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A Connection Between Clone Theory and FCA Provided by Duality Theory

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The aim of this paper is to show how Formal Concept Analysis can be used for the benefit of clone theory. More precisely, we show how a recently developed duality theory for clones can be used to dualize clones over bounded lattices into the framework of Formal Concept Analysis, where they can be investigated with techniques very different from those that universal algebraists are usually armed with. We also illustrate this approach with some small examples.

1 Introduction

In this paper, we show how a duality theory from [Ker11] can be used to connect clone theory with Formal Concept Analysis [GW99].

A clone is a set of (finitary) operations over a set A that is closed under composition and contains all the projection mappings. The interest in clones is driven by the fact that clones represent the behaviour of algebras. However, as long as A contains at least three elements, very little is known about the structure of all clones on A, despite intensive research for several decades.

The principle of Duality is "a very pervasive and important concept in (modern) mathematics" [Haz95] and "an important general theme that has manifestations in almost every area of mathematics" [GBGL08]. When it comes to dualizing clones, the usual approach is to consider a clone as the set of term functions of a suitable algebra and then try to dualize this algebra, which may, or may not, be possible. Another approach, applicable for all clones, was introduced in [Ker11], where clones, inspired by an idea from [Maš06], are dualized by treating them in a more general way as sets of morphisms in a category.

In this paper, we will use the duality theory from [Ker11] (recalled in Section 3 after the preliminaries) and put it to work in Section 4, where we apply it to clones over bounded lattices (also called centralizer clones of bounded lattices), i.e., clones in which every operation is a homomorphism from a finite power of a bounded lattice to the lattice itself. Since the category of bounded lattices can be dualized to the category of standard topological contexts ([Har93], see Subsection 2.3), we can dualize the clones to certain sets of context morphisms. This allows us to investigate the clones from a different angle, namely in the setting of Formal Concept Analysis. To show that this method is in fact a helpful technique to investigate clones over lattices, we choose a few small examples in Section 5 and put the duality to work, producing some concrete results.

2 Preliminaries

In the preliminaries, we will introduce all the ingredients that we need to set up a duality for clones over bounded lattices, except that we will assume the reader to be familiar with the basic notions from Formal Concept Analysis [GW99]. We start with the necessary terminology from category theory, recall the rudimentary basics of clone theory, and end by outlining the dual equivalence for lattices from [Har93] that we are about to incorporate into our clone duality.

2.1 Category theory

We assume that the reader is familiar with the rudimentary basics of category theory. By that, we mean that the reader should be familiar with the definitions of categories, functors, natural transformations, products and coproducts. In this section, we only introduce our notation and the terminology of duality. For an object $\mathbf{A}$ in a category $\mathcal{C}$, we denote by $\mathbf{A}^n$ the n-th power of $\mathbf{A}$ (provided it exists) and by $\pi_i^n\colon \mathbf{A}^n \to \mathbf{A}$ $(i \in \{1,\dots,n\})$ the associated projection morphisms. For morphisms $f_1,\dots,f_n\colon \mathbf{B} \to \mathbf{A}$, we denote by $\langle f_1,\dots,f_n\rangle\colon \mathbf{B} \to \mathbf{A}^n$ the *tupling* of $f_1,\dots,f_n$. Dually, for an object $\mathbf{X} \in \mathcal{C}$, we denote by $n\cdot\mathbf{X}$ the n-th copower of $\mathbf{X}$ (provided it exists) and by $\iota_i^n\colon \mathbf{X} \to n\cdot\mathbf{X}$ $(i \in \{1,\dots,n\})$ the associated injection morphisms. For morphisms $h_1,\dots,h_n\colon \mathbf{X} \to \mathbf{Y}$, we denote by $[h_1,\dots,h_n]\colon n\cdot\mathbf{X} \to \mathbf{Y}$ the *cotupling of* $h_1,\dots,h_n$.[1]

A *dual equivalence* between categories $\mathcal{A}$ and $\mathcal{X}$ is a quadruple $\langle D, E, e, \epsilon\rangle$ where $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$ are contravariant functors (i.e., functors that reverse the direction of the morphisms) and $e\colon id_{\mathcal{A}} \to ED$ and $\epsilon\colon id_{\mathcal{X}} \to DE$ are natural isomorphisms. The notion "dual equivalence" is justified since D and E are full, faithful and preserve all purely category-theoretic properties, except that they reverse the direction of the morphisms. For instance, monomorphisms become epimorphisms and products become coproducts. In particular, we have $\mathbf{A}^n \in \mathcal{A}$ if and only if $n\cdot D(\mathbf{A}) \in \mathcal{X}$.

2.2 Clones

Let A be a (not necessarily finite) non-empty set. For $n \in \mathbb{N}_+$ and a set B, we say that the i-th argument of a function $f\colon A^n \to B$ is *nonessential* if

$$f(x_1,\dots,x_n) \approx f(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n).$$

[1] We will not use the letter g for morphisms since we want to reserve this letter for objects in contexts.

If the i-th argument of f is not nonessential, then it is called *essential*. We say that f is *essentially k-ary* if it has exactly k essential arguments.

Now let $O_A := \bigcup_{n\geq 1} A^{A^n}$ be the set of all finitary, non-nullary operations over A. A subset $C \subseteq O_A$ is a *clone on A* if it contains all the projection mappings

$$\pi_i^n \colon A^n \to A : (x_1, \ldots, x_n) \mapsto x_i \ (1 \leq i \leq n)$$

(also called *trivial operations*) and is closed with respect to superposition of operations in the following sense: For an n-ary operation $f \in C$ and k-ary operations $f_1, \ldots, f_n \in C$, the k-ary operation $f(f_1, \ldots, f_n)$ defined by

$$f(f_1, \ldots, f_n)(x_1, \ldots, x_k) := f(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k))$$

is also in C. Given an algebra, the set of its non-nullary term functions is a clone. Conversely, every clone can be realized as the set of term functions of a suitable algebra. Hence, clones on a set A represent all possible different behaviours of algebras with carrier set A. Roughly speaking, if one understands all clones on a set A, one understands all algebras on A. This is the main motivation behind clone theory.

The set of all clones on a set A forms a lattice with inclusion, which we denote by $\mathcal{L}_A$. The lattice is countable and completely known for $|A| \leq 2$. However, for $|A| \geq 3$, there are continuum many clones in $\mathcal{L}_A$, and very little is known about the structure of this lattice.

2.3 Hartung's Duality for Lattices

A topological representation theorem for lattices seems to have first appeared in [Urq78]. Since then, there has been put much work into lifting this representation theorem to a dual equivalence of categories (see for example [Geh06], [HD97]). Here, we will look at the duality presented in [Har93], where the dual equivalence is set up between the category of bounded lattices with homomorphisms (i.e., functions that commute with $\vee$ and $\wedge$ and preserve the bottom and the top of the lattice) and the category of so-called standard topological contexts with so-called multivalued standard morphisms, described as in the remainder of this subsection.

A standard topological context is a standard context where the set of objects and the set of attributes are equipped with suitable topologies. To explain this more precisely, let $\mathbb{K}^\tau = ((G, \rho), (M, \sigma), \mathcal{I})$ be a triple where (G, ρ) and (M, σ) are topological spaces and $(G, M, \mathcal{I})$ is a context. A concept $(A, B) \in \mathfrak{B}(G, M, \mathcal{I})$ is said to be *closed* if A and B are closed with respect to ρ and σ, respectively. Denote the set of all closed concepts of $\mathfrak{B}(G, M, \mathcal{I})$ by $\mathfrak{B}^\tau(\mathbb{K}^\tau)$. To define a topological context, recall that, for a topological space $(X, \mathcal{T})$, a subcollection $\mathfrak{S} \subseteq \mathcal{T}$ is said to be a *subbasis of* $(X, \mathcal{T})$ if $\mathcal{T}$ is generated by $\mathfrak{S}$, i.e., if $\mathcal{T}$ is the smallest topology on X containing $\mathfrak{S}$.

Definition 2.1. The structure $\mathbb{K}^\tau$ is called a *topological context* if

(i) $A \in \rho \Rightarrow A'' \in \rho$ and $B \in \sigma \Rightarrow B'' \in \sigma$,

(ii) $\mathfrak{S}_\rho := \{A \subseteq G \mid (A, A') \in \mathfrak{B}^\tau(\mathbb{K}^\tau)\}$ is a subbasis of (G, ρ) and
$\mathfrak{S}_\sigma := \{B \subseteq M \mid (B', B) \in \mathfrak{B}^\tau(\mathbb{K}^\tau)\}$ is a subbasis of (M, σ).

A topological context is called a *standard topological context* if, in addition, the following three conditions hold:

(a) $(G, M, \mathcal{I})$ is a standard context,
(b) for every $(g, m) \in \mathcal{I}$, there exists some $(A, B) \in \mathfrak{B}^\tau(\mathbb{K}^\tau)$ such that $g \in A$ and $m \in B$,
(c) $(\mathcal{I}^c, (\rho \times \sigma)|_{\mathcal{I}^c})$ is a compact space[2] where $\mathcal{I}^c := (G \times M) \setminus \mathcal{I}$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

We will now explain that there is indeed a one-to-one correspondence between bounded lattices and standard topological spaces.

First, let $\mathbb{K}^\tau = ((G, \rho), (M, \sigma), \mathcal{I})$ be a standard topological context, and set

$$\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau) := \langle \mathfrak{B}^\tau(\mathbb{K}^\tau), \leq \rangle$$

where $\leq$ is the restriction of the usual order-relation on $\mathfrak{B}(G, M, \mathcal{I})$. Then, $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau)$ is a bounded lattice. In fact, it is a bounded (but not necessarily complete) sublattice of $\underline{\mathfrak{B}}(G, M, \mathcal{I})$.

For the other direction, we need to introduce the notion of I-maximal filters and F-maximal ideals: For a bounded lattice $\mathbf{A}$, denote by $\mathfrak{F}(\mathbf{A})$ and $\mathfrak{I}(\mathbf{A})$ the set of non-empty (but not necessarily proper) lattice filters and lattice ideals of $\mathbf{A}$, respectively. For $F \in \mathfrak{F}(\mathbf{A})$ and $I \in \mathfrak{I}(\mathbf{A})$, we say that F is I-maximal whenever $F \cap I = \emptyset$ and every proper superfilter $F^* \supsetneq F$ already intersects I. Similarly, we say that I is F-maximal if $F \cap I = \emptyset$ and every proper superideal $I^* \supsetneq I$ already intersects F. Now, set

$$\begin{aligned}
\mathfrak{F}_0(\mathbf{A}) &:= \{F \in \mathfrak{F}(\mathbf{A}) \mid \exists I \in \mathfrak{I}(\mathbf{A}) : F \text{ is } I\text{-maximal}\}, \\
\mathfrak{I}_0(\mathbf{A}) &:= \{I \in \mathfrak{I}(\mathbf{A}) \mid \exists F \in \mathfrak{F}(\mathbf{A}) : I \text{ is } F\text{-maximal}\}, \\
\mathfrak{R}(\mathbf{A}) &:= \{(F, I) \in \mathfrak{F}_0(\mathbf{A}) \times \mathfrak{I}_0(\mathbf{A}) \mid F \cap I \neq \emptyset\}.
\end{aligned}$$

With this notation, we can now define a standard topological context $\mathbb{K}^\tau(\mathbf{A})$ such that $\underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau(\mathbf{A})) \cong \mathbf{A}$. This standard topological context can be defined as follows:

$$\mathbb{K}^\tau(\mathbf{A}) := ((\mathfrak{F}_0(\mathbf{A}), \rho_0), (\mathfrak{I}_0(\mathbf{A}), \sigma_0), \mathfrak{R}(\mathbf{A})),$$

where ρ_0 and σ_0 are given by the subbases

$$\begin{aligned}
\mathfrak{S}_{\rho_0} &:= \{\{F \in \mathfrak{F}_0(\mathbf{A}) \mid a \in F\} \mid a \in A\}, \\
\mathfrak{S}_{\sigma_0} &:= \{\{I \in \mathfrak{I}_0(\mathbf{A}) \mid a \in I\} \mid a \in A\},
\end{aligned}$$

respectively.

Since we will use this fact in the remainder of this paper, let us note the following (obvious) proposition:

[2]By a compact space, we mean what is sometimes also called a quasicompact space. That is, a topological space in which all open covers have finite subcovers.

Proposition 2.2. *For $X \subseteq \mathfrak{F}_0(\mathbf{A})$, we have $g \in X''$ if and only if g is a superfilter of some $x \in X$. Similarly, for $X \subseteq \mathfrak{I}_0(\mathbf{A})$, we have $m \in X''$ if and only if m is a superideal of some $x \in X$.*

Let us now turn our attention to the morphism part of the duality. Therefor, we need to define multivalued standard morphisms and their composition.

A *multivalued function* $F\colon X \to Y$ from a set X to a set Y is a binary relation $F \subseteq X \times Y$ such that $\pi_1(F) = X$. For $x \in X$, $A \subseteq X$ and $B \subseteq Y$, we define

$$\begin{aligned} F(x) &:= \{y \in Y \mid (x,y) \in F\}, \\ F[A] &:= \{y \in Y \mid \exists a \in A : (a,y) \in F\}, \\ F^{[-1]}[B] &:= \{x \in X \mid F(x) \subseteq B\}. \end{aligned}$$

Definition 2.3. Let $\mathbb{K}_1^\tau = ((G_1,\rho_1),(M_1,\sigma_1),\mathcal{I}_1)$, $\mathbb{K}_2^\tau = ((G_2,\rho_2),(M_2,\sigma_2),\mathcal{I}_2)$ be standard topological contexts. A *multivalued standard morphism* $h\colon \mathbb{K}_1^\tau \to \mathbb{K}_2^\tau$ is a pair (R_h, S_h) of multivalued functions $R_h\colon G_1 \to G_2$ and $S_h\colon M_1 \to M_2$ such that

(i) $(R_h^{[-1]}[A], S_h^{[-1]}[B]) \in \mathfrak{B}^\tau(\mathbb{K}_1^\tau)$ for every $(A,B) \in \mathfrak{B}^\tau(\mathbb{K}_2^\tau)$,

(ii) $R_h(x) = R_h(x)'' = \overline{R_h(x)}$ for every $x \in G_1$ and
$S_h(x) = S_h(x)'' = \overline{S_h(x)}$ for every $x \in M_1$.

For $j \in \{1,2,3\}$, let $\mathbb{K}_j^\tau = ((G_j,\rho_j),(M_j,\sigma_j),\mathcal{I}_j)$, be standard topological contexts. We define the composition $h_2 \circ h_1$ of two multivalued standard morphisms $h_1\colon \mathbb{K}_1^\tau \to \mathbb{K}_2^\tau$ and $h_2\colon \mathbb{K}_2^\tau \to \mathbb{K}_3^\tau$ by setting:

$$\begin{array}{rlcll} R_{h_2\circ h_1}\colon & G_1 \to G_3 & : & R_{h_2\circ h_1}(x) & := R_{h_2}[R_{h_1}(x)]'', \\ S_{h_2\circ h_1}\colon & M_1 \to M_3 & : & S_{h_2\circ h_1}(x) & := S_{h_2}[S_{h_1}(x)]''. \end{array}$$

For two bounded lattices $\mathbf{A}$, $\mathbf{B}$ and a homomorphism $f\colon \mathbf{A} \to \mathbf{B}$, we define the multivalued standard morphism $(R^f, S^f)\colon \mathbb{K}^\tau(\mathbf{B}) \to \mathbb{K}^\tau(\mathbf{A})$ by setting:

$$\begin{array}{rlcl} R^f\colon & \mathfrak{F}_0(\mathbf{B}) \to \mathfrak{F}_0(\mathbf{A}) & : & R^f(x) := \{y \in \mathfrak{F}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq y\}, \\ S^f\colon & \mathfrak{I}_0(\mathbf{B}) \to \mathfrak{I}_0(\mathbf{A}) & : & S^f(x) := \{y \in \mathfrak{I}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq y\}. \end{array}$$

It is important to note that, for f being surjective, the preimage of each $F \in \mathfrak{F}_0(\mathbf{B})$ and each $I \in \mathfrak{I}_0(\mathbf{B})$ is an element of $\mathfrak{F}_0(\mathbf{A})$ and $\mathfrak{I}_0(\mathbf{A})$, respectively. For arbitrary homomorphisms, this is not necessarily true.

Now let $\mathcal{X}$ be the category with standard topological contexts as objects, multivalued standard morphisms as morphisms and $\circ$ as composition. Note that, for a given standard topological context $\mathbf{X} = ((G,\rho),(M,\sigma),\mathcal{I}) \in \mathcal{X}$, the identity morphism $id_{\mathbf{X}}$ is given as follows:

$$\begin{array}{rlcll} R_{id_{\mathbf{X}}}\colon & G \to G & : & R_{id_{\mathbf{X}}}(x) & = x'', \\ S_{id_{\mathbf{X}}}\colon & M \to M & : & S_{id_{\mathbf{X}}}(x) & = x''. \end{array}$$

Theorem 2.4 ([Har93])**.** *Let $\mathscr{A}$ be the category of bounded lattices with homomorphisms as morphisms and the usual composition of functions. Then, $\mathscr{A}$ and $\mathscr{X}$ are dually equivalent via the two contravariant functors $D\colon \mathscr{A} \to \mathscr{X}$ and $E\colon \mathscr{X} \to \mathscr{A}$ that are given as follows:*

$$\begin{aligned} D(\mathbf{A}) &:= \mathbb{K}^\tau(\mathbf{A}) = ((\mathfrak{F}_0(\mathbf{A}), \rho_0), (\mathfrak{I}_0(\mathbf{A}), \sigma_0), \mathfrak{R}(\mathbf{A})), \\ D(f) &:= (R^f, S^f), \\ E(\mathbb{K}^\tau) &:= \underline{\mathfrak{B}}^\tau(\mathbb{K}^\tau), \\ E(h) &:= (R_h^{[-1]}[-_1], S_h^{[-1]}[-_2]) : (A, B) \mapsto (R_h^{[-1]}[A], S_h^{[-1]}[B]). \end{aligned}$$

3 Duality Theory for Clones

In this section, we will explain how we can dualize arbitrary clones. This theory will be the foundation of our work in Section 4, where we will use the machinery to dualize clones over bounded lattice into the framework of Formal Concept Analysis. To obtain this duality theory for clones, we will use a more general notion of a clone:

Definition 3.1. Let $n \in \mathbb{N}_+$. A morphism $f\colon \mathbf{A}^n \to \mathbf{A}$ is called an *n-ary operation over* $\mathbf{A}$. Denote by $O_\mathbf{A}^{(n)}$ the set of all n-ary operations over $\mathbf{A}$, define $O_\mathbf{A} := \bigcup_{n\in\mathbb{N}_+} O_\mathbf{A}^{(n)}$ and, for $F \subseteq O_\mathbf{A}$, set $F^{(n)} := F \cap O_\mathbf{A}^{(n)}$.

Definition 3.2. A subset $C \subseteq O_\mathbf{A}$ is called a *clone of operations*, written $C \leq O_\mathbf{A}$, if C contains all the projection morphisms $\pi_i^n\colon \mathbf{A}^n \to \mathbf{A}$ and, for each $f \in C^{(n)}$ and $f_1, \dots, f_n \in C^{(k)}$, the *superposition* $f \circ \langle f_1, \dots, f_n\rangle$ is also in C.

If $\mathscr{A}$ is the category of sets, then this definition coincides with the usual notion of a clone. It is easy to verify that the clones over an object $\mathbf{A}$ form a complete lattice with respect to inclusion. We call this lattice the *lattice of clones over* $\mathbf{A}$, and we denote it by $\mathcal{L}_\mathbf{A}$. The top element of $\mathcal{L}_\mathbf{A}$ is the *full clone* $O_\mathbf{A}$, and the bottom element is the clone that contains only the projection morphisms.

Since clones are closed under arbitrary intersection, we can define the closure operator Clo that assigns to each subset $F \subseteq O_\mathbf{A}$ the least clone of operations over $\mathbf{A}$ that contains F. It is called the clone *generated by* F. For a single operation f, we write $\mathrm{Clo}(f)$ to mean $\mathrm{Clo}(\{f\})$.

Examples 3.3.

(i) If $\mathscr{A} = \mathscr{Set}$, then $O_\mathbf{A}$ is the full clone on the set A and $\mathcal{L}_\mathbf{A}$ is the usual clone lattice.
(ii) If $\mathscr{A}$ is a variety (or a quasivariety) of algebras, then $O_\mathbf{A}$ is the centralizer clone of the algebra $\mathbf{A}$ and $\mathcal{L}_\mathbf{A}$ is the lattice of subclones of $O_\mathbf{A}$. Centralizer clones are of particular interest in universal algebra (see [MMT87], for instance).
(iii) For each clone C on a finite set A, we obtain $C = O_\mathbf{A}$ if we define $\mathbf{A}$ to be a relational structure $\langle A, R\rangle$ in a variety of relational structures such that C is the set of polymorphisms of R (that is, the set of operations that preserve each $\sigma \in R$).

Such a set of relations R can always be found. In this case, $\mathcal{L}_{\mathbf{A}}$ is the lattice of subclones of C.

These examples show that one can investigate clones over sets by treating them as clones over objects in (abstract or concrete) categories different from $\mathcal{S}et$.

We can lift every notion from clone theory to our setting as long as we can write it in purely category-theoretic terms. For instance, we can write all kinds of identities. E.g., we can define essential arguments of an operation as follows:

Definition 3.4. For $n \in \mathbb{N}_+$ and $i \in \{1, \dots, n\}$, the i-th argument of an operation $f \in O_{\mathbf{A}}^{(n)}$ is said to be *nonessential* if

$$f \circ \langle \pi_1^{n+1}, \dots, \pi_n^{n+1} \rangle = f \circ \langle \pi_1^{n+1}, \dots, \pi_{i-1}^{n+1}, \pi_{n+1}^{n+1}, \pi_{i+1}^{n+1}, \dots, \pi_n^{n+1} \rangle.$$

An argument is called *essential* if it is not nonessential. Moreover, we say that an operation is *essentially k-ary* if it has exactly k essential arguments.

This definition coincides with the usual definition of (non-)essential arguments as presented at the beginning of Subsection 2.2 whenever the latter is applicable (that is, if the powers of $\mathbf{A}$ are Cartesian powers and the morphisms are set-functions). Having written operations and clones in purely category-theoretic terms, we can dualize all these notions:

Definition 3.5. Let $n \in \mathbb{N}_+$. An *n-ary dual operation over* $\mathbf{X}$ (or *cooperation over* $\mathbf{X}$) is a morphism from $\mathbf{X}$ to $n \cdot \mathbf{X}$. Denote by $\overline{O}_{\mathbf{X}}^{(n)}$ the set of all n-ary dual operations over $\mathbf{X}$, define $\overline{O}_{\mathbf{X}} := \bigcup_{n \in \mathbb{N}_+} \overline{O}_{\mathbf{X}}^{(n)}$ and, for a set of dual operations $H \subseteq \overline{O}_{\mathbf{X}}$, set $H^{(n)} := H \cap \overline{O}_{\mathbf{X}}^{(n)}$.

Definition 3.6. A subset $C \subseteq \overline{O}_{\mathbf{X}}$ is called a *clone of dual operations (or coclone)*, written $C \leq \overline{O}_{\mathbf{X}}$, if it contains all the injection morphisms and, for each $h \in C^{(n)}$ and $h_1, \dots, h_n \in C^{(k)}$, the *superposition* $[h_1, \dots, h_n] \circ h$ is also in C.

If $\mathbf{X}$ is a set in the category of sets, then a clone of dual operations over $\mathbf{X}$ is a coclone as introduced in [Csá85].

Definition 3.7. For $n \in \mathbb{N}_+$ and $i \in \{1, \dots, n\}$, the i-th argument of a dual operation $h \in \overline{O}_{\mathbf{X}}^{(n)}$ is said to be *nonessential* if

$$[\iota_1^{n+1}, \dots, \iota_n^{n+1}] \circ h = [\iota_1^{n+1}, \dots, \iota_{i-1}^{n+1}, \iota_{n+1}^{n+1}, \iota_{i+1}^{n+1}, \dots, \iota_n^{n+1}] \circ h.$$

An argument is called *essential* if it is not nonessential. Moreover, we say that an operation is *essentially k-ary* if it has exactly k essential arguments.

Again, clones of dual operations form a complete lattice, which we will denote by $\overline{\mathcal{L}}_{\mathbf{X}}$ and call the *lattice of clones of dual operations over* $\mathbf{X}$.

Analogue to the closure operator Clo on sets of operations, we can define $\overline{\mathrm{Clo}}$: For a set of dual operations $H \subseteq \overline{O}_{\mathbf{X}}$, we denote by $\overline{\mathrm{Clo}}(H)$ the least clone of dual operations that contains H. Again, for a single dual operation, we write $\overline{\mathrm{Clo}}(h)$ instead of $\overline{\mathrm{Clo}}(\{h\})$.

We will now describe how to dualize clones. For this, let $\langle D, E, e, \epsilon\rangle$ be a dual equivalence between two arbitrary categories $\mathcal{A}$ and $\mathcal{X}$, and let $\mathbf{A} \in \mathcal{A}$ such that all finite non-empty powers of $\mathbf{A}$ are also in $\mathcal{A}$. Set $\mathbf{X} := D(\mathbf{A})$. Since $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent, $\mathcal{X}$ contains all finite non-empty copowers of $\mathbf{X}$. The functor D carries $\mathbf{A}$ to $\mathbf{X}$ and reverses the order of the morphisms, so wishful thinking suggests that it should map a morphism $f \in O_{\mathbf{A}}$ to a morphism in $\overline{O}_{\mathbf{X}}$. Unfortunately, this is not always the case as D maps f to a morphism from $\mathbf{X}$ to $D(\mathbf{A}^n)$ and the latter is only isomorphic and not necessarily equal to $n \cdot \mathbf{X}$.[3] However, we can get around this technical problem by finding a family of isomorphisms $(\eta_n)_{n\in\mathbb{N}_+}$ such that $f \mapsto \eta_{ar(f)} \circ D(f)$ becomes a clone isomorphism from $O_{\mathbf{A}}$ to $\overline{O}_{\mathbf{X}}$ (recall that $ar(f)$ denotes the arity of f).

Lemma 3.8 ([Ker11])**.** *There exists a unique family of isomorphisms*

$$(\eta_n \colon D(\mathbf{A}^n) \to n \cdot \mathbf{X})_{n\in\mathbb{N}_+}$$

such that the mapping

$$(-)^{\partial} \colon O_{\mathbf{A}} \to \overline{O}_{\mathbf{X}} : f \mapsto \eta_{\mathrm{ar}(f)} \circ D(f)$$

has the following properties:

(i) $(-)^{\partial} \colon O_{\mathbf{A}}^{(n)} \to \overline{O}_{\mathbf{X}}^{(n)}$ *is a bijection for each* $n \in \mathbb{N}_+$,

(ii) $(\pi_i^n)^{\partial} = \iota_i^n$ *and* $(f \circ \langle h_1, \ldots, h_n\rangle)^{\partial} = [h_1^{\partial}, \ldots, h_n^{\partial}] \circ f^{\partial}$ *for all* $n, k \in \mathbb{N}_+$, $f \in O_{\mathbf{A}}^{(n)}$ *and* $h_1, \ldots, h_n \in O_{\mathbf{A}}^{(k)}$.

In fact, $\eta_n = [D(\pi_1^n), \ldots, D(\pi_n^n)]^{-1}$.

By this lemma, it follows immediately that C is a clone of operations over $\mathbf{A}$ if and only if C^{∂} is a clone of dual operations over $\mathbf{X}$. Moreover, the family $(\eta_n)_{n\in\mathbb{N}_+}$ and hence the construction of $(-)^{\partial}$ only depends on the choice of the dual equivalence. Thus, the following definition is justified:

Definition 3.9. The mapping $(-)^{\partial} \colon O_{\mathbf{A}} \to \overline{O}_{\mathbf{X}}$ is called the *clone duality with respect to D*. For $F \subseteq O_{\mathbf{A}}$, set $F^{\partial} := \{f^{\partial} \mid f \in F\}$.

By Lemma 3.8, we immediately obtain the following theorem:

Theorem 3.10. $\mathcal{L}_{\mathbf{A}} \cong \overline{\mathcal{L}}_{\mathbf{X}}$, *where an isomorphism between* $\mathcal{L}_{\mathbf{A}}$ *and* $\overline{\mathcal{L}}_{\mathbf{X}}$ *is given by* $C \mapsto C^{\partial}$.

[3]Of course, we could avoid the trouble by defining $n \cdot \mathbf{X} := D(\mathbf{A}^n)$ for all $n \in \mathbb{N}_+$. But then, the copowers of $\mathbf{X}$ might not be canonical and they would depend on the choice of the dual equivalence. One usually wants to avoid both.

Moreover, it is an obvious consequence of Lemma 3.8 that an identity holds in C if and only if its dualized version holds in C^∂:

Lemma 3.11. *Let $f_1 \in O_{\mathbf{A}}^{(k)}$, $f_2 \in O_{\mathbf{A}}^{(l)}$. For $i_1, \dots, i_k, j_1, \dots, j_l \in \{1, \dots, n\}$, we have*

$$f_1 \circ \langle \pi^n_{i_1}, \dots, \pi^n_{i_k} \rangle = f_2 \circ \langle \pi^n_{j_1}, \dots, \pi^n_{j_l} \rangle \iff [\iota^n_{i_1}, \dots, \iota^n_{i_k}] \circ f_1^\partial = [\iota^n_{j_1}, \dots, \iota^n_{j_l}] \circ f_2^\partial.$$

In particular, this lemma evidently implies the statement that the i-th argument of some $f \in O_{\mathbf{A}}$ is nonessential if and only if the i-th argument of $f^\partial \in \overline{O}_{\mathbf{X}}$ is nonessential.

In [Ker11], clone dualities are used to obtain new results for clones over finite sets and in particular for clones over classical algebraic structures such as Boolean algebras, distributive lattices, median algebras or Boolean groups. In the next section, we will present a new example and discuss how clones over (not necessarily finite) bounded lattices dualize to clones of dual operations in an FCA-framework.

4 Clones over Bounded Lattices

From now on until the end of this paper, let $\mathbf{A} = \langle A, \vee, \wedge, 0, 1 \rangle$ be a bounded lattice, and let $\mathcal{A}$ be the category of bounded lattices with all homomorphisms as morphisms. Recall that, in this scenario, $O_{\mathbf{A}}$ is the centralizer clone of the lattice $\mathbf{A}$ (cf. Example 3.3(ii)). Our goal is to investigate $\mathcal{L}_{\mathbf{A}}$, that is, the lattice of subclones of $O_{\mathbf{A}}$.

We will now construct a clone duality for $O_{\mathbf{A}}$. By Theorem 2.4, $\mathcal{A}$ is dually equivalent to the category $\mathcal{X}$ of standard topological contexts with multivalued standard morphisms. Recall that the corresponding functor $D\colon \mathcal{A} \to \mathcal{X}$ is given as follows:

$$\begin{aligned} D(\mathbf{A}) &:= \mathbb{K}^\tau(\mathbf{A}) = ((\mathfrak{F}_0(\mathbf{A}), \rho_0), (\mathfrak{I}_0(\mathbf{A}), \sigma_0), \mathfrak{R}(\mathbf{A})), \\ D(f) &:= (R^f, S^f). \end{aligned}$$

From now on, let $\mathbf{X} := D(\mathbf{A})$. To obtain the clone duality $(-)^\partial\colon O_{\mathbf{A}} \to \overline{O}_{\mathbf{X}}$, we need to observe how the powers of $\mathbf{A}$ dualize under D.

Lemma 4.1. *For $n \in \mathbb{N}_+$, we have*

$$\begin{aligned} \mathfrak{F}_0(\mathbf{A}^n) &= \left\{ A^{i-1} \times x \times A^{n-i} \mid i \in \{1, \dots, n\}, x \in \mathfrak{F}_0(\mathbf{A}) \right\}, \\ \mathfrak{I}_0(\mathbf{A}^n) &= \left\{ A^{i-1} \times x \times A^{n-i} \mid i \in \{1, \dots, n\}, x \in \mathfrak{I}_0(\mathbf{A}) \right\}. \end{aligned}$$

Proof. We only show the first equality, since the part for $\mathfrak{I}_0(\mathbf{A}^n)$ is similar.

"$\subseteq$". We will show this direction in three steps. First we show that each $F \in \mathfrak{F}_0(\mathbf{A}^n)$ must be the Cartesian product of n filters $F_1, \dots, F_n \in \mathfrak{F}(\mathbf{A})$, then we show that exactly $n-1$ of the sets $F_1, \dots, F_n$ equal A, and finally we show that $F_i \neq A$ implies $F_i \in \mathfrak{F}_0(\mathbf{A})$. For the first part, let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in F$. Since F is a filter, it is closed under $\wedge$. Thus, for $c_i \in \{a_i, b_i\}$, we have

$$(c_1, \dots, c_n) \geq (a_1 \wedge b_1, \dots, a_n \wedge b_n) = (a_1, \dots, a_n) \wedge (b_1, \dots, b_n) \in F,$$

and consequently $(c_1, \ldots, c_n) \in F$ since F is also an increasing set. This proves that F can be written as $F_1 \times \ldots \times F_n$ for some $F_1, \ldots, F_n \subseteq A$. If F_i is not an increasing set for some $i \in \{1, \ldots, n\}$, then F is not an increasing set. If F_i is not closed under $\wedge$ for some $i \in \{1, \ldots, n\}$, then F is not closed under $\wedge$. Hence, $F_1, \ldots, F_n \in \mathfrak{F}(\mathbf{A})$.
For the second part, let us first note that we cannot have $F = A^n$, so $F_i \neq A$ holds for at least one $i \in \{1, \ldots, n\}$. Now, let us assume the existence of two integers $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $F_i \neq A$ and $F_j \neq A$. Without loss of generality we can assume $i = 1$ and $j = 2$. Since $F_1, \ldots, F_n$ are filters, we also have that $F^* := F_1 \times A \times F_3 \times \ldots \times F_n$ and $F^{**} := A \times F_2 \times \ldots \times F_n$ are filters. Moreover, they both properly contain F. Since $F \in \mathfrak{F}_0(\mathbf{A}^n)$, there must exist an ideal $I \in \mathfrak{I}(\mathbf{A}^n)$ that is disjoint to F but intersects F^* as well as F^{**}. Let $x_1 \in F^* \cap I$ and $x_2 \in F^{**} \cap I$. But now, we have $x_1 \vee x_2 \in I$ since I is an ideal, and we have $x_1 \vee x_2 \in F$ by construction of F^* and F^{**}. Thus, $x_1 \vee x_2 \in I \cap F$, which is impossible.
For the third part, let us assume $F_i \neq A$ for some $i \in \{1, \ldots, n\}$. Since we already know that F_i is a filter, we can finish the proof by showing that there exists $I_i \in \mathfrak{I}(\mathbf{A})$ such that F_i is I_i-maximal. Recall that F is I-maximal for some $I \in \mathfrak{I}(\mathbf{A}^n)$. By arguments analogue to above, I can be written as $I_1 \times \ldots \times I_n$ where $I_1, \ldots, I_n \in \mathfrak{I}(\mathbf{A})$. In particular, I_i is an ideal. Let us show that F_i is I_i-maximal. By

$$F_1 = \ldots = F_{i-1} = F_{i+1} = \ldots = F_n = A,$$

we can conclude $F_i \cap I_i = \emptyset$ since otherwise it would follow $F \cap I \neq \emptyset$, a contradiction to the I-maximality of F. It remains to show that there cannot exist a proper superfilter $F_i^* \supsetneq F_i$ that is disjoint to I_i: The existence of such $F_i^* \in \mathfrak{F}(\mathbf{A})$ would imply that I does not intersect the filter $A^{i-1} \times F_i^* \times A^{n-i} \supsetneq F$, which would again contradict the I-maximality of F.

"$\supseteq$". Let $i \in \{1, \ldots, n\}$ and $x \in \mathfrak{F}_0(\mathbf{A})$. Clearly, $F := A^{i-1} \times x \times A^{n-i}$ is a filter. Since $x \in \mathfrak{F}_0(\mathbf{A})$, there exists $I \in \mathfrak{I}(\mathbf{A})$ such that x is I-maximal. But now, $A^{i-1} \times I \times A^{n-i}$ is an ideal, and we will finish the proof by showing that F is $(A^{i-1} \times I \times A^{n-i})$-maximal. Clearly, $A^{i-1} \times I \times A^{n-i}$ is disjoint to F. Let $F^* \supsetneq F$ be a proper superfilter. By arguments from above, F^* is of the form $A^{i-1} \times y \times A^{n-i}$ for some filter $y \supsetneq x$. But now, y intersects I, and so F^* intersects $A^{i-1} \times I \times A^{n-i}$. Thus, $F \in \mathfrak{F}_0(\mathbf{A}^n)$. $\square$

It remains to understand how the projection morphisms dualize. For $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n\}$, we have $(\pi_i^n)^{-1}[x] = A^{i-1} \times x \times A^{n-i}$ for each $x \in \mathfrak{F}_0(\mathbf{A})$ and each $x \in \mathfrak{I}_0(\mathbf{A})$. Thus, the multivalued standard morphism

$$D(\pi_i^n) = (R^{\pi_i^n}, S^{\pi_i^n})\colon D(\mathbf{A}) \to D(\mathbf{A}^n)$$

is given as follows:

$$R^{\pi_i^n}(x) = \{y \in \mathfrak{F}_0(\mathbf{A}^n) \mid A^{i-1} \times x \times A^{n-i} \subseteq y\},$$
$$S^{\pi_i^n}(x) = \{y \in \mathfrak{I}_0(\mathbf{A}^n) \mid A^{i-1} \times x \times A^{n-i} \subseteq y\}.$$

We will now look at the following canonical definition of copowers in $\mathcal{X}$: Let

$$\mathbf{Y} := ((G_{\mathbf{Y}}, \rho_{\mathbf{Y}}), (M_{\mathbf{Y}}, \sigma_{\mathbf{Y}}), \mathcal{I}_{\mathbf{Y}})$$

be an object in $\mathcal{X}$. Then, the n-th copower of $\mathbf{Y}$ is defined by setting

$$n \cdot \mathbf{Y} := ((n \cdot G_{\mathbf{Y}}, \rho_{n\cdot\mathbf{Y}}), (n \cdot M_{\mathbf{Y}}, \sigma_{n\cdot\mathbf{Y}}), \mathcal{I}_{n\cdot\mathbf{Y}}),$$

where

$$\begin{aligned} n \cdot G_{\mathbf{Y}} &:= \{\langle i, g\rangle \mid i \in \{1, \ldots, n\}, g \in G_{\mathbf{Y}}\}, \\ n \cdot M_{\mathbf{Y}} &:= \{\langle i, m\rangle \mid i \in \{1, \ldots, n\}, m \in M_{\mathbf{Y}}\}, \end{aligned}$$

$\rho_{n\cdot\mathbf{Y}}$ and $\sigma_{n\cdot\mathbf{Y}}$ are the disjoint union topologies (that is, the finest topologies for which all canonical injections $g \mapsto \langle i, g\rangle$ and $m \mapsto \langle i, m\rangle$ are continuous) and

$$\langle i, g\rangle \mathcal{I}_{n\cdot\mathbf{Y}} \langle j, m\rangle :\Longleftrightarrow i \neq j \text{ or } g\mathcal{I}_{\mathbf{Y}}m.$$

The associated injection morphisms ι_i^n are given by

$$\begin{aligned} R_{\iota_i^n}\colon \ & G_{\mathbf{Y}} \to n \cdot G_{\mathbf{Y}} &:\ & R_{\iota_i^n}(y) &= \{\langle i, g\rangle \mid g \in y''\} = \langle i, y\rangle'', \\ S_{\iota_i^n}\colon \ & M_{\mathbf{Y}} \to n \cdot M_{\mathbf{Y}} &:\ & S_{\iota_i^n}(y) &= \{\langle i, m\rangle \mid m \in y''\} = \langle i, y\rangle''. \end{aligned}$$

Moreover, for a standard topological context $\mathbf{Z} = ((G_{\mathbf{Z}}, \rho_{\mathbf{Z}}), (M_{\mathbf{Z}}, \sigma_{\mathbf{Z}}), \mathcal{I}_{\mathbf{Z}})$ and morphisms $h_1, \ldots, h_n\colon \mathbf{Y} \to \mathbf{Z}$, the cotupling $[h_1, \ldots, h_n]\colon n \cdot \mathbf{Y} \to \mathbf{Z}$ is given as follows:

$$\begin{aligned} R_{[h_1,\ldots,h_n]}\colon \ & n \cdot G_{\mathbf{Y}} \to G_{\mathbf{Z}} &:\ & R_{[h_1,\ldots,h_n]}(\langle i, y\rangle) &= R_{h_i}(y), \\ S_{[h_1,\ldots,h_n]}\colon \ & n \cdot M_{\mathbf{Y}} \to M_{\mathbf{Z}} &:\ & S_{[h_1,\ldots,h_n]}(\langle i, y\rangle) &= S_{h_i}(y). \end{aligned}$$

Recall that $\mathbf{X} = D(\mathbf{A}) = ((\mathfrak{F}_0(\mathbf{A}), \rho_0), (\mathfrak{I}_0(\mathbf{A}), \sigma_0), \mathfrak{R}(\mathbf{A}))$. For the copowers of $\mathbf{X}$, we can give a more concrete characterization of the injection morphisms and their cotuplings:

Lemma 4.2. *Let $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n\}$. Then,*

$$\begin{aligned} R_{\iota_i^n}(x) &= \{\langle i, y\rangle \mid y \in \mathfrak{F}_0(\mathbf{A}), x \subseteq y\}, \\ S_{\iota_i^n}(x) &= \{\langle i, y\rangle \mid y \in \mathfrak{I}_0(\mathbf{A}), x \subseteq y\}. \end{aligned}$$

Consequently, for $i_1, \ldots, i_k \in \{1, \ldots, n\}$, we obtain

$$\begin{aligned} R_{[\iota_{i_1}^n,\ldots,\iota_{i_k}^n]}(\langle j, x\rangle) &= \{\langle i_j, y\rangle \mid y \in \mathfrak{F}_0(\mathbf{A}), x \subseteq y\}, \\ S_{[\iota_{i_1}^n,\ldots,\iota_{i_k}^n]}(\langle j, x\rangle) &= \{\langle i_j, y\rangle \mid y \in \mathfrak{I}_0(\mathbf{A}), x \subseteq y\}. \end{aligned}$$

Proof. As described above, we have $R_{\iota_{i_j}^n}(x) = \{\langle i_j, y\rangle \mid y \in x''\}$, and it is a direct consequence of Proposition 2.2 that we also have

$$\{\langle i_j, y\rangle \mid y \in x''\} = \{\langle i_j, y\rangle \mid y \in \mathfrak{F}_0(\mathbf{A}), x \subseteq y\}.$$

The part for $S_{\iota_i^n}$ follows in the same way. $\square$

Let us now turn back to constructing our duality. By Lemma 3.8, there exists a unique family of isomorphism $(\eta_n\colon D(\mathbf{A}^n) \to n \cdot \mathbf{X})_{n \in \mathbb{N}_+}$ with $\iota_i^n = \eta_n \circ D(\pi_i^n)$ for all $n \in \mathbb{N}_+$ and $i \in \{1, \ldots, n\}$. Moreover, the lemma states that this family is obtained by setting $\eta_n := [D(\pi_1^n), \ldots, D(\pi_n^n)]^{-1}$ for all $n \in \mathbb{N}_+$. In the following proposition, we will describe this family more concretely:

Proposition 4.3. *For $n \in \mathbb{N}_+$, the unique isomorphism $\eta_n\colon D(\mathbf{A}^n) \to n \cdot \mathbf{X}$ from Proposition 3.8 is given as follows:*

$$\begin{aligned} &\text{For } x \in \mathfrak{F}_0(\mathbf{A}^n)\text{:} && R_{\eta_n}(x) = \{\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid x \subseteq A^{i-1} \times y \times A^{n-i}\},\\ &\text{for } x \in \mathfrak{I}_0(\mathbf{A}^n)\text{:} && S_{\eta_n}(x) = \{\langle i, y\rangle \in n \cdot \mathfrak{I}_0(\mathbf{A}) \mid x \subseteq A^{i-1} \times y \times A^{n-i}\}. \end{aligned}$$

Proof. In view of Lemma 3.8, we need to show $\eta_n = [D(\pi_1^n), \ldots, D(\pi_n^n)]^{-1}$. For brevity, let us set $h := [D(\pi_1^n), \ldots, D(\pi_n^n)]$. Then,

$$\begin{aligned} R_h\colon &\ n \cdot \mathfrak{F}_0(\mathbf{A}) \to \mathfrak{F}_0(\mathbf{A}^n) &:&\ R_h(\langle i, y\rangle) &= \{x \in \mathfrak{F}_0(\mathbf{A}^n) \mid A^{i-1} \times y \times A^{n-i} \subseteq x\},\\ S_h\colon &\ n \cdot \mathfrak{I}_0(\mathbf{A}) \to \mathfrak{I}_0(\mathbf{A}^n) &:&\ S_h(\langle i, y\rangle) &= \{x \in \mathfrak{I}_0(\mathbf{A}^n) \mid A^{i-1} \times y \times A^{n-i} \subseteq x\}. \end{aligned}$$

On the one hand, for $x \in \mathfrak{F}_0(\mathbf{A}^n)$, we have

$$\begin{aligned} R_{h \circ \eta_n}(x) = R_h[R_{\eta_n}(x)]'' &= R_h[\{\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid x \subseteq A^{i-1} \times y \times A^{n-i}\}]''\\ &= \{z \in \mathfrak{F}_0(\mathbf{A}^n) \mid x \subseteq z\}'' = x'' = R_{id_{D(\mathbf{A}^n)}}, \end{aligned}$$

where the last but one step follows directly from Proposition 2.2. On the other hand, for $\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A})$, we have

$$\begin{aligned} R_{\eta_n \circ h}(\langle i, y\rangle) = R_{\eta_n}[R_h(\langle i, y\rangle)]'' &= R_{\eta_n}[\{x \in \mathfrak{F}_0(\mathbf{A}^n) \mid A^{i-1} \times y \times A^{n-i} \subseteq x\}]''\\ &= \{\langle i, z\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid y \subseteq z\}'' = \langle i, y\rangle'' = R_{id_{n \cdot \mathbf{X}}}(\langle i, y\rangle), \end{aligned}$$

where the fourth step is again due to Proposition 2.2. In the same way, it follows that we have $S_{h \circ \eta_n} = S_{id_{D(\mathbf{A}^n)}}$ and $S_{\eta_n \circ h} = S_{id_{n \cdot \mathbf{X}}}$. Thus, $\eta_n = h^{-1}$. □

As outlined in Section 3, we now obtain the clone duality $(-)^\partial\colon O_{\mathbf{A}} \to \overline{O}_{\mathbf{X}}$ by setting $f^\partial := \eta_{ar(f)} \circ D(f)$ for $f \in O_{\mathbf{A}}$. The following proposition states $(-)^\partial$ explicitly:

Proposition 4.4. *For $f \in O_{\mathbf{A}}$ with $ar(f) = n$, the multivalued standard morphism $f^\partial \in \overline{O}_{\mathbf{X}}^{(n)}$ is given as follows:*

$$\begin{aligned} &\text{For } x \in \mathfrak{F}_0(\mathbf{A})\text{:} && R_{f^\partial}(x) = \{\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq A^{i-1} \times y \times A^{n-i}\},\\ &\text{for } x \in \mathfrak{I}_0(\mathbf{A})\text{:} && S_{f^\partial}(x) = \{\langle i, y\rangle \in n \cdot \mathfrak{I}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq A^{i-1} \times y \times A^{n-i}\}. \end{aligned}$$

Proof. For $x \in \mathfrak{F}_0(\mathbf{A})$, we have

$$\begin{aligned} R_{f^\partial}(x) &= R_{\eta_n \circ D(f)}(x)\\ &= R_{\eta_n}[R^f(x)]''\\ &= R_{\eta_n}[\{z \in \mathfrak{F}_0(\mathbf{A}^n) \mid f^{-1}[x] \subseteq z\}]''\\ &= \{\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq A^{i-1} \times y \times A^{n-i}\}''\\ &= \{\langle i, y\rangle \in n \cdot \mathfrak{F}_0(\mathbf{A}) \mid f^{-1}[x] \subseteq A^{i-1} \times y \times A^{n-i}\}. \end{aligned}$$

The part for S_{f^∂} follows in the same way. □

As already noted in the preliminaries, surjective homomorphisms play a special role in the dual equivalence. For them, we can state the following proposition:

Proposition 4.5. *Let $f \in O_{\mathbf{A}}^{(n)}$ be surjective. Then, for each $x \in \mathfrak{F}_0(\mathbf{A})$ there exist $i \in \{1, \dots, n\}$, $y \in \mathfrak{F}_0(\mathbf{A})$ such that $R_{f^\partial}(x) = R_{\iota_i^n}(y)$, and similarly, for each $x \in \mathfrak{I}_0(\mathbf{A})$ there exist $i \in \{1, \dots, n\}$, $y \in \mathfrak{I}_0(\mathbf{A})$ such that $S_{f^\partial}(x) = S_{\iota_i^n}(y)$.*

Proof. Let $x \in \mathfrak{F}_0(\mathbf{A})$. As noted in the preliminaries, f being surjective implies that we have $f^{-1}[x] \in \mathfrak{F}_0(\mathbf{A}^n)$. Thus, there exist $i \in \{1, \dots, n\}$ and $y \in \mathfrak{F}_0(\mathbf{A})$ such that $f^{-1}[x] = A^{i-1} \times y \times A^{n-i}$. Hence, $R_{f^\partial}(x) = \{\langle i, z\rangle \mid z \in \mathfrak{F}_0(\mathbf{A}), y \subseteq z\} = R_{\iota_i^n}(y)$. As usual, the part for S_{f^∂} follows in the same way. $\square$

Let us summarize the work of this section: We have constructed a clone duality $(-)^\partial$ that dualizes clones over bounded lattices (of arbitrary cardinality) to clones of dual operations that consist of multivalued standard morphisms between standard topological contexts. Thus, we have obtained a technique that allows us to transfer problems from clone theory to the field of Formal Concept Analysis. In the next section, we will put this duality to work and give a small illustration of how this connection can be a useful tool to investigate clones over bounded lattices.

5 A Small Illustration of the Duality

Let us now illustrate that the duality can be used to obtain (new) results for clones over bounded lattices that would be much harder to obtain without the duality. Recall that the categories $\mathcal{A}$ and $\mathcal{X}$, the objects $\mathbf{A}$ and $\mathbf{X}$, the functor D and the clone duality $(-)^\partial \colon O_{\mathbf{A}} \to \overline{O}_{\mathbf{X}}$ still denote what they denoted in the last section (they were all introduced on page 9).

First, we deal with essential arguments. Since the morphisms in our category $\mathcal{A}$ are homomorphisms and therefore set-functions and the products in $\mathcal{A}$ are the Cartesian products, the i-th argument of a morphism $f \in O_{\mathbf{A}}$ is essential in the sense of Definition 3.4 if and only if the i-th argument of f is essential in the usual sense that we have presented at the beginning of Subsection 2.2. Furthermore, as we have noted in Lemma 3.11, the i-th argument of f is nonessential if and only if the i-th argument of f^∂ is nonessential. Thus, we can investigate the essentiality of the arguments of an operation $f \in O_{\mathbf{A}}$ by investigating the arguments of its dual $f^\partial \in \overline{O}_{\mathbf{X}}$. To do the latter, we can use the following lemma:

Lemma 5.1. *Let $n \in \mathbb{N}_+$. For an at least binary multivalued standard morphism $h \in \overline{O}_{\mathbf{X}}^{(n)}$, the following two statements are equivalent:*

(1) the t-th argument of h is nonessential,
(2) $R_h[\mathfrak{F}_0(\mathbf{A})] \subseteq \{\langle i, y\rangle \mid i \in \{1, \dots, t-1, t+1, \dots, n\}, y \in \mathfrak{F}_0(\mathbf{A})\}$, and $S_h[\mathfrak{I}_0(\mathbf{A})] \subseteq \{\langle i, y\rangle \mid i \in \{1, \dots, t-1, t+1, \dots, n\}, y \in \mathfrak{I}_0(\mathbf{A})\}$.

Proof. Without loss of generality, we can assume $t = 1$.

(1) $\Longrightarrow$ (2). By assumption, h does not depend on its first argument. Hence,

$$[\iota_1^{n+1}, \ldots, \iota_n^{n+1}] \circ h = [\iota_{n+1}^{n+1}, \iota_2^{n+1}, \ldots, \iota_n^{n+1}] \circ h,$$

and so the claim follows by using Lemma 4.2.

(2) $\Longrightarrow$ (1). We have $R_h(x) \subseteq \{\langle i, y\rangle \mid i \in \{2, \ldots, n\}, y \in \mathfrak{F}_0(\mathbf{A})\}$ for each $x \in \mathfrak{F}_0(\mathbf{A})$. Hence, Lemma 4.2 yields

$$\begin{aligned} R_{[\iota_1^{n+1},\ldots,\iota_n^{n+1}]\circ h}(x) &= R_{[\iota_1^{n+1},\iota_2^{n+1},\ldots,\iota_n^{n+1}]}[R_h(x)] \\ &= R_{[\iota_{n+1}^{n+1},\iota_2^{n+1},\ldots,\iota_{n+1}^{n+1}]}[R_h(x)] \\ &= R_{[\iota_{n+1}^{n+1},\iota_2^{n+1},\ldots,\iota_n^{n+1}]\circ h}(x). \end{aligned}$$

The equation $S_{[\iota_1^{n+1},\ldots,\iota_n^{n+1}]\circ h} = S_{[\iota_{n+1}^{n+1},\iota_2^{n+1},\ldots,\iota_n^{n+1}]\circ h}$ follows in the same way. $\square$

As this lemma shows, one only needs to look at the images of R_{f^∂} and S_{f^∂} to determine which arguments of an operation $f \in O_\mathbf{A}$ are essential and which are nonessential. In most cases, this is much easier than trying to investigate the essentiality of an argument in the usual way. In fact, with this lemma, it becomes remarkably easy to infer many results about the essential arity of operations among $O_\mathbf{A}$. For instance, we can now almost trivially deduce the fact that the essential arity of operations among $\overline{O}_\mathbf{X}$, and hence $O_\mathbf{A}$, is bounded if $\mathbf{X}$ is finite (note that $\mathbf{X}$ is finite if and only if $\mathbf{A}$ is finite). This result is known and usually derived from the fact that lattice-homomorphisms satisfy the strong term condition [McK83], so what we have obtained is an alternative proof, where the lemma above replaces the arguments from universal algebra. A much more ambitious goal would be to use this lemma to obtain a sharp bound on the essential arity of operations over a given finite lattice, which, to the best knowledge of the author, is an open problem. It seems promising that this problem can be solved with the help of the lemma above and some work with the multivalued standard morphisms. However, it would be beyond the scope of this paper.

Let us instead conclude this section with some results about idempotent operations. Recall that a function f is said to be *idempotent* if $f(x, \ldots, x) \approx x$. Writing this equivalently in category-theoretic notation, we can say that an operation $f \in O_\mathbf{A}^{(n)}$ is idempotent if and only if $f \circ \langle id_\mathbf{A}, \ldots, id_\mathbf{A}\rangle = id_\mathbf{A}$. Clearly, by Lemma 3.11, $f \in O_\mathbf{A}$ is idempotent if and only if $f^\partial \in \overline{O}_\mathbf{X}$ is a *dual idempotent operation*, that is, $[id_\mathbf{X}, \ldots, id_\mathbf{X}] \circ f^\partial = id_\mathbf{X}$. A clone of (dual) operations is called *idempotent* if it contains only idempotent (dual) operations.

We start our small investigation of idempotent operations by providing the following characterization of the dual idempotent operations among $\overline{O}_\mathbf{X}$:

Lemma 5.2. *Let $h \in \overline{O}_\mathbf{X}^{(n)}$. The following two statements are equivalent:*

(1) h is idempotent.

(2) For all $x \in \mathfrak{F}_0(\mathbf{A})$, there exists $i \in \{1, \ldots, n\}$ such that $R_h(x) = R_{\iota_i^n}(x)$, and for all $x \in \mathfrak{I}_0(\mathbf{A})$ there exists $i \in \{1, \ldots, n\}$ such that $S_h(x) = S_{\iota_i^n}(x)$.

Proof. (1) $\Longrightarrow$ (2). As usual, we only need to show the part for R_h since the statement for S_h follows in the same way. First, let $x \in \mathfrak{F}_0(\mathbf{A})$. There exists $f \in O_{\mathbf{A}}$ such that $f^\partial = h$. Since h is idempotent, so is f. This also implies that f is surjective. Therefore, we can apply Proposition 4.5, and it follows that there exists $i \in \{1, \ldots, n\}$ and $y \in \mathfrak{F}_0(\mathbf{A})$ such that $R_h(x) = R_{\iota_i^n}(y)$. Moreover, the idempotency of h implies $\iota_i^n = [\iota_i^n, \ldots, \iota_i^n] \circ h$. Hence,

$$\begin{aligned} R_{\iota_i^n}(x) &= R_{[\iota_i^n, \ldots, \iota_i^n] \circ h}(x) = R_{[\iota_i^n, \ldots, \iota_i^n]}[R_h(x)]'' = R_{[\iota_i^n, \ldots, \iota_i^n]}[R_{\iota_i^n}(y)]'' \\ &= R_{[\iota_i^n, \ldots, \iota_i^n] \circ \iota_i^n}(y) = R_{\iota_i^n}(y) = R_h(x). \end{aligned}$$

(2) $\Longrightarrow$ (1). We have to show $id_{\mathbf{X}} = [id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ h$. For each $x \in \mathfrak{F}_0(\mathbf{A})$, there exists $i \in \{1, \ldots, n\}$ such that $R_h(x) = R_{\iota_i^n}(x)$. Hence,

$$\begin{aligned} R_{id_{\mathbf{X}}}(x) &= R_{[id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ \iota_i^n}(x) = R_{[id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}]}[R_{\iota_i^n}(x)]'' \\ &= R_{[id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}]}[R_h(x)]'' = R_{[id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ h}(x). \end{aligned}$$

Analogously, it follows $S_{id_{\mathbf{X}}} = S_{[id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ h}$, so $id_{\mathbf{X}} = [id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ h$. □

With this lemma, we can establish a close connection between the dual idempotent operations over $\mathbf{X}$ (and hence the idempotent operations over $\mathbf{A}$) and certain partitions.

Definition 5.3. For a dual idempotent operation $h \in \overline{O}_{\mathbf{X}}^{(n)}$, we denote by $\Pi(h)$ the partition of $\mathfrak{F}_0(\mathbf{A}) \cup \mathfrak{I}_0(\mathbf{A})$ obtained by setting $\Pi(h) := \{X_1, \ldots, X_n\} \setminus \{\emptyset\}$ where $X_1, \ldots, X_n$ are defined as follows:

$$\begin{aligned} &\text{For } x \in \mathfrak{F}_0(\mathbf{A})\text{: } x \in X_i :\Longleftrightarrow R_h(x) = R_{\iota_i^n}(x), \\ &\text{for } x \in \mathfrak{I}_0(\mathbf{A})\text{: } x \in X_i :\Longleftrightarrow S_h(x) = S_{\iota_i^n}(x). \end{aligned}$$

Note that $\Pi(h)$ is a well-defined partition due to Lemma 5.2. Thus, every dual idempotent operation on $\mathbf{X}$ can be uniquely assigned to a partition of $\mathfrak{F}_0(\mathbf{A}) \cup \mathfrak{I}_0(\mathbf{A})$ (but not necessarily vice versa). Moreover, denoting by $\preccurlyeq$ the finer-than relation for partitions, we can use Lemma 5.2 to easily deduce the following statement (its proof will be omitted due to limitation of space):

Lemma 5.4.

(a) For two idempotent dual operations $h_1, h_2 \in \overline{O}_{\mathbf{X}}$, we have $h_2 \in \overline{\mathrm{Clo}}(h_1)$ if and only if $\Pi(h_1) \preccurlyeq \Pi(h_2)$. Consequently, $\overline{\mathrm{Clo}}(h_1) = \overline{\mathrm{Clo}}(h_2)$ if and only if $\Pi(h_1) = \Pi(h_2)$.
(b) Each idempotent $C \leq \overline{O}_{\mathbf{X}}$ is generated by a single dual operation.

Note that the second part of the lemma clearly also holds in its dualized version, that is, each idempotent $C \leq O_{\mathbf{A}}$ is determined by a single operation. The first part of this lemma makes it very easy to decide whether two dual idempotent operations generate each other, and with a little bit more work, we can also establish a close connection between the lattice of partitions of $\mathfrak{F}_0(\mathbf{A}) \cup \mathfrak{I}_0(\mathbf{A})$ (given by $\preccurlyeq$) and the lattice of idempotent clones over $\mathbf{A}$ (one of the ideals of the clone lattice that is of particular interest).

Proposition 5.5. *The lattice of idempotent clones of operations over* $\mathbf{A}$ *can be embedded into the lattice of partitions of the set* $\mathfrak{F}_0(\mathbf{A}) \cup \mathfrak{I}_0(\mathbf{A})$.

Proof. By Lemma 5.4, the desired lattice-embedding φ can be obtained by setting $\varphi(C) := \Pi(f^\partial)$ where f is one of the single operations that generate C. $\square$

For future research, it would be an interesting task to further investigate the lattice of idempotent clones of operations over $\mathbf{A}$ by characterizing the sublattice of the lattice of partitions of $\mathfrak{F}_0(\mathbf{A}) \cup \mathfrak{I}_0(\mathbf{A})$ to which it is isomorphic.

6 Conclusion

We used the dual equivalence from [Har93] and the results from [Ker11] to construct a duality between clones over bounded lattices and so-called clones of dual operations over standard topological contexts. We gave some small examples of how this connection between clone theory and Formal Concept Analysis can be used to simplify clone theoretic problems and to produce concrete results. In the process, we also stated some open problems for which an application of the duality seems promising.

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