TECHNISCHE UNIVERSITÄT DRESDEN
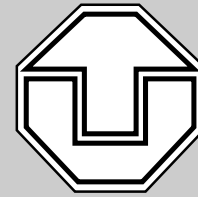
Fakultät Informatik
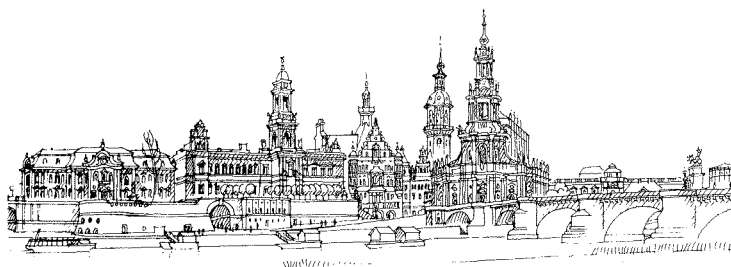
TUD-FI00-12 - November 2000

Peter Buchholz, Igor V. Tarasyuk

Dresden University of Technology,
Institute of Informatics Systems, Novosibirsk

**A Class of Stochastic Petri Nets
with Step Semantics
and Related Equivalence Notions**

A Class of Stochastic Petri Nets with Step Semantics and Related Equivalence Notions

PETER BUCHHOLZ
Department of Computer Science
Technical University of Dresden
01069 Dresden, Germany
p.buchholz@inf.tu-dresden.de

IGOR V. TARASYUK *
A.P. Ershov Institute of Informatics Systems
Siberian Division of the Russian Academy of Sciences
6, Acad. Lavrentiev pr., 630090 Novosibirsk, Russia
itar@iis.nsk.su

Abstract

This paper presents a class of Stochastic Petri Nets with concurrent transition firings. It is assumed that transitions occur in steps and that for every step each enabled transition decides probabilistically whether it wants to participate in the step or not. Among the transitions which want to participate in a step, a maximal number is chosen to perform the firing step. The observable behavior of a net is described by labels associated with transitions. For this class of nets the dynamic behavior is defined and equivalence relations are introduced. The equivalence relations extend the well-known trace and bisimulation equivalences for systems with step semantics to Stochastic Petri Nets with concurrent transition firing. It is shown that the equivalence notions form a lattice of interrelations.

Keywords: Stochastic Petri Nets, Step Semantics, Equivalence Relations, Bisimulation.

1 Introduction

Stochastic Petri Nets (SPNs) are an established model type for the quantitative analysis of Discrete Event Dynamic Systems (DEDSs). SPNs have been proposed about twenty years ago [9, 15] and are mainly considered on a continuous time scale which usually means that exponential or phase type distributions are associated with transitions. In this way, the stochastic process underlying an SPN is a Continuous Time Markov Chain (CTMC) which can be generated and analyzed with well-known methods [20]. One particular characterization of this class of SPNs is that only single transitions fire, such that the well-known interleaving semantics is the basic approach for defining the dynamic behavior of SPNs. This interleaving behavior is also used for Generalized Stochastic Petri Nets (GSPNs) [1, 6] which include transitions with exponential firing delay and that with zero firing delay. Even for such immediate transitions with instantaneous firing interleaving semantics is commonly considered. For SPNs and GSPNs, labeling of transitions has been introduced recently [3, 4]. After definition of transition labeling it is possible to define bisimulation equivalence for SPNs and GSPNs such that equivalent nets behave identically from a stochastic point of view. Details about the approach which introduces bisimulation for CTMCs with labeled transitions can be found in [2, 3, 11, 12].

Apart from continuous time distributions also discrete time distributions can be assigned to transitions of Petri nets. Usually geometric distributions or mixtures of geometric distributions are used. First approaches have been published about 15 years ago [16], but also more recent extensions of the basic class of nets with discrete time steps have been proposed [23, 24]. To distinguish continuous and discrete time SPNs, we denote the former as CTSPNs and the latter as DTSPNs. DTSPNs describe an underlying Discrete Time Markov Chain (DTMC). The major problem with this model class is that transitions fire concurrently such that steps instead of interleavings have to be considered. This makes the interpretation and analysis of the model class more complex. For DTSPNs labeling of transitions and an adequate definition of equivalence has not been introduced yet.

In this paper, we present an introduction of a new class of DTSPNs with labeled transitions. The dynamic behavior of this class of nets is characterized by steps instead of single transitions. The underlying stochastic process is still a DTMC, however, transitions of the DTMC describe sets of transitions that fire concurrently.

Thus, commonly used notions defining bisimulation or trace equivalence of probabilistic processes [7, 14] are not adequate for this type of model.

The outline of the rest of the paper is as follows. In the next Section 2 a new class of DTSPNs and the underlying stochastic process is introduced. Afterwards some examples are presented. Then, in Section 3, equivalence relations are defined for the presented class of nets, and interrelations between the different equivalence relations are outlined. Section 4 introduces briefly the long run behavior of DTSPNs and describes which behavior is preserved by which equivalence relation. In the concluding Section 5 we remind the main results of the paper and propose some directions of future research.

2 A class of Discrete Time Stochastic Petri Nets

In this section, we introduce basic notions used throughout the paper and present several examples.

2.1 Formal definitions of the model and its behavior

DTSPNs which are the basic net class considered in this paper are defined as follows.

Definition 2.1 *A DTSPN is a seven tuple* $N = (P, T, W, \Lambda, \Omega, L, M_{in})$ *where:*

- *P and T are finite sets of* places *and* transitions *respectively such that* $P \cup T \neq \emptyset$ *and* $P \cap T = \emptyset$;

- *$W : (P \times T) \cup (T \times P) \to \mathbf{N}$ is function describing the* weights of arcs *between places and transitions and vice versa;*

- *$\Lambda : T \to \mathbf{R}^+$ is the* transition weight *function;*

- *$\Omega : T \to (0, 1]$ is the* transition probability *function;*

- *$L : T \to Act_\tau$ is the* transition labeling *function assigning labels from a finite set of visible actions Act or an invisible action τ to transitions (i.e., $Act_\tau = Act \cup \{\tau\}$);*

- *$M_{in} : P \to \mathbf{N}$ is the* initial marking.

The initial marking M_{in} is a specific case of a marking which assigns natural numbers to places. The marking of the net is modified by firing transitions. A transition $t \in T$ is enabled at marking M if $M(p) \geq W(p, t)$ for all $p \in P$. Let $Ena(M)$ be the set of *all transitions that are enabled at marking M*. Firings of transitions are atomic operations, and transitions may fire concurrently. We assume that firings of transitions take place in steps. A transition $t \in Ena(M)$ tries to fire in the next step with probability $\Omega(t)$. Let $U \subseteq Ena(M)$ be a set of transitions that try to fire in the next step. The probability that transitions from the set U try to fire is given by:

$$PF[U] = \prod_{t \in U} \Omega(t) \cdot \prod_{t \in Ena(M) \setminus U} (1 - \Omega(t)) \,. \tag{1}$$

However, not necessarily the whole batch U can fire concurrently because transitions may be in conflict such that only a subset of transitions is able to fire. All transitions from a set U can fire if:

$$\forall p \in P : M(p) \geq \sum_{t \in U} W(p, t) \,. \tag{2}$$

If not all transitions from U can fire, then maximal subsets are chosen.

A set $V \subseteq Ena(M)$ is a *maximal fireable subset at marking M* if (2) holds for V and no more transitions from $Ena(M) \setminus V$ can be added when (2) has to hold. By $MaxFire(M)$ we denote the set of *all maximal fireable subsets at marking M*.

Similarly, a set $V \subseteq U$ is a *maximal fireable subset of U at marking M* if (2) holds for V and no more transitions from $U \setminus V$ can be added when (2) has to hold. By $MaxFire(U, M)$ we denote the set of *all maximal fireable subsets of U at marking M*.

We extend the weight function to sets of transitions. If $V \subseteq T$ then:

$$\Lambda(V) = \sum_{t \in V} \Lambda(t) \,.$$

If transitions from the set U try to fire, but cannot fire concurrently since (2) does not hold, then a maximal fireable subset of transitions, i.e., one element from $MaxFire(U, M)$, is chosen. Subsets are chosen according to the normalized weights. I.e., a subset $V \in MaxFire(U, M)$ is chosen with probability:

$$PC[V, U] = \Lambda(V) \left/ \left(\sum_{W \in MaxFire(U,M)} \Lambda(W) \right) \right. . \tag{3}$$

For each $V \in MaxFire(M)$ let $SubEna(V, M)$ be the set of *all subsets of $Ena(M)$ that include V*. The probability of observing $V \in MaxFire(M)$ is given by:

$$PT[V, M] = \sum_{U \in SubEna(V,M)} PF[U] \cdot PC[V, U]. \tag{4}$$

Observe that (3) defines a probability distribution over all sets of transitions from $MaxFire(M)$. Sets of transitions that do not belong to $MaxFire(M)$ cannot fire concurrently at marking M and thus receive zero probability.

We have not considered the labeling of transitions yet. However, the idea of labeling is that transitions receive the same label if they are indistinguishable for an external observer. We assume that the set of labels Act_τ contains a specific label τ that is not visible. Thus, transitions labeled with τ cannot be observed and called *invisible*.

We define the *visible labeling* function $VisL$ on sets of transitions which associates with them multisets of visible actions. If $V \subseteq T$ then:

$$VisL(V) = \sum_{(t \in V) \wedge (L(t) \neq \tau)} L(t).$$

Denote a set of all multisets over a set X by $\mathcal{M}(X)$. Let A be a multiset of visible transition labels, i.e., $A \in \mathcal{M}(Act)$. Then

$$Trans(A) = \{V \subseteq T \mid VisL(V) = A\}$$

is the set of *all subsets of transitions which are labeled with A*.

The probability of observing A at marking M is then given by:

$$PL[A, M] = \sum_{V \in Trans(A) \cap MaxFire(M)} PT[V, M]. \tag{5}$$

Firing of sets of transitions yields a successor marking. If V fires in M, then the successor marking $\widetilde{M}$ is defined componentwise as:

$$\widetilde{M}(p) = M(p) - \sum_{t \in V} W(p, t) + \sum_{t \in V} W(t, p).$$

Let V be a set of transitions which can fire concurrently at marking M resulting to $\widetilde{M}$ and $\mathcal{P} = PT[V, M]$. We use the shorthand notation $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$ for such a firing step. We shall write $M \xrightarrow{V} \widetilde{M}$ if $M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element set of transitions $V = \{t\}$ we write $M \xrightarrow{t}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{t} \widetilde{M}$.

By considering only the labels and not the concrete transitions, we obtain steps described by multisets of transition labels. Thus, $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ describes a step starting at marking M, performing transitions labeled with A and ending at $\widetilde{M}$. The probability of the step $\mathcal{P} = PS[A, M, \widetilde{M}]$ is computed as:

$$PS[A, M, \widetilde{M}] = \sum_{\{V \in Trans(A) \mid M \xrightarrow{V}_{\mathcal{P}} \widetilde{M}\}} \mathcal{P}.$$

We shall write $M \xrightarrow{A} \widetilde{M}$ if $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a} \widetilde{M}$.

Definition 2.2 *For a DTSPN N we define:*

- *The* reachability set $RS(N)$ *as the minimal set of markings M for which the following conditions hold:*

  - $M_{in} \in RS(N)$;

– if $M \in RS(N)$ and $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for $\mathcal{P} > 0$, then $\widetilde{M} \in RS(N)$.

- **The** reachability graph $RG(N)$ **as a directed labeled graph with a set of nodes** $RS(N)$ **and an arc labeled with** A, $\mathcal{P}$ **between nodes** M **and** $\widetilde{M}$ **whenever** $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ **holds.**

- **The underlying** Discrete Time Markov Chain (DTMC) $DT(N)$ **with state space** $RS(N)$ **and a transition** $M \longrightarrow_{\mathcal{P}} \widetilde{M}$ **whenever at least one arc between** M **and** $\widetilde{M}$ **exists in** $RG(N)$. **In this case, the probability** $\mathcal{P}$ **is computed as:**

$$\mathcal{P} = \sum_{A \in \mathcal{M}(Act)} PS[A, M, \widetilde{M}].$$

The previous definition proposes the set of reachable markings, the corresponding reachability graph which preserves transition labels and probabilities and the underlying Discrete Time Markov Chain. Observe that the reachability graph may include arcs with non-zero probability which correspond to the empty multiset. In this case, a marking is modified by firing internal transition labeled with τ. An external observer who can only see visible transitions labeled with some action from Act cannot notice such a step. At the level of the DTMC, transition steps can no longer be distinguished, and we observe the stochastic process as usual for discrete time models like SPNs in discrete time [16, 23, 24].

If we assume that an observer does not know when a step takes place, (s)he cannot see firing of a set of internal transitions resulting in an empty multiset of transition labels. This behavior can be described by transforming the reachability graph by skipping unobservable transitions. The approach is similar to building the observational graph in untimed models [8]. A step $M \xrightarrow{\emptyset}_{\mathcal{P}} \widetilde{M}$ with $\mathcal{P} > 0$ takes place when $\widetilde{M}$ is reachable from M by firing a set of internal transitions. To skip steps of internal transitions, we use the following recursive definition of internal transition probabilities:

$$PS^k[\emptyset, M, \widetilde{M}] = \begin{cases} \sum_{\overline{M} \in RS(N)} PS^{k-1}[\emptyset, M, \overline{M}] \cdot PS[\emptyset, \overline{M}, \widetilde{M}] & \text{if } k \geq 1; \\ 1 & \text{if } k = 0 \text{ and } M = \widetilde{M}; \\ 0 & \text{otherwise.} \end{cases}$$

$PS^k[\emptyset, M, \widetilde{M}]$ describes the *probability of reaching* $\widetilde{M}$ *from* M *by* k *steps of internal transitions*. Furthermore we define:

$$PS^*[\emptyset, M, \widetilde{M}] = \sum_{k=0}^{\infty} PS^k[\emptyset, M, \widetilde{M}]$$

which is the *probability of reaching* $\widetilde{M}$ *from* M *by steps of internal transitions* and:

$$PS^*[A, M, \widetilde{M}] = \sum_{\overline{M} \in RS(N)} PS^*[\emptyset, M, \overline{M}] \cdot PS[A, \overline{M}, \widetilde{M}]$$

which is the *probability of reaching* $\widetilde{M}$ *from* M *by an arbitrary number of internal steps, followed by an observable step* A.

A *trap* is a loop of internal transitions starting and ending at some marking M which occurs with probability 1. If $RG(N)$ contains a trap, then the net stucks in a sequence of internal transitions which cannot be left. $PS^*[\emptyset, M, \widetilde{M}]$ is finite as long as no traps exist which will be assumed in the sequel. If $PS^*[\emptyset, M, \widetilde{M}]$ is finite, then $PS^*[A, M, \widetilde{M}]$ defines a probability distribution, i.e.:

$$\sum_{A \in \mathcal{M}(Act) \setminus \emptyset} \sum_{\widetilde{M} \in RS^*(N)} PS^*[A, M, \widetilde{M}] = 1.$$

The result follows from standard results on absorbing Markov chains [13]. Thus, we can define a new transition system with the transition relation $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ where $\mathcal{P} = PS^*[A, M, \widetilde{M}]$ and $A \neq \emptyset$.

We shall write $M \xrightarrow{A} \widetilde{M}$ if $M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}$ for some $\mathcal{P} > 0$. For one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}_{\mathcal{P}} \widetilde{M}$ and $M \xrightarrow{a} \widetilde{M}$.

We denote by $RS^*(N)$ and $RG^*(N)$ the *observable reachability set* and *graph* respectively. Note that $RS(N) \neq RS^*(N)$ whenever markings exist that are entered by invisible steps only (see also the examples given below). $RG^*(N)$ describes the viewpoint of an person who observes steps only if they include visible transitions.
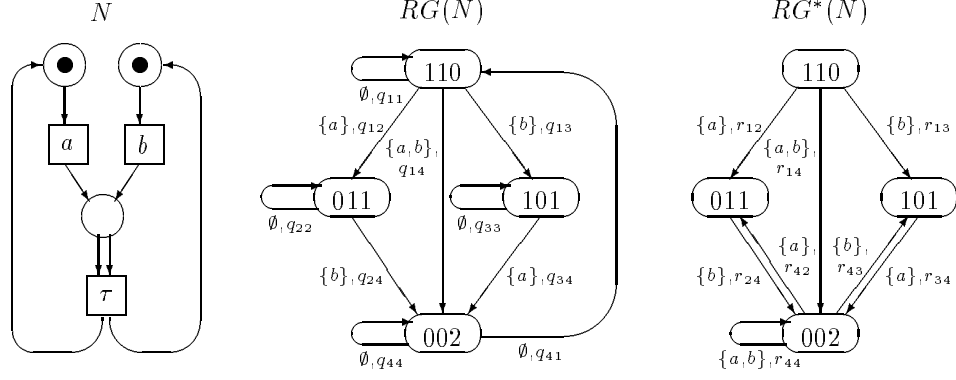
Figure 1: First example net and the corresponding reachability graphs

Transition probabilities $PS^*[\cdot]$ define a DTMC with state space $RS^*(N)$ and transition probabilities:

$$PS^*[M, \widetilde{M}] = \sum_{A \in \mathcal{M}(Act)} PS^*[A, M, \widetilde{M}]$$

which will be denoted as the *embedded DTMC*.

Following the terminology of [10], we have introduced a generative model. However, in contrast to other stochastic models [7, 10, 14] which are based on some form of stochastic automata where only single events occur, we consider here the concurrent execution of different transitions. This is a very natural view for Petri nets which allow distributed state descriptions and parallel executions of transitions. Let us also note that we do not allow selfconcurrency, i.e., concurrent firing of a transition with itself (multisets of transitions).

2.2 Examples of DTSPNs

A first example is shown in Figure 1. It describes simple net with two observable transitions t_1 (labeled by a), t_2 (labeled by b) and one τ-labeled transition t_3. The reachability graph $RG(N)$ and the observable reachability graph $RG^*(N)$ are also depicted in the figure. To define probabilities we use the following numbering of markings: 1. $(110), 2. (011), 3. (101), 4. (002)$. The values q_{ij} and r_{ij} are probabilities which receive the values shown below. Weights of transitions are not relevant in this example because the net contains no conflict. For convenience we use the following notation: $\overline{\Omega}(t_i) = 1 - \Omega(t_i)$ $(1 \le i \le 3)$. Now we present the probabilities q_{ij} $(1 \le i, j \le 4)$:

$$
\begin{array}{lllll}
q_{11} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2) & q_{12} = \Omega(t_1) \cdot \overline{\Omega}(t_2) & q_{13} = \overline{\Omega}(t_1) \cdot \Omega(t_2) & q_{14} = \Omega(t_1) \cdot \Omega(t_2) & q_{22} = \overline{\Omega}(t_2) \\
q_{24} = \Omega(t_2) & q_{33} = \overline{\Omega}(t_1) & q_{34} = \Omega(t_3) & q_{41} = \Omega(t_3) & q_{44} = \overline{\Omega}(t_3)
\end{array}
$$

For the definition of r_{kl} $(1 \le k, l \le 4)$ the values q_{ij} defined above are used:

$$r_{12} = r_{42} = \frac{q_{12}}{1 - q_{11}} \quad r_{13} = r_{43} = \frac{q_{13}}{1 - q_{11}} \quad r_{14} = r_{44} = \frac{q_{14}}{1 - (1 - q_{11})} = \frac{q_{14}}{q_{11}} \quad r_{24} = 1 \quad r_{34} = 1$$

The second example is shown in Figure 2. It describes a net with two observable transitions t_1 (labeled by a), t_2 (labeled by b) and two τ-labeled transitions t_3 and t_4. To avoid an overloading of notations, if two arcs with different labels exist in $RG(N)$ or $RG^*(N)$, then only one arc is shown, and both labels are printed beneath the arc (i.e., $\{a\}, \{b\}$ describes that one arc labeled with $\{a\}$ and one arc $\{b\}$ are present). To define probabilities we use the following numbering of markings: 1. $(110), 2. (011), 3. (101), 4. (002), 5. (020)$ and 6. (200). Observe that $RS^*(N)$ contains only the markings 1–4. Markings 5 and 6 are not reachable, i.e., after an observable event, the net cannot be in one of these markings. We use the notation q_{ij}^A for the probability of the transition in $RG(N)$ between i and j which is labeled with set A (for one-element multisets like $A = \{a\}$ we shall omit the curly braces). If only one transition between i and j exists, then label A is suppressed. Similarly r_{ij}^A is used for transition probabilities in $RG^*(N)$. For the presentation of the probabilities we use the abbreviations:

$$\Lambda_{34} = \frac{\Lambda(t_3)}{\Lambda(t_3) + \Lambda(t_4)} \text{ and } \Lambda_{43} = \frac{\Lambda(t_4)}{\Lambda(t_3) + \Lambda(t_4)}.$$

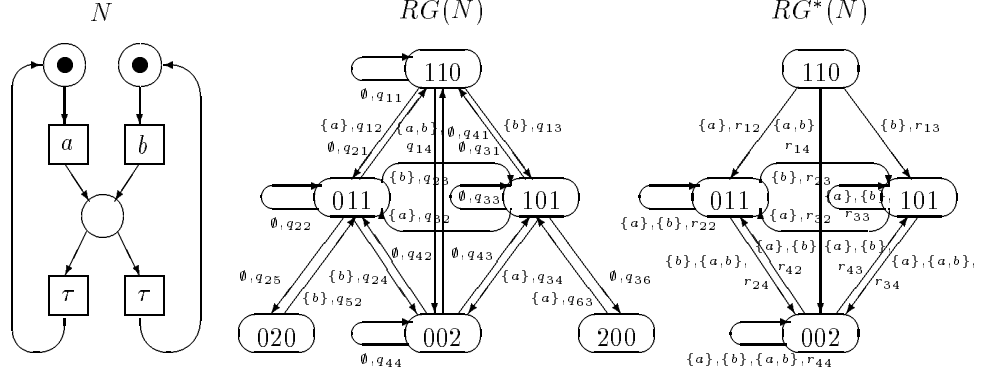Thus, we obtain the probabilities q_{ij}^A $(1 \le i, j \le 6)$:

Figure 2: Second example net and the corresponding reachability graphs

$$q_{11} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_2) \qquad\qquad q_{12} = \Omega(t_1) \cdot \overline{\Omega}(t_2)$$
$$q_{13} = \overline{\Omega}(t_1) \cdot \Omega(t_2) \qquad\qquad q_{14} = \Omega(t_1) \cdot \Omega(t_2)$$
$$q_{21} = \overline{\Omega}(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4)) \qquad q_{22}^{\emptyset} = \overline{\Omega}(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{22}^{b} = \Omega(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \qquad q_{23} = \Omega(t_2) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{24} = \Omega(t_2) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \qquad q_{25} = \overline{\Omega}(t_2) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q_{31} = \overline{\Omega}(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3)) \qquad q_{32} = \Omega(t_1) \cdot \Omega(t_4) \cdot (\Lambda_{43} \cdot \Omega(t_3) + \overline{\Omega}(t_3))$$
$$q_{33}^{\emptyset} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \qquad q_{33}^{a} = \overline{\Omega}(t_1) \cdot \overline{\Omega}(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{34} = \Omega(t_1) \cdot \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4) \qquad q_{36} = \overline{\Omega}(t_1) \cdot \Omega(t_3) \cdot (\Lambda_{34} \cdot \Omega(t_4) + \overline{\Omega}(t_4))$$
$$q_{41} = \Omega(t_3) \cdot \Omega(t_4) \qquad\qquad q_{42} = \overline{\Omega}(t_3) \cdot \Omega(t_4)$$
$$q_{43} = \Omega(t_3) \cdot \overline{\Omega}(t_4) \qquad\qquad q_{42} = \overline{\Omega}(t_3) \cdot \overline{\Omega}(t_4)$$
$$q_{52} = \Omega(t_2) \qquad\qquad q_{55} = \overline{\Omega}(t_2)$$
$$q_{63} = \Omega(t_1) \qquad\qquad q_{66} = \overline{\Omega}(t_1)$$

For the definition of probabilities r_{kl}^{A} $(1 \leq k, l \leq 4)$, we use the probabilities q_{ij}^{A}:

$$r_{12} = q_{12}/(1 - q_{11}) \qquad r_{13} = q_{13}/(1 - q_{11}) \qquad r_{14} = q_{14}/(1 - q_{11})$$
$$r_{22}^{a} = q_{21} \cdot r_{12}/(1 - q_{22}^{\emptyset}) \qquad r_{22}^{b} = (q_{22}^{b} + q_{25})/(1 - q_{22}^{\emptyset}) \qquad r_{23} = (q_{23} + q_{21} \cdot r_{13})/(1 - q_{22}^{\emptyset})$$
$$r_{24}^{b} = q_{24}/(1 - q_{22}^{\emptyset}) \qquad r_{24}^{\{a,b\}} = q_{21} \cdot r_{14}/(1 - q_{22}^{\emptyset}) \qquad r_{32} = (q_{32} + q_{31} \cdot r_{12})/(1 - q_{33}^{\emptyset})$$
$$r_{33}^{a} = (q_{33}^{a} + q_{36})/(1 - q_{33}^{\emptyset}) \qquad r_{33}^{b} = q_{31} \cdot r_{13}/(1 - q_{33}^{\emptyset}) \qquad r_{34}^{a} = q_{34}/(1 - q_{33}^{\emptyset})$$
$$r_{34}^{\{a,b\}} = q_{31} \cdot r_{14}/(1 - q_{33}^{\emptyset}) \qquad r_{42}^{a} = (q_{41} \cdot r_{12} + q_{43} \cdot r_{32})/(1 - q_{44}) \qquad r_{42}^{b} = q_{42} \cdot r_{22}^{b}/(1 - q_{44})$$
$$r_{43}^{a} = q_{43} \cdot r_{33}^{a}/(1 - q_{44}) \qquad r_{43}^{b} = (q_{41} \cdot r_{13} + q_{42} \cdot r_{23})/(1 - q_{44}) \qquad r_{44}^{a} = q_{43} \cdot r_{34}^{a}/(1 - q_{44})$$
$$r_{44}^{b} = q_{42} \cdot r_{24}^{b}/(1 - q_{44}) \qquad r_{44}^{\{a,b\}} = q_{41} \cdot r_{14}^{\{a,b\}}/(1 - q_{44})$$

3 Equivalence Relations for DTSPNs

Different equivalences have been proposed in the context of Petri nets [19, 21]. Furthermore relations have been defined for probabilistic systems [7, 14]. However, in the probabilistic case usually some sort of probabilistic interleaving is assumed such that only single transitions occur and not sets of transitions. A widely used class of equivalence relations which have been defined in different settings are trace and bisimulation equivalences. Consequently, we propose the corresponding notions for DTSPNs.

3.1 Trace equivalences

Trace equivalences are the simplest ones. In trace semantics, a behavior of a system is associated with the set of all possible sequences of activities, i.e., protocols of work or computations. Thus, the points of choice of an external observer between several extensions of a particular computation are not taken into account.

Let us introduce formal definitions of the trace relations. These notions resemble that of trace relations for standard Petri nets from [21], but additionally have to take into account probabilities of occurrences of sequences of (multisets of) actions. For this reason we have to collect probabilities of happening (multisets of) actions along *all possible* paths which correspond to our sequence in the observable reachability graphs $RG^*(N)$ and

$RG^*(N')$ of two compared nets N and N'. Since we have already abstracted from particular transitions in such graphs, the paths differ only by markings belonging to them. Thus, we should calculate a sum of probabilities for all paths according our sequence and differentiating at least by one marking.

Definition 3.1 *An* interleaving trace *of a DTSPN N is a pair $(\sigma, \mathcal{P})$, where $\sigma = a_1 \cdots a_n \in Act^*$ and:*

$$\mathcal{P} = \sum_{\{M_1, \ldots, M_n \mid M_{in} \xrightarrow{a_1}_{\mathcal{P}_1} M_1 \xrightarrow{a_2}_{\mathcal{P}_2} \cdots \xrightarrow{a_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of all interleaving traces of a DTSPN N by $IntTraces(N)$. Two DTSPNs N and N' are interleaving trace equivalent, *denoted by $N \equiv_i N'$, if:*

$$IntTraces(N) = IntTraces(N').$$

Definition 3.2 *A* step trace *of a DTSPN N is a pair $(\Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in \mathcal{M}(Act)^*$ and:*

$$\mathcal{P} = \sum_{\{M_1, \ldots, M_n \mid M_{in} \xrightarrow{A_1}_{\mathcal{P}_1} M_1 \xrightarrow{A_2}_{\mathcal{P}_2} \cdots \xrightarrow{A_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^{n} \mathcal{P}_i.$$

We denote a set of all step traces of a DTSPN N by $StepTraces(N)$. Two DTSPNs N and N' are step trace equivalent, *denoted by $N \equiv_s N'$, if:*

$$StepTraces(N) = StepTraces(N').$$

3.2   Bisimulation equivalences

Bisimulation equivalences completely respect points of choice of an external observer in the behavior of a modeled system, unlike trace ones.

To define probabilistic bisimulation equivalences, we have to consider a bisimulation as an *equivalence* relation which partitions states of the *union* of the observable reachability graphs $RG^*(N)$ and $RG^*(N')$ of two compared nets N and N'. For nets N and N' to be bisimulation equivalent, their initial markings M_{in} and M'_{in} should be related by the bisimulation having the following transfer property: two markings are related if at each of them the same (multisets of) actions can occur, and the resulting markings *belong to the same equivalence class*. In addition, sums of probabilities for all such occurrences should be the same for both compared markings. Thus, for our definitions, we follow the approach of [14]. Hence, the difference of bisimulation from trace equivalences is that we do not consider *all possible* occurrences of (multisets of) actions from the initial markings, but only such that lead (stepwise) to markings *belonging to the same equivalence class*.

First we introduce several helpful notations. Let for a DTSPN N $\mathcal{L} \subseteq RS^*(N)$. For some $M \in RS^*(N)$ and $A \in \mathcal{M}(Act)$ we write $M \xrightarrow{A}_{\mathcal{Q}} \mathcal{L}$ if:

$$\sum_{\{\widetilde{M} \in \mathcal{L} \mid M \xrightarrow{A}_{\mathcal{P}} \widetilde{M}\}} \mathcal{P} = \mathcal{Q}.$$

We shall write $M \xrightarrow{A} \mathcal{L}$ if $M \xrightarrow{A}_{\mathcal{Q}} \mathcal{L}$ for some $\mathcal{Q} > 0$. For one-element multiset of actions $A = \{a\}$ we write $M \xrightarrow{a}_{\mathcal{Q}} \mathcal{L}$ and $M \xrightarrow{a} \mathcal{L}$.

Let X be some set. The number of elements in X is denoted as $|X|$. We denote the cartesian product of X with itself $X \times X$ by X^2. Let $\mathcal{E} \subseteq X^2$ be an equivalence relation on X. Then an *equivalence class* (w.r.t. $\mathcal{E}$) of an element $x \in X$ is defined by $[x]_{\mathcal{E}} = \{y \in X \mid (x, y) \in \mathcal{E}\}$. The equivalence $\mathcal{E}$ partitions X by the *set of equivalence classes* $X/_{\mathcal{E}} = \{[x]_{\mathcal{E}} \mid x \in X\}$.

Definition 3.3 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is an* interleaving bisimulation *between two markings M_1 and M_2 of N (i.e., $(M_1, M_2) \in \mathcal{R}$), denoted by $\mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$, if $\forall a \in Act \; \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \xrightarrow{a}_{\mathcal{Q}} \mathcal{L} \;\; \Leftrightarrow \;\; M_2 \xrightarrow{a}_{\mathcal{Q}} \mathcal{L}.$$

Two markings M_1 and M_2 are interleaving bisimulation equivalent, *denoted by $M_1 \underline{\leftrightarrow}_i M_2$, if $\exists \mathcal{R} : M_1 \underline{\leftrightarrow}_i M_2$.*

To introduce a bisimulation between two DTSPNs N and N' we should consider a "composite" set or reachable states, i.e., $RS^*(N) \cup RS^*(N')$.

Definition 3.4 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is an* interleaving bisimulation *between N and N', denoted by $\mathcal{R} : N\underline{\leftrightarrow}_i N'$, if $\mathcal{R} : M_{in}\underline{\leftrightarrow}_i M'_{in}$.*

Two DTSPNs N and N' are interleaving bisimulation equivalent, *denoted by $N\underline{\leftrightarrow}_i N'$, if $\exists \mathcal{R} : N\underline{\leftrightarrow}_i N'$.*

Definition 3.5 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is a* step bisimulation *between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1\underline{\leftrightarrow}_s M_2$, if $\forall A \in \mathcal{M}(Act) \ \forall \mathcal{L} \in RS^*(N)/_\mathcal{R}$:*

$$M_1 \xrightarrow{\ A\ }_\mathcal{Q} \mathcal{L} \quad \Leftrightarrow \quad M_2 \xrightarrow{\ A\ }_\mathcal{Q} \mathcal{L}.$$

Two markings M_1 and M_2 are step bisimulation equivalent, *denoted by $M_1\underline{\leftrightarrow}_s M_2$, if $\exists \mathcal{R} : M_1\underline{\leftrightarrow}_s M_2$.*

Definition 3.6 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is a* step bisimulation *between N and N', denoted by $\mathcal{R} : N\underline{\leftrightarrow}_s N'$, if $\mathcal{R} : M_{in}\underline{\leftrightarrow}_s M'_{in}$.*

Two DTSPNs N and N' are step bisimulation equivalent, *denoted by $N\underline{\leftrightarrow}_s N'$, if $\exists \mathcal{R} : N\underline{\leftrightarrow}_s N'$.*

It is straightforward to show that the union of two (interleaving or step) bisimulations is also an (interleaving or step) bisimulation such that the largest bisimulation relation exists uniquely up to the ordering of equivalence classes. Consequently, for a given DTSPN equivalent nets with a minimal state space exist.

3.3 Backward bisimulation equivalences

For untimed systems apart from bisimulation in forward direction, also bisimulation in backward direction has been defined [17, 18]. However, the definition introduced in [17] is not a straightforward extension of forward bisimulation which would simply mean to define a backward bisimulation as a bisimulation on the transition graph after reversing the direction of arcs. The authors in [17] argue why such a definition is not useful in their context of untimed systems and define backward bisimulation based on paths preserving the history that brought the system to a state. This definition cannot be transferred to our viewpoint of stochastic systems. Instead we define here backward bisimulation by extending forward bisimulation using two additional conditions on the initial marking and on outgoing transition probabilities. The latter implies that we define some form of back and forth bisimulation. However, we use the notation *backward bisimulation* for the resulting equivalence which has shown to be useful for stochastic automata networks [5] and can be transferred naturally to DTSPNs.

Like bisimulation, which will from now on also be denoted as *forward bisimulation*, backward bisimulation is defined using equivalence relations. For $\mathcal{L} \subseteq RS^*(N)$, $M \in RS^*(N)$ and $A \in \mathcal{M}(Act)$ we define $\mathcal{L} \xrightarrow{\ A\ }_\mathcal{Q} M$ as follows:

$$\sum_{\{\widetilde{M} \in \mathcal{L} \mid \widetilde{M} \xrightarrow{\ A\ }_\mathcal{P} M\}} \mathcal{P} = \mathcal{Q}.$$

We shall write $\mathcal{L} \xrightarrow{\ A\ } M$ if $\mathcal{L} \xrightarrow{\ A\ }_\mathcal{Q} M$ for some $\mathcal{Q} > 0$. For one-element multiset of actions $A = \{a\}$ we write $\mathcal{L} \xrightarrow{\ a\ }_\mathcal{Q} M$ and $\mathcal{L} \xrightarrow{\ a\ } M$.

Definition 3.7 *Let N be a DTSPN. An* equivalence *relation $\mathcal{R} \subseteq RS^*(N)^2$ is an* interleaving backward bisimulation *between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1\underline{\leftrightarrow}_{ib} M_2$, if $\forall a \in Act \ \forall \mathcal{L} \in RS^*(N)/_\mathcal{R}$:*

$$M_1 \xrightarrow{\ a\ }_\mathcal{Q} RS^*(N) \ \Leftrightarrow \ M_2 \xrightarrow{\ a\ }_\mathcal{Q} RS^*(N), \ \mathcal{L} \xrightarrow{\ a\ }_\mathcal{Q} M_1 \ \Leftrightarrow \ \mathcal{L} \xrightarrow{\ a\ }_\mathcal{Q} M_2 \ and \ [M_{in}]_\mathcal{R} = \{M_{in}\}.$$

Two markings M_1 and M_2 are interleaving backward bisimulation equivalent, *denoted by $M_1\underline{\leftrightarrow}_{ib} M_2$, if $\exists \mathcal{R} : M_1\underline{\leftrightarrow}_{ib} M_2$.*

Observe that backward bisimulation has a part looking forward in the future due to identical probability sums of leaving a marking via a-labeled transitions and a part looking backwards due to identical probabilities of incoming transitions from each other equivalence class. The definition of backward bisimulation for two nets looks a little bit more complicated than the corresponding definition for forward bisimulation because we cannot assume that incoming transition probabilities are the same for equivalent markings from different nets. Instead it has to be assured that the probability flow from one equivalence class to another is the same in both nets and for each net separately the flow into each marking of an equivalence class has to be the same. To simplify the mentioned definitions we propose the following *indicator function* Γ which recovers a DTSPN by a marking belonging to it. Let N be a DTSPN and $M \in RS^*(N)$, then $\Gamma(M) = N$. Thus, this is just a convenient notation allowing one to avoid a treatment of different cases when markings of two nets are considered together.

Definition 3.8 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is an interleaving backward bisimulation between N and N', denoted by $\mathcal{R} : N\underline{\leftrightarrow}_{ib}N'$, if $\forall a \in Act \;\; \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}} \;\; \forall M_1, M_2 \in \mathcal{L}$:*

$$M_1 \overset{a}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(\Gamma(M_1)) \;\; \Leftrightarrow \;\; M_2 \overset{a}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(\Gamma(M_2)), \; [M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\} \;\; and$$

$$\mathcal{K} \overset{a}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q} \frac{|\mathcal{L} \cap RS^*(\Gamma(M_1))|}{|\mathcal{K} \cap RS^*(\Gamma(M_1))|}} M_1 \;\; \Leftrightarrow \;\; \mathcal{K} \overset{a}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q} \frac{|\mathcal{L} \cap RS^*(\Gamma(M_2))|}{|\mathcal{K} \cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are interleaving backward bisimulation equivalent, denoted by $N\underline{\leftrightarrow}_{ib}N'$, if $\exists \mathcal{R} : N\underline{\leftrightarrow}_{ib}N'$.

For markings M_1 and M_2 belonging to the same net, the conditions on incoming probabilities reduce to the requirement of identical incoming probabilities.

Definition 3.9 *Let N be a DTSPN. An equivalence relation $\mathcal{R} \subseteq RS^*(N)^2$ is a step backward bisimulation between two markings M_1 and M_2 of N, denoted by $\mathcal{R} : M_1\underline{\leftrightarrow}_{sb}M_2$, if $\forall A \in \mathcal{M}(Act) \;\; \forall \mathcal{L} \in RS^*(N)/_{\mathcal{R}}$:*

$$M_1 \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(N) \;\; \Leftrightarrow \;\; M_2 \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(N), \; \mathcal{L} \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} M_1 \;\; \Leftrightarrow \;\; \mathcal{L} \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} M_2 \;\; and \;\; [M_{in}]_{\mathcal{R}} = \{M_{in}\}.$$

Two markings M_1 and M_2 are step backward bisimulation equivalent, denoted by $M_1\underline{\leftrightarrow}_{sb}M_2$, if $\exists \mathcal{R} : M_1\underline{\leftrightarrow}_{sb}M_2$.

Definition 3.10 *Let N and N' be two DTSPNs. A relation $\mathcal{R} \subseteq (RS^*(N) \cup RS^*(N'))^2$ is a step backward bisimulation between N and N', denoted by $\mathcal{R} : N\underline{\leftrightarrow}_{sb}N'$, if $\forall A \in \mathcal{M}(Act) \;\; \forall \mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}} \;\; \forall M_1, M_2 \in \mathcal{L}$:*

$$M_1 \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(\Gamma(M_1)) \;\; \Leftrightarrow \;\; M_2 \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q}} RS^*(\Gamma(M_2)), \; [M_{in}]_{\mathcal{R}} = \{M_{in}, M'_{in}\} \;\; and$$

$$\mathcal{K} \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q} \frac{|\mathcal{L} \cap RS^*(\Gamma(M_1))|}{|\mathcal{K} \cap RS^*(\Gamma(M_1))|}} M_1 \;\; \Leftrightarrow \;\; \mathcal{K} \overset{A}{\longrightarrow\!\!\!\!\!\twoheadrightarrow}_{\mathcal{Q} \frac{|\mathcal{L} \cap RS^*(\Gamma(M_2))|}{|\mathcal{K} \cap RS^*(\Gamma(M_2))|}} M_2.$$

Two DTSPNs N and N' are step backward bisimulation equivalent, denoted by $N\underline{\leftrightarrow}_{sb}N'$, if $\exists \mathcal{R} : N\underline{\leftrightarrow}_{sb}N'$.

As before the union of backward bisimulations is a backward bisimulation.

3.4 Back and forth bisimulation equivalences

A natural way of defining a new equivalence is to combine backward and forward bisimulation. We define here only back and forth bisimulation equivalences for two nets, the remaining definitions can be transferred similarly.

First, we define an interleaving relation.

Definition 3.11 *Two DTSPNs N and N' are interleaving back and forth bisimulation equivalent, denoted by $N\underline{\leftrightarrow}_{ibf}N'$, if $N\underline{\leftrightarrow}_{i}N'$ and $N\underline{\leftrightarrow}_{ib}N'$.*

A definition of a step equivalence is introduced similarly.

Definition 3.12 *Two DTSPNs N and N' are step back and forth bisimulation equivalent, denoted by $N\underline{\leftrightarrow}_{sbf}N'$, if $N\underline{\leftrightarrow}_{s}N'$ and $N\underline{\leftrightarrow}_{sb}N'$.*

3.5 Examples of the equivalences

Let us present some examples of equivalence relations.

As we have seen, one can consider bisimulation between a net and itself, i.e., a bisimulation between markings of the net and bisimulation between different nets. Let us first consider equivalence of markings of a single net for the net shown in Figure 1. Markings (110) and (002) of N are forward bisimilar, if $r_{12} = r_{42}$, $r_{13} = r_{43}$ and $r_{44} = r_{14}$ which holds by definition of the transition probabilities. If we assume that a and b are identical symbols, then (011) and (101) are forward bisimulation equivalent independently of $\Lambda(t_1)$ and $\Lambda(t_2)$ as long as both values are non-zero which has been assumed when $RS^*(N)$ has been generated. Observe that the bisimulation is not a backward bisimulation.
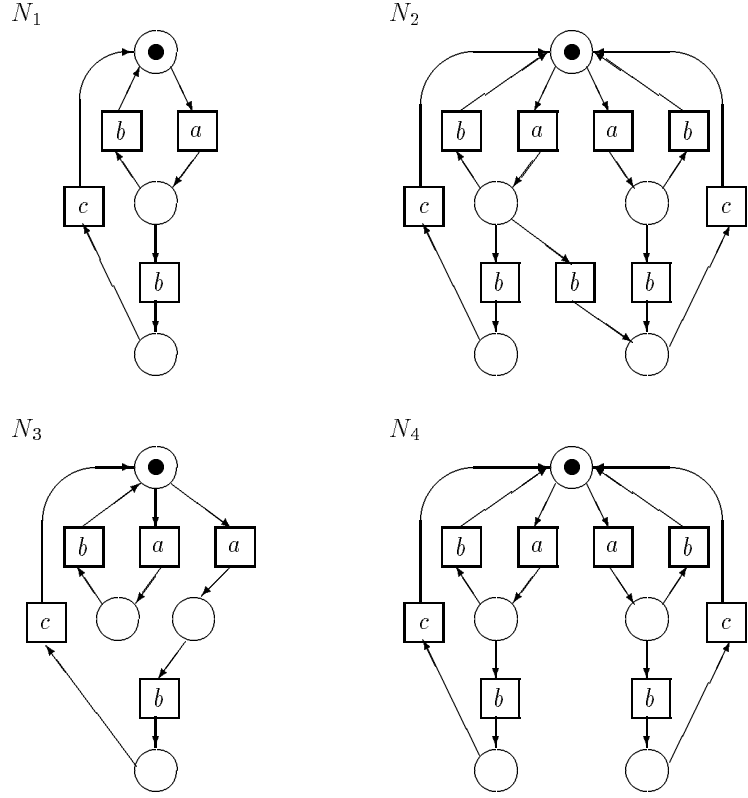
Figure 3: Nets related via different equivalences

For bisimulation between different nets we consider the example shown in Figure 3. We assume that conflicting transitions have the same weights and firing probabilities. All nets have a very simple structure without concurrently enabled transitions such that interleaving behavior is identical to the step one.

The following equivalence relations exist between the nets:

$$N_1 \equiv_s N_2 \equiv_s N_3 \equiv_s N_4 \quad N_1 \underline{\leftrightarrow}_s N_2 \underline{\leftrightarrow}_s N_4 \quad N_1 \underline{\leftrightarrow}_{sb} N_3 \underline{\leftrightarrow}_{sb} N_4 \quad N_1 \underline{\leftrightarrow}_{sbf} N_4$$

Observe that there is no bisimulation relation between N_2 and N_3, i.e., $N_2 \underline{\not\leftrightarrow}_i N_3$ and $N_2 \underline{\not\leftrightarrow}_{ib} N_3$.

3.6 Interrelations of the equivalences

In this section, we compare the introduced equivalences and obtain the lattice of their interrelations.

Proposition 3.1 *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_\star N' \ \Rightarrow \ N \equiv_\star N'.$$

Proof. See Appendix A. ☐

In a similar way we show that backward bisimulation implies trace equivalence.

Proposition 3.2 *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_{\star b} N' \ \Rightarrow \ N \equiv_\star N'.$$

Proof. See Appendix B. ☐

The following proposition concerns relations of back and forth bisimulations with other ones.
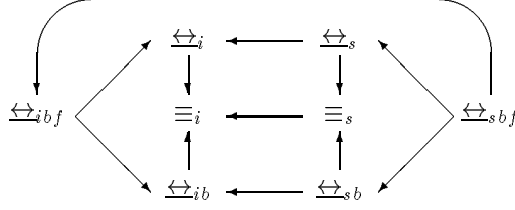
Figure 4: Interrelations of the equivalences

Proposition 3.3 *Let $\star \in \{i, s\}$. For DTSPNs N and N' the following holds:*

$$N \underline{\leftrightarrow}_{\star bf} N' \;\Rightarrow\; N \underline{\leftrightarrow}_{\star} N' \text{ and } N \underline{\leftrightarrow}_{\star b} N'.$$

Proof. The result follows from the definitions of back and forth bisimulations. □

Thus, we obtained several important results for our equivalences stating that bisimulation (forward or backward) relations imply trace ones. This helps us to establish interrelations of the introduced equivalence notions.

Theorem 3.1 *Let $\leftrightarrow, \leftarrow\!\!\!\!\!\rightarrow \in \{\equiv, \underline{\leftrightarrow}\}$ and $\star, \star\star \in \{i, s, ib, sb, ibf, sbf\}$. For DTSPNs N and N' the following holds:*

$$N \leftrightarrow_{\star} N' \;\Rightarrow\; N \leftarrow\!\!\!\!\!\rightarrow_{\star\star} N'$$

iff in the graph in Figure 4 there exists a directed path from $\leftrightarrow_{\star}$ to $\leftarrow\!\!\!\!\!\rightarrow_{\star\star}$.

Proof. ($\Leftarrow$) Let us check the validity of the implications in the graph in Figure 4.

- The implications $\leftrightarrow_s \rightarrow \leftrightarrow_i$, $\leftrightarrow \in \{\equiv, \underline{\leftrightarrow}\}$, and $\underline{\leftrightarrow}_{sb} \rightarrow \underline{\leftrightarrow}_{ib}$, $\underline{\leftrightarrow}_{sbf} \rightarrow \underline{\leftrightarrow}_{ibf}$, are valid since actions are one-element multisets.

- The implications $\underline{\leftrightarrow}_{\star} \Rightarrow \equiv_{\star}$, $\underline{\leftrightarrow}_{\star b} \Rightarrow \equiv_{\star}$, $\star \in \{i, s\}$, are valid by Proposition 3.1 and Proposition 3.2 respectively.

- The implications $\underline{\leftrightarrow}_{\star bf} \Rightarrow \underline{\leftrightarrow}_{\star}$, $\underline{\leftrightarrow}_{\star bf} \Rightarrow \underline{\leftrightarrow}_{\star b}$, $\star \in \{i, s\}$, are valid by Proposition 3.3.

($\Rightarrow$) An absence of additional nontrivial arrows in the graph in Figure 4 is proved by the following examples. As in the previous examples we assume that conflicting transitions have equal weights and probabilities.

- In Figure 5(a), $N \underline{\leftrightarrow}_{ibf} N'$, but $N \not\equiv_s N'$, since only in the DTSPN N' actions a and b cannot happen concurrently.

- In Figure 5(b), $N \equiv_s N'$, but $N \not\underline{\leftrightarrow}_i N'$ and $N \not\underline{\leftrightarrow}_{ib} N'$, since only in the DTSPN N' an action a can happen so that no action b can happen afterwards.

- In Figure 3, $N_1 \underline{\leftrightarrow}_s N_2$, but $N_1 \not\underline{\leftrightarrow}_{ib} N_2$, since only in N_2 there is a place with two input transitions labeled by b. Hence, the probability for a token to go to this place is always more than for that with only one input b-labeled transition.

- In Figure 3, $N_1 \underline{\leftrightarrow}_{sb} N_3$, but $N_1 \not\underline{\leftrightarrow}_i N_3$, since only in the DTSPN N_1 an action a can happen so that a sequence of actions bc cannot happen just after it. □
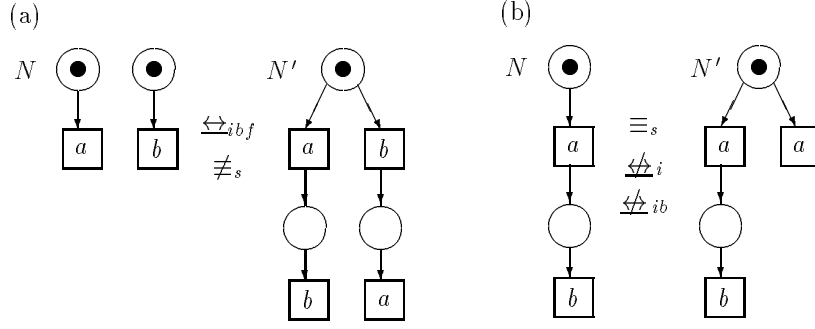
Figure 5: Examples of the equivalences

4 Stationary Behavior of DTSPNs

A natural observation of the behavior of a dynamic system is the observation of traces starting from the initial marking of the DTSPN. Depending on the chosen viewpoint steps or only single transitions are observed. Traces have been used to define trace equivalence. Consequently, trace equivalent DTSPNs have the same traces, and since trace equivalence is the weakest relation we have defined, all other equivalences also preserve traces.

An alternative and commonly used viewpoint in stochastic systems is to consider the DTSPN in its steady state. For this behavior we consider only nets with an infinite behavior and assume that the embedded DTMC is irreducible or contains at least only one irreducible subset of markings. The embedded steady state distribution after the observation of a visible event is the unique solution of the set linear equation:

$$ps^*(M) = \sum_{\widetilde{M} \in RS^*(N)} ps^*(\widetilde{M}) \cdot PS^*[\widetilde{M}, M]$$

subject to $\sum_{M \in RS^*(N)} ps^*(M) = 1$.

We consider in the following only step behavior but the results can be easily formulated for interleaving behavior as well. First, extend the notion of step traces by defining step traces starting at some marking $M \in RS^*(N)$ as $(M, \Sigma, \mathcal{P})$, where $\Sigma = A_1 \cdots A_n \in Act^*$ and:

$$\mathcal{P} = \sum_{\{M_1, \ldots, M_n \mid M \xrightarrow{A_1}_{\mathcal{P}_1} M_1 \xrightarrow{A_2}_{\mathcal{P}_2} \cdots \xrightarrow{A_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^n \mathcal{P}_i.$$

Thus, in the definition of $StepTraces(N)$ we replace M_{in} by M. Let $StepTraces(N, M)$ be the set of *all step traces of DTSPN N starting at marking M*. The set of *all step traces in steady state* is defined as:

$$StStepTraces(N) = \{(M, \Sigma, ps(M) \cdot \mathcal{P}) \mid M \in RS^*(N) \wedge (\Sigma, \mathcal{P}) \in StepTraces(N, M)\}.$$

Now we show that forward or backward bisimulation equivalent nets have the same steady state traces, whereas trace equivalence does not preserve steady state traces.

Proposition 4.1   *1. Let N and N' be two forward bisimulation equivalent DTSPNs, then $\forall \mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_\mathcal{R}$:*

$$\sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) = \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M').$$

2. Let N and N' be two backward bisimulation equivalent DTSPNs, then $\forall \mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_\mathcal{R}$:

$$\sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) = \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M');$$

$\forall M, \widetilde{M} \in \mathcal{L} \cap RS^*(N), \forall M', \widetilde{M'} \in \mathcal{L} \cap RS^*(N')$:

$$ps^*(M) = ps^*(\widetilde{M}) \ and \ ps^*(M') = ps^*(\widetilde{M'}).$$
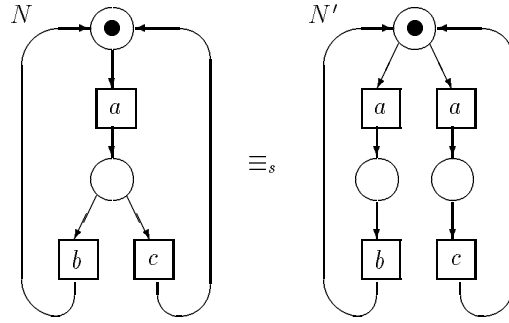
12

Figure 6: Two step trace equivalent nets with $StStepTraces(N) \neq StStepTraces(N')$

Proof. The proof is an extension of the corresponding results for the continuous time case [2, 3]. □

Theorem 4.1 *Let N and N' be backward or forward bisimulation equivalent DTSPNs, then:*

$$StStepTraces(N) = StStepTraces(N').$$

Proof. See Appendix C. □

The implication stated in the previous theorem cannot be reversed, since for step trace equivalent nets N and N', we may have $StStepTrace(N) \neq StStepTrace(N')$. This can be seen by the two nets shown in Figure 6. For net N, the probability of being in one of both possible markings is 1/2. Consequently, a trace starts with probability 1/2 with an a. For net N' the probability of being in one of the three possible markings after observation of a transition equals 1/3. Consequently, the probability of observing a trace starting with a equals 1/3.

One should note that the stationary distribution is defined here according to the embedded distribution after observing a step of visible transitions. This distribution differs from the stationary distribution of the net at an arbitrary time. The latter behavior has to be analyzed on $RS(N)$ instead of $RS^*(N)$ and is not preserved by any of the proposed equivalences even if we restrict the observation to visible transitions.

5 Conclusion

In this paper, we introduced a new class of Stochastic Petri Nets with labeled transitions and a step semantics for transition firing. For this class of nets we proposed several equivalence relations and showed that these equivalences preserve interesting aspects of system behavior. Equivalence relations can be used to compare different systems and to compute for a given system a minimal equivalent representation [3]. The latter aspect is especially interesting for bisimulation equivalences, for which efficient algorithms exist to compute the largest bisimulation for a given net. By representation each equivalence class of this relation by a single marking we obtain a minimal representation at the state transition level. As a result of comparing the equivalences in accordance to differentiating power, we obtained a lattice of implications. Thus, we provided the new variant of Stochastic Petri Nets with step semantics, and this naturally corresponds to non-interleaving character of the model. This can be considered as the main contribution of the paper.

Possible extension of this work can be an attempt to define other bisimulation equivalences in interleaving and step semantics. For example, branching bisimulation [19] can be considered as well as variants of back-forth equivalences defined in [17, 18]. For these equivalences we cannot use observable state graphs, since we may need lower level information. For example, to define branching relations, we should respect occurrences of invisible transitions and states where they conflict with other ones. Thus, we cannot just abstract of invisible transitions from very beginning. To propose notions of back-forth bisimulations, we need an information about the path of events which came to the present state. Hence, it is not enough even to consider paths of transitions which led from the initial marking to the present one, since the same transitions can happen concurrently or sequentially resulting the same marking (in non-safe nets). In such a case, we should have something like processes for stochastic nets and collect events for out of paths from such processes. We may also define true concurrent equivalences for stochastic nets such that partial word or pomset ones [19, 22]. Step semantics proposed in the present paper can be the first stage to true concurrent semantics for stochastic nets. These directions are left for future research.

13

References

[1] M. AJMONE-MARSAN, G. BALBO, AND G. CONTE. *A class of generalized stochastic Petri nets for performance evaluation of multiprocessor systems. ACM Transactions on Computer Systems* **2(2)**, pages 93–122, 1984.

[2] P. BUCHHOLZ. *Markovian process algebra: composition and equivalence.* In U. Herzog and M. Rettelbach, editors, *Proc. of the 2nd Work. on Process Algebras and Performance Modelling, Arbeitsberichte des IMMD* **27**, pages 11–30, University of Erlangen, 1994.

[3] P. BUCHHOLZ. *A notion of equivalence for stochastic Petri nets.* In G. De Michelis and M. Diaz, editors, *Application and Theory of Petri Nets 1995, Springer LNCS* **935**, pages 161–180, 1995.

[4] P. BUCHHOLZ. *Iterative decomposition and aggregation of labelled GSPNs.* In J. Desel and M. Silva, editors, *Application and Theory of Petri Nets 1998, Springer LNCS* **1420**, pages 226–245, 1998.

[5] P. BUCHHOLZ. *Exact performance equivalence — an equivalence relation for stochastic automata. Theoretical Computer Science* **215(1/2)**, pages 263–287, 1999.

[6] G. CHIOLA, M. AJMONE-MARSAN, G. BALBO, AND G. CONTE. *Generalized stochastic Petri nets: a definition at the net level and its implications. IEEE Transactions on Software Engineering* **19(2)**, pages 89–107, 1993.

[7] I. CHRISTOFF. *Testing equivalence and fully abstract models of probabilistic processes.* In J. C. M. Baeten and J. W. Klop, editors, *Proc. CONCUR'90, Springer LNCS* **458**, pages 128–140, 1990.

[8] R. CLEAVELAND, J. PARROW, AND B. STEFFEN. *The concurrency workbench: a semantics based tool for the verification of concurrent systems. ACM Transactions on Programming Languages and Systems* **15(1)**, pages 36–72, 1993.

[9] G. FLORIN AND S. NATKIN. *Les reseaux de Petri stochastiques. Technique et Science Informatique* **4(1)**, 1985.

[10] R. J. VAN GLABBEEK, S. SMOLKA, B. STEFFEN, AND C. TOFTS. *Reactive, generative and stratified models for probabilistic processes.* In *Proc. LICS'90*, pages 130–141, 1990.

[11] H. HERMANNS AND M. RETTELBACH. *Syntax, semantics, equivalences, and axioms for MTIPP.* In U. Herzog and M. Rettelbach, editors, *Proc. of the 2nd Work. on Process Algebras and Performance Modelling. Arbeitsberichte des IMMD* **27**, University of Erlangen, 1994.

[12] J. HILLSTON. *A compositional approach for performance modelling. Ph.D. thesis*, University of Edinburgh, Dep. of Comp. Sc., 1994.

[13] J. G. KEMENY AND J. L. SNELL. *Finite Markov Chains.* Springer, 1976.

[14] K. LARSEN AND A. SKOU. *Bisimulation through probabilistic testing. Information and Computation* **94**, pages 1–28, 1991.

[15] M. MOLLOY. *Performance analysis using stochastic Petri nets. IEEE Transactions on Software Engineering* **31(9)**, pages 913–917, 1982.

[16] M. MOLLOY. *Discrete time stochastic Petri nets. IEEE Transactions on Software Engineering* **11(4)**, pages 417–423, 1985.

[17] R. DE NICOLA, U. MONTANARI, AND F. W. VAANDRAGER. *Back and forth bisimulations.* In *CONCUR'90, Springer LNCS* **458**, pages 152–165, 1990.

[18] S. PINCHINAT. *Bisimulations for the semantics of reactive systems. Ph.D. Thesis*, Institut National Politechnique de Grenoble, January 1993 (in French).

[19] L. POMELLO, G. ROZENBERG, AND C. SIMONE. *A survey of equivalence relations for net based systems.* In G. Rozenberg, editor, *Advances in Petri Nets 1992, Springer LNCS* **609**, pages 410–472, 1992.

[20] W. J. STEWART. *Introduction to the numerical solution of Markov chains.* Princeton University Press, 1994.

[21] I. V. Tarasyuk. *Place bisimulation equivalences for design of concurrent and sequential systems. Electronic Notes in Theoretical Computer Science* **18**, 16 pages, 1998.

[22] W. Vogler. *Modular construction and partial order semantics of Petri nets. Springer LNCS* **625**, 252 pages, 1992.

[23] R. Zijal, G. Ciardo, and G. Hommel. *Discrete deterministic and stochastic Petri nets.* In K. Irmscher, Ch. Mittasch, and K. Richter, editors, *MMB'97, Aktuelle Probleme der Informatik*: Band **1**. VDE Verlag, 1997.

[24] R. Zijal and R. German. *A new approach to discrete time stochastic Petri nets.* In G. Cohen and J. P. Quadrat, editors, 11th *Int. Conference on Analysis and Optimizations of Systems, Springer LNIS* **199**, pages 198–204, 1994.

A   Proof of Proposition 3.1

It is enough to prove for $\star = s$, since $\star = i$ is a particular case of the previous one with one-element multisets of actions.

Let $\mathcal{R} : N \underleftrightarrow{}_s N'$ and $(M_1, M_2) \in \mathcal{R}$. By the definition of step bisimulation we have $\forall A \in \mathcal{M}(Act)$ $\forall \widetilde{\mathcal{L}} \in (RS^*(N) \cup RS^*(N'))/_\mathcal{R}$:

$$M_1 \xrightarrow{A}_{\mathcal{Q}} \widetilde{\mathcal{L}} \Leftrightarrow M_2 \xrightarrow{A}_{\mathcal{Q}} \widetilde{\mathcal{L}}.$$

Let $\mathcal{L} = [M_1]_\mathcal{R} = [M_2]_\mathcal{R}$. Then we can rewrite the above identity as:

$$\mathcal{L} \xrightarrow{A}_{\mathcal{Q}} \widetilde{\mathcal{L}},$$

since for all markings from equivalence class $\mathcal{L}$ their probabilities of moving into $\widetilde{\mathcal{L}}$ as a result of occurrence of multiset of actions A coincide (they are equal to $\mathcal{Q}$).

Let $(A_1 \cdots A_n, \mathcal{P}) \in StepTraces(N)$. Since $\mathcal{R} : N \underleftrightarrow{}_s N'$ and taking into account the previous identity, we have:

$$M_{in} \xrightarrow{A_1}_{\mathcal{Q}_1} \mathcal{L}_1 \xrightarrow{A_2}_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}_{\mathcal{Q}_n} \mathcal{L}_n \Leftrightarrow M'_{in} \xrightarrow{A_1}_{\mathcal{Q}_1} \mathcal{L}_1 \xrightarrow{A_2}_{\mathcal{Q}_2} \cdots \xrightarrow{A_n}_{\mathcal{Q}_n} \mathcal{L}_n.$$

Let us also note that starting from markings of N (N') to some set of markings $\mathcal{L} \subseteq (RS^*(N) \cup RS^*(N'))$ we can reach only markings of the same net, since observable state graphs of two nets do not communicate.

Now we intend to show that the sum of probabilities of all paths going through markings from $\mathcal{L}_1, \ldots, \mathcal{L}_n$ coincides with the product of $Q_1, \ldots, Q_n$, which is essentially the probability of the path going through $\mathcal{L}_1, \ldots, \mathcal{L}_n$ in $RG^*(N)/_\mathcal{R}$.

Lemma A.1 *For DTSPN N and all n ($1 \leq n \leq |RG^*(N)/_\mathcal{R}|$) the following holds:*

$$\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_n \in \mathcal{L}_n | M_{in} \xrightarrow{A_1}_{\mathcal{P}_1} M_1 \xrightarrow{A_2}_{\mathcal{P}_2} \cdots \xrightarrow{A_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^n \mathcal{P}_i = \prod_{i=1}^n \mathcal{Q}_i.$$

Proof. (of lemma) We shall prove by induction on n.

- $n = 1$:

    We have to prove that:

$$\sum_{\{M_1 \in \mathcal{L}_1 | M_{in} \xrightarrow{A_1}_{\mathcal{P}_1} M_1\}} \mathcal{P}_1 = \mathcal{Q}_1.$$

    This follows from the definition of transition relation between markings and sets of markings.

- $n \rightarrow (n+1)$:

    By induction hypothesis, we have the following equality:

$$\sum_{\{M_1 \in \mathcal{L}_1, \ldots, M_n \in \mathcal{L}_n | M_{in} \xrightarrow{A_1}_{\mathcal{P}_1} M_1 \xrightarrow{A_2}_{\mathcal{P}_2} \cdots \xrightarrow{A_n}_{\mathcal{P}_n} M_n\}} \prod_{i=1}^n \mathcal{P}_i = \prod_{i=1}^n \mathcal{Q}_i.$$

In addition, we have:

$$\sum_{\{M_{n+1}\in\mathcal{L}_{n+1}|M_n\xrightarrow{A_{n+1}}_{\mathcal{P}_{n+1}}M_{n+1}\}}\mathcal{P}_{n+1}=\mathcal{Q}_{n+1},$$

again by the definition of transition relation between markings and sets of markings. Let us note that the sum above does not depend on particular $M_n\in\mathcal{L}_n$, i.e., it is the *same for all paths* of $SG^*(N)$ starting at M_{in} and going through $\mathcal{L}_1,\ldots,\mathcal{L}_n$.

As a result of multiplying left and right parts of the two equalities above, we obtain:

$$\sum_{\{M_1\in\mathcal{L}_1,\ldots,M_n\in\mathcal{L}_n|M_{in}\xrightarrow{A_1}_{\mathcal{P}_1}M_1\xrightarrow{A_2}_{\mathcal{P}_2}\cdots\xrightarrow{A_n}_{\mathcal{P}_n}M_n\}}\left(\sum_{\{M_{n+1}\in\mathcal{L}_{n+1}|M_n\xrightarrow{A_{n+1}}_{\mathcal{P}_{n+1}}M_1\}}\mathcal{P}_{n+1}\right)\cdot\prod_{i=1}^{n}\mathcal{P}_i=$$

$$\left(\prod_{i=1}^{n}\mathcal{Q}_i\right)\cdot\mathcal{Q}_{n+1}.$$

By distributivity law and with the use of the above note on independence of the sum of current probabilities on concrete marking M_n, we conclude:

$$\sum_{\{M_1\in\mathcal{L}_1,\ldots,M_{n+1}\in\mathcal{L}_{n+1}|M_{in}\xrightarrow{A_1}_{\mathcal{P}_1}M_1\xrightarrow{A_2}_{\mathcal{P}_2}\cdots\xrightarrow{A_{n+1}}_{\mathcal{P}_{n+1}}M_{n+1}\}}\prod_{i=1}^{n+1}\mathcal{P}_i=\prod_{i=1}^{n+1}\mathcal{Q}_i.$$

This ends a proof of the lemma. □

Let us note that the result of this lemma can also be applied to N'.

Now we have only to note that summation by *all equivalence classes* is the same as summation by *all markings*, i.e.:

$$\sum_{\{M_1,\ldots,M_n|M_{in}\xrightarrow{A_1}_{\mathcal{P}_1}M_1\xrightarrow{A_2}_{\mathcal{P}_2}\cdots\xrightarrow{A_n}_{\mathcal{P}_n}M_n\}}\prod_{i=1}^{n}\mathcal{P}_i=\sum_{\{\mathcal{L}_1,\ldots,\mathcal{L}_n|M_{in}\xrightarrow{A_1}_{\mathcal{Q}_1}\mathcal{L}_1\xrightarrow{A_2}_{\mathcal{Q}_2}\cdots\xrightarrow{A_n}_{\mathcal{Q}_n}\mathcal{L}_n\}}\prod_{i=1}^{n}\mathcal{Q}_i=$$

$$\sum_{\{\mathcal{L}_1,\ldots,\mathcal{L}_n|M'_{in}\xrightarrow{A_1}_{\mathcal{Q}_1}\mathcal{L}_1\xrightarrow{A_2}_{\mathcal{Q}_2}\cdots\xrightarrow{A_n}_{\mathcal{Q}_n}\mathcal{L}_n\}}\prod_{i=1}^{n}\mathcal{Q}_i=\sum_{\{M'_1,\ldots,M'_n|M'_{in}\xrightarrow{A_1}_{\mathcal{P}'_1}M'_1\xrightarrow{A_2}_{\mathcal{P}'_2}\cdots\xrightarrow{A_n}_{\mathcal{P}'_n}M'_n\}}\prod_{i=1}^{n}\mathcal{P}'_i.$$

Hence, $(A_1\cdots A_n,\mathcal{P})\in StepTraces(N')$, and we obtain $StepTraces(N)\subseteq StepTraces(N')$. The reverse inclusion is proved by symmetry. □

B Proof of Proposition 3.2

As before it is enough to prove that $StepTraces(N)\subseteq StepTraces(N')$.

Let $\mathcal{R}:N\underline{\leftrightarrow}_{sb}N'$. We prove the inclusion by induction over the length of traces.

- $n=1$:

  Since the initial markings are the only markings in their equivalence class we have $\forall A\in\mathcal{M}(Act)$ $\forall\mathcal{L}\in RS^*(N)/_{\mathcal{R}}$:

  $$M_{in}\xrightarrow{A}_{\mathcal{Q}}\mathcal{L}\Leftrightarrow M'_{in}\xrightarrow{A}_{\mathcal{Q}}\mathcal{L}.$$

  However, $\mathcal{Q}$ is in this case exactly the probability of observing A in the first step or the probability of trace A. Furthermore, let $ps^*[A,M]$ be the probability of being at marking M after observing A from M_{in}. Then $\forall\mathcal{L}\in RS^*(N)/_{\mathcal{R}}$ the following relation holds (see [5]):

$$ps^*[A, \mathcal{L} \cap RS^*(N)] = \sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*[A, M] = \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*[A, M'] = ps^*[A, \mathcal{L} \cap RS^*(N')]$$

In addition, $ps^*[A, M_1] = ps^*[A, M_2]$ for $M_1, M_2 \in \mathcal{L} \cap RS^*(N)$ and $ps^*[A, M_1'] = ps^*[A, M_2']$ for $M_1', M_2' \in \mathcal{L} \cap RS^*(N')$. I.e., the equalities hold for any two markings of the same net such that they are from one equivalence class.

Consequently, we have $ps^*[A, M] = ps^*[A, \mathcal{L}]/|\mathcal{L} \cap RS^*(N)|$ for $M \in RS^*(N)$ and $ps^*[A, M'] = ps^*[A, \mathcal{L}]/|\mathcal{L} \cap RS^*(N')|$ for $M' \in RS^*(N')$.

- $n \to (n+1)$:

  Assume that the above relations are proved for all traces of length n. Let $A_1 \cdots A_n$ be the trace of length n and let A_{n+1} be the multiset of actions observed in step $n+1$. The probability of observing A_{n+1} in N equals:

$$\sum_{M \in RS^*(N)} ps^*[A_1 \cdots A_n, M] \cdot \sum_{\widetilde{M} \in RS^*(N)} PS^*[A_{n+1}, M, \widetilde{M}]$$

  Due to equality of probabilities in an equivalence class this probability can be rewritten as:

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \frac{ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N)] \cdot PS^*[A_{n+1}, \mathcal{L} \cap RS^*(N), \mathcal{K} \cap RS^*(N)]}{|\mathcal{L} \cap RS^*(N)|}$$

  where the summation ranges over all $\mathcal{L}, \mathcal{K} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$. By definition this equals:

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \frac{ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N')] \cdot PS^*[A_{n+1}, \mathcal{L} \cap RS^*(N'), \mathcal{K} \cap RS^*(N')]}{|\mathcal{L} \cap RS^*(N')|}$$

  which is the probability of observing A_{n+1} in N'. The probabilities of being in $M \in \mathcal{K} \in RS^*(N)/_{\mathcal{R}}$ after observing A_{n+1} are computed as:

$$ps^*[A_1 \cdots A_n, M] = \sum_{\mathcal{L}} \frac{ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N)]}{|\mathcal{L} \cap RS^*(N)|} \cdot \frac{PS^*[A_{n+1}, \mathcal{L} \cap RS^*(N), \mathcal{K} \cap RS^*(N)]}{|\mathcal{K} \cap RS^*(N)|}$$

  which is the same for a all $M \in \mathcal{K} \in RS^*(N)/_{\mathcal{R}}$. Since the above relation holds both for N and N', it is easy to show that also

$$ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N)] = ps^*[A_1 \cdots A_n, \mathcal{L} \cap RS^*(N')]$$

  holds for all $\mathcal{L} \in (RS^*(N) \cup RS^*(N'))/_{\mathcal{R}}$ which completes the induction step. $\qquad\square$

C Proof of Theorem 4.1

We prove the theorem for backward bisimulation equivalence the proof for forward bisimulation equivalence is similar.

We prove the theorem by induction over the length n of a trace.

- $n = 1$:

  The following relations hold for the probability of observing A_1 in steady state:

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \sum_{M \in \mathcal{L} \cap RS^*(N)} ps^*(M) \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} PS^*[A_1, M, \widetilde{M}] = \sum_{\mathcal{L}} ps^*(\mathcal{L}) \sum_{\mathcal{K}} PS^*[A_1, \mathcal{L}, \mathcal{K}] =$$

$$\sum_{\mathcal{L}} \sum_{\mathcal{K}} \sum_{M' \in \mathcal{L} \cap RS^*(N')} ps^*(M') \sum_{\widetilde{M}' \in \mathcal{K} \cap RS^*(N')} PS^*[A_1, M', \widetilde{M}']$$

  where:

$$PS^*[A, \mathcal{L}, \mathcal{K}] \;=\; \sum_{M \in \mathcal{L} \cap RS^*(N)} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} PS^*[A, M, \widetilde{M}]$$

$$=\; \sum_{M' \in \mathcal{L} \cap RS^*(N')} \sum_{\widetilde{M'} \in \mathcal{K} \cap RS^*(N')} PS^*[A, M', \widetilde{M'}]$$

- $n \to (n+1)$:

The proof for $n = 1$ is based on equal probabilities of the equivalence classes and equal probabilities of states inside the equivalence classes. Thus, we only have to prove that the identity holds after observing an arbitrary step. Together with the proof for $n = 1$ this proves the required identity of traces. Both equalities hold after observing a step A if they hold before observing a the step since we have:

$$\sum_{\mathcal{K}} \sum_{M \in \mathcal{K} \cap RS^*(N)} ps^*(M) \sum_{\widetilde{M} \in \mathcal{L} \cap RS^*(N)} PS^*[A, M, \widetilde{M}] \qquad =$$

$$\sum_{\mathcal{K}} ps^*(\mathcal{K}) \sum_{\mathcal{L}} PS^*[A, \mathcal{L}, \mathcal{K}] \qquad\qquad =$$

$$\sum_{\mathcal{K}} \sum_{M' \in \mathcal{K} \cap RS^*(N')} ps^*(M') \sum_{\widetilde{M'} \in \mathcal{L} \cap RS^*(N')} PS^*[A, M', \widetilde{M'}]$$

which implies that probabilities of being in equivalence class $\mathcal{L}$ are identical for N and N'.

Let $ps_A^*(M)$ be the probability of being in $M \in \mathcal{L} \cap RS^*(N)$ after observing A starting with probabilities ps^*:

$$ps_A^*(M) \;=\; \sum_{\mathcal{K}} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} ps^*(\widetilde{M}) PS^*[A, \widetilde{M}, M]$$

$$=\; \sum_{\mathcal{K}} ps^*(\mathcal{K}) \cdot \frac{PS^*[A, \mathcal{K}, \mathcal{L}]}{|\mathcal{L} \cap RS^*(N)|}$$

$$=\; \sum_{\mathcal{K}} \sum_{\widetilde{M} \in \mathcal{K} \cap RS^*(N)} ps^*(\widetilde{M}) PS^*[A, \widetilde{M}, \overline{M}] \;=\; ps_A^*(\overline{M})$$

which shows that $\forall M, \overline{M} \in \mathcal{L} \cap RS^*(N) : \; ps_A^*(M) = ps_A^*(\overline{M})$. By a symmetric argument the equality of probabilities in an equivalence class for states from $RS^*(N')$ can be proved. $\qquad\square$