-hand side contains the fundamental operations of $\mathbf{U}|_U$, whereas the set on the right-hand side contains those of $\mathbf{A}|_U$. As the mentioned restricted algebras also share the same universe, they are identical. □

It is a general aim of RST to study an algebra $\mathbf{A}$ via its restrictions $\mathbf{A}|_U$ to neighbourhoods $U \in \operatorname{Neigh} \mathbf{A}$. In order to obtain restrictions of preferably small cardinality, it seems a good strategy to iterate this procedure. Thus, it is useful to characterise the role of neighbourhoods of a restricted algebra $\mathbf{A}|_U$ in the original algebra $\mathbf{A}$. The following result is what can be expected from a reasonable localisation theory. We mention that the characterisation of the neighbourhoods of an induced algebra as the subneighbourhoods of the inducing neighbourhood (in the global algebra) is already contained as Lemma 3.3.5 in [Beh09].

3.3.7 Lemma. *For an algebra* $\mathbf{A}$ *and an idempotent* $e \in \operatorname{Idem} \mathbf{A}$ *inducing a neighbourhood* $U = e[A] \in \operatorname{Neigh} \mathbf{A}$ *, we have*

$$\begin{aligned}\operatorname{Idem} \mathbf{A}|_U &= \left\{ (e \circ f \circ e)|_U \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge e \circ f \circ e \in \operatorname{Idem} A \right\},\\ \operatorname{Neigh} \mathbf{A}|_U &= (\operatorname{Neigh} \mathbf{A}) \cap \mathfrak{P}(U).\end{aligned}$$

That is to say, the neighbourhoods of the restricted algebra $\mathbf{A}|_U$ *are precisely the subneighbourhoods of the neighbourhood* U *in the global algebra* $\mathbf{A}$*.*

Proof: First, we prove the characterisation of the idempotent clone operations. Clearly, by Lemma 3.3.4, for every $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ the restriction $(e \circ f \circ e)|_U$ is a member of $e(\operatorname{Clo}(\mathbf{A}))^{(1)} = \operatorname{Clo}^{(1)}(\mathbf{A}|_U)$. Since $\operatorname{im} e \circ f \circ e \subseteq \operatorname{im} e = U$, the operation $e \circ f \circ e$ belongs to $\operatorname{Pol}_A U$, so Lemma 3.1.1(b) applied to the unary algebra $\tilde{\mathbf{A}} = \langle A; e \circ f \circ e \rangle$, yields that restriction to U is a clone homomorphism from $\operatorname{Pol}_A \operatorname{Inv}_A \{e \circ f \circ e\}$ to O_U. In particular, it is a semigroup homomorphism from $\operatorname{Pol}_A^{(1)} \operatorname{Inv}_A \{e \circ f \circ e\}$ to $\left\langle U^U; \circ \right\rangle$, whence idempotency of $e \circ f \circ e$ carries over to its restriction. Thus, $(e \circ f \circ e)|_U$ belongs to $\operatorname{Idem} \mathbf{A}|_U$.

Conversely, by Lemma 3.3.4, every $\hat{e} \in \operatorname{Idem} \mathbf{A}|_U$ can be written as a restriction $(e \circ f)|_U$ for some $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$. Now for every $u \in U$, it is

$$e \circ f \circ e(u) = e \circ f(e(u)) = e \circ f(u) = \hat{e}(u)$$

as e is idempotent and U is its image. Therefore, $\hat{e} = (e \circ f)|_U = (e \circ f \circ e)|_U$. Furthermore, letting $u := e(x) \in U$ for $x \in A$, we have

$$\begin{aligned}(e \circ f \circ e) \circ (e \circ f \circ e)(x) &= (e \circ f \circ e) \circ (e \circ f \circ e) \circ e(x)\\ &= (e \circ f \circ e) \circ (e \circ f \circ e)(e(x))\\ &= (e \circ f \circ e)|_U \circ (e \circ f \circ e)|_U(u)\\ &= \hat{e} \circ \hat{e}(u) = \hat{e}(u)\\ &= e \circ f \circ e(u) = e \circ f \circ e(e(x))\\ &= e \circ f \circ e \circ e(x) = e \circ f \circ e(x),\end{aligned}$$

exploiting idempotency of e and $\hat{e}$. This shows that $e \circ f \circ e$ is an idempotent operation restricting to $\hat{e}$, where $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$, as desired.

We can use the previous result to characterise the neighbourhoods of $\mathbf{A}|_U$. By definition, we have

$$\begin{aligned}\operatorname{Neigh} \mathbf{A}|_U &= \{ \hat{e}[U] \mid \hat{e} \in \operatorname{Idem} \mathbf{A}|_U \}\\ &= \left\{ (e \circ f \circ e)|_U [U] \;\middle|\; f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge e \circ f \circ e \in \operatorname{Idem} A \right\}.\end{aligned}$$

Using idempotency of e, we can write such images as

$$(e \circ f \circ e)|_U [U] = (e \circ f \circ e)[e[A]] = (e \circ f \circ e) \circ e[A] = e \circ f \circ e[A] \subseteq \operatorname{im} e = U.$$

Since $e \circ f \circ e$ belongs to Idem $\mathbf{A}$, such sets are neighbourhoods of $\mathbf{A}$, lying in $\mathfrak{P}(U)$. Hence, Neigh $\mathbf{A}|_U \subseteq \mathfrak{P}(U) \cap \text{Neigh}\,\mathbf{A}$.

For the converse inclusion let us consider any unary operation $f \in \text{Idem}\,\mathbf{A}$ satisfying $V := \text{im}\, f \subseteq \text{im}\, e = U$. By Lemma 3.1.3, this is equivalent to $e \circ f = f$. Thus, we have $f = f \circ f = e \circ f \circ e \circ f$ and $e \circ f \circ e \circ e = e \circ f \circ e = f \circ e$. Therefore, it is

$$\begin{aligned} f[A] &= e \circ f \circ e \circ f[A] = e \circ f \circ e[f[A]] = e \circ f \circ e[V] \subseteq e \circ f \circ e[U] \\ &= e \circ f \circ e[e[A]] = e \circ f \circ e \circ e[A] = f \circ e[A] = f[e[A]] = f[U] \subseteq f[A], \end{aligned}$$

and all these sets are identical. Especially, we can write the neighbourhood V as

$$V = f[A] = e \circ f \circ e[U] = (e \circ f \circ e)|_U[U].$$

Since f belongs to $\text{Clo}^{(1)}(\mathbf{A})$, by the characterisation of Neigh $\mathbf{A}|_U$ obtained above, it only remains to show that $e \circ f \circ e$ is idempotent. This follows again from the equalities $e \circ e = e$, $f \circ f = f$ and $e \circ f = f$:

$$e \circ f \circ (e \circ e) \circ f \circ e = (e \circ f) \circ (e \circ f) \circ e = (f \circ f) \circ e = f \circ e = (e \circ f) \circ e. \quad \square$$

In the next section, we are now turning our attention to some special collections of neighbourhoods of an algebra that will be called *covers*.

3.4 Covers

As mentioned in the introduction to this chapter, we would like to present a localisation theory that pays special attention to the relational aspect of an algebra, i.e. to its invariant relations, which determine the algebra up to local term equivalence. Since every function in the clone of an algebra has to preserve all invariant relations of that algebra, those can be interpreted as constraints that select which functions can be composed from the fundamental operations of the algebra and projections, and which cannot. A richness of invariant relations corresponds to a small clone of operations and vice versa. However, when restricting an algebra to a neighbourhood U, it can happen that one loses some diversity among invariant relations, that is, $S\restriction_U = T\restriction_U$ for some invariants $S \neq T$. Like this some part of the information about the structure of the algebra is lost. For instance, the most extreme case would be the restriction to a one-element subset U. Then Inv $\mathbf{A}|_U$ would only consist of trivial (diagonal) relations plus possibly the empty relation, and, accordingly, $\mathbf{A}|_U$ would admit all at least unary operations in its clone and, hence, as fundamental operations.

As we want to study an algebra by its restrictions to neighbourhoods, we are interested in avoiding this kind of loss of information. The previous arguments have made plausible that simply one restricted structure will usually not be sufficient for this purpose. Hence, we want to study collections $\mathcal{U}$ of neighbourhoods having the following property: whenever we have two different invariants $S \neq T$ of $\mathbf{A}$ of common arity, we want to be able to distinguish them in at least one restricted algebra belonging to one neighbourhood from $\mathcal{U}$. That is, for every $m \in \mathbb{N}$ and all $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ the implication

$$S \neq T \implies \exists U \in \mathcal{U}\colon S\restriction_U \neq T\restriction_U \tag{3.9}$$

should hold. Collections of neighbourhoods satisfying this condition will be called *covers* of $\mathbf{A}$. At the end of this section, we will even show that the concept of a cover of an algebra is so powerful that it allows a complete reconstruction of the algebra up to equality of its clone, i.e. local term equivalence, from its restrictions to neighbourhoods in a cover. For finite algebras we will see that this can always be done via a product-retract construction on the relational side. In the infinite case we will need to be slightly more ingenious and "localise" this idea.

Before we shall state a formal definition, we want to give a bit more of motivation why covers might be useful. In this respect we will pick up the theme introduced in the discussion at the beginning of the previous section, explaining the advantages of our choice of neighbourhoods and of the induced algebra. There we focussed on the fact that restriction of invariant relations to a neighbourhood U is a surjective clone homomorphism from $\operatorname{Inv}\mathbf{A}$ to $\operatorname{Inv}\mathbf{A}|_U$, which allows us to transport the truth of universally quantified identities $s \approx t$ of relational clones, as in equation (3.8) on page 68, from $\operatorname{Inv}\mathbf{A}$ to $\operatorname{Inv}\mathbf{A}|_U$. So whenever an identity $s \approx t$ holds in $\operatorname{Inv}\mathbf{A}$, it also holds in $\operatorname{Inv}\mathbf{A}|_U$ for every $U \in \mathcal{U}$ and any collection $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ of neighbourhoods.

Now covers enable us to take this way in the opposite direction. This works, first of all, for individual relations. Suppose, $\mathcal{U}$ covers $\mathbf{A}$ and $(S)_{i\in I}$, $(T)_{j\in J}$ are invariant relations of $\mathbf{A}$ such that their restrictions to U fulfil a property expressible by evaluation of an identity $\prod^{\beta}_{(\alpha_i)_{i\in I}} (x_i)_{i\in I} \approx \prod^{\delta}_{(\gamma_j)_{j\in J}} (y_j)_{j\in J}$ at $(S\restriction_U)_{i\in I}$, $(T\restriction_U)_{j\in J}$ for every $U \in \mathcal{U}$. This means

$$\left(\prod^{\beta}_{(\alpha_i)_{i\in I}} (S_i)_{i\in I}\right)\restriction_U = \prod^{\beta}_{(\alpha_i)_{i\in I}} (S_i\restriction_U)_{i\in I} = \prod^{\delta}_{(\gamma_j)_{j\in J}} (T_j\restriction_U)_{j\in J} = \left(\prod^{\delta}_{(\gamma_j)_{j\in J}} (T_j)_{j\in J}\right)\restriction_U$$

for every $U \in \mathcal{U}$, where the first and the second equality hold by the homomorphism property of restriction to U. However, the displayed equality says that the invariant relations $S := \prod^{\beta}_{(\alpha_i)_{i\in I}} (S_i)_{i\in I}$ and $T := \prod^{\delta}_{(\gamma_j)_{j\in J}} (T_j)_{j\in J}$ have identical restrictions for every $U \in \mathcal{U}$. By the contrapositive of the cover property described above (cf. Definition 3.4.2), they must be globally identical, i.e.

$$\prod^{\beta}_{(\alpha_i)_{i\in I}} (S_i)_{i\in I} = S = T = \prod^{\delta}_{(\gamma_j)_{j\in J}} (T_j)_{j\in J}\,.$$

Hence, the property represented by this equality, which held for every restriction $\underset{\sim}{\mathbf{A}}\restriction_U$, $U \in \mathcal{U}$, is true in $\mathbf{A}$, too.

Now, second, let us consider the situation when such a property holds universally quantified in every restricted relational structure belonging to a cover $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ of an algebra $\mathbf{A}$. In detail, we assume that for every $U \in \mathcal{U}$ the equality

$$\prod_{(\alpha_i)_{i\in I}}^{\beta} \left(\tilde{S}_i\right)_{i\in I} = \prod_{(\gamma_j)_{j\in J}}^{\delta} \left(\tilde{T}_j\right)_{j\in J}$$

holds for every choice of relations $\left(\tilde{S}_i\right)_{i\in I}$, $\left(\tilde{T}_j\right)_{j\in J}$ in $\operatorname{Inv} \mathbf{A}|_U$ of appropriate arity, i.e., the arity of $\tilde{S}_i$ matches the arity of the symbol x_i for every $i \in I$, and similarly for $\tilde{T}_j$ and y_j, $j \in J$. We claim that then the equality

$$\prod_{(\alpha_i)_{i\in I}}^{\beta} (S_i)_{i\in I} = \prod_{(\gamma_j)_{j\in J}}^{\delta} (T_j)_{j\in J}$$

holds for every choice of relations $(S_i)_{i\in I}$, $(T_j)_{j\in J}$ in $\operatorname{Inv} \mathbf{A}$ of appropriate arity. Indeed, for every $U \in \mathcal{U}$ the restrictions $(S_i\restriction_U)_{i\in I}$ and $(T_j\restriction_U)_{j\in J}$ are invariant relations of $\mathbf{A}|_U$ of appropriate arity, whence the assumption yields that

$$\prod_{(\alpha_i)_{i\in I}}^{\beta} (S_i\restriction_U)_{i\in I} = \prod_{(\gamma_j)_{j\in J}}^{\delta} (T_j\restriction_U)_{j\in J}$$

is true for every $U \in \mathcal{U}$. The argument for individual relations above now implies that $\prod^{\beta}_{(\alpha_i)_{i\in I}} (S_i)_{i\in I} = \prod^{\delta}_{(\gamma_j)_{j\in J}} (T_j)_{j\in J}$ holds in $\mathbf{A}$.

Summing up and combining the explained ideas with the remarks in Section 3.3, we can say that for covers $\mathcal{U}$ of an algebra $\mathbf{A}$ any universally quantified identity $\prod^{\beta}_{(\alpha_i)_{i\in I}} (x_i)_{i\in I} \approx \prod^{\delta}_{(\gamma_j)_{j\in J}} (y_j)_{j\in J}$ holds in $\operatorname{Inv} \mathbf{A}$ if and only if it holds in every $\operatorname{Inv} \mathbf{A}|_U$, $U \in \mathcal{U}$. In other words, the relational clone $\operatorname{Inv} \mathbf{A}$ generates[12] the same "variety" as the set of clones $\{\operatorname{Inv} \mathbf{A}|_U \mid U \in \mathcal{U}\}$.

We want to illustrate this once more harking back to the example of permutability of arbitrary binary invariant relations used in the previous section. If an algebra $\mathbf{A}$ satisfies $S \circ T = T \circ S$ for all $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$, then by surjectivity of $\restriction_U$ for U in a collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$, for every $\tilde{S}, \tilde{T} \in \operatorname{Inv}^{(2)}\mathbf{A}|_U$ there exist some $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$ such that $\tilde{S} = S\restriction_U$ and $\tilde{T} = T\restriction_U$. That means, exploiting the homomorphism property, we have

$$\tilde{T} \circ \tilde{S} = T\restriction_U \circ S\restriction_U = (T \circ S)\restriction_U = (S \circ T)\restriction_U = S\restriction_U \circ T\restriction_U = \tilde{S} \circ \tilde{T},$$

i.e. binary invariants of $\mathbf{A}|_U$ commute for every $U \in \mathcal{U}$.

Now, for the converse let us assume that precisely this condition holds and $\mathcal{U}$ is a cover of $\mathbf{A}$. Then for all $S, T \in \operatorname{Inv}^{(2)}\mathbf{A}$ and every $U \in \mathcal{U}$ the relations $\tilde{S} := S\restriction_U$ and $\tilde{T} := T\restriction_U$ belong to $\operatorname{Inv}^{(2)}\mathbf{A}|_U$, and it is

$$(T \circ S)\restriction_U = T\restriction_U \circ S\restriction_U = \tilde{T} \circ \tilde{S} = \tilde{S} \circ \tilde{T} = S\restriction_U \circ T\restriction_U = (S \circ T)\restriction_U.$$

[12] in the sense of closure against all models of satisfied identities

By the cover property of $\mathcal{U}$ we can now infer that $T \circ S$ cannot be different from $S \circ T$ since these relations could not be distinguished by restriction to any neighbourhood in $\mathcal{U}$. Consequently, the equality $T \circ S = S \circ T$ holds for all binary invariants $S, T \in \operatorname{Inv}^{(2)} \mathbf{A}$.

Examples of more useful properties involve congruences. In this respect we recall the following notions: for $k \in \mathbb{N}$, $k \geq 2$, an algebra is said to have *k-permuting congruences* (or to be *congruence k-permutable*) if $\theta \circ_k \psi = \psi \circ_k \theta$ holds for all congruences $\theta, \psi \in \operatorname{Con} \mathbf{A}$, where $\theta \circ_k \psi = \theta \circ \psi \circ \theta \circ \cdots$ denotes the alternating relation product starting with θ and involving altogether k factors from $\{\theta, \psi\}$. Furthermore, $\mathbf{A}$ is *congruence distributive* or *congruence modular* if $\operatorname{Con} \mathbf{A}$ is a distributive or modular lattice, respectively.

3.4.1 Example. For $k \in \mathbb{N}$, $k \geq 2$, an algebra $\mathbf{A}$ and a cover $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ the following implications are true:

(a) If $\mathbf{A}|_U$ is congruence k-permutable for every $U \in \mathcal{U}$, then so is $\mathbf{A}$.

(b) If $\mathbf{A}|_U$ is congruence distributive for every $U \in \mathcal{U}$, then so is $\mathbf{A}$.

(c) If $\mathbf{A}|_U$ is congruence modular for every $U \in \mathcal{U}$, then so is $\mathbf{A}$. ♢

We remark that the idea for the proof of item (a) is implicitly present in the proof of Lemma 3.1 in [KL10]. The purpose of this example is to be explicitly illustrative. Therefore, the selected properties of algebras are not particularly technical, and we still bother with their rather easy proof.

Proof: For the proof we rely on the fact, proven in item (d) of Lemma 3.3.3, that restriction of congruences of $\mathbf{A}$ to a neighbourhood U maps to $\operatorname{Con} \mathbf{A}|_U$, and moreover is a homomorphism w.r.t. lattice and relational clone operations. In the context of our example we may exploit this for any $U \in \mathcal{U}$.

(a) To show that $\mathbf{A}$ is congruence k-permutable, we consider arbitrary congruences θ and ψ in $\operatorname{Con} \mathbf{A}$. We let $\alpha := \theta \circ_k \psi = \theta \circ \psi \circ \theta \circ \cdots$ and $\beta := \psi \circ_k \theta$. These are again invariant relations, and, exploiting the homomorphism property of restriction, we obtain for every $U \in \mathcal{U}$ that $\alpha{\restriction_U} = (\theta \circ_k \psi){\restriction_U} = \theta{\restriction_U} \circ_k \psi{\restriction_U}$, and, similarly, $\beta{\restriction_U} = \psi{\restriction_U} \circ_k \theta{\restriction_U}$. Now $\theta{\restriction_U}$ and $\psi{\restriction_U}$ are congruences of $\mathbf{A}|_U$, and this algebra is congruence k-permutable for any $U \in \mathcal{U}$. Hence, it is $\alpha{\restriction_U} = \theta{\restriction_U} \circ_k \psi{\restriction_U} = \psi{\restriction_U} \circ_k \theta{\restriction_U} = \beta{\restriction_U}$ for every $U \in \mathcal{U}$. By the cover property, α and β must be identical invariant relations, i.e. $\theta \circ_k \psi = \alpha = \beta = \psi \circ_k \theta$. Since θ and ψ were arbitrarily chosen, the algebra $\mathbf{A}$ has k-permuting congruences.

(b) In the previous item we have used the homomorphism property of restriction to neighbourhoods w.r.t. an operation of the relational clone. Now we do the same for lattice operations of $\operatorname{Con} \mathbf{A}$. For $\theta, \varphi, \psi \in \operatorname{Con} \mathbf{A}$ we put now

$\alpha := \theta \wedge (\varphi \vee \psi)$ and $\beta := (\theta \wedge \varphi) \vee (\theta \wedge \psi)$. These relations belong again to $\operatorname{Con} \mathbf{A}$, and their restrictions to some $U \in \mathcal{U}$ are

$$\alpha{\restriction}_U = (\theta \wedge (\varphi \vee \psi)){\restriction}_U = \theta{\restriction}_U \wedge (\varphi{\restriction}_U \vee \psi{\restriction}_U)\,,$$
$$\beta{\restriction}_U = ((\theta \wedge \varphi) \vee (\theta \wedge \psi)){\restriction}_U = (\theta{\restriction}_U \wedge \varphi{\restriction}_U) \vee (\theta{\restriction}_U \wedge \psi{\restriction}_U)\,.$$

Since $\theta{\restriction}_U$, $\varphi{\restriction}_U$ and $\psi{\restriction}_U$ are members of $\operatorname{Con} \mathbf{A}|_U$, and this lattice is distributive by assumption, we may infer $\alpha{\restriction}_U = \beta{\restriction}_U$ for every $U \in \mathcal{U}$. Now by the contrapositive of (3.9), the equality $\alpha = \beta$ must hold, which shows that $\wedge$ distributes over $\vee$ in $\operatorname{Con} \mathbf{A}$. It is well-known that this suffices to prove $\operatorname{Con} \mathbf{A}$ to be distributive; alternatively, the dual distributive law follows by analogous considerations.

(c) Although modularity of a lattice can be expressed by satisfaction of the identity $x \wedge (y \vee (x \wedge z)) \approx (x \wedge y) \vee (x \wedge z)$, we shall use the following implicational formulation of the modular law to reveal a bit more the scope of our method:

$$x \geq z \;\longrightarrow\; x \wedge (y \vee z) \approx (x \wedge y) \vee z.$$

To show that this expression is (universally) satisfied in $\operatorname{Con} \mathbf{A}$, let us again consider relations θ, φ and ψ in $\operatorname{Con} \mathbf{A}$, this time subject to the constraint $\theta \supseteq \psi$. We need to prove that $\theta \wedge (\varphi \vee \psi) = (\theta \wedge \varphi) \vee \psi$. The trick here is simply that restriction (as an intersection) preserves set inclusions (between relations). Apart from this, everything else works as in the previous item. Explicitly, for every $U \in \mathcal{U}$ we we have $\theta{\restriction}_U \supseteq \psi{\restriction}_U$. Using the homomorphism property of restriction for the congruences $\alpha := \theta \wedge (\varphi \vee \psi)$ and $\beta := (\theta \wedge \varphi) \vee \psi$, we obtain

$$\alpha{\restriction}_U = \theta{\restriction}_U \wedge (\varphi{\restriction}_U \vee \psi{\restriction}_U)$$
$$\beta{\restriction}_U = (\theta{\restriction}_U \wedge \varphi{\restriction}_U) \vee \psi{\restriction}_U$$

for every $U \in \mathcal{U}$. Since $\operatorname{Con} \mathbf{A}|_U$ universally satisfied the implication above, and $\theta{\restriction}_U$ and $\psi{\restriction}_U$ are congruences of $\mathbf{A}|_U$ satisfying its assumption, we have that the conclusion $\theta{\restriction}_U \wedge (\varphi{\restriction}_U \vee \psi{\restriction}_U) = (\theta{\restriction}_U \vee \varphi{\restriction}_U) \vee \psi{\restriction}_U$ is true. In other words, the equality $\alpha{\restriction}_U = \beta{\restriction}_U$ holds for all $U \in \mathcal{U}$, whence, by the cover property, it must hold for α and β. However, this was the claim to be shown. $\square$

Item (c) hints at the fact that actually more complicated properties than just identities can be transported via covers from localisations to the global algebra. Certainly, quasi-identities are possible, even more general versions where the antecedent of the quasi-identity can be replaced by a conjunction of inequalities (that have to be interpreted by set inclusion of relations).

Now we formally define the concept of a *cover of an algebra* and, more generally, covers of neighbourhoods and whole collections of neighbourhoods (see also Definitions 3.1 and 5.1 in [Kea01], Definition 3.4.1 in [Beh09] and Definition 2.3 and 2.8(2) in [KL10]). However, we are not going to use the implication (3.9) above, but its contrapositive, which has been proven to be useful in the previous considerations.

3.4.2 Definition. For an algebra $\mathbf{A}$ and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ of neighbourhoods, we define the following relations.

(i) A neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ is said to *be covered by* $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, or $\mathcal{V}$ *covers* U, (w.r.t. $\mathbf{A}$)[13] if the following implication

$$(\forall V \in \mathcal{V}\colon \quad S{\restriction_V} = T{\restriction_V}) \implies S{\restriction_U} = T{\restriction_U} \tag{3.10}$$

holds for all invariant relations $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ of any arity $m \in \mathbb{N}$. We denote this by $U \leq_{\mathrm{cov}} \mathcal{V}$ and introduce $\operatorname{Cov}_{\mathbf{A}}(U) := \{\mathcal{W} \subseteq \operatorname{Neigh} \mathbf{A} \mid U \leq_{\mathrm{cov}} \mathcal{W}\}$ as the *set of all covers of the neighbourhood* U.

(ii) We say that $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ covers a whole collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$, and write $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$, if $\mathcal{V}$ covers every $U \in \mathcal{U}$. Furthermore, we define that a neighbourhood $V \in \operatorname{Neigh} \mathbf{A}$ covers $U \in \operatorname{Neigh} \mathbf{A}$ if the singleton collection $\{V\}$ covers U. This is denoted by $U \leq_{\mathrm{cov}} V$, as well.

For convenience and particular importance, we introduce the concept of a *cover of an algebra*: a collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ is called a *cover of* $\mathbf{A}$ if it is a cover of the full neighbourhood $A \in \operatorname{Neigh} \mathbf{A}$. The set of all covers of $\mathbf{A}$ is consequently $\operatorname{Cov}(\mathbf{A}) := \operatorname{Cov}_{\mathbf{A}}(A)$.

(iii) Neighbourhoods $U, V \in \operatorname{Neigh} \mathbf{A}$ are *covering equivalent*, in symbols $U \equiv_{\mathrm{cov}} V$, if they cover one another, i.e. if $U \leq_{\mathrm{cov}} V$ and $V \leq_{\mathrm{cov}} U$. Likewise, we call collections $\mathcal{U}$ and $\mathcal{V}$ in $\mathfrak{P}(\operatorname{Neigh} \mathbf{A})$ *covering equivalent* if $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V} \leq_{\mathrm{cov}} \mathcal{U}$. This fact is similarly denoted by $\mathcal{U} \equiv_{\mathrm{cov}} \mathcal{V}$. $\diamondsuit$

3.4.3 Remark. Using the contrapositive of the definition of the cover relation (see (3.9)), it is easy to see that non-empty covers actually distinguish *all* pairs of invariant relations separated by a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$. That is, a non-empty collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ covers a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ if and only if for all $S, T \in \operatorname{Inv} \mathbf{A}$ the condition $S{\restriction_U} \neq T{\restriction_U}$ implies $S{\restriction_V} \neq T{\restriction_V}$ for some $V \in \mathcal{V}$. This is true because invariant relations $S, T \in \operatorname{Inv} \mathbf{A}$ of distinct arity are separated by every neighbourhood of $\mathbf{A}$, unless they are both empty. In particular, we have $S{\restriction_V} \neq T{\restriction_V}$ for every $V \in \mathcal{V}$, and as $\mathcal{V}$ was not the empty collection, implication (3.9) is valid even for all pairs of invariants. $\diamondsuit$

The concept of cover is intimately related to and has a neat reformulation in terms of separation sets introduced in Definition 3.2.20.

3.4.4 Lemma. *For an algebra* $\mathbf{A}$*, neighbourhoods* $U, V \in \operatorname{Neigh} \mathbf{A}$ *and collections* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ *the following is true:*

[13] Most of the times we are going to omit the reference to the algebra $\mathbf{A}$ because it is going to be clear from the context. However, in some cases it is necessary to mention it since the underlying algebra cannot always be guessed from just mentioning the neighbourhoods in $\mathcal{V}$ and U (for instance, if different algebras with the same carrier set A are considered).

(a) *$U \leq_{\text{cov}} \mathcal{V}$ if and only if $\{U\} \leq_{\text{cov}} \mathcal{V}$; especially, the condition $U \leq_{\text{cov}} V$ is equivalent to $U \leq_{\text{cov}} \{V\}$, to $\{U\} \leq_{\text{cov}} \{V\}$ and to $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(V)$.*

(b) *$U \leq_{\text{cov}} \mathcal{V}$ if and only if $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.*

(c) *$\mathcal{U} \leq_{\text{cov}} \mathcal{V}$ if and only if $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$ for every $U \in \mathcal{U}$ if and only if $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.*

Proof: (a) The first item is an immediate consequence of the definition. The last part of the characterisation of $U \leq_{\text{cov}} V$ follows by applying item (b) for $\mathcal{V} = \{V\}$ and keeping in mind Remark 3.2.21.

(b) The proof of this item works by equivalently transforming the definition of $U \leq_{\text{cov}} \mathcal{V}$ stated as in (3.9). So we start with the condition that for all $S, T \in \operatorname{Inv} \mathbf{A}$ of common arity the fact $S{\restriction}_U \neq T{\restriction}_U$ implies $S{\restriction}_V \neq T{\restriction}_V$ for at least one neighbourhood $V \in \mathcal{V}$. This means that for any pair $(S,T) \in (\operatorname{Inv} \mathbf{A})^2$ the assumption $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$ implies the existence of some $V \in \mathcal{V}$ for which $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(V)$, or equivalently that (S,T) belongs to

$$\bigcup \{\operatorname{Sep}_{\mathbf{A}}(V) \mid V \in \mathcal{V}\} = \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}).$$

As this is true for every pair $(S,T) \in (\operatorname{Inv} \mathbf{A})^2$, it is equivalent to the inclusion $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.

(c) The third item directly follows from item (b). Namely, $\mathcal{U} \leq_{\text{cov}} \mathcal{V}$ is equivalent to $U \leq_{\text{cov}} \mathcal{V}$ for every $U \in \mathcal{U}$. Using the previous item, we can rewrite this as $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$ for every $U \in \mathcal{U}$, or equivalently as

$$\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) = \bigcup \{\operatorname{Sep}_{\mathbf{A}}(U) \mid U \in \mathcal{U}\} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}). \qquad \square$$

We see that if a neighbourhood V covers some neighbourhood U, then V is at least as powerful w.r.t. separating invariant relations as U. The analogous statement holds, of course, for collections of neighbourhoods being in covering relation. So if (sets of) neighbourhoods mutually cover each other, then they are equally powerful in the sense above. This is the content of the following corollary.

3.4.5 Corollary. *For any algebra $\mathbf{A}$ the monorelational structures $(\operatorname{Neigh} \mathbf{A}; \leq_{\text{cov}})$ and $(\mathfrak{P}(\operatorname{Neigh} \mathbf{A}); \leq_{\text{cov}})$ are quasiordered sets. The corresponding equivalence relation is covering equivalence $\equiv_{\text{cov}}$, and it is the kernel $\ker \operatorname{Sep}_{\mathbf{A}}$ of the respective mapping associating separation sets.*

Proof: It is a general fact that given sets A and B, a mapping $f\colon A \longrightarrow B$ and a quasiorder $\leq$ on B, the following definition yields a quasiorder on A:

$$\sqsubseteq \; := \left\{ (a_1, a_2) \in A^2 \;\middle|\; f(a_1) \leq f(a_2) \right\}.$$

If $\leq$ is a partial order on B, then the associated equivalence relation $\equiv := \sqsubseteq \cap \sqsubseteq^{-1}$ is the kernel of f. Indeed, for all $a_1, a_2 \in A$ we have

$$\begin{aligned} a_1 \equiv a_2 \iff a_1 \sqsubseteq a_2 \wedge a_2 \sqsubseteq a_1 &\iff f(a_1) \leq f(a_2) \wedge f(a_2) \leq f(a_1) \\ &\iff f(a_1) = f(a_2) \iff (a_1, a_2) \in \ker f. \end{aligned}$$

In our case $(B, \leq)$ is the poset $\left(\mathfrak{P}\left((\operatorname{Inv}\mathbf{A})^2\right), \subseteq\right)$, f equals $\operatorname{Sep}_{\mathbf{A}}$ mapping neighbourhoods or sets of neighbourhoods to their set of separated invariant relations, and the set A is $\operatorname{Neigh}\mathbf{A}$ or $\mathfrak{P}(\operatorname{Neigh}\mathbf{A})$, respectively. By items (a) and (c) of Lemma 3.4.4, the quasiorder that $f = \operatorname{Sep}_{\mathbf{A}}$ induces on A is $\leq_{\text{cov}}$, and therefore, according to Definition 3.4.2, the associated equivalence relation $\equiv_{\text{cov}}$ is the kernel of $\operatorname{Sep}_{\mathbf{A}}$. $\square$

So neighbourhoods U and V or collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods of an algebra are covering equivalent if and only if they separate precisely the same pairs of invariant relations.

Using the two previous results, we can now rewrite Corollary 3.2.23 in terms of the covering relation and covering equivalence:

3.4.6 Corollary. *For any algebra* $\mathbf{A}$ *the following inclusions hold between binary relations on* $\operatorname{Neigh}\mathbf{A}$*: the implications*[14]

(a) $U \subseteq V \implies U \precapprox V \implies U \leq_{\text{cov}} V$ *and*

(b) $U \cong V \implies U \precapprox \cap \succapprox V \implies U \equiv_{\text{cov}} V$

are valid for all $U, V \in \operatorname{Neigh}\mathbf{A}$.

Proof: The first item follows from Corollary 3.2.23(a) and Lemma 3.4.4(a). The second one is implied by Corollary 3.2.12(b), symmetry of the isomorphism relation, Corollary 3.2.23(c) and Corollary 3.4.5. $\square$

We can extend the statement of this corollary to the covering relation on the powerset of $\operatorname{Neigh}\mathbf{A}$ if we introduce the following isomorphism notion for collections of neighbourhoods (see also Definition 3.4.5 of [Beh09] and Definition 2.8(1) of [KL10]). Later it will mainly become important on its own as isomorphism notion for covers of algebras.

3.4.7 Definition. Let $\mathbf{A}$ be an algebra and $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ be two systems of neighbourhoods of $\mathbf{A}$. $\mathcal{U}$ is said to be *isomorphic to* $\mathcal{V}$, written as $\mathcal{U} \cong \mathcal{V}$, if there exists a bijective mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ such that

$$\forall U \in \mathcal{U}: \quad U \cong \varphi(U).$$

Any such bijection φ will be called an *isomorphism* between $\mathcal{U}$ and $\mathcal{V}$. $\diamondsuit$

[14] To avoid confusion we have refrained from writing this as $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq \precapprox \subseteq \leq_{\text{cov}}$ and $\cong \subseteq \precapprox \cap \succapprox \subseteq \equiv_{\text{cov}}$.

Since isomorphism of neighbourhoods (cf. 3.2.1) is an equivalence relation, obviously isomorphism of sets of neighbourhoods is an equivalence relation, as well.

The notion of isomorphism between collections of neighbourhoods is a natural choice and will further be justified by the fact that it fits well into the general picture in Section 3.7. For a homomorphism concept there are many possible choices, and, at present, we see none, which is particularly distinguished. Therefore, we refrain here from defining what a homomorphism should be. However, for an analogy of Corollary 3.4.6, we need to know what embedding of collections of neighbourhoods should mean. Again, there is room to manoeuvre, and hence, we present two possible embedding notions, a weak and a stronger one. Of course, nuances in between are conceivable.

3.4.8 Definition. For an algebra $\mathbf{A}$ and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ we define the following notions.

(i) A mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is called *weak embedding* if it fulfils $U \precapprox \varphi(U)$ for every $U \in \mathcal{U}$. We say that $\mathcal{U}$ *weakly embeds into* $\mathcal{V}$, in symbols $\mathcal{U} \precapprox_{\mathrm{w}} \mathcal{V}$, if there exists a weak embedding of $\mathcal{U}$ into $\mathcal{V}$.

(ii) A mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is called *strong embedding*, or *embedding* for short, *of* $\mathcal{U}$ *into* $\mathcal{V}$ if it is injective and satisfies $U \cong \varphi(U)$ for every $U \in \mathcal{U}$. We say that $\mathcal{U}$ is *(strongly) embedded into* $\mathcal{V}$ and write $\mathcal{U} \precapprox \mathcal{V}$, if there exists a strong embedding of $\mathcal{U}$ into $\mathcal{V}$. ◊

Obviously, a collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ weakly embeds into a set $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ if and only if for every $U \in \mathcal{U}$ there exists some neighbourhood $V \in \mathcal{V}$, namely $V = \varphi(U)$, such that $U \precapprox V$. In Section 3.5 we will state this relationship as $\mathcal{U} \sqsubseteq (\precapprox) \mathcal{V}$. A weak embedding can also be seen as an $\precapprox$-morphisms to be introduced in Definition 3.7.7. Furthermore, it is equally easy to see that every isomorphism of collections of neighbourhoods is a strong embedding, and every strong embedding is also a weak embedding (cf. Corollary 3.2.12(b)).

In complete analogy to Corollary 3.4.6, we now obtain the following.

3.4.9 Lemma. *For any algebra* $\mathbf{A}$ *the following implications hold for every choice of collections* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$*:*

(a) $\mathcal{U} \subseteq \mathcal{V} \implies \mathcal{U} \precapprox \mathcal{V} \implies \mathcal{U} \precapprox_{\mathrm{w}} \mathcal{V} \implies \mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ *and*

(b) $\mathcal{U} \cong \mathcal{V} \implies \mathcal{U} \precapprox \cap \succapprox \mathcal{V} \implies \mathcal{U} \precapprox_{\mathrm{w}} \cap \succapprox_{\mathrm{w}} \mathcal{V} \implies \mathcal{U} \equiv_{\mathrm{cov}} \mathcal{V}$.

Proof: (a) If $\mathcal{U}$ is a subset of $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, then the identical embedding φ of $\mathcal{U}$ into $\mathcal{V}$ shows $\mathcal{U} \precapprox \mathcal{V}$ since $U = \varphi(U)$ certainly implies that φ is injective and that U and $\varphi(U)$ are isomorphic. Every strong embedding is also a weak one as isomorphism of neighbourhoods implies that they are (mutually) embeddable into each other. Thus, the second implication holds. For the third one suppose that $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a weak embedding and that $U \in \mathcal{U}$ is an arbitrary neighbourhood. We know $U \precapprox \varphi(U)$, so $U \leq_{\mathrm{cov}} \varphi(U)$ by Corollary 3.4.6(a). As $\varphi(U)$ belongs to $\mathcal{V}$, we have $\mathrm{Sep}_{\mathbf{A}}(\varphi(U)) \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$ by definition, so $\varphi(U) \leq_{\mathrm{cov}} \mathcal{V}$ (see Lemma 3.4.4(b)). So we have $\{U\} \leq_{\mathrm{cov}} \{\varphi(U)\} \leq_{\mathrm{cov}} \mathcal{V}$, whence we get $U \leq_{\mathrm{cov}} \mathcal{V}$ by transitivity of $\leq_{\mathrm{cov}}$ (cf. Lemma 3.4.4(a) and Corollary 3.4.5). As U was an arbitrary member of $\mathcal{U}$, we have proven $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$.

(b) We noted above that isomorphism of collections of neighbourhoods implies strong embeddability. Now everything else follows from symmetry of the isomorphism relation and the implications of the previous item. □

Next, we continue with a few trivial observations. The first one remarks that the covering concept is not a property of an algebra, but rather only depends on its associated clone.

3.4.10 Remark. The notion of cover depends only on the clone of invariant relations. This means, if $\mathbf{A}$ and $\mathbf{B}$ are algebras on the same carrier set $A = B$ that share the same clone of invariant relations $\mathrm{Inv}\,\mathbf{A} = \mathrm{Inv}\,\mathbf{B}$ (equivalently the same clone of operations $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(\mathbf{B})$), then they have the same covering relation

$$(\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}), \leq_{\mathrm{cov}}) = (\mathfrak{P}(\mathrm{Neigh}\,\mathbf{B}), \leq_{\mathrm{cov}}).$$

Hereby, we obtain especially $\mathrm{Cov}_{\mathbf{A}}(U) = \mathrm{Cov}_{\mathbf{B}}(U)$ for any neighbourhood U in the shared set $\mathrm{Neigh}\,\mathbf{A} = \mathrm{Neigh}\,\mathbf{B}$.

In particular, $\mathbf{A}$ and the corresponding saturated algebra $\mathbf{A}|_A$ possess the same covering relation, whence we have the equality $\mathrm{Cov}_{\mathbf{A}}(U) = \mathrm{Cov}_{\mathbf{A}|_A}(U)$ for every $U \in \mathrm{Neigh}\,\mathbf{A} = \mathrm{Neigh}\,\mathbf{A}|_A$. ◇

Second, we observe that a cover of a neighbourhood automatically covers all its subneighbourhoods.

3.4.11 Lemma. *For an algebra $\mathbf{A}$, a neighbourhood $U \in \mathrm{Neigh}\,\mathbf{A}$ and a collection $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ the following equivalences hold:*

$$U \leq_{\mathrm{cov}} \mathcal{V} \iff \forall W \in \mathrm{Neigh}\,\mathbf{A}|_U : W \leq_{\mathrm{cov}} \mathcal{V} \iff \mathrm{Neigh}\,\mathbf{A}|_U \leq_{\mathrm{cov}} \mathcal{V}.$$

In particular, $\mathcal{V}$ covers $\mathbf{A}$ if and only if it covers all its neighbourhoods.

Proof: The second equivalence holds by definition of the covering relation (see Definition 3.4.2(ii)). The first one follows from the fact that $\mathrm{Neigh}\,\mathbf{A}|_U$ contains precisely all subneighbourhoods of U (see Lemma 3.3.7). Indeed, by Corollary 3.4.6(a) the condition $W \subseteq U$ implies $W \leq_{\mathrm{cov}} U$, i.e. $\{W\} \leq_{\mathrm{cov}} \{U\}$ (see Lemma 3.4.4(a)) for every neighbourhood $W \in \mathrm{Neigh}\,\mathbf{A}$. So if $U \leq_{\mathrm{cov}} \mathcal{V}$, i.e. $\{U\} \leq_{\mathrm{cov}} \mathcal{V}$, then by transitivity (see Corollary 3.4.5), we obtain $\{W\} \leq_{\mathrm{cov}} \mathcal{V}$, i.e. $W \leq_{\mathrm{cov}} \mathcal{V}$ for every $W \in \mathrm{Neigh}\,\mathbf{A}|_U$. The converse implication is trivial since U is one of the neighbourhoods in $\mathrm{Neigh}\,\mathbf{A}|_U$.

The final claim about covers of algebras is just a special case where U is the full neighbourhood A. □

From this lemma we make the simple observation that every set $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ containing the full neighbourhood A covers $\mathbf{A}$ (for instance, by Lemmas 3.4.9(a) and 3.4.4(a)) and hence covers $\operatorname{Neigh} \mathbf{A}$, i.e. every possible neighbourhood of $\mathbf{A}$. Consequently, we have $\mathcal{U} \leq_{\text{cov}} \{A\}$ for any $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$. Thus, covers of $\mathbf{A}$ precisely form one equivalence class with regard to covering equivalence, namely $\operatorname{Cov}(\mathbf{A}) = [\{A\}]_{\equiv_{\text{cov}}}$.

The third lemma tells us that only trivial neighbourhoods are covered by an empty collection of neighbourhoods.

3.4.12 Lemma. *For an algebra $\mathbf{A}$ and a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ the following facts are equivalent:*

(a) U is covered by the empty collection of neighbourhoods.

(b) The separation set $\operatorname{Sep}_{\mathbf{A}}(U)$ is empty.

(c) For every $m \in \mathbb{N}$ and every invariant relation $S \in \operatorname{Inv}^{(m)} \mathbf{A}$ its restriction to U is $S{\restriction_U} = U^m$, i.e. $\operatorname{Inv}^{(m)} \mathbf{A}|_U = \{U^m\}$.

(d) U is a singleton set $\{u\}$ and the nullary constant $c_u^{(0)}$ belongs to $\operatorname{Clo}^{(0)}(\mathbf{A})$, or equivalently to $\operatorname{Term}^{(0)}(\mathbf{A})$ since $\operatorname{Clo}^{(0)}(\mathbf{A}) = \operatorname{Term}^{(0)}(\mathbf{A})$.

(e) U is a singleton set and $\mathbf{A}$ contains at least one nullary constant among its fundamental operations.

Proof: We will show the implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c).

"(a) $\Leftrightarrow$ (b)" By Lemma 3.4.4(b), the assumption that the empty collection covers U is equivalent to $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(\emptyset) = \emptyset$, i.e. to $\operatorname{Sep}_{\mathbf{A}}(U) = \emptyset$.

"(b) $\Leftrightarrow$ (c)" The assumption

$$\emptyset = \operatorname{Sep}_{\mathbf{A}}(U) = \bigcup_{m \in \mathbb{N}} \left\{ (S, T) \in \left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2 \;\middle|\; S{\restriction_U} \neq T{\restriction_U} \right\},$$

is equivalent to the equality $S{\restriction_U} = T{\restriction_U}$ for every pair $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ of any arity $m \in \mathbb{N}$. In particular, for any $S \in \operatorname{Inv}^{(m)} \mathbf{A}$ and $T = A^m$ we get $S{\restriction_U} = A^m{\restriction_U} = U^m$. That is, for any arity $m \in \mathbb{N}$ there exists precisely one m-ary relation in $\operatorname{Inv}^{(m)} \mathbf{A}|_U$, namely U^m. Conversely, we obtain from this condition that $S{\restriction_U} = U^m = T{\restriction_U}$ holds for all $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ and all $m \in \mathbb{N}$.

"(c) $\Rightarrow$ (e)" Using the given fact for $S = \Delta_A$, we get $\Delta_U = U^2 = \nabla_U$, i.e. $u = v$ for any two elements $u, v \in U$. This means that U contains at most one element. If $\mathbf{A}$ did not have any nullary operations, then $S = \emptyset$ would be an invariant (nullary) relation. By assumption, one would then obtain the contradiction $\emptyset = \emptyset{\restriction_U} = S{\restriction_U} = U^0 = \{\emptyset\}$. Hence, $\mathbf{A}$ must have nullary operations, and so be different from the empty algebra, which cannot carry nullary operations by definition. Thus, the neighbourhood U is also non-empty, since empty neighbourhoods can only arise inside the empty algebra. Consequently, U is a singleton set.

"(e) $\Rightarrow$ (d)" If U is a singleton, say $\{u\}$, then it is given as the image $e[A] = \{u\}$ of some idempotent unary operation $e \in \operatorname{Idem} \mathbf{A}$. This implies that e must be the constant unary operation $c_u^{(1)}$ with value u, i.e. $c_u^{(1)} \in \operatorname{Clo}(A)$. By assumption, for some $a \in A$ we have some nullary constant $c_a^{(0)}$ amidst the fundamental operations of $\mathbf{A}$, whence $c_a^{(0)}$ also belongs to $\operatorname{Clo}(\mathbf{A})$. As this set is a clone, it is closed under composition and must contain $c_u^{(0)} = c_u^{(1)} \circ c_a^{(0)}$ in its nullary part $\operatorname{Clo}^{(0)}(\mathbf{A})$.

The alternative statement about having $c_u^{(0)}$ as a nullary term operation is a general fact: for any set of operations $F \subseteq \mathrm{O}_A$, we have the equality

$$\operatorname{Loc}_A^{(0)} F = \operatorname{Loc}_A F^{(0)} = F^{(0)}$$

since the local closure operator works arity-wise and any nullary operation interpolated by a function in $F^{(0)}$ on any finite subset of its domain must already coincide with the interpolant belonging to $F^{(0)}$ (cf. also Corollary 3.20 of [Beh11]). Thus, $\operatorname{Clo}^{(0)}(\mathbf{A}) = \operatorname{Loc}_A^{(0)} \operatorname{Term}(\mathbf{A}) = \operatorname{Term}^{(0)}(\mathbf{A})$.

"(d) $\Rightarrow$ (c)" Suppose that U is of the form $U = \{u\}$ and $c_u^{(0)}$ belongs to $\operatorname{Clo}^{(0)}(\mathbf{A})$. Then this constant has to be preserved by every relation $S \in \operatorname{Inv} \mathbf{A}$, i.e. S has to contain the tuple $(u, \dots, u)$. In other words, we have $U^{\operatorname{ar} S} \subseteq S$ for any $S \in \operatorname{Inv} \mathbf{A}$, equivalently, $S{\restriction}_U = S \cap U^{\operatorname{ar} S} = U^{\operatorname{ar} S}$ □

Putting the result of the previous lemma differently, covers are almost never empty, except for the special case, when the covered neighbourhood is a singleton. This allows us to infer the following corollary about empty algebras. These constitute a rare, pathological case, which, however, may be encountered when forming subalgebras while generating varieties from non-empty algebras.

3.4.13 Corollary. *The unique initial algebra* $\mathbf{A}$ *on the empty carrier set* $A = \emptyset$ *satisfies* $\operatorname{Neigh} \mathbf{A} = \{A\}$ *and* $\operatorname{Cov}(\mathbf{A}) = \{\{A\}\}$.

Proof: Clearly, there is only one unary operation on the empty set, namely the identity, which at the same time forms the only idempotent clone operation of the initial algebra. Thus, $\operatorname{Neigh} \mathbf{A}$ is the singleton set only containing of the image of the identity, which is A. So covers have to be subsets of $\{A\}$, and there are only two of them: first, $\{A\}$, which clearly is a cover, and second, $\emptyset$ which is fails to be a cover by Lemma 3.4.12 as $U = A$ does not contain any element. Hence, $\operatorname{Cov}(\mathbf{A}) = \{\{A\}\}$. □

A simple but important observation about covers is the following: it actually suffices to check the cover condition (3.10) only for such pairs $S, T \in \operatorname{Inv} \mathbf{A}$ of invariant relations where S is a subrelation of T. First, we shall prove a general, but technical lemma about this, then we will state its main case as a corollary.

3.4.14 Lemma. *Let $m \in \mathbb{N}$ be an arity, $\mathbf{A}$ any algebra and $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ collections of neighbourhoods. Furthermore, assume that for every $U \in \mathcal{U}$, we are given a subset $\top_U \subseteq U^m$, which we use to define*

$$\downarrow \top_{\mathcal{U}} := \left\{ (S,T) \in \left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2 \;\middle|\; \forall U \in \mathcal{U} \colon S{\restriction_U}, T{\restriction_U} \subseteq \top_U \right\}.$$

Then the separation condition $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \downarrow \top_{\mathcal{U}} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$ is equivalent to the inclusion $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \downarrow \top_{\mathcal{U}} \cap \subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$, where $\subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}}$ denotes the partial order relation on $\operatorname{Inv}^{(m)} \mathbf{A}$ given by set inclusion.

Proof: The left-to-right implication of the equivalence is trivial and we only need to verify that $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \downarrow \top_{\mathcal{U}} \cap \subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$ implies the more general inclusion $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \downarrow \top_{\mathcal{U}} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$. That is to say, we have to check that implication (3.11) (see below) is correct for all $(S,T) \in \downarrow \top_{\mathcal{U}}$, knowing its validity only for such invariants S and T where $(S,T) \in \downarrow \top_{\mathcal{U}}$ and $S \subseteq T$. We shall benefit from the fact that restriction to neighbourhoods is a $\cap$-homomorphism. Let us consider relations $(S,T) \in \downarrow \top_{\mathcal{U}}$ of arity $m \in \mathbb{N}$ such that $S{\restriction_V} = T{\restriction_V}$ holds for all $V \in \mathcal{V}$. Then $Y := S \cap T$ also belongs to $\operatorname{Inv}^{(m)} \mathbf{A}$ and satisfies $Y \subseteq X$ for any $X \in \{S,T\}$. Since $(S,T) \in \downarrow \top_{\mathcal{U}}$, we have $X{\restriction_U} \subseteq \top_U$ for all $U \in \mathcal{U}$, which implies $Y{\restriction_U} \subseteq X{\restriction_U} \subseteq \top_U$ for all $U \in \mathcal{U}$. Thus, we have $(Y,X) \in \downarrow \top_{\mathcal{U}}$ and $Y \subseteq X$. From our assumption we get $Y{\restriction_V} = (S \cap T){\restriction_V} = S{\restriction_V} \cap T{\restriction_V} = X{\restriction_V}$ for all $V \in \mathcal{V}$ and $X \in \{S,T\}$. Using (3.11) for the invariants $Y \subseteq X$, yields $Y{\restriction_U} = X{\restriction_U}$ for all $U \in \mathcal{U}$ and both $X \in \{S,T\}$. Therefore, we have $S{\restriction_U} = Y{\restriction_U} = T{\restriction_U}$ for every $U \in \mathcal{U}$ as desired. □

The following corollary addresses the special case where for every $U \in \mathcal{U}$ the subset $\top_U \subseteq U^m$ is chosen as the full power U^m.

3.4.15 Corollary. *For every $m \in \mathbb{N}$, an algebra $\mathbf{A}$ and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ the condition $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2 \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2$ is equivalent to the inclusion $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}}$, where $\subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}}$ denotes the partial order relation on $\operatorname{Inv}^{(m)} \mathbf{A}$ given by set inclusion.*

Therefore, $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ is equivalent to $\operatorname{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \subseteq_{\mathrm{R}_A} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \subseteq_{\mathrm{R}_A}$, where $\subseteq_{\mathrm{R}_A}$ denotes set inclusion on R_A. Explicitly, this condition means that the implication

$$(\forall V \in \mathcal{V} \colon \quad S{\restriction_V} = T{\restriction_V}) \implies (\forall U \in \mathcal{U} \colon \quad S{\restriction_U} = T{\restriction_U}) \tag{3.11}$$

holds for all $S, T \in \operatorname{Inv} \mathbf{A}$ such that $S \subseteq T$.

Proof: If we define $\top_U := U^m$ for every $U \in \mathcal{U}$, then the set $\downarrow \top_{\mathcal{U}}$ defined in Lemma 3.4.14 becomes nothing but $\left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2$. As the order relation $\subseteq_{\operatorname{Inv}^{(m)} \mathbf{A}}$ is a subset of $\left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2$, the statement claimed in the first paragraph of the lemma is a direct consequence of Lemma 3.4.14. The second claim can be derived from this in the following way.

According to Lemma 3.4.4(c) the assumption $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ is equivalent to the inclusion $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$, which clearly implies $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \subseteq_{\mathrm{R}_A} \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \subseteq_{\mathrm{R}_A}$, and further $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \subseteq_{\mathrm{Inv}^{(k)}\mathbf{A}} \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \subseteq_{\mathrm{Inv}^{(k)}\mathbf{A}}$ for every $k \in \mathbb{N}$, because the order $\subseteq_{\mathrm{Inv}^{(k)}\mathbf{A}}$ is a subrelation of $\subseteq_{\mathrm{R}_A}$. Now, by what we showed above, we may infer $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) \cap \left(\mathrm{Inv}^{(k)}\mathbf{A}\right)^2 \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V}) \cap \left(\mathrm{Inv}^{(k)}\mathbf{A}\right)^2$ for every $k \in \mathbb{N}$, and this is equivalent to $\mathrm{Sep}_{\mathbf{A}}(\mathcal{U}) \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$. □

It is by design that the relation $(\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}); \leq_{\mathrm{cov}})$ is completely determined by those pairs $(\mathcal{U}, \mathcal{V}) \in (\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}))^2$ where $\mathcal{U}$ is a singleton, i.e. by the relation $\leq_{\mathrm{cov}} \subseteq \mathrm{Neigh}\,\mathbf{A} \times \mathfrak{P}(\mathrm{Neigh}\,\mathbf{A})$. As explained at the beginning of this section, we intend to use the latter relation to determine when we can transfer joint properties from algebras $\mathbf{A}|_V$, $V \in \mathcal{V}$, to a localisation $\mathbf{A}|_U$. Subsequently, we even want to exploit $U \leq_{\mathrm{cov}} \mathcal{V}$ to reconstruct $\mathbf{A}|_U$. The quasiorder $(\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}); \leq_{\mathrm{cov}})$ rather is an auxiliary construct that was introduced to simplify reasoning about the covering relation, using e.g. transitivity arguments. Consequently, in what follows, we will mainly consider the covering relation between neighbourhoods and collections of neighbourhoods.

In Definition 3.4.2(i) it was not excluded that a neighbourhood U is covered by a collection $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ of neighbourhoods that my partially lie outside of U. To study such a situation is an unnatural question for a localisation theory as the truth of $U \leq_{\mathrm{cov}} \mathcal{V}$ depends upon information that cannot be seen from the perspective of the local algebra $\mathbf{A}|_U$. Therefore, in the following we will mainly focus on the constellation where $\mathcal{V} \subseteq \mathfrak{P}(U)$, i.e. $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}|_U$. We suspect that this essentially boils down to examining, when an algebra is covered by a collection of its neighbourhoods. The following lemma, being essentially contained as Lemma 3.4.2 in [Beh09], confirms our expectations.

3.4.16 Lemma. *For an algebra* $\mathbf{A}$*, a neighbourhood* $U \in \mathrm{Neigh}\,\mathbf{A}$ *and a set of neighbourhoods* $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$*, the following three statements are equivalent:*

(a) $\mathcal{V} \in \mathrm{Cov}(\mathbf{A}|_U)$.

(b) $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}|_U$ *and* $\mathcal{V}$ *is a cover of* $\mathbf{A}|_U$.

(c) $\mathcal{V} \subseteq \mathfrak{P}(U)$ *and* $\mathcal{V}$ *covers* U *w.r.t.* $\mathbf{A}$.

Hence, we have

$$\begin{aligned}\mathrm{Cov}(\mathbf{A}|_U) &= \mathrm{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}|_U) = \mathrm{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\mathrm{Neigh}\,\mathbf{A} \cap \mathfrak{P}(U)) \\ &= \mathrm{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\mathfrak{P}(U)).\end{aligned}$$

Proof: By Lemma 3.3.7, one knows that for a set of neighbourhoods $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ the following equivalence is valid

$$\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}|_U \iff \mathcal{V} \subseteq \mathfrak{P}(U).$$

Moreover, by Definition 3.4.2(ii), the set $\mathcal{V}$ is a cover of $\mathbf{A}|_U$ if and only if $\mathcal{V}$ covers the full neighbourhood $U \in \operatorname{Neigh} \mathbf{A}|_U$ w.r.t. $\mathbf{A}|_U$. The definition of $\mathbf{A}|_U$ implies that the relational clone $\operatorname{Inv} \mathbf{A}|_U$ equals $\{S{\restriction}_U \mid S \in \operatorname{Inv} \mathbf{A}\}$ (cf. Remark 3.3.2). Thus, the equivalences

$$\begin{aligned}
&\mathcal{V} \text{ covers } U \text{ w.r.t. } \mathbf{A}|_U \\
&\qquad \iff \forall\, S, T \in \operatorname{Inv} \mathbf{A}\colon\ \ (\forall\, V \in \mathcal{V}\colon\ (S{\restriction}_U){\restriction}_V = (T{\restriction}_U){\restriction}_V) \implies S{\restriction}_U = T{\restriction}_U \\
&\qquad \iff \forall\, S, T \in \operatorname{Inv} \mathbf{A}\colon\ \ (\forall\, V \in \mathcal{V}\colon\ S{\restriction}_V = T{\restriction}_V) \implies S{\restriction}_U = T{\restriction}_U \\
&\qquad \iff \mathcal{V} \text{ covers } U \text{ w.r.t. } \mathbf{A}
\end{aligned}$$

hold. The combination of the two shown equivalences confirms that item (b) is true if and only item (c) holds. Furthermore, item (a) is equivalent to item (b) by definition of the Cov-operator.

For the additional equalities we note that, by Lemma 3.3.7, we have the inclusion $\operatorname{Neigh} \mathbf{A}|_U = \operatorname{Neigh} \mathbf{A} \cap \mathfrak{P}(U) \subseteq \mathfrak{P}(U)$, which implies

$$\operatorname{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\operatorname{Neigh} \mathbf{A}|_U) \subseteq \operatorname{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\mathfrak{P}(U)).$$

From the implications "(a) ⇒ (c)" and "(a) ⇒ (b)" we infer the inclusion

$$\operatorname{Cov}(\mathbf{A}|_U) \subseteq \operatorname{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\operatorname{Neigh} \mathbf{A}|_U),$$

and from "(c) ⇒ (a)", we get $\operatorname{Cov}_{\mathbf{A}}(U) \cap \mathfrak{P}(\mathfrak{P}(U)) \subseteq \operatorname{Cov}_{\mathbf{A}}(\mathbf{A}|_U)$. These three inclusions prove that the three sets occurring in them are all equal to $\operatorname{Cov}(\mathbf{A}|_U)$.□

After having clarified the relationship between covers of a restricted algebra $\mathbf{A}|_U$ and covers of a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ w.r.t. the global algebra $\mathbf{A}$, we can try to further simplify the criterion for covers of local algebras.

For this we recall a few order theoretic facts. Every completely join-irreducible element $y \in L$ of a complete lattice $\mathbf{L} = \langle L; \bigwedge_{\mathbf{L}}, \bigvee_{\mathbf{L}} \rangle$ possesses a unique lower cover, namely

$$x := \bigvee\nolimits_{\mathbf{L}} ({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\} = \bigvee\nolimits_{\mathbf{L}} \{a \in L \mid a < y\}.$$

It is clear that $x \leq y$, so by complete join-irreducibility of y, the join x must be different from y, i.e. $x < y$. Obviously, by definition of x, every element $a \in L$ being strictly smaller than y is bounded above by x, so x is a lower cover of y and uniquely so.

We remark here that the condition of having a precisely one lower cover is even equivalent[15] to complete join-irreducibility, if the underlying complete lattice is

[15]Lower covers of y are precisely the maximal elements in $({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\}$, and since there is exactly one of them, say x, the set $({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\}$ contains maximal elements and is non-empty, in particular. Using the ascending chain condition, or equivalently, co-well-foundedness, one can show that every element of $({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\}$ lies below one maximal element of this set (see Lemma 3.5.5), i.e. below the unique one, namely x. Thus, $x \leq \bigvee_{\mathbf{L}} ({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\} \leq x$, and so $y > x = \bigvee_{\mathbf{L}} ({\downarrow_{\mathbf{L}}}\, y) \setminus \{y\}$, which proves that y is completely join-irreducible.

assumed to fulfil the *ascending chain condition* (see also page 137), the dual of the axiom explained in the next paragraph. However, we do not need this equivalence here.

A partially ordered set $(P;\leq)$ is said to satisfy the *descending chain condition (DCC)* if every countable, monotone decreasing sequence in P eventually stabilises, i.e. if for every $(p_i)_{i\in\mathbb{N}}\in P^{\mathbb{N}}$ with $p_{i+1}\leq p_i$ for all $i\in\mathbb{N}$, there is some $i\in\mathbb{N}$ such that $p_j=p_i$ for every $j\in\mathbb{N}$, $j\geq i$. So DCC says that $(P;\leq)$ contains only finite descending chains, but it does not prescribe a common upper bound on the lengths of these finite chains. Clearly, every finite poset even possesses such an upper bound and hence fulfils DCC.

A poset $(P;\leq)$ is called *well-founded* if every non-empty subposet of $(P;\leq)$ contains a minimal element. One can easily see that every well-founded poset satisfies DCC. Conversely, under the assumption of the axiom of choice, if $(P;\leq)$ fails to be well-founded, one can construct a countably infinite, strictly decreasing sequence of elements in P, violating DCC. Thus, satisfaction of the seemingly weaker descending chain condition and well-foundedness of a poset are equivalent. Since we think that DCC is easier to check than well-foundedness, we shall use the former condition in Definition 3.4.18 and hence in the assumptions of Corollaries 3.4.20 and 3.4.21. In the following lemma, which is the basis for these corollaries, we will, however, exploit a particular instance of well-foundedness.

3.4.17 Lemma. *Let $m\in\mathbb{N}$, $\mathbf{A}$ be an algebra, $U\in\operatorname{Neigh}\mathbf{A}$ one of its neighbourhoods, $\mathcal{V}\subseteq\operatorname{Neigh}\mathbf{A}|_U$ a collection of subneighbourhoods of U and $\top\subseteq U^m$ a fixed subset (not necessarily an invariant relation).*

(a) Suppose that for every pair of relations $S,T\in\operatorname{Inv}^{(m)}\mathbf{A}$ such that $S\subsetneq T$ and $S{\restriction_U}\neq T{\restriction_U}\subseteq\top$, i.e. $S{\restriction_U}\subsetneq T{\restriction_U}\subseteq\top$, the set

$$\left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U}T{\restriction_U}\right)\setminus\left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U}S{\restriction_U}\right)=\left\{R\in\operatorname{Inv}^{(m)}\mathbf{A}|_U\;\middle|\;R\subseteq T{\restriction_U}\text{ and }R\not\subseteq S{\restriction_U}\right\}$$

contains minimal elements. Then implication (3.10) holds for all invariants $S,T\in\operatorname{Inv}^{(m)}\mathbf{A}$ satisfying $S{\restriction_U},T{\restriction_U}\subseteq\top$ if and only if it holds for all relations $S,T\in\operatorname{Inv}^{(m)}\mathbf{A}$ such that $S\subseteq T$, $T{\restriction_U}\subseteq\top$ and $T{\restriction_U}$ is completely join-irreducible in the lattice $\operatorname{Inv}^{(m)}\mathbf{A}|_U=\operatorname{Sub}\left((\mathbf{A}|_U)^m\right)$ having $S{\restriction_U}$ as its unique lower cover. This is equivalent to say that $S{\restriction_V}\neq T{\restriction_V}$ for some $V\in\mathcal{V}$ for all aforementioned relations S and T.

(b) If the set $\left(\downarrow_{\operatorname{Inv}^{(k)}\mathbf{A}|_U}T{\restriction_U}\right)\setminus\left(\downarrow_{\operatorname{Inv}^{(k)}\mathbf{A}|_U}S{\restriction_U}\right)$ contains minimal elements for all $k\in\mathbb{N}$ and all $S,T\in\operatorname{Inv}^{(k)}\mathbf{A}$ satisfying $S\subsetneq T$, $S{\restriction_U}\subsetneq T{\restriction_U}$, then the condition $\mathcal{V}\in\operatorname{Cov}(\mathbf{A}|_U)$ is equivalent to the following assertion: for every $k\in\mathbb{N}$ and all $S,T\in\operatorname{Inv}^{(k)}\mathbf{A}$ such that $S\subseteq T$ and $T{\restriction_U}$ is completely join-irreducible in $\operatorname{Inv}^{(k)}\mathbf{A}|_U$ having $S{\restriction_U}$ as its unique lower cover, there is some $V\in\mathcal{V}$ such that $S{\restriction_V}\neq T{\restriction_V}$.

Proof: (a) It is evident that the stated condition is necessary, because if implication (3.10) holds of any pair of invariant relations $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ such that $S\restriction_U, T\restriction_U \subseteq \top$, it must especially be true for all invariant relations $S \subseteq T$ where $S\restriction_U \prec T\restriction_U$ form a covering pair in the lattice $\operatorname{Inv}^{(m)}\mathbf{A}|_U$ and $T\restriction_U \subseteq \top$. For such pairs we have, of course, $S\restriction_U \neq T\restriction_U$, whereby the validity of (3.10) is equivalent to non-satisfaction of its premise, i.e. to $S\restriction_V \neq T\restriction_V$ for some $V \in \mathcal{V}$.

So the main task of the proof lies in demonstrating sufficiency of the stated condition. We shall tackle this by transforming the question into a local problem for the algebra $\mathbf{A}|_U$, whereby we can then consider without loss of generality the special case $U = A$. It is clear that any two invariant relations $\tilde{S}, \tilde{T} \in \operatorname{Inv}^{(m)}\mathbf{A}|_U$ of arity $m \in \mathbb{N}$ are induced as restrictions of some global invariants $S', T \in \operatorname{Inv}^{(m)}\mathbf{A}$ (cf. Remark 3.3.2). If $\tilde{S} \subsetneq \tilde{T} \subseteq \top$, then we may consider $S := S' \cap T \in \operatorname{Inv}^{(m)}\mathbf{A}$ instead of S' and exploit the homomorphism property of $\restriction_U$ w.r.t. $\cap$, to get $S\restriction_U = (S' \cap T)\restriction_U = S'\restriction_U \cap T\restriction_U = \tilde{S} \cap \tilde{T} = \tilde{S}$, $T\restriction_U = \tilde{T} \subseteq \top$ and $S \subsetneq T$. By assumption of the lemma, we have minimal elements in $\left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U} \tilde{T}\right) \setminus \left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U} \tilde{S}\right)$. Thus, the restricted algebra $\mathbf{A}|_U$ fulfils the assumption of the lemma for its full neighbourhood U. Furthermore, the condition on invariants of $\mathbf{A}$ whose restrictions are covering pairs in $\operatorname{Inv}^{(m)}\mathbf{A}|_U$ such that the larger restriction is completely join-irreducible and contained in $\top$ means precisely that any covering pair of invariant relations of $\mathbf{A}|_U$ whose upper cover is completely join-irreducible and a subset of $\top$ can be distinguished in $\mathcal{V}$. So if we knew sufficiency of the stated separation property in the case of full algebras, we could infer that all m-ary invariants of $\mathbf{A}|_U$ contained in $\top$ could be separated in $\mathcal{V}$, i.e. (3.10) would hold for all $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ with $S\restriction_U, T\restriction_U \subseteq \top$.

Hence, let us now consider an algebra $\mathbf{A}$ with a subset $\top \subseteq A^m$, where

$$\downarrow T \setminus \downarrow S := \left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}} T\right) \setminus \left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}} S\right)$$

contains minimal elements w.r.t. inclusion for any $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, satisfying $S \subsetneq T \subseteq \top$. Let us suppose, furthermore, that $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ is able to distinguish every covering pair in $\operatorname{Inv}^{(m)}\mathbf{A}$, where the upper cover is completely join-irreducible and a subset of $\top$ (condition (3.10) for $U = A$ and only special relations). We need to prove that $\mathcal{V}$ separates any pair of non-identical m-ary invariants of $\mathbf{A}$ contained in $\top$. Letting $\mathcal{U} = \{A\}$ and $\top_A := \top$ in Lemma 3.4.14, we know that we only need to separate such pairs of invariants $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ in $\mathcal{V}$, where $S \subsetneq T \subseteq \top$ (cf. the contrapositive of condition (3.11)). To do this, we pick some minimal[16] element $R \in \downarrow T \setminus \downarrow S$. Evidently, $R \cap S \in \operatorname{Inv}^{(m)}\mathbf{A}$ and $R \cap S \subsetneq R$ because $R \nsubseteq S$. Moreover, we have $R \cap S \prec R$, because for every $R' \in \operatorname{Inv}^{(m)}\mathbf{A}$ satisfying $R \cap S \subseteq R' \subseteq R$ one of the following cases is true: either $R' \subseteq S$, whence $R' \subseteq R \cap S$ and so $R' = R \cap S$. Otherwise, $R' \nsubseteq S$

[16]The author is grateful to Martin Schneider for hinting at using minimality to obtain this elegant formulation of the proof.

and $R' \subseteq R \subseteq T$, which, by minimality of R, yields $R' = R$. Furthermore, we have $R \subseteq T \subseteq \top$.

We will show now that R is completely join-irreducible with (unique) lower cover $R \cap S$. Consider the set $Q' := \downarrow R \setminus \{R\} = \left\{ \varrho \in \mathrm{Inv}^{(m)}\mathbf{A} \;\middle|\; \varrho \subsetneq R \right\}$. By minimality of R, every member of Q' must satisfy $\varrho \subseteq S$, and it trivially satisfies $\varrho \subseteq R$, i.e. $\varrho \subseteq R \cap S$. Hence, $\bigvee_{\mathrm{Inv}^{(m)}\mathbf{A}} Q' \subseteq R \cap S$. As $R \cap S \in Q'$, we obtain

$$R \cap S \subseteq \bigvee\nolimits_{\mathrm{Inv}^{(m)}\mathbf{A}} Q' \subseteq R \cap S,$$

and consequently, $\bigvee_{\mathrm{Inv}^{(m)}\mathbf{A}} Q' = R \cap S \prec R$.

By assumption, there is some $V \in \mathcal{V}$ such that $(R \cap S){\restriction_V} \subsetneq R{\restriction_V}$. If $S{\restriction_V} = T{\restriction_V}$ held, then $R \subseteq T$ would imply $R{\restriction_V} \subseteq T{\restriction_V} = S{\restriction_V}$, whereby we had the equality $(R \cap S){\restriction_V} = R{\restriction_V} \cap S{\restriction_V} = R{\restriction_V}$, contradicting what was inferred above for V. Therefore, we must have $S{\restriction_V} \neq T{\restriction_V}$, finishing the proof of the first item.

(b) If the assumption of the existence of minimal elements in the sets of relations mentioned in the lemma is true for all arities $k \in \mathbb{N}$ simultaneously (for $\top := U^m$), then the equivalence proven in item (a) also holds for every arity $k \in \mathbb{N}$. Separating all pairs of relations $\mathrm{Inv}^{(k)}\mathbf{A}$ for any $k \in \mathbb{N}$, by Definition 3.4.2, means precisely that $\mathcal{V}$ covers the neighbourhood U. According to Lemma 3.4.16, this is equivalent to $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A}|_U)$ since $\mathcal{V}$ was assumed to be a subcollection of $\mathrm{Neigh}\,\mathbf{A}|_U \subseteq \mathfrak{P}\,(U)$. $\square$

Of course, the most natural choice for the subset $\top \subseteq U^m$ in the assumptions of the previous lemma is $\top = U^m$, which means that the condition and hence the equivalence in item (a) is true for all m-ary invariant relations. However, we shall see an application of this lemma for some value of $\top$ that is different from U^m in a corollary to the main result of this section.

Furthermore, one can clearly imagine that a more general version of item (b) is possible, too, where one has such a top relation $\top_k \subseteq U^k$ for every arity $k \in \mathbb{N}$ and only requires the minimality condition for $T \in \mathrm{Inv}^{(k)}\mathbf{A}$ where $T{\restriction_U} \subseteq \top_k$. The claim would then reduce the separation of all k-ary invariant relations S, T with $S{\restriction_U}, T{\restriction_U} \subseteq \top_k$ for any $k \in \mathbb{N}$ to the separation of completely join-irreducible restrictions contained in $\top_k$ from their lower covers. Yet, we do not see that this is very likely to be applied, which is why we have suppressed this result here.

Certainly, the technical assumption, occurring in Lemma 3.4.17, that inclusion minimal relations in the set-theoretical difference of certain principal downsets generated by invariant relations have to exist, will follow from the easier condition that the lattice $\mathrm{Inv}^{(m)}\mathbf{A}|_U$ is well-founded, i.e. satisfies DCC, for any arity $m \in \mathbb{N}$.

For modules the property that the poset of submodules under inclusion, i.e. the unary invariant relations, fulfils DCC has been named ARTIN*ian*. Likewise, but less frequently, a group is called ARTIN*ian*, if its lattice of subgroups forms a well-founded poset. Furthermore, a ring is said to be *right (left)* ARTIN*ian* if the associated poset of right (left) ideals under inclusion is well-founded, a condition which is equivalent to forming a right (left) module over itself.

In view of these notions we would have liked to call a universal algebra *strongly* Artin*ian* if the poset of m-ary invariant relations under inclusion was well-founded for every $m \in \mathbb{N}$. However, the term *strongly* Artin*ian* has already been used for rings to mean that its additive subgroup is Artinian in the sense above (cf. [KW70, §3. Streng artinsche Ringe, p. 9] or [VH79, 2. Preliminaries, p. 546]). In order to avoid colliding with well-established notions, we therefore propose the following terminology:

3.4.18 Definition. Let $m \in \mathbb{N}$ be a natural number. An algebra $\mathbf{A}$ should be called m-Artin*ian*, or Artin*ian of degree* m, if the poset $\left(\operatorname{Inv}^{(m)}\mathbf{A}, \subseteq\right)$ of its m-ary invariant relations satisfies DCC. It should be called *poly*-Artin*ian* if it is Artinian of every degree $k \in \mathbb{N}$.

Furthermore, we say that a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ is m-Artinian or poly-Artinian if the restricted algebra $\mathbf{A}|_U$ has the respective property. $\Diamond$

In our terminology an Artinian group or module would be 1-Artinian. We remark, however, that any right (left) Artinian ring (with unit) is already right (left) poly-Artinian. This follows because any finitely generated right (left) module over an Artinian ring $\mathbf{R}$ is an Artinian right (left) $\mathbf{R}$-module (cf. e.g. [Lam91, Proposition 1.21, p. 21]). Using this for the finitely generated right (left) $\mathbf{R}$-module $\mathbf{R}^n$, $n \in \mathbb{N}$, we have that it is Artinian as a right (left) $\mathbf{R}$-module, and hence as a right (left) $\mathbf{R}^n$-module (whenever $N \subseteq M$ are right (left) $\mathbf{R}^n$-submodules, then they are right (left) $\mathbf{R}$-submodules, as well). Thus, for every $n \in \mathbb{N}$, the direct power $\mathbf{R}^n$ is an Artinian right (left) module over itself, i.e. an Artinian ring.

Clearly, every finite algebra $\mathbf{A}$ is poly-Artinian since the lattices $\operatorname{Inv}^{(m)}\mathbf{A}$ are all finite for every $m \in \mathbb{N}$. In particular, all finite neighbourhoods of an algebra are thus poly-Artinian since they correspond to finite restricted algebras.

It is also easy to see that an m-Artinian algebra is also k-Artinian for every $k \in \mathbb{N}$, $k \leq m$. Indeed, for all $S, T \in \operatorname{Inv}^{(k)}\mathbf{A}$ such that $S \subseteq T$, we also have $S \times A^{m-k} \subseteq T \times A^{m-k}$ and $S \times A^{m-k}, T \times A^{m-k} \in \operatorname{Inv}^{(m)}\mathbf{A}$. So every descending chain $(S_i)_{i\in\mathbb{N}} \in \left(\operatorname{Inv}^{(k)}\mathbf{A}^{\mathbb{N}}\right)$ yields a descending chain

$$\left(S_i \times A^{m-k}\right)_{i\in\mathbb{N}} \in \left(\operatorname{Inv}^{(m)}\mathbf{A}\right)^{\mathbb{N}},$$

which must become stationary at some $j \in \mathbb{N}$ for $\mathbf{A}$ was supposed to be Artinian of degree m. Thus, we have $S_i \times A^{m-k} = S_j \times A^{m-k}$ for every $i \in \mathbb{N}$, $i \geq j$, implying for such $i \geq j$ that $S_i = \operatorname{pr}_{0,\dots,k-1}\left(S_i \times A^{k-m}\right) = \operatorname{pr}_{0,\dots,k-1}\left(S_j \times A^{k-m}\right) = S_j$ if $A \neq \emptyset$. Since every finite algebra, especially the empty one, is poly-Artinian, we may assume this condition without loss of generality. This shows that also $(S_i)_{i\in\mathbb{N}}$ becomes eventually constant, such that $\mathbf{A}$ is also k-Artinian.

Lemma 3.4.17 identifies a certain type of pairs of invariant relations whose separation suffices to conclude the covering condition. For this reason they will play an important role later on and therefore, are named *crucial pairs*.

3.4.19 Definition. Let $\mathbf{A}$ be an algebra and $m \in \mathbb{N}$ any arity. A pair (S,T) is called *m-crucial pair of* $\mathbf{A}$ if $S,T \in \operatorname{Inv}^{(m)}\mathbf{A}$, T is completely join-irreducible in the lattice $\operatorname{Inv}^{(m)}\mathbf{A}$ and $S \prec T$ is its unique lower cover[17]. We denote by

$$\operatorname{Cruc}^{(m)}(\mathbf{A}) := \left\{ (S,T) \in \left(\operatorname{Inv}^{(m)}\mathbf{A}\right)^2 \;\middle|\; (S,T) \text{ is an } m\text{-crucial pair of } \mathbf{A} \right\}$$

the *set of all m-crucial pairs of* $\mathbf{A}$.

We say that a pair (S,T) is a *crucial pair of* $\mathbf{A}$ if it is an ℓ-crucial pair for some $\ell \in \mathbb{N}$. The *set of all crucial pairs of* $\mathbf{A}$ is written as

$$\operatorname{Cruc}(\mathbf{A}) = \bigcup_{\ell \in \mathbb{N}} \operatorname{Cruc}^{(\ell)}(\mathbf{A}). \qquad \Diamond$$

Using the newly introduced terminology, we can now formulate the following corollaries to Lemma 3.4.17. We also want to mention that the fact in 3.4.20(b) provides a generalisation of Lemma 3.4.3(c) of [Beh09] from finite to poly-Artinian algebras.

3.4.20 Corollary. *Let $m \in \mathbb{N}$, $\mathbf{A}$ be an algebra, $U \in \operatorname{Neigh}\mathbf{A}$ one of its neighbourhoods, $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}|_U$ a collection of subneighbourhoods of U and $\top \subseteq U^m$ a fixed subset.*

(a) If the neighbourhood U is m-Artinian, then implication (3.10) holds for all $S,T \in \operatorname{Inv}^{(m)}\mathbf{A}$ satisfying $S{\restriction_U}, T{\restriction_U} \subseteq \top$ if and only if it holds for those invariant relations $S \subseteq T$ for which $(S{\restriction_U}, T{\restriction_U})$ forms an m-crucial pair of $\mathbf{A}|_U$ and $T{\restriction_U} \subseteq \top$. Equivalently, for such relations we have $S{\restriction_V} \subsetneq T{\restriction_V}$ for at least one $V \in \mathcal{V}$.

(b) If U is poly-Artinian, then the condition $\mathcal{V} \in \operatorname{Cov}(\mathbf{A}|_U)$ is equivalent to the following assertion: for all $S,T \in \operatorname{Inv}\mathbf{A}$ such that $S \subseteq T$ and $(S{\restriction_U}, T{\restriction_U})$ belongs to $\operatorname{Cruc}(\mathbf{A}|_U)$ there is at least one $V \in \mathcal{V}$ such that $S{\restriction_V} \subsetneq T{\restriction_V}$.

Proof: For U the condition of being m-Artinian means that $\operatorname{Inv}^{(m)}\mathbf{A}|_U$ satisfies DCC, and it has been argued above that this is equivalent to well-foundedness of this poset. So any of its non-empty subsets contains inclusion minimal relations. In particular, if $S{\restriction_U} \subsetneq T{\restriction_U}$, then $T{\restriction_U} \in \left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U} T{\restriction_U}\right) \setminus \left(\downarrow_{\operatorname{Inv}^{(m)}\mathbf{A}|_U} S{\restriction_U}\right)$, so the minimality conditions among the assumptions of Lemma 3.4.17 are fulfilled. The remainder of Corollary 3.4.20 can be obtained by rewording the statements of Lemma 3.4.17 using the concept of crucial pairs. $\square$

As a further corollary we wish to point out, particularly, the special case of full algebras (see also Corollary 3.4.4 of [Beh09] for the special case of finite algebras w.r.t. item (c)):

3.4.21 Corollary. *Let $m \in \mathbb{N}$, $\mathbf{A}$ be an algebra, $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ a collection of neighbourhoods of $\mathbf{A}$ and $\top \subseteq A^m$ any subset.*

[17] Such pairs of relations are special instances of so-called *prime quotients* (cf. [HM88, p. 28]).

(a) *If* $\mathbf{A}$ *is* m-ARTIN*ian, then* $\mathcal{V}$ *separates every non-identical pair of* m*-ary invariants* $S, T \subseteq \top$ *if and only if it separates all* m*-crucial pairs* $S \prec T$ *of* $\mathbf{A}$ *where* $T \subseteq \top$.

(b) *If* $\mathbf{A}$ *is* m-ARTIN*ian, then* $\mathcal{V}$ *separates every non-identical pair of* m*-ary invariants if and only if it separates all* m*-crucial pairs of* $\mathbf{A}$.

(c) *If* $\mathbf{A}$ *is poly-*ARTIN*ian, then the condition* $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *is equivalent to the inclusion* $\operatorname{Cruc}(\mathbf{A}) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.

Proof: The claim immediately follows from Corollary 3.4.20 by letting $U = A$ and taking into account the definitions of crucial pair and separation set. Item (b) is the special case of item (a) for $\top := A^m$. $\square$

As mentioned briefly in the introduction to this section, it is one of its main goals to characterise covers of finite algebras using a product-retract construction between the associated relational structures. One part of this is a condition which is sufficient for covering also in the case of infinite algebras: whenever we have a retraction from any product of restricted relational duals $\underset{\sim}{\mathbf{A}}\restriction_V$, $V \in \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, to $\underset{\sim}{\mathbf{A}}\restriction_U$, then $\mathcal{V}$ covers a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$. To see this we first present a few simple facts about retracts and products of relational structures in general.

3.4.22 Lemma. *Let* I *be an index set and* $\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}$ *and* $\underset{\sim}{\mathbf{B}}_i$ *for* $i \in I$ *be relational structures of common signature* Σ*. Then the following assertions are true:*

(a) *A relational structure and a mapping which is a retraction completely determine the retract: if* $\Lambda\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ *is a retraction, then* $\underset{\sim}{\mathbf{B}}$ *and* Λ *fully determine* $\underset{\sim}{\mathbf{A}} = \left\langle \Lambda[B];\, \left(\Lambda \circ \left[\underset{\sim}{\varrho}^{\mathbf{B}}\right]\right)_{\varrho \in \Sigma} \right\rangle$*. In particular, for any* $\varrho, \sigma \in \Sigma$ *we have*

$$\underset{\sim}{\varrho}^{\mathbf{B}} = \underset{\sim}{\sigma}^{\mathbf{B}} \implies \underset{\sim}{\varrho}^{\mathbf{A}} = \underset{\sim}{\sigma}^{\mathbf{A}}.$$

(b) *The factors of a product fully determine their product: if* $\underset{\sim}{\mathbf{B}} = \prod_{i \in I} \underset{\sim}{\mathbf{B}}_i$*, then* $\underset{\sim}{\mathbf{B}} = \left\langle \prod_{i \in I} B_i;\, \left(\coprod_{i \in I} \underset{\sim}{\varrho}^{\mathbf{B}_i}\right)_{\varrho \in \Sigma} \right\rangle$*. In particular, for any* $\varrho, \sigma \in \Sigma$ *we have*

$$\left(\underset{\sim}{\varrho}^{\mathbf{B}_i}\right)_{i \in I} = \left(\underset{\sim}{\sigma}^{\mathbf{B}_i}\right)_{i \in I} \implies \underset{\sim}{\varrho}^{\mathbf{B}} = \underset{\sim}{\sigma}^{\mathbf{B}}.$$

Furthermore, for $\varrho \in \Sigma$*, the conditions* $\underset{\sim}{\varrho}^{\mathbf{B}} \neq \emptyset$*,* $\prod_{i \in I} \underset{\sim}{\varrho}_i^{\mathbf{B}_i} \neq \emptyset$ *and* $\underset{\sim}{\varrho}_i^{\mathbf{B}_i} \neq \emptyset$ *for each index* $i \in I$ *are equivalent. If they are fulfilled, then the displayed implication is actually an equivalence.*

(c) *Suppose that for every* $i \in I$ *there is a retraction* $e_i\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}_i$ *and* $\underset{\sim}{\mathbf{A}}$ *is a retract of* $\prod_{i \in I} \underset{\sim}{\mathbf{B}}_i$ *via some retraction* $\Lambda\colon \prod_{i \in I} \underset{\sim}{\mathbf{B}}_i \longrightarrow \underset{\sim}{\mathbf{A}}$*, then via these retractions* $\underset{\sim}{\mathbf{A}}$ *and* $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i \in I}$ *determine each other uniquely. That is, given* $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i \in I}$ *and the retraction* Λ*, the structure* $\underset{\sim}{\mathbf{A}}$ *is uniquely determined as*

$$\underset{\sim}{\mathbf{A}} = \left\langle \Lambda\left[\prod_{i \in I} B_i\right];\, \left(\Lambda \circ \left[\coprod_{i \in I} \underset{\sim}{\varrho}^{\mathbf{B}_i}\right]\right)_{\varrho \in \Sigma} \right\rangle,$$

and $\left(\varrho^{\mathbf{B}_i}\right)_{i\in I} = \left(\sigma^{\mathbf{B}_i}\right)_{i\in I}$ *implies* $\varrho^{\mathbf{A}} = \sigma^{\mathbf{A}}$ *for every* $\varrho, \sigma \in \Sigma$.

Conversely, if $\underset{\sim}{\mathbf{A}}$ *and the retractions* $(e_i)_{i\in I}$ *are known, then the structures* $\left(\underset{\sim}{\mathbf{B}}_{i\in I}\right)$ *are uniquely determined as* $\underset{\sim}{\mathbf{B}}_i = \left\langle e_i\,[A]\,;\,\left(e_i \circ \left[\varrho^{\mathbf{A}}\right]\right)_{\varrho\in\Sigma}\right\rangle$*, such that* $\varrho^{\mathbf{A}} = \sigma^{\mathbf{A}}$ *implies* $\varrho^{\mathbf{B}_i} = \sigma^{\mathbf{B}_i}$ *for any* $i \in I$ *and all* $\varrho, \sigma \in \Sigma$*. Thus, we have*

$$\left(\varrho^{\mathbf{B}_i}\right)_{i\in I} = \left(\sigma^{\mathbf{B}_i}\right)_{i\in I} \iff \varrho^{\mathbf{A}} = \sigma^{\mathbf{A}}$$

for all $\varrho, \sigma \in \Sigma$.

Proof: (a) Let us denote the co-retraction corresponding to Λ by $M\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$, i.e. $\Lambda \circ M = \mathrm{id}_{\underset{\sim}{\mathbf{A}}}$. Since both mappings are homomorphisms between relational structures, for any $\varrho \in \Sigma$ we have $\Lambda \circ \left[\varrho^{\mathbf{B}}\right] \subseteq \varrho^{\mathbf{A}}$ and $M \circ \left[\varrho^{\mathbf{A}}\right] \subseteq \varrho^{\mathbf{B}}$. Sticking these inclusions together and using monotonicity of the action of Λ on relations on B, we obtain

$$\varrho^{\mathbf{A}} = \mathrm{id}_{\underset{\sim}{\mathbf{A}}} \circ \left[\varrho^{\mathbf{A}}\right] = \Lambda \circ M \circ \left[\varrho^{\mathbf{A}}\right] = \Lambda \circ \left[M \circ \left[\varrho^{\mathbf{A}}\right]\right] \subseteq \Lambda \circ \left[\varrho^{\mathbf{B}}\right] \subseteq \varrho^{\mathbf{A}},$$

hence $\Lambda \circ \left[\varrho^{\mathbf{B}}\right] = \varrho^{\mathbf{A}}$ for every $\varrho \in \Sigma$. Moreover, as Λ must be surjective, we have $\Lambda\,[B] = A$, such that $\underset{\sim}{\mathbf{A}}$ indeed looks as claimed.

(b) In Section 2 we had actually defined the interpretation of every relational symbol $\varrho \in \Sigma$ in the product structure $\underset{\sim}{\mathbf{B}}$ as $\varrho^{\mathbf{B}} = \prod\!\!\!\prod_{i\in I} \varrho^{\mathbf{B}_i}$; further, the carrier of $\underset{\sim}{\mathbf{B}}$ indeed is the Cartesian product $\prod_{i\in I} B_i$ of the carriers of its factors. Thus, the first claim and the stated implication follow directly by definition.

For the additional remark that the function $\prod\!\!\!\prod_{i\in I}$ is injective on non-empty relations, we relate $\prod\!\!\!\prod_{i\in I} \varrho^{\mathbf{B}_i}$ to the Cartesian product $\prod_{i\in I} \varrho^{\mathbf{B}_i}$. For this we quickly observe that for any $m \in \mathbb{N}$ the operation

$$\begin{array}{rccc} {}^{\top}\colon & \prod_{i\in I} (B_i)^m & \longrightarrow & \left(\prod_{i\in I} B_i\right)^m \\ & \mathbf{x} = \begin{pmatrix} x_1(i) \\ \vdots \\ x_m(i) \end{pmatrix}_{i\in I} & \longmapsto & \mathbf{x}^{\top} := \begin{pmatrix} (x_1(i))_{i\in I} \\ \vdots \\ (x_m(i))_{i\in I} \end{pmatrix} \end{array}$$

is a bijection. For any m-ary symbol $\varrho \in \Sigma$, it may be applied to the subset $\prod_{i\in I} \varrho^{\mathbf{B}_i} \subseteq \prod_{i\in I} (B_i)^m$ having as image

$$\left[\prod_{i\in I} \varrho^{\mathbf{B}_i}\right]^{\top} = \left\{ \mathbf{x}^{\top} \;\middle|\; \mathbf{x} \in \prod_{i\in I} \varrho^{\mathbf{B}_i} \right\} = \prod\!\!\!\prod_{i\in I} \varrho^{\mathbf{B}_i}.$$

Since ${}^{\top}$ is bijective, injectivity of $\prod\!\!\!\prod_{i\in I}$ depends on injectivity of the Cartesian product. It is clear, that for any $j \in I$ one can reconstruct $\varrho^{\mathbf{B}_j}$ from $\prod_{i\in I} \varrho^{\mathbf{B}_i}$ via projection if all factors $\varrho^{\mathbf{B}_i}$, $i \in I$ are non-empty sets, namely $\varrho^{\mathbf{B}_j} = \mathrm{pr}_j \left[\prod_{i\in I} \varrho^{\mathbf{B}_i}\right]$. Thus, $\prod_{i\in I}$ is injective for non-empty relations. Moreover, assuming the axiom of choice, the condition $\varrho^{\mathbf{B}_i} \neq \emptyset$ for all $i \in I$ is equivalent to $\prod_{i\in I} \varrho^{\mathbf{B}_i} \neq \emptyset$, which is equivalent to $\prod\!\!\!\prod_{i\in I} \varrho^{\mathbf{B}_i} \neq \emptyset$ since these products are related via being the image of an operation.

(c) This statement is a combination of the two previous items. If $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ and the retraction $\Lambda\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ from the product $\underset{\sim}{\mathbf{B}} := \prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$ is given, then we have

$$\underset{\sim}{\mathbf{A}} \overset{(a)}{=} \left\langle \Lambda\left[B\right];\ \left(\Lambda\circ\left[\underset{\sim}{\varrho}^{\mathbf{B}}\right]\right)_{\varrho\in\Sigma}\right\rangle \overset{(b)}{=} \left\langle \Lambda\left[\prod_{i\in I} B_i\right];\ \left(\Lambda\circ\left[\coprod_{i\in I} \underset{\sim}{\varrho}^{\mathbf{B}_i}\right]\right)_{\varrho\in\Sigma}\right\rangle.$$

If $\underset{\sim}{\mathbf{A}}$ and the retractions $e_i\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}_i$ are given, then by item (a) each structure $\underset{\sim}{\mathbf{B}}_i$, $i\in I$, can be written as $\left\langle e_i\left[A\right];\ \left(e_i\circ\left[\underset{\sim}{\varrho}^{\mathbf{A}}\right]\right)_{\varrho\in\Sigma}\right\rangle$. Furthermore, for relational symbols $\varrho,\sigma\in\Sigma$, item (a) via the retraction e_i yields $\underset{\sim}{\varrho}^{\mathbf{B}_i} = \underset{\sim}{\sigma}^{\mathbf{B}_i}$ for every $i\in I$ if $\underset{\sim}{\varrho}^{\mathbf{A}} = \underset{\sim}{\sigma}^{\mathbf{A}}$ holds. So the latter equality implies $\left(\underset{\sim}{\varrho}^{\mathbf{B}_i}\right)_{i\in I} = \left(\underset{\sim}{\sigma}^{\mathbf{B}_i}\right)_{i\in I}$. Conversely, if this is true, then item (b) implies $\underset{\sim}{\varrho}^{\mathbf{B}} = \underset{\sim}{\sigma}^{\mathbf{B}}$ in the product $\underset{\sim}{\mathbf{B}} = \prod_{i\in I}\underset{\sim}{\mathbf{B}}_i$, whence we can infer $\underset{\sim}{\varrho}^{\mathbf{A}} = \underset{\sim}{\sigma}^{\mathbf{A}}$ from item (a) applied to the retraction Λ. □

The following corollary shows how we can use the structural connections established in the previous lemma to infer the cover relation.

3.4.23 Corollary. *Let I be any index set, $\mathbf{A}$ an algebra and $U \in \operatorname{Neigh}\mathbf{A}$ one of its neighbourhoods. Furthermore, assume that $V\colon I \longrightarrow \mathcal{V}$ is a mapping in a collection $\mathcal{V}\subseteq \operatorname{Neigh}\mathbf{A}$. If $\underset{\sim}{\mathbf{A}}\restriction_U$ is a retract of $\prod_{i\in I}\underset{\sim}{\mathbf{A}}\restriction_{V(i)}$, then for any $m\in\mathbb{N}$ and every pair $S,T\in\operatorname{Inv}^{(m)}\mathbf{A}$ the following implication holds:*

$$\left(\forall i\in I\colon S\restriction_{V(i)} = T\restriction_{V(i)}\right) \implies S\restriction_U = T\restriction_U.$$

Especially, it follows that $\mathcal{V}\in\operatorname{Cov}_{\mathbf{A}}(U)$. Moreover, if $\mathcal{V}\subseteq\operatorname{Neigh}\mathbf{A}|_U$ is a collection of subneighbourhoods of U, then the displayed implication is actually an equivalence. This is true in particular if $U = A$, i.e. if one is looking for covers of the whole algebra $\mathbf{A}$.

Proof: We have proven in the first part of item (c) of Lemma 3.4.22 that equality of relations in $\underset{\sim}{\mathbf{A}}\restriction_U$ is determined by equality of the corresponding relations in every factor $\underset{\sim}{\mathbf{A}}\restriction_{V(i)}$, $i\in I$. This shows the validity of the displayed implication. Furthermore, we have $\mathcal{V}\in\operatorname{Cov}_{\mathbf{A}}(U)$ since for $S,T\in\operatorname{Inv}^{(m)}\mathbf{A}$, $m\in\mathbb{N}$, the assumption $S\restriction_{\tilde{V}} = T\restriction_{\tilde{V}}$ for all $\tilde{V}\in\mathcal{V}$ implies the truth of the left-hand side of this implication, due to V mapping to $\mathcal{V}$.

Additionally, if we suppose $\mathcal{V}\subseteq\operatorname{Neigh}\mathbf{A}|_U$, then for each $\tilde{V}\in\mathcal{V}\overset{3.3.7}{\subseteq}\mathfrak{P}(U)$, the relational structure $\underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}} = (\underset{\sim}{\mathbf{A}}\restriction_U)\restriction_{\tilde{V}}$ is a retract of $\underset{\sim}{\mathbf{A}}\restriction_U$ (cf. Remark 3.2.3). Then Lemma 3.4.22(c) yields the remaining implication. □

3.4.24 Remark. From Lemma 3.4.22(a) we may infer also another proof for the implication

$$U\subseteq V \implies U\leq_{\mathrm{cov}} V$$

between neighbourhoods $U, V \in \operatorname{Neigh} \mathbf{A}$ of an algebra $\mathbf{A}$, which we already saw in Corollary 3.4.6.

Namely, $U \subseteq V$ implies by Lemma 3.3.7 that $U \in \operatorname{Neigh} \mathbf{A}|_V$ and hence, the restricted structure $\underset{\sim}{\mathbf{A}}\restriction_U = (\underset{\sim}{\mathbf{A}}\restriction_V)\restriction_U$ is a retract of the relational counterpart $\underset{\sim}{\mathbf{A}}\restriction_V$ of $\mathbf{A}|_V$ (cf. Remark 3.2.3). This ensures that the assumptions of Lemma 3.4.22(a) are fulfilled, and so $S\restriction_U \neq T\restriction_U$ implies $S\restriction_V \neq T\restriction_V$ for all $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ and any arity $m \in \mathbb{N}$. However, this expresses precisely $\operatorname{Sep}_{\mathbf{A}}(U) \subseteq \operatorname{Sep}_{\mathbf{A}}(V)$, which is equivalent to $U \leq_{\text{cov}} V$ using Lemma 3.4.4(b). ♢

We have seen in Corollary 3.4.23 that having a retraction from a product of restrictions of the relational counterpart $\underset{\sim}{\mathbf{A}}$ of an algebra to a restriction $\underset{\sim}{\mathbf{A}}\restriction_U$ is sufficient for deducing the cover relation. We anticipate already one of the main results of this section saying that for finite algebras, this situation is even equivalent to covering a neighbourhood, and moreover to an operational characterisation of the cover relation, which was originally defined in purely relational terms.

Although we would like to have such a characterisation also in general, that is, for infinite algebras, we cannot expect it to be true in this form. First of all, we will see from the proof of our characterisation (see Theorem 3.4.31 and the subsequent corollaries) that we would need to allow relational clones with infinitary relations of arity at least $|A|$ and extend the cover property to these relations to make the proof work in general. This argument however, does not exclude the possibility that a different proof might work. Yet, the following tells us why it is not likely that the cover property (in the present form, using only finitary invariants) could be sufficient for getting a retraction from a product of restricted relational counterparts: covering certainly is a limited property, always determined by one m-tuple contained in the symmetric difference of two m-ary relations that have to be separated. Since the arity m is finite, this touches only finitely many elements of A, and hence is a local property. On the contrary, having a retraction is a global property, enabling us to reconstruct all of $\underset{\sim}{\mathbf{A}}\restriction_U$ simultaneously.

This observation already hints at a suitable way to remedy the asymmetry of the strength of the concepts of cover and retract in the case of infinite algebras: instead of retract we shall introduce the notion of *local retract*. This will then allow us to derive a working characterisation of the cover property for general algebras.

3.4.25 Definition. Let $m \in \mathbb{N}$ be a finite cardinal. We call a relational structure $\underset{\sim}{\mathbf{A}}$ an $m^{=}$*-local retract* of a structure $\underset{\sim}{\mathbf{B}}$ if for every subset $X \subseteq A$ of cardinality $|X| = m$ there exists a pair of morphisms $\Lambda\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ and $M\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ such that $(\Lambda \circ M)\,|_X^A = \operatorname{id}_A|_X^A$. Moreover, $\underset{\sim}{\mathbf{A}}$ is said to be an m*-local retract* of $\underset{\sim}{\mathbf{B}}$ if it is a $k^{=}$-local retract for every $k \leq m$, $k \in \mathbb{N}$. It is a *local retract* of $\underset{\sim}{\mathbf{B}}$ if it is an m-local retract for every $m \in \mathbb{N}$.

For an index set I and structures $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ we say that $\underset{\sim}{\mathbf{A}}$ is a *jointly finite* $m^{=}$*-local retract* of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ if for every subset $X \subseteq A$ of cardinality $|X| = m$ there is a finite subset $J \subseteq I$ and there are relational morphisms $\Lambda\colon \prod_{j\in J} \underset{\sim}{\mathbf{B}}_j \longrightarrow \underset{\sim}{\mathbf{A}}$ and $M\colon \underset{\sim}{\mathbf{A}} \longrightarrow \prod_{j\in J} \underset{\sim}{\mathbf{B}}_j$ satisfying $(\Lambda \circ M)\,|_X^A = \operatorname{id}_A|_X^A$. The structure $\underset{\sim}{\mathbf{A}}$ is a *jointly* *finite m-local retract* of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ if it is a jointly finite $k^=$-local retract of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ for every $k \leq m$, $k \in \mathbb{N}$, and it is a *jointly finite local retract* of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ if it is a jointly finite m-local retract of these structures for all $m \in \mathbb{N}$. ♢

It is an immediate consequence of the definition that being an m-local retract of some structure implies being a k-local retract of the same structure for every $k \leq m \in \mathbb{N}$. The analogous implication holds for jointly finite m-local (and k-local) retracts.

Furthermore, it is evident from the definition that the m-local versions of the two introduced concepts just mean that the respective condition is fulfilled for every at most m-ary subset X of the carrier of the structure $\underset{\sim}{\mathbf{A}}$, and the local versions mean that this is the case for every finite subset X. This situation is, of course, the one, which is of main interest. However, as our Theorem 3.4.31 characterising the cover condition will work arity-wise, we have introduced the parametric notions. In this way, we can also deal with the constellation, when only relations of a certain arity are known to be separated by a collection of neighbourhoods. We admit that from a philosophical point of view, the notion of (jointly finite) $m^=$-local retract is not a very well-chosen one. Namely, if $\underset{\sim}{\mathbf{A}}$ is finite and m exceeds the cardinality of its carrier, then $\underset{\sim}{\mathbf{A}}$ is vacuously a (jointly finite) $m^=$-local retract of any structure as there are no subsets $X \subseteq A$ with $|A| \geq |X| = m > |A|$. The concept of (jointly finite) $m^=$-local retracts is a technical one which is only justified by the fact that it shortens the formulation of some of our later results and proofs, because it is a defining component for (jointly finite) m-local retracts. The latter and their non-parametric twins are the important concepts in Definition 3.4.25.

Since the category of relational structures of a given signature has products, both notions from Definition 3.4.25 are, in fact, related. Indeed, the finiteness condition on the subset $J \subseteq I$ in the definition of jointly finite m-local retract of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ makes the latter notion slightly stronger than just being an m-local retract of the product $\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$, unless such a retraction is impossible for trivial reasons. So we readily prove the following lemma:

3.4.26 Lemma. *Let $m \in \mathbb{N}$ and I be any index set. Furthermore let $\underset{\sim}{\mathbf{A}}$, and $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ be relational structures of the same type such that* $\operatorname{Hom}\left(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}_i\right)$ *is non-empty*[18] *for every $i \in I$. If $\underset{\sim}{\mathbf{A}}$ is a jointly finite $m^=$-local retract of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$, then it is an $m^=$-local retract of $\underset{\sim}{\mathbf{B}} := \prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$. Consequently, if $\underset{\sim}{\mathbf{A}}$ is a jointly finite m-local retract, respectively jointly finite local retract, of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$, then it is an m-local retract, respectively local retract, of $\underset{\sim}{\mathbf{B}}$.*

[18] This condition necessarily follows from $\underset{\sim}{\mathbf{A}}$ being an m-local retract of $\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$ via composing the existing local co-retraction $M : \underset{\sim}{\mathbf{A}} \longrightarrow \prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$ for the subset $X = \emptyset$ with the projection morphisms belonging to the product $\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$. Thus, it is a natural requirement excluding cases where the claim must obviously fail.

Proof: For $J \subseteq I$ let us denote by $\mathrm{pr}_J\colon \prod_{i\in I} \underset{\sim}{\mathbf{B}}_i \longrightarrow \prod_{j\in J} \underset{\sim}{\mathbf{B}}_j$ the projection morphism to the indices belonging to J. That is, we have $\mathrm{pr}_J\left((x)_{i\in I}\right) = (x_j)_{j\in J}$ for all $(x)_{i\in I} \in \prod_{i\in I} B_i$. Furthermore, for $j \in J$ we write pr'_j for the canonical projection morphism $\mathrm{pr}'_j\colon \prod_{k\in J} \underset{\sim}{\mathbf{B}}_k \longrightarrow \underset{\sim}{\mathbf{B}}_j$.

Now assume that $\underset{\sim}{\mathbf{A}}$ is a jointly finite $m^{=}$-local retract of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$ and consider any subset $X \subseteq A$ having cardinality m. By assumption we can find a finite subset $J \subseteq I$ and relational morphisms $\Lambda\colon \prod_{j\in J} \underset{\sim}{\mathbf{B}}_j \longrightarrow \underset{\sim}{\mathbf{A}}$ and $M\colon \underset{\sim}{\mathbf{A}} \longrightarrow \prod_{j\in J} \underset{\sim}{\mathbf{B}}_j$ such that $\Lambda \circ M|_X^A = \mathrm{id}_A|_X^A$. For $j \in J$ the mapping $M_j := \mathrm{pr}'_j \circ M$ is a relational morphism between $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}_j$. Moreover, straining the axiom of choice, we may pick other relational morphisms $M_i\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}_i$ for every $i \in I \setminus J$ because of the precondition $\mathrm{Hom}\left(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}}_i\right) \neq \emptyset$. Defining $\tilde{M} := (M_i)_{i\in I}$ to be the tupling of all these morphisms, we get a morphism $\tilde{M}\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}} = \prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$ satisfying $\mathrm{pr}_{\{j\}} \circ \tilde{M} = M_j$ for every $j \in J$. Hence, we have $\mathrm{pr}_J \circ \tilde{M} = M$. Moreover, letting $\tilde{\Lambda} := \Lambda \circ \mathrm{pr}_J$, we have established another morphism $\tilde{\Lambda}\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ fulfilling

$$\tilde{\Lambda} \circ \tilde{M} = \Lambda \circ \mathrm{pr}_J \circ \tilde{M} = \Lambda \circ M.$$

Consequently, we obtain $\left(\tilde{\Lambda} \circ \tilde{M}\right)|_X^A = (\Lambda \circ M)|_X^A = \mathrm{id}_A|_X^A$ as needed.

If we assume $\underset{\sim}{\mathbf{A}}$ to be a jointly finite local (m-local) retract of $\left(\underset{\sim}{\mathbf{B}}_i\right)_{i\in I}$, then we can carry out this argument for every finite cardinal $k \in \mathbb{N}$ ($k \leq m$) and hence infer that $\underset{\sim}{\mathbf{A}}$ is a local (m-local) retract of $\underset{\sim}{\mathbf{B}}$. $\square$

If a relational structure $\underset{\sim}{\mathbf{A}}$ is a retract of a structure $\underset{\sim}{\mathbf{B}}$ of the same similarity type, then by knowing a retraction $\Lambda\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ and the co-retract $\underset{\sim}{\mathbf{B}}$, one can completely reconstruct $\underset{\sim}{\mathbf{A}}$ as demonstrated in Lemma 3.4.22(a). For local retracts a similar property holds, but the situation is not as easy.

3.4.27 Lemma. *For $m \in \mathbb{N}$, relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ of the same signature Σ and $\mu := \min\{m, |A|\}$, the following statements hold:*

(a) If $\underset{\sim}{\mathbf{A}}$ is a μ-local retract of $\underset{\sim}{\mathbf{B}}$, then for every $\sigma \in \Sigma$ of arity $k \leq m$, one can reconstruct $\sigma^{\underset{\sim}{\mathbf{A}}}$ from $\underset{\sim}{\mathbf{B}}$. Namely, if we fix for every subset $X \subseteq A$ of cardinality at most m a relational morphism $M_X\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ such that there is another morphism $\Lambda_X\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ satisfying $(\Lambda_X \circ M_X)|_X^A = \mathrm{id}_A|_X^A$, then

$$\sigma^{\underset{\sim}{\mathbf{A}}} = \left\{ x \in A^k \;\middle|\; M_{\mathrm{im}\, x} \circ x \in \sigma^{\underset{\sim}{\mathbf{B}}} \right\}.$$

In particular, for all symbols $\varrho, \sigma \in \Sigma$ of common arity $k \leq m$ the implication $\varrho^{\underset{\sim}{\mathbf{B}}} = \sigma^{\underset{\sim}{\mathbf{B}}} \implies \varrho^{\underset{\sim}{\mathbf{A}}} = \sigma^{\underset{\sim}{\mathbf{A}}}$ holds.

(b) If $\underset{\sim}{\mathbf{A}}$ is a local retract of $\underset{\sim}{\mathbf{B}}$, then $\underset{\sim}{\mathbf{A}}$ is reconstructible from $\underset{\sim}{\mathbf{B}}$. Namely, if we fix for every finite subset $X \subseteq A$ a relational morphism $M_X\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ such that there is another morphism $\Lambda_X\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$ satisfying $(\Lambda_X \circ M_X)|_X^A = \mathrm{id}_A|_X^A$, then

$$\underset{\sim}{\mathbf{A}} = \left\langle A; \left(\left\{ x \in A^{\mathrm{ar}\,\sigma} \;\middle|\; M_{\mathrm{im}\, x} \circ x \in \sigma^{\underset{\sim}{\mathbf{B}}} \right\}\right)_{\sigma\in\Sigma} \right\rangle.$$

Therefore, for all $\sigma, \varrho \in \Sigma$ of common arity we have $\varrho^{\underset{\sim}{\mathbf{B}}} = \sigma^{\underset{\sim}{\mathbf{B}}} \implies \varrho^{\underset{\sim}{\mathbf{A}}} = \sigma^{\underset{\sim}{\mathbf{A}}}$.

(c) *If, for an index set I and structures $\underset{\sim}{\mathbf{B}}_i$, $i \in I$, $\underset{\sim}{\mathbf{A}}$ is a μ-local retract of the product $\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i$, then for all symbols $\varrho, \sigma \in \Sigma$ of common arity at most m, we have $\left(\varrho^{\underset{\sim}{\mathbf{B}}_i}\right)_{i\in I} = \left(\sigma^{\underset{\sim}{\mathbf{B}}_i}\right)_{i\in I} \implies \varrho^{\underset{\sim}{\mathbf{A}}} = \sigma^{\underset{\sim}{\mathbf{A}}}$. If each of the structures $\underset{\sim}{\mathbf{B}}_i$ is a* $\min\{m, |B_i|\}$*-local retract of $\underset{\sim}{\mathbf{A}}$, then this implication is an equivalence. An analogous statement holds for local retracts and all relational symbols.*

Proof: Claim (b) is a simple consequence of item (a) because being a local retract is defined by being an m-local retract for every $m \in \mathbb{N}$. Hence, using (a) one can reconstruct relations of arbitrary finite arity $m \in \mathbb{N}$.

Now we demonstrate item (a) by showing both inclusions. Suppose σ is a k-ary symbol in Σ and let $x \in \sigma^{\underset{\sim}{\mathbf{A}}}$. As $M_{\operatorname{im} x}$ is relation preserving, we trivially have $M_{\operatorname{im} x} \circ x \in \sigma^{\underset{\sim}{\mathbf{B}}}$. For the converse inclusion consider $x \in A^k$ fulfilling $M_{\operatorname{im} x} \circ x \in \sigma^{\underset{\sim}{\mathbf{B}}}$. This implies that $x = \Lambda_{\operatorname{im} x} \circ M_{\operatorname{im} x} \circ x \in \sigma^{\underset{\sim}{\mathbf{A}}}$ since $\Lambda_{\operatorname{im} x}$ is a morphism and every entry of x occurs in the at most k-element subset $\operatorname{im} x$ ($k \leq m$ and $k \leq |A|$). Finally, the stated implication follows from the functional dependence of $\sigma^{\underset{\sim}{\mathbf{A}}}$ upon $\sigma^{\underset{\sim}{\mathbf{B}}}$ for symbols of arity at most m.

For (c) consider $\varrho, \sigma \in \Sigma$ of arity at most m satisfying $\left(\varrho^{\underset{\sim}{\mathbf{B}}_i}\right)_{i\in I} = \left(\sigma^{\underset{\sim}{\mathbf{B}}_i}\right)_{i\in I}$. This assumption implies $\varrho^{\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i} = \prod\!\!\prod_{i\in I} \varrho^{\underset{\sim}{\mathbf{B}}_i} = \prod\!\!\prod_{i\in I} \sigma^{\underset{\sim}{\mathbf{B}}_i} = \sigma^{\prod_{i\in I} \underset{\sim}{\mathbf{B}}_i}$, so by (a) we get $\varrho^{\underset{\sim}{\mathbf{A}}} = \sigma^{\underset{\sim}{\mathbf{A}}}$. In a similar way we can show the converse implication if all $\underset{\sim}{\mathbf{B}}_i$, $i \in I$, are m-local retracts of $\underset{\sim}{\mathbf{A}}$. Using (b) instead of (a), one can prove the statements about local retracts. □

From this lemma one could derive a similar corollary as 3.4.23, however we will not do this immediately but leave this as a part of Theorem 3.4.31 and Corollary 3.4.35. There the structures $\underset{\sim}{\mathbf{B}}_i$ will be restricted relational counterparts of some algebra, i.e. (idempotent) retracts of the relational counterpart.

More generally, the previous lemma only uses the assumption of *local* retracts to derive a very close relationship between the invariant relations of the algebra and its restrictions. This hints at a possible generalisation of neighbourhoods: instead one might use local idempotent retracts, i.e. subsets $U \subseteq A$ where for every finite subset $X \subseteq U$ (of cardinality at most m) there exists a morphism $e_X : \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_U$ being the identity on X. This means e_X satisfies $e_X(x) = x$ for $x \in X$ and is given as restriction $e_X = e|_A^U$ of an endomorphism $e \in \operatorname{End} \underset{\sim}{\mathbf{A}} = \operatorname{Clo}^{(1)}(\mathbf{A})$ fulfilling $\operatorname{im} e \subseteq U$.

Before we finally come to the main result of this section, we need two more lemmas providing parts the subsequent theorem.

3.4.28 Lemma. *For $m \in \mathbb{N}$, an algebra $\mathbf{A}$, a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ and a collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ the following hold:*

(a) *if U and $\mathcal{V}$ fulfil the separation property (3.10) for all invariants $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$, then they also do this for all $S, T \in \operatorname{Inv}^{(k)} \mathbf{A}$ for every arity $1 \leq k \leq m$;*

(b) U and $\mathcal{V}$ satisfy (3.10) for all pairs $S, T \in \operatorname{Inv} \mathbf{A}$ of at most m-ary relations if and only if they do this for pairs of arity m and 0, respectively;

(c) $\mathcal{V}$ separates all pairs of nullary invariants that are distinguished by U if and only if it separates all distinct pairs of nullary invariant relations if and only if $\emptyset \in \operatorname{Inv} \mathbf{A}$ implies $\mathcal{V} \neq \emptyset$.

Proof: Before we begin to consider the individual statements, we recall that the contrapositive of implication (3.10) is

$$S{\restriction}_U \neq T{\restriction}_U \implies \exists V \in \mathcal{V}\colon S{\restriction}_V \neq T{\restriction}_V,$$

which is the separation property we will actually be working with in this proof.

(a) Let us suppose that condition (3.10) holds for invariants of arity m and let $1 \leq k \leq m$. For a relation $S \in \mathrm{R}_A$, $\ell \in \mathbb{N}$ and a neighbourhood $W \in \operatorname{Neigh} \mathbf{A}$ we have $\left(S \times A^\ell\right){\restriction}_W = S{\restriction}_W \times W^\ell$. Considering now $S, T \in \operatorname{Inv}^{(k)} \mathbf{A}$ separated in U with $S \subseteq T$, we can find some tuple $x \in U^k$ belonging to $T \setminus S$. Since $k > 0$, this means $U \neq \emptyset$, such that also $U^{m-k} \neq \emptyset$. So we may pick some $z \in U^{m-k}$ showing that the invariant relations $S \times A^{m-k} \subseteq T \times A^{m-k}$ are separated in U. Namely, $(x, z) \in U^k \times U^{m-k} = U^m$ and $x \in T \setminus S$, so

$$(x, z) \in \left(T{\restriction}_U \times U^{m-k}\right) \setminus \left(S{\restriction}_U \times U^{m-k}\right) = \left(T \times A^{m-k}\right){\restriction}_U \setminus \left(S \times A^{m-k}\right){\restriction}_U.$$

Hence, these m-ary relations must be distinguished by some $V \in \mathcal{V}$, i.e. there exists some element

$$(y, w) \in \left(T \times A^{m-k}\right){\restriction}_V \setminus \left(S \times A^{m-k}\right){\restriction}_V = \left(T{\restriction}_V \times V^{m-k}\right) \setminus \left(S{\restriction}_V \times V^{m-k}\right).$$

Thus, we have $y \in T{\restriction}_V \setminus S{\restriction}_V$, proving condition (3.10) for k-ary invariants $S \subseteq T$. By Corollary 3.4.15 for $\mathcal{U} = \{U\}$, we are done.

(b) This is an obvious consequence of the previous item.

(c) The only nullary relations on A are $\emptyset$ and A^0. Hence, the only possible pair of distinct nullary invariants can be $S = \emptyset$ and $T = A^0$. These two relations are separated by any neighbourhood $W \in \operatorname{Neigh} \mathbf{A}$, since $S{\restriction}_W = \emptyset \neq W^0 = T{\restriction}_W$. Especially, this holds for U, and so the first equivalence of item (c) is proven.

Now assume that $\mathcal{V}$ separates all distinct nullary invariants, and suppose that $S = \emptyset \in \operatorname{Inv} \mathbf{A}$. Then S is properly contained in $T = A^0$, and the assumption on $\mathcal{V}$ cannot be vacuously true. Thus, $\mathcal{V}$ really separates S from T, which implies $\mathcal{V} \neq \emptyset$. Conversely, if $\emptyset \in \operatorname{Inv} \mathbf{A}$ implies $\mathcal{V} \neq \emptyset$ and S and T are distinct nullary invariants, then $\{S, T\} = \{\emptyset, A^0\}$, i.e. $\emptyset \in \operatorname{Inv} \mathbf{A}$. Therefore, we can find some neighbourhood $V \in \mathcal{V}$, which then separates S from T because every neighbourhood does this. $\square$

To understand the second lemma, we recall from Remark 3.1.11 the symbol $\mathrm{pr}^I_J = \prod^{\beta}_{\mathrm{id}_I}$ for sets $J \subseteq I$ and $\beta\colon J \hookrightarrow I$ being the natural mapping given by identical inclusion. This symbol stands for projection of subsets of A^I to their J-coordinates, i.e. the action of the canonical projection mapping $A^I \longrightarrow A^J$ on subsets of A^I. Viewed as a projection homomorphism, it is clear that this action transforms subalgebras of the power $\mathbf{A}^I$ into subalgebras of $\mathbf{A}^J$. In the lemma, we shall use this operation pr^I_J for the special case where I equals the carrier set of $\mathbf{A}$ and J is some subset $X \subseteq A$.

3.4.29 Lemma. *Let* $\mathbf{A}$ *be an algebra,* $e \in \mathrm{Idem}\,\mathbf{A}$ *an idempotent unary operation belonging to* $\mathrm{Clo}\,(\mathbf{A})$ *and* $E \subseteq \mathrm{Idem}\,\mathbf{A}$ *a collection of such idempotents. Denote by* $U := \mathrm{im}\, e \in \mathrm{Neigh}\,\mathbf{A}$ *and* $\mathcal{V} := \{\mathrm{im}\, e' \mid e' \in E\} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *the corresponding neighbourhoods. Furthermore, let* $T_0 := \mathrm{Clo}^{(1)}\,(\mathbf{A})$ *and* $S_0 := \langle F\rangle_{\mathbf{A}^A}$*, where* $F := \left\{ f \in \mathrm{Clo}^{(1)}\,(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V}\colon\ \mathrm{im}\, f \subseteq V\right\}$*. Moreover, we fix a choice function* $V\colon F \longrightarrow \mathcal{V}$ *satisfying* $\mathrm{im}\, f \subseteq V(f)$ *for every* $f \in F$*. For any finite cardinal* $m \in \mathbb{N}$ *the implications* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j) *are true, where the conditions* (a) *through* (j) *are the following:*

(a) The collection $\mathcal{V}$ *separates all* m*-ary invariant relations of* $\mathbf{A}$ *that are separated in* U*.*

(b) The collection $\mathcal{V}$ *separates all pairs* $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$*,* $S \subseteq T$ *of* m*-ary invariant relations that are separated in* U*.*

(c) For every $X \subseteq U$ *of cardinality* $|X| = m$ *the collection* $\mathcal{V}$ *separates the invariant relations* $\mathrm{pr}^A_X\, S_0$ *and* $\mathrm{pr}^A_X\, T_0$ *belonging to* $\mathrm{Inv}^{(|X|)}\mathbf{A}$ *(via some arbitrary but fixed indexing bijection between* X *and its finite cardinality* $m \in \mathbb{N}$*) if their restrictions to* U *are distinct.*

(d) For every $X \subseteq U$ *of cardinality* $|X| = m$*, we have* $\left(\mathrm{pr}^A_X\, S_0\right)\restriction_U = \left(\mathrm{pr}^A_X\, T_0\right)\restriction_U$*.*

(e) For every $X \subseteq U$ *of cardinality* $|X| = m$*, we have* $e|^A_X \in \mathrm{pr}^A_X\,(e \circ [S_0])$*.*

(f) For every $X \subseteq U$ *of cardinality* $|X| = m$ *there exists some arity* $n \in \mathbb{N}$*, some* $\lambda \in \mathrm{Term}^{(n)}\,(\mathbf{A})$ *and* $(f_1, \dots, f_n) \in F^n$ *such that*

$$(e \circ \lambda \circ (f_1, \dots, f_n))\,|^A_X = e|^A_X.$$

(g) For every $X \subseteq U$ *of cardinality* $|X| = m$ *there exists some arity* $n \in \mathbb{N}$*, some* $\lambda \in \mathrm{Clo}^{(n)}\,(\mathbf{A})$ *and* $(f_1, \dots, f_n) \in F^n$ *such that* $(\lambda \circ (f_1, \dots, f_n))\,|^A_X = e|^A_X$*.*

(h) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite* $m^=$*-local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f\in F}$*.*

(i) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is an* $m^=$*-local retract of* $\prod_{f\in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$*.*

(j) There is an index set Φ *and a mapping* $\tilde{V}\colon \Phi \longrightarrow \mathcal{V}$ *such that* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is an* $m^=$*-local retract of* $\prod_{\varphi\in\Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$*.*

Proof: Before we begin the actual proof, we will collect a few facts about S_0 and T_0. For any set I and a set $G \subseteq A^I$ we know from Chapter 2 that the subpower generated by G can be written as

$$\langle G\rangle_{\mathbf{A}^I} = \bigcup_{n\in\mathbb{N}} \left\{ h \circ (\mathbf{g}) \;\middle|\; h \in \operatorname{Term}^{(n)}(\mathbf{A}) \wedge \mathbf{g} \in G^n \right\}$$

Especially, we obtain for $I = A$ and $G = F$ that

$$S_0 = \langle F\rangle_{\mathbf{A}^A} = \bigcup_{n\in\mathbb{N}} \left\{ h \circ (\mathbf{f}) \;\middle|\; h \in \operatorname{Term}^{(n)}(\mathbf{A}) \wedge \mathbf{f} \in F^n \right\}.$$

Furthermore, $T_0 = \operatorname{Clo}^{(1)}(\mathbf{A})$ is a subalgebra of $\mathbf{A}^A$ since every fundamental operation of $\mathbf{A}$ belongs to $\operatorname{Clo}(\mathbf{A})$ and the latter is closed under composition. As F is a subset of $\operatorname{Clo}^{(1)}(\mathbf{A}) = T_0$, monotonicity of subalgebra closure yields the inclusion $S_0 = \langle F\rangle_{\mathbf{A}^A} \subseteq \langle T_0\rangle_{\mathbf{A}^A} = T_0$. These two subpowers have been chosen in such a way that we can now show $e_V \circ [T_0] = e_V \circ [S_0]$ for any idempotent $e_V \in \operatorname{Idem}\mathbf{A}$ such that $V := \operatorname{im} e_V \in \mathcal{V}$. For the argument it is important to note that, by definition, we have $e_V \circ [T_0] \subseteq \operatorname{Clo}^{(1)}(\mathbf{A}) = T_0$. That is, T_0 is closed under the action of such a clone operation e_V whereas S_0 generally is not. This is due to the fact that composing a unary operation from $\operatorname{Clo}(\mathbf{A})$ with a term operation of $\mathbf{A}$, in general, only yields an operation in $\operatorname{Clo}(\mathbf{A})$, not a term function. So, according to Lemma 3.1.4, we have $T_0{\restriction_V} = \{ f \in T_0 \mid \operatorname{im} f \subseteq V\} = e_V \circ [T_0]$, but only $S_0{\restriction_V} = \{ f \in S_0 \mid \operatorname{im} f \subseteq V\} \subseteq e_V \circ [S_0]$. Using $T_0 = \operatorname{Clo}^{(1)}(\mathbf{A})$ and the definition of the set F, it is now clear that

$$\begin{aligned} e_V \circ [T_0] = \{ f \in T_0 \mid \operatorname{im} f \subseteq V\} = \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \operatorname{im} f \subseteq V \right\} \\ \subseteq F \subseteq \langle F\rangle_{\mathbf{A}^A} = S_0, \end{aligned}$$

and, furthermore, the image of any operation $f \in e_V \circ [T_0]$ lies in V. Thus,

$$e_V \circ [T_0] \subseteq \{ f \in S_0 \mid \operatorname{im} f \subseteq V\} \subseteq e_V \circ [S_0] \subseteq e_V \circ [T_0],$$

where the last inclusion follows from $S_0 \subseteq T_0$. Thus, we have indeed demonstrated $e_V \circ [S_0] = e_V \circ [T_0]$.

Now Lemma 3.1.8 allows us to permute the action of e_V and arbitrary projection operations. Hence, in particular for any finite set $X \subseteq A$, we get

$$\begin{aligned} \left(\operatorname{pr}^A_X S_0\right){\restriction_V} &\overset{3.1.4}{=} e_V \circ \left[\operatorname{pr}^A_X S_0\right] \overset{3.1.8}{=} \operatorname{pr}^A_X (e_V \circ [S_0]) \\ &= \operatorname{pr}^A_X (e_V \circ [T_0]) \overset{3.1.8}{=} e_V \circ \left[\operatorname{pr}^A_X T_0\right] \overset{3.1.4}{=} \left(\operatorname{pr}^A_X T_0\right){\restriction_V}, \end{aligned}$$

where Lemma 3.1.4 was applicable here since the arguments to the restriction were finitary subpowers, i.e. invariant relations, that need to be preserved by e_V.

Note that these considerations are correct for any finite subset $X \subseteq A$. Therefore, whenever implication (3.10) is true for a certain subset of invariant relations of $\mathbf{A}$

containing the invariants corresponding to $\mathrm{pr}^A_X S_0$ and $\mathrm{pr}^A_X T_0$ (via some bijection between the set X and its finite cardinality), then $\left(\mathrm{pr}^A_X S_0\right)\restriction_U = \left(\mathrm{pr}^A_X T_0\right)\restriction_U$ must be true.

Now the proof of the lemma is pretty straightforward: we will show the stated implications in the order of occurrence in the lemma. The first ones (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are evidently specialisations: in each step the set of pairs of invariant relations for which condition (3.10) is assumed to hold is simply reduced.

The set remaining in (c) contains the pairs $\left(\mathrm{pr}^A_X S_0, \mathrm{pr}^A_X T_0\right)$ for any finite m-element subset $X \subseteq U \subseteq A$. So by what was explained above, we have the equality $\left(\mathrm{pr}^A_X S_0\right)\restriction_U = \left(\mathrm{pr}^A_X T_0\right)\restriction_U$, establishing the statement of (d). Starting with this assumption, we can again use Lemmas 3.1.4 and 3.1.8 to get

$$\begin{aligned}&\mathrm{pr}^A_X\left(e \circ [S_0]\right) \overset{3.1.8}{=} e \circ \left[\mathrm{pr}^A_X S_0\right] \overset{3.1.4}{=} \left(\mathrm{pr}^A_X S_0\right)\restriction_U \\ &= \left(\mathrm{pr}^A_X T_0\right)\restriction_U \overset{3.1.4}{=} e\circ\left[\mathrm{pr}^A_X T_0\right] \overset{3.1.8}{=} \mathrm{pr}^A_X\left(e \circ [T_0]\right) = \left\{ \left(e \circ f\right)|^A_X \;\middle|\; f \in \mathrm{Clo}^{(1)}\left(\mathbf{A}\right)\right\}.\end{aligned}$$

Since id_A belongs to $\mathrm{Clo}^{(1)}(\mathbf{A})$, we can infer $e|^A_X \in \mathrm{pr}^A_X\left(e \circ [S_0]\right)$, which is the claim of (e).

Now we suppose this and consider any subset $X \subseteq U$ of cardinality m to verify (f). Using the assumption from (e), we get $e|^A_X \in \mathrm{pr}^A_X\left(e \circ [S_0]\right)$, that is, there is an operation $g \in S_0$ such that $e|^A_X = (e \circ g)|^A_X$. Looking back at the characterisation of S_0 we stated at the beginning of the proof, we can infer that g is of the form $\lambda \circ (f_1, \dots, f_n)$ for some $n \in \mathbb{N}$, some tuple $(f_1, \dots, f_n) \in F^n$ and some n-ary term operation $\lambda \in \mathrm{Term}^{(n)}(\mathbf{A})$. Therefore, we have

$$\left(e \circ \lambda \circ (f_1, \dots, f_n)\right)|^A_X = (e \circ g)|^A_X = e|^A_X,$$

proving (f).

This equality clearly implies that $\left(\tilde{\lambda} \circ (f_1, \dots, f_n)\right)|^A_X = e|^A_X$, where we have put $\tilde{\lambda} := e \circ \lambda$, which belongs to $\mathrm{Clo}^{(n)}(\mathbf{A})$ as a composition of $e \in \mathrm{Clo}^{(1)}(\mathbf{A})$ with $\lambda \in \mathrm{Term}^{(n)}(\mathbf{A}) \subseteq \mathrm{Clo}^{(n)}(\mathbf{A})$. Thus, item (f) implies (g).

Assuming the truth of item (g), we have a decomposition equation

$$\left(\lambda \circ (f_1, \dots, f_n)\right)|^A_X = e|^A_X$$

for all subsets $X \subseteq U$ of cardinality $|X| = m$, where $\lambda \in \mathrm{Clo}^{(n)}(\mathbf{A})$ and the tuple $(f_1, \dots, f_n) \in F^n$ may depend on X. To show (h) we consider any subset $X \subseteq U$ of cardinality m. We shall construct relational morphisms $\Lambda\colon \prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_U$ and $M\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)}$ such that Λ applied to M composes to the identity everywhere on X. Obviously, for every $f \in F$ the restriction $f|^{V(f)}_A\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$ is a relational morphism, as for every relation $S \in \mathrm{Inv}\,\mathbf{A}$ we have the inclusions $f \circ [S] \subseteq (\mathrm{im}\, f)^{\mathrm{ar}\, S} \subseteq (V(f))^{\mathrm{ar}\, S}$ and $f \circ [S] \subseteq S$ due to $f \in F \subseteq \mathrm{Clo}(\mathbf{A})$. Thus, $f \circ [S] \subseteq S \cap (V(f))^{\mathrm{ar}\, S} = S\restriction_{V(f)}$. Since by definition the fundamental relations of $\underset{\sim}{\mathbf{A}}\restriction_U$ are subrelations of those of $\underset{\sim}{\mathbf{A}}$, we can restrict further and get that the function $f|_U^{V(f)}\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$ is a relational morphism, too. Consequently, the tupling $M := \left(f_1|_U^{V(f_1)}, \ldots, f_n|_U^{V(f_n)}\right)$ is one between $\underset{\sim}{\mathbf{A}}\restriction_U$ and $\prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)}$. By their nature of being polymorphisms in $\operatorname{Clo}(\mathbf{A}) = \operatorname{Pol}_A \operatorname{Inv} \mathbf{A} = \operatorname{Pol} \underset{\sim}{\mathbf{A}}$, we have that $\lambda\colon \underset{\sim}{\mathbf{A}}^n \longrightarrow \underset{\sim}{\mathbf{A}}$ and $e\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}$ are relational morphisms. Since $U = \operatorname{im} e$, Remark 3.2.3 yields that $\underset{\sim}{\mathbf{A}}\restriction_U = \underset{\sim}{\mathbf{A}}\restriction_{\operatorname{im} e} = e\,[\underset{\sim}{\mathbf{A}}]$, and so $e|_A^U\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_U$ is a relation preserving map, as well. Furthermore, for every $i \in \{1, \ldots, n\}$ the structure $\underset{\sim}{\mathbf{A}}\restriction_{V(f_i)}$ is a substructure of $\underset{\sim}{\mathbf{A}}$, whence $\prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)}$ is one of $\underset{\sim}{\mathbf{A}}^n$. Thus, the morphism $\lambda\colon \underset{\sim}{\mathbf{A}}^n \longrightarrow \underset{\sim}{\mathbf{A}}$ can be restricted to $\lambda|^A_{\prod_{i=1}^n V(f_i)}\colon \prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)} \longrightarrow \underset{\sim}{\mathbf{A}}$. Composing these morphisms, we can define the morphism $\Lambda := e|_A^U \circ \lambda|^A_{\prod_{i=1}^n V(f_i)}$ between $\prod_{i=1}^n \underset{\sim}{\mathbf{A}}\restriction_{V(f_i)}$ and $\underset{\sim}{\mathbf{A}}\restriction_U$. Now the assumed decomposition equation together with idempotency of e implies $\Lambda(M(x)) = e\left(\lambda\left(f_1(x), \ldots, f_n(x)\right)\right) = e(e(x)) = e(x) = x$ for every $x \in X$ as desired. This establishes item (h).

We have demonstrated above that $f|_U^{V(f)}\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$ is a relational morphism for any $f \in F$. Hence, $\operatorname{Hom}\left(\underset{\sim}{\mathbf{A}}\restriction_U, \underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right) \neq \emptyset$ for $f \in F$, and we can apply Lemma 3.4.26 to infer from item (h) that $\underset{\sim}{\mathbf{A}}\restriction_U$ is an $m^=$-local retract of $\prod_{f\in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$. This is the content of item (i).

Finally, the implication (i) $\Rightarrow$ (j) follows by letting $\Phi := F$ and $\tilde{V} := V$. □

From this lemma we immediately derive the following consequence:

3.4.30 Corollary. *Under the assumptions of Lemma 3.4.29 the implications stated in this lemma remain true if the text blocks "all m-ary invariant relations", "of cardinality $|X| = m$" and "$m^=$-local retract" are replaced by "all at most m-ary invariant relations", "of cardinality $|X| \leq m$" and "m-local retract", respectively.*

We remark that it will be shown in Corollary 3.4.32 that all implications in the previous Corollary are actually equivalences.

Proof: This follows since the natural number $m \in \mathbb{N}$ could be chosen arbitrarily in Lemma 3.4.29. Thus, if we have one of the claims for every arity, cardinality or parameter k less than or equal to m, then we can use the corresponding implication of Lemma 3.4.29 for each $k \leq m$ separately, to obtain the conclusion for every arity, cardinality or parameter $k \leq m$. □

That Corollary 3.4.30 can be strengthened to equivalence is a consequence of the following theorem, which is also the main result of this section. It is the base for subsequent characterisations that will be formulated as corollaries. Several of them will, of course, deal with the cover relation. The fundamental ideas are already suggested by Lemma 3.4.29 and Corollary 3.4.30. The covering condition can be seemingly weakened including less and less relations that have to be separated, finally enabling an equivalent operational characterisation. This, in turn, can be translated into an equivalent condition involving a product-local-retract construction. By Lemma 3.4.27, the latter can be interpreted as a means of reconstruction (in relational language), at least up to local term equivalence.

3.4.31 Theorem. *Let* $\mathbf{A}$ *be an algebra,* $e \in \operatorname{Idem}\mathbf{A}$ *an idempotent unary operation belonging to* $\operatorname{Clo}(\mathbf{A})$ *and* $E \subseteq \operatorname{Idem}\mathbf{A}$ *a collection of such idempotents. Denote by* $U := \operatorname{im} e \in \operatorname{Neigh}\mathbf{A}$ *and* $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\} \subseteq \operatorname{Neigh}\mathbf{A}$ *the corresponding neighbourhoods. Furthermore, let* $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ *and* $S_0 := \langle F \rangle_{\mathbf{A}^A}$*, where* $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V}\colon\ \operatorname{im} f \subseteq V \right\}$*. Moreover, we fix a choice function* $V\colon F \longrightarrow \mathcal{V}$ *satisfying* $\operatorname{im} f \subseteq V(f)$ *for every* $f \in F$*. For a fixed finite cardinal* $m \in \mathbb{N}$ *we let* $\mu := \min\{m, |U|\}$*. Then the following facts are equivalent:*

(a) *The collection* $\mathcal{V}$ *separates all at most* m*-ary invariant relations of* $\mathbf{A}$ *that are separated in* U*.*

(b) *The collection* $\mathcal{V}$ *separates all pairs* $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$*,* $S \subseteq T$ *of* m*-ary invariant relations that are separated in* U*, and if* $\emptyset \in \operatorname{Inv}\mathbf{A}$*, then* $\mathcal{V} \neq \emptyset$*.*

(c) *The collection* $\mathcal{V}$ *separates all pairs* $S, T \in \operatorname{Inv}^{(\mu)}\mathbf{A}$*,* $S \subseteq T$ *of* μ*-ary*[19] *invariant relations that are separated in* U*.*

(d) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$ *the collection* $\mathcal{V}$ *separates the invariant relations* $\operatorname{pr}^A_X S_0$ *and* $\operatorname{pr}^A_X T_0$ *belonging to* $\operatorname{Inv}^{(\mu)}\mathbf{A}$ *(via some arbitrary but fixed indexing bijection between* X *and its finite cardinality* $\mu \in \mathbb{N}$*) if their restrictions to* U *are distinct.*

(e) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$*, we have* $\left(\operatorname{pr}^A_X S_0\right)\restriction_U = \left(\operatorname{pr}^A_X T_0\right)\restriction_U$*.*

(f) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$*, we have* $e|^A_X \in \operatorname{pr}^A_X (e \circ [S_0])$*.*

(g) *For every* $X \subseteq A$ *of cardinality* $|X| \leq m$ *there exists some arity* $n \in \mathbb{N}$*, some* $\lambda \in \operatorname{Term}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that*

$$(e \circ \lambda \circ (f_1, \ldots, f_n))\,|^A_X = e|^A_X.$$

(h) *For every* $X \subseteq U$ *of cardinality* $|X| = \mu$ *there exists some arity* $n \in \mathbb{N}$*, some* $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that* $(\lambda \circ (f_1, \ldots, f_n))\,|^A_X = e|^A_X$*.*

(i) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite* m*-local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$*.*

(j) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a* μ*-local retract of* $\prod_{f \in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$*.*

(k) *There is an index set* Φ *and a mapping* $\tilde{V}\colon \Phi \longrightarrow \mathcal{V}$ *such that* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a* μ*-local retract of* $\prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$*.*

[19] This claim is not equivalent to the one arising if we replace μ by m. Namely, if $A = U = \mathcal{V} = \emptyset$ and $m \in \mathbb{N}_+$, the claim is vacuously true for m-ary relations, but it fails for nullary invariant relations due to $\mathcal{V}$ being empty.

Clearly, it follows from item (g) that the operation λ in item (h) can be chosen as a *term* operation, whenever e is an idempotent *term* operation.

Furthermore, items (a), (g) and (i) are to be seen as conditions one can exploit if one has the assertion of this theorem as an assumption. The other equivalent (but weaker looking) formulations are intended to be used for proving that a collection $\mathcal{V}$ has the separation property w.r.t. a neighbourhood U and at most m-ary invariant relations.

Proof: Using Lemma 3.4.29 the proof of the theorem is not too difficult. We show (b) $\Leftrightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) $\Rightarrow$ (j) $\Rightarrow$ (k) $\Rightarrow$ (a). Equivalence of items (a) and (b) is a direct consequence of Lemma 3.4.28 in combination with Corollary 3.4.15, to see that separation of pairs of subrelations is enough. The implications (a) $\Rightarrow$ (c) $\Rightarrow$ (d) are evidently specialisations: in each step the set of pairs of invariant relations for which condition (3.10) is assumed to hold is simply reduced.

The implications (d) $\Rightarrow$ (e) $\Rightarrow$ (f) follow by the implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) of Lemma 3.4.29. Using (e) $\Rightarrow$ (f) of the same lemma, we obtain the claim of item (g) for subsets of U with precisely μ elements. To prove (g), we consider any subset $X \subseteq A$ of cardinality at most m. Letting $X' := e\,[X] \subseteq U$, we can infer $|X'| \leq |U|$, which implies $|X'| \leq \min\{m, |U|\} = \mu \leq |U|$. Hence, there exists a finite subset $Y \subseteq U$, having μ elements and containing $X' \subseteq Y$. Thus, by what we know about μ-element subsets of U, we can find $n \in \mathbb{N}$, a tuple $(f'_1, \dots, f'_n) \in F^n$ and some n-ary term operation $\lambda \in \mathrm{Term}^{(n)}(\mathbf{A})$ such that $(e \circ \lambda \circ (f'_1, \dots, f'_n))\,|_Y^A = e|_Y^A$. Defining $f_i := f'_i \circ e$ for $i \in \{1, \dots, n\}$, we obtain an operation in $\mathrm{Clo}^{(1)}(\mathbf{A})$, satisfying $\operatorname{im} f_i \subseteq \operatorname{im} f'_i \subseteq V$ for some neighbourhood $V \in \mathcal{V}$. In other words, we have $(f_1, \dots, f_n) \in F^n$. Now for every $x \in X$, it is $e(x) =: x' \in X' \subseteq Y$, so by the equality for e on Y and idempotency of e the following holds:

$$
\begin{aligned}
(e \circ \lambda \circ (f_1, \dots, f_n))\,(x) &= (e \circ \lambda \circ (f'_1 \circ e, \dots, f'_n \circ e))\,(x) \\
&= (e \circ \lambda \circ (f'_1, \dots, f'_n) \circ e)\,(x) \\
&= (e \circ \lambda \circ (f'_1, \dots, f'_n))\,(e(x)) \\
&= e(x') = e(e(x)) = e(x).
\end{aligned}
$$

Therefore, we have $(e \circ \lambda \circ (f_1, \dots, f_n))\,|_X^A = e|_X^A$, proving (g).

The claim of (g) for subsets $X \subseteq A$ of cardinality at most m means that it is of course true for those subsets of $U \subseteq A$ having exactly μ elements. Then (f) $\Rightarrow$ (g) of Lemma 3.4.29 entails the truth of (h).

Assuming this, the implication (g) $\Rightarrow$ (h) of the same lemma yields that $\underset{\sim}{\mathbf{A}}\restriction_U$ is a jointly finite $\mu^{=}$-local retract of $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$. To prove (i) we consider any subset $Y \subseteq U$ of cardinality at most m. Of course, the inclusion $Y \subseteq U$ implies $|Y| \leq |U|$, so $|Y| \leq \min\{m, |U|\} = \mu$. Since $\mu \leq |U|$, there is a μ-element subset $X \subseteq U$ containing Y. As $\underset{\sim}{\mathbf{A}}\restriction_U$ is a jointly finite $\mu^{=}$-local retract, we can find a finite subset $F' \subseteq F$ and relational morphisms $\Lambda\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}\restriction_U$ and $M\colon \underset{\sim}{\mathbf{A}}\restriction_U \longrightarrow \underset{\sim}{\mathbf{B}}$, where $\underset{\sim}{\mathbf{B}} := \prod_{f\in F'} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$, such that $\Lambda(M(x)) = x$ for every $x \in X$ and thus for all $y \in Y$. Hence, we have demonstrated item (i).

Since $\mu \leq m$ we obtain from this that $\underset{\sim}{\mathbf{A}}\restriction_U$ also is a jointly finite μ-local retract of $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f\in F}$ (see the paragraph following Definition 3.4.25). Now by Corollary 3.4.30 we may infer (j). The implication (j) $\Rightarrow$ (k) is immediate by the same corollary or via $\Phi := F$ and $\tilde{V} := V$.

To come back to item (a), we assume that $\underset{\sim}{\mathbf{A}}\restriction_U$ is a μ-local retract of some product $\underset{\sim}{\mathbf{B}} := \prod_{\varphi\in\Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$. To show the contrapositive of the claim in (a), we consider arbitrary at most m-ary invariants $S, T \in \operatorname{Inv}^{(k)}\mathbf{A}$, $0 \leq k \leq m$, such that $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$. By definition of the separation set, this implies $S\restriction_W = T\restriction_W$ for every $W \in \mathcal{V}$, in particular $S\restriction_{\tilde{V}(\varphi)} = T\restriction_{\tilde{V}(\varphi)}$ for all $\varphi \in \Phi$. By the implication proven in Lemma 3.4.27(c), we directly obtain $S\restriction_U = T\restriction_U$, which establishes $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(U)$. □

The following result is a simple consequence of the previous theorem:

3.4.32 Corollary. *The implications mentioned in Corollary 3.4.30 are actually equivalences.*

Proof: We only need to check that $\mathcal{V}$ separates all at most m-ary invariant relations of $\mathbf{A}$ that are separated in U provided that $\underset{\sim}{\mathbf{A}}\restriction_U$ is an m-local retract of some product $\prod_{\varphi\in\Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$ where $\tilde{V} : \Phi \longrightarrow \mathcal{V}$. Letting $\mu := \min\{m, |U|\}$, we have $\mu \leq m$. Reading the paragraph following Definition 3.4.25 tells us that our assumption implies that $\underset{\sim}{\mathbf{A}}\restriction_U$ also is a μ-local retract of $\prod_{\varphi\in\Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$, which is item (k) of Theorem 3.4.31. According to the theorem, this is equivalent to item (a), which is what we wanted to show. □

From Theorem 3.4.31, we will now derive a number of further corollaries, namely, by adding one or several of the following three assumptions: first $U = A$, second U being finite and third, assuming the separation condition in item (a) for all arities simultaneously, which is equivalent to $U \leq_{\mathrm{cov}} \mathcal{V}$.

The first corollary is just the special case, when $U = A$ is the full neighbourhood, characterising the separation of m-ary invariants without any further assumptions.

3.4.33 Corollary. *Let $\mathbf{A}$ be an algebra and $E \subseteq \operatorname{Idem}\mathbf{A}$ a collection of unary idempotent clone operations. Denote by $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\} \subseteq \operatorname{Neigh}\mathbf{A}$ the corresponding neighbourhoods. Furthermore, let $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ and $S_0 := \langle F\rangle_{\mathbf{A}^A}$, where $F := \left\{f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V} \colon \operatorname{im} f \subseteq V\right\}$. Moreover, we fix a choice function $V : F \longrightarrow \mathcal{V}$ satisfying $\operatorname{im} f \subseteq V(f)$ for every $f \in F$. For a fixed finite cardinal $m \in \mathbb{N}$ we let $\mu := \min\{m, |A|\}$. Then the following facts are equivalent:*

(a) The collection $\mathcal{V}$ separates all pairs of distinct at most m-ary invariant relations of $\mathbf{A}$.

(b) The collection $\mathcal{V}$ separates all pairs $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ of m-ary invariant relations satisfying $S \subsetneq T$, and if $\emptyset \in \operatorname{Inv}\mathbf{A}$, then $\mathcal{V} \neq \emptyset$.

(c) *The collection $\mathcal{V}$ separates all pairs $S, T \in \operatorname{Inv}^{(\mu)} \mathbf{A}$ of μ-ary invariant relations satisfying $S \subsetneq T$.*

(d) *For every $X \subseteq A$ of cardinality $|X| = \mu$ the collection $\mathcal{V}$ separates the invariant relations $\operatorname{pr}^A_X S_0$ and $\operatorname{pr}^A_X T_0$ belonging to $\operatorname{Inv}^{(\mu)} \mathbf{A}$ (via some arbitrary but fixed indexing bijection between X and its finite cardinality $\mu \in \mathbb{N}$) in case they are distinct.*

(e) *For every $X \subseteq A$ of cardinality $|X| = \mu$, we have $\operatorname{pr}^A_X S_0 = \operatorname{pr}^A_X T_0$.*

(f) *For every $X \subseteq A$ of cardinality $|X| = \mu$, we have $\operatorname{id}_A|^A_X \in \operatorname{pr}^A_X S_0$.*

(g) *For every $X \subseteq A$ of cardinality $|X| \leq m$ there exists some arity $n \in \mathbb{N}$, some $\lambda \in \operatorname{Term}^{(n)}(\mathbf{A})$ and $(f_1, \ldots, f_n) \in F^n$ such that*

$$(\lambda \circ (f_1, \ldots, f_n))|^A_X = \operatorname{id}_A|^A_X.$$

(h) *For every $X \subseteq A$ of cardinality $|X| = \mu$ there exists some arity $n \in \mathbb{N}$, some $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ and $(f_1, \ldots, f_n) \in F^n$ such that $(\lambda \circ (f_1, \ldots, f_n))|^A_X = \operatorname{id}_A|^A_X$.*

(i) *$\underset{\sim}{\mathbf{A}}$ is a jointly finite m-local retract of $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.*

(j) *$\underset{\sim}{\mathbf{A}}$ is a μ-local retract of $\prod_{f \in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.*

(k) *There is an index set Φ and a mapping $\tilde{V} : \Phi \longrightarrow \mathcal{V}$ such that $\underset{\sim}{\mathbf{A}}$ is a μ-local retract of $\prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$.*

*If $\mathbf{A}$ is assumed to be μ-*Artin*ian, then these conditions are equivalent to the following:*

(l) $\operatorname{Cruc}^{(\mu)}(\mathbf{A}) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.

(m) *For every $X \subseteq A$ of cardinality $|X| = \mu$, the collection $\mathcal{V}$ separates all μ-crucial pairs $S \prec T \subseteq \operatorname{pr}^A_X T_0$.*

Proof: Almost everything follows directly from Theorem 3.4.31 by letting $e = \operatorname{id}_A$ and, consequently setting $U = A$. So we only need to deal with the additional claims under the assumption that $\mathbf{A}$ is Artinian of degree μ. We shall argue that (a) $\Leftrightarrow$ (l) $\Rightarrow$ (m) $\Rightarrow$ (d).

Clearly, item (a) implies that all distinct pairs of μ-ary invariant relations of $\mathbf{A}$ are separated in $\mathcal{V}$. By item (b) of Corollary 3.4.21, this is equivalent to item (l) since $\mathbf{A}$ was μ-Artinian. It is obvious that (m) follows from item (l). To achieve equivalence we derive (d) from (m). For this consider any $X \subseteq A$ of cardinality μ. Using Corollary 3.4.21(a) for $\top \subseteq A^\mu$ corresponding to $\operatorname{pr}^A_X T_0 \subseteq A^X$ (via some fixed bijection between μ and X), we obtain from (m) that $\mathcal{V}$ separates all distinct μ-ary invariants (subuniverses of $\mathbf{A}^X$) contained in $\operatorname{pr}^A_X T_0$. As $X \subseteq A$ with $|X| = \mu$ was arbitrary, this condition in particular entails the truth of item (d). $\square$

In the second corollary we address a useful observation in case that U is finite.

3.4.34 Corollary. *Suppose the assumptions of Theorem 3.4.31 hold and the considered neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ is finite. Let furthermore $m \in \mathbb{N}$ such that $m \geq |U|$, then it is $\mu = |U|$ and the condition $U \leq_{\mathrm{cov}} \mathcal{V}$ is equivalent to each of the statements listed in Theorem 3.4.31 for $\mu = |U|$. This means in particular, that in items (d), (e), (f) and (h) of Theorem 3.4.31 only $X = U$ needs to be considered.*

In particular, the collection $\mathcal{V}$ separates all pairs of invariant relations distinguished by U if and only if it separates those of arity $|U|$. This means that for finite U, discussing the separation property for special arities in general is the same as discussing it for all arities simultaneously.

Proof: It is clear that $U \leq_{\mathrm{cov}} \mathcal{V}$ entails the separation property (3.10) for all pairs of invariant relations $S \subseteq T$ of arity $|U|$. This is item (c) of Theorem 3.4.31.

For the converse we assume the condition in 3.4.31(c) and need to verify (3.10) for arbitrary relations $S, T \in \operatorname{Inv}^{(k)} \mathbf{A}$ of any arity $k \in \mathbb{N}$. Define $m' := \max\{|U|, k\}$, then $m' \geq k$ and $\mu' := \min\{m', |U|\} = |U| = \mu$. By Theorem 3.4.31 applied to m' and μ', the assumed separation condition for invariant relations $S \subseteq T$ of arity $\mu' = |U| = \mu$ is equivalent to having (3.10) for invariants of any arity less than or equal to m'. In particular, we obtain that it is true for relations of arity $k \leq m'$ as to be shown. □

The previous corollary also entails that the full cover property (for relations of every arity) only needs to be discussed for infinite neighbourhoods U. In this case we have $\mu = \min\{m, |U|\} = m$ for every $m \in \mathbb{N}$, whereby the content of the following result becomes a direct consequence of Theorem 3.4.31. Using a slightly different argument, we can see that the statement remains true also for finite neighbourhoods U, and hence for any $U \in \operatorname{Neigh} \mathbf{A}$ in general.

For finite neighbourhoods Corollary 3.4.34 yields better characterisations of the cover relation, but for general (infinite) neighbourhoods the following corollary is as good as one can get. We mention as well that parts of this result also occur as Theorem 3.4.6 in [Beh09] for finite algebras, and as parts of Theorem 3.6.7 in [Sch12] for the discrete case of topological algebras on possibly infinite carrier sets.

3.4.35 Corollary. *Suppose everything that was assumed in Theorem 3.4.31 but ignore the fixed cardinal $m \in \mathbb{N}$. Then the following facts[20] are equivalent:*

(a) The collection $\mathcal{V}$ covers U.

(b) For every $m \in \mathbb{N}$, [$m \leq |U|$,] the collection $\mathcal{V}$ separates all m-ary invariant relations of $\mathbf{A}$ that are separated in U.

[20] The additions in square brackets yield alternative formulations that are equivalent to all the other statements with or without additions in square brackets.

(c) *For every* $m \in \mathbb{N}_+$, *[*$m \leq |U|$*,] the collection* $\mathcal{V}$ *separates all pairs* $S \subseteq T$ *of* m*-ary invariant relations* $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ *that are separated in* U*, and we have that* $\emptyset \in \operatorname{Inv} \mathbf{A}$ *implies* $\mathcal{V} \neq \emptyset$.

(d) *For every* $m \in \mathbb{N}$, *[*$m \leq |U|$*,] the collection* $\mathcal{V}$ *separates all pairs* $S \subseteq T$ *of* m*-ary invariant relations* $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ *that are separated in* U.

(e) *For every finite subset* $X \subseteq U$ *the collection* $\mathcal{V}$ *separates the invariant relations* $\operatorname{pr}_X^A S_0$ *and* $\operatorname{pr}_X^A T_0$ *belonging to* $\operatorname{Inv}^{(|X|)} \mathbf{A}$ *(via some arbitrary but fixed indexing bijection between* X *and its finite cardinality) if their restrictions to* U *are distinct.*

(f) *For every finite subset* $X \subseteq U$*, we have* $\left(\operatorname{pr}_X^A S_0\right)\restriction_U = \left(\operatorname{pr}_X^A T_0\right)\restriction_U$.

(g) *For every finite subset* $X \subseteq U$*, we have* $e|_X^A \in \operatorname{pr}_X^A (e \circ [S_0])$.

(h) *For every finite subset* $X \subseteq A$*, there exists some arity* $n \in \mathbb{N}$*, some operation* $\lambda \in \operatorname{Term}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that*

$$(e \circ \lambda \circ (f_1, \ldots, f_n))\,|_X^A = e|_X^A.$$

(i) *For every finite subset* $X \subseteq U$*, there exists some arity* $n \in \mathbb{N}$*, some operation* $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ *and* $(f_1, \ldots, f_n) \in F^n$ *such that* $(\lambda \circ (f_1, \ldots, f_n))\,|_X^A = e|_X^A$.

(j) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.

(k) *For every* $m \in \mathbb{N}$, *[*$m \leq |U|$*,] the structure* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a jointly finite* m*-local retract of* $\left(\underset{\sim}{\mathbf{A}}\restriction_{V(f)}\right)_{f \in F}$.

(l) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a local retract of* $\prod_{f \in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.

(m) *For every* $m \in \mathbb{N}$, *[*$m \leq |U|$*,] the structure* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is an* m*-local retract of the product* $\prod_{f \in F} \underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.

(n) *For every* $m \in \mathbb{N}$, $m \leq |U|$*, there is an index set* Φ_m *and a map* $\tilde{V}_m \colon \Phi_m \longrightarrow \mathcal{V}$ *such that the structure* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is an* m*-local retract of the product* $\prod_{\varphi \in \Phi_m} \underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}_m(\varphi)}$.

If U *is poly-*ARTIN*ian*[21]*, and* $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}|_U$ *is a collection of subneighbourhoods, then these conditions are equivalent to each of the following:*

(o) *For every* $m \in \mathbb{N}$, $m \leq |U|$*, and all* $S, T \in \operatorname{Inv}^{(m)} \mathbf{A}$ *such that* $S \subseteq T$ *and the restrictions* $S\restriction_U \prec T\restriction_U$ *form an* m*-crucial pair of* $\mathbf{A}|_U$*, the collection* $\mathcal{V}$ *separates* S *from* T.

(p) *For every finite subset* $X \subseteq U$ *and all invariants* $S, T \in \operatorname{Inv}^{(|X|)} \mathbf{A}$*, where* $S \subseteq T$ *and* $S\restriction_U \prec T\restriction_U \subseteq \operatorname{pr}_X^A T_0$ *form an* $|X|$*-crucial pair of* $\mathbf{A}|_U$*, the collection* $\mathcal{V}$ *separates* S *and* T.

[21] For a finite neighbourhood U this is no condition at all.

If the neighbourhood U is finite, then these facts are equivalent to:

(q) $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a (global) retract of* $\prod_{f\in F}\underset{\sim}{\mathbf{A}}\restriction_{V(f)}$.

(r) *There is a [finite] index set* Φ *and a mapping* $\tilde{V}\colon \Phi \longrightarrow \mathcal{V}$ *such that* $\underset{\sim}{\mathbf{A}}\restriction_U$ *is a (global) retract of* $\prod_{\varphi\in\Phi}\underset{\sim}{\mathbf{A}}\restriction_{\tilde{V}(\varphi)}$.

If (x) is a statement of this corollary containing an optional text block OPT in square brackets, we shall refer to the statement obtained from (x) by assuming the addition OPT as [x]. Of course, these additions are only of interest if the neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ is finite. For infinite U, all statements (x) are trivially equivalent to [x].

Proof: Let us first discuss the unadorned statements without square brackets. Equivalence of items (a) and (b) is just the definition (see 3.4.2). Equivalence of (c) and (d) holds by Lemma 3.4.28(c). The equivalence of items (b), (d), (e), (f), (g), (h), (i), (j), (k), (l) and (m) follows from Corollary 3.4.32 together with the definition of (jointly finite) local retract because $\mathbb{N} = \bigcup_{m\in\mathbb{N}}\{k\in\mathbb{N} \mid k \leq m\}$.

Now we consider the alternative formulations. If (x) is a (plain) statement of the corollary, then [x] denotes the statement obtained from (x) by adding the text block in square brackets. Since every finite cardinal lies below any infinite one, the additions in square brackets are no restrictions of the universal quantifications in all adorned statements if U is infinite. Thus let us assume that U is finite. Using $m = |U|$ in Corollary 3.4.32, we may infer that items [b], [d], (e), [k] and [m] are equivalent (for [k] and [m] keep in mind the remark after Definition 3.4.25). Furthermore, by Lemma 3.4.28(c) statements [c] and [d] are equivalent. Moreover, [m] trivially implies item (n), even in such a way that $\Phi_m = F$ and $\tilde{V}_m = V$ do not depend on $m \in \mathbb{N}$. Finally, another application of Corollary 3.4.32 yields (n) $\Rightarrow$ [*b*].

It remains to discuss the equivalence of items (o) and (p) in case that $\mathcal{V} \subseteq \mathfrak{P}(U)$ and U is poly-Artinian. One could infer this from Corollary 3.4.33 but we think that a direct proof is simpler to understand. The implications (b) $\Rightarrow$ (o) $\Rightarrow$ (p) are evidently specialisations. We only need to prove that (p) entails the truth of item (e). Thus, let us consider any finite subset $X \subseteq U$ and set $m := |X| \leq |U|$. Since $\mathbf{A}|_U$ is m-Artinian, we can use Corollary 3.4.20(a) for $\top = \operatorname{pr}^A_X T_0$ to show that all m-ary invariants $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ whose restrictions are subject to the conditions $S\restriction_U, T\restriction_U \subseteq \operatorname{pr}^A_X T_0$ (via some fixed bijection between X and m) and $S\restriction_U \neq T\restriction_U$ are separated in $\mathcal{V}$. This implies that item (e) holds.

From item (m) we infer that $\underset{\sim}{\mathbf{A}}\restriction_U$ is an m-local retract of $\prod_{f\in F}\underset{\sim}{\mathbf{A}}\restriction_{V(f)}$ for every $m \in \mathbb{N}$. For finite U this is especially true for $m = |U|$. Consequently, for $X = U$ of cardinality $|X| = |U| = m$, one can find relational morphisms that compose to the identity on $X = U$. In other words, this pair of local retraction and co-retraction is actually global, proving (q). Conversely, if item (q) holds, then there is a pair of global retraction and co-retraction between $\underset{\sim}{\mathbf{A}}\restriction_U$ and $\prod_{f\in F}\underset{\sim}{\mathbf{A}}\restriction_{V(f)}$. Restricting its composition to every finite subset of U, we may infer the truth of (l).

Similarly, for finite U, item (k) implies that $\underset{\approx}{\mathbf{A}}\restriction_U$ is a jointly finite $|U|$-local retract of $\left(\underset{\approx}{\mathbf{A}}\restriction_{V(f)}\right)_{f\in F}$. This means for the subset $X = U$ there is a finite subset $\Phi \subseteq F$ such that we have relational morphisms between $\underset{\approx}{\mathbf{A}}\restriction_U$ and $\prod_{\varphi\in\Phi} \underset{\approx}{\mathbf{A}}\restriction_{V(\varphi)}$ whose composition is the identity on $X = U$. This establishes that these morphisms are a pair of global retraction and co-retraction, proving [r] for $\tilde{V} := V|_{\Phi}^{\mathcal{V}}$. Clearly, item [r] implies (r), and the latter entails the truth of (n) by putting $\Phi_m := \Phi$ and $\tilde{V}_m := \tilde{V}$ for all $m \in \mathbb{N}$ (the composition of the global morphisms can simply be restricted to finite subsets to get the local condition). $\square$

We refrain from giving an explicit corollary of this statement for the case $U = A$. Instead, as a last consequence, we celebrate the important case of covers of *finite* algebras. Here all previous conditions can be combined. The resulting characterisation subsumes Theorem 3.3 in [Kea01], Corollary 3.4.7 in [Beh09], Theorem 3.1 in [Beh12] and partially Corollary 3.6.8 in [Sch12] for the case of discrete algebras.

3.4.36 Corollary. *Suppose everything that was assumed in Corollary 3.4.33 and let $m \in \mathbb{N}$ satisfy $m \geq |A|$, in particular $\mathbf{A}$ must be finite. Then the following facts are equivalent:*

(a) $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$.

(b) The collection $\mathcal{V}$ separates all pairs of distinct at most m-ary invariant relations of $\mathbf{A}$.

(c) The collection $\mathcal{V}$ separates all pairs $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$ of m-ary invariant relations satisfying $S \subsetneq T$, and if $\emptyset \in \mathrm{Inv}\,\mathbf{A}$, then $\mathcal{V} \neq \emptyset$.

(d) The collection $\mathcal{V}$ separates all pairs $S, T \in \mathrm{Inv}^{(|A|)}\mathbf{A}$ of $|A|$-ary invariant relations satisfying $S \subsetneq T$.

(e) $\mathrm{Cruc}^{(|A|)}(\mathbf{A}) \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$.

(f) $\mathcal{V}$ separates all $|A|$-crucial pairs $S \prec T \subseteq T_0$.

(g) If $S_0 \neq T_0$, then $(S_0, T_0) \in \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$.

(h) $S_0 = T_0$.

(i) $\mathrm{id}_A \in S_0$.

(j) There exists some term operation $\lambda \in \mathrm{Term}^{(|F|)}(\mathbf{A})$ such that $\lambda \circ (f)_{f\in F} = \mathrm{id}_A$.

(k) There exists some arity $n \in \mathbb{N}$, some $\lambda \in \mathrm{Clo}^{(n)}(\mathbf{A})$ and $(f_1, \ldots, f_n) \in F^n$ such that $\lambda \circ (f_1, \ldots, f_n) = \mathrm{id}_A$.

(l) $\underset{\approx}{\mathbf{A}}$ is a jointly finite local retract of $\left(\underset{\approx}{\mathbf{A}}\restriction_{V(f)}\right)_{f\in F}$.

(m) *$\underset{\sim}{\mathbf{A}}$ is a retract of the finite product $\prod_{f\in F}\underset{\sim}{\mathbf{A}}{\restriction_{V(f)}}$.*

(n) *There is a [finite] index set Φ and a mapping $\tilde{V}\colon \Phi \longrightarrow \mathcal{V}$ such that $\underset{\sim}{\mathbf{A}}$ is a retract of $\prod_{\varphi\in\Phi}\underset{\sim}{\mathbf{A}}{\restriction_{\tilde{V}(\varphi)}}$.*

Proof: We derive this corollary by letting $U = A$ in Corollary 3.4.34. By choice of $m \geq |A|$ we then have $\mu = |A|$. Using this information, it is easy to see that items (b), (c), (d), (g), (h), (i), (k) are mutually respectively (in this order) equivalent to items (a), (b), (c), (d), (e), (f), (h) of Corollary 3.4.34, which characterise the condition $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$, i.e. item (a). Since $\mathbf{A}$ was assumed to be finite, it is of course poly-Artinian, such that (b) is equivalent to each of items (e) and (f) by Corollary 3.4.33. Item (l) is equivalent to (a) by letting $U = A$ in Corollary 3.4.35.

From the same result we infer that (a) implies that $\underset{\sim}{\mathbf{A}}$ is a retract of $\prod_{f\in F}\underset{\sim}{\mathbf{A}}{\restriction_{V(f)}}$, which is item (m). Since $\mathbf{A}$ is finite, also F is a finite set of functions. So (m) implies (n) with finite index set $\Phi := F$ and $\tilde{V} := V$. Of course, item (n) with finiteness condition on Φ implies the same item without it. This is exactly item (r) of Corollary 3.4.35 for $U = A$. Using the same corollary, we know that this is equivalent to $A = U \leq_{\mathrm{cov}} \mathcal{V}$, i.e. item (a). Thus, we have demonstrated the implications (a) $\Rightarrow$ (m) $\Rightarrow$ (n) $\Rightarrow$ (a).

Finally, we show (i) $\Rightarrow$ (j) $\Rightarrow$ (k). The argument is very similar to the one used in the proof of Lemma 3.4.29, but for finite sets, we can derive a stronger condition. Namely, let $n := |F|$ be the cardinality of the finite set F. Thus, the relation S_0 is finitely generated by precisely n elements, and this implies $S_0 = \langle F\rangle_{\mathbf{A}^A} = \left\{ g\circ (f(i))_{i\in n} \;\middle|\; g \in \mathrm{Term}^{(n)}\,(\mathbf{A})\right\}$, where $f\colon n \longrightarrow F$ is any fixed bijection between F and its cardinality. By (i), we have $\mathrm{id}_A \in S_0$, i.e. there exists some n-ary term operation $\lambda \in \mathrm{Term}^{(n)}\,(\mathbf{A})$ such that $\mathrm{id}_A = \lambda\circ (f(i))_{i\in n}$, which we have stated in (j) as $\mathrm{id}_A = \lambda \circ (f)_{f\in F}$, suppressing the bijection for brevity. Certainly, item (j) implies (k) since $\mathrm{Term}\,(\mathbf{A}) \subseteq \mathrm{Clo}\,(\mathbf{A})$ (in the finite case we even have equality here) and F is finite. $\square$

In the case of a finite algebra we just saw that we do not only get a jointly finite local retraction as in the general case but a real retraction from the cover property. In view of Lemma 3.4.22 this is certainly a desirable situation. Moreover, if $e\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ is a retraction between relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ with corresponding co-retraction $m\colon \underset{\sim}{\mathbf{B}} \longrightarrow \underset{\sim}{\mathbf{A}}$, then $m\circ e\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{A}}$ is an idempotent endomorphism of $\underset{\sim}{\mathbf{A}}$, corresponding to a neighbourhood $U := \operatorname{im} m\circ e \subseteq A$ of the operational counterpart of $\underset{\sim}{\mathbf{A}}$. It is easy to see that $\underset{\sim}{\mathbf{A}}{\restriction_U}$ and $\underset{\sim}{\mathbf{B}}$ are isomorphic via restrictions of e and m as isomorphisms. In the special case arising from covers in finite algebras, the role of $\underset{\sim}{\mathbf{A}}$ is played by a product of restricted relational counterparts of the original algebra, whose relational counterpart takes the position of $\underset{\sim}{\mathbf{B}}$ above. Thus, getting a retraction instead of just a jointly finite local retraction can be seen as a strong form of global reconstruction. In this connection, we call an equality between clone operations as in item (k) of Corollary 3.4.36, which is responsible for such a reconstruction, a *decomposition equation*.

Motivated by the finite case, in a further corollary we establish a more general condition than finiteness, also guaranteeing this strong way of reconstruction. For a better understanding of this result, let us quickly revive the notion of n-local finiteness (cp. p. 44). An algebra is said to be n-locally finite if all its n-generated subuniverses are finite. It is *locally finite* if it is n-locally finite for every $n \in \mathbb{N}$. Moreover, a class of algebras is called *n-locally finite* (*locally finite*, respectively) if each of its members has this property.

3.4.37 Corollary. *Let $\mathbf{A}$ be an algebra, $e \in \operatorname{Idem}\mathbf{A}$, $E \subseteq \operatorname{Idem}\mathbf{A}$, $U := \operatorname{im} e$ and $\mathcal{V} := \{\operatorname{im} e' \mid e' \in E\}$, and put $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V}\colon \operatorname{im} f \subseteq V \right\}$ as in Theorem 3.4.31. Assume moreover, that $|F| = N \in \mathbb{N}$, that $\mathbf{A}$ is N-locally finite and let $\{f_1, \ldots, f_N\} = F$ be an enumeration of F. Then $\mathcal{V}$ covers U if and only if there exists a clone operation $\lambda \in \operatorname{Clo}^{(N)}(\mathbf{A})$ such that $\lambda \circ (f_1, \ldots, f_N) = e$.*

Proof: The stated condition is clearly sufficient for covering. Namely, if an operation $\lambda \in \operatorname{Clo}^{(N)}(\mathbf{A})$ satisfies $\lambda \circ (f_1, \ldots, f_N) = e$, then we certainly have the equality $(\lambda \circ (f_1, \ldots, f_N))|_X^A = e|_X^A$ for any finite subset $X \subseteq U$. Thus, Corollary 3.4.35(i) implies $U \leq_{\mathrm{cov}} \mathcal{V}$.

For the converse implication we first note that item (h) of the same corollary states that for every finite subset $X \subseteq A$, there exist an arity $n \in \mathbb{N}$, a term operation $\tilde{\lambda}_X \in \operatorname{Term}^{(n)}(\mathbf{A})$ and a tuple $(g_1, \ldots, g_n) \in F^n$ such that the restriction $\left(e \circ \tilde{\lambda}_X \circ (g_1, \ldots, g_n)\right)|_X^A$ equals $e|_X^A$. Putting $\hat{\lambda}_X := e \circ \tilde{\lambda}_X$ we hence obtain a clone operation $\hat{\lambda}_X \in \operatorname{Clo}^{(n)}(\mathbf{A})$ satisfying $\left(\hat{\lambda}_X \circ (g_1, \ldots, g_n)\right)|_X^A = e|_X^A$. Now the arity $n \in \mathbb{N}$ and the tuple $(g_1, \ldots, g_n)$ still depend on the subset X. However, by appropriately identifying coordinates in $\hat{\lambda}_X$, then adding fictitious variables as needed and finally permuting coordinates, we can construct an N-ary operation λ_X fulfilling

$$(\lambda_X \circ (f_1, \ldots, f_N))|_X^A = \left(\hat{\lambda}_X \circ (g_1, \ldots, g_n)\right)|_X^A = e|_X^A.$$

Since $\operatorname{Clo}(\mathbf{A})$ is a clone and $\hat{\lambda}_X \in \operatorname{Clo}(\mathbf{A})$, it follows that λ_X belongs to $\operatorname{Clo}^{(N)}(\mathbf{A})$.

Abbreviating the tupling $(f_1, \ldots, f_N) : A \longrightarrow A^N$ by $\mathbf{f}$, we have shown by now that the subset $G_X := \left\{ \lambda \in \operatorname{Clo}^{(N)}(\mathbf{A}) \;\middle|\; (\lambda \circ \mathbf{f})|_X^A = e|_X^A \right\}$ of $\operatorname{Clo}^{(N)}(\mathbf{A})$ is nonempty for any finite $X \subseteq A$.

Moreover, G_X is locally closed, i.e. $\operatorname{Loc}_A G_X = G_X$. Indeed, if some operation $\xi \in A^{A^N}$ belongs to $\operatorname{Loc}_A G_X$, then $\xi \in \operatorname{Loc}_A G_X \subseteq \operatorname{Loc}_A \operatorname{Clo}^{(N)}(\mathbf{A}) = \operatorname{Clo}^{(N)}(\mathbf{A})$ as $G_X \subseteq \operatorname{Clo}^{(N)}(\mathbf{A})$, Loc_A is monotone and $\operatorname{Clo}(\mathbf{A})$ is a locally closed clone. We know that ξ is interpolated by functions from G_X on any finite subset of its domain A^N. In particular, if $X \subseteq A$ is finite, then $\mathbf{f}[X] \subseteq A^N$ is finite, too, and hence there exists an interpolant $\lambda \in G_X$ for this set. This means

$$(\xi \circ \mathbf{f})|_X^A = \xi|_{\mathbf{f}[X]}^A \circ \mathbf{f}|_X^{\mathbf{f}[X]} = \lambda|_{\mathbf{f}[X]}^A \circ \mathbf{f}|_X^{\mathbf{f}[X]} = (\lambda \circ \mathbf{f})|_X^A = e|_X^A,$$

where the last equality holds due to $\lambda \in G_X$. So we have established $\xi \in G_X$, and hence $\operatorname{Loc}_A G_X \subseteq G_X$. The converse inclusion is true as Loc_A is a closure operator and thus extensive.

Furthermore, if $k \in \mathbb{N}$ and $X_1, \ldots, X_k \subseteq A$ are finitely many finite subsets of A, then so is $X := \bigcup_{i=1}^{k} X_i$. Since $G_X \subseteq \bigcap_{i=1}^{k} G_{X_i}$ and $G_X \neq \emptyset$, the latter intersection is non-empty, too. Thus, the collection $\mathcal{G} := \{G_X \mid X \subseteq A \text{ finite}\}$ has the finite intersection property.

Since $\mathbf{A}$ is N-locally finite, Remark 5.2.9 of [Sch12] states that $\mathrm{Clo}^{(N)}(\mathbf{A})$ is a compact subspace of A^{A^N} when A is understood as a discrete topological space. It is a well-known fact from topology that compactness of a topological space is equivalent to the assertion that arbitrary intersections of collections of closed subsets having the finite intersection property are non-empty. Since for A understood as a discrete space, the closed subsets of A^{A^N} are precisely the locally closed sets of operations from A^N into A, we can conclude that $\bigcap \mathcal{G} = \bigcap_{X \subseteq A, X \text{ finite}} G_X \neq \emptyset$.

Thus, let us pick some $\lambda \in \bigcap \mathcal{G}$. By definition, it satisfies $\lambda \in G_X$ for any finite subset $X \subseteq A$, in particular, we have $\lambda \in G_{\{x\}}$ for any $x \in A$. This means that λ belongs to $\mathrm{Clo}^{(N)}(\mathbf{A})$ and that $(\lambda \circ \mathbf{f})|_{\{x\}}^{A} = e|_{\{x\}}^{A}$ holds for all $x \in A$, i.e. $\lambda(\mathbf{f}(x)) = e(x)$. This demonstrates $\lambda \circ \mathbf{f} = e$ as desired. □

We shall see in Collary 3.5.13 that for algebras in a 1-locally finite variety[22] the set $\mathrm{Clo}^{(1)}(\mathbf{A})$ is finite. Therefore, also the set F occurring in the previous corollary is finite. This means that Corollary 3.4.37 is particularly useful for locally finite algebras in 1-locally finite varieties. A special case are of course algebras in locally finite varieties, including e.g. all varieties generated by finite algebras.

In Corollaries 3.4.35 and 3.4.36 we have seen that, using a cover $\mathcal{V}$ of $\mathbf{A}$, we can reconstruct the algebra $\mathbf{A}|_A = \langle A; \mathrm{Clo}(\mathbf{A})\rangle$ from its restrictions $\mathbf{A}|_V$ to neighbourhoods $V \in \mathcal{V}$: namely, we take the relational counterparts $\underset{\sim}{\mathbf{A}}\restriction_V$ of $\mathbf{A}|_V$ for all $V \in \mathcal{V}$. Then the relational counterpart $\underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle$ is a local retract of some product $\underset{\sim}{\mathbf{P}}$ of relational structures in $\{\underset{\sim}{\mathbf{A}}\restriction_V \mid V \in \mathcal{V}\}$. Hence, in principle, $\underset{\sim}{\mathbf{A}}$ can be derived from the product $\underset{\sim}{\mathbf{P}}$ together with morphisms witnessing the local retract property (cf. Lemma 3.4.27). Note that in this context not for all $V \in \mathcal{V}$ the restriction $\underset{\sim}{\mathbf{A}}\restriction_V$ necessarily occurs as a factor of this product. Those which do not can be omitted from $\mathcal{V}$ without destroying the cover property. However, conversely, it is definitely possible (and usually happens) that one restriction $\underset{\sim}{\mathbf{A}}\restriction_V$ occurs several times as a factor of $\underset{\sim}{\mathbf{P}}$. Now from the local retract $\underset{\sim}{\mathbf{A}} = \langle A; \mathrm{Inv}\,\mathbf{A}\rangle$ of this product we can, by applying the operator Pol_A, (re-)obtain the algebra $\mathbf{A}|_A = \langle A; \mathrm{Clo}(\mathbf{A})\rangle$, which is locally term equivalent to $\mathbf{A}$.

So, instead of working with the algebra $\mathbf{A}$ or $\mathbf{A}|_A$, or $\underset{\sim}{\mathbf{A}}$, we can in principle also study $\underset{\sim}{\mathbf{P}}$ or its operational dual. The construction of the latter algebra motivates the following definition (cf. Definition 3.4 in [Kea01] and Definition 3.4.8 in [Beh09]):

3.4.38 Definition. Let $\mathbf{A}$ be an algebra and $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ be a set of neighbourhoods of $\mathbf{A}$. Furthermore, let Φ be an index set and $V\colon \Phi \longrightarrow \mathcal{V}$ be an enumeration of some not necessarily different neighbourhoods of $\mathbf{A}$. Denote by

$$\underset{\sim}{\mathbf{P}} = \left\langle P; \left(\left(S^{\underset{\sim}{\mathbf{P}}}\right)_{S \in \mathrm{Inv}^{(m)}\mathbf{A}}\right)_{m \in \mathbb{N}}\right\rangle := \prod_{\varphi \in \Phi} \underset{\sim}{\mathbf{A}}\restriction_{V(\varphi)}$$

[22]This is actually equivalent to generating a 1-locally finite variety.

the product of the restricted relational counterparts of $\mathbf{A}$, where $S^{\underset{\sim}{\mathbf{P}}} = \coprod_{\varphi\in\Phi} S{\restriction}_{V(\varphi)}$ for $S \in \operatorname{Inv} \mathbf{A}$. Then

$$\underset{\varphi\in\Phi}{\boxtimes} \mathbf{A}|_{V(\varphi)} := \left\langle P; \operatorname{Pol}_P \left\{ S^{\underset{\sim}{\mathbf{P}}} \;\middle|\; S \in \operatorname{Inv} \mathbf{A} \right\} \right\rangle$$

is called *matrix product* of $\left(\mathbf{A}|_{V(\varphi)}\right)_{\varphi\in\Phi}$. $\Diamond$

Note that the matrix product is not only defined for covers but for an arbitrary set of neighbourhoods of $\mathbf{A}$. Although the index set Φ can be arbitrary, it will mostly be finite in our applications. In case of a finite index set, we will restrict ourselves w.l.o.g. to sets of the form $\Phi = \{1, \ldots, m\}$ where $m \in \mathbb{N}$. Then we are going to write U_i for $U(i)$, $(1 \leq i \leq m)$, and $\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m}$ for $\boxtimes_{\varphi\in\Phi} \mathbf{A}|_{U(\varphi)}$.

If the index set Φ in Definition 3.4.38 is empty, then the relational structure $\underset{\sim}{\mathbf{P}}$ is the one-element terminal structure, where $P = \{\emptyset\}$ and $S^{\underset{\sim}{\mathbf{P}}} = P^m$ for every $S \in \operatorname{Inv}^{(m)} \mathbf{A}$ and all $m \in \mathbb{N}$. Therefore, the empty matrix product of (the empty collection of) any algebra is the one-element structure equipped with the full clone O_P of operations.

There is a related notion, called *matrix power*, closely corresponding to the case, when $V\colon \Phi \longrightarrow \mathcal{V}$ in Definition 3.4.38 is constant, i.e. has a singleton range $\{U\}$ (see also Lemma 3.4.39 below). Of course, no generality is lost in assuming $U = A$ for this kind of construction. Particularly, *finite* matrix powers have gained importance in general algebra via R. McKenzie's famous result characterising categorical equivalence of algebras[23] without nullary operations ([McK96]). His characterisation theorem states that one of them has to be isomorphic to an algebra being term equivalent to a finite matrix power of the other restricted to a neighbourhood V (of the matrix power) given as the image of an idempotent invertible *term* operation. In this context, invertibility means that a decomposition equation as in Corollary 3.4.36(k) for $\mathcal{V} = \{V\}$ holds. In case of finite algebras, this is equivalent to saying that $\{V\}$ covers the matrix power.

McKenzie's article triggered a number of other publications dealing with categorical equivalence of algebras or varieties. Among them are, for instance, [BB96] using the notion of matrix power and McKenzie's result to characterise categorical equivalence of finite subalgebra-primal, congruence-primal and automorphism-primal algebras. In [BB99] the same is done for finitely generated varieties of modes (idempotent, entropic algebras), and for semilattices, in particular. In [BB98] a detailed analysis of matrix powers and invertible terms is used to obtain an algorithm checking for two finite algebras if they are categorically equivalent. In [Zád97b], L. Zádori draws on McKenzie's theorem to prove a different characterisation of categorical equivalence of algebras placing a condition on associated relational

[23] Two algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *categorically equivalent* if the varieties they generate are equivalent as full subcategories of the categories of all algebras of the same type as $\mathbf{A}$ and $\mathbf{B}$, respectively, with the additional requirement that one of the equivalence functors has to send $\mathbf{A}$ to $\mathbf{B}$. A more precise definition can be obtained from the cited references and is not necessary here, as we are not going to work with categorical equivalences in detail.

structures. On the same basis Denecke and Lüders show in [DL01] that two finite algebras are categorically equivalent if and only if their relational clones are isomorphic.

However, matrix powers have also found other applications, such as in [KKV98], where they are used while developing a coordinatisation theory for multitraces, or in [KKV99] and [KKSW02].

We emphasise that the notion of matrix power in all these publications is defined in a different way than we introduced it above. The commonly used definition is as in the following lemma, but letting all unary idempotents be the identity operation on A and replacing $\mathrm{Clo}\,(\mathbf{A})$ by $\mathrm{Term}\,(\mathbf{A})$. Thus, for algebras with a locally closed clone of term operations, especially finite ones, Lemma 3.4.39 shows that our matrix product is truly related to matrix powers as occurring in the literature. Conversely, via McKenzie's theorem, it reveals an intimate connection between covers (particularly of finite algebras) and categorical equivalence of algebras.

3.4.39 Lemma. *Let* $\mathbf{A}$ *be an algebra and* $(e_1,\dots,e_m) \in (\mathrm{Idem}\,\mathbf{A})^m$ *be a finite enumeration of* $m \in \mathbb{N}$ *not necessarily different idempotent clone operations of* $\mathbf{A}$*. Defining* $U_i := \mathrm{im}\, e_i$ *for every* $1 \le i \le m$*, we have*

$$\mathbf{A}|_{U_1} \boxtimes \dots \boxtimes \mathbf{A}|_{U_m} = \langle U_1 \times \dots \times U_m;\, e_1(\mathrm{Clo}\,(\mathbf{A})) \boxtimes \dots \boxtimes e_m(\mathrm{Clo}\,(\mathbf{A}))\rangle\,,$$

where for $n \in \mathbb{N}$ *the* n*-ary fundamental operations*[24] *are*

$$(e_1(\mathrm{Clo}\,(\mathbf{A})) \boxtimes \dots \boxtimes e_m(\mathrm{Clo}\,(\mathbf{A})))^{(n)} :=$$
$$\left\{ \begin{array}{lccc|c} \mathbf{f}\colon & \left(\prod_{i=1}^m U_i\right)^n & \longrightarrow & \prod_{i=1}^m U_i & \\ & (\mathbf{u}_1,\dots,\mathbf{u}_n) & \longmapsto & \begin{pmatrix} e_1 \circ f_1\,(\mathbf{u}_1 \cdots \mathbf{u}_n) \\ \vdots \\ e_m \circ f_m\,(\mathbf{u}_1 \cdots \mathbf{u}_n) \end{pmatrix} & f_1,\dots,f_m \in \mathrm{Clo}^{(m\times n)}\,(\mathbf{A}) \end{array} \right\}.$$

This result is mentioned as Lemma 3.5 in [Kea01] and occurs as Lemma 3.4.9 in [Beh09] for non-nullary operations. Although matrix products were defined as untyped algebras in Definition 3.4.38, this lemma shows that it is possible to view an m-fold matrix product as an indexed structure where the set of n-ary operation symbols is given by all m-tuples of $(m \cdot n)$-ary operations in the clone of $\mathbf{A}$. If $\mathrm{Term}\,(\mathbf{A})$ is locally closed, one can of course replace clone operations by term operations in this type set. This means that it is even possible to use all m-tuples of $(m \cdot n)$-ary terms over the signature of $\mathbf{A}$ as n-ary symbols in this

[24] Here the notation $f \in \mathrm{Clo}^{(m\times n)}\,(\mathbf{A})$ means just an $(m \cdot n)$-ary operation (i.e. a function $\tilde{f} \in \mathrm{Clo}^{(m\cdot n)}\,(\mathbf{A})$) with the convention that the arguments are not provided as a long list with $m \cdot n$ entries but as columns of an $(m \times n)$-matrix. Differently stated, one constructs a vector of length $m \cdot n$ by successively concatenating the columns of the matrix and supplies this vector to the function $\tilde{f} \in \mathrm{Clo}^{(m\cdot n)}\,(\mathbf{A})$ in order to calculate the value of f at an $(m \times n)$-matrix of arguments.

case. Although this implies yet another tremendous blow-up of the signature, it is a purely syntactical solution requiring no further knowledge. Thus, one avoids the problem to need to know $\operatorname{Clo}(\mathbf{A})$ in order to write down our matrix product as an indexed algebra, which seems paradoxical, given the fact that we want to use localisations and matrix products to obtain a better understanding of $\mathbf{A}$ up to equality of $\operatorname{Clo}(\mathbf{A})$.

Proof: Let $\underset{\sim}{\mathbf{P}} = \left\langle P; \left(\left(\underset{\sim}{S}^{\mathbf{P}}\right)_{S\in\operatorname{Inv}^{(m)}\mathbf{A}}\right)_{m\in\mathbb{N}}\right\rangle := \prod_{i=1}^{m} \underset{\sim}{\mathbf{A}}\restriction_{U_i}$, then, in particular, we have $P = \prod_{i=1}^{m} U_i$, and, by Definition 3.4.38,

$$\mathbf{A}|_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}|_{U_m} = \left\langle P; \operatorname{Pol}_P \left\{ \underset{\sim}{S}^{\mathbf{P}} \;\middle|\; S \in \operatorname{Inv}\mathbf{A} \right\}\right\rangle.$$

So, obviously, the carrier set of the matrix product has the right form. It remains to be shown that

$$e_1(\operatorname{Clo}(\mathbf{A})) \boxtimes \cdots \boxtimes e_m(\operatorname{Clo}(\mathbf{A})) = \operatorname{Pol}_P \left\{ \underset{\sim}{S}^{\mathbf{P}} \;\middle|\; S \in \operatorname{Inv}\mathbf{A} \right\}.$$

First, we discuss the case that $m = 0$, i.e. that the matrix product has no factors. We have noted earlier that in this case the matrix product is the one-element algebra on $P = \{\emptyset\}$ equipped with all finitary operations. As $m = 0$, every operation $\mathbf{f}$ mentioned in the definition of $e_1(\operatorname{Clo}(\mathbf{A})) \boxtimes \cdots \boxtimes e_m(\operatorname{Clo}(\mathbf{A}))$ maps the n-tuple $(\emptyset, \ldots, \emptyset)$ to the empty tuple, i.e. the unique element of $P = \prod_{i\in\emptyset} U_i$. Hence, it is the unique constant n-ary operation on P, which means that we got all operations from O_P in $e_1(\operatorname{Clo}(\mathbf{A})) \boxtimes \cdots \boxtimes e_m(\operatorname{Clo}(\mathbf{A}))$.

Before we are going to discuss the case of $m \geq 1$ factors, we would like to remind the reader that, by definition of the product of relational structures, for all $k \in \mathbb{N}$ and every k-ary relation $S \in \operatorname{Inv}^{(k)}\mathbf{A}$ and all tuples $(x_1, \ldots, x_k) \in P^k$ we have that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \in \underset{\sim}{S}^{\mathbf{P}} \iff \begin{array}{ccccc} (x_1(1), & \ldots, & x_1(i), & \ldots, & x_1(m)) \\ (x_2(1), & \ldots, & x_2(i), & \ldots, & x_2(m)) \\ \vdots & & \vdots & & \vdots \\ (x_k(1), & \ldots, & x_k(i), & \ldots, & x_k(m)) \\ \Cap & & \Cap & & \Cap \\ S\restriction_{U_1} & & S\restriction_{U_i} & & S\restriction_{U_m}. \end{array} \tag{†}$$

Now we are going to show the above equality by dealing with both set inclusions separately. At the end of each part, we will make a remark on how the proof can be simplified in the case of nullary operations. We add this because the general proof presented below is, of course, valid also in the special case $n = 0$, but contains many steps that are void for nullary operations and can be omitted.

"$\subseteq$" Fix some arity $n \in \mathbb{N}$, choose $f_1, \ldots, f_m \in \operatorname{Clo}^{(m\times n)}(\mathbf{A})$ arbitrarily and construct out of these the function $\mathbf{f} \in \mathrm{O}_P^{(n)}$ as it occurs in the definition of $(e_1(\operatorname{Clo}(\mathbf{A})) \boxtimes \cdots \boxtimes e_m(\operatorname{Clo}(\mathbf{A})))^{(n)}$ in Lemma 3.4.39. It will be shown that $\mathbf{f} \in \operatorname{Pol}_P \left\{ \underset{\sim}{S}^{\mathbf{P}} \;\middle|\; S \in \operatorname{Inv}\mathbf{A} \right\}$. To this end, we consider an arbitrary $k \in \mathbb{N}$

and a relation $S \in \mathrm{Inv}^{(k)}\mathbf{A}$ and verify $\mathbf{f} \rhd S^{\mathbf{P}}_{\sim}$: Let $\mathbf{x}^1, \ldots, \mathbf{x}^n \in S^{\mathbf{P}}_{\sim}$. Then for every $1 \leq j \leq n$ the tuple $\mathbf{x}^j \in P^k$ is of the form

$$\mathbf{x}^j = \begin{pmatrix} x_1^j \\ x_2^j \\ \vdots \\ x_k^j \end{pmatrix},$$

where every $x_\nu^j = (x_\nu^j(i))_{1\leq i\leq m} \in P = \prod_{i=1}^m U_i$ for all $1 \leq \nu \leq k$. It has to be shown that the tuple

$$\begin{pmatrix} \mathbf{f}\left(x_1^1, \ldots, x_1^j, \ldots, x_1^n\right) \\ \mathbf{f}\left(x_2^1, \ldots, x_2^j, \ldots, x_2^n\right) \\ \vdots \\ \mathbf{f}\left(x_k^1, \ldots, x_k^j, \ldots, x_k^n\right) \end{pmatrix} \in S^{\mathbf{P}}_{\sim}.$$

Using (†), we have to show for all $1 \leq i \leq m$ that

$$\begin{pmatrix} \left(\mathbf{f}\left(x_1^1, \ldots, x_1^j, \ldots, x_1^n\right)\right)(i) \\ \left(\mathbf{f}\left(x_2^1, \ldots, x_2^j, \ldots, x_2^n\right)\right)(i) \\ \vdots \\ \left(\mathbf{f}\left(x_k^1, \ldots, x_k^j, \ldots, x_k^n\right)\right)(i) \end{pmatrix} \in S{\restriction_{U_i}}.$$

Applying (†) to $\mathbf{x}^j \in S^{\mathbf{P}}_{\sim}$ yields

$$\begin{pmatrix} x_1^j(i) \\ x_2^j(i) \\ \vdots \\ x_k^j(i) \end{pmatrix} \in S{\restriction_{U_i}} \subseteq S \tag{‡}$$

for $1 \leq j \leq n$ and all $1 \leq i \leq m$. Thus, for every $1 \leq i \leq m$

$$\begin{pmatrix} \left(\mathbf{f}\left(x_1^1, \ldots, x_1^j, \ldots, x_1^n\right)\right)(i) \\ \left(\mathbf{f}\left(x_2^1, \ldots, x_2^j, \ldots, x_2^n\right)\right)(i) \\ \vdots \\ \left(\mathbf{f}\left(x_k^1, \ldots, x_k^j, \ldots, x_k^n\right)\right)(i) \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} (e_i \circ f_i)\left(x_1^1 \cdots x_1^j \cdots x_1^n\right) \\ (e_i \circ f_i)\left(x_2^1 \cdots x_2^j \cdots x_2^n\right) \\ \vdots \\ (e_i \circ f_i)\left(x_k^1 \cdots x_k^j \cdots x_k^n\right) \end{pmatrix}$$

$$\overset{(2)}{=} \begin{pmatrix} (e_i \circ f_i)\left(x_1^1(1), \ldots, x_1^1(m), \ldots, x_1^j(1), \ldots, x_1^j(m), \ldots, x_1^n(1), \ldots, x_1^n(m)\right) \\ (e_i \circ f_i)\left(x_2^1(1), \ldots, x_2^1(m), \ldots, x_2^j(1), \ldots, x_2^j(m), \ldots, x_2^n(1), \ldots, x_2^n(m)\right) \\ \vdots \\ (e_i \circ f_i)\left(x_k^1(1), \ldots, x_k^1(m), \ldots, x_k^j(1), \ldots, x_k^j(m), \ldots, x_k^n(1), \ldots, x_k^n(m)\right) \end{pmatrix}$$

$$\overset{(3)}{\in} S{\restriction_{U_i}},$$

where (1) holds by definition of $\mathbf{f}$, (2) by definition of $\mathrm{Clo}^{(m\times n)}(\mathbf{A})$ and (3) because of (‡), $e_i \circ f_i \rhd S$ and $\operatorname{im} e_i = U_i$.

For $n = 0$ explicitly, we have to show that $(\mathbf{f}(\emptyset), \dots, \mathbf{f}(\emptyset)) \in S^{\mathbf{P}}_{\sim}$. By (†), this is equivalent to $((e_i \circ f_i)(\emptyset), \dots, (e_i \circ f_i)(\emptyset)) \in S\restriction_{U_i}$ for all $1 \leq i \leq m$. This is indeed true, because the nullary operation $e_i \circ f_i$ preserves S and $\operatorname{im} e_i \circ f_i \subseteq \operatorname{im} e_i = U_i$.

"⊇" Conversely, let $n \in \mathbb{N}$ and take some arbitrary n-ary compatible operation $\mathbf{f} \in \mathrm{Pol}_P^{(n)}\left\{ S^{\mathbf{P}}_{\sim} \;\middle|\; S \in \operatorname{Inv} \mathbf{A} \right\}$. It has to be shown that this function $\mathbf{f}$ belongs to $e_1(\mathrm{Clo}(\mathbf{A})) \boxtimes \cdots \boxtimes e_m(\mathrm{Clo}(\mathbf{A}))^{(n)}$. Denote for $1 \leq \iota \leq m$ the ι-th projection mapping by $pr_\iota \colon P \longrightarrow U_\iota$. Then we define

$$\begin{array}{rccc} f_\iota\colon & A^{(m\times n)} & \longrightarrow & A \\ & \begin{pmatrix} x_{1,1} & & x_{1,n} \\ \vdots & \dots & \vdots \\ x_{m,1} & & x_{m,n} \end{pmatrix} & \longmapsto & (pr_\iota \circ \mathbf{f}) \left(\begin{pmatrix} e_1(x_{1,1}) \\ \vdots \\ e_m(x_{m,1}) \end{pmatrix}, \dots, \begin{pmatrix} e_1(x_{1,n}) \\ \vdots \\ e_m(x_{m,n}) \end{pmatrix} \right). \end{array}$$

By definition of f_ι and pr_ι, we have $f_\iota = e_\iota \circ f_\iota$ for all $1 \leq \iota \leq m$. Furthermore, it follows for all

$$\left(\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{m,1} \end{pmatrix}, \cdots, \begin{pmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{pmatrix} \right) \in P^n$$

that

$$f_\iota \begin{pmatrix} u_{1,1} & & u_{1,n} \\ \vdots & \dots & \vdots \\ u_{m,1} & & u_{m,n} \end{pmatrix} = (pr_\iota \circ \mathbf{f}) \left(\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{m,1} \end{pmatrix}, \cdots, \begin{pmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{pmatrix} \right)$$

for all $1 \leq \iota \leq m$, that is to say,

$$\begin{aligned} \mathbf{f} \left(\begin{pmatrix} u_{1,1} \\ \vdots \\ u_{m,1} \end{pmatrix}, \dots, \begin{pmatrix} u_{1,n} \\ \vdots \\ u_{m,n} \end{pmatrix} \right) &= \left(f_\iota \begin{pmatrix} u_{1,1} & & u_{1,n} \\ \vdots & \dots & \vdots \\ u_{m,1} & & u_{m,n} \end{pmatrix} \right)_{1\leq\iota\leq m} \\ &= \left((e_\iota \circ f_\iota) \begin{pmatrix} u_{1,1} & & u_{1,n} \\ \vdots & \dots & \vdots \\ u_{m,1} & & u_{m,n} \end{pmatrix} \right)_{1\leq\iota\leq m}. \end{aligned}$$

It remains to be proven that $f_\iota \in \mathrm{Clo}^{(m\times n)}(\mathbf{A})$ for all $1 \leq \iota \leq m$. So for a fixed integer $1 \leq \iota \leq m$, we are going to verify that $f_\iota \in \mathrm{Pol}_A \operatorname{Inv} \mathbf{A} = \mathrm{Clo}(\mathbf{A})$. Hence, we take some $k \in \mathbb{N}$ and some $S \in \operatorname{Inv}^{(k)} \mathbf{A}$ in order to see that $f_\iota \rhd S$. For this purpose let us choose an arbitrary list of $m \cdot n$ tuples

$$x^1(1), \dots, x^1(m), x^2(1), \dots, x^2(m), \dots, x^n(1), \dots, x^n(m) \in S,$$

i.e. for all $1 \leq j \leq n$ and all $1 \leq i \leq m$ the tuple $x^j(i)$ satisfies

$$x^j(i) = \begin{pmatrix} x_1^j(i) \\ \vdots \\ x_k^j(i) \end{pmatrix} \in S. \tag{$\star$}$$

We have to show that the k-tuple

$$\left((pr_\iota \circ \mathbf{f}) \left(\begin{pmatrix} e_1(x_\nu^1(1)) \\ \vdots \\ e_m(x_\nu^1(m)) \end{pmatrix}, \dots, \begin{pmatrix} e_1(x_\nu^n(1)) \\ \vdots \\ e_m(x_\nu^n(m)) \end{pmatrix} \right) \right)_{1 \leq \nu \leq k}$$
$$= \left(f_\iota \begin{pmatrix} x_\nu^1(1) & & x_\nu^n(1) \\ \vdots & \dots & \vdots \\ x_\nu^1(m) & & x_\nu^n(m) \end{pmatrix} \right)_{1 \leq \nu \leq k}$$

belongs to S. For every $1 \leq j \leq n$ we define the k-tuple

$$\mathbf{y}_j := \begin{pmatrix} \left(e_i\left(x_1^j(i)\right)\right)_{1 \leq i \leq m} \\ \vdots \\ \left(e_i\left(x_k^j(i)\right)\right)_{1 \leq i \leq m} \end{pmatrix} \in \left(\prod_{i=1}^m U_i \right)^k = P^k.$$

Using ($\star$), we can infer that for every $1 \leq j \leq n$ it is

$$e_i \circ x^j(i) = \begin{pmatrix} e_i\left(x_1^j(i)\right) \\ \vdots \\ e_i\left(x_k^j(i)\right) \end{pmatrix} \in S{\restriction}_{U_i}$$

for all $1 \leq i \leq m$ because $e_i \rhd S$ and $\operatorname{im} e_i = U_i$. Hence, by ($\dagger$), we obtain $\mathbf{y}_j \in S^{\mathbf{P}}_{\sim}$ for every $1 \leq j \leq n$. As, by assumption, $\mathbf{f} \rhd S^{\mathbf{P}}_{\sim}$, it follows

$$\begin{pmatrix} \mathbf{f}\left(\mathbf{y}_1(1), \dots, \mathbf{y}_n(1)\right) \\ \vdots \\ \mathbf{f}\left(\mathbf{y}_1(k), \dots, \mathbf{y}_n(k)\right) \end{pmatrix} \in S^{\mathbf{P}}_{\sim},$$

which is, by ($\dagger$), equivalent to

$$\begin{pmatrix} (pr_i \circ \mathbf{f})\left(\mathbf{y}_1(1), \dots, \mathbf{y}_n(1)\right) \\ \vdots \\ (pr_i \circ \mathbf{f})\left(\mathbf{y}_1(k), \dots, \mathbf{y}_n(k)\right) \end{pmatrix} \in S{\restriction}_{U_i}$$

for all $1 \leq i \leq m$. Especially, one obtains

$$\left((pr_\iota \circ \mathbf{f}) \left(\begin{pmatrix} e_1(x_\nu^1(1)) \\ \vdots \\ e_m(x_\nu^1(m)) \end{pmatrix}, \dots, \begin{pmatrix} e_1(x_\nu^n(1)) \\ \vdots \\ e_m(x_\nu^n(m)) \end{pmatrix} \right) \right)_{1 \leq \nu \leq k}$$
$$= \left((pr_\iota \circ \mathbf{f})\left(\mathbf{y}_1(\nu), \dots, \mathbf{y}_n(\nu)\right) \right)_{1 \leq \nu \leq k} \in S{\restriction}_{U_\iota} \subseteq S.$$

Consequently, $f_\iota \rhd S$, and, therefore, $f_\iota \in \operatorname{Clo}(\mathbf{A})$.

The proof also works for $n = 0$, but can be shortened. Namely, if $\mathbf{f} \in \operatorname{Pol}^{(0)}\underset{\sim}{\mathbf{P}}$, then for $S \in \operatorname{Inv}^{(k)}\mathbf{A}$ we have $(\mathbf{f}(\emptyset), \dots, \mathbf{f}(\emptyset)) \in S^{\underset{\sim}{\mathbf{P}}}$ since $f \rhd S^{\underset{\sim}{\mathbf{P}}}$. By (†) we have equivalently

$$(f_\iota(\emptyset), \dots, f_\iota(\emptyset)) = ((pr_\iota \circ \mathbf{f})(\emptyset), (pr_\iota \circ \mathbf{f})(\emptyset)) \in S{\restriction_{U_\iota}} \subseteq S$$

for all $1 \leq \iota \leq m$. Hence f_ι preserves S, and thus belongs to $\operatorname{Clo}(\mathbf{A})$. □

We want to finish this section with a few comments about the importance of covers and the related matrix products for possible applications of our localisation theory. Due to scope restrictions we will not give too many details, here. As a more in-depth presentation of this material should appear in a subsequent publication, the following rather gives an overview of ideas and results than educing them step by step.

We start with a collection of neighbourhoods $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ of an algebra $\mathbf{A}$, define $F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathcal{V} \colon \operatorname{im} f \subseteq V \right\}$ as in Theorem 3.4.31 and moreover, fix a choice function $V \colon F \longrightarrow \mathcal{V}$ satisfying $\operatorname{im} f \subseteq V(f)$ for every $f \in F$ as in that theorem. Let us furthermore suppose that we know a decomposition equation $\lambda \circ (f_1, \dots, f_n) = \mathrm{id}_A$ as in Corollary 3.4.36(k), and assume, in addition, that $f_1, \dots, f_n \in F \cap \operatorname{Term}^{(1)}(\mathbf{A})$, $\lambda \in \operatorname{Term}(\mathbf{A})$ and that the neighbourhoods $V(f_1), \dots, V(f_n)$ are given by idempotent unary *term* operations of $\mathbf{A}$. This additional requirement will, for instance, be fulfilled automatically if the clone of term operations of $\mathbf{A}$ is locally closed, i.e. if $\operatorname{Term}(\mathbf{A}) = \operatorname{Clo}(\mathbf{A})$.

Under these assumptions one can show that $\mathbf{A}$ is categorically equivalent to the matrix product $\mathbf{A}|_{V(f_1)} \boxtimes \cdots \boxtimes \mathbf{A}|_{V(f_n)}$ in the term sense, i.e. as characterised in Lemma 3.4.39 but everywhere using $\operatorname{Term}(\mathbf{A})$ instead of $\operatorname{Clo}(\mathbf{A})$ (cf. Theorem 4.1 in [Kea01]). Again, if we use the stronger assumption that $\operatorname{Term}(\mathbf{A})$ is locally closed, then the last mentioned, minor technical difference vanishes.

In the following paragraphs we will sketch two proofs for this result. One way to see it is to apply similar ideas as presented before Corollary 3.4.37, and then to exploit the fact that the matrix product in the claim is compatible with the matrix powers occurring in McKenzie's theorem due to using $\operatorname{Term}(\mathbf{A})$ instead of $\operatorname{Clo}(\mathbf{A})$. In fact, it can be proven without difficulties that the matrix product mentioned above is equal to a restricted algebra of the form $\mathbf{A}^{[n]}|_U$ where $\mathbf{A}^{[n]}$ denotes the n-th matrix power in McKenzie's sense. If for $1 \leq i \leq n$ the term operation $e_i \in \operatorname{Idem}\mathbf{A} \cap \operatorname{Term}^{(1)}(\mathbf{A})$ is an idempotent describing $V(f_i)$ as $V(f_i) = \operatorname{im} e_i$, then the operation $e_1 \times \cdots \times e_n \colon A^n \longrightarrow A^n \quad (x_1, \dots, x_n) \mapsto (e_1(x_1), \dots, e_n(x_n))$ is an idempotent unary term operation of $\mathbf{A}^{[n]}$ whose image is exactly the carrier set U of the matrix product above. Finally, an application of McKenzie's theorem finishes the argument, where one has to use the given decomposition equation to show that $e_1 \times \cdots \times e_n$ is indeed an invertible unary term operation.

Another way to prove this result is to explicitly construct the equivalence functors needed for the categorical equivalence. In Section 3.5 of [Beh09] this approach has been carried out for varieties of algebras satisfying a decomposition identity as given above (ignoring nullary operations but this only constitutes a minor difference). Varieties generated by a finite algebra for which a cover is known are the prototypical examples generating such a situation: clones of term operations of finite algebras are always locally closed, and Corollary 3.4.36 asserts that covers entail decomposition equations as needed for our argument.

However, finite algebras are not the only ones, where these ideas work. We will prove in Corollary 3.5.13 that all algebras generating locally finite varieties (see page 121 for a definition) have locally closed clones of term operations. Moreover, we have argued earlier that such algebras fulfil the assumptions of Corollary 3.4.37. The latter then yields precisely the sort of decomposition equation we are looking for.

As a general consequence, covers provide a tool to establish categorical equivalences between algebras in locally finite varieties. Understanding covers, and therefore Relational Structure Theory in detail, helps to understand such algebras up to categorical equivalence.

At least for finite algebras, even more is possible. Based on the categorical equivalence between algebras and their matrix products belonging to covers that was established above, one can even show a characterisation theorem for categorical equivalence of finite algebras, cf. [Iza13, Theorem 4.4]. A full understanding of the precise formulation would require to give more definitions than we intend to do here. However, we can say that the result involves so-called non-refinable covers (to be introduced in Definition 3.5.2(v)), and, as characterising statement, that the matrix products belonging to the respective non-refinable covers need to be weakly isomorphic, i.e. isomorphic up to term equivalence.

We claim that we can further extend Izawa's characterisation on the relational side as follows: two finite algebras $\mathbf{A}$ and $\mathbf{B}$ are categorically equivalent if and only if any two non-refinable covers $\{U_1, \ldots, U_m\}$ of $\mathbf{A}$ and $\{V_1, \ldots, V_n\}$ of $\mathbf{B}$ can be bijectively matched in such a way that restricted relational counterparts belonging to associated neighbourhoods are isomorphic w.r.t. to some indexing that is compatible with the clones of invariant relations of $\mathbf{A}$ and $\mathbf{B}$. The existence of such a compatible signature in case of categorical equivalence is not absurd in view of Denecke and Lüders' result stating that $\operatorname{Inv}\mathbf{A}$ and $\operatorname{Inv}\mathbf{B}$ must be isomorphic as relational clones. Moreover, we think of the advertised extended characterisation as an RST-analogy of Theorem 2.5 given in [Zád97b, p. 572].

From the previous remarks we already see that the mentioned applications touch parts of Relational Structure Theory we have not even looked at so far and combine them e.g. with the notions of matrix product introduced above.

Indeed, in the following section, we introduce the concepts of refinement and non-refinability, and furthermore examine conditions ensuring the existence of non-refinable covers. What is more, we even provide concrete algorithms to obtain them. As a consequence of Corollary 3.5.14 and Lemma 3.5.32, we shall get that these methods are applicable to algebras in 1-locally finite varieties, and especially to those in locally finite varieties.

Existence of non-refinable covers is fundamental for the cited characterisation of categorical equivalence, but unfortunately it is not enough. It is the aim of Sections 3.6 and 3.7 to establish criteria implying uniqueness of non-refinable covers in a certain sense. These conditions will, in particular, be fulfilled by all finite algebras, for which we can, more specifically, prove uniqueness of non-refinable covers up to isomorphism (see Corollary 3.7.23).

Without knowing about these two facts, existence and uniqueness, it would not even make sense to formulate or prove a criterion for categorical equivalence using the notion of non-refinable cover as a characterising invariant. This is one of the reasons, why we have not (and will not) give more details on the characterisation of categorical equivalence via non-refinable covers. Yet, we hope that we have nevertheless provided an impression of why covers and their connection to matrix products can be useful.

3.5 Refinement

Starting with a cover $\mathcal{U}$ of an algebra $\mathbf{A}$, one can try to construct new covers consisting of smaller neighbourhoods by replacing a neighbourhood $U \in \mathcal{U}$ by the members of a cover of $\mathbf{A}|_U$. In particular for finite algebras, iterating this construction seems to be a feasible strategy to obtain covers with neighbourhoods of manageable size, i.e. a decomposition into factors that are easier to understand than the original algebra. Since the singleton collection $\{A\}$ containing the image of the identity operation always covers $\mathbf{A}$, this procedure has a definite starting point (however, it may not terminate for infinite algebras).

The intermediate collections of neighbourhoods obtained in such an iteration have the following two characteristic properties: first, each member of the new collection is a subset of a neighbourhood of the previous one. Second, collectively, the new generation of neighbourhoods is at least as strong w.r.t. to separation of invariant relations, i.e. w.r.t. covering, as the one from which it was derived.

These characteristics lead to the concept of refinement of sets of neighbourhoods, which occurs as Definition 5.2 in [Kea01], as Definition 3.6.1 in [Beh09] and in Definition 2.8 of [KL10]. In particular, this notion is of course applicable to covers of an algebra (cf. also [Beh12, p. 236, bottom]).

For a better understanding of the definition, let us first have a look at the following easy fact about quasiorders. If (N, q) is a quasiordered set, then one can define a quasiorder on the powerset of N in the following canonical way:

$$\sqsubseteq(q) := \left\{ (U, V) \in (\mathfrak{P}(N))^2 \;\middle|\; \forall u \in U\, \exists v \in V\colon (u, v) \in q \right\}.$$

A moment of reflection shows that for $U, V \subseteq N$ the condition $U \sqsubseteq(q)\, V$ is equivalent[25] to the inclusion $\downarrow_{(N,q)} U \subseteq \downarrow_{(N,q)} V$ of the downsets generated by the two subsets w.r.t. to the quasiorder q.

[25] The author is grateful to Bernhard Ganter for pointing out this observation.

3.5.1 Remark. If we let $N = \operatorname{Neigh}\mathbf{A}$ and $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ be a subquasiorder of $\leq_{\mathrm{cov}}$, then $\sqsubseteq(q)$ is a quasiorder on $\mathfrak{P}(\operatorname{Neigh}\mathbf{A})$ that is contained in $\leq_{\mathrm{cov}}$. ♢

Proof: Namely, if we have $\mathcal{U} \sqsubseteq(q)\, \mathcal{V}$, then for every $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ such that $(U, V) \in q$, which implies $U \leq_{\mathrm{cov}} V$. Obviously, $V \leq_{\mathrm{cov}} \mathcal{V}$, so by transitivity, we have $U \leq_{\mathrm{cov}} \mathcal{V}$ for all $U \in \mathcal{U}$. This means by definition that $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$. □

In the standard definition of refinement ([Kea01, Beh09, KL10, Beh12]) a very simple example of a quasiorder q as in the previous remark is used, scilicet set inclusion of neighbourhoods. In the remainder of this section, and especially for finite algebras, we are mainly interested in using this quasiorder. Nevertheless, in the following definition we will introduce a slightly generalised version of refinement which is parametrised by an arbitrary quasiorder $q \subseteq {\leq_{\mathrm{cov}}} \subseteq (\operatorname{Neigh}\mathbf{A})^2$.

This is not done for the sheer fun of abstraction. Our definition is motivated by the fact that in Section 3.7 we shall prove a uniqueness theorem for so-called q-non-refinable covers, which more generally works for $q = \precapprox$, but not always for $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$. However, there are some situations, including the important case of finite algebras, where the two notions arising from both choices of the parameter q coincide (cp. Corollary 3.7.21). In Remark 3.5.7 we shall comment a little bit more on the advantages and disadvantages of different choices of the quasiorder q w.r.t. refinement.

3.5.2 Definition. For an algebra $\mathbf{A}$, a quasiorder $q \subseteq {\leq_{\mathrm{cov}}} \subseteq (\operatorname{Neigh}\mathbf{A})^2$ and sets of neighbourhoods $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ we define:

(i) $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ if $\mathcal{V} \sqsubseteq(q)\, \mathcal{U}$ and $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ hold in conjunction, and we say that $\mathcal{V}$ *q-refines* $\mathcal{U}$ or $\mathcal{V}$ *is a q-refinement of* $\mathcal{U}$.

(ii) We put $\mathcal{V} \equiv_{\mathrm{ref}}(q)\, \mathcal{U} :\Longleftrightarrow \mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U} \wedge \mathcal{U} \leq_{\mathrm{ref}}(q)\, \mathcal{V}$
and say that $\mathcal{U}$ and $\mathcal{V}$ are *q-refinement equivalent*[26].

(iii) A q-refinement $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ is called *proper* if $\mathcal{V} \not\equiv_{\mathrm{ref}}(q)\, \mathcal{U}$. We denote this relation by $\mathcal{V} <_{\mathrm{ref}}(q)\, \mathcal{U}$.

(iv) $\mathcal{U}$ is called *q-refinement minimal* if it does not have proper q-refinements, that is, for all collections $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ the condition $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ implies $\mathcal{U} \leq_{\mathrm{ref}}(q)\, \mathcal{V}$, i.e. $\mathcal{U} \equiv_{\mathrm{ref}}(q)\, \mathcal{V}$.

(v) $\mathcal{U}$ is called *q-non-refinable* if for every collection $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ the condition $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ implies $\mathcal{U} \sqsubseteq(q \cap q^{-1})\, \mathcal{V}$, and $\mathcal{U}$ consists of mutually incomparable[27] elements w.r.t. $q \cap q^{-1}$. Otherwise, it is called *q-refinable.*

[26] The justification for this terminology will be given in Lemma 3.5.4 below.

[27] Since the quasiorder $q \cap q^{-1}$ is an equivalence relation, this means that $\mathcal{U}$ forms a transversal of $\mathcal{U}/q \cap q^{-1}$. In other words, for all $U_1, U_2 \in \mathcal{U}$ the condition $U_1 \; q \; U_2 \; q \; U_1$ implies $U_1 = U_2$.

(vi) We stipulate that w.r.t. the notions introduced in (i) through (v), omission of the prefix referring to the quasiorder is supposed to implicitly mean $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$. In this case, we shall also write $\leq_{\mathrm{ref}}$, $\equiv_{\mathrm{ref}}$ and $\sqsubseteq(\subseteq)$ instead of the clumsy expressions $\leq_{\mathrm{ref}}(\subseteq_{\mathrm{Neigh}\,\mathbf{A}})$, $\equiv_{\mathrm{ref}}(\subseteq_{\mathrm{Neigh}\,\mathbf{A}})$ and $\sqsubseteq(\subseteq_{\mathrm{Neigh}\,\mathbf{A}})$.

(vii) A cover $\mathcal{V}$ of a collection $\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}$ (or of $\mathbf{A}$) is called *irredundant* if it is minimal w.r.t. set inclusion among all covers of $\mathcal{U}$ (of $\mathbf{A}$, respectively), that is, if every proper subcollection $\mathcal{W} \subset \mathcal{V}$ of neighbourhoods fails to cover $\mathcal{U}$ (or the algebra $\mathbf{A}$). We agree on saying that the term *irredundant cover* is supposed to mean irredundant cover of $\mathbf{A}$. ♢

It is evident, that the special case of $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$ in Definition 3.5.2 yields the notions of refinement, proper refinement and refinement-equivalence as introduced in [Kea01, Beh09, KL10]. The relationship of refinement-minimality and non-refinability w.r.t. notions established in the literature is discussed separately in Remark 3.5.3 below.

We just mention in passing that for q being set inclusion, the equivalence relation $q \cap q^{-1}$ becomes the equality relation, whence the second requirement of $\subseteq_{\mathrm{Neigh}\,\mathbf{A}}$-non-refinability is trivially true. As $\mathcal{U} \sqsubseteq (q \cap q^{-1})\,\mathcal{V}$, i.e. $\mathcal{U} \sqsubseteq (\Delta_{\mathrm{Neigh}\,\mathbf{A}})\,\mathcal{V}$, is precisely expressing that $\mathcal{U} \subseteq \mathcal{V}$, the remaining condition for non-refinability of $\mathcal{U}$ becomes that $\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$ implies $\mathcal{U} \subseteq \mathcal{V}$ for all $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$. This is the way how non-refinability was formulated in Definition 2.8 of [KL10].

As noted there, non-refinability is generally stronger than just refinement minimality. We shall see in a moment (in item (e) of Lemma 3.5.9) that for covers of $\mathbf{A}$ non-refinability has got a catchy characterisation in terms of refinement minimality *plus* irredundancy.

3.5.3 Remark. With Definition 3.5.2 we deviate a little from how *non-refinability* was defined in [Beh09, Definition 3.6.1] and [Beh12]. What we have called *non-refinable* is in accordance with Definition 2.8 of [KL10] and is referred to as *irredundant and non-refinable* in [Beh09]. The notion of non-refinability occurring in [Beh09] and [Beh12] is precisely what we have called *refinement minimality* here. This change of terminology is motivated by the fact that it allows a more concise expression of the concept of an irredundant and refinement minimal cover of an algebra that will become important in subsequent sections for it is unique up to isomorphism for any finite algebra and, more generally, for any poly-Artinian algebra in a 1-locally finite variety (cp. Corollary 3.7.22(c)). ♢

The following is an assortment of different remarks and easy observations concerning the q-refinement relation. Some of the statements are immediate consequences of the involved definitions. We mention them nevertheless, in order to be able to reference them later, and to clarify some relationships between the previously defined notions. One should note that many of the following facts are generalisations of the statements in Lemma 3.6.2 of [Beh09] to the more general setting of q-refinement.

3.5.4 Lemma. *For any algebra* $\mathbf{A}$ *and a quasiorder* $q \subseteq \leq_{\mathrm{cov}} \subseteq (\operatorname{Neigh}\mathbf{A})^2$ *the following facts are true.*

(a) *The q-refinement relation* $\leq_{\mathrm{ref}}(q) \subseteq (\mathfrak{P}(\operatorname{Neigh}\mathbf{A}))^2$ *is a quasiorder. It is contained in covering equivalence* $\equiv_{\mathrm{cov}}$.

(b) *If* $q' \subseteq \leq_{\mathrm{cov}} \subseteq (\operatorname{Neigh}\mathbf{A})^2$ *is another quasiorder,* $\mathcal{U}, \mathcal{V}, \mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ *are collections of neighbourhoods, then the following implications hold*

$$\begin{aligned} \mathcal{U} \sqsubseteq (q)\, \mathcal{V} \sqsubseteq (q')\, \mathcal{W} &\implies \mathcal{U} \sqsubseteq (q \circ q')\, \mathcal{W} &&\implies \mathcal{U} \sqsubseteq (q \vee q')\, \mathcal{W} \\ \mathcal{U} \leq_{\mathrm{ref}}(q)\, \mathcal{V} \leq_{\mathrm{ref}}(q')\, \mathcal{W} &\implies \mathcal{U} \leq_{\mathrm{ref}}(q \circ q')\, \mathcal{W} &&\implies \mathcal{U} \leq_{\mathrm{ref}}(q \vee q')\, \mathcal{W}, \end{aligned}$$

implying

$$\sqsubseteq(q') \circ \sqsubseteq(q) \quad \subseteq \quad \sqsubseteq(q \circ q') \quad \subseteq \quad \sqsubseteq(q \vee q')$$

and

$$\leq_{\mathrm{ref}}(q') \circ \leq_{\mathrm{ref}}(q) \quad \subseteq \quad \leq_{\mathrm{ref}}(q \circ q') \quad \subseteq \quad \leq_{\mathrm{ref}}(q \vee q').$$

Moreover, for $q' \subseteq q$, *it is* $\sqsubseteq(q') \subseteq \sqsubseteq(q)$ *and* $\leq_{\mathrm{ref}}(q') \subseteq \leq_{\mathrm{ref}}(q)$.

(c) *For collections* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$, *the condition* $\mathcal{V} \sqsubseteq (\tilde{q})\, \mathcal{U}$ *holds for all quasiorders* $\tilde{q} \subseteq (\operatorname{Neigh}\mathbf{A})^2$ *if and only if* $\mathcal{V} \subseteq \mathcal{U}$.

(d) *Suppose* $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ *for collections* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$. *Then it is* $\mathcal{V} \equiv_{\mathrm{ref}}(q)\, \mathcal{U}$, *i.e.* $\mathcal{U} \leq_{\mathrm{ref}}(q)\, \mathcal{V}$, *if and only if* $\mathcal{U} \sqsubseteq (q)\, \mathcal{V}$. *Thus, the q-refinement* $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ *is proper if and only if* $\mathcal{U} \sqsubseteq (q)\, \mathcal{V}$ *fails.*

(e) *The empty collection q-refines a set* $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ *if and only if* $\mathcal{U} = \emptyset$, *or every member of* $\mathcal{U}$ *is a singleton and* $\mathbf{A}$ *contains nullary operations.*

(f) *If* $(\operatorname{Neigh}\mathbf{A}, q)$ *has* A *as a largest element (e.g. if* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q$*), then the q-refinement relation is in general not an order relation. If the algebra* $\mathbf{A}$ *has got a proper subneighbourhood* $U \in \operatorname{Neigh}\mathbf{A}$, $U \subset A$, *satisfying* $U\ q\ A$ *(this follows automatically if* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q$*), then* $\mathcal{U} := \{A\}$ *and* $\mathcal{V} := \{A, U\}$ *are different covers satisfying* $\mathcal{U} \equiv_{\mathrm{ref}}(q)\, \mathcal{V}$. *Indeed, for q extending set inclusion, the structure* $(\mathfrak{P}(\operatorname{Neigh}\mathbf{A}), \leq_{\mathrm{ref}}(q))$ *is an ordered set if and only if* $\mathbf{A}$ *does not have proper subneighbourhoods, i.e. if* $\operatorname{Neigh}\mathbf{A} = \{A\}$.

If it is an ordered set, then it is a two-element chain precisely if $|A| = 1$ *and* $\mathbf{A}$ *has nullary constants; otherwise, it is a two-element antichain.*

(g) *If* $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ *is a cover of* $\mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ *and* $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ *q-refines* $\mathcal{U}$, *then* $\mathcal{V}$ *is a cover of* $\mathcal{W}$, *as well. Especially, every q-refinement* $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ *of a cover* $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ *is again a cover of* $\mathbf{A}$.

This implies that to check a cover of $\mathbf{A}$ *for q-refinement minimality, only other q-refining covers, not arbitrary q-refining sets of neighbourhoods, have to be considered. That is to say*

$$\mathrm{Cov}\,(\mathbf{A}) \cap \mathrm{Min}\,(\mathfrak{P}\,(\mathrm{Neigh}\,\mathbf{A})\,, \leq_{\mathrm{ref}}(q)) = \mathrm{Min}\left(\mathrm{Cov}\,(\mathbf{A})\,, \leq_{\mathrm{ref}}(q)\restriction_{\mathrm{Cov}(\mathbf{A})}\right).$$

(h) For every $\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *and a cover* $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *of* $\mathcal{U}$*, we have that* $\mathcal{V}$ *is a q-refinement of* $\mathcal{U}$ *if and only if* $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$*, i.e. if every* $V \in \mathcal{V}$ *lies below some* $U \in \mathcal{U}$ *w.r.t. q. In particular, this is true if* $\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}|_W$ *for some* $W \in \mathrm{Neigh}\,\mathbf{A}$ *and* $\mathcal{V} \in \mathrm{Cov}_{\mathbf{A}}(W)$*. Thus, without further conditions on* $\mathcal{U}$*, the stated equivalence holds for covers* $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$*. Moreover, any subset* $\mathcal{V} \subseteq \mathcal{U}$ *that covers* $\mathcal{U}$ *or* $\mathbf{A}$ *satisfies* $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$*.*

(i) For a cover $\mathcal{U} \in \mathrm{Cov}\,(\mathbf{A})$ *and a set of neighbourhoods* $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *satisfying* $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$*, we have* $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ *if and only if* $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$*.*

(j) For collections $\mathcal{U}, \mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *such that* $\mathcal{V} \subseteq \mathcal{U}$*, we have* $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ *if and only if* $\mathcal{U} \setminus \mathcal{V} \leq_{\mathrm{cov}} \mathcal{V}$*.*

(k) If $(\mathrm{Neigh}\,\mathbf{A}, q)$ *has* A *as a largest element, then for every set of neighbourhoods* $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$*, it is* $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\{A\}$ *if and only if* $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$*, and inversely* $\{A\} \leq_{\mathrm{ref}}(q)\,\mathcal{V}$ *if and only if* $A\;q\;V\;q\;A$ *for some* $V \in \mathcal{V}$ *(if q is an order relation, this happens exactly if* $A \in \mathcal{V}$*).*

Proof: (a) By Definition 3.5.2(i) we have $\leq_{\mathrm{ref}}(q) = (\leq_{\mathrm{cov}})^{-1} \cap \sqsubseteq (q)$. This is a quasiorder as an intersection of quasiorders (see also Corollary 3.4.5). It follows from Remark 3.5.1 that $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$ implies $\mathcal{V} \leq_{\mathrm{cov}} \mathcal{U}$. Hence, $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ entails $\mathcal{V} \leq_{\mathrm{cov}} \mathcal{U}$ and $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$, i.e. $\mathcal{V} \equiv_{\mathrm{cov}} \mathcal{U}$.

(b) First, we check the last statement of this item. If $q' \subseteq q$ are quasiorders, then for $\mathcal{U}, \mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ the following implications hold: if $\mathcal{U} \sqsubseteq (q')\,\mathcal{V}$, then any $U \in \mathcal{U}$ satisfies $U\;q'\;V$, i.e. $U\;q\;V$, for some $V \in \mathcal{V}$. Hence, we have $\mathcal{U} \sqsubseteq (q)\,\mathcal{V}$. This means $\sqsubseteq (q') \subseteq \sqsubseteq (q)$. Intersecting this inclusion on both sides with the quasiorder $(\leq_{\mathrm{cov}})^{-1}$ yields the inclusion $\leq_{\mathrm{ref}}(q') \subseteq \leq_{\mathrm{ref}}(q)$.

In order to prove the first two lines of implications, we now suppose that q and q' are unrelated quasiorders. If τ is any binary relation containing q and q', then we get $q \circ q' \subseteq \tau \circ \tau$ since relation product is monotone. If we assume further, that τ is transitive (e.g. if it is a quasiorder), then we can infer $q \circ q' \subseteq \tau \circ \tau \subseteq \tau$. In particular, this works for the quasiorder $\tau = q \vee q'$. Therefore, the last two implications in the two displayed formulæ follow from what we showed in the previous paragraph. Finally, it only remains to verify that $\mathcal{U} \sqsubseteq (q)\,\mathcal{V} \sqsubseteq (q')\,\mathcal{W}$ implies $\mathcal{U} \sqsubseteq (q \circ q')\,\mathcal{W}$. This is true because for every $U \in \mathcal{U}$ we can find some $V \in \mathcal{V}$ for which in turn there is some $W \in \mathcal{W}$ such that $U\;q\;V\;q'\;W$. Since $V \in \mathcal{V}$ is in particular some neighbourhood, this shows $(U, W) \in q \circ q'$. As we could find such a $W \in \mathcal{W}$ for any $U \in \mathcal{U}$, we have shown $\mathcal{U} \sqsubseteq (q \circ q')\, \mathcal{W}$. Using this fact in conjunction with $\mathcal{W} \leq_{\text{cov}} \mathcal{V} \leq_{\text{cov}} \mathcal{U}$ and transitivity of covering, establishes that $\mathcal{U} \leq_{\text{ref}} (q)\, \mathcal{V} \leq_{\text{ref}} (q')\, \mathcal{W}$ implies $\mathcal{U} \leq_{\text{ref}} (q \circ q')\, \mathcal{W}$.

(c) If $\mathcal{V} \sqsubseteq (\tilde{q})\, \mathcal{U}$ holds for any quasiorder $\tilde{q}$ between neighbourhoods, then it is certainly true for $\tilde{q}$ being the identity relation. Then every $V \in \mathcal{V}$ is equal to some $U \in \mathcal{U}$, or equivalently, $\mathcal{V} \subseteq \mathcal{U}$. Conversely, if we start with this inclusion and $\tilde{q} \subseteq (\operatorname{Neigh} \mathbf{A})^2$ is any quasiorder, then reflexivity implies that every $V \in \mathcal{V}$ is $\tilde{q}$-related to itself, i.e. to some element of $\mathcal{U}$.

(d) Assuming $\mathcal{V} \leq_{\text{ref}} (q)\, \mathcal{U}$, q-refinement equivalence of the two collections is of course equivalent to $\mathcal{U} \leq_{\text{ref}} (q)\, \mathcal{V}$. From the assumption and item (a), we infer $\mathcal{V} \equiv_{\text{cov}} \mathcal{U}$, in particular $\mathcal{V} \leq_{\text{cov}} \mathcal{U}$, which is one part of the refinement condition $\mathcal{U} \leq_{\text{ref}} (q)\, \mathcal{V}$. Therefore, the latter is logically equivalent to the second requirement, which is $\mathcal{U} \sqsubseteq (q)\, \mathcal{V}$. The statement about the q-refinement being proper is an immediate consequence of this.

(e) For $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ we have $\emptyset \leq_{\text{ref}} (q)\, \mathcal{U}$ if and only if $\mathcal{U} \leq_{\text{cov}} \emptyset$ since $\emptyset \sqsubseteq (q)\, \mathcal{U}$ trivially holds for the empty collection. Now by Definition 3.4.2(ii), $\mathcal{U} \leq_{\text{cov}} \emptyset$ means that every $U \in \mathcal{U}$ is covered by the empty collection of neighbourhoods. According to Lemma 3.4.12 this is equivalent to the conjunction of $|U| = 1$ and $\mathbf{A}$ having nullary operations for all members $U \in \mathcal{U}$. In other words, we may say that $\mathcal{U}$ can be empty or all neighbourhoods of $\mathcal{U}$ are singleton sets and $\mathbf{A}$ contains indeed nullary operations (for there is at least one neighbourhood $U \in \mathcal{U}$ allowing us to infer the existence of nullary constants in $\mathbf{A}$).

(f) As $A \in \mathcal{U} \cap \mathcal{V}$, both collections of neighbourhoods are covers of $\mathbf{A}$. As $U \mathrel{q} A$ holds by assumption, it follows that $\mathcal{U}$ is a system of $(q \cap q^{-1})$-representatives for $\operatorname{Max}(\mathcal{V}, q{\restriction}_{\mathcal{V}})$. Hence, we obtain from Lemma 3.5.6(d) below that $\mathcal{V} \equiv_{\text{ref}} (q)\, \mathcal{U}$. For $U \subset A$ is assumed to be a proper subset, $\mathcal{U}$ and $\mathcal{V}$ are distinct, which shows that $(\mathfrak{P}(\operatorname{Neigh} \mathbf{A}), \leq_{\text{ref}} (q))$ cannot be an ordered set whenever proper subneighbourhoods $U \mathrel{q} A$ exist.

Conversely, assume that $\mathbf{A}$ fails to have proper subneighbourhoods, i.e. that $\operatorname{Neigh} \mathbf{A} = \{A\}$. For every subset $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, the first condition of q-refinement entails that $\mathcal{V} \leq_{\text{ref}} (q)\, \emptyset$ only holds if $\mathcal{V} = \emptyset$. Hence, $\equiv_{\text{ref}} (q)$ is the equality relation on $\mathfrak{P}(\operatorname{Neigh} \mathbf{A})$, which means that q-refinement is an order relation. Item (e) implies that it is an antichain in any case but A being a singleton and having the unique nullary constant as a fundamental operation. In the latter case we truly have $\emptyset \leq_{\text{ref}} (q)\, \{A\} = \operatorname{Neigh} \mathbf{A}$, i.e. a two-element chain.

(g) If we have $\mathcal{W} \leq_{\text{cov}} \mathcal{U}$ and $\mathcal{V} \leq_{\text{ref}} (q)\, \mathcal{U}$, then, from the definition of q-refinement, we get $\mathcal{W} \leq_{\text{cov}} \mathcal{U} \leq_{\text{cov}} \mathcal{V}$. The rest is transitivity of covering (see Corollary 3.4.5). Furthermore, the second mentioned fact follows by letting $\mathcal{W} = \{A\}$.

From this we may infer the inclusion

$$\operatorname{Min}\left(\operatorname{Cov}(\mathbf{A}), \leq_{\mathrm{ref}}(q)\restriction_{\operatorname{Cov}(\mathbf{A})}\right) \subseteq \operatorname{Cov}(\mathbf{A}) \cap \operatorname{Min}(\mathfrak{P}(\operatorname{Neigh}\mathbf{A}), \leq_{\mathrm{ref}}(q)).$$

The converse is true because $\operatorname{Cov}(\mathbf{A}) \subseteq \mathfrak{P}(\operatorname{Neigh}\mathbf{A})$.

(h) If $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ is assumed, then the definition of $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ reduces to checking for $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$. For $W \in \operatorname{Neigh}\mathbf{A}$, any cover $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(W)$, by Lemma 3.4.11, fulfils $\operatorname{Neigh}\mathbf{A}|_W \leq_{\mathrm{cov}} \mathcal{V}$. If $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}|_W$, we have $\mathcal{U} \leq_{\mathrm{cov}} \operatorname{Neigh}\mathbf{A}|_W$ due to Lemma 3.4.9(a). Finally, transitivity yields $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$. In particular, the statement is true for covers $\mathcal{V}$ of $\mathbf{A}$ because $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A} = \operatorname{Neigh}\mathbf{A}|_A$ holds without additional assumptions. Furthermore, it is evident that $\mathcal{V} \subseteq \mathcal{U}$ implies $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$ due to q being reflexive. Hence, covers $\mathcal{V} \subseteq \mathcal{U}$ satisfy $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$.

(i) Since $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$, every refinement $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ is a cover of $\mathbf{A}$ due to item (g). Conversely, if $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$, then by item (h) we have $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ if and only if $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$, which holds by assumption.

(j) As q is reflexive, the precondition $\mathcal{V} \subseteq \mathcal{U}$ clearly implies $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$, whence $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\mathcal{U}$ is equivalent to $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ by definition. This means that we have $U \leq_{\mathrm{cov}} \mathcal{V}$ for every $U \in \mathcal{U}$. However, every neighbourhood $U \in \mathcal{U} \cap \mathcal{V}$ belongs to $\mathcal{V}$, and so $\{U\} \subseteq \mathcal{V}$ implies $\{U\} \leq_{\mathrm{cov}} \mathcal{V}$ (Lemma 3.4.9(a)), i.e. $U \leq_{\mathrm{cov}} \mathcal{V}$. Therefore, $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ happens if and only if all neighbourhoods of $\mathcal{U}$ that do not belong to $\mathcal{V}$ are covered by $\mathcal{V}$, i.e. if $\mathcal{U} \setminus \mathcal{V} \leq_{\mathrm{cov}} \mathcal{V}$.

(k) Since A is a largest element of $(\operatorname{Neigh}\mathbf{A}, q)$, every $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ trivially satisfies $\mathcal{V} \sqsubseteq (q)\,\{A\}$, and $\{A\} \in \operatorname{Cov}(\mathbf{A})$. Hence, letting $\mathcal{U} = \{A\}$ in item (i) yields the first equivalence.

For the special case of q being set inclusion, we just noted that $\mathcal{V} \sqsubseteq (\subseteq)\,\{A\}$, so we know $\mathcal{V} \leq_{\mathrm{cov}} \{A\}$ due to Remark 3.5.1. So according to item (h), we have $\{A\} \leq_{\mathrm{ref}}(q)\,\mathcal{V}$ if and only if $\{A\} \sqsubseteq (q)\,\mathcal{V}$. The latter condition is clearly equivalent to $A\ q\ V\ q\ A$, for some $V \in \mathcal{V}$ which proves the second equivalence. If q is an order relation, then $A\ q\ V\ q\ A$ means $A = V \in \mathcal{V}$, i.e. $A \in \mathcal{V}$. □

In passing by, we observe that item (h) of Lemma 3.5.4 justifies the name *refinement* given to the quasiorder $\leq_{\mathrm{ref}}$: for covers $\mathcal{U}, \mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ the condition $\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$ is equivalent to $\mathcal{V} \sqsubseteq (\subseteq)\,\mathcal{U}$, and the latter is exactly the notion used in topology for refinement of covers of subsets of topological spaces or for refinement of filter bases.

Next, we want to present a few methods how to construct q-refining subsets. Since q-refinement implies covering equivalence (see Lemma 3.5.4(a)), these constructions also provide a way of reducing covers to subcollections, a fact gaining importance later on.

Generalising the situation encountered in finite algebras, we want to assume a chain condition for two of these reductions. Namely, items (c) and (d) of the subsequent Lemma 3.5.6 require the *ascending chain condition* (*ACC*), the dual of the property occurring in Section 3.4. Formally, a poset $(P, \leq)$ satisfies ACC if the dual poset $(P, \geq)$ satisfies DCC as described on page 95. This means that every countable, monotone increasing sequence of elements from P eventually becomes constant. It is well-known that, similarly as for DCC, the ascending chain condition is equivalent to co-well-foundedness of $(P, \leq)$, requiring that every non-empty subposet of $(P, \leq)$ contains maximal elements.

The following lemma presents a well-understood order-theoretic result, that follows from co-well-foundedness and will find application in the proof of Lemma 3.5.6.

3.5.5 Lemma. *Every member of a partially ordered set $(P, \leq)$ satisfying ACC lies below some maximal element of $(P, \leq)$.*

Proof: We define the subset $C := \{x \in P \mid \forall\, m \in \operatorname{Max}(P, \leq)\colon x \nleq m\}$ of P, containing all counterexamples to our claim. In order to obtain a contradiction, we assume that C is non-empty. Since $(P, \leq)$ satisfies ACC, or equivalently is co-well-founded, C must contain a maximal element x. Since $\leq$ is reflexive, we can infer that $x \notin \operatorname{Max}(P, \leq)$. As x is not maximal in P, there exists some $y \in P$ such that $y \geq x$ but $y \nleq x$. Thus, $y \notin C$ because $x \in \operatorname{Max}(C, \leq\restriction_C)$. By definition of C, this implies, that there is some $m \in \operatorname{Max}(P, \leq)$ such that $m \geq y \geq x$. However, this proves $x \notin C$, in contradiction to our hypothesis on x. $\square$

The following lemma presents the promised constructions of q-refining subcollections. We mention that a precursor to Lemma 3.5.6(d) can also be found in item (c) of Lemma 3.6.2 from [Beh09].

3.5.6 Lemma. *Let $\mathbf{A}$ be an algebra, $(\operatorname{Neigh}\mathbf{A}, q)$ a quasiordered set where $q \subseteq \leq_{\mathrm{cov}}$, $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ a set of neighbourhoods, $\theta \subseteq \leq_{\mathrm{cov}} \cap\, \mathcal{U}^2$ an equivalence relation and $q', \tilde{q} \subseteq \leq_{\mathrm{cov}} \cap\, \mathcal{U}^2$ quasiorders such that the poset[28] $\left(\mathcal{U}/\left(\tilde{q} \cap \tilde{q}^{-1}\right), \tilde{q}/\left(\tilde{q} \cap \tilde{q}^{-1}\right)\right)$ satisfies the ascending chain condition (ACC). Let us denote by $\langle q' \rangle$, $\langle \tilde{q} \rangle$ and $\langle \theta \rangle$ the least quasiorders on $\operatorname{Neigh}\mathbf{A}$ containing q', $\tilde{q}$ and $\tilde{\theta}$, respectively, then the following facts are true.*

(a) If $\mathcal{V} \subseteq \mathcal{U}$ is a subcollection satisfying $\mathcal{U} \sqsubseteq (\langle q' \rangle)\, \mathcal{V}$, then we have $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$, and moreover $\mathcal{V} \equiv_{\mathrm{ref}} (\langle q' \rangle \vee q)\, \mathcal{U}$. Especially, it is $\mathcal{V} \equiv_{\mathrm{cov}} \mathcal{U}$, wherefore we have $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ if and only if $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$.

(b) If $\mathcal{V}_\theta \subseteq \mathcal{U}$ is a transversal[29] of $\mathcal{U}/\theta$, then we have $\mathcal{V}_\theta \leq_{\mathrm{ref}} (q)\, \mathcal{U}$, and moreover, it is $\mathcal{V}_\theta \equiv_{\mathrm{ref}} (\langle \theta \rangle \vee q)\, \mathcal{U}$. Especially, $\mathcal{V}_\theta \equiv_{\mathrm{cov}} \mathcal{U}$, so we have $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ if and only if $\mathcal{V}_\theta \in \operatorname{Cov}(\mathbf{A})$.

(c) Denoting by $\mathcal{M} := \bigcup \operatorname{Max}\left(\mathcal{U}/\left(\tilde{q} \cap \tilde{q}^{-1}\right), \tilde{q}/\left(\tilde{q} \cap \tilde{q}^{-1}\right)\right)$ the set of all maximal neighbourhoods w.r.t. $\tilde{q}$, it is $\mathcal{M} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ and moreover $\mathcal{M} \equiv_{\mathrm{ref}} (\langle \tilde{q} \rangle \vee q)\, \mathcal{U}$. Especially, we have $\mathcal{M} \equiv_{\mathrm{cov}} \mathcal{U}$, so $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ if and only if $\mathcal{M} \in \operatorname{Cov}(\mathbf{A})$.

[28] By $\tilde{q}/(\tilde{q} \cap \tilde{q}^{-1})$ we mean that two blocks $B, C \in \mathcal{U}/\left(\tilde{q} \cap \tilde{q}^{-1}\right)$ are ordered if and only if $(b, c) \in \tilde{q}$ for all $b \in B$ and $c \in C$.

[29] system of representatives for $\mathcal{U}/\theta$

(d) *Let $\mathcal{M} \subseteq \mathcal{U}$ be a system of representatives for the maximal blocks of the poset $(\mathcal{U}/(\tilde{q} \cap \tilde{q}^{-1}), \tilde{q}/(\tilde{q} \cap \tilde{q}^{-1}))$, then we have $\mathcal{M} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ and $\mathcal{M} \equiv_{\mathrm{ref}} (q \vee \langle \tilde{q} \rangle)\, \mathcal{U}$. In particular, for $\tilde{q} = q{\restriction}_{\mathcal{U}}$, it is $\mathcal{M} \equiv_{\mathrm{ref}} (q)\, \mathcal{U}$. Furthermore, $\mathcal{M} \equiv_{\mathrm{cov}} \mathcal{U}$, whence $\mathcal{M} \in \mathrm{Cov}\,(\mathbf{A})$ holds if and only if $\mathcal{U} \in \mathrm{Cov}\,(\mathbf{A})$.*

Proof: (a) Since $q' \subseteq \leq_{\mathrm{cov}}$, it follows $\langle q' \rangle \subseteq \leq_{\mathrm{cov}}$, and hence $\mathcal{U} \sqsubseteq (\langle q' \rangle)\, \mathcal{V}$ implies $\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$ by Remark 3.5.1. As $\mathcal{V} \subseteq \mathcal{U}$, Lemma 3.5.4(h) directly implies $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$. Now, using monotonicity from Lemma 3.5.4(b), we obtain $\mathcal{V} \leq_{\mathrm{ref}} (\langle q' \rangle \vee q)\, \mathcal{U}$, and likewise $\mathcal{U} \sqsubseteq (\langle q' \rangle \vee q)\, \mathcal{V}$ from $\mathcal{U} \sqsubseteq (\langle q' \rangle)\, \mathcal{V}$. So according to Lemma 3.5.4(d), it follows $\mathcal{V} \equiv_{\mathrm{ref}} (\langle q' \rangle \vee q)\, \mathcal{U}$. Furthermore, both sets are covering equivalent by Lemma 3.5.4(a), whereupon the statement about being covers of $\mathbf{A}$ is an easy consequence.

(b) Since $\mathcal{V}_\theta$ is a transversal of $\mathcal{U}/\theta$, every $U \in \mathcal{U}$ satisfies $U\, \theta\, V$, and hence $U\, \langle \theta \rangle\, V$, for some $V \in \mathcal{V}_\theta$. In other words, we have $\mathcal{U} \sqsubseteq (\langle \theta \rangle)\, \mathcal{V}_\theta$. This yields the assumption of item (a) for the case $q' = \theta$ and $\mathcal{V} = \mathcal{V}_\theta$, wherefore all claims follow from there.

(c) Since $(\mathcal{U}/(\tilde{q} \cap \tilde{q}^{-1}), \tilde{q}/(\tilde{q} \cap \tilde{q}^{-1}))$ satisfies ACC, every member of $\mathcal{U}$ lies below (w.r.t. $\tilde{q}$, and thus w.r.t. $\langle \tilde{q} \rangle$) some element of $\mathcal{M}$ (see Lemma 3.5.5). This shows that $\mathcal{U} \sqsubseteq (\langle \tilde{q} \rangle)\, \mathcal{M}$, providing the assumption of item (a) for the case $q' = \tilde{q}$ and $\mathcal{V} = \mathcal{M}$. Thus, all remaining facts are again a consequence of item (a).

(d) This fact can be inferred using a combination of items (c) and (b) and putting $\theta := (\tilde{q} \cap \tilde{q}^{-1}){\restriction}_{\mathrm{Max}\,(\mathcal{U}, \tilde{q})}$, or once more from item (a): as the poset associated with $(\mathcal{U}, \tilde{q})$ satisfies ACC, every member of $\mathcal{U}$ lies below (w.r.t. $\tilde{q}$) some maximal element of $(\mathcal{U}, \tilde{q})$ (see Lemma 3.5.5) and hence below some representative in $\mathcal{M}$. In other words, we have $\mathcal{U} \sqsubseteq (\tilde{q})\, \mathcal{M}$, whereupon a further application of item (a) finishes the proof. $\square$

3.5.7 Remark. In view of item (d) of Lemma 3.5.6, studying notions of q-refinement other than the ordinary one given by $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$ can be valuable. A natural choice for the respective quasiorder $q \subseteq (\mathrm{Neigh}\,\mathbf{A})^2$ would be $\subseteq_{\mathrm{Neigh}\,\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$. Generalising set inclusion is a reasonable and desirable requirement for q, and being contained in $\leq_{\mathrm{cov}}$ is motivated by Remark 3.5.1. Choosing $q \supset \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$, e.g. $q = \leq_{\mathrm{cov}}$ or $q = \precapprox$, in 3.5.6(d) allows a further reduction of the collection $\mathcal{U}$ than just taking the maximal neighbourhoods w.r.t. set inclusion while keeping the same strength regarding separation of invariants.

On the other hand, the condition for general q-refinements is not as easy to check as that for $\sqsubseteq (\subseteq_{\mathrm{Neigh}\,\mathbf{A}})$, and, moreover, the corresponding notion of q-non-refinability generally would be different. However, in important special cases (see e.g. Corollary 3.7.21) there is no difference at all. So, from a practical perspective, especially if one intends to devise an algorithm to determine q-non-refinable covers for finite algebras, the present definition of refinement as $\leq_{\mathrm{ref}} (\subseteq_{\mathrm{Neigh}\,\mathbf{A}})$ seems to be the preferable choice. This also explains why we will focus in this section on the classical definition stemming from [Kea01] from Lemma 3.5.15 onwards. $\diamondsuit$

In Lemma 3.5.9(e) it is our aim to characterise non-refinable covers in terms of irredundancy. We prepare this fact by connecting non-refinability and refinement minimality.

3.5.8 Lemma. *For an algebra* $\mathbf{A}$ *and a quasiorder* $q \subseteq \leq_{\mathrm{cov}} \subseteq (\mathrm{Neigh}\,\mathbf{A})^2$*, a collection* $\mathcal{U} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *is* q*-non-refinable if and only if it is* q*-refinement minimal and an antichain*[30] *w.r.t.* q*.*

Proof: Suppose that $\mathcal{U}$ is a q-non-refinable collection. First, we show that it is q-refinement minimal. If $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ is a collection satisfying $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$, then by our assumption of q-non-refinability, we obtain $\mathcal{U} \sqsubseteq (q \cap q^{-1})\, \mathcal{V}$, and hence $\mathcal{U} \sqsubseteq (q)\, \mathcal{V}$ by Lemma 3.5.4(b). Using item (d) of the same lemma, we can now infer $\mathcal{V} \equiv_{\mathrm{ref}} (q)\, \mathcal{U}$. Second, we check that $\mathcal{U}$ cannot contain non-identical comparable pairs w.r.t. q. For a contradiction let us assume, it does. So let $U, U' \in \mathcal{U}$ be distinct such that $U \mathrel{q} U'$ and put $\mathcal{U}' := \mathcal{U} \setminus \{U\} \subset \mathcal{U}$. By the assumption on q, we get $U \leq_{\mathrm{cov}} U' \leq_{\mathrm{cov}} \mathcal{U}'$ from $U \mathrel{q} U'$ and $U \neq U'$. Therefore, $\mathcal{U} \setminus \mathcal{U}' = \{U\} \leq_{\mathrm{cov}} \mathcal{U}'$, and so we have $\mathcal{U}' \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ due to $\mathcal{U}' \subseteq \mathcal{U}$ and Lemma 3.5.4(j). Now, q-non-refinability implies $\mathcal{U} \sqsubseteq (q \cap q^{-1})\, \mathcal{U}'$, whence there is some $V \in \mathcal{U}'$ such that $(U, V) \in q \cap q^{-1}$. Since $V \in \mathcal{U}'$, we have $U, V \in \mathcal{U}$ and $V \neq U$, a contradiction to the second condition of q-non-refinability. Thus, we have that $(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ is an antichain.

For the converse, assume that $\mathcal{U}$ is q-refinement minimal and an antichain w.r.t. q. Therefore, it certainly is an antichain w.r.t. $q \cap q^{-1}$, and it remains to show the first requirement of q-non-refinability. For this let us consider a collection $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ such that $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$. Now q-refinement minimality of $\mathcal{U}$ tells us that $\mathcal{U} \equiv_{\mathrm{ref}} (q)\, \mathcal{V}$, in particular that $\mathcal{U} \sqsubseteq (q)\, \mathcal{V} \sqsubseteq (q)\, \mathcal{U}$. So for every $U \in \mathcal{U}$, there is some $V \in \mathcal{V}$ for which again there is some $U' \in \mathcal{U}$ such that $U \mathrel{q} V \mathrel{q} U'$. Transitivity of q implies $U \mathrel{q} U'$, from which we get $U = U'$ due to $(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ being an antichain. Thus, we have $U \mathrel{q} V \mathrel{q} U' = U$, i.e. $(U, V) \in q \cap q^{-1}$. Since this works for every $U \in \mathcal{U}$, we have shown that $\mathcal{U} \sqsubseteq (q \cap q^{-1})\, \mathcal{V}$, concluding the proof. □

The subsequent lemma collects basic facts about irredundant covers. In particular, it characterises non-refinable covers by being irredundant refinement minimal covers as announced earlier. As for Lemma 3.5.4 the following statements can be seen as generalisations of facts occurring in [Beh09, Lemma 3.6.2].

3.5.9 Lemma. *For an algebra* $\mathbf{A}$*, a quasiorder* $q \subseteq \leq_{\mathrm{cov}} \subseteq (\mathrm{Neigh}\,\mathbf{A})^2$ *and collections* $\mathcal{U}, \mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *the following facts are true:*

(a) A cover $\mathcal{V}$ *of* $\mathcal{U}$ *is irredundant if and only if* $\mathcal{U} \not\leq_{\mathrm{cov}} \mathcal{V} \setminus \{V\}$ *for every* $V \in \mathcal{V}$*. Especially, covers* $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$ *are irredundant, precisely if* $\mathcal{V} \setminus \{V\} \notin \mathrm{Cov}\,(\mathbf{A})$ *for all* $V \in \mathcal{V}$*.*

(b) Every irredundant cover $\mathcal{V}$ *of* $\mathcal{U}$ *consists of mutually incomparable neighbourhoods w.r.t. (any quasiorder)* q *as above. In particular,* $\mathcal{V}$ *is an antichain w.r.t.*

[30] This means $q{\restriction}_{\mathcal{U}} = \Delta_{\mathcal{U}}$.

set inclusion, but also w.r.t. embedding. Especially, this holds for $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *being irredundant.*

(c) *Suppose* $\mathcal{V}$ *is q-refinement minimal and* $\mathcal{V} \equiv_{\text{cov}} \mathcal{U}$*. The collection* $\mathcal{V}$ *is an irredundant cover of* $\mathcal{U}$ *if and only if* $(\mathcal{V}, q{\restriction}_{\mathcal{V}})$ *is an antichain, i.e.* $V\; q\; V'$ *implies* $V = V'$ *for all* $V, V' \in \mathcal{V}$*. Especially, this is true for q-refinement minimal covers* $\mathcal{V}$ *of* $\mathbf{A}$*.*

(d) *If* $\mathcal{V}$ *is an irredundant cover of* $\mathcal{U}$ *and* $\mathcal{W} \subseteq \operatorname{Neigh} \mathbf{A}$ *a collection of neighbourhoods with* $\mathcal{V} \equiv_{\text{ref}} (q)\, \mathcal{W}$*, then* $\mathcal{V} \sqsubseteq (q \cap q^{-1})\, \mathcal{W}$*. In particular, for any poset* $(\operatorname{Neigh} \mathbf{A}, q)$ *subject to the condition* $q \subseteq\, \leq_{\text{cov}}$*, every two irredundant, q-refinement equivalent covers are identical.*

(e) *Suppose* $\mathcal{V} \leq_{\text{cov}} \mathcal{U}$*. The collection* $\mathcal{V}$ *is a q-non-refinable cover of* $\mathcal{U}$ *if and only if it is q-refinement minimal and an irredundant cover of* $\mathcal{U}$*, equivalently if it is a q-refinement minimal cover of* $\mathcal{U}$ *and an antichain w.r.t. q.*

In particular, the q-non-refinable covers of $\mathbf{A}$ *are exactly the q-refinement minimal irredundant covers of* $\mathbf{A}$*, equivalently the q-refinement minimal covers that consist of pairwise q-incomparable neighbourhoods.*

Proof: The special statements about covers of $\mathbf{A}$ follow always by letting $\mathcal{U} = \{A\}$. Thus, in the following explanations, we only consider the general case.

(a) The given condition is obviously necessary for irredundancy. It is also sufficient, because for every proper subcollection $\mathcal{W} \subset \mathcal{V}$ there is some $V \in \mathcal{V} \setminus \mathcal{W}$, i.e. $\mathcal{W} \subseteq \mathcal{V} \setminus \{V\} \subset \mathcal{V}$. Using Lemma 3.4.9(a), we get $\mathcal{W} \leq_{\text{cov}} \mathcal{V} \setminus \{V\}$. Transitivity of covering implies that $\mathcal{U} \leq_{\text{cov}} \mathcal{W}$ does not hold, since $\mathcal{U} \leq_{\text{cov}} \mathcal{V} \setminus \{V\}$ is false by assumption. Hence, $\mathcal{V}$ is an irredundant cover of $\mathcal{U}$.

(b) Suppose that $\mathcal{V}$ is an irredundant cover of $\mathcal{U}$, and, in order to obtain a contradiction, that $\mathcal{V}$ contains distinct neighbourhoods V and V' such that $(V, V') \in q$, i.e. $V \leq_{\text{cov}} V'$. We prove that $\mathcal{W} := \mathcal{V} \setminus \{V\}$ also is a cover of $\mathcal{U}$, which contradicts irredundancy of $\mathcal{V}$. For every $W \in \mathcal{W}$ it is clear that $W \leq_{\text{cov}} \mathcal{W}$, in particular, we have $V \leq_{\text{cov}} V' \leq_{\text{cov}} \mathcal{W}$ since $V' \in \mathcal{W}$. Transitivity of the covering relation yields $V \leq_{\text{cov}} \mathcal{W}$, consequently, every member of $\mathcal{V}$ is covered by $\mathcal{W}$. This shows $\mathcal{V} \leq_{\text{cov}} \mathcal{W}$, and, because of $\mathcal{U} \leq_{\text{cov}} \mathcal{V}$, again transitivity of covering implies $\mathcal{U} \leq_{\text{cov}} \mathcal{W}$. This establishes the desired contradiction. Using $q = \subseteq$ we obtain that $(\mathcal{V}, \subseteq)$ must be an inclusion antichain. Similarly, for $q = \precapprox$, we see that $\mathcal{V}$ cannot contain distinct neighbourhoods where one is embeddable into the other.

(c) Suppose $\mathcal{V}$ is q-refinement minimal and covering equivalent to $\mathcal{U}$. If it is irredundant w.r.t. covering $\mathcal{U}$, then, by item (b), the structure $(\mathcal{V}, q{\restriction}_{\mathcal{V}})$ must be an antichain.

Conversely, assume $(\mathcal{V}, q{\restriction_{\mathcal{V}}})$ to be an antichain, and suppose for a contradiction that $\mathcal{V}$ is not irredundant. That is, by (a) there is some neighbourhood $V \in \mathcal{V}$, such that $\mathcal{V}' := \mathcal{V} \setminus \{V\}$ also covers $\mathcal{U}$. By assumption we have $V \leq_{\mathrm{cov}} \mathcal{V} \equiv_{\mathrm{cov}} \mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}'$, so $\mathcal{V} \setminus \mathcal{V}' = \{V\} \leq_{\mathrm{cov}} \mathcal{V}'$. Together with $\mathcal{V}' \subseteq \mathcal{V}$, it follows $\mathcal{V}' \leq_{\mathrm{ref}} (q)\ \mathcal{V}$ by Lemma 3.5.4(j). Since $\mathcal{V}$ is q-refinement minimal, one obtains $\mathcal{V} \leq_{\mathrm{ref}} (q)\ \mathcal{V}'$. Hence, there is some $V' \in \mathcal{V}' = \mathcal{V} \setminus \{V\}$ such that $V\ q\ V'$, and $V \neq V'$, contradicting the assumption that $(\mathcal{V}, q{\restriction_{\mathcal{V}}})$ forms an antichain. Thus, $\mathcal{V}$ has to be irredundant, indeed.

(d) Since $\mathcal{V} \leq_{\mathrm{ref}} (q)\ \mathcal{W}$, every $V \in \mathcal{V}$ satisfies $V\ q\ W$ for some $W \in \mathcal{W}$. That in turn fulfils $W\ q\ V'$ for some $V' \in \mathcal{V}$ as $\mathcal{W} \leq_{\mathrm{ref}} (q)\ \mathcal{V}$. For $\mathcal{V}$ is irredundant, we obtain by (b) that $(\mathcal{V}, q{\restriction_{\mathcal{V}}})$ forms an antichain. Therefore, $V\ q\ W\ q\ V'$ implies $V = V'$, i.e. $V\ q\ W\ q\ V$, and consequently, $(V, W) \in q \cap q^{-1}$. As this was true for every $V \in \mathcal{V}$, we have proven $\mathcal{V} \sqsubseteq (q \cap q^{-1})\ \mathcal{W}$. If, moreover, q is an order relation, then we have $\mathcal{V} \sqsubseteq (\Delta_{\mathrm{Neigh}\,\mathbf{A}})\ \mathcal{W}$, i.e. $\mathcal{V} \subseteq \mathcal{W}$.

(e) We generally assume that $\mathcal{V}$ is covered by $\mathcal{U}$. First, we additionally suppose that $\mathcal{V}$ is a q-non-refinable cover of $\mathcal{U}$. By Lemma 3.5.8 we know that $\mathcal{V}$ is q-refinement minimal and an antichain w.r.t. q. As it is a cover of $\mathcal{U}$, we have $\mathcal{U} \equiv_{\mathrm{cov}} \mathcal{V}$. So from item (c), we see that $\mathcal{V}$ also is an irredundant cover of $\mathcal{U}$ since it was an antichain w.r.t. q. The converse implication is slightly easier to obtain. As $\mathcal{V}$ is an irredundant cover of $\mathcal{U}$ by assumption, it must be an antichain w.r.t. q due to item (b). Since it was also supposed to be q-refinement minimal, Lemma 3.5.8 yields that it is a q-non-refinable cover of $\mathcal{U}$.

For $\mathcal{U} = \{A\}$ the assumption $\mathcal{V} \leq_{\mathrm{cov}} \mathcal{U}$ is trivially true. Hence, regarding covers of $\mathbf{A}$, the stated characterisation holds without further assumptions. □

3.5.10 Remark. For an ordered[31] set $(\mathrm{Neigh}\,\mathbf{A}, q)$ where $q \subseteq \leq_{\mathrm{cov}}$, we want to comment a little bit more on the consequences of item (d) of the previous lemma. For every algebra $\mathbf{A}$, the structure $(\mathrm{Cov}(\mathbf{A}), \leq_{\mathrm{ref}}(q))$ is a quasiordered set, and the canonically associated equivalence relation is q-refinement equivalence $\equiv_{\mathrm{ref}}(q)$ of covers. The second statement in Lemma 3.5.9(d) means that each block in the poset $\mathrm{Cov}(\mathbf{A}) / \equiv_{\mathrm{ref}}(q)$ contains at most one irredundant cover. Especially, if $(\mathrm{Cov}(\mathbf{A}), \subseteq)$ satisfies DCC, then this poset is well-founded and hence every block $[\mathcal{U}]_{\equiv_{\mathrm{ref}}(q)}$ contains an inclusion minimal cover, because it is non-empty. Combining this with the observation above, every block of $\mathrm{Cov}(\mathbf{A}) / \equiv_{\mathrm{ref}}(q)$ contains precisely one irredundant cover of $\mathbf{A}$ as a representative. So the irredundant covers of $\mathbf{A}$ (ordered by q-refinement) form a distinguished system of representatives of the poset on $\mathrm{Cov}(\mathbf{A}) / \equiv_{\mathrm{ref}}(q)$ whose order is given by q-refinement of any two representatives.

This special situation takes place, for instance, if $(\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}), \subseteq)$ fulfils DCC, which is equivalent to $\mathrm{Neigh}\,\mathbf{A}$ being finite. In particular, this happens for finite algebras, but also for those in 1-locally finite varieties (cp. Corollary 3.5.14). ◇

[31] We mainly have in mind $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$, here.

The characterisation in item (e) of Lemma 3.5.9 explains why we are interested in finding non-refinable covers of algebras. Since they are exactly those refinement minimal covers that are irredundant, they provide an optimal solution to the task of determining covers containing small neighbourhoods. Refinement minimality ensures smallness of neighbourhoods (at least for finite algebras), and irredundancy guarantees that the considered cover does not contain any junk, i.e. superfluous neighbourhoods.

For the remainder of this section, the main goal will be to approach the task of obtaining non-refinable covers (meaning $\subseteq_{\mathrm{Neigh}\,\mathbf{A}}$-non-refinable covers) via different algorithms. These will surely work well for finite algebras. However, to treat infinite structures, too, we shall impose certain chain conditions on various posets, as already done in Lemma 3.5.6 and Remark 3.5.10. These assumptions will always be fulfilled if $\mathrm{Neigh}\,\mathbf{A}$ is a finite set, and especially for finite algebras.

In order to motivate our abstract assumptions, in the following four results we shall exhibit a large and important class of algebras having only finitely many neighbourhoods and therefore satisfying all chain conditions stated afterwards.

We begin with a fundamental fact about local closures of sets of operations which is certainly well-known.

3.5.11 Lemma. *Every finite set $F \subseteq \mathrm{O}_A$ of finitary operations on any set A is locally closed, i.e. satisfies $\mathrm{Loc}_A\, F = F$.*

Proof: It is a well-known fact that local closure operators work arity-wise. Therefore, to prove the claim, it suffices to verify the equality labelled by a question mark in the formula

$$\mathrm{Loc}_A\, F = \bigcup_{n\in\mathbb{N}} \mathrm{Loc}_A^{(n)} F = \bigcup_{n\in\mathbb{N}} \mathrm{Loc}_A\, F^{(n)} \overset{(?)}{=} \bigcup_{n\in\mathbb{N}} F^{(n)} = F.$$

Hence, no generality is lost in assuming that $F \subseteq \mathrm{O}_A^{(n)}$ for a fixed arity $n \in \mathbb{N}$. As Loc_A is a closure operator, it clearly suffices to demonstrate the inclusion $\mathrm{Loc}_A\, F \subseteq F$.

If this were not the case, then we could find an n-ary operation $g \in \mathrm{Loc}_A\, F \setminus F$. Thus, for any $f \in F$, we have $g \neq f$, i.e. there is a tuple $x_f \in A^n$ such that $g(x_f) \neq f(x_f)$. Since F was assumed to be finite, the set $\{x_f \mid f \in F\} \subseteq A^n$ is a finite subset of the domain of g. As g belonged to the local closure of F, there should be some operation $f \in F$ interpolating g on this set. Yet, this is impossible because every $f \in F$ violates the interpolation condition on x_f. This contradiction proves that g must be in F, and so $\mathrm{Loc}_A\, F \subseteq F$. $\square$

From this basic statement we can derive two important facts.

3.5.12 Corollary. *For an algebra $\mathbf{A}$ and any $n \in \mathbb{N}$ the following three facts are equivalent:*

(a) $\mathrm{Clo}^{(n)}(\mathbf{A})$ *is finite.*

(b) $\mathrm{Term}^{(n)}(\mathbf{A})$ *is finite.*

(c) $\mathrm{Clo}^{(n)}(\mathbf{A}) = \mathrm{Term}^{(n)}(\mathbf{A})$ *and both sets are finite.*

Proof: Evidently, item (c) implies item (a), and because $\mathrm{Term}^{(n)}(\mathbf{A}) \subseteq \mathrm{Clo}^{(n)}(\mathbf{A})$, the latter also implies item (b). The remaining implication, i.e. that (b) is sufficient for (c), follows from Lemma 3.5.11 since $\mathrm{Clo}^{(n)}(\mathbf{A}) = \mathrm{Loc}_A \mathrm{Term}^{(n)}(\mathbf{A})$. $\square$

For the second consequence of Lemma 3.5.11, let us recall the notion of n-local finiteness (cp. p. 44). An algebra was said to be n-locally finite if all its n-generated subuniverses are finite. Moreover, a class of algebras was called n-locally finite if each of its members had this property.

3.5.13 Corollary. *For every $n \in \mathbb{N}$ and every algebra $\mathbf{A}$ belonging to an n-locally finite variety*[32] *$\mathcal{V}$, we have the equality*

$$\mathrm{Clo}^{(n)}(\mathbf{A}) = \mathrm{Term}^{(n)}(\mathbf{A})$$

and the latter set is finite.

Proof: Suppose that $\mathbf{A} \in \mathcal{V}$ is an algebra in an n-locally finite variety $\mathcal{V}$. It is well-known that the set of n-ary term operations of $\mathbf{A}$ is generated by the n projection operations of arity n as a subpower in $\mathbf{A}^{A^n}$:

$$\mathrm{Term}^{(n)}(\mathbf{A}) = \left\langle \left\{ e_i^{(n)} \;\middle|\; 1 \leq i \leq n \right\} \right\rangle_{\mathbf{A}^{A^n}}.$$

As $\mathbf{A} \in \mathcal{V}$ and $\mathcal{V}$ is closed under taking powers of algebras, also $\mathbf{A}^{A^n} \in \mathcal{V}$. Since $\mathrm{Term}^{(n)}(\mathbf{A})$ is n-generated, it is finite because $\mathbf{A}^{A^n}$ is n-locally finite as a member of the n-locally finite class $\mathcal{V}$. Now, Corollary 3.5.12 yields the claim. $\square$

From here we can infer that algebras generating a 1-locally finite variety fall under the scope of the algorithms we are going present soon. In particular this includes algebras lying in locally finite varieties, which are by definition n-locally finite for every $n \in \mathbb{N}$. Important instances of such varieties are given by varieties generated by a finite algebra, such as BOOLEan algebras or distributive lattices.

3.5.14 Corollary. *Every algebra $\mathbf{A}$ belonging to a 1-locally finite variety has got only a finite number of (idempotent) unary clone operations and neighbourhoods. Therefore, the sets $\mathrm{Idem}\,\mathbf{A}$, $\mathrm{Neigh}\,\mathbf{A}$, $\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A})$, $\mathrm{Cov}(\mathbf{A})$ and all of their factor sets are finite, whence they will fulfil any sort of chain condition in particular.*

Furthermore, such algebras have the FIP, and therefore are neighbourhood self-embedding simple. In particular for them mutual embeddability of neighbourhoods and isomorphism is the same.

[32]This assumption is equivalent to $\mathrm{Var}\,\mathbf{A}$ being n-locally finite.

Proof: From Corollary 3.5.13 for the case $n = 1$, it is immediate that the sets $\operatorname{Idem} \mathbf{A} \subseteq \operatorname{Clo}^{(1)}(\mathbf{A}) = \operatorname{Term}^{(1)}(\mathbf{A})$ are finite. Therefore, we can only have finitely many neighbourhoods in an algebra $\mathbf{A}$ as given. The remaining statements about chain conditions are obvious consequences of this.

Furthermore, since $\operatorname{Clo}^{(1)}(\mathbf{A})$ is finite, we can infer from Corollary 3.2.17 that algebras in 1-locally finite varieties have the FIP, whence they are neighbourhood self-embedding simple by Lemma 3.2.18(b). Thus, according to Lemma 3.2.19, we have $\precapprox \cap \succapprox = \cong$. $\square$

Next, we return to the problem of determining non-refinable covers of algebras. In fact, Lemma 3.5.9 already prescribes a rough method how one could attack the problem of isolating the non-refinable ones among all covers of an algebra. Let us suppose that the quasiordered set $(\operatorname{Cov}(\mathbf{A}), \leq_{\mathrm{ref}})$ contains minimal elements, and we have an idea how to descend in this quasiordered set to its minimal elements. Assume further that $\mathcal{U} \in \operatorname{Neigh} \mathbf{A}$ is one of them, then item (e) tells us, we only need to stay in the same block w.r.t. refinement equivalence and look for a cover that consists of mutually incomparable neighbourhoods. This will indeed be a non-refinable cover of $\mathbf{A}$, especially irredundant. Besides, item (d) asserts that such a representative in the refinement equivalence class of $\mathcal{U}$ is unique, which should help to find it. Indeed, if $(\mathcal{U}, \subseteq)$ satisfies ACC, then the set $\operatorname{Max}(\mathcal{U}, \subseteq)$ of inclusion maximal members of $\mathcal{U}$ forms a cover (see items 3.5.6(d) and 3.5.4(g)) as desired because inclusion maximal sets are mutually incomparable. We summarise this thought in the following lemma.

3.5.15 Lemma. *Let $\mathbf{A}$ be an algebra such that $(\operatorname{Neigh} \mathbf{A}, \subseteq)$ fulfils ACC and that the poset on $\operatorname{Cov}(\mathbf{A}) / \equiv_{\mathrm{ref}}$ that is canonically associated with $(\operatorname{Cov}(\mathbf{A}), \leq_{\mathrm{ref}})$ satisfies DCC. Then refinement minimal covers of $\mathbf{A}$ exist; let $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ be one of them. The collection $\operatorname{Max}(\mathcal{U}, \subseteq)$ forms a non-refinable cover of $\mathbf{A}$. The assumptions of this lemma are especially fulfilled if $\operatorname{Neigh} \mathbf{A}$ is finite, e.g. for finite algebras.*

In particular Lemma 3.5.15 exhibits sufficient conditions for the existence of non-refinable covers.

Proof: Since the poset on $\operatorname{Cov}(\mathbf{A}) / \equiv_{\mathrm{ref}}$, fulfils DCC, refinement minimal covers exist, they are just the members of any minimal block in this poset. Finally, ACC for $(\operatorname{Neigh} \mathbf{A}, \subseteq)$ implies ACC for $(\mathcal{U}, \subseteq)$, so everything else follows from the explanations in the paragraph preceding the lemma. $\square$

The previous result immediately gives rise to Algorithm 1, which may also be tried for algebras that are not subject to the conditions in Lemma 3.5.15. However, in such a general case it is not guaranteed that it will ever terminate with a satisfactory answer.

Regarding Algorithm 1, the first part, to obtain refinement minimal covers of an algebra, is still unclear and needs further attention. In order to understand this step

Algorithm 1: Determining non-refinable covers in general

Data: An algebra $\mathbf{A}$ satisfying the assumptions of Lemma 3.5.15
Result: A non-refinable cover of $\mathbf{A}$
begin

```
Determine a refinement minimal cover 𝒰 of A;
// e.g. by iteratively refining a given cover of A
Compute Max (𝒰, ⊆);
return Max (𝒰, ⊆)
```

in more detail, we want to dwell a little bit more upon the procedure mentioned in the introductory paragraph of this section. The central idea there was to replace one neighbourhood U in a collection (e.g. a cover) $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ by the members of a cover $\mathcal{V}$ of $\mathbf{A}|_U$. Of course, this only makes sense if this cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A}|_U)$ does not contain U itself, i.e. if it consists of proper subneighbourhoods of U (cf. Lemma 3.3.7). Otherwise, one would only unnecessarily add a number of redundant (w.r.t. whatever is originally covered by $\mathcal{U}$) neighbourhoods. If such a cover $\mathcal{V} \subseteq \mathfrak{P}(U) \setminus \{U\}$ exists, then by Lemma 3.4.9(a), clearly every superset is a cover, too, so in particular the set $\operatorname{Neigh} \mathbf{A}|_U \setminus \{U\}$ of all proper subneighbourhoods of U is.

These considerations motivate Definitions 3.5.16 and 3.5.24. The first, concerning *irreducibility*, will play a major role in the following section, and we shall only list a few basic observations here (3.5.17–3.5.23), which are direct consequences of the definition. The second (3.5.24), will be important for the algorithms developed in the remainder of this section.

For later use we already introduce a quite general version of irreducibility, which is parametrised by a quasiorder $q \subseteq (\operatorname{Neigh} \mathbf{A})^2$, $q \subseteq \leq_{\mathrm{cov}}$, similarly as the refinement relation. Of all choices of q, the motivation above only addresses the most important case $q = \subseteq_{\operatorname{Neigh} \mathbf{A}}$, which also occurs in Definitions 5.4 of [Kea01] and 3.7.1 of [Beh09].

3.5.16 Definition. Let $q \subseteq (\operatorname{Neigh} \mathbf{A})^2$ be a quasiorder where $q \subseteq \leq_{\mathrm{cov}}$. An algebra $\mathbf{A}$ is called *q-irreducible* if every cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ contains a neighbourhood $V \in \mathcal{V}$ such that $A \mathrel{q} V$, otherwise *q-reducible*.

Furthermore, a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ of an algebra $\mathbf{A}$ is called *q-irreducible* if the restricted algebra $\mathbf{A}|_U$ is $q{\restriction}_{\operatorname{Neigh} \mathbf{A}|_U}$-irreducible, and *$q$-reducible* otherwise.

We use $\operatorname{Irr}_q(\mathbf{A}) := \{U \in \operatorname{Neigh} \mathbf{A} \mid U \text{ } q\text{-irreducible}\}$ to denote the *set of all q-irreducible neighbourhoods of an algebra* $\mathbf{A}$.

We agree on *irreducible* and *reducible* without any prefix to mean $\subseteq_{\operatorname{Neigh} \mathbf{A}}$-irreducible and $\subseteq_{\operatorname{Neigh} \mathbf{A}}$-reducible, respectively. Correspondingly, we define the set of all irreducible neighbourhoods of $\mathbf{A}$ as $\operatorname{Irr}(\mathbf{A}) := \operatorname{Irr}_{\subseteq_{\operatorname{Neigh} \mathbf{A}}}(\mathbf{A})$. ◊

Let us note that this definition nicely fits into the framework of our localisation theory since q-irreducibility of neighbourhoods $U \in \operatorname{Neigh} \mathbf{A}$ is defined in terms of the local structure $\mathbf{A}|_U$. The following remark makes the notion more explicit by avoiding the use of $\mathbf{A}|_U$ and reformulates it using q-refinement.

3.5.17 Remark. We observe that a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ is q-irreducible for a quasiorder $q \subseteq \leq_{\text{cov}}$ if and only if every collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A} \cap \mathfrak{P}(U)$ covering U w.r.t. $\mathbf{A}$ contains a member $V \in \mathcal{V}$ such that $U \mathrel{q} V$. Equivalently, we can express this by requiring that every $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ satisfying $\mathcal{V} \subseteq \mathfrak{P}(U)$ must fulfil $\{U\} \sqsubseteq (q)\, \mathcal{V}$. Of course, for any collection of neighbourhoods $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, the condition $\mathcal{V} \subseteq \mathfrak{P}(U)$ implies $\mathcal{V} \sqsubseteq (\subseteq)\, \{U\}$, and hence, by Remark 3.5.1, also $\mathcal{V} \leq_{\text{cov}} \{U\}$. Therefore, we can slightly strengthen the conclusion of the implication above and obtain the following characterisation: $U \in \operatorname{Neigh} \mathbf{A}$ is q-irreducible if and only if every $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ satisfying $\mathcal{V} \subseteq \mathfrak{P}(U)$, fulfils $\{U\} \leq_{\text{ref}} (q)\, \mathcal{V}$, as well.

Under the assumption that U is a largest element w.r.t. q in $\operatorname{Neigh} \mathbf{A}|_U$, i.e. if $\operatorname{Neigh} \mathbf{A}|_U \sqsubseteq (q)\, \{U\}$, then, using Lemma 3.5.4(d), the characterisation can be strengthened once more: a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ is q-irreducible if and only if every $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}|_U$ covering U satisfies $\{U\} \equiv_{\text{ref}} (q)\, \mathcal{V}$. ♢

Another indication confirming the impression that q-irreducibility is compatible with localisation is given by the next remark.

3.5.18 Remark. For a quasiorder $q \subseteq \leq_{\text{cov}}$, an algebra $\mathbf{A}$ is q-irreducible if and only if its full neighbourhood $A \in \operatorname{Neigh} \mathbf{A}$ is q-irreducible, furthermore, this is the case if and only if $\mathbf{A}|_A$ is q-irreducible. Combining this with Remark 3.5.17, we get that $\mathbf{A}$ is q-irreducible, if and only if every $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ satisfies $\{A\} \leq_{\text{ref}} (q)\, \mathcal{V}$.

If A is a largest element w.r.t. q in $\operatorname{Neigh} \mathbf{A}$, then $\mathbf{A}$ is q-irreducible if and only if every cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ is q-refinement equivalent to the singleton collection $\{A\}$. ♢

Proof: The second stated equivalence is exactly the definition of q-irreducibility of the neighbourhood A. Hence, we only need to see, why q-irreducibility of $\mathbf{A}$ and of the corresponding saturated algebra $\mathbf{A}|_A$ are equivalent. It has been explained in Remark 3.4.10 that $\operatorname{Cov}(\mathbf{A}) = \operatorname{Cov}(\mathbf{A}|_A)$. Since both algebras have the same carrier set A, it follows that $\mathbf{A}$ is q-irreducible if and only if $\mathbf{A}|_A$ is q-irreducible.

Using the characterisations of q-irreducibility of A from the previous remark, we can directly read off the last stated characterisations. □

The following statement clears up trivial cases[33] and follows directly from the definition and previous observations.

3.5.19 Corollary. *Let $q \subseteq \leq_{\text{cov}}$ be a quasiorder on neighbourhoods. The initial algebra $\mathbf{E}$ on the empty set is always q-irreducible. A singleton neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ of an algebra $\mathbf{A}$, in particular a one-element algebra itself, is q-irreducible if and only if $\mathbf{A}$ does not contain nullary fundamental operations.*

[33] The observation concerning the classical version of irreducibility of one-element neighbourhoods also occurs on page 4 of [KL10].

Proof: We read from Corollary 3.4.13 that the algebra $\mathbf{E}$ on the empty set satisfies $\operatorname{Cov}(\mathbf{E}) = \{\{\emptyset\}\}$, and hence is q-irreducible due to reflexivity of q. Furthermore, if $U \in \operatorname{Neigh}\mathbf{A}$ is a singleton, then $\operatorname{Neigh}\mathbf{A}|_U = \{U\}$, and thus we obtain $\mathfrak{P}(\operatorname{Neigh}\mathbf{A}|_U) = \{\emptyset, \{U\}\}$. Therefore, U is q-irreducible if and only if $\emptyset \notin \operatorname{Cov}_{\mathbf{A}}(U)$, which, by Lemma 3.4.12, is equivalent to $\mathbf{A}$ not having nullary operations. □

It is also helpful to know how irreducibility notions for different quasiorders on neighbourhoods relate to each other. The answer is the following evident lemma.

3.5.20 Lemma. *Suppose for an algebra $\mathbf{A}$ that $q_1, q_2 \subseteq (\operatorname{Neigh}\mathbf{A})^2$ are quasiorders such that $q_1 \subseteq q_2 \subseteq \leq_{\mathrm{cov}}$, then we have $\operatorname{Irr}_{q_1}(\mathbf{A}) \subseteq \operatorname{Irr}_{q_2}(\mathbf{A})$. Especially, if $\mathbf{A}$ is q_1-irreducible, then it is also q_2-irreducible.*

Proof: Under the assumptions in the lemma let $U \in \operatorname{Neigh}\mathbf{A}$ be q_1-irreducible. Using Remark 3.5.17, we know that every cover $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A} \cap \mathfrak{P}(U)$ of U contains a neighbourhood V for which $U \; q_1 \; V$. Since $q_1 \subseteq q_2$, it follows automatically $U \; q_2 \; V$, which by the same remark implies that U is also q_2-irreducible.

Via Remark 3.5.18, the statement about neighbourhoods entails the one about whole algebras. □

3.5.21 Corollary. *Among all parametric irreducibility notions, classical irreducibility is the strongest. That is to say, for an algebra $\mathbf{A}$ and any quasiorder $q \subseteq \leq_{\mathrm{cov}}$ on neighbourhoods of $\mathbf{A}$, we have $\operatorname{Irr}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A})$, and irreducibility of $\mathbf{A}$ entails any form of q-irreducibility of $\mathbf{A}$.*

Proof: The least quasiorder on neighbourhoods is given by the equality relation. Thus, by Lemma 3.5.20, we get $\operatorname{Irr}_{\Delta_{\operatorname{Neigh}\mathbf{A}}}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A})$. Since irreducibility of neighbourhoods is defined via subneighbourhoods, it follows that every irreducible neighbourhood U has the property that every cover $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A} \cap \mathfrak{P}(U)$ contains a neighbourhood $V \in \mathcal{V}$ such that $U \subseteq V$, implying $U = V$. Thus, we have $\operatorname{Irr}(\mathbf{A}) \subseteq \operatorname{Irr}_{\Delta_{\operatorname{Neigh}\mathbf{A}}}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A})$. Finally, the statement about irreducibility of algebras follows from that about neighbourhoods. □

We have observed above how reducible algebras can be characterised. We now record this simple fact for later reference (cf. also Theorem 2.7 of [KL10], but note that in Definition 2.6 of that article, irreducibility has been defined in a different but equivalent way, see Theorem 5.5 of [Kea01] and Proposition 3.6.15 below). At the same time we extend the mentioned results to q-irreducibility.

3.5.22 Lemma. *Given an algebra $\mathbf{A}$, and a quasiorder $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ such that $q \subseteq \leq_{\mathrm{cov}}$, a set $U \in \operatorname{Neigh}\mathbf{A}$ is q-reducible if and only if $\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_q \{U\}$ belongs to $\operatorname{Cov}_{\mathbf{A}}(U)$.*

In case that the quasiorder q is an equivalence relation on neighbourhoods, we have of course $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\} = \operatorname{Neigh}\mathbf{A}|_U \setminus [U]_q$, such that U is q-reducible exactly if the subneighbourhoods of U that are not q-equivalent to U suffice to cover U. Moreover, for q being set inclusion we get the set of all proper subneighbourhoods $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\} = \operatorname{Neigh}\mathbf{A}|_U \setminus \{U\}$.

Proof: If $U \in \operatorname{Neigh}\mathbf{A}$ is q-reducible, then according to Remark 3.5.17, it has got a cover $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}|_U$ where $U \mathrel{q} V$ fails for all $V \in \mathcal{V}$, i.e. $V \notin {\uparrow_q}\{U\}$. This means, $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$, and $U \leq_{\text{cov}} \mathcal{V}$ (see Lemma 3.4.16). Using Lemma 3.4.9(a), we infer $\mathcal{V} \leq_{\text{cov}} \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$, and via transitivity of $\leq_{\text{cov}}$ we conclude that $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\} \in \operatorname{Cov}_{\mathbf{A}}(U)$.

The converse implication is trivial in view of Lemma 3.4.16 and the fact that $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$ evidently does not contain neighbourhoods lying above (w.r.t. q) of the full neighbourhood U of $\mathbf{A}|_U$. $\square$

With the following simple corollary we provide a sort of converse to the implication / inclusion proven in Corollary 3.5.21.

3.5.23 Corollary. *For an algebra $\mathbf{A}$, quasiorders $q, q_1, q_2 \subseteq \leq_{\text{cov}}$ on the set of neighbourhoods of $\mathbf{A}$ and a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ satisfying the condition ${\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\} = {\uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\}$, we have that U is q_1-irreducible if and only if U is q_2-irreducible.*

Special instances of this statement include the following cases: If $U \in \operatorname{Neigh}\mathbf{A}$ has got the property that for all $V \in \operatorname{Neigh}\mathbf{A}$ the condition $U \mathrel{q} V \subseteq U$ implies $V = U$, i.e. if ${\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\} = \{U\}$, then U is q-irreducible if and only if U is irreducible. In particular, if $q \subseteq \precsim$ and $\mathbf{A}$ is neighbourhood self-embedding simple, then ${\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_V}}}\{V\} = \{V\}$ holds for all $V \in \operatorname{Neigh}\mathbf{A}$, whence we get $\operatorname{Irr}(\mathbf{A}) = \operatorname{Irr}_q(\mathbf{A})$.

Furthermore, if ${\uparrow_q}\{A\} = \{A\}$, then q-irreducibility of $\mathbf{A}$ is equivalent to irreducibility of $\mathbf{A}$. Especially, this equivalence holds for $\mathbf{A}$ if $(\operatorname{Neigh}\mathbf{A}, q)$ is a poset with largest element A.

Proof: Using Lemma 3.5.22, the actual statement of this corollary is almost trivial: for $U \in \operatorname{Neigh}\mathbf{A}$ and any quasiorder $q \subseteq \leq_{\text{cov}}$, we certainly have the equality $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\} = \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\}$. Since Lemma 3.5.22 provides a characterisation of q-irreducibility of neighbourhoods, where the parameter q only occurs in the expression $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$, we see that, under the assumption ${\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\} = {\uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\}$, the neighbourhood U is q_1-irreducible precisely if it is q_2-irreducible.

If q is set inclusion, then, evidently, we have ${\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}}\{U\} = \{U\}$, explaining the first special case of the corollary.

Again as a specialisation of this statement, we have the following: if $\mathbf{A}$ is neighbourhood self-embedding simple, $q \subseteq \precsim$ and $V, W \in \operatorname{Neigh}\mathbf{A}$ satisfy $V \mathrel{q} W \subseteq V$, then it follows $V \precsim W \subseteq V$, and hence $W = V$. Consequently, we obtain the equality ${\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_V}}}\{V\} = \{V\}$ for all $V \in \operatorname{Neigh}\mathbf{A}$. By the statement that was verified in the previous paragraph, this means, for any neighbourhood $V \in \operatorname{Neigh} \mathbf{A}$, irreducibility and q-irreducibility are equivalent. This proves that $\operatorname{Irr}(\mathbf{A}) = \operatorname{Irr}_q(\mathbf{A})$.

A second specialisation arises by letting $U = A$. Then assuming $\uparrow_q \{A\} = \{A\}$ implies that the full neighbourhood A is q-irreducible if and only if it is irreducible. Combining this with Remark 3.5.18, we get the analogous equivalence for the algebra $\mathbf{A}$.

In the very last special case we suppose that q is an order relation with A as a top element. Then, of course every $V \in \operatorname{Neigh} \mathbf{A}$ with $A \, q \, V$ must be equal to A. Hence, we obtain $\uparrow_q \{A\} = \{A\}$ and the desired conclusion follows from the considerations above. $\square$

Next, we state the second definition that was already announced before Definition 3.5.16 and plays an important role in iteratively refining covers of algebras. In this respect, we deliberately do not deal with the parametric versions of refinement and (ir)reducibility, mainly since irreducibility is the strongest irreducibility notion (see Corollary 3.5.21) and for the reasons explained in Remark 3.5.7.

3.5.24 Definition. For an algebra $\mathbf{A}$ we define the following operation on collections of neighbourhoods

$$\begin{aligned} \pitchfork\colon \quad \operatorname{Neigh} \mathbf{A} \times \mathfrak{P}(\operatorname{Neigh} \mathbf{A}) &\longrightarrow \mathfrak{P}(\operatorname{Neigh} \mathbf{A}) \\ (U,\mathcal{U}) &\longmapsto \pitchfork(U,\mathcal{U}) := (\mathcal{U} \cup \operatorname{Neigh} \mathbf{A}|_U) \setminus \{U\}, \end{aligned}$$

replacing the first argument in the second argument by all its proper subneighbourhoods. So for $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ and $U \in \operatorname{Neigh} \mathbf{A}$, it is

$$\pitchfork(U,\mathcal{U}) = \mathcal{U} \setminus \{U\} \cup \operatorname{Neigh} \mathbf{A}|_U \setminus \{U\},$$

and this is to be read as U *replaced by proper subneighbourhoods in* $\mathcal{U}$ or unofficially as U *hacked into pieces inside* $\mathcal{U}$. $\diamondsuit$

Using this terminology, we can now formulate the procedure outlined before Definition 3.5.16 as Algorithm 2. We also mention that by studying the operation $\pitchfork$ and by making it part of algorithms whose goal it is to determine certain sorts of covers, e.g. non-refinable ones, we somehow give an answer to the first open research question in [Beh09, p. 147]. The hedging "somehow" in the previous sentence reflects the fact that the mentioned open question was (deliberately) formulated in a rather imprecise form.

So far, we have not discussed if Algorithm 2 is correct, nor if it ever terminates. This is the purpose of the following simple lemma.

3.5.25 Lemma. *For an algebra* $\mathbf{A}$*, a collection* $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ *and a neighbourhood* $U \in \mathcal{U}$ *the following holds:*

(a) We have $\pitchfork(U,\mathcal{U}) \sqsubseteq (\subseteq) \, \mathcal{U}$*, so* $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$ *if and only if* $\mathcal{U} \leq_{\mathrm{cov}} \pitchfork(U,\mathcal{U})$*.*

(b) If U *is a reducible neighbourhood or* $\pitchfork(U,\mathcal{U}) \in \operatorname{Cov}(\mathbf{A})$*, then* $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$*.*

Algorithm 2: Producing covers with irreducible neighbourhoods

Data: A cover $\mathcal{U}$ of an algebra $\mathbf{A}$

Result: A cover $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$ that refines $\mathcal{U}$ and contains only irreducible neighbourhoods

begin

  Initialise $\mathcal{V} \leftarrow \mathcal{U}$;

  while $\mathcal{V}$ *contains reducible neighbourhoods* **do** `// i.e. while` $\mathcal{V} \nsubseteq \mathrm{Irr}(\mathbf{A})$

    Choose a reducible $V \in \mathcal{V}$;

      `// preferably of maximum cardinality or` $\subseteq$`-maximal`

    $\mathcal{V} \leftarrow \pitchfork(V, \mathcal{V})$;

  return $\mathcal{V}$

(c) *Furthermore, if* $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$*, then we have* $\pitchfork(U,\mathcal{U}) \equiv_{\mathrm{ref}} \mathcal{U}$ *if and only if there exists some* $V \in \mathcal{U}$ *such that* $U \subset V$*. Consequently, if* $(\mathcal{U}, \subseteq)$ *is an antichain, then the refinement* $\pitchfork(U,\mathcal{U}) <_{\mathrm{ref}} \mathcal{U}$ *is proper.*

(d) *If there exist sequences* $(\mathcal{V}_i)_{i\in\mathbb{N}} \in (\mathfrak{P}(\mathrm{Neigh}\,\mathbf{A}))^{\mathbb{N}}$ *and* $(V_i)_{i\in\mathbb{N}} \in (\mathrm{Neigh}\,\mathbf{A})^{\mathbb{N}}$ *such that* $V_i \in \mathcal{V}_i$ *and* $\mathcal{V}_{i+1} = \pitchfork(V_i, \mathcal{V}_i)$ *for every* $i \in \mathbb{N}$*, then* $\mathrm{Neigh}\,\mathbf{A}$ *must be an infinite set.*

So, if Algorithm 2 terminates after finitely many steps, then its result is correct. Furthermore, if $\mathrm{Neigh}\,\mathbf{A}$ *is finite*[34]*, then the algorithm is guaranteed to stop after a finite number of steps.*

Proof: First we verify the four initial statements, then we discuss their consequences for Algorithm 2.

(a) By definition it is $\pitchfork(U,\mathcal{U}) = \mathcal{U} \setminus \{U\} \cup \mathrm{Neigh}\,\mathbf{A}|_U \setminus \{U\}$, and every member $U' \in \mathcal{U} \setminus \{U\}$ is trivially contained in itself. Furthermore, $\mathrm{Neigh}\,\mathbf{A}|_U \subseteq \mathfrak{P}(U)$ with $U \in \mathcal{U}$ holds by Lemma 3.3.7, wherefore we have $\pitchfork(U,\mathcal{U}) \sqsubseteq (\subseteq)\,\mathcal{U}$. The remaining characterisation is a direct consequence of Definition 3.5.2(i).

(b) Suppose that U is reducible, so we get $U \leq_{\mathrm{cov}} \mathrm{Neigh}\,\mathbf{A}|_U \setminus \{U\} \subseteq \pitchfork(U,\mathcal{U})$ by Lemma 3.5.22. It follows $U \leq_{\mathrm{cov}} \mathrm{Neigh}\,\mathbf{A}|_U \setminus \{U\} \leq_{\mathrm{cov}} \pitchfork(U,\mathcal{U})$, upon application of Lemma 3.4.9(a), i.e. $U \leq_{\mathrm{cov}} \pitchfork(U,\mathcal{U})$. Any other neighbourhood $U' \in \mathcal{U} \setminus \{U\}$ belongs to $\pitchfork(U,\mathcal{U})$ since $\mathcal{U} \setminus \{U\} \subseteq \pitchfork(U,\mathcal{U})$. So, we have $U' \leq_{\mathrm{cov}} \pitchfork(U,\mathcal{U})$. Altogether we have demonstrated that $\pitchfork(U,\mathcal{U})$ covers every member of $\mathcal{U}$, i.e. $\mathcal{U} \leq_{\mathrm{cov}} \pitchfork(U,\mathcal{U})$. Using item (a) the latter is equivalent to $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$.

   If $\pitchfork(U,\mathcal{U}) \in \mathrm{Cov}(\mathbf{A})$, then $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$ if and only if $\pitchfork(U,\mathcal{U}) \sqsubseteq (\subseteq)\,\mathcal{U}$ due to Lemma 3.5.4(h). The truth of the latter condition is asserted by item (a), so $\pitchfork(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$ follows.

[34] This already provides a slight generalisation of first item of Theorem 2.9 in [KL10].

(c) According to Lemma 3.5.4(d), such a refinement $\Axe(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$ is actually an equivalence if and only if $\mathcal{U} \sqsubseteq (\subseteq) \Axe(U,\mathcal{U})$. Since $\mathcal{U} \setminus \{U\} \subseteq \Axe(U,\mathcal{U})$, we trivially have $\mathcal{U} \setminus \{U\} \sqsubseteq (\subseteq) \Axe(U,\mathcal{U})$. Hence, the condition $\mathcal{U} \sqsubseteq (\subseteq) \Axe(U,\mathcal{U})$ is equivalent to $\{U\} \sqsubseteq (\subseteq) \Axe(U,\mathcal{U})$. From here it follows that there exists some neighbourhood $V \in \Axe(U,\mathcal{U}) = (\mathcal{U} \cup \operatorname{Neigh} \mathbf{A}|_U) \setminus \{U\}$ being a superset of U. Since $U \notin \Axe(U,\mathcal{U})$, this implies $U \subset V$ and excludes $V \subseteq U$ and, in particular, $V \in \operatorname{Neigh} \mathbf{A}|_U \subseteq \mathfrak{P}(U)$. The only remaining possibility is $V \in \mathcal{U}$ and $U \subset V$. Conversely, if this is true, then we have $U \subseteq V \in \mathcal{U} \setminus \{U\} \subseteq \Axe(U,\mathcal{U})$, which shows $\{U\} \sqsubseteq (\subseteq) \Axe(U,\mathcal{U})$ and, by the above, $\mathcal{U} \equiv_{\mathrm{ref}} \Axe(U,\mathcal{U})$.

Certainly, if $(\mathcal{U}, \subseteq)$ forms an antichain, then such a proper inclusion is impossible, wherefore the refinement $\Axe(U,\mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$ is not an equivalence, i.e. a proper one.

(d) The proof of this fact will be by contradiction, so let us suppose that $\operatorname{Neigh} \mathbf{A}$ is finite and sequences as described exist. To carry out the proof we need to introduce a little bit of terminology. For a subset $\mathcal{X} \subseteq \operatorname{Neigh} \mathbf{A}$ let us call any sequence $(\mathcal{V}_i)_{i\in\mathbb{N}} \in \mathfrak{P}(\mathcal{X})^{\mathbb{N}}$ where $\mathcal{V}_{i+1} = \Axe(V_i, \mathcal{V}_i)$ for some neighbourhood $V_i \in \mathcal{V}_i$ and all $i \in \mathbb{N}$ an illegal $\mathcal{X}$-sequence. We say that $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ is an illegal $\mathcal{X}$-input if there is some illegal $\mathcal{X}$-sequence $(\mathcal{V}_i)_{i\in\mathbb{N}}$ such that $\mathcal{V}_0 = \mathcal{U}$. It is easy to see, that for every illegal $\mathcal{X}$-sequence $(\mathcal{V}_i)_{i\in\mathbb{N}}$ and an integer $k \in \mathbb{N}$ also $(\mathcal{V}_{k+i})_{i\in\mathbb{N}}$ is an illegal $\mathcal{X}$-sequence. Hence, ever entry of an illegal $\mathcal{X}$-sequence is again an illegal $\mathcal{X}$-input.

By our assumption there exists an illegal $\operatorname{Neigh} \mathbf{A}$-sequence. Consequently, the set

$$\mathbb{I} := \left\{ \mathcal{X} \subseteq \operatorname{Neigh} \mathbf{A} \;\middle|\; \left\{ (\mathcal{V}_i)_{i\in\mathbb{N}} \in \mathfrak{P}(\mathcal{X})^{\mathbb{N}} \;\middle|\; (\mathcal{V}_i)_{i\in\mathbb{N}} \text{ is an illegal } \mathcal{X}\text{-sequence} \right\} \neq \emptyset \right\}$$

contains $\operatorname{Neigh} \mathbf{A}$, so it is non-empty. As $\operatorname{Neigh} \mathbf{A}$ is finite, we may choose a set $\mathcal{Y} \in \operatorname{Min}(\mathbb{I}, \subseteq)$. Let $(\mathcal{V}_i)_{i\in\mathbb{N}} \in \mathfrak{P}(\mathcal{Y})^{\mathbb{N}}$ be an illegal $\mathcal{Y}$-sequence. Since $\mathbb{N}$ is infinite and $\mathcal{Y} \subseteq \operatorname{Neigh} \mathbf{A}$ is finite, this sequence cannot be injective, thus it contains repetitions. That is, we can find indices $i, j \in \mathbb{N}$ such that $i < j$ and $\mathcal{V}_i = \mathcal{V}_j$. By construction, we have $V_i \in \mathcal{V}_i \setminus \mathcal{V}_{i+1}$ since $\mathcal{V}_{i+1} = \Axe(V_i, \mathcal{V}_i)$ and $V_i \in \mathcal{V}_i$. Moreover, $V_i \in \mathcal{V}_i = \mathcal{V}_j$ and $j > i$. Therefore, the set

$$\mathcal{N} := \{ W \in \operatorname{Neigh} \mathbf{A} \mid \exists i, j \in \mathbb{N} \colon i < j \wedge W \in \mathcal{V}_i \setminus \mathcal{V}_{i+1} \wedge W \in \mathcal{V}_j \}$$

contains V_i and hence is non-empty. Let $V \in \operatorname{Max}(\mathcal{N}, \subseteq)$ and $i \in \mathbb{N}$ such that $V \in \mathcal{V}_i \setminus \mathcal{V}_{i+1}$ and $V \in \mathcal{V}_j$ for some $j > i$. Set $k := \min \{ i < \nu \leq j \mid V \in \mathcal{V}_\nu \}$, then $V \in \mathcal{V}_k \setminus \mathcal{V}_{i+1}$, so $k > i + 1$. Thus, $k - 1 > i$, and by minimality of k, we have $V \notin \mathcal{V}_{k-1}$. Together with

$$V \in \mathcal{V}_k = \Axe(V_{k-1}, \mathcal{V}_{k-1}) = \mathcal{V}_{k-1} \setminus \{V_{k-1}\} \cup \operatorname{Neigh} \mathbf{A}|_{V_{k-1}} \setminus \{V_{k-1}\}$$

this implies $V \in \operatorname{Neigh} \mathbf{A}|_{V_{k-1}} \setminus \{V_{k-1}\}$, i.e. $V \subset V_{k-1}$. So by maximality of $V \in \mathcal{N}$, we get $V_{k-1} \notin \mathcal{N}$. Since $V_{k-1} \in \mathcal{V}_{k-1} \setminus \mathcal{V}_k$, this means that there does not exist an index $\ell \in \mathbb{N}$, $\ell > k-1$, such that $V_{k-1} \in \mathcal{V}_\ell$. In other words, for every index $\ell \geq k$ we have $V_{k-1} \notin \mathcal{V}_\ell$, i.e. $\mathcal{V}_\ell \subseteq \mathcal{Y} \setminus \{V_{k-1}\}$. Therefore, $(\mathcal{V}_{\ell+k})_{\ell\in\mathbb{N}}$ is an illegal $\mathcal{Y} \setminus \{V_{k-1}\}$-sequence, and $\mathcal{Y} \setminus \{V_{k-1}\}$ is a proper subset of $\mathcal{Y}$ as $V_{k-1} \in \mathcal{V}_{k-1} \subseteq \mathcal{Y}$. This contradicts that $\mathcal{Y}$ was minimal in $\mathbb{I}$ w.r.t. set inclusion and shows that for a finite set Neigh $\mathbf{A}$, illegal Neigh $\mathbf{A}$-sequences do not exist.

Let us now see how these facts apply to Algorithm 2. Assuming that upon input $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ Algorithm 2 terminates after $n \in \mathbb{N}$ repetitions of the loop, it produces a sequence of collections of neighbourhoods $\mathcal{U} =: \mathcal{V}_0, \ldots, \mathcal{V}_n$ where $\mathcal{V}_{i+1} = \Axe(V_i, \mathcal{V}_i)$ for some reducible neighbourhood $V_i \in \mathcal{V}_i$ and all $0 \leq i < n$. By item (b), we get $\mathcal{V}_{i+1} \leq_{\text{ref}} \mathcal{V}_i$ for each $0 \leq i < n$, so using transitivity we have $\mathcal{V}_n \leq_{\text{ref}} \mathcal{V}_0 = \mathcal{U}$. The collection $\mathcal{V}_n$ is returned by the algorithm causing it to stop. Hence, it must have failed to satisfy the loop condition, which means that $\mathcal{V}_n$ contains only irreducible neighbourhoods. According to Lemma 3.5.4(g), the refinement $\mathcal{V}_n \leq_{\text{ref}} \mathcal{U}$ is a cover of $\mathbf{A}$ because the algorithm assumes $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$. Therefore, Algorithm 2 is correct.

Next, we suppose that Neigh $\mathbf{A}$ is finite to show that the algorithm will always terminate after a finite number of steps. If it would not, then some cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A}) \subseteq \operatorname{Neigh}\mathbf{A}$ would be an illegal Neigh $\mathbf{A}$-input generating sequences as in item (d) (even special ones, where V_i is a reducible neighbourhood for every $i \in \mathbb{N}$). However, we have shown above that this contradicts finiteness of Neigh $\mathbf{A}$, wherefore Algorithm 2 eventually has to terminate on any input. □

From this lemma we can immediately observe that non-refinable covers must consist of irreducible neighbourhoods[35]. This makes Algorithm 2 seem a useful tool on our way to non-refinable covers. This impression, however, is misleading as we shall see soon.

3.5.26 Corollary. *If $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ is a non-refinable cover of an algebra $\mathbf{A}$, then $\mathcal{U} \subseteq \operatorname{Irr}(\mathbf{A})$. In this case it follows in particular that $\operatorname{Irr}(\mathbf{A}) \in \operatorname{Cov}(\mathbf{A})$.*

Proof: If $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ is non-refinable, then Lemma 3.5.9(e) implies that $(\mathcal{U}, \subseteq)$ is an antichain. If some $U \in \mathcal{U}$ were reducible, then $\Axe(U, \mathcal{U}) \leq_{\text{ref}} \mathcal{U}$ would be a proper refinement due to items (b) and (c) of Lemma 3.5.25 and $(\mathcal{U}, \subseteq)$ being an antichain. However, this would contradict refinement minimality of $\mathcal{U}$, which follows from non-refinability by Lemma 3.5.9(e). Thus, every $U \in \mathcal{U}$ must be an irreducible neighbourhood.

Clearly, if $\mathcal{U} \subseteq \operatorname{Irr}(\mathbf{A})$ and we know $A \leq_{\text{cov}} \mathcal{U}$, then $A \leq_{\text{cov}} \mathcal{U} \leq_{\text{cov}} \operatorname{Irr}(\mathbf{A})$, i.e. $\operatorname{Irr}(\mathbf{A}) \in \operatorname{Cov}(\mathbf{A})$, holds by Lemma 3.4.9(a) and transitivity of the covering relation. □

The question remains, does Algorithm 2 help to obtain refinement minimal covers? That is to say, does it produce (enough) proper refinements? The answer is

[35]This fact also occurs as Proposition 2.8 in [Iza13] with essentially the same proof.

"not necessarily", as the following warning example shows. At the same time it makes clear that for algebras with infinitely many neighbourhoods, the algorithm does not have to terminate.

3.5.27 Example. Let $\mathbf{A} = \langle A; F\rangle$ be the (non-indexed) algebra having carrier set $A = \mathbb{N}$ and fundamental operations $F := \{\, f_k \mid k \in \mathbb{N}\} \cup \{\, \lambda_{k,\ell} \mid k, \ell \in \mathbb{N}\}$ where

$$\begin{aligned} f_k\colon \quad \mathbb{N} &\longrightarrow \mathbb{N} \\ x &\longmapsto f_k(x) := \begin{cases} x + k & \text{for } x < k \\ x & \text{else, i.e. for } x \geq k \end{cases} \end{aligned}$$

and

$$\begin{aligned} \lambda_{k,\ell}\colon \quad \mathbb{N} &\longrightarrow \mathbb{N} \\ x &\longmapsto \lambda_{k,\ell}(x) := \begin{cases} x & \text{for } x < \ell \\ x - \ell & \text{for } \ell \leq x < 2k + \ell \\ x - k - \ell & \text{else, i.e. for } x \geq 2k + \ell \end{cases} \end{aligned}$$

for $k, \ell \in \mathbb{N}$. Clearly, we have $\{\, f_k \mid k \in \mathbb{N}\} \subseteq \operatorname{Idem} \mathbf{A}$ and $\operatorname{im} f_k = \mathbb{N}_{\geq k} \in \operatorname{Neigh} \mathbf{A}$ for every $k \in \mathbb{N}$. Obviously, $\mathbb{N}_{\geq k_2} \subset \mathbb{N}_{\geq k_1}$ is a proper subneighbourhood whenever $k_2 > k_1$. It follows $\{\, \mathbb{N}_{\geq k} \mid k \in \mathbb{N}\} \subseteq \operatorname{Neigh} \mathbf{A} \setminus \operatorname{Irr}(\mathbf{A})$ since for every $k \in \mathbb{N}$ the neighbourhood $\mathbb{N}_{\geq k}$ is covered by the collection $\{\, \mathbb{N}_{\geq \ell} \mid \ell > k\}$ of proper subneighbourhoods (see proof below).

Upon the input cover $\mathcal{V}_0 := \{\mathbb{N}\} \in \operatorname{Cov}(\mathbf{A})$, Algorithm 2 produces one proper refinement in the first step, namely $V_0 \in \mathcal{V}_0$ must be chosen as $V_0 = \mathbb{N}$, and hence $\mathcal{V}_1 := \Axe(V_0, \mathcal{V}_0)$ contains all proper subneighbourhoods of $\mathbb{N}$. In particular it contains the subset $\{\, \mathbb{N}_{\geq k} \mid k > 0\}$. For $i \geq 1$ we define $V_i := \mathbb{N}_{\geq i+1}$ and iteratively $\mathcal{V}_{i+1} := \Axe(V_i, \mathcal{V}_i)$. Using this definition, one can show (v.i.) the inclusions

$$\begin{aligned} \{\, \mathbb{N}_{\geq k} \mid 1 < k \leq i\} &\subseteq \operatorname{Neigh} \mathbf{A} \setminus \mathcal{V}_i \\ \{\mathbb{N}_{\geq 1}\} \cup \{\, \mathbb{N}_{\geq k} \mid k > i\} &\subseteq \mathcal{V}_i, \end{aligned}$$

such that V_i is indeed a reducible neighbourhood in $\mathcal{V}_i$ for every $i \geq 1$. Therefore, it is possible for the algorithm to choose these $(V_i)_{i\geq 1}$, whereupon it produces the infinite sequence $(\mathcal{V}_i)_{i\geq 1}$. If $j > i \geq 1$, then $V_i \in \mathcal{V}_i \setminus \mathcal{V}_j$, so this sequence has pairwise distinct entries, i.e. contains no repetitions. Furthermore, $V_i = \mathbb{N}_{\geq i+1}$ is a proper subset of $\mathbb{N}_{\geq 1} \in \mathcal{V}_i$ for every $i \geq 1$, so Lemma 3.5.25(c) shows that $\mathcal{V}_i \equiv_{\mathrm{ref}} \Axe(V_i, \mathcal{V}_i) = \mathcal{V}_{i+1}$. Thus, $(\mathcal{V}_i)_{i\geq 1}$ is an infinite sequence of distinct refinement equivalent covers of $\mathbf{A}$. Consequently, we have established a run where the algorithm does not terminate and never changes the refinement equivalence class but in the very first step.

We observe furthermore, that the neighbourhoods $\mathcal{V}_i$ $(i \geq 1)$ nevertheless have a proper refinement. For instance, we can show that $\mathcal{V}_* := \mathcal{V}_1 \setminus \{\mathbb{N}_{\geq 1}\}$ refines $\mathcal{V}_1$ and therefore every collection in the sequence: as $\mathcal{V}_* \subseteq \mathcal{V}_1$, Lemma 3.5.4(j) implies that it suffices to show $\{\mathbb{N}_{\geq 1}\} = \mathcal{V}_1 \setminus \mathcal{V}_* \leq_{\mathrm{cov}} \mathcal{V}_*$. We know that $\mathbb{N}_{\geq 1}$ is covered by $\{\, \mathbb{N}_{\geq k} \mid k > 1\} \subseteq \mathcal{V}_1 \setminus \{\mathbb{N}_{\geq 1}\} = \mathcal{V}_*$, whence we infer $\mathbb{N}_{\geq 1} \leq_{\mathrm{cov}} \mathcal{V}_*$. Thus, we have established $\mathcal{V}_* \leq_{\mathrm{ref}} \mathcal{V}_1$. This refinement is proper because $\mathcal{V}_1 \sqsubseteq (\subseteq) \mathcal{V}_*$ fails. Namely, the neighbourhood $\mathbb{N}_{\geq 1} \in \mathcal{V}_1$ cannot be a subset of any member of $\mathcal{V}_*$: the only subsets of $\mathbb{N}$ it can be contained in are $\mathbb{N}$ and itself. However, neither of these belongs to the refining collection $\mathcal{V}_* = \mathcal{V}_1 \setminus \{\mathbb{N}_{\geq 1}\} \subseteq \mathcal{V}_1 \subseteq \mathfrak{P}(\mathbb{N}) \setminus \{\mathbb{N}\}$. $\diamondsuit$

Proof: A few statements in the example still need some words of explanation. Let us first show why for every $k \in \mathbb{N}$ the neighbourhood $\mathbb{N}_{\geq k}$ is indeed covered by the set $\{\mathbb{N}_{\geq \ell} \mid \ell > k\}$. We shall see that for every finite subset $X \subseteq \mathbb{N}$, there exist a unary term operation λ of $\mathbf{A}$, in fact even a fundamental operation, and some $m > k$ such that $\lambda \circ f_m(x) = f_k(x)$ for all $x \in X$. According to item (i) of Corollary 3.4.35, this implies $\mathbb{N}_{\geq k} \leq_{\mathrm{cov}} \{\mathbb{N}_{\geq \nu} \mid \nu > k\}$. Let us consider a finite subset $X \subseteq \mathbb{N}$ and define $\ell := 1 + \max(X, \leq)$ and $m := k + \ell$, then $\ell > 0$ and $m > k$. Every $x \in X$ satisfies $x < \ell \leq k + \ell = m$, whence we have $f_m(x) = x + m = x + k + \ell \geq \ell$. So,

$$\lambda_{k,\ell}(f_m(x)) = \lambda_{k,\ell}(x + k + \ell) = \begin{cases} x + k & \text{if } x < k \\ x & \text{if } x \geq k \end{cases} = f_k(x)$$

holds for all $x \in X$ as was to be shown.

Moreover, we prove the two inclusions

$$\begin{aligned} \{\mathbb{N}_{\geq k} \mid 1 < k \leq i\} &\subseteq \operatorname{Neigh} \mathbf{A} \setminus \mathcal{V}_i \\ \{\mathbb{N}_{\geq 1}\} \cup \{\mathbb{N}_{\geq k} \mid k > i\} &\subseteq \mathcal{V}_i, \end{aligned}$$

by induction on $i \in \mathbb{N}_{\geq 1}$. The base is clear as $\{\mathbb{N}_{\geq k} \mid 1 < k \leq 1\}$ is empty and $\{\mathbb{N}_{\geq 1}\} \cup \{\mathbb{N}_{\geq k} \mid k > i\} = \{\mathbb{N}_{\geq k} \mid k > 0\} \subseteq \mathcal{V}_1$ as we already observed above. Now suppose the inclusions hold for $i \geq 1$, we demonstrate their truth for $i + 1$. First, if $1 < k \leq i$, then $\mathbb{N}_{\geq k} \not\subseteq \mathbb{N}_{\geq i+1} = V_i$ because $k \not\geq i + 1$. Moreover, $\mathbb{N}_{\geq k} \notin \mathcal{V}_i$ by the inductive hypothesis, which means that $\mathbb{N}_{\geq k} \notin \mathcal{V}_i \cup \operatorname{Neigh} \mathbf{A}|_{V_i} \supseteq \text{🪓}(V_i, \mathcal{V}_i) = \mathcal{V}_{i+1}$. So $\mathbb{N}_{\geq k}$ cannot belong to $\mathcal{V}_{i+1}$. Finally, $\mathbb{N}_{\geq i+1} = V_i$ does obviously not belong to $\mathcal{V}_{i+1} = \text{🪓}(V_i, \mathcal{V}_i)$ by definition of 🪓. Thus, we have verified the first inclusion $\{\mathbb{N}_{\geq k} \mid 1 < k \leq i + 1\} \subseteq \operatorname{Neigh} \mathbf{A} \setminus \mathcal{V}_{i+1}$.

Now second let $k > i + 1$. Then $\mathbb{N}_{\geq k} \neq \mathbb{N}_{\geq i+1} = V_i$, and also $\mathbb{N}_{\geq 1} \neq \mathbb{N}_{\geq i+1} = V_i$ since $1 < i + 1$. So using the inductive hypothesis, we immediately obtain that $\mathbb{N}_{\geq 1}, \mathbb{N}_{\geq k} \in \mathcal{V}_i \setminus \{V_i\} \subseteq \text{🪓}(V_i, \mathcal{V}_i) = \mathcal{V}_{i+1}$, and this settles the second inclusion. $\square$

The previous example has made clear that Algorithm 2 needs further modifications to be of use. Namely, we have to ensure that in each repetition of the loop, we construct a proper refinement. Luckily, item (c) of Lemma 3.5.25 already contains a good sufficient condition for this purpose. Namely, if in every step we refine the cover we obtained in the previous iteration to a cover which is an antichain w.r.t. inclusion, and only afterwards hack one of its reducible neighbourhoods into pieces, then we always obtain proper refinements.

There are different possibilities to get such refining antichain covers. We list two sufficient conditions and corresponding approaches in the following lemma.

3.5.28 Lemma. *Let* $\mathbf{A}$ *be an algebra and* $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *a cover.*

(a) If $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ *satisfies ACC, then* $\operatorname{Max}(\mathcal{V}, \subseteq) =: \mathcal{W} \subseteq \mathcal{V}$ *is a cover of* $\mathbf{A}$ *that is refinement equivalent to* $\mathcal{V}$ *such that* $(\mathcal{W}, \subseteq)$ *is an antichain.*

(b) If $(\operatorname{Cov}(\mathbf{A}), \subseteq)$ *fulfils DCC, then there exists an irredundant subcover* $\mathcal{W} \subseteq \mathcal{V}$*, which refines* $\mathcal{V}$ *and, besides, is an antichain w.r.t. set inclusion. It can be found by iteratively removing redundant neighbourhoods from* $\mathcal{V}$*.*

Proof: (a) If $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ satisfies ACC, then by items (d) of Lemma 3.5.6 and (g) of Lemma 3.5.4 the subset $\mathcal{W} = \operatorname{Max}(\mathcal{V}, \subseteq)$ is a cover that is refinement equivalent to $\mathcal{V}$ and consists of mutually incomparable neighbourhoods w.r.t. set inclusion.

(b) Adding a little bit more freedom, we can also suppose that $(\operatorname{Cov}(\mathbf{A}), \subseteq)$ fulfils DCC. Then the set $\{\mathcal{W} \in \operatorname{Cov}(\mathbf{A}) \mid \mathcal{W} \subseteq \mathcal{V}\}$ is non-empty, so it contains minimal collections w.r.t. inclusion, i.e. irredundant subcovers of $\mathcal{V}$. Since there are no infinite properly descending chains in $(\operatorname{Cov}(\mathbf{A}), \subseteq)$, we may find one of these by removing one redundant neighbourhood from $\mathcal{V}$ after the other, until none of the remaining ones can be deleted any more without losing the cover property. Since the resulting cover $\mathcal{W} \in \operatorname{Cov}(\mathbf{A})$ is a subset of $\mathcal{V}$, Lemma 3.5.4(h) tells us that $\mathcal{W} \leq_{\text{ref}} \mathcal{V}$. Finally, item (b) of Lemma 3.5.9 implies that $\mathcal{W}$ is an antichain, since it is an irredundant cover of $\mathbf{A}$. □

We do not discuss how to find such refinements in more detail. Instead, we state the corresponding (generic) modification of Algorithm 2 as Algorithm 3. At the same time we already mention that both algorithms still share a major weakness that we have not addressed so far. This explains why we have labelled Algorithm 3 "Not to be used". We shall see the exact reason in a moment.

In the next lemma we quickly discuss that Algorithm 3 works, sometimes even quite well. We note that we thereby obtain a further generalisation of Theorem 2.9(1) of [KL10] than was already given by Lemma 3.5.25 and Algorithm 2 above. Subsequently, we will see that Algorithm 3 is not always helpful, and how we can improve this situation.

3.5.29 Lemma. *Suppose that* $\mathbf{A}$ *is an algebra with a cover* $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ *where Algorithm 3 is practicable and terminates, then its result is correct.*

Moreover, if the poset canonically associated with $(\operatorname{Cov}(\mathbf{A}), \leq_{\text{ref}})$ *has DCC, and every cover* $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *has a refinement* $\mathcal{W} \leq_{\text{ref}} \mathcal{V}$ *that is an antichain w.r.t. set inclusion, then the algorithm is realisable and terminates after finitely many steps. Especially, if* $(\operatorname{Cov}(\mathbf{A}), \subseteq)$ *has DCC or* $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ *has ACC, then the realisability condition is fulfilled and Algorithm 3 finally terminates.*

Furthermore, if $\mathbf{A}$ *has the additional property that each of its covers having a proper refinement also contains a reducible neighbourhood, then Algorithm 3 produces a non-refinable cover.*

Algorithm 3: Not to be used

Data: A cover $\mathcal{U}$ of an algebra $\mathbf{A}$

Result: A cover $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$ that refines $\mathcal{U}$ and contains only irreducible neighbourhoods which are mutually incomparable w.r.t. set inclusion

begin

  Choose a refinement $\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$ such that $(\mathcal{V}, \subseteq)$ is an antichain;

  `// e.g. a subcollection by one of the methods explained above`

  while $\mathcal{V} \nsubseteq \mathrm{Irr}(\mathbf{A})$ **do**

    Choose a reducible $V \in \mathcal{V}$;

    `// preferably of maximum cardinality or` $\subseteq$`-maximal`

    $\mathcal{V}' \leftarrow \Axe(V, \mathcal{V})$;

    Choose a refinement $\mathcal{W} \leq_{\mathrm{ref}} \mathcal{V}'$ such that $(\mathcal{W}, \subseteq)$ is an antichain;

    `// e.g.` $\mathcal{W} \subseteq \mathcal{V}'$ `by one of the methods explained above`

    $\mathcal{V} \leftarrow \mathcal{W}$;

  return $\mathcal{V}$

Proof: Assume that Algorithm 3 terminates, then it produces a finite sequence $\mathcal{V}_0, \ldots, \mathcal{V}_k$, where for each $0 \leq i < k$ the collection $\mathcal{V}_i \subseteq \mathrm{Neigh}\,\mathbf{A}$ contains a reducible neighbourhood $V_i \in \mathcal{V}_i$, for which $\mathcal{V}_{i+1} \leq_{\mathrm{ref}} \mathcal{V}'_i := \Axe(V_i, \mathcal{V}_i)$. Furthermore, all collections $\mathcal{V}_0, \ldots, \mathcal{V}_k$ are antichains w.r.t. set inclusion, $\mathcal{V}_0 \leq_{\mathrm{ref}} \mathcal{U}$ and $\mathcal{V}_k$ is the return value. Since $\mathcal{V}_k$ was output by the algorithm, it must have been a set of neighbourhoods violating the loop condition, which implies $\mathcal{V}_k \subseteq \mathrm{Irr}(\mathbf{A})$. Moreover, according to Lemma 3.5.25(b), we have $\mathcal{V}'_i \leq_{\mathrm{ref}} \mathcal{V}_i$ for every $0 \leq i < k$ since $V_i \in \mathcal{V}_i$ is reducible. These refinements are proper as all $(\mathcal{V}_i, \subseteq)$ are antichains, see Lemma 3.5.25(c). Consequently, we obtain

$$\mathcal{V}_k \leq_{\mathrm{ref}} \mathcal{V}'_{k-1} <_{\mathrm{ref}} \mathcal{V}_{k-1} \leq_{\mathrm{ref}} \mathcal{V}'_{k-2} <_{\mathrm{ref}} \mathcal{V}_{k-2} \leq_{\mathrm{ref}} \cdots <_{\mathrm{ref}} \mathcal{V}_1 \leq_{\mathrm{ref}} \mathcal{V}'_0 <_{\mathrm{ref}} \mathcal{V}_0 \leq_{\mathrm{ref}} \mathcal{U},$$

so $\mathcal{V}_k \leq_{\mathrm{ref}} \mathcal{U}$ by transitivity. By Lemma 3.5.4(g), this implies that $\mathcal{V}_k \in \mathrm{Cov}(\mathbf{A})$ since $\mathcal{U}$ was a cover of $\mathbf{A}$ by assumption. Thus, the return value $\mathcal{V}_k$ satisfies everything what was proposed by the algorithm.

Next, we prove that under the assumptions stated in the second paragraph of the lemma Algorithm 3 really stops. They literally guarantee that the necessary refining $\subseteq$-antichains can be chosen in every step of the algorithm. Especially, these conditions follow via Lemma 3.5.28 from $(\mathrm{Neigh}\,\mathbf{A}, \subseteq)$ having ACC or $(\mathrm{Cov}(\mathbf{A}), \subseteq)$ having DCC. Again, we use a proof by contradiction to show that the procedure suggested in Algorithm 3 terminates. If it would not, the finite sequence $\mathcal{V}_0, \ldots, \mathcal{V}_k$ examined above would just never end, i.e. for no index $k \in \mathbb{N}$ we had $\mathcal{V}_k \subseteq \mathrm{Irr}(\mathbf{A})$. Apart from this, all arguments given above remain true for any finite initial segment $\mathcal{V}_0, \ldots, \mathcal{V}_k$ of the infinite sequence of sets of neighbourhoods the algorithm traverses. However, this means we had an infinite chain $(\mathcal{V}_i)_{i \in \mathbb{N}} \in \mathrm{Cov}(\mathbf{A})^{\mathbb{N}}$ of proper refinements

$$\cdots <_{\mathrm{ref}} \mathcal{V}_k <_{\mathrm{ref}} \mathcal{V}_{k-1} <_{\mathrm{ref}} \mathcal{V}_{k-2} <_{\mathrm{ref}} \cdots <_{\mathrm{ref}} \mathcal{V}_0,$$

which obviously contradicts the assumption that $(\operatorname{Cov}(\mathbf{A}), \leq_{\mathrm{ref}})$ factored by $\equiv_{\mathrm{ref}}$ satisfies DCC.

Under the additional hypothesis that all covers $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ having proper refinements contain reducible neighbourhoods, we know that every cover $\mathcal{V} \subseteq \operatorname{Irr}(\mathbf{A})$ cannot have proper refinements, i.e. it must be refinement minimal. Therefore, the return value $\mathcal{V}_k$ is a refinement minimal cover (because it satisfied $\mathcal{V} \subseteq \operatorname{Irr}(\mathbf{A})$), and it is an antichain w.r.t. set inclusion. So, by Lemma 3.5.9(e), the collection $\mathcal{V}_k$ is a non-refinable cover. ☐

In view of the last condition of Lemma 3.5.29, there are cases, when Algorithm 3 can be used to determine non-refinable covers. Unfortunately, this condition is not always fulfilled, not even for finite algebras. In fact, in Section 3.8 we shall see an algebra on a four-element set covered by a collection $\mathcal{U}$ of two irreducible and $\subseteq$-incomparable neighbourhoods, which is properly refined by a (non-refinable) cover (see Lemma 3.8.6(f)). Thus, this algebra violates the sufficient condition, and constitutes an example, where upon input $\mathcal{U}$ Algorithm 3 does not have a chance of finding the non-refinable cover, which is actually unique in this case. Of course there is the chance that the algorithm incidentally picks the non-refinable cover in the first step if it really chooses one antichain refinement $\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$ by random. However, in any reasonable deterministic implementation of Algorithm 3 one would probably stay with the input cover $\mathcal{U}$ if it already is an antichain w.r.t. inclusion. Since $\mathcal{U} \subseteq \operatorname{Irr}(\mathbf{A})$, the loop is not entered at all and hence the input $\mathcal{U}$ is returned unchanged. In particular, the answer of the algorithm is not the non-refinable cover.

This shows that the entrance condition of the loop indeed renders Algorithms 2 and 3 both inappropriate for determining non-refinable covers in general, not even for the class of all finite algebras. The example demonstrates furthermore, that to descend properly to the bottom of $(\operatorname{Cov}(\mathbf{A}), \leq_{\mathrm{ref}})$ it can, generally, be necessary to replace irreducible neighbourhoods by subneighbourhoods. This brings up the question, how the entrance condition "$\mathcal{V} \nsubseteq \operatorname{Irr}(\mathbf{A})$" should be modified. Moreover, is the intuitive approach of iteratively using the operation $\Axe$ feasible, at all?

A partial answer to the latter question is contained in the following lemma, which presents a necessary condition for a collection of neighbourhoods to have a proper refinement. The most important case of item (c) is also contained in [Iza13] as Lemma 4.6.

3.5.30 Lemma. *For an algebra $\mathbf{A}$ and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ the following facts are true:*

(a) If $U \in \mathcal{U}$ is a neighbourhood such that there is no $V \in \mathcal{V}$ satisfying $U \subseteq V$ and $\mathcal{V} \leq_{\mathrm{ref}} \mathcal{U}$, then $\Axe(U, \mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$.

(b) If $\mathcal{U}$ has a proper refinement, then there exists some neighbourhood $U \in \mathcal{U}$ such that $\Axe(U, \mathcal{U}) \leq_{\mathrm{ref}} \mathcal{U}$.

(c) *If $\mathcal{V} \leq_{\text{cov}} \mathcal{U}$ and $\mathcal{U}$ has a proper refinement, then there is some $U \in \mathcal{U}$ such that $\mathcal{V} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$. Especially, if a cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ has a proper refinement, then there exists a neighbourhood $U \in \mathcal{U}$ such that $\Lsh(U,\mathcal{U}) \in \operatorname{Cov}(\mathbf{A})$, i.e. we have $\{U \in \mathcal{U} \mid \Lsh(U,\mathcal{U}) \in \operatorname{Cov}(\mathbf{A})\} \neq \emptyset$.*

Note that in the example discussed before the previous lemma, we saw a cover $\mathcal{U} \subseteq \operatorname{Irr}(\mathbf{A})$ having a proper refinement. This makes clear that one cannot expect that the neighbourhood which is guaranteed to exist by items (b) and (c) of Lemma 3.5.30 can always be chosen as a reducible one.

Proof: Everything basically follows from the first item.

(a) Let $U \in \mathcal{U}$ be a neighbourhood which is not contained as a subneighbourhood in any $V \in \mathcal{V}$ and assume that $\mathcal{V} \leq_{\text{ref}} \mathcal{U}$. We want to prove $\mathcal{W} := \Lsh(U,\mathcal{U}) \leq_{\text{ref}} \mathcal{U}$. According to Lemma 3.5.25(a), this is equivalent to $\mathcal{U} \leq_{\text{cov}} \mathcal{W}$. Since we assumed $\mathcal{V} \leq_{\text{ref}} \mathcal{U}$, Lemma 3.5.4(a) tells us that we have $\mathcal{V} \equiv_{\text{cov}} \mathcal{U}$. Hence, $\mathcal{U} \leq_{\text{cov}} \mathcal{W}$ is equivalent to $\mathcal{V} \leq_{\text{cov}} \mathcal{W}$. So, using item (c) of Lemma 3.4.4, we consider an arbitrary neighbourhood $V \in \mathcal{V}$ and need to show the inclusion $\operatorname{Sep}_{\mathbf{A}}(V) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{W})$. First, if $V \subseteq U$, then $V \subset U$, because $V \in \mathcal{V}$ implies $U \neq V$ by assumption on U not to be included in any neighbourhood of $\mathcal{V}$. Hence, in this case we have $V \in \operatorname{Neigh} \mathbf{A}|_U \setminus \{U\} \subseteq \mathcal{W}$, which entails $\operatorname{Sep}_{\mathbf{A}}(V) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{W})$. Second, we consider the possibility that $V \nsubseteq U$. Since $\mathcal{V} \leq_{\text{ref}} \mathcal{U}$, we have $\mathcal{V} \sqsubseteq (\subseteq) \mathcal{U}$, and thus $V \subseteq U'$ for some $U' \in \mathcal{U}$. As $V \nsubseteq U$, we may infer that $U' \in \mathcal{U} \setminus \{U\} \subseteq \mathcal{W}$. Now $V \subseteq U'$ implies $V \leq_{\text{cov}} U'$, which means $\operatorname{Sep}_{\mathbf{A}}(V) \subseteq \operatorname{Sep}_{\mathbf{A}}(U') \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{W})$.

(b) Assume that $\mathcal{U}$ has a proper refinement, say $\mathcal{V} <_{\text{ref}} \mathcal{U}$. Then, referring to Lemma 3.5.4(d), we have that $\mathcal{U} \sqsubseteq (\subseteq) \mathcal{V}$ fails. This implies that there exists some neighbourhood $U \in \mathcal{U}$ for which there is no $V \in \mathcal{V}$ such that $U \subseteq V$. Now, item (a) states that $\Lsh(U,\mathcal{U}) \leq_{\text{ref}} \mathcal{U}$.

(c) If we suppose that $\mathcal{V} \leq_{\text{cov}} \mathcal{U}$ and $\mathcal{U}$ has a proper refinement, then by item (b) we can find some neighbourhood $U \in \mathcal{U}$ such that $\Lsh(U,\mathcal{U}) \leq_{\text{ref}} \mathcal{U}$. Hence, by definition of refinement, we infer $\mathcal{V} \leq_{\text{cov}} \mathcal{U} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$, i.e. $\mathcal{V} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$ due to transitivity of the covering relation. The additional statement about covers of $\mathbf{A}$ follows by letting $\mathcal{V} := \{A\}$. $\square$

The contrapositive of the statements in the previous lemma immediately yields the following sufficient conditions for refinement minimality and non-refinability.

3.5.31 Corollary. *For an algebra $\mathbf{A}$ and a set of neighbourhoods $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ the following holds:*

(a) *If no $U \in \mathcal{U}$ suffices in order that $\mathcal{U}$ is covered by $\Lsh(U,\mathcal{U})$, then $\mathcal{U}$ is refinement minimal.*

(b) *If $\mathcal{U}$ is a cover of a set $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ and $\{U \in \mathcal{U} \mid \mathcal{V} \leq_{\text{cov}} \Lsh(U,\mathcal{U})\} = \emptyset$, then $\mathcal{U}$ is a non-refinable cover of $\mathcal{V}$. Especially, if $\mathcal{U}$ is a cover of $\mathbf{A}$ and $\{U \in \mathcal{U} \mid \Lsh(U,\mathcal{U}) \in \operatorname{Cov}(\mathbf{A})\} = \emptyset$, then $\mathcal{U}$ is a non-refinable cover of $\mathbf{A}$.*

Proof: (a) If for every $U \in \mathcal{U}$ the condition $\mathcal{U} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$ fails, then so does $\Lsh(U,\mathcal{U}) \leq_{\text{ref}} \mathcal{U}$. According to Lemma 3.5.30(b) this means that $\mathcal{U}$ cannot have any proper refinements, i.e. it is refinement minimal.

(b) Suppose $\mathcal{U}$ covers $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$. If there is no $U \in \mathcal{U}$ for which $\mathcal{V} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$, then Lemma 3.5.30(c) asserts that $\mathcal{U}$ has no proper refinements, i.e. it is a refinement minimal cover of $\mathcal{V}$. We want to derive that $\mathcal{U}$ is non-refinable, so, in view of Lemma 3.5.8, it remains to show that $\mathcal{U}$ is an antichain w.r.t. set inclusion. This will follow from Lemma 3.5.9(b), if we can demonstrate irredundancy of $\mathcal{U}$. Let us suppose the contrary, i.e. that $\mathcal{U}$ is a redundant cover of $\mathcal{V}$. Then, by Lemma 3.5.9(a) there is some $U \in \mathcal{U}$ such that $\mathcal{V} \leq_{\text{cov}} \mathcal{U} \setminus \{U\}$. Since $\mathcal{U} \setminus \{U\} \subseteq \Lsh(U,\mathcal{U})$ by definition, Lemma 3.4.9(a) implies $\mathcal{V} \leq_{\text{cov}} \mathcal{U} \setminus \{U\} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$. Therefore, we have $\mathcal{V} \leq_{\text{cov}} \Lsh(U,\mathcal{U})$ since covering is transitive. This contradicts the assumptions of the lemma, wherefore $\mathcal{U}$ is irredundant and hence non-refinable. Finally, the additional statement about covers of $\mathbf{A}$ follows by letting $\mathcal{V} := \{A\}$. □

The idea of replacing reducible neighbourhoods by proper subneighbourhoods until all neighbourhoods were irreducible was central to Algorithm 3. Even though, above, we saw an example which condemns this approach to failure, item (c) of Lemma 3.5.30 contains a condition which proposes to be algorithmically useful without completely abandoning the plan of employing the replacement operator $\Lsh$. The details are contained in Algorithm 4.

We remark that this algorithm is closely related to what was proposed in the proof of Theorem 2.9(2) in [KL10]. The procedure suggested there is for finite algebras only and does not employ refining antichains, which ensure proper refinements in every step of Algorithm 4. The strategy developed in [KL10] essentially amounts to what is described in Algorithm 5 below.

3.5.32 Lemma. *Suppose that $\mathbf{A}$ is an algebra with a cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ where Algorithm 4 is practicable and terminates, then its result is correct.*

Moreover, if the poset canonically associated with $(\operatorname{Cov}(\mathbf{A}), \leq_{\text{ref}})$ has DCC, and every cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ has a refinement $\mathcal{W} \leq_{\text{ref}} \mathcal{V}$ that is an antichain w.r.t. set inclusion, then the algorithm is realisable and terminates after finitely many steps. Especially, if $(\operatorname{Cov}(\mathbf{A}), \subseteq)$ has DCC or $(\operatorname{Neigh} \mathbf{A}, \subseteq)$ has ACC, then the realisability condition is fulfilled and Algorithm 4 eventually terminates yielding a non-refinable cover of $\mathbf{A}$.

Let us note that the conditions given in this lemma for realisability and termination of Algorithm 4 represent a substantial generalisation of items (1) and (2) of Theorem 2.9 from [KL10], which state that every cover of a finite algebra can be

Algorithm 4: Finding non-refinable covers

Data: A cover $\mathcal{U}$ of an algebra $\mathbf{A}$

Result: A non-refinable cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ refining $\mathcal{U}$

begin

  Choose a refinement $\mathcal{V} \leq_{\text{ref}} \mathcal{U}$ such that $(\mathcal{V}, \subseteq)$ is an antichain;

  `// e.g. a subcollection using Lemma 3.5.28`

  $\mathcal{C} \leftarrow \left\{\tilde{V} \in \mathcal{V} \,\middle|\, \Axe\left(\tilde{V}, \mathcal{V}\right) \in \operatorname{Cov}(\mathbf{A})\right\}$;

  `// set of replacement candidates`

  while $\mathcal{C} \neq \emptyset$ **do**

    Choose a neighbourhood $V \in \mathcal{C}$;

      `// preferably of maximum cardinality or ⊆-maximal`

    $\mathcal{V}' \leftarrow \Axe(V, \mathcal{V})$;

    Choose a refinement $\mathcal{W} \leq_{\text{ref}} \mathcal{V}'$ such that $(\mathcal{W}, \subseteq)$ is an antichain;

    `// e.g. W ⊆ V' using Lemma 3.5.28`

    $\mathcal{V} \leftarrow \mathcal{W}$;

    $\mathcal{C} \leftarrow \left\{\tilde{V} \in \mathcal{V} \,\middle|\, \Axe\left(\tilde{V}, \mathcal{V}\right) \in \operatorname{Cov}(\mathbf{A})\right\}$;

  return $\mathcal{V}$

refined to a non-refinable one, implying, in particular, that there is a cover consisting of irreducible neighbourhoods (see also Corollary 3.5.26). With our result we can ensure this not only for finite algebras, but at least for all algebras in 1-locally finite varieties (see Corollary 3.5.14).

Proof: Assume that Algorithm 4 terminates, then it produces a finite sequence $\mathcal{V}_0, \ldots, \mathcal{V}_k$, where for each $0 \leq i < k$ the set $\mathcal{C}_i := \{V \in \mathcal{V}_i \mid \Axe(V, \mathcal{V}_i) \in \operatorname{Cov}(\mathbf{A})\}$ contains at least one neighbourhood $V_i \in \mathcal{C}_i$, for which $\mathcal{V}_{i+1} \leq_{\text{ref}} \mathcal{V}_i' := \Axe(V_i, \mathcal{V}_i)$. Furthermore, all collections $\mathcal{V}_0, \ldots, \mathcal{V}_k$ are antichains w.r.t. set inclusion, $\mathcal{V}_0 \leq_{\text{ref}} \mathcal{U}$ and $\mathcal{V}_k$ is the return value. Since $\mathcal{V}_k$ was output by the algorithm, it must have been a set of neighbourhoods for which the loop condition was not satisfied. This means $\mathcal{C}_k := \{V \in \mathcal{V}_k \mid \Axe(V, \mathcal{V}_k) \in \operatorname{Cov}(\mathbf{A})\} = \emptyset$, so, if we can show that $\mathcal{V}_k$ covers $\mathbf{A}$, then Corollary 3.5.31(b) implies that $\mathcal{V}_k$ is non-refinable.

By choice of $V_i \in \mathcal{C}_i$, we have that $V_i \in \mathcal{V}_i$ and $\mathcal{V}_i' = \Axe(V_i, \mathcal{V}_i) \in \operatorname{Cov}(\mathbf{A})$. Therefore, Lemma 3.5.25(b) asserts that $\mathcal{V}_i' = \Axe(V_i, \mathcal{V}_i) \leq_{\text{ref}} \mathcal{V}_i$, for $0 \leq i < k$. Now, according to Lemma 3.5.25(c) these refinements are proper since all $(\mathcal{V}_i, \subseteq)$ are antichains. Therefore, we have

$$\mathcal{V}_k \leq_{\text{ref}} \mathcal{V}_{k-1}' <_{\text{ref}} \mathcal{V}_{k-1} \leq_{\text{ref}} \mathcal{V}_{k-2}' <_{\text{ref}} \mathcal{V}_{k-2} \leq_{\text{ref}} \cdots <_{\text{ref}} \mathcal{V}_1 \leq_{\text{ref}} \mathcal{V}_0' <_{\text{ref}} \mathcal{V}_0 \leq_{\text{ref}} \mathcal{U},$$

so $\mathcal{V}_k \leq_{\text{ref}} \mathcal{U}$ by transitivity. By Lemma 3.5.4(g), this implies that $\mathcal{V}_k \in \operatorname{Cov}(\mathbf{A})$ as $\mathcal{U}$ was a cover of $\mathbf{A}$ by assumption. Thus, with the return value $\mathcal{V}_k$ the algorithm indeed fulfils its contract.

Having discussed correctness of Algorithm 4, realisability and termination can be shown using almost literally the same arguments as were applied in the corresponding part of the proof of Lemma 3.5.29 w.r.t. Algorithm 3. The only piece of text that needs to be replaced is "$\mathcal{V}_k \subseteq \operatorname{Irr}(\mathbf{A})$" by "$\mathcal{C}_k = \emptyset$". □

With Lemma 3.5.32 we have actually reached the goal of this section. For completeness we state a simplified version of Algorithm 4 that one can extract from [KL10] by carefully reading the proof of their Theorem 2.9 and taking into account Corollary 3.5.31(b). Of course the scope where this procedure is applicable is not as broad as for Algorithm 4 and one should expect that it needs more loop iterations than Algorithm 4.

Algorithm 5: Finding non-refinable covers (simplified)

Data: A cover $\mathcal{U}$ of an algebra $\mathbf{A}$
Result: A non-refinable cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ refining $\mathcal{U}$
begin
  $\mathcal{V} \leftarrow \mathcal{U}$;
  $\mathcal{C} \leftarrow \left\{\tilde{V} \in \mathcal{V} \,\middle|\, \Lsh\left(\tilde{V}, \mathcal{V}\right) \in \operatorname{Cov}(\mathbf{A})\right\}$;
  `// set of replacement candidates`
  while $\mathcal{C} \neq \emptyset$ **do**
    Choose a neighbourhood $V \in \mathcal{C}$;
      `// preferably of maximum cardinality or ⊆-maximal`
    $\mathcal{V} \leftarrow \Lsh(V, \mathcal{V})$;
    $\mathcal{C} \leftarrow \left\{\tilde{V} \in \mathcal{V} \,\middle|\, \Lsh\left(\tilde{V}, \mathcal{V}\right) \in \operatorname{Cov}(\mathbf{A})\right\}$;
  return $\mathcal{V}$

3.5.33 Lemma. *Suppose that* $\mathbf{A}$ *is an algebra with a cover* $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ *where Algorithm 5 terminates, then its result is correct.*

Moreover, if $\operatorname{Neigh}\mathbf{A}$ *is finite, then for any input cover the algorithm returns, after a finite number of iterations, a non-refinable cover refining the input.*

We observe that, as for Algorithm 2, Example 3.5.27 demonstrates that Algorithm 5 does not always have to stop for algebras with infinitely many neighbourhoods.

Proof: The correctness argument uses similar ideas as before. If Algorithm 5 terminates after a finite number of steps, then it traverses a finite sequence of collections $\mathcal{V}_0, \ldots, \mathcal{V}_k \subseteq \operatorname{Neigh}\mathbf{A}$, where $\mathcal{V}_0 := \mathcal{U}$ is the input and $\mathcal{V}_k$ is the output. Furthermore, for $0 \leq i < k$ these collections are linked via some neighbourhood $V_i \in \mathcal{C}_i := \{V \in \mathcal{V}_i \mid \Lsh(V, \mathcal{V}_i) \in \operatorname{Cov}(\mathbf{A})\}$ for which $\mathcal{V}_{i+1} = \Lsh(V_i, \mathcal{V}_i)$. Since $\mathcal{V}_k$ was returned by the algorithm, it must have caused the loop to stop by violating the entrance condition, i.e. $\emptyset = \mathcal{C}_k := \{V \in \mathcal{V}_k \mid \Lsh(V, \mathcal{V}_k) \in \operatorname{Cov}(\mathbf{A})\}$. By choice of $V_i \in \mathcal{C}_i$ for every $0 \leq i < k$ we have $\mathcal{V}_{i+1} = \Lsh(V_i, \mathcal{V}_i) \in \operatorname{Cov}(\mathbf{A})$. In particular, $\mathcal{V}_k$ covers $\mathbf{A}$, so by Corollary 3.5.31(b) it is a non-refinable cover of $\mathbf{A}$. The only fact that remains open is that $\mathcal{V}_k$ is a refinement of the input $\mathcal{U}$. Again this can be proven using a finite chain of refinements. Namely, for $0 \leq i < k$, the fact $V_i \in \mathcal{C}_i$ implies $V_i \in \mathcal{V}_i$ and $\mathcal{V}_{i+1} = \text{🪓}(V_i, \mathcal{V}_i) \in \operatorname{Cov}(\mathbf{A})$. By Lemma 3.5.25(b), we can now infer $\mathcal{V}_{i+1} \leq_{\text{ref}} \mathcal{V}_i$ for $0 \leq i < k$. Consequently, we obtain the chain $\mathcal{V}_k \leq_{\text{ref}} \mathcal{V}_{k-1} \leq_{\text{ref}} \cdots \leq_{\text{ref}} \mathcal{V}_0 = \mathcal{U}$, whereupon transitivity of refinement concludes the argument.

If Algorithm 5 would not terminate for some input $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$, then we had infinite sequences $(\mathcal{V}_i)_{i \in \mathbb{N}} \in (\operatorname{Cov}(\mathbf{A}))^{\mathbb{N}} \subseteq (\mathfrak{P}(\operatorname{Neigh}\mathbf{A}))^{\mathbb{N}}$ and $(V_i)_{i \in \mathbb{N}} \in \operatorname{Neigh}\mathbf{A}^{\mathbb{N}}$ where $V_i \in \mathcal{C}_i \subseteq \mathcal{V}_i$ and $\mathcal{V}_{i+1} = \text{🪓}(V_i, \mathcal{V}_i)$ for all $i \in \mathbb{N}$. Quoting Lemma 3.5.25(d), this necessitates that $\operatorname{Neigh}\mathbf{A}$ is infinite. Therefore, if $\mathbf{A}$ only has finitely many neighbourhoods, then the algorithm finally terminates. □

3.6 Irreducibility notions

The understanding of various notions of composition and decomposition is a topic of general interest throughout classical and general algebra. For example, regarding commutative rings with unity one considers ring multiplication and divisibility; for general algebras one is interested in constructing direct products or subdirect products and in representing algebras as such. Associated with these studies we find four major problems, which we shall explain in the following paragraphs and afterwards discuss in the context of Relational Structure Theory.

The first of them is to provide an appropriate concept of *indecomposables*, i.e. of elements that do not allow a further non-trivial decomposition. This means the only way to obtain them as a composition is by choosing at least one factor in a way obviously guaranteeing the desired result. In the case of commutative rings (or slightly more specific, integral domains) the indecomposable elements are the irreducible elements of the ring, i.e. such non-unit elements that when written as a product of two factors one of them must be a unit. For algebraic structures the indecomposables are the directly or subdirectly irreducible algebras, respectively.

Related to the definition of indecomposable elements is the second problem, imposing a further requirement on them: the *completeness problem* is the question if every element of interest can really be obtained as a composition of indecomposables. Conversely stated, is it possible to decompose any element into indecomposables? Of course, the answer is not always true, certainly not for all integral domains. Furthermore, not all algebras can be written as a direct product of directly indecomposables, but, for instance, finite algebras can. In contrast, every algebra can be written as a product of subdirectly irreducibles coming from the same variety.

The third question is the *uniqueness problem*, asking if representations by indecomposables are in some sense unique. Again, for integral domains, the answer is generally no. That is why ring theory has come up with an axiomatic treatment of this question. Namely, unique factorisation domains are by definition precisely those commutative rings with unity where every non-zero non-unit can be written as a product of irreducible elements, uniquely up to the order of the factors and a unit. This generalises the familiar situation encountered in the ring of integers and provides a positive answer to the uniqueness and completeness problem. For general algebras, representations as direct products of directly irreducible algebras are essentially unique by the universal property of direct products, whereas subdirect representations are not. So in universal algebra the uniqueness problem has a positive answer w.r.t. direct products, but not w.r.t. subdirect products, a situation which is quite opposite to the completeness problem w.r.t. both notions of decomposition.

The fourth and last problem we want to mention is the *characterisation problem*. This is the task to give an intrinsic description of indecomposability which is usually defined in terms of the composition notion. For example, in the case of algebras, characterisations of direct and subdirect irreducibility in terms of conditions on the congruences of the algebra are known.

Let us now examine these four problems with regard to Relational Structure Theory. Considering $U = A$ in Corollary 3.4.35 establishes covers of algebras not only as a means of decomposition but also as a means of reconstruction, i.e. structural composition. With respect to this notion of composition, there is a natural indecomposability concept that we have already encountered in Definition 3.5.16 in the previous section, while searching for non-refinable covers. The notion of irreducible algebra reveals a strong analogy with that of subdirectly irreducible algebra. Irreducible algebras in the sense of 3.5.16 are such where every cover necessarily contains the full neighbourhood A. In view of item (1) of Corollary 3.4.35 this means that whenever the relational counterpart $\underset{\sim}{\mathbf{A}}$ can be expressed as a local retract of a product of restricted relational structures $\underset{\sim}{\mathbf{A}}\restriction_V$ where $V \in \operatorname{Neigh}\mathbf{A}$, then it must be contained among the factors. In parallel, subdirectly irreducible algebras have the property that they occur up to isomorphism as a factor of any subdirect representation.

Let us also note here that for infinite structures such an irreducibility notion may be too strong. One may want to call a structure irreducible if each of its covers contains a neighbourhood being somehow related to the full carrier set. For instance, for topological algebras one could say that a structure is irreducible if every cover contains at least one neighbourhood, where A (densely) embeds into. For this reason, we have provided the more flexible notion of q-irreducibility where $q \subseteq \leq_{\mathrm{cov}}$ is some quasiorder on neighbourhoods.

So if one understands covers of algebras as a means of decomposition, then irreducible algebras (or neighbourhoods), and more generally q-irreducible algebras, are a reasonable answer to the first question after indecomposables. Let us now see how this choice of indecomposable elements behaves with regard to the remaining problems.

In the previous section we have already established sufficient criteria for the existence of non-refinable covers (see Lemmas 3.5.15 and 3.5.32), and large classes of algebras (e.g. every algebra generating a 1-locally finite variety) satisfying these. Using Corollary 3.5.26 we can see that such algebras have got a cover consisting of irreducible neighbourhoods, i.e. a decomposition into irreducible algebras (in particular q-irreducible algebras, see Corollary 3.5.21). So we have already given a positive answer to the completeness problem for interesting classes of algebras, such as, for example, distributive lattices or BOOLEan algebras.

From looking at examples it becomes quite evident that the uniqueness problem w.r.t. decomposition into irreducible neighbourhoods generally has a negative answer. For instance, considering an irreducible algebra $\mathbf{A}$, the singleton set $\{A\}$ forms a cover consisting of irreducible neighbourhoods. As soon as there is another irreducible neighbourhood $U \subset A$ in $\operatorname{Neigh}\mathbf{A}$—and this happens quite frequently, e.g. if A has got at least two elements and a unary constant operation in its clone, but no nullary ones—then $\{A, U\}$ forms another such cover, which is not isomorphic to $\{A\}$. Yet, both covers are refinement equivalent. Therefore, a better example may be the four-element algebra anticipated in the discussion after Lemma 3.5.29 on page 158. There we can find two covers consisting of irreducible neighbourhoods, that are neither isomorphic nor refinement equivalent since one properly refines the other. This means, that in order to obtain a satisfactory answer to the uniqueness problem, we need to find a suitable subclass of all irreducible neighbourhoods, which still solves the completeness problem for a sufficiently large class of algebras. This is one of the main tasks of this and the following section.

Furthermore, we shall also make progress here on the characterisation problem, which has so far only slightly been touched in Lemma 3.5.22.

A general task directly related to the characterisation problem is the following: we have established irreducible algebras as "building blocks" of (finite) algebras. Furthermore, irreducibility of neighbourhoods can be reduced to irreducibility of algebras since $U \in \operatorname{Neigh}\mathbf{A}$ is irreducible if and only if $\mathbf{A}|_U$ is an irreducible algebra (see Definition 3.5.16). The converse is true, as well, namely $\mathbf{A}$ is an irreducible algebra if and only if $A \in \operatorname{Neigh}\mathbf{A}$ is an irreducible neighbourhood (see also Remark 3.6.10). Consequently, it is interesting to ask for a description of irreducible algebras (up to local term equivalence). Generally this question is wide open and a complete classification seems very unlikely, at least to the present knowledge of the author. Therefore, it would be desirable to have a list of important examples of irreducible algebras. Unfortunately, this is still not too easy to achieve since we will not have a strong irreducibility criterion until a bit later.

The following examples are mainly taken from Section 4[36] of [Beh09] and we only give proofs where irreducibility can be derived by simple arguments checking the truth of Definition 3.5.16. For a few more complicated examples we just state the result or give references for further reading.

3.6.1 Example. The following classes of algebras contain almost exclusively irreducible members:

[36] There the results are slightly different, because the framework in [Beh09] does not allow nullary operations, whence constants have been modelled as unary operations there. Here, we have the possibility to use nullary operations and to consider structures with constants as usual in universal algebra, which results in a small difference regarding irreducibility of one-element algebras.

(a) Our first example are *essentially at most unary algebras*. We say that an algebraic structure $\mathbf{A}$ is essentially at most unary if it is locally term equivalent to an algebra having fundamental operations of arity at most one. That is to say that $\operatorname{Clo}(\mathbf{A}) = \operatorname{Pol}_A \operatorname{Inv}_A F$ for some set $F \subseteq \mathrm{O}_A^{(0)} \cup \mathrm{O}_A^{(1)}$. We claim that any *finite* algebra of this sort is irreducible with the exception of a one-element algebra having a nullary constant (cf. Lemma 4.2.3 in [Beh09]).

Since irreducibility solely depends on neighbourhoods and the covering relation, it is completely determined by $\operatorname{Clo}(\mathbf{A})$. Therefore, no generality is lost in assuming that actually $\mathbf{A} = \langle A; F\rangle$ with $F \subseteq \mathrm{O}_A^{(0)} \cup \mathrm{O}_A^{(1)}$. Let us denote by $M \subseteq A^A$ the transformation monoid generated by the set

$$F^{(1)} \cup \left\{ c_a^{(1)} \;\middle|\; c_a^{(0)} \in F^{(0)} \wedge a \in A \right\}.$$

It is easy to see, e.g. by induction, that

$$\operatorname{Term}^{(n)}(\mathbf{A}) = \left\{ f \circ e_i^{(n)} \;\middle|\; f \in M \wedge 1 \leq i \leq n \right\}$$

for $n \in \mathbb{N}_+$ (cf. Lemma 4.2.2 in [Beh09]). Furthermore, if $F^{(0)} \neq \emptyset$, we have $\operatorname{Term}^{(0)}(\mathbf{A}) = \left\{ c_a^{(0)} \;\middle|\; c_a^{(1)} \in M \wedge a \in A \right\}$, and $\operatorname{Term}^{(0)}(\mathbf{A}) = \emptyset$, otherwise. We assume that $\mathbf{A}$ is not a singleton algebra with a nullary constant. Then we can prove that every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ contains the set A. According to Corollary 3.4.36(k) there exists some integer $n \in \mathbb{N}$ and clone operations $\lambda \in \operatorname{Clo}^{(n)}(A) = \operatorname{Term}^{(n)}(\mathbf{A})$, $(f_1, \dots, f_n) \in \left(\operatorname{Clo}^{(1)}(A)\right)^n$ satisfying a decomposition equation $\lambda \circ (f_1, \dots, f_n) = \mathrm{id}_A$. If we had $n = 0$, then the equality $\lambda \circ () = \mathrm{id}_A$ would imply that A is a singleton set containing the image of $\lambda \in \operatorname{Term}^{(0)}(\mathbf{A})$. This implied that $F^{(0)} \neq \emptyset$ in contradiction to our assumption. Therefore, it is $n > 0$, and thus $\lambda = f \circ e_i^{(n)}$ for some index $1 \leq i \leq n$ and a function $f \in M$. Consequently, the decomposition equation becomes

$$\mathrm{id}_A = \lambda \circ (f_1, \dots, f_n) = f \circ e_i^{(n)} \circ (f_1, \dots, f_n) = f \circ f_i,$$

which proves that f_i is an injective function. Since A is finite, f_i must be surjective, as well. So, according to the choice of functions $f_1, \dots, f_n$ guaranteed by Corollary 3.4.36(k), its image $A = \operatorname{im} f_i$ is a subset of one of the neighbourhoods in $\mathcal{U}$, which means that A belongs to $\mathcal{U}$. Hence, $\mathbf{A}$ is irreducible.

We remark that Example 3.5.27 describes a unary algebra on an infinite carrier set which is not irreducible. Indeed, there, it was discussed that the full neighbourhood $\mathbb{N}$, and therefore the whole algebra, is covered by the collection $\{\mathbb{N}_{\geq \ell} \mid \ell > 0\}$ of proper subneighbourhoods.

We also mention that Theorem 6.1.8 of [Sch12] provides a characterisation of irreducible neighbourhoods for essentially unary topological algebras living on a compact Hausdorff space and satisfying a certain compactness condition.

(b) The second example are *idempotent* algebras, i.e. such structures $\mathbf{A} = \langle A; F \rangle$, where every fundamental operation $f \in F$ satisfies $f(a, \ldots, a) = a$ for all $a \in A$. This implies in particular that $F^{(0)} = \emptyset$, unless $|A| = 1$. Idempotent algebras can be equivalently defined by requiring that every singleton set $\{a\}$ $(a \in A)$ is a subuniverse (see also Lemma 4.3.2 in [Beh09]). These subuniverses stay invariant w.r.t. the operations in $\mathrm{Clo}(\mathbf{A})$ (cf. again Lemma 4.3.2 in [Beh09]), whence it follows directly, that $\mathrm{Clo}^{(1)}(\mathbf{A}) = \{\mathrm{id}_A\}$. Therefore, idempotent algebras have got only one neighbourhood, namely the full carrier set A, and so possible collections of neighbourhoods are $\{A\}$ and $\emptyset$. The empty collection never covers $\mathbf{A}$ apart from A being a singleton and $\mathbf{A}$ containing the unique nullary operation on A (see Lemma 3.4.12(e)). Therefore, except for this very special case, we have $\mathrm{Cov}(\mathbf{A}) = \{\{A\}\}$ for idempotent algebras, which shows their irreducibility. Indeed, the exceptional one-element idempotent algebra including the unique nullary operation is reducible since Lemma 3.4.12 tells us that it is covered by $\emptyset$.

The class of idempotent algebras includes of course important examples such as lattices, modes (i.e. idempotent entropic algebras), bands (i.e. idempotent semigroups), and in particular semilattices, which are precisely the commutative bands.

(c) An algebra containing at most one element is irreducible if and only if it does not contain nullary operations. This follows directly from Corollary 3.5.19 and the fact that there are no nullary operations on the empty set.

(d) In [Beh09, Lemma 4.3.4] the unary term operations of (finite) bounded and partially bounded lattices and semilattices, as well as ortholattices (i.e. bounded lattices with an antitone involutive complementation, forming a generalisation of Boolean algebras that forgets about distributivity) have been described. The arguments given there extend to infinite algebras without any change: it is obvious that the operations listed below are indeed unary term operations, and it is easy to check from the identities assumed for the algebras that the given sets of unary operations are closed under the respective fundamental operations.

Lemma 4.3.4 of [Beh09] contains the following results: let $\mathbf{S}_0 = \left\langle S_0; \wedge, c_0^{(0)} \right\rangle$ be a partially (lower) bounded semilattice, $\mathbf{S}_{0,1} = \left\langle S_{0,1}; \wedge, c_0^{(0)}, c_1^{(0)} \right\rangle$ a (doubly) bounded semilattice, $\mathbf{L}_0 = \left\langle L_0; \wedge, \vee, c_0^{(0)} \right\rangle$ a partially (lower) bounded lattice, $\mathbf{L}_{0,1} = \left\langle L_{0,1}; \wedge, \vee, c_0^{(0)}, c_1^{(0)} \right\rangle$ a bounded lattice and $\mathbf{O} = \left\langle O; \wedge, \vee, ', c_0^{(0)}, c_1^{(0)} \right\rangle$ an ortholattice, then we have

$$\mathrm{Term}^{(1)}(\mathbf{S}_0) = \left\{\mathrm{id}_{S_0}, c_0^{(1)}\right\}$$
$$\mathrm{Term}^{(1)}(\mathbf{S}_{0,1}) = \left\{\mathrm{id}_{S_{0,1}}, c_0^{(1)}, c_1^{(1)}\right\}$$

$$\mathrm{Term}^{(1)}(\mathbf{L}_0) = \left\{\mathrm{id}_{L_0}, c_0^{(1)}\right\}$$
$$\mathrm{Term}^{(1)}(\mathbf{L}_{0,1}) = \left\{\mathrm{id}_{L_0}, c_0^{(1)}, c_1^{(1)}\right\}$$
$$\mathrm{Term}^{(1)}(\mathbf{O}) = \left\{\mathrm{id}_O, ', c_0^{(1)}, c_1^{(1)}\right\}.$$

Since these sets are finite, they are automatically locally closed (see Corollary 3.5.12), and hence this list actually presents the unary part of the locally closed clone of the given algebras, not only their unary term operations.

It follows from this description that the only neighbourhoods of these structures are the full carrier set and the singleton sets containing the constants occurring in the signature of the respective algebras. Since these signatures contain nullary symbols, all algebras considered above have non-empty carrier sets. Furthermore, we can infer from the previous item that the respective algebras having a singleton base set are reducible. It is our claim that this is the only case when this happens, i.e. that the mentioned structures are irreducible if (and only if) they contain at least two elements.

To prove this, we assume our algebra to be reducible, and we need to show that it contains only one element. For this end let us denote the carrier of the algebra by A, and consider an arbitrary element $x \in A$. Put $X := \{0, x\}$. Since the algebra is reducible, its full neighbourhood A is reducible, which means, by Lemma 3.5.22, that the set $\mathcal{V}$ of proper subneighbourhoods of the algebra covers it. Now for our algebras, the collection of proper subneighbourhoods is a collection of singletons. Exploiting item (h) of Corollary 3.4.35 for the case $e = \mathrm{id}_A$, we obtain an integer $n \in \mathbb{N}$, an n-ary term operation λ and a tuple $(f_1, \dots, f_n)$ of unary clone operations having their images in neighbourhoods belonging to $\mathcal{V}$ such that $(\lambda \circ (f_1, \dots, f_n))|_X^A = \mathrm{id}_A|_X^A$. Since the functions $f_1, \dots, f_n$ have images in neighbourhoods of $\mathcal{V}$, they must be constant operations, i.e. we have some tuple $(a_1, \dots, a_n) \in A^n$ such that $\mathrm{id}_A|_X^A = \left(\lambda \circ \left(c_{a_1}^{(1)}, \dots, c_{a_n}^{(1)}\right)\right)|_X^A = c_{\lambda(a_1,\dots,a_n)}^{(1)}|_X^A$. Therefore, it is

$$\{0, x\} = X = \mathrm{id}_A[X] = c_{\lambda(a_1,\dots,a_n)}^{(1)}[X] = \{\lambda(a_1, \dots, a_n)\},$$

which means that $0 = \lambda(a_1, \dots, a_n) = x$. As x was an arbitrary element of the carrier A, we have shown $|A| = 1$.

(e) Finite irreducible semigroups have been characterised in 2009 by Tamás Waldhauser, thereby answering a question in [Beh09] after the irreducible monoids among $\{\langle \mathbb{Z}_m; \cdot \rangle \mid m \in \mathbb{N}\}$. The manuscript is still unpublished and also contains a complete description of all neighbourhoods of finite semigroups, as well as a characterisation of the irreducible ones among them. A joint paper with the author of this thesis is in preparation, where these results are used to characterise categorical equivalence of finite semigroups by the fact that both semigroups need to be weakly isomorphic. In detail, this means that one of them must be isomorphic to a semigroup which is term equivalent to the other. This characterisation generalises a similar result for finite groups obtained earlier by László Zádori (see Corollary 3.3 on p. 407 of [Zád97a]).

It exceeds the scope of this list of examples to prove which finite semigroups are irreducible. However, we may state the result here. For this we recall that a semigroup is said to be *completely regular* if and only if it is a union of groups, i.e. if for any of its elements the generated monogenic subsemigroup forms a (semigroup reduct) of a group.

Waldhauser proved (see Theorem 6 of [Wal09]) that a finite semigroup is irreducible if and only if it either fails to be completely regular or it is a union of (subsemigroups which are reducts of) groups whose sizes are all powers of a common prime. This result provides a generalisation of Theorem 4.4.3 of [Beh09] where the finite irreducible groups were characterised as those having prime power exponent.

(f) It is not hard to show (cf. Corollary 4.5.3 in [Beh09]) that the unary operations in the clone of a finite left module $\mathbf{M}$ over a ring are precisely the left multiplications by scalars from the ring. If the module is faithful (i.e. $r \cdot x = 0$ for every $x \in M$ implies $r = 0$) and the ring has a unity and no non-zero zero-divisors (such rings are also called *domains*), then the only idempotent operations among the scalar multiplications are those belonging to the zero and the one of the ring. In other words, one has $\operatorname{Idem}\mathbf{M} = \left\{\mathrm{id}_M, c^{(1)}_{0_\mathbf{M}}\right\}$ and so only two neighbourhoods $\operatorname{Neigh}\mathbf{M} = \{M, \{0_\mathbf{M}\}\}$. Since we consider a module here as a structure with a nullary constant $0_\mathbf{M}$, the one-element trivial module is reducible. All other finite faithful left modules over domains are irreducible, which can be seen from similar arguments as employed in the case of bounded lattices. In particular, finite vector spaces are included here as special cases.

This means that w.r.t. decompositions (covers) these classes of modules are uninteresting. However, their polynomial expansions (obtained by adding all possible nullary constants) have a much richer structure. We mention that the article [CK04] examines minimal neighbourhoods in polynomial expansions of finite rings by also studying neighbourhoods of polynomial expansions of finite modules and bimodules.

More generally, in Section 6.3 of [Sch12] the irreducible ones among the compact neighbourhoods of polynomial expansions of Hausdorff topological modules over compact Hausdorff unitary topological rings are characterised. In particular, this yields a description of all irreducible polynomial expansions of compact Hausdorff modules over unitary compact Hausdorff rings.

(g) We saw above that lattices have a poor structure w.r.t. covering. Again, polynomial expansions change the situation dramatically. In the beginning of 2012 the author of this thesis, in collaboration with Friedrich Martin Schneider, described all irreducible neighbourhoods and non-refinable covers for polynomial expansions derived from finite distributive lattices.

The result is easiest understood via the characterisation of irreducible polynomial expansions of finite distributive lattices. These algebras are irreducible precisely if the bottom element is completely meet-irreducible and the top element is completely join-irreducible in the underlying lattice, or if they are empty.

A non-empty neighbourhood of such an expansion is irreducible if and only if it is a non-empty interval of the underlying lattice, such that the induced sublattice polynomially expands into an irreducible algebra as described before, i.e. such that the bottom element of the interval is completely meet-irreducible and the top element is completely join-irreducible within the sublattice.

This work has been generalised by a complete description of all closed irreducible neighbourhoods in bounded distributive compact HAUSDORFF topological lattices in Section 6 of [Sch12]. The result is the almost verbatim generalisation of the characterisation in the finite case (cp. Theorem 6.2.17, Corollary 6.2.18 and Lemma 6.2.16(3) of [Sch12]). $\diamondsuit$

3.6.2 Remark. It has been explained before Example 3.6.1 that in general the study of irreducible algebras is equivalent to the study of irreducible neighbourhoods within algebras. For this being true the term "in general" needs to be understood as "for all algebras" or at least as "for all algebras up to local term equivalence". This is, of course, a very ambitious aim, in fact, it seems too ambitious to be realisable.

This is why it is commonly easier, and more useful, to characterise all neighbourhoods, and after that all irreducible ones among them, arising from a specific sort of structure. It is more useful, because one can then directly apply the decomposition results derived in Section 3.4. It is usually easier, because one can stay within a limited part of the mathematical world. The precise meaning of this statement is probably best understood via an example.

If one wants to determine irreducible neighbourhoods in finite groups, one needs to deal with unary term operations of groups, and afterwards to check for idempotency etc. All of this can be done employing group theoretic thoughts. If one wants to achieve the same task via a general list of irreducible algebras, one cannot stay within the world of groups. This is so because the neighbourhoods of groups are generally not carrier sets of subgroups. Therefore, the induced algebras $\mathbf{A}|_U$ do not need to have clones with group operations, nor clones generated by a group operation. So, if one wants to check irreducibility of a neighbourhood U via irreducibility of $\mathbf{A}|_U$, then a precomputed database of irreducible algebras used for the check needs to comprise more entries than just all irreducible groups (up to local term equivalence). In fact, it would usually have to contain less natural structures that one perhaps has not thought about before.

However, there are algebras where the abstract approach works. For instance, polynomial expansions $\mathbf{A}_A$ of finite distributive lattices $\mathbf{A}$ have the property that idempotent unary clone operations are algebraic endomorphisms of the underlying structure $\mathbf{A}$. This ensures that neighbourhoods of $\mathbf{A}_A$ live on subuniverses of $\mathbf{A}$, and the corresponding induced algebras $\mathbf{A}_A|_U$ are locally term equivalent to polynomial expansions $\mathbf{U}_U$ of subalgebras $\mathbf{U}$ of the underlying algebra. Therefore, in this case a database of irreducible algebras only has to provide information about algebras from a clean-cut class of well-known structures, namely polynomial expansions of distributive lattices. $\diamondsuit$

Before we start working towards better q-irreducibility criteria for neighbourhoods, we want to take a short detour and discuss an aspect of Definition 3.5.16 that has not been looked at and may lead to a different notion of q-irreducibility.

In Definition 3.5.16 q-irreducibility of a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ has been formulated in terms of the restricted algebra $\mathbf{A}|_U$. Therefore, in this regard only covers $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ are considered that are subsets of $\mathfrak{P}(U)$. This is done on purpose, to support our localisation theory. However, one may raise the question if it is possible to define a reasonable concept of irreducibility (or q-irreducibility) for neighbourhoods that somehow takes into account all collections $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$. The subsequent remark contains a few ideas on what could be done in this respect.

3.6.3 Remark. The following straightforward "global variant" of irreducibility of neighbourhoods would imply irreducibility, but is a concept which is much too strong, i.e. too restrictive. Namely, if one would call a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ *globally irreducible in* $\mathbf{A}$ provided that every cover $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ contains U as an element, then only $U = A$ could possibly be globally irreducible because certainly $\{A\} \in \operatorname{Cov}_{\mathbf{A}}(U)$.

This issue could be dealt with by calling $U \in \operatorname{Neigh} \mathbf{A}$ globally irreducible if every cover $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ contained a neighbourhood $V \in \mathcal{V}$ such that $U \subseteq V$. This notion would still imply irreducibility since every cover of $\mathbf{A}|_U$ is a subcollection of $\operatorname{Neigh} \mathbf{A}|_U \subseteq \mathfrak{P}(U)$ (see Lemma 3.3.7).

However, regarding this definition the following problematic situation is plausible. One could have two (small) disjoint neighbourhoods $U_0, U_1 \in \operatorname{Neigh} \mathbf{A}$ that are isomorphic and hence covering equivalent (see Corollary 3.4.6(b)). Moreover, suppose that U_0 and U_1 are irreducible in the sense of Definition 3.5.16. For instance, U_0 and U_1 could be singletons if $\operatorname{Clo}^{(1)}(\mathbf{A})$ contains two distinct unary constant operations. These would always be isomorphic, and irreducible provided $\mathbf{A}$ does not have nullary operations (see Corollary 3.5.19). In a non-empty algebra without nullary operations, one can hardly think of any neighbourhood that could more deserve to be called irreducible, in any reasonable sense, than a singleton. Nevertheless, such neighbourhoods are not globally irreducible since $U_0 \leq_{\mathrm{cov}} \{U_1\}$ and $U_0 \nsubseteq U_1$ (and vice versa).

We can solve this conflict by introducing the following more flexible notion of *global q-irreducibility* which does not any more imply irreducibility, but however q-irreducibility (see Lemma 3.6.5). As it turns out, in our example situation the neighbourhoods U_0 and U_1 may still be globally $\leq_{\mathrm{cov}}$-irreducible. $\diamondsuit$

3.6.4 Definition. For an algebra $\mathbf{A}$ and a quasiorder $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ such that $q \subseteq \leq_{\mathrm{cov}}$, a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ is said to be *globally q-irreducible in $\mathbf{A}$* if every cover $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ contains a neighbourhood $V \in \mathcal{V}$ such that $U\, q\, V$. ◊

Clearly, global q-irreducibility of a neighbourhood generally depends on the surrounding algebra. Therefore, this notion cannot be expected to be fully compatible with our localisation approach. The following lemma collects the evident observations one makes from the definition.

3.6.5 Lemma. *For an algebra $\mathbf{A}$, a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ and quasiorders $q, q_1, q_2 \subseteq (\operatorname{Neigh}\mathbf{A})^2$ where $q, q_1, q_2 \subseteq \leq_{\mathrm{cov}}$ and $q_1 \subseteq q_2$, the following implications hold:*

(a) If U is globally q_1-irreducible in $\mathbf{A}$, then it also is globally q_2-irreducible in $\mathbf{A}$.

(b) The algebra $\mathbf{A}$ is q-irreducible if and only if its full neighbourhood A is globally q-irreducible.

(c) If U is globally q-irreducible in $\mathbf{A}$, then it is q-irreducible, i.e. $\mathbf{A}|_U$ is a q-irreducible algebra.

Proof: If the neighbourhood U is globally q_1-irreducible, then every $\mathcal{V} \in \operatorname{Cov}_{\mathbf{A}}(U)$ contains a member $V \in \mathcal{V}$ satisfying $U\, q_1\, V$ and hence $U\, q_2\, V$ due to the assumption $q_1 \subseteq q_2$. This establishes (a).

Comparing Definition 3.5.16 with Definition 3.6.4 for $U = A$ directly implies claim (b).

Global q-irreducibility of $U \in \operatorname{Neigh}\mathbf{A}$ contains a universal quantification over all covers in $\operatorname{Cov}_{\mathbf{A}}(U)$, whereas, according to Remark 3.5.17, the notion of q-irreducibility of a neighbourhood considers just fewer covers, namely those contained in $\mathfrak{P}(U)$. As both conditions otherwise impose the same requirement, the implication in (c) holds. □

The considerations in Remark 3.6.3 suggest to use global $\leq_{\mathrm{cov}}$-irreducibility of neighbourhoods as defined in 3.6.4 if one is interested in a non-trivial and non-paradoxical irreducibility notion that involves all possible covers of a neighbourhood inside the surrounding algebra. Such a wish deliberately violates our localisation maxim. Therefore, we do not further study global q-irreducibility here, even though it may be useful.

Instead, we turn back to our aim of providing easier sufficient criteria for q-irreducibility in the normal sense. For this we remember that q-irreducibility is tied to covers and these have to do with separation of pairs of distinct invariant relations of $\mathbf{A}$, i.e. $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ where $m \in \mathbb{N}$ and $S{\restriction_U} \neq T{\restriction_U}$ holds for the full neighbourhood $U = A$. Now, let us imagine, that, for reasons of efficiency, we reduce the neighbourhood U further and further in size until we cannot do so any more without losing the separation property $S{\restriction_U} \neq T{\restriction_U}$. Such a process successfully terminates if the poset $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ fulfils DCC, and what we have done is simply choosing an element of $\operatorname{Min}\left(\{U \in \operatorname{Neigh}\mathbf{A} \mid S{\restriction_U} \neq T{\restriction_U}\}, \subseteq\right)$. We shall call such neighbourhoods that are minimal w.r.t. separating a specific pair S, T of invariant relations (S,T)-irreducible, a notion coined in [Kea01, Definition 5.4] and also to be found in Definitions 3.7.1 of [Beh09] and 2.6 of [KL10]. Of course, we can proceed similarly for a quasiordered set $(\operatorname{Neigh}\mathbf{A}, q)$, where $q \subseteq \leq_{\mathrm{cov}}$ and the canonically associated poset on the factor set $\operatorname{Neigh}\mathbf{A}/q \cap q^{-1}$ has DCC. Provided that $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q$, any neighbourhood U from $\operatorname{Min}\left(\left\{\tilde{U} \in \operatorname{Neigh}\mathbf{A} \,\middle|\, S{\restriction_{\tilde{U}}} \neq T{\restriction_{\tilde{U}}}\right\}, q\right)$ fulfils $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$ and $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\})$ (see item (e) of Remark 3.6.10 below; if q actually equals set inclusion, i.e. for (S,T)-irreducibility, even the converse implication is valid, too). So, in generalisation of (S,T)-irreducibility such neighbourhoods are called (S,T)-q-irreducible.

We record this idea, alongside with a few related concepts in the next definition. After that, we will immediately see that (S,T)-q-irreducible neighbourhoods rightfully deserve their name, i.e. that they are indeed q-irreducible neighbourhoods.

3.6.6 Definition. For an algebra $\mathbf{A}$, a quasiorder $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ where $q \subseteq \leq_{\mathrm{cov}}$, an arity $m \in \mathbb{N}$, m-ary invariant relations $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$ and a set $U \in \operatorname{Neigh}\mathbf{A}$ we define:

(i) The algebra $\mathbf{A}$ is called *(S,T)-q-irreducible* if $S \neq T$ and for every proper subneighbourhood $V \in \operatorname{Neigh}\mathbf{A} \setminus {\uparrow_q}\{A\}$ one has $S{\restriction_V} = T{\restriction_V}$.

(ii) The set $U \in \operatorname{Neigh}\mathbf{A}$ is called *(S,T)-q-irreducible* if it is $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$, but $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\})$

(iii) We say that $U \in \operatorname{Neigh}\mathbf{A}$ is *m-strictly q-irreducible* if it is (S,T)-q-irreducible for some m-crucial pair $(S,T) \in \operatorname{Cruc}^{(m)}(\mathbf{A})$. We denote by

$$\operatorname{Irr}_q^{*(m)}(\mathbf{A}) := \left\{V \in \operatorname{Neigh}\mathbf{A} \,\middle|\, \exists (S,T) \in \operatorname{Cruc}^{(m)}(\mathbf{A}) : V \ (S,T)\text{-}q\text{-irreducible}\right\}$$

the *set of all m-strictly q-irreducible neighbourhoods of* $\mathbf{A}$.

(iv) We say that $U \in \operatorname{Neigh}\mathbf{A}$ is *strictly q-irreducible* if it is ℓ-strictly q-irreducible for some $\ell \in \mathbb{N}$. We denote the *set of all strictly q-irreducible neighbourhoods* by $\operatorname{Irr}_q^{*}(\mathbf{A}) := \bigcup_{\ell \in \mathbb{N}} \operatorname{Irr}_q^{*(\ell)}(\mathbf{A})$.

(v) We say that $U \in \operatorname{Neigh}\mathbf{A}$ is *m-crucially q-irreducible* if it is (S,T)-q-irreducible for some m-crucial pair $(S,T) \in \operatorname{Cruc}^{(m)}(\mathbf{A})$, where[37] $T \subseteq \operatorname{pr}_X^A \operatorname{Clo}^{(1)}(\mathbf{A})$

[37]The inclusion $T \subseteq \operatorname{pr}_X^A \operatorname{Clo}^{(1)}(\mathbf{A})$ here and everywhere else in this section is to be understood via an implicitly fixed indexing bijection between X and $|X| = m$. Alternatively, one may also explicitly use injective maps $\beta\colon m \longrightarrow A$ and $\prod_{\mathrm{id}_A}^{\beta}$ instead of pr_X^A.

for some finite subset $X \subseteq A$ of cardinality m. We denote by

$$\operatorname{Irr}_q^{**(m)}(\mathbf{A}) := \left\{ V \in \operatorname{Neigh}\mathbf{A} \;\middle|\; \begin{array}{l} \exists X \subseteq A\, \exists (S,T) \in \operatorname{Cruc}^{(m)}(\mathbf{A}) : |X| = m \wedge \\ T \subseteq \operatorname{pr}_X^A \operatorname{Clo}^{(1)}(\mathbf{A}) \wedge V\ (S,T)\text{-}q\text{-irreducible} \end{array} \right\}$$

the *set of all m-crucially q-irreducible neighbourhoods of* $\mathbf{A}$.

(vi) We say that $U \in \operatorname{Neigh}\mathbf{A}$ is *crucially q-irreducible* if it is ℓ-crucially q-irreducible for some $\ell \in \mathbb{N}$. We collect all crucially q-irreducible neighbourhoods in the set $\operatorname{Irr}_q^{**}(\mathbf{A}) := \bigcup_{\ell\in\mathbb{N}} \operatorname{Irr}_q^{**(\ell)}(\mathbf{A})$.

Furthermore, we agree on the convention that omission of the parameter q in all notions and denotions defined above is implicitly meant to stand for $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$. ♢

3.6.7 Remark. For an algebra $\mathbf{A}$ and a quasiorder $q \subseteq \leq_{\mathrm{cov}}$, $m \in \mathbb{N}$ and invariants $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ by definition is (S,T)-q-irreducible if and only if $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U) \setminus \operatorname{Sep}_{\mathbf{A}}\left(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}} \{U\}\right)$. This can be reformulated as the following implication: U is (S,T)-q-irreducible if and only if $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$ and for every neighbourhood $V \in \operatorname{Neigh}\mathbf{A}|_U$ the separation condition $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(V)$ implies $U\ q\ V$. ♢

For the classical case $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$, the following simple observation can also be found as part of Theorem 5.5 of [Kea01] and Lemma 3.7.3 of [Beh09].

3.6.8 Lemma. *Let* $\mathbf{A}$ *be an algebra,* $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ *a quasiorder where* $q \subseteq \leq_{\mathrm{cov}}$, $U \in \operatorname{Neigh}\mathbf{A}$ *a neighbourhood,* $m \in \mathbb{N}$ *and* $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$*. If* U *is* (S,T)*-*q*-irreducible, then it is a* q*-irreducible neighbourhood of* $\mathbf{A}$*.*

Proof: Due to (S,T)-q-irreducibility of U, we have $S\restriction_U \neq T\restriction_U$, wherefore covers of $\mathbf{A}|_U$ need to separate the pair of invariants $(S\restriction_U, T\restriction_U) \in \operatorname{Inv}^{(m)}\mathbf{A}|_U$. If U were q-reducible, then, by Lemma 3.5.22, we had $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\} \in \operatorname{Cov}_{\mathbf{A}}(U)$. According to Lemma 3.4.16 this would mean that $\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$ were a cover of $\mathbf{A}|_U$, whence we got $(S\restriction_U, T\restriction_U) \in \operatorname{Sep}_{\mathbf{A}|_U}(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\})$. Thus, for some $V \in \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$ we had $S\restriction_V = (S\restriction_U)\restriction_V \neq (T\restriction_U)\restriction_V = T\restriction_V$, which were to say, $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(V) \subseteq \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\})$. This would certainly contradict (S,T)-q-irreducibility of U, wherefore U must be a q-irreducible neighbourhood. □

3.6.9 Remark. For every algebra and a quasiorder $q \subseteq \leq_{\mathrm{cov}}$, we have the inclusions

$$\operatorname{Irr}_q^{**}(\mathbf{A}) \subseteq \operatorname{Irr}_q^{*}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A}).$$

The first one is evident since crucially q-irreducible neighbourhoods are strictly q-irreducible ones that are (S,T)-q-irreducible for a special sort of crucial pair (S,T). The second inclusion is a consequence of the previous lemma. ♢

We continue with a few basic remarks, exploring e.g. the connection between (S,T)-q-irreducibility of neighbourhoods and (S,T)-q-irreducibility of algebras. The last of them also contains a slightly simpler condition for (S,T)-q-irreducibility than the one used in Definition 3.6.6(ii).

3.6.10 Remark (cf. Remark 3.7.2 in [Beh09]). For an algebra $\mathbf{A}$, a quasiorder $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$, fulfilling $q \subseteq \leq_{\mathrm{cov}}$, an arity $m \in \mathbb{N}$ and m-ary invariant relations $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, as well as any set $U \in \operatorname{Neigh}\mathbf{A}$ the following facts are true.

(a) The neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ is (S,T)-q-irreducible if and only if it is (T,S)-q-irreducible.

(b) If the neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ is (S,T)-q-irreducible, it is $(S\cap T, T)$-q-irreducible or $(S \cap T, S)$-q-irreducible.

(c) The algebra $\mathbf{A}$ is (S,T)-q-irreducible if and only if its largest neighbourhood $A \in \operatorname{Neigh}\mathbf{A}$ is (S,T)-q-irreducible, if and only if $\mathbf{A}|_A$ is (S,T)-q-irreducible.

(d) A subneighbourhood $V \in \operatorname{Neigh}\mathbf{A}$, $V \subseteq U$ is (S,T)-q-irreducible w.r.t. $\mathbf{A}$ if and only if it is $(S{\restriction_U}, T{\restriction_U})$-$q{\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}$-irreducible w.r.t. $\mathbf{A}|_U$.

Especially, $U \in \operatorname{Neigh}\mathbf{A}$ is (S,T)-q-irreducible exactly if the algebra $\mathbf{A}|_U$ is $(S{\restriction_U}, T{\restriction_U})$-$q{\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}$-irreducible.

(e) If $U \in \operatorname{Min}\left(\left\{\tilde{U} \in \operatorname{Neigh}\mathbf{A} \;\middle|\; (S,T) \in \operatorname{Sep}_{\mathbf{A}}\left(\tilde{U}\right)\right\}, q\right)$ and $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q$, then U is (S,T)-q-irreducible. For $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$ this implication actually is an equivalence, providing the missing link to the definition of (S,T)-irreducibility given in [Kea01, Beh09].

(f) Suppose, $(\operatorname{Neigh}\mathbf{A}|_U, \tilde{q})$ is a quasiordered set[38] whose factor poset satisfies ACC, let $\tilde{q} \subseteq \leq_{\mathrm{cov}}$, and let $\mathcal{M} \subseteq \operatorname{Neigh}\mathbf{A}|_U \setminus \{U\}$ be a set of $(\tilde{q}\cap\tilde{q}^{-1})$-representatives for the $\tilde{q}$-maximal elements in $\left(\operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}, \tilde{q}{\restriction_{\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_q\{U\}}}\right)$. Then U is (S,T)-q-irreducible if and only if $S{\restriction_U} \neq T{\restriction_U}$ and $S{\restriction_V} = T{\restriction_V}$ for all $V \in \mathcal{M}$. $\diamondsuit$

Proof: (a) Clearly, (S,T)-q-irreducibility of a neighbourhood is a symmetric notion since equality is a symmetric relation.

(b) Note that for all neighbourhoods $V \in \operatorname{Neigh}\mathbf{A}$ it is $(S\cap T){\restriction_V} = S{\restriction_V} \cap T{\restriction_V}$. Since $U \in \operatorname{Neigh}\mathbf{A}$ is (S,T)-q-irreducible, one has $S{\restriction_U} \neq T{\restriction_U}$. Hence, it follows $S{\restriction_U} \not\subseteq T{\restriction_U}$, implying $(S\cap T){\restriction_U} \not\supseteq S{\restriction_U}$, or possibly $T{\restriction_U} \not\subseteq S{\restriction_U}$, implying $(S\cap T){\restriction_U} \not\supseteq T{\restriction_U}$. That is, $(S\cap T){\restriction_U} \neq S{\restriction_U}$ or $(S\cap T){\restriction_U} \neq T{\restriction_U}$. Furthermore, for every proper subneighbourhood $V \in \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$, we have $S{\restriction_V} = T{\restriction_V}$, so $S{\restriction_V} = (S\cap T){\restriction_V} = T{\restriction_V}$.

[38]The most natural choice for $\tilde{q}$ is of course set inclusion.

(c) Since $\operatorname{Neigh} \mathbf{A}|_A = \operatorname{Neigh} \mathbf{A}$, the first equivalence follows directly from Definition 3.6.6, the second one from the second statement of part (d).

(d) Let us consider any neighbourhood $V \in \operatorname{Neigh} \mathbf{A}|_U \overset{3.3.7}{=} (\operatorname{Neigh} \mathbf{A}) \cap \mathfrak{P}(U)$. For this argument two facts are essential. First, as just noted, that neighbourhoods of the restricted algebra $\mathbf{A}|_U$ are precisely the subneighbourhoods of U in the global algebra. Second, that restriction of relations to subsets is compatible with set inclusion, i.e. that $(R{\restriction_U}){\restriction_W} = R{\restriction_W}$ holds for any $W \subseteq U$ and all relations $R \in \operatorname{Inv} \mathbf{A}$.

By definition, $(S{\restriction_U}, T{\restriction_U})$-$q{\restriction_{\operatorname{Neigh} \mathbf{A}|_U}}$-irreducibility of V in $\mathbf{A}|_U$ means that we have $(S{\restriction_U}){\restriction_V} \neq (T{\restriction_U}){\restriction_V}$, and that $(S{\restriction_U}){\restriction_W} = (T{\restriction_U}){\restriction_W}$ holds for every $W \in \operatorname{Neigh} (\mathbf{A}|_U)|_V$, $V \not q W$. By the above this is equivalent to say that $S{\restriction_V} \neq T{\restriction_V}$, and $S{\restriction_W} = T{\restriction_W}$ holds for every $W \in \operatorname{Neigh} \mathbf{A}|_V$, $V \not q W$. Again, by definition, this is equivalent to V being (S,T)-q-irreducible in $\mathbf{A}$.

The second statement follows as a specialisation of the first via $V = U$.

(e) Suppose that $U \in \operatorname{Neigh} \mathbf{A}$ is q-minimal among all neighbourhoods of $\mathbf{A}$ separating the pair (S,T). Then, clearly, we have $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$. If we had $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(V)$ for some $V \in \operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\}$, then we had of course $V \subseteq U$, and thus $V \, q \, U$. Now, by minimality of U, we obtained $U \, q \, V$, in contradiction to $V \notin \uparrow_q \{U\}$. Hence, $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\})$, and so U is (S,T)-q-irreducible.

For the converse suppose that q equals set inclusion, and that U is (S,T)-irreducible. So certainly, we have $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$, we only need to check that U is minimal w.r.t. inclusion among all neighbourhoods of $\mathbf{A}$ having this property. This is evident, since $\operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\}$ in this special case is just $\operatorname{Neigh} \mathbf{A}|_U \setminus \{U\}$, and so no proper subneighbourhood of U separates S from T. This is exactly the notion of (S,T)-irreducibility used in the literature so far.

(f) If U is (S,T)-q-irreducible, then we have $S{\restriction_U} \neq T{\restriction_U}$ and $S{\restriction_V} = T{\restriction_V}$ for every $V \in \operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\}$. Since $\mathcal{M} \subseteq \operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\}$, the condition stated in the remark is clearly necessary. Next, we prove that it is also sufficient for (S,T)-q-irreducibility. The part that (S,T) belongs to $\operatorname{Sep}_{\mathbf{A}}(U)$ is evident. Now consider any $V \in \operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\}$. Since the poset canonically associated with $(\operatorname{Neigh} \mathbf{A}|_U, \tilde{q})$ satisfies ACC, we can infer $V \, \tilde{q} \, M$ for some $M \in \mathcal{M}$ using Lemma 3.5.5. Thus, we have $V \leq_{\mathrm{cov}} M$, and so $\operatorname{Sep}_{\mathbf{A}}(V) \subseteq \operatorname{Sep}_{\mathbf{A}}(M)$. Therefore, $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(M)$, which is part of the assumption, implies that $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(V)$, i.e. $S{\restriction_V} = T{\restriction_V}$. This concludes the proof that U is indeed (S,T)-q-irreducible. □

It is evident from Definition 3.6.6(ii) that the way in which (S,T)-q-irreducibility depends on the parameter q is completely determined by set

$$\operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_q \{U\} = \operatorname{Neigh} \mathbf{A}|_U \setminus \uparrow_{q{\restriction_{\operatorname{Neigh} \mathbf{A}|_U}}} \{U\} .$$

For this reason, we get similar results for (S,T)-q-irreducibility of neighbourhoods as established in Lemma 3.5.20 and Corollaries 3.5.21 and 3.5.23 for q-irreducibility. In particular, item (c) of the following lemma and Corollary 3.5.21 imply that the assumption $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q$ does not impose any restriction on the quasiorder parameter q of irreducibility and (S,T)-irreducibility.

3.6.11 Lemma. *For an algebra $\mathbf{A}$, quasiorders $q, q_1, q_2 \subseteq \leq_{\mathrm{cov}}$ on neighbourhoods of $\mathbf{A}$, an arity $m \in \mathbb{N}$, a pair of invariants $S,T \in \operatorname{Inv}^{(m)}\mathbf{A}$ and a neighbourhood $U \in \operatorname{Neigh}\mathbf{A}$ the following facts are true:*

(a) If $\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} \subseteq \uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\}$, then (S,T)-q_1-irreducibility of U implies (S,T)-q_2-irreducibility of this neighbourhood. Especially, this is the case (for all neighbourhoods $U \in \operatorname{Neigh}\mathbf{A}$) if we globally have $q_1 \subseteq q_2$. Thus, under this stronger assumption we obtain $\operatorname{Irr}^*_{q_1}(\mathbf{A}) \subseteq \operatorname{Irr}^*_{q_2}(\mathbf{A})$ and $\operatorname{Irr}^{**}_{q_1}(\mathbf{A}) \subseteq \operatorname{Irr}^{**}_{q_2}(\mathbf{A})$.

(b) If $\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\}$, then U is (S,T)-q_1-irreducible if and only if it is (S,T)-q_2-irreducible.

(c) If U is (S,T)-irreducible, then it is also (S,T)-q-irreducible. Consequently, we have $\operatorname{Irr}^*(\mathbf{A}) \subseteq \operatorname{Irr}^*_q(\mathbf{A})$ and $\operatorname{Irr}^{**}(\mathbf{A}) \subseteq \operatorname{Irr}^{**}_q(\mathbf{A})$.

(d) If U has got the property that for any $V \in \operatorname{Neigh}\mathbf{A}$ the condition $U\ q\ V \subseteq U$ implies $V = U$, i.e. if $\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \{U\}$, then U is (S,T)-q-irreducible if and only if it is (S,T)-irreducible. Especially, if $q \subseteq \precapprox$ and $\mathbf{A}$ is neighbourhood self-embedding simple, then we have $\uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \{U\}$ for all $U \in \operatorname{Neigh}\mathbf{A}$, implying $\operatorname{Irr}^*(\mathbf{A}) = \operatorname{Irr}^*_q(\mathbf{A})$ and $\operatorname{Irr}^{**}(\mathbf{A}) = \operatorname{Irr}^{**}_q(\mathbf{A})$.

Proof: (a) If U is (S,T)-q_1-irreducible, then we have of course $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U)$. Our assumption $\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} \subseteq \uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\}$, certainly implies the inclusion

$$\begin{aligned}\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_2}\{U\} &= \operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} \\ &\subseteq \operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_1}\{U\}.\end{aligned}$$

Hence, it follows $\operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_2}\{U\}) \subseteq \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_1}\{U\})$, so that one can exclude $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_2}\{U\})$ using that (S,T) does not belong to $\operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh}\mathbf{A}|_U \setminus \uparrow_{q_1}\{U\})$ by the assumed (S,T)-q_1-irreducibility of U. This proves that U is (S,T)-q_2-irreducible. In particular, if the inclusion $q_1 \subseteq q_2$ holds globally, then restriction yields $q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U} \subseteq q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}$ and hence $\uparrow_{q_1\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} \subseteq \uparrow_{q_2\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\}$ for any set $U \in \operatorname{Neigh}\mathbf{A}$. So the implication we just proved holds independently of $U \in \operatorname{Neigh}\mathbf{A}$, and for an arbitrary pair of invariant relations. As a consequence, we get the inclusions $\operatorname{Irr}^*_{q_1}(\mathbf{A}) \subseteq \operatorname{Irr}^*_{q_2}(\mathbf{A})$ and $\operatorname{Irr}^{**}_{q_1}(\mathbf{A}) \subseteq \operatorname{Irr}^{**}_{q_2}(\mathbf{A})$.

(b) This equivalence is an obvious consequence of item (a) exploiting both inclusions.

(c) Suppose that U is (S,T)-irreducible, i.e. (S,T)-$\subseteq_{\operatorname{Neigh}\mathbf{A}}$-irreducible. We verify the assumption of item (a) to obtain that U is (S,T)-q-irreducible. Indeed, if $V \in \operatorname{Neigh}\mathbf{A}|_U$ satisfies $U \subseteq V$, then, it must be $V = U$. Thus, we have shown $\uparrow_{\subseteq_{\operatorname{Neigh}\mathbf{A}}\restriction_{\operatorname{Neigh}\mathbf{A}|_U}} \{U\} = \{U\} \subseteq \uparrow_{q\restriction_{\operatorname{Neigh}\mathbf{A}|_U}} \{U\}$, which ensures that U is (S,T)-q-irreducible by item (a). As this argument was again valid for arbitrary $U \in \operatorname{Neigh}\mathbf{A}$, we can infer that classical strict and crucial irreducibility of neighbourhoods is the strongest of all parametric strict and crucial irreducibility concepts: $\operatorname{Irr}^*(\mathbf{A}) \subseteq \operatorname{Irr}^*_q(\mathbf{A})$ and $\operatorname{Irr}^{**}(\mathbf{A}) \subseteq \operatorname{Irr}^{**}_q(\mathbf{A})$.

(d) The first statement of this item follows directly via an application of (b) since we have $\uparrow_{\subseteq_{\operatorname{Neigh}\mathbf{A}}\restriction_{\operatorname{Neigh}\mathbf{A}|_U}} \{U\} = \{U\}$ as shown in the proof of the previous item.

It was demonstrated in Corollary 3.5.23, that, if $q \subseteq \precapprox$ and $\mathbf{A}$ is neighbourhood self-embedding simple, then for any $U, V \in \operatorname{Neigh}\mathbf{A}$ the condition $U \mathrel{q} V \subseteq U$ entails $U = V$. This fact ensures the assumption of the first part of this item for any set $U \in \operatorname{Neigh}\mathbf{A}$. Thus, we can infer $\operatorname{Irr}^*(\mathbf{A}) = \operatorname{Irr}^*_q(\mathbf{A})$ and $\operatorname{Irr}^{**}(\mathbf{A}) = \operatorname{Irr}^{**}_q(\mathbf{A})$. $\square$

From this result, we can obtain the following lemma about how (S,T)-$\precapprox$-irreducibility spreads among neighbourhoods. Subsequently, we are going to present a sufficient condition for irreducibility of algebras whose monoid of unary clone operations is commutative, before we finally deal with a characterisation connecting q-irreducibility of neighbourhoods and (S,T)-q-irreducibility.

3.6.12 Lemma. *Let $\mathbf{A}$ be an algebra, $q \subseteq (\operatorname{Neigh}\mathbf{A})^2$ a quasiorder satisfying the inclusion $q \subseteq \leq_{\mathrm{cov}}$ and let $U, W \in \operatorname{Neigh}\mathbf{A}$ be neighbourhoods such that $U \precapprox W$. If W is (S,T)-q-irreducible for some m-ary invariants $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, $m \in \mathbb{N}$, that are still distinguishable in U, i.e. $S\restriction_U \neq T\restriction_U$, then it is $U \mathrel{\tilde{q} \cap \tilde{q}^{-1}} W$, and U is (S,T)-$(\tilde{q} \cap \tilde{q}^{-1})$-irreducible, where $\tilde{q} := q \vee \precapprox$.*

In particular, the following specialisations of this result hold:

(a) if W is (S,T)-$\precapprox$-irreducible, $\uparrow_{\precapprox\restriction_{\operatorname{Neigh}\mathbf{A}|_U}} \{U\} = \{U\}$ and $S\restriction_U \neq T\restriction_U$, then U is (S,T)-irreducible and $U \mathrel{\precapprox \cap \succapprox} W$.

(b) if W is (S,T)-$\precapprox$-irreducible, $\mathbf{A}|_U$ is neighbourhood self-embedding simple and $S\restriction_U \neq T\restriction_U$, then U is (S,T)-irreducible and $U \mathrel{\precapprox \cap \succapprox} W$.

(c) if $W \mathrel{\precapprox \cap \succapprox} U$ and W is (S,T)-$\precapprox$-irreducible, $\mathbf{A}|_U$ is neighbourhood self-embedding simple, then U is (S,T)-irreducible.

(d) if $W \mathrel{\precapprox \cap \succapprox} U$ and both $\mathbf{A}|_U$ and $\mathbf{A}|_W$ are neighbourhood self-embedding simple, then W is (S,T)-irreducible if and only if U is.

Especially, in this case U is irreducible exactly if W is.

Proof: By assumption we know that $S\restriction_U \neq T\restriction_U$. Now consider a subneighbourhood $U' \in \operatorname{Neigh}\mathbf{A}$, $U' \subseteq U$ such that $S\restriction_{U'} \neq T\restriction_{U'}$. According to Remark 3.6.7, we need to show that $U\ \tilde{q}\cap\tilde{q}^{-1}\ U'$. From the assumption $U \precapprox W$ and Proposition 3.2.10(c) we get unary clone operations $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ such that $f[U] \subseteq W$ and $g(f(u)) = u$ for all $u \in U$. Corollary 3.2.11 implies that $V' := f[U']$ and $V := f[U]$ are neighbourhoods of $\mathbf{A}$ satisfying $U' \cong V'$, $U \cong V$ and $V' \subseteq V \subseteq W$. Therefore, we have $U' \cong V' \subseteq W$, implying $U' \leq_{\mathrm{cov}} V' \subseteq W$ due to Corollary 3.4.6. Hence, we can infer $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(U') \subseteq \operatorname{Sep}_{\mathbf{A}}(V')$ for the subneighbourhood V' of W. Now the assumption of (S,T)-q-irreducibility of W together with Remark 3.6.7 yields $W\ q\ V'$, such that we get $U \cong V \subseteq W\ q\ V' \cong U' \subseteq U$. Using Corollary 3.2.12 we get $\subseteq, \cong\ \subseteq\ \precapprox\ \subseteq \tilde{q}$, whence it follows $U\ \tilde{q}\ V\ \tilde{q}\ W\ \tilde{q}\ V'\ \tilde{q}\ U'\ \tilde{q}\ U$. This proves $U\ \tilde{q}\cap\tilde{q}^{-1}\ U'$, concluding the argument that U is (S,T)-$(\tilde{q}\cap\tilde{q}^{-1})$-irreducible, and also shows $U\ \tilde{q}\cap\tilde{q}^{-1}\ W$.

Now let us treat the listed special cases:

(a) Here, we simply have instantiated $q = \precapprox$, and we have used the additional condition $\uparrow_{\precapprox\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \{U\}$ and Lemma 3.6.11 to get that (S,T)-$(q\cap q^{-1})$-irreducibility of U implies (S,T)-q-irreducibility, which again implies (S,T)-irreducibility.

(b) Here we have used that neighbourhood self-embedding simplicity of $\mathbf{A}|_U$ implies the assumption $\uparrow_{\precapprox\restriction_{\operatorname{Neigh}\mathbf{A}|_U}}\{U\} = \{U\}$ to be used in item (a).

(c) Here, we exploit the fact that $W \precapprox U$ implies $\operatorname{Sep}_{\mathbf{A}}(W) \subseteq \operatorname{Sep}_{\mathbf{A}}(U)$ (see Corollary 3.2.23(a)), enabling us to infer the assumption $S\restriction_U \neq T\restriction_U$ of (b) via $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(W) \subseteq \operatorname{Sep}_{\mathbf{A}}(U)$ from (S,T)-$\precapprox$-irreducibility of W.

(d) In the last part, we have used that (S,T)-irreducibility implies (S,T)-$\precapprox$-irreducibility (see Lemma 3.6.11), and we have symmetrised the assumptions on U and W to turn item (c) into an equivalence.

The last statement about irreducibility of neighbourhoods being invariant under isomorphism, will follow from the characterisation of irreducibility via (S,T)-irreducibility in the subsequent Proposition 3.6.15. □

The following sufficient criterion for irreducibility of algebras with commuting unary clone operations has been conjectured by Tamás Waldhauser [Wal12] for the case of finite algebras and the relations $q = \tilde{q}$ being set inclusion. It can for instance be applied to show that all finite non-completely regular semigroups are irreducible. However, we do not provide details concerning this fact, as describing the distribution of neighbourhoods in finite semigroups would exceed the scope of this thesis.

3.6.13 Lemma. *For every algebra $\mathbf{A}$ such that its monoid $\left(\operatorname{Clo}^{(1)}(\mathbf{A}), \circ\right)$ of unary clone operations is commutative the following statements hold:*

(a) For every $V \in \operatorname{Neigh}\mathbf{A}$ the condition $A \leq_{\mathrm{cov}} V$ implies $V = A$.

(b) *If for some quasiorders $q, \tilde{q} \subseteq (\operatorname{Neigh} \mathbf{A})^2$ satisfying $q, \tilde{q} \subseteq \leq_{\mathrm{cov}}$ there is a proper neighbourhood $V \in \operatorname{Neigh} \mathbf{A} \setminus \{A\}$ such that $\operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\} \sqsubseteq (\tilde{q}) \{V\}$, then $\mathbf{A}$ is q-irreducible. In particular, if the poset $(\operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\}, \subseteq)$ of proper neighbourhoods has a largest element $V \subset A$, then $\mathbf{A}$ must be q-irreducible. Especially, if $(\operatorname{Neigh} \mathbf{A} \setminus \{A\}, \subseteq)$ contains a largest element, then $\mathbf{A}$ is irreducible.*

Proof: (a) Suppose that for some idempotent operation $e \in \operatorname{Idem} \mathbf{A}$ the neighbourhood $V := \operatorname{im} e$ satisfies $A \leq_{\mathrm{cov}} \{V\}$. Now exploiting condition (i) of Corollary 3.4.35 for $A = U$, for every finite subset $X \subseteq A$ there exists an arity $n \in \mathbb{N}$, an operation $\lambda \in \operatorname{Clo}^{(n)}(\mathbf{A})$ and a tuple $(f_1, \ldots, f_n) \in \left(\operatorname{Clo}^{(1)}(A)\right)^n$ subject to $\operatorname{im} f_i \subseteq V$ for all $1 \leq i \leq n$ such that $(\lambda \circ (f_1, \ldots, f_n))|_X^A = \operatorname{id}_A|_X^A$. Using Lemma 3.1.3, we know that the condition $\operatorname{im} f_i \subseteq V = \operatorname{im} e$ can equivalently be expressed by the equality $e \circ f_i = f_i$. Thus, together with the assumed commutativity of unary clone operations w.r.t. composition, we have $f_i = e \circ f_i = f_i \circ e$ for all $1 \leq i \leq n$. Substituting this into the decomposition equation obtained earlier, we get

$$\begin{aligned}\operatorname{id}_A|_X^A &= (\lambda \circ (f_1, \ldots, f_n))|_X^A = (\lambda \circ (f_1 \circ e, \ldots, f_n \circ e))|_X^A \\ &= (\lambda \circ (f_1, \ldots, f_n) \circ e)|_X^A = (f_X \circ e)|_X^A,\end{aligned}$$

where $f_X := \lambda \circ (f_1, \ldots, f_n) \in \operatorname{Clo}^{(1)}(\mathbf{A})$. Putting $X := \{x, y\}$ for $x, y \in A$, this implies $x = f_X(e(x)) = f_X(e(y)) = y$, i.e. that e is injective. Now, due to idempotency, we have $e(e(a)) = e(a)$ for all $a \in A$, implying $e(a) = a$ for all $a \in A$ because e is injective. However, this is saying that $e = \operatorname{id}_A$, and thus it follows $V = \operatorname{im} e = \operatorname{im} \operatorname{id}_A = A$.

(b) Suppose that for quasiorders $q, \tilde{q} \subseteq (\operatorname{Neigh} \mathbf{A})^2$ subject to $q, \tilde{q} \subseteq \leq_{\mathrm{cov}}$, we have $\operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\} \sqsubseteq (\tilde{q}) \{V\}$ where $V \in \operatorname{Neigh} \mathbf{A} \setminus \{A\}$. Using Remark 3.5.1, this implies $\operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\} \leq_{\mathrm{cov}} \{V\}$. If the algebra $\mathbf{A}$ were q-reducible, then, according to Lemma 3.5.22, we had $A \leq_{\mathrm{cov}} \operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\} \leq_{\mathrm{cov}} \{V\}$. Hence, by transitivity we got $A \leq_{\mathrm{cov}} \{V\}$, which by (a) implied $V = A$, in contradiction to the choice of $V \subset A$. Therefore, $\mathbf{A}$ must be q-irreducible.

Especially, if the poset $(\operatorname{Neigh} \mathbf{A} \setminus \uparrow_q \{A\}, \subseteq)$ has got some $V \subset A$ as a greatest element, then the assumptions used above are fulfilled for $\tilde{q}$ being set inclusion of neighbourhoods, wherefore $\mathbf{A}$ is q-irreducible in this case. Letting $q = \subseteq_{\operatorname{Neigh} \mathbf{A}}$ in this last argument, we obtain that $\mathbf{A}$ is irreducible provided that $\operatorname{Neigh} \mathbf{A} \setminus \{A\}$ has got a largest element. $\square$

The subsequent example demonstrates that in general the converse implication in item (b) of Lemma 3.6.13 has to fail. That is to say there exists a finite, nontrivial algebra $\mathbf{A}$ having a commutative monoid of unary term operations and being irreducible despite of having two distinct coatoms in $(\operatorname{Neigh} \mathbf{A}, \subseteq)$.

3.6.14 Example. Let $A = \{0,1,2\}$ and consider the unary algebra $\mathbf{A} = \langle A; e_1, e_2\rangle$ where $e_i(x) := x$ for all $x \in \{0,i\}$ and $e_i(x) = 0$ for $x \notin \{0,i\}$, $i \in \{1,2\}$. Then $\mathrm{Clo}^{(1)}(\mathbf{A}) = \{\mathrm{id}_A, e_1, e_2, c\}$ where c is the unary constant operation taking on value 0. It forms a commutative monoid w.r.t. composition:

$$\begin{array}{c|cccc} \circ & \mathrm{id}_A & e_1 & e_2 & c \\ \hline \mathrm{id}_A & \mathrm{id}_A & e_1 & e_2 & c \\ e_1 & e_1 & e_1 & c & c \\ e_2 & e_2 & c & e_2 & c \\ c & c & c & c & c. \end{array}$$

Furthermore, $\mathbf{A}$ is finite and essentially unary and has more than one element, hence it is irreducible (cf. Example 3.6.1(a)). Its poset of neighbourhoods is $(\{A, \{0,1\}, \{0,2\}, \{0\}\}, \subseteq)$ having the two coatoms $\{0,1\}$ and $\{0,2\}$. $\diamondsuit$

With the next result, we provide a sort of converse to the implication stated in Lemma 3.6.8. Thereby, we obtain a quite detailed answer to the characterisation problem (even on the general level of q-irreducibility) advertised in the introduction to this section. We first prove a general description for arbitrary neighbourhoods in arbitrary algebras, in this way extending the the following results given for finite algebras and classical irreducibility: Theorem 5.5 in [Kea01], Lemma 3.7.3 in [Beh09] and Theorem 2.7 in [KL10]. With regard to the latter the characterisation in 3.6.15 also shows that our definition of irreducibility coincides with the one given in [KL10, Definition 2.6]. Subsequently, we will see how our result can be sharpened for finite neighbourhoods. Note that in both propositions we avoid to speak explicitly about indexing bijections between finite subsets of A and their cardinalities in connection with projections.

3.6.15 Proposition. *Let $\mathbf{A}$ be an algebra and $q \subseteq \leq_{\mathrm{cov}}$ be a quasiorder on neighbourhoods of $\mathbf{A}$. For some neighbourhood $U \in \mathrm{Neigh}\,\mathbf{A}$ let us define the sets*

$$F := \left\{ f \in \mathrm{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \mathrm{Neigh}\,\mathbf{A}|_U \setminus {\uparrow_q}\{U\} : \mathrm{im}\, f \subseteq V \right\},$$

$S_0 := \langle F\rangle_{\mathbf{A}^A}$ and $T_0 := \mathrm{Clo}^{(1)}(\mathbf{A})$ as in Theorem 3.4.31, then the following statements are equivalent:

(a) U is q-irreducible.

(b) $\mathrm{Neigh}\,\mathbf{A}|_U \setminus {\uparrow_q}\{U\} \notin \mathrm{Cov}_{\mathbf{A}}(U)$.

(c) There is $m \in \mathbb{N}$ and $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$ such that U is (S,T)-q-irreducible.

(d) There is $m \in \mathbb{N}$, $m \leq |U|$, and $S, T \in \mathrm{Inv}^{(m)}\mathbf{A}$, $S \subset T$, for which the neighbourhood U is (S,T)-q-irreducible.

(e) There is a finite subset $X \subseteq U$ such that $\mathrm{pr}^A_X S_0 \subset \mathrm{pr}^A_X T_0$ and the neighbourhood U is $\left(\mathrm{pr}^A_X S_0, \mathrm{pr}^A_X T_0\right)$-$q$-irreducible.

*If U is poly-*Artin*ian, then the mentioned facts are furthermore equivalent to each of the following:*

(f) There is a finite subset $X \subseteq U$ and invariant relations $S, T \in \operatorname{Inv}^{(|X|)}\mathbf{A}$ such that $S \subset T$, $S{\restriction_U} \prec T{\restriction_U} \subseteq \operatorname{pr}_X^A T_0$ form an $|X|$-crucial pair of $\mathbf{A}|_U$ and U is (S,T)-q-irreducible.

(g) There exists some $m \in \mathbb{N}$, $m \leq |U|$ and $S, T \in \operatorname{Inv}^{(m)}\mathbf{A}$, $S \subset T$, such that $(S{\restriction_U}, T{\restriction_U}) \in \operatorname{Cruc}^{(m)}(\mathbf{A}|_U)$ and U is (S,T)-q-irreducible.

Let us mention that the statement in item (c) is useful as a sufficient condition, i.e., in case one wants to prove by hand that a neighbourhood is q-irreducible. Then it is a good idea to look for invariant relations S, T of small arity for which the neighbourhood is (S,T)-q-irreducible. The other characterisations should be considered as necessary conditions that follow from q-irreducibility. In other words, their negations should be seen as sufficient conditions to prove q-reducibility of a neighbourhood.

Proof: For the sake of brevity, let us put $\mathcal{V} := \operatorname{Neigh}\mathbf{A}|_U \setminus {\uparrow_q}\{U\}$ as the set of all proper subneighbourhoods of U not lying above U w.r.t. q. First, we are going to demonstrate the implications "(a) $\Leftrightarrow$ (b) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (a)".

"(a) $\Leftrightarrow$ (b)" This equivalence is exactly the contrapositive of the characterisation obtained in Lemma 3.5.22.

"(b) $\Rightarrow$ (e)" Suppose that U is not covered by the collection $\mathcal{V}$. This fact has been characterised in Corollary 3.4.35. Namely, item (e), more precisely its negation, tells us that there exists a finite subset $X \subseteq U$ for which the restrictions $\left(\operatorname{pr}_X^A S_0\right){\restriction_U}$ and $\left(\operatorname{pr}_X^A T_0\right){\restriction_U}$ are distinct, but $\left(\operatorname{pr}_X^A S_0, \operatorname{pr}_X^A T_0\right) \notin \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$. According to Definition 3.6.6, this is precisely expressing that the neighbourhood U is $\left(\operatorname{pr}_X^A S_0, \operatorname{pr}_X^A T_0\right)$-$q$-irreducible. The inclusion $\operatorname{pr}_X^A S_0 \subseteq \operatorname{pr}_X^A T_0$ follows directly from the definition since $F \subseteq T_0$ implies $S_0 = \langle F \rangle_{\mathbf{A}^A} \subseteq T_0$. It is a proper one, since these two relations are separated by U.

"(e) $\Rightarrow$ (d)" This implication follows from the fact that the arity of the relations exhibited in (e) is $m = |X| \leq |U|$.

"(d) $\Rightarrow$ (c)" This is evident.

"(c) $\Rightarrow$ (a)" Lemma 3.6.8 contains precisely the fact needed to complete the first part of the proof.

For the remaining characterisations we suppose that U is poly-Artinian. For a finite neighbourhood U this yields of course no further condition. We will demonstrate "(b) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (c)".

"(b) ⇒ (f)" Suppose that the set U is not covered by $\mathcal{V}$. Hence, item (p) of Corollary 3.4.35 implies that there exists a finite subset $X \subseteq U$ and $|X|$-ary relations $S, T \in \operatorname{Inv} \mathbf{A}$ such that $S \subseteq T$, $(S{\restriction}_U, T{\restriction}_U) \in \operatorname{Cruc}^{(|X|)}(\mathbf{A}|_U)$ and $T{\restriction}_U \subseteq \operatorname{pr}_X^A T_0$, but $(S, T) \notin \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$. Since $S{\restriction}_U \prec T{\restriction}_U$ is a crucial pair, U separates S from T, whence U is (S, T)-q-irreducible, and in particular, the inclusion $S \subseteq T$ is a proper one.

"(f) ⇒ (g)" As for "(e) ⇒ (d)" this implication is true since the arity of the relations found in (f) is the finite number $|X| \leq |U|$.

"(g) ⇒ (c)" This implication is obvious. □

For finite neighbourhoods, the statements in the previous result can be strengthened a little bit.

3.6.16 Proposition. *For an algebra $\mathbf{A}$, a quasiorder $q \subseteq \leq_{\mathrm{cov}}$ and a finite neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$, we define*

$$F := \left\{ f \in \operatorname{Clo}^{(1)}(\mathbf{A}) \;\middle|\; \exists V \in \operatorname{Neigh} \mathbf{A}|_U \setminus {\uparrow_q} \{U\} : \ \operatorname{im} f \subseteq V \right\},$$

$S_0 := \langle F \rangle_{\mathbf{A}^A}$ and $T_0 := \operatorname{Clo}^{(1)}(\mathbf{A})$ as in Proposition 3.6.15. Then the following statements are equivalent:

(a) U is q-irreducible.

(b) There are $S, T \in \operatorname{Inv}^{(|U|)} \mathbf{A}$ such that $S \subset T$ and U is (S, T)-q-irreducible.

(c) The neighbourhood U is $\left(\operatorname{pr}_U^A S_0, \operatorname{pr}_U^A T_0\right)$-$q$-irreducible for the $|U|$-ary invariants $\operatorname{pr}_U^A S_0 \subset \operatorname{pr}_U^A T_0$.

(d) There are invariant relations $S, T \in \operatorname{Inv}^{(|U|)} \mathbf{A}$ such that $S \subset T$, $T{\restriction}_U \subseteq \operatorname{pr}_U^A T_0$, $(S{\restriction}_U, T{\restriction}_U) \in \operatorname{Cruc}^{(|U|)}(\mathbf{A}|_U)$ and U is (S, T)-q-irreducible.

(e) There exist $S, T \in \operatorname{Inv}^{(|U|)} \mathbf{A}$, $S \subset T$, such that $(S{\restriction}_U, T{\restriction}_U) \in \operatorname{Cruc}^{(|U|)}(\mathbf{A}|_U)$ and U is (S, T)-q-irreducible.

Proof: The proof functions similarly to the previous one. For the sake of brevity, let us again put $\mathcal{V} := \operatorname{Neigh} \mathbf{A}|_U \setminus {\uparrow_q} \{U\}$. Statement (e) is a relaxation of (d), and (b) is one of (e). Therefore, the implications "(d) ⇒ (e) ⇒ (b)" are obvious, and "(b) ⇒ (a)" follows from Proposition 3.6.15. Therefore, in order to prove that all five conditions are equivalent, we only need to show that (a) implies (c) and that (c) implies (d).

If $U \in \operatorname{Irr}_q(\mathbf{A})$, then $\mathcal{V}$ fails to cover U by Lemma 3.5.22. According to Corollary 3.4.34, this is equivalent to the negation of statement (d) of Theorem 3.4.31 for $\mu := |U|$. That is to say, there exists a subset $X \subseteq U$ of cardinality $|X| = \mu = |U|$ such that $\left(\operatorname{pr}_X^A S_0, \operatorname{pr}_X^A T_0\right) \in \operatorname{Sep}_{\mathbf{A}}(U) \setminus \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$. As U is finite, the subset X

must be U, and hence, we have $\left(\mathrm{pr}_U^A S_0, \mathrm{pr}_U^A T_0\right) \in \mathrm{Sep}_{\mathbf{A}}(U) \setminus \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$, proving that U is $\left(\mathrm{pr}_U^A S_0, \mathrm{pr}_U^A T_0\right)$-$q$-irreducible. This automatically entails $\mathrm{pr}_U^A S_0 \neq \mathrm{pr}_U^A T_0$, the inclusion stated in (c) follows again by definition of S_0 and T_0 as in the proof of Proposition 3.6.15. Thus, item (c) is a consequence of (a).

To show that item (d) is implied by (c), we put $m := |U|$ and $\top := \left(\mathrm{pr}_U^A T_0\right){\restriction_U}$ in Corollary 3.4.20. If we had that implication (3.10) held for all $S, T \in \mathrm{Inv}^{(|U|)}\mathbf{A}$, where $S \subseteq T$, $(S{\restriction_U}, T{\restriction_U}) \in \mathrm{Cruc}^{(|U|)}(\mathbf{A}|_U)$ and $T{\restriction_U} \subseteq \left(\mathrm{pr}_U^A T_0\right){\restriction_U}$, then we had it for all $S, T \in \mathrm{Inv}^{(|U|)}\mathbf{A}$, where $S{\restriction_U}, T{\restriction_U} \subseteq \left(\mathrm{pr}_U^A T_0\right){\restriction_U}$. In particular, this implication were true for $S = \mathrm{pr}_U^A S_0$ and $T = \mathrm{pr}_U^A T_0$, yet this constitutes a contradiction to $\left(\mathrm{pr}_U^A S_0, \mathrm{pr}_U^A T_0\right)$-$q$-irreducibility guaranteed by item (c). Therefore, there exist invariants $S, T \in \mathrm{Inv}^{(|U|)}\mathbf{A}$ such that $S \subseteq T$, $(S{\restriction_U}, T{\restriction_U}) \in \mathrm{Cruc}^{(|U|)}(\mathbf{A}|_U)$, $T{\restriction_U} \subseteq \left(\mathrm{pr}_U^A T_0\right){\restriction_U} \subseteq \mathrm{pr}_U^A T_0$ where (3.10) fails, i.e. $(S, T) \in \mathrm{Sep}_{\mathbf{A}}(U) \setminus \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$. The latter condition describes precisely (S, T)-q-irreducibility of U and, besides, implies that the inclusion $S \subseteq T$ is proper. This establishes the truth of item (d).□

In the next section, we want to provide an internal method to construct non-refinable covers, that is, a way not based on iteratively refining covers as in Section 3.5. In order to do this, we will collect a few helpful properties of crucially and strictly q-irreducible neighbourhoods. A first, and very simple case to consider is that of q-irreducible algebras treated in the following lemma (cf. also Corollary 3.8.2 of [Beh09]).

3.6.17 Lemma. *For an algebra $\mathbf{A}$ and a quasiorder $q \subseteq (\mathrm{Neigh}\,\mathbf{A})^2$, such that $q \subseteq \leq_{\mathrm{cov}}$, we have the following facts: if $\mathbf{A}$ is q-irreducible, then $\{A\}$ is a q-non-refinable cover of $\mathbf{A}$. Moreover, if q has A as one of its largest elements, then $\mathbf{A}$ is q-irreducible if and only if $\{A\}$ is a q-refinement minimal set of neighbourhoods. If, in addition, q is an order relation, then $\mathbf{A}$ is irreducible if and only if $\{A\}$ is a q-refinement minimal set of neighbourhoods.*

Proof: First, let us suppose that $\mathbf{A}$ is a q-irreducible algebra. We are going to show that $\{A\}$ is then a q-non-refinable cover, proving in particular that it is a q-refinement minimal set of neighbourhoods (cp. Lemma 3.5.8). Of course, $\{A\}$ is a cover of $\mathbf{A}$, and it certainly forms an antichain w.r.t. q because it is a singleton collection. Now, by Lemma 3.5.4(g), every collection $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\{A\}$ is a cover of $\mathbf{A}$. According to Remark 3.5.18, the assumption of q-irreducibility directly yields $\{A\} \leq_{\mathrm{ref}}(q)\,\mathcal{V}$, showing that $\{A\}$ is q-refinement minimal. Via Lemma 3.5.8, we conclude that $\{A\}$ is q-non-refinable.

This proves already one direction of the equivalence stated at the end of the lemma. For the converse, we need the additional assumption that $V\;q\;A$ holds for all $V \in \mathrm{Neigh}\,\mathbf{A}$. We suppose now that $\{A\}$ is q-refinement minimal. In order to check q-irreducibility of $\mathbf{A}$, we consider any $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$. Under our assumption on q, this is equivalent to $\mathcal{V} \leq_{\mathrm{ref}}(q)\,\{A\}$ by Lemma 3.5.4(k). As $\{A\}$ is q-refinement minimal, we obtain $\{A\} \leq_{\mathrm{ref}}(q)\,\mathcal{V}$, which by Remark 3.5.18 is equivalent to q-irreducibility of $\mathbf{A}$.

Finally, if, in addition, q is an order relation, then Corollary 3.5.23 states that q-irreducibility of $\mathbf{A}$, that was characterised above, is equivalent to its irreducibility. $\square$

The following results show that strictly q-irreducible neighbourhoods, i.e. such being (S,T)-q-irreducible for some crucial pair (S,T), have stronger properties than just q-irreducible neighbourhoods. Specifically, the fact obtained in the corollary to the next lemma will become important for constructing q-non-refinable covers in the subsequent section.

3.6.18 Lemma. *Suppose that $\mathbf{A}$ is an algebra having the FIP and $q \subseteq \leq_{\mathrm{cov}}$ is a quasiorder on neighbourhoods of $\mathbf{A}$. For any crucial pair $(S,T) \in \mathrm{Cruc}(\mathbf{A})$ and any two (S,T)-q-irreducible neighbourhoods $U, V \in \mathrm{Neigh}\,\mathbf{A}$, given as images $U = \mathrm{im}\, e_U$ and $V = \mathrm{im}\, e_V$ of idempotents $e_U, e_V \in \mathrm{Idem}\,\mathbf{A}$, there exist operations $e_Z, e_W \in \mathrm{Idem}\,\mathbf{A}$, $F, G \in \mathrm{Clo}^{(1)}(\mathbf{A})$ such that $V \mathrel{q} \mathrm{im}\, e_W \subseteq V$, $U \mathrel{q} \mathrm{im}\, e_Z \subseteq U$ and the following equations hold*

$$
\begin{aligned}
e_Z \circ e_U &= e_Z & \qquad e_W \circ e_V &= e_W\\
e_V \circ F &= F & e_U \circ G &= G\\
F \circ G \circ e_V &= e_W & G \circ F \circ e_U &= e_Z.
\end{aligned}
$$

Proof: Let $m \in \mathbb{N}$, $(S,T) \in \mathrm{Cruc}^{(m)}(\mathbf{A})$ and assume that $e_U, e_V \in \mathrm{Idem}\,\mathbf{A}$ yield neighbourhoods $U := \mathrm{im}\, e_U$ and $V := \mathrm{im}\, e_V$ such that both are (S,T)-q-irreducible. Thus, both neighbourhoods separate the pair (S,T), and hence, there exist tuples $x_V \in T{\restriction_V} \setminus S$ and $y_U \in T{\restriction_U} \setminus S$. As $S \prec T$, we have $\langle x\rangle_{\mathbf{A}^m} = T$ for all $x \in T \setminus S$ because $\langle x\rangle_{\mathbf{A}^m} \subseteq T$ and the assumption $\langle x\rangle_{\mathbf{A}^m} \subset T$ would imply $\langle x\rangle_{\mathbf{A}^m} \subseteq S$, contradicting $x \notin S$. In particular, we can infer that $T = \langle x_V\rangle_{\mathbf{A}^m} = \langle y_U\rangle_{\mathbf{A}^m}$. So, as $x_V, y_U \in T$, we can find some $f, g \in \mathrm{Term}^{(1)}(\mathbf{A})$ such that $y_U = g \circ x_V$ and $x_V = f \circ y_U$. Thus, we have $y_U = e_U \circ y_U = e_U \circ g \circ x_V$ since $y_U \in U$, and similarly, $x_V = e_V \circ x_V = e_V \circ f \circ y_U$ due to $x_V \in V$. Therefore,

$$
\begin{aligned}
x_V &= e_V \circ f \circ y_U = e_V \circ f \circ e_U \circ g \circ x_V = e_V \circ f \circ e_U \circ g \circ e_V \circ x_V,\\
y_U &= e_U \circ g \circ x_V = e_U \circ g \circ e_V \circ f \circ y_U = e_U \circ g \circ e_V \circ f \circ e_U \circ y_U,
\end{aligned}
$$

and, hence, for all $n \in \mathbb{N}$

$$
\begin{aligned}
x_V &= (e_V \circ f \circ e_U \circ g \circ e_V)^n \circ x_V,\\
y_U &= (e_U \circ g \circ e_V \circ f \circ e_U)^n \circ y_U.
\end{aligned}
$$

Now, as $\mathbf{A}$ has the FIP, one can choose some integer $n \in \mathbb{N} \setminus \{0\}$ such that

$$
\begin{aligned}
e_W &:= (e_V \circ f \circ e_U \circ g \circ e_V)^n \in \mathrm{Clo}^{(1)}(\mathbf{A})\\
e_Z &:= (e_U \circ g \circ e_V \circ f \circ e_U)^n \in \mathrm{Clo}^{(1)}(\mathbf{A})
\end{aligned}
$$

are idempotent (cp. Remark 3.2.15). Obviously, we have $W := \operatorname{im} e_W \subseteq V$ and $Z := \operatorname{im} e_Z \subseteq U$ (see also Lemma 3.1.3). Assuming $S\restriction_W = T\restriction_W$ would yield

$$x_V = e_W \circ x_V \in T\restriction_W = S\restriction_W \subseteq S,$$

a contradiction to the choice of $x_V \in T\restriction_V \setminus S$. Thus, $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(W)$, and since we know $W \in \operatorname{Neigh} \mathbf{A}|_V$ and $(S,T) \notin \operatorname{Sep}_{\mathbf{A}}(\operatorname{Neigh} \mathbf{A}|_V \setminus \uparrow_q \{V\})$, it follows that $W \in \uparrow_q \{V\}$, i.e. $V \mathrel{q} W \subseteq V$. Likewise, as $y_U = e_Z \circ y_U \in T\restriction_Z$, one obtains $U \mathrel{q} Z \subseteq U$. Since $n > 0$ and e_V is idempotent, we get $e_W \circ e_V = e_W$, and similarly $e_Z \circ e_U = e_Z$ due to $e_U^2 = e_U$. To demonstrate the remaining equalities, let us define operations $F, G \in \operatorname{Clo}^{(1)}(\mathbf{A})$ by $F := (e_V \circ f \circ e_U \circ g \circ e_V)^{n-1} \circ e_V \circ f \circ e_U$ and $G := e_U \circ g \circ e_V$. It is again clear (Lemma 3.1.3) that $\operatorname{im} F \subseteq \operatorname{im} e_V = V$ and $\operatorname{im} G \subseteq \operatorname{im} e_U = U$, i.e. $e_V \circ F = F$ and $e_U \circ G = G$. Now a straightforward induction, using $e_U^2 = e_U$ and $e_V^2 = e_V$, proves that

$$e_U \circ g \circ e_V \circ (e_V \circ f \circ e_U \circ g \circ e_V)^{k-1} \circ e_V \circ f \circ e_U = (e_U \circ g \circ e_V \circ f \circ e_U)^k$$

holds for all $k \in \mathbb{N}_+$. Hence, exploiting once more $e_U^2 = e_U$ and $e_V^2 = e_V$, we have

$$\begin{aligned} F \circ G \circ e_V &= (e_V \circ f \circ e_U \circ g \circ e_V)^{n-1} \circ e_V \circ f \circ e_U \circ e_U \circ g \circ e_V \circ e_V \\ &= (e_V \circ f \circ e_U \circ g \circ e_V)^n = e_W, \end{aligned}$$

and, via the equality obtained by induction, we can infer

$$\begin{aligned} G \circ F \circ e_U &= e_U \circ g \circ e_V \circ (e_V \circ f \circ e_U \circ g \circ e_V)^{n-1} \circ e_V \circ f \circ e_U \circ e_U \\ &= (e_U \circ g \circ e_V \circ f \circ e_U)^n = e_Z \end{aligned}$$

as desired. □

Using this lemma, we can derive the following three conclusions. We remark that the last statement in the third of them has already been proven in Lemma 3.7.4 of [Beh09] for the case of finite algebras.

3.6.19 Corollary. *For an algebra $\mathbf{A}$ having the FIP, a quasiorder $q \subseteq \leq_{\text{cov}}$ on neighbourhoods of $\mathbf{A}$ and any crucial pair $(S,T) \in \operatorname{Cruc}(\mathbf{A})$ the following implications hold:*

(a) Any two (S,T)-q-irreducible neighbourhoods $U, V \in \operatorname{Neigh} \mathbf{A}$ fulfil $U \mathrel{\tilde{q} \cap \tilde{q}^{-1}} V$ where $\tilde{q} = q \vee \precapprox$ is the join of the quasiorders q and $\precapprox$.

(b) If for all $U, V \in \operatorname{Neigh} \mathbf{A}$ the condition $U \mathrel{q} V \subseteq U$ implies $U = V$, then any two (S,T)-q-irreducible neighbourhoods of $\mathbf{A}$ are isomorphic.

(c) If $q \subseteq \precapprox$, then any two (S,T)-q-irreducible neighbourhoods of $\mathbf{A}$ are isomorphic. Especially[39], any two (S,T)-irreducible neighbourhoods of $\mathbf{A}$ are isomorphic.

[39] and equivalently

Proof: (a) Suppose that $e_U, e_V \in \operatorname{Idem} \mathbf{A}$ are idempotents describing the given (S,T)-q-irreducible neighbourhoods $U = \operatorname{im} e_U$ and $V = \operatorname{im} e_V$. Furthermore, let $e_Z, e_W \in \operatorname{Idem} \mathbf{A}$, $F, G \in \operatorname{Clo}^{(1)}(\mathbf{A})$ be operations satisfying the conditions claimed by Lemma 3.6.18 and define $f, g \in \operatorname{Clo}^{(1)}(\mathbf{A})$ by $f := e_W \circ F$ and $g := G \circ e_V$. The equalities

$$\begin{aligned} e_W \circ f &= e_W \circ e_W \circ F = e_W \circ F = f \\ e_U \circ g &= e_U \circ G \circ e_V = G \circ e_V = g \\ f \circ g \circ e_W &= (e_W \circ F) \circ (G \circ e_V) \circ e_W = e_W \circ (F \circ G \circ e_V) \circ e_W \\ &= e_W \circ e_W \circ e_W = e_W \end{aligned}$$

are an immediate consequence of those listed in Lemma 3.6.18. An application of Proposition 3.2.10(f) now shows that $W \precapprox U$. From the conclusions of Lemma 3.6.18, we additionally get $V \mathrel{q} W \precapprox U$, so $V \mathrel{\tilde{q}} W \mathrel{\tilde{q}} U$ since $q, \precapprox \subseteq q \vee \precapprox = \tilde{q}$, and finally $V \mathrel{\tilde{q}} U$ due to transitivity of $\tilde{q}$. As the assumptions we made here were symmetric in U and V, we also obtain $V \mathrel{\tilde{q}} U$, such that we can conclude $U \mathrel{\tilde{q} \cap \tilde{q}^{-1}} V$.

(b) Again let $e_U, e_V \in \operatorname{Idem} \mathbf{A}$ be such that the images $U = \operatorname{im} e_U$ and $V = \operatorname{im} e_V$ are (S,T)-q-irreducible, and fetch $e_Z, e_W \in \operatorname{Idem} \mathbf{A}$, $F, G \in \operatorname{Clo}^{(1)}(\mathbf{A})$ fulfilling the conclusions of Lemma 3.6.18. Among them, we have $V \mathrel{q} W \subseteq V$ and $U \mathrel{q} Z \subseteq U$, which under our assumptions imply $W = V$ and $Z = U$. Now, using the equalities $e_W \circ e_V = e_W$ and $e_Z \circ e_U = e_Z$ we got from Lemma 3.6.18, together with $W = V$, $Z = U$ and Lemma 3.1.3, we can infer the equalities $e_W = e_W \circ e_V = e_V$ and $e_Z = e_Z \circ e_U = e_U$. Combining these with the remaining equalities from the previous lemma, we obtain

$$\begin{aligned} e_V \circ F &= F & e_U \circ G &= G \\ F \circ G \circ e_V &= e_W = e_V & G \circ F \circ e_U &= e_Z = e_U. \end{aligned}$$

According to Proposition 3.2.8(e), these allow us to infer $U \cong V$ as desired.

(c) We assume here that $q \subseteq \precapprox$, so any two neighbourhoods $U, V \in \operatorname{Neigh} \mathbf{A}$ subject to $U \mathrel{q} V \subseteq U$ will satisfy $U \precapprox V \subseteq U$. Since $\mathbf{A}$ has the FIP, it is neighbourhood self-embedding simple (see Lemma 3.2.18(b)), and consequently, we can infer $U = V$. This establishes the additional condition needed to apply item (b), in order to derive the desired conclusion.

The last statement about (S,T)-irreducible neighbourhoods is just the special case where q is set inclusion, which is clearly contained in the neighbourhood embedding relation. We should not here that this case is indeed the only one, since for neighbourhood self-embedding simple algebras and $q \subseteq \precapprox$, a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ is (S,T)-q-irreducible if and only if it is (S,T)-irreducible (cp. Lemma 3.6.11(d)). □

The next proposition shows how (S,T)-q-irreducible neighbourhoods can be used to explicitly describe covers of algebras.

3.6.20 Proposition. *Let* $\mathbf{A}$ *be an algebra,* $(\operatorname{Neigh}\mathbf{A}, q)$ *a quasiordered set whose canonically associated poset satisfies DCC, and suppose* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$. *Consider a subset*[40] $\mathcal{C} \subseteq \operatorname{Sep}_{\mathbf{A}}(A)$ *such that* $\operatorname{Cov}(\mathbf{A}) = \{\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A} \mid \mathcal{C} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})\}$, *then*

$$\{V \in \operatorname{Neigh}\mathbf{A} \mid \exists (S,T) \in \mathcal{C}\colon V\ (S,T)\text{-}q\text{-}\textit{irreducible}\} \in \operatorname{Cov}(\mathbf{A}).$$

Proof: Let us abbreviate

$$\mathcal{V} := \{V \in \operatorname{Neigh}\mathbf{A} \mid \exists (S,T) \in \mathcal{C}\colon V\ (S,T)\text{-}q\text{-irreducible}\} \subseteq \operatorname{Neigh}\mathbf{A}.$$

We have assumed that it suffices to verify the inclusion $\mathcal{C} \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$ in order to demonstrate $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$. Consequently, let us consider an arbitrary pair $(S,T) \in \mathcal{C}$. As $\mathcal{C} \subseteq \operatorname{Sep}_{\mathbf{A}}(A)$, the set $\mathcal{S} := \{V \in \operatorname{Neigh}\mathbf{A} \mid (S,T) \in \operatorname{Sep}_{\mathbf{A}}(V)\}$ is non-empty. Using the chain condition assumed for the quasiorder q, we can now choose a neighbourhood $W \in \operatorname{Min}(\mathcal{S}, q{\restriction}_{\mathcal{S}})$. Since q extends set inclusion, according to item (e) of Remark 3.6.10, we get that W is (S,T)-q-irreducible. Yet, by definition of $\mathcal{V}$ this says exactly that W belongs to $\mathcal{V}$. As (S,T)-q-irreducibility of W moreover implies $(S,T) \in \operatorname{Sep}_{\mathbf{A}}(W) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$, all of a sudden, the proof is finished. □

The condition assumed for the set $\mathcal{C}$ in the preceding proposition makes this result still quite abstract. In order to obtain an applicable instance, we assume that our algebra is poly-Artinian.

3.6.21 Corollary. *Suppose that* $\mathbf{A}$ *is an algebra and* $(\operatorname{Neigh}\mathbf{A}, q)$ *a quasiordered set whose factor poset satisfies DCC, and assume* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$. *If* $\mathbf{A}$ *is poly-Artinian, then the following three collections are covers of* $\mathbf{A}$*:*

$$\operatorname{Irr}_q^{**}(\mathbf{A}) \subseteq \operatorname{Irr}_q^{*}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A}).$$

Moreover, if $\mathbf{A}$ *is finite, then we can exhibit smaller subsets as covers, namely*

$$\operatorname{Irr}_q^{**(|A|)}(\mathbf{A}) \subseteq \operatorname{Irr}_q^{*(|A|)}(\mathbf{A}) \subseteq \operatorname{Irr}_q(\mathbf{A}).$$

Proof: Due to Lemma 3.4.9(a) it suffices of course to verify that the respective smallest collection covers $\mathbf{A}$, i.e. $\operatorname{Irr}_q^{**}(\mathbf{A}) \in \operatorname{Cov}(\mathbf{A})$, and $\operatorname{Irr}_q^{**(|A|)}(\mathbf{A}) \in \operatorname{Cov}(\mathbf{A})$ for finite $\mathbf{A}$. We do this by exhibiting suitable sets $\mathcal{C}, \mathcal{C}' \subseteq \operatorname{Sep}_{\mathbf{A}}(A)$ to be used in the previous lemma. In the general case,

$$\mathcal{C} := \left\{(S,T) \in \operatorname{Cruc}(\mathbf{A}) \;\middle|\; \exists X \subseteq A\colon |X| < \aleph_0 \wedge T \subseteq \operatorname{pr}_X^A \operatorname{Clo}^{(1)}(\mathbf{A})\right\}$$

does the job by Corollary 3.4.35(p) and the assumption that $\mathbf{A}$ is poly-Artinian. For finite algebras $\mathbf{A}$, we may use

$$\mathcal{C}' := \left\{(S,T) \in \operatorname{Cruc}^{(|A|)}(\mathbf{A}) \;\middle|\; T \subseteq \operatorname{Clo}^{(1)}(\mathbf{A})\right\}$$

[40] Such subsets exist since by Lemma 3.4.4(b), a collection $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ is a cover of A if and only if $\operatorname{Sep}_{\mathbf{A}}(A) \subseteq \operatorname{Sep}_{\mathbf{A}}(\mathcal{V})$.

due to Corollary 3.4.36(f). It is straightforward to check with Definition 3.6.6 that the collections of neighbourhoods defined in Proposition 3.6.20 via these collections of pairs are indeed $\mathrm{Irr}^{**}_q(\mathbf{A})$ and $\mathrm{Irr}^{**(|A|)}_q(\mathbf{A})$, respectively.

We remark that again we have left indexing bijections between X and $|X|$ (and A and $|A|$) implicit in the definitions of $\mathcal{C}$ and $\mathcal{C}'$. □

Under the assumptions of the previous corollary, we hence obtain that crucially and strictly q-irreducible neighbourhoods form a subcollection of indecomposables (q-irreducible neighbourhoods) that still solve the completeness problem, at least for the class of poly-Artinian algebras, including e.g. all finite ones. Consequently, we have accomplished one step that, at the beginning of this section, we have argued to be important in order to achieve a positive answer to the uniqueness problem w.r.t. decomposition via covers.

The last proposition of this section establishes that strictly q-irreducible neighbourhoods somehow form the cores of all covers of algebras satisfying FIP. We remark that precursors of this and the previous result, namely the special case of q being set inclusion and $\mathbf{A}$ being a finite algebra, have been obtained in close collaboration with Friedrich Martin Schneider and presented at the 82. Arbeitstagung Allgemeine Algebra (82nd Workshop on General Algebra) in Potsdam, on 24th June, 2011.

3.6.22 Proposition. *Let $\mathbf{A}$ be an algebra having the FIP and $q \subseteq (\mathrm{Neigh}\,\mathbf{A})^2$ a quasiorder on neighbourhoods of $\mathbf{A}$ such that $\subseteq_{\mathrm{Neigh}\,\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$ and the factor poset associated with $(\mathrm{Neigh}\,\mathbf{A}, q)$ satisfies DCC. An arbitrary cover $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$ has to satisfy the following condition w.r.t. strictly q-irreducible neighbourhoods:*

$$\forall\, U \in \mathrm{Irr}^*_q(\mathbf{A})\ \exists\, V \in \mathcal{V}\colon \quad U\,(q \vee \precapprox)\,V.$$

In view of Lemma 3.6.11(d) and the fact that the FIP implies neighbourhood self-embedding simplicity (see Lemma 3.2.18(b)), for $q \subseteq \precapprox$ the statement of the proposition slightly collapses because $\mathrm{Irr}^*_q(\mathbf{A}) = \mathrm{Irr}^*(\mathbf{A})$. It then reduces to the claim that every strictly irreducible neighbourhood embeds into some neighbourhood of any cover of $\mathbf{A}$, provided that $(\mathrm{Neigh}\,\mathbf{A}, \subseteq)$ satisfies DCC.

Proof: Consider an arbitrary neighbourhood $U \in \mathrm{Irr}^*_q(\mathbf{A})$. By definition there exists a pair $(S,T) \in \mathrm{Cruc}(\mathbf{A})$ such that U is (S,T)-q-irreducible. Since $\mathcal{V} \in \mathrm{Cov}(\mathbf{A})$, we have $A \leq_{\mathrm{cov}} \mathcal{V}$, i.e., using Lemma 3.4.4(b), $\mathrm{Sep}_{\mathbf{A}}(A) \subseteq \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$. Combining this with $(S,T) \in \mathrm{Cruc}(\mathbf{A}) \subseteq \mathrm{Sep}_{\mathbf{A}}(A)$, we get $(S,T) \in \mathrm{Sep}_{\mathbf{A}}(\mathcal{V})$, that is to say, $(S,T) \in \mathrm{Sep}_{\mathbf{A}}(V)$ for some neighbourhood $V \in \mathcal{V}$. Therefore, the collection $\mathcal{S} := \left\{\tilde{V} \in \mathrm{Neigh}\,\mathbf{A} \;\middle|\; \tilde{V}\ q\ V \wedge (S,T) \in \mathrm{Sep}_{\mathbf{A}}\left(\tilde{V}\right)\right\}$ is non-empty. According to the assumed chain condition for $(\mathrm{Neigh}\,\mathbf{A}, q)$, we may now choose some $W \in \mathrm{Min}(\mathcal{S}, q{\restriction_{\mathcal{S}}})$. Due to transitivity of q, it follows that the set W also belongs to $\mathrm{Min}(\mathcal{S}', q{\restriction_{\mathcal{S}'}})$, where $\mathcal{S}' := \left\{\tilde{V} \in \mathrm{Neigh}\,\mathbf{A} \;\middle|\; (S,T) \in \mathrm{Sep}_{\mathbf{A}}\left(\tilde{V}\right)\right\} \supseteq \mathcal{S}$. Applying Remark 3.6.10(e) we conclude that W also is (S,T)-q-irreducible, whence item (a) of Corollary 3.6.19 implies $U\,(q \vee \precapprox)\,W$. Since $W \in \mathcal{S}$, we have $W\ q\ V$, and thus $W\,(q \vee \precapprox)\,V$. By transitivity of $q \vee \precapprox$ we finally get $U\,(q \vee \precapprox)\,V$. □

3.7 Intrinsic description of non-refinable covers

It is the aim of this section to give, under some mild assumptions on the algebra, an intrinsic construction of q-non-refinable covers for certain quasiorders q on the set of neighbourhoods. At the same time we are going to show a sort of uniqueness result for q-non-refinable covers, which thus answers the third problem posed in the beginning of Section 3.6.

Motivated by the results 3.6.20 through 3.6.22, we intend to educe the constructive approach towards q-non-refinable covers on a conceptual level, rather than sticking with the mentioned concrete propositions and the specific collections of neighbourhoods contained therein. We believe that in doing so, the properties really needed to obtain a unique decomposition via covers become more evident and that our proofs become tidier. Hence, we pose the following definition.

3.7.1 Definition. For an algebra $\mathbf{A}$ and a quasiorder $q \subseteq \leq_{\mathrm{cov}}$ we say that a collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ forms a *cover prebase of* $\mathbf{A}$ *w.r.t.* q (a *q-cover prebase* for short), if we have $\mathcal{V} \sqsubseteq (q)\, \mathcal{U}$ for every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$. A set $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ is called a *cover base of* $\mathbf{A}$ *w.r.t.* q (or *q-cover base of* $\mathbf{A}$) if $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ and it is a cover prebase w.r.t. q. Moreover, a q-cover base is said to be an *irredundant q-cover base* if it is irredundant as a cover of $\mathbf{A}$. ◊

As a consequence of the results established at the end of the previous section, we obtain that under certain assumptions, the defined concepts of q-cover prebase and q-cover base are actually non-void, and moreover, we have non-trivial collections of neighbourhoods forming concrete instances.

3.7.2 Corollary. *Let* $\mathbf{A}$ *be an algebra satisfying FIP and suppose* $(\operatorname{Neigh} \mathbf{A}, q)$ *is a quasiordered set whose canonically associated poset fulfils DCC, and we have the inclusions* $\subseteq_{\operatorname{Neigh} \mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$. *It then follows that the collections* $\operatorname{Irr}_q^{**}(\mathbf{A})$ *and* $\operatorname{Irr}_q^{*}(\mathbf{A})$ *are* $(q \vee \precapprox)$*-cover prebases of* $\mathbf{A}$. *Moreover, if* $\mathbf{A}$ *also is poly-*Artin*ian, then these sets form* $(q \vee \precapprox)$*-cover bases.*

Furthermore, for any finite algebra $\mathbf{A}$ *and any quasiorder* $\subseteq_{\operatorname{Neigh} \mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$ *on its set of neighbourhoods, the collections* $\operatorname{Irr}_q^{**(|A|)}(\mathbf{A})$ *and* $\operatorname{Irr}_q^{*(|A|)}(\mathbf{A})$ *are examples of* $(q \vee \precapprox)$*-cover bases.*

Proof: All collections $\mathcal{U}$ of neighbourhoods listed in the corollary are actually subsets of all strictly q-irreducible neighbourhoods $\operatorname{Irr}_q^{*}(\mathbf{A})$. The given assumptions guarantee furthermore, that Proposition 3.6.22 is applicable to them. Its conclusion says that for every cover $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$, we have the property that every $U \in \mathcal{U}$ satisfies $U\,(q \vee \precapprox)\,V$ for some $V \in \mathcal{V}$. This means precisely $\mathcal{U} \sqsubseteq (q \vee \precapprox)\, \mathcal{V}$ for all $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$, and consequently shows that $\mathcal{U}$ is a $(q \vee \precapprox)$-cover prebase of $\mathbf{A}$.

If we know in addition, that the algebra $\mathbf{A}$ is poly-Artinian, then, by Corollary 3.6.21 the mentioned sets $\mathcal{U}$ also form covers of $\mathbf{A}$, whence they are actually $(q \vee \precapprox)$-cover bases.

Furthermore, for a finite algebra, the chain condition on the quasiorder q and poly-ARTINianness are trivially fulfilled, and Corollary 3.6.21 tells us that $\mathcal{U}$ can be chosen in $\left\{\operatorname{Irr}_q^{**(|A|)}(\mathbf{A}), \operatorname{Irr}_q^{*(|A|)}(\mathbf{A})\right\}$ without making the previous arguments false. Thus, these two collections present examples of $(q \vee \precapprox)$-cover bases for finite algebras. □

In the proof of the corollary we already used the fact that subcollections of q-cover prebases are again q-cover prebases. Easy observations of this kind are collected in the following lemma. The last item also provides a characterisation of q-cover bases in terms of the q-refinement relation.

3.7.3 Lemma. *For an algebra $\mathbf{A}$, quasiorders $q, q' \subseteq \leq_{\mathrm{cov}}$ on $\operatorname{Neigh}\mathbf{A}$ and a collection $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ the following holds:*

(a) If $q \subseteq q'$ and $\mathcal{V}$ is a q-cover prebase (base, respectively) of $\mathbf{A}$, then it is also a q'-cover prebase (base, respectively).

(b) If $\mathcal{V}$ is a q-cover prebase and $\mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ satisfies $\mathcal{W} \sqsubseteq (q')\, \mathcal{V}$, then $\mathcal{W}$ is a $(q \vee q')$-cover prebase.

Especially, if $\mathcal{V}$ is a q-cover prebase and $\mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ satisfies $\mathcal{W} \sqsubseteq (q)\, \mathcal{V}$, then $\mathcal{W}$ is a q-cover prebase, too.

In particular, every q-refinement $\mathcal{W} \leq_{\mathrm{ref}} (q)\, \mathcal{V}$ and every subset $\mathcal{W} \subseteq \mathcal{V}$ of a q-cover prebase $\mathcal{V}$ is again a q-cover prebase.

(c) Every q'-refinement $\mathcal{W} \leq_{\mathrm{ref}} (q')\, \mathcal{V}$ of a q-cover base $\mathcal{V}$, is a $(q \vee q')$-cover base. In particular every q-refinement of a q-cover base is again a q-cover base of $\mathbf{A}$.

(d) Every cover $\mathcal{W} \in \operatorname{Cov}(\mathbf{A})$ satisfying $\mathcal{W} \sqsubseteq (q)\, \mathcal{V}$ w.r.t. a q-cover prebase $\mathcal{V}$, especially every subset $\mathcal{W} \subseteq \mathcal{V}$, $\mathcal{W} \in \operatorname{Cov}(\mathbf{A})$, forms a q-cover base of $\mathbf{A}$.

(e) The collection $\mathcal{V}$ is a q-cover base if and only if we have $\mathcal{V} \leq_{\mathrm{ref}} (q)\, \mathcal{U}$ for all $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$.

Proof: (a) If $q \subseteq q'$, then Lemma 3.5.4(b) implies that $\sqsubseteq (q) \subseteq \sqsubseteq (q')$. Hence, provided that $\mathcal{V}$ is a q-cover (pre)base, for every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ we have $\mathcal{V} \sqsubseteq (q)\, \mathcal{U}$, implying $\mathcal{V} \sqsubseteq (q')\, \mathcal{U}$. Therefore, $\mathcal{V}$ is also a q'-cover (pre)base.

(b) If $\mathcal{V}$ is a q-cover prebase and $\mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ fulfils $\mathcal{W} \sqsubseteq (q')\, \mathcal{V}$, then for every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ we have $\mathcal{W} \sqsubseteq (q')\, \mathcal{V} \sqsubseteq (q)\, \mathcal{U}$, which, by Lemma 3.5.4(b), implies $\mathcal{W} \sqsubseteq (q \vee q')\, \mathcal{U}$. Thus, we have shown that $\mathcal{W}$ is a $(q \vee q')$-cover prebase of $\mathbf{A}$.

The remaining assertions follow by letting $q = q'$, whence every $\mathcal{W} \sqsubseteq (q)\, \mathcal{V}$ where $\mathcal{V}$ is a q-cover prebase, also is a q-cover prebase. Especially, every q-refinement $\mathcal{W} \leq_{\mathrm{ref}} (q)\, \mathcal{V}$ and every subset $\mathcal{W} \subseteq \mathcal{V}$ of a q-cover prebase $\mathcal{V}$ fulfils the precondition $\mathcal{W} \sqsubseteq (q)\, \mathcal{V}$, and therefore the conclusion that it is a q-cover prebase, as well.

(c) If $\mathcal{V}$ is a q-cover base, and $\mathcal{W} \leq_{\mathrm{ref}} (q') \, \mathcal{V}$, then we have $\mathcal{W} \sqsubseteq (q') \, \mathcal{V}$ for the q-cover prebase $\mathcal{V}$ and can thus apply the previous item. In conclusion, we get that $\mathcal{W}$ is a $(q \vee q')$-cover prebase of $\mathbf{A}$. Moreover, since $\mathcal{V}$, as a q-cover base, was assumed to be a cover of $\mathbf{A}$ and $\mathcal{W} \leq_{\mathrm{ref}} (q) \, \mathcal{V}$, Lemma 3.5.4(g) implies that $\mathcal{W}$ is a cover, too, and therefore a $(q \vee q')$-cover base of $\mathbf{A}$.

(d) If we have $\mathcal{W} \sqsubseteq (q) \, \mathcal{V}$, in particular, if $\mathcal{W} \subseteq \mathcal{V}$, and $\mathcal{V}$ is a q-cover prebase, then according to item (b) the collection $\mathcal{W}$ is another q-cover prebase. Since we have assumed here in addition that $\mathcal{W}$ covers A, we can infer indeed that $\mathcal{W}$ forms a q-cover base of $\mathbf{A}$.

(e) Suppose that $\mathcal{V}$ is a q-cover base of $\mathbf{A}$, then we have $\mathcal{V} \sqsubseteq (q) \, \mathcal{U}$ for every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ since $\mathcal{V}$ is in particular a q-cover prebase of $\mathbf{A}$. Using Lemma 3.5.4(h) together with the fact that the q-cover base $\mathcal{V}$ is in particular a cover of $\mathbf{A}$, we get that $\mathcal{V} \sqsubseteq (q) \, \mathcal{U}$ is equivalent to $\mathcal{V} \leq_{\mathrm{ref}} (q) \, \mathcal{U}$, for every $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$.

For the converse, let us suppose that $\mathcal{V}$ q-refines every cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$. As $\mathcal{U} := \{A\}$ actually is a cover of $\mathbf{A}$, this universally quantified statement is not vacuously true, whence $\mathcal{V}$ as a q-refinement of a cover is again a cover of $\mathbf{A}$ (see Lemma 3.5.4(g)). Moreover, by definition of q-refinement, $\mathcal{V} \leq_{\mathrm{ref}} (q) \, \mathcal{U}$ for all covers $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ implies $\mathcal{V} \sqsubseteq (q) \, \mathcal{U}$ for all covers, i.e. that $\mathcal{V}$ is a q-cover prebase of $\mathbf{A}$. This completes the proof that $\mathcal{V}$ is a q-cover base. $\square$

As a consequence of Lemma 3.5.6, we obtain the following possibilities to simplify given q-cover bases:

3.7.4 Corollary. *Let $\mathbf{A}$ be an algebra, $q \subseteq \leq_{\mathrm{cov}}$ a quasiorder on neighbourhoods, $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ a q-cover prebase, $\theta \subseteq \leq_{\mathrm{cov}} \cap \, \mathcal{U}^2$ an equivalence relation on $\mathcal{U}$ and $\tilde{q} \subseteq \leq_{\mathrm{cov}} \cap \, \mathcal{U}^2$ a quasiorder on $\mathcal{U}$ such that the canonically associated poset on $\mathcal{U}/\tilde{q} \cap \tilde{q}^{-1}$ has ACC. For $\mathcal{V} \subseteq \mathcal{U}$ being any one of the subsequent three subsets*

(i) a transversal of $\mathcal{V}/\theta$,

(ii) the set $\operatorname{Max}(\mathcal{U}, \tilde{q})$ of all $\tilde{q}$-maximal neighbourhoods in $\mathcal{U}$,

(iii) a set of $\tilde{q} \cap \tilde{q}^{-1}$representatives of the $\tilde{q}$-maximal neighbourhoods in $\mathcal{U}$,

the following facts are equivalent:

(a) $\mathcal{V}$ is a q-cover base of $\mathbf{A}$.

(b) $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$.

(c) $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$.

(d) $\mathcal{U}$ is a q-cover base of $\mathbf{A}$.

The most useful statement of this corollary certainly is the implication “(d) $\Rightarrow$ (a)”.

Proof: The three subcollections $\mathcal{V} \subseteq \mathcal{U}$ mentioned in items (i), (ii) and (iii) correspond, in this order, to the subsets treated in items (b) through (d) of Lemma 3.5.6. The only conclusion from this lemma we use here is that $\mathcal{V}$ covers $\mathbf{A}$ if and only if $\mathcal{U}$ does this. Under the assumption that $\mathcal{U}$ is a q-cover prebase, the latter is of course equivalent to $\mathcal{U}$ being a q-cover base. Moreover, if $\mathcal{V}$ is a cover, then, as a subset of the q-cover prebase $\mathcal{U}$, it is a q-cover base (see Lemma 3.7.3(d)). Conversely, every q-cover base is a cover of the underlying algebra by definition. $\square$

The previous result allows us to construct irredundant q-cover bases out of known q-cover bases.

3.7.5 Proposition. *Let* $\mathbf{A}$ *be an algebra and* $q \subseteq \leq_{\mathrm{cov}}$ *a quasiorder on* $\operatorname{Neigh}\mathbf{A}$*,* $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ *a* q*-cover base of* $\mathbf{A}$ *and* $(\mathcal{V}, \tilde{q})$ *a quasiordered set such that* $\tilde{q} \subseteq \leq_{\mathrm{cov}}$ *and* $q{\restriction_{\mathcal{V}}} \subseteq \tilde{q}$*. Furthermore, let* $\mathcal{U} \subseteq \operatorname{Max}(\mathcal{V}, \tilde{q})$ *be a subcollection still covering*[41] $\mathbf{A}$*,* $\theta \subseteq \leq_{\mathrm{cov}} \cap\, \mathcal{U}^2$ *an equivalence relation on* $\mathcal{U}$ *satisfying* $(\tilde{q} \cap \tilde{q}^{-1}){\restriction_{\mathcal{U}}} \subseteq \theta$ *and* $\mathcal{V}' \subseteq \mathcal{U}$ *be a* θ*-transversal of* $\mathcal{U}$*. Then it follows that* $\mathcal{V}' \subseteq \mathcal{V}$ *is an irredundant* q*-cover base of* $\mathbf{A}$*, especially, it is an irredundant cover of* $\mathbf{A}$*.*

Proof: First, we argue that $\mathcal{V}'$, which by construction is a subset of $\mathcal{V}$, remains a q-cover base of $\mathbf{A}$. First, we note that the cover $\mathcal{U} \subseteq \operatorname{Max}(\mathcal{V}, \tilde{q}) \subseteq \mathcal{V}$ is a q-cover base due to Lemma 3.7.3(d) and $\mathcal{V}$ being a q-cover (pre)base. Second, employing construction (i) of Corollary 3.7.4, we obtain that $\mathcal{V}' \subseteq \mathcal{U}$ is a q-cover base of $\mathbf{A}$, as claimed. In particular, it is a cover of $\mathbf{A}$.

The interesting part of the proof is to show that this collection is indeed irredundant. Assume, it were not. Then, according to Lemma 3.5.9(a), we had some $U \in \mathcal{V}'$ such that $\mathcal{V}'' := \mathcal{V}' \setminus \{U\}$ still belonged to $\operatorname{Cov}(\mathbf{A})$. Therefore, we got $\mathcal{V}' \sqsubseteq (q)\, \mathcal{V}''$ as $\mathcal{V}'$ was a q-cover prebase of $\mathbf{A}$. Since $U \in \mathcal{V}'$, this meant, we could find some $V \in \mathcal{V}''$ such that $U\; q\; V$, implying $U\; \tilde{q}\; V$ by assumption and the fact that U and V both belonged to $\mathcal{V}' \subseteq \mathcal{U} \subseteq \mathcal{V}$. Now V was a member of $\mathcal{V}$ and $U \in \mathcal{U} \subseteq \operatorname{Max}(\mathcal{V}, \tilde{q})$, so the condition $U\; \tilde{q}\; V$ implied $U\; \tilde{q}\; V\; \tilde{q}\; U$, i.e. $U\; \tilde{q} \cap \tilde{q}^{-1}\; V$. Since both neighbourhoods belonged to $\mathcal{U}$, we had $U\; \theta\; V$ by assumption. Combining this with $U, V \in \mathcal{V}'$ and the fact that $\mathcal{V}' \subseteq \mathcal{U}$ was a system of representatives of θ-equivalence classes, we had to have $U = V \in \mathcal{V}'' = \mathcal{V} \setminus \{U\}$, an obvious contradiction. $\square$

We remark that this proposition is the first result to provide a way to obtain irredundant covers without relying on the definition as a minimal subcollection of a cover w.r.t. set inclusion, regarding the property of being a cover of $\mathbf{A}$. That is to say, we have here a sketch of an algorithm not just removing neighbourhoods from a cover until one cannot continue without violating the cover property. This has been bargained of course by using a stronger assumption, i.e. the method presented in Proposition 3.7.5 does not allow us to reduce all covers to irredundancy, but is only applicable to q-cover bases of $\mathbf{A}$. Yet, we will see that this actually suffices to construct q-non-refinable covers.

[41] If the factor poset associated with $(\mathcal{V}, \tilde{q})$ satisfies ACC, one may, for instance, take $\mathcal{U} = \operatorname{Max}(\mathcal{V}, \tilde{q})$ due to Lemma 3.5.6(c).

First, however, we remove a few of the technical assumptions from Proposition 3.7.5 by specialising some of the general conditions.

3.7.6 Corollary. *Let* $\mathbf{A}$ *be an algebra,* $(\operatorname{Neigh}\mathbf{A}, q)$ *be a quasiordered set such that* $q \subseteq \leq_{\mathrm{cov}}$*,* $\mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ *a* q*-cover base of* $\mathbf{A}$ *and* $\mathcal{U} \subseteq \operatorname{Max}(\mathcal{V}, q{\restriction_{\mathcal{V}}})$ *be a subcollection still covering* $\mathbf{A}$*. Now any* $(q \cap q^{-1})$*-transversal* $\mathcal{V}' \subseteq \mathcal{U}$ *yields an irredundant* q*-cover base of* $\mathbf{A}$*, in particular an irredundant cover.*

Proof: We let $\tilde{q} := q{\restriction_{\mathcal{V}}}$ and $\theta := (q \cap q^{-1}){\restriction_{\mathcal{U}}}$ in the assumptions of the previous Proposition 3.7.5. Then clearly, $\tilde{q} \subseteq q \subseteq \leq_{\mathrm{cov}}$, and furthermore, $\theta \subseteq q{\restriction_{\mathcal{U}}} \subseteq \leq_{\mathrm{cov}}{\restriction_{\mathcal{U}}}$ is an equivalence relation on $\mathcal{U}$, satisfying $(\tilde{q} \cap \tilde{q}^{-1}){\restriction_{\mathcal{U}}} = \theta$. Moreover, $\mathcal{V}' \subseteq \mathcal{U}$ really is a θ-transversal of $\mathcal{U}$ as required in the proposition above, and thus an irredundant q-cover base of $\mathbf{A}$. □

Proposition 3.7.5 constitutes the last ingredient needed to show existence of q-non-refinable covers. In order to also prove a corresponding uniqueness result, we first need to clarify what notion of uniqueness we want to apply. To approach this matter, with the following definition we extend what was defined in 3.4.8 to arbitrary quasiorders on neighbourhoods.

3.7.7 Definition. For any quasiorder $q \subseteq \leq_{\mathrm{cov}}$ on the set of neighbourhoods of an algebra $\mathbf{A}$, we call a mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ between two collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh}\mathbf{A}$ of neighbourhoods a *q-morphism* if $U \; q \; \varphi(U)$ holds for all $U \in \mathcal{U}$. ◇

It is evident that letting $q = \precapprox$ in Definition 3.7.7 yields precisely the notion of weak embedding proposed in Definition 3.4.8.

Next, we see that for any algebra $\mathbf{A}$ and every quasiorder $q \subseteq \leq_{\mathrm{cov}}$, the previous definition induces a small category on the powerset of $\operatorname{Neigh}\mathbf{A}$.

3.7.8 Remark. Assume $\mathbf{A}$ is an algebra and $q \subseteq \leq_{\mathrm{cov}}$ is a fixed quasiorder on its set of neighbourhoods. Using transitivity of q, it is easy to see that for collections $\mathcal{U}, \mathcal{V}, \mathcal{W} \subseteq \operatorname{Neigh}\mathbf{A}$ and q-morphisms $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ and $\psi\colon \mathcal{V} \longrightarrow \mathcal{W}$ also their composition $\psi \circ \varphi\colon \mathcal{U} \longrightarrow \mathcal{W}$ is a q-morphism. Moreover, due to reflexivity of q, also the identical mapping $\mathrm{id}_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathcal{U}$ on every set $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ is a q-morphism.

Thus, with Definition 3.7.7 we have turned $\mathfrak{P}(\operatorname{Neigh}\mathbf{A})$ into a small category whose objects are just all subcollections of neighbourhoods of $\mathbf{A}$, morphisms are q-morphisms, which are composed in the natural way, and whose identical morphisms are given by the identical mappings. ◇

Having this interpretation at hand, it now makes sense to speak of category theoretic properties of collections of neighbourhoods w.r.t. some quasiorder $q \subseteq \leq_{\mathrm{cov}}$. Important examples are, for instance, isomorphisms and isomorphic objects. We fix the corresponding terminology in the following definition:

3.7.9 Definition. For an algebra $\mathbf{A}$ and a quasiorder $q \subseteq \leq_{\mathrm{cov}}$, we say that a mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a *q-isomorphism* between collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ if it is a q-morphism and an isomorphism in the category introduced in Remark 3.7.8. Explicitly, this is to say that there exists an inverse q-morphism $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$ such that $\varphi \circ \psi = \mathrm{id}_{\mathcal{V}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathcal{U}}$.

Two collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ are called *q-isomorphic* if they are isomorphic objects in the category just mentioned, i.e. if there exists some q-isomorphism $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ between them. We denote this relationship by $\mathcal{U} \cong_q \mathcal{V}$. ♢

3.7.10 Remark. It is clear that $\cong_q \subseteq (\mathfrak{P}(\operatorname{Neigh} \mathbf{A}))^2$ is an equivalence relation on collections of neighbourhoods of an algebra $\mathbf{A}$, whatever quasiorder $q \subseteq \leq_{\mathrm{cov}}$ we choose. ♢

With the following lemma, we want to give a characterisation of q-isomorphisms that does not require to speak about other q-morphisms. This also simplifies the check if two collections of neighbourhoods are q-isomorphic.

3.7.11 Lemma. *For an algebra $\mathbf{A}$, a quasiorder $q \subseteq \leq_{\mathrm{cov}}$ and two collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, a mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a q-isomorphism between $\mathcal{U}$ and $\mathcal{V}$ if and only if φ is a bijective $(q \cap q^{-1})$-morphism, i.e. satisfies $U \; q \; \varphi(U) \; q \; U$ for all $U \in \mathcal{U}$.*

Proof: If φ is a q-isomorphism and $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$ is the corresponding inverse q-isomorphism, then we have $\varphi \circ \psi = \mathrm{id}_{\mathcal{V}}$ and $\psi \circ \varphi = \mathrm{id}_{\mathcal{U}}$, which, first of all, proves that φ is bijective. Moreover, exploiting the properties of φ and ψ and the identity $\psi \circ \varphi = \mathrm{id}_{\mathcal{U}}$, we get $U \; q \; \varphi(U) \; q \; \psi(\varphi(U)) = U$ for every $U \in \mathcal{U}$, i.e. $U \; q \cap q^{-1} \; \varphi(U)$ as claimed.

Conversely, if we know that φ is a bijective $(q \cap q^{-1})$-morphism, then it is of course also a q-morphism as $q \cap q^{-1} \subseteq q$. Let us denote the inverse mapping of φ by ψ. We have to verify that $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$ is a q-morphism. Using the condition $\varphi \circ \psi = \mathrm{id}_{\mathcal{V}}$ and the assumption for φ on $\psi(V)$ for any neighbourhood $V \in \mathcal{V}$, we obtain $\psi(V) \; q \cap q^{-1} \; \varphi(\psi(V)) = V$ and hence $V \; q \; \psi(V)$ as desired. □

The previous characterisation also reveals that $\cong$-isomorphism of sets of neighbourhoods is exactly what we have defined as the canonical isomorphism notion in Definition 3.4.7. The next result relates this concept to $\lessapprox$-isomorphism of collections of neighbourhoods, that will turn up naturally later in a consequence of our main theorem.

3.7.12 Corollary. *For a neighbourhood self-embedding simple algebra $\mathbf{A}$ and sets $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, a mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is an $\lessapprox$-isomorphism if and only it is an isomorphism between these collections. Thus, we have $\mathcal{U} \cong_{\lessapprox} \mathcal{V}$ if and only if $\mathcal{U} \cong \mathcal{V}$.*

In particular, this characterisation holds for algebras having the FIP, thus algebras, where $\operatorname{Clo}^{(1)}(\mathbf{A})$ is finite, algebras in 1-locally finite varieties and especially finite ones.

Proof: By Lemma 3.2.19, in neighbourhood self-embedding simple algebras, we have $\precapprox \cap \succapprox = \cong$, providing the key to this observation. Using Lemma 3.7.11, we know that the mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is an $\precapprox$-isomorphism if and only if it is a bijective $(\precapprox \cap \succapprox)$-morphism, which by the above is a bijective $\cong$-morphism, i.e. an isomorphism in the sense of Definition 3.4.7.

The remaining statements are a consequence of Corollary 3.5.14 and the fact that finite algebras generate a 1-locally finite variety. □

In the next lemma we characterise the q-isomorphism relation among collections of neighbourhoods in terms of being transversals describing the same $(q \cap q^{-1})$-equivalence classes. This will become useful in the proof of the subsequent uniqueness theorem.

3.7.13 Lemma. *For an algebra $\mathbf{A}$, a quasiorder $q \subseteq \leq_{\mathrm{cov}}$, an equivalence relation $\theta \subseteq \leq_{\mathrm{cov}}$ and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ the following facts hold:*

(a) The condition $\mathcal{U} \sqsubseteq (\theta)\, \mathcal{V}$ is equivalent to the inclusion $\mathcal{U}/\theta \subseteq \mathcal{V}/\theta$, so the relationship $\mathcal{U} \sqsubseteq (\theta)\, \mathcal{V} \sqsubseteq (\theta)\, \mathcal{U}$ can be characterised by $\mathcal{U}/\theta = \mathcal{V}/\theta$.

(b) There exists a θ-morphism $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ if and only if $\mathcal{U}/\theta \subseteq \mathcal{V}/\theta$. Therefore, we have $\mathcal{U}/\theta = \mathcal{V}/\theta$ if and only if there exist θ-morphisms $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ and $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$.

(c) If $\mathcal{U} \cong_q \mathcal{V}$, then we have $\mathcal{U}/\left(q \cap q^{-1}\right) = \mathcal{V}/\left(q \cap q^{-1}\right)$.

(d) If $\mathcal{U}$ and $\mathcal{V}$ consist of pairwise non-equivalent neighbourhoods w.r.t. $q \cap q^{-1}$, i.e. if they are $(q \cap q^{-1})$-antichains, then it is $\mathcal{U} \cong_q \mathcal{V}$ if and only if the factor sets $\mathcal{U}/\left(q \cap q^{-1}\right) = \mathcal{V}/\left(q \cap q^{-1}\right)$ are equal.

(e) If $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$ are systems of representatives for the factors $\mathcal{U}/\left(q \cap q^{-1}\right)$ and $\mathcal{V}/\left(q \cap q^{-1}\right)$, respectively, then $\mathcal{U}/\left(q \cap q^{-1}\right) = \mathcal{V}/\left(q \cap q^{-1}\right)$ holds if and only if $\mathcal{U}' \cong_q \mathcal{V}'$.

Proof: (a) Of course, we only need to prove that $\mathcal{U} \sqsubseteq (\theta)\, \mathcal{V}$ is equivalent to $\mathcal{U}/\theta \subseteq \mathcal{V}/\theta$. By definition, the condition $\mathcal{U} \sqsubseteq (\theta)\, \mathcal{V}$ is equivalent to asserting that for every $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ such that $U \mathrel{\theta} V$, i.e. $[U]_\theta = [V]_\theta$. This means that $[U]_\theta \in \mathcal{V}/\theta$ for every $U \in \mathcal{U}$, i.e. that $\mathcal{U}/\theta \subseteq \mathcal{V}/\theta$.

(b) If $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ is a θ-morphism, then for every $U \in \mathcal{U}$ we have $U \mathrel{\theta} \varphi(U)$, i.e. $[U]_\theta = [\varphi(U)]_\theta \in \mathcal{V}/\theta$. This implies $\mathcal{U}/\theta \subseteq \mathcal{V}/\theta$. Conversely, if this inclusion holds, then for every $U \in \mathcal{U}$ there exists some $V \in \mathcal{V}$ such that $[U]_\theta = [V]_\theta$, or equivalently, $U \mathrel{\theta} V$. Via the axiom of choice, we can obtain a choice function $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ such that $U \mathrel{\theta} \varphi(U)$ holds for every $U \in \mathcal{U}$, i.e. a θ-morphism φ between $\mathcal{U}$ and $\mathcal{V}$.

Applying this characterisation to both inclusions yields the second statement of this item.

(c) If $\mathcal{U} \cong_q \mathcal{V}$, then by definition there exist mutually inverse q-isomorphisms $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$ and $\psi\colon \mathcal{V} \longrightarrow \mathcal{U}$. By Lemma 3.7.11, both of these mappings are $(q \cap q^{-1})$-morphisms, whence item (b) for $\theta = q \cap q^{-1}$ implies the equality $\mathcal{U}/(q \cap q^{-1}) = \mathcal{V}/(q \cap q^{-1})$.

(d) The "only if" direction of the equivalence is covered by the previous item. So let us now assume that $\mathcal{U}$ and $\mathcal{V}$ are both $(q \cap q^{-1})$-antichains and that $\mathcal{U}/(q \cap q^{-1}) = \mathcal{V}/(q \cap q^{-1})$. This means for every block $Z \in \mathcal{V}/(q \cap q^{-1})$ there exists exactly one member $V \in \mathcal{V}$ such that $Z = [V]_{q\cap q^{-1}}$, and similarly for every $W \in \mathcal{U}/(q \cap q^{-1})$ there exists a unique $U \in \mathcal{U}$ such that $W = [U]_{q\cap q^{-1}}$. For $U \in \mathcal{U}$, the block $[U]_{q\cap q^{-1}} \in \mathcal{U}/(q \cap q^{-1})$ belongs to $\mathcal{V}/(q \cap q^{-1})$ by the assumption that both factor sets are equal. Let $\varphi(U) \in \mathcal{V}$ be the unique member of $\mathcal{V}$ satisfying $[U]_{q\cap q^{-1}} = [\varphi(U)]_{q\cap q^{-1}}$, i.e. $U\ (q \cap q^{-1})\ \varphi(U)$. This yields a mapping $\varphi\colon \mathcal{U} \longrightarrow \mathcal{V}$. It is surjective, because for every $V \in \mathcal{V}$, we have $[V]_{q\cap q^{-1}} \in \mathcal{V}/(q \cap q^{-1}) = \mathcal{U}/(q \cap q^{-1})$, i.e. there is some $U \in \mathcal{U}$ such that $[V]_{q\cap q^{-1}} = [U]_{q\cap q^{-1}}$. By definition of φ, this means that $V = \varphi(U)$, showing surjectivity of φ. Moreover, this map is injective as for $U_1, U_2 \in \mathcal{U}$ the condition $\varphi(U_1) = \varphi(U_2)$ entails $[U_1]_{q\cap q^{-1}} = [\varphi(U_1)]_{q\cap q^{-1}} = [\varphi(U_2)]_{q\cap q^{-1}} = [U_2]_{q\cap q^{-1}}$, i.e. $U_1\ (q \cap q^{-1})\ U_2$. Since $\mathcal{U}$ was assumed to be a $(q \cap q^{-1})$-antichain, we can infer $U_1 = U_2$. We have now proven that φ is a bijective $(q \cap q^{-1})$-morphism from $\mathcal{U}$ to $\mathcal{V}$. Using Lemma 3.7.11 we may conclude that φ is a q-isomorphism, and hence that $\mathcal{U} \cong_q \mathcal{V}$.

(e) Since $\mathcal{U}'$ and $\mathcal{V}'$ are systems of representatives of $(q \cap q^{-1})$-classes, they contain only pairwise $(q \cap q^{-1})$-non-equivalent neighbourhoods. Furthermore, we know $\mathcal{U}/(q \cap q^{-1}) = \mathcal{U}'/(q \cap q^{-1})$ and $\mathcal{V}/(q \cap q^{-1}) = \mathcal{V}'/(q \cap q^{-1})$ such that $\mathcal{U}/(q \cap q^{-1}) = \mathcal{V}/(q \cap q^{-1})$ is equivalent to $\mathcal{U}'/(q \cap q^{-1}) = \mathcal{V}'/(q \cap q^{-1})$, that by item (d) can be characterised by $\mathcal{U}' \cong_q \mathcal{V}'$. ☐

We are now ready to prove the anticipated main theorem of this section, constructing q-non-refinable covers and giving an answer to the uniqueness problem posed in the introduction to Section 3.6.

3.7.14 Theorem. *Let* $\mathbf{A}$ *be an algebra and* $q \subseteq \leq_{\mathrm{cov}}$ *a quasiorder on the set of neighbourhoods. Every irredundant q-cover base* $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *of* $\mathbf{A}$ *satisfies:*

(a) $\mathcal{V}$ *is q-non-refinable.*

(b) Every other q-non-refinable cover $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ *is q-isomorphic to* $\mathcal{V}$*, i.e. fulfils* $\mathcal{U} \cong_q \mathcal{V}$*.*

Proof: (a) Since $\mathcal{V}$ is an irredundant q-cover base, it is an irredundant cover of $\mathbf{A}$, and hence it is a q-antichain (see Lemma 3.5.9(b)). According to Lemma 3.5.8, it only remains to show that $\mathcal{V}$ is $\leq_{\mathrm{ref}}(q)$-minimal. For this let $\mathcal{U} \subseteq \operatorname{Neigh}\mathbf{A}$ be such that $\mathcal{U} \leq_{\mathrm{ref}}(q)\ \mathcal{V}$. Since $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$, Lemma 3.5.4(g) implies that $\mathcal{U}$ is a cover of $\mathbf{A}$, too, whence we get $\mathcal{V} \sqsubseteq (q)\ \mathcal{U}$ as $\mathcal{V}$ is a q-cover prebase. Now Lemma 3.5.4(d) yields $\mathcal{V} \leq_{\mathrm{ref}}(q)\ \mathcal{U}$ as needed.

(b) Let $\mathcal{U} \in \mathrm{Cov}\,(\mathbf{A})$ be another q-non-refinable cover of $\mathbf{A}$. Since $\mathcal{V}$ is a q-cover prebase, we get $\mathcal{V} \sqsubseteq (q)\,\mathcal{U}$. Moreover, as $\mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$, Lemma 3.5.4(h) allows us to infer $\mathcal{V} \leq_{\mathrm{ref}} (q)\,\mathcal{U}$. Hence, q-non-refinability of $\mathcal{U}$ implies $\mathcal{U} \sqsubseteq (q \cap q^{-1})\,\mathcal{V}$. Together with $\mathcal{V} \leq_{\mathrm{ref}} (q)\,\mathcal{U}$, we obtain that for every $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$, for which in turn there is some $V' \in \mathcal{V}$ such that $V\ q\ U\ q \cap q^{-1}\ V'$. Transitivity of q now implies that $V\ q\ V'$, upon which irredundancy of $\mathcal{V}$ via Lemma 3.5.9(b) implies $V = V'$ since $\mathcal{V}$ had to be an antichain w.r.t. q. In other words, we got $V\ q\ U\ q\ V' = V$, and thus $V\ q \cap q^{-1}\ U$, which proves $\mathcal{V} \sqsubseteq (q \cap q^{-1})\,\mathcal{U}$. From above we recall the relation $\mathcal{U} \sqsubseteq (q \cap q^{-1})\,\mathcal{V}$, whence Lemma 3.7.13(a) yields the equality $\mathcal{U}/\,(q \cap q^{-1}) = \mathcal{V}/\,(q \cap q^{-1})$. As $\mathcal{U}$ and $\mathcal{V}$ are both q-non-refinable, they are $(q \cap q^{-1})$-antichains by definition. Therefore, Lemma 3.7.13(d) now implies $\mathcal{U} \cong_q \mathcal{V}$ and so completes the proof. □

Our theorem states that irredundant q-cover bases of algebras are q-non-refinable covers, and that the latter are uniquely determined up to q-isomorphism provided that irredundant q-cover bases exist.

As a first application of our theorem, we infer that irredundant q-cover bases of algebras are uniquely determined up to q-isomorphism, a fact that one could also have deduced earlier on its own.

3.7.15 Corollary. *Let* $\mathbf{A}$ *be an algebra and* $q \subseteq \leq_{\mathrm{cov}}$ *a quasiorder on its set of neighbourhoods, then any two irredundant* q*-cover bases* $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}\,(\mathbf{A})$ *of* $\mathbf{A}$ *satisfy* $\mathcal{U} \cong_q \mathcal{V}$.

Proof: By item (a) of Theorem 3.7.14, the collection $\mathcal{U}$ is a q-non-refinable cover. Hence, by item (b) of the same result, $\mathcal{U}$ is q-isomorphic to $\mathcal{V}$ since $\mathcal{V}$ was an irredundant q-cover base of $\mathbf{A}$. □

In the second corollary we combine the theorem with Corollary 3.7.6 to replace the assumption of an existing irredundant q-cover base by the existence of some q-cover base plus a more accessible condition on the quasiorder q.

3.7.16 Corollary. *Let* $\mathbf{A}$ *be an algebra,* $(\mathrm{Neigh}\,\mathbf{A}, q)$ *be a quasiordered set such that* $q \subseteq \leq_{\mathrm{cov}}$*, and let* $\mathcal{V} \subseteq \mathrm{Neigh}\,\mathbf{A}$ *be any* q*-cover base of* $\mathbf{A}$*. Suppose that* $\mathcal{V}' \subseteq \mathcal{U}$ *is any* $(q \cap q^{-1})$*-transversal of a subcollection* $\mathcal{U} \subseteq \mathrm{Max}\,(\mathcal{V}, q{\restriction_{\mathcal{V}}})$ *still covering* $\mathbf{A}$*. Then* $\mathcal{V}'$ *is a* q*-non-refinable cover and* q*-non-refinable covers of* $\mathbf{A}$ *are unique up to* q*-isomorphism.*

Proof: Under the assumptions listed above, we can apply Corollary 3.7.6 to infer that $\mathcal{V}'$ is an irredundant q-cover base of $\mathbf{A}$. Then Theorem 3.7.14 implies that this collection is q-non-refinable, and moreover, that any other q-non-refinable ecover of $\mathbf{A}$ is q-isomorphic to it. Thus, if $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Cov}\,(\mathbf{A})$ are q-non-refinable, then it is $\mathcal{U}_1 \cong_q \mathcal{V}' \cong_q \mathcal{U}_2$, and via transitivity (see Remark 3.7.10) we get $\mathcal{U}_1 \cong_q \mathcal{U}_2$. □

In the third instance of the main theorem, we give a concrete description of q-non-refinable covers.

3.7.17 Corollary. *Let* $\mathbf{A}$ *be a poly-*ARTIN*ian algebra having the FIP,* q *a quasiorder on* $\operatorname{Neigh}\mathbf{A}$ *such that* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$ *and the factor poset associated with* $(\operatorname{Neigh}\mathbf{A}, q)$ *fulfils DCC. Moreover, put* $\tilde{q} := q \vee \precsim$*.*

For $\mathcal{V} \in \left\{\operatorname{Irr}^{**}_q(\mathbf{A}), \operatorname{Irr}^{*}_q(\mathbf{A})\right\}$*, every* $(\tilde{q} \cap \tilde{q}^{-1})$*-transversal* $\mathcal{V}' \subseteq \operatorname{Max}(\mathcal{V}, \tilde{q}{\restriction_{\mathcal{V}}})$ *of the* $\tilde{q}$*-maximal members of* $\mathcal{V}$ *is a* $\tilde{q}$*-non-refinable cover of* $\mathbf{A}$*, and such covers are uniquely determined up to* $\tilde{q}$*-isomorphism, all subject to the condition*[42] *that the factor poset associated with* $(\mathcal{V}, \tilde{q}{\restriction_{\mathcal{V}}})$ *satisfies ACC.*

If $\mathbf{A}$ *is finite, only the requirement* $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$ *is needed to obtain the same for* $\mathcal{V} \in \left\{\operatorname{Irr}^{**(|A|)}_q(\mathbf{A}), \operatorname{Irr}^{*(|A|)}_q(\mathbf{A})\right\}$*.*

Proof: According to Corollary 3.7.2, under the assumptions on $\mathbf{A}$ and q, the given collections $\mathcal{V}$ are all $\tilde{q}$-cover bases of $\mathbf{A}$. Since $(\mathcal{V}, \tilde{q}{\restriction_{\mathcal{V}}})$ fulfils ACC, we know by Lemma 3.5.6(c) that $\mathcal{U} := \operatorname{Max}(\mathcal{V}, \tilde{q}{\restriction_{\mathcal{V}}})$ is a cover of $\mathbf{A}$ since $\mathcal{V}$ was one by assumption. Using the conditions on $\tilde{q}$, Corollary 3.7.16 then yields that $\mathcal{V}'$ indeed is a $\tilde{q}$-non-refinable cover and that such covers are unique up to $\tilde{q}$-isomorphism. □

In the end, we are interested in the standard case, that is crucially and strictly irreducible neighbourhoods of $\mathbf{A}$. This means, we want to put $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$. For this, as a first step, we are interested in the special case $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \precsim$ of the previous corollary. We keep in mind that since the FIP implies neighbourhood self-embedding simplicity (see Lemma 3.2.18(b)), we have $\operatorname{Irr}^{*}_q(\mathbf{A}) = \operatorname{Irr}^{*}(\mathbf{A})$ for all quasiorders $q \subseteq \precsim$ due to Lemma 3.6.11(d). Thus, the case $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq q \subseteq \precsim$ reduces to the case $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$ treated in the following result.

3.7.18 Corollary. *Let* $\mathbf{A}$ *be a poly-*ARTIN*ian algebra having the FIP such that the poset* $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ *fulfils DCC and the one associated with* $\left(\operatorname{Irr}^{*}(\mathbf{A}), \precsim{\restriction_{\operatorname{Irr}^{*}(\mathbf{A})}}\right)$ *satisfies ACC.*

For both $\mathcal{V} \in \{\operatorname{Irr}^{**}(\mathbf{A}), \operatorname{Irr}^{*}(\mathbf{A})\}$*, every* $\cong$*-transversal* $\mathcal{V}' \subseteq \operatorname{Max}(\mathcal{V}, \precsim{\restriction_{\mathcal{V}}})$ *of the* $\precsim$*-maximal members of* $\mathcal{V}$ *is an* $\precsim$*-non-refinable cover of* $\mathbf{A}$*, and such covers are uniquely determined up to isomorphism (in the sense of Definition 3.4.7).*

If $\mathbf{A}$ *is finite, all the previously listed assumptions are fulfilled and we obtain the same for* $\mathcal{V} \in \left\{\operatorname{Irr}^{**(|A|)}(\mathbf{A}), \operatorname{Irr}^{*(|A|)}(\mathbf{A})\right\}$*.*

Proof: We additionally assume that $q = \subseteq_{\operatorname{Neigh}\mathbf{A}}$ in Corollary 3.7.17, then it follows $\tilde{q} = q \vee \precsim = \precsim$. Moreover, we keep in mind that $\cong_{\tilde{q}} = \cong_{\precsim} = \cong$ holds for collections of neighbourhoods by Corollary 3.7.12 since $\mathbf{A}$ has the FIP. For the same reason, we have $\tilde{q} \cap \tilde{q}^{-1} = \precsim \cap \succsim = \cong$ for neighbourhoods due to Lemma 3.2.19. Substituting these observations into the wording of Corollary 3.7.17 and generally assuming the ascending chain condition for the largest set $\mathcal{V} = \operatorname{Irr}^{*}(\mathbf{A})$, we precisely obtain Corollary 3.7.18. □

[42] This follows of course, if the posets associated with $\left(\operatorname{Irr}^{*}_q(\mathbf{A}), \tilde{q}{\restriction_{\operatorname{Irr}^{*}_q(\mathbf{A})}}\right)$ or $(\operatorname{Neigh}\mathbf{A}, \tilde{q})$ fulfil ACC.

For algebras fulfilling the assumptions in the previous corollary, in particular poly-Artinian algebras in 1-locally finite varieties, we have now solved the uniqueness and completeness problem w.r.t. crucially and strictly irreducible neighbourhoods in a concrete way. Every such algebra has a unique decomposition into crucially irreducible neighbourhoods, up to isomorphism. For finite algebras, it is even enough to use $|A|$-crucially irreducible neighbourhoods. The only imperfection that remains in Corollary 3.7.18 is that it speaks about $\precapprox$-non-refinable covers of $\mathbf{A}$, where we actually would like to see a statement about non-refinable covers.

It is the next goal of this section to provide a remedy in this respect. Our first observation connecting $\precapprox$-non-refinability and non-refinability is that $\precapprox$-non-refinability is a necessary condition for classical non-refinability. This fact helps in removing the stumbling block from the uniqueness part of Corollary 3.7.18.

3.7.19 Lemma. *For any collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ of any algebra the following implications hold:*

(a) If $\mathcal{U}$ is $\leq_{\text{ref}}$-minimal, then it is $\leq_{\text{ref}} (\precapprox)$-minimal, as well.

(b) If $\mathcal{U}$ is non-refinable, then it is $\precapprox$-non-refinable.

Proof: (a) Suppose that $\mathcal{U}$ is refinement minimal. In order to demonstrate that it is also $\precapprox$-refinement minimal, we consider a collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ such that $\mathcal{V} \leq_{\text{ref}} (\precapprox) \mathcal{U}$. By definition, this entails $\mathcal{V} \sqsubseteq (\precapprox) \mathcal{U}$, so for every $V \in \mathcal{V}$ there exists some $U_V \in \mathcal{U}$ such that $V \precapprox U_V$. Using Proposition 3.2.10(b), this means in detail that for all $V \in \mathcal{V}$ there are some $W_V \in \operatorname{Neigh} \mathbf{A}$ and $U_V \in \mathcal{U}$ satisfying $V \cong W_V \subseteq U_V$. Let $\mathcal{W} := \{W_V \mid V \in \mathcal{V}\}$. For $V \in \mathcal{V}$, we have $W_V \subseteq U_V \in \mathcal{U}$, thus $\mathcal{W} \sqsubseteq (\subseteq) \mathcal{U}$ holds by construction. Since for $V \in \mathcal{V}$ it is $V \cong W_V \in \mathcal{W}$, we may also infer $\mathcal{V} \sqsubseteq (\cong) \mathcal{W}$, so that Remark 3.5.1 yields $\mathcal{V} \leq_{\text{cov}} \mathcal{W}$. Moreover, using Lemma 3.5.4(a), the condition $\mathcal{V} \leq_{\text{ref}} (\precapprox) \mathcal{U}$ implies $\mathcal{U} \equiv_{\text{cov}} \mathcal{V} \leq_{\text{cov}} \mathcal{W}$; so we have $\mathcal{U} \leq_{\text{cov}} \mathcal{W}$ and $\mathcal{W} \sqsubseteq (\subseteq) \mathcal{U}$. Hence, it is $\mathcal{W} \leq_{\text{ref}} \mathcal{U}$, upon which refinement minimality yields $\mathcal{U} \leq_{\text{ref}} \mathcal{W}$, especially $\mathcal{U} \sqsubseteq (\subseteq) \mathcal{W}$. This means, for every $U \in \mathcal{U}$, there is some $V \in \mathcal{V}$ such that $U \subseteq W_V \cong V$, implying $U \precapprox W_V \precapprox V$, i.e. $U \precapprox V$, due to Corollary 3.2.12. Thus, $\mathcal{U} \sqsubseteq (\precapprox) \mathcal{V}$, showing $\mathcal{U} \leq_{\text{ref}} (\precapprox) \mathcal{V}$ because of $\mathcal{V} \leq_{\text{ref}} (\precapprox) \mathcal{U}$ and Lemma 3.5.4(d). This finally proves $\precapprox$-refinement minimality of $\mathcal{U}$.

(b) Since $\mathcal{U}$ is non-refinable, it is refinement minimal by Lemma 3.5.8. Applying item (a), it is thus $\precapprox$-refinement minimal. In order to obtain a contradiction, we assume that $\mathcal{U}$ fails to be an antichain w.r.t. $\precapprox$. Then there exist $U, V \in \mathcal{U}$ such that $U \precapprox V$ and $U \neq V$. Let us define $\mathcal{V} := \mathcal{U} \setminus \{U\} \subseteq \mathcal{U}$ and $q' := \precapprox\restriction_{\mathcal{U}}$. By the condition on U and V, we have $\mathcal{U} \sqsubseteq (\langle q' \rangle) \mathcal{V}$, and hence Lemma 3.5.6(a) for $q = \subseteq_{\operatorname{Neigh} \mathbf{A}}$ implies $\mathcal{V} \leq_{\text{ref}} \mathcal{U}$. Since $\mathcal{U}$ was non-refinable, this entails $\mathcal{U} \sqsubseteq (\subseteq \cap \supseteq) \mathcal{V}$, i.e. $U \in \mathcal{U} \subseteq \mathcal{V}$, a contradiction to the choice of $\mathcal{V}$. Therefore, $\mathcal{U}$ must be an $\precapprox$-antichain, such that we can use Lemma 3.5.8 to infer $\precapprox$-non-refinability of $\mathcal{U}$. $\square$

Of course, we are now after the converse implication of Lemma 3.7.19(b). In comparison with general q-non-refinability, classical non-refinability of a collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ really is a quite strong condition since $\mathcal{U}$ has to be a subcollection of any of its refinements. Therefore, it is plausible, that the implication we want to achieve cannot be obtained without further preconditions. However, with these preconditions, not only can we prove that $\precsim$-non-refinability implies $\subseteq$-non-refinability, but we can do this on the more general level of two quasiorders $q' \subseteq q$.

3.7.20 Lemma. *Let* $\mathbf{A}$ *be an algebra and* $q' \subseteq q \subseteq \leq_{\mathrm{cov}}$ *quasiorders on its set of neighbourhoods such that* $U\ q\ V\ q'\ U$ *implies* $U\ q'\ V$ *for all* $U, V \in \operatorname{Neigh} \mathbf{A}$*. Then any collection* $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ *that is* q*-non-refinable, is* q'*-non-refinable, too.*

Proof: Since $\mathcal{U}$ is q-non-refinable by assumption, it is $\leq_{\mathrm{ref}}(q)$-minimal and a q-antichain by Lemma 3.5.8. We will first show that $\mathcal{U}$ is q'-refinement minimal, and then that it is a q'-antichain. Using the same lemma again, this proves that $\mathcal{U}$ is q'-non-refinable.

For q'-refinement minimality let us consider any collection $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ satisfying $\mathcal{V} \leq_{\mathrm{ref}}(q')\, \mathcal{U}$. Item (b) of Lemma 3.5.4 now implies $\mathcal{V} \leq_{\mathrm{ref}}(q)\, \mathcal{U}$ since $q' \subseteq q$. As $\mathcal{U}$ is q-refinement minimal, we get $\mathcal{U} \leq_{\mathrm{ref}}(q)\, \mathcal{V}$. So for every $U \in \mathcal{U}$ there exists some $V \in \mathcal{V}$, for which there is some $U' \in \mathcal{U}$, such that $U\ q\ V\ q'\ U'$. Since $q' \subseteq q$, this implies $U\ q\ V\ q\ U'$, i.e. $U\ q\ U'$ and so $U = U'$ due to $\mathcal{U}$ being a q-antichain. Hence, we have $U\ q\ V\ q'\ U' = U$, which by the special assumption of the lemma entails $U\ q'\ V$. Hence, for $U \in \mathcal{U}$ we have constructed $V \in \mathcal{V}$ satisfying $U\ q'\ V$, showing $\mathcal{U} \sqsubseteq (q')\, \mathcal{V}$. Via Lemma 3.5.4(d) we have thus proven $\mathcal{U} \leq_{\mathrm{ref}}(q')\, \mathcal{V}$, which demonstrates that $\mathcal{U}$ is indeed q'-refinement minimal.

Moreover, for $U, U' \in \mathcal{U}$ the condition $U\ q'\ U'$ implies $U\ q\ U'$ as $q' \subseteq q$. Since $\mathcal{U}$ was an antichain w.r.t. q, this implies $U = U'$, proving that $\mathcal{U}$ also is one w.r.t. q'.

Using Lemma 3.5.8, we conclude from the facts established in the two previous paragraphs that $\mathcal{U}$ is indeed q'-non-refinable. $\square$

Putting the two preceding lemmas together, we can show that for neighbourhood self-embedding simple algebras, the notions of $\precsim$-non-refinability and classical non-refinability are actually the same thing.

3.7.21 Corollary. *For a neighbourhood self-embedding simple algebra* $\mathbf{A}$*, a collection* $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ *is non-refinable if and only if it is* $\precsim$*-non-refinable.*

Proof: Non-refinability always implies $\precsim$-non-refinability by Lemma 3.7.19(b). For the converse, every neighbourhood self-embedding simple algebra satisfies the additional assumption on $q' = \subseteq_{\operatorname{Neigh} \mathbf{A}}$ and $q = \precsim$ in Lemma 3.7.20, whence $\precsim$-non-refinability implies $\subseteq_{\operatorname{Neigh} \mathbf{A}}$-non-refinability, i.e. non-refinability. $\square$

In the next result we first combine Theorem 3.7.14 with Lemma 3.7.20 and second with Corollary 3.7.21, for the special case of neighbourhood embedding and neighbourhood self-embedding simple algebras. Finally, we can then also fix the small deficiency of Corollary 3.7.18 concerning $\precsim$-non-refinability vs. non-refinability.

3.7.22 Corollary. *For an algebra* $\mathbf{A}$ *and a cover* $\mathcal{V} \in \operatorname{Cov}(\mathbf{A})$ *the following facts hold:*

(a) If $\mathcal{V}$ *is an irredundant* q*-cover base for a quasiorder* $q \subseteq \leq_{\mathrm{cov}}$ *and we have that* $U \mathrel{q} V \mathrel{q'} U$ *implies* $U \mathrel{q'} V$ *for all* $U, V \in \operatorname{Neigh}\mathbf{A}$ *and some other quasiorder* $q' \subseteq q$*, then* $\mathcal{V}$ *is a* q'*-non-refinable cover of* $\mathbf{A}$*.*

(b) If $\mathcal{V}$ *is an irredundant* $\precapprox$*-cover base of* $\mathbf{A}$ *and the algebra* $\mathbf{A}$ *is neighbourhood self-embedding simple, then* $\mathcal{V}$ *is a non-refinable cover (equivalently an* $\precapprox$*-non-refinable cover), and every other non-refinable (equivalently* $\precapprox$*-non-refinable) cover* $\mathcal{U} \in \operatorname{Cov}(\mathbf{A})$ *is isomorphic to* $\mathcal{V}$*.*

(c) Suppose that $\mathbf{A}$ *is poly-*ARTIN*ian, has the FIP and the poset* $(\operatorname{Neigh}\mathbf{A}, \subseteq)$ *satisfies DCC, then every irredundant cover* $\mathcal{V} \subseteq \operatorname{Irr}^*(\mathbf{A})$ *is an* $\precapprox$*-cover base of* $\mathbf{A}$ *and hence a non-refinable cover. Moreover, every other non-refinable cover is isomorphic to it.*

If, in addition, the poset naturally associated with $\left(\operatorname{Irr}^*(\mathbf{A}), \precapprox\restriction_{\operatorname{Irr}^*(\mathbf{A})}\right)$ *fulfils ACC, then the collections* $\mathcal{V}'$ *constructed in Corollary 3.7.18 are irredundant covers contained in* $\operatorname{Irr}^*(\mathbf{A})$*, non-refinable, and the latter are unique up to isomorphism.*

Proof: (a) According to Theorem 3.7.14(a) every irredundant q-cover base $\mathcal{V}$ is a q-non-refinable cover. The additional condition on q' then via Lemma 3.7.20 implies that $\mathcal{V}$ is also a q'-non-refinable cover of $\mathbf{A}$.

(b) By Corollary 3.7.21 non-refinable covers of a neighbourhood self-embedding simple algebra are the same as $\precapprox$-non-refinable covers. From this point of view Theorem 3.7.14 states that irredundant $\precapprox$-cover bases, such as $\mathcal{V}$, are non-refinable covers. Furthermore, it claims that every other non-refinable cover must be $\precapprox$-isomorphic to $\mathcal{V}$, and hence isomorphic to $\mathcal{V}$ due to Corollary 3.7.12 and neighbourhood self-embedding simplicity of $\mathbf{A}$.

(c) Under the assumptions made in the first paragraph of this item, the set $\operatorname{Irr}^*(\mathbf{A})$ is an $\precapprox$-cover base of $\mathbf{A}$ as one can read in Corollary 3.7.2. If $\mathcal{V} \subseteq \operatorname{Irr}^*(\mathbf{A})$ is an irredundant cover, then it is an irredundant $\precapprox$-cover base of $\mathbf{A}$ by item (d) of Lemma 3.7.3. Since the FIP entails neighbourhood self-embedding simplicity (see Lemma 3.2.18(b)) item (b) is now applicable. It states that $\mathcal{V}$ is a non-refinable cover and that every other non-refinable cover of $\mathbf{A}$ is isomorphic to $\mathcal{V}$.

If additionally the factor poset of $\left(\operatorname{Irr}^*(\mathbf{A}), \precapprox\restriction_{\operatorname{Irr}^*(\mathbf{A})}\right)$ has ACC, then one can construct collections $\mathcal{V}' \subseteq \operatorname{Irr}^*(\mathbf{A})$ as in Corollary 3.7.18, and they are $\precapprox$-non-refinable covers as mentioned in this corollary. Lemma 3.5.9(e) characterises such covers as being $\precapprox$-refinement minimal and, in particular, an irredundant cover of $\mathbf{A}$. The remaining statements have thus already been proven in the previous paragraph. $\square$

For finite algebras item (c) of the previous corollary has already been obtained earlier in collaboration with Friedrich Martin Schneider in 2011. For special importance, we formulate the finite case explicitly in the following corollary.

3.7.23 Corollary. *Every finite algebra* $\mathbf{A}$ *has got a unique non-refinable cover, up to isomorphism.*

It consists of a set of $\cong$*-representatives of the* $\precapprox$*-maximal* $|A|$*-strictly (or* $|A|$*-crucially) or strictly (or crucially) irreducible neighbourhoods of* $\mathbf{A}$*.*

Proof: Every finite algebra is poly-ARTINian, has the FIP and has a finite set of neighbourhoods. Thus, the assumptions of Corollary 3.7.22(c) are fulfilled, and the claim follows as a special case by re-enacting the constructions contained in Corollary 3.7.18. □

We acknowledge here that the statement in the first sentence of the previous corollary has already been known as of 2001 due to work by Keith Kearnes and Ágnes Szendrei, see [Kea01, Theorem 5.3]. To the author's best knowledge the first detailed proof of this fact occurs in [Beh09, Theorem 3.8.1], where it is also stated that the neighbourhoods in non-refinable covers are strictly irreducible–although not using this terminology. So the last mentioned result can be considered as a direct precursor to Corollary 3.7.23. Besides, the same fact is also cited in Theorem 3.2 of [Beh12], but without proof.

The real progress that has been made here consists in the way how the result is deduced, yielding in particular a concrete constructive description of non-refinable covers that has been missing so far in all previously published work. Thereby, we also solve the third open problem from [Beh09, p. 147], and furthermore, our new method of determining non-refinable covers makes answering the second open question, posed in the same place, obsolete.

Moreover, the second paragraph of Corollary 3.7.23 corrects Theorem 2.9(3) of [KL10] stating that a (unique) non-refinable cover can be given by choosing an $\cong$-transversal of $\operatorname{Max}\left(\operatorname{Irr}(\mathbf{A}), \precapprox\restriction_{\operatorname{Irr}(\mathbf{A})}\right)$, i.e. of the $\precapprox$-maximal irreducible neighbourhoods of a finite algebra $\mathbf{A}$. To be fair, we have not yet justified that our result really is a correction, because neither have we seen that the isomorphism classes of $\precapprox$-maximal irreducible neighbourhoods are different from those of $\precapprox$-maximal strictly irreducible neighbourhoods, nor have we provided any evidence that the result stated in Theorem 2.9(3) of [KL10] is incorrect. However, we will see this in an example in the following section (cf. Lemma 3.8.6(f) and Remark 3.8.9).

The last aspect we want to address in this section is that Corollaries 3.7.17 and 3.7.18 generally propose two, and for finite algebras even four, ways to construct $\tilde{q}$-non-refinable covers (non-refinable covers, respectively). In the final step of the construction presented there, one takes a system $\mathcal{V}'$ of $(\tilde{q} \cap \tilde{q}^{-1})$-representatives of sets from the collection of all $\tilde{q}$-maximal neighbourhoods in one of two/four initial collections $\mathcal{V}$. Since $\mathcal{V}'$ is shown there to be a $\tilde{q}$-non-refinable cover, and such covers are unique up to $\tilde{q}$-isomorphism, the sets of $(\tilde{q} \cap \tilde{q}^{-1})$-equivalence classes

$\operatorname{Max}(\mathcal{V}, \tilde{q}\restriction_{\mathcal{V}}) / (\tilde{q} \cap \tilde{q}^{-1})$ belonging to any two choices for $\mathcal{V}$ offered in the mentioned corollaries are identical (cp. Lemma 3.7.13(e)).

However, with regard to efficiency, the question remains, if some simplification, e.g. size reduction of $\operatorname{Max}(\mathcal{V}, \tilde{q}\restriction_{\mathcal{V}})$, can be gained from playing with the different choices for $\mathcal{V}$.

To answer this question, the property of being closed w.r.t. $\tilde{q} \cap \tilde{q}^{-1}$ (generally w.r.t. an equivalence relation) becomes important.

3.7.24 Definition. For an algebra $\mathbf{A}$ and an equivalence relation $\theta \subseteq (\operatorname{Neigh} \mathbf{A})^2$, a collection $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ is said to be *closed under* θ if $\bigcup \mathcal{U}/\theta \subseteq \mathcal{U}$, i.e. if $[U]_\theta \subseteq \mathcal{U}$ holds for all $U \in \mathcal{U}$. $\Diamond$

With the following sequence of small results we will exhibit a condition on covers $\mathcal{V}$, under which the sets of q-maximal elements of $\mathcal{V}$ do not depend on the choice of the collection $\mathcal{V}$. One way to interpret this is that no benefit can be gained from letting $\mathcal{V}$ vary within these limitations. Another way to understand this result is that covers $\mathcal{V}$ being subject to those conditions are really closely related: presume that for instance $\mathcal{V}_1 := \operatorname{Irr}^{**}(\mathbf{A})$ and $\mathcal{V}_2 := \operatorname{Irr}^{*}(\mathbf{A})$ are two covers of $\mathbf{A}$, then we have argued above that $\operatorname{Max}(\mathcal{V}_1, \tilde{q}\restriction_{\mathcal{V}_1})$ is the same as $\operatorname{Max}(\mathcal{V}_2, \tilde{q}\restriction_{\mathcal{V}_2})$, up to $\tilde{q} \cap \tilde{q}^{-1}$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ fulfil the conditions to be established below, then these collections of $\tilde{q}$-maximal neighbourhoods would really be equal, i.e. no harm were done in preferring one collection over the other.

3.7.25 Lemma. *For an algebra $\mathbf{A}$, a quasiorder $q \subseteq \leq_{\mathrm{cov}}$ on its set of neighbourhoods and collections $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ such that $\mathcal{U} \sqsubseteq (q)\, \mathcal{V} \sqsubseteq (q)\, \mathcal{U}$ the following implications are true:*

(a) If $q\restriction_{\mathcal{U}} \subseteq q \cap q^{-1}$ and $\mathcal{V}$ is closed under $q \cap q^{-1}$, then $\mathcal{U} \subseteq \mathcal{V}$.

(b) If $q\restriction_{\mathcal{U}} \subseteq q \cap q^{-1}$, $q\restriction_{\mathcal{V}} \subseteq q \cap q^{-1}$, and $\mathcal{U}$ and $\mathcal{V}$ are closed under $q \cap q^{-1}$, then $\mathcal{U} = \mathcal{V}$.

Proof: As the assumptions of (b) are symmetric in $\mathcal{U}$ and $\mathcal{V}$ and contain the assumptions of item (a), one can use the latter to prove both set inclusions, i.e. equality, in the former.

Consequently, we only deal with the proof of item (a). We want to prove $\mathcal{U} \subseteq \mathcal{V}$, so we consider an arbitrary neighbourhood $U \in \mathcal{U}$. By the assumption of the lemma there is some $V \in \mathcal{V}$, for which there is again some $U' \in \mathcal{U}$, such that $U \mathrel{q} V \mathrel{q} U'$. Thus, by transitivity, we have $U \mathrel{q\restriction_{\mathcal{U}}} U'$, implying $U \mathrel{q \cap q^{-1}} U'$ by the first assumption of item (a). Therefore, we have $U \mathrel{q} V \mathrel{q} U' \mathrel{q} U$, whence $U \mathrel{q} V \mathrel{q} U$, i.e. $U \mathrel{q \cap q^{-1}} V$ follows. Since $V \in \mathcal{V}$ and this collection was assumed to be closed under $q \cap q^{-1}$, we obtain $U \in [V]_{q \cap q^{-1}} \subseteq \mathcal{V}$ as needed. $\square$

In the next lemma we see that one of the two conditions in Lemma 3.7.25 is automatically fulfilled if the collection $\mathcal{U}$ is chosen as the set of all q-maximal members of some other collection. Moreover, we look at the consequence of adding the assumption of being q-cover bases.

3.7.26 Lemma. *For an algebra* $\mathbf{A}$*, a quasiorder* $q \subseteq \leq_{\mathrm{cov}}$ *and two collections* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ *of neighbourhoods the following assertions hold.*

(a) $q{\restriction}_{\operatorname{Max}(\mathcal{U}, q\restriction_{\mathcal{U}})} \subseteq q \cap q^{-1}$.

(b) If $\mathcal{U}' := \operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ *and* $\mathcal{V}' := \operatorname{Max}(\mathcal{V}, q{\restriction}_{\mathcal{V}})$ *are* q*-cover bases of* $\mathbf{A}$ *and both are closed under* $q \cap q^{-1}$*, then* $\mathcal{U}' = \mathcal{V}'$*.*

Proof: (a) The first stated inclusion is nothing but a simple consequence of maximality: if $U, V \in \operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ satisfy $U\ q\ V$, then we certainly have $V\ q\ U$ since U is $q{\restriction}_{\mathcal{U}}$-maximal in $\mathcal{U}$ and V belongs to $\operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}}) \subseteq \mathcal{U}$. Consequently, we have $U\ q\ V\ q\ V$, i.e. $U\ q \cap q^{-1}\ V$.

(b) From item (a), we can infer that $q{\restriction}_{\mathcal{U}'} \subseteq q \cap q^{-1}$ and similarly, $q{\restriction}_{\mathcal{V}'} \subseteq q \cap q^{-1}$. Moreover, $\mathcal{U}'$ and $\mathcal{V}'$ are closed under $q \cap q^{-1}$ by assumption. In order to apply Lemma 3.7.25, we only need to prove that $\mathcal{U}' \sqsubseteq (q)\ \mathcal{V}' \sqsubseteq (q)\ \mathcal{U}'$. As $\mathcal{V}'$ is a q-cover base, it is in particular a cover of $\mathbf{A}$. Furthermore, $\mathcal{U}'$ is a q-cover prebase due to being a q-cover base. Hence, it follows $\mathcal{U}' \sqsubseteq (q)\ \mathcal{V}'$, and similarly, by swapping the roles of $\mathcal{U}'$ and $\mathcal{V}'$, we get $\mathcal{V}' \sqsubseteq (q)\ \mathcal{U}'$. Now we can use Lemma 3.7.25(b) to infer $\mathcal{U}' = \mathcal{V}'$. ☐

Next we record that closedness under $q \cap q^{-1}$ is inherited by the q-maximal members of a collection of neighbourhoods.

3.7.27 Lemma. *If, for an algebra* $\mathbf{A}$ *and a quasiorder* $q \subseteq (\operatorname{Neigh} \mathbf{A})^2$*, a collection* $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ *is closed under* $q \cap q^{-1}$*, then so is* $\mathcal{V} := \operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}})$*.*

Proof: This is again a simple consequence of the fact that q-maximal elements are $(q \cap q^{-1})$-equivalent. Let $V \in \mathcal{V}$ and $U \in \operatorname{Neigh} \mathbf{A}$ such that $U\ q \cap q^{-1}\ V$. Since $V \in \mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{U}$ is closed under $q \cap q^{-1}$, we get $U \in [V]_{q \cap q^{-1}} \subseteq \mathcal{U}$. Since V is q-maximal in $\mathcal{U}$, we can now show that the same holds for U. Namely, if $W \in \mathcal{U}$ satisfies $U\ q{\restriction}_{\mathcal{U}}\ W$, then we get $V\ q{\restriction}_{\mathcal{U}}\ U\ q{\restriction}_{\mathcal{U}}\ W$, and so $W\ q{\restriction}_{\mathcal{U}}\ V\ q{\restriction}_{\mathcal{U}}\ U$ by maximality of V in $(\mathcal{U}, q{\restriction}_{\mathcal{U}})$. Thus, we have $W\ q{\restriction}_{\mathcal{U}}\ U$, showing that $U \in \operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}}) = \mathcal{V}$. ☐

From the previous results, we conclude that collections of q-maximal elements of q-cover bases which are $(q \cap q^{-1})$-closed are unique. This is exactly the type of result we have been looking for.

3.7.28 Corollary. *Let* $\mathbf{A}$ *be an algebra,* $q \subseteq \leq_{\mathrm{cov}}$ *a quasiorder on neighbourhoods of* $\mathbf{A}$ *and* $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ *be* q*-cover bases that are closed under* $q \cap q^{-1}$*. If the posets canonically associated with* $(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ *and* $(\mathcal{V}, q{\restriction}_{\mathcal{V}})$ *satisfy ACC, then we have the equality* $\operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}}) = \operatorname{Max}(\mathcal{V}, q{\restriction}_{\mathcal{V}})$*.*

Proof: By Lemma 3.7.27, the sets $\mathcal{U}' := \operatorname{Max}(\mathcal{U}, q{\restriction}_{\mathcal{U}})$ and $\mathcal{V}' := \operatorname{Max}(\mathcal{V}, q{\restriction}_{\mathcal{V}})$ are closed under $q \cap q^{-1}$ since $\mathcal{U}$ and $\mathcal{V}$ were. Since $\mathcal{U}$ and $\mathcal{V}$ are q-cover bases, they are covers, in particular, and hence $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{V}' \subseteq \mathcal{V}$ are covers, too, by Lemma 3.5.6(c). Hence, Lemma 3.7.3(d) implies that $\mathcal{U}'$ and $\mathcal{V}'$ are q-cover bases. Finally, Lemma 3.7.26(b) entails $\mathcal{U}' = \mathcal{V}'$. ☐

We would like to apply this corollary to the $(q \vee \precapprox)$-cover bases $\mathrm{Irr}^{*}_{q}(\mathbf{A})$ and $\mathrm{Irr}^{**}_{q}(\mathbf{A})$ exhibited in Corollary 3.7.2 for poly-ARTINian algebras having the FIP and $\subseteq_{\mathrm{Neigh}\,\mathbf{A}} \subseteq q \subseteq \leq_{\mathrm{cov}}$. To do this we would need to show that these $(q \vee \precapprox)$-cover bases are closed under the associated equivalence relation. This is feasible for $q \subseteq \precapprox$, but then only the case $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$ remains as explained before Corollary 3.7.18. So, in correspondence to the latter result we now finally obtain:

3.7.29 Corollary. *Let* $\mathbf{A}$ *be a poly-*ARTIN*ian algebra having the FIP such that the poset* $(\mathrm{Neigh}\,\mathbf{A}, \subseteq)$ *fulfils DCC and the one associated with* $\left(\mathrm{Irr}^{*}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{A})}\right)$ *satisfies ACC.*

Then we have $\mathrm{Max}\left(\mathrm{Irr}^{*}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{A})}\right) = \mathrm{Max}\left(\mathrm{Irr}^{**}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{**}(\mathbf{A})}\right)$*, and moreover, if* $\mathbf{A}$ *is finite, it is*

$$\begin{aligned}\mathrm{Max}\left(\mathrm{Irr}^{*}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{A})}\right) &= \mathrm{Max}\left(\mathrm{Irr}^{**}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{**}(\mathbf{A})}\right)\\ &= \mathrm{Max}\left(\mathrm{Irr}^{**(|A|)}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{**(|A|)}(\mathbf{A})}\right) = \mathrm{Max}\left(\mathrm{Irr}^{*(|A|)}(\mathbf{A}), \precapprox\restriction_{\mathrm{Irr}^{*(|A|)}(\mathbf{A})}\right).\end{aligned}$$

Proof: Under the mentioned conditions, all four collections

$$\mathcal{V} \in \left\{\mathrm{Irr}^{*}(\mathbf{A}), \mathrm{Irr}^{**}(\mathbf{A}), \mathrm{Irr}^{*(|A|)}(\mathbf{A}), \mathrm{Irr}^{**(|A|)}(\mathbf{A})\right\}$$

are $\precapprox$-cover bases (of $\mathbf{A}$ or a finite $\mathbf{A}$, respectively) due to Corollary 3.7.2. By Lemma 3.6.12(d), these cover bases $\mathcal{V}$ are closed under mutual embeddability, and the factor posets of $(\mathcal{V}, \precapprox\restriction_{\mathcal{V}})$ fulfil ACC, since this is true for the largest of them, $\mathcal{V} = \mathrm{Irr}^{*}(\mathbf{A})$. Hence, we have derived all the assumptions of Corollary 3.7.28, which finally implies the desired equalities. □

Thus, we have obtained a negative answer to the question posed before Definition 3.7.24. The set of $\precapprox$-maximal strictly irreducible neighbourhoods does not reduce by using crucially irreducible (or $|A|$-crucially irreducible neighbourhoods for finite algebras) instead. This should not be a cause for disappointment. We interpret the result in the following way: with the smaller sets of strictly irreducibles, i.e. $\mathrm{Irr}^{**}(\mathbf{A})$, and $\mathrm{Irr}^{*(|A|)}(\mathbf{A})$ or $\mathrm{Irr}^{**(|A|)}(\mathbf{A})$ for finite algebras, one has to check fewer pairs of invariant relations in order to prove that an (irreducible) neighbourhood indeed is strictly irreducible ((S,T)-irreducible for a pair of distinct invariants from a special class of relations). Nevertheless, the resulting $\precapprox$-maximal neighbourhoods are the same, so absolutely no information is lost by the mentioned simplification.

At last, it is now high time for a concrete example, to see our theory work in practice.

3.8 Elaborated example

In this section we want to present the example of a small algebra on four elements which shows that, generally, the notions of irreducible neighbourhood and that of strictly irreducible neighbourhood fall apart. At the same time, it is our goal to demonstrate in detail, how the notions developed in the previous sections work together, and how Corollary 3.7.23 can be used to determined non-refinable covers of finite algebras.

We will first define the example algebra, then we start our analysis by determining its unary clone operations. Subsequently, we will have a look at the unary part of the corresponding relational clone, i.e. the lattice of subuniverses of the algebra. Based on this information, we can describe all neighbourhoods, order them by inclusion and embedding, and furthermore, identify those among them that are irreducible and strictly irreducible, in particular. We can then use this knowledge to prove that this algebra actually has got a unique non-refinable cover, and show how it looks like.

Throughout this section the symbol E_4 designates the four-element set $\{0,1,2,3\}$, which will be the carrier of our example algebra. The latter will possess one binary and three idempotent, unary fundamental operations.

3.8.1 Definition. Let $\mathbf{E}_4 = \langle E_4; F\rangle$ be the algebra on $\{0,1,2,3\}$ having the following set $F = \{*, f, h_1, g_1\}$ of fundamental operations that are given by

$$f\colon\; E_4 \longrightarrow E_4,\quad x \longmapsto \begin{cases} x & \text{if } x\in\{0,1,2\}\\ 0 & \text{otherwise,}\end{cases} \qquad h_1\colon\; E_4 \longrightarrow E_4,\quad x \longmapsto \begin{cases} x & \text{if } x\in\{0,3\}\\ 0 & \text{otherwise,}\end{cases}$$

$$g_1\colon\; E_4 \longrightarrow E_4,\quad x \longmapsto \begin{cases} 0 & \text{if } x\in\{0,3\}\\ 1 & \text{otherwise,}\end{cases}$$

$*$	0	1	2	3
0	0	1	3	0
1	0	1	1	0
2	0	1	2	0
3	3	2	2	3,

where the rows of the operation table correspond to the first argument of $*$ and the columns to the second one. ◊

3.8.2 Remark. We rely on the computer package `UACalc` [FKV11] for the computation of the free algebra on one generator id_{E_4} in the variety $\operatorname{Var}\mathbf{E}_4$, i.e. the subpower generated by id_{E_4} in $\mathbf{E}_4^{E_4}$. Its carrier set equals $\mathrm{Term}^{(1)}(\mathbf{E}_4)$ and contains the following twelve unary operations, given by their values in Table 3.1. Due to Corollary 3.5.12 these functions coincide with the members of $\mathrm{Clo}^{(1)}(\mathbf{E}_4)$.

In passing by we note the following equalities, which can easily be checked using Definition 3.8.1. The ones in the left block show how the functions c_0, g_2 and h_2 can be generated from the fundamental operations of $\mathbf{E}_4$, the other ones will be used later:

$$\begin{aligned} c_0 &= f * h_1, & g_2 * f &= c_0 * \mathrm{id}_{E_4},\\ g_2 &= c_0 * (c_0 * \mathrm{id}_{E_4}), & (g_2 * f) * c_0 &= h_2 \circ f,\\ h_2 &= (h_1 * \mathrm{id}_{E_4}) * h_1, & f \circ h_1 &= c_0. \end{aligned}$$

id_{E_4}	f	h_1	h_2	g_1	g_2	c_0	$\mathrm{id}_{E_4} * g_2$	$\mathrm{id}_{E_4} * g_1$	$h_1 * \mathrm{id}_{E_4}$	$c_0 * \mathrm{id}_{E_4}$	$h_2 \circ f$
0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	1	1	1	1	0
2	2	0	3	1	0	0	0	1	3	3	3
3	0	3	3	0	0	0	3	3	3	0	0

Table 3.1: Unary term operations of $\mathbf{E}_4$.

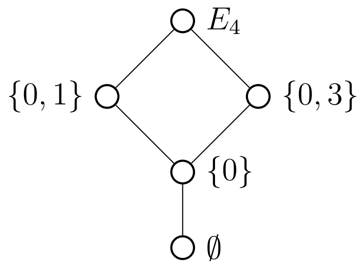

Figure 3.1: The lattice of subuniverses of $\mathbf{E}_4$ w.r.t. inclusion.

We have now gathered enough information to verify that all the functions listed in Table 3.1 are indeed contained in $\mathrm{Clo}^{(1)}(\mathbf{E}_4)$. So the only fact that we did not prove (and where we trust computers) is that the set of quadruples in Table 3.1 is closed under component-wise application of the fundamental operations of $\mathbf{E}_4$. $\diamondsuit$

For irreducibility arguments we shall use some knowledge about the unary invariants of our example algebra. Therefore, in the next lemma we have a brief look at the lattice of all subuniverses of $\mathbf{E}_4$.

3.8.3 Lemma. *The algebra $\mathbf{E}_4$ given in Definition 3.8.1 has the following set of subuniverses:*

$$\operatorname{Sub}\mathbf{E}_4 = \{\emptyset, \{0\}, \{0,1\}, \{0,3\}, E_4\}.$$

An order diagram of the lattice of subuniverses w.r.t. set inclusion can be seen in Figure 3.1.

Proof: First, it is straightforward to see that the mentioned subsets are indeed closed under the fundamental operations in F. As $c_0 \in \mathrm{Clo}^{(1)}(\mathbf{E}_4)$, every non-empty subuniverse has to contain 0 as an element. Therefore, for $x \in \{1,3\}$ the generated subuniverse $\langle\{x\}\rangle_{\mathbf{E}_4}$ must be the two-element set $\{0,x\}$ because the latter is closed, as seen before. Using our knowledge about $\mathrm{Clo}^{(1)}(\mathbf{E}_4)$, we see that $E_4 = \{h_1(2), g_1(2), 2, h_2(2)\} \subseteq \langle\{2\}\rangle_{\mathbf{E}_4}$. Hence, every subuniverse containing 2 must be the full carrier set E_4. So for non-empty generating sets of subuniverses, we can now focus on such neither containing 0 (because it is generated anyway) nor 2 (because it generates E_4). There are four possibilities for generating sets of this sort, depending on which of the elements 1 or 3 they contain. Three of them have already been discussed, the only remaining case is when the generating set contains both elements. It is easy to check that $E_4 = \{1 * 3, 1, 3 * 1, 3\} \subseteq \langle\{1, 3\}\rangle_{\mathbf{E}_4}$. So the join of the subuniverses $\{0, 1\}$ and $\{0, 3\}$ is E_4, and we are done. $\square$

Next, we examine neighbourhoods and their properties. First, we list all neighbourhoods of $\mathbf{E}_4$ and verify that they are all pairwise non-isomorphic. We conclude that the embedding quasiorder is just the partial order given by set inclusion. Then we determine which neighbourhoods are irreducible, and which are not. After that we have a look at two covers, one of which is non-refinable.

3.8.4 Lemma. *The set of idempotent unary operations in the clone of* $\mathbf{E}_4$ *is the ten-element set* $\mathrm{Clo}^{(1)}(\mathbf{E}_4) \setminus \{c_0 * \mathrm{id}_{E_4}, h_2 \circ f\}$. *These functions give rise to the following set of neighbourhoods*

$$\mathrm{Neigh}\,\mathbf{E}_4 = \{\{0\}, \{0, 1\}, \{0, 3\}, \{0, 1, 3\}, \{0, 1, 2\}, E_4\},$$

all of which are pairwise non-isomorphic.

Proof: Using Table 3.1 and Lemma 3.1.3, it is evident that all unary term operations apart from $c_0 * \mathrm{id}_{E_4}$ and $h_2 \circ f$ are idempotent. For all $i \in \{1, 2\}$ we have

$$\begin{aligned} \mathrm{im}\, g_i &= \{0, 1\}, & \mathrm{im}\, \mathrm{id}_{E_4} &= E_4, \\ \mathrm{im}\, h_i &= \{0, 3\}, & \mathrm{im}\, f &= \{0, 1, 2\}, \\ \mathrm{im}\, \mathrm{id}_{E_4} * g_i = \mathrm{im}\, h_1 * \mathrm{id}_{E_4} &= \{0, 1, 3\}, & \mathrm{im}\, c_0 &= \{0\}, \end{aligned}$$

so these sets are precisely the neighbourhoods of $\mathbf{E}_4$. We shall exclude the possibility that the two two-element ones or the two three-element ones are isomorphic. The argument is by contradiction. From Proposition 3.2.8(c), we know that if two neighbourhoods $U, V \in \mathrm{Neigh}\,\mathbf{E}_4$ are isomorphic then, among other conditions, there must exist a unary term operation $\varphi \in \mathrm{Clo}^{(1)}(\mathbf{E}_4)$ such that $\mathrm{im}\,\varphi = \varphi[U] = V$. This is impossible in both cases: if $(U, V) = (\{0, 1, 3\}, \{0, 1, 2\})$, then $\mathrm{im}\,\varphi = \{0, 1, 2\}$ implies $\varphi = f$, but $f[\{0, 1, 3\}] = \{0, 1\} \subset \{0, 1, 2\}$. If, in the second case, $(U, V) = (\{0, 3\}, \{0, 1\})$, then $\mathrm{im}\,\varphi = \{0, 1\}$ implies $\varphi \in \{g_1, g_2\}$, but $g_i[\{0, 3\}] = \{0\} \subset \{0, 1\}$ for both indices $i \in \{1, 2\}$. $\square$

3.8.5 Corollary. *For any two neighbourhoods* $U, V \in \mathrm{Neigh}\,\mathbf{E}_4$, *we have*

$$U \precapprox V \iff U \subseteq V,$$

i.e. if U is contained in V up to isomorphism, it is actually a subset. Consequently, the embedding quasiorder $(\mathrm{Neigh}\,\mathbf{E}_4, \precapprox)$ *is an order* $(\mathrm{Neigh}\,\mathbf{E}_4, \subseteq)$, *in this case even a lattice order (see Figure 3.2).*

Proof: For neighbourhoods $U, V \in \mathrm{Neigh}\,\mathbf{E}_4$, the condition $U \precapprox V$ holds if and only if there exists a subneighbourhood $U' \subseteq V$ that is isomorphic to U (see Proposition 3.2.10(b)). By Lemma 3.8.4 we obtain $U = U'$ from $U \cong U'$, so $U = U' \subseteq V$. The reverse implication is generally clear by Lemma 3.2.12(a). $\square$

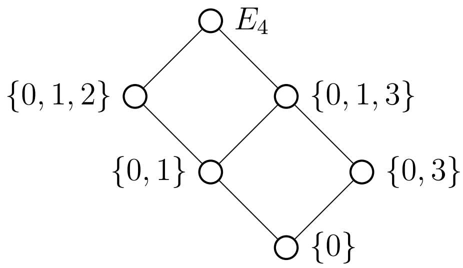

Figure 3.2: The lattice $(\mathrm{Neigh}\,\mathbf{E}_4, \subseteq)$ of neighbourhoods of $\mathbf{E}_4$, which equals $(\mathrm{Neigh}\,\mathbf{E}_4, \precapprox)$.

We can now combine the previous observations to obtain different covers of $\mathbf{E}_4$, especially the non-refinable one.

3.8.6 Lemma. *For the algebra $\mathbf{E}_4$ as given in Definition 3.8.1 the following assertions are true.*

(a) The equation $h_2 * g_1 = \mathrm{id}_{E_4}$ is a decomposition equation.

(b) The collections $\{\{0,1\}, \{0,3\}\}$ and $\{\{0,1,2\}, \{0,3\}\}$ are covers of $\mathbf{E}_4$. Moreover, we have $\{\{0,1\}, \{0,3\}\} \in \mathrm{Cov}_{\mathbf{E}_4}(\{0,1,3\})$.

(c) The neighbourhood $\{0\}$ is $(\emptyset, \{0\})$-irreducible, $\{0,1\}$ is $(\{0\}, \{0,1\})$-irreducible, $\{0,3\}$ is $(\{0\}, \{0,3\})$-irreducible, and $\{0,1,2\}$ is $(\{0,1\}, E_4)$-irreducible.

(d) The set of irreducible neighbourhoods is

$$\mathrm{Irr}(\mathbf{E}_4) = \{\{0\}, \{0,1\}, \{0,3\}, \{0,1,2\}\},$$

the $\precapprox$-maximal irreducible ones are

$$\mathrm{Max}\left(\mathrm{Irr}(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}(\mathbf{E}_4)}\right) = \{\{0,3\}, \{0,1,2\}\}.$$

The latter set is an irredundant cover consisting of pairwise non-isomorphic $\precapprox$-maximal irreducible neighbourhoods.

(e) The collection of 1-strictly irreducible neighbourhoods equals the set of strictly irreducible ones, and it is $\mathrm{Irr}^*(\mathbf{E}_4) = \mathrm{Irr}^{*(1)}(\mathbf{E}_4) = \{\{0\}, \{0,1\}, \{0,3\}\}$. The $\precapprox$-maximal neighbourhoods among them are

$$\mathrm{Max}\left(\mathrm{Irr}^*(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}^*(\mathbf{E}_4)}\right) = \{\{0,1\}, \{0,3\}\},$$

and they are pairwise non-isomorphic, so the latter set is the unique (not only up to isomorphism) non-refinable cover.

(f) *The non-refinable cover is a proper refinement of the (any) irredundant subcover of the collection of all $\precsim$-maximal irreducible neighbourhoods:*

$$\operatorname{Max}\left(\operatorname{Irr}^*(\mathbf{E}_4), \precsim\restriction_{\operatorname{Irr}^*(\mathbf{E}_4)}\right) = \{\{0,1\},\{0,3\}\}$$
$$<_{\mathrm{ref}} \operatorname{Max}\left(\operatorname{Irr}(\mathbf{E}_4), \precsim\restriction_{\operatorname{Irr}(\mathbf{E}_4)}\right) = \{\{0,3\},\{0,1,2\}\}.$$

Proof: (a) The equation $h_2 * g_1 = \mathrm{id}_{E_4}$ can be checked using Definition 3.8.1. It is a decomposition equation since h_2 and g_1 are idempotent unary terms belonging to the neighbourhoods $\{0,3\}$ and $\{0,1\}$, and $*$ is a binary term operation.

(b) From item (a) we can immediately conclude $\{\{0,1\},\{0,3\}\} \in \operatorname{Cov}(\mathbf{E}_4)$ (cp. Corollary 3.4.36). This directly implies that also $\{\{0,1,2\},\{0,3\}\}$ covers $\mathbf{E}_4$ because every pair of relations that is separated by $\{0,1\}$ is also separated by $\{0,1,2\}$. This is, in fact, an instance of Remark 3.5.1 saying that the condition $\{\{0,1\},\{0,3\}\} \sqsubseteq (\subseteq) \{\{0,1,2\},\{0,3\}\}$ implies

$$E_4 \leq_{\mathrm{cov}} \{\{0,1\},\{0,3\}\} \leq_{\mathrm{cov}} \{\{0,1,2\},\{0,3\}\},$$

i.e. $\{\{0,1,2\},\{0,3\}\} \in \operatorname{Cov}(\mathbf{E}_4)$.

Moreover, as $\{\{0,1\},\{0,3\}\} \in \operatorname{Cov}(\mathbf{E}_4)$, we obtain that this set also covers $\{0,1,3\}$, because

$$\{0,1,3\} \leq_{\mathrm{cov}} E_4 \leq_{\mathrm{cov}} \{\{0,1\},\{0,3\}\}$$

due to the inclusion $\{0,1,3\} \subseteq E_4$ and Corollary 3.4.6(a).

(c) A pair of invariant relations $S,T \in \operatorname{Inv}^{(m)}\mathbf{E}_4$ such that $S \subseteq T$ is separated by a neighbourhood $U \in \operatorname{Neigh}\mathbf{E}_4$ if and only if $(T \setminus S) \cap U^m \neq \emptyset$. For unary relations this condition becomes $(T \setminus S) \cap U \neq \emptyset$.

If $(S,T) = (\emptyset,\{0\})$, then $U \in \operatorname{Neigh}\mathbf{E}_4$ separates if and only if $0 \in U$. So the unique $(\emptyset,\{0\})$-irreducible neighbourhood is $\{0\}$. Choosing $(\{0\},\{0,x\})$ for the pair (S,T), $x \in \{1,3\}$, we get that a neighbourhood $U \in \operatorname{Neigh}\mathbf{E}_4$ separates it if and only if $x \in U$. Thus, the unique $(\{0\},\{0,x\})$-irreducible neighbourhood is the set $\{0,x\}$. In case (S,T) equals $(\{0,1\},E_4)$, the pair is separated by $U \in \operatorname{Neigh}\mathbf{E}_4$ if and only if $2 \in U$ or $3 \in U$. So the $(\{0,1\},E_4)$-irreducible neighbourhoods of $\mathbf{E}_4$ are $\{0,1,2\}$ and $\{0,3\}$.

(d) We have seen in item (b) that E_4 and $\{0,1,3\}$ are covered by the collection $\{\{0,1\},\{0,3\}\}$ consisting of proper subneighbourhoods of the respective sets. So the neighbourhoods E_4 and $\{0,1,3\}$ are both reducible. By item (c) the other four neighbourhoods are (S,T)-irreducible for unary invariant relations S and T. Therefore, they are irreducible (cp. Lemma 3.6.8) and, consequently, $\operatorname{Irr}(\mathbf{E}_4) = \{\{0\},\{0,1\},\{0,3\},\{0,1,2\}\}$. Applying Corollary 3.8.5, we get that the $\precsim$-maximal irreducible ones are given by

$$\operatorname{Max}\left(\operatorname{Irr}(\mathbf{E}_4), \precsim\restriction_{\operatorname{Irr}(\mathbf{E}_4)}\right) = \{\{0,3\},\{0,1,2\}\},$$

and these sets are trivially non-isomorphic since they have different cardinalities. We have verified in item (b) that the last mentioned collection is a cover. Irredundancy is quickly shown by finding pairs of relations that are not separated by its proper subcollections: $\{\{0,3\}\}$ does not distinguish the pair $(\{0\},\{0,1\})$ of unary invariants, and $\{\{0,1,2\}\}$ does not suffice to separate $(\{0\},\{0,3\})$.

(e) A combination of the statements in item (c) and a look at Figure 3.1 yields

$$\{\{0\},\{0,1\},\{0,3\}\} \subseteq \mathrm{Irr}^{*(1)}(\mathbf{E}_4) \subseteq \mathrm{Irr}^{*}(\mathbf{E}_4) \subseteq \mathrm{Irr}(\mathbf{E}_4).$$

We shall prove that $\{0,1,2\} \notin \mathrm{Irr}^{*}(\mathbf{E}_4)$. Therefore, we get

$$\mathrm{Irr}^{*}(\mathbf{E}_4) \subseteq \mathrm{Irr}(\mathbf{E}_4) \setminus \{\{0,1,2\}\} = \{\{0\},\{0,1\},\{0,3\}\},$$

which together with the previous chain of inclusions shows

$$\mathrm{Irr}^{*}(\mathbf{E}_4) = \mathrm{Irr}^{*(1)}(\mathbf{E}_4) = \{\{0\},\{0,1\},\{0,3\}\}.$$

If the neighbourhood $\{0,1,2\}$ were strictly irreducible, then it would have to embed into at least one member of every cover of the algebra $\mathbf{E}_4$ (Proposition 3.6.22 for $q = \subseteq_{\mathrm{Neigh}\,\mathbf{A}}$). In fact, Corollary 3.8.5 implies that it would have to be contained as a subneighbourhood of some member of an arbitrary cover of $\mathbf{E}_4$. However, in item (b) we saw that $\{\{0,1\},\{0,3\}\}$ is a cover of $\mathbf{E}_4$ violating this condition. Hence, $\{0,1,2\} \in \mathrm{Irr}(\mathbf{E}_4) \setminus \mathrm{Irr}^{*}(\mathbf{E}_4)$, as claimed.

Using again Corollary 3.8.5, we see

$$\mathrm{Max}\left(\mathrm{Irr}^{*}(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{E}_4)}\right) = \{\{0,1\},\{0,3\}\},$$

and these neighbourhoods are pairwise non-isomorphic by Lemma 3.8.4. By the general description of non-refinable covers in Corollary 3.7.23, we can conclude that $\mathrm{Max}\left(\mathrm{Irr}^{*}(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{E}_4)}\right)$ is a non-refinable cover, and that, for this special algebra, it is actually unique.

(f) Clearly, the non-refinable cover $\{\{0,1\},\{0,3\}\}$ is a refinement of the cover $\mathrm{Max}\left(\mathrm{Irr}(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}(\mathbf{E}_4)}\right)$ The converse refinement relation does not hold since the three-element neighbourhood $\{0,1,2\}$ is not contained in any member of $\mathrm{Max}\left(\mathrm{Irr}^{*}(\mathbf{E}_4), \precapprox\restriction_{\mathrm{Irr}^{*}(\mathbf{E}_4)}\right)$, cp. Lemma 3.5.4(d). □

3.8.7 Remark. The proof of irredundancy of the cover $\{\{0,3\},\{0,1,2\}\}$ (see Lemma 3.8.6(d)) can also be done in a different way. In the proof as seen above, we refuted the cover property of the proper subcollections by finding pairs of invariant relations that are not separated. Alternatively, exploiting Corollary 3.4.36, one could show that a decomposition equation is impossible. For this one has to check that the subpower closure of the 4-ary relation given by the operations of the form

$e \circ f$ where $f \in \mathrm{Clo}^{(1)}(\mathbf{E}_4)$ and e is an idempotent term operation belonging to a neighbourhood in the subcollection does not generate the identity function on E_4. In other words, the generators we consider are precisely all term operations whose images are contained in one of the neighbourhoods of the subcollection.

In our case the two subcollections are $\{\{0,3\}\}$ and $\{\{0,1,2\}\}$. The unary operations in $\mathrm{Clo}^{(1)}(\mathbf{E}_4)$ having their images in $\{0,3\}$ are h_1, h_2 and $h_2 \circ f$. They generate

$$\langle\{h_1, h_2, h_2 \circ f\}\rangle_{\mathbf{E}_4^{E_4}} = \{h_1, h_2, h_2 \circ f, f \circ h_2 = c_0\},$$

which does not contain id_{E_4}.

Likewise, for $\{\{0,1,2\}\}$ the generators are f, g_1, g_2 and c_0. Their generated subpower is

$$\langle\{f, g_1, g_2, c_0\}\rangle_{\mathbf{E}_4^{E_4}} = \{f, g_1, g_2, c_0, g_2 * f, (g_2 * f) * c_0\},$$

which neither contains id_{E_4} since by Remark 3.8.2 the equalities $g_2 * f = c_0 * \mathrm{id}_{E_4}$ and $(g_2 * f) * c_0 = h_2 \circ f$ hold.

So in both cases it is impossible to obtain a decomposition equation, whence the cover $\{\{0,3\}, \{0,1,2\}\}$ is irredundant.

Generating subpowers instead of finding invariant relations that are not separated can be more effort if done by hand. However, we have presented this alternative approach because for an automatic check of the cover property, the method used in this remark seems to be more feasible. $\diamondsuit$

With the following we illustrate a second example, and at the same time we demonstrate, that one has quite careful to set up an algebra witnessing the inequality $\mathrm{Irr}^*(\mathbf{A}) \neq \mathrm{Irr}(\mathbf{A})$. A small modification of $\mathbf{E}_4$, which does not even change the set of neighbourhoods, already implies that there is no difference between strictly irreducible neighbourhoods and ordinary irreducible ones, any more. In point of fact, practical experience tells that this is often the case for small finite algebras. Therefore, it usually is a good idea, to start with a cover consisting of $\precapprox$-maximal irreducible neighbourhoods and then to derive from this a non-refinable cover.

3.8.8 Remark. It is interesting to see what happens if one removes the fundamental operation g_1. Although only small changes for $\mathrm{Clo}^{(1)}(\mathbf{E}_4)$ and $\mathrm{Sub}\,\mathbf{E}_4$ occur, the result is quite different. Let us call the new algebra $\mathbf{F}_4 = \langle E_4; f, h_1, *\rangle$. Since we have not changed the operation $*$, we can use the equations in Remark 3.8.2 to see that the following functions are unary term operations of $\mathbf{F}_4$:

$$\begin{aligned}\mathrm{Clo}^{(1)}(\mathbf{F}_4) &= \{\mathrm{id}_{E_4}, f, h_1, h_2, g_2, c_0, \mathrm{id}_{E_4} * g_2, h_1 * \mathrm{id}_{E_4}, c_0 * \mathrm{id}_{E_4}, h_2 \circ f\}\\ &\subseteq \mathrm{Clo}^{(1)}(\mathbf{E}_4).\end{aligned}$$

Again, we trust `UACalc` that this list is complete.

Since all fundamental operations of $\mathbf{F}_4$ are term operations of $\mathbf{E}_4$, we obtain the inclusion $\mathrm{Sub}\,\mathbf{E}_4 \subseteq \mathrm{Sub}\,\mathbf{F}_4$. In addition, we see, that all fundamental operations of

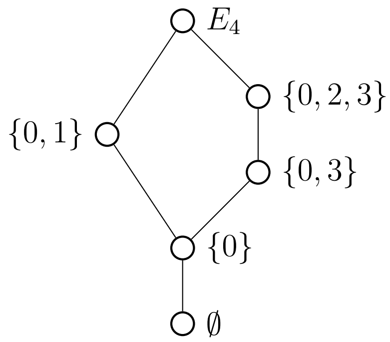

Figure 3.3: The lattice of subuniverses of $\mathbf{F}_4$.

$\mathbf{F}_4$ preserve the subset $\{0,2,3\}$, so $\langle\{2\}\rangle_{\mathbf{F}_4} = \{0,2,3\}$. This is the only modification of the proof of Lemma 3.8.3 that has to be made to get that

$$\operatorname{Sub}\mathbf{F}_4 = \{\emptyset, \{0\}, \{0,1\}, \{0,3\}, \{0,2,3\}, E_4\},$$

see Figure 3.3 for an order diagram of $(\operatorname{Sub}\mathbf{F}_4, \subseteq)$.

Using the description of $\operatorname{Clo}^{(1)}(\mathbf{F}_4)$, it is clear that

$$\operatorname{Neigh}\mathbf{F}_4 = \operatorname{Neigh}\mathbf{E}_4 = \{\{0\}, \{0,1\}, \{0,3\}, \{0,1,3\}, \{0,1,2\}, E_4\},$$

and with almost literally the same arguments as in the proof of Lemma 3.8.4, one can see that all of them are pairwise non-isomorphic.

As in Lemma 3.8.6(c) we verify that the neighbourhood $\{0\}$ is $(\emptyset, \{0\})$-irreducible, $\{0,1\}$ is $(\{0\}, \{0,1\})$-irreducible, $\{0,3\}$ is $(\{0\}, \{0,3\})$-irreducible, and $\{0,1,2\}$ is $(\{0,3\}, \{0,2,3\})$-irreducible. Besides, the equation $h_1 * g_2 = \mathrm{id}_{E_4} * g_2$ shows that $\{0,1,3\}$ is covered by its proper subneighbourhoods $\{0,3\}$ and $\{0,1\}$ (cp. Corollary 3.4.36). Thus, $\{0,1,3\}$ is a reducible neighbourhood. Similarly, from $h_2 * f = \mathrm{id}_{E_4}$ we obtain that $\{\{0,3\}, \{0,1,2\}\} \in \operatorname{Cov}(\mathbf{F}_4)$, whence E_4 is reducible.

From the previous paragraph we can immediately conclude that

$$\{\{0\}, \{0,1\}, \{0,3\}, \{0,1,2\}\} = \operatorname{Irr}^{*(1)}(\mathbf{F}_4) = \operatorname{Irr}^*(\mathbf{F}_4) = \operatorname{Irr}(\mathbf{F}_4).$$

Consequently, the collection

$$\operatorname{Max}\left(\operatorname{Irr}(\mathbf{F}_4), \precsim\restriction_{\operatorname{Irr}(\mathbf{F}_4)}\right) = \operatorname{Max}\left(\operatorname{Irr}^*(\mathbf{F}_4), \precsim\restriction_{\operatorname{Irr}^*(\mathbf{F}_4)}\right) = \{\{0,3\}, \{0,1,2\}\}$$

of pairwise non-isomorphic neighbourhoods is a non-refinable cover, which is again unique for the algebra $\mathbf{F}_4$. ♢

3.8.9 Remark. Our main example algebra $\mathbf{E}_4$ shows that the concepts of strictly irreducible neighbourhood and irreducible neighbourhood are truly distinct. Indeed, we proved in Lemma 3.8.6(d) and (e) that $\{0,1,2\} \in \operatorname{Irr}(\mathbf{E}_4) \setminus \operatorname{Irr}^*(\mathbf{E}_4)$. While the notions of strict irreducibility and crucial irreducibility agree if neighbourhoods are considered that are maximal among their kind w.r.t. the embedding quasiorder $\precapprox$, our example even proves that this does not happen in the case of strict irreducibility and ordinary irreducibility. This is because we have

$$\{0,1,2\} \in \operatorname{Max}\left(\operatorname{Irr}\left(\mathbf{E}_4\right), \precapprox\restriction_{\operatorname{Irr}(\mathbf{E}_4)}\right) \setminus \operatorname{Irr}^*\left(\mathbf{E}_4\right),$$

which is equivalent to the condition

$$\{0,1,2\} \in \operatorname{Max}\left(\operatorname{Irr}\left(\mathbf{E}_4\right), \precapprox\restriction_{\operatorname{Irr}(\mathbf{E}_4)}\right) \setminus \operatorname{Max}\left(\operatorname{Irr}^*\left(\mathbf{E}_4\right), \precapprox\restriction_{\operatorname{Irr}^*(\mathbf{E}_4)}\right).$$

Combining this observation with the fact that all neighbourhoods in the two mentioned collections are pairwise non-isomorphic, we can see that the description of non-refinable covers as a set of representatives of isomorphism classes of $\precapprox$-maximal irreducible neighbourhoods given in Theorem 2.9(3) of [KL10] does indeed not work for the algebra $\mathbf{E}_4$. A second way to see this is, of course, item (f) of Lemma 3.8.6. ◇

The importance of the differences exhibited in the previous remark lies in the connection between maximal strictly irreducible neighbourhoods and the computation of non-refinable covers for finite algebras. Corollary 3.7.23 states that a collection of representatives for the isomorphism classes of maximal strictly (alternatively: crucially) irreducible neighbourhoods is a non-refinable cover, which is unique up to isomorphism of covers.

Even though this is a nice description from the theoretical point of view, checking for strict irreducibility causes a lot of computational effort in the practical case. For small examples, which, as it is the case here, can be studied using the paper-and-pencil-method, it is usually a good idea to start with determining the lattices of invariants of small arity to find the m-crucial pairs for small arities $m = 1, 2, 3, \ldots$. Then one can examine, which neighbourhoods are 1-strictly, 2-strictly, 3-strictly irreducible etc. However, an automatic check, that can be run, for instance, on a computer, requires consideration of the worst-case scenario. It occurs if one wants to exclude the possibility that a neighbourhood is strictly irreducible. How can one prove that an irreducible neighbourhood is not strictly irreducible? We ask the question in this way, because refuting irreducibility of neighbourhoods can be done using the irreducibility criterion given in Proposition 3.6.15(b) in conjunction with the characterisation of covers of finite neighbourhoods in Corollary 3.4.34 and Theorem 3.4.31(f). This means, it is not too hard to identify the irreducible neighbourhoods within the collection of all neighbourhoods of a finite algebra[43]. The more difficult task is to find out, which of them are strictly (or crucially) irreducible. At present knowledge, the only answer that can be given to the question posed above is: one would have to check for crucial pairs of arity $|A|$ (contained in the principal downset generated by $\operatorname{Clo}^{(1)}(\mathbf{A})$) that the neighbourhood either does not separate the pair, or it is not minimal under inclusion w.r.t. this property.

[43] Alternatively, one could also use the criterion established in Theorem 3.9.8 of [Beh09], see also Corollary 5.9 of [Kea01].

Due to the sheer number of partial tasks, this answer is often not suitable for a computer.

Frightened by the high computational effort, one could be tempted to use an irredundant subcover of the collection of all $\precapprox$-maximal irreducible neighbourhoods (or an irredundant subcover of its isomorphism representatives). In many small examples the cover obtained in this way is already non-refinable, often even without taking an irredundant subcollection w.r.t. being a cover once isomorphic copies have been excluded. So the behaviour of small algebras could support the (false) conjecture that an irredundant subcover of all $\precapprox$-maximal irreducible neighbourhoods generally would be a non-refinable cover. If it had been true, the computation of non-refinable covers would have been much easier.

Yet our example $\mathbf{E}_4$ unmistakably makes clear that, in general, one cannot hope for such a simplification. Lemma 3.8.6(f) explicitly refutes the conjecture. Its only remaining use could be to start with the computation of an irredundant subcover of all $\precapprox$-maximal irreducible neighbourhoods and to further refine this collection until one obtains a non-refinable cover.

4 Problems and Prospects for Future Research

Finally, a few open problems are presented that emerged during the writing process of this thesis but, unfortunately, could not be solved or written down in due time. Furthermore, we are going to have a look at interesting topics for future research.

The first problem touches a concrete piece of RST that has not been looked at in this thesis. Items (2) to (4) deal with applications of RST to categorical equivalence of algebras, while problems (5) to (9) ask for extensions of different parts of the theory. Question (10) is about a possible application to the clone lattice, and the last question addresses a generalisation of RST, and in particular the results of this thesis, to a much broader context.

(1) The first problem we state here should be quite a routine task and simply has not been included in this thesis for lack of time. In Section 3.9 of [Beh09] the notion of (S,T)-minimality of neighbourhoods and algebras is developed, where S and T are invariant relations of equal arity in a finite algebra without nullary operations. Since (S,T)-minimality of a neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ can be formulated in terms of the restricted algebra $\mathbf{A}|_U$ (see Definition 3.9.1(ii) of [Beh09]), it suffices to give the definition of an (S,T)-minimal algebra: $\mathbf{A}$ is (S,T)-minimal if and only if every unary operation $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ is a permutation, or its action on tuples maps S into T and vice versa. In case that S is a subrelation of T, this definition reduces to requiring that every non-permutation $f \in \operatorname{Clo}^{(1)}(\mathbf{A})$ maps T into S. This idea is contained in Definition 5.6 of [Kea01] and corresponds to the notion of (δ,θ)-minimal algebra from [HM88, Definition 2.12 and Lemma 2.13(1), p. 32], where δ, θ are congruences, forming a so-called congruence quotient, i.e. satisfying $\delta \subseteq \theta$.

The main results from Section 3.9 of [Beh09] are the following: (S,T)-minimality of a neighbourhood implies (S,T)-irreducibility, and moreover, irreducibility, i.e. (S,T)-irreducibility for some pair (S,T) of invariants of identical arity, is equivalent to (S,T)-minimality for some, in general different, pair of invariants.

In this sense (S,T)-minimality, generalising the well-recognised concept of (δ,θ)-minimality from classical TCT, and the notion of irreducibility, which is more naturally arising in RST (see Section 3.6), are equivalent for finite algebras.

Moreover, Theorem 3.9.8 of [Beh09] (cf. also Corollary 5.9 in [Kea01]) provides a characterisation of irreducibility of neighbourhoods U in finite algebras by the fact that the unary non-surjective operations in $\mathrm{Clo}^{(1)}(\mathbf{A}|_U)$ form a subpower of $(\mathbf{A}|_U)^U$. This can be turned into the equivalent criterion requiring that

$$\mathrm{id}_U \notin \left\langle \left\{ f|_U \;\middle|\; f \in \mathrm{Clo}^{(1)}(\mathbf{A}) \wedge \mathrm{im}\, f \subseteq U \wedge f[U] \subset U \right\} \right\rangle_{\mathbf{A}^U}.$$

This characterisation is important for at least two reasons. First, the condition mentioned in the previous displayed formula is a practicable tool to check irreducibility of neighbourhoods in finite algebras by computer. Second, the characterisation of irreducibility in [Beh09, Theorem 3.9.8] establishes a connection between $\subseteq$-irreducibility as in Definition 3.5.16 and irreducibility notions differently defined in the literature (cf. [Zád98, Theorem 2.2], [Zád97b, Proposition 1.11, Corollary 1.12] and implicitly in Section 4 of [BB98, e.g. Lemma 4.2]).

For the mentioned reasons the author of this thesis considers it a valuable task to generalise and / or modify the definitions and results discussed above to infinite algebras as far as this is possible. At least a generalisation to algebras in locally finite varieties seems to be desirable.

(2) The second question is of similar nature. In Section 3.5 of [Beh09], it has been elaborated that every cover of a finite algebra gives rise to a localisation functor mapping the algebra to the matrix product of restricted algebras corresponding to the neighbourhoods in the cover. As suggested at the end of Section 3.4, the decomposition equation belonging to the cover can be exploited to show that this functor is part of a categorical equivalence, and this fact has been proven in Section 3.5 of [Beh09] for algebras without nullary operations. However, the inverse functor has not been given explicitly there, which leads to the open problem: extend the results in [Beh09], at least to all algebras generating locally finite varieties (cf. Corollary 3.4.37), by explicitly describing both parts of the categorical equivalence. We are aware of the fact that such a description is implicit in the proofs of [McK96]. However, we think that a concrete description is nevertheless helpful for generalisations (see below), and moreover, we want to escape the caveat made w.r.t. nullary operations in McKenzie's paper.

Algebras generating locally finite varieties possess locally closed clones of term operations (see Corollary 3.5.13), wherefore the decomposition equation belonging to a cover of such an algebra can be transferred to all members of the generated variety via the terms belonging to the functions in such an equation. Based on a concrete description of the functors involved in the categorical equivalence above, it should be examined if, from a decomposition equation as in Corollary 3.4.37, one can also obtain a categorical equivalence between the variety generated by an algebra $\mathbf{A}$ and the one generated by a matrix product in the sense of Definition 3.4.38, even without the requirement that $\mathrm{Clo}(\mathbf{A}) = \mathrm{Term}(\mathbf{A})$.

(3) Is it possible to obtain some sort of categorical equivalence from a cover if one has only got a decomposition via a jointly-finite local retract as in Corollary 3.4.35(j), but not a global retract as e.g. in the finite case (see Corollary 3.4.36)?

(4) The relational characterisation result for categorical equivalence of finite algebras announced at the end of Section 3.4 still requires a detailed proof. Building on the previous two items, is it possible to extend this characterisation beyond the realm of finite algebras?

(5) Give a formal description, e.g. in terms of first order formulæ, of properties that can be transferred via covers from restricted structures to the global algebra as hinted at in connection with Example 3.4.1

(6) It should be explored further, how algebras satisfying certain Mal'cev conditions can be studied via RST. For the global to local correspondence, see Remark 3.3.5, for the converse direction the ideas presented at the beginning of Section 3.4.

(7) It has been mentioned in Remark 3.3.5 that the fact that a unary idempotent clone operation $e \in \operatorname{Idem} \mathbf{A}$ is a homomorphism $e\colon \mathbf{A} \longrightarrow \mathbf{A}$ w.r.t. the fundamental operations of $\mathbf{A}$ seems to be a helpful condition. This special case should be looked at and studied in more detail. One example where this happens is if one considers idempotent unary polynomials of finite distributive lattices. All of them are lattice homomorphisms.

(8) This thesis has mainly focussed on the study of single (finite) algebras. A good localisation theory however, should provide more than just that. Suppose that $\mathbf{A}$ is an algebra generating a locally finite variety, and $\mathbf{B}$ is a member of the variety generated by $\mathbf{A}$. Then one should have an explicit description of how one can transfer knowledge about $\mathbf{A}$, which can be obtained using e.g. the results of this thesis, to $\mathbf{B}$.

    Since $\mathbf{A}$ and $\mathbf{B}$ belong to a locally finite variety, they have locally closed clones of term operations, i.e. $\operatorname{Clo}(\mathbf{A}) = \operatorname{Term}(\mathbf{A})$ and $\operatorname{Clo}(\mathbf{B}) = \operatorname{Term}(\mathbf{B})$. As $\mathbf{B}$ fulfils all identities that hold in $\mathbf{A}$, there is a canonical surjective clone homomorphism h between $\operatorname{Term}(\mathbf{A})$ and $\operatorname{Term}(\mathbf{B})$ given by evaluation of the underlying term. This implies, we automatically get a full description of $\operatorname{Clo}(\mathbf{B})$ from $\operatorname{Clo}(\mathbf{A})$, of $\operatorname{Idem} \mathbf{B}$ as $\{ h(e) \mid e \in \operatorname{Idem} \mathbf{A} \}$ and of $\operatorname{Neigh} \mathbf{B}$ as $\{ \operatorname{im} h(e) \mid e \in \operatorname{Idem} \mathbf{A} \}$.

    The author of this thesis has obtained numerous (unpublished) partial results how certain properties as, for instance, reducibility, irreducibility, (S,T)-irreducibility, embedding and isomorphism of neighbourhoods translate back and forth between $\mathbf{A}$ and $\mathbf{B}$. However, the overall picture is still missing, concerning in particular the transfer of non-refinable covers. We therefore state it as an open question to complete this part of the theory. Especially, the connection between non-refinable covers of $\mathbf{A}$ and that of $\mathbf{B} \in \operatorname{Var} \mathbf{A}$ should be elucidated further.

(9) The next open problem is similar to the previous one, but concerns polynomial expansions of algebras. Let $\mathcal{K}$ be a class of algebras of the same type, e.g. a variety, then one is interested in the class $\mathcal{C} := [\mathbf{A}_A \mid \mathbf{A} \in \mathcal{K}]$ of polynomial expansions of algebras from $\mathcal{K}$. Examining structures in $\mathcal{C}$ using RST corresponds to examining the local closures of clones of polynomial operations from algebras in $\mathcal{K}$, and is hence more closely related to classical TCT.

Let us consider the case that $\mathcal{K} := \operatorname{Var} \mathbf{A}$ is the variety generated by $\mathbf{A}$, and we have got a good understanding of the RST-related concepts of $\mathbf{A}_A$. How can one use this knowledge in order to understand e.g. covers of other members in $\mathcal{C}$? It should be noted that this problem cannot be treated as an instance of the previous one because the algebras belonging to $\mathcal{C}$ do not even have the same signatures (due to different nullary constants depending on the respective carrier set).

(10) Classify all algebras on a three-element carrier set based on the distribution of their neighbourhoods and, in particular, the shape of their unique non-refinable cover. In connection with the comments at the end of Section 3.4, this asks in fact for a description of three-element algebras up to categorical equivalence. Does this help to understand the structure of the clone lattice on a three-element set?

(11) A usable implementation of RST-concepts for finite algebras in `UACalc`, see [FKV11], is still missing. We state this as a project for the future, and any contribution is highly appreciated.

(12) The author is strongly convinced that, using the ideas presented in this thesis, it is possible to develop a general decomposition theory for structures based on sets of locally manageable interesting properties of a structure and an axiomatic treatment of the localisation process.

In the following paragraphs we will sketch the decomposition framework we have in mind, without going into too much detail about the axioms we require for the localisation part. Let $\mathcal{C}$ be a class of structures together with an assignment $|\cdot| \colon \mathcal{C} \longrightarrow \mathcal{S}et$ of a set for each structure, which we interpret as a carrier set. We shall denote the members of $\mathcal{C}$ by boldface letters, such as $\mathbf{A}$, and, for convenience, we then write A for $|\mathbf{A}|$. Furthermore, each structure $\mathbf{A} \in \mathcal{C}$ is associated with a set $\operatorname{Neigh} \mathbf{A} \subseteq \mathfrak{P}(A)$ of local subsets, called neighbourhoods, for which we require $A \in \operatorname{Neigh} \mathbf{A}$. Moreover, with every structure $\mathbf{A} \in \mathcal{C}$ we associate a set $\operatorname{Things} \mathbf{A}$ of *properties* and a set $\operatorname{Int} \mathbf{A} \subseteq \operatorname{Things} \mathbf{A}$ of *interesting properties*. In the case of RST, we may choose $\operatorname{Things} \mathbf{A} := \bigcup_{m \in \mathbb{N}} \left(\operatorname{Inv}^{(m)} \mathbf{A}\right)^2$ and the interesting things as all distinct pairs of invariant relations of identical arity, i.e. $\operatorname{Int} \mathbf{A} := \{ (S, T) \in \operatorname{Things} \mathbf{A} \mid S \neq T \}$. For every neighbourhood

$U \in \operatorname{Neigh} \mathbf{A}$ we need to specify a set $\operatorname{Man}_{\mathbf{A}}(U) \subseteq \operatorname{Int} \mathbf{A}$ of *manageable properties* that can be handled by the neighbourhood U. In order not to have redundancy, we require that $\operatorname{Man}_{\mathbf{A}}(A) = \operatorname{Int} \mathbf{A}$, i.e. that the full neighbourhood is capable of managing all interesting properties. With regard to RST, we use $\operatorname{Man}_{\mathbf{A}}(U) := \operatorname{Sep}_{\mathbf{A}}(U)$ as the set of properties managed by a neighbourhood.

This definition can be extended to collections $\mathcal{U} \subseteq \operatorname{Neigh} \mathbf{A}$ of neighbourhoods via

$$\operatorname{Man}_{\mathbf{A}}(\mathcal{U}) := \bigcup \{ \operatorname{Man}_{\mathbf{A}}(U) \mid U \in \mathcal{U} \},$$

and based on this, for $\mathcal{U}, \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$, one can define a *covering quasiorder* $\mathcal{U} \leq_{\mathrm{cov}}{}^{\mathbf{A}} \mathcal{V}$ by the inclusion $\operatorname{Man}_{\mathbf{A}}(\mathcal{U}) \subseteq \operatorname{Man}_{\mathbf{A}}(\mathcal{V})$. Putting $U \leq_{\mathrm{cov}}{}^{\mathbf{A}} \mathcal{V}$ for $U \in \operatorname{Neigh} \mathbf{A}$ if and only if $\{U\} \leq_{\mathrm{cov}} \mathcal{V}$, the *set of all covers of a structure* $\mathbf{A}$ is defined as $\operatorname{Cov}(\mathbf{A}) := \left\{ \mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A} \;\middle|\; A \leq_{\mathrm{cov}}{}^{\mathbf{A}} \mathcal{V} \right\}$. In view of Lemma 3.4.4, this definition corresponds exactly to the covering concept used in RST.

Having defined the covering quasiorder, one can now define the notions of q-refinement, q-refinement minimality and q-non-refinability, as we have done it in this thesis. Most of the theory of refinement developed here remains true in this more general setting, as long as it does not concern facts explicitly following from $\subseteq_{\operatorname{Neigh} \mathbf{A}} \subseteq \leq_{\mathrm{cov}}$ or $\precapprox \subseteq \leq_{\mathrm{cov}}$. Moreover, as in Definition 3.5.16, we can speak of q-irreducible algebras, and using the characterisation in Remark 3.5.17 as a definition, we also have q-irreducible neighbourhoods at hand. Again, most of the general observations about q-irreducibility can be retained. Finally, after having dealt with the localisation process below, we can also keep the definition of X-q-irreducible neighbourhood for $X \in \operatorname{Int} \mathbf{A}$, just substituting Sep by Man in the wording used in Definition 3.6.6. If we simply replace $\operatorname{Neigh} \mathbf{A}|_U$ by $\operatorname{Neigh} \mathbf{A} \cap \mathfrak{P}(U)$ in that definition, we can also obtain a reasonable concept of X-q-irreducibility for neighbourhoods even without caring about localisation. Besides, the notions of q-cover prebase and q-cover base still make sense, and the results established in Section 3.7, especially how q-non-refinable covers can be constructed from q-cover bases, and uniqueness of q-non-refinable covers up to q-isomorphism, still apply in the general setting. A problem that needs to be thought about anew is the existence of q-cover bases. This is related to finding a subclass $\operatorname{Cruc}(\mathbf{A}) \subseteq \operatorname{Int} \mathbf{A}$ of *crucially interesting properties* which suffice to ensure the covering relation: this means for $\mathcal{V} \subseteq \operatorname{Neigh} \mathbf{A}$ the inclusion $\operatorname{Cruc}(\mathbf{A}) \subseteq \operatorname{Man}_{\mathbf{A}}(\mathcal{V})$ should already imply $\operatorname{Int} \mathbf{A} \subseteq \operatorname{Man}_{\mathbf{A}}(\mathcal{V})$, i.e. $A \leq_{\mathrm{cov}}{}^{\mathbf{A}} \mathcal{V}$ (cf. Proposition 3.6.20). This set $\operatorname{Cruc}(\mathbf{A})$ needs to be chosen small enough to ensure that X-q-irreducible neighbourhoods, where $X \in \operatorname{Cruc}(\mathbf{A})$, have a property similar to the one shown in Proposition 3.6.22.

The theory we have sketched so far allows to recover properties from local subsets of a structure. Let us call this a *decomposition framework*. However, we still have not combined this with a *localisation framework*, i.e. a true localisation process. For this, we require for every structure $\mathbf{A} \in \mathcal{C}$ and neighbourhood $U \in \operatorname{Neigh} \mathbf{A}$ two restriction operations $|_U$, one on structures, and one on interesting properties. A reasonable assumption to wish for is that $\mathbf{A}|_U$ belongs again to $\mathcal{C}$ and that $|\mathbf{A}|_U| = U$. Abstracting from Lemma 3.3.7, we want to have that $\operatorname{Neigh}\mathbf{A}|_U = \operatorname{Neigh}\mathbf{A} \cap \mathfrak{P}(U)$. Moreover, for $V \in \operatorname{Neigh}\mathbf{A}|_U$ we require $(\mathbf{A}|_U)|_V = \mathbf{A}|_V$. The second restriction operation that we need (and denote with the same symbol) is $|_U \colon \operatorname{Things}\mathbf{A} \longrightarrow \operatorname{Things}\mathbf{A}|_U$ with the property that $|_A$ is the identical operation (implying implicitly that $\operatorname{Things}\mathbf{A}|_A = \operatorname{Things}\mathbf{A}$). Again, it is natural to ask for the equality $(X|_U)|_V = X|_V$ for $X \in \operatorname{Things}\mathbf{A}$ and $V \in \operatorname{Neigh}\mathbf{A}|_U$.

In addition to the specifications stated until now, we additionally demand the following two axioms to be satisfied: for $\mathbf{A} \in \mathcal{C}$ and $U \in \operatorname{Neigh}\mathbf{A}$ we postulate

$$\operatorname{Int}\mathbf{A}|_U \subseteq \{\, X|_U \mid X \in \operatorname{Int}\mathbf{A}\}$$
$$\operatorname{Man}_{\mathbf{A}}(U) = \{\, X \in \operatorname{Int}\mathbf{A} \mid X|_U \in \operatorname{Int}\mathbf{A}|_U\}$$

Moreover, we require the inclusion $\subseteq_{\operatorname{Neigh}\mathbf{A}} \subseteq {\leq_{\mathrm{cov}}}^{\mathbf{A}}$ between quasiorders on the set of neighbourhoods of structures $\mathbf{A} \in \mathcal{C}$. This ensures compatibility of the covering condition with the natural order on the localising subsets.

Considering RST, we use restriction of algebras to neighbourhoods as the restriction process on structures and restriction of pairs of invariant relations for localisation of $\operatorname{Things}\mathbf{A}$. In view of these definitions, the axioms we have asked for above are quite natural.

We note that, as soon as for structures in $\mathcal{C}$ neighbourhoods, interesting properties and both restrictions have been specified, by the second displayed axiom, there is no choice any more for manageable properties of neighbourhoods. This means a localisation framework automatically necessitates a decomposition framework, whereas it makes sense to consider a decomposition framework on its own. In other words, a localisation framework should be seen as a specialisation that can be put on top as a particular feature of a decomposition framework.

From this more general perspective on RST, it becomes also clear that many parts of this theory most probably just depend on the decomposition framework, and not on the localisation process. We have tried to argue above that the important definitions of RST can be given within the axioms of its underlying decomposition framework, and we hope the reader is slightly convinced that most of their relationships can be proven without actually making use of the localisation framework. Putting the reason for this in more concrete terms, one can just look at the neighbourhoods of an algebra without considering the restricted algebra belonging to a neighbourhood and exploiting the fact that this algebra again has neighbourhoods, invariant relations, covers etc. For instance, it should be plausible from Lemma 3.5.8 that a structure $\mathbf{A}$ with a finite collection $\operatorname{Neigh}\mathbf{A}$ certainly will have q-non-refinable covers (and that this fact does not require considering structures of the form $\mathbf{A}|_U$). Furthermore, if we rewrite Definition 3.6.6(ii) in a way that avoids to speak of

Neigh $\mathbf{A}|_U$ (this was indicated above), then one can speak about X-q-irreducible neighbourhoods (in case of RST of (S,T)-q-irreducible neighbourhoods) without considering restricted structures, too. Besides, from what we have seen above, we believe that,—modulo a few problems that need to be solved in any particular decomposition framework, and have been solved in this thesis for the case of RST—under suitable assumptions, one can obtain a uniqueness result for q-non-refinable covers. Again, the deduction of this main result does not seem to need restricted algebras (or the localisation framework in the general case).

This implies that it should be possible to reuse big branches of the theory elaborated in this thesis under the more general assumption of a decomposition framework, possibly extended by the axiom $\subseteq_{\mathrm{Neigh}\,\mathbf{A}} \subseteq \leq_{\mathrm{cov}}{}^{\mathbf{A}}$ from the localisation framework.

If the particular decomposition framework under consideration admits a localisation framework, then the developed decomposition theory describing properties via local subsets becomes a real localisation theory. Namely, in this case it is possible to really reduce the structures that need to be examined in size because one can work with structures on carrier sets that are (ideally) proper subsets of the whole carrier set. For practical considerations this can be a big advantage. For example, if in RST one is interested in the subneighbourhoods of a three-element neighbourhood U, one has two choices given by reasoning within the decomposition or within the localisation framework: working on the level of the decomposition framework, one has to consider all idempotent unary clone operations and one has to check, which of them have images lying in U. However, if one has the chance to work with the localisation framework, i.e. if knowledge about the restricted algebra $\mathbf{A}|_U$ is available, then working inside this structure can be much easier, as it has e.g. at most $3^3 = 9$ unary operations in its clone.

We take the previous remarks as a motivation to state the following task for future research: Provide further substantiation of the thoughts presented above. In particular, educe a general localisation theory for structures based on the axioms above, try to deduce as many results of RST as possible from these general assumptions (with possibly small modifications if necessary), such that the main body of RST becomes an instance of a more general theory. In this respect it should be highlighted, which parts of RST really exploit the localisation framework, and which follow already from the decomposition framework.

If this attempt succeeds, find further examples for this general approach. In particular, we suggest to look at a modified variant of RST, where one is only interested in separating a special class of invariant relations, such as e.g. congruences or quasiorders. A second example to try could be localisation of closure systems via subsets.

Acknowledgements

This is the place to thank all those people who directly or indirectly contributed to the origination of this thesis. First of all, I am immensely obliged to Reinhard Pöschel for giving me the opportunity to do research in his group, and more specifically within the DFG-research project mentioned below, allowing me to gain experience and to establish valuable contacts with other scientists around the world. Moreover, I thank him for his constant support, for patiently guiding but not in any way restricting my scientific curiosity, and for encouraging me to go mostly my own ways, concerning the content of this thesis but also any other piece of research I became interested in.

Second, I am also grateful to my colleague Friedrich Martin Schneider for the close and uncomplicated collaboration, for countless inspiring, almost always productive, albeit mostly technical discussions, for many critical questions and helpful remarks.

Not lesser is my gratitude towards Keith Kearnes and Ágnes Szendrei, for organisational and financial support regarding my visit at the University of Colorado at Boulder in 2010, for the fruitful co-operation, for generously sharing scientific ideas and for showing a continuous interest in my work.

Furthermore, I would like to thank Tamás Waldhauser and László Zádori, both from the University of Szeged; the former for the profitable co-operation illuminating the behaviour of finite semigroups w.r.t. Relational Structure Theory, the latter for pointing out to me, already in 2008, the important connection of RST and categorical equivalence of varieties. This advice lead me to a much deeper understanding of one of the most important parts of RST.

Besides, my gratitude is also due to Ulrike Püschmann to whose hands I could faithfully entrust an unfinished version of the text for proofreading. Implicitly, she is as well contained in the following list of people whose moral support during the process of writing I greatly appreciated: Daniel Borchmann, Edith Vargas-García, Karsten Ebenhan and Cécile Germain.

Further, I do not wish to forget to thank the whole Institute of Algebra at Technische Universität Dresden for providing a very sociable and friendly research environment over the last years.

Finally, I am indebted to my family for how they have always supported me over all the years and covered my back whenever needed.

Last, but not least I would like to acknowledge funding support of the present work by Deutsche Forschungsgemeinschaft [German Research Foundation], grant number PO 434/6-1.

Index of Notation

$f \circ \mathbf{a}$ .... composition of a function $f\colon A \longrightarrow B$ with an m-tuple $\mathbf{a} \in A^m$, $m \in \mathbb{N}$

$f \circ (g_1, \ldots, g_n)$ .... general composition of $f \in \mathrm{O}_A^{(n)}$ with $g_1, \ldots, g_n \in \mathrm{O}_A^{(m)}$

$f \circ g$ .... composition of functions, application of f after g

$f \circ [S]$ .... application of a function $f \in \mathrm{O}_A^{(1)}$ to a relation $S \in \mathrm{R}_A$

$(g_1, \ldots, g_n)$ .... tupling of the mappings $g_j\colon A \longrightarrow B_j$, $1 \leq j \leq n$

$f\,[U]$ .... image of a set $U \subseteq A$ under a mapping $f\colon A \longrightarrow B$

$f^{-1}\,[V]$ .... preimage of a set $V \subseteq B$ under a mapping $f\colon A \longrightarrow B$

$f|_U^V$ .... restriction of a function $f\colon A \longrightarrow B$ to subsets $U \subseteq A$ and $V \subseteq B$ such that $f\,[U] \subseteq V$

$f|_U$ .... restriction of an operation $f \in \mathrm{O}_A$ to a preserved subset $U \subseteq A$

$\varrho \circ \sigma$ .... composition of binary relations

$S{\restriction}_U$ .... restriction of a relation $S \in \mathrm{R}_A$ to a subset $U \subseteq A$

${\downarrow}_{(Q,q)}\, V$ .... downset generated by a subset $V \subseteq Q$ of a quasiordered set (Q, q)

${\uparrow}_{(Q,q)}\, V$ .... upset generated by a subset $V \subseteq Q$ of a quasiordered set (Q, q)

🪓 $(U,\mathcal{U})$ .... set of neighbourhoods arising by replacing U in $\mathcal{U}$ by its proper subneighbourhoods, “U hacked into pieces inside $\mathcal{U}$”, $\mathcal{U} \setminus \{U\} \cup \operatorname{Neigh} \mathbf{A}|_U \setminus \{U\}$

$\prod\!\!\!\prod_{i\in I} \varrho_i$ .... product relation as in a product structure (isomorphic to the Cartesian product of the relations $(\varrho_i)_{i\in I}$)

$\bigsqcap^{\beta}_{(\alpha_i)_{i\in I}} (\varrho_i)_{i\in I}$ .... general composition of relations $(\varrho_i)_{i\in I}$

$\subseteq$ .... set inclusion

$\subset$	proper set inclusion
$\lvert A\rvert$	cardinality of a set A
$\aleph_0$	least infinite cardinal number, $\lvert\mathbb{N}\rvert$
$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc.	algebras
$\mathbf{A}_A$	polynomial expansion of an algebra $\mathbf{A}$, i.e. extension with all nullary constants
$\mathbf{A} = \langle A; F\rangle$	non-indexed algebra with basic operations $F \subseteq \mathrm{O}_A$
$\mathbf{A}\lvert_A$	saturated algebra $\langle A;\ \mathrm{Clo}\,(\mathbf{A})\rangle$
$\mathbf{A}\lvert_{U_1} \boxtimes \cdots \boxtimes \mathbf{A}\lvert_{U_m}$	matrix product w.r.t. neighbourhoods $U_1, \ldots, U_m \subseteq A$
$\mathbf{A}\lvert_U$	algebra induced by $\mathbf{A}$ on a neighbourhood $U \subseteq A$
$\underset{\sim}{\mathbf{A}}$, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{C}}$, etc	relational structures
$\underset{\sim}{\mathbf{A}} = \langle A; Q\rangle$	non-indexed relational structure with basic relations $Q \subseteq \mathrm{R}_A$
$\underset{\sim}{\mathbf{A}}\restriction_U$	induced substructure of a relational structure $\underset{\sim}{\mathbf{A}}$ on a subset $U \subseteq A$, in the context of RST usually restricted relational counterpart of an algebra to a neighbourhood U
$\mathbf{A} \models s \approx t$	an identity $s \approx t$ holds in an algebra $\mathbf{A}$
$\underset{\sim}{\mathbf{A}} \precsim \underset{\sim}{\mathbf{B}}$	embeddability of relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ of identical signature
$\underset{\sim}{\mathbf{A}} \cong \underset{\sim}{\mathbf{B}}$	isomorphism relation between relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ of identical signature
$\mathrm{ar}(f)$	arity of a finitary operation f or an operation symbol
$\mathrm{ar}(S)$	arity of a finitary relation S or a relation symbol
$\mathrm{Clo}\,(\mathbf{A})$	clone of operations ($\mathrm{Inv}_A \,\mathrm{Pol}\,\mathbf{A}$) of an algebra $\mathbf{A}$
$\mathrm{Clo}^{(n)}\,(\mathbf{A})$	set of n-ary operations in $\mathrm{Clo}\,(\mathbf{A})$
$\mathrm{Con}\,\mathbf{A}$	set of all congruences of an algebra $\mathbf{A}$
$\mathrm{Cov}\,(\mathbf{A})$	set of all covers of an algebra $\mathbf{A}$, $\mathrm{Cov}_{\mathbf{A}}\,(A)$
$\prec$	covering relation in an ordered set

$\mathrm{Cov}_{\mathbf{A}}(U)$ .... set of all covers of a neighbourhood U in an algebra $\mathbf{A}$

$\mathrm{Cruc}(\mathbf{A})$ .... set of all crucial pairs of an algebra $\mathbf{A}$

$\mathrm{Cruc}^{(m)}(\mathbf{A})$ .... set of all m-crucial pairs of an algebra $\mathbf{A}$, i.e. crucial pairs whose entries both have arity equal to m

Δ_A .... binary identity relation on A

$e(\mathbf{A})$ .... restricted algebra to the image of an operation $e \in \mathrm{Idem}\,\mathbf{A}$, $\mathbf{A}|_{e[A]}$

$e(\mathrm{Clo}(\mathbf{A}))$ .... set of basic operations of $e(\mathbf{A})$

$\emptyset$ .... empty set

$\mathrm{End}\,\underset{\sim}{\mathbf{A}}$ .... set of endomorphisms of a relational structure $\underset{\sim}{\mathbf{A}}$

$e_i^{(n)}$ .... n-ary projection on the i-th coordinate ($1 \leq i \leq n$)

$\mathrm{Eq}\,A$ .... set of all equivalence relations on a set A

$\equiv_{\mathrm{ref}}$ .... refinement equivalence

$\langle F \rangle_{\mathrm{O}_A}$ .... clone generated by $F \subseteq \mathrm{O}_A$

$F^{(n)}$ .... set of all n-ary operations in a set $F \subseteq \mathrm{O}_A$

$f \rhd S$ .... finitary operation $f \in \mathrm{O}_A$ preserves a finitary relation $S \in \mathrm{R}_A$

$\mathrm{graph}(f)$ .... graph of a function $f\colon A \longrightarrow B$

$h[\mathbf{A}]$ .... homomorphic image of an algebra w.r.t. a homomorphism $h\colon \mathbf{A} \longrightarrow \mathbf{B}$ between algebras of identical signature

$\mathrm{Hom}(\underset{\sim}{\mathbf{A}}, \underset{\sim}{\mathbf{B}})$ .... set of all (homo)morphisms between relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$

$h[\underset{\sim}{\mathbf{A}}]$ .... image substructure w.r.t. a homomorphism $h\colon \underset{\sim}{\mathbf{A}} \longrightarrow \underset{\sim}{\mathbf{B}}$ between relational structures $\underset{\sim}{\mathbf{A}}$ and $\underset{\sim}{\mathbf{B}}$ of identical signature

id_A .... identity mapping on a set A

$\mathrm{Idem}\,\mathbf{A}$ .... set of idempotent unary operations in $\mathrm{Clo}(\mathbf{A})$

$\mathrm{Idem}\,A$ .... set of idempotent unary operations on A

$\mathrm{im}\,\mathbf{a}$ .... set of all entries of a tuple $\mathbf{a} \in A^m$, $m \in \mathbb{N}$

$\mathrm{R}_A^{(m)}$	set of all m-ary relations on a set A
$\mathrm{Sep}_{\mathbf{A}}(U)$	separation set of a neighbourhood U in an algebra $\mathbf{A}$
$\mathrm{Sep}_{\mathbf{A}}(\mathcal{U})$	separation set of a collection of neighbourhoods $\mathcal{U}$ in an algebra $\mathbf{A}$
$s \approx t$	an identity between terms s and t using the same algebraic signature
$\mathrm{Sub}\,\mathbf{A}$	set of all subuniverses of an algebra $\mathbf{A}$
$t^{\mathbf{A}}$	term operation belonging to a term t over the same signature as an algebra $\mathbf{A}$
$\mathrm{Term}(\mathbf{A})$	set of all term operations of an algebra $\mathbf{A}$
$\mathrm{Term}^{(n)}(\mathbf{A})$	set of all n-ary operations in $\mathrm{Term}(\mathbf{A})$
$\langle\theta\rangle^{\mathrm{trans}}$	transitive closure of a binary relation $\theta \in \mathrm{R}_A^{(2)}$
$\langle U\rangle_{\mathbf{A}}$	subuniverse generated by $U \subseteq A$ in an algebra $\mathbf{A}$
$U \leq_{\mathrm{cov}} V$	covering quasiorder between neighbourhoods
$U \leq_{\mathrm{cov}} \mathcal{V}$	covering relation between neighbourhoods and collection of neighbourhoods
$\mathcal{U} \leq_{\mathrm{cov}} \mathcal{V}$	covering quasiorder between collections of neighbourhoods
$U \equiv_{\mathrm{cov}} V$	covering equivalence of neighbourhoods, $\leq_{\mathrm{cov}} \cap \leq_{\mathrm{cov}}^{-1}$
$\mathcal{U} \equiv_{\mathrm{cov}} \mathcal{V}$	covering equivalence of collections of neighbourhoods, $\leq_{\mathrm{cov}} \cap \leq_{\mathrm{cov}}^{-1}$
$U \precapprox V$	embedding of neighbourhoods U and V
$\mathcal{U} \cong_q \mathcal{V}$	q-isomorphism of collections of neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ w.r.t. a quasiorder q
$U \cong V$	isomorphism of neighbourhoods U and V
$\mathcal{U} \cong \mathcal{V}$	isomorphism of collections of neighbourhoods $\mathcal{U}$ and $\mathcal{V}$
$\mathcal{U} <_{\mathrm{ref}}(q)\, \mathcal{V}$	proper q-refinement of collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods w.r.t. a quasiorder q
$\mathcal{U} \equiv_{\mathrm{ref}}(q)\, \mathcal{V}$	q-refinement equivalence of collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods w.r.t. a quasiorder q

$\mathcal{U} \leq_{\mathrm{ref}} (q)\, \mathcal{V}$ .... q-refinement quasiorder between collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods w.r.t. a quasiorder q

$\mathcal{U} <_{\mathrm{ref}} \mathcal{V}$ .... proper refinement of collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods, $\mathcal{U} <_{\mathrm{ref}} (\subseteq)\, \mathcal{V}$

$\mathcal{U} \equiv_{\mathrm{ref}} \mathcal{V}$ .... refinement equivalence of collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods $\mathcal{U} \equiv_{\mathrm{ref}} (\subseteq)\, \mathcal{V}$

$\mathcal{U} \leq_{\mathrm{ref}} \mathcal{V}$ .... refinement quasiorder between collections $\mathcal{U}$ and $\mathcal{V}$ of neighbourhoods, $\mathcal{U} \leq_{\mathrm{ref}} (\subseteq)\, \mathcal{V}$

$\mathcal{U} \precapprox \mathcal{V}$ .... strong embeddability of collections of neighbourhoods $\mathcal{U}$ and $\mathcal{V}$

$\mathcal{U} \precapprox_{\mathrm{w}} \mathcal{V}$ .... weak embeddability of collections of neighbourhoods $\mathcal{U}$ and $\mathcal{V}$

$\mathcal{U} \sqsubseteq (q)\, \mathcal{V}$ .... q-downset quasiorder on collections of neighbourhoods induced by a quasiorder q on neighbourhoods, see the definition on page 131

$\mathbf{U} \leq \mathbf{A}$ .... $\mathbf{U}$ is a subalgebra of $\mathbf{A}$, usually only for algebras of identical signature

$U \leq \mathbf{A}$ .... U is a subuniverse of $\mathbf{A}$, $U \in \mathrm{Inv}^{(1)}\mathbf{A}$

$\mathrm{Var}\, \mathcal{K}$ .... variety generated by a class $\mathcal{K}$ of algebras

Index of Terms

Bibliography

[AHS06] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and concrete categories: the joy of cats*, Repr. Theory Appl. Categ. (2006), no. 17, 1–507 (electronic), Reprint of the 1990 original [Wiley, New York; MR1051419]. MR 2240597

[Awo10] Steve Awodey, *Category theory*, second ed., Oxford Logic Guides, vol. 52, Oxford University Press, Oxford, 2010. MR 2668552 (2011g:18001)

[BB96] Clifford Bergman and Joel Berman, *Morita equivalence of almost-primal clones*, J. Pure Appl. Algebra **108** (1996), no. 2, 175–201. MR 1382247 (97a:08004)

[BB98] ———, *Algorithms for categorical equivalence*, Math. Structures Comput. Sci. **8** (1998), no. 1, 1–15. MR 1617307 (99j:08003)

[BB99] ———, *Categorical equivalence of modes*, Discuss. Math. Algebra Stochastic Methods **19** (1999), no. 1, 41–62, Modes, modals, related structures and applications (Warsaw, 1997). MR 1701735 (2000f:18004)

[Beh09] Mike Behrisch, *Relational Tame Congruence Theory and subalgebra primal algebras*, Diplomarbeit [diploma thesis], TU Dresden [Dresden University of Technology], September 2009, online available at http://www.math.tu-dresden.de/~mbehri/documents/subalgebraPrimal_links.pdf.

[Beh11] ———, *A note on clones with nullary operations. How clones should be*, Tech. Report MATH-AL-11-2011, TU Dresden, Institut für Algebra, Technische Universität Dresden, 01062 Dresden, Germany, December 2011, online available at http://www.math.tu-dresden.de/~mbehri/documents/clones_with_nullary_operations.pdf.

[Beh12] ———, *Irreducible subalgebra primal algebras*, Order **29** (2012), no. 2, 231–244, 10.1007/s11083-011-9220-6. MR 2926310

[BIM+10] Joel Berman, Paweł Idziak, Petar Marković, Ralph N. McKenzie, Matthew A. Valeriote, and Ross Willard, *Varieties with few subalgebras of powers*, Trans. Amer. Math. Soc. **362** (2010), no. 3, 1445–1473. MR 2563736 (2010k:08010)

[BK12] Libor Barto and Marcin Kozik, *Absorbing subalgebras, cyclic terms, and the constraint satisfaction problem*, Log. Methods Comput. Sci. **8** (2012), no. 1, 1:07, 27. MR 2893395

[BKKR69] V. G. Bodnarčuk, Lev Arkaďevič Kalužnin, V. N. Kotov, and Boris A. Romov, *Galois theory for Post algebras. I, II*, Kibernetika (Kiev) (1969), no. 3, 1–10; ibid. 1969, no. 5, 1–9. MR 0300895 (46 #55)

[BKM+09] Libor Barto, Marcin Kozik, Miklós Maróti, Ralph N. McKenzie, and Todd Niven, *Congruence modularity implies cyclic terms for finite algebras*, Algebra Universalis **61** (2009), no. 3-4, 365–380. MR 2565859 (2011a:08009)

[BS81] Stanley Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, vol. 78, Springer-Verlag, New York, 1981, online available at http://www.math.uwaterloo.ca/~snburris/htdocs/UALG/univ-algebra.pdf. MR 648287 (83k:08001)

[CK04] LeAnne Conaway and Keith A. Kearnes, *Minimal sets in finite rings*, Algebra Universalis **51** (2004), no. 1, 81–109. MR 2067151 (2005d:16035)

[Coh65] Paul Moritz Cohn, *Universal algebra*, Harper & Row Publishers, New York, 1965. MR 0175948 (31 #224)

[CV02] Matthias Clasen and Matthew Valeriote, *Tame congruence theory*, Lectures on algebraic model theory, Fields Inst. Monogr., vol. 15, Amer. Math. Soc., Providence, RI, 2002, pp. 67–111. MR 1873595 (2003e:08008)

[Day69] Alan Day, *A characterization of modularity for congruence lattices of algebras.*, Canad. Math. Bull. **12** (1969), 167–173. MR 0248063 (40 #1317)

[DL01] Klaus Denecke and Otfried Lüders, *Categorical equivalence of varieties and invariant relations*, Algebra Universalis **46** (2001), no. 1-2, 105–118, The Viktor Aleksandrovich Gorbunov memorial issue. MR 1835788 (2002e:08004)

[FKV11] Ralph Freese, Emil Kiss, and Matthew Valeriote, *Universal Algebra Calculator*, 2011, Available at: http://www.uacalc.org.

[Gei68] David Geiger, *Closed systems of functions and predicates*, Pacific J. Math. **27** (1968), 95–100. MR 0234893 (38 #3207)

[Gum81] Heinz-Peter Gumm, *Congruence modularity is permutability composed with distributivity*, Arch. Math. (Basel) **36** (1981), no. 6, 569–576. MR 629294 (82j:08009)

[HM88] David Hobby and Ralph N. McKenzie, *The structure of finite algebras*, Contemporary Mathematics, vol. 76, American Mathematical Society, Providence, RI, 1988. MR 958685 (89m:08001)

[Iza13] Shohei Izawa, *Composition of matrix products and categorical equivalence*, Algebra Universalis **69** (2013), no. 4, 327–356. MR 3061092

[Jón67] Bjarni Jónsson, *Algebras whose congruence lattices are distributive*, Math. Scand. **21** (1967), 110–121 (1968). MR 0237402 (38 #5689)

[Kea01] Keith A. Kearnes, *Tame Congruence Theory is a localization theory*, Lecture Notes from "A Course in Tame Congruence Theory" Workshop, Paul Erdős Summer Research Center of Mathematics, Budapest, 2001, available at http://www.math.u-szeged.hu/confer/algebra/2001/lec+ex1.ps, July 2001.

[Kis97] Emil W. Kiss, *An introduction to tame congruence theory*, Algebraic model theory (Toronto, ON, 1996), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 496, Kluwer Acad. Publ., Dordrecht, 1997, pp. 119–143. MR 1481441 (99d:03028)

[KK03] Keith A. Kearnes and Emil W. Kiss, *Residual smallness and weak centrality*, Internat. J. Algebra Comput. **13** (2003), no. 1, 35–59. MR 1970866 (2004c:08012)

[KKSW02] Keith A. Kearnes, Emil W. Kiss, Ágnes Szendrei, and Ross D. Willard, *Chief factor sizes in finitely generated varieties*, Canad. J. Math. **54** (2002), no. 4, 736–756. MR 1913917 (2003h:08001)

[KKV98] Keith A. Kearnes, Emil W. Kiss, and Matthew A. Valeriote, *Minimal sets and varieties*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 1–41. MR 1348152 (98d:08001)

[KKV99] ———, *A geometric consequence of residual smallness*, Ann. Pure Appl. Logic **99** (1999), no. 1-3, 137–169. MR 1708150 (2000j:08006)

[KL10] Keith A. Kearnes and Yingwei Li, *Finitely generated varieties with small $\langle R, S\rangle$-irreducible sets*, Research paper, pre-print available at http://spot.colorado.edu/~kearnes/Papers/kbounded5.pdf, 2010.

[KS12] Keith A. Kearnes and Ágnes Szendrei, *Clones of algebras with parallelogram terms*, Internat. J. Algebra Comput. **22** (2012), no. 1, 1250005, 30. MR 2900858

[KSKM01] Keith A. Kearnes, Ágnes. Szendrei, Emil W. Kiss, and Ralph N. McKenzie (eds.), *Sixty four problems in universal algebra*, July 2001, Lecture

Notes from "A Course in Tame Congruence Theory" Workshop, Budapest, available at http://www.math.u-szeged.hu/confer/algebra/2001/64problems.ps.

[KW70] Andor Kertész and Alfred Widiger, *Artinsche Ringe mit artinschem Radikal*, J. Reine Angew. Math. **242** (1970), 8–15. MR 0262276 (41 #6886)

[Lam91] Tsit Yuen Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 1991. MR 1125071 (92f:16001)

[Lau06] Dietlinde Lau, *Function algebras on finite sets*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2006, A basic course on many-valued logic and clone theory. MR 2254622 (2007m:03001)

[Mal′54] Anatoliĭ Ivanovič Mal′cev, *On the general theory of algebraic systems*, Mat. Sb. N.S. **35(77)** (1954), 3–20. MR 0065533 (16,440e)

[McK83] Ralph N. McKenzie, *Finite forbidden lattices*, Universal algebra and lattice theory (Puebla, 1982), Lecture Notes in Math., vol. 1004, Springer, Berlin, 1983, pp. 176–205. MR 716183 (85b:06006)

[McK96] ———, *An algebraic version of categorical equivalence for varieties and more general algebraic categories*, Logic and algebra (Pontignano, 1994), Lecture Notes in Pure and Appl. Math., vol. 180, Dekker, New York, 1996, pp. 211–243. MR 1404941 (98a:08010)

[MMT87] Ralph N. McKenzie, George F. McNulty, and Walter F. Taylor, *Algebras, lattices, varieties. Vol. I*, The Wadsworth & Brooks/Cole Mathematics Series, vol. I, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987. MR 883644 (88e:08001)

[Pix63] Alden F. Pixley, *Distributivity and permutability of congruence relations in equational classes of algebras*, Proc. Amer. Math. Soc. **14** (1963), 105–109. MR 0146104 (26 #3630)

[PK79] Reinhard Pöschel and Lev Arkad′evič Kalužnin, *Funktionen- und Relationenalgebren*, Mathematische Monographien [Mathematical Monographs], vol. 15, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979, Ein Kapitel der diskreten Mathematik. [A chapter in discrete mathematics]. MR 543839 (81f:03075)

[Pös79] Reinhard Pöschel, *Concrete representation of algebraic structures and a general Galois theory*, Contributions to general algebra (Proc. Klagenfurt Conf., Klagenfurt, 1978), Heyn, Klagenfurt, 1979, pp. 249–272. MR 537425 (80h:08008)

[Pös80] ______, *A general Galois theory for operations and relations and concrete characterization of related algebraic structures*, Report 1980, vol. 1, Akademie der Wissenschaften der DDR Institut für Mathematik, Berlin, 1980, With German and Russian summaries. MR 568709 (81h:08003)

[Ros70] Ivo G. Rosenberg, *Über die funktionale Vollständigkeit in den mehrwertigen Logiken. Struktur der Funktionen von mehreren Veränderlichen auf endlichen Mengen*, vol. 80, Rozpravy Československé Akademie Věd: Řada matematickych a Přírodních Věd, no. 4, Academia, 1970. MR 0292647 (45 #1732)

[Sch12] Friedrich Martin Schneider, *A relational localisation theory for topological algebras*, Doktorarbeit [dissertation], TU Dresden [Dresden University of Technology], May 2012, online available at http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-93424.

[Sig10] Mark H. Siggers, *A strong Mal'cev condition for locally finite varieties omitting the unary type*, Algebra Universalis **64** (2010), no. 1-2, 15–20. MR 2745482 (2011k:08008)

[Sze86] Ágnes Szendrei, *Clones in universal algebra*, Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 99, Presses de l'Université de Montréal, Montreal, QC, 1986. MR 859550 (87m:08005)

[Tay77] Walter Taylor, *Varieties obeying homotopy laws*, Canad. J. Math. **29** (1977), no. 3, 498–527. MR 0434928 (55 #7891)

[VH79] Dinh Van Huynh, *A note on artinian rings*, Archiv der Mathematik **33** (1979), 546–553, 10.1007/BF01222798.

[Wal09] Tamás Waldhauser, *Irreducible semigroups*, Preprint, July 2009.

[Wal12] Tamás Waldhauser, personal communication, August 2012.

[Zád97a] László Zádori, *Categorical equivalence of finite groups*, Bull. Austral. Math. Soc. **56** (1997), no. 3, 403–408. MR 1490657 (98m:20002)

[Zád97b] ______, *Relational sets and categorical equivalence of algebras*, Internat. J. Algebra Comput. **7** (1997), no. 5, 561–576. MR 1470353 (99a:08006)

[Zád98] ______, *Characterizing finite irreducible relational sets*, Acta Sci. Math. (Szeged) **64** (1998), no. 3-4, 455–462. MR 1666014 (2000b:08020)

Versicherung

(a) Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

(b) Die vorliegende Dissertation wurde seit Dezember 2010 am Institut für Algebra, Fachrichtung Mathematik, Fakultät Mathematik und Naturwissenschaften an der Technischen Universität Dresden unter der Betreuung von Prof. Dr. rer. nat. habil. Reinhard Pöschel angefertigt.

(c) Es wurden zuvor keine Promotionsvorhaben unternommen.

(d) Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der Technischen Universität Dresden vom 23. Februar 2011, in der geänderten Fassung mit Gültigkeit vom 15. Juni 2011 an.

Dresden, den 26. April 2013