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Axiomatizing Confident $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-GCIs of Finite Interpretations

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Axiomatizing Confident $\mathcal{EL}^{\bot}_{\text{gfp}}$-GCIs of Finite Interpretations

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Abstract

Constructing description logic ontologies is a difficult task that is normally conducted by experts. Recent results show that parts of ontologies can be constructed from description logic interpretations. However, these results assume the interpretations to be free of errors, which may not be the case for real-world data. To provide some mechanism to handle these errors, the notion of confidence from data mining is introduced into description logics, yielding confident general concept inclusions (confident GCIs) of finite interpretations. The main focus of this work is to prove the existence of finite bases of confident GCIs and to describe some of theses bases explicitly.

1 Introduction and Motivation

Description logic ontologies are a widely appreciated method to formalize large amounts of knowledge. Examples for large-scale ontologies used in practice are SNOMED-CT [17], the Systematized Nomenclature of Medicine Clinical Terms, GALEN [15], a bio-medical ontology, and the Gene Ontology [2].

The construction of such ontologies is an expensive task, in terms of both time and money. This is because of the complexity of the ontologies which requires its construction to be conducted by human experts. In addition, the information which is to be formalized in an ontology is most often only available in formats not accessible by machines, e. g. as textual publications.

However, there are efforts to overcome the latter problem of machine-inaccessible data. One such effort is to publish data as so-called *linked data* [7], as it is promoted by the W3C SWEO Linking Open Data community project[1]. Intuitively, one can understand linked data as a directed graph with labeled edges. The format used to store these labeled graphs are *RDF Triples*. As of September 2011, the above mentioned project has managed to publish over 295 interlinked data sets with an overall amount 31 billion RDF triples.

This growing amount of machine-readable information motivates the question whether it is possible to automate the process of ontology construction. Surely, we cannot assume that ontology construction can be done completely by machines. This would require data that both contains all relevant information and is free of errors. This cannot be assumed for real-world data. Still, one can ask whether it is possible to design algorithms that *assist* domain experts in designing ontologies.

[1] http://www.w3.org/wiki/SweoIG/TaskForces/CommunityProjects/LinkingOpenData

From the point of view of description logics this idea has an additional appeal. On the one hand, linked data can be understood as labeled, directed graphs. On the other hand, *interpretations*, which are used to define semantics for description logics, can as well be regarded as labeled graphs. Therefore, from the point of view of description logics, asking how we can construct ontologies from RDF triples can be understood as the question on how to construct ontologies from description logic interpretations.

In an ongoing collaboration between description logics and formal concept analysis, various attempts have been made to approach the problem of constructing ontologies from interpretations [4, 5, 10, 16]. Using ideas from formal concept analysis seems natural here. This is because the method of *attribute exploration* from formal concept analysis achieves a goal similar to constructing an ontology from an interpretation. Attribute exploration roughly works as follows: given data in the form of a *formal context*, attribute exploration interactively explores the *valid implications* of this formal context. Here, a formal context can simply be understood as a set of objects, a set of attributes and the information on which attributes which object possesses. Then, during the exploration process, an expert is asked questions of the following kind: do all objects that posses all attributes from a set A also posses all attributes from a set B? Equivalently, we can ask whether the attributes A *imply* the attributes B in our formal context. We write those statements as $A \longrightarrow B$ and call it an *implication*. If the implication $A \longrightarrow B$ is true, then the expert confirms it. If it is false, the expert has to provide a *counterexample*, i. e. an object g together with its attributes C such that $A \subseteq C$ and $B \nsubseteq C$. The attribute exploration algorithm stops when there are no more implications left to ask.

The resulting set of implications can then be regarded as a formalization of the knowledge from both the initial data (the formal context) and the participating expert. If attribute exploration could be generalized to also work with interpretations, we would be able to construct from a finite interpretation a large part of ontologies, namely the *terminological* part, in a semi-automatic way. As a result of such an algorithm we would obtain a set of *general concept inclusions (GCIs)* that express connections between certain concept descriptions in our given data.

There have been two attempts for such a generalization [10, 16]. In this work, we shall focus on the approach of [10]. Therein, the description logic $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ [3] is used, which is an extension of $\mathcal{EL}^{\bot}$ and is able to express *cyclic concept descriptions*. The usage of this description logic allows for an analogous definition of the *derivation operators* from formal concept analysis in this description logic. Based on this, the attribute exploration algorithm can be generalized to the description logic $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ and finite interpretations, as it has been done in [10].

In [9], this generalized attribute exploration algorithm has been applied to real-world data, in particular to parts of the DBpedia Project [8]. This experiment shows that the algorithm itself is applicable to modest-size interpretations. However, the resulting set of general concept inclusions reveals another problem: the exploration algorithm assumes the given interpretation to be *free of errors*. This is a legitimate assumption for theoretical considerations, but does not hold in practice. It would therefore be desirable to be able to somehow handle errors in interpretations. This is the main motivation of this work.

To achieve some control over errors we shall transfer the notion of *confidence* from data mining [1] to description logics. Roughly speaking, we allow general concept inclusions to have a limited amount of *exceptions*. The motivation for this is the following: if our initial data contains some errors, some connections between concept descriptions in this data may have been invalidated by these errors. These connections are expressed as general concept inclusions. But if we allow these general concept inclusions to have some exceptions, then few errors may not invalidate these general concept inclusions, provided that not too of these errors occurred. We shall discuss this in more detail in the corresponding section.

This publication is structured as follows. In the first two sections we shall introduce the necessary definitions and facts from the fields of formal concept analysis and description logics. This will also include the definition of *model-based most-specific concept descriptions*, which will play a major role in our considerations. In particular, as we shall see in Section 4, model-based most-specific concept descriptions are in a one-to-one correspondence to the intents of a particular formal context. The notion of model-based most-specific concept-descriptions has been introduced in [10].

Afterwards, we shall introduce *confident general concept inclusions* in Section 5. This will be done providing a detailed motivation, which also includes experimental results from [9].

Having defined confident general concept inclusions, we turn our focus to the following question: given a finite interpretation, can we find a *finite base* for the confident GCIs of this interpretation? Such a base would compromise all the information we can express using confident GCIs. But as the number of GCIs is normally infinite, it is not clear whether we can actually find such a finite base. Still, as we shall see in Section 6, finite bases always exists and can be computed effectively. To show this, we shall use ideas from formal concept analysis that have been found by Luxenburger during his work on *partial implications* [12]. What we can not achieve using Luxenburger's ideas is to obtain a *non-redundant* base of all confident GCIs of a finite interpretation. But as we shall see in Section 7, we can use other ideas from Formal Concept Analysis to reduce the size of the base.

2 Formal Concept Analysis

In this section we want to introduce the necessary definitions from formal concept analysis [11] needed in this work. This will include definitions for the notions already mentioned in the introduction, such as the ones of *formal contexts* and *implications*. We shall also give a short introduction to the *canonical base* of a formal context and its computation. However, as we are not going to present an exploration algorithm for confident GCIs in this work, we will not give a description of attribute exploration here. See [10] for more details on this.

2.1 Formal Contexts and Contextual Derivation Operators

Let us start by defining the notion of a formal context.

2.1 Definition Let G, M be two sets and let $I \subseteq G \times M$. Then the triple $\mathbb{K} = (G, M, I)$ is called a *formal context*, whereas the set G is denoted as the set of *objects* of $\mathbb{K}$ and the set M is denoted as the set of *attributes* of $\mathbb{K}$. For $g \in G, m \in M$ we read $(g, m) \in I$ as "object g has attribute m" and write $g\ I\ m$ in this case. ♢

If a formal context $\mathbb{K} = (G, M, I)$ is finite, i. e. if the sets G and M are finite, it is sometimes convenient to depict $\mathbb{K}$ as a *cross table*, as shown in the following example.

2.2 Example Let $G = \{ 2, 3, 5, 7 \}, M = \{ 1, \dots, 10 \}$ and

$$I = \{ (g, m) \in G \times M \mid g \text{ divides } m \}.$$

Then $\mathbb{K} = (G, M, I)$ is a formal context, which is depicted as a cross table in Figure 1. Here, we have a table where the rows are labeled with elements from G and the rows are labeled with elements from M. In a cell corresponding to a pair $(g, m) \in G \times M$ we write a cross "×" if and only if $(g, m) \in I$. Otherwise, we leave this cell blank or write a single dot "." in it. ♢

Given a formal context $\mathbb{K} = (G, M, I)$ and some set $A \subseteq G$ of objects one can ask what the largest set of attributes is that all objects in A share. Likewise, one can ask for a set $B \subseteq M$ of attributes what the largest set of objects is that have all attributes in B. To answer this question we introduce the *derivation operators* for a formal context $\mathbb{K}$.

	1	2	3	4	5	6	7	8	9	10
2	.	×	.	×	.	×	.	×	.	×
3	.	.	×	.	.	×	.	.	×	.
5	.	.	.	.	×	.	.	.	.	×
7	.	.	.	.	.	.	×	.	.	.

Figure 1: A formal context depicted as cross table

2.3 Definition Let $\mathbb{K} = (G, M, I)$ and $A \subseteq G, B \subseteq M$. Then we define the *derivations in the formal context* $\mathbb{K}$ as

$$A' := \{\, m \in M \mid \forall g \in A : g \; I \; m \,\},$$
$$B' := \{\, g \in G \mid \forall m \in B : g \; I \; m \,\}.$$

The set A is called an *extent* of $\mathbb{K}$ if and only if $A = (A')'$. The set B is called an *intent* of $\mathbb{K}$ if and only if $B = (B')'$. $\diamondsuit$

For convenience, we shall drop the extra parentheses and write shorter $(A')' = A''$ and $(B')' = B''$, respectively.

As a first observation on the derivation operators let us note that the functions

$$\cdot' \colon \mathfrak{P}(G) \longrightarrow \mathfrak{P}(M),$$
$$\cdot' \colon \mathfrak{P}(M) \longrightarrow \mathfrak{P}(G)$$

form a so called *Galois connection*. For this let us recall that for a set P an *order relation* $\leqslant_P$ is just a set $\leqslant_P \subseteq P \times P$ such that $\leqslant_P$ is *reflexive*, *antisymmetric* and *transitive*.

2.4 Definition Let P, Q be two sets and let $\leqslant_P$ and $\leqslant_Q$ be order relations on P and Q, respectively. Then the two mappings

$$\varphi \colon P \longrightarrow Q,$$
$$\psi \colon Q \longrightarrow P$$

form an *antitone Galois connection* between $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ if and only if for all $x \in P, y \in Q$ holds

$$x \leqslant_P \psi(y) \iff y \leqslant_Q \varphi(x). \qquad \diamondsuit$$

2.5 Proposition *Let $\mathbb{K} = (G, M, I)$ be a formal context, $A_1, A_2 \subseteq G, B_1, B_2 \subseteq M$. Then the following conditions hold:*

- $A_1 \subseteq A_2 \implies A_1' \supseteq A_2'$,
- $B_1 \subseteq B_2 \implies B_1' \supseteq B_2'$,
- $A_1 \subseteq A_1''$,
- $B_1 \subseteq B_1''$,
- $A_1' = A_1'''$,

- $B_1' = B_1'''$,
- $A_1' \subseteq B_1 \iff A_1 \supseteq B_1'$.

Another easy observation regarding derivation operators is the following: If $A \subseteq M$ and $(B_i \mid i \in I)$ is a family of subsets of A such that $\bigcup_{i \in I} B_i = A$, then

$$A' = \bigcap \{ B_i' \mid i \in I \}, \tag{2.1}$$

because for all $g \in G$

$$\begin{aligned} g \in A' &\iff \forall m \in A \colon g \ I \ m \\ &\iff \forall m \in \bigcup_{i \in I} B_i \colon g \ I \ m \\ &\iff \forall i \in I \colon g \in B_i'. \end{aligned}$$

Finally, we can see easily that for $\mathcal{A} \subseteq \mathfrak{P}(M)$ we always have

$$\bigcap_{A \in \mathcal{A}} A' = (\bigcup_{A \in \mathcal{A}} A'). \tag{2.2}$$

This is because for each $g \in G$ it is true that

$$\begin{aligned} g \in \bigcap_{A \in \mathcal{A}} A' &\iff \forall A \in \mathcal{A} \colon g \in A' \\ &\iff \forall A \in \mathcal{A} \forall m \in A \colon g \ I \ m \\ &\iff \forall m \in \bigcup_{A \in \mathcal{A}} A \colon g \ I \ m \\ &\iff g \in (\bigcup_{A \in \mathcal{A}} A'). \end{aligned}$$

Of course, this also holds for sets $\mathcal{B} \subseteq \mathfrak{P}(G)$, i. e.

$$\bigcap_{B \in \mathcal{B}} B' = (\bigcup_{B \in \mathcal{B}} B)'.$$

2.2 Implications

If we have given a formal context $\mathbb{K} = (G, M, I)$, it may very well be that for all objects that have certain attributes $A \subseteq M$ always have the attributes $B \subseteq M$ in addition. We say may say that the attributes from A *imply* the attributes from B in the formal context $\mathbb{K}$.

2.6 Definition Let M be a set. An *implication* $A \longrightarrow B$ on M is a pair (A, B) where $A, B \subseteq M$. In this case, A is called the *premise* and B is called the *conclusion* of the implication $A \longrightarrow B$. We shall denote the set of all implications on M by $\operatorname{Imp}(M)$.

Let $\mathbb{K} = (G, M, I)$ be a formal context. An *implication* $A \longrightarrow B$ of $\mathbb{K}$ is an implication on M. The set of all implications of $\mathbb{K}$ is denoted by $\operatorname{Imp}(\mathbb{K})$, i. e.

$$\operatorname{Imp}(\mathbb{K}) = \operatorname{Imp}(M).$$

The implication $A \longrightarrow B$ *holds in* $\mathbb{K}$ (or is *valid* in $\mathbb{K}$) if $B \subseteq A''$. We then write $\mathbb{K} \models (A \longrightarrow B)$. If $\mathcal{J}$ is a set of implications of $\mathbb{K}$ such that each implication in $\mathcal{J}$ holds in $\mathbb{K}$, then we may denote this with $\mathbb{K} \models \mathcal{J}$. The set of all implications of $\mathbb{K}$ that hold in $\mathbb{K}$ is denoted by $\operatorname{Th}(\mathbb{K})$. $\diamondsuit$

Note that the condition $B \subseteq A''$ is equivalent to $A' \subseteq B'$ by Proposition 2.5, i. e. an implication $A \longrightarrow B$ holds in $\mathbb{K} = (G, M, I)$ if and only if every object $g \in G$ that has all attributes in A also has all attribute in B.

2.7 Definition Let $\mathbb{K} = (G, M, I)$ be a formal context and let $\mathcal{J}$ be a set of implications of $\mathbb{K}$. Then an implication $A \longrightarrow B$ *is entailed by* $\mathcal{J}$ if for every context $\tilde{\mathbb{K}}$, in which all implications from $\mathcal{J}$ hold, the implication $A \longrightarrow B$ holds as well. ♢

Implications on a set M give rise to a certain class of mappings on the powerset lattices $(\mathfrak{P}(M), \subseteq)$, namely *closure operators on* M. Abstractly, these are mappings

$$c \colon \mathfrak{P}(M) \longrightarrow \mathfrak{P}(M)$$

such that

- $A \subseteq c(A)$, i. e. c is *extensive*,
- $A \subseteq B \Rightarrow c(A) \subseteq c(B)$, i. e. c is *monotone*, and
- $c(c(A)) = c(A)$, i. e. c is *idempotent*,

holds for all sets $A, B \subseteq M$. A set $A \subseteq M$ is said to be *closed under* c if and only if $c(A) = A$.

Now every closure operator on the lattice $(\mathfrak{P}(M), \subseteq)$ can be characterized by implications in the way as it is described by the following definition.

2.8 Definition Let M be a set and $\mathcal{L} \subseteq \operatorname{Imp}(M)$. Then define for $A \subseteq M$

$$\begin{aligned} \mathcal{L}^1(A) &:= \bigcup \{ Y \mid (X \longrightarrow Y) \in \mathcal{L}, X \subseteq A \}, \\ \mathcal{L}^{i+1}(A) &:= \mathcal{L}(\mathcal{L}^i(A)) \quad (i \in \mathbb{N}_{>0}), \\ \mathcal{L}(A) &:= \bigcup_{i \in \mathbb{N}_{>0}} \mathcal{L}^i(A). \end{aligned}$$

The mapping $\mathcal{L} : \mathfrak{P}(M) \longrightarrow \mathfrak{P}(M)$ with $A \longmapsto \mathcal{L}(A)$ is then called the *closure operator induced by* $\mathcal{L}$. A set $A \subseteq M$ is said to be *closed under* $\mathcal{L}$ if and only if $\mathcal{L}(A) = A$. ♢

It is easy to see that every closure operator induced by a set of implications on a set M is indeed a closure operator on M in the sense of the aforementioned definition.

Entailment for implications can be rephrased in terms of the induced closure operators. To show this, we shall first prove two auxiliary claims before we consider the characterization of Lemma 2.11.

2.9 Proposition *Let* $\mathbb{K} = (G, M, I)$ *be a formal context and let* $\mathcal{L} \subseteq \operatorname{Imp}(\mathbb{K})$*. If* $\mathbb{K} \models \mathcal{L}$*, then*

$$\mathcal{L}(A) \subseteq A''$$

holds for each $A \subseteq M$.

Proof We show $A' \subseteq \mathcal{L}(A)'$, as then

$$\mathcal{L}(A) \subseteq \mathcal{L}(A)'' \subseteq A''$$

by Proposition 2.5.

Let $g \in A'$. If then $(X \longrightarrow Y) \in \mathcal{L}$ with $X \subseteq A$, then $g \in A' \subseteq X' \subseteq Y'$, since $X \longrightarrow Y$ holds in $\mathbb{K}$. Therefore,

$$g \in \bigcap \{ Y' \mid (X \longrightarrow Y) \in \mathcal{L}, X \subseteq A \} = (\mathcal{L}^1(A))'$$

by Equation (2.1). With the same argumentation, we see that

$$g \in (\mathcal{L}^i(A))'$$

holds for all $i \in \mathbb{N}_{>0}$, and hence we obtain

$$g \in \bigcap_{i \in I} (\mathcal{L}^i(A))' = (\bigcup_{i \in I} \mathcal{L}^i(A))' = (\mathcal{L}(A))'$$

using Equations (2.1) and (2.2). □

2.10 Proposition *Let M be a set and $\mathcal{L} \subseteq \mathrm{Imp}(M)$. Then the formal context*

$$\mathbb{K}_{\mathcal{L}} := (\{ \mathcal{L}(A) \mid A \subseteq M \}, M, \ni)$$

satisfies $X'' = \mathcal{L}(X)$ for all $X \subseteq M$.

Proof From Proposition 2.9 we already know the inclusion $\mathcal{L}(X) \subseteq X''$. For the converse inclusion we shall show that $(\mathcal{L}(X))'' = \mathcal{L}(X)$, since this together with $X \subseteq \mathcal{L}(X)$ then implies $X'' \subseteq \mathcal{L}(X)$, using Proposition 2.5.

We compute

$$(\mathcal{L}(X))' = \{ \mathcal{L}(A) \mid A \subseteq M, \mathcal{L}(A) \supseteq \mathcal{L}(X) \}.$$

In particular, $\mathcal{L}(X) \in (\mathcal{L}(X))'$. Then

$$\begin{aligned}(\mathcal{L}(X))'' &= \{ \mathcal{L}(A) \mid A \subseteq M, \mathcal{L}(A) \supseteq \mathcal{L}(X) \}' \\ &= \bigcap \{ \mathcal{L}(A) \mid A \subseteq M, \mathcal{L}(A) \supseteq \mathcal{L}(X) \} \\ &= \mathcal{L}(X)\end{aligned}$$

as required. □

2.11 Lemma *Let M be a set and let $\mathcal{L} \subseteq \mathrm{Imp}(M)$, $(A \longrightarrow B) \in \mathrm{Imp}(M)$. Then*

$$\mathcal{L} \models (A \longrightarrow B) \iff B \subseteq \mathcal{L}(A).$$

Proof Suppose that $\mathcal{L} \models (A \longrightarrow B)$ and let $\mathbb{K} = \mathbb{K}_{\mathcal{L}}$ be a formal context such as described in Proposition 2.10. Then

$$\mathcal{L}(X) = X'' \tag{2.3}$$

holds for all $X \subseteq M(\mathbb{K})$. Since $\mathcal{L} \models (A \longrightarrow B)$, the implication $A \longrightarrow B$ holds in $\mathbb{K}$ as well and therefore $A' \subseteq B'$. Then Proposition 2.5 together with Equation (2.3) implies

$$B \subseteq A'' = \mathcal{L}(A).$$

Conversely, let $B \subseteq \mathcal{L}(A)$ and let $\mathbb{K}$ be a formal context such that $\mathbb{K} \models \mathcal{L}$. Then

$$\mathcal{L}(A) \subseteq A''.$$

by Proposition 2.9. But then

$$B \subseteq \mathcal{L}(A) \subseteq A'',$$

hence $A' \subseteq B'$ by Proposition 2.5 and therefore $A \longrightarrow B$ holds in $\mathbb{K}$. □

2.3 Bases of Implications

Implications can be understood as logical objects for which we can decide validity in formal contexts. This automatically yields the following definition of *implicational bases*, which results in a way to represent all valid implications of a formal context in a compact way.

2.12 Definition Let $\mathbb{K}$ be a formal context. A set $\mathcal{J}$ of implications of $\mathbb{K}$ is an *implicational base* (or just a *base*) of $\mathbb{K}$ if the following conditions hold:

1) $\mathcal{J}$ is *sound* for $\mathbb{K}$, i. e. every implication in $\mathcal{J}$ holds in $\mathbb{K}$,

2) $\mathcal{J}$ is *complete* for $\mathbb{K}$, i. e. every implication holding in $\mathbb{K}$ follows from $\mathcal{J}$,

Moreover, a base $\mathcal{J}$ of $\mathbb{K}$ is said to be *non-redundant* if each proper subset of $\mathcal{J}$ is not a base of $\mathbb{K}$. ♢

An obvious base is the following.

2.13 Theorem *Let $\mathbb{K}$ be a formal context. Then the set*

$$\mathcal{L} := \{ A \longrightarrow A'' \mid A \subseteq M_{\mathbb{K}} \}$$

is a base of $\mathbb{K}$.

Proof Obviously, $\mathcal{L}$ is sound for $\mathbb{K}$. To see that $\mathcal{L}$ is also complete for $\mathbb{K}$, let $X \longrightarrow Y$ be an implication holding in $\mathbb{K}$. By Lemma 2.11 we only have to show that $Y \subseteq \mathcal{L}(X)$. However, since $X \longrightarrow Y$ holds in $\mathbb{K}$, we obtain $Y \subseteq X''$. On the other hand, $X'' = \mathcal{L}(X)$ and the claim follows. □

Checking completeness of a set $\mathcal{L}$ of implications may be a tedious task, as one has to consider all valid implications of $\mathbb{K}$. However, completeness of $\mathcal{L}$ can be verified by considering the intents of $\mathbb{K}$, as the following lemma shows.

2.14 Lemma *Let $\mathbb{K} = (G, M, I)$ be a formal context and let $\mathcal{L} \subseteq \operatorname{Imp}(M)$. Then $\mathcal{L}$ is complete for $\mathbb{K}$ if and only if*

$$\forall U \subseteq M \colon \mathcal{L}(U) = U \implies U = U'',$$

i. e. the closed sets of $\mathcal{L}$ are intents of $\mathbb{K}$.

Proof Suppose that $\mathcal{L}$ is complete for $\mathbb{K}$. Let $U \subseteq M$ be such that $U \neq U''$. Then because of $U \subseteq U''$ we have $U \nsupseteq U''$. But then the implication $U \longrightarrow U''$ is valid in $\mathbb{K}$. Since $\mathcal{L}$ is complete for $\mathbb{K}$,

$$\mathcal{L} \models (U \longrightarrow U''),$$

i. e. by Lemma 2.11,

$$U'' \subseteq \mathcal{L}(U).$$

This implies $U \neq \mathcal{L}(U)$ as required.

Now suppose that for each $U \subseteq M$ that $\mathcal{L}(U) = U$ implies $U = U''$. Since $\mathcal{L}(\mathcal{L}(U)) = \mathcal{L}(U)$, we obtain

$$\mathcal{L}(U) = \mathcal{L}(U)''$$

for each $U \subseteq M$. But this implies

$$U'' \subseteq \mathcal{L}(U)'' = \mathcal{L}(U)$$

and therefore $\mathcal{L} \models (U \longrightarrow U'')$ again for each $U \subseteq M$. By Theorem 2.13, $\mathcal{L}$ is complete for $\mathbb{K}$. □

It is easy to see that if we reverse the direction of the implication in the previous lemma, that we then obtain a characterization for $\mathcal{L}$ to be sound for $\mathbb{K}$.

The base that is described in Theorem 2.13 is not very practical, as it always contains an exponentially many implications measured in the size of M. Luckily, we can explicitly describe a base that always has *minimal cardinality* among all bases of a formal context.

2.15 Definition ($\mathcal{K}$-pseudo-intent) Let $\mathbb{K}$ be a finite formal context and let $\mathcal{K} \subseteq \operatorname{Imp}(M)$. A set $P \subseteq M$ is said to be a *$\mathcal{K}$-pseudo-intent* of $\mathbb{K}$ if and only if

i. $P \neq P''$,

ii. $\mathcal{K}(P) = P$ and

iii. for all $\mathcal{K}$-pseudo-intents $Q \subsetneq P$ it holds that $Q'' \subseteq P$.

If $\mathcal{K} = \emptyset$, then P is also called a *pseudo-intent* of $\mathbb{K}$. ◇

Let us define for a formal context $\mathbb{K}$ and $\mathcal{K} \subseteq \operatorname{Th}(M)$ the *canonical base of* $\mathbb{K}$ *with background knowledge* $\mathcal{K}$ to be the set

$$\operatorname{Can}(\mathbb{K}, \mathcal{K}) := \{\, P \longrightarrow P'' \mid P \text{ a } \mathcal{K}\text{-pseudo-intent of } \mathbb{K} \,\}.$$

We can consider the canonical base for $\mathbb{K}$ with background knowledge $\mathcal{K}$ as the smallest set of valid implications of $\mathbb{K}$ such that $\operatorname{Can}(\mathbb{K}, \mathcal{K}) \cup \mathcal{K}$ is a base for $\mathbb{K}$. Intuitively, if we assume that we already know the implications of $\mathcal{K}$ but want to learn all valid implications of $\mathbb{K}$, then $\operatorname{Can}(\mathbb{K}, \mathcal{K})$ is the smallest set of valid implications that we need to add.

2.16 Theorem (Theorem 3.8 from [10]) *Let $\mathbb{K}$ be a finite formal context and $\mathcal{K} \subseteq \operatorname{Th}(M)$. Then the set $\operatorname{Can}(\mathbb{K}, \mathcal{K}) \cup \mathcal{K}$ is base of $\mathbb{K}$ having the least number of elements among all bases of $\mathbb{K}$ containing $\mathcal{K}$.*

This theorem assumes the background knowledge $\mathcal{K}$ to contain only valid implications of $\mathbb{K}$. However, this is not necessary. As we are going to consider invalid implications in this work as well, we shall therefore formulate and prove the following theorem. Indeed, the same proofs as for the case of valid background knowledge apply and we shall repeat them here for completeness. The first part of the proof is the same as in [18]. The part of the proof that shows minimal cardinality is taken from [10].

2.17 Theorem *Let $\mathbb{K} = (G, M, I)$ be a formal context and let $\mathcal{K} \subseteq \operatorname{Imp}(M)$. Then $\operatorname{Can}(\mathbb{K}, \mathcal{K})$ is the set of valid implications with minimal cardinality such that $\operatorname{Can}(\mathbb{K}, \mathcal{K}) \cup \mathcal{K}$ is complete for $\mathbb{K}$.*

Proof Let us write

$$\mathcal{L} := \operatorname{Can}(\mathbb{K}, \mathcal{K}) \cup \mathcal{K}.$$

Then we need so show three statements:

i. $\mathcal{L}$ is a set of valid implications of $\mathbb{K}$.

ii. $\mathcal{L}$ is complete for $\mathbb{K}$.

iii. $\mathcal{L}$ has minimal cardinality for all sets $\mathcal{M}$ of valid implications of $\mathbb{K}$ such that $\mathcal{M} \cup \mathcal{K}$ is complete for $\mathbb{K}$.

It is apparent that $\mathcal{L}$ contains only valid implications of $\mathbb{K}$.

We shall go on and show completeness of $\mathcal{L}$ for $\mathbb{K}$. For this we use Lemma 2.14 and show that every set $U \subseteq M$ with $\mathcal{L}(U) = U$ is already an intent of $\mathbb{K}$.

Let $U \subseteq M$ be such that $\mathcal{L}(U) = U$. If then $V \subsetneq U$ is an $\mathcal{K}$-pseudo intent, then $V'' \subseteq U$, as $(V \longrightarrow V'') \in \mathrm{Can}(\mathbb{K}, \mathcal{K})$. Furthermore, $\mathcal{K}(U) = U$. Hence, if we assume $U \neq U''$, then U is an $\mathcal{K}$-pseudo intent of $\mathbb{K}$. But then $(U \longrightarrow U'') \in \mathcal{L}$, contradicting the fact that $U \neq U''$. Therefore, $U = U''$ and $\mathcal{L}$ is complete for $\mathbb{K}$.

We shall now show the minimal cardinality of $\mathrm{Can}(\mathbb{K}, \mathcal{K})$. For this, let $\mathcal{M}$ be another set of valid implications of $\mathbb{K}$ such that $\mathcal{M} \cup \mathcal{K}$ is complete for $\mathbb{K}$. Without loss of generality we may assume that $\mathcal{M}$ only contains implications of the form $U \longrightarrow U''$ for some $U \subseteq M$.

We now show that for each $\mathcal{K}$-pseudo intent P of $\mathbb{K}$ there exists a set $U_P \subseteq M$ such that $(U_P \longrightarrow U_P'') \in \mathcal{M}$. In addition, we shall show that if P and Q are two $\mathcal{K}$-pseudo intents of $\mathbb{K}$ such that $P \neq Q$, then $U_P \neq U_Q$. From these claim it immediately follows that $|\mathcal{M}| \geqslant |\mathrm{Can}(\mathbb{K}, \mathcal{K})|$, as we then have an injective mapping $P \longmapsto U_P$ from $\mathrm{Can}(\mathbb{K}, \mathcal{K})$ to $\mathcal{M}$.

Now let P be a $\mathcal{K}$-pseudo intent of $\mathbb{K}$. Then $P \neq P''$. As $\mathcal{M} \cup \mathcal{K}$ is complete for $\mathbb{K}$ and $\mathcal{K}(P) = P$, there exists an implication $(X \longrightarrow X'') \in \mathcal{M}$ such that $X \subseteq P$ and $X'' \nsubseteq P$. Define $U_P := X$.

Now let P and Q be two different $\mathcal{K}$-pseudo intents of $\mathbb{K}$ and assume $U_P = U_Q =: U$. Then $U \subseteq P$ and $U \subseteq Q$, hence $U \subseteq P \cap Q$. Therefore, $U'' \subseteq (P \cap Q)''$. But then $U'' = U_P'' \nsubseteq P$ and $U'' \subseteq (P \cap Q)''$ imply $(P \cap Q)'' \nsubseteq P$. This implies $(P \cap Q)'' \nsubseteq P \cap Q$ and thus

$$(P \cap Q)'' \neq P \cap Q.$$

Since $\mathcal{K}(P) = P$ and $\mathcal{K}(Q) = Q$, we obtain $\mathcal{K}(P \cap Q) = P \cap Q$. But since $(P \cap Q)'' \neq P \cap Q$ and $\mathcal{K}(P \cap Q) = P \cap Q$, there must exist an implication $(R \longrightarrow R'') \in \mathrm{Can}(\mathbb{K}, \mathcal{K})$ such that $R \subseteq P \cap Q$ and $R'' \nsubseteq (P \cap Q)$. Without loss of generality, we assume $R'' \nsubseteq Q$. But since R is an $\mathcal{K}$-pseudo intent and $R \subseteq Q$, $R \neq Q$ cannot hold. Thus $R = Q$ and $Q \subseteq P \cap Q$. This implies $Q \subseteq P$ and $Q = P \cap Q$. But $Q'' \nsubseteq P$ as $(P \cap Q)'' \nsubseteq P$. Therefore, as Q and P are $\mathcal{K}$-pseudo intents, Q cannot be different from P and we obtain $P = Q$ as desired. □

2.4 Computing the Canonical Base for Arbitrary Background Knowledge

Albeit its quite incomprehensible and recursive definition, the canonical base of $\mathbb{K}$ with background knowledge $\mathcal{K}$ allows an easy algorithm for its computation, see Algorithm 2.21. To describe this algorithm, we shall first introduce some definitions and preliminary results.

Please note that the following considerations are not necessary for the understanding of the rest of the work and may therefore be skipped. They are included to show how we can compute the canonical base of a formal context with arbitrary (in particular invalid) background knowledge.

2.18 Definition Let M be a set and $<$ a strict linear order on M. Then the *lectic order on M induced by* $<$ is the relation $\prec$ defined as

$$A \prec B \iff \min_{<}(A \triangle B) \in B,$$

for $A, B \subseteq M$, where

$$A \triangle B := (A \setminus B) \cup (B \setminus A)$$

is the *symmetric difference* of A and B.

For $i \in M$ we may say that A *is lectically smaller then B at position i*, written as $A \prec_i B$, if and only if

$$i = \min_{<}(A \triangle B) \text{ and } i \in B. \qquad \diamondsuit$$

2.20 Algorithm (Next Closure)

```
define next-closure(M, <, A, c)
  let (C := { i ∈ M | A ≺_i A ⊕_c i })
    if C = ∅
      return nil
    else
      return A ⊕_c max_<(C)
    end if
  end let
end
```

Note that $A \prec B$ if and only if there exists an $i \in M$ such that $A \prec_i B$. Also note that we shall denote the reflexive closure of $\prec$ by $\preceq$, i. e.

$$A \preceq B \iff A = B \text{ or } A \prec B$$

for $A, B \subseteq M$.

Let c be some closure operator on the set M. One of the remarkable properties of lectic orders is that they admit a simple but efficient algorithm to compute for an arbitrary set $A \subseteq M$ the lectically next closed set after A that is closed under c, i. e. the set

$$\min_{\prec}\{\, B \subseteq M \mid c(B) = B, A \prec B \,\},$$

if it exists.

To ease the formulation, let us define for a set $A \subseteq M$ and $i \in M$

$$A \oplus_c i := \mathcal{L}(\{\, a \in A \mid a < i \,\} \cup \{\, i \,\}).$$

Then the following theorem holds, which is well-known in the field of formal concept analysis as the Next-Closure algorithm.

2.19 Theorem (Theorem 5 of [11]) *Let M be a set, $<$ a strict linear order relation on M, $\prec$ the lectic order on $\mathfrak{P}(M)$ induced by $<$, c a closure operator on M and $A \subseteq M$. Then if there exists a set B such that $A \prec B$ and B is closed under c, then the lectically smallest such set is $A \oplus_c i$, where i is $<$-maximal with the property $A \prec_i A \oplus_c i$.*

This theorem immediately gives rise an effective algorithm as given in Algorithm 2.20. Under the prerequisites of Theorem 2.19 it is true that

$$\texttt{next-closure}(M, <, A, c) = \min_{\prec}\{\, B \subseteq M \mid A \prec B, c(B) = B \,\} \tag{2.4}$$

if this minimum exists and $\texttt{next-closure}(M, <, A, c) = \textbf{nil}$ otherwise. This algorithm will turn out to be very useful in the following considerations.

We can now prove that Algorithm 2.21 can indeed be used to compute the canonical base for arbitrary background knowledge.

2.22 Theorem *Let $\mathbb{K} = (G, M, I)$ be a finite formal context, $\mathcal{K} \subseteq \operatorname{Imp}(\mathbb{K})$ and $<$ a linear order on M. Then*

$$\operatorname{Can}(\mathbb{K}, \mathcal{K}) = \texttt{canonical-base}(\mathbb{K}, <, \mathcal{K}).$$

2.21 Algorithm (Computing the Canonical Base with Background Knowledge)

```
define next-closed-non-intent (𝕂, <, A, c)
  let (P := next-closure(M(𝕂), <, A, c))
    if P = nil then
      return nil
    else if P ≠ P'' then
      return P
    else
      return next-closed-non-intent(M(𝕂), <, P, c)
    end if
  end let
end

define first-closed-non-intent(𝕂, <, c)
  if ∅ ≠ ∅'' and c(∅) = ∅ then
    return ∅
  else
    return next-closed-non-intent(𝕂, <, ∅, c)
  end if
end

define canonical-base(𝕂, <, 𝒦)
  let (i := 0,
       P_i := first-closed-non-intent(𝕂, <, 𝒦),
       ℒ_i := ∅)
    if P_i = nil then
      return ℒ
    else
      recur (ℒ_{i+1} := ℒ_i ∪ { P_i ⟶ P_i'' },
             P_{i+1} := next-closed-non-intent(𝕂, <, P_i, ℒ_{i+1} ∪ 𝒦),
             i := i + 1)
  end let
end define
```

Before we are going to prove this theorem, let us first argue that the auxiliary functions given in Algorithm 2.21 yield what their names suggest.

2.23 Proposition *Let $\mathbb{K} = (G, M, I)$ be a finite formal context, $<$ a strict linear order on M and c a closure operator on M. Denote with $\prec$ the lectic order on $\mathfrak{P}(M)$ induced by $<$.*

i. Let $A \subseteq M$ and define

$$S := \texttt{next-closed-non-intent}(\mathbb{K}, <, A, c).$$

Then

$$S = \min_{\prec}\{ B \subseteq M \mid A \prec B, B = c(B), B \neq B'' \}$$

if this minimum exists and $S = \textbf{nil}$ otherwise.

ii. Define

$$T := \texttt{first-closed-non-intent}(\mathbb{K}, <, c).$$

Then

$$T = \min_{\prec}\{ B \subseteq M \mid B = c(B), B \neq B'' \}$$

if this minimum exists and $T = \textbf{nil}$ otherwise.

Proof For the first statement observe that the algorithm considers in lectic order all sets $C \subseteq M$, $A \prec C$ that are closed under c. Now, if a lectically smallest set $C \subseteq M$ with $A \prec C$ exists such that $C \neq C''$, then it is finally found by the algorithm and returned as the resulting value. If, however, no such set exists, the variable P in the algorithm will finally obtain the value M in the subsequent iteration will return **nil**, as

$$\texttt{next-closure}(M, <, M, c) = \textbf{nil}.$$

For the second statement suppose that $\varnothing \neq \varnothing''$ and $c(\varnothing) = \varnothing$. Then clearly

$$\min_{\prec}\{ B \subseteq M \mid B = c(B), B \neq B'' \} = \varnothing = \texttt{first-closed-non-intent}(\mathbb{K}, <, c).$$

If $\varnothing = \varnothing''$ or $c(\varnothing) \neq \varnothing$, then

$$\varnothing \prec \min_{\prec}\{ B \subseteq M \mid B = c(B), B \neq B'' \}$$

and hence

$$\begin{aligned}\min_{\prec}\{ B \subseteq M \mid B = c(B), B \neq B'' \} &= \min_{\prec}\{ B \subseteq M \mid \varnothing \prec B, B = c(B), B \neq B \} \\ &= \texttt{next-closed-non-intent}(\mathbb{K}, <, \varnothing, c) \\ &= \texttt{first-closed-non-intent}(\mathbb{K}, <, c).\end{aligned}$$

□

We are now prepared for the proof of Theorem 2.22. Again, the proof is the same as for the classical case, where the set $\mathcal{K}$ is only allowed to contain valid implications of $\mathbb{K}$. We shall give the proof nevertheless to show that it yields the desired claim.

The main line of argumentation of the proof is the following: using induction we shall prove that in each iteration i the set $\mathcal{L}_i$ contains implications $P_j \longrightarrow P_j''$, $j = 0, \dots, i-1$, where $P_0, \dots, P_{i-1}$ are the lectically first $\mathcal{K}$-pseudo intents of $\mathbb{K}$. Hence, if the algorithm finishes in iteration k, the variable $\mathcal{L}_k$ contains all implications $P \longrightarrow P''$ where P is a $\mathcal{K}$-pseudo intent of $\mathbb{K}$.

A simple argument used in these considerations is the following: if $P, Q \subseteq M$ are two arbitrary sets such that $P \subsetneq Q$, then $P \prec Q$. This is immediate from the definition of $\prec$, as $\min_<(P \triangle Q)$ is a non-empty subset of Q, hence

$$\min_<(P \triangle Q) \in Q$$

and therefore $P \prec Q$.

Proof (Theorem 2.22) Let $\prec$ be the lectic order induced by $<$. We shall prove by induction that in each iteration i, the variables

$$P_0, \dots, P_{i-1}$$

are the first i (with respect to $\prec$) $\mathcal{K}$-pseudo intents of $\mathbb{K}$.

Let $i = 0$. By Proposition 2.23, the value of P_0 is the $\prec$-smallest subset P of M such that $P = \mathcal{K}(P)$ and $P \neq P''$. Hence, if $Q \prec P_0$ is such that $Q = \mathcal{K}(Q)$, then $Q = Q''$. In addition, if $Q \subsetneq P_0$, then $Q \prec P_0$. Hence, there are no $\mathcal{K}$-pseudo intents properly contained in P_0, and therefore P_0 is the $\prec$-smallest $\mathcal{K}$-pseudo intent of $\mathbb{K}$.

Now suppose that we are in iteration $i > 0$ of the algorithm. Suppose that $P_i \neq$ **nil**. By Proposition 2.23, P_i is the $\prec$-smallest subset P of M such that $(\mathcal{L}_i \cup \mathcal{K})(P) = P$ and $P \neq P''$. Let $Q \subsetneq P_i$ be a $\mathcal{K}$-pseudo intent of $\mathbb{K}$. Then $Q \prec P_i$. By the induction hypotheses, $Q = P_j$ for some $j < i$ and therefore $(Q \longrightarrow Q'') \in \mathcal{L}_i$. As $\mathcal{L}_i(P_i) = P_i$ and $Q \subseteq P_i$, it is true that $Q'' \subseteq P_i$. Together with the observation that $\mathcal{K}(P_i) = P_i$ we see that P_i is a $\mathcal{K}$-pseudo intent of $\mathbb{K}$. Since P_i is $\prec$-minimal with respect to $(\mathcal{L}_i \cup \mathcal{K})(P_i) = P_i$ and $P_i \neq P_i''$, it is also the $\prec$-smallest $\mathcal{K}$-pseudo intent lectically larger then P_{i-1}, i. e.

$$P_0, \dots, P_i$$

are the first $i + 1$ $\mathcal{K}$-pseudo intents of $\mathbb{K}$. This completes the inductive step.

If $P_i =$ **nil**, then each set $Q \subseteq M$ with $P_{i-1} \prec Q$ and $(\mathcal{L}_i \cup \mathcal{K})(Q) = Q$ satisfies $Q = Q''$. Therefore, there do not exist any further $\mathcal{K}$-pseudo intents after P_{i-1}. Since $P_0, \dots, P_{i-1}$ are the first i $\mathcal{K}$-pseudo intents, they are all $\mathcal{K}$-pseudo intents of $\mathbb{K}$. Therefore, $\mathcal{L}_i = \mathrm{Can}(\mathbb{K}, \mathcal{K})$ and the theorem is proven. □

3 The Description Logics $\mathcal{EL}^\perp$ and $\mathcal{EL}^\perp_{\mathrm{gfp}}$

Description logics are part of the field of Knowledge Representation, a branch of Artificial Intelligence. Its main focus lies in the representation of knowledge using well-defined semantics. For this, description logics provide the notion of *ontologies*. These ontologies can be understood as a collection of axioms. More specifically, description logic ontologies consist of *assertional axioms* and *terminological axioms*. Examples for an assertional axioms are “Tom is a cat” and “Jerry is a mouse”, written in description logic syntax as

$$\mathsf{Cat}(\mathsf{Tom}) \quad \text{and} \quad \mathsf{Mouse}(\mathsf{Jerry}).$$

An example for terminological axiom would be to say that “every cat hunts a mouse”, written as

$$\mathsf{Cat} \sqsubseteq \exists \mathsf{hunts}.\mathsf{Mouse}.$$

The use of the existential quantifier may be a bit surprising here, but it can be explained as follows. Consider the reformulation of “every cat hunts a mouse” to “whenever there is a cat, there exists a mouse it hunts.” The above statement should be read with this reformulation in mind.

Another example would be to say that "nothing is both a cat and a mouse", written as

$$\mathsf{Cat} \sqcap \mathsf{Mouse} \sqsubseteq \bot.$$

Again, a reformulation may clarify the used syntax. The phrase "nothing is both a cat and a mouse" can be understood as "whenever there is something that is both a cat and a mouse, we have a contradiction." The bottom sign $\bot$ denotes this contradiction.

These examples are formulated in the description logic $\mathcal{EL}^{\bot}$, the logic we shall mainly use in this work. The constructors used in $\mathcal{EL}^{\bot}$ are *conjunction* $\sqcap$, *existential restriction* $\exists$ and the *bottom concept* $\bot$.

During the course of our considerations, however, it shall turn out that $\mathcal{EL}^{\bot}$ does not suffice for all our purposes. We shall therefore latter on introduce another description logic called $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ that can be understood as an extension of $\mathcal{EL}^{\bot}$ that allows for cyclic concept descriptions. The main motivation to consider this description logic shall become clear when we introduce *model-based most-specific concept descriptions*, which allow us to reformulate notions from FCA in the language of description logics.

3.1 The Description Logic $\mathcal{EL}^{\bot}$

We are now going to introduce the syntax and semantics of the description logic $\mathcal{EL}^{\bot}$.

3.1 Definition Let N_C and N_R be two disjoint sets. The set $\mathcal{C}$ of $\mathcal{EL}$*-concept description with signature* (N_C, N_R) is defined as follows:

i. If $A \in N_C$, then $A \in \mathcal{C}$.

ii. If $C, D \in \mathcal{C}$, then $C \sqcap D \in \mathcal{C}$.

iii. If $C \in \mathcal{C}$ and $r \in N_R$, then $\exists r.C \in \mathcal{C}$.

iv. $\top \in \mathcal{C}$.

v. $\mathcal{C}$ is minimal with these properties.

The elements of the set N_C are called *concept names* and the elements of N_R are called *role names*.

An $\mathcal{EL}^{\bot}$*-concept description over the signature* (N_C, N_R) is either $\bot$ or an $\mathcal{EL}$-concept description over the signature (N_C, N_R). $\diamondsuit$

For convenience we may sometimes omit to explicitly mention the signature of an $\mathcal{EL}^{\bot}$-concept description. We may also talk about *concept descriptions* if the description logic used is clear from the context.

We have already seen some examples for $\mathcal{EL}^{\bot}$-concept descriptions, but let us consider one more example, this time a bit more formally.

3.2 Example As an example, let us consider the sets

$$N_C = \{ \mathsf{Cat}, \mathsf{Mouse}, \mathsf{Animal} \},$$
$$N_R = \{ \mathsf{hunts} \}.$$

Then

$$\mathsf{Cat} \sqcap \exists \mathsf{hunts}.\mathsf{Mouse}$$

is a valid $\mathcal{EL}^{\bot}$-concept description. Informally, it can be understood as the set of all cats that are (at this very moment) hunting a mouse. $\diamondsuit$

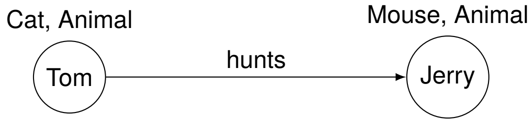

Figure 2: An example interpretation

Intuitively associating a meaning with an $\mathcal{EL}^{\bot}$-concept description is not sufficient for a knowledge representation formalism. Therefore, description logics define the semantics of concept descriptions in terms of *interpretations*. Suppose we have given three pairwise disjoint sets N_C of concept names, N_R of role names and N_I of *individual names*. Then an interpretation can be understood as a directed graph where the vertices are labeled with concept names from N_C and edges are labeled with role names from N_R. Additionally, some of the vertices are explicitly named with elements from N_I and no vertex has more than one name.

3.3 Definition Let N_C, N_R and N_I be pairwise disjoint sets. An *interpretation* $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ over the signature (N_C, N_R, N_I) consists of a set $\Delta_{\mathcal{I}}$ and an *interpretation function* $\cdot^{\mathcal{I}}$ such that

$$\begin{aligned} A^{\mathcal{I}} &\subseteq \Delta_{\mathcal{I}} \quad \text{for all } A \in N_C, \\ r^{\mathcal{I}} &\subseteq \Delta_{\mathcal{I}} \quad \text{for all } r \in N_R, \\ a^{\mathcal{I}} &\in \Delta_{\mathcal{I}} \quad \text{for all } a \in N_I. \end{aligned}$$

In addition, the *unique name assumption* holds: If $a, b \in N_I, a \neq b$, then $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. ◇

We may sometimes omit to explicitly mention the signature of an interpretation. Moreover, we may also talk about interpretations with signature (N_C, N_R) if the set of individuals is not important for the current context.

3.4 Example Let us choose again $N_C = \{\mathsf{Cat}, \mathsf{Mouse}, \mathsf{Animal}\}$, $N_R = \{\mathsf{hunts}\}$ and in addition $N_I = \{\mathsf{Tom}, \mathsf{Jerry}\}$. An interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ over the signature (N_C, N_R, N_I) would then be given by

$$\begin{aligned} \Delta_{\mathcal{I}} &= \{x_1, x_2\}, \\ \cdot^{\mathcal{I}} &= \{(\mathsf{Cat}, \{x_1\}), (\mathsf{Mouse}, \{x_2\}), (\mathsf{Animal}, \{x_1, x_2\})\}, \\ \mathsf{Tom}^{\mathcal{I}} &= x_1, \\ \mathsf{Jerry}^{\mathcal{I}} &= x_2, \end{aligned}$$

where we have specified the interpretation function $\cdot^{\mathcal{I}}$ through its graph. Figure 2 shows the interpretation $\mathcal{I}$ as a directed and labeled graph. ◇

Given an interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$, we can extend the interpretation function $\cdot^{\mathcal{I}}$ to the set of all $\mathcal{EL}^{\bot}$-concept descriptions as follows. Let C be an $\mathcal{EL}^{\bot}$-concept description with signature (N_C, N_R).

- If $C = \top$, then $C^{\mathcal{I}} = \Delta_{\mathcal{I}}$.
- If $C = \bot$, then $C^{\mathcal{I}} = \emptyset$.
- If $C = C_1 \sqcap C_2$, then $C^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$.

- If $C = \exists r.C_1$ with $r \in N_R$, then

$$C^{\mathcal{I}} = \{\, x \in \Delta_{\mathcal{I}} \mid \exists y \in \Delta_{\mathcal{I}} \colon (x,y) \in r^{\mathcal{I}} \text{ and } y \in C_1^{\mathcal{I}} \,\}.$$

Note that if $C \in N_C$, then $C^{\mathcal{I}}$ is already defined.

3.5 Definition If C is an $\mathcal{EL}^{\perp}$-concept description with signature (N_C, N_R) and $\mathcal{I}$ is an interpretation over the signature (N_C, N_R), then $C^{\mathcal{I}}$ is said the be the *extension of* C *in* $\mathcal{I}$. The elements of $C^{\mathcal{I}}$ are said to *satisfy* the concept description C and the elements of $\Delta_{\mathcal{I}} \backslash C^{\mathcal{I}}$ are said to *not satisfy* the concept description C. ♢

The notion of interpretations also allows us to speak of concept descriptions that are *more specific* than other concept descriptions.

3.6 Definition Let C, D be two $\mathcal{EL}^{\perp}$-concept descriptions over the signature (N_C, N_R). Then C is said to be *more specific* then D (or C is *subsumed by* D), written as $C \sqsubseteq D$, if and only if for all interpretations $\mathcal{I}$ with a suitable signature it holds

$$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}.$$

Two $\mathcal{EL}^{\perp}$-concept descriptions C and D are *equivalent*, written as $C \equiv D$, if and only if C is more specific than D and D is more specific than C. In other words,

$$C \equiv D \iff (C \sqsubseteq D) \text{ and } (D \sqsubseteq C). \qquad ♢$$

We are now going to define the notion of an *ontology* more formally. For this, we shall introduce the notions of *assertional axioms* and *terminological axioms*.

We start with the definition of assertional axioms and ABoxes.

3.7 Definition Let N_C, N_R and N_I be pairwise disjoint sets. Then an *assertional axiom over the signature* (N_C, N_R, N_I) is of the form

$$A(a) \quad \text{or} \quad r(a,b)$$

for $A \in N_C, r \in N_R$ and $a, b \in N_I$. An assertional axiom of the form $A(a)$ is called a *concept assertion*, an axiom of the form $r(a,b)$ is called a *role assertion*.

An *ABox* (short for *assertion box*) over the signature (N_C, N_R, N_I) is a finite set of assertional axioms over the signature (N_C, N_R, N_I).

Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R, N_I). Then a concept assertion $A(a)$ *holds* in $\mathcal{I}$ if and only if $a^{\mathcal{I}} \in A^{\mathcal{I}}$. A role assertion $r(a,b)$ *holds* in $\mathcal{I}$ if and only if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. We shall write

$$\mathcal{I} \models A(a) \quad \text{and} \quad \mathcal{I} \models r(a,b)$$

if $A(a)$ and $r(a,b)$ hold in $\mathcal{I}$, respectively. Finally, $\mathcal{I}$ is a *model* of $\mathcal{A}$ if and only if all assertional axioms hold in $\mathcal{I}$. ♢

Again, we may not name the signature explicitly if it is clear from the context or not relevant.

We have already seen some examples for assertional axioms in the beginning of this section. There, the ABox

$$\mathcal{A} = \{\, \mathsf{Cat}(\mathsf{Tom}), \mathsf{Mouse}(\mathsf{Jerry}) \,\}$$

represents the fact that in every model of $\mathcal{A}$, the element associated with the individual Tom must satisfy Cat and likewise the element associated with the individual Jerry must satisfy Mouse. Indeed, the interpretation from Figure 2 is a model of $\mathcal{A}$.

Besides assertional axioms, description logic ontologies allow for terminological axioms. These axioms are able to formulate constraints between different concept descriptions.

3.8 Definition Let N_C, N_R be disjoint sets. Then an *terminological axiom over the signature* (N_C, N_R) is of the form

$$C \sqsubseteq D \quad \text{or} \quad A \equiv D,$$

where $A \in N_C$ and C, D are $\mathcal{EL}^{\perp}$-concept descriptions over the signature (N_C, N_R). Terminological axioms of the form $C \sqsubseteq D$ are called *general concepts inclusions (GCIs)*, axioms of the form $A \equiv D$ are called *concept definitions*. If $C \sqsubseteq D$ is a GCI, then C is called the *subsumee* and D is called the *subsumer* of $C \sqsubseteq D$.

Let N_I be a set disjoint to both N_C and N_R and let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R, N_I). Then a general concept inclusion $C \sqsubseteq D$ *holds* in $\mathcal{I}$ if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. A concept definition $A \equiv C$ *holds* in $\mathcal{I}$ if and only if $A^{\mathcal{I}} = C^{\mathcal{I}}$. An interpretation $\mathcal{I}$ is a *model* of a set $\mathcal{T}$ of terminological axioms if and only if all axioms in $\mathcal{T}$ hold in $\mathcal{I}$. ♢

3.9 Example We can define the notion of a *hunting cat* by the concept definition

$$\mathsf{HuntingCat} \equiv \mathsf{Cat} \sqcap \exists \mathsf{hunts}.\top.$$

A general concept inclusions which expresses that every Cat is also an Animal would be

$$\mathsf{Cat} \sqsubseteq \mathsf{Animal}.$$ ♢

A word of caution is appropriate here. We have introduced the symbol $\sqsubseteq$ for denoting both subsumption and general concept inclusions. This may cause some confusions, but is an established convention in the world of description logics. It may even sometimes be that both meanings of this sign occur together. In those situations we have to exercise some extra care on clearly distinguishing both meanings of $\sqsubseteq$.

The analogue of an ABox for terminological axioms is the notion of a *TBox* (*terminological box*). We shall define two types of TBoxes, namely *cyclic TBoxes* and *general TBoxes*.

3.10 Definition Let N_P, N_R, N_D be three disjoint sets. Let $\mathcal{T}$ be a set of concept definitions of the form $A \equiv C$, where $A \in N_D$ and C is an $\mathcal{EL}$-concept description over the signature $(N_P \cup N_D, N_R)$. $\mathcal{T}$ is called a *cyclic TBox* over the signature (N_P, N_R) if and only if for each concept name $A \in N_P$ there exists exactly one concept definition $A \equiv C \in \mathcal{T}$. The set N_P is then called the set of *primitive concept names* of $\mathcal{T}$ and the set N_D is called the set of *defined concept names* of $\mathcal{T}$. ♢

3.11 Example In the case of Tom and Jerry, it is often not really clear who hunts whom. We can therefore define

$$\mathsf{HuntingCat} \equiv \mathsf{Cat} \sqcap \exists \mathsf{hunts}.\mathsf{HuntingMouse},$$
$$\mathsf{HuntingMouse} \equiv \mathsf{Mouse} \sqcap \exists \mathsf{hunts}.\mathsf{HuntingCat}.$$

Clearly, these definitions depend on each other. The set containing these two concept definitions is a cyclic TBox. Its defined concept names are $\{\, \mathsf{HuntingMouse}, \mathsf{HuntingCat} \,\}$. ♢

Concept definitions are not really necessary if we can use general concept inclusions. To see this, let us recall the definition of a concept definition to hold in an interpretation $\mathcal{I}$. A concept definition $A \equiv C$ holds in $\mathcal{I}$ if and only if $A^{\mathcal{I}} = C^{\mathcal{I}}$. But this is the case if and only if $A^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ and $A^{\mathcal{I}} \supseteq C^{\mathcal{I}}$.

Hence $A \equiv C$ holds in $\mathcal{I}$ if and only if $A \sqsubseteq C$ and $C \sqsubseteq A$ both hold in $\mathcal{I}$. Therefore, general concept inclusions can express concept definitions. Thus, if we are given a cyclic TBox $\mathcal{T}_1$ that contains concept definitions, we can always transform it into a set $\mathcal{T}_2$ containing only general concept inclusions such that the models of $\mathcal{T}_1$ are precisely the models of $\mathcal{T}_2$. In this respect, sets containing only general concept inclusions are a generalization of cyclic TBoxes. We shall call such sets *general TBoxes*.

3.12 Definition Let N_C, N_R be two disjoint sets. A *general TBox* over the signature (N_C, N_R) is then a set of general concept inclusions $C \sqsubseteq D$, where C, D are $\mathcal{EL}^{\bot}$-concept descriptions over the signature (N_C, N_R). ♢

We have just defined the semantics of both cyclic and general TBoxes. If $\mathcal{T}$ is such a TBox, then an interpretation $\mathcal{I}$ is a model of $\mathcal{T}$ if and only if all definitions in $\mathcal{T}$ hold in $\mathcal{I}$. This semantics is called *descriptive semantics*. As we shall see later, there are also other kinds of semantics for TBoxes. As a particular example, we shall introduce *greatest fixpoint semantics* when we discuss the description logic $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$.

We are now able to finally give a formal definition of the notion of an ontology.

3.13 Definition Let N_C, N_R, N_I be pairwise disjoint sets, $\mathcal{A}$ an ABox over the signature (N_C, N_R, N_I) and $\mathcal{T}$ a cyclic or general TBox over the signature (N_C, N_R). Then the pair $(\mathcal{A}, \mathcal{T})$ is called an *ontology*.

An interpretation $\mathcal{I}$ over the signature (N_C, N_R, N_I) is said to be a *model* of the ontology $(\mathcal{A}, \mathcal{T})$ if and only if $\mathcal{I}$ is a model of both $\mathcal{A}$ and $\mathcal{T}$. ♢

We have already argued that description logic ontologies are a useful tool to formalize knowledge. Moreover, we have suggested that the construction of ontologies may be an expensive and time-consuming task. To ease the construction, various attempts have been made to incorporate ideas from formal concept analysis into the world of description logics [4, 5, 10, 16]. In the sequel, we shall present the approach from [10].

In this work, various parallels between the fields of formal concept analysis and description logics are noted. In particular, in both areas certain elements can be *described*. Let $\mathbb{K} = (G, M, I)$ be a formal context. Then an object $g \in G$ can be *described* by a set $A \subseteq M$ of attributes if $x \in A'$. The same is true for an interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$. An element $x \in \Delta_{\mathcal{I}}$ is *described* by a concept description C if $x \in C^{\mathcal{I}}$. Furthermore, in both $\mathbb{K}$ and $\mathcal{I}$ we can obtain for a description A and C the set of objects A' and elements $C^{\mathcal{I}}$ described by it.

However, in $\mathbb{K}$ we can associate for g a *most-specific description* $B := \{ g \}'$. By Proposition 2.5, $g \in B'$, i. e. B describes g. If then $g \in A'$, then $\{ g \} \subseteq A'$, i. e. $\{ g \}'' \subseteq A''' = A'$. But then $B' \subseteq A'$, and hence B describes the fewest objects of all sets $A \subseteq M$ that describe g. In other words, B describes g in the most specific way.

An analogous notion of a *most-specific concept-description* with respect to an interpretation $\mathcal{I}$ has been introduced in [10] as *model-based most-specific concept description*.

3.14 Definition Let $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a interpretation over a signature (N_C, N_R, N_I) and let $X \subseteq \Delta_{\mathcal{I}}$. Then a *model-based most-specific concept description* for X over $\mathcal{I}$ is a concept description C over the signature (N_C, N_R) such that

- $X \subseteq C^{\mathcal{I}}$ and
- for all concept descriptions D with $X \subseteq D^{\mathcal{I}}$ it holds $C \sqsubseteq D$. ♢

Intuitively speaking, a model-based most-specific concept description for $X \subseteq \Delta_{\mathcal{I}}$ is a most-specific concept description that describes all elements in X.

Model-based most-specific concept descriptions may not exist. We shall see in the next example an interpretation $\mathcal{I}$ where some elements do not have model-based most-specific concept descriptions

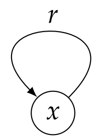

Figure 3: An interpretation where $\{ x \}$ has no model-based most-specific concept description in $\mathcal{EL}^{\perp}$.

in $\mathcal{EL}^{\perp}$. To compensate for this we shall introduce the description logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ that allows for cyclic concept descriptions. In this logic, model-based most-specific concept descriptions always exist.

The following example also occurs in a minor variation in [10].

3.15 Example Let $N_C = \varnothing$ and $N_R = \{ r \}$. We consider the interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ with $\Delta_{\mathcal{I}} = \{ x \}$ and $r^{\mathcal{I}} = \{ (x, x) \}$. The interpretation depicted as a graph is shown in Figure 3.

Now suppose that C is an $\mathcal{EL}^{\perp}$-concept description that is at the same time a model-based most-specific concept description for $X = \{ x \}$ over $\mathcal{I}$. Because $N_C = \varnothing$ and $N_R = \{ r \}$, C is equivalent to one of the concept descriptions

$$\top, \exists r.\top, \exists r.\exists r.\top, \ldots,$$

i. e.

$$C \equiv \underbrace{\exists r. \ldots \exists r.}_{n \text{ times}} \top$$

for some $n \in \mathbb{N}$. Then define

$$D := \underbrace{\exists r. \ldots \exists r.}_{n+1 \text{ times}} \top.$$

Then $D^{\mathcal{I}} = \{ x \}$ and $D \sqsubseteq C, D \not\equiv C$, contradicting the fact that C is a model-based most-specific concept description of X over $\mathcal{I}$. ♢

On the other hand, if model-based most-specific concept descriptions exist, they are necessarily equivalent. Therefore, if X is a set of elements of an interpretation $\mathcal{I}$, we can denote the model-based most-specific concept description of X over $\mathcal{I}$ by the special name $X^{\mathcal{I}}$. This notation has been used to stress the similarity to the derivation operators from formal concept analysis.

3.2 The Description Logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$

As it was shown in Example 3.15, model-based most-specific concept descriptions need not necessarily exist. However, it has been shown in [4, 10] that one can extend the description logic $\mathcal{EL}^{\perp}$ to the description logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ which always has model-based most-specific concept descriptions.

3.16 Definition Let N_P, N_R be two disjoint sets and let $\mathcal{T}$ be a cyclic TBox over the signature (N_P, N_R). A concept definition $A \equiv C \in \mathcal{T}$ is said to be *normalized*, if C is of the form

$$C = B_1 \sqcap \ldots \sqcap B_m \sqcap \exists r_1.A_1 \sqcap \ldots \sqcap \exists r_n.A_n$$

where $m, n \in \mathbb{N}$, $B_1, \ldots, B_m \in N_P$ and $A_1, \ldots, A_n \in N_D$. If $n = m = 0$, then $C = \top$. We call $\mathcal{T}$ normalized if and only if it contains only normalized concept definitions.

An $\mathcal{EL}_{\mathrm{gfp}}$-concept description over the signature (N_P, N_R) is of the form

$$C = (A, \mathcal{T})$$

where $\mathcal{T}$ is a normalized TBox over the signature (N_P, N_R) and A is a defined concept name of $\mathcal{T}$. An $\mathcal{EL}^{\bot}_{\text{gfp}}$-concept description is either $\bot$ or an $\mathcal{EL}_{\text{gfp}}$-concept description. ♢

3.17 Example Let us reconsider the TBox from Example 3.11, i. e.

$$\begin{aligned}\mathcal{T} := \{\, &\mathsf{HuntingCat} \equiv \mathsf{Cat} \sqcap \exists\mathsf{hunts}.\mathsf{HuntingMouse},\\ &\mathsf{HuntingMouse} \equiv \mathsf{Mouse} \sqcap \exists\mathsf{hunts}.\mathsf{HuntingCat}\,\}.\end{aligned}$$

Then $\mathcal{T}$ is a normalized cyclic TBox and the pair

$$(\mathsf{HuntingMouse}, \mathcal{T})$$

is a valid $\mathcal{EL}^{\bot}_{\text{gfp}}$-concept description. ♢

We have already defined the notion of $\mathcal{EL}^{\bot}$-GCIs. Of course, this definition can be easily modified to yield the notion of $\mathcal{EL}^{\bot}_{\text{gfp}}$-GCIs. These are just expressions of the form $C \sqsubseteq D$, where C and D are $\mathcal{EL}^{\bot}_{\text{gfp}}$-concept descriptions.

We shall sometimes omit the logic and call an $\mathcal{EL}^{\bot}_{\text{gfp}}$-concept description just a concept description and likewise shall call an $\mathcal{EL}^{\bot}_{\text{gfp}}$-GCIs just a GCI.

As we have defined the syntax of $\mathcal{EL}^{\bot}_{\text{gfp}}$, the natural next step is to define the semantics of $\mathcal{EL}^{\bot}_{\text{gfp}}$. This, however, is not as straight forward as in the case of $\mathcal{EL}^{\bot}$, as we have to deal with circular concept descriptions. As we shall see shortly, semantics can be defined using *fixpoint semantics*. This has been done in [3, 13].

Let C be an $\mathcal{EL}^{\bot}_{\text{gfp}}$-concept description over the signature (N_C, N_R) and let $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation over the same signature. If $C = \bot$, then certainly $C^{\mathcal{I}} = \varnothing$. Hence let $C = (A, \mathcal{T})$ and let N_D be the set of defined concept names of $\mathcal{T}$. Then $A \in N_D$.

The idea to define the set of elements of $\mathcal{I}$ that satisfy C now works as follows: Let us suppose that we can extend the interpretation function $\cdot^{\mathcal{I}}$ to a function $\cdot^{\mathcal{I}_1}$ that is also defined on the set N_D of defined concept names such that

$$A^{\mathcal{I}_1} = C^{\mathcal{I}_1}$$

holds for all concept definitions $A \equiv C \in \mathcal{T}$. Then $\mathcal{I}_1$ is a model of $\mathcal{T}$ and we could define

$$C^{\mathcal{I}_1} = (A, \mathcal{T})^{\mathcal{I}_1} := A^{\mathcal{I}_1}.$$

Intuitively speaking, we extend $\cdot^{\mathcal{I}}$ to all defined concept descriptions of $\mathcal{T}$ such that this extension is a model of $\mathcal{T}$. Then we take the interpretation of the symbol A and use it to define the extension of the whole description C.

To use this approach to actually define an extension of C we have to address two issues. The first is to guarantee that such extensions of $\cdot^{\mathcal{I}}$ actually exist. Secondly, if such extensions exist, we have to choose one of them to actually define the extension of C. We shall address these two issues in what follows.

3.18 Definition Let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R, N_I) and let $\mathcal{T}$ be a TBox over (N_C, N_R) with defined symbols N_D. Then an interpretation $\mathcal{J}$ over the signature $(N_C \cup N_D, N_R, N_I)$ is an *extension* of the interpretation $\mathcal{I}$ if and only if

- $\forall A \in N_C \colon A^{\mathcal{I}} = A^{\mathcal{J}}$,

- $\forall r \in N_R \colon r^{\mathcal{I}} = r^{\mathcal{J}}$ and
- $\forall a \in N_I \colon a^{\mathcal{I}} = a^{\mathcal{J}}$.

We shall denote with $\mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$ the set of all extensions of $\mathcal{I}$. ♢

We can define an order relation $\preceq$ on $\mathrm{Ext}(\mathcal{I})$ by

$$\mathcal{I}_1 \preceq \mathcal{I}_2 \iff A^{\mathcal{I}_1} \subseteq A^{\mathcal{I}_2} \quad \text{for all } A \in N_D$$

for $\mathcal{I}_1, \mathcal{I}_2 \in \mathrm{Ext}(\mathcal{I})$. It is clear that $(\mathrm{Ext}_{\mathcal{T}}(\mathcal{I}), \preceq)$ is an ordered set.

3.19 Proposition *For each interpretation $\mathcal{I}$ over the signature (N_C, N_R, N_I) and TBox $\mathcal{T}$ over the signature (N_C, N_R), the ordered set*

$$(\mathrm{Ext}_{\mathcal{T}}(\mathcal{I}), \preceq)$$

is a complete lattice.

Not all extensions $\mathcal{J} \in \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$ are actually models of $\mathcal{T}$. These are only those that satisfy

$$A^{\mathcal{J}} = C^{\mathcal{J}}$$

for all $(A \equiv C) \in \mathcal{T}$.

We can view this fact from another perspective. Let us define a mapping $f \colon \mathrm{Ext}_{\mathcal{T}}(\mathcal{I}) \longrightarrow \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$ by

$$A^{f(\mathcal{J})} := C^{\mathcal{J}}$$

for all $(A \equiv C) \in \mathcal{T}$ and $\mathcal{J} \in \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$. Since for each $A \in N_D$, there is exactly one concept definition $(A \equiv C) \in \mathcal{T}$, the function f is well-defined. Furthermore, it is sufficient to define $f(\mathcal{J})$ only on defined concept names. Thus $f(\mathcal{J}) \in \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$. Moreover, this mapping is monotone, i. e.

$$\mathcal{I}_1 \preceq \mathcal{I}_2 \implies f(\mathcal{I}_1) \preceq f(\mathcal{I}_2)$$

for all $\mathcal{I}_1, \mathcal{I}_2 \in \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$. This is easy to see if one recalls that the concept description C is normalized, i. e.

$$C = B_1 \sqcap \ldots \sqcap B_m \sqcap \exists r_1.A_1 \sqcap \ldots \sqcap \exists r_n.A_n$$

where $B_1, \ldots, B_m \in N_C$ and $A_1, \ldots, A_n \in N_D$.

We can now see that the extensions of $\mathcal{I}$ that are models of $\mathcal{T}$ are actually *fixpoints* of f. This is because $\mathcal{J} \in \mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$ is a model of $\mathcal{T}$ if and only if

$$A^{\mathcal{J}} = C^{\mathcal{J}} \quad \text{for all } A \equiv C \in \mathcal{T}.$$

But this means that

$$A^{f(\mathcal{J})} = C^{\mathcal{J}} = A^{\mathcal{J}},$$

i. e. $f(\mathcal{J}) = \mathcal{J}$. Hence to show that there exist extensions of $\mathcal{I}$ that are models of $\mathcal{T}$ it is sufficient to show that f has fixpoints. To do this, we use the fact that f is monotone and the following, well-known theorem by Tarski [20].

3.20 Theorem *Let $(L, \leqslant)$ be a complete lattice and let $h\colon L \longrightarrow L$ be a monotone mapping on $(L, \leqslant)$, i. e.*

$$x \leqslant y \implies h(x) \leqslant h(y)$$

holds for all $x, y \leqslant L$. Then the set

$$F := \{ z \in L \mid h(z) = z \}$$

is such that $(F, \leqslant)$ is a complete sublattice of $(L, \leqslant)$. In particular, $F \neq \emptyset$ and there exists a least and greatest fixpoint of h.

As a corollary we obtain the fact that the mapping f has fixpoints in $\mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$ and that there exists a greatest fixpoint of f in $\mathrm{Ext}_{\mathcal{T}}(\mathcal{I})$. We call this fixpoint the *greatest fixpoint model (gfp-model)* of $\mathcal{T}$ in $\mathcal{I}$. Having this, we are finally able to define the extension of the concept description C.

3.21 Definition Let C be an $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept description over a signature (N_C, N_R) and let $\mathcal{I}$ be an interpretation over the signature (N_C, N_R, N_I). Then

$$C^{\mathcal{I}} := \begin{cases} \emptyset & \text{if } C = \perp \\ A^{\mathcal{J}} & \text{if } C = (A, \mathcal{T}) \text{ and } \mathcal{J} \text{ is the gfp-model of } \mathcal{T} \text{ in } \mathcal{I}. \end{cases} \qquad \diamondsuit$$

The main result about $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ is now the following theorem from [4, 10].

3.22 Theorem (Theorem 4.7 of [10]) *Let $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation and $X \subseteq \Delta_{\mathcal{I}}$. Then there exists a model-based most-specific $\mathcal{EL}^{\perp}_{gfp}$-concept description of X over $\mathcal{I}$.*

Now that we can guarantee the existence of model-based most-specific concept descriptions we can consider some first properties. The following result can also be found in [4].

3.23 Lemma (Lemma 4.1 of [10]) *Let $\mathcal{I}$ be a finite interpretation. Then for each $\mathcal{EL}^{\perp}_{gfp}$-concept description D and every $X \subseteq \Delta_{\mathcal{I}}$, it holds*

$$X \subseteq D^{\mathcal{I}} \iff X^{\mathcal{I}} \sqsubseteq D.$$

Proof Suppose $X \subseteq D^{\mathcal{I}}$. Then $X^{\mathcal{I}} \sqsubseteq D$ holds by the definition of model-based most-specific concept descriptions (Definition 3.14). This shows the direction from left to right.

Suppose conversely that $X^{\mathcal{I}} \sqsubseteq D$. Then $X^{\mathcal{I}}$ is a concept description that is satisfied by all elements of X, therefore

$$X \subseteq (X^{\mathcal{I}})^{\mathcal{I}} \subseteq D^{\mathcal{I}},$$

as $X^{\mathcal{I}} \sqsubseteq D$ implies $(X^{\mathcal{I}})^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. This shows the converse direction. □

This lemma may remind one of the definition of a Galois connection, however the relation $\sqsubseteq$ is not an order relation on the set of all model-based most-specific concept descriptions. This is because model-based most-specific concept descriptions are only unique up to equivalence.

Yet, most of the properties of a Galois connection are still valid. More precisely, if $\mathcal{I}$ is a finite interpretation, C, D are concept descriptions and $X, Y \subseteq \Delta_{\mathcal{I}}$, then the following statements are true.

i. $X \subseteq Y \implies X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}$,

ii. $C \sqsubseteq D \implies C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,

iii. $X \subseteq (X^{\mathcal{I}})^{\mathcal{I}}$,

iv. $(C^{\mathcal{I}})^{\mathcal{I}} \sqsubseteq C$,

v. $X^{\mathcal{I}} \equiv ((X^{\mathcal{I}})^{\mathcal{I}})^{\mathcal{I}}$,

vi. $C^{\mathcal{I}} = ((C^{\mathcal{I}})^{\mathcal{I}})^{\mathcal{I}}$.

They can be proven in the same way as for any Galois connection. We shall often write $X^{\mathcal{I}\mathcal{I}}$ instead of $(X^{\mathcal{I}})^{\mathcal{I}}$.

Another property that was already claimed is that $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ can be considered as an extension of the description logic $\mathcal{EL}^{\perp}$. This may not be obvious at a first glance, since the definition of $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept descriptions is quite different from the one of $\mathcal{EL}^{\perp}$-concept descriptions. Still, $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ can be understood as an extension of $\mathcal{EL}^{\perp}$. To see this we shall first define conjunction and existential restriction for $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept descriptions.

Let C, D be two $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept descriptions over the signature (N_C, N_R). If $C = \perp$, then $C \sqcap D := \perp$ and $\exists r.C := \perp$. Likewise for $D = \perp$. Hence we may assume that both C, D are not the $\perp$ concept description. Then $C = (A_C, \mathcal{T}_C), D = (A_D, \mathcal{T}_D)$ and we can assume that the defined concept names of $\mathcal{T}_C$ and $\mathcal{T}_D$ are disjoint. Then let us define

$$C \sqcap D := (A, \mathcal{T}_C \cup \mathcal{T}_D \cup \{ A \equiv A_C \sqcap A_D \}),$$

where A is a new name. Furthermore, if $r \in N_R$, then

$$\exists r.C := (A, \mathcal{T}_C \cup \{ A \equiv \exists r.A_C \})$$

where again A is a new name. These definitions preserve the semantics, i. e. for each interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ it holds

$$\begin{aligned} (C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}}, \\ (\exists r.C)^{\mathcal{I}} &= \{ x \in \Delta_{\mathcal{I}} \mid \exists y \in \Delta_{\mathcal{I}} \colon (x, y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}} \}. \end{aligned}$$

We can use these definitions to see that $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ can indeed be regarded as an extension of $\mathcal{EL}^{\perp}$. For this we assign for the $\mathcal{EL}^{\perp}$-concept description $\top$ the $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept description $(A, \{ A \equiv \top \})$. Furthermore, if B is a concept name, then it is equivalent to the $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept description $(A, \{ A \equiv B \})$. Using the definitions for conjunction and existential restriction for $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept descriptions, we can inductively assign for each $\mathcal{EL}^{\perp}$-concept description an equivalent $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept description. As these constructors preserve the semantics, $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ can be seen as an extension of $\mathcal{EL}^{\perp}$.

3.3 Bases for GCIs of Interpretations

In the case of formal contexts, we were able to extract bases of implications form them. As we view GCIs as the description logic analogue of implications, we want to do the same for GCIs and finite interpretations.

In [10], the algorithm for computing the canonical base has been generalized to the description logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$. This generalized algorithm is then able to compute *bases of valid GCIs* of a finite interpretation $\mathcal{I}$. In this short subsection we want to introduce the notion of a base and some related definitions.

3.24 Definition Let $\mathcal{I}$ be a finite interpretation over the signature (N_C, N_R, N_I). The set of valid GCIs of $\mathcal{I}$ that consist of $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept descriptions is denoted by $\mathrm{Th}(\mathcal{I})$. ♢

One of the main results of [10] was to find a finite set of valid GCIs of $\mathcal{I}$ such that every valid GCI of $\mathcal{I}$ was already entail by this finite set. These finite sets are then called *bases* of $\mathcal{I}$.

We are going to introduce the notion of base in a more general setting, namely for arbitrary sets of GCIs.

3.25 Definition Let $\mathcal{C}$ be a set of GCIs. Let $\mathcal{D}$ be a set of GCIs.

i. $\mathcal{D}$ is said to be *sound for* $\mathcal{C}$ if and only if $\mathcal{C} \models \mathcal{D}$, i. e. every GCI in $\mathcal{D}$ is entailed by $\mathcal{C}$;

ii. $\mathcal{D}$ is said to be *complete for* $\mathcal{C}$ if and only if $\mathcal{D} \models \mathcal{C}$, i. e. every GCI in $\mathcal{C}$ is entailed by $\mathcal{D}$;

iii. $\mathcal{D}$ is said to be a *base for* $\mathcal{C}$ if and only if $\mathcal{D}$ is both sound and complete for $\mathcal{C}$.

If $\mathcal{D}$ is a base of $\mathcal{C}$, then $\mathcal{D}$ is said to be a *non-redundant base* of $\mathcal{C}$ if and only if no proper subset of $\mathcal{D}$ is a base of $\mathcal{C}$. $\diamondsuit$

It is clear that if $\mathcal{D}$ is a base of $\mathcal{C}$, then the set of GCIs entailed by $\mathcal{D}$ and $\mathcal{C}$ are the same.

3.26 Definition Let $\mathcal{I}$ be a finite interpretation. Then a set $\mathcal{B}$ of GCIs is said to be a *base for* $\mathcal{I}$ if and only if $\mathcal{B}$ is a base for $\operatorname{Th}(\mathcal{I})$. $\diamondsuit$

Equivalently, $\mathcal{B}$ is a base for $\mathcal{I}$ if and only if it contains only valid GCIs of $\mathcal{I}$ and every valid GCI of $\mathcal{I}$ is already entailed from $\mathcal{B}$.

On of the main results of [10] is to prove the existence of finite bases for finite interpretations. The following result shows one of these bases.

3.27 Theorem (Theorem 5.10 of [10]) *Let $\mathcal{I}$ be a finite interpretation. Then the set*

$$\mathcal{B}_2 := \{ \textstyle\bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid U \subseteq M_{\mathcal{I}} \}$$

is a finite base for $\mathcal{I}$.

4 A Contextual Representation of All Model-Based Most-Specific Concepts of a Finite Interpretation

We have already seen some parallels between the description logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ and formal concept analysis. In [10], *induced contexts* are used to further investigate the connection between the operator $\cdot^{\mathcal{I}}$ in $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ and the derivations operators in FCA. This approach has also been used before, for example in [6, 14, 16]. However, [10] uses induced contexts to provide a general framework to combine formal concept analysis with description logics. Using this framework in the particular case of $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$, [10] was able to obtain a formal context $\mathbb{K}_{\mathcal{I}}$ that describes all model-based most-specific concept descriptions of a finite interpretation $\mathcal{I}$. Although not mentioned explicitly in [10], with this context it is easy to see that the lattice of intents of $\mathbb{K}_{\mathcal{I}}$ is dually order-isomorphic to the lattice of equivalence classes of the model-based most-specific concept descriptions of $\mathcal{I}$.

It is the main purpose of this section to introduce this result in a way suitable for this work. For this, we shall start with introducing the notions of *induced contexts* and *projections*. Following this, we shall present the formal context $\mathbb{K}_{\mathcal{I}}$ as it has been introduced in [10]. The final goal is then to obtain an order-isomorphism between that lattice $(\operatorname{Int}(\mathbb{K}_{\mathcal{I}}), \subseteq)$ and the set of all model-based most-specific concept descriptions up to equivalence, ordered by $\sqsupseteq$. Here, $\sqsupseteq$ denotes the reverse subsumption relation.

4.1 Induced Contexts and Projections

In [10], one of the main definition used to investigate description logics from a formal concept analysis perspective is the definition of *induced contexts*. Given a finite interpretation $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ and a set M of concept descriptions, we can say that an element $x \in \Delta_{\mathcal{I}}$ *has* a concept description C as attribute if and only if $x \in C^{\mathcal{I}}$, i. e. if x satisfies C.

4.1 Definition Let M be a set of concept descriptions and let $\mathcal{I}$ be a finite interpretation. Then the formal context *induced by* M *and* $\mathcal{I}$ is defined to be $(\Delta_{\mathcal{I}}, M, J)$, where

$$(x, C) \in J \iff x \in C^{\mathcal{I}}. \qquad \diamondsuit$$

Indeed, induced contexts already allow us to consider extensions of certain concept descriptions as extents of a suitable formal context. Under certain circumstances, we may even be able to represent model-based most-specific concept descriptions as intents of an induced context.

To make this more precise, we need to be able to construct from a set U of concept descriptions a new concept description. As sets of attributes are normally understood as describing objects that have *all* attributes from this set, it is natural to assign to U the concept description that is obtained by a conjunction of all elements of U.

4.2 Definition Let M be a set of concept descriptions and let $U \subseteq M$. Then define

$$\bigsqcap U := \begin{cases} \bigsqcap_{D \in U} D & \text{if } U \neq \varnothing \\ \top & \text{otherwise} \end{cases}$$

as the *concept description defined by* U.

A concept description C is said to be *expressible in terms of* M if there exists $U \subseteq M$ such that

$$C \equiv \bigsqcap U. \qquad \diamondsuit$$

Dually, we can construct from a given concept description C a set of concept descriptions from M that is, in a certain sense, the best *approximation* of C in terms of M.

4.3 Definition Let M be a set of concept descriptions and let C be another concept description. Then the set

$$\mathrm{pr}_M(C) := \{ D \in M \mid C \sqsubseteq D \}$$

is said to be the *projection of* C *onto* M. $\diamondsuit$

Projections indeed capture some notion of approximation. More precisely, let us consider the set of all concept descriptions that we can obtain by conjunctions of elements of M. Under these concept descriptions we search for a concept description $U \subseteq M$ that is $\sqsubseteq$-minimal with $C \sqsubseteq \bigsqcap U$. In other words, we look for a best *upper approximation* within the set of concept descriptions defined by subsets of M. With the notion of projections it is easy to see that

$$U = \mathrm{pr}_M(C).$$

As an interesting matter of fact, the mappings $U \longmapsto \bigsqcap U$ and $C \longmapsto \mathrm{pr}_M(C)$ satisfy the main property of a Galois connection. But note that because of $\sqsubseteq$ not being an order relation on the set of all concept descriptions, these mappings actually cannot form a Galois connection.

4.4 Lemma *Let M be a set of concept descriptions. Then for each $U \subseteq M$ and for each concept description C it is true that*

$$C \sqsubseteq \bigsqcap U \iff U \subseteq \operatorname{pr}_M(C).$$

Proof Let us first show the direction from left to right. From $C \sqsubseteq \bigsqcap U$ we can conclude $\operatorname{pr}_M(\bigsqcap U) \subseteq \operatorname{pr}_M(C)$, since every concept description $D \in M$ satisfying $\bigsqcap U \sqsubseteq D$ also satisfies $C \sqsubseteq D$. Furthermore, for each $F \in U$ we have $\bigsqcap U \sqsubseteq F$, therefore $U \subseteq \operatorname{pr}_M(\bigsqcap U)$ and hence

$$U \subseteq \operatorname{pr}_M(\bigsqcap U) \subseteq \operatorname{pr}_M(C)$$

as desired.

For the other direction let us suppose that $U \subseteq \operatorname{pr}_M(C)$. Then $\bigsqcap U \sqsupseteq \bigsqcap \operatorname{pr}_M(C)$. Now since each $D \in \operatorname{pr}_M(C)$ satisfies $D \sqsupseteq C$, $\bigsqcap \operatorname{pr}_M(C) \sqsupseteq C$ holds as well. Therefore,

$$C \sqsubseteq \bigsqcap \operatorname{pr}_M(C) \sqsubseteq \bigsqcap U$$

as desired. □

Although the mappings $U \longmapsto \bigsqcap U$ and $C \longmapsto \operatorname{pr}_M(C)$ do not form a Galois connection they still possess some similar properties. For example, the following properties hold:

i. $C \sqsubseteq D \implies \operatorname{pr}_M(D) \subseteq \operatorname{pr}_M(C)$,

ii. $U \subseteq V \implies \bigsqcap V \sqsubseteq \bigsqcap U$,

iii. $C \sqsubseteq \bigsqcap \operatorname{pr}_M(C)$ and

iv. $U \subseteq \operatorname{pr}_M(\bigsqcap U)$

for all concept descriptions C, D and sets $U, V \subseteq M$. The proofs for these claims have already been given along the lines of the proof of the previous lemma.

Our main motivation for the following considerations is now to find a contextual representation of the model-based most-specific concept descriptions of a finite interpretation $\mathcal{I}$. For this, we need some more preliminary results.

The first result gives a simple characterization of when a concept description C is expressible in terms of M.

4.5 Proposition ([10]) *Let M be a set of $\mathcal{EL}^{\bot}_{gfp}$-concept descriptions and let C be an $\mathcal{EL}^{\bot}_{gfp}$-concept description. Then C is expressible in terms of M if and only if*

$$C \equiv \bigsqcap \operatorname{pr}_M(C).$$

Proof If $C \equiv \bigsqcap \operatorname{pr}_M(C)$, then clearly C is expressible in terms of M. Conversely, let $N \subseteq M$ such that $C \equiv \bigsqcap N$. Then $C \sqsubseteq D$ for each $D \in N$ and hence

$$N \subseteq \operatorname{pr}_M(C),$$

which implies $C \sqsupseteq \bigsqcap \operatorname{pr}_M(C)$. On the other hand, $C \sqsubseteq \bigsqcap \operatorname{pr}_M(C)$ by Lemma 4.4 and hence $C \equiv \bigsqcap \operatorname{pr}_M(C)$ follows as required. □

The following two propositions give some first results on how the mappings $U \longmapsto \bigsqcap U, C \longmapsto \operatorname{pr}_M(C)$ connect the description logic $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ and formal concept analysis.

4.6 Proposition (Lemma 4.11 and Lemma 4.12 from [10]) *Let $\mathcal{I}$ be a finite interpretation and M a set of concept descriptions. Let C be an concept description expressible in terms of M. Then*

$$C^{\mathcal{I}} = \operatorname{pr}_M(C)'$$

where the derivation are computed within the induced context of $\mathcal{I}$ and M. Furthermore, every set $O \subseteq \Delta_{\mathcal{I}}$ satisfies

$$O' = \operatorname{pr}_M(O^{\mathcal{I}}).$$

Proof Since C is expressible in terms of M, $C \equiv \bigsqcap \operatorname{pr}_M(C)$ by Proposition 4.5. Therefore

$$\begin{aligned} x \in C^{\mathcal{I}} &\iff x \in (\bigsqcap \operatorname{pr}_M(C))^{\mathcal{I}} \\ &\iff \forall D \in \operatorname{pr}_M(C) : x \in D^{\mathcal{I}} \\ &\iff x \in \operatorname{pr}_M(C)' \end{aligned}$$

as $\operatorname{pr}_M(C)' = \{ x \in \Delta_{\mathcal{I}} \mid \forall D \in \operatorname{pr}_M(C) : x \in D^{\mathcal{I}} \}$.

For the second claim we observe

$$\begin{aligned} D \in O' &\iff \forall g \in O \colon g \in D^{\mathcal{I}} \\ &\iff O \subseteq D^{\mathcal{I}} \\ &\iff O^{\mathcal{I}} \sqsubseteq D \\ &\iff D \in \operatorname{pr}_M(O^{\mathcal{I}}), \end{aligned}$$

where $O \subseteq D^{\mathcal{I}} \iff O^{\mathcal{I}} \sqsubseteq D$ holds due to Lemma 3.23. □

4.7 Proposition (Lemma 4.10 and 4.11 from [10]) *Let $\mathcal{I}$ be a finite interpretation and let M be a set of concept descriptions. Let $\mathbb{K}$ be the formal context induced by M and $\mathcal{I}$. Then each $B \subseteq M$ satisfies*

$$B' = (\bigsqcap B)^{\mathcal{I}}.$$

Let $A \subseteq \Delta_{\mathcal{I}}$. If $A^{\mathcal{I}}$ is expressible in terms of M, then

$$\bigsqcap A' \equiv A^{\mathcal{I}}.$$

Proof Remember that an object $g \in G(\mathbb{K}_{\mathcal{I}})$ has an attribute $m \in M(\mathbb{K}_{\mathcal{I}})$ if and only if $g \in m^{\mathcal{I}}$. Hence

$$g \in B' \iff \forall m \in B : g \in m^{\mathcal{I}} \iff g \in (\bigsqcap B)^{\mathcal{I}}.$$

Let $A \subseteq \Delta_{\mathcal{I}}$ such that $A^{\mathcal{I}}$ is expressible in terms of M. By Proposition 4.5,

$$A^{\mathcal{I}} \equiv \bigsqcap \operatorname{pr}_M(A^{\mathcal{I}}).$$

By Proposition 4.6, $\operatorname{pr}_M(A^{\mathcal{I}}) = A'$ and hence the claim follows. □

In particular, if the set B in the previous proposition has the form $B = \{ D \}$, then $\bigsqcap B = D$ and hence $\{ D \}' = D^{\mathcal{I}}$ for each $D \in M$.

We now give the construction of the formal context $\mathbb{K}_{\mathcal{I}}$ from [10]. This formal context will finally allow us to understand model-based most-specific concept descriptions as intents of $\mathbb{K}_{\mathcal{I}}$.

4.8 Definition Let $\mathcal{I}$ be a finite interpretation, N_C be a set of concept names and N_R be a set of role names. Then define

$$M_{\mathcal{I}} := \{\bot\} \cup N_C \cup \{\exists r.X^{\mathcal{I}\mathcal{I}} \mid X \subseteq \Delta_{\mathcal{I}}, r \in N_R\}. \qquad \diamondsuit$$

4.9 Theorem (Lemma 5.9 from [10]) *Let $\mathcal{I}$ be a finite interpretation and let C be an concept description. Then $C^{\mathcal{I}\mathcal{I}}$ is expressible in terms of $M_{\mathcal{I}}$.*

4.10 Definition Let $\mathcal{I}$ be a finite interpretation. Then the formal context $\mathbb{K}_{\mathcal{I}}$ is the formal context induced by $M_{\mathcal{I}}$ and $\mathcal{I}$. $\diamondsuit$

A first simple result shows that the extents of $\mathbb{K}_{\mathcal{I}}$ correspond to sets of elements of $\mathcal{I}$ that are of the form $X^{\mathcal{I}\mathcal{I}}$.

4.11 Lemma *Let $\mathcal{I}$ be a finite interpretation and let $X \subseteq \Delta_{\mathcal{I}}$. Then $X^{\mathcal{I}\mathcal{I}} = X''$, where the derivations are computed in $\mathbb{K}_{\mathcal{I}}$.*

Proof By Theorem 4.9, $X^{\mathcal{I}}$ is expressible in terms of $M_{\mathcal{I}}$, hence by Proposition 4.5

$$X^{\mathcal{I}} \equiv \bigsqcap \operatorname{pr}_{M_{\mathcal{I}}}(X^{\mathcal{I}}).$$

This implies

$$\begin{aligned} X^{\mathcal{I}\mathcal{I}} &= \left(\bigsqcap \operatorname{pr}_{M_{\mathcal{I}}}(X^{\mathcal{I}})\right)^{\mathcal{I}} \\ &= \operatorname{pr}_{M_{\mathcal{I}}}(X^{\mathcal{I}})' \\ &= X'' \end{aligned}$$

by Proposition 4.7. □

4.2 Model-Based Most-Specific Concept Descriptions as Intents of a Formal Context

Having defined the formal context $\mathbb{K}_{\mathcal{I}}$, we are now going to show that this formal context indeed allows us to view model-based most-specific concept descriptions as intents of a formal context. Indeed, we have already seen that all model-based most-specific concept descriptions are expressible in terms of $M_{\mathcal{I}}$, the set of attributes of $\mathbb{K}_{\mathcal{I}}$. It is therefore not surprising that the lattice of intents of $\mathbb{K}_{\mathcal{I}}$ and the equivalence classes of model-based most-specific concept descriptions ordered by $\sqsupseteq$ are order-isomorphic.

Before we can define the corresponding Theorem 4.13 we need to show one more auxiliary result.

4.12 Proposition *Let $\mathcal{I}$ be a finite interpretation and let $X \subseteq M_{\mathcal{I}}$. Then*

$$X \subseteq \operatorname{pr}_{M_{\mathcal{I}}}(\bigsqcap X) \subseteq X'',$$

where the derivation is computed in $\mathbb{K}_{\mathcal{I}}$.

Proof By Lemma 4.4, $X \subseteq \operatorname{pr}_{M_{\mathcal{I}}}(\bigsqcap X)$ holds. Now

$$\begin{aligned} D \in \operatorname{pr}_{M_{\mathcal{I}}}(\bigsqcap X) &\iff \bigsqcap X \sqsubseteq D \\ &\implies (\bigsqcap X)^{\mathcal{I}} \subseteq D^{\mathcal{I}} \\ &\iff X' \subseteq \{D\}' \\ &\iff X'' \supseteq \{D\}'' \ni D \\ &\implies D \in X'' \end{aligned}$$

as required. ◻

We are now ready to formulate the main theorem of this section. In this theorem we shall describe the desired contextual representation of model-based most-specific concept descriptions as intents of $\mathbb{K}_{\mathcal{I}}$. However, before we do so we have to deal with a technical detail. This is because model-based most-specific concept descriptions are only unique up to equivalence. In particular, $\sqsubseteq$ is in general not an order relation on the set of all model-based most-specific concept descriptions. To overcome this we use the standard trick of considering classes of equivalent concept descriptions instead.

Let M be a set of concept descriptions. Then let us define

$$M/\equiv \; := \{ [X] \mid X \in M \}$$

where

$$[X] := \{ Y \in M \mid X \equiv Y \}.$$

Furthermore, for $X, Y \in M$ we set

$$[X] \sqsubseteq [Y] \iff X \sqsubseteq Y.$$

Note that this is well-defined because if $\hat{X} \in [X], \hat{Y} \in [Y]$, then $\hat{X} \equiv X, \hat{Y} \equiv Y$ and hence $X \sqsubseteq Y \iff \hat{X} \sqsubseteq \hat{Y}$. With this definition it is easy to see that $(M/\equiv, \sqsubseteq)$ is an ordered set.

4.13 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $\mathcal{M}$ be the set of all model-based most-specific concept descriptions of $\mathcal{I}$. Then the mappings*

$$\bigsqcap : \mathfrak{P}(M_{\mathcal{I}}) \longrightarrow \mathcal{M} \qquad \text{and} \qquad \mathrm{pr}_{M_{\mathcal{I}}} : \mathcal{M} \longrightarrow \mathfrak{P}(M_{\mathcal{I}})$$

describe an order-isomorphism between the ordered sets $(\mathfrak{P}(M_{\mathcal{I}}), \subseteq)$ and $(\mathcal{M}/\equiv, \sqsupseteq)$ via

$$\begin{array}{rccc} \varphi : & \mathfrak{P}(M_{\mathcal{I}}) & \longrightarrow & \mathcal{M}/\equiv \\ & N & \longmapsto & [\bigsqcap N] \end{array}$$

and $\varphi^{-1}([X]) = \mathrm{pr}_{M_{\mathcal{I}}}(X)$. More precisely, the following statements hold:

i. $\bigsqcap U \in \mathcal{M}$ *for each* $U \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$.

ii. $\mathrm{pr}_{M_{\mathcal{I}}}(C) \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$ *for each* $C \in \mathcal{M}$.

iii. $U \subseteq V$ *implies* $\mathrm{pr}_{M_{\mathcal{I}}}(U) \sqsupseteq \mathrm{pr}_{M_{\mathcal{I}}}(V)$ *for all* $U, V \subseteq M_{\mathcal{I}}$.

iv. $C \sqsubseteq D$ *implies* $\bigsqcap C \sqsupseteq \bigsqcap D$ *for all* $C, D \in \mathcal{M}$.

v. $\mathrm{pr}_{M_{\mathcal{I}}}(\bigsqcap U) = U$ *for each* $U \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$.

vi. $\bigsqcap \mathrm{pr}_{M_I}(C) \equiv C$ *for each* $C \in \mathcal{M}$.

Additionally, $U'' = \mathrm{pr}_{M_{\mathcal{I}}}((\bigsqcap U)^{\mathcal{I}\mathcal{I}})$ and $C^{\mathcal{I}\mathcal{I}} \equiv \bigsqcap(\mathrm{pr}_{M_{\mathcal{I}}}(C))''$ for each set $U \subseteq M_{\mathcal{I}}$ and each concept description C expressible in terms of $M_{\mathcal{I}}$, where the derivations are computed in $\mathbb{K}_{\mathcal{I}}$.

Proof We show each claim step by step.

For i, let $U \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$, i. e. $U = U''$. Then

$$\bigsqcap U = \bigsqcap U'' \equiv (U')^{\mathcal{I}} = (\bigsqcap U)^{\mathcal{I}\mathcal{I}}$$

by Proposition 4.7. Hence $\bigsqcap U \equiv (\bigsqcap U)^{\mathcal{I}\mathcal{I}}$ and therefore $\bigsqcap U \in \mathcal{M}$.

For ii, let $C \in \mathcal{M}$, i. e. $C \equiv C^{\mathcal{I}\mathcal{I}}$. By Theorem 4.9, C is expressible in terms of $M_{\mathcal{I}}$ and hence by Proposition 4.6

$$\begin{aligned}\mathrm{pr}_{M_{\mathcal{I}}}(C) &= \mathrm{pr}_{M_{\mathcal{I}}}(C^{\mathcal{I}\mathcal{I}}) \\ &= (C^{\mathcal{I}})' \\ &= (\mathrm{pr}_{M_{\mathcal{I}}}(C))'',\end{aligned}$$

thus $\mathrm{pr}_{M_{\mathcal{I}}}(C) \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$.

Claims iii and iv are already contained in Lemma 4.4.

For v we need to show that

$$\mathrm{pr}_{M_{\mathcal{I}}}(\bigsqcap U) = U$$

for $U \in \mathrm{Int}(\mathbb{K}_{\mathcal{I}})$. By Proposition 4.12, $U \subseteq \mathrm{pr}_{M_{\mathcal{I}}}(\bigsqcap U) \subseteq U''$, and since $U = U''$, equality follows. Claim vi follows from Proposition 4.5, as $C \in \mathcal{M}$ is expressible in terms of $M_{\mathcal{I}}$ by Theorem 4.9.

Finally for $U \subseteq M_{\mathcal{I}}$

$$\begin{aligned}\mathrm{pr}_{M_{\mathcal{I}}}((\bigsqcap U)^{\mathcal{I}\mathcal{I}}) &= \mathrm{pr}_{M_{\mathcal{I}}}(U'^{\mathcal{I}}) \\ &= U''\end{aligned}$$

by Proposition 4.7 and Proposition 4.6, and

$$\begin{aligned}\bigsqcap(\mathrm{pr}_{M_{\mathcal{I}}}(C))'' &\equiv (\mathrm{pr}_{M_{\mathcal{I}}}(C)')^{\mathcal{I}} \\ &= C^{\mathcal{I}\mathcal{I}}\end{aligned}$$

for every $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-concept description C, again by Proposition 4.7 and Proposition 4.6. □

One has to be careful with the equivalence $\bigsqcap(\mathrm{pr}_{M_{\mathcal{I}}}(C))'' \equiv C^{\mathcal{I}\mathcal{I}}$, which in general is only true if C is expressible in terms of $M_{\mathcal{I}}$, as the following example shows.

4.14 Example We shall consider a trivial example to show that $\bigsqcap(\mathrm{pr}_{M_{\mathcal{I}}}(C))'' \equiv C^{\mathcal{I}\mathcal{I}}$ is not necessarily true if C is not expressible in terms of $M_{\mathcal{I}}$. Let $N_C = \emptyset$, $N_R = \{ r \}$ and $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta_{\mathcal{I}} = \{ x \}$ and $r^{\mathcal{I}} = \emptyset$. Then the model-based most-specific concept descriptions of $\mathcal{I}$ are, up to equivalence, just $\bot$ and $\top$, because

$$\begin{aligned}\emptyset^{\mathcal{I}} &= \bot, \\ \{ x \}^{\mathcal{I}} &= \top.\end{aligned}$$

Therefore,

$$M_{\mathcal{I}} = \{ \bot, \exists r.\top \}$$

and

$$\mathbb{K}_{\mathcal{I}} = \begin{array}{c||c|c} & \bot & \exists r.\top \\ \hline\hline x & . & . \end{array}$$

Now consider $C = \exists r.\exists r.\top$. Then

$$C^{\mathcal{I}\mathcal{I}} = \emptyset^{\mathcal{I}} = \bot.$$

On the other hand,

$$\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(C)'' = \bigsqcap \emptyset'' = \bigsqcap \emptyset = \top.$$

Therefore, $C^{\mathcal{I}\mathcal{I}} \not\equiv \bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(C)''$. ♢

5 Confident GCIs of Finite Interpretations

Recall our original motivation: we are interested in completely describing the knowledge of a finite interpretation $\mathcal{I}$ that can be formulated using GCIs. However, it is easy to see that just considering the set of all valid GCIs is not practical, as it is infinite in general. More precisely, if $C \sqsubseteq D$ is valid in $\mathcal{I}$, then for each $r \in N_R$, the GCI $\exists r.C \sqsubseteq \exists r.D$ is also valid in $\mathcal{I}$. But what we can do is to look for a *base* of all valid $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs that are valid in $\mathcal{I}$. Such a base then captures all knowledge representable by valid GCIs of $\mathcal{I}$ and is therefore sufficient for our application. As it has been shown in [10], such a finite base of all valid $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs of $\mathcal{I}$ always exists.

To consider only the valid knowledge of an interpretation $\mathcal{I}$ has one disadvantage. The main motivation why we want to represent this knowledge contained in $\mathcal{I}$ is to obtain some knowledge about *the real world* that is represented by $\mathcal{I}$. But this then requires $\mathcal{I}$ to be *perfect*: each $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCI valid in $\mathcal{I}$ corresponds to a valid statement in the real world and each true statement expressible in $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$ actually corresponds to some $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCI valid in $\mathcal{I}$. This assumption is reasonable for theoretical considerations but has practical limitations.

5.1 Example Let us consider the DBpedia data set [8], which collects data in form of RDF triples from Wikipedia Infoboxes. The structure of these boxes is not formally specified and therefore the collection of data has to be made on a heuristic basis. This in turn leads to a number of errors, both randomly and systematically.

In [9] first experiments have been conducted to apply the theory of axiomatizing the $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs of finite interpretations from [10] to a small fraction of the DBpedia data set. For this, all individuals in the DBpedia data set that occur in a child relation with some other individual haven been collected in a set $\Delta_{\mathrm{DBpedia}}$. This set contained 5626 individuals. Together with their properties and the child relation, this gives rise to the interpretation $\mathcal{I}_{\mathrm{DBpedia}}$, which was used in [9] as an example. One has to note, however, that the child relation in DBpedia is more abstract one might assume. For example, in $\mathcal{I}_{\mathrm{DBpedia}}$, not only humans can have humans as children, but also authors can have theirs works as children. Even populated places or music bands occur as children, a fact most likely due to the free form of Wikipedia's infoboxes. One can see this as an systematic error made during the collection.

Applying the theory of [10], a base $\mathcal{L}_{\mathrm{DBpedia}}$ of GCIs valid in $\mathcal{I}_{\mathrm{DBpedia}}$ has been obtained. However, some of the GCIs found described facts that one may have assumed to be true more generally. Examples for this are

$$
\begin{gathered}
\exists \mathsf{c}.\exists \mathsf{c}.\top \sqsubseteq \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\top),\\
\exists \mathsf{c}.\mathsf{Work} \sqsubseteq \mathsf{Person},\\
\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\top))))\\
\sqsubseteq \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\exists \mathsf{c}.\mathsf{Person})))),\\
\mathsf{OfficeHolder} \sqcap \exists \mathsf{c}.\exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\top) \sqsubseteq \exists \mathsf{c}.(\mathsf{Person} \sqcap \exists \mathsf{c}.\exists \mathsf{c}.\mathsf{Person}),\\
\mathsf{Person} \sqcap \exists \mathsf{c}.\mathsf{Criminal} \sqsubseteq \mathsf{Criminal},
\end{gathered}
$$

where we have abbreviated the child relation with a singleton c for better readability.

Now, if one would add the GCI $\exists \mathsf{c}.\top \sqsubseteq \mathsf{Person}$ to $\mathcal{L}_{\mathrm{DBpedia}}$, the first two GCIs turn out to be dispensable (i. e. they are trivial) and the other GCIs would become much simpler:

$$
\begin{gathered}
\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\top \sqsubseteq \exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\mathsf{Person},\\
\mathsf{OfficeHolder} \sqcap \exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\top \sqsubseteq \exists \mathsf{c}.\exists \mathsf{c}.\exists \mathsf{c}.\mathsf{Person},\\
\exists \mathsf{c}.\mathsf{Criminal} \sqsubseteq \mathsf{Criminal},
\end{gathered}
$$

given the fact that Criminal $\sqsubseteq$ Person is contained in $\mathcal{L}_{\text{DBpedia}}$.

Indeed, the fact that only persons can have children sounds quite natural even in the quite abstract understanding of the child relation of DBpedia. However, the GCI $\exists \mathsf{c}.\top \sqsubseteq$ Person does not hold in $\mathcal{I}_{\text{DBpedia}}$, as there are 4 individuals that satisfy $\exists \mathsf{c}.\top$ but not Person, namely `Teresa_Carpio`, `Charles_Heung`, `Adam_Cheng` and `Lydia_Shum`. However, these individuals denote real persons[2] and therefore should satisfy Person, an error in our interpretation $\mathcal{I}_{\text{DBpedia}}$. ♢

To understand the occurrence of errors we shall propose the following point of view on our given data: The relevant part of the reality we are interested, our *domain of interest*, can be understood as a finite interpretation $\mathcal{I}_{\text{perfect}}$. However, we do not have direct access to $\mathcal{I}_{\text{perfect}}$ but to another interpretation $\mathcal{I}$ that originates from $\mathcal{I}_{\text{perfect}}$ by errors. Our aim is then to find as much valid GCIs from $\mathcal{I}_{\text{perfect}}$ as possible by examining GCIs from $\mathcal{I}$.

We can then regard the interpretation $\mathcal{I}$ as an *approximation* to $\mathcal{I}_{\text{perfect}}$ that contains some errors. The following definitions formulates this view more precisely.

5.2 Definition Let $\mathcal{I}_1, \mathcal{I}_2$ be two finite interpretations over the signature (N_C, N_R) such that $\Delta_{\mathcal{I}_1} = \Delta_{\mathcal{I}_2} := \Delta$. Then we may call $\mathcal{I}_2$ an *approximation* of $\mathcal{I}_1$ and $\mathcal{I}_1$ the *origin* of $\mathcal{I}_2$. In this case, an *error in $\mathcal{I}_2$ with respect to $\mathcal{I}_1$* is

i. either a pair $(x, C) \in \Delta \times N_C$ such that $x \in (C^{\mathcal{I}_1} \triangle C^{\mathcal{I}_2})$ or

ii. a pair $(x, r) \in \Delta \times N_R$ such that there exists an individual $y \in \Delta$ with $(x, y) \in (r^{\mathcal{I}_1} \triangle r^{\mathcal{I}_2})$.

If the interpretation $\mathcal{I}_1$ is clear from the context, we may more informally speak of *errors in $\mathcal{I}_2$*. ♢

Errors in interpretations may occur for a variety of reasons and it is not reasonable to assume that an algorithm can automatically detect errors in general unless given some other source of information on the origin. This is due to the simple fact that an error in an approximation $\mathcal{I}$ of an interpretation $\mathcal{I}_1$ may not necessarily be an error of $\mathcal{I}$ if regarded as an approximation of $\mathcal{I}_2$. Hence, when we are given no additional information we cannot decide which of the interpretations $\mathcal{I}_1$ or $\mathcal{I}_2$ we are actually considering as correct and so additional information is needed.

For the remainder of this section let $\mathcal{I}_{\text{perfect}}$ be a finite interpretation and let $\mathcal{I}$ be an approximation of $\mathcal{I}_{\text{perfect}}$. Our idea of handling errors in $\mathcal{I}$ is now as follows: instead of considering valid GCIs of $\mathcal{I}$ only, we may also consider GCIs that are "almost" valid in $\mathcal{I}$. With other words, we may consider GCIs $C \sqsubseteq D$ such that the number of *negative examples* (or *counterexamples*) is "small" compared to the number of *positive examples*. Here, a negative example for the GCI $C \sqsubseteq D$ is an element $x \in \Delta_{\mathcal{I}}$ satisfying $x \in C^{\mathcal{I}}$ but $x \notin D^{\mathcal{I}}$. Analogously, a positive example for $C \sqsubseteq D$ is an element $y \in \Delta_{\mathcal{I}}$ satisfying $y \notin C^{\mathcal{I}}$ or $y \in D^{\mathcal{I}}$. Furthermore, we shall express "small" using some chosen threshold $c \in [0, 1]$, i. e. the number of positive examples should be at least c times the number of all individuals $z \in \Delta_{\mathcal{I}}$ satisfying the premise C of the GCI.

Considering those GCIs is motivated by the assumption that if not too many random errors have been made between $\mathcal{I}_{\text{perfect}}$ and $\mathcal{I}$, the valid GCIs $\mathcal{I}_{\text{perfect}}$ that do not hold in $\mathcal{I}$ may still be GCIs which are almost valid in $\mathcal{I}$. On the other hand, GCIs that are almost true in $\mathcal{I}$ need not necessarily be GCIs that are true in $\mathcal{I}_{\text{perfect}}$, so some extra caution should be exercised and the resulting GCIs should be checked for usefulness afterwards. This is the point where an external source of information is needed to validate the resulting GCIs.

It is the aim of this section is to introduce an approach to this idea based on the notion of *confidence* as it is used in data mining [1]. We transfer this notion to the setting of $\mathcal{EL}^{\perp}_{\text{gfp}}$-GCIs to obtain *confident GCIs* and discuss what we shall understand by a *base* of confident GCIs for a finite interpretation.

But before we do so, let us consider an introductory example, which we shall use throughout this section to illustrate our definitions and motivations.

[2] Curiously enough, these are all artists from Hong Kong.

5.3 Example Let us consider the programming language family Lisp and its development history. As an important and still actively used programming language, it has affected the design of many modern programming languages. According to the Wikipedia article[3], those languages include ML, Haskell, Logo, Tcl, Forth, Smalltalk, Perl, Python, Ruby, Dylan and Lua. Furthermore, the family of Lisp languages contains a lot of dialects, the most important of them being Scheme and Common Lisp. Finally, there has been the programming language IPL, which influenced the design of Lisp.

We shall take these languages as individuals for our example interpretation. As role names we choose influenced and dialectOf and as concept names we use programming paradigms supported by programming languages such as Imperative, Functional, Lazy and so on. These paradigms are listed on the Wikipedia articles for the corresponding languages. The resulting interpretation $\mathcal{I}_{\mathrm{Lisp}}$ is shown in Figure 4.

When we now apply the theory of [10], we obtain a base $\mathcal{L}$ of all valid $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs of $\mathcal{I}_{\mathrm{Lisp}}$. Now $\mathcal{L}$ contains 89 GCIs, some of which are very special as $\mathsf{Concatenative} \sqsubseteq \mathsf{StackOriented}$ (this only applies to Forth) or $\mathsf{Prototyping} \sqsubseteq \mathsf{Scripting}$ (this only applies to Lua). But there are of course also GCIs in $\mathcal{L}$ that apply to more then one individual. Examples for this are the GCIs

$$\begin{aligned}
\mathsf{Imperative} &\sqsubseteq \mathsf{Functional}\\
\exists\mathsf{influenced}.\mathsf{Imperative} &\sqsubseteq \exists\mathsf{influenced}.(\mathsf{Imperative} \sqcap \mathsf{ObjectOriented} \sqcap \mathsf{Reflective})\\
\exists\mathsf{influenced}.\mathsf{Procedural} &\sqsubseteq \exists\mathsf{influenced}.(\mathsf{Functional} \sqcap \exists\mathsf{influenced}.(\mathsf{ObjectOriented}\\
&\qquad \sqcap \mathsf{Reflective} \sqcap \mathsf{Imperative}))\\
&\qquad \sqcap \exists\mathsf{influenced}.(\mathsf{Functional} \sqcap \mathsf{Procedural}\\
&\qquad \sqcap \exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional}))
\end{aligned}$$

which apply to at least 6 individuals in $\mathcal{I}_{\mathrm{Lisp}}$, and

$$\begin{gathered}
\exists\mathsf{influenced}.\mathsf{Functional} \sqcap \exists\mathsf{influenced}.\mathsf{ObjectOriented}\\
\sqsubseteq \exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional}),\\
\exists\mathsf{influenced}.\mathsf{Reflective} \sqsubseteq \exists\mathsf{influenced}(\mathsf{Functional} \sqcap \mathsf{Reflective}),\\
\exists\mathsf{influenced}.\exists\mathsf{influenced}.\top\\
\sqsubseteq \exists\mathsf{influenced}.\exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Reflective} \sqcap \mathsf{Imperative}),
\end{gathered}$$

which apply to 8, 8 and 10 individuals, respectively.

Now let us suppose that we are interested in axiomatizing $\mathcal{I}_{\mathrm{Lisp}}$ without having access to it. Instead, we only have given an interpretation $\mathcal{I}'_{\mathrm{Lisp}}$ that differs from $\mathcal{I}_{\mathrm{Lisp}}$ in the following way:

- Lisp did not influence Haskell anymore, likewise with Smalltalk and Ruby and Dylan and Python;
- Common Lisp is not ObjectOriented anymore, likewise with Perl and Reflective, Lua and Functional and Lisp and Functional;
- IPL now influenced ML, likewise with Forth and Common Lisp;
- Lisp is now ObjectOriented, likewise with Dylan and Meta.

The resulting interpretation $\mathcal{I}'_{\mathrm{Lisp}}$ is shown in Figure 5.

Now, in $\mathcal{I}'_{\mathrm{Lisp}}$, all of the previously given GCIs are not valid anymore. However, for some of them the number of positive examples in $\mathcal{I}'_{\mathrm{Lisp}}$ is still quite high compared to the number of negative

[3] http://en.wikipedia.org/wiki/Lisp_(programming_language)

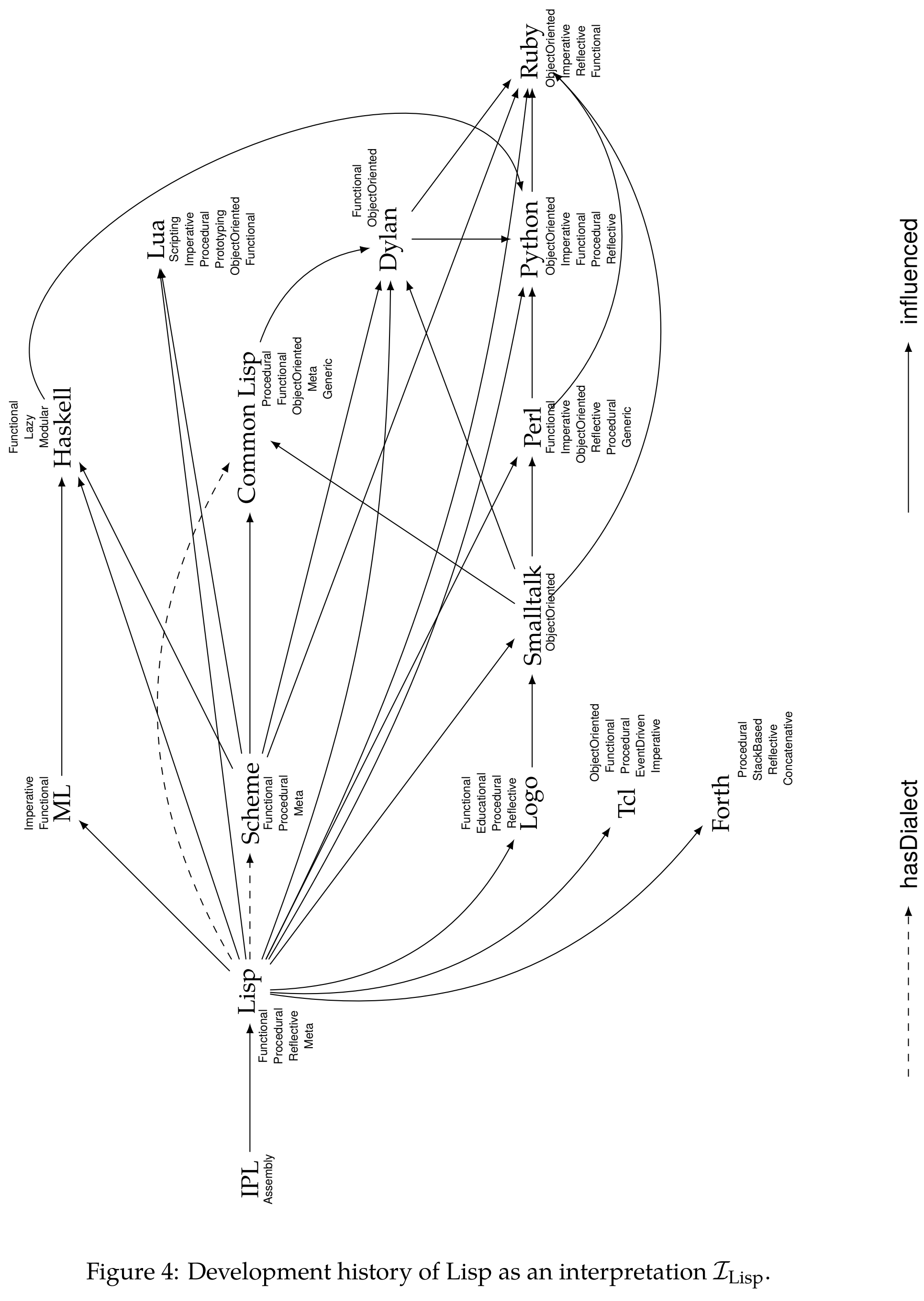

Figure 4: Development history of Lisp as an interpretation $\mathcal{I}_{\text{Lisp}}$.

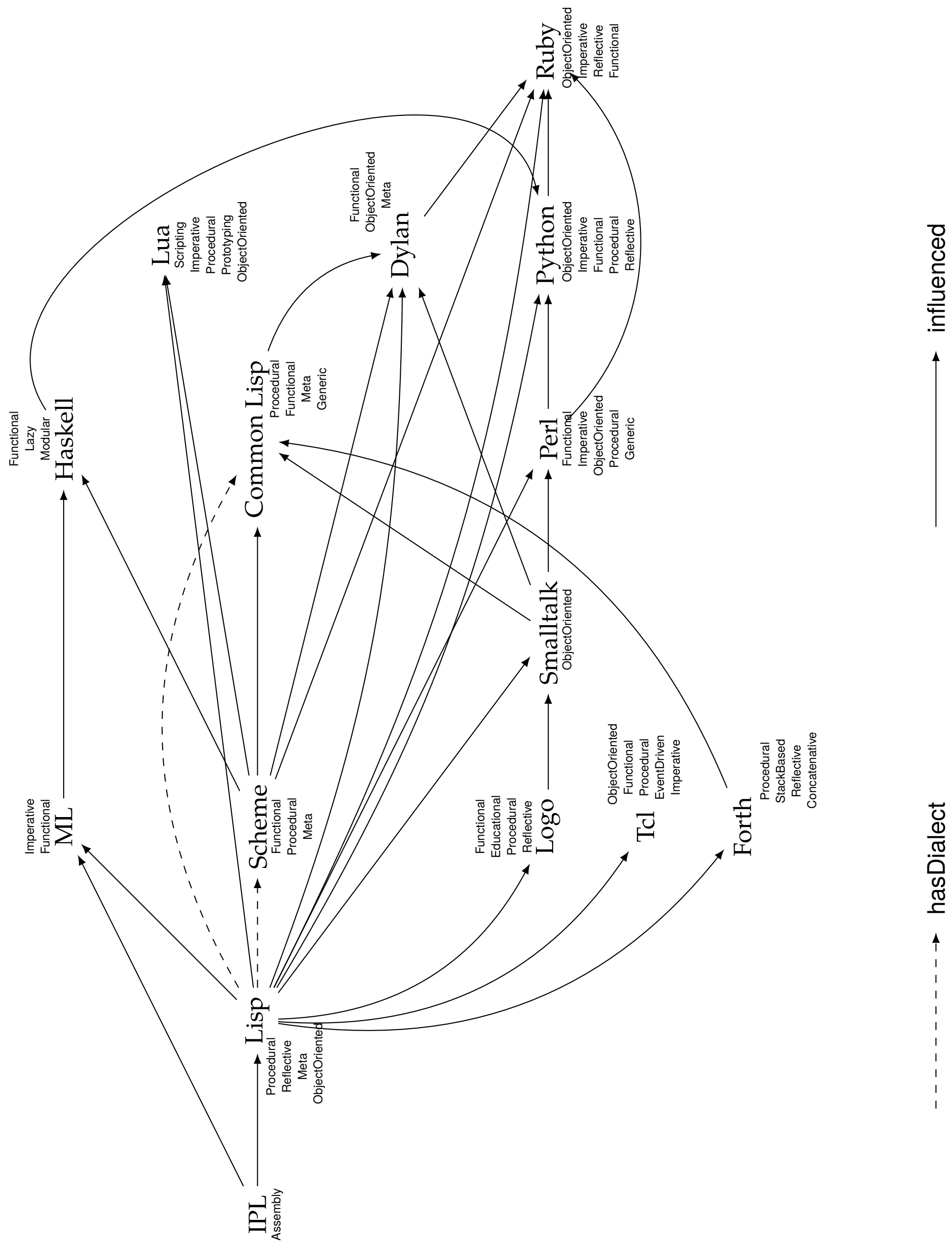

Figure 5: Disturbed interpretation $\mathcal{I}'_{\text{Lisp}}$.

examples. For example, the GCI

$$\exists \mathsf{Influenced}.\mathsf{Reflective} \sqsubseteq \exists \mathsf{Influenced}.(\mathsf{Functional} \sqcap \mathsf{Reflective})$$

now has 6 positive examples (namely Lisp, Scheme, Perl, Python, Haskell, Dylan) and 1 negative one (namely IPL). Likewise, the GCI

$$\begin{aligned} &\exists \mathsf{Influenced}.\mathsf{Functional} \sqcap \exists \mathsf{Influenced}.\mathsf{ObjectOriented} \\ &\quad \sqsubseteq \exists \mathsf{Influenced}(\mathsf{ObjectOriented} \sqcap \mathsf{Funcational}), \end{aligned}$$

which is valid in $\mathcal{I}_{\text{Lisp}}$ has in $\mathcal{I}'_{\text{Lisp}}$ 8 positive and 1 negative examples. ♢

As we have argued before and seen in the previous example, it might be worthwhile to consider GCIs of an interpretation which might not necessarily be true but hold in "a large number of cases". Thereby we understand a GCI $C \sqsubseteq D$ to hold in "a large number of cases" if the number of counterexamples of $C \sqsubseteq D$ is below a predefined percentage of all individuals to which $C \sqsubseteq D$ is applicable. To make this into a formal definition we introduce the notion of *confidence* of GCIs as follows.

5.4 Definition (Confidence of GCIs) Let $\mathcal{I}$ be a finite interpretation and let C, D be $\mathcal{EL}^{\perp}_{\text{gfp}}$-concept descriptions. The *confidence of* $C \sqsubseteq D$ *in* $\mathcal{I}$ is defined as

$$\operatorname{conf}_{\mathcal{I}}(C \sqsubseteq D) := \begin{cases} \frac{|(C \sqcap D)^{\mathcal{I}}|}{|C^{\mathcal{I}}|} & \text{if } C^{\mathcal{I}} \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Let $c \in [0, 1]$. Then the GCI $C \sqsubseteq D$ is a *confident GCI of* $\mathcal{I}$ *with minimal confidence* c (or is said to *hold confidently in* $\mathcal{I}$ *with minimal confidence* c) if and only if $\operatorname{conf}_{\mathcal{I}}(C \sqsubseteq D) \geqslant c$. The set of all confident $\mathcal{EL}^{\perp}_{\text{gfp}}$-GCIs with minimal confidence c is denoted by $\operatorname{Th}_c(\mathcal{I})$. ♢

5.5 Example i. For the interpretation $\mathcal{I}_{\text{DBpedia}}$ we have

$$\operatorname{conf}_{\mathcal{I}_{\text{DBpedia}}}(\exists \mathsf{child} \top \sqsubseteq \mathsf{Person}) = \frac{2547}{2551}.$$

ii. For the interpretation $\mathcal{I}_{\text{Lisp}}$ we have

$$\begin{aligned} \operatorname{conf}_{\mathcal{I}_{\text{Lisp}}}(\exists \mathsf{Influenced}.\mathsf{Reflective} \sqsubseteq \exists \mathsf{Influenced}.(\mathsf{Functional} \sqcap \mathsf{Reflective})) = \frac{6}{7}, \\ \operatorname{conf}_{\mathcal{I}_{\text{Lisp}}}(\exists \mathsf{Influenced}.\mathsf{Functional} \sqcap \exists \mathsf{Influenced}.\mathsf{ObjectOriented} \\ \sqsubseteq \exists \mathsf{Influenced}(\mathsf{ObjectOriented} \sqcap \mathsf{Funcational})) = \frac{8}{9}. \end{aligned}$$ ♢

With the definition of confidence of GCIs we can now formulate our initial idea of handling errors more formally. For this, let us assume that we are given a number $c \in [0, 1]$. Then the GCIs that are "almost" true in $\mathcal{I}$ are now the GCIs in the set $\operatorname{Th}_c(\mathcal{I})$ of confident $\mathcal{EL}^{\perp}_{\text{gfp}}$-GCIs with minimal confidence c. The heuristic idea now is that within $\operatorname{Th}_c(\mathcal{I})$ there are interesting GCIs that are valid in $\mathcal{I}_{\text{perfect}}$. This is due to the idea that if a GCI $C \sqsubseteq D$ is valid in $\mathcal{I}_{\text{perfect}}$ and if not too many errors exist in $\mathcal{I}$, that then the confidence $\operatorname{conf}_{\mathcal{I}}(C \sqsubseteq D)$ is higher than c (under certain constraints on the minimal confidence c) and hence $(C \sqsubseteq D) \in \operatorname{Th}_c(\mathcal{I})$.

On the other hand, if $C \sqsubseteq D$ is valid in $\mathcal{I}$ but $(C \sqsubseteq D) \notin \operatorname{Th}_c(\mathcal{I})$, then strictly more than $c \cdot |C^{\mathcal{I}}|$ individuals from $C^{\mathcal{I}}$ must have been affected by errors. If, however, $c \cdot |C^{\mathcal{I}}|$ is not too small compared to the errors that occurred, then $(C \sqsubseteq D) \notin \operatorname{Th}_c(\mathcal{I})$ is very unlikely. This however implies that our approach is reasonable only for GCIs $C \sqsubseteq D$ for which $|C^{\mathcal{I}}|$ is not too small, for otherwise random errors may affect the confidence $\operatorname{conf}_{\mathcal{I}}(C \sqsubseteq D)$ enormously. This motivates the following definition.

5.6 Definition (Support of GCIs) Let $\mathcal{I}$ be a finite interpretation and let $C \sqsubseteq D$ be an $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCI. Then the *support of* $C \sqsubseteq D$ *in* $\mathcal{I}$ is defined to be

$$\mathrm{supp}_{\mathcal{I}}(C \sqsubseteq D) := \begin{cases} \frac{|C^{\mathcal{I}}|}{|\Delta_{\mathcal{I}}|} & \text{if } \Delta_{\mathcal{I}} \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Let $s \in [0,1]$. Then we say that $C \sqsubseteq D$ has *minimal support* s if and only if $\mathrm{supp}_{\mathcal{I}}(C \sqsubseteq D) \geqslant s$. $\diamondsuit$

If we now assume that the differences between $\mathcal{I}_{\mathrm{perfect}}$ and $\mathcal{I}$ are not too big, then we can find valid GCIs of $\mathcal{I}_{\mathrm{perfect}}$ as confident GCIs in $\mathcal{I}$.

5.7 Lemma *Let N_C, N_R be two finite sets and let $\mathcal{I}_1, \mathcal{I}_2$ be two finite interpretations over the signature (N_C, N_R) with $\Delta_{\mathcal{I}_1} = \Delta_{\mathcal{I}_2} \neq \varnothing$. Let $k \in \mathbb{N}$ be such that for each concept description E over the signature (N_C, N_R) it is true that*

$$|E^{\mathcal{I}_1} \triangle E^{\mathcal{I}_2}| \leqslant k.$$

If then $C \sqsubseteq D$ is an $\mathcal{EL}^{\bot}_{gfp}$-GCI such that $\mathrm{conf}_{\mathcal{I}_1}(C \sqsubseteq D) \geqslant c$ and $\mathrm{supp}_{\mathcal{I}_1}(C \sqsubseteq D) \geqslant s$ for some $s, c \in [0,1]$, then

$$\mathrm{conf}_{\mathcal{I}_2}(C \sqsubseteq D) \geqslant c - k \cdot \frac{1+c}{s \cdot |\Delta_{\mathcal{I}_1}| + k}$$
$$\mathrm{supp}_{\mathcal{I}_2}(C \sqsubseteq D) \geqslant s - \frac{k}{|\Delta_{\mathcal{I}_1}|}.$$

Proof By the prerequisites of the Lemma,

$$|C^{\mathcal{I}_1} \triangle C^{\mathcal{I}_2}| \leqslant k,$$
$$|(C \sqcap D)^{\mathcal{I}_1} \triangle C^{\mathcal{I}_2}| \leqslant k.$$

In particular,

$$|C^{\mathcal{I}_2}| \leqslant |C^{\mathcal{I}_1}| + k,$$
$$|(C \sqcap D)^{\mathcal{I}_1}| \geqslant |(C \sqcap D)^{\mathcal{I}_2}| - k.$$

With this we can now argue as follows:

$$\begin{aligned} \mathrm{conf}_{\mathcal{I}_2}(C \sqsubseteq D) &= \frac{|(C \sqcap D)^{\mathcal{I}_2}|}{|C^{\mathcal{I}_2}|} \\ &\geqslant \frac{|(C \sqcap D)^{\mathcal{I}_1}| - k}{|C^{\mathcal{I}_1}| + k} \end{aligned}$$

and as $\mathrm{conf}_{\mathcal{I}_1}(C \sqsubseteq D) \geqslant c$ implies $|(C \sqcap D)^{\mathcal{I}_1}| \geqslant c \cdot |C^{\mathcal{I}_1}|$, we furthermore obtain

$$\begin{aligned} \mathrm{conf}_{\mathcal{I}_2}(C \sqsubseteq D) &\geqslant \frac{c \cdot |C^{\mathcal{I}_1}| - k}{|C^{\mathcal{I}_1}| + k} \\ &= \frac{c \cdot |C^{\mathcal{I}_1}| + c \cdot k}{|C^{\mathcal{I}_1}| + k} - \frac{k + c \cdot k}{|C^{\mathcal{I}_1}| + k} \\ &= c - k \cdot \frac{1+c}{|C^{\mathcal{I}_1}| + k} \\ &\geqslant c - k \cdot \frac{1+c}{s \cdot |\Delta_{\mathcal{I}_1}| + k} \end{aligned}$$

since $|C^{\mathcal{I}_1}| \geqslant s \cdot |\Delta_{\mathcal{I}_1}|$ due to $\text{supp}_{\mathcal{I}_1}(C \sqsubseteq D) \geqslant s$. This yields the claim about $\text{conf}_{\mathcal{I}_1}(C \sqsubseteq D)$.

For $\text{supp}_{\mathcal{I}_2}(C \sqsubseteq D)$ we conclude likewise:

$$\begin{aligned}\text{supp}_{\mathcal{I}_2}(C \sqsubseteq D) &= \frac{|C^{\mathcal{I}_2}|}{|\Delta_{\mathcal{I}_2}|} \\ &\geqslant \frac{|C^{\mathcal{I}_1}| - k}{|\Delta_{\mathcal{I}_1}} \\ &= \frac{|C^{\mathcal{I}_1}|}{|\Delta_{\mathcal{I}_1}|} - \frac{k}{|\Delta_{\mathcal{I}_1}|} \\ &\geqslant s - \frac{k}{|\Delta_{\mathcal{I}_1}|}\end{aligned}$$

as required. □

So far we have only considererd GCIs valid in $\mathcal{I}_{\text{perfect}}$ and argued that they are likely to occur as confident GCIs of $\mathcal{I}$. On the other hand, errors in $\mathcal{I}$ may very well lead to GCIs confident in $\mathcal{I}$ that are not valid in $\mathcal{I}_{\text{perfect}}$.

5.8 Example In the case of $\mathcal{I}'_{\text{Lisp}}$ we have seen that some valid GCIs of $\mathcal{I}_{\text{Lisp}}$ with high support are still confident GCIs of $\mathcal{I}'_{\text{Lisp}}$ with suitable parameters c. On the other hand, there exist GCIs that are confident in $\mathcal{I}'_{\text{Lisp}}$ with the same parameter but that are not valid in $\mathcal{I}_{\text{Lisp}}$. An example for one of those GCIs is

$$\begin{gathered}\mathsf{StackOriented} \sqsubseteq \\ \mathsf{Concatenative} \sqcap \exists\mathsf{influenced}.(\mathsf{Meta} \sqcap \mathsf{Generic} \sqcap \\ \exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional} \sqcap \mathsf{Meta} \sqcap \\ \exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional} \sqcap \mathsf{Reflective} \sqcap \mathsf{Imperative}))),\end{gathered}$$

as we have added an influenced-successor for Forth, the only individual satisfying this GCI. Another example is

$$\begin{gathered}\exists\mathsf{influenced}.\exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional} \sqcap \mathsf{Reflective} \sqcap \mathsf{Imperative}) \\ \sqcap \mathsf{Functional} \sqsubseteq \exists\mathsf{influenced}.(\mathsf{Functional} \\ \sqcap \exists\mathsf{influenced}.(\mathsf{ObjectOriented} \sqcap \mathsf{Functional} \sqcap \mathsf{Reflective} \sqcap \mathsf{Imperative})),\end{gathered}$$

which has support $8/15$ and confidence $7/8$ in $\mathcal{I}_{\text{Lisp}}$. ◇

GCIs in $\text{Th}_c(\mathcal{I})$ that are not valid in $\mathcal{I}_{\text{perfect}}$ are not the ones we are interested in and they need to be distinguished from the valid GCIs of $\mathcal{I}_{\text{perfect}}$ in $\text{Th}_c(\mathcal{I})$ by means of additional information, for example a human expert.

However, we can easily obtain some additional insights into this problem, as the definition of approximations is *symmetrical*: Instead of considering $\mathcal{I}$ as an erroneous approximation of $\mathcal{I}_{\text{perfect}}$, we can likewise consider $\mathcal{I}_{\text{perfect}}$ as an erronous approximation of $\mathcal{I}$. Now, using Lemma 5.7 and considering a GCI $(C \sqsubseteq D) \in \text{Th}_c(\mathcal{I})$, we can may still deduce that the confidence of $C \sqsubseteq D$ in $\mathcal{I}_{\text{perfect}}$ is bounded from below. Hence GCIs in $\text{Th}_c(\mathcal{I})$ may not be valid in $\mathcal{I}_{\text{perfect}}$, but are at least of bounded confidence.

We shall now turn our attention to the following question: as we have already argued it might be a reasonable idea to consider the set $\text{Th}_c(\mathcal{I})$ of confident GCIs of $\mathcal{I}$ in the presence of errors. However, this set itself is in general an infinite set. To make this a practical approach we somehow need to be able to represent $\mathrm{Th}_c(\mathcal{I})$ in a finite way. For this we shall use the definition of *bases of confident GCIs*, i. e. sets $\mathcal{B}$ that are bases of $\mathrm{Th}_c(\mathcal{I})$. Recall from Definition 3.26 that this means that $\mathcal{B} \subseteq \mathrm{Th}_c(\mathcal{I})$ and that $\mathcal{B}$ is complete for $\mathrm{Th}_c(\mathcal{I})$, i. e. that every GCI $(C \sqsubseteq D) \in \mathrm{Th}_c(\mathcal{I})$ already follows from $\mathcal{B}$. To emphasize the description logic we are using, we may also say that $\mathcal{B}$ is an $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-base of $\mathrm{Th}_c(\mathcal{I})$.

In the case of $c = 1$, bases of $\mathrm{Th}_1(\mathcal{I})$ are just bases of $\mathcal{I}$. We therefore see that the definition generalizes the classical notion of an $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-base of valid GCIs of a finite interpretation. However, in contrast to the case of valid GCIs, in the general case of confident GCIs we have to face the fact that the set $\mathrm{Th}_c(\mathcal{I})$ is not necessarily closed under entailment.

5.9 Example Let us consider the interpretation $\mathcal{I}'_{\mathrm{Lisp}}$ again. For this interpretation we obtain

$$\begin{aligned}\mathrm{conf}_{\mathcal{I}'_{\mathrm{Lisp}}}(\mathsf{ObjectOriented} \sqsubseteq \mathsf{Reflective}) &= \frac{3}{8}\\ \mathrm{conf}_{\mathcal{I}'_{\mathrm{Lisp}}}(\mathsf{ObjectOriented} \sqsubseteq \mathsf{Functional}) &= \frac{5}{8},\end{aligned}$$

thus both GCIs are elements of $\mathrm{Th}_{3/8}(\mathcal{I}'_{\mathrm{Lisp}})$. However, the GCI

$$\mathsf{ObjectOriented} \sqsubseteq \mathsf{Reflective} \sqcap \mathsf{Functional}$$

has only confidence $1/4$ in $\mathcal{I}'_{\mathrm{Lisp}}$ and hence, albeit entailed by the above mentioned GCIs, is not an element of $\mathrm{Th}_{3/8}(\mathcal{I}'_{\mathrm{Lisp}})$. ♢

However, as our overall goal is to find an $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-base $\mathcal{L}$ of $\mathrm{Th}_c(\mathcal{I})$ such that an external expert identifies all elements of $\mathcal{L}$ as valid in the original interpretation $\mathcal{I}_{\mathrm{perfect}}$, the fact that $\mathrm{Th}_c(\mathcal{I})$ might not be closed under entailment is not an issue. Indeed, if all GCIs in $\mathcal{L}$ are valid in $\mathcal{I}_{\mathrm{perfect}}$, then all GCIs entailed by $\mathcal{L}$ (and hence by $\mathrm{Th}_c(\mathcal{I})$) are valid in $\mathcal{I}_{\mathrm{perfect}}$ as well.

6 A Finite $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-Base for the Confident GCIs of a Finite Interpretation

We shall now consider the question whether the set $\mathrm{Th}_c(\mathcal{I})$ always has a finite $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-base. As we shall see, this question has an affirmative answer.

To see that $\mathrm{Th}_c(\mathcal{I})$ always has a finite $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-base we shall use ideas of M. Luxenburger [12]. In his work, Luxenburger considered *partial implications*. These are implications with an additional numeric parameter controlling its confidence. As this is quite similar to our setting of confident GCIs it is only natural to ask whether Luxenburgers results can be generalized in this direction.

We shall start this section by repeating some ideas from [12]. After this we shall show that these ideas can indeed be used to find a finite base for $\mathrm{Th}_c(\mathcal{I})$. Finally, we shall discuss how this finite base can be computed directly in $\mathbb{K}_{\mathcal{I}}$.

6.1 Partial Implications of Formal Contexts

The main focus of Luxenburgers work [12] lies in the investigation of *partial implications*. Essentially, these are just implications equipped with a numerical parameter denoting its confidence. To define this rigorously we shall first repeat the notion of confidence for implications.

6.1 Definition Let $\mathbb{K} = (G, M, I)$ be a finite formal context and let $c \in [0,1]$. Then for $A, B \subseteq M$, we define the *confidence* of $A \longrightarrow B$ in $\mathbb{K}$ as

$$\operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) := \begin{cases} 1 & \text{if } A' = \emptyset \\ \frac{|(A \cup B)'|}{|A'|} & \text{otherwise.} \end{cases} \qquad \diamondsuit$$

The definition of confidence of implications is only a special case of the definition of confidence for GCIs. Indeed, every finite formal context $\mathbb{K} = (G, M, I)$ can be understood as an interpretation $\mathcal{I}_{\mathbb{K}}$. This has already been noted in the general framework devised in [4, 10] to combine formal concept analysis and description logics. To see this connection in our special case we can just define $N_C = M$, $N_R = \emptyset$ and $\mathcal{I}_{\mathbb{K}} = (G, \cdot^{\mathcal{I}})$, where

$$m^{\mathcal{I}} := \{\, m \,\}'.$$

Then the only concept descriptions that are possible are essentially concept descriptions of the form

$$C = m_1 \sqcap \ldots \sqcap m_n$$

for some $n \in \mathbb{N}$. But then

$$C^{\mathcal{I}} = \prod_{i=1}^{n} m_i^{\mathcal{I}} = \bigcap_{i=1}^{n} \{\, m_i \,\}' = \{\, m_1, \ldots, m_n \,\}'$$

and we can identify sets $A \subseteq M$ with concept descriptions $\bigsqcap A$ over the signature (N_C, N_R). Furthermore, the GCIs that can be constructed in this way are in the same way in a one-to-one correspondence to implications of $\mathbb{K}$. It is easy to see that then for $A, B \subseteq M$,

$$\operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) = \operatorname{conf}_{\mathcal{I}_{\mathbb{K}}}(\bigsqcap A \longrightarrow \bigsqcap B).$$

Using the notion of confidence for implications we can now define the notion of a *partial implication*.

6.2 Definition Let M be a set. Then a *partial implication* $(A \longrightarrow B, c)$ over the set M consists of two sets $A, B \subseteq M$ and a number $c \in [0,1]$. The partial implication $(A \longrightarrow B, c)$ is called a *proper partial implication* if $c \neq 1$.

Let $\mathbb{K} = (G, M, I)$ be a finite formal context. Then a partial implication $(A \longrightarrow B, c)$ over M is said to *hold* in $\mathbb{K}$ if and only if

$$\operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) = c.$$

In this case we shall write $\mathbb{K} \models (A \longrightarrow B, c)$. If $\mathcal{J}$ is a set of partial implications over M, then we shall write $\mathbb{K} \models \mathcal{J}$ if and only if every partial implication in $\mathcal{J}$ holds in $\mathbb{K}$. In this case, $\mathbb{K}$ is called a *model* of $\mathcal{J}$. $\diamondsuit$

Let $\mathbb{K} = (G, M, I)$ be a formal context. Then we shall consider the set

$$\mathcal{J}^{<1}(\mathbb{K}) := \{\, (A \longrightarrow B, c) \mid A, B \subseteq M, c = \operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) < 1 \,\}$$

of all *proper partial implications of* $\mathbb{K}$. In [12], the notion of *bases* $\mathcal{L}$ of $\mathcal{J}^{<1}(\mathbb{K})$ is considered. Here $\mathcal{L}$ is a base of $\mathcal{J}^{<1}(\mathbb{K})$ if and only if

i. $\mathcal{L} \subseteq \mathcal{J}^{<1}(\mathbb{K})$,

ii. $\mathcal{L} \models (A \longrightarrow B, c)$ holds for all $(A \longrightarrow B, c) \in \mathcal{J}^{<1}(\mathbb{K})$ and

iii. $\mathcal{L}$ is $\subseteq$-minimal with this property.

Here we write $\mathcal{L} \models (A \longrightarrow B, c)$ if and only if for every model $\mathbb{L}$ of $\mathcal{L}$, it holds $\mathbb{L} \models (A \longrightarrow B, c)$, i. e.

$$\mathbb{L} \models \mathcal{L} \implies \mathbb{L} \models (A \longrightarrow B, c)$$

for all formal contexts $\mathbb{L}$ with attribute set M.

Luxenburger restricts himself to finding a base of $\mathcal{J}^{<1}(\mathbb{K})$ instead of $\mathcal{J}(\mathbb{K})$. This is because for the implications with confidence 1, i. e. valid implications of $\mathbb{K}$, one can explicitly describe a minimal base, as we have seen in Section 2.3.

Luxenburger does not explicitly describe a base for $\mathcal{J}^{<1}(\mathbb{K})$. However, Stumme et. al. [19] later used results from Luxenburger to describe a small set of *association rules* that can be understood as bases. Association rules can be seen as a variant of partial implications that are equipped with two parameters specifying their support and their confidence, respectively. However, we are not going to introduce association rules here. Instead, we restrict ourselves to formulating the basic problem of [19] in terms of partial implications and ignore the support of these implications. The problem then reads as follows: given a formal context $\mathbb{K} = (G, M, I)$, a number $c \in [0, 1]$ and an implication $A \longrightarrow B$, is $\operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) \geqslant c$? In other words, is there a number $c' \geqslant c$ such that $(A \longrightarrow B, c')$ is a partial implication of $\mathbb{K}$? Of course, one could simply compute the confidence of $A \longrightarrow B$ in $\mathbb{K}$. However, in [19] this is regarded as too expensive, as $\mathbb{K}$ may be represented as a data base and accessing *all* items in the data base is expensive; but this would be necessary to compute the confidence of $A \longrightarrow B$, at least if done in an naïve way. Instead, one likes to have a small set $\mathcal{L}$ of partial implications that already determine the confidence of all partial implications of $\mathbb{K}$ with confidence at least c. In addition, we are provided with a mechanism to compute the closure X'' of X in $\mathbb{K}$ for each $X \subseteq M$.

To find such a set, [19] uses the following result. The second claim of the lemma has already been mentioned in [12, Proposition 1].

6.3 Lemma *Let $\mathbb{K}$ be a finite formal context and let $c \in [0, 1]$. Then for all $A, B, C \subseteq M$ the following statements hold.*

i. $\operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) = \operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow B'')$.

ii. $\operatorname{conf}_{\mathbb{K}}(A \longrightarrow C) = \operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) \cdot \operatorname{conf}_{\mathbb{K}}(B \longrightarrow C)$ *if* $A \subseteq B \subseteq C$.

Based upon this lemma, we shall now describe a set $\mathcal{L}$ of proper partial implications such that $\operatorname{conf}_{\mathbb{K}}(X \longrightarrow Y)$ is determined by $\mathcal{L}$ if $1 > \operatorname{conf}_{\mathbb{K}}(X \longrightarrow Y) \geqslant c$. In other words, for every model $\mathbb{L}$ of $\mathcal{L}$, the confidence of $X \longrightarrow Y$ in $\mathbb{L}$ is the same.

We shall do this as follows. From the first statement of the lemma we can deduce that it is sufficient to know the confidence of implications of the form $A'' \longrightarrow B''$, i. e. $\mathcal{L}$ only needs to contain partial implications of the form $(A'' \longrightarrow B'', c)$ for some sets $A, B \subseteq M$ and $c \in [0, 1]$.

Then, the second statement of the above lemma provides use with some kind of *multiplicativity* for confidence of implications under certain circumstances. We utilize this fact in the following way: at first, if $A'' \longrightarrow B''$ is a partial implication, then we can assume without loss of generality that $B'' \supseteq A''$. This is because

$$\operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow B'') = \operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow (A'' \cup B'')'').$$

Now we use the second statement of the above lemma. From this we can see that we only need to consider partial implications $A'' \longrightarrow B''$ with $A'' \subseteq B''$ as elements of $\mathcal{L}$ where A'' and B'' are *directly neighbored intents*. In other words, there does not exist a set C such that $A'' \subsetneq C'' \subsetneq B''$. If such a set C'' would exist, then

$$\operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow B'') = \operatorname{conf}_{\mathbb{K}}(A'' \longrightarrow C'') \cdot \operatorname{conf}_{\mathbb{K}}(C'' \longrightarrow B'').$$

Hence the confidence of $A'' \longrightarrow B''$ is already determined by the confidence of $A'' \longrightarrow C''$ and $C'' \longrightarrow B''$.

The set $\mathcal{L}$ containing only partial implications of the form $(A'' \longrightarrow B'', c)$ where $A'' \subseteq B''$ are directly neighbored is now as desired: if $(X \longrightarrow Y, c')$ is a partial implication of $\mathbb{K}$ with $c' \geqslant c$ and if $\mathbb{L}$ is a model of $\mathcal{L}$, then

$$\operatorname{conf}_{\mathbb{K}}(X \longrightarrow Y) = \operatorname{conf}_{\mathbb{L}}(X \longrightarrow Y).$$

In other words, $\mathcal{L}$ determines the confidence of $X \longrightarrow Y$ in $\mathbb{K}$.

6.4 Theorem *Let $\mathbb{K} = (G, M, I)$ be a finite formal context and let $c \in [0, 1)$. Define*

$$\mathcal{L}(\mathbb{K}, c) := \{ (A \longrightarrow B, c') \mid A, B \subseteq M, A'' = A, B'' = B, \\ c' = \operatorname{conf}_{\mathbb{K}}(A \longrightarrow B) \in [c, 1), \nexists C \subseteq M \colon A'' \subsetneq C'' \subsetneq B'' \}.$$

Let $X, Y \subseteq M$ and let $1 \geqslant d \geqslant c$. Let $\mathbb{L}$ be a model of $\mathcal{L}(\mathbb{K}, c)$ such that for each $A \subseteq M$, A'' is the same in both $\mathbb{K}$ and $\mathbb{L}$. Then $(X \longrightarrow Y, d)$ is a proper partial implication of $\mathbb{K}$ if and only if $(X \longrightarrow Y, d)$ is a proper partial implication of $\mathbb{L}$.

Proof Let $(X \longrightarrow Y, d)$ be a proper partial implication of $\mathbb{K}$. Then we can assume without loss of generality that $Y = Y'', X = X''$ and that $Y \supseteq X$. Since $\mathbb{L}$ is finite, there exists a sequence $X = C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq C_n = Y$ of directly neighbored intents of $\mathbb{L}$ (and also of $\mathbb{K}$). Then

$$\operatorname{conf}_{\mathbb{L}}(X \longrightarrow Y) = \prod_{i=1}^{n-1} \operatorname{conf}_{\mathbb{L}}(C_i \longrightarrow C_{i+1}).$$

But then

$$(C_i \longrightarrow C_{i+1}, \operatorname{conf}_{\mathbb{K}}(C_i \longrightarrow C_{i+1})) \in \mathcal{L}(\mathbb{K}, c),$$

as $\operatorname{conf}_{\mathbb{K}}(C_i \longrightarrow C_{i+1}) \geqslant d \geqslant c$. Since $\mathbb{L}$ is a model of $\mathcal{L}(\mathbb{K}, c)$ it holds

$$\operatorname{conf}_{\mathbb{K}}(C_i \longrightarrow C_{i+1}) = \operatorname{conf}_{\mathbb{L}}(C_i \longrightarrow C_{i+1})$$

and therefore $d = \operatorname{conf}_{\mathbb{L}}(X \longrightarrow Y)$. Reversing the roles of $\mathbb{K}$ and $\mathbb{L}$ yields the other direction of the claim, as $\mathbb{K}$ is a model of $\mathcal{L}(\mathbb{K}, c)$. □

6.2 A First Base

As the first statement of Lemma 6.3 implies, we can restrict our attention to partial implications consisting of intents only. The next theorem shows that the same holds for confident GCIs. The crucial observation here is that the GCI $A \sqsubseteq A^{\mathcal{I}\mathcal{I}}$ is valid in $\mathcal{I}$ for each finite interpretation $\mathcal{I}$ and concept description A. This is due to the fact that $A^{\mathcal{I}\mathcal{I}\mathcal{I}} = A^{\mathcal{I}}$ by Lemma 3.23. Then

$$A^{\mathcal{I}} \subseteq A^{\mathcal{I}} = A^{\mathcal{I}\mathcal{I}\mathcal{I}}$$

and therefore $A \sqsubseteq A^{\mathcal{I}\mathcal{I}}$ is valid in $\mathcal{I}$.

6.5 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $\mathcal{B}$ be a finite $\mathcal{EL}^{\perp}_{gfp}$-base of $\mathcal{I}$. Let $c \in [0, 1)$ and*

$$\mathcal{C} := \{ A^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}} \mid A, B \text{ concept descriptions}, 1 > \operatorname{conf}_{\mathcal{I}}(A^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}}) \geqslant c \}.$$

Then $\mathcal{B} \cup \mathcal{C}$ is a base of $\operatorname{Th}_c(\mathcal{I})$.

Proof Clearly $\mathcal{B} \cup \mathcal{C} \subseteq \mathrm{Th}_c(\mathcal{I})$ and it only remains to be shown that $\mathcal{B} \cup \mathcal{C}$ entails all $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs with confidence at least c in $\mathcal{I}$.

Let $A \sqsubseteq B$ be an $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCI with $\mathrm{conf}_{\mathcal{I}}(A \sqsubseteq B) \geqslant c$. We have to show that $\mathcal{B} \cup \mathcal{C} \models A \sqsubseteq B$. If $A \sqsubseteq B$ is already valid in $\mathcal{I}$, then $\mathcal{B} \models A \sqsubseteq B$ and nothing remains to be shown. We therefore assume that $1 > \mathrm{conf}_{\mathcal{I}}(A \sqsubseteq B) \geqslant c$.

As $A \sqsubseteq A^{\mathcal{II}}$ is valid in $\mathcal{I}$, $\mathcal{B} \models A \sqsubseteq A^{\mathcal{II}}$. Furthermore, $\mathrm{conf}_I(A \sqsubseteq B) = \mathrm{conf}_{\mathcal{I}}(A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}})$ and hence $(A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}}) \in \mathcal{C}$. Finally, $\varnothing \models B^{\mathcal{II}} \sqsubseteq B$. We therefore obtain

$$\mathcal{B} \cup \mathcal{C} \models A \sqsubseteq A^{\mathcal{II}}, A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}}, B^{\mathcal{II}} \sqsubseteq B$$

and hence $\mathcal{B} \cup \mathcal{C} \models A \sqsubseteq B$ as required. □

The set $\mathcal{C}$ defined in the previous theorem shall play some role in our further considerations. Therefore, we shall give it an extra name and define

$$\mathrm{Conf}(\mathcal{I}, c) := \{\, A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}} \mid A, B \text{ concept descriptions}, 1 > \mathrm{conf}_{\mathcal{I}}(A^{\mathcal{II}} \sqsubseteq B^{\mathcal{I}}) \geqslant c \,\}.$$

The base $\mathcal{B} \cup \mathrm{Conf}(\mathcal{I}, c)$ of the previous theorem is always a finite base of $\mathrm{Th}_c(\mathcal{I})$. To see this we observe that model-based most specific concepts $A^{\mathcal{II}}$ arise as concept descriptions $X^{\mathcal{I}}$ for $X \subseteq \Delta_{\mathcal{I}}$. Since there are only finitely many such X, there are only finitely many model-based most-specific concept descriptions of $\mathcal{I}$, up to equivalence.

6.6 Corollary *Let $\mathcal{I}$ be a finite interpretation and let $c \in [0, 1)$. Then*

$$\mathrm{Conf}(\mathcal{I}, c) \subseteq \{\, X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}} \mid X, Y \subseteq \Delta_{\mathcal{I}}, 1 > \mathrm{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \geqslant c \,\}.$$

In particular, if $\mathcal{B}$ is a finite base of $\mathcal{I}$, then $\mathcal{B} \cup \mathrm{Conf}(\mathcal{I}, c)$ is a finite base of $\mathrm{Th}_c(\mathcal{I})$.

Proof Let $A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}} \in \mathrm{Conf}(\mathcal{I}, c)$. Then define $X := A^{\mathcal{I}}, Y := B^{\mathcal{I}}$. Obviously, $X, Y \subseteq \Delta_{\mathcal{I}}$ and $\mathrm{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) = \mathrm{conf}_{\mathcal{I}}(A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}}) \in [c, 1)$. This shows the claimed inclusion. The rest of the corollary follows from Theorem 6.5. □

In [10] it has been shown that for finite interpretations $\mathcal{I}$ there always exists a finite base. Hence we obtain the following corollary.

6.7 Corollary *If $\mathcal{I}$ is a finite interpretation and $c \in [0, 1]$, then there always exists a finite base of $\mathrm{Th}_c(\mathcal{I})$.*

6.3 Using the Neighborhood Relation

We now know that finite bases of $\mathrm{Th}_c(\mathcal{I})$ always exist. However, we can make the set $\mathrm{Conf}(\mathcal{I}, c)$ smaller by using the idea of Theorem 6.4. For this, we shall first prove the multiplicativity property for the confidence of GCIs.

6.8 Lemma *Let $\mathcal{I}$ be a finite interpretation and let $(C_i \mid i = 0, \dots, n), n \in \mathbb{N}$, be a finite sequence of concept descriptions such that $C_{i+1}^{\mathcal{I}} \subseteq C_i^{\mathcal{I}}$ for all $i = 1, \dots, n-1$. Then*

$$\mathrm{conf}_{\mathcal{I}}(C_0 \sqsubseteq C_n) = \prod_{i=0}^{n-1} \mathrm{conf}_{\mathcal{I}}(C_i \sqsubseteq C_{i+1}).$$

Proof Let us first assume that the set $\{ i \mid C_i^{\mathcal{I}} = \emptyset \}$ is not empty and let

$$i := \min\{ i \mid C_i^{\mathcal{I}} = \emptyset \}.$$

If $i = 0$, then $C_j^{\mathcal{I}} = \emptyset$ for all $j \in \{ 0, \dots, n \}$, hence $\operatorname{conf}_{\mathcal{I}}(C_0 \sqsubseteq C_n) = 1$ and $\operatorname{conf}_{\mathcal{I}}(C_j \sqsubseteq C_{j+1}) = 1$ for all $j \in \{ 0, \dots, n \}$.

Otherwise, $0 < i \leqslant n$. But then $C_n^{\mathcal{I}} = \emptyset$ and hence $\operatorname{conf}_{\mathcal{I}}(C_0 \sqsubseteq C_n) = 0$. Furthermore, $\operatorname{conf}_{\mathcal{I}}(C_{i-1} \sqsubseteq C_i) = 0$ since $C_{i-1}^{\mathcal{I}} \neq \emptyset$ and $C_i^{\mathcal{I}} = \emptyset$. Therefore,

$$\prod_{i=1}^{n-1} \operatorname{conf}_{\mathcal{I}}(C_i \sqsubseteq C_{i+1}) = 0 = \operatorname{conf}_{\mathcal{I}}(C_0 \sqsubseteq C_n).$$

Finally, let us consider the case when $\{ i \mid C_i^{\mathcal{I}} = \emptyset \}$ is empty. Then we can calculate

$$\begin{aligned}
\prod_{i=1}^{n-1} \operatorname{conf}_{\mathcal{I}}(C_i \sqsubseteq C_{i+1}) &= \prod_{i=1}^{n-1} \frac{|C_i^{\mathcal{I}} \cap C_{i+1}^{\mathcal{I}}|}{|C_i^{\mathcal{I}}|} \\
&= \prod_{i=1}^{n-1} \frac{|C_{i+1}^{\mathcal{I}}|}{|C_i^{\mathcal{I}}|} \\
&= \frac{|C_n^{\mathcal{I}}|}{|C_0^{\mathcal{I}}|} \\
&= \frac{|C_0^{\mathcal{I}} \cap C_n^{\mathcal{I}}|}{|C_0^{\mathcal{I}}|} \\
&= \operatorname{conf}_{\mathcal{I}}(C_0 \sqsubseteq C_n).
\end{aligned}$$

□

Using this lemma we can now formulate and prove the an analog to Theorem 6.4.

6.9 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $c \in [0,1]$. Then the set*

$$\begin{aligned}
\mathcal{D} := \{ X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}} \mid\ & Y \subseteq X \subseteq \Delta_{\mathcal{I}}, 1 > \operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \geqslant c, \\
& \nexists Z \subseteq \Delta_{\mathcal{I}} \colon Y \subseteq Z \subseteq X \text{ and } Y^{\mathcal{I}} \not\equiv Z^{\mathcal{I}} \not\equiv X^{\mathcal{I}} \}
\end{aligned}$$

satisfies $\mathcal{D} \models \operatorname{Conf}(\mathcal{I}, c)$. In particular, if $\mathcal{B}$ is a finite base of $\mathcal{I}$, then $\mathcal{B} \cup \mathcal{D}$ is a finite base of $\operatorname{Th}_c(\mathcal{I})$.

Proof Let $A^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}} \in \operatorname{Conf}(\mathcal{I}, c)$. As $(A \sqcap B)^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}}$ always holds, $A^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}}$ follows from $A^{\mathcal{I}\mathcal{I}} \sqsubseteq (A \sqcap B)^{\mathcal{I}\mathcal{I}}$. Furthermore, since $(A^{\mathcal{I}\mathcal{I}} \sqcap B^{\mathcal{I}\mathcal{I}})^{\mathcal{I}} = A^{\mathcal{I}\mathcal{I}\mathcal{I}} \cap B^{\mathcal{I}\mathcal{I}\mathcal{I}} = A^{\mathcal{I}} \cap B^{\mathcal{I}} = (A \cap B)^{\mathcal{I}}$, we obtain

$$\begin{aligned}
\operatorname{conf}_{\mathcal{I}}(A^{\mathcal{I}\mathcal{I}} \sqsubseteq B^{\mathcal{I}\mathcal{I}}) &= \frac{|(A^{\mathcal{I}\mathcal{I}} \sqcap B^{\mathcal{I}\mathcal{I}})^{\mathcal{I}}|}{|A^{\mathcal{I}\mathcal{I}\mathcal{I}}|} \\
&= \frac{|(A \sqcap B)^{\mathcal{I}}|}{|A^{\mathcal{I}\mathcal{I}\mathcal{I}}|} \\
&= \frac{|(A \sqcap B)^{\mathcal{I}\mathcal{I}\mathcal{I}}|}{|A^{\mathcal{I}\mathcal{I}\mathcal{I}}|} \\
&= \operatorname{conf}_{\mathcal{I}}(A^{\mathcal{I}\mathcal{I}} \sqsubseteq (A \sqcap B)^{\mathcal{I}\mathcal{I}})
\end{aligned}$$

since $|A^{\mathcal{III}}| \neq 0$, as otherwise $\operatorname{conf}_{\mathcal{I}}(A^{\mathcal{II}} \sqsubseteq B^{\mathcal{II}}) = 1$. Therefore, $A^{\mathcal{II}} \sqsubseteq (A \sqcap B)^{\mathcal{II}} \in \operatorname{Conf}(\mathcal{I}, c)$ and we shall show now that $\mathcal{D} \models (A^{\mathcal{II}} \sqsubseteq (A \sqcap B)^{\mathcal{II}})$.

Let us define $X := A^{\mathcal{I}}$ and $Y := (A \sqcap B)^{\mathcal{I}}$. Then $Y \subseteq X$. As $\mathcal{I}$ is finite, $\Delta_{\mathcal{I}}$ is finite and therefore the set

$$\{ Z^{\mathcal{I}} \mid Y \subseteq Z \subseteq X, Y^{\mathcal{I}} \not\equiv Z^{\mathcal{I}} \not\equiv X^{\mathcal{I}} \}$$

is finite as well. Hence we can find a finite sequence $(C_i \mid 0 \leqslant i \leqslant n)$ for some $n \in \mathbb{N}$ of sets $C_i \subseteq \Delta_{\mathcal{I}}$ such that

i. $Y := C_n, X := C_0$,

ii. $C_{i+1} \subsetneq C_i$ for $0 \leqslant i < n$,

iii. $C_i^{\mathcal{I}} \not\equiv C_{i+1}^{\mathcal{I}}$ for $0 \leqslant i < n$,

iv. $C_i^{\mathcal{II}} = C_i$ for $0 \leqslant i \leqslant n$,

v. $C_{i+1} \subseteq Z \subseteq C_i$ implies $C_i^{\mathcal{I}} \equiv Z^{\mathcal{I}}$ or $C_{i+1}^{\mathcal{I}} \equiv Z^{\mathcal{I}}$ for $0 \leqslant i < n$.

Then by Lemma 6.8

$$\operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) = \prod_{i=0}^{n-1} \operatorname{conf}_{\mathcal{I}}(C_i^{\mathcal{I}} \sqsubseteq C_{i+1}^{\mathcal{I}})$$

and therefore $\operatorname{conf}_{\mathcal{I}}(C_i^{\mathcal{I}} \sqsubseteq C_{i+1}^{\mathcal{I}}) \in [c, 1]$. As $C_i^{\mathcal{II}} \subseteq C_{i+1}^{\mathcal{II}}$ would imply $C_i \subseteq C_{i+1}$ and so $C_i \subseteq C_i$, we obtain $\operatorname{conf}_{\mathcal{I}}(C_i^{\mathcal{I}} \sqsubseteq C_{i+1}^{\mathcal{I}}) \neq 1$. Hence, $C_i^{\mathcal{I}} \sqsubseteq C_{i+1}^{\mathcal{II}} \in \mathcal{D}$ for $0 \leqslant i < n$. Thus

$$\mathcal{D} \models C_i^{\mathcal{I}} \sqsubseteq C_{i+1}^{\mathcal{I}}, \quad (0 \leqslant i < n)$$

and therefore $\mathcal{D} \models (X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) = (A^{\mathcal{II}} \sqsubseteq (A \sqcap B)^{\mathcal{II}})$ as required. □

To ease further discussions, we shall give the set $\mathcal{D}$ of the previous theorem an extra name. As this definition is inspired by ideas from Luxenburger, let us define

$$\begin{aligned} \operatorname{Lux}(\mathcal{I}, c) := \{ X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}} \mid{} & Y \subseteq X \subseteq \Delta_{\mathcal{I}}, 1 > \operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) \geqslant c, \\ & \nexists Z \subseteq \Delta_{\mathcal{I}} \colon Y \subseteq Z \subseteq X \text{ and } Y^{\mathcal{I}} \not\equiv Z^{\mathcal{I}} \not\equiv X^{\mathcal{I}} \}. \end{aligned}$$

6.4 Computing the Base from the Formal Context $\mathbb{K}_{\mathcal{I}}$

Recall the main results of Section 4. In this section we have shown certain similarities between the description logic $\mathcal{EL}^{\perp}_{\text{gfp}}$ on the one hand and formal concept analysis on the other hand. In particular, in Theorem 4.13 we have shown that the lattice of intents of the formal context $\mathbb{K}_{\mathcal{I}}$ is order-isomorphic to the lattice of model-based most-specific concept descriptions (up to equivalence) of $\mathcal{I}$. In particular, this means the directly neighbored intents of $\mathbb{K}_{\mathcal{I}}$ correspond to directly neighbored model-based most-specific concept descriptions of $\mathcal{I}$. But directly neighbored model-based most-specific concept descriptions are needed in the computation of the set $\operatorname{Lux}(\mathcal{I}, c)$. It is therefore obvious that we can make use of this order-isomorphism to do computations directly in $\mathbb{K}_{\mathcal{I}}$. To show this is the purpose of this subsection.

We start by showing that computing confidence of a GCI in an interpretation $\mathcal{I}$ can just as well be done in the formal context $\mathbb{K}_{\mathcal{I}}$. Thereafter we show that indeed directly neighbored model-based most-specific concept descriptions of $\mathcal{I}$ correspond to directly-neighbored intents of $\mathbb{K}_{\mathcal{I}}$. We then use these two facts to prove Theorem 6.13.

6.10 Lemma *Let $\mathcal{I}$ be a finite interpretation and let $X, Y \subseteq \Delta_{\mathcal{I}}$. Then*

$$\operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) = \operatorname{conf}_{\mathbb{K}_{\mathcal{I}}}(X' \longrightarrow Y').$$

Proof By Lemma 4.11,

$$\begin{aligned}\operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) &= \frac{|(X^{\mathcal{I}} \sqcap Y^{\mathcal{I}})^{\mathcal{I}}|}{|X^{\mathcal{I}\mathcal{I}}|} \\ &= \frac{|X^{\mathcal{I}\mathcal{I}} \cap Y^{\mathcal{I}\mathcal{I}}|}{|X^{\mathcal{I}\mathcal{I}}|} \\ &= \frac{|X'' \cap Y''|}{|X''|} \\ &= \frac{|(X' \cup Y')'|}{|X''|} \\ &= \operatorname{conf}_{\mathbb{K}_{\mathcal{I}}}(X' \longrightarrow Y')\end{aligned}$$

if $|X^{\mathcal{I}\mathcal{I}}| \neq 0$. Otherwise, $|X''| = 0$ and hence $\operatorname{conf}_{\mathcal{I}}(X^{\mathcal{I}} \sqsubseteq Y^{\mathcal{I}}) = 1 = \operatorname{conf}_{\mathbb{K}_{\mathcal{I}}}(X' \longrightarrow Y')$. ◻

We now show that direct neighborhood is preserved under the order-isomorphism from Theorem 4.13. This result is formulated in Lemma 6.12. To show it we shall start with the following proposition.

6.11 Proposition *Let $\mathcal{I}$ be a finite interpretation and let $A \subseteq B \subseteq \Delta_{\mathcal{I}}$. Then*

1. *$A^{\mathcal{I}} \sqsubsetneq B^{\mathcal{I}}$ implies $\operatorname{pr}_{M_{\mathcal{I}}}(A^{\mathcal{I}}) \supsetneq \operatorname{pr}_{M_{\mathcal{I}}}(B^{\mathcal{I}})$ and*
2. *$A' \supsetneq B'$ implies $\bigsqcap A' \sqsubsetneq \bigsqcap B'$.*

Proof It directly follows from Theorem 4.13 that $A^{\mathcal{I}} \sqsubsetneq B^{\mathcal{I}}$ implies $\operatorname{pr}_{M_{\mathcal{I}}}(A^{\mathcal{I}}) \supseteq \operatorname{pr}_{M_{\mathcal{I}}}(B^{\mathcal{I}})$ and that $A' \supsetneq B'$ implies $\bigsqcap A' \sqsubseteq \bigsqcap B'$.

Let us assume that $A^{\mathcal{I}} \not\equiv B^{\mathcal{I}}$ but $\operatorname{pr}_{M_{\mathcal{I}}}(A^{\mathcal{I}}) = \operatorname{pr}_{M_{\mathcal{I}}}(B^{\mathcal{I}})$. Then by Theorem 4.13,

$$A^{\mathcal{I}} \equiv \bigsqcap \operatorname{pr}_{M_{\mathcal{I}}}(A^{\mathcal{I}}) = \bigsqcap \operatorname{pr}_{M_{\mathcal{I}}}(B^{\mathcal{I}}) \equiv B^{\mathcal{I}},$$

as both $A^{\mathcal{I}}$ and $B^{\mathcal{I}}$ are model-based most specific concepts of $\mathcal{I}$, a contradiction.

Conversely, if $A' \neq B'$ but $\bigsqcap A' \equiv \bigsqcap B'$, then again by Theorem 4.13,

$$A' = \operatorname{pr}_{M_{\mathcal{I}}}(\bigsqcap A') = \operatorname{pr}_{M_{\mathcal{I}}}(\bigsqcap B') = B'$$

as $A', B' \in \operatorname{Int}(\mathbb{K}_{\mathcal{I}})$, again a contradiction. ◻

6.12 Lemma *Let $\mathcal{I}$ be a finite interpretation and let $Y \subseteq Z \subseteq X \subseteq \Delta_{\mathcal{I}}$. Then Z satisfies $Y^{\mathcal{I}} \not\equiv Z^{\mathcal{I}} \not\equiv X^{\mathcal{I}}$ if and only if $Y' \neq Z' \neq X'$, where the derivations are computed in $\mathbb{K}_{\mathcal{I}}$.*

Proof Suppose $Y^{\mathcal{I}} \not\equiv Z^{\mathcal{I}} \not\equiv X^{\mathcal{I}}$, i. e. $Y^{\mathcal{I}} \sqsubsetneq Z^{\mathcal{I}} \sqsubsetneq X^{\mathcal{I}}$. Then by Proposition 6.11, $\operatorname{pr}_{M_{\mathcal{I}}}(Y^{\mathcal{I}}) \supsetneq \operatorname{pr}_{M_{\mathcal{I}}}(Z^{\mathcal{I}}) \supsetneq \operatorname{pr}_{M_{\mathcal{I}}}(X^{\mathcal{I}})$ and by Proposition 4.6, $Y' \supsetneq Z' \supsetneq X'$.

Conversely, let $Y' \supsetneq Z' \supsetneq X'$. Then by Proposition 6.11, $\bigsqcap Y' \sqsupsetneq \bigsqcap Z' \bigsqcap X'$ and by Proposition 4.7, $Y^{\mathcal{I}} \sqsubsetneq Z^{\mathcal{I}} \sqsubsetneq X^{\mathcal{I}}$ as required. ◻

With this result in conjunction with Lemma 6.10, Proposition 4.7 and Theorem 6.9, we immediately obtain the following theorem.

6.13 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $c \in [0,1]$. Then the set*

$$\mathcal{E} := \{ \bigsqcap X' \sqsubseteq \bigsqcap Y' \mid Y \subseteq X \subseteq \Delta_{\mathcal{I}}, 1 > \operatorname{conf}_{\mathbb{K}_{\mathcal{I}}}(X' \longrightarrow Y') \geqslant c, \\ \nexists Z \subseteq \Delta_{\mathcal{I}} \colon Y \subseteq Z \subseteq X \text{ and } Y' \neq Z' \neq X' \}$$

is equal to $\operatorname{Lux}(\mathcal{I}, c)$ *up to equivalence of the concept descriptions contained in the GCIs. In particular, if $\mathcal{B}$ is a finite base of $\mathcal{I}$, then $\mathcal{B} \cup \mathcal{E}$ is a finite base of* $\operatorname{Th}_c(\mathcal{I})$.

With this theorem we see that we can do all of the actual computation directly in the formal context $\mathbb{K}_{\mathcal{I}}$.

7 Reducing the Size of the Base

We have seen in the previous section that finite bases for $\operatorname{Th}_c(\mathcal{I})$ always exist. For this we have used ideas from Luxenburger for partial implications and transferred them to confident GCIs. This allowed us to not only prove the existence of finite bases of $\operatorname{Th}_c(\mathcal{I})$. Indeed, using these ideas from formal concept analysis we were able to explicitly describe such a base, which can therefore be computed effectively.

In this section we shall exploit the correspondence between the description logic $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$ and formal concept analysis a bit further. For this, we are going to make use of the results we have obtained in Section 4.

More precisely, we shall use formal concept analysis to remove some redundancies from the bases of $\operatorname{Th}_c(\mathcal{I})$ we have discussed so far. These bases where of the form $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$ as defined in Theorem 6.5 or $\mathcal{B} \cup \operatorname{Lux}(\mathcal{I}, c)$ as defined in Theorem 6.9. For the following considerations, we shall concentrate on a base of $\operatorname{Th}_c(\mathcal{I})$ of the form $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$. This base does not need to be non-redundant. Indeed, it may be the case that some GCIs in $\mathcal{B}$ are already entailed by GCIs from $\operatorname{Conf}(\mathcal{I}, c)$. As the computation of $\mathcal{B}$ might be quite expensive, we shall show how we can make use of the set $\operatorname{Conf}(\mathcal{I}, c)$ as *background knowledge* to allow for the computation of a smaller set $\hat{\mathcal{B}} \subseteq \mathcal{B}$ such that $\hat{\mathcal{B}} \cup \operatorname{Conf}(\mathcal{I}, c)$ is a base of $\operatorname{Th}_c(\mathcal{I})$. We shall call such a set $\hat{\mathcal{B}}$ a *completing sets* for $\operatorname{Conf}(\mathcal{I}, c)$ and $\mathcal{I}$.

Moreover, it is easy to see that we do not really need all GCIs in the set $\operatorname{Conf}(\mathcal{I}, c)$. As we have already discussed in Theorem 6.9, the set $\operatorname{Lux}(\mathcal{I}, c) \subseteq \operatorname{Conf}(\mathcal{I}, c)$ suffices. More generally, it is sufficient to consider a base $\mathcal{C}$ of $\operatorname{Conf}(\mathcal{I}, c)$. If then $\mathcal{B}$ is a completing set for $\mathcal{C}$, $\mathcal{B} \cup \mathcal{C}$ is a base for $\operatorname{Th}_c(\mathcal{I})$. This will be shown in Section 7.1.

Using the notion of the canonical base introduced in Section 2 we shall then show that we are even able to find a completing set of $\mathcal{C}$ that is of minimal cardinality. To show this we shall adapt the proof of [10, Theorem 5.18]. This theorem shows minimal cardinality of a finite base of $\mathcal{I}$. As we shall see, we can use its proof to show that the canonical base of $\mathbb{K}_{\mathcal{I}}$ with some suitable chosen background knowledge yields a completing set of $\mathcal{C}$ of minimal cardinality.

7.1 Completing Sets

We have already mentioned that the base $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$ for $\operatorname{Th}_c(\mathcal{I})$ as described in Theorem 6.5 may be redundant. This even holds for the base $\mathcal{B} \cup \operatorname{Lux}(\mathcal{I}, c)$ constructed in Theorem 6.9. This redundancy might be undesired, because an ontology that is constructed from such a base then contains redundant information. This in turn makes the ontology unnecessarily larger and reasoning with this ontology more expensive. Therefore, one would like to construct ontologies as small as possible.

In theory, this problem can be handled very easily. As we already have given a finite base $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$ of $\operatorname{Th}_c(\mathcal{I})$, we can successively search for GCIs in this set that are already entailed by others. If such a GCIs exists, it is removed and the procedure starts anew. If no such GCI exists, the set is obviously non-redundant.

This naïve algorithm however is not really suitable for handling large bases of $\operatorname{Th}_c(\mathcal{I})$. Instead, it might be preferable to explicitly describe non-redundant bases. Doing this is an open problem for bases of $\operatorname{Th}_c(\mathcal{I})$. Therefore, in this section we shall focus on removing some obvious redundancies from our base $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$ of $\operatorname{Th}_c(\mathcal{I})$.

We start with a simple observation. Let us consider the base $\mathcal{B} \cup \operatorname{Conf}(\mathcal{I}, c)$ of $\operatorname{Th}_c(\mathcal{I})$ from Theorem 6.5. In this base both sets $\mathcal{B}$ and $\operatorname{Conf}(\mathcal{I}, c)$ can potentially be made smaller. Indeed, if suffices to consider subsets $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$ such that all GCIs from $\operatorname{Conf}(\mathcal{I}, c)$ already follow from $\mathcal{C}$. In addition, it is not necessary for the set $\mathcal{B}$ to be a base of $\mathcal{I}$; it merely suffices if $\mathcal{B}$ is a set of valid GCIs such that $\mathcal{B} \cup \mathcal{C}$ is complete for $\operatorname{Th}_c(\mathcal{I})$.

7.1 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $c \in [0,1]$. Let $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$ such that $\mathcal{C} \models \operatorname{Conf}(\mathcal{I}, c)$ and let $\mathcal{B}$ be a set of valid GCIs of $\mathcal{I}$ such that $\mathcal{B} \cup \mathcal{C}$ entails all valid GCIs of $\mathcal{I}$. Then $\mathcal{B} \cup \mathcal{C}$ is a base of $\operatorname{Th}_c(\mathcal{I})$.*

Proof Let $\hat{\mathcal{B}}$ be a finite base of $\mathcal{I}$. Then by Theorem 6.5 the set $\hat{\mathcal{B}} \cup \operatorname{Conf}(\mathcal{I}, c)$ is a base of $\operatorname{Th}_c(\mathcal{I})$. We show that the set $\mathcal{B} \cup \mathcal{C}$ entails all GCIs from $\hat{\mathcal{B}} \cup \operatorname{Conf}(\mathcal{I}, c)$.

If $(C \sqsubseteq D) \in \operatorname{Conf}(\mathcal{I}, c)$, then by the prerequisites of the theorem it holds $\mathcal{C} \models (C \sqsubseteq D)$. Therefore, $\mathcal{B} \cup \mathcal{C} \models \operatorname{Conf}(\mathcal{I}, c)$.

If $C \sqsubseteq D$ is a GCI that holds in $\mathcal{I}$ then $\mathcal{B} \cup \mathcal{C} \models C \sqsubseteq D$, as $\mathcal{B} \cup \mathcal{C}$ entails all valid GCIs of $\mathcal{I}$. In particular, $\mathcal{B} \cup \mathcal{C} \models \hat{\mathcal{B}}$. □

This theorem now motivates the following definition.

7.2 Definition Let $\mathcal{I}$ be an interpretation and let $\mathcal{C}$ be a set of GCIs. Then a set $\mathcal{B}$ of valid GCIs is called a *completing set* for $\mathcal{C}$ and $\mathcal{I}$ if and only if $\mathcal{B} \cup \mathcal{C}$ is complete for $\mathcal{I}$. ◇

The above theorem is not very helpful in finding sets the $\mathcal{B}$ and $\mathcal{C}$. Indeed, explicitly describing a minimal set $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$ that is complete for the set $\operatorname{Conf}(\mathcal{I}, c)$ is an open problem. However, we shall see shortly that we can utilize formal concept analysis to find completing sets for $\mathcal{C}$.

The idea behind this construction is the following: from formal concept analysis we know how to find for a set $\mathcal{L}$ of implications of a formal context $\mathbb{K}$ a set $\mathcal{K}$ of valid implications of $\mathbb{K}$ such that $\mathcal{L} \cup \mathcal{K}$ is complete for $\mathbb{K}$, see Theorem 2.22. This theorem can be transferred to GCIs to yield Theorem 7.3.

Before we shall formulate this theorem let us give some motivation. Let $\mathcal{C}$ be a set of GCIs. Then we want to find a $\mathcal{B}$ of GCIs such that $\mathcal{B} \cup \mathcal{C}$ is complete for $\mathcal{I}$. To use formal concept analysis here we shall make use of the mappings $C \longmapsto \operatorname{pr}_{M_{\mathcal{I}}}(C)$ and $U \longmapsto \bigsqcap U$ that we have introduced in Section 4. More precisely, we associate to $\mathcal{C}$ the set

$$\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C}) := \{ \operatorname{pr}_{M_{\mathcal{I}}}(X) \longrightarrow \operatorname{pr}_{M_{\mathcal{I}}}(Y) \mid (X \sqsubseteq Y) \in \mathcal{C} \}.$$

Then $\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C})$ is a set of implications. Therefore, we can compute a set $\mathcal{L}$ of valid implications of $\mathbb{K}_{\mathcal{I}}$ such that $\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C}) \cup \mathcal{L}$ is complete for $\mathbb{K}_{\mathcal{I}}$. Additionally, implications in $\mathcal{L}$ are of the form $U \longrightarrow U''$. It then follows that the set

$$\{ \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid (U \longrightarrow U'') \in \mathcal{L} \}$$

together with $\mathcal{C}$ is complete for $\mathcal{I}$, i. e. entails all valid GCIs of $\mathcal{I}$.

This connection has already been used in [10, Theorem 5.12], however only for the case where the set $\mathcal{C}$ contains valid GCIs. We shall generalize this theorem and its proof to also cover the case where $\mathcal{C}$ may contain arbitrary GCIs.

7.3 Theorem *Let $\mathcal{I}$ be a finite interpretation and let $\mathcal{C}$ be a set of GCIs. Let $\mathcal{L} \subseteq \mathrm{Th}(\mathbb{K}_{\mathcal{I}})$ be such that*

i. $\mathcal{L} \cup \{ \mathrm{pr}_{M_{\mathcal{I}}}(X) \longrightarrow \mathrm{pr}_{M_{\mathcal{I}}}(Y) \mid (X \sqsubseteq Y) \in \mathcal{C} \}$ is complete for $\mathrm{Th}(\mathbb{K}_{\mathcal{I}})$ and

ii. $\mathcal{L}$ only contains implications of the form $A \longrightarrow A''$ with $A \subseteq M_{\mathcal{I}}$.

Define

$$\mathcal{B} := \{ \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid (U \longrightarrow U'') \in \mathcal{L} \}.$$

Then $\mathcal{B} \cup \mathcal{C}$ is complete for $\mathcal{I}$.

To show completeness for the set $\mathcal{B} \cup \mathcal{C}$ from the above theorem let us recall the definition of the set $\mathcal{B}_2$ from Theorem 3.27. There, $\mathcal{B}_2$ has been defined as

$$\mathcal{B}_2 := \{ \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid U \subseteq M_{\mathcal{I}} \}.$$

$\mathcal{B}_2$ is a base for $\mathcal{I}$. In the proof of Theorem 7.3 we shall utilize this fact and show that $\mathcal{B} \cup \mathcal{C}$ is complete for the set $\mathcal{B}_2$.

Proof (of Theorem 7.3) We show that

$$\mathcal{B} \cup \mathcal{C} \models \mathcal{B}_2$$

with the set $\mathcal{B}_2$ as defined in Theorem 3.27. For this, let $\mathcal{J}$ be a finite interpretation such that $\mathcal{J} \models \mathcal{B} \cup \mathcal{C}$.

Let $\mathbb{K}$ now denote the formal context induced by $M_{\mathcal{I}}$ and $\mathcal{J}$. Since we are dealing with two contexts $\mathbb{K}_{\mathcal{I}}$ and $\mathbb{K}$ now, we shall distinguish the derivation operators by writing $\cdot^{\prime_{\mathcal{I}}}$ and $\cdot^{\prime_{\mathcal{J}}}$, respectively. For brevity, let us furthermore define

$$\mathcal{L}_{\mathcal{C}} := \{ \mathrm{pr}_{M_{\mathcal{I}}}(X) \longrightarrow \mathrm{pr}_{M_{\mathcal{I}}}(Y) \mid (X \sqsubseteq Y) \in \mathcal{C} \}.$$

We shall now show the following claims

i. $\mathbb{K} \models \mathcal{L} \cup \mathcal{L}_{\mathcal{C}}$,

ii. $\mathbb{K} \models (V \longrightarrow V^{\prime_{\mathcal{I}}\prime_{\mathcal{I}}})$ for each $V \subseteq M_{\mathcal{I}}$, and

iii. $\mathcal{J} \models (\bigsqcap V \sqsubseteq (\bigsqcap V)^{\mathcal{I}\mathcal{I}})$ for each $V \subseteq M_{\mathcal{I}}$.

From the last claim we can conclude $\mathcal{B} \cup \mathcal{C} \models \mathcal{B}_2$.

To prove the first claim, we shall start by showing a connection between the operators $\cdot^{\mathcal{I}}$, $\cdot^{\mathcal{J}}$, $\cdot^{\prime_{\mathcal{I}}}$ and $\cdot^{\prime_{\mathcal{J}}}$. For this, let $U \subseteq M_{\mathcal{I}}$. Then by Proposition 4.7

$$(\bigsqcap U)^{\mathcal{J}} = U^{\prime_{\mathcal{J}}}. \tag{7.1}$$

Since $(\bigsqcap U)^{\mathcal{I}\mathcal{I}}$ is expressible in terms of $M_{\mathcal{I}}$ as of Theorem 4.9, Proposition 4.5 yields

$$(\bigsqcap U)^{\mathcal{I}\mathcal{I}} \equiv (\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}((\bigsqcap U)^{\mathcal{I}\mathcal{I}})).$$

Therefore, we obtain with Theorem 4.13

$$((\bigsqcap U)^{\mathcal{I}\mathcal{I}})^{\mathcal{J}} = (\mathrm{pr}_{M_{\mathcal{I}}}((\bigsqcap U)^{\mathcal{I}\mathcal{I}}))^{\mathcal{J}} = U'^{\mathcal{I}'\mathcal{I}'\mathcal{J}}. \tag{7.2}$$

Now let $(U \sqsubseteq U'^{\mathcal{I}'\mathcal{I}}) \in \mathcal{L}$. Then $J \models (U \sqsubseteq U'^{\mathcal{I}'\mathcal{I}}$ by the choice of $\mathcal{J}$. Hence $(\bigsqcap U)^{\mathcal{J}} \subseteq ((\bigsqcap U)^{\mathcal{I}\mathcal{I}})^{\mathcal{J}}$ and by (7.1) and (7.2)

$$U'^{\mathcal{J}} \subseteq U'^{\mathcal{I}'\mathcal{I}'\mathcal{J}}.$$

This is the same as $\mathbb{K} \models (U \longrightarrow U'^{\mathcal{I}'\mathcal{I}})$.

Let $(X \sqsubseteq Y) \in \mathcal{C}$. Then $\mathcal{J} \models (X \sqsubseteq Y)$. Therefore,

$$X^{\mathcal{J}} \subseteq Y^{\mathcal{J}}.$$

As both $X^{\mathcal{I}\mathcal{I}}$ and $Y^{\mathcal{I}\mathcal{I}}$ are expressible in terms of $M_{\mathcal{I}}$, Proposition 4.5 yields

$$(\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(X))^{\mathcal{J}} \subseteq (\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(Y))^{\mathcal{J}}.$$

With (7.1) we obtain from this

$$(\mathrm{pr}_{M_{\mathcal{I}}}(X))'^{\mathcal{J}} \subseteq (\mathrm{pr}_{M_{\mathcal{I}}}(Y))^{\mathcal{J}}$$

and thus

$$\mathbb{K} \models (\mathrm{pr}_{M_{\mathcal{I}}}(X) \longrightarrow \mathrm{pr}_{M_{\mathcal{I}}}(Y)).$$

Therefore, we have shown that

$$\mathbb{K} \models \mathcal{L} \cup \mathcal{L}_{\mathcal{C}}$$

as required by the first claim.

Now let $V \subseteq M_{\mathcal{I}}$. Then certainly

$$\mathbb{K}_{\mathcal{I}} \models (V \longrightarrow V'^{\mathcal{I}'\mathcal{I}}).$$

Therefore, $\mathcal{L} \cup \mathcal{L}_{\mathcal{C}} \models (V \longrightarrow V'^{\mathcal{I}'\mathcal{I}})$ as $\mathcal{L} \cup \mathcal{L}_{\mathcal{C}}$ is complete for $\mathrm{Th}(\mathbb{K}_{\mathcal{I}})$. Now $\mathbb{K} \models \mathcal{L} \cup \mathcal{L}_{\mathcal{C}}$ and therefore

$$\mathbb{K} \models (V \longrightarrow V'^{\mathcal{I}'\mathcal{I}}),$$

which means nothing else but $V'^{\mathcal{J}} \subseteq V'^{\mathcal{I}'\mathcal{I}'\mathcal{J}}$. Then (7.1) and (7.2) yield

$$(\bigsqcap V)^{\mathcal{J}} \subseteq (\bigsqcap V^{\mathcal{I}\mathcal{I}})^{\mathcal{J}}$$

which amounts to $\mathcal{J} \models (\bigsqcap V \sqsubseteq (\bigsqcap V)^{\mathcal{I}\mathcal{I}})$, as it was claimed. □

We now apply this theorem to our special setting. For this we assume that we are given a complete set of GCIs of the finite set $\mathrm{Conf}(\mathcal{I}, c)$. The idea is then to use Theorem 7.3 to compute a finite set $\mathcal{B}$ such that $\mathcal{B} \cup \mathrm{Conf}(\mathcal{I}, c)$ is complete for $\mathcal{I}$. By Theorem 7.1 $\mathcal{B} \cup \mathrm{Conf}(\mathcal{I}, c)$ is a finite base of $\mathrm{Th}_c(\mathcal{I})$.

However, the set $\mathcal{B}$ may contain GCIs with redundant descriptions. These descriptions may be redundant because some parts of it are subsumed by others. To see what is meant by this, let us consider the following example.

7.4 Example Consider the interpretation $\mathcal{I}$ in Figure 6. There, $N_C = \emptyset$, $N_R = \{ r \}$ and $N_I = \{ x, y \}$.

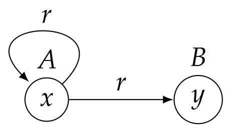

Figure 6: Example interpretation $\mathcal{I}$ that contains subsumption dependencies between elements of $M_{\mathcal{I}}$

For this interpretation $\mathcal{I}$ we can compute

$$\varnothing^{\mathcal{I}} = \bot \qquad \{x\}^{\mathcal{I}} = (A, \{A \equiv \exists r.A\})$$
$$\{y\}^{\mathcal{I}} = \top \qquad \{x, y\}^{\mathcal{I}} = \top.$$

Therefore,

$$M_{\mathcal{I}} = \{\bot, \exists r.\top, (A, \{A \equiv \exists r.A\})\}.$$

Note that $\exists r.(A, \{A \equiv \exists r.A\}) \equiv (A, \{A \equiv \exists r.A\})$.

If we then choose $\mathcal{C} = \varnothing$ we can compute a set $\mathcal{B}$ using Theorem 7.3 that is complete for $\mathcal{I}$, i. e. that is a base for $\mathcal{B}$. More precisely, the context $\mathbb{K}_{\mathcal{I}}$ has the following form:

$\mathbb{K}_{\mathcal{I}} =$

	$\bot$	$\exists r.\top$	$(A, \{A \equiv \exists r.A\})$
x		$\times$	$\times$
y			

The following implications then yield a non-redundant base of $\mathbb{K}_{\mathcal{I}}$:

$$\{\bot\} \longrightarrow \{(A, \{A \equiv \exists r.A\}), \exists r.\top\}$$
$$\{(A, \{A \equiv \exists r.A\})\} \longrightarrow \{\exists r.\top\}$$
$$\{\exists r.\top\} \longrightarrow \{(A, \{A \equiv \exists r.A\})\}$$

The second implication gives rise to the GCI

$$(A, \{A \equiv \exists r.A\}) \sqsubseteq \exists r.\top \tag{7.3}$$

which holds in every interpretation, as $(A, \{A \equiv \exists r.A\})$ is subsumed by $\exists r.\top$. This GCI is therefore redundant. ◇

The phenomenon shown in the previous example arises from the fact that the subsumption relations between concept descriptions in $M_{\mathcal{I}}$ is not present in the formal context $\mathbb{K}_{\mathcal{I}}$. However, we can avoid this by explicitly including the knowledge about the subsumption relation. More precisely, we can define

$$\mathcal{S}_{\mathcal{I}} := \{C \sqsubseteq D \mid C, D \in M_{\mathcal{I}}, C \sqsubseteq D\}.$$

(Note that the first $\sqsubseteq$ denotes a GCI while the second denotes the fact that C is subsumed by D.) Then we can apply Theorem 7.3 to the set $\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}$ and obtain a set $\mathcal{B}$ of GCIs that do not contain GCIs as in (7.3).

The following theorem makes use of this. Recall that we have defined for a set $\mathcal{D}$ of GCIs the set $\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{D})$ to be

$$\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{D}) := \{\operatorname{pr}_{M_{\mathcal{I}}}(C) \longrightarrow \operatorname{pr}_{M_{\mathcal{I}}}(Y) \mid (X \sqsubseteq Y) \in \mathcal{D}\}.$$

7.5 Corollary *Let $\mathcal{I}$ be a finite interpretation and $c \in [0,1]$. Let $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$ be complete for $\operatorname{Conf}(\mathcal{I}, c)$. Define*

$$\mathcal{L} := \operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{\mathbb{K}_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))$$

and

$$\mathcal{B} := \{ \bigsqcap A \sqsubseteq (\bigsqcap A)^{\mathcal{I}\mathcal{I}} \mid (A \longrightarrow A'') \in \mathcal{L} \}.$$

Then $\mathcal{B} \cup \mathcal{C}$ is a base of $\operatorname{Th}_c(\mathcal{I})$.

Proof By Theorem 2.17, the set $\operatorname{pr}_{\mathbb{K}_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}) \cup \mathcal{L}$ is complete for $\mathbb{K}_{\mathcal{I}}$ and $\mathcal{L}$ only contains implications of the form $A \longrightarrow A''$ for some $A \subseteq M_{\mathcal{I}}$. Then by Theorem 7.3, the set $\mathcal{B} \cup \mathcal{C} \cup \mathcal{S}_{\mathcal{I}}$ is complete for $\mathcal{I}$. As $\mathcal{S}_{\mathcal{I}}$ holds in every interpretation, $\mathcal{B} \cup \mathcal{C} \models \mathcal{S}_{\mathcal{I}}$ and therefore $\mathcal{B} \cup \mathcal{C}$ is also complete for $\mathcal{I}$. By Theorem 7.1, $\mathcal{B} \cup \mathcal{C}$ is a base of $\operatorname{Th}_c(\mathcal{I})$. □

7.2 Minimal Cardinality

The foregone considerations suggest that for a set $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$, the completing set

$$\mathcal{B}(\mathcal{C}) := \{ \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid (U \longrightarrow U'') \in \operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})) \}$$

may be small. Indeed, we shall show in this section that this completing set is of minimal cardinality. More generally, we shall show that for an arbitrary set $\hat{\mathcal{C}}$ of GCIs, the set $\mathcal{B}(\hat{\mathcal{C}})$ is a completing set of $\hat{\mathcal{C}}$ of minimal cardinality.

To show this we are going to adapt the proof from [10, Theorem 5.18]. In this theorem it is shown that the base

$$\mathcal{B}_{\mathcal{DG}} := \{ \bigsqcap U \sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}} \mid (U \longrightarrow U'') \in \operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{S}_{\mathcal{I}})) \}$$

of $\mathcal{I}$ has minimal cardinality among all bases of $\mathcal{I}$. It is obvious that

$$\mathcal{B}_{\mathcal{DG}} = \mathcal{B}(\varnothing)$$

and hence [10, Theorem 5.18] is the special case $\mathcal{C}_2 = \varnothing$ of our claim that $\mathcal{B}(\mathcal{C}_2)$ is a completing set of minimal cardinality for $\mathcal{C}_2$ and $\mathcal{I}$.

Before we can prove our claim we have to introduce some preliminary results from [10], which have also been used in the proof of [10, Theorem 5.18]. We shall however not give proofs for these results as these proofs are not relevant for our considerations.

The first lemma shows another similarity between $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs and formal concept analysis. More precisely, if $\mathbb{K}$ is a formal context and $A \longrightarrow B$ is a valid implication of $\mathbb{K}$, then $A \longrightarrow B$ already follows from $A \longrightarrow A''$. The same holds for the case of $\mathcal{EL}^{\bot}_{\mathrm{gfp}}$-GCIs.

7.6 Lemma (Lemma 4.3 of [10]) *Let $\mathcal{I}$ be a finite interpretation and let $C \sqsubseteq D$ be a GCI valid in $\mathcal{I}$. Then $C \sqsubseteq D$ follows from $C \sqsubseteq C^{\mathcal{I}\mathcal{I}}$.*

The second lemma presents a description logic version of an argument we have used in Theorem 2.17. In this proof, we have considered a pseudo-intent P of a formal context $\mathbb{K}$. For this set P it is true that $P \neq P''$. If now $\mathcal{L}$ is a base for $\mathbb{K}$, then $P \longrightarrow P''$ follows from $\mathcal{L}$. This means that there is an implication $(X \longrightarrow Y) \in \mathcal{L}$ such that

$$X \subseteq P \quad \text{and} \quad Y \nsubseteq P,$$

since otherwise $\mathcal{L}(P) = P \neq P''$. More generally, if we have a set $\mathcal{K}$ of implications and another implication $S \longrightarrow T$, $T \nsubseteq S$ such that $\mathcal{K} \models (S \longrightarrow T)$, then there must exist an implication $(X \longrightarrow Y) \in \mathcal{K}$ such that $X \subseteq S$ and $Y \nsubseteq S$.

It has been noted in [10] that this property does not holds in general for $\mathcal{EL}^{\perp}_{\mathrm{gfp}}$-GCIs. However, we can obtain the following restriction of the argument.

7.7 Lemma (Lemma 5.16 of [10]) *Let $\mathcal{I}$ be a finite interpretation and let $\mathcal{B}$ be a set of GCIs valid in $\mathcal{I}$. Let $C = \bigsqcap U$ where $U \subseteq M_{\mathcal{I}}$ and let D be a concept description such that $C \not\sqsubseteq D$. Then if $C \sqsubseteq D$ follows from $\mathcal{B}$, there exists a GCI $(E \sqsubseteq F) \in \mathcal{B}$ such that*

$$C \sqsubseteq E \quad \text{and} \quad C \not\sqsubseteq F.$$

Another notion that we shall use for our proof is again concerned with *approximating* concept descriptions. Recall that for a concept description C and a finite interpretation $\mathcal{I}$ we always have

$$C \sqsubseteq \bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(C).$$

Moreover, $\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(C)$ is the most specific concept description expressible in terms of $M_{\mathcal{I}}$ that subsumes C. We can therefore regard $\bigsqcap \mathrm{pr}_{M_{\mathcal{I}}}(C)$ as an *upper approximation* of C in terms of $M_{\mathcal{I}}$.

Having defined a notion for an upper approximation it is naturally to ask whether it is also possible to define a *lower approximation*. A definition for the analogous notion of a *lower approximation* is the following.

7.8 Definition Let $\mathcal{I}$ be a finite interpretation and let C be an concept description. If $C \neq \perp$, then there exists a set $U \subseteq N_C$ and a set Π of pairs of role names and concept descriptions such that

$$C \equiv \bigsqcap U \sqcap \bigsqcap_{(r,E)\in\Pi} \exists r.E.$$

Then we define the *lower approximation* of C as

$$\mathrm{approx}_{\mathcal{I}}(C) := \bigsqcap U \sqcap \bigsqcap_{(r,E)\in\Pi} \exists r.E^{\mathcal{I}\mathcal{I}}.$$

If $C = \perp$, then $\mathrm{approx}_{\mathcal{I}}(C) := \perp$. ◇

Of course, $\mathrm{approx}_{\mathcal{I}}(C)$ is expressible in terms of $M_{\mathcal{I}}$. Moreover, $\mathrm{approx}_{\mathcal{I}}(C)$ is subsumed by C.

7.9 Lemma (Lemma 5.8 of [10]) *Let $\mathcal{I}$ be a finite interpretation and let C be a concept description. Then*

$$\mathrm{approx}_{\mathcal{I}}(C) \sqsubseteq C.$$

One can show that $\mathrm{approx}_{\mathcal{I}}(C)$ is the least specific concept description that is subsumed by C and expressible in terms of $M_{\mathcal{I}}$. Therefore, $\mathrm{approx}_{\mathcal{I}}(C)$ can indeed be regarded as another approximation in terms of $M_{\mathcal{I}}$.

7.10 Lemma (Lemma 5.17 of [10]) *Let $\mathcal{I}$ be a finite interpretation, C be a concept description and $U \subseteq M_{\mathcal{I}}$. Then $\bigsqcap U \sqsubseteq C$ implies $\bigsqcap U \sqsubseteq \mathrm{approx}_{\mathcal{I}}(C)$.*

We are now prepared to prove our main claim formulated in the following theorem. Note that its proof is an adaption from the proof of [10, Theorem 5.18].

7.11 Theorem *Let $\mathcal{I} = (\Delta_{\mathcal{I}}, \cdot^{\mathcal{I}})$ be a finite interpretation and let $c \in [0,1]$. Let $\mathcal{C} \subseteq \operatorname{Conf}(\mathcal{I}, c)$ be complete for $\operatorname{Conf}(\mathcal{I}, c)$. Then*

$$\mathcal{B} := \{ \bigsqcap A \sqsubseteq (\bigsqcap A)^{\mathcal{I}\mathcal{I}} \mid (A \longrightarrow A'') \in \operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})) \}$$

has minimal cardinality among all completing sets of $\mathcal{C}$.

The following proof is complex and long, so let us sketch the main line of thoughts to ease its understandability.

We want to show that $\mathcal{B}$ has minimal cardinality. From our previous considerations it is clear that $\mathcal{B}$ is a completing set for $\mathcal{C}$. Thus, we focus on showing that $\mathcal{B}$ has minimal cardinality.

For this we consider another completing set $\mathcal{D}$ of $\mathcal{C}$. The main idea now is to utilize Theorem 2.17, which states that $\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))$ has minimal cardinality among all sets of valid implications that complete $\operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})$ to a complete set of implications for $\mathbb{K}_{\mathcal{I}}$.

For this, we associate to $\mathcal{D}$ the set

$$\mathcal{L}_{\mathcal{D}} := \{ \operatorname{pr}_{M_{\mathcal{I}}}(\operatorname{approx}_{\mathcal{I}}(E)) \longrightarrow \operatorname{pr}_{M_{\mathcal{I}}}(E^{\mathcal{I}\mathcal{I}}) \mid (E \sqsubseteq E^{\mathcal{I}\mathcal{I}}) \in \mathcal{D} \}$$

of implications of $\mathbb{K}_{\mathcal{I}}$ and show that $\mathcal{L}_{\mathcal{D}}$ is a set of valid implications of $\mathbb{K}_{\mathcal{I}}$ and that $\mathcal{L}_{\mathcal{D}} \cup \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})$ is complete for $\mathbb{K}_{\mathcal{I}}$. Then by Theorem 2.17, $|\mathcal{L}_{\mathcal{D}}| \geqslant |\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})$. Because of $|\mathcal{L}_{\mathcal{D}}| \leqslant |\mathcal{D}|$ and $|\mathcal{B}| = |\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))|$, we obtain $|\mathcal{D}| \geqslant |\mathcal{C}|$ and the claim follows.

One may ask whether we cannot consider the set

$$\{ \operatorname{pr}_{M_{\mathcal{I}}}(E) \longrightarrow \operatorname{pr}_{M_{\mathcal{I}}}(E^{\mathcal{I}\mathcal{I}}) \mid (E \sqsubseteq E^{\mathcal{I}\mathcal{I}}) \in \mathcal{D} \}$$

instead of $\mathcal{L}_{\mathcal{D}}$ as given above, i. e. why we cannot leave out the lower approximation $\operatorname{approx}_{\mathcal{I}}(E)$. This is because we cannot guarantee that the concept description E is expressible in terms of $M_{\mathcal{I}}$. But we can guarantee that $\operatorname{approx}_{\mathcal{I}}(E)$ is expressible in terms of $M_{\mathcal{I}}$ — this follows easily from its definition. As we shall see in the following proof we need the fact that both premise and conclusion of implications in $\mathcal{L}_{\mathcal{D}}$ are expressible in terms of $M_{\mathcal{I}}$. Therefore, we cannot use the set of implications mentioned above.

Proof (of Theorem 7.11) By Theorem 7.1 and Theorem 7.3 we know that $\mathcal{B}$ is a completing set for $\mathcal{C}$. It therefore remains to show that $\mathcal{B}$ has minimal cardinality among all completing sets of $\mathcal{C}$.

Let $\mathcal{D}$ be a set of valid GCIs of $\mathcal{I}$ such that $\mathcal{D} \cup \mathcal{C}$ is a base of $\operatorname{Th}_c(\mathcal{I})$. By Lemma 7.6 we can assume that $\mathcal{D}$ contains only GCIs of the form $C \sqsubseteq C^{\mathcal{I}\mathcal{I}}$.

The number of GCIs in $\mathcal{B}$ is equal to the number of implications in $\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))$, formally

$$|\mathcal{B}| = |\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))|. \tag{7.4}$$

Now let

$$\mathcal{L}_{\mathcal{D}} := \{ \operatorname{pr}_{M_{\mathcal{I}}}(\operatorname{approx}_{\mathcal{I}}(E)) \longrightarrow \operatorname{pr}_{M_{\mathcal{I}}}(E^{\mathcal{I}\mathcal{I}}) \mid (E \sqsubseteq E^{\mathcal{I}\mathcal{I}}) \in \mathcal{D} \}.$$

Then $\mathcal{L}_{\mathcal{D}}$ contains at most as many implications as $\mathcal{D}$ contains GCIs, i. e.

$$|\mathcal{L}_{\mathcal{D}}| \leqslant |\mathcal{D}|. \tag{7.5}$$

We shall now going to show that $\mathcal{L}_{\mathcal{D}}$ contains only valid implications of $\mathbb{K}_{\mathcal{I}}$ and that $\mathcal{L}_{\mathcal{D}} \cup \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}})$ is complete for $\mathbb{K}_{\mathcal{I}}$. Then by Theorem 2.17 it follows that

$$|\operatorname{Can}(\mathbb{K}_{\mathcal{I}}, \operatorname{pr}_{M_{\mathcal{I}}}(\mathcal{C} \cup \mathcal{S}_{\mathcal{I}}))| \leqslant |\mathcal{L}_{\mathcal{D}}|$$

and therefore with (7.4) and (7.5)

$$|\mathcal{B}| \leqslant |\mathcal{D}|.$$

Let us show that $\mathcal{L}_\mathcal{D}$ contains only valid implications of $\mathbb{K}_\mathcal{I}$. For this, let

$$(\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)) \longrightarrow \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})) \in \mathcal{L}_\mathcal{D}.$$

As $\mathrm{approx}_\mathcal{I}(E)$ is expressible in terms of $M_\mathcal{I}$,

$$(\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)))' = \mathrm{approx}_\mathcal{I}(E)^\mathcal{I}$$

by Proposition 4.6. By Lemma 7.9, $\mathrm{approx}_\mathcal{I}(E) \sqsubseteq E$ and thus

$$\mathrm{approx}_\mathcal{I}(E)^\mathcal{I} \subseteq E^\mathcal{I} \equiv (E^{\mathcal{I}\mathcal{I}})^\mathcal{I}.$$

Since $E^{\mathcal{I}\mathcal{I}}$ can be expressed in terms of $M_\mathcal{I}$, $(E^{\mathcal{I}\mathcal{I}})^\mathcal{I} = (\mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}}))'$ again by Proposition 4.6. Thus

$$(\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)))' \subseteq (\mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}}))'$$

which shows that the implication $\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)) \longrightarrow \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})$ holds in $\mathbb{K}_\mathcal{I}$.

Let us now show that $\mathcal{L}_\mathcal{D} \cup \mathrm{pr}_{M_\mathcal{I}}(\mathcal{C} \cup \mathcal{S}_\mathcal{I})$ is complete for $\mathbb{K}_\mathcal{I}$. To ease readability, let us define

$$\mathcal{K} := \mathcal{L}_\mathcal{D} \cup \mathrm{pr}_{M_\mathcal{I}}(\mathcal{C} \cup \mathcal{S}_\mathcal{I}).$$

To show that $\mathcal{K}$ is complete for $\mathbb{K}_\mathcal{I}$, we shall use Lemma 2.14. By this lemma it suffices to show that for each $U \subseteq M_\mathcal{I}$, if $U \neq U''$, then $\mathcal{K}(U) \neq U$.

Hence let $U \subseteq M_\mathcal{I}$ such that $U \neq U''$ and let us assume that U respects all implications in $\mathrm{pr}_{M_\mathcal{I}}(\mathcal{C} \cup \mathcal{S}_\mathcal{I})$. As $U \neq U''$, there exists a concept description $D \in M_\mathcal{I} \setminus U$ such that all $x \in U'$ satisfy $x\ I\ D$ in the formal context $\mathbb{K}_\mathcal{I}$. Since

$$x\ I\ D \iff x \in D^\mathcal{I},$$

we obtain $U' \subseteq D^\mathcal{I}$. By Proposition 4.6 we have $U' = (\bigsqcap U)^\mathcal{I}$. Hence $(\bigsqcap U)^\mathcal{I} \subseteq D^\mathcal{I}$ and thus by Lemma 3.23

$$(\bigsqcap U)^{\mathcal{I}\mathcal{I}} \sqsubseteq D.$$

Because U is closed under all implications from $\mathrm{pr}_{M_\mathcal{I}}(\mathcal{S}_\mathcal{I})$ and because of $D \in U$, we obtain $F \not\sqsubseteq D$ for all $F \in U$, since otherwise $F \sqsubseteq D$ would imply $D \in U$. But then

$$\bigsqcap U \not\sqsubseteq D$$

as D is either a concept name or of the form $D = \exists r.X^{\mathcal{I}\mathcal{I}}$ for some $X \subseteq \Delta_\mathcal{I}$. From $(\bigsqcap U)^{\mathcal{I}\mathcal{I}} \sqsubseteq D$ thus follows

$$\bigsqcap U \not\sqsubseteq (\bigsqcap U)^{\mathcal{I}\mathcal{I}}.$$

Now Lemma 7.7 yields a GCI $(E \sqsubseteq F) \in \mathcal{D} \cup \mathcal{C}$ such that

$$\bigsqcap U \sqsubseteq E \text{ and } \bigsqcap U \not\sqsubseteq F. \tag{7.6}$$

Suppose that $(E \sqsubseteq F) \in \mathcal{C}$. Then $E = G^{\mathcal{I}\mathcal{I}}, F = H^{\mathcal{I}\mathcal{I}}$ for some concept descriptions G, H. Then U respects the implication

$$\mathrm{pr}_{M_\mathcal{I}}(G^{\mathcal{I}\mathcal{I}}) \longrightarrow \mathrm{pr}_{M_\mathcal{I}}(H^{\mathcal{I}\mathcal{I}}),$$

i. e.

$$\mathrm{pr}_{M_\mathcal{I}}(G^{\mathcal{I}\mathcal{I}}) \subseteq U \implies \mathrm{pr}_{M_\mathcal{I}}(H^{\mathcal{I}\mathcal{I}}) \subseteq U.$$

By Lemma 4.4 this is equivalent to

$$\bigsqcap U \sqsubseteq G^{\mathcal{I}\mathcal{I}} \implies \bigsqcap U \sqsubseteq H^{\mathcal{I}\mathcal{I}}.$$

But this contradicts (7.6) and therefore $(E \sqsubseteq F) \notin \mathcal{C}$.

Hence, $(E \sqsubseteq F) \in \mathcal{D}$ and thus $F = E^{\mathcal{I}\mathcal{I}}$. We shall now show that U does not respect the implication $\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)) \longrightarrow \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})$. This then shows $\mathcal{K}(U) \neq U$ and the proof is finished.

From Lemma 7.10 and $\bigsqcap U \sqsubseteq E$ we obtain $\bigsqcap U \sqsubseteq \mathrm{approx}_\mathcal{I}(E)$. Lemma 4.4 implies $\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)) \subseteq U$.

Now assume that $\mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}}) \subseteq U$. Then $\bigsqcap U \sqsubseteq \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})$. As $E^{\mathcal{I}\mathcal{I}}$ is expressible in terms of $M_\mathcal{I}$, Proposition 4.5 yields $E^{\mathcal{I}\mathcal{I}} \equiv \bigsqcap \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})$. But then $\bigsqcap U \sqsubseteq E^{\mathcal{I}\mathcal{I}}$, contradicting (7.6). Thus $\mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}}) \nsubseteq U$ and hence U does not respect the implication $\mathrm{pr}_{M_\mathcal{I}}(\mathrm{approx}_\mathcal{I}(E)) \longrightarrow \mathrm{pr}_{M_\mathcal{I}}(E^{\mathcal{I}\mathcal{I}})$. □

Using Theorem 7.1 we immediately obtain the following corollary.

7.12 Corollary *Let $\mathcal{I} = (\Delta_\mathcal{I}, \cdot^\mathcal{I})$ be a finite interpretation and let $c \in [0,1]$. Let $\mathcal{C} \subseteq \mathrm{Conf}(\mathcal{I}, c)$ be complete for $\mathrm{Conf}(\mathcal{I}, c)$. Then*

$$\mathcal{B} := \{ \bigsqcap A \sqsubseteq (\bigsqcap A)^{\mathcal{I}\mathcal{I}} \mid (A \longrightarrow A'') \in \mathrm{Can}(\mathbb{K}_\mathcal{I}, \mathrm{pr}_{M_\mathcal{I}}(\mathcal{C} \cup \mathcal{S}_\mathcal{I})) \}$$

has minimal cardinality among all sets of valid GCIs of $\mathcal{I}$ with respect to $\mathcal{B} \cup \mathcal{C}$ being a base of $\mathrm{Th}_c(\mathcal{I})$.

8 Conclusions and Further Work

The results and considerations presented in this work are a first step in adapting the results from [10] for interpretations containing errors. For this, we defined and motivated the notion of *confident GCIs* of finite interpretations. Using ideas from formal concept analysis, we were able to explicitly describe finite bases of confident GCIs. Even more, given a set $\mathcal{C} \subseteq \mathrm{Conf}(\mathcal{I}, c)$ that is complete for $\mathrm{Conf}(\mathcal{I}, c)$, we were able to show how to compute minimal sets $\mathcal{B}$ of valid GCIs such that $\mathcal{B} \cup \mathcal{C}$ is a base for $\mathrm{Th}_c(\mathcal{I})$. Here, we again used ideas from formal concept analysis.

However, the results as presented here are not sufficient to achieve our initial goal of constructing ontologies. For this, some further investigations are needed, some of which are the following.

An Exploration Algorithm for Confident GCIs Suppose that we are given a finite interpretation $\mathcal{I}$ from which we want to construct the terminological part of an ontology. We have argued that the interpretation $\mathcal{I}$ may contain errors. However, it may also be the case that $\mathcal{I}$ might be *incomplete*. Let us makes this more concrete: as argued in Section 5, the interpretation $\mathcal{I}$ can be seen as an approximation of a *perfect* interpretation $\mathcal{I}_{\mathrm{perfect}}$. From the interpretation $\mathcal{I}$ we want to construct the terminological knowledge that is present in the interpretation $\mathcal{I}_{\mathrm{perfect}}$. However, it might be that $\mathcal{I}$ contains some terminological knowledge that is not valid in $\mathcal{I}_{\mathrm{perfect}}$. In other words, errors in $\mathcal{I}$ have removed some *valid counterexamples* from $\mathcal{I}_{\mathrm{perfect}}$ such that invalid GCIs are now valid ones. Within this respect we can say that the interpretation $\mathcal{I}$ is incomplete as it misses some crucial counterexamples to invalid GCIs.

Even worse, our approach of considering GCIs with a minimal confidence makes it likely that invalid GCIs in $\mathcal{I}_{\text{perfect}}$ may turn out as confident GCIs of $\mathcal{I}$, if only the number of valid counterexamples is small enough. We have briefly discussed that in Section 5.

Let us consider a simple example to illustrate the point. Suppose that the interpretation $\mathcal{I}$ contains various kinds of birds. Then we may ask whether all birds fly, i. e. whether

$$\mathcal{I} \models (\mathsf{Bird} \sqsubseteq \mathsf{CanFly}).$$

$\mathcal{I}$ may contain some errors which prevent this GCIs from being true. Then we can try to handle this by considering the confident GCIs.

However, there are birds which do not fly (penguins). They are not errors in the data. Thus, $\mathsf{Bird} \sqsubseteq \mathsf{CanFly}$ is simply not correct. But the GCI $\mathsf{Bird} \sqsubseteq \mathsf{CanFly}$ may still have enough confidence in $\mathcal{I}$. It is even possible that penguins are not present in $\mathcal{I}$. Therefore we may say that $\mathcal{I}$ is *incomplete for describing birds*. Our heuristic idea that GCIs with high confidence are actually valid GCIs invalidated by errors fails here. Also the notion of *approximations of interpretations* is not useful anymore, as it does not allow the set of individuals to change.

An idea to overcome this issue is to use an external expert. This expert has to be able to distinguish between GCIs that have been invalidated by errors and invalid GCIs. An exploration algorithm may then use this expert to explore the confident GCIs of $\mathcal{I}$. More precisely, the expert is asked GCIs $C \sqsubseteq D$ that have enough confidence in $\mathcal{I}$. She then has to decide whether $C \sqsubseteq D$ is true in the domain of interest. If $C \sqsubseteq D$ is true and $C \sqsubseteq D$ is not valid in $\mathcal{I}$, the counterexamples in $\mathcal{I}$ are errors and the expert may correct them. If $C \sqsubseteq D$ is not valid, the expert has to provide a real counterexample for it. Alternatively, if there are already counterexamples for $C \sqsubseteq D$, then the expert may confirm some of them as being valid counterexamples (i. e. as not being errors.)

A major problem that arises from this description is that we now have to deal with two kinds of counterexamples in $\mathcal{I}$: those that have been confirmed or provided by the expert and those that have not. In other words, providing one counterexample for $C \sqsubseteq D$ may not change the confidence of $C \sqsubseteq D$ substantially, i. e. $C \sqsubseteq D$ may still be a confident GCI. But the counterexample provided by the expert has another quality, because we know that it is true. Thus we have to work with *possible counterexamples* and *confirmed counterexamples*.

A Finite $\mathcal{EL}^{\perp}$-base In the bases we have discussed so far we have always allowed GCIs with $\mathcal{EL}^{\perp}_{\text{gfp}}$-concept descriptions. This may be a problem if we want to use these GCIs as elements of a TBox of an ontology. As discussed in [10] and as defined in this work, TBoxes use descriptive semantics. However, the TBoxes in $\mathcal{EL}^{\perp}_{\text{gfp}}$-concept descriptions use greatest-fixpoint semantics. Handling these two different semantics together is difficult, and two approaches are mentioned in [10].

As a solution to this problem, [10] shows that for a finite $\mathcal{EL}^{\perp}_{\text{gfp}}$-base of $\mathcal{I}$ there always exists a finite, equivalent $\mathcal{EL}^{\perp}$-base of $\mathcal{I}$ that can effectively be computed. Providing a similar results for bases of $\mathrm{Th}_c(\mathcal{I})$ would also resolve the conflict between the two different kinds of semantics.

An Explicit Connection between Errors and Confidence In Section 5 we have shown in Lemma 5.7 how the confidence behaves if the interpretations of concept descriptions between two interpretations does not vary too much. We have used this result to motivate that considering the confidence of GCIs may be a good heuristic to identify GCIs that have been invalidated by errors. However, Lemma 5.7 does not involve the notion of errors as defined in Definition 5.2. A direct connection between errors and the confidence of GCIs would be desirable.

Non-Redundant Bases of $\mathrm{Conf}(\mathcal{I}, c)$ and $\mathrm{Th}_c(\mathcal{I})$ In Section 7.2 we have shown how to find minimal completing sets for bases $\mathcal{C}$ of $\mathrm{Conf}(\mathcal{I}, c)$. But we have not discussed how such bases can be found.

Taking the whole set $\text{Conf}(\mathcal{I}, c)$ does not seem very practical. The adaption of Luxenburgers idea in Theorem 6.9 may provide a smaller set, but this is still unsatisfactory. Instead, it would be better to explicitly describe some non-redundant base of $\text{Conf}(\mathcal{I}, c)$ or even some base for $\text{Conf}(\mathcal{I}, c)$ of minimal cardinality.

It would be even better to be able to describe non-redundant bases of $\text{Th}_c(\mathcal{I})$ explicitly. This is because a non-redundant base $\mathcal{C}$ of $\text{Conf}(\mathcal{I}, c)$ together with a minimal completing set $\mathcal{B}$ may yield a redundant base $\mathcal{B} \cup \mathcal{C}$ of $\text{Th}_c(\mathcal{I})$.

References

[1] R. Agrawal, T. Imielinski, and A. Swami. Mining Association Rules between Sets of Items in Large Databases. In *Proceedings of the ACM SIGMOD International Conference on Management of Data*, pages 207–216, May 1993.

[2] Michael Ashburner, Catherine A. Ball, Judith A. Blake, David Botstein, Heather Butler, J. Michael Cherry, Allan P. Davis, Kara Dolinski, Selina S. Dwight, Janan T. Eppig, Midori A. Harris, David P. Hill, Laurie Issel-Tarver, Andrew Kasarskis, Suzanna Lewis, John C. Matese, Joel E. Richardson, Martin Ringwald, Gerald M. Rubin, and Gavin Sherlock. Gene ontology: tool for the unification of biology. *Nature Genetics*, 25:25–29, May 2000.

[3] F. Baader. Least common subsumers, most specific concepts, and role-value-maps in a description logic with existential restrictions and terminological cycles. LTCS-Report LTCS-02-07, Chair for Automata Theory, Institute for Theoretical Computer Science, Dresden University of Technology, Germany, 2002. See `http://lat.inf.tu-dresden.de/research/reports.html`.

[4] Franz Baader and Felix Distel. A Finite Basis for the Set of $\mathcal{EL}$-Implications Holding in a Finite Model. In Raoul Medina and Sergei Obiedkov, editors, *Proceedings of the 6th International Conference on Formal Concept Analysis, (ICFCA 2008)*, volume 4933 of *Lecture Notes in Artificial Intelligence*, pages 46–61. Springer Verlag, 2008.

[5] Franz Baader and Felix Distel. Exploring finite models in the description logic $\mathcal{EL}_{\text{gfp}}$. In Sébastien Ferré and Sebastian Rudolph, editors, *Proceedings of the 7th International Conference on Formal Concept Analysis, (ICFCA 2009)*, volume 5548 of *Lecture Notes in Artificial Intelligence*, pages 146–161. Springer Verlag, 2009.

[6] Franz Baader, Bernhard Ganter, Ulrike Sattler, and Baris Sertkaya. Completing description logic knowledge bases using formal concept analysis. In *Proceedings of the Twentieth International Joint Conference on Artificial Intelligence (IJCAI-07)*, pages 230–235. AAAI Press, 2007.

[7] Christian Bizer, Tom Heath, and Tim Berners-Lee. Linked Data - The Story So Far. *International Journal on Semantic Web and Information Systems (IJSWIS)*, 5(3):1–22, March 2009.

[8] Christian Bizer, Jens Lehmann, Gergi Kobilarov, Sören Auer, Christian Becker, Richard Cyganiak, and Sebastian Hellmann. DBpedia - a Crystallization Point of the Web of Data. *Web Semantics: Science, Services and Agents on the World Wide Web*, 7(3):154–165, 9 2009.

[9] Daniel Borchmann and Felix Distel. Mining of $\mathcal{EL}$-GCIs. In Myra Spiliopoulou, Haixun Wang, Diane J. Cook, Jian Pei, Wei Wang, Osmar R. Zaïane, and Xindong Wu, editors, *ICDM Workshops*, pages 1083–1090. IEEE, 2011.

[10] Felix Distel. *Learning Description Logic Knowledge Bases from Data Using Methods from Formal Concept Analysis*. PhD thesis, TU Dresden, 2011.

[11] Bernhard Ganter and Rudolph Wille. *Formal Concept Analysis: Mathematical Foundations*. Springer, Berlin-Heidelberg, 1999.

[12] M. Luxenburger. Implications partielles dans un contexte. *Mathématiques, Informatique et Sciences Humaines*, 29(113):35–55, 1991.

[13] Bernhard Nebel. Terminological Cycles: Semantics and Computational Properties. In *Principles of Semantic Networks*, pages 331–362. Morgan Kaufmann, 1991.

[14] Susanne Prediger. Logical scaling in formal concept analysis. In Dickson Lukose, Harry S. Delugach, Mary Keeler, Leroy Searle, and John F. Sowa, editors, *ICCS*, volume 1257 of *Lecture Notes in Computer Science*, pages 332–341. Springer, 1997.

[15] A. L. Rector, W. A. Nowlan, and Galen Consortium. The galen project. *Computer Methods and Programs in Biomedicine*, 45(1-2):75 – 78, 1994.

[16] Sebastian Rudolph. *Relational exploration: combining description logics and formal concept analysis for knowledge specification*. PhD thesis, TU Dresden, 2006.

[17] Kent A. Spackman, Ph. D, Keith E. Campbell, Ph. D, Roger A. Côté, and D. Sc. (hon. Snomed rt: A reference terminology for health care. In *J. of the American Medical Informatics Association*, pages 640–644, 1997.

[18] Gerd Stumme. Attribute exploration with background implications and exceptions. In H.-H. Bock and W. Polasek, editors, *Data Analysis and Information Systems. Statistical and Conceptual approaches. Proc. GfKl'95. Studies in Classification, Data Analysis, and Knowledge Organization 7*, pages 457–469, Heidelberg, 1996. Springer.

[19] Gerd Stumme, Rafik Taouil, Yves Bastide, Nicolas Pasquier, and Lotfi Lakhal. Intelligent structuring and reducing of association rules with formal concept analysis. In Franz Baader, Gerhard Brewka, and Thomas Eiter, editors, *KI/ÖGAI*, volume 2174 of *Lecture Notes in Computer Science*, pages 335–350. Springer, 2001.

[20] Alfred Tarski. A Lattice-Theoretical Fixpoint Theorem and Its Applications. *Pacific Journal of Mathematics*, 1955.