Heterogeneous environment on examples*

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Abstract. We propose a running example for heterogeneous approach based on new type of fuzzification that diversifies fuzziness of every object, fuzziness of every attribute and fuzziness of every table value in a formal context. Moreover we suggest another working examples on heterogeneous environment and provide additional utilization and illustration of this new model that allows to use Formal Concept Analysis also for heterogenenous data. An interpretation of heterogeneous formal concepts and the resulting concept lattice is included.

Keywords: heterogeneous context, longterm preferences, shortterm preferences

1 Introduction

Formal concepts consisting of developing countries in supranational groups is one of the earliest example in which has been applied classical Boolean approach of Formal Concept Analysis and appears in [14]. By attribute fuzzification proposed independently by Ben Yahia [11], Bělohlavek [3] and Krajči [16] is possible to think about students and their evaluation in more than two degrees. Such method to process data tables is called one-sided fuzzy approach. Another Krajči's generalized approach [18], [19] diversifies fuzziness of objects and fuziness of attribute. Medina, Ojeda-Aciego and Ruiz-Calviño [22] utilize personal preferences to choose suitable journal for a paper submitting and use different adjoint triples to find the best object.

An additional level of generalization based on diversification of every object, every attribute and every table value is proposed in [1]. In this paper we would like to clarify that it has some natural motivation to consider such level of generalization. Also an interpretation of both concept-forming operators and the notions of longterm and shortterm preferences are included in addition to

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environment introduced in [2]. The cottage example is introduced in more detail here and computed on two larger contexts in compare to [1]. It gives better intuition for the underlying structures. Similarly other applications of heterogeneous Formal Concept Analysis are discussed.

The structure of this paper is as follows. Section 2 recalls the basic notion of heterogeneous approach and details heterogeneous formal context on proposed running cottage example. Section 3 gives appropriate interpretation of heterogeneous concepts. Section 4 describes overview of another working examples of our environment that works also with heterogenenous data. Section 5 briefly explains how to construct a heterogeneous formal concept lattice and gives the result for our proposed cottage example. Section 6 concludes the paper and describes future work.

2 Heterogeneous formal context

Consider the following situation as a motivation. People (friends, colleagues, classmates) are going to stay at some cottage. In fact, every person can have different requirements and preferences connected with cottage conditions depending on number of days spending at the cottage. One can prefer hot water, other necessarily expect internet connection. Natural requirements based on some actual preferences can be formulated:

- ⋄ Eva admits full discomfort on water conditions, partial discomfort on internet/TV and full discomfort on a lake available.
- ⋄ Joe accepts half discomfort on water conditions, admits great discomfort on internet/TV and no discomfort on a lake available.
- ⋄ Ken allows full discomfort on water conditions, admits discomfort on services and half discomfort on a lake available.

Realize that every person feels full discomfort diverse in general. For instance, full discomfort on water conditions is for Eva connected with absence of hot water even though one arbitrary day at the cottage. Joe is more adaptive and full discomfort is connected with absence of hot water only at second day. Ken is the most adaptive and full discomfort corresponds to two days absence of hot water. So it is natural to inquire which cottage conditions have to be fulfilled to satisfy all people staying at cottage even though different number of days.

In follows we define heterogeneous formal context and formally describe mentioned situation. Let A and B be non-empty sets. Let $\mathcal{P} = ((P_{a,b}, \leq_{P_{a,b}}) : a \in A, b \in B)$ be a system of posets and let R be a function from $A \times B$ such that $R(a,b) \in P_{a,b}$, for all $a \in A$ and $b \in B$. Let $\mathcal{C} = ((C_a, \leq_{C_a}) : a \in A)$ and $\mathcal{D} = ((D_b, \leq_{D_b}) : b \in B)$ be systems of complete lattices. (For simplicity, we will omit the indices of all noticed $\leq_?$, it will be always clear which of one is used.)

Let $\odot = (\bullet_{a,b} : a \in A, b \in B)$ be a system of operations such that $\bullet_{a,b}$ is from $C_a \times D_b$ to $P_{a,b}$ and it is isotone and left-continuous in both arguments, i. e.

1a) $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2 \in C_a$ and $d \in D_b$,
1b) $d_1 \leq d_2$ implies $c \bullet_{a,b} d_1 \leq c \bullet_{a,b} d_2$ for all $c \in C_a$ and $d_1, d_2 \in D_b$,

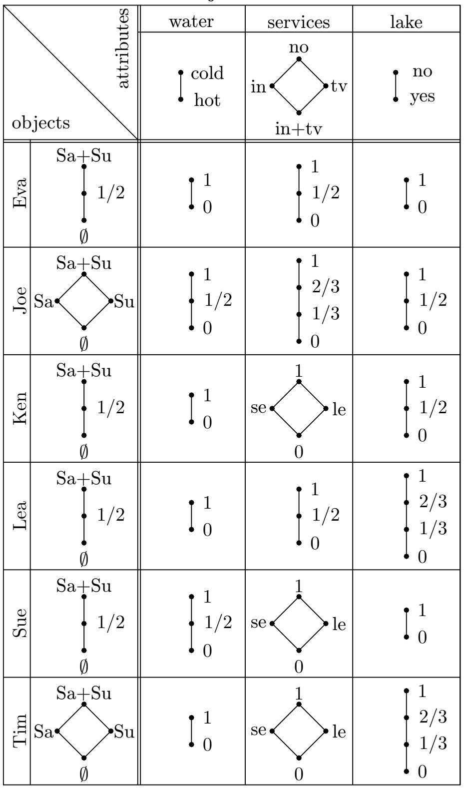
2a) if $c \bullet_{a,b} d \leq p$ for some $d \in D_b$, $p \in P_{a,b}$ and for all $c \in X \subseteq C_a$ then $\sup X \bullet_{a,b} d \leq p$,
2b) if $c \bullet_{a,b} d \leq p$ for some $c \in C_a$, $p \in P_{a,b}$ and for all $d \in Y \subseteq D_b$ then $c \bullet_{a,b} \sup Y \leq p$.

Then the tuple $\langle A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot \rangle$ will be called a *heterogeneous formal context*. Notice that if $C_a = D_b$ and $\bullet_{a,b}$ is commutative these conditions can be reduced to these two:

1) $c_1 \leq c_2$ implies $c_1 \bullet_{a,b} d \leq c_2 \bullet_{a,b} d$ for all $c_1, c_2, d \in C_a = D_b$,
2) if $c \bullet_{a,b} d \leq p$ for some $d \in C$, $p \in P_{a,b}$ and for all $c \in X \subseteq C_a = D_b$ then $\sup X \bullet_{a,b} d \leq p$.

Figure 1 illustrates the notions of heterogeneous formal context. Let $B = \{\text{Eva}, \text{Joe}, \text{Ken}, \ldots\}$, set of objects, consists of six people thinking about staying at some cottage. Let $A = \{\text{water}, \text{services}, \text{lake}\}$, set of attributes, responds to water conditions, services conditions and lake availability at cottage. In next paragraph, illustration of the notions of heterogeneous formal context for three people and three cottage conditions from Figure 1 is included.

Fig. 1. List of possible values for objects and attributes in heterogenous case



To go beyond objects, Eva's preferences contain staying in three degrees: not at all, one day (it does not matter which one) or all two days (D_{Eva}); Joe in four degrees: not at all, only Saturday, only Sunday or all two days (D_{Joe}); Ken again in three degrees (D_{Ken}). Beyond attributes, water conditions contain two degrees: hot or cold (there are also cottages having cold water only in facilities) corresponding to C_{water}; services include four degrees: internet and television, only internet connection, only television or nothing at all (C_{services}); lake availability contains two degrees: yes or no (C_{lake}). Finally in case of table values $\mathcal{P} = (P_{\mathrm{water,Eva}}, P_{\mathrm{services,Eva}}, P_{\mathrm{lake,Eva}}, P_{\mathrm{water,Joe}}, P_{\mathrm{services,Joe}}, P_{\mathrm{lake,Joe}}, P_{\mathrm{water,Ken}}, P_{\mathrm{services,Ken}}, P_{\mathrm{lake,Ken}})$ expresses different scales of degrees for discomfort of every person and every condition. For instance $P_{\mathrm{services,Eva}} = \{0, 1/2, 1\}$ expresses Eva's comfort, partial discomfort, full discomfort, respectively. Further $P_{\mathrm{services,Ken}} = \{0, \mathrm{le}, \mathrm{se}, 1\}$ correspond to comfort, discomfort on length of staying, discomfort on services and full discomfort, respectively. And this completes description of Figure 1.

Having expressed list of possible values for every person and every condition, further we will consider some concrete **longterm preferences** (diverse perception of discomfort by different conditions in Figure 2) and **shortterm preferences** (actual degree of discomfort that person admits in Figure 3).

Fig. 2. Longterm preferences in heterogeneous approach

$\bullet_{\mathrm{water,Eva}}$	hot	cold
∅	0	0
1/2	0	1
Sa+Su	0	1

$\bullet_{\mathrm{services,Eva}}$	in+tv	in	tv	no
∅	0	0	0	0
1/2	0	1/2	1	1
Sa+Su	0	1	1	1

$\bullet_{\mathrm{lake,Eva}}$	yes	no
∅	0	0
1/2	0	1
Sa+Su	0	1

$\bullet_{\mathrm{water,Joe}}$	hot	cold
∅	0	0
Sa	0	1/2
Su	0	1
Sa+Su	0	1

$\bullet_{\mathrm{services,Joe}}$	in+tv	in	tv	no
∅	0	0	0	0
Sa	0	1/3	2/3	2/3
Su	0	1/3	1	1
Sa+Su	0	1/3	1	1

$\bullet_{\mathrm{lake,Joe}}$	yes	no
∅	0	0
Sa	0	1
Su	0	1
Sa+Su	0	1

$\bullet_{\mathrm{water,Ken}}$	hot	cold
∅	0	0
1/2	0	0
Sa+Su	0	1

$\bullet_{\mathrm{services,Ken}}$	in+tv	in	tv	no
∅	0	0	0	0
1/2	0	se	le	1
Sa+Su	0	1	1	1

$\bullet_{\mathrm{lake,Ken}}$	yes	no
∅	0	0
1/2	0	1/2
Sa+Su	0	1

Starting with longterm preferences, notice that every person can have different perception of discomfort depending on cottage conditions and length of stay. In effort to express these longterm preferences, every person has own behavior that expresses $\odot = (\bullet_{\mathrm{water,Eva}}, \bullet_{\mathrm{services,Eva}}, \bullet_{\mathrm{lake,Eva}}, \bullet_{\mathrm{water,Joe}}, \bullet_{\mathrm{services,Joe}}, \bullet_{\mathrm{lake,Joe}}, \bullet_{\mathrm{water,Ken}}, \bullet_{\mathrm{services,Ken}}, \bullet_{\mathrm{lake,Ken}})$. That means for instance $\bullet_{\mathrm{services,Eva}}$ is from $C_{\mathrm{services}} \times D_{\mathrm{Eva}}$ to $P_{\mathrm{services,Eva}}$.

Values of isotone and left-continuous operations $\odot$ (by assumptions of our approach) with respect to the number of days and cottage conditions is known and for our example included in Figure 2. First of all, notice that higher table values correspond to worse situation (0 as no discomfort, i.e. good situation, 1 as full discomfort, i.e. bad case) that might be in opposite with natural expectation, but this follows from assuptioms of our heterogeneous approach.

We describe some remarks and interpretation on these longterm preferences:

a) Notice that $c \bullet_{\text{services,Eva}} \emptyset = 0$ for all $c \in C_{\text{services}}$, because no staying at the cottage and arbitrary conditions respond to no discomfort.
b) Notice that $\text{hot} \bullet_{\text{water,Eva}} d = 0$ for all $d \in D_{\text{Eva}}$, because presence of hot water and arbitrary number of days respond to no discomfort.
c) Notice that $\text{in} + \text{tv} \bullet_{\text{services,Joe}} d = 0$ for all $d \in D_{\text{Joe}}$, because presence of all services and arbitrary number of days respond to no discomfort.
d) To see monotonicity, staying on Saturday and cold water represent half discomfort for Joe, but two days and cold water lead to big discomfort.
e) Similarly one day staying and internet only represent half discomfort for Eva, but two days and internet only, or missing internet lead to full discomfort.
f) Only internet and Saturday represent one third discomfort for Joe, only television and Saturday two third discomfort, only television and Sunday or two days lead to full discomfort.
g) To see left-continuity, Saturday or Sunday and internet only represent one third discomfort for Joe, but supremum of these days (Saturday+Sunday) and internet only also lead to one third discomfort.

Having known longterm preferences of people, now we would like to express some shortterm preferences corresponding to some actual circumstances or actual sentiment of every person connected with actual staying. So every person appoints degree of discomfort that accepts or admits at the actual situation, i.e. $R(\text{water}, \text{Eva}) \in P(\text{water}, \text{Eva})$, $R(\text{services}, \text{Eva}) \in P(\text{services}, \text{Eva})$, $R(\text{lake}, \text{Eva}) \in P(\text{lake}, \text{Eva})$, $R(\text{water}, \text{Joe}) \in P(\text{water}, \text{Joe})$, $R(\text{services}, \text{Joe}) \in P(\text{services}, \text{Joe})$, $R(\text{lake}, \text{Joe}) \in P(\text{lake}, \text{Joe})$, $R(\text{water}, \text{Ken}) \in P(\text{water}, \text{Ken})$, $R(\text{services}, \text{Ken}) \in P(\text{services}, \text{Ken})$ and $R(\text{lake}, \text{Ken}) \in P(\text{lake}, \text{Ken})$. For example, Eva admits full discomfort on water conditions, half discomfort on services conditions and full discomfort on lake availability. Joes allows half discomfort on water conditions, great discomfort on services conditions and no discomfort on lake availability. Ken admits full discomfort on water conditions, discomfort on services by services conditions and half discomfort on lake availability as it is shown in Figure 3.

Fig. 3. Shortterm preferences in heterogeneous approach

	water	services	lake
Eva	1	1/2	1
Joe	1/2	2/3	0
Ken	1	se	1/2

Full discomfort on water conditions for Eva (table value 1) means that by the first table from Figure 2, Eva admits arbitrary number of days and hot or cold water, because all cases from Eva's water table are less or equal to 1 in Figure 2. Partial discomfort on services for Eva (table value 1/2) admits neither presence of all services or maximal one arbitrary day at the cottage and internet only, because these cases from Eva's services table are less or equal to 1/2 in second table of Figure 2. Similarly Ken permit neither all services or only one day at the cottage with internet connection only as you can see from Figure 2 and the eighth table in Figure 3.

Eventually in effort to identify necessary cottage conditions that fulfill all personal requirements we define following mappings $\nearrow$ and $\swarrow$.

Let G be the set of all functions g with the domain B such that $g(b) \in D_b$, for all $b \in B$. (i. e. $G = \Pi_{b\in B} D_b$). Each function g corresponds to particular person's length of stay (e. g. $g(\text{Eva}) = 1/2$, $g(\text{Joe}) = \text{Sa}$, $g(\text{Ken}) = 1/2$).

And let F be the set of all functions f with the domain A such that $f(a) \in C_a$, for all $a \in A$ (i. e., more formally, $F = \Pi_{a\in A} C_a$). Each function f corresponds to particular cottage conditions (e. g. $f(\text{water}) = \text{hot}$, $f(\text{services}) = \text{in}$, $f(\text{lake}) = \text{yes}$).

Define the following mapping $\nearrow : G \to F$: If $g \in G$ then $\nearrow(g) \in F$ is defined by

$$(\nearrow(g))(a) = \sup\{c \in C_a : (\forall b \in B) c \bullet_{a,b} g(b) \leq R(a, b)\}.$$

Mapping $(\nearrow(g))(a)$ expresses requirement to the worst water or services conditions at the cottage by specific number of staying days that return at most degree of discomfort admitted by people. For instance, if $g(\text{Eva}) = 1/2$, $g(\text{Joe}) = \text{Sa}$, $g(\text{Ken}) = 1/2$, then for water we get $(\nearrow(g))(\text{water}) = \text{cold}$, that means that one day staying for Eva, staying on Saturday for Joe and one day staying for Ken correspond to the possibility for cold water at the cottage. Another example, if $g(\text{Eva}) = \text{Sa} + \text{Su}$, $g(\text{Joe}) = \text{Sa}$, $g(\text{Ken}) = \text{Sa} + \text{Su}$, then for services we get $(\nearrow(g))(\text{services}) = \text{in} + \text{tv}$, that admitted only cottage with internet connection and tv as the worst possible cottage in case of Eva's staying on Saturday and Sunday, Joe's staying on Saturday and Ken's staying on Saturday and Sunday.

Symmetrically define the mapping $\swarrow : F \to G$: If $f \in F$ then $\swarrow(f) \in G$ is defined as following:

$$(\swarrow(f))(b) = \sup\{d \in D_b : (\forall a \in A) f(a) \bullet_{a,b} d \leq R(a, b)\}.$$

Mapping $(\swarrow(f))(b)$ expresses natural requirement to maximalize number of days spent at the cottage by specific water and services conditions that return at most degree of discomfort admitted by a person. For instance, for $f(\text{water}) = \text{hot}$, $f(\text{services}) = \text{in}$, $f(\text{lake}) = \text{yes}$ we get $(\swarrow(f))(\text{Eva}) = 1/2$, that means that hot water and internet only correspond to maximal one day staying at the cottage for Eva.

We proved in [2] that the concept-forming mappings defined in this way have worthwile properties. Here we give some natural interpretation for this theorem written below.

Theorem 1. *Let $f \in F$ and $g \in G$. Then the following conditions are equivalent:*

1 $f \leq \nearrow(g)$.
2 $g \leq \swarrow(f)$.
3 $f(a) \bullet_{a,b} g(b) \leq R(a,b)$ *for all* $a \in A$ *and* $b \in B$.

First part of the theorem can be interpreted as too much superflous or equal conditions than conditions corresponding to concrete lengths of stay for people. Second part expresses that lengths of stay for people is less or equal than lengths corresponding to concrete cottage conditions. And third part represents that this concrete conditions and lengths of stay certainly satisfy all shortterm preferences.

Corollary 1. *Mappings* $\nearrow$ *and* $\swarrow$ *form a Galois connection.*

Proof. It follows from the equivalency of conditions 1 and 2 of the previous theorem.

3 Heterogeneous formal concept

We use a Galois connection $(\nearrow, \swarrow)$ for the concept lattice construction via classical Ganter-Wille's approach from [14].

By a *concept* we will understand a pair $\langle g, f \rangle$ from $G \times F$ such that $\nearrow(g) = f$ and $\swarrow(f) = g$.

Lemma 1. *If* $\langle g_1, f_1 \rangle$ *and* $\langle g_2, f_2 \rangle$ *are concepts then* $g_1 \leq g_2$ *iff* $f_1 \geq f_2$.

Proof. It is a simple consequency of the Collorary 2 (namely, parts 3a and 3b).

This lemma allows to define the following ordering of concepts: $\langle g_1, f_1 \rangle \leq \langle g_2, f_2 \rangle$ iff $g_1 \leq g_2$ (or equivalently $f_1 \geq f_2$).

In summary by previous consideration we observed eight concepts in our running cottage example for three people and three cottage conditions shown in Figure 4.

Intents correspond to the worst cottage conditions that fulfill all personal requirements for specific number of days noticed in extent of concept. For example cold water, no services and lake available at the cottage are connected with no staying for Eva, staying on Saturday for Joe and no staying for Ken (second concept). In contrary, hot water, full services and lake available indicates maximal number of days spent for all people (last concept). Similarly one can interpret further concepts. For instance seventh concept shows that one day spend at the cottage by Eva, both days by Joe and one day by Ken requires the worst possible condition with hot water, internet connection and lake available.

Notice that intents do not include posibility of hot water and no services simultaneously. In this case we obtain $\swarrow(\text{hot}, \text{no}, \text{yes}) = (\emptyset, \text{Sa}, \emptyset)$ and subsequently $\nearrow(\emptyset, \text{Sa}, \emptyset) = (\text{cold}, \text{no}, \text{yes})$. It can be interpreted as too much superflous cottage conditions for Joe's stay on Saturday and maybe we can choose cheaper cottage.

Fig. 4. Heterogenous formal concepts for 3 people and 3 attributes

extents			intents		
Eva	Joe	Ken	water	services	lake
∅	∅	∅	cold	no	no
∅	Sa	∅	cold	no	yes
1/2	∅	1/2	cold	in	no
1/2	Sa	1/2	cold	in	yes
Sa+Su	∅	1/2	cold	in+tv	no
Sa+Su	Sa	Sa+Su	cold	in+tv	yes
1/2	Sa+Su	1/2	hot	in	yes
Sa+Su	Sa+Su	Sa+Su	hot	in+tv	yes

All computations in our cottage example are done for three people, but it is fruitful to consider more complex example as in Figure 1 for six people water conditions, services conditions and lake for swimming available. Also it is possible that two people have the same lattice structures, for instance Eva and Lea have the same water and services lattices. Nevertheless behavior of Eva and Lea by the same condition should be diverse. One day and cold water should correspond to discomfort for Eva, but comfort for Lea or vice versa.

In this sense we make computation of all concepts for cottage example on six people and three cottage conditions and number of concepts was nine. The results are shown in Figure 5.

Fig. 5. Heterogenous formal concepts for 6 people and 3 attributes

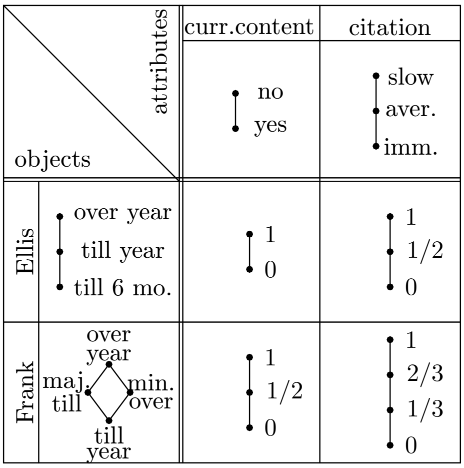
extents						intents		
Eva	Joe	Ken	Lea	Sue	Tim	water	services	lake
∅	∅	∅	∅	1/2	Su	cold	no	no
∅	Sa	∅	∅	1/2	Su	cold	no	yes
1/2	∅	1/2	1/2	1/2	Su	cold	in	no
1/2	Sa	1/2	Sa+Su	1/2	Su	cold	in	yes
Sa+Su	∅	1/2	1/2	1/2	Su	cold	in+tv	no
Sa+Su	Sa	Sa+Su	Sa+Su	1/2	Sa+Su	cold	in+tv	yes
1/2	Sa+Su	1/2	Sa+Su	1/2	Su	hot	in	yes
Sa+Su	∅	1/2	1/2	Sa+Su	Su	hot	in+tv	no
Sa+Su	Sa+Su	Sa+Su	Sa+Su	Sa+Su	Sa+Su	hot	in+tv	yes

4 Another working examples

Medina, Ojeda-Aciego and Ruiz Calviño in [22] consider situation that we have written a scientific paper and have to decide which journal to choose for submitting. Set of objects consists of particular scientific journal (AMC, CAMWA, FSS, ...) and set of attributes includes journal properties as impact factor, immediacy index, cited half-life and best position. Furthermore, problem consists in finding a multi-adjoint concept which represent the suitable journal to submit.

We provide the analogous analysis for our heterogeneous approach and propose following situation. People writing a scientific paper have to conclude which attributes of scientific journal is required to satisfy all researchers. Let $B = \{\text{Ellis}, \text{Frank}, \ldots\}$, set of objects, consists of people writing a mutual paper with specification of willingness to wait some time period to accepting of an article (till 6 months, till year, till year in case of science is a major job, over year in case of science is a minor job, over year). And let $A = \{\text{current content}, \text{citation}, \ldots\}$, set of attributes, includes specific properties of journals. Table values correspond to dissatisfaction with overall process of paper accepting by actual conditions. For example waiting till 6 month and current content means for Ellis no dissatisfaction (notated as table value 0), but waiting over year and uncurrent content full dissatisfaction (notated as table value 1).

Fig. 6. List of possible values for objects and attributes in journal example



Having expressed complete longterm and shortterm preferences of people working together on a paper, obtained concepts correspond to necessary attributes of journal satisfying all preferences. For instance consider that Ellis requires to publish till 6 months and Frank wishes to publish till one year and research represents his major job. In that case is necessary to submit to uncurrent content journal with medium immediacy index for citation.

Another example is based on a job background. Consider people applying for a job in the same company. The purpose is to specify conditions satisfying all personal requirements and effort to work together. Let $B = \{\text{Peter}, \text{Paul}, \ldots\}$, set of objects, consists of people applying for no job, part-time job, job on performance contract or full time job. And let $A = \{\text{salary}, \text{language skills}, \text{start date}, \ldots\}$, set of attributes, includes specific job conditions. Salary conditions contain three degrees: high, medium and low; start date includes two degrees: immediately or at a later date; foreign languages requirements contain three degrees: no, one or two foreign language required. Table values express dissatisfaction with conditions connected with type of contract and job properties. Higher value corresponds to more dissatisfaction.

Fig. 7. List of possible values for objects and attributes in heterogenous job example

objects \ attributes		salary	start	languages
		low med. high	late imm.	two one no
Peter	full-time ∅	1 0	1 1/2 0	1 0
Paul	full-time part, cont. ∅	1 1/2 0	1 2/3 1/3 0	1 1/2 0

Resulting concepts have the following interpretation. By consideration that Peter requires full-time job and Paul claims for job on performance contract, it is for instance necessary to find job with medium salary, immediate start date and most one spoken language.

We do not introduce particular longterm and shortterm preferences for this running examples on journal and job, but we give some motivation about usefulness of this heterogeneous approach in such area.

5 Heterogeneous formal concept lattice

The poset of all concepts ordered by $\leq$ will be called a *heterogeneous concept lattice* and denoted by $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$.

The following theorem shows that the word *lattice* in its name corresponds with reality. The proofs of analogous theorems in previous approaches are included in different papers ([13], [14], [18]).

Theorem 2. *(The Basic Theorem on Heterogeneous Concept Lattices)*

1 *A heterogeneous concept lattice* $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ *is a complete lattice in which*

$$\bigwedge_{i\in I} \langle g_i, f_i \rangle = \left\langle \bigwedge_{i\in I} g_i, \nearrow \left(\swarrow \left(\bigvee_{i\in I} f_i \right) \right) \right\rangle$$

and

$$\bigvee_{i\in I}\langle g_i, f_i\rangle = \left\langle \swarrow\left(\nearrow\left(\bigvee_{i\in I} g_i\right)\right), \bigwedge_{i\in I} f_i\right\rangle .$$

2 *For each* $a \in A$*,* $b \in B$*, let* $P_{a,b}$ *have the least element* $0_{P_{a,b}}$ *such that* $0_{C_a} \bullet_{a,b} d = c \bullet_{a,b} 0_{D_b} = 0_{P_{a,b}}$*, for all* $c \in C_a$*,* $d \in D_b$*. Then a complete lattice* L *is isomorphic to* $\mathrm{HCL}(A, B, \mathcal{P}, R, \mathcal{C}, \mathcal{D}, \odot, \swarrow, \nearrow, \leq)$ *if and only if there are mappings* $\alpha : \bigcup_{a\in A}(\{a\} \times C_a) \to L$ *and* $\beta : \bigcup_{b\in B}(\{b\} \times D_b) \to L$ *such that:*

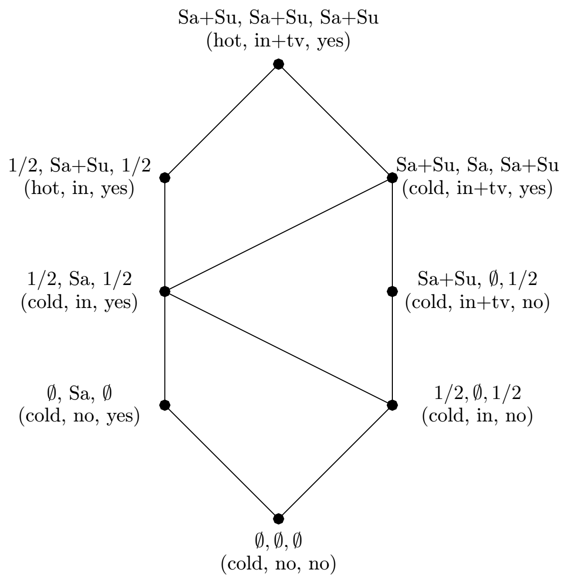
1a) α *does not increase in the second argument (for the fixed first one).*
1b) β *does not decrease in the second argument (for the fixed first one).*
2a) $\mathrm{Rng}(\alpha)$ *is* inf*-dense in* L*,* [1]
2b) $\mathrm{Rng}(\beta)$ *is* sup*-dense in* L*.*
3) For every $a \in A$*,* $b \in B$ *and* $c \in C_a$*,* $d \in D_b$

$$\alpha(a, c) \geq \beta(b, d) \qquad \text{if and only if} \qquad c \bullet_{a,b} d \leq R(a, b).$$

Proof. For self-contained proof see [2].

Figure 8 represents the resulting heterogeneous concept lattice for our cottage example with ordered concepts for 3 people and 3 cottage conditions.

Fig. 8. Heterogenous formal concept lattice for 3 people and 3 attributes



[1] $\mathrm{Rng}(\alpha)$ denotes range of mapping α.

6 Conclusions and possible future works

In this paper we introduce some running examples on heterogeneous environment of the Formal Concept Analysis based on cottage, journal or job context. The main idea of heterogenenous approach is to diversify all that can be diversified and it is interesting that process of concept lattice construction still works. Hence, intuitively, it allows to use the Formal Concept Analysis also for tables with data of different types.

Bělohlavek shows how to deal with the problem of generating all concepts of a fuzzy concept lattice in [4]. A fast bottom-up algorithm to compute all concepts of a fuzzy closure operator is presented in [7]. We would like to modify and generalize these algorithms for our heterogeneous approach, too. And in this way we will make assumption of not linearly ordered set of truth degrees. Then it is fruitful to apply it on real-world data.

We would like to put emphasis that there is an similarly called approach working with multi-adjoint concept lattices based on heterogeneous conjunctors. This is done by Medina and Ojeda-Aciego in [21]. The difference is following. Multi-adjoint concept lattices work with different lattices too, but only for sets of attributes and objects. Objects and attributes are evaluated in two different lattices and on heterogeneous conjunctors, finally both different lattices are embedded to new so-called connected lattice and thus resulting concept lattice utilizes the same lattice for objects and attributes.

The next interesting connection is clarifying the relationship of our heterogeneous approach to Bělohlávek & Vychodil's fuzzification working with truth-stressers, so-called hedges (in [9] and [10]). In [19] it is shown that generalized concept lattices cover them in some sense but it seems that this new approach make this relationship more immediate.

In [15] is hedges used as a tool to reduce the size of multi-adjoint concept lattices with heterogeneous conjunctors as unifying of [21] and [10]. Another relationship that seems to be interesting for future work is heterogenity in multi-adjoint concept multilattices that are more general structures as lattices [26].

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