Multioperator Weighted Monadic Datalog

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Contents

CHAPTER 1

Introduction

One of the core aspects of computer science is the development of models of computation. Computation in this sense is the process of transforming input data into output data.

Usually data have an underlying structure. One of the most general, common, and broadly applicable ways to structure data is the hierarchical scheme, also called tree structure. Tree structured data have many applications in computer science:

- The rise of the Web has lead to new ways of representing data and storing data by means of semi-structured databases [1, 82] and eXtensible Markup Language (for short: XML) [24]. Semi-structured databases are, essentially, data having a tree structure. The popularity of XML led to an immense increase of research concerning information extraction techniques for tree structured data.

- Sentences or words of both artificial and natural language are typically denoted as strings. However, languages usually have a structure, which is given by the grammar of the language. Hence, strings are a convenient but not the natural way to represent elements of the language. By adding the structure, which is implied by the grammar, to such an element, one obtains a tree structure. Hence, processing natural or artificial languages is most successfully accomplished by using a tree structured representation of the language elements.

As there are numerous fields of research in computer science, there are multiple definitions of trees. One of the most broadly used variants are trees such that their nodes are labeled and the children of every node are totally ordered. There are two flavors of such trees: (i) for ranked trees, the rank of every node, i.e., the number of its children, is unambiguously determined by its label and (ii) for unranked trees every node can have an arbitrary rank. In some sense both variants are interchangeable, because every ranked tree is an unranked tree and every unranked tree can be encoded as a ranked tree. Therefore, let us restrict our focus to ranked trees in this motivation because their restricted structure usually allows for simpler computation models.

There are a variety of methods to model computation processes. Common examples are Turing machines, programming languages, neural networks, or finite automata, i.e., devices having a bounded amount of memory. Research in the field of finite automata has led to many fruitful discoveries and applications because often finite automata have a simple structure, can be processed efficiently, behave "nicely", i.e., have many closure properties and decidability properties, the classes of computations they define are robust, i.e., often coincide for similar finite specifications, and, quite often, have satisfactory computational power.

Let us have a closer look at finite devices for processing ranked trees. In particular, let us discuss the developments of two branches of such models.

Finite state tree automata and transducers The theory of tree automata [37, 49, 65, 66, 128] emerged in the middle of the 1960s and generalizes the concept of finite automata over strings. Finite state tree automata (for short: *fta*) were introduced independently by Doner [37] and Thatcher and Wright [128].

Essentially, an fta consists of a finite state set Q, a set δ of allowed transitions, and a set F of final states. The semantics of an fta is a tree language; it is usually defined in terms of *runs* and *transitions*. Given an input tree, a run on that tree is a decoration of its nodes with states of the fta such that the root node is associated with a final state. Given a run, every position w of the input tree determines a transition, which is the triple $(q_1 \cdots q_k, \sigma, q)$ consisting of the sequence $q_1 \cdots q_k$ of states that the direct children of w are decorated with, the label σ of the input tree at w, and the state q at w. The fta accepts the input tree if there is a run on that tree such that every transition in the input tree for that run is allowed, i.e., it is in the set δ. Now the set of accepted input trees is the recognized language of the fta.

The set δ of allowed transitions can be considered as a mapping from the set of possible transitions into the set $\mathbb{B}$ of Boolean values; let us consider this point of the view in the following discussion. It is natural to extend this model by assigning more sophisticated weights to each transition. In order to be able to calculate with these weights, an algebraic structure is required; usually, semirings [67, 72] are most appropriate for this purpose. Roughly speaking, a semiring is an algebra containing two binary operations, where the first operation is called the semiring addition and the second operation is called the semiring multiplication; moreover, the semiring multiplication is required to distribute over the semiring addition. This extension results in the concept of weighted tree automata [15, 95, 52, 53] (for short: *wta*). Wta are defined similarly to fta; however, in addition every wta involves a semiring and the mapping δ of allowed transitions is replaced by a mapping μ which associates every possible transition with a semiring element. The semantics of a wta is a mapping from the set of input trees to the carrier set of the semiring of the wta; this mapping is called the recognized (formal) tree series [97, 51] of the wta. Now let us explain how this tree series is computed. For every input tree t the image of t under the recognized tree series is the sum of the weights of every run of the wta on t, where the weight of every run is defined to be the product of the weight of every transition (note that this weight is given by μ) in the input tree for that run; if the considered semiring is not commutative, then one has to agree on a certain order of the factors in that product. Weighted tree automata have broad applications in computer science, e.g., for code selection in compilers [54, 21], tree pattern matching [120], and natural language processing [87, 27]. Note that in [114] the concept of wta has been extended to strong bimonoids [44], which, roughly speaking, are semirings without distributivity.

In the previous paragraph we have described a tree automaton model that is derived from fta by allowing the mapping δ to map into an arbitrary given semiring. Other models that originate from fta by employing similar extensions, have been proposed, investigated, and applied fruitfully. We will just mention two such models.

- The concept of *tree transducers* [117, 130, 8, 48, 9] (for short: *tt*) can be considered to originate from the concept of fta by replacing the mapping δ of allowed transitions by a mapping μ which associates every possible transition with a finite set of segments of output trees; such a segment is modeled as a tree containing output symbols and variables. The semantics of a tt is a mapping from input trees to sets of output trees; such a mapping is called a tree transformation. The set of output trees for a given input tree is the union of the induced tree languages of every run of the tt on t; the tree language that is induced by a run is obtained by applying the tree substitution operation to the output tree segment sets of every transition (which are given by μ) in the input tree for that run. Note that usually the semantics of tt is defined in terms of derivations instead of runs and transitions [48]. Tree transducers have been applied in practice for, e.g., computational linguistics [110, 91, 107], query languages of semi-structured databases [16], and generation of pictures [39, 40].

- Weighted tree transducers [97, 51, 61, 57, 62, 105, 63] (for short: *wtt*) are a combination of wta and tree transducers. Every wtt defines a mapping from input trees to tree series over output trees and a given semiring; such a mapping is called a tree series transformation. In particular, wtt and extended models have applications in the field of translation of natural languages (cf., e.g., [86, 106]).

Since fta, wta, tt, and wtt share a similar structure, a unifying model has been proposed that subsumes all of the former four models. This automaton, called the weighted multioperator tree automaton (for short: *wmta*), has been introduced in [96] and is obtained from fta by replacing the mapping δ of allowed transitions by a mapping μ that associates every possible transition with operations of a multioperator monoid (for short: m-monoid) [95, 96] (also see [103, 58, 123]). The latter is an algebraic structure which consists of a commutative monoid and a Δ-algebra [70, 135] (where Δ is a given signature) such that the carrier sets of the monoid and the Δ-algebra are required to coincide. More precisely, the weight of the transition at some k-ary input symbol is a k-ary symbol in Δ. The semantics is defined similarly to the semantics of wta: it is a tree series which maps every input tree t to an element of the m-monoid; this m-monoid element results from summing up over the weights of every run of the wmta on t. The weight of a run is obtained as follows: first t is transformed into a tree s over Δ by replacing at every position w of t the label σ of t at w by the symbol $\mu(q_1 \cdots q_k, \sigma, q)$, where $(q_1 \cdots q_k, \sigma, q)$ is the transition of t for the considered run. Then the weight of the run is the result of the initial homomorphism for the Δ-algebra of the considered m-monoid applied to s.

Roughly speaking, by employing particular m-monoids, wmta can simulate fta, wta, tt, and wtt. Hence, in general investigations and results of wmta subsume investigations and results of all the former models. For examples of such generalizing results we refer the reader to [58, 121, 123, 59].

Monadic datalog *Monadic datalog* [68, 69], a syntactically restricted fragment of standard datalog [2], is a means of formally specifying tree languages (and node selection queries on trees). Let us roughly sketch the underlying idea. Essentially, a monadic datalog program (for short: *md*) consists of a set P of user-defined predicates, a finite set R of rules, and a query predicate $q \in P$. For a given tree t the predicates in P are instantiated for every node of t, yielding *user-defined atom instances*. The finite set R of rules is then used to restrict the set of *interpretations*, that is, mappings from these atom instances to the set of Boolean values. Every rule consists of a left- and a right-hand side, where the left-hand side is a *user-defined atom* (a user-defined predicate applied to a variable) and the right-hand side is a finite sequence of user-defined atoms and *structural atoms*, where structural atoms have a special syntactic form and express properties of the input tree. An example of such a rule looks as follows:

$$p(x) \leftarrow q(y), s(z), \mathrm{label}_\sigma(x), \mathrm{child}_1(x, y) \ .$$

In this rule $p(x)$, $q(y)$, and $s(z)$ are user-defined atoms and $\mathrm{label}_\sigma(x)$ as well as $\mathrm{child}_1(x, y)$ are structural atoms. Intuitively, this rule means that for all nodes w, v, u of t, if the atom instance $q(v)$ is true, the atom instance $s(u)$ is true, the node w is labeled with σ, and v is the first child node of w, then the atom instance $p(w)$ needs to be true, too. Of all interpretations that satisfy this condition (and satisfy the other rules in R) we choose the smallest interpretation, where an interpretation is considered to be smaller than another interpretation if the former one maps less atom instances to the value 'true'. Finally, the input tree is accepted if the designated interpretation maps the atom instance $q(\varepsilon)$ (where q is the query predicate and ε is the root of the input tree) to 'true'.

Monadic datalog can naturally be used in the setting of ranked trees as well as in the setting of unranked trees. It has, thus, applications in the field of semi-structured databases and XML. Its potential for practical applications is due to the following reasons: (i) the use of rules for specifying properties of tree languages is intuitive and natural, (ii) the time complexity of evaluating the semantics of monadic datalog is linear in the size of the considered input tree and the size of the considered md, and (iii) the class of tree languages that can be specified by means of monadic datalog has been proved to be the class of tree languages that can be recognized by means of fta (cf. [68, 69]).

Similarly to the extension of fta to wta and tt, the concept of monadic datalog has been extended to models for specifying tree series (called weighted monadic datalog [122]) and models for specifying tree transformations (called monadic datalog tree transducers [28]).

In resemblance to wta, a weighted monadic datalog program (for short: *wmd*) involves a semiring. Moreover, the syntax of rules in wmd is similar to the syntax of rules in md; however, their right-hand sides are allowed to contain semiring elements. Let us consider an example of such a rule:

$$p(x) \leftarrow q(y), 3, s(z), 2, \mathrm{label}_\sigma(x), \mathrm{child}_1(x, y) \ .$$

In the setting of weighted monadic datalog every interpretation is a mapping from user-defined atom instances to the carrier set of the semiring. Again, we restrict the set of all possible interpretations to those interpretations that satisfy the rules in the wmd; an interpretation is said to satisfy a rule if for every instance of the variables in that rule, the resulting atom instance on the left-hand side of the rule is the product of the values of the atom instances and semiring elements on the right-hand side of the rule, where a structural atom instance is interpreted as the neutral element of the semiring addition if it is false and as the neutral element of the semiring multiplication if it is true. If there are multiple combinations of rules and instantiations of variables in these rules that yield the same user-defined atom instance on their left-hand sides, then the value of this atom instance needs to be the sum of all the resulting products of the respective right-hand sides. Finally, for a given order on interpretations, we choose the smallest interpretation among all interpretations that satisfy the rules in the md. Then the resulting semiring element of the considered input tree is the image of the atom instance $q(\varepsilon)$ under the designated interpretation.

Weighted monadic datalog has been shown to be strongly more expressive than wta and, similarly to monadic datalog, to have the property that it can be evaluated in linear time (both in the size of the input tree and the wmd) (see [122]). A concept that is similar to wmd, called semiring-based constraint logic programming, has been introduced and studied in [18]

Monadic datalog tree transducer programs (for short: *mdtt*) are defined likewise. In this setting interpretations map user-defined atom instances to sets of output trees. Since there is no natural way to represent Boolean values as sets of output trees, the syntax of rules in mdtt is slightly different from the syntax rules in md and wmd: the right-hand side of rules consist of two parts, called *body* and *guard*, where the body is a tree over output symbols with user-defined atoms as indices and the guard is a finite set of structural atoms. Consider the following example:

$$p(x) \leftarrow \sigma\big(\alpha, \gamma(q(y)), s(z)\big) \;;\; \{\mathrm{label}_\sigma(x), \mathrm{child}_1(x,y)\} \;.$$

The guard is used to restrict the set of possible variable assignments. Intuitively, this rule means that for all nodes w, v, u of the input tree such that the node w is labeled with σ and v is the first child node of w, we have that the value of $p(w)$ needs to be the set of all trees of the form $\sigma\big(\alpha, s_1, s_2\big)$ where s_1 is in the set $q(v)$ and s_2 is in the set $s(u)$. If there are multiple combinations of rules and variable assignments that yield that same user-defined atom instance in their left-hand side, then the value of this atom instance needs to be the union of all the resulting tree languages of the respective bodies. The output tree language of the considered input tree is the image of the atom instance $q(\varepsilon)$ under the smallest interpretation that satisfies the rules in the mdtt.

Monadic datalog tree transducers can be used to specify finite tree transformations (i.e., for every input tree the set of output trees is finite) and infinite tree transformations (i.e., for a given input tree the set of output trees can be infinite). Mdtt have been shown to be at least as expressive as attributed tree transducers [56, 11, 12] (see [28]).

The purpose of this thesis

In this thesis we study a generalization of md, wmd, and mdtt that resembles the generalization of fta, wta, and tt to wmta. We call this model multioperator weighted monadic datalog (for short: *mwmd*); Table 1.1 illustrates how this formalism fits into the landscape of models that we discussed above.

	tree language	tree series	tree transformation	generalization
tree automata	fta	wta	tt	wmta
monadic datalog	md	wmd	mdtt	*mwmd*

Table 1.1: An overview of the presented formalisms.

Intuitively, the semantics of an mwmd is evaluated by means of the operations of a given m-monoid and, by employing particular m-monoids, mwmd can simulate md, wmd, and mdtt. In fact, the syntax of mwmd is reminiscent of the syntax of mdtt; i.e., rules consist of a left-hand side (called *head*), a body, and a guard, where the body is a tree over some ranked alphabet Δ with user-defined atoms as indices. Then the considered m-monoid that is used for the computation of the semantics is a commutative monoid with a Δ-algebra, i.e., the symbols of Δ in the body of rules are interpreted as operations of the m-monoid.

In this thesis we will introduce the syntax and semantics of mwmd. In order to develop a rich theory we will define multiple versions of semantics; as a preliminary this requires a detailed investigation of the theory of m-monoids. Moreover, we will study normal forms and decidability results of mwmd. We will show that, by employing particular m-monoids, the theory of mwmd subsumes the theory of both wmd and mdtt. We conclude this thesis by showing that mwmd even contain wmta (and, roughly speaking, fta, wta, and tt) as a syntactic subclass and present results concerning this subclass.

Overview

Now let us give an overview over the contents of this thesis.

Chapter 2. In this preliminary chapter we will recall basic notions that are central to this thesis. First we will deal with sets, relations, directed graphs, mappings, and operations. In particular, we will cover operations of infinite arity [72, 95, 42, 63]. We proceed with a discussion of signatures, algebras, homomorphisms and free algebras [30, 70, 135]. Moreover, we will deal with basic concepts of strings, trees, and define and study properties of tree substitutions [66, 51, 63, 105]. Finally, we will give an introduction into the topic of hypergraphs and hyperpaths [6, 64]. In that section we will put particular emphasis on the study of basic results of hyperpath segments, dependence relations, and decomposition and composition of hyperpaths and hyperpath segments.

Chapter 3. In this chapter we present the definition of the core algebra of this thesis, called multioperator monoids (for short: m-monoids). Our definition of m-monoids is based on the definition of *distributive m-monoids* (or *distributive* Ω*-monoids*) that have been introduced by Kuich [95, 96, 98]. The concept of distributive m-monoids is a generalization of distributive F-magmas defined by Courcelle [33, Section 10] and of K–Γ-algebras by Bozapalidis [22]. M-monoids that are not necessarily distributive have been studied in [121, 123, 59].

An m-monoids is a combination of a commutative monoid and a Δ-algebra (for some ranked alphabet Δ). They are a crucial component of the definition of the semantics of mwmd programs because it is computed by means of the monoid operation as well as the Δ-algebra operations of the m-monoid.

As a preliminary we give an introduction into the theory of semigroups, monoids, and tree series. Afterwards, we will define m-monoids and study basic properties of m-monoids and homomorphisms.

It turns out that in general the operations that m-monoids provide are not sufficient to compute the semantics of mwmd. This happens whenever the considered mwmd exhibits circular behavior and, thus, loops indefinitely. We will cope with this problem by employing extended definitions of m-monoids that are capable of computing a well-defined output value for such mwmd. To this end we introduce ω-complete and ω-continuous m-monoids. We will give examples and study their properties in detail. In particular, we will investigate relationships between ω-complete and ω-continuous m-monoids and present important results concerning this relationship.

Chapter 4. In this chapter we present the core definitions of this thesis, i.e., both syntax and semantics of mwmd. We will define the syntactic structure of mwmd in resemblance to the syntax of monadic datalog tree transducers [28] and weighted monadic datalog [122].

We will put particular emphasis on the definition of the semantics of mwmd. We will provide two different types of semantics, which we call fixpoint semantics and hypergraph semantics. The fixpoint semantics is reminiscent of the initial algebra semantics of bottom-up weighted tree automata [15, 63], whereas the hypergraph semantics is related to the run semantics of weighted tree automata (or similar concepts such as m-weighted tree automata [103, 123]). The fixpoint semantics is inspired by the definition of the semantics of mdtt, wmd, and md whereas the hypergraph semantics is novel.

Each type of semantics requires three types of inputs: an mwmd, a tree, and an m-monoid. The semantics are defined in such a way that they evaluate the input tree according to the mwmd by applying operations from the m-monoid, and afterwards return an element of the m-monoid. Thus, when keeping the mwmd and the m-monoid fixed, the semantics are mappings from input trees to m-monoid elements, i.e., a tree series.

For both the fixpoint and the hypergraph semantics we will introduce two variants, which we call the finitary and the infinitary semantics. In general, the finitary variants of the semantics are only applicable for a certain class of mwmd, which we call weakly non-circular mwmd. Mwmd, that do not belong to this class, exhibit circular behavior when computing their semantics; as a consequence they require an ω-continuous m-monoid (for the fixpoint semantics) or an ω-complete m-monoid (for the hypergraph semantics) as input. The infinitary versions of the semantics use the strength of ω-continuous and ω-complete m-monoids and are applicable to all mwmd (even mwmd that are not weakly non-circular). Hence, we will define and study four different variants of semantics (see Definitions 4.20, 4.29, 4.40, and 4.43).

We conclude this chapter with a comparison of the fixpoint and the hypergraph semantics and provide conditions that guarantee that the finitary fixpoint and hypergraph semantics coincide and that the infinitary fixpoint and hypergraph semantics coincide (see Theorem 4.53).

Chapter 5. In this chapter we will we will study four syntactic subclasses of mwmd, called restricted, connected, local, and proper mwmd (see Definitions 5.1, 5.4, and 5.5). The connected normal form has been introduced by Gottlob and Koch [69, Theorem 4.2]; it has also been studied in [122, 28]. The remaining three syntactic classes have first been investigated in [28].

We will present conditions that guarantee that these classes coincide, i.e., conditions that allow any of this subclasses (and intersections of them) to be considered to be a normal form of mwmd (see Theorem 5.8). To this end we investigate when a given mwmd can be transformed into a semantically equivalent mwmd belonging to a particular syntactic subclass. What are constructions that exhibit 'semantic equivalence'? In fact, we aim for the strongest definition of semantics equivalence; more precisely, we will present constructions that preserve all four kinds of semantics, that we will have introduced in Chapter 4, simultaneously. Since equivalence proofs for such constructions are very laborious, we will, as a preliminary step, first prove a generic equivalence result that we will employ for (almost every) of the normal form constructions later in this chapter. The constructions that we present in this section are based on the constructions in [28].

Chapter 6. In this chapter we prove that there is an effective procedure that decides whether an mwmd is weakly non-circular (see Theorem 6.1).

The definition of weak non-circularity is inspired by and adapted from the definitions of non-circularity for attribute grammars [32, 50], attributed tree transducers [56, 60], and weighted monadic datalog [122]. A decision procedure for the non-circularity of attribute grammars, called a circularity test, has first been studied by Knuth [90, 89] (also see [3] and [94, Figure 3.6, Lemma 3.25]). A similar circularity test for attributed tree transducers has been proposed in [56] and investigated in [60, Figure 5.7, Lemma 5.17]. Both of these circularity tests are based on the inductive construction of a finite set of graphs, called is-graphs, that are checked for cycles.

In this thesis we will not follow the approach to develop a circularity test that is based on the construction of is-graphs. Instead we will employ the following idea. Let M be an mwmd and let L_M be the set of input trees t such that M exhibits a circular behavior for the input t. Then M is weakly non-circular iff L_M is empty. We will show that there effectively is an MSO-logic formula [131, 38] that defines L_M; this implies that L_M is a recognizable tree language. The decidability of weak non-circularity of mwmd follows from the fact that the emptiness problem of recognizable tree languages is decidable [38, Corollary 1.12(i)] or [131, Theorem 7].

Chapter 7. In this chapter we show that mwmd are capable of simulating weighted monadic datalog. More precisely, we study the semantics of mwmd for a certain class of m-monoids, viz. the class of m-monoids that behave like semirings [67, 72]. In order to develop a richer theory, we even study m-monoids that behave like strong bimonoids [44]. This chapter is a revised and extended version of [122]; note that the scope of investigation in [122] is weighted monadic datalog over semirings and unranked trees [127], whereas we study weighted monadic datalog over strong bimonoids and ranked trees in this thesis.

We will study the expressive power of wmd. In particular, we will compare the finitary and infinitary semantics of wmd (Lemma 7.15), we show that wmd can simulate md when using the Boolean semiring (Lemma 7.16), and that wmd is strictly more expressive than wta (Theorem 7.18). We will conclude this chapter by proving that wmd can be evaluated efficiently (Theorem 7.21).

Chapter 8. In this chapter we show that mwmd are capable of simulating mdtt. This task is accomplished by, roughly speaking, abstracting from the semantic domain. Hence, we will not evaluate the semantics of such mwmd in an arbitrary given m-monoid but will use an m-monoid instead, that behaves like the term algebra. This chapter is a revised and extended version of [28], where monadic datalog tree transducers have first been investigated.

We will show that there is a sharp boundary between those mdtt that can be applied for practical purposes and those mdtt that can not. We will call mdtt of the former kind *executable* and show that it is decidable whether a given mdtt is executable (see Lemma 8.12).

Mdtt and (nondeterministic) attributed tree transducers [56, 11, 12] (for short:att) share conceptual ideas. We will prove that the class of tree transformations that are definable by attributed tree transducers coincides with the class of tree transformations that are computed by restricted mdtt (see Theorem 8.21).

Chapter 9. In this chapter we will show that mwmd are capable of simulating wmta. More precisely, will study a semantic class of mwmd such that mwmd in this class behave precisely like wmta.

This chapter is a revised version of the most important results of [123, 59]. We will restrict ourselves to proving the following two main results.

1. We will show that, for a given absorptive m-monoid satisfying some additional condition, the class of tree series recognized by wmta over $\mathcal{A}$ can be decomposed into the class of relabeling tree transformations, followed by the class of characteristic

tree transformations of recognizable tree languages, and followed by the class of tree series recognized by homomorphism wmta over $\mathcal{A}$, where a homomorphism wmta is a wmta having precisely one state (see Theorem 9.17).

2. We will give an alternative characterization of the class of tree series recognized by wmta. This characterization is based on m-expressions, which form a new kind of weighted MSO-logic. This characterization is a Büchi-like theorem [26, 46] for the class of tree series recognized by wmta (see Theorem 9.26).

CHAPTER 2

Preliminaries

In this chapter we recall notions and establish notations that we will use throughout this thesis.

2.1 Notation

We will abbreviate “if and only if” by “iff” and “with respect to” by “wrt”. We will indicate the end of definitions, examples, and remarks by the symbol “□” and we will finish every proof by using the HALMOS end mark “■”.

2.2 Sets and Relations

Sets

Set theory is the foundation of most areas of mathematics and therefore also of this thesis. A naive approach to set theory [47, 55] is sufficient for most of our purposes. However, we will occassionally need to draw on an axiomatic definition of sets. In this case we use the *Zermelo-Fraenkel set theory with the Axiom of Choice* [13, 93, 112, 124] (abbreviated *ZFC*) (or any similar theory). Note that we will not employ ZFC on an axiomatic level; we merely require a non-naive treatment of set theory for the following two reasons. Firstly, in the field of formal language theory we sometimes need to deal with proper classes (i.e., classes that are no sets). A theoretic foundation that does not distinguish between sets and proper classes can give rise to an inconsistent theory (for an example see Section A.1 in the appendix); in this thesis we will always take care not to refer to proper classes as sets. Secondly, we will follow the convention to point out whenever any result relies on the Axiom of Choice[1].

We assume that the reader is familiar with the *empty set* $\emptyset$, the *membership* relation $\in$, the *subset* and *strict subset* relations $\subseteq$ and $\subset$, respectively, the operations *union* $\cup$, *intersection* $\cap$, *set difference* $\setminus$, and *Cartesian product* $\times$, and the notions of *disjoint sets*, *cardinality* of a set, and *finite* and *infinite* sets. For a thorough introduction to set theory we refer the reader to [80].

In the sequel let A, B, C, and D be sets. By $\mathcal{P}(A)$ we denote the power set of A, by $\mathcal{P}_{\text{fin}}(A)$ we denote the set of finite subsets of A, and by $|A|$ the cardinality of A.

By $\mathbb{N}$, the set of ***natural numbers***, we denote the set of non-negative integers and by $\mathbb{N}_+$ the set of positive integers. The set A is called ***countable*** if $|A| \leq |\mathbb{N}|$. For every finite non-empty subset N of $\mathbb{N}$ we denote the maximal natural number in N by $\max(N)$. We let $\max(\emptyset) = 0$. For every $n \in \mathbb{N}$ we abbreviate the set $\{i \in \mathbb{N}_+ \mid i \leq n\}$ by $[n]$; observe

[1] We will use the Axiom of Choice [73, 109, 136] only very rarely in this thesis. The main reference to it is in Section 8.1.

that $[0] = \emptyset$. An ***initial segment of*** $\mathbb{N}$ is a set $N \subseteq \mathbb{N}$ such that for every $n \in N$ and $n' \in \mathbb{N}$ with $n' \leq n$ also $n' \in N$. Note that if N is an initial segment of $\mathbb{N}$, then either $N = \mathbb{N}$ or there is an $n \in \mathbb{N}$ with $N = \{i \in \mathbb{N} \mid i < n\}$. Moreover, if A is countable, then there is a unique initial segment N of $\mathbb{N}$ such that $|A| = |N|$.

Relations

A *(binary)* ***relation*** *from A to B* is a subset of $A \times B$. In particular, also the empty set $\emptyset$ is a relation from A to B. Let ρ be a relation from A to B. For every $a \in A$ and $b \in B$ we usually write $a\,\rho\,b$ instead of $(a, b) \in \rho$. We define the ***inverse relation*** ρ^{-1} *of ρ* as the relation from B to A defined by $\rho^{-1} = \{(b, a) \in B \times A \mid a\,\rho\,b\}$. For every subset $A' \subseteq A$ we abbreviate the set $\{b \in B \mid \exists\, a \in A' : a\,\rho\,b\}$ by $\rho(A')$. The ***domain*** $\mathrm{dom}(\rho)$ *of ρ* is defined as the set $\rho^{-1}(B)$ and the ***range*** $\mathrm{ran}(\rho)$ *of ρ* is defined as the set $\rho(A)$. For every $A' \subseteq A$ we define the ***restriction*** *of ρ to A'*, denoted by $\rho|_{A'}$, as the binary relation from A' to B defined by $\rho|_{A'} = \rho \cap (A' \times B)$; furthermore, we say that ρ ***extends*** $\rho|_{A'}$. Let τ be a relation from C to D. The ***composite relation*** $\rho\,;\tau$ *of ρ and τ* is the relation from A to D defined by

$$\rho\,;\tau = \{(a, d) \in A \times D \mid \exists\, b \in B \cap C : a\,\rho\,b \wedge b\,\tau\,d\}\ .$$

A relation from A to A is also called a ***relation*** *on A*. A particular relation on A is the ***identity relation*** id_A *on A*, defined by $\mathrm{id}_A = \{(a, a) \mid a \in A\}$. An excellent introduction to relations can be found in, e.g., [119].

Graphs and diagrams

A pair (V, E) is called a ***directed graph*** [10, 36, 119] (for short: ***digraph***) if V is a set and E is a relation on V. We refer to the elements of V as ***vertices*** and to the elements of E as ***edges***.

If V is finite, then the directed graph (V, E) can be represented graphically by means of a ***diagram***. Every vertex $v \in V$ is depicted by a node (or circle, etc.) having the label v. Every edge $(v, w) \in E$ is represented by an arrow that starts in the vertex for v, ends in the vertex for w, and does not intersect with any vertex other than v and w. The diagram is called ***planar*** if no arrows cross.

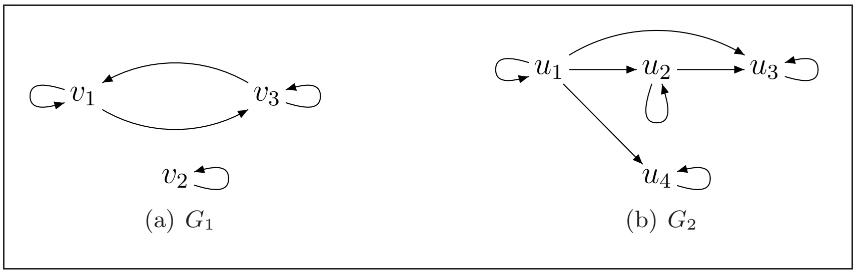


Figure 2.1: Diagrams of directed graphs.

As an example consider the directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where we let $V_1 = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_3), (v_3, v_1)\} \cup \mathrm{id}_{V_1}$, $V_2 = \{u_1, u_2, u_3, u_4\}$, and $E_2 = \{(u_1, u_2), (u_1, u_3), (u_1, u_4), (u_2, u_3)\} \cup \mathrm{id}_{V_2}$. The diagrams of G_1 and G_2 are given in Figure 2.1. Both diagrams are planar.

Special relations

Let τ be a relation on A. For every $A' \subseteq A$ we call an element $a \in A'$ ***minimal*** *(in A' wrt τ)* if $a' \; \tau \; a$ implies $a' = a$ for every $a' \in A'$. We say that τ is

- ***reflexive*** if $\mathrm{id}_A \subseteq \tau$,
- ***irreflexive*** if $\tau \cap \mathrm{id}_A = \emptyset$,
- ***symmetric*** if $\tau = \tau^{-1}$,
- ***antisymmetric*** if $\tau \cap \tau^{-1} \subseteq \mathrm{id}_A$, and
- ***transitive*** if $\tau \, ; \tau \subseteq \tau$.

The ***transitive closure*** *of* τ, denoted by τ^+, is the smallest transitive relation on A containing τ. The ***transitive reflexive closure*** *of* τ, denoted by τ^*, is defined as the relation $\tau^* = \tau^+ \cup \mathrm{id}_A$ on A. If $G = (V, E)$ is a digraph, then we say that G is ***acyclic*** if E^+ is irreflexive; otherwise it is called ***cyclic***.

We call τ an ***equivalence relation*** *(on A)* if τ is reflexive, symmetric, and transitive. If τ is an equivalence relation on A, then for every $a \in A$ we call the set $\{b \in A \mid b \; \tau \; a\}$ the ***equivalence class*** *of a modulo* τ and denote it by $[a]_\tau$; the ***quotient set*** *of A modulo* τ, denoted by A/τ, is the set $\{[a]_\tau \mid a \in A\}$. For an example consider the digraph $G_1 = (V_1, E_1)$ of Figure 2.1(a). Clearly, E_1 is an equivalence relation on V_1 and $V_1/E_1 = \big\{[v_1]_{E_1}, [v_2]_{E_1}, [v_3]_{E_1}\big\} = \big\{\{v_1, v_3\}, \{v_2\}\big\}$.

The relation τ is a ***partial order*** [34] *(on A)* if τ is reflexive, antisymmetric, and transitive; it is a ***strict order*** *(on A)* if τ is irreflexive and transitive. If τ is a partial order on A, then $\tau \setminus \mathrm{id}_A$, called the ***strict part*** *of* τ, is a strict order an A. As usual we will denote the strict part of, e.g., $\leq$ and $\sqsubseteq$ by $<$ and $\sqsubset$, respectively.

Let τ be a partial order on A. Then (A, τ) is called a ***partially ordered set*** (for short: ***poset***). The ***covering relation*** *of* τ is the relation τ^{cov} on A such that for every $a, b \in A$ we have: $a \; \tau^{\mathrm{cov}} \; b$ iff (i) $a \neq b$, (ii) $a \; \tau \; b$, and (ii) there is no $c \in A \setminus \{a, b\}$ with $a \; \tau \; c$ and $c \; \tau \; b$. Note that if A is finite, then $(\tau^{\mathrm{cov}})^* = \tau$.

A subset $A' \subseteq A$ is called ***totally ordered*** *(wrt τ)* if $A' \times A' \subseteq \tau \cup \tau^{-1}$. We call τ a ***total order*** *(on A)* if A is totally ordered wrt τ. The digraph $G_2 = (V_2, E_2)$ of Figure 2.1(b) is an example of a poset. Note that E_2 is not a total order on V_2; however, the subset $\{u_1, u_2, u_3\} \subseteq V_2$ is totally ordered wrt E_2. The covering relation of E_2 is given by $E_2^{\mathrm{cov}} = \{(u_1, u_2), (u_1, u_4), (u_2, u_3)\}$.

An example of a totally ordered poset is $(\mathbb{R}, \leq)$, where $\leq$ is the set of all pairs $(n, m) \in \mathbb{R}^2$ such that there is a nonnegative real number r with $n + r = m$; in this thesis we will refer to this order $\leq$ on $\mathbb{R}$ as the *natural order on real numbers* (and likewise for other familiar sets of numbers like $\mathbb{N}$ and $\mathbb{Q}$).

Partial orders on finite sets can graphically be represented by means of Hasse diagrams [34, Page 11]. The Hasse diagram of a given partial order τ on a finite set A is the diagram of the digraph (A, τ^{cov}) such that for every $a, b \in A$ with $a \neq b$ and $a \; \tau \; b$ the vertex for a is positioned below the vertex of b. Due to this convention it suffices to represent edges by simple arcs instead of arrows; their orientations are clear from the positions of the vertices. The Hasse diagram of the partial order G_2 from Figure 2.1(b) is shown in Figure 2.2(a). Although Hasse diagrams can only represent partial orders on finite sets in a meaningful way, we will extend their use to infinite partial orders by means of ellipsis (...); an example of such a Hasse diagram is given Figure 2.2(b), it depicts the poset $(\mathbb{N} \cup \{\infty\}, \leq)$, where $\leq$ is the natural order on the natural numbers with infinity.

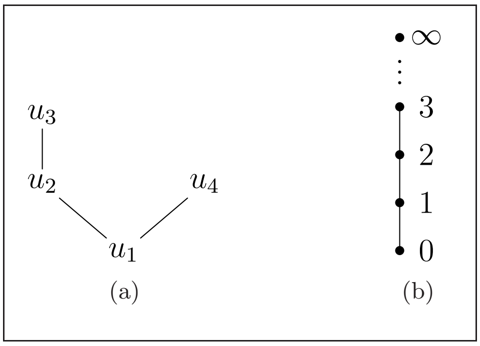


Figure 2.2: (a) Hasse diagram of G_2. (b) Hasse diagram of an infinite poset.

Let τ be an irreflexive relation on A. We call τ ***well-founded*** [124, 112] (*on* A) if every nonempty $B \subseteq A$ contains a minimal element. Note that if A is finite, then τ is well-founded iff τ^+ is irreflexive. Let τ be well-founded on A and $B \subseteq A$. We call B ***closed*** *under* τ if $\{a \in A \mid \tau^{-1}(\{a\}) \subseteq B\} \subseteq B$. If B is closed under τ, then it is easy to see that $A \setminus B$ has no minimal element; hence, $A \setminus B = \emptyset$ or, equivalently, $B = A$. Thus, in order to prove that a property $\varphi(x)$ holds for every $a \in A$ it suffices to show that the set $\{a \in A \mid \varphi(a)\}$ is closed under τ. This is called the principle of *proof by well-founded* (or: *Noetherian*) *induction on* τ [7, 112]. The principle of well-founded induction is most commonly used when $A = \mathbb{N}$ and $\tau = \{(n, n+1) \mid n \in \mathbb{N}\}$ is the successor relation of natural numbers; in this case a proof by well-founded induction on τ is referred to as a *proof by* (*mathematical*) *induction*.

2.3 Mappings and Operations

Mappings

A relation ρ from A to B is called a ***partial mapping*** *from* A *to* B if for all $a \in A$ and $b_1, b_2 \in B$, $a\ \rho\ b_1$ and $a\ \rho\ b_2$ imply $b_1 = b_2$. Let ρ be a partial mapping from A to B. We call ρ ***injective*** if ρ^{-1} is a partial mapping from B to A; moreover, ρ is ***surjective*** *(onto* B*)* if $\mathrm{ran}(\rho) = B$. For every $a \in \mathrm{dom}(\rho)$ we denote the unique $b \in \rho(\{a\})$ by $\rho(a)$ and call b the ***image*** *of* a *under* ρ. Note that if τ is a partial mapping from C to D, then $\rho\,;\tau$ is a partial mapping from A to D.

If $\mathrm{dom}(\rho) = A$, then ρ is called a ***mapping*** (or: *function*) *from* A *to* B; in this case we write $\rho : A \to B$.[2] We denote the set of all mappings from A to B by B^A. Clearly, $B^\emptyset = \{\emptyset\}$; in this context the empty set $\emptyset$ is also called the ***empty mapping***. We follow the convention that function application is left associative, i.e., for every $f : A \to C^B$, $a \in A$, and $b \in B$ we write $f(a)(b)$ instead of $(f(a))(b)$. Let $\rho : A \to B$. For every set $A' \subseteq A$ we have $\rho|_{A'} : A' \to B$. If $B \subseteq C$ and $\tau : C \to D$, then $\rho\,;\tau : A \to D$; moreover, $(\rho\,;\tau)|_{A'} = (\rho|_{A'})\,;\tau$ for every $A' \subseteq A$.

[2]In the literature mappings from A to B are sometimes defined as triples (A, B, ρ), where ρ is called the *graph* of the mapping. In this thesis a mapping from A to B is always a particular relation from A to B.

Let $\rho : A \to B$ and let a, b be arbitrary objects (that may be elements of A or B or not). The ***extension*** *of* ρ *with* (a, b) is the mapping $\rho[a \mapsto b]$ from $A \cup \{a\}$ to $B \cup \{b\}$ that is defined by $\rho[a \mapsto b] = \big(\rho \setminus (\{a\} \times B)\big) \cup \{(a, b)\}$. Note that $\rho[a \mapsto b](a) = b$ and that $\rho|_{A\setminus\{a\}} = (\rho[a \mapsto b])|_{A\setminus\{a\}}$, i.e., ρ and $\rho[a \mapsto b]$ agree on $A \setminus \{a\}$. For every $n \in \mathbb{N}$ and objects $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ we will abbreviate $\rho[a_1 \mapsto b_1] \cdots [a_n \mapsto b_n]$ by $\rho[a_1 \mapsto b_1, \ldots, a_n \mapsto b_n]$; if ρ is the empty mapping, we simply write $[a_1 \mapsto b_1, \ldots, a_n \mapsto b_n]$ instead of $\rho[a_1 \mapsto b_1, \ldots, a_n \mapsto b_n]$. This notation enables us to enumerate every mapping on a finite domain as follows: if there is an $n \in \mathbb{N}$ and pairwise distinct $a_1, \ldots, a_n$ such that $A = \{a_1, \ldots, a_n\}$, then $\rho = [a_1 \mapsto \rho(a_1), \ldots, a_n \mapsto \rho(a_n)]$.

We call $\rho : A \to B$ a ***bijection*** *(from A to B)* if ρ is injective and surjective onto B. An example of a bijection from A to A is the identity relation id_A. Note that if ρ is a bijection, then $\rho^{-1} : B \to A$. Furthermore, if $|A| = |B|$, then there exists a bijection $\rho : A \to B$. For more details on mappings we refer the reader to [124, 126, 135].

Let $f : A \to A$. An element $a \in A$ with $f(a) = a$ is called ***fixpoint*** *of* f. For every $n \in \mathbb{N}$ we define the *n-fold* ***composition*** $f^n : A \to A$ *of* f as follows: (i) $f^0 = \mathrm{id}_A$ and (ii) $f^{n+1} = f^n \,;\, f$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be sets. For every set B, $f : A_1 \times \cdots \times A_n \to B$, and $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ we will write $f(a_1, \ldots, a_n)$ instead of $f((a_1, \ldots, a_n))$ if no confusions arise. Let $i \in [n]$. The ***projection*** *to the i-th component* (*wrt* $A_1, \ldots, A_n$) is the mapping $\mathrm{pr}_i^{A_1,\ldots,A_n} : A_1 \times \cdots \times A_n \to A_i$ which is defined by $\mathrm{pr}_i^{A_1,\ldots,A_n}(a_1, \ldots, a_n) = a_i$ for every $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. If $A_1, \ldots, A_n$ are clear from the context, we simply write pr_i instead of $\mathrm{pr}_i^{A_1,\ldots,A_n}$.

Let I be a set. An *(I-indexed)* ***family*** *over* A is a mapping $f : I \to A$. In this context the set I is called an ***index set***. We usually write f_i instead of $f(i)$ for every $i \in I$ and we write $(f_i \mid i \in I)$ instead of f. If I is empty, then $(f_i \mid i \in I)$ is called the ***empty family***.

A ***generalized partition*** [72, Chapter IV] *of* A is a family $(A_i \mid i \in I)$ over $\mathcal{P}(A)$ such that $\bigcup_{i\in I} A_i = A$ and for every $i, j \in I$ with $i \neq j$: $A_i \cap A_j = \emptyset$. A ***partition*** *of* A is a generalized partition $(A_i \mid i \in I)$ of A such that for every $i \in I$: $A_i \neq \emptyset$.

Let τ be a well-founded relation on A and for every $a \in A$ let $A_a = \tau^{-1}(\{a\})$ and $f_a : B^{A_a} \to B$ be a mapping. Then there is precisely one mapping $g : A \to B$ such that $g(a) = f_a(g|_{A_a})$ for every $a \in A$; thus, in order to define this uniquely determined g it suffices to define f_a for every $a \in A$. This is called the principle of *definition by well-founded recursion on* τ [112].

Operations

Let $n \in \mathbb{N}$. An *n-**tuple*** *over* A is a mapping $t : [n] \to A$. As usual, we denote t by $(t(1), t(2), \ldots, t(n))$. For the set of all n-tuples over A we simply write A^n instead of $A^{[n]}$; then $A^0 = A^{\emptyset} = \{\emptyset\}$. As usual, we identify A^n with the n-fold Cartesian product $A \times \cdots \times A$ of A.

An *n-ary* ***operation*** [135] *over* A is a mapping $\nu : A^n \to A$; the number n is called the ***arity*** of ν. Operations of arity 2, 1, and 0 are also called ***binary***, ***unary***, and ***nullary*** operations, respectively. The set of all n-ary operations on A is denoted by $\mathrm{Ops}(A)^{(n)}$. Usually, the operations in $\mathrm{Ops}^{(0)}(A)$ are called ***constants***; obviously there is a one-to-one correspondence between the sets $\mathrm{Ops}^{(0)}(A)$ and A. We abbreviate $\bigcup_{n\in\mathbb{N}} \mathrm{Ops}^{(n)}(A)$ by $\mathrm{Ops}(A)$.

Let $n \in \mathbb{N}$ and $\nu \in \mathrm{Ops}^{(n)}(A)$. For every $B \subseteq A$ we say that B is ***closed*** *under* ν if $\nu(B^n) \subseteq B$. The operation ν is called ***idempotent*** if $\nu(a, \ldots, a) = a$ for every $a \in A$.

An element $a \in A$ is called ***absorbing*** *wrt* ν if for every $a_1, \ldots, a_n \in A$ and $i \in [n]$: $a_i = a$ implies $\nu(a_1, \ldots, a_n) = a$. Let τ be a relation on A. The operation ν is called ***monotone*** (*wrt* τ) if for every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ with $a_i \mathrel{\tau} b_i$ for every $i \in [n]$, we have $\nu(a_1, \ldots, a_n) \mathrel{\tau} \nu(b_1, \ldots, b_n)$.

Let $\circ \in \mathrm{Ops}^{(2)}(A)$. We usually write $a \circ b$ instead of $\circ(a, b)$, for every $a, b \in A$. We call $\circ$ ***commutative*** if $a \circ b = b \circ a$ for every $a, b \in A$. Moreover, we call $\circ$ ***associative*** if $a \circ (b \circ c) = (a \circ b) \circ c$ for every $a, b, c \in A$. Let $a \in A$. We say that a is ***neutral*** *wrt* $\circ$ if for every $a' \in A$, $\nu(a, a') = a' = \nu(a', a)$. For more details on operations we refer the reader to [135].

Operations of infinite arity

In this paragraph we will deal with operations of infinite arity. First let us briefly motivate this concept. In the previous paragraph we showed that a 4-ary operation over a set A, i.e., an operation that always takes 4 arguments, can be modeled as a mapping over the domain A^4. Likewise, we can model an operation that may take 4 or 2 arguments, i.e., an operation that behaves both like a 4- and a 2-ary operation, as a mapping over the domain $A^4 \cup A^2$ because A^4 and A^2 are disjoint sets. In this manner we can define an operation that takes an arbitrary finite number of elements as a mapping over the domain $\bigcup_{n \in \mathbb{N}} A^n$. It is natural to extend this definition to operations that take even infinitely many arguments. For our purposes it suffices to restrict ourselves to a countably infinite number of operands. The set $A^{\mathbb{N}}$ is the set of countably infinite sequences over A; hence, an operation that takes an arbitrary countable number of arguments can be modeled as an operation over the domain $A^{\mathbb{N}} \cup \bigcup_{n \in \mathbb{N}} A^n$. This domain is rather heterogeneous. Instead we will use an alternative, more uniform approach to define the domain of an operation of arbitrary countable arity.

By Fam^{ω}_A we denote the set of pairs $(I, (a_i \mid i \in I))$ such that $I \subseteq \mathbb{N}$ and $(a_i \mid i \in I)$ is a family over A.[3] In this paragraph we will consider mappings $\sum$ whose domain is the set Fam^{ω}_A; in order to simplify notation we will write $\sum_{i \in I} a_i$ instead of $\sum(I, (a_i \mid i \in I))$, for every $(I, (a_i \mid i \in I)) \in \mathrm{Fam}^{\omega}_A$. A mapping $\sum : \mathrm{Fam}^{\omega}_A \to A$ is called an ***ω-infinitary operation*** *on* A if the following holds for every $n \in \mathbb{N}$, $I, J \subseteq \mathbb{N}$, generalized partition $(I_j \mid j \in J)$ of I, and families $(a_i \mid i \in I)$ and $(b_i \mid i \in \{n\})$ over A:

$$\sum\nolimits_{i \in \{n\}} b_i = b_n \; , \tag{2.1}$$

$$\sum\nolimits_{i \in I} a_i = \sum\nolimits_{j \in J} \Big(\sum\nolimits_{i \in I_j} a_i \Big) \; . \tag{2.2}$$

Let $\sum$ be an ω-infinitary operation on A. By Equations (2.1) and (2.2) we have that for every index set $I \subseteq \mathbb{N}$, index set $J \subseteq \mathbb{N}$, bijection $\pi : J \to I$, and family $(a_i \mid i \in I)$ (see [72, Lemma IV.1.6(b)] or [81, Equation (1.6)]):

$$\sum\nolimits_{i \in I} a_i = \sum\nolimits_{j \in J} a_{\pi(j)} \; . \tag{2.3}$$

In this thesis an ω-infinitary operation is only defined for families on index sets that are sets of natural numbers. However, in the literature (e.g., [72, 95, 42, 63]) infinitary

[3] Note that in order to give an appropriate definition of the domain Fam^{ω}_A of an operation that takes an arbitrary countable number of elements, it suffices to restrict ourselves to those index sets $I \subseteq \mathbb{N}$ that are initial segments of $\mathbb{N}$; however, for technical reasons we allowed for index sets I that are arbitrary sets of natural numbers.

operations are usually defined for families on arbitrary (countable) index sets. The author avoided this definition because the class of all countable families over A is a proper class (i.e., it is not a set) and, thus, cannot be the domain of a mapping; this implies that infinitary operations that are defined for arbitrary countable families are no proper mappings, which is undesirable. We can remedy our restriction to index sets of natural numbers and simulate the extension to arbitrary countable index sets as follows.

Equation (2.3) allows for a convenient notational extension of ω-infinitary operations to arbitrary countable families over $\mathcal{A}$. Let I be a countable index set and $(a_i \mid i \in I)$ be a family over A. Then there is a set $J \subseteq \mathbb{N}$ and a bijection $\pi : J \to I$; we will denote $\sum_{j\in J} a_{\pi(j)}$ simply by $\sum_{i\in I} a_i$. Note that the notation $\sum_{i\in I} a_i$ (omitting J and π) is justified because the value of $\sum_{i\in I} a_i$ is independent of the choice of J and π due to Equation (2.3). Moreover, note that Equations (2.1) and (2.2) carry over to arbitrary countable families over A; more precisely, for every singleton set $K = \{k\}$, countable sets I, J, generalized partition $(I_j \mid j \in J)$ of I, and families $(a_i \mid i \in I)$ and $(b_i \mid i \in \{k\})$ over A:

$$\sum\nolimits_{i\in\{k\}} b_i = b_k \ , \tag{2.4}$$

$$\sum\nolimits_{i\in I} a_i = \sum\nolimits_{j\in J}\Big(\sum\nolimits_{i\in I_j} a_i\Big) \ . \tag{2.5}$$

2.4 Signatures and Algebras

A ***signature*** is a pair (Σ, r), where Σ is a (possibly infinite) set and $r : \Sigma \to \mathbb{N}$. Let (Σ, r) be a signature. If Σ is finite, then (Σ, r) is called a ***ranked alphabet***. For every $\sigma \in \Sigma$ the natural number $r(\sigma)$ is called the ***rank*** *of* σ, and is denoted by $\mathrm{rk}(\sigma)$ if r is clear from the context. As usual, we identify the signature (Σ, r) with the set Σ. For every $k \in \mathbb{N}$ we denote the set $\{\sigma \in \Sigma \mid \mathrm{rk}(\sigma) = k\}$ by $\Sigma^{(k)}$. By $\mathrm{maxrk}(\Sigma)$ we denote the maximal rank of symbols in Σ, i.e., $\mathrm{maxrk}(\Sigma) = \max\{k \in \mathbb{N} \mid \Sigma^{(k)} \neq \emptyset\}$. Σ is called ***monadic*** if $\mathrm{maxrk}(\Sigma) \leq 1$, i.e., $\Sigma = \Sigma^{(0)} \cup \Sigma^{(1)}$.

Example 2.1. Consider the signature $(\Sigma_{\mathrm{mon}}, r)$ with $\Sigma_{\mathrm{mon}} = \{e, \circ\}$, $r(e) = 0$, $r(\circ) = 2$. Then $\Sigma^{(0)}_{\mathrm{mon}} = \{e\}$, $\Sigma^{(2)}_{\mathrm{mon}} = \{\circ\}$, and $\Sigma^{(k)}_{\mathrm{mon}} = \emptyset$ for every $k \in \mathbb{N} \setminus \{0, 2\}$. In the sequel we will define signatures in a less cumbersome way and simply write $\Sigma_{\mathrm{mon}} = \{e^{(0)}, \circ^{(2)}\}$. □

A Σ-***algebra*** [70, 135] is a pair $\mathcal{A} = (A, \theta_A)$ where A is a set (called ***carrier set*** *of* $\mathcal{A}$) and $\theta_A : \Sigma \to \mathrm{Ops}(A)$ such that $\theta_A(\sigma) \in \mathrm{Ops}^{(k)}(A)$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. Let $A' \subseteq A$. We say that $\mathcal{A}$ is ***generated*** *by* A' if for every set A'' such that $A' \subseteq A'' \subseteq A$ and A'' is closed under $\theta_A(\sigma)$ for every $\sigma \in \Sigma$, we have $A = A''$.

Example 2.2. The pair $\mathcal{N} = (\mathbb{N}, \theta)$, where $\theta(e) = +$ is the conventional addition on natural numbers and $\theta(\circ) = 0$, is a Σ_{mon}-algebra. The algebra $\mathcal{N}$ is generated by the set $\{1\}$, which is a direct consequence of the induction axiom of natural numbers. □

Let $\mathcal{A} = (A, \theta_A)$ and $\mathcal{B} = (B, \theta_B)$ be Σ-algebras. A Σ-***homomorphism*** *from* $\mathcal{A}$ *to* $\mathcal{B}$ is a mapping $h : A \to B$ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $a_1, \ldots, a_k \in A$, $h(\theta_A(\sigma)(a_1, \ldots, a_k)) = \theta_B(\sigma)(h(a_1), \ldots, h(a_k))$. A bijective Σ-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is also called a Σ-***isomorphism*** *from* $\mathcal{A}$ *to* $\mathcal{B}$; if there is a Σ-isomorphism from $\mathcal{A}$ to $\mathcal{B}$, we say that $\mathcal{A}$ and $\mathcal{B}$ are Σ-***isomorphic***. If Σ is clear from the context, we simply write *homomorphism*, *isomorphism*, and *isomorphic*.

Let $\mathfrak{C}$ be a class of Σ-algebras, $\mathcal{C} = (C, \theta_C)$ be a Σ-algebra in $\mathfrak{C}$, and $C' \subseteq C$ such that $\mathcal{C}$ is generated by C'. We say that $\mathcal{C}$ is ***freely generated*** *by C' for $\mathfrak{C}$* if for every Σ-algebra $\mathcal{D} = (D, \theta_D)$ in $\mathfrak{C}$ and mapping $f : C' \to D$ there is a Σ-homomorphism h from $\mathcal{C}$ to $\mathcal{D}$ with $h|_{C'} = f$.[4] If $C' = \emptyset$, then $\mathcal{C}$ is also called ***initial*** *for $\mathfrak{C}$*. For a thorough introduction to universal algebra we refer the reader to [30, 70, 135].

2.5 Strings and Trees

Strings

Let A be a (possibly infinite) set. We abbreviate $\bigcup_{n\in\mathbb{N}} A^n$ by A^*. An element of A^* is also called a ***string*** *over A*. In this context we refer to the unique element in A^0 as the ***empty string*** and denote it by ε; furthermore, we simply write $a_1 \cdots a_n$ instead of $(a_1, \ldots, a_n)$ for every $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in A$; we call n the ***length*** *of $a_1 \cdots a_n$* and denote it by $|a_1 \cdots a_n|$. As usual, we identify the sets A and A^1.

For two strings $w = a_1 \cdots a_n$ and $v = b_1 \cdots b_m$ over A we denote by $w \cdot v$ (or simply wv) the ***concatenation*** *of w and v*, which is defined by $w \cdot v = a_1 \cdots a_n b_1 \cdots b_m$. If $v, w \in A^*$, then v is called a ***prefix*** *of w* if there is a $v' \in A^*$ with $vv' = w$; v is called a ***proper prefix*** *of w* if v is a prefix of w and $v \neq w$. Let $W \subseteq A^*$ be nonempty. The ***longest common prefix*** *of W* is the string $w \in A^*$ such that w is a prefix of every $v \in W$ and the following implication holds for every $w' \in A^*$: if w' is a prefix of every $v \in W$, then w' is a prefix of w. Note that the longest common prefix of W does always exist and is unique. For an excellent introduction to the theory of strings and formal string languages we refer the reader to [76].

Trees

Let Σ be a signature and D be a set. The set $T_\Sigma(D)$ of ***trees*** [66] *(over Σ indexed by D)* is the smallest set T such that (i) $D \subseteq T$ and (ii) for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T$ also $\sigma(t_1, \ldots, t_k) \in T$. Every tree $t \in D$ is called an ***index***. Note that even if Σ and D are not disjoint it is always clear whether a tree $t \in T_\Sigma(D)$ is an index or not; e.g., in the former case we have $t = d$ and in the latter case $t = d()$, where $d \in \Sigma \cap D$. However, we will occasionally denote the tree $d()$ by d if no confusions arise. We follow the convention to denote, for every $t \in T_\Sigma(D)$, $\gamma \in \Sigma^{(1)}$, and $n \in \mathbb{N}$, the tree $\gamma(\cdots(\gamma(t))\cdots)$ (with n consecutive occurrences of γ) by $\gamma^n(t)$. We abbreviate $T_\Sigma(\emptyset)$ by T_Σ. Moreover, we call any subset of $T_\Sigma(D)$ a ***tree language*** *(over Σ and D)*.

Example 2.3. Let $\Sigma_{\text{ex}} = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$ be a ranked alphabet and $D_{\text{ex}} = \{x, y\}$. Then $T_{\Sigma_{\text{ex}}}(D_{\text{ex}}) = \{x, y, \alpha, \gamma(x), \gamma(\alpha), \gamma(\gamma(x)), \sigma(\alpha, \gamma(y)), \ldots\}$. The set $L = \{\gamma^n(x) \mid n \in \mathbb{N}\}$ is a tree language over Σ_{ex} and D_{ex}. □

Let us consider the relation $<$ on $T_\Sigma(D)$ which is defined as follows: for every $t, t' \in T_\Sigma(D)$ we let $t < t'$ iff there are $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $t_1, \ldots, t_k \in T_\Sigma(D)$, and $i \in [k]$ such that $t = t_i$ and $t' = \sigma(t_1, \ldots, t_k)$. The relation $<$ is well-founded on $T_\Sigma(D)$ and we call a proof by well-founded induction on $<$ a *proof by structural induction*. Moreover, we call a definition by well-founded recursion on $<$ a *definition by structural recursion*. For further details on trees we refer to [66].

[4] The Σ-homomorphism h is even unique (see [7, Lemma 3.3.1] or [70, Corollary 24.1])

Term algebras

Let $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. The ***σ-top concatenation*** *(wrt Σ and D)* is the operation $\mathrm{top}_\sigma^{\Sigma,D} \in \mathrm{Ops}^{(k)}(T_\Sigma(D))$ defined by $\mathrm{top}_\sigma^{\Sigma,D}(t_1,\ldots,t_k) = \sigma(t_1,\ldots,t_k)$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1,\ldots,t_k \in T_\Sigma(D)$. If Σ and D are clear from the context, then we write top_σ instead of $\mathrm{top}_\sigma^{\Sigma,D}$. We lift the operation top_σ to the operation $\mathrm{top}_\sigma^{\mathrm{lang}} \in \mathrm{Ops}^{(k)}\big(\mathcal{P}(T_\Sigma(D))\big)$, called ***σ-language top concatenation***, by letting

$$\mathrm{top}_\sigma^{\mathrm{lang}}(L_1,\ldots,L_k) = \{\mathrm{top}_\sigma(t_1,\ldots,t_k) \mid t_1 \in L_1,\ldots,t_k \in L_k\}$$

for every $L_1,\ldots,L_k \in \mathcal{P}(T_\Sigma(D))$. If there is no chance of confusion, then we write top_σ instead of $\mathrm{top}_\sigma^{\mathrm{lang}}$.

The (Σ, D)-***term algebra*** is the Σ-algebra $\mathcal{T}_\Sigma(D) = (T_\Sigma(D), \theta_\Sigma)$ such that $\theta_\Sigma(\sigma) = \mathrm{top}_\sigma$ for every $\sigma \in \Sigma$. The $(\Sigma,\emptyset)$-term algebra is also called the Σ-***term algebra*** and denoted by $\mathcal{T}_\Sigma$. For every Σ-algebra $\mathcal{A} = (A, \theta_A)$ and mapping $f : D \to A$ there is a unique Σ-homomorphism from $\mathcal{T}_\Sigma(D)$ to $\mathcal{A}$ extending f (for a proof of this statement we refer to [135, Theorem 1.2.3.4]); this unique Σ-homomorphisms is also called the ***evaluation homomorphism*** *of $\mathcal{A}$ and f*. Therefore, $\mathcal{T}_\Sigma(D)$ is freely generated by D for the class of all Σ-algebras, and $\mathcal{T}_\Sigma$ is initial for the class of all Σ-algebras.

Example 2.4. Consider the Σ-algebra $\mathcal{A} = (\mathcal{P}(T_\Sigma), \theta)$, where for every $\sigma \in \Sigma$, $\theta(\sigma) = \mathrm{top}_\sigma^{\mathrm{lang}}$ is the σ-language top concatenation. Then the mapping $f : T_\Sigma \to \mathcal{P}(T_\Sigma)$ with $f(t) = \{t\}$, for every $t \in T_\Sigma$, is the unique Σ-homomorphism from $\mathcal{T}_\Sigma$ to $\mathcal{A}$. □

Operations on trees

We define the mappings $\mathrm{pos} : T_\Sigma(D) \to \mathcal{P}((\mathbb{N}_+)^*)$, $\mathrm{ind} : T_\Sigma(D) \to \mathcal{P}(D)$, and $\mathrm{indyield} : T_\Sigma(D) \to D^*$ by structural recursion as follows:

- for every $d \in D$ we let $\mathrm{pos}(d) = \{\varepsilon\}$, $\mathrm{ind}(d) = \{d\}$, and $\mathrm{indyield}(d) = d$, and
- for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1,\ldots,t_k \in T_\Sigma(D)$ we let

$$\begin{aligned} \mathrm{pos}(\sigma(t_1,\ldots,t_k)) &= \{\varepsilon\} \cup \{iw \mid i \in [k], w \in \mathrm{pos}(t_i)\}\ , \\ \mathrm{ind}(\sigma(t_1,\ldots,t_k)) &= \mathrm{ind}(t_1) \cup \cdots \cup \mathrm{ind}(t_k)\ , \\ \mathrm{indyield}(\sigma(t_1,\ldots,t_k)) &= \mathrm{indyield}(t_1) \cdot \ldots \cdot \mathrm{indyield}(t_k)\ . \end{aligned}$$

Let $t \in T_\Sigma(D)$. We call $\mathrm{pos}(t)$ the *set of* ***positions*** *in t* and $\mathrm{ind}(t)$ the *set of* ***indices*** *occurring in t*. The ***size*** *of t* and the ***height*** *of t* is defined by $\mathrm{size}(t) = |\mathrm{pos}(t)|$ and $\mathrm{height}(t) = \max\{|w| \mid w \in \mathrm{pos}(t)\}$, respectively. The ***root*** *of t* is the position $\varepsilon \in \mathrm{pos}(t)$ and a ***leaf*** *of t* is a position $v \in \mathrm{pos}(t)$ such that for every $w \in \mathrm{pos}(t)$, v is not a proper prefix of w.

Example 2.5 (*Continuation of Example 2.3***).** Recall the ranked alphabet Σ_{ex} and the set D_{ex} from Example 2.3. Consider the tree $t_{\mathrm{ex}} = \gamma\big(\sigma(\sigma(\alpha, y), \gamma(x))\big)$. Then $\mathrm{pos}(t_{\mathrm{ex}}) = \{\varepsilon, 1, 11, 111, 112, 12, 121\}$, $\mathrm{ind}(t_{\mathrm{ex}}) = \{x, y\}$, $\mathrm{indyield}(t_{\mathrm{ex}}) = yx$, and $\mathrm{size}(t_{\mathrm{ex}}) = 7$ and $\mathrm{height}(t_{\mathrm{ex}}) = |111| = 3$. □

Lemma 2.6. *Let Σ be finite (i.e., Σ is a ranked alphabet) and for every $n \in \mathbb{N}$ let $T_n = \{t \in T_\Sigma \mid \mathrm{height}(t) \leq n\}$. Then for every $n \in \mathbb{N}$, $|T_n| \leq |\Sigma|^{((b+1)^n)}$, where $b = \mathrm{maxrk}(\Sigma)$.*

Proof. We give a proof by induction on n.

Induction base. Clearly, $T_0 = \{\alpha() \mid \alpha \in \Sigma^{(0)}\}$ and, hence, $|T_0| \leq |\Sigma| = |\Sigma|^1 = |\Sigma|^{((b+1)^0)}$.

Induction step. Let $n \in \mathbb{N}$ and assume that $|T_n| \leq |\Sigma|^{((b+1)^n)}$. Observe that $T_{n+1} = \{\sigma(t_1, \ldots, t_k) \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, t_1, \ldots, t_k \in T_n\}$; this implies that we have $|T_{n+1}| \leq |\Sigma| \cdot |T_n|^b \leq |\Sigma| \cdot (|\Sigma|^{((b+1)^n)})^b = |\Sigma|^{(1+(b+1)^n \cdot b)} \leq |\Sigma|^{((b+1)^n + (b+1)^n \cdot b)} = |\Sigma|^{((b+1)^n \cdot (b+1))} = |\Sigma|^{((b+1)^{n+1})}$. ■

Let $t \in T_\Sigma(D)$. For every $w \in \operatorname{pos}(t)$ and $t' \in T_\Sigma(D)$ we define the ***label*** $t(w) \in \Sigma \cup D$ *of* t *at position* w, the ***subtree*** $t|_w \in T_\Sigma(D)$ *of* t *at position* w, and the ***substitution*** $t[t']_w \in T_\Sigma(D)$ *of* t' *in* t *at position* w by recursion on the length of w as follows. If $t \in D$, then $w = \varepsilon$ and we let $t(w) = t|_w = t$ and $t[t']_w = t'$. If $t = \sigma(t_1, \ldots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \ldots, t_k \in T_\Sigma$, then

- if $w = \varepsilon$, then we let $t(w) = \sigma$, $t|_w = t$, and $t[t']_w = t'$.
- if $w = iw'$ for some $i \in [k]$ and $w' \in \operatorname{pos}(t_i)$, then we let $t(w) = t_i(w')$, $t|_w = t_i|_{w'}$, and $t[t']_w = \sigma(t_1, \ldots, t_{i-1}, t_i[t']_{w'}, t_{i+1}, \ldots, t_k)$.

We say that $t' \in T_\Sigma(D)$ is a ***subtree*** (respectively ***proper subtree***) *of* t if there is a $w \in \operatorname{pos}(t)$ (respectively $w \in \operatorname{pos}(t) \setminus \{\varepsilon\}$) such that $t' = t|_w$.

Trees can be represented graphically as follows. Let $t \in T_\Sigma(D)$ and define the partial order $\leq$ on $\operatorname{pos}(t)$ by letting for every $v, w \in \operatorname{pos}(t)$: $v \leq w$ iff w is a prefix of v (note that the order of v and w is inverted in this definition). Then a diagram of t is a Hasse diagram of $(\operatorname{pos}(t), \leq)$ that is (i) planar and (ii) for every $v \in \operatorname{pos}(t)$ and $i, j \in \mathbb{N}_+$ with $vi, vj \in \operatorname{pos}(t)$ and $i < j$ we have that the vertex for vi is left of the vertex of vj; moreover, in this Hasse diagram we label, for every $v \in \operatorname{pos}(t)$, the vertex for v with $t(v)$ instead of v. A graphical representation of the tree t_{ex} of Example 2.5 is given in Figure 2.3 (note that we added the positions of t_{ex} in this figure for clarity).

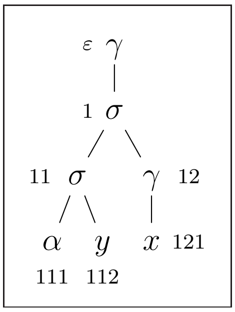


Figure 2.3: Diagram of the tree t_{ex} of Example 2.5.

The following observation can easily be proved by induction on the length of w and u.

Observation 2.7. *Let* $t, t' \in T_\Sigma(D)$, $w \in \operatorname{pos}(t)$, $w' \in \operatorname{pos}(t')$, *and* $u, v \in (\mathbb{N}_+)^*$ *such that* $uv = w$. *Then* $v \in \operatorname{pos}(t|_u)$, $ww' \in \operatorname{pos}(t[t']_w)$, $\operatorname{ind}(t') \subseteq \operatorname{ind}(t[t']_w)$, *and*

$$t(w) = t|_u(v) , \qquad t'(w') = (t[t']_w)(ww') .$$

Now we will define the notion of substitution of a *sequence of trees* $\bar{t}$ into a given *source tree* t. In general we need to provide two kinds of information in order to carry out such a substitution: (i) what are the substitution positions, i.e., at what positions of the source tree t shall we substitute trees from the sequence of trees $\bar{t}$ and (ii) for every substitution position, what tree of the sequence of trees shall we choose for substitution at this particular position? The tree substitution that we present below is defined in such a way that the set of substitution positions is precisely the set of index positions in t, i.e., every index in the source tree is replaced by one tree from $\bar{t}$. Now let us explain how we provide the second kind of information (i.e., what trees to substitute). In the literature [51, 63, 105] this is sometimes accomplished by using special sets of indices of the form $D = \{z_1, z_2, z_3, \dots\}$; then the tree substitution is defined in such a way that every index, say z_i, is replaced by the i-th tree in the sequence of trees $\bar{t}$. We will not proceed along these lines in this thesis, instead our definition of tree substitution simply substitutes the i-th index in the tree t (read from left to right) by the i-th tree in $\bar{t}$; this definition presupposes, however, that the number of indices in t is equal to the number of trees in $\bar{t}$.

Let $t \in T_\Sigma(D)$ and $\bar{t} \in \big(T_\Sigma(D)\big)^*$ such that $|\mathrm{indyield}(t)| = |\,\bar{t}\,|$. The ***tree substitution*** *of $\bar{t}$ into t* is the tree $t \leftarrow \bar{t}$ in $T_\Sigma(D)$ defined by recursion as follows:

- if $t \in D$, then $|\,\bar{t}\,| = |\mathrm{indyield}(t)| = 1$ (i.e, $\bar{t} \in T_\Sigma(D)$) and we let $t \leftarrow \bar{t} = \bar{t}$,
- if $t = \sigma(t_1, \dots, t_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $t_1, \dots, t_k \in T_\Sigma(D)$, then there are unique $\bar{t}_1, \dots, \bar{t}_k \in \big(T_\Sigma(D)\big)^*$ with $\bar{t}_1 \cdots \bar{t}_k = \bar{t}$ and $|\mathrm{indyield}(t_i)| = |\,\bar{t}_i\,|$ for every $i \in [k]$; we let $t \leftarrow \bar{t} = \sigma(t_1 \leftarrow \bar{t}_1, \dots, t_k \leftarrow \bar{t}_k)$.

The definition of tree substitution is illustrated in Figure 2.4.

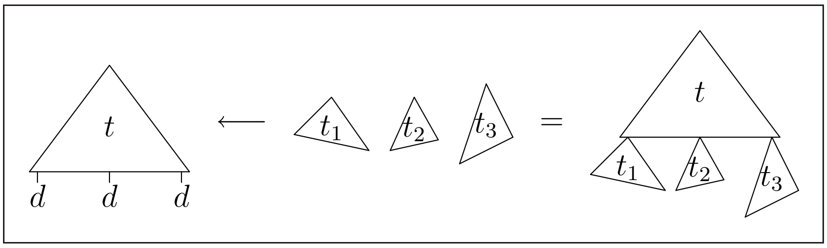


Figure 2.4: Illustration of the tree substitution of $t_1 t_2 t_3$ into t. The tree t has three occurrences of the index d and no occurrences of any other index.

Example 2.8 (*Continuation of Example 2.5*)**.** Let $t_1 = x$ and $t_2 = \gamma(\alpha)$. Then $t_{\mathrm{ex}} \leftarrow t_1 t_2 = \gamma\big(\sigma(\sigma(\alpha, x), \gamma(\gamma(\alpha)))\big)$. □

We conclude this section with two simple properties regarding the substitution of trees.

Observation 2.9. *Let $t \in T_\Sigma(D)$, $k \in \mathbb{N}$, and $t_1, \dots, t_k \in T_\Sigma(D)$ such that* $|\mathrm{indyield}(t)| = k$*. Then* $\mathrm{indyield}(t \leftarrow t_1 \cdots t_k) = \mathrm{indyield}(t_1) \cdots \mathrm{indyield}(t_k)$.

The substitution operation is "associative" in the following sense.

Lemma 2.10. *Let* $t \in T_\Sigma(D)$*,* $k = |\mathrm{indyield}(t)|$*, and* $t_1, \dots, t_k \in T_\Sigma(D)$*. Moreover, for every* $i \in [k]$ *let* $\bar{s}_i \in \big(T_\Sigma(D)\big)^*$ *such that* $|\bar{s}_i| = |\mathrm{indyield}(t_i)|$*. Then*

$$t \leftarrow (t_1 \leftarrow \bar{s}_1) \cdots (t_k \leftarrow \bar{s}_k) = (t \leftarrow t_1 \cdots t_k) \leftarrow \bar{s}_1 \cdots \bar{s}_k\ . \tag{2.6}$$

Proof. First observe that the right-hand side of Equation (2.6) is well-defined because Observation 2.9 implies $|\mathrm{indyield}(t \leftarrow t_1 \cdots t_k)| = |\mathrm{indyield}(t_1)| + \cdots + |\mathrm{indyield}(t_k)| = |\bar{s}_1| + \cdots + |\bar{s}_k| = |\bar{s}_1 \cdots \bar{s}_k|$. Now we give a proof by structural induction on t.

Induction base. If $t \in D$, then $k = 1$ and we have $t \leftarrow (t_1 \leftarrow \bar{s}_1) = t_1 \leftarrow \bar{s}_1 = (t \leftarrow t_1) \leftarrow \bar{s}_1$.

Induction step. Let $t = \sigma(t'_1, \dots, t'_l)$ for some $l \in \mathbb{N}$, $\sigma \in \Sigma^{(l)}$, and $t'_1, \dots, t'_l \in T_\Sigma(D)$. For every $i \in [l]$ let $n_i = |\mathrm{indyield}(t'_i)|$. Then $n_1 + \cdots + n_l = k$. Moreover, for every $i \in [l]$ let

- $t^a_i = t'_i \leftarrow (t_{n_1+\cdots+n_{i-1}+1} \leftarrow \bar{s}_{n_1+\cdots+n_{i-1}+1}) \cdots (t_{n_1+\cdots+n_i} \leftarrow \bar{s}_{n_1+\cdots+n_i})$,
- $t^b_i = (t'_i \leftarrow t_{n_1+\cdots+n_{i-1}+1} \cdots t_{n_1+\cdots+n_i}) \leftarrow \bar{s}_{n_1+\cdots+n_{i-1}+1} \cdots \bar{s}_{n_1+\cdots+n_i}$,
- $t^c_i = t'_i \leftarrow t_{n_1+\cdots+n_{i-1}+1} \cdots t_{n_1+\cdots+n_i}$.

Then $t \leftarrow (t_1 \leftarrow \bar{s}_1) \cdots (t_k \leftarrow \bar{s}_k) = \sigma(t'_1, \dots, t'_l) \leftarrow (t_1 \leftarrow \bar{s}_1) \cdots (t_k \leftarrow \bar{s}_k) = \sigma(t^a_1, \dots, t^a_l)$. Due to the induction hypothesis this is equal to $\sigma(t^b_1, \dots, t^b_l) = \sigma(t^c_1, \dots, t^c_l) \leftarrow \bar{s}_1 \cdots \bar{s}_k = (\sigma(t'_1, \dots, t'_l) \leftarrow t_1 \cdots t_k) \leftarrow \bar{s}_1 \cdots \bar{s}_k = (t \leftarrow t_1 \cdots t_k) \leftarrow \bar{s}_1 \cdots \bar{s}_k$. ■

2.6 Hypergraphs

Directed hypergraphs [6, 64] are a generalization of the usual concept of directed graphs, where an edge in a hypergraph, called hyperedge, is allowed to connect any number of vertices, i.e., a hyperedge can have multiple input and multiple output vertices. In this thesis we focus on functional hypergraphs (also called B-graphs in the literature [64, 84]); these are hypergraphs where every hyperedge has precisely one output vertex. We follow along the lines of Huang and Chiang [77] and define hypergraphs in such a way that the input vertices of every hyperedge are given by a string over the set of vertices; such a hypergraph is called an ordered hypergraph.

Hypergraphs are our main tool to describe the behavior of a multioperator weighted monadic datalog program for a given input tree. In this section we will study basic properties of hypergraphs that will be useful in later chapters.

A (*finite, functional, ordered, and directed*) ***hypergraph*** is a triple (V, E, μ) such that V and E are finite sets, and $\mu : E \to V^* \times V$. We refer to the elements of N as ***vertices*** and to the elements of E as ***hyperedges***. We consider E as a ranked alphabet, where for every $e \in E$ and $(w, v) \in V^* \times V$ with $\mu(e) = (w, v)$, the rank of e is $|w|$. Let $G = (V, E, \mu)$ be a hypergraph, $k \in \mathbb{N}$, $e \in E^{(k)}$, and $v_1, \dots, v_k, v \in V$ such that $\mu(e) = (v_1 \cdots v_k, v)$. We will denote v by $\mathrm{out}(e)$, called ***output vertex*** *of* e, and for every $i \in [k]$ we will denote v_i by $\mathrm{in}_i(e)$ and call it the *i-th* ***input vertex*** *of* e.

Example 2.11. Consider the hypergraph $G_{\mathrm{ex}} = (V_{\mathrm{ex}}, E_{\mathrm{ex}}, \mu_{\mathrm{ex}})$ whose components are defined as follows: $V_{\mathrm{ex}} = \{v_1, v_2\}$, $E_{\mathrm{ex}} = \{e_1, e_2, e_3, e_4, e_5\}$, $\mu_{\mathrm{ex}}(e_1) = (\varepsilon, v_1)$, $\mu_{\mathrm{ex}}(e_2) = (v_2, v_1)$, $\mu_{\mathrm{ex}}(e_3) = (v_2 v_1, v_2)$, $\mu_{\mathrm{ex}}(e_4) = (v_1 v_1, v_1)$, and $\mu_{\mathrm{ex}}(e_5) = (\varepsilon, v_2)$.

Then $\mathrm{rk}(e_1) = \mathrm{rk}(e_5) = 0$, $\mathrm{rk}(e_2) = 1$, and $\mathrm{rk}(e_3) = \mathrm{rk}(e_4) = 2$. Moreover, we have $\mathrm{out}(e_1) = \mathrm{out}(e_2) = \mathrm{out}(e_4) = v_1$ and $\mathrm{out}(e_3) = \mathrm{out}(e_5) = v_2$. The input vertices of e_3 are $\mathrm{in}_1(e_3) = v_2$ and $\mathrm{in}_2(e_3) = v_1$. □

A hypergraph (V, E, μ) can be represented graphically by means of a ***hypergraph diagram***. Every vertex $v \in V$ is depicted by a circle containing the label 'v' and every hyperedge $e \in E$ by a small box with several incoming and one outgoing edge (called ***tentacles***). There is one incoming tentacle for every input vertex $\mathrm{in}_i(e)$ ($i \in [\mathrm{rk}(e)]$) and one outgoing tentacle for the output vertex $\mathrm{out}(e)$. The incoming tentacles start at the input vertices $\mathrm{in}_1(e), \ldots, \mathrm{in}_{\mathrm{rk}(e)}(e)$ and are ordered counter-clockwise starting from the outgoing tentacle; the outgoing tentacle ends at the output vertex of e. The label e is written next to the small box that represents the hyperedge e. The hypergraph G_{ex} from Example 2.11 is shown in Figure 2.5.

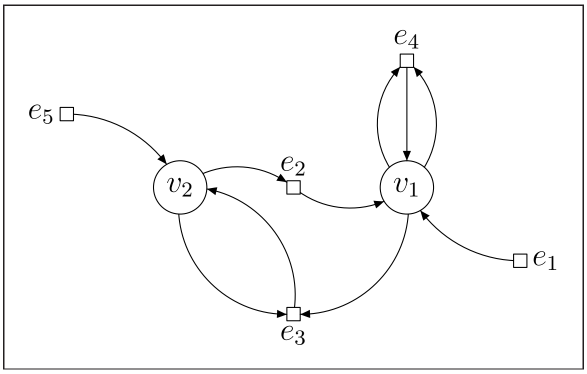


Figure 2.5: Hypergraph diagram of the hypergraph G_{ex} of Example 2.11.

Hyperpaths and hyperpath segments

A path in a digraph is a sequence of edges that fit together; more precisely, the output vertex of the first edge has to be the input vertex of the second edge and so on. Hyperpaths are the counterpart of paths for hypergraphs. Since hyperedges can have multiple input vertices it is not possible to describe hyperpaths by means of sequences of hyperedges. However, hyperpaths can adequately be represented by trees over the ranked alphabet of hyperedges. In this sense the hyperpath ends in the output vertex of the hyperedge that labels the root of the hyperpath, i.e., the label of the root is the final hyperedge in the hyperpath. The hyperpath might have multiple initial hyperedges; these initial hyperedges are the leafs of the tree representation of the hyperpath. Note that since the label at every leaf in a tree is nullary, the initial hyperedges of the hyperpath do not have any input vertex; hence, the hyperpath does have a unique final vertex but no initial vertices.

For our purposes it turns out useful to generalize this definition of hyperpaths. In fact, we will allow hyperpaths to have initial vertices and refer to such hyperpaths as hyperpath segments. A hyperpath segment is (similarly to a hyperpath) defined as a tree over the ranked alphabet of hyperedges, where we allow the use of copies of elements of the set of vertices as indices of the tree.

Similarly to paths in digraphs we require the hyperedges that occur in the tree representation to fit together, e.g., the output vertex of the i-th subtree must be the i-th input vertex of the hyperedge at the root of the tree.

In the sequel let $G = (V, E, \mu)$ be a hypergraph and $U \subseteq V$.

Let $v \in V$. A tree $\eta \in T_E(U) \setminus U$ is called a ***hyperpath segment*** *of G starting in U and ending in v* if $\text{out}(\eta(\varepsilon)) = v$ and for every $w \in \text{pos}(\eta)$ and $i \in \mathbb{N}_+$ with $wi \in \text{pos}(\eta)$ we have

- if $\eta|_{wi} \in U$, then $\text{in}_i(\eta(w)) = \eta|_{wi}$,
- if $\eta|_{wi} \notin U$, then $\text{in}_i(\eta(w)) = \text{out}(\eta(w \cdot i))$ and $\text{in}_i(\eta(w)) \notin U$.

We denote the set of all hyperpath segments of G starting in U and ending in v by $\mathrm{H}_G^{v,U}$. We lift the notion of input and output vertices to hyperpath segments as follows. Let $\eta \in \mathrm{H}_G^{v,U}$; then v is called the ***output vertex*** *of* η and $\text{indyield}(\eta)$ is the sequence of ***input vertices*** *of* η. Moreover, for every $w \in \text{pos}(\eta) \setminus \{\varepsilon\}$ with $\eta|_w \notin U$ we call $\text{out}(\eta(w))$ and ***inner vertex*** *of* η.

Roughly speaking, we can generate the hyperpath segments in $\mathrm{H}_G^{v,U}$ as follows. We start at vertex v, choose a hyperpath whose output vertex is v, and move backwards along the hyperedge to its input vertices. Then we proceed with this stepwise "unfolding" of the hyperpath for each of the input vertices in parallel. This process finishes when we reach nullary hyperedges or vertices that are in the set U. Note that due to our definition we have to stop at any vertex that is in the set U; thus, for any hyperpath segment in $\mathrm{H}_G^{v,U}$ the only hyperedge that is allowed to have an output vertex in the set U is the hyperedge at the root of the hyperpath.

Example 2.12 (*Continuation of Example 2.11*). The sets of hyperpath segments for the hypergraph G_{ex} are shown in Table 2.1. The diagrams of one example hyperpath segment in each of the sets $\mathrm{H}_{G_{\text{ex}}}^{v_1,\emptyset}$ and $\mathrm{H}_{G_{\text{ex}}}^{v_1,\{v_2\}}$ is given in Figure 2.6; for reasons of clarity we have, for every hyperedge, indicated the input vertices (below the hyperedge) and the output vertex (above the hyperedge); this illustrates how input and output vertices fit together.

The output vertex of the hyperpath segment in Figure 2.6(b) is v_1, its sequence of input vertices is $v_2 v_2$ and $\{v_1\}$ is the set of its inner vertices. The set of inner vertices of the hyperpath segment $e_3(v_2, v_1)$, which is in $\mathrm{H}_{G_{\text{ex}}}^{v_2,\{v_1,v_2\}}$, is empty. □

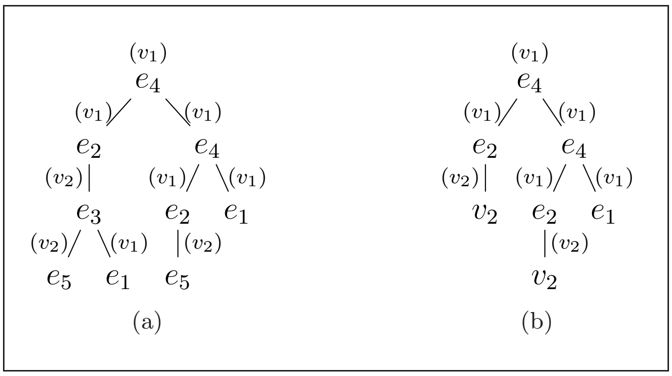


Figure 2.6: Two diagrams of hyperpath segments of the hypergraph G_{ex} of Example 2.11. Figure (a) shows a hyperpath segment in the set $\mathrm{H}_{G_{\text{ex}}}^{v_1,\emptyset}$ and Figure (b) a hyperpath segment in $\mathrm{H}_{G_{\text{ex}}}^{v_1,\{v_2\}}$.

U	$\mathrm{H}^{v_1,U}_{G_{\mathrm{ex}}}$	$\mathrm{H}^{v_2,U}_{G_{\mathrm{ex}}}$
$\emptyset$	$\{e_1, e_2(e_5), e_2(e_3(e_5, e_1)),$ $e_4(e_1, e_1), e_4(e_1, e_2(e_5)), \dots\}$	$\{e_5, e_3(e_5, e_1),$ $e_3(e_5, e_2(e_5)), \dots\}$
$\{v_1\}$	$\{e_1, e_4(v_1, v_1), e_2(e_5),$ $e_2(e_3(e_5), v_1), \dots\}$	$\{e_5, e_3(e_5, v_1),$ $e_3(e_3(e_5, v_1), v_1), \dots\}$
$\{v_2\}$	$\{e_1, e_2(v_2), e_4(e_1, e_1), e_4(e_2(v_2), e_1),$ $e_4(e_2(v_2), e_4(e_1, e_1)), \dots\}$	$\{e_5, e_3(v_2, e_1),$ $e_3(v_2, e_2(v_2)), \dots\}$
$\{v_1, v_2\}$	$\{e_1, e_2(v_2), e_4(v_1, v_1)\}$	$\{e_3(v_2, v_1), e_5\}$

Table 2.1: Hyperpath segments for the hypergraph G_{ex} of Example 2.11.

Observation 2.13. *Let $v \in V$, $\eta \in \mathrm{H}^{v,U}_G$, $w \in \mathrm{pos}(\eta)$, and $i \in \mathbb{N}_+$ such that $wi \in \mathrm{pos}(\eta)$. Then $\eta|_{wi} \in U$ iff $\mathrm{in}_i(\eta(w)) \in U$.*

Above we described that one can generate the set $\mathrm{H}^{v,U}_G$ of hyperpath segments (for some $v \in V$) by starting at v and tracing backwards along hyperedges. It is obvious that when one reaches a vertex $v' \notin U$ during this process, then the set of subhyperpaths that one can generate when proceeding at v' is the set $\mathrm{H}^{v',U}_G$. This relationship between the sets in the family $(\mathrm{H}^{v,U}_G \mid v \in V)$ is made precise by the following two lemmas.

Lemma 2.14. *For every $k \in \mathbb{N}$, $e \in E^{(k)}$, and $i \in [k]$ let $H^e_i = \mathrm{H}^{\mathrm{in}_i(e),U}_G$ if $\mathrm{in}_i(e) \notin U$ and $H^e_i = \{\mathrm{in}_i(e)\}$ otherwise. Then for every $v \in V$,*

$$\mathrm{H}^{v,U}_G = \left\{ e(\eta_1, \dots, \eta_k) \mid k \in \mathbb{N}, e \in E^{(k)}, \mathrm{out}(e) = v, \eta_1 \in H^e_1, \dots, \eta_k \in H^e_k \right\} .$$

Proof. Let $v \in V$.

"$\subseteq$": Let $\eta \in \mathrm{H}^{v,U}_G$. Then $\eta \notin U$. Therefore, there is a $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1, \dots, \eta_k \in T_E(U)$ such that $\eta = e(\eta_1, \dots, \eta_k)$ and $\mathrm{out}(e) = v$. Let $i \in [k]$. It remains to show that $\eta_i \in H^e_i$.

First we consider the case that $\mathrm{in}_i(e) \notin U$. We need to show that $\eta_i \in \mathrm{H}^{\mathrm{in}_i(e),U}_G$. By Observation 2.13, $\eta_i \notin U$. Thus, $\eta_i \in T_E(U) \setminus U$. Moreover, $\mathrm{out}(\eta_i(\varepsilon)) = \mathrm{out}(\eta(i)) = \mathrm{in}_i(\eta(\varepsilon)) = \mathrm{in}_i(e)$ because $\eta \in \mathrm{H}^{v,U}_G$. The remainder of the proof that $\eta_i \in \mathrm{H}^{\mathrm{in}_i(e),U}_G$ is trivial.

Now we consider the case that $\mathrm{in}_i(e) \in U$. We show $\eta_i = \mathrm{in}_i(e)$. By Observation 2.13, $\eta_i \in U$. Then $\mathrm{in}_i(e) = \mathrm{in}_i(\eta(\varepsilon)) = \eta(i) = \eta_i$.

"$\supseteq$": Let $k \in \mathbb{N}$ and $e \in E^{(k)}$ such that $\mathrm{out}(e) = v$. Moreover, for every $i \in [k]$ let $\eta_i \in \mathrm{H}^{\mathrm{in}_i(e),U}_G$ if $\mathrm{in}_i(e) \notin U$ and $\eta_i = \mathrm{in}_i(e)$ otherwise. Let $\eta = e(\eta_1, \dots, \eta_k)$. We show that $\eta \in \mathrm{H}^{v,U}_G$.

For every $i \in [k]$, $\eta_i \in T_E(U)$; hence, $\eta \in T_E(U)$. Clearly, $\eta \notin U$ and $\mathrm{out}(\eta(\varepsilon)) = \mathrm{out}(e) = v$. Let $w \in \mathrm{pos}(\eta)$ and $i \in \mathbb{N}_+$ such that $wi \in \mathrm{pos}(\eta)$.

First let us consider the case $w = \varepsilon$. If $\eta|_i \in U$, then $\eta_i \notin \mathrm{H}^{\mathrm{in}_i(e),U}_G$ due to the definition of $\mathrm{H}^{\mathrm{in}_i(e),U}_G$; hence, $\eta_i = \mathrm{in}_i(e) = \mathrm{in}_i(\eta(\varepsilon))$. If $\eta|_i \notin U$, then $\mathrm{in}_i(e) \notin U$ and $\eta_i \in \mathrm{H}^{\mathrm{in}_i(e),U}_G$; we obtain $\mathrm{out}(\eta_i(\varepsilon)) = \mathrm{in}_i(e)$, which implies $\mathrm{in}_i(\eta(\varepsilon)) = \mathrm{out}(\eta(i)) \notin U$.

Now consider the case that $w = jw'$ for some $j \in [k]$ and $w' \in \mathrm{pos}(\eta_j)$. Then $\eta_j \in \mathrm{H}^{\mathrm{in}_j(e),U}_G$; for otherwise $\eta_j = \mathrm{in}_j(e) \in U$ which contradicts $jw'i = wi \in \mathrm{pos}(\eta)$. Therefore the remainder of the proof is trivial. ∎

Lemma 2.15. *Let $v \in V$, $\eta \in \mathrm{H}_G^{v,U}$, and $w \in \mathrm{pos}(\eta)$ such that $\eta|_w \notin U$. Then $\eta|_w \in \mathrm{H}_G^{\mathrm{out}(\eta(w)),U}$.*

Proof. We give a proof by induction on the length of w.

Induction base. If $|w| = 0$, i.e., $w = \varepsilon$, then $\eta|_w = \eta \in \mathrm{H}_G^{v,U} = \mathrm{H}_G^{\mathrm{out}(\eta(\varepsilon)),U}$.

Induction step. Now assume that $|w| > 1$. Let $e = \eta(\varepsilon)$. Then there is an $i \in [\mathrm{rk}(e)]$ and $w' \in \mathrm{pos}(\eta|_i)$ such that $w = iw'$. Since $(\eta|_i)|_{w'} = \eta|_w \notin U$ we obtain $\eta|_i \notin U$. By Lemma 2.14 we have that (i) $\eta|_i = \mathrm{in}_i(e)$ if $\mathrm{in}_i(e) \in U$ and (ii) $\eta|_i \in \mathrm{H}_G^{\mathrm{in}_i(e),U}$ if $\mathrm{in}_i(e) \notin U$. Since $\eta|_i \notin U$ we obtain that $\eta|_i \in \mathrm{H}_G^{\mathrm{in}_i(e),U}$. Then the induction hypothesis together with the fact that $(\eta|_i)|_{w'} \notin U$ yields that $\eta|_w = (\eta|_i)|_{w'} \in \mathrm{H}_G^{\mathrm{out}(\eta|_i(w')),U} = \mathrm{H}_G^{\mathrm{out}(\eta(w)),U}$. ■

We put $\mathrm{H}_G^v = \mathrm{H}_G^{v,\emptyset}$ and call any element η in H_G^v a ***hyperpath*** *(of G ending in v)*. The following corollary is an immediate consequence of Lemma 2.14.

Corollary 2.16. *Let $v \in V$. Then*

$$\mathrm{H}_G^v = \left\{e(\eta_1, \ldots, \eta_n) \mid k \in \mathbb{N}, e \in E^{(k)}, \mathrm{out}(e) = v, \eta_1 \in \mathrm{H}_G^{\mathrm{in}_1(e)}, \ldots, \eta_k \in \mathrm{H}_G^{\mathrm{in}_k(e)}\right\} .$$

The previous two lemmas characterized how the sets in the family $(\mathrm{H}_G^{v,U} \mid v \in V)$ are related. Now let us analyze how the sets H_G^v and $\mathrm{H}_G^{v,U}$, for a given $v \in V$, are connected. Roughly speaking, we can transform every hyperpath η in H_G^v into a hyperpath segment η' in $\mathrm{H}_G^{v,U}$ by the following method: for every leaf position w in η we trace along the path from the root of η to w and, at the first occurrence of a vertex u in U, we cut the path at this occurrence, remove the subhyperpath that starts at this vertex and replace it by the index u. An example of this transformation is given in Figure 2.6, where the hyperpath segment in Figure (b) is the result of transforming the hyperpath in Figure (a) for $U = \{v_2\}$. Now we define this operation formally.

Definition 2.17. Let $v \in V$ and $\eta \in \mathrm{H}_G^v$. We define the ***top decomposition*** *of η wrt U and G*, denoted by $\mathrm{dec}{\uparrow}(\eta, U, G) \in \mathrm{H}_G^{v,U}$, by recursion on the structure of η as follows. By Corollary 2.16 there are uniquely determined $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1 \in \mathrm{H}_G^{\mathrm{in}_1(e)}, \ldots, \eta_k \in \mathrm{H}_G^{\mathrm{in}_k(e)}$ such that $\mathrm{out}(e) = v$ and $\eta = e(\eta_1, \ldots, \eta_k)$. We define

$$\mathrm{dec}{\uparrow}(\eta, U, G) = e(\eta'_1, \ldots, \eta'_k) ,$$
$$\text{where } \forall\, i \in [k] : \eta'_i = \begin{cases} \mathrm{dec}{\uparrow}(\eta_i, U, G) , & \text{if } \mathrm{in}_i(e) \notin U, \\ \mathrm{in}_i(e) , & \text{otherwise.} \end{cases}$$ □

By means of Lemma 2.14 it is easy to check that $\mathrm{dec}{\uparrow}(\eta, U, G)$ is well-defined. If G is clear from the context, then we write $\mathrm{dec}{\uparrow}(\eta, U)$ instead of $\mathrm{dec}{\uparrow}(\eta, U, G)$. The following lemma captures some basic properties of the top decomposition.

Lemma 2.18. *Let $\eta \in \mathrm{H}_G^v$ and $\eta' = \mathrm{dec}{\uparrow}(\eta, U)$.*

1. *Let $w \in \mathrm{pos}(\eta) \cap \mathrm{pos}(\eta')$. If both $\mathrm{out}(\eta(w)) \in U$ and $|w| > 0$, then $\mathrm{out}(\eta(w)) = \eta'|_w$; otherwise $\eta(w) = \eta'(w)$.*
2. *For every $w \in (\mathbb{N}_+)^*$ the following statements are equivalent: (i) $w \in \mathrm{pos}(\eta')$ and (ii) $w \in \mathrm{pos}(\eta)$ and for every proper prefix w' of w with $w' \neq \varepsilon$ we have $\mathrm{out}(\eta(w')) \notin U$.*

3. *Let* $w \in \text{pos}(\eta)$ *such that* $\text{out}(\eta(w')) \notin U$ *for every prefix* w' *of* w *with* $w' \neq \varepsilon$*. Then* $\eta'|_w = \text{dec}{\uparrow}(\eta|_w, U)$*.*

Proof. We prove Statements 1 to 3 simultaneously be structural induction on η. By Corollary 2.16 there are uniquely determined $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1 \in \text{H}_G^{\text{in}_1(e)}, \ldots, \eta_k \in \text{H}_G^{\text{in}_k(e)}$ such that $\text{out}(e) = v$ and $\eta = e(\eta_1, \ldots, \eta_k)$. Moreover, by the definition of $\text{dec}{\uparrow}(\eta, U)$ we have $\eta' = e(\eta'_1, \ldots, \eta'_k)$, where for every $i \in [k]$: (i) $\eta'_i = \text{dec}{\uparrow}(\eta_i, U)$ if $\text{in}_i(e) \notin U$ and (ii) $\eta'_i = \text{in}_i(e)$ otherwise.

First we prove Statement 1. Let $w \in \text{pos}(\eta) \cap \text{pos}(\eta')$. If $w = \varepsilon$, then obviously $|w| = 0$ and $\eta(w) = e = \eta'(w)$. Now assume that $w = iw_0$ for some $i \in [\text{rk}(e)]$ and $w_0 \in \text{pos}(\eta_i) \cap \text{pos}(\eta'_i)$. First we consider the case that $\text{in}_i(e) \in U$. Then $\eta'_i = \text{in}_i(e)$ and, thus, $w_0 = \varepsilon$; hence, $|w| > 0$, $\text{out}(\eta(w)) = \text{out}(\eta(i)) = \text{in}_i(\eta(\varepsilon)) = \text{in}_i(e) \in U$, and $\text{out}(\eta(w)) = \text{in}_i(e) = \eta'_i = \eta'|_w$.

It remains to consider the case that $\text{in}_i(e) \notin U$. Then $\eta'_i = \text{dec}{\uparrow}(\eta_i, U)$. First assume that $\text{out}(\eta(w)) \in U$ and $|w| > 0$. If $|w| = 1$, then $w = i$ and $\text{in}_i(e) = \text{in}_i(\eta(\varepsilon)) = \text{out}(\eta(i)) = \text{out}(\eta(w)) \in U$, a contradiction. Hence, $|w| > 1$, i.e., $|w_0| > 0$. Then the induction hypothesis together with the facts that $\text{out}(\eta_i(w_0)) = \text{out}(\eta(w)) \in U$ and $|w_0| > 0$ yields $\text{out}(\eta(w)) = \text{out}(\eta_i(w_0)) = \eta'_i|_{w_0} = \eta'|_w$. Now assume that $\text{out}(\eta(w)) \notin U$ or $|w| \not> 0$. Thus, $\text{out}(\eta(w)) \notin U$ because $|w| = |iw_0| > 0$. Then the induction hypothesis together with the fact that $\text{out}(\eta_i(w_0)) = \text{out}(\eta(w)) \notin U$ implies $\eta(w) = \eta_i(w_0) = \eta'_i(w_0) = \eta'(w)$.

Next we prove Statement 2. Let $w \in (\mathbb{N}_+)^*$. If $w = \varepsilon$, then Statements (i) and (ii) are both true and therefore equivalent. Now assume that $w = iw_0$ for some $i \in [\text{rk}(e)]$ and $w_0 \in \text{pos}(\eta_i) \cap \text{pos}(\eta'_i)$.

First we consider the case that $\text{in}_i(e) \notin U$. Then $\eta'_i = \text{dec}{\uparrow}(\eta_i, U)$ and the induction hypothesis yields that the following two statements are equivalent: (i') $w_0 \in \text{pos}(\eta'_i)$ and (ii') $w_0 \in \text{pos}(\eta_i)$ and for every proper prefix w'_0 of w_0 with $w'_0 \neq \varepsilon$ we have $\text{out}(\eta_i(w'_0)) \notin U$. It is easy to check that the equivalence of Statements (i') and (ii'), and the fact that $\text{out}(\eta(i)) = \text{in}_i(\eta(\varepsilon)) = \text{in}_i(e) \notin U$ imply that Statements (i) and (ii) are equivalent.

Now consider the case that $\text{in}_i(e) \in U$. Then $\eta'_i = \text{in}_i(e)$. First we show that Statement (i) implies Statement (ii). Assume that Statement (i) holds. Then $w_0 \in \text{pos}(\eta'_i)$ and, therefore, $w_0 = \varepsilon$; thus, $w = i \in \text{pos}(\eta)$ and $\text{out}(\eta(w')) \notin U$ for every proper prefix w' of w with $w' \neq \varepsilon$. Thus, Statement (ii) holds as well. Next we show that Statement (ii) implies Statement (i). Assume that Statement (ii) holds. Since i is a prefix of w and $\text{out}(\eta(i)) = \text{in}_i(\eta(\varepsilon)) = \text{in}_i(e) \in U$, we obtain that $i = w$. Clearly, $w = i \in \text{pos}(\eta')$. Thus, Statement (i) holds as well.

Now we prove Statement 3. This is trivial if $w = \varepsilon$. For the remainder of the proof let us assume that $w = iw_0$ for some $i \in [k]$ and $w_0 \in \text{pos}(\eta_i)$. Since $\text{out}(\eta(w')) \notin U$ for every prefix w' of w with $w' \neq \varepsilon$, we obtain that (i) $\text{in}_i(e) = \text{in}_i(\eta(\varepsilon)) = \text{out}(\eta(i)) \notin U$ and that (ii) $\text{out}(\eta_i(w'_0)) = \text{out}(\eta_{iw'_0}) \notin U$ for every prefix w'_0 of w_0 with $w'_0 \neq \varepsilon$. Then Condition (i) asserts that $\eta'_i = \text{dec}{\uparrow}(\eta_i, U)$; moreover, Condition (ii) together with the induction hypothesis implies that $\eta'|_w = (\eta'|_i)|_{w_0} = (\eta'_i)|_{w_0} = (\text{dec}{\uparrow}(\eta_i, U))|_{w_0} = \text{dec}{\uparrow}((\eta_i)|_{w_0}, U) = \text{dec}{\uparrow}(\eta|_w, U)$. ■

Dependence Relation

Now we are going to define a relation on the set of vertices, the direct dependence relation, that turns out to be very useful for our purposes. Roughly speaking, we say that a vertex

v_2 directly depends on a vertex v_1 if there is a hyperpath such that v_1 is a direct predecessor of v_2 in this hyperpath.

We define the relation $\prec_G$ on V, called the ***direct dependence relation*** *of* G, as follows for every $v_1, v_2 \in V$: $v_1 \prec_G v_2$ iff there are $\eta \in \mathrm{H}_G^{v_2}$ and $i \in [\mathrm{rk}(\eta(\varepsilon))]$ such that $\eta|_i \in \mathrm{H}_G^{v_1}$.

Example 2.19 (*Continuation of Example 2.11*)**.** Consider the hyperpath η that is depicted in Figure 2.6(a). Clearly, $\eta \in \mathrm{H}_{G_{\mathrm{ex}}}^{v_1}$. By Lemma 2.15 we obtain $\eta|_1 \in \mathrm{H}_{G_{\mathrm{ex}}}^{v_1}$; hence, $v_1 \prec_{G_{\mathrm{ex}}} v_1$. By the same reasoning we have that $\eta|_{11} \in \mathrm{H}_{G_{\mathrm{ex}}}^{v_2}$, $\eta|_{111} \in \mathrm{H}_{G_{\mathrm{ex}}}^{v_2}$, and $\eta|_{112} \in \mathrm{H}_{G_{\mathrm{ex}}}^{v_1}$. Thus, we obtain that also $v_1 \prec_{G_{\mathrm{ex}}} v_2$, $v_2 \prec_{G_{\mathrm{ex}}} v_1$, and $v_2 \prec_{G_{\mathrm{ex}}} v_2$ hold.

Assume that we construct a hypergraph G' that originates from G_{ex} by removing e_4. Then we still have $v_1 \prec_{G'} v_2$, $v_2 \prec_{G'} v_1$, and $v_2 \prec_{G'} v_2$; however, $v_1 \prec_{G'} v_1$ does not hold anymore. □

Now we state two alternative characterizations of the direct dependence relation.

Lemma 2.20. *Let $v_1, v_2 \in V$. Then the following statements are equivalent.*

1. $v_1 \prec_G v_2$.
2. *There are $v \in V$, $\eta \in \mathrm{H}_G^v$, $w, w' \in \mathrm{pos}(\eta)$ such that w' is a prefix of w, $|w| - |w'| = 1$, $\mathrm{out}(\eta(w)) = v_1$, and $\mathrm{out}(\eta(w')) = v_2$.*
3. *There are $k \in \mathbb{N}$, $e \in E^{(k)}$, and $i \in [k]$ such that $\mathrm{out}(e) = v_2$, $\mathrm{in}_i(e) = v_1$, and $\mathrm{H}_G^{\mathrm{in}_j(e)} \neq \emptyset$ for every $j \in [k]$.*

Proof. "1 ⇒ 2": Since $v_1 \prec_G v_2$, there are $\eta \in \mathrm{H}_G^{v_2}$ and $i \in [\mathrm{rk}(\eta(\varepsilon))]$ such that $\eta|_i \in \mathrm{H}_G^{v_1}$. Let $v = v_2$, $w = i$, and $w' = \varepsilon$. Then $|w| - |w'| = 1$, $\mathrm{out}(\eta(w)) = \mathrm{out}(\eta|_i(\varepsilon)) = v_1$, and $\mathrm{out}(\eta(w')) = v_2$.

"2 ⇒ 3": Suppose that there are $v \in V$, $\eta \in \mathrm{H}_G^v$, $w, w' \in \mathrm{pos}(\eta)$ such that $|w| - |w'| = 1$, $\mathrm{out}(\eta(w)) = v_1$, and $\mathrm{out}(\eta(w')) = v_2$. Let $e = \eta(w')$ and $k = \mathrm{rk}(e)$. Then there is an $i \in [k]$ such that $w'i = w$. Since $\eta \in \mathrm{H}_G^v$, we have $\mathrm{in}_i(e) = \mathrm{in}_i(\eta(w')) = \mathrm{out}(\eta(w'i)) = \mathrm{out}(\eta(w)) = v_1$. It remains to show that $\mathrm{H}_G^{\mathrm{in}_j(e)} \neq \emptyset$ for every $j \in [k]$; this follows from the fact that $\eta|_{w'j} \in \mathrm{H}_G^{\mathrm{out}(\eta(w'j))} = \mathrm{H}_G^{\mathrm{in}_j(\eta(w'))} = \mathrm{H}_G^{\mathrm{in}_j(e)}$, which is implied by Lemma 2.15.

"3 ⇒ 1": For every $j \in [k]$ choose $\eta_j \in \mathrm{H}_G^{\mathrm{in}_j(e)}$. Then $\eta = e(\eta_1, \dots, \eta_k) \in \mathrm{H}_G^{v_2}$ by Corollary 2.16. Clearly, $\eta|_i = \eta_i \in \mathrm{H}_G^{\mathrm{in}_i(e)} = \mathrm{H}_G^{v_1}$. We obtain $v_1 \prec_G v_2$. ■

For every $v, v' \in V$ we have $v \prec_G v'$ iff there is a hyperpath η such that v' is the output vertex of the root hyperedge of η and v is the output vertex of some hyperedge that is the child of the root hyperedge. If we are given a sequence $v_0, \dots, v_n \in V$ with $v_0 \prec_G v_1 \prec_G \cdots \prec_G v_n$, then we can even assume that there is one hyperpath η such that v_n is the output vertex of the root of η, v_{n-1} is the output vertex of some child hyperedge of the root of η, v_{n-2} is the output vertex of some child hyperedge of this child hyperedge, and so on, i.e., the sequence $v_0, \dots, v_n$ can be embedded into one hyperpath η. Lemma 2.23 will make this explicit. Before we are going to present Lemma 2.23, let us state a similar but stronger property.

Lemma 2.21. *Let $n \in \mathbb{N}$, $v_0, \dots, v_n \in V$, $e_1, \dots, e_n \in E$, and $i_1, \dots, i_n \in \mathbb{N}_+$ such that for every $j \in [n]$ we have $\mathrm{out}(e_j) = v_j$, $i_j \in [\mathrm{rk}(e_j)]$, $\mathrm{in}_{i_j}(e_j) = v_{j-1}$, and $\mathrm{H}_G^{\mathrm{in}_l(e_j)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e_j)]$. Then*

1. *(i)* $n = 0$ *or (ii)* $\mathrm{H}_G^{v_0} \neq \emptyset$ *and for every* $\eta' \in \mathrm{H}_G^{v_0}$ *there is an* $\eta \in \mathrm{H}_G^{v_n}$ *with:*
   a) $w \in \mathrm{pos}(\eta)$,
   b) $\eta|_w = \eta'$, *and*
   c) $\eta(w') = e_{n-|w'|}$ *for every proper prefix* w' *of* w,

   where $w = i_n \cdots i_1$.

2. *Suppose that there is a* $j \in [n]$ *such that* $v_0 = v_j$. *Then for every* $m \in \mathbb{N}_+$ *there are* $\eta \in \mathrm{H}_G^{v_n}$ *and* $w \in \mathrm{pos}(\eta)$ *such that: (i)* $|w| = j(m-1)+n$ *and (ii) for every proper prefix* w' *of* w *we have* $\eta(w') = e_{f(|w|-|w'|)}$, *where the mapping* $f : [j(m-1)+n] \to [n]$ *is defined as follows for every* $l \in [j(m-1)+n]$:

$$f(l) = \begin{cases} ((l-1) \bmod j) + 1\ , & \text{if } l \leq j(m-1)\ , \\ l - j(m-1)\ , & \text{otherwise.} \end{cases}$$

PROOF. 1. We give a proof by induction on n.

Induction base. For $n = 0$ this implication holds trivially.

Induction step. Let $n \in \mathbb{N}$, $v_0, \ldots, v_{n+1} \in V$, $e_1, \ldots, e_{n+1} \in E$, and $i_1, \ldots, i_{n+1} \in \mathbb{N}_+$ such that for every $j \in [n+1]$ we have $\mathrm{out}(e_j) = v_j$, $i_j \in [\mathrm{rk}(e_j)]$, $\mathrm{in}_{i_j}(e_j) = v_{j-1}$, and $\mathrm{H}_G^{\mathrm{in}_l(e_j)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e_j)]$.

If $n \geq 1$, then the induction hypothesis yields that for every $\eta' \in \mathrm{H}_G^{v_0}$ there is an $\eta \in \mathrm{H}_G^{v_n}$ with such that $w \in \mathrm{pos}(\eta)$, $\eta|_w = \eta'$, and $\eta(w') = e_{n-|w'|}$ for every proper prefix w' of w, where $w = i_n \cdots i_1$.

Lemma 2.20(3 $\Rightarrow$ 1) implies that $v_{j-1} \prec_G v_j$ holds for every $j \in [n+1]$. In particular, we have $v_0 \prec_G v_1$. This implies that $\mathrm{H}_G^{v_0} \neq \emptyset$.

Now let $\eta' \in \mathrm{H}_G^{v_0}$. We show that there is an $\tilde{\eta} \in \mathrm{H}_G^{v_{n+1}}$ such that the following properties are satisfied: (a) $\tilde{w} \in \mathrm{pos}(\tilde{\eta})$, (b) $\tilde{\eta}|_{\tilde{w}} = \eta'$, and (c) $\tilde{\eta}(w') = e_{n+1-|w'|}$ for every proper prefix w' of $\tilde{w}$, where $\tilde{w} = i_{n+1} \cdots i_1$.

Let $w = i_n \cdots i_1$. First let us define the hypergraph $\eta \in \mathrm{H}_G^{v_n}$ satisfying the following properties: (a') $w \in \mathrm{pos}(\eta)$, (b') $\eta|_w = \eta'$, and (c') $\eta(w') = e_{n-|w'|}$ for every proper prefix w' of w. If $n = 0$, then we let $\eta = \eta'$; it is obvious that η satisfies Properties (a'), (b'), and (c'). If $n \geq 1$, then we can apply the induction hypothesis and obtain that there is such an η satisfying Properties (a'), (b'), and (c').

For every $l \in [\mathrm{rk}(e_{n+1})]$ choose an $\eta_l \in \mathrm{H}_G^{\mathrm{in}_l(e_{n+1})}$; such an η_l exists by assumption. Moreover, $v_n = \mathrm{in}_{i_{n+1}}(e_{n+1})$ and, hence, $\eta \in \mathrm{H}_G^{\mathrm{in}_{i_{n+1}}(e_{n+1})}$. We put

$$\tilde{\eta} = e_{n+1}\big(\eta_1, \ldots, \eta_{i_{n+1}-1}, \eta, \eta_{i_{n+1}+1}, \ldots, \eta_{\mathrm{rk}(e_{n+1})}\big)\ .$$

Then Corollary 2.16 yields that $\tilde{\eta} \in \mathrm{H}_G^{v_{n+1}}$ because $\mathrm{out}(e_{n+1}) = v_{n+1}$. It remains to prove that Properties (a), (b), and (c) hold. These properties follow immediately from the definition of $\tilde{\eta}$ and Properties (a'), (b'), and (c').

2. Let $m \in \mathbb{N}_+$. For every $l \in [j(m-1)+n]$ we define $v'_l \in V$, $e'_l \in E$, and $i'_l \in \mathbb{N}_+$ by letting $(v'_l, e'_l, i'_l) = (v_{f(l)}, e_{f(l)}, i_{f(l)})$. Let $v'_0 = v_0$. Clearly, for every $l \in [j(m-1)+n]$ we have that $\mathrm{out}(e'_l) = \mathrm{out}(e_{f(l)}) = v_{f(l)} = v'_l$, $i'_l = i_{f(l)} \in [\mathrm{rk}(e_{f(l)})] = [\mathrm{rk}(e'_l)]$, and $\mathrm{H}_G^{\mathrm{in}_k(e'_l)} = \mathrm{H}_G^{\mathrm{in}_k(e_{f(l)})} \neq \emptyset$ for every $k \in [\mathrm{rk}(e'_l)]$; moreover, it is easy to check that $v_{f(l)-1} = v_{f(l-1)}$ because $v_0 = v_j$; therefore, $\mathrm{in}_{i'_l}(e'_l) = \mathrm{in}_{i_{f(l)}}(e_{f(l)}) = v_{f(l)-1} = v_{f(l-1)} = v'_{l-1}$. Thus, we can apply Statement 1 to the sequences $v'_0, \ldots, v'_{n'}$, $e'_1, \ldots, e'_{n'}$, and $i'_1, \ldots, i'_{n'}$,

where $n' = j(m-1)+n$, and obtain that there are $\eta' \in \mathrm{H}_G^{v'_0}$ and $\eta \in \mathrm{H}_G^{v'_{n'}} = \mathrm{H}_G^{v_n}$ such that $w \in \mathrm{pos}(\eta)$ (where $w = i'_{n'} \cdots i'_1$) and for every proper prefix w' of w we have that $\eta(w') = e'_{n'-|w'|} = e'_{|w|-|w'|} = e_{f(|w|-|w'|)}$. ■

Corollary 2.22. *Let $n \in \mathbb{N}$ and $v_0, \ldots, v_n \in V$ such that $v_{j-1} \prec_G v_j$ for every $j \in [n]$. Then there are $w \in (\mathbb{N}_+)^n$ and $e_1, \ldots, e_n \in E$ such that $\mathrm{out}(e_j) = v_j$, for every $j \in [n]$, and for every $\eta' \in \mathrm{H}_G^{v_0}$ there is an $\eta \in \mathrm{H}_G^{v_n}$ with: (a) $w \in \mathrm{pos}(\eta)$, (b) $\eta|_w = \eta'$, and (c) $\eta(w') = e_{n-|w'|}$ for every proper prefix w' of w.*

Proof. If $n = 0$, then we put $w = \varepsilon$; the proof is trivial in this case. Now assume that $n \geq 1$. Lemma 2.20(1 ⇒ 3) yields that for every $j \in [n]$ there are $e_j \in E$ and $i_j \in [\mathrm{rk}(e_j)]$ such that $\mathrm{out}(e_j) = v_j$, $\mathrm{in}_{i_j}(e_j) = v_{j-1}$, and $\mathrm{H}_G^{\mathrm{in}_l(e_j)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e_j)]$. We put $w = i_n \cdots i_1$. Then the assertion follows immediately from Lemma 2.21(1). ■

Lemma 2.23. *Let $n \in \mathbb{N}_+$ and $v_0, \ldots, v_n \in V$. Then the following statements are equivalent.*

1. *$v_{j-1} \prec_G v_j$ for every $j \in [n]$.*
2. *There are $\eta \in \mathrm{H}_G^{v_n}$ and $w \in \mathrm{pos}(\eta)$ such that $|w| = n$ and $v_{n-|w'|} = \mathrm{out}(\eta(w'))$ for every prefix w' of w.*

Proof. “1 ⇒ 2”: Suppose that Statement 1 holds. Then Lemma 2.20(1 ⇒ 3) yields that for every $j \in [n]$ there are $e_j \in E$ and $i_j \in [\mathrm{rk}(e_j)]$ such that $\mathrm{out}(e_j) = v_j$, $\mathrm{in}_{i_j}(e_j) = v_{j-1}$, and $\mathrm{H}_G^{\mathrm{in}_l(e_j)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e_j)]$. Then Lemma 2.21(1) yields that there is an $\eta' \in \mathrm{H}_G^{v_0}$ and an $\eta \in \mathrm{H}_G^{v_n}$ such that $w \in \mathrm{pos}(\eta)$, $\eta|_w = \eta'$, and $\eta(w') = e_{n-|w'|}$ for every proper prefix w' of w, where $w = i_n \cdots i_1$. Clearly, $|w| = n$. Let w' be a prefix of w. If w' is a proper prefix of w, then $\mathrm{out}(\eta(w')) = \mathrm{out}(e_{n-|w'|}) = v_{n-|w'|}$. It remains to show that $\mathrm{out}(\eta(w')) = v_{n-|w'|}$ for the case that $w' = w$. This is an immediate consequence of the facts that $\eta(w) = \eta'(\varepsilon)$ and $\eta' \in \mathrm{H}_G^{v_0}$.

“2 ⇒ 1”: Let $\eta \in \mathrm{H}_G^{v_n}$ and $w \in \mathrm{pos}(\eta)$ such that $|w| = n$ and $v_{n-|w'|} = \mathrm{out}(\eta(w'))$ for every prefix w' of w. Let $j \in [n]$. We show that $v_{j-1} \prec_G v_j$. Let w_1 and w_2 be the unique prefixes of w of length $n-j$ and $n-j+1$, respectively. Since $\mathrm{out}(\eta(w_2)) = v_{n-|w'_2|} = v_{j-1}$ and, likewise, $\mathrm{out}(\eta(w_1)) = v_j$, Lemma 2.20(2 ⇒ 1) yields $v_{j-1} \prec_G v_j$. ■

Lemma 2.23 turns out to be useful for giving a measure of the number and size of hyperpaths depending on whether the transitive closure of the direct dependence relation is reflexive, i.e., whether the hypergraph contains loops of dependency. Roughly speaking, if it contains such a loop, then there are infinitely many hyperpaths and, thus, their heights are not bounded from above. The converse implication holds as well. For an example consider the hypergraph G_{ex} from Example 2.11. In Example 2.19 we have shown that both $v_1 \prec_{G_{\mathrm{ex}}} v_1$ and $v_2 \prec_{G_{\mathrm{ex}}} v_2$ hold; in fact, Table 2.1 demonstrates that both $\mathrm{H}_{G_{\mathrm{ex}}}^{v_1}$ and $\mathrm{H}_{G_{\mathrm{ex}}}^{v_2}$ are infinite.

These relationships are captured by the following three lemmas.

Lemma 2.24. *Assume that there is an $\eta \in \mathrm{H}_G^{v}$ such that $\mathrm{height}(\eta) \geq |V|$. Then there is a $u \in V$ such that $u \prec_G^+ u$ and $u \prec_G^* v$.*

Proof. There is a $w \in \mathrm{pos}(\eta)$ such that $|w| = |V|$. Let W be the set of prefixes of w. Clearly, $|W| = |w| + 1 > |V|$. Then there are $w_1, w_2 \in W$ such that w_1 is a proper prefix of w_2 and $\mathrm{out}(\eta(w_1)) = \mathrm{out}(\eta(w_2))$. Let $n = |w_2|$ and for every $i \in \{0, \ldots, n\}$ let $v_i = \mathrm{out}(\eta(w_i'))$ where w_i' is the unique prefix of w_2 of length $n - i$. Lemma 2.23($2 \Rightarrow 1$) yields $v_{i-1} \prec_G v_i$ for every $i \in [n]$. Hence, $v_0 \prec_G^+ v_{n-|w_1|} \prec_G^* v_n$ and, thus, $\mathrm{out}(\eta(w_2)) \prec_G^+ \mathrm{out}(\eta(w_1)) \prec_G^* \mathrm{out}(\eta(\varepsilon))$. The assertion follows from the facts that $\mathrm{out}(\eta(w_2)) = \mathrm{out}(\eta(w_1))$ and $\mathrm{out}(\eta(\varepsilon)) = v$. ■

Lemma 2.25. *Let $\lhd = \prec_G \setminus (U \times V)$ and $v \in V$. Then the following Statement 1 implies Statement 2.*

1. *There is a $u \in V$ such that $u \lhd^+ u$ and $u \lhd^* v$.*
2. *$\mathrm{H}_G^{v,U}$ is infinite.*

If $U = \emptyset$ (and, thus, $\lhd = \prec_G$), then Statements 1 and 2 are equivalent.

Proof. "$1 \Rightarrow 2$": Let $u \in V$ such that $u \lhd^+ u$ and $u \lhd^* v$ and assume that $\mathrm{H}_G^{v,U}$ is finite. We derive a contradiction. Let $m = \max\{\mathrm{height}(\eta) \mid \eta \in \mathrm{H}_G^{v,U}\} + 1$.

Clearly, there are $j, n \in \mathbb{N}_+$ and $v_0, \ldots, v_n \in V$ such that $j \in [n]$, $v_0 = v_j = u$, $v_n = v$, $v_0, \ldots, v_{\max(j,n-1)} \notin U$, and $v_{i-1} \prec_G v_i$ for every $i \in [n]$. By Lemma 2.20($1 \Rightarrow 3$) we obtain that for every $k \in [n]$ there are $e_k \in E$ and $i_k \in [\mathrm{rk}(e_k)]$ such that $\mathrm{out}(e_k) = v_k$, $\mathrm{in}_{i_k}(e_k) = v_{k-1}$, and $\mathrm{H}_G^{\mathrm{in}_l(e_k)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e_k)]$. Thus, we can apply Lemma 2.21(2) and obtain that there are $\eta \in \mathrm{H}_G^{v_n} = \mathrm{H}_G^{v}$ and $w \in \mathrm{pos}(\eta)$ such that $|w| = j(m-1) + n$ and for every proper prefix w' of w we have $\eta(w') = e_{f(|w|-|w'|)}$, where the mapping $f : [j(m-1)+n] \to [n]$ is as in Lemma 2.21(2).

Let $\eta' = \mathrm{dec}{\uparrow}(\eta, U)$. Then $\eta' \in \mathrm{H}_G^{v,U}$. It is easy to check that for every index $k \in \{1, \ldots, j(m-1)+n-1\}$ we have that $f(k) \in \{1, \ldots, \max(j, n-1)\}$, i.e., $v_{f(k)} \notin U$. Hence, Lemma 2.18(2) yields that $w \in \mathrm{pos}(\eta')$ because for every proper prefix w' of w with $w' \neq \varepsilon$ we have $\mathrm{out}(\eta(w')) = \mathrm{out}(e_{f(|w|-|w'|)}) = v_{f(|w|-|w'|)} \notin U$. Thus, $\mathrm{height}(\eta') \geq |w| = j(m-1) + n \geq m - 1 + 1 = m$ because $j, n \geq 1$; this contradicts the assumption that $m = \max\{\mathrm{height}(\eta) \mid \eta \in \mathrm{H}_G^{v,U}\} + 1$.

Now we assume that $U = \emptyset$ and show that Statement 2 implies Statement 1. Clearly, $\lhd = \prec_G$. Due to our definition of hypergraphs, V and E are finite. Thus, there is an $\eta \in \mathrm{H}_G^{v,U} = \mathrm{H}_G^{v}$ such that $\mathrm{height}(\eta) \geq |V|$. Then Statement 2 follows from Lemma 2.24. ■

Lemma 2.26. *Let $v \in V$, $\lhd = \prec_G \setminus (U \times V)$, and $V' = \{u \in V \mid u \lhd^* v\}$. Moreover, let $\sqsubset = \lhd \cap (V' \times V')$. If $\mathrm{H}_G^{v,U}$ is finite, then $\sqsubset^+$ is irreflexive.*

Proof. Let $\mathrm{H}_G^{v,U}$ be finite. Let $u \in V$ and assume, contrary to our claim, that $u \sqsubset^+ u$. Since $\sqsubset^+ = (\lhd \cap (V' \times V'))^+ \subseteq \lhd^+ \cap (V' \times V')$, we obtain that $u \lhd^+ u$ and $u \in V'$, i.e., $u \lhd^* v$. Then by Lemma 2.25 $\mathrm{H}_G^{v,U}$ is infinite, a contradiction. ■

Decomposition of hyperpaths

Now we will define the complement of the top decomposition of hyperpaths. When applied to a hyperpath, the result of this complementing operation, called bottom decomposition, is a sequence of hyperpaths, namely the sequence of hyperpaths that are cut off when performing the top decomposition. Let us consider the example from Figure 2.6. The hyperpath segment in Figure (b) is the result of the top decomposition applied to the hyperpath η in Figure (a). During this process we removed the hyperpaths $\eta_1 = e_3(e_5, e_1)$ and $\eta_2 = e_5$. Hence, the string $\eta_1\eta_2$ is the result of the bottom decomposition when applied to η. Now let us define this operation formally.

Definition 2.27. Let $v \in V$ and $\eta \in \mathrm{H}_G^v$. We define the ***bottom decomposition*** *of η wrt U and G*, denoted by $\mathrm{dec}{\downarrow}(\eta, U, G) \in (\bigcup_{u\in U} \mathrm{H}_G^u)^*$, by recursion on the structure of η as follows. By Corollary 2.16 there are uniquely determined $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1 \in \mathrm{H}_G^{\mathrm{in}_1(e)}, \ldots, \eta_k \in \mathrm{H}_G^{\mathrm{in}_k(e)}$ such that $\mathrm{out}(e) = v$ and $\eta = e(\eta_1, \ldots, \eta_k)$. We define

$$\mathrm{dec}{\downarrow}(\eta, U, G) = \bar{\eta}_1 \cdots \bar{\eta}_k \ ,$$

$$\text{where } \forall\, i \in [k] : \bar{\eta}_i = \begin{cases} \mathrm{dec}{\downarrow}(\eta_i, U, G)\ , & \text{if } \mathrm{in}_i(e) \notin U, \\ \eta_i\ , & \text{otherwise.} \end{cases} \qquad \square$$

By means of Lemma 2.14 it is easy to check that $\mathrm{dec}{\downarrow}(\eta, U, G)$ is well-defined. If G is clear from the context, then we write $\mathrm{dec}{\downarrow}(\eta, U)$ instead of $\mathrm{dec}{\downarrow}(\eta, U, G)$.

The combination of top decomposition and bottom decomposition is an inverse operation to tree substitution. More precisely, if η' is the result of the top decomposition of a given hyperpath η and $\hat{\eta}_1 \cdots \hat{\eta}_l$ is the string from the bottom decomposition applied to η, then $\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_k = \eta$. Roughly speaking, top and bottom decomposition followed by tree substitution is the identity mapping. It turns out that in some restricted sense the inverse holds as well, i.e., tree substitution followed by top and bottom decomposition is the identity mapping. These two properties are stated formally by the following two lemmas.

Lemma 2.28. *Let $v \in V$, and $\eta \in \mathrm{H}_G^v$. Let $l \in \mathbb{N}$ and $\hat{\eta}_1, \ldots, \hat{\eta}_l \in \bigcup_{u\in U} \mathrm{H}_G^u$ such that $\mathrm{dec}{\downarrow}(\eta, U) = \hat{\eta}_1 \cdots \hat{\eta}_l$. Moreover, let $j \in \mathbb{N}$ and $u_1, \ldots, u_j \in U$ such that $u_1 \cdots u_j = \mathrm{indyield}(\mathrm{dec}{\uparrow}(\eta, U))$.*

Then $j = l$ and for every $m \in [l]$ we have that $\hat{\eta}_m \in \mathrm{H}_G^{u_m}$ and that $\hat{\eta}_m$ is a proper subtree of η. Furthermore, $\mathrm{dec}{\uparrow}(\eta, U) \leftarrow \mathrm{dec}{\downarrow}(\eta, U) = \eta$.

Proof. Throughout this proof we abbreviate $\bigcup_{u\in U} \mathrm{H}_G^u$ by H. We give a proof by induction on the structure of η. By Corollary 2.16 there are uniquely determined $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1 \in \mathrm{H}_G^{\mathrm{in}_1(e)}, \ldots, \eta_k \in \mathrm{H}_G^{\mathrm{in}_k(e)}$ such that $\mathrm{out}(e) = v$ and $\eta = e(\eta_1, \ldots, \eta_k)$. For every $i \in [k]$ let

$$(\eta_i', \bar{\eta}_i) = \begin{cases} \big(\mathrm{dec}{\uparrow}(\eta_i, U), \mathrm{dec}{\downarrow}(\eta_i, U)\big)\ , & \text{if } \mathrm{in}_i(e) \notin U, \\ (\mathrm{in}_i(e), \eta_i)\ , & \text{otherwise.} \end{cases}$$

Then $\mathrm{dec}{\uparrow}(\eta, U) = e(\eta_1', \ldots, \eta_k')$ and $\mathrm{dec}{\downarrow}(\eta, U) = \bar{\eta}_1 \cdots \bar{\eta}_k = \hat{\eta}_1 \cdots \hat{\eta}_l$.

For every $i \in [k]$ let $l_i \in \mathbb{N}$ and $\hat{\eta}_1^i, \ldots, \hat{\eta}_{l_i}^i \in H$ such that $\bar{\eta}_i = \hat{\eta}_1^i \cdots \hat{\eta}_{l_i}^i$; moreover, let $j_i \in \mathbb{N}$ and $u_1^i, \ldots, u_{j_i}^i \in U$ such that $u_1^i \cdots u_{j_i}^i = \mathrm{indyield}(\eta_i')$. Before we proceed with the main proof, we show that for every $i \in [k]$ we have that $j_i = l_i$ and, for every $m \in [l_i]$, $\hat{\eta}_m^i \in \mathrm{H}_G^{u_m^i}$ and $\hat{\eta}_m^i$ is a subtree of η_i. Let $i \in [k]$. First we consider the case that $\mathrm{in}_i(e) \in U$. Then $(\eta_i', \bar{\eta}_i) = (\mathrm{in}_i(e), \eta_i)$; hence, $j_i = 1 = l_i$, $\hat{\eta}_1^i = \eta_i \in \mathrm{H}_G^{\mathrm{in}_i(e)} = \mathrm{H}_G^{\eta_i'} = \mathrm{H}_G^{u_1^i}$, and $\hat{\eta}_1^i$ is a subtree of $\eta_i = \hat{\eta}_1^i$. Next we consider the case that $\mathrm{in}_i(e) \notin U$. Then $(\eta_i', \bar{\eta}_i) = \big(\mathrm{dec}{\uparrow}(\eta_i, U), \mathrm{dec}{\downarrow}(\eta_i, U)\big)$ and the induction hypothesis yields that $j_i = l_i$ and, for every $m \in [l_i]$, $\hat{\eta}_m^i \in \mathrm{H}_G^{u_m^i}$ and $\hat{\eta}_m^i$ is a subtree of η_i.

Clearly, the following two identities hold: $\hat{\eta}_1 \cdots \hat{\eta}_l = \bar{\eta}_1 \cdots \bar{\eta}_k = \hat{\eta}^1_1 \cdots \hat{\eta}^1_{l_1} \cdots \hat{\eta}^k_1 \cdots \hat{\eta}^k_{l_k}$ and $u_1 \cdots u_j = \mathrm{indyield}(\mathrm{dec}{\uparrow}(\eta, U)) = \mathrm{indyield}(e(\eta'_1, \ldots, \eta'_k)) = \mathrm{indyield}(\eta'_1) \cdots \mathrm{indyield}(\eta'_k) = u^1_1 \cdots u^1_{j_1} \cdots u^k_1 \cdots u^k_{j_k}$. We obtain $j = j_1 + \cdots + j_k = l_1 + \cdots + l_k = l$. Moreover, for every $m \in [l]$ there is an $i \in [k]$ and $m' \in [l_i]$ such that $\hat{\eta}_m = \hat{\eta}^i_{m'}$ and $u_m = u^i_{m'}$; then $\hat{\eta}_m = \hat{\eta}^i_{m'} \in \mathrm{H}_G^{u^i_{m'}} = \mathrm{H}_G^{u_m}$ and $\hat{\eta}_m = \hat{\eta}^i_{m'}$ is a proper subtree of $\eta = e(\eta_1, \ldots, \eta_k)$ because it is a subtree of η_i.

It remains to show that $\mathrm{dec}{\uparrow}(\eta, U) \leftarrow \mathrm{dec}{\downarrow}(\eta, U) = \eta$. First we prove that for every $i \in [k]$, $\eta'_i \leftarrow \bar{\eta}_i = \eta_i$. If $\mathrm{in}_i(e) \in U$, then $\eta'_i \leftarrow \bar{\eta}_i = \mathrm{in}_i(e) \leftarrow \eta_i = \eta_i$. If $\mathrm{in}_i(e) \notin U$, then $(\eta'_i, \bar{\eta}_i) = \big(\mathrm{dec}{\uparrow}(\eta_i, U), \mathrm{dec}{\downarrow}(\eta_i, U)\big)$ and, thus, $\eta'_i \leftarrow \bar{\eta}_i = \eta_i$ follows from the induction hypothesis. We obtain

$$\begin{aligned}\mathrm{dec}{\uparrow}(\eta, U) \leftarrow \mathrm{dec}{\downarrow}(\eta, U) &= e(\eta'_1, \ldots, \eta'_k) \leftarrow \bar{\eta}_1 \cdots \bar{\eta}_k \\ &= e(\eta'_1, \ldots, \eta'_k) \leftarrow \hat{\eta}^1_1 \cdots \hat{\eta}^1_{l_1} \cdots \hat{\eta}^k_1 \cdots \hat{\eta}^k_{l_k} \\ &= e(\eta'_1 \leftarrow \hat{\eta}^1_1 \cdots \hat{\eta}^1_{l_1}, \ldots, \eta'_k \leftarrow \hat{\eta}^k_1 \cdots \hat{\eta}^k_{l_k}) \\ &= e(\eta'_1 \leftarrow \bar{\eta}_1, \ldots, \eta'_k \leftarrow \bar{\eta}_k) = e(\eta_1, \ldots, \eta_k) = \eta\ . \end{aligned}$$

■

Lemma 2.29. *Let* $v \in V$, $\eta' \in \mathrm{H}_G^{v,U}$, $l \in \mathbb{N}$, *and* $u_1, \ldots, u_l \in U$ *such that* $u_1 \cdots u_l = \mathrm{indyield}(\eta')$. *Moreover, for every* $i \in [l]$, *let* $\hat{\eta}_i \in \mathrm{H}_G^{u_i}$.

Then $\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l \in \mathrm{H}_G^v$, $\mathrm{dec}{\uparrow}(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U) = \eta'$, *and* $\mathrm{dec}{\downarrow}(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U) = \hat{\eta}_1 \cdots \hat{\eta}_l$.

Proof. We give a proof by structural induction on η'. By Lemma 2.14 there are $k \in \mathbb{N}$, $e \in E^{(k)}$, and $\eta_1, \ldots, \eta_k$ such that $\mathrm{out}(e) = v$, $\eta' = e(\eta_1, \ldots, \eta_k)$, and, for every $i \in [k]$, $\eta_i \in \mathrm{H}_G^{\mathrm{in}_i(e),U}$ if $\mathrm{in}_i(e) \notin U$ and $\eta_i = \mathrm{in}_i(e)$ otherwise. For every $i \in [k]$ let $j_i \in \mathbb{N}$ and $u^i_1, \ldots, u^i_{j_i}$ such that $u^i_1 \cdots u^i_{j_i} = \mathrm{indyield}(\eta_i)$. Clearly, $u^1_1 \cdots u^1_{j_1} \cdots u^k_1 \cdots u^k_{j_k} = u_1 \cdots u_l$. Observe that for every $i \in [k]$ and $m \in [j_i]$ there is an $\hat{\eta}^i_m \in \mathrm{H}_G^{u^i_m}$ such that $\hat{\eta}^1_1 \cdots \hat{\eta}^1_{j_1} \cdots \hat{\eta}^k_1 \cdots \hat{\eta}^k_{j_k} = \hat{\eta}_1 \cdots \hat{\eta}_l$.

First we show that $\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l \in \mathrm{H}_G^v$. To this end we prove for every $i \in [k]$ that $\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i} \in \mathrm{H}_G^{\mathrm{in}_i(e)}$. Let $i \in [k]$. If $\mathrm{in}_i(e) \in U$, then $\eta_i = \mathrm{in}_i(e)$ and we have $j_i = 1$ and $\mathrm{indyield}(\eta_i) = u^i_1 = \mathrm{in}_i(e)$; thus, $\eta_i \leftarrow \hat{\eta}^i_1 = \hat{\eta}^i_1 \in \mathrm{H}_G^{u^i_1} = \mathrm{H}_G^{\mathrm{in}_i(e)}$. If $\mathrm{in}_i(e) \notin U$, then $\eta_i \in \mathrm{H}_G^{\mathrm{in}_i(e),U}$ and we obtain $\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i} \in \mathrm{H}_G^{\mathrm{in}_i(e)}$ due to the induction hypothesis. Therefore Corollary 2.16 yields

$$\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l = e(\eta_1 \leftarrow \hat{\eta}^1_1 \cdots \hat{\eta}^1_{j_1}, \ldots, \eta_k \leftarrow \hat{\eta}^k_1 \cdots \hat{\eta}^k_{j_k}) \in \mathrm{H}_G^v\ .$$

Next we prove that $\mathrm{dec}{\uparrow}(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U) = \eta'$ and $\mathrm{dec}{\downarrow}(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U) = \hat{\eta}_1 \cdots \hat{\eta}_l$. For every $i \in [k]$ let

$$(\eta'_i, \bar{\eta}_i) = \begin{cases} \big(\mathrm{dec}{\uparrow}(\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i}, U), \mathrm{dec}{\downarrow}(\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i}, U)\big)\ , & \text{if } \mathrm{in}_i(e) \notin U, \\ (\mathrm{in}_i(e), \eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i})\ , & \text{otherwise.} \end{cases}$$

The fact $\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l = e(\eta_1 \leftarrow \hat{\eta}^1_1 \cdots \hat{\eta}^1_{j_1}, \ldots, \eta_k \leftarrow \hat{\eta}^k_1 \cdots \hat{\eta}^k_{j_k})$ and the definitions of $\mathrm{dec}{\uparrow}$ and $\mathrm{dec}{\downarrow}$ imply $\mathrm{dec}{\uparrow}\big(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U\big) = e(\eta'_1, \ldots, \eta'_k)$ and $\mathrm{dec}{\downarrow}\big(\eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l, U\big) = \bar{\eta}_1 \cdots \bar{\eta}_k$. Hence, it suffices to show that $(\eta'_i, \bar{\eta}_i) = (\eta_i, \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i})$ for every $i \in [k]$. Let $i \in [k]$. If $\mathrm{in}_i(e) \in U$, then $\mathrm{in}_i(e) = \eta_i$ and $j_i = 1$; thus, $(\eta'_i, \bar{\eta}_i) = (\mathrm{in}_i(e), \eta_i \leftarrow \hat{\eta}^i_1) = (\eta_i, \hat{\eta}^i_1)$. If $\mathrm{in}_i(e) \notin U$, then $(\eta'_i, \bar{\eta}_i) = \big(\mathrm{dec}{\uparrow}(\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i}, U), \mathrm{dec}{\downarrow}(\eta_i \leftarrow \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i}, U)\big) = (\eta_i, \hat{\eta}^i_1 \cdots \hat{\eta}^i_{j_i})$ due to the induction hypothesis. ■

Lemma 2.30. *Let* $v \in V$ *and* $\eta' \in \mathrm{H}_G^{v,U}$ *such that* $\mathrm{H}_G^u \neq \emptyset$ *for every* $u \in \mathrm{ind}(\eta')$*. Then there is an* $\eta \in \mathrm{H}_G^v$ *such that* $\mathrm{dec}\!\uparrow\!(\eta, U) = \eta'$*.*

PROOF. Let $l \in \mathbb{N}$ and $u_1, \dots, u_l \in U$ such that $u_1 \cdots u_l = \mathrm{indyield}(\eta')$. For every $i \in [l]$ choose an $\hat{\eta}_i \in \mathrm{H}_G^{u_i}$, which exists by assumption. Let $\eta = \eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l$. Then $\eta \in \mathrm{H}_G^v$ and $\mathrm{dec}\!\uparrow\!(\eta, U) = \eta'$ by Lemma 2.29. ∎

Reduction to directed graphs

Every hypergraph can be reduced to a digraph by constructing, for every hyperedge and pair of input vertex u and output vertex v of that hyperedge, an edge (u, v). Formally, the ***digraph reduct*** *of* G is the digraph (V, E') such that

$$E' = \{(u, v) \in V \times V \mid \exists\, e \in E : \exists\, i \in [\mathrm{rk}(e)] : \mathrm{in}_i(e) = u, \mathrm{out}(e) = v\} \, .$$

Lemma 2.31. *Let* (V, E') *be the digraph reduct of* G *and* $u, v \in V$*. Then* $u \prec_G v$ *implies* $(u, v) \in E'$*.*

PROOF. Assume that $u \prec_G v$. By Lemma 2.20(1 $\Rightarrow$ 3) there are $k \in \mathbb{N}$, $e \in E^{(k)}$, and $i \in [k]$ such that $\mathrm{out}(e) = v$ and $\mathrm{in}_i(e) = u$. Hence, $(u, v) \in E'$. ∎

M-monoids

In this chapter we present the definition of multioperator monoids (for short: m-monoids) and study their extension to complete and continuous m-monoids. An m-monoid is an algebraic structure consisting of a commutative monoid and arbitrary additional operations. M-monoids are a crucial component of the definition of the semantics of m-weighted monadic datalog programs, that we will introduce in the next chapter: the operations of a specific m-monoid are used for computing the output value of a given input tree.

Our definition of m-monoids is based on the definition of *distributive m-monoids* (or *distributive Ω-monoids*) that have been introduced by Kuich [95, 96, 98]. The concept of distributive m-monoids is a generalization of distributive F-magmas defined by Courcelle [33, Section 10] and of K–Γ-algebras by Bozapalidis [22]. M-monoids that are not necessarily distributive have been studied in [121, 123, 59].

The results in this chapter are a generalization of the work of Kuich [95] for (not necessarily distributive) m-monoids.

3.1 Semigroups and monoids

First let us recall the notions of semigroups and monoids [99, 100].

Semigroups

A semigroup is an algebraic structure that consists of an associative binary operation.

Definition 3.1. A ***semigroup*** is a pair $(A, +)$ such that A is a set and $+$ is an associative binary operation on A, i.e., $a + (b + c) = (a + b) + c$ for every $a, b, c \in A$.

Let $\mathcal{A} = (A, +)$ be a semigroup. An equivalence relation $\equiv$ on A is called ***semigroup congruence*** (*on* $\mathcal{A}$) if $a \equiv a'$ and $b \equiv b'$ implies $(a+b) \equiv (a'+b')$ for every $a, a', b, b' \in A$. Let $\equiv$ be a semigroup congruence on $\mathcal{A}$. The ***quotient semigroup*** *of* $\mathcal{A}$ *modulo* $\equiv$, denoted by $\mathcal{A}/{\equiv}$, is the pair $(A/{\equiv}, \circ)$ where $\circ \in \mathrm{Ops}^{(2)}(A/{\equiv})$ such that $[a]_\equiv \circ [b]_\equiv = [a+b]_\equiv$ for every $a, b \in A$. Note that the operation $\circ$ is well-defined.

Let $\mathcal{A} = (A, +_\mathcal{A})$ and $\mathcal{B} = (B, +_\mathcal{B})$ be semigroups. The ***direct product*** *of* $\mathcal{A}$ *and* $\mathcal{B}$ is the pair $\mathcal{A} \times \mathcal{B} = (A \times B, \circ)$ with $\circ \in \mathrm{Ops}^{(2)}(A \times B)$ such that $(a, b) \circ (a', b') = (a +_\mathcal{A} a', b +_\mathcal{B} b')$ for every $a, a' \in A$ and $b, b' \in B$. □

The following lemma is a folklore result (cf. [72, Section 2.1.2 and 3.1.3]).

Lemma 3.2. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be semigroups and let* $\equiv$ *be a semigroup congruence on* $\mathcal{A}$. *Then* $\mathcal{A} \times \mathcal{B}$ *and* $\mathcal{A}/{\equiv}$ *are semigroups.*

Monoids

A monoid is a semigroup having a neutral element.

Definition 3.3. A ***monoid*** is a tuple $(A, +, \mathbf{0})$ such that $(A, +)$ is a semigroup and $\mathbf{0}$ is neutral wrt $+$. The monoid $(A, +, \mathbf{0})$ is called ***zero-sum free*** if $a + b = \mathbf{0}$ implies $a = b = \mathbf{0}$ for every $a, b \in A$. Moreover, $(A, +, \mathbf{0})$ is ***commutative*** if $a + b = b + a$ for every $a, b \in A$. If $(A, +, \mathbf{0})$ is commutative, we extend the operation $+$ to arbitrary finite families over A as follows. Let $I = \{i_1, \dots, i_n\}$ be a finite index set and $(a_i \mid i \in I)$ be a family over A. Then we put $\sum_{i \in I} a_i = a_{i_1} + \dots + a_{i_n}$; in particular $\sum_{i \in \emptyset} a_i = \mathbf{0}$. □

Remark 3.4. Every semigroup can be extended to a monoid as follows. Let $(A, +)$ be a semigroup and $\mathbf{0}$ be an element that is not in A. Then $(A \cup \{\mathbf{0}\}, \circ, \mathbf{0})$ is a monoid, where $\circ|_{A \times A} = +$ and $\mathbf{0} \circ a = a = a \circ \mathbf{0}$ for every $a \in A \cup \{\mathbf{0}\}$. □

Remark 3.5. Every monoid $(A, +, \mathbf{0})$ can be considered as a particular Σ_{mon}-algebra (recall the definition of Σ_{mon} from Example 2.1), where e is interpreted as $\mathbf{0}$ and $\circ$ is interpreted as the operation $+$. This allows to carry over notions defined for general algebras to monoids. For every set B the monoid $(B^*, \cdot, \varepsilon)$, where $\cdot$ is the concatenation of strings, is freely generated by B for the class of all monoids. □

Example 3.6 (*Continuation of Example 2.2*). Since the addition $+$ of natural numbers is an associative and commutative operation and 0 is neutral wrt $+$, we can consider the Σ_{mon}-algebra $\mathcal{N} = (\mathbb{N}, \theta)$ as the monoid $(\mathbb{N}, +, 0)$. □

Tree series

A (formal) tree series [97, 51] is a mapping from the set of trees over a given alphabet into a commutative monoid.

Definition 3.7. Let $\mathcal{A} = (A, +, \mathbf{0})$ be a commutative monoid, Δ be a signature, and D be a set. A ***tree series*** (*over* Δ, D, *and* $\mathcal{A}$) is a mapping $\lambda : T_\Delta(D) \to A$; if $D = \emptyset$, then we call λ a ***tree series*** (*over* Δ *and* $\mathcal{A}$). We denote the set of all tree series over Δ, D, and $\mathcal{A}$ by $\mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$. Let $\lambda \in \mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$. For every $t \in T_\Delta(D)$ we call $\lambda(t)$ the ***coefficient*** *of* t *under* λ.[1] The ***support*** *of* λ is defined as $\mathrm{supp}(\lambda) = \lambda^{-1}(A \setminus \{\mathbf{0}\})$. If $\mathrm{supp}(\lambda)$ is finite, then λ is called a ***polynomial tree series***. The set of polynomial tree series over Δ, D, and $\mathcal{A}$ is denoted by $\mathcal{A}\langle T_\Delta(D)\rangle$. The tree series $\lambda \in \mathcal{A}\langle T_\Delta(D)\rangle$ with $\mathrm{supp}(\lambda) = \emptyset$ is the ***empty tree series*** and denoted by $\tilde{\mathbf{0}}$.

For two tree series $\lambda_1, \lambda_2 \in \mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ we define the ***sum*** $\lambda_1 + \lambda_2 \in \mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ *of* λ_1 *and* λ_2 pointwise, i.e., for every $t \in T_\Delta(D)$, $(\lambda_1 + \lambda_2)(t) = \lambda_1(t) + \lambda_2(t)$. Let I be an index set. We call a family $(\lambda_i \mid i \in I)$ over $\mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ ***locally finite*** if for every $t \in T_\Delta(D)$ there are only finitely many $i \in I$ with $t \in \mathrm{supp}(\lambda_i)$. We extend the sum of tree series to locally finite families as follows. If the family $(\lambda_i \mid i \in I)$ over $\mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ is locally finite, then we denote by $\sum_{i \in I} \lambda_i \in \mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ the tree series such that for every $t \in T_\Delta(D)$:

$$\Big(\sum_{i \in I} \lambda_i\Big)(t) = \sum_{\substack{i \in I \\ t \in \mathrm{supp}(\lambda_i)}} \lambda_i(t) \ .$$

[1] Note that some authors, e.g., [51], denote the coefficient of t under λ by (λ, t) instead of $\lambda(t)$. However, we will use the standard notation $\lambda(t)$ for the application of mappings in this thesis.

Let $a \in A$ and $t \in T_\Delta(D)$. The ***monomial tree series*** *over* a *and* t is the tree series $a.t \in \mathcal{A}\langle T_\Delta(D)\rangle$ such that $(a.t)(t) = a$ and $(a.t)(t') = \mathbf{0}$ for every $t' \in T_\Delta(D) \setminus \{t\}$. Note that for every $\lambda \in \mathcal{A}\langle\langle T_\Delta(D)\rangle\rangle$ we have $\sum_{t \in T_\Delta(D)} \lambda(t).t = \lambda$. □

For more information on tree series we refer the reader to [51, Section 2.5].

3.2 General m-monoids

Now we are prepared to define the main algebraic structure of this thesis. Our definition is based on the definition of m-monoids in [95], however, we do not require the additional operations of the m-monoid to distribute over the monoid. Moreover, unlike the definitions of m-monoids in the literature [95, 58, 103, 123], we do not allow the set of additional operations of the m-monoid to be chosen arbitrarily; instead we require them to form a Δ-algebra for a given signature Δ.

Definition 3.8. Let Δ be a signature. A ***multioperator monoid*** (for short: ***m-monoid***) *over* Δ is a tuple $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ such that $(A, +, \mathbf{0})$ is a commutative monoid and (A, θ) is a Δ-algebra.

We say that $\mathcal{A}$ is ***idempotent*** if $+$ is idempotent, i.e., if for every $a \in A$, $a + a = a$. The m-monoid $\mathcal{A}$ is called ***absorptive*** if for every $\delta \in \Delta$ the element $\mathbf{0}$ is absorbing wrt $\theta(\delta)$. For every $\delta \in \Delta$ we say that $\theta(\delta)$ is ***supportive*** if $\mathrm{ran}(\theta(\delta)) \neq \{\mathbf{0}\}$. □

> *In the sequel we fix an arbitrary signature* Δ *and an m-monoid* $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ *over* Δ*.*

Since every m-monoid $\mathcal{A}$ over Δ contains a Δ-algebra, we can carry over the concept of Δ-homomorphisms to $\mathcal{A}$. One useful application of Δ-homomorphisms is to evaluate a tree over Δ in the Δ-algebra of $\mathcal{A}$. The next two lemmas capture basic properties of such evaluation homomorphisms. The first lemma states how certain elements and sets of elements of A propagate when applying the homomorphism. The second lemma states how relationships between elements of A are preserved during the application of the evaluation homomorphism.

Lemma 3.9. *Let* D *be a set,* $h : T_\Delta(D) \to A$ *be a* Δ*-homomorphism from* $\mathcal{T}_\Delta(D)$ *to* (A, θ)*, and* $s \in T_\Delta(D)$*.*

1. *Let* $\mathcal{A}$ *be absorptive. Then* $\mathbf{0} \in h(\mathrm{ind}(s))$ *implies* $h(s) = \mathbf{0}$*.*
2. *Let* $A' \subseteq A$ *such that* A' *is closed under* $\theta(\delta)$ *for every* $\delta \in \Delta$*. Then* $h(\mathrm{ind}(s)) \subseteq A'$ *implies* $h(s) \in A'$*.*

Proof. 1. We prove this statement by structural induction on s. Assume that $\mathbf{0} \in h(\mathrm{ind}(s))$.

Induction base. Let $s \in D$. Since $\mathrm{ind}(s) = \{s\}$, $h(s) = \mathbf{0}$.

Induction step. Let $s = \delta(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, and trees $s_1, \ldots, s_k \in T_\Delta(D)$. There is an $i \in [k]$ such that $\mathbf{0} \in h(\mathrm{ind}(s_i))$. Then

$$h(s) = \theta(\delta)(h(s_1), \ldots, h(s_{i-1}), \mathbf{0}, h(s_{i+1}), \ldots, h(s_k)) = \mathbf{0}$$

by the induction hypothesis and the fact that $\mathcal{A}$ is absorptive.

2. We prove this statement by structural induction on s. Let $A' \subseteq A$ such that A' is closed under $\theta(\delta)$ for every $\delta \in \Delta$, and assume that $h(\text{ind}(s)) \subseteq A'$.

Induction base. Let $s \in D$. Since $\text{ind}(s) = \{s\}$, $h(s) \in A'$.

Induction step. Let $s = \delta(s_1, \dots, s_k)$ for some $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, and trees $s_1, \dots, s_k \in T_\Delta(D)$. By the induction hypothesis and since A is closed under $\theta(\delta)$, we have $h(s) = \theta(\delta)(h(s_1), \dots, h(s_k)) \in A'$. ■

Lemma 3.10. *Let D be a set, $h, h' : T_\Delta(D) \to A$ be Δ-homomorphisms from $\mathcal{T}_\Delta(D)$ to (A, θ), and τ be a relation on A such that $\theta(\delta)$ is monotone wrt τ for every $\delta \in \Delta$. Moreover, let $s \in T_\Sigma(D)$ such that $h(d)\ \tau\ h'(d)$ for every $d \in \text{ind}(s)$. Then $h(s)\ \tau\ h'(s)$.*

Proof. We give a proof by structural induction on s. Let $s \in T_\Delta(D)$ such that $h(d)\ \tau\ h'(d)$ for every $d \in \text{ind}(s)$.

Induction base. Let $s \in D$. Since $\text{ind}(s) = \{s\}$, $h(s)\ \tau\ h'(s)$.

Induction step. Let $s = \delta(s_1, \dots, s_k)$ for some $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, and trees $s_1, \dots, s_k \in T_\Delta(D)$. By the induction hypothesis $h(s_i)\ \tau\ h'(s_i)$ for every $i \in [k]$. Then $h(s) = \theta(\delta)(h(s_1), \dots, h(s_k))\ \tau\ \theta(\delta)(h'(s_1), \dots, h'(s_k)) = h'(s)$ because $\theta(\delta)$ is monotone wrt τ. ■

Corollary 3.11. *Let D be a set, $h, h' : T_\Delta(D) \to A$ be Δ-homomorphisms from $\mathcal{T}_\Delta(D)$ to (A, θ), and $s \in T_\Sigma(D)$ such that $h|_{\text{ind}(s)} = h'|_{\text{ind}(s)}$. Then $h(s) = h'(s)$.*

Remark 3.12 (see [59, Section 2.3]). Every M-monoid $\mathcal{A}$ can easily be extended to an absorptive one. The ***absorptive extension*** *of* $\mathcal{A}$ is defined to be the m-monoid $\mathcal{A}_\perp = (A \cup \{\perp\}, +', \perp, \theta')$, where $+'$ is the extension of $+$ to the set $A \cup \{\perp\}$ defined by $a +' \perp = \perp +' a = a$ for every $a \in A \cup \{\perp\}$, and for every $\delta \in \Delta$ the operation $\theta'(\delta)$ is the extension of $\theta(\delta)$ to $A \cup \{\perp\}$ such that $\theta'(\delta)(\dots, \perp, \dots) = \perp$. Obviously, $\mathcal{A}_\perp$ is absorptive. □

Now we define distributive m-monoids [95, 103, 58].

Definition 3.13. An absorptive m-monoid is called ***distributive*** if for every $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, $a, b, a_1, \dots, a_k \in A$, and $i \in [k]$ we have

$$\begin{aligned} &\theta(\delta)(a_1, \dots, a_{i-1}, a + b, a_{i+1}, \dots, a_k) \\ &\quad = \theta(\delta)(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_k) + \theta(\delta)(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_k)\ . \end{aligned} \tag{3.1}$$

A distributive m-monoid is also called a ***dm-monoid***. □

Before we finish this section, let us state another technical lemma that combines evaluation homomorphisms, tree substitutions, and distributive m-monoids. Let us give a brief explanation of this lemma. Every tree s over Δ having k indices can be considered as a k-ary operation over A in the following sense: for every sequence $a_1, \dots, a_k \in A$ we apply the evaluation homomorphism of $\mathcal{A}$ to s, where we "plug in" the elements $a_1, \dots, a_k \in A$ at the indices of s; this yields a single element $a \in A$. The main statement of the following lemma is that such an operation is distributive if $\mathcal{A}$ is distributive.

Lemma 3.14. *Let D be a set and h be a Δ-homomorphism from $\mathcal{T}_\Delta(D)$ to (A, θ). Let $s \in T_\Delta(D)$, $k \in \mathbb{N}$, and $d_1, \dots, d_k \in D$ such that* $\text{indyield}(s) = d_1 \cdots d_k$.

1. *For every $d \in \mathrm{ind}(s)$ let I_d be a finite set and $(s_i^d \mid i \in I_d)$ be a family over $T_\Delta(D)$. If $\mathcal{A}$ is distributive or $|I_d| = 1$ for every $d \in \mathrm{ind}(s)$, then*

$$\sum\nolimits_{(i_1,\ldots,i_k)\in I_{d_1}\times\cdots\times I_{d_k}} h(s \leftarrow s_{i_1}^{d_1}\cdots s_{i_k}^{d_k}) = h'(s) \ ,$$

where h' is the unique Δ-homomorphism from $\mathcal{T}_\Delta(\mathrm{ind}(s))$ to (A,θ) such that $h'(d) = \sum_{i\in I_d} h(s_i^d)$ for every $d \in \mathrm{ind}(s)$.

2. *For every $g : \mathrm{ind}(s) \to T_\Delta(D)$ we obtain $h(s \leftarrow g(d_1)\cdots g(d_k)) = h'(s)$, where h' is the unique Δ-homomorphism from $\mathcal{T}_\Delta(\mathrm{ind}(s))$ to (A,θ) extending $g\,;h$.*

Proof. 1. We give a proof by structural induction on s.

Induction base. Let $s \in D$. Then $k = 1$, $s = d_1$, and $\sum_{i_1\in I_{d_1}} h(s \leftarrow s_{i_1}^{d_1}) = \sum_{i_1\in I_{d_1}} h(s_{i_1}^{d_1}) = h'(d_1) = h'(s)$.

Induction step. Let $l \in \mathbb{N}$, $\delta \in \Delta^{(l)}$, and $s_1,\ldots,s_l \in T_\Delta(D)$ such that $s = \sigma(s_1,\ldots,s_l)$. For every $j \in [l]$ let $\mathrm{indyield}(s_j) = d_1^j \cdots d_{k_j}^j$ for some $k_j \in \mathbb{N}$ and $d_1^j,\ldots,d_{k_j}^j \in D$; moreover, we introduce the following abbreviations (where m stands for k_j): we let $I^j = I_{d_1^j} \times \cdots \times I_{d_m^j}$ and for every $\vec{i} = i_1\cdots i_m \in I^j$ we abbreviate $s_{i_1}^{d_1^j}\cdots s_{i_m}^{d_m^j}$ by $\vec{s}_{\vec{i}}^{\,j}$. Clearly, $d_1\cdots d_k = d_1^1\cdots d_{k_1}^1\cdots d_1^l\cdots d_{k_l}^l$ and, therefore, $I_{d_1}\times\cdots\times I_{d_k} = I^1\times\cdots\times I^l$. Then

$$\begin{aligned}
&\sum\nolimits_{(i_1,\ldots,i_k)\in I_{d_1}\times\cdots\times I_{d_k}} h(s \leftarrow s_{i_1}^{d_1}\cdots s_{i_k}^{d_k})\\
&= \sum\nolimits_{(\vec{i}_1,\ldots,\vec{i}_l)\in I^1\times\cdots\times I^l} h(s \leftarrow \vec{s}_{\vec{i}_1}^{\,1}\cdots \vec{s}_{\vec{i}_l}^{\,l})\\
&= \sum\nolimits_{(\vec{i}_1,\ldots,\vec{i}_l)\in I^1\times\cdots\times I^l} h\big(\delta(s_1 \leftarrow \vec{s}_{\vec{i}_1}^{\,1},\ldots,s_l \leftarrow \vec{s}_{\vec{i}_l}^{\,l})\big)\\
&\qquad\qquad (\text{because } |\vec{s}_{\vec{i}_j}^{\,j}| = k_j = |\mathrm{indyield}(s_j)| \text{ for every } j\in[l] \text{ and } i_j \in I^j)\\
&= \sum\nolimits_{(\vec{i}_1,\ldots,\vec{i}_l)\in I^1\times\cdots\times I^l} \theta(\delta)\big(h(s_1 \leftarrow \vec{s}_{\vec{i}_1}^{\,1}),\ldots,h(s_l \leftarrow \vec{s}_{\vec{i}_l}^{\,l})\big)\\
&= \theta(\delta)\big(\sum\nolimits_{\vec{i}_1\in I^1} h(s_1 \leftarrow \vec{s}_{\vec{i}_1}^{\,1}),\ldots,\sum\nolimits_{\vec{i}_l\in I^l} h(s_l \leftarrow \vec{s}_{\vec{i}_l}^{\,l})\big)\\
&\qquad\qquad (\text{since } \mathcal{A} \text{ is distributive or } |I_d| = 1 \text{ for every } d \in \mathrm{ind}(s))\\
&= \theta(\delta)(h_1'(s_1),\ldots,h_l'(s_l)) \ , \qquad\qquad (\text{by the induction hypothesis})
\end{aligned}$$

where, for every $j \in [l]$, h_j' is the unique Δ-homomorphism from $\mathcal{T}_\Delta(\mathrm{ind}(s_j))$ to (A,θ) such that $h_j'(d) = \sum_{i\in I_d} h(s_i^d)$ for every $d \in \mathrm{ind}(s_j)$; clearly $h_j'(s_j) = h'(s_j)$; hence,

$$\begin{aligned}
&= \theta(\delta)(h'(s_1),\ldots,h'(s_l))\\
&= h'(\delta(s_1,\ldots,s_l)) = h'(s) \ .
\end{aligned}$$

2. This statement follows from Statement 1 by instantiating $I_d = \{1\}$ and $s_1^d = g(d)$ for every $d \in \mathrm{ind}(s)$. ∎

3.3 M-monoids with infinite behavior

In the previous section we have shown that m-monoids are useful for evaluating a finite set of trees and adding up their resulting values. In the next chapter we will employ m-monoids for computing the semantics of m-weighted monadic datalog programs. It will turn out that in general the power of m-monoids is too weak for this task. Roughly speaking, we instead require m-monoids to be able to (i) sum up over infinitely many values or (ii) evaluate infinite trees.

In this section we will study how to extend m-monoids with the required capabilities. We will call an m-monoid that has Capability (i) a complete m-monoid and an m-monoid that has Capability (ii) a continuous monoid. Note that not every m-monoid can be extended in this manner. Both complete and continuous m-monoids have been introduced by Kuich [95].

For our purposes it suffices to restrict ourselves to countably infinite behavior of both complete and continuous m-monoids. Therefore, we will refer to them as ω-complete and ω-continuous m-monoids in this thesis.

3.3.1 Complete m-monoids

An ω-complete m-monoid is an m-monoid with an additional ω-infinitary operation that extends the monoid operation $+$ to countably infinite families.

Definition 3.15. An ***ω-infinitary sum operation*** *for $\mathcal{A}$* is an ω-infinitary operation $\sum'$ on A such that for every family $(b_i \mid i \in \{j,k\})$ over A:

$$\sum\nolimits'_{i\in\{j,k\}} b_i = b_j + b_k \; . \tag{3.2}$$

Let $\sum'$ be an ω-infinitary sum operation for $\mathcal{A}$. Then we call $(\mathcal{A}, \sum')$ an ***ω-complete m-monoid***. □

We will give examples of ω-complete m-monoids in Example 3.24. In Definition 3.15 we took great care not to use the symbol $\sum$ for ω-infinitary sum operations in order to not confuse infinitary sum operations with the extension of the operation $+$ to finite families (see Definition 3.3). However, this distinction is unnecessary according to the following well-known proposition (cf. [72, Lemma IV.1.17]).

Proposition 3.16. *Let $\sum'$ be an ω-infinitary sum operation for $\mathcal{A}$. Then for every finite set I and family $(a_i \mid i \in I)$ over A:*

$$\sum\nolimits'_{i\in I} a_i = \sum\nolimits_{i\in I} a_i \; .$$

Due to Proposition 3.16 we will henceforth allow to use the symbol $\sum$ for ω-infinitary sum operations. The following lemma states basic properties of ω-infinitary sum operations.

Lemma 3.17. *Let $(\mathcal{A}, \sum)$ be an ω-complete m-monoid, I be a countable set, and let $(a_i \mid i \in I)$ be a family over A.*

1. *If $a_i = \mathbf{0}$ for every $i \in I$, then $\sum_{i\in I} a_i = \mathbf{0}$. In particular, $\sum_{i\in\emptyset} a_i = \mathbf{0}$.*
2. *Let $a \in A$ such that $\{i \in I \mid a_i = a\}$ is infinite. Then $a + \sum_{i\in I} a_i = \sum_{i\in I} a_i$.*

Proof. 1. First assume that $I = \emptyset$. Let $a_1 = \mathbf{0}$, $I' = \{1\}$, $J = \{1,2\}$, $I_1 = \{1\}$, $I_2 = \emptyset$. Then $(I_j \mid j \in J)$ is a generalized partition of I'. Thus, $\sum_{i\in I} a_i = \sum_{i\in I_2} a_i = \mathbf{0} + \sum_{i\in I_2} a_i = \sum_{i\in I_1} a_i + \sum_{i\in I_2} a_i = \sum_{j\in J}\sum_{i\in I_j} a_i = \sum_{i\in I'} a_i = \mathbf{0}$ by Equations (2.4), (2.5), and (3.2). Now let I be arbitrary and assume $a_i = \mathbf{0}$ for every $i \in I$. Let $K = \emptyset$ and for every $i \in I$ let $K_i = \emptyset$. Then $(K_i \mid i \in I)$ is a generalized partition of K and we obtain $\sum_{i\in I} a_i = \sum_{i\in I} \mathbf{0} = \sum_{i\in I} \sum_{k\in K_i} a_k = \sum_{k\in K} a_k = \mathbf{0}$.

2. Choose $k \in I$ such that $a_k = a$. Let $J = \{1, 2\}$, $I_1 = \{k\}$, $I_2 = I \setminus \{k\}$. Then $\sum_{i\in I} a_i = \sum_{j\in J} \sum_{i\in I_j} a_i = \sum_{i\in I_1} a_i + \sum_{i\in I_2} a_i = a + \sum_{i\in I_2} a_i$ by Equations (2.4), (2.5), and (3.2). Clearly, there is a bijection $\pi : I \to I_2$ such that $a_i = a_{\pi(i)}$, for every $i \in I$, because $\{i \in I \mid a_i = a\}$ is infinite. Then $a + \sum_{i\in I_2} a_i = a + \sum_{i\in I} a_{\pi(i)} = a + \sum_{i\in I} a_i$ by Equation (2.3). ■

If an m-monoid is distributive, then this property does not need to carry over to ω-infinitary sum operations. However, if it does, then we call the m-monoid ω-distributive.

Definition 3.18. We call an ω-complete m-monoid $(\mathcal{A}, \sum)$ ***ω-distributive*** if for every $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, $a_1, \ldots, a_k \in A$, $j \in [k]$, countable index set I, and family $(b_i \mid i \in I)$ over A:

$$\begin{aligned} &\theta(\delta)\big(a_1, \ldots, a_{j-1}, \sum\nolimits_{i\in I} b_i, a_{j+1}, \ldots, a_k\big) \\ &= \sum\nolimits_{i\in I} \theta(\delta)(a_1, \ldots, a_{j-1}, b_i, a_{j+1}, \ldots, a_k) \, . \end{aligned} \tag{3.3}$$

□

If an ω-complete m-monoid is ω-distributive, then its underlying m-monoid is distributive.

Lemma 3.19. *Let $\sum$ be an ω-infinitary sum operation for $\mathcal{A}$ such that $(\mathcal{A}, \sum)$ is ω-distributive. Then $\mathcal{A}$ is a dm-monoid.*

Proof. First we show that $\mathcal{A}$ is absorptive. Let $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, $a_1, \ldots, a_k \in A$, and $i \in [k]$ with $a_i = \mathbf{0}$. Consider the family $(b_j \mid j \in \emptyset)$. Then $\theta(\delta)(a_1, \ldots, a_k) = \theta(\delta)(a_1, \ldots, a_{i-1}, \sum_{j\in\emptyset} b_j, a_{i+1}, a_k) = \sum_{j\in\emptyset} \theta(\delta)(a_1, \ldots, a_{i-1}, b_j, a_{i+1}, a_k) = \mathbf{0}$.

Next we prove that Equation (3.1) holds. Let $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, $a, b, a_1, \ldots, a_k \in A$, and $i \in [k]$. Consider the family $(b_j \mid j \in \{k, l\})$ with $b_k = a$ and $b_l = b$. Then

$$\begin{aligned} &\theta(\delta)(a_1, \ldots, a_{i-1}, a + b, a_{i+1}, \ldots, a_k) \\ &= \theta(\delta)\big(a_1, \ldots, a_{i-1}, \sum\nolimits_{j\in\{k,l\}} b_j, a_{i+1}, \ldots, a_k\big) \\ &= \sum\nolimits_{j\in\{k,l\}} \theta(\delta)(a_1, \ldots, a_{i-1}, b_j, a_{i+1}, \ldots, a_k) \\ &= \theta(\delta)(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k) + \theta(\delta)(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \, . \end{aligned}$$

■

Similarly to distributivity, the property of the m-monoid to be idempotent does not need to carry over to ω-infinitary sum operations. If it does, then we call the m-monoid ω-idempotent.

Definition 3.20. Let $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an m-monoid over Δ and let $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. We say that $(\mathcal{A}, \sum)$ is ***ω-idempotent*** if for every nonempty countable index set I and $a \in A$ we have $\sum_{i\in I} a = a$. Note that $\mathcal{A}$ is idempotent whenever $(\mathcal{A}, \sum)$ is ω-idempotent. □

The following observation is easy to prove.

Observation 3.21. *Let $(\mathcal{A}, \sum)$ be an ω-complete m-monoid that is ω-idempotent. Moreover, let I and K be countable index sets and let $(a_i \mid i \in I)$ and $(b_k \mid k \in K)$ be families over A, where A is the carrier set of $\mathcal{A}$. Suppose that $\{a_i \mid i \in I\} = \{b_k \mid k \in K\}$. Then $\sum_{i\in I} a_i = \sum_{k\in K} b_k$.*

We will now present some examples of ω-complete m-monoids. In order to do this in a concise way, we introduce an auxiliary notion first.

Definition 3.22. Let $a \in A$ and $\sum$ be an ω-infinitary sum operation for $\mathcal{A}$. We call $\sum$ a-***canonical*** if for every countable index set I and family $(a_i \mid i \in I)$ over A we have

$$\sum\nolimits_{i\in I} a_i = \begin{cases} a_{i_1} + \cdots + a_{i_n}\ , & \text{if } \{i \in I \mid a_i \neq \mathbf{0}\} = \{i_1, \ldots, i_n\} \text{ for some } n \in \mathbb{N} \\ & \text{and pairwise distinct } i_1, \ldots, i_n \in I\ , \\ a\ , & \text{otherwise.} \end{cases} \tag{3.4}$$

□

Proposition 3.23. *The $\mathbf{0}$-canonical ω-infinitary sum operation exists iff $A = \{\mathbf{0}\}$. Moreover, for every $a \in A \setminus \{\mathbf{0}\}$, the a-canonical ω-infinitary sum operation exists iff a is absorbing wrt to $+$ and the monoid $(A, +, \mathbf{0})$ is zero-sum free.*

Proof. First we prove the first equivalence. The direction "⇐" is trivial. We show direction "⇒". For every $a \in A \setminus \{\mathbf{0}\}$, $\mathbf{0} = \sum_{i\in\mathbb{N}} a = a + \sum_{i\in\mathbb{N}} a = a + \mathbf{0} = a$ due to Lemma 3.17(2).

Now we prove the second equivalence. To this end let $a \in A \setminus \{\mathbf{0}\}$.

"⇒": Let $\sum$ be a-canonical. First we show that a is absorbing wrt $+$. Clearly, $\mathbf{0}+a = a$ and for every $b \in A \setminus \{\mathbf{0}\}$, $b + a = b + \sum_{i\in\mathbb{N}} b = \sum_{i\in\mathbb{N}} b = a$ by Lemma 3.17(2).

Next we show that $(A, +, \mathbf{0})$ is zero-sum free. Let $b, c \in A\setminus\{\mathbf{0}\}$ and assume that $b+c = \mathbf{0}$. For every $n \in \mathbb{N}$ let $d_n = b$ if n is even and $d_n = c$ otherwise. Furthermore, for every $j \in \mathbb{N}$ let $I_j = \{2j, 2j + 1\}$. Then $(I_j \mid j \in \mathbb{N})$ is a partition of $\mathbb{N}$. By Equations (2.2), and (3.2), $a = \sum_{n\in\mathbb{N}} d_n = \sum_{j\in\mathbb{N}} \sum_{i\in I_j} d_i = \sum_{j\in\mathbb{N}}(d_{2j} + d_{2j+1}) = \sum_{j\in\mathbb{N}} \mathbf{0} = \mathbf{0}$, a contradiction.

"⇐": Let $\sum$ be defined as in Equation (3.4). We show that $\sum$ is an ω-infinitary sum operation. Since Equations (2.4) and (3.2) are obvious, we only prove Equation (2.5). Let I and J be countable sets, $(I_j \mid j \in J)$ be a generalized partition of I, and $(a_i \mid i \in I)$ be a family over A. If $\{i \in I \mid a_i \neq \mathbf{0}\}$ is finite, then for every $j \in J$ the sets $\{i \in I_j \mid a_i \neq \mathbf{0}\}$ and $\{j \in J \mid \sum_{i\in I_j} a_i \neq \mathbf{0}\}$ are finite as well; Equation (2.5) follows immediately.

Now assume that $\{i \in I \mid a_i \neq \mathbf{0}\}$ is infinite; hence, we have $\sum_{i\in I} a_i = a$. If we have that $\{j \in J \mid \sum_{i\in I_j} a_i \neq \mathbf{0}\}$ is infinite, then $\sum_{j\in J} \sum_{i\in I_j} a_i = a$ and we are done. Now assume that there is an $n \in \mathbb{N}$ and pairwise distinct $j_1, \ldots, j_n \in J$ such that $\{j_1, \ldots, j_n\} = \{j \in J \mid \sum_{i\in I_j} a_i \neq \mathbf{0}\}$. We show that $\sum_{i\in I_{j_1}} a_i + \cdots + \sum_{i\in I_{j_n}} a_i = a$. For every $j \in J$ with $\sum_{i\in I_j} a_i = \mathbf{0}$ we obtain $a_i = \mathbf{0}$ for every $i \in I_j$ because $(A, +, \mathbf{0})$ is zero-sum free. Hence, there is a $k \in [n]$ such that $\{i \in I_{j_k} \mid a_i \neq \mathbf{0}\}$ is infinite because the set $\{i \in I \mid a_i \neq \mathbf{0}\}$ is infinite by assumption. Thus, $\sum_{i\in I_{j_k}} a_i = a$ and therefore, $\sum_{i\in I_{j_1}} a_i + \cdots + \sum_{i\in I_{j_n}} a_i = a$ because a is absorbing wrt $+$. ■

Now we are prepared to give examples of ω-complete m-monoids.

Example 3.24. Here we list examples of ω-complete m-monoids. Let $\Delta = \emptyset$.

1. According to Proposition 3.23, every zero-sum free monoid $(A, +, \mathbf{0})$ which has an absorbing element a, can be extended to an ω-complete m-monoid $((A, +, \mathbf{0}, \theta), \sum)$ over Δ, where $\sum$ is a-canonical.

   Specific examples of such monoids are

   - $(\mathbb{N} \cup \{\infty\}, +, 0)$,
   - $(\mathbb{N}_+ \cup \{\infty\}, \cdot, 1)$,
   - $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0)$, where $\mathbb{R}_{\geq 0}$ are the non-negative real numbers,
   - $(\{r \in \mathbb{R} \mid r \geq 1\} \cup \{\infty\}, \cdot, 1)$,
   - $(\mathbb{R} \cup \{-\infty, \infty\}, \max, -\infty)$, and
   - $([0, 1], \cdot, 1)$.

2. Consider the m-monoid $\mathcal{A}_\Omega = (A_\Omega, \max, 0, \theta)$ over Δ, where A_Ω is the set of all countable ordinal numbers [31, 112, 124] and max is the maximum operation of countable ordinal numbers. Then the supremum operation $\vee_\Omega$ of countable ordinal numbers is an ω-infinitary sum operation for $\mathcal{A}_\Omega$.[2] Note that there is no $a \in A_\Omega$ such that the a-canonical ω-infinitary sum operation exists due to Proposition 3.23 and the fact that A_Ω has no maximal element. Hence, $\mathcal{A}_\Omega$ is an example of an m-monoid that admits an ω-infinitary sum operation but not an a-canonical one, for any $a \in A$.

3. Every complete lattice $(S, \vee, \wedge, \mathbf{0}, \mathbf{1})$ can be embedded into an ω-complete m-monoid $((S, \vee, \mathbf{0}, \theta), \bigvee)$ over Δ, where $\bigvee$ is the supremum operation. If $(S, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is even a completely distributive lattice[3], then the ω-complete m-monoid $((S, \vee, \mathbf{0}, \theta'), \bigvee)$ over Δ', where $\Delta' = \{\sigma^{(2)}\}$ and $\theta'(\sigma) = \{\wedge\}$, is ω-distributive.

4. Consider the m-monoid $\mathcal{A} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta)$ over Δ. We define an ω-infinitary sum operation $\sum$ for $\mathcal{A}$ as follows. Let I be a countable set, $(a_i \mid i \in I)$ be a family over $\mathbb{R}_{\geq 0}$, N be an initial segment of $\mathbb{N}$ with $|N| = |I|$, and $\pi : N \to I$ be an arbitrary bijection. Then

$$\sum\nolimits_{i \in I} a_i = \begin{cases} 0 & \text{if } N = \emptyset \\ a_{\pi(0)} + \cdots + a_{\pi(n)} \ , & \text{if } N \neq \emptyset \text{ is finite and } n = \max(N) \\ \lim\limits_{n \to \infty} a_{\pi(0)} + \cdots + a_{\pi(n)} \ , & \text{otherwise.} \end{cases}$$

   The operation $\sum$ is well-defined (see [72, Example IV.1.3(d)] and [118]).

5. Let Γ be a signature and consider the m-monoid $\mathcal{A} = (\mathcal{P}(T_\Gamma), \cup, \emptyset, \theta)$ over Γ, where, for every $\gamma \in \Gamma$, $\theta(\gamma)$ is the γ-language top concatenation. Then $(\mathcal{A}, \bigcup)$ is an ω-complete m-monoid which is ω-distributive. □

[2] This is implied by the fact that the supremum of countably many countable ordinal numbers is still a countable ordinal number (see Remark 3.27).

[3] i.e., it is a distributive lattice and for all sets I, J and family $(s_{(i,j)} \mid (i,j) \in I \times J)$ over S:

$$\bigwedge\nolimits_{i \in I} \bigvee\nolimits_{j \in J} s_{(i,j)} = \bigvee\nolimits_{f \in J^I} \bigwedge\nolimits_{i \in I} s_{(i, f(i))} \ .$$

A thorough introduction into completely distributive lattices can be found, e.g., in [34, 115].

Example 3.24 shows that there are m-monoids that admit more than one ω-infinitary sum operation. In fact, the ω-infinitary sum operations for the m-monoid $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta)$ that are given in item 1 and 4 of the example are distinct[4]. Hence, in general the underlying monoid of an m-monoid does not uniquely determine its ω-infinitary extension (if any ω-infinitary sum operation exists at all).

3.3.2 Continuous m-monoids

In the literature [95, 103] continuous m-monoids are defined as special complete m-monoids, namely complete m-monoids that are naturally ordered (i.e., the relation $\sqsubseteq$ on A is a partial order, where for every $a, b \in A$ we have $a \sqsubseteq b$ iff $a + c = b$ for some $c \in A$), and for every index set I, family $(a_i \mid i \in I)$ over A, and $c \in A$ we have

$$\text{if } \sum\nolimits_{i \in E} a_i \sqsubseteq c \text{ for every } E \in \mathcal{P}_{\text{fin}}(I), \text{ then } \sum\nolimits_{i \in I} a_i \sqsubseteq c\,.$$

In this thesis we will define continuous m-monoids differently. Our definition is independent from ω-infinitary sum operations and uses partial orders instead; it is based on the definition of ω-cpo semirings in [122]. It turns out that our definition and the definition of continuous m-monoids given by Kuich are related (see Lemmas 3.40 and 3.43). We begin this section with recalling complete partial orders and related concepts.

Complete partial orders

The order theoretic notions in this section are taken from [34, 71].

Let $(A, \leq)$ be a poset and $B \subseteq A$. A set $C \subseteq B$ is called a ***cofinal subset*** *of* B (*wrt* $\leq$) if for every $b \in B$ there is a $c \in C$ with $b \leq c$. Let $B' \subseteq A$. We say that B and B' are ***mutually cofinal*** (*wrt* $\leq$) if for every $b \in B$ there is $b' \in B'$ with $b \leq b'$ and for every $b' \in B'$ there is a $b \in B$ with $b' \leq b$.

An element $a \in A$ is called ***upper bound*** (respectively ***lower bound***) *of* B (*wrt* $\leq$) if $b \leq a$ ($a \leq b$) for every $b \in B$. If $a \in A$ is a lower bound (respectively upper bound) of A, then it is called the ***least element*** (***greatest element***) *of* A (*wrt* $\leq$). A subset D of A is called ***directed*** (*wrt* $\leq$) if for every $d, d' \in D$ the set $\{d, d'\}$ has an upper bound in D. An upper bound (respectively lower bound) a of B wrt $\leq$ is called the ***supremum*** (***infimum***) *of* B (*wrt* $\leq$) if $a \leq a'$ ($a' \leq a$) for every upper bound (lower bound) a' of B. If the order $\leq$ is understood, we denote the supremum and the infimum of B by $\vee B$ and $\wedge B$, respectively; if we use the notation $\vee B$ in any term in the sequel, we imply that the supremum of B exists, and likewise for $\wedge B$.

The poset $(A, \leq)$ is called a ***complete lattice*** if every subset of A has both a supremum and an infimum.

A mapping $c : \mathbb{N} \to A$ is called an ***ω-chain*** (*wrt* $\leq$) if $n_1 \leq n_2$ implies $c(n_1) \leq c(n_2)$ for every $n_1, n_2 \in \mathbb{N}$. Occasionally we will denote c as the family $(c(n) \mid n \in \mathbb{N})$. We call c ***ultimately constant*** if there is an $n \in \mathbb{N}$ such that $c(n) = c(n+m)$ for every $m \in \mathbb{N}$. We refer to the supremum of the range of an ω-chain c as the ***supremum*** *of* c. Observe that the range of every ω-chain is totally ordered and that every totally ordered subset of A is directed.

Lemma 3.25. *Let* $(A, \leq)$ *be a partial order.*

[4]For example, the former one yields $\sum_{n \in \mathbb{N}} 2^{-n} = \infty$, whereas for the latter one we obtain $\sum_{n \in \mathbb{N}} 2^{-n} = 2$.

1. *Let $B \subseteq A$ be nonempty, countable and directed. Then there is an ω-chain $c : \mathbb{N} \to B$ such that* $\mathrm{ran}(c)$ *is a cofinal subset of B.*

2. *Let $B, B' \subseteq A$ be mutually cofinal. If B or B' has a supremum, then $\vee B = \vee B'$.*

3. *Let $B \subseteq A$ and $C \subseteq B$ such that C is a cofinal subset of B. Moreover, let B or C have a supremum. Then $\vee C = \vee B$.*

Proof. 1. It is easy to show by induction that every finite subset of B has an upper bound in B.

Let N be the initial segment of $\mathbb{N}$ such that $|B| = |N|$ and choose a bijection $\pi : N \to B$. Let $c : \mathbb{N} \to B$ be defined by recursion as follows: we put $c(0) = \pi(0)$ and for every $n \in N$ observe that the set $\{\pi(m) \mid m \in N, m \leq n+1\} \cup \{c(n)\}$ is a finite subset of B and, thus, has an upper bound b in B; choose such a b and put $c(n+1) = b$. Clearly, c is an ω-chain and $\pi(n) \leq c(n)$ for every $n \in N$. Moreover, $\mathrm{ran}(c)$ is a cofinal subset of B because for every $b \in B$ we have $b = \pi(\pi^{-1}(b)) \leq c(\pi^{-1}(b))$.

2. It suffices to show that every upper bound of B is an upper bound of B' and vice versa. If a is an upper bound of B, then it is also an upper bound of B' because for every $b' \in B'$ there is a $b \in B$ with $b' \leq b$; hence, $b' \leq a$. Likewise, every upper bound of B' is an upper bound of B.

3. Clearly, B and C are mutually cofinal. The assertion follows from Statement 2. ■

We call $(A, \leq)$ an ***ω-complete partial order*** (for short: ***ω-cpo***) if A has a least element and every ω-chain wrt $\leq$ has a supremum.

Corollary 3.26. *Let $(A, \leq)$ be an ω-cpo and $B \subseteq A$ be countable and directed. Then B has a supremum.*

Proof. If B is empty, then the least element of A is the supremum of B. Otherwise the statement follows immediately from Lemma 3.25. ■

Remark 3.27. The restriction to countable sets B in Lemma 3.25 and Corollary 3.26 is crucial. For example, consider the poset $(A, \leq)$ where A is the set of countable ordinal numbers and $\leq$ is the natural ordering of countable ordinal numbers. Using the von Neumann definition of ordinal and cardinal numbers [134], the set A is the smallest uncountable ordinal number, i.e., the cardinal number $\aleph_1$ (see [124, Chapters 7 and 9]).

Consider the set $B = A = \aleph_1$, which is directed (it is even totally ordered) but not countable. However, $\aleph_1$ has no countable cofinal subset, i.e., $\aleph_1$ is a regular cardinal number[5]; in particular, there is no ω-chain whose range is a cofinal subset of B.

We conclude further that the supremum of any ω-chain of countable ordinal numbers is still a countable ordinal number, therefore, $(A, \leq)$ is an ω-cpo. However, B has no supremum in A. □

Let $(A, \leq)$ be an ω-cpo, $k \in \mathbb{N}$, and $\nu \in \mathrm{Ops}^{(k)}(A)$. For every ω-chain c wrt $\leq$ we say that ν is ***c-continuous*** (*wrt* $\leq$) if for every $a_1, \ldots, a_k \in A$, and $i \in [k]$ we have

$$\begin{aligned} &\nu\big(a_1, \ldots, a_{i-1}, \vee\{c(n) \mid n \in \mathbb{N}\}, a_{i+1}, \ldots, a_k\big) \\ &= \vee\big\{\nu(a_1, \ldots, a_{i-1}, c(n), a_{i+1}, \ldots, a_k) \mid n \in \mathbb{N}\big\} . \end{aligned} \tag{3.5}$$

[5] See [112, Satz 38.8] or [124, Theorem 11.13]. This result requires the Axiom of (countable) Choice. It is a corollary of the fact that any countable union of countable sets is still a countable set, i.e., $\bigcup_{i \in I} S_i$ is countable if I is countable and S_i is countable for every $i \in I$ (see [47, Theorem 6Q] or [112, Satz 31.8]).

We call ν ***ω-continuous*** (*wrt* $\leq$) if it is c-continuous wrt $\leq$ for every ω-chain wrt $\leq$.

Observation 3.28. *Let $k \in \mathbb{N}$ and $\nu \in \mathrm{Ops}^{(k)}(A)$.*

1. *ν is monotone iff ν is c-continuous for every ultimately constant ω-chain c.*
2. *ν is ω-continuous iff ν is monotone and ν is c-continuous for every ω-chain c that is not ultimately constant.*

The following fixpoint theorem is well-known[6] (cf. [5] and [135, Theorem 1.5.7]).

Theorem 3.29. *Let $(A, \leq)$ be an ω-cpo and let $f : A \to A$ be ω-continuous. Then $\vee\{f^n(\bot) \mid n \in \mathbb{N}\}$ is the least fixpoint of f, where $\bot$ is the least element of A.*

Before we proceed, we present three auxiliary lemmas about ω-complete partial orders. To this end we fix an arbitrary ω-cpo $(A, \leq)$. The first lemma states that Equation (3.5) can be extended from ω-chains to nonempty, countable, and directed sets.

Lemma 3.30. *Let $k \in \mathbb{N}$, $a_1, \ldots, a_k \in A$, $i \in [k]$, and $\nu \in \mathrm{Ops}^{(k)}(A)$ be ω-continuous. Then for every nonempty, countable, and directed set $B \subseteq A$ we have*

$$\begin{aligned} &\nu\big(a_1, \ldots, a_{i-1}, \vee B, a_{i+1}, \ldots, a_k\big) \\ &= \vee\big\{\nu(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \mid b \in B\big\} . \end{aligned} \tag{3.6}$$

Proof. Let $B \subseteq A$ be nonempty, countable, and directed. By Lemma 3.25(1) there is an ω-chain $c : \mathbb{N} \to B$ such that $\mathrm{ran}(c)$ is a cofinal subset of B. Since ν is monotone, we obtain that the set $\{\nu(a_1, \ldots, a_{i-1}, c(n), a_{i+1}, \ldots, a_k) \mid n \in \mathbb{N}\}$ is a cofinal subset of $\big\{\nu(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \mid b \in B\big\}$. Hence,

$$\begin{aligned} &\nu\big(a_1, \ldots, a_{i-1}, \vee B, a_{i+1}, \ldots, a_k\big) \\ &= \nu\big(a_1, \ldots, a_{i-1}, \vee\{c(n) \mid n \in \mathbb{N}\}, a_{i+1}, \ldots, a_k\big) && \text{(by Lemma 3.25(3))} \\ &= \vee\big\{\nu(a_1, \ldots, a_{i-1}, c(n), a_{i+1}, \ldots, a_k) \mid n \in \mathbb{N}\big\} && (\nu \text{ is } \omega\text{-continuous}) \\ &= \vee\big\{\nu(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k) \mid b \in B\big\} . && \text{(by Lemma 3.25(3))} \end{aligned}$$

■

Lemma 3.31. *Let $k \in \mathbb{N}$, $\nu \in \mathrm{Ops}^{(k)}(A)$ be an ω-continuous operation, let $B_1, \ldots, B_k$ be nonempty, countable, and directed subsets of A, and let $a \in A$ such that for every $(b_1, \ldots, b_k) \in B_1 \times \cdots \times B_k$, $\nu(b_1, \ldots, b_k) \leq a$. Then $\nu\big(\vee B_1, \ldots, \vee B_k\big) \leq a$.*

Proof. For every $l \in [k+1]$ and $(b_l, \ldots, b_k) \in B_l \times \cdots \times B_k$ we define an element $a^l_{b_l,\ldots,b_k} \in A$ as follows: $a^l_{b_l,\ldots,b_k} = \nu\big(\vee B_1, \ldots, \vee B_{l-1}, b_l, \ldots, b_k\big)$. We need to show that $a^{k+1}_\varepsilon \leq a$. To this end we show by induction on l that for every $l \in [k+1]$ and $(b_l, \ldots, b_k) \in B_l \times \cdots \times B_k$ we have $a^l_{b_l,\ldots,b_k} \leq a$.

Induction base. The statement holds trivially for $l = 1$ because $a^1_{b_1,\ldots,b_k} = \nu(b_1, \ldots, b_k) \leq a$ by assumption.

[6] In the literature this fixpoint theorem is sometimes attributed to Kleene [83, Theorem XXVI] and sometimes attributed to (Knaster and) Tarski [125, Theorem 1]. However, the attribution to Tarski is slightly incorrect as his fixpoint theorem [125, Theorem 1] is different from Theorem 3.29; for a thorough discussion on this topic the reader is referred to [101]. Tarski's fixpoint theorem occurs in this thesis in a later chapter (see Theorem 6.6).

Induction step. Let $l \in [k+1]$ such that $l > 1$. We obtain

$$\begin{aligned} a^l_{b_l,\dots,b_k} &= \nu\big(\vee B_1, \dots, \vee B_{l-2}, \vee B_{l-1}, b_l, \dots, b_k\big) \\ &= \vee\big\{\nu(\vee B_1, \dots, \vee B_{l-2}, b_{l-1}, b_l, \dots, b_k) \mid b_{l-1} \in B_{l-1}\big\} && \text{(by Lemma 3.30)} \\ &= \vee\{a^{l-1}_{b_{l-1},b_l,\dots,b_k} \mid b_{l-1} \in B_{l-1}\} \,. \end{aligned}$$

The induction hypothesis yields $a^{l-1}_{b_{l-1},b_l,\dots,b_k} \leq a$ for every $b_{l-1} \in B_{l-1}$. This implies our assertion. ■

Lemma 3.32. *Let $k \in \mathbb{N}$, $\nu \in \mathrm{Ops}^{(k)}(A)$ be ω-continuous, and for every $i \in [k]$ let $c_i : \mathbb{N} \to A$ be an ω-chain. Then $\big(\nu(c_1(n), \dots, c_k(n)) \mid n \in \mathbb{N}\big)$ is an ω-chain and*

$$\nu\big(\vee\{c_1(n) \mid n \in \mathbb{N}\}, \dots, \vee\{c_k(n) \mid n \in \mathbb{N}\}\big) = \vee\big\{\nu(c_1(n), \dots, c_k(n)) \mid n \in \mathbb{N}\big\} \,.$$

Proof. This statement holds trivially if $k = 0$. For the remainder of the proof we assume $k > 0$.

Clearly, $\big(\nu(c_1(n), \dots, c_k(n)) \mid n \in \mathbb{N}\big)$ is an ω-chain because ν is monotone. Let lhs be the left-hand side and rhs be the right-hand side of the equation.

First we show rhs $\leq$ lhs. For every $n \in \mathbb{N}$, we have $c_i(n) \leq \vee\{c_i(n') \mid n' \in \mathbb{N}\}$ for every $i \in [k]$; therefore, $\nu(c_1(n), \dots, c_k(n)) \leq$ lhs. This yields rhs $\leq$ lhs.

Next we show lhs $\leq$ rhs. By Lemma 3.31 it suffices to show that for every $(n_1, \dots, n_k) \in \mathbb{N}^k$ we have $\nu(c_1(n_1), \dots, c_k(n_k)) \leq$ rhs; this is clearly true because $\nu(c_1(n_1), \dots, c_k(n_k)) \leq \nu(c_1(n), \dots, c_k(n)) \leq$ rhs, where $n = \max\{n_1, \dots, n_k\}$. ■

Continuous m-monoids and their properties

Now we are prepared to define the main notion of this section.

Definition 3.33. Let $(A, \leq)$ be an ω-cpo. We call $(\mathcal{A}, \leq)$ an ***ω-continuous m-monoid*** if:

- $\mathbf{0}$ is the least element of A wrt $\leq$,
- $+$ is ω-continuous wrt $\leq$, and
- for every $\delta \in \Delta$, $\theta(\delta)$ is ω-continuous wrt $\leq$. □

First let us study examples of ω-continuous m-monoids.

Example 3.34. Now we list some examples of ω-continuous m-monoids. Let $\Delta = \emptyset$.

1. The m-monoid $(\mathbb{N} \cup \{\infty\}, +, 0, \theta)$ over Δ together with the natural order on $\mathbb{N} \cup \{\infty\}$ is an ω-continuous m-monoid.
2. The m-monoid $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta)$ over Δ together with the natural order on the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is an ω-continuous m-monoid.
3. Let $\Delta' = \{\sigma^{(2)}\}$. The m-monoid $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta')$ over Δ' together with the natural order is an ω-continuous m-monoid for, e.g., each of the following three definitions of $\theta'(\sigma)$:
   - $\theta'(\sigma) = \cdot$ is the conventional multiplication,

- $\theta'(\sigma) = \max$,
- $\theta'(\sigma) = \min$.

4. Let $(L, \leq)$ be an ω-cpo that is additionally a join-semilattice, i.e., for every $a, b \in L$ the set $\{a, b\}$ has a supremum wrt $\leq$, denoted by $a \vee b$. Then $((L, \vee, \bot, \theta), \leq)$ is an ω-continuous m-monoid over Δ, where $\bot$ denotes the least element of L wrt $\leq$.

   Particular instances of such ω-cpos $(L, \leq)$ are totally ordered ω-cpos; e.g., well-ordered sets having a greatest element[7].

5. Let Γ be a signature and let $\mathcal{A}$ be defined as in Example 3.24(5). Then $(\mathcal{A}, \subseteq)$ is an ω-continuous m-monoid.

6. Let $A = \mathbb{N} \cup \{\infty_1, \infty_2, \infty_3\}$ and define the operations $\nu \in \mathrm{Ops}^{(1)}(A)$ and $\circ \in \mathrm{Ops}^{(2)}(A)$ as follows:
   - $\forall\, n \in \mathbb{N} : \nu(n) = n + 1$ and $\forall\, a \in \{\infty_1, \infty_2, \infty_3\} : \nu(a) = a$
   - 0 is neutral wrt $\circ$, and $a \circ b = \infty_3$ for every $a, b \in A \setminus \{0\}$.

   Clearly, $\circ$ is commutative and associative. Consider the signature $\Delta' = \{\gamma^{(1)}\}$ and the m-monoid $\mathcal{A} = (A, \circ, 0, \theta)$ over Δ', where $\theta(\gamma) = \nu$. It is easy to check that there are precisely two partial orders $\leq_1$ and $\leq_2$ on A such that $(\mathcal{A}, \leq_i)$ is an ω-continuous m-monoid (for $i \in \{1, 2\}$), namely

   $$\leq_1 \,=\, \leq \cup\, (\mathbb{N} \times \{\infty_1, \infty_2, \infty_3\}) \cup \left(\{(\infty_1, \infty_2), (\infty_2, \infty_3)\}^*\right) ,$$
   $$\leq_2 \,=\, \leq \cup\, (\mathbb{N} \times \{\infty_1, \infty_2, \infty_3\}) \cup \left(\{(\infty_2, \infty_1), (\infty_1, \infty_3)\}^*\right) ,$$

   where $\leq$ is the natural order on natural numbers.

   Now consider the operation $\nu' \in \mathrm{Ops}^{(1)}(A)$ which is defined as follows:
   - $\forall\, n \in \mathbb{N}_+ : \nu'(n) = n + 1$ and $\forall\, a \in \{0, \infty_1, \infty_2, \infty_3\} : \nu'(a) = a$.

   Then the m-monoid $\mathcal{A}' = (A, \circ, 0, \theta')$ over Δ', where $\theta'(\gamma) = \nu'$, is distributive. Observe that $(\mathcal{A}', \leq_1)$ and $(\mathcal{A}', \leq_2)$ are ω-continuous m-monoids, too. However, there are other partial orders $\leq$ such that $(\mathcal{A}', \leq)$ is an ω-continuous m-monoid, e.g., $\leq\, = (\{0\} \times A) \cup (A \times \{\infty_3\})$. □

Example 3.34(6) shows that there are m-monoids that admit more than one extension to an ω-continuous m-monoid. Hence, in general an m-monoid does not uniquely determine its extension to an ω-continuous m-monoid (if such an extension exists at all).

The following observation is a consequence of the fact that $\mathbf{0}$ is the least element in an ω-continuous monoid and that addition is monotone (due to Observation 3.28).

Observation 3.35. *Let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then for every $a, b \in A$, $a \leq a + b$. Moreover, for all finite sets I, J with $I \subseteq J$ and every family $(a_j \mid j \in J)$ over A we have $\sum_{i \in I} a_i \leq \sum_{j \in J} a_j$.*

We conclude this section with a technical lemma that connects evaluation homomorphisms and suprema of ω-chains.

[7] A totally ordered poset $(L, \leq)$ is a *well-ordered set* if every nonempty subset of L has a least element. An example of a well-ordered set with a greatest element is the set of all ordinal numbers less or equal to a given limit ordinal (e.g., $\omega + \omega$), together with the natural order of ordinal numbers. A thorough introduction into well-ordered sets and ordinal numbers can be found in, e.g., [13, 31, 93, 124].

Lemma 3.36. *Let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid, C be a set, and for every $n \in \mathbb{N}$ let $f_n : C \to A$ such that, for every $c \in C$, $(f_n(c) \mid n \in \mathbb{N})$ is an ω-chain. Moreover, let $f : C \to A$ be defined by $f(c) = \vee\{f_n(c) \mid n \in \mathbb{N}\}$ for every $c \in C$. Then for every $t \in T_\Delta(C)$,*

1. $(g_n(t) \mid n \in \mathbb{N})$ *is an ω-chain and*
2. $g(t) = \vee\{g_n(t) \mid n \in \mathbb{N}\}$,

where g and, for every $n \in \mathbb{N}$, g_n is the unique Δ-homomorphism from $\mathcal{T}_\Delta(C)$ to (A, θ) extending f and f_n, respectively.

Proof. We give a proof by structural induction on t.

Induction base. If $t \in C$, then clearly $(g_n(t) \mid n \in \mathbb{N}) = (f_n(t) \mid n \in \mathbb{N})$ is an ω-chain and we have $g(t) = f(t) = \vee\{f_n(t) \mid n \in \mathbb{N}\} = \vee\{g_n(t) \mid n \in \mathbb{N}\}$.

Induction step. Now let $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, and $t_1, \ldots, t_k \in T_\Delta(C)$ such that $t = \delta(t_1, \ldots, t_k)$. Then $(g_n(t) \mid n \in \mathbb{N}) = \big(\theta(\delta)(g_n(t_1), \ldots, g_n(t_k)) \mid n \in \mathbb{N}\big)$ is an ω-chain due to Lemma 3.32 and the first part of the induction hypothesis. Now we show Statement 2:

$$\begin{aligned}
g(t) &= \theta(\delta)(g(t_1), \ldots, g(t_k)) \\
&= \theta(\delta)\big(\vee\{g_n(t_1) \mid n \in \mathbb{N}\}, \ldots, \vee\{g_n(t_k) \mid n \in \mathbb{N}\}\big) && \text{(by ind. hyp.)} \\
&= \vee\big\{\theta(\delta)(g_n(t_1), \ldots, g_n(t_k)) \mid n \in \mathbb{N}\big\} && \text{(by Lemma 3.32)} \\
&= \vee\big\{g_n(\delta(t_1, \ldots, t_k)) \mid n \in \mathbb{N}\big\} \; .
\end{aligned}$$

■

3.3.3 Relationships

It turns out that some m-monoids $\mathcal{A}$ can both be extended to an ω-continuous m-monoid $(\mathcal{A}, \leq)$ and an ω-complete m-monoid $(\mathcal{A}, \sum)$. We are particularly interested in such extensions $\leq$ and $\sum$ that are related in a specific way, namely that the sum of any given family is the supremum of the set of finite partial sums of that family.

Definition 3.37. Let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid and $\sum$ be an ω-infinitary sum operation for $\mathcal{A}$. Then the ω-continuous m-monoid $(\mathcal{A}, \leq)$ and the ω-complete m-monoid $(\mathcal{A}, \sum)$ are called ***related*** if for every countable index set I and family $(a_i \mid i \in I)$ over A the following holds:

$$\sum_{i \in I} a_i = \vee\Big\{\sum_{j \in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\Big\} \; .$$

□

This relationship is reminiscent of the principle used in the theory of real number series, where the sum value of a series is defined to be the limit of the sequence of partial sums of the series (for more information about the theory of series the interested reader is referred to, e.g., [23, 25, 88]). This fact is illustrated by the following example.

Example 3.38. Let $\Delta = \emptyset$ and consider the m-monoid $\mathcal{A} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta)$ over Δ. Moreover, let

- $(\mathcal{A}, \leq)$ be the ω-continuous m-monoid from Example 3.34(2) and
- $(\mathcal{A}, \sum)$ be the ω-complete m-monoid from Example 3.24(4).

Then $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related.

Let $\sum'$ be the ∞-canonical ω-infinitary sum operation for the m-monoid $\mathcal{A}$ (see Example 3.24(1)). Clearly, $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum')$ are not related, which is witnessed by the family $(2^{-n} \mid n \in \mathbb{N})$: $\sum'_{n\in\mathbb{N}} 2^{-n} = \infty \neq 2 = \vee\{\sum_{n\in N} 2^{-n} \mid N \in \mathcal{P}_{\text{fin}}(\mathbb{N})\}$. □

The last paragraph of Example 3.38 shows that there are ω-continuous m-monoids and ω-complete m-monoids (over the same underlying m-monoid) which are not related. We will now investigate the notion of related ω-continuous and ω-complete m-monoids in more detail.

Lemma 3.39. *Let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid.*

1. *Let $\sum$ and $\sum'$ be ω-infinitary sum operations for $\mathcal{A}$ such that $(\mathcal{A}, \leq)$ is related both to $(\mathcal{A}, \sum)$ and $(\mathcal{A}, \sum')$. Then $\sum = \sum'$.*

2. *Let I be a countable set and $(a_i \mid i \in I)$ be family over A. Then we have that the set $\{\sum_{j\in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\}$ is nonempty, countable, and directed. Moreover, for every initial segment N of $\mathbb{N}$ and bijection $\pi : N \to I$:*

$$\vee\{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in N\} = \vee\Big\{\sum\nolimits_{j\in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\Big\} .$$

3. *Let $(\mathcal{A}, \sum)$ be an ω-complete m-monoid that is related to $(\mathcal{A}, \leq)$. If $\mathcal{A}$ is idempotent, then $(\mathcal{A}, \sum)$ is ω-idempotent. Moreover, if $\mathcal{A}$ is distributive, then $(\mathcal{A}, \sum)$ is ω-distributive.*

Proof. 1. This statement follows from Definition 3.37.

2. Let $B = \{\sum_{j\in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\}$. Clearly, $\mathcal{P}_{\text{fin}}(I)$ is nonempty and countable. Furthermore, for every $J, J' \in \mathcal{P}_{\text{fin}}(I)$ also $J \cup J' \in P_{\text{fin}}(I)$ and $\sum_{j\in J} a_j \leq \sum_{j\in J\cup J'} a_j$ and $\sum_{j\in J'} a_j \leq \sum_{j\in J\cup J'} a_j$ by Observation 3.35. Hence, B is nonempty, countable, and directed; therefore, B has a supremum by Corollary 3.26.

Let N be an initial segment of $\mathbb{N}$ and $\pi : N \to I$ be a bijection. We define the set $C = \{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in N\}$. We need to show that $\vee C = \vee B$. This is trivial if $I = \emptyset$. For the remainder of this proof assume $I \neq \emptyset$. Due to Lemma 3.25(3) it suffices to show that C is a cofinal subset of B. Clearly, C is a subset of B. Let $J \in \mathcal{P}_{\text{fin}}(I)$. If $J = \emptyset$, then $\sum_{j\in J} a_j \leq a_{\pi(0)}$. If J is nonempty, then for $n = \max(\pi^{-1}(J))$ we obtain $\sum_{j\in J} a_j \leq a_{\pi(0)} + \cdots + a_{\pi(n)}$ by Observation 3.35.

3. First suppose that $\mathcal{A}$ is idempotent. Let I be a nonempty countable index set and $a \in A$. Then $\sum_{i\in I} a = \vee\{\sum_{j\in J} a \mid J \in \mathcal{P}_{\text{fin}}(I)\} = \vee\big(\{\mathbf{0}\} \cup \{a \mid J \in \mathcal{P}_{\text{fin}}(I), J \neq \emptyset\}\big) = a$ because I is non-empty.

Now suppose that $\mathcal{A}$ is distributive. Let $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, $a_1, \ldots, a_k \in A$, $j \in [k]$, I be a countable index set, and $(b_i \mid i \in I)$ be a family over A. We show that Equation (3.3) is satisfied. By Statement 2 the set $\{\sum_{i\in J} b_i \mid J \in \mathcal{P}_{\text{fin}}(I)\}$ is nonempty, countable, and directed. Then

$$\begin{aligned}
&\theta(\delta)\big(a_1, \ldots, a_{j-1}, \sum\nolimits_{i\in I} b_i, a_{j+1}, \ldots, a_k\big) \\
&= \theta(\delta)\big(a_1, \ldots, a_{j-1}, \vee\big\{\sum\nolimits_{i\in J} b_i \mid J \in \mathcal{P}_{\text{fin}}(I)\big\}, a_{j+1}, \ldots, a_k\big) && \text{(by Def. 3.37)} \\
&= \vee\big\{\theta(\delta)(a_1, \ldots, a_{j-1}, \sum\nolimits_{i\in J} b_i, a_{j+1}, \ldots, a_k) \mid J \in \mathcal{P}_{\text{fin}}(I)\big\} && \text{(by Lemma 3.30)}
\end{aligned}$$

$$= \vee\left\{\sum\nolimits_{i \in J} \theta(\delta)(a_1, \ldots, a_{j-1}, b_i, a_{j+1}, \ldots, a_k) \mid J \in \mathcal{P}_{\text{fin}}(I)\right\} \qquad (\mathcal{A} \text{ is distributive})$$
$$= \sum\nolimits_{i \in I} \theta(\delta)(a_1, \ldots, a_{j-1}, b_i, a_{j+1}, \ldots, a_k)\ . \qquad (\text{by Def. 3.37})$$

■

For the remainder of this section we will deal with the problem when a given ω-continuous m-monoid admits a related ω-complete m-monoid and vice versa.

Lemma 3.40 (cf. [52, Prop. 2.2]). *Let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then there is an ω-infinitary sum operation $\sum^{\leq}$ for $\mathcal{A}$ such that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum^{\leq})$ are related.*

PROOF. First we define $\sum^{\leq}$. Let I be a countable index set and $(a_i \mid i \in I)$ be a family over A. Due to Lemma 3.39(2), the set $\left\{\sum_{j \in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\right\}$ is countable and directed. Thus, it has a supremum by Corollary 3.26. We put $\sum^{\leq}_{i \in I} a_i = \vee\left\{\sum_{j \in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\right\}$.

Clearly, if $\sum^{\leq}$ is an ω-infinitary sum operation for $\mathcal{A}$, then $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum^{\leq})$ are related. We show that $\sum^{\leq}$ is an ω-infinitary sum operation for $\mathcal{A}$. Therefore, we need to show that Equations (2.4), (2.5), and (3.2) hold. Equations (2.4) and (3.2) hold trivially. We prove that Equation (2.5) holds as well. Let I and J be countable sets, $(I_j \mid j \in J)$ be a generalized partition of I, and $(a_i \mid i \in I)$ be a family over A. We show that $\sum^{\leq}_{i \in I} a_i = \sum^{\leq}_{j \in J} \sum^{\leq}_{i \in I_j} a_i$. To this end let $b_j = \sum^{\leq}_{i \in I_j} a_i$ for every $j \in J$.

First we show that $\sum^{\leq}_{i \in I} a_i \leq \sum^{\leq}_{j \in J} b_j$. Let $I' \in \mathcal{P}_{\text{fin}}(I)$. We need to prove that $\sum_{i \in I'} a_i \leq \sum^{\leq}_{j \in J} b_j$ (note that the sum $\sum$ on the left-hand side is the finite extension of the operation $+$). Let $J' = \{j \in J \mid I_j \cap I' \neq \emptyset\}$. Then $J' \in \mathcal{P}_{\text{fin}}(J)$ and $I' \subseteq \bigcup_{j \in J'} I_j$. For every $j \in J'$ the set $I_j \cap I'$ is finite, hence, $\sum_{i \in I_j \cap I'} a_i \leq \sum^{\leq}_{i \in I_j} a_i = b_j$. We obtain $\sum_{i \in I'} a_i = \sum_{j \in J'} \sum_{i \in I_j \cap I'} a_i \leq \sum_{j \in J'} b_j$ because $+$ is monotone wrt $\leq$. Finally, the definition of $\sum^{\leq}$ yields $\sum_{j \in J'} b_j \leq \sum^{\leq}_{j \in J} b_j$.

It remains to show $\sum^{\leq}_{j \in J} b_j \leq \sum^{\leq}_{i \in I} a_i$. Let $J' \in \mathcal{P}_{\text{fin}}(J)$. We need to prove that $\sum_{j \in J'} b_j \leq \sum^{\leq}_{i \in I} a_i$. Let $k \in \mathbb{N}$ and $j_1, \ldots, j_k \in J$ be pairwise distinct such that $J' = \{j_1, \ldots, j_k\}$. Let the operation $\nu \in \text{Ops}^{(k)}(A)$ be defined by $\nu(a'_1, \ldots, a'_k) = a'_1 + \cdots + a'_k$ for every $a'_1, \ldots, a'_k \in A$. Clearly, ν is ω-continuous. Moreover, for every $l \in [k]$ let $B_l = \{\sum_{i \in I'} a_i \mid I' \in \mathcal{P}_{\text{fin}}(I_{j_l})\}$; hence, $b_{j_l} = \vee B_l$ and B_l is nonempty, countable, and directed due to Lemma 3.39(2). It remains to show that $\nu(\vee B_1, \ldots, \vee B_k) \leq \sum^{\leq}_{i \in I} a_i$. Observe that Lemma 3.31 implies that it suffices to show that for every tuple $(I'_1, \ldots, I'_k) \in \mathcal{P}_{\text{fin}}(I_{j_1}) \times \cdots \times \mathcal{P}_{\text{fin}}(I_{j_k})$ we have $\nu(\sum_{i \in I'_1} a_i, \ldots, \sum_{i \in I'_k} a_i) \leq \sum^{\leq}_{i \in I} a_i$. Let $(I'_1, \ldots, I'_k) \in \mathcal{P}_{\text{fin}}(I_{j_1}) \times \cdots \times \mathcal{P}_{\text{fin}}(I_{j_k})$. Since $j_1, \ldots, j_k$ are pairwise distinct, the sets $I_{j_1}, \ldots, I_{j_k}$ are pairwise disjoint and, thus, also $I'_1, \ldots, I'_k$ are pairwise disjoint. Moreover, $I'_1 \cup \cdots \cup I'_k \in \mathcal{P}_{\text{fin}}(I)$. Thus, $\nu(\sum_{i \in I'_1} a_i, \ldots, \sum_{i \in I'_k} a_i) = \sum_{l \in [k]} \sum_{i \in I'_l} a_i = \sum_{i \in I'_1 \cup \cdots \cup I'_k} a_i \leq \sum^{\leq}_{i \in I} a_i$. ■

According to Lemma 3.40 for every ω-continuous m-monoid there is a related ω-complete m-monoid. The converse does not hold in general. More precisely, let $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$ be the class of ω-complete m-monoids that admit a related ω-continuous m-monoid; then there are ω-complete m-monoids that are not in the class $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$. This is witnessed by the following two examples.

Example 3.41. Let $\Delta = \emptyset$. Consider the ω-complete m-monoid $\big((A, +, 0, \theta), \sum\big)$ over Δ, where $A = \{0, 1, 2, \infty\}$, ∞ is absorbing wrt $+$, for every $a, b \in \{1, 2\}$ we have $a + b = (a + b - 1) \bmod 2 + 1$, and $\sum$ is the ∞-canonical ω-infinitary sum operation.

Assume that there is a partial order $\leq$ on A such that $\big((A, +, 0, \theta), \leq\big)$ is an ω-continuous m-monoid. Then by Observation 3.35 we obtain both $1 \leq 1 + 1 = 2$ and $2 \leq 2 + 1 = 1$, a contradiction. Thus, $\big((A, +, 0, \theta), \sum\big) \notin \mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$. □

Example 3.42. Let $\Delta = \emptyset$ and consider the ω-complete m-monoid $(\mathcal{A}, \sum')$ over Δ from Example 3.38, i.e., $\mathcal{A} = (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta)$ and $\sum'$ is the ∞-canonical ω-infinitary sum operation for $\mathcal{A}$.

Assume that there is an ω-continuous m-monoid $(\mathcal{A}, \leq)$ such that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum')$ are related. By Observation 3.35, $\leq$ is the natural order on $\mathbb{R}_{\geq 0} \cup \{\infty\}$. However, $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum')$ are not related as we have already discussed in Example 3.38. Thus, $\big(\mathcal{A}, \sum'\big) \notin \mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$. □

Let us briefly analyze Examples 3.41 and 3.42. Let $(\mathcal{A}, \sum) = \big((A, +, 0, \theta), \sum\big)$ be an ω-complete m-monoid and consider the relation $\leq\, = \{(a, a + b) \mid a, b \in A\}$ on A. We say that $(\mathcal{A}, \sum)$ has property (R1) if $\leq$ is antisymmetric. Moreover, we say that $(\mathcal{A}, \sum)$ has property (R2) if for every $a \in A$, countable set I, and family $(a_i \mid i \in I)$ over A: if $a < \sum_{i \in I} a_i$, then there is a finite $J \subseteq I$ with $\sum_{i \in J} a_i \not\leq a$. Obviously, the ω-complete m-monoid in Example 3.41 does not have property (R1) and the one in Example 3.42 does not have property (R2). It is easy to see that every ω-complete m-monoid in $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$ has properties (R1) and (R2), i.e., (R1) and (R2) are necessary conditions for an ω-complete m-monoid to be in $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$. It turns out, however, that (R1) and (R2) together are no sufficient conditions. This is witnessed by the more elaborate counterexample (Example A.1) that is given in Appendix A.2.

Particularly Example A.1 indicates that there is no simple characterization (in terms of necessary and sufficient conditions) of the class $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$. Therefore, we restrict ourselves to giving sufficient conditions that an ω-complete m-monoids is in $\mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$.

Lemma 3.43. *Let $(\mathcal{A}, \sum)$ be an ω-distributive ω-complete m-monoid such that for every $a \in A$ and family $(a_n \mid n \in \mathbb{N})$ over A:*

- *if $a_0 + \cdots + a_n \leq a$ for every $n \in \mathbb{N}$, then $\sum_{n \in \mathbb{N}} a_n \leq a$,*

where $\leq\, = \{(b, b + c) \mid b, c \in A\}$. Then the following statements hold.

1. *$(A, \leq)$ is a poset.*
2. *For every countable set I, family $(a_i \mid i \in I)$ over A, initial segment N of $\mathbb{N}$, and bijection $\pi : N \to I$, $\sum_{i \in I} a_i = \bigvee\{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in N\}$.*
3. *$(\mathcal{A}, \leq)$ is an ω-continuous m-monoid that is related to $(\mathcal{A}, \sum)$.*
4. *$(\mathcal{A}, \sum) \in \mathfrak{A}_{\mathrm{s} \rightsquigarrow \mathrm{c}}$.*

Proof. 1. Clearly, $\leq$ is transitive and reflexive. We show that $\leq$ is antisymmetric. Let $a, b \in A$ such that $a \leq b$ and $b \leq a$. Thus, there are $a', b' \in A$ such that $a + a' = b$ and $b + b' = a$. Let $a_0 = a$ and for every $n \in \mathbb{N}_+$ let $a_n = a'$ if n is odd and $a_n = b'$ otherwise. Clearly, for every $n \in \mathbb{N}$, $a_0 + \cdots + a_n = a$ if n is even and $a_0 + \cdots + a_n = b$ otherwise. Hence, $a_0 + \cdots + a_n \leq a$ for every $n \in \mathbb{N}$. By assumption, we obtain that $\sum_{n \in \mathbb{N}} a_n \leq a$. Lemma 3.17(2) yields $\sum_{n \in \mathbb{N}} a_n = a' + \sum_{n \in \mathbb{N}} a_n$. Since $\sum_{n \in \mathbb{N}} a_n \leq a$, there is a $c \in A$ such that $c + \sum_{n \in \mathbb{N}} a_n = a$. Hence, $a = c + \sum_{n \in \mathbb{N}} a_n = a' + c + \sum_{n \in \mathbb{N}} a_n = a' + a = b$.

2. Let I be a countable set, $(a_i \mid i \in I)$ be a family over A, N be an initial segment of $\mathbb{N}$, and $\pi : N \to I$ be a bijection. The statement is trivial if I is finite. Now assume that I is infinite, i.e., $N = \mathbb{N}$. For every $n \in \mathbb{N}$ let $b_n = a_{\pi(n)}$. Equation (2.3) yields $\sum_{i\in I} a_i = \sum_{n\in\mathbb{N}} a_{\pi(n)} = \sum_{n\in\mathbb{N}} b_n$. Clearly, $\sum_{n\in\mathbb{N}} b_n$ is an upper bound of $\{b_0 + \cdots + b_n \mid n \in \mathbb{N}\}$ due to Equations (2.2) and (3.2) and Observation 3.35. Now let b be an upper bound of $\{b_0 + \cdots + b_n \mid n \in \mathbb{N}\}$. Our assumption yields $\sum_{n\in\mathbb{N}} b_n \le b$. Thus, $\sum_{n\in\mathbb{N}} b_n = \bigvee\{b_0 + \cdots + b_n \mid n \in \mathbb{N}\} = \bigvee\{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in \mathbb{N}\}$.

3. Observe that for every ω-chain $c : \mathbb{N} \to A$ there is a family $(a_n \mid n \in \mathbb{N})$ over A such that $c(n) = a_0 + \cdots + a_n$ for every $n \in \mathbb{N}$. We will use this decomposition of ω-chains throughout this proof.

We show that $(A, \le)$ is an ω-cpo. Clearly, $\mathbf{0}$ is the least element of A. Due to the above decomposition of ω-chains it suffices to show that for every family $(a_n \mid n \in \mathbb{N})$ over A, $\sum_{n\in\mathbb{N}} a_n$ is the supremum of the set $\{a_0 + \cdots + a_n \mid n \in \mathbb{N}\}$. This follows from Statement 2.

Next we show that $(\mathcal{A}, \le)$ is an ω-continuous m-monoid. First we prove that $+$ is ω-continuous wrt $\le$. Let $a \in A$ and $c : \mathbb{N} \to A$ be an ω-chain. Consider the families $(a_n \mid n \in \mathbb{N})$ and $(a'_n \mid n \in \mathbb{N})$ over A such that $c(n) = a_0 + \cdots + a_n$ for every $n \in \mathbb{N}$, $a'_0 = a + a_0$, and $a'_n = a_n$ for every $n \in \mathbb{N}_+$.

$$\begin{aligned} a + \bigvee\{c(n) \mid n \in \mathbb{N}\} &= a + \bigvee\{a_0 + \ldots + a_n \mid n \in \mathbb{N}\} = a + \sum_{n\in\mathbb{N}} a_n \\ &= \sum_{n\in\mathbb{N}} a'_n = \bigvee\{a'_0 + \cdots + a'_n \mid n \in \mathbb{N}\} = \bigvee\{a + c(n) \mid n \in \mathbb{N}\} . \end{aligned}$$

Let $k \in \mathbb{N}$ and $\delta \in \Delta^{(k)}$. We prove that $\theta(\delta)$ is ω-continuous wrt $\le$. Let $a_1, \ldots, a_k \in A$, $i \in [k]$, and $c : \mathbb{N} \to A$ be an ω-chain. Let $(b_n \mid n \in \mathbb{N})$ and $(b'_n \mid n \in \mathbb{N})$ be families over A such that $c(n) = b_0 + \cdots + b_n$ and $b'_n = \theta(\delta)(a_1, \ldots, a_{i-1}, b_n, a_{i+1}, \ldots, a_k)$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned} &\theta(\delta)(a_1, \ldots, a_{i-1}, \bigvee\{c(n) \mid n \in \mathbb{N}\}, a_{i+1}, \ldots, a_k) \\ &= \theta(\delta)(a_1, \ldots, a_{i-1}, \sum_{n\in\mathbb{N}} b_n, a_{i+1}, \ldots, a_k) \\ &= \sum_{n\in\mathbb{N}} \theta(\delta)(a_1, \ldots, a_{i-1}, b_n, a_{i+1}, \ldots, a_k) && ((\mathcal{A}, \textstyle\sum) \text{ is } \omega\text{-distributive}) \\ &= \sum_{n\in\mathbb{N}} b'_n = \bigvee\{b'_0 + \cdots + b'_n \mid n \in \mathbb{N}\} \\ &= \bigvee\big\{\theta(\delta)(a_1, \ldots, a_{i-1}, b_0 + \cdots + b_n, a_{i+1}, \ldots, a_k) \mid n \in \mathbb{N}\big\} \\ &&& (\text{since } \mathcal{A} \text{ is distributive by Lemma 3.19}) \\ &= \bigvee\big\{\theta(\delta)(a_1, \ldots, a_{i-1}, c(n), a_{i+1}, \ldots, a_k) \mid n \in \mathbb{N}\big\} . \end{aligned}$$

We conclude that $(\mathcal{A}, \le)$ is an ω-continuous m-monoid.

It remains to prove that $(\mathcal{A}, \le)$ and $(\mathcal{A}, \sum)$ are related. Let I be a countable set and $(a_i \mid i \in I)$ be a family over A. There is an initial segment N of $\mathbb{N}$ and a bijection $\pi : N \to I$. By Statement 2 and Lemma 3.39(2) $\sum_{i\in I} a_i = \bigvee\{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in N\} = \bigvee\{\sum_{j\in J} a_j \mid J \in \mathcal{P}_{\text{fin}}(I)\}$.

4. This statement is a direct consequence of the third statement. ∎

3.4 Conclusion and open problems

In the present chapter we introduced the main algebraic notions that we are going to employ in the definition of the semantics of m-weighted monadic datalog programs. In the next chapter we will define different kinds of semantics; some of them are only applicable when a given m-monoid can be extended to an ω-continuous or an ω-complete m-monoid. A characterization of m-monoids that admit such extensions or a characterization of ω-complete m-monoids that admit a related ω-continuous m-monoid could therefore potentially benefit the theory of m-weighed monadic datalog programs.

CHAPTER 4

M-weighted monadic datalog programs

In this chapter we present the definition of m-weighted monadic datalog programs (for short: mwmd), the device that we will investigate in the remainder of this thesis. The syntactic structure of our definition is similar to the syntax of monadic datalog tree transducers [28], which are based on weighted monadic datalog [122]. Weighted monadic datalog is a combination of the concepts of monadic datalog [68, 69] and semiring-based constraint logic programming [18].

We will define two different types of semantics, which we call fixpoint semantics and hypergraph semantics. The fixpoint semantics is reminiscent of the initial algebra semantics of bottom-up weighted tree automata [15, 63], whereas the hypergraph semantics is related to the run semantics of weighted tree automata (or similar concepts such as m-weighted tree automata [103, 123]). The fixpoint semantics is inspired by the definition of the semantics of monadic datalog tree transducers [28], weighted monadic datalog [122], and monadic datalog [68, 69]. The concept of the hypergraph semantics is novel.

Roughly speaking, each type of semantics of mwmd takes three inputs: an mwmd, a tree, and an m-monoid. The semantics are defined in such a way that they evaluate the input tree according to the mwmd by applying operations from the m-monoid, and afterwards return an element of the m-monoid. Thus, when keeping the mwmd and the m-monoid fixed, the semantics are mappings from input trees to m-monoid elements.

It turns out that there are mwmd such that their semantics cannot be evaluated for arbitrary m-monoids. Such mwmd exhibit circular behavior when computing their semantics; as a consequence they require an ω-continuous m-monoid (for the fixpoint semantics) or an ω-complete m-monoid (for the hypergraph semantics) as input; moreover, we need to develop alternate variants of the fixpoint and the hypergraph semantics that can employ the strength of ω-continuous m-monoids and ω-complete m-monoids, respectively. Hence, we will define, study, and compare four different variants of semantics in this chapter (see Definitions 4.20, 4.29, 4.40, and 4.43).

This chapter is organized as follows. In Section 4.1 we will introduce the syntax and in Section 4.2 the semantics of mwmd. Section 4.2 consists of the following subsections: first we will deal with instantiations of mwmd programs (Section 4.2.1), then we will define the fixpoint semantics (Section 4.2.2) and the hypergraph semantics (Section 4.2.3). We conclude Section 4.2 by a comparison of fixpoint and hypergraph semantics (Section 4.2.4).

4.1 Syntax

In this section we define the syntactic structure of mwmd. An mwmd consists of three parts. The first component is a ranked alphabet, that does only contain nullary and unary elements; the elements of this alphabet are called user-defined predicates. The third component is a unary user-defined predicates and is called the query predicate.

The core of the mwmd is the second component, a finite collection of rules. Every rule consists of three parts, called head, body, and guard, and it is denoted in the form "head $\leftarrow$ body ; guard". We will explain and define the syntax of these three components below. First let us define their basic building blocks, which we call atoms.

For the remainder of this chapter we fix a ranked alphabet Σ and a signature Δ.

Definition 4.1. Let Γ be a ranked alphabet and let H be a set. We define $\Gamma(H) = \{\gamma(h_1, \ldots, h_k) \mid k \in \mathbb{N}, \gamma \in \Gamma^{(k)}, h_1, \ldots, h_k \in H\}$; obviously, $\Gamma(H) \subseteq T_\Gamma(H)$.

Throughout this thesis, we fix an infinite set V, the elements of which being called ***variables***; in examples we will use $x, y, z, x_1, x_2, \ldots$ as variables. We call $\Gamma(\mathrm{V})$ the *set of **atoms** (over Γ).* □

Definition 4.2. We define the ranked alphabet sp_Σ by letting

$$\mathrm{sp}_\Sigma = \{\mathrm{root}^{(1)}, \mathrm{leaf}^{(1)}\} \cup \{\mathrm{label}_\sigma^{(1)} \mid \sigma \in \Sigma\} \cup \{\mathrm{child}_i^{(2)} \mid i \in [\mathrm{maxrk}(\Sigma)]\} \ .$$

We refer to the elements of sp_Σ as ***structural predicates*** *over* Σ. □

Example 4.3. Consider the ranked alphabet $\Sigma_{\mathrm{ex}} = \{\alpha^{(0)}, \beta^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$. Then $\mathrm{sp}_{\Sigma_{\mathrm{ex}}} = \{\mathrm{root}^{(1)}, \mathrm{leaf}^{(1)}, \mathrm{label}_\alpha^{(1)}, \mathrm{label}_\beta^{(1)}, \mathrm{label}_\gamma^{(1)}, \mathrm{label}_\sigma^{(1)}, \mathrm{child}_1^{(2)}, \mathrm{child}_2^{(2)}\}$. Moreover, we obtain that the set $\mathrm{sp}_{\Sigma_{\mathrm{ex}}}(\mathrm{V})$ of atoms over $\mathrm{sp}_{\Sigma_{\mathrm{ex}}}$ contains, amongst other elements, $\mathrm{leaf}(x)$, $\mathrm{label}_\sigma(y)$, $\mathrm{child}_2(y, x)$, and $\mathrm{child}_1(z, z)$. □

Now let us explain the syntax of heads, bodies, and guards of rules. The head is always an atom over the set of used-defined predicates. The body is a tree over Δ that is allowed to have atoms over user-defined predicates as indices. Finally, the guard is a finite set of atoms over structural predicates. Now we present the core definition of this thesis.

Definition 4.4. A triple (P, R, q) is called an ***m-weighted monadic datalog program*** (for short: ***mwmd***) *over* Σ *and* Δ if

- P is a monadic ranked alphabet,
- $R \subseteq P(\mathrm{V}) \times T_\Delta(P(\mathrm{V})) \times \mathcal{P}_{\mathrm{fin}}(\mathrm{sp}_\Sigma(\mathrm{V}))$ is a finite set,
- $q \in P^{(1)}$.

Note that $T_\Delta(P(\mathrm{V}))$ is the set of trees over symbols from Δ which are indexed by elements of $P(\mathrm{V})$. We call the members of P ***user-defined predicates***, the members of R ***rules***, and the predicate q ***query predicate***.

Let $r = (h', b', G') \in R$. We denote r by $h' \leftarrow b' \,;\, G'$. Moreover, we call h', b', and G' the ***head***, ***body***, and ***guard*** *of* r, and denote them by r_h, r_b, and r_G, respectively. Thus, $r' = r'_\mathrm{h} \leftarrow r'_\mathrm{b} \,;\, r'_\mathrm{G}$ for every $r' \in R$. By $\mathrm{var}(r_\mathrm{h})$, $\mathrm{var}(\mathrm{r_b})$, and $\mathrm{var}(\mathrm{r_G})$ we denote the sets of ***variables occurring*** *in* r_h, r_b, *and* r_G, respectively. More precisely, we let $\mathrm{var}(r_\mathrm{h})$ be the smallest set $V' \subseteq \mathrm{V}$ such that $r_\mathrm{h} \in P(V')$. Likewise, we let $\mathrm{var}(r_\mathrm{b})$ and $\mathrm{var}(r_\mathrm{G})$ be the smallest sets $V' \subseteq \mathrm{V}$ and $V'' \subseteq \mathrm{V}$ such that $r_\mathrm{b} \in T_\Delta(P(V'))$ and $r_\mathrm{G} \in \mathcal{P}_{\mathrm{fin}}(\mathrm{sp}_\Sigma(V''))$, respectively. We put $\mathrm{var}(r) = \mathrm{var}(r_\mathrm{h}) \cup \mathrm{var}(r_\mathrm{b}) \cup \mathrm{var}(r_\mathrm{G})$.

We refer to the elements in $P(\mathrm{V})$ and $\mathrm{sp}_\Sigma(\mathrm{V})$ as ***user-defined atoms*** and ***structural atoms***, respectively. □

Example 4.5 (*Continuation of Example 4.3*)**.** Let $\Delta_{\text{ex}} = \Sigma_{\text{ex}}$ be a ranked alphabet. Then $M_1 = (P_1, R_1, q)$ is an mwmd over Σ_{ex} and Δ_{ex}, where $P_1 = \{q^{(1)}, p^{(0)}\}$ and $R_1 = \{r_1, r_2, r_3\}$ such that

$$\begin{aligned}
r_1 &= q(x) \leftarrow \sigma\big(q(y), p()\big) \,;\, \{\text{child}_2(x, y)\} \;, \\
r_2 &= p() \leftarrow \alpha \,;\, \{\text{label}_\gamma(y)\} \;, \\
r_3 &= q(x) \leftarrow \beta \,;\, \{\text{leaf}(x), \text{root}(z), \text{label}_\sigma(z)\} \;.
\end{aligned}$$

Clearly, $\text{var}(r_1) = \{x, y\}$, $\text{var}(r_2) = \{y\}$, and $\text{var}(r_3) = \{x, z\}$. Another more elaborate example is the mwmd $M_2 = (P_2, R_2, q)$, where $P_2 = \{q^{(1)}, p^{(1)}, r^{(1)}\}$, and R_2 consists of the following rules:

$$\begin{aligned}
r_4 &= q(x) \leftarrow q(y) \,;\, \{\text{child}_2(x, y)\} \;, \\
r_5 &= q(x) \leftarrow \sigma\big(p(y), r(x)\big) \,;\, \{\text{leaf}(x), \text{child}_2(y, x)\} \;, \\
r_6 &= p(x) \leftarrow r(z) \,;\, \{\text{root}(x), \text{child}_1(x, z)\} \;, \\
r_7 &= p(x) \leftarrow \sigma\big(p(y), r(z)\big) \,;\, \{\text{child}_2(y, x), \text{child}_1(x, z)\} \;, \\
r_8 &= r(x) \leftarrow \alpha \,;\, \{\text{label}_\alpha(x)\} \;, \\
r_9 &= r(x) \leftarrow \beta \,;\, \{\text{label}_\beta(x)\} \;.
\end{aligned}$$

We will study the behavior of the mwmd M_1 and M_2 in the next section. □

4.2 Semantics

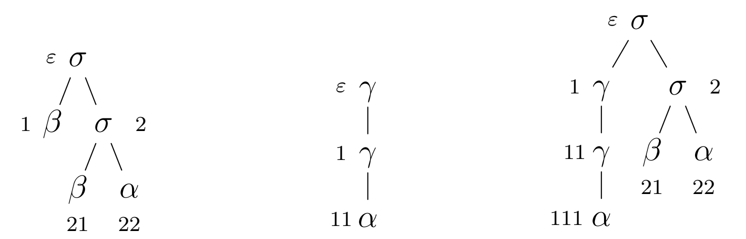
In this section we present the semantics of mwmd. We are going to define four variants of semantics as we have already mentioned in the introduction of this chapter. Roughly speaking, each of them associate with every valid combination of an mwmd M and an m-monoid $\mathcal{A}$ a mapping from the set T_Σ of input trees to the carrier set of $\mathcal{A}$. All of them use the concept of rule instances, that we need to define and study as a preliminary step before we can define the semantics formally.

In the sequel let $M = (P, R, q)$ be an mwmd over Σ and Δ.

4.2.1 Instantiations

Given an input tree $t \in T_\Sigma$, the variables from the set V that occur in rules of M are to be interpreted as positions of the input tree. Such an assignment of variables to positions of t can be modeled as a mapping from the set of variables to $\text{pos}(t)$. We will apply variable assignments at the level of rules; i.e., all occurrences of the same variable within one rule always represent the same position in t whereas occurrences of the same variable in different rules may stand for different positions.

When computing the semantics of mwmd we are only interested in certain combinations of a rule and an assignment of variables occurring in it; these are, roughly speaking, those combinations such that the instantiated guard of the rule satisfies the input tree t. As an example assume that the guard contains the atoms $\text{label}_\alpha(x)$ and $\text{leaf}(y)$; then we are only interested in those variable assignments that map x to a position that is labeled α and y to a leaf position in t. These concepts are specified formally in the following two definitions.



(a) t_1 (b) t_2 (c) t_3

Figure 4.1: Diagrams of the trees from Example 4.8.

Definition 4.6. Let $t \in T_\Sigma$. We refer to the elements in $P(\mathrm{pos}(t))$ and $\mathrm{sp}_\Sigma(\mathrm{pos}(t))$ as ***user-defined atom instances*** and ***structural atom instances*** (*over* t), respectively. The tree t constitutes a set $\mathrm{B}_t \subseteq \mathrm{sp}_\Sigma(\mathrm{pos}(t))$ of t-***compatible structural atom instances*** defined as:

$$\begin{aligned}\mathrm{B}_t = &\big\{\mathrm{root}(\varepsilon)\big\} \cup \big\{\mathrm{leaf}(w) \mid w \in \mathrm{pos}(t), t(w) \in \Sigma^{(0)}\big\}\\ &\cup \big\{\mathrm{child}_i(w, wi) \mid w \in \mathrm{pos}(t), i \in [\mathrm{rk}(t(w))]\big\}\\ &\cup \big\{\mathrm{label}_\sigma(w) \mid w \in \mathrm{pos}(t), t(w) = \sigma\big\}\,.\end{aligned}$$ □

Definition 4.7. Let $t \in T_\Sigma$ and $r \in R$. An r, t-***variable assignment*** is a mapping $\rho : \mathrm{var}(r) \to \mathrm{pos}(t)$. Let ρ be an r, t-variable assignment. For every ranked alphabet Γ we lift ρ to three mappings $\rho_{1,\Gamma} : \Gamma(\mathrm{var}(r)) \to \Gamma(\mathrm{pos}(t))$, $\rho_2 : T_\Delta\big(P(\mathrm{var}(r))\big) \to T_\Delta\big(P(\mathrm{pos}(t))\big)$, and $\rho_3 : \mathcal{P}\big(\mathrm{sp}_\Sigma(\mathrm{var}(r))\big) \to \mathcal{P}\big(\mathrm{sp}_\Sigma(\mathrm{pos}(t))\big)$ as follows:

- $\rho_{1,\Gamma}$ is the restriction of ρ' to $\Gamma(\mathrm{var}(r))$, where ρ' is the unique Γ-homomorphism from $\mathcal{T}_\Gamma(\mathrm{var}(r))$ to $\mathcal{T}_\Gamma(\mathrm{pos}(t))$ extending ρ.
- ρ_2 is the unique Δ-homomorphism from $\mathcal{T}_\Delta(P(\mathrm{var}(r)))$ to $\mathcal{T}_\Delta(P(\mathrm{pos}(t)))$ extending the mapping $\rho_{1,P}$.
- $\rho_3(G) = \{\rho_{1,\mathrm{sp}_\Sigma}(g) \mid g \in G\}$ for every $G \subseteq \mathrm{sp}_\Sigma(\mathrm{var}(r))$.

For the sake of simplicity we write ρ instead of $\rho_{1,\Gamma}$, ρ_2, and ρ_3. If $\rho(r_\mathrm{G}) \subseteq \mathrm{B}_t$, then we call ρ ***valid***.

The set of ***rule instances*** *of* M *and* t is the set $\Phi_{M,t}$ of pairs (r', ρ') such that $r' \in R$ and ρ' is a valid r', t-variable assignment. A rule instance (r', ρ') of M and t is called ***ε-rule instance*** if $\rho'(r'_\mathrm{b}) \in P(\mathrm{pos}(t))$. For every $c \in P(\mathrm{pos}(t))$ we define the set $\Phi_{M,t,c}$ of ***rule instances*** *of* M *and* t *for* c by letting $\Phi_{M,t,c} = \{(r', \rho') \in \Phi_{M,t} \mid \rho'(r'_\mathrm{h}) = c\}$. □

Example 4.8 (*Continuation of Example 4.5*)**.** In this example we consider three example trees $t_1, t_2, t_3 \in T_{\Sigma_\mathrm{ex}}$. These trees are shown in Figure 4.1. Let $t_1 = \sigma(\beta, \sigma(\beta, \alpha))$ and let us determine the set Φ_{M_1,t_1}. Clearly, an r_1, t_1-variable assignment is a mapping $\rho : \{x, y\} \to \mathrm{pos}(t_1)$; if ρ is valid, then it must map y to the second child of x; hence, $\rho = [x \mapsto \varepsilon, y \mapsto 2]$ or $\rho = [x \mapsto 2, y \mapsto 22]$. There are no valid r_2, t_1-variable assignments because there is no position labeled γ in t_1. Every valid r_3, t_1-variable assignments

ρ maps x to a leaf position in t and z to the root of t; hence, $\rho = [x \mapsto 1, z \mapsto \varepsilon]$, $\rho = [x \mapsto 21, z \mapsto \varepsilon]$, or $\rho = [x \mapsto 22, z \mapsto \varepsilon]$. We obtain that

$$\begin{aligned}\Phi_{M_1,t_1} &= \big\{(r_1, [x \mapsto \varepsilon, y \mapsto 2]), (r_1, [x \mapsto 2, y \mapsto 22])\big\} \\ &\cup \big\{(r_3, [x \mapsto 1, z \mapsto \varepsilon]), (r_3, [x \mapsto 21, z \mapsto \varepsilon]), (r_3, [x \mapsto 22, z \mapsto \varepsilon])\big\} .\end{aligned}$$

For the pair $(r, \rho) = (r_1, [x \mapsto \varepsilon, y \mapsto 2])$ we have that $\rho(r_{\mathrm{h}}) = q(\varepsilon)$; likewise, for $(r, \rho) = (r_1, [x \mapsto 2, y \mapsto 22])$ we obtain $\rho(r_{\mathrm{h}}) = q(2)$. Observe that for every $v \in \mathrm{pos}(t_1) = \{\varepsilon, 1, 2, 21, 22\}$ there is a unique $(r, \rho) \in \Phi_{M_1,t_1}$ such that $\rho(r_{\mathrm{h}}) = q(v)$. Hence, for every $v \in \mathrm{pos}(t_1)$ we have

$$\Phi_{M_1,t_1,q(v)} = \begin{cases} \big\{(r_1, [x \mapsto v, y \mapsto v \cdot 2])\big\} , & \text{if } v \in \{\varepsilon, 2\}, \\ \big\{(r_3, [x \mapsto v, z \mapsto \varepsilon])\big\} , & \text{otherwise.} \end{cases}$$

Moreover, $\Phi_{M_1,t_1,p()} = \emptyset$. Now let us consider the tree $t_2 = \gamma(\gamma(\alpha))$. Then there is no valid r_1, t_2-variable assignment because there is no position in t_2 that is the second child of any other position in t. There are precisely two valid r_2, t_2-variable assignments, namely $\rho_1 = [y \mapsto \varepsilon]$ and $\rho_2 = [y \mapsto 1]$. Furthermore, there is no valid r_3, t_2-variable assignment because the root position is not labeled σ. Hence, we obtain that $\Phi_{M_1,t_2} = \Phi_{M_1,t_2,p()} = \big\{(r_2, [y \mapsto \varepsilon]), (r_2, [y \mapsto 1])\big\}$.

Finally, let $t_3 = \sigma(\gamma(\gamma(\alpha)), \sigma(\beta, \alpha))$. Then we obtain for every $v \in \mathrm{pos}(t_3)$:

$$\begin{aligned}\Phi_{M_1,t_3} &= \big\{(r_1, [x \mapsto \varepsilon, y \mapsto 2]), (r_1, [x \mapsto 2, y \mapsto 22])\big\} \\ &\cup \big\{(r_2, [y \mapsto 1]), (r_2, [y \mapsto 11]), (r_3, [x \mapsto 111, z \mapsto \varepsilon])\big\} \\ &\cup \big\{(r_3, [x \mapsto 21, z \mapsto \varepsilon]), (r_3, [x \mapsto 22, z \mapsto \varepsilon])\big\} , \\ \Phi_{M_1,t_3,q(v)} &= \begin{cases} \big\{(r_1, [x \mapsto v, y \mapsto v \cdot 2])\big\} , & \text{if } v \in \{\varepsilon, 2\}, \\ \emptyset , & \text{if } v \in \{1, 11\}, \\ \big\{(r_3, [x \mapsto v, z \mapsto \varepsilon])\big\} , & \text{otherwise,} \end{cases} \\ \Phi_{M_1,t_3,p()} &= \big\{(r_2, [y \mapsto 1]), (r_2, [y \mapsto 11])\big\} .\end{aligned}$$

These examples show that rule r_2 is only applicable (i.e., has an instance) if there is a γ-labeled position in the input tree, and rule r_3 can only be applied if the root of the input tree is labeled σ.

Let us conclude this example with determining the rule instances Φ_{M_2,t_1} of the mwmd M_2 and t_1. We have

$$\begin{aligned}\Phi_{M_2,t_1} &= \big\{(r_4, [x \mapsto \varepsilon, y \mapsto 2]), (r_4, [x \mapsto 2, y \mapsto 22])\big\} \\ &\cup \big\{(r_5, [x \mapsto 22, y \mapsto 2])\big\} \\ &\cup \big\{(r_6, [x \mapsto \varepsilon, z \mapsto 1])\big\} \\ &\cup \big\{(r_7, [x \mapsto 2, y \mapsto \varepsilon, z \mapsto 21])\big\} \\ &\cup \big\{(r_8, [x \mapsto 22]), (r_9, [x \mapsto 1]), (r_9, [x \mapsto 21])\big\} .\end{aligned}$$

Observe that the three rule instances $(r_4, [x \mapsto \varepsilon, y \mapsto 2])$, $(r_4, [x \mapsto 2, y \mapsto 22])$, and $(r_6, [x \mapsto \varepsilon, z \mapsto 1])$ are ε-rule instances. □

Dependency graphs and dependency hypergraphs

All kinds of the semantics that we will define in this thesis make direct use of the set of rule instances that we defined in the previous section. Intuitively, the user-defined atom instances are associated with elements of the carrier set of the considered m-monoid, and the rule instances provide the means to compute this association. As an example consider the following rule instance:

$$\big(p(x) \leftarrow \sigma(p(y), q(x))\,;\{\text{child}_1(x,y)\}\,,\ [x \mapsto 2, y \mapsto 21]\big)\ .$$

Roughly speaking, this rule instance signifies that the value of the atom instance $p(2)$ is obtained by applying the operation σ (evaluated in the considered m-monoid) to the values of the atom instances of $p(21)$ and $q(2)$; note that the guard $\{\text{child}_1(x,y)\}$ can be disregarded in this phase of the computation of the semantics; in fact, it was only required for determining the set of valid variable assignments and, thus, the set of rules instances. Therefore, the above rule instance asserts that the associated value of $p(2)$ depends on the values of $p(21)$ and $q(2)$. The collection of dependencies between user-defined atom instances that are induced by the rule instances in this manner can be represented by a hypergraph, called the dependency hypergraph. The dependency hypergraph will turn out to be a useful tool when dealing with the semantics that we will define later in this chapter.

Definition 4.9. Let $t \in T_\Sigma$. The ***dependency hypergraph*** *of M and t* is the hypergraph $\text{G}^{\text{dep}}_{M,t} = \big(P(\text{pos}(t)), \Phi_{M,t}, \mu\big)$ such that, for every $(r,\rho) \in \Phi_{M,t}$, $\mu((r,\rho)) = \big(\text{indyield}(\rho(r_\text{b})), \rho(r_\text{h})\big)$. Note that (due to our definition of hypergraphs) in this context we consider $\Phi_{M,t}$ as a ranked alphabet, where the rank of every element $(r,\rho) \in \Phi_{M,t}$ is $|\text{indyield}(\rho(r_\text{b}))|$. Let $G = \text{G}^{\text{dep}}_{M,t}$ and $c \in P(\text{pos}(t))$. We refer to the elements in H^c_G as ***derivations*** *of M and t ending in c.*

The digraph reduct of the dependency hypergraph of M and t is called the ***dependency graph*** *of M and t.* □

The definition of the dependency graph is inspired by the definition of dependency graphs for attribute grammars [32, 50] and attributed tree transducers [56, 60]. We will use dependency graphs only rarely in this thesis because dependency hypergraphs contain more information (and, thus, are more useful for our purposes) and allow for a more sophisticated definition of non-circularity (see Definition 4.12). Note that both dependency graphs and dependency hypergraphs are independent from any particular m-monoids.

Example 4.10 (*Continuation of Example 4.8*)**.** In Figure 4.2 we present the dependency hypergraphs of the combinations of mwmd and input trees that we considered in Example 4.8. Note that, for reasons of simplicity, we only depicted those vertices (atom instances) that are the output or an input vertex for some hyperedge. We will follow this convention throughout this thesis. □

Clearly, when using dependency hypergraphs, we are allowed to employ the generic terminology of hypergraphs. For example, we can consider some rule instance (r,ρ) as a hyperedge whose output vertex is the atom instance $\rho(r_\text{h})$ that is obtained by instantiating the head r_h by the variable assignment ρ. This is stated formally by the following Observation. We will use these terminologies interchangeably whenever it suits our purposes. Note that the last statement of Observation 4.11 follows from Corollary 2.16.

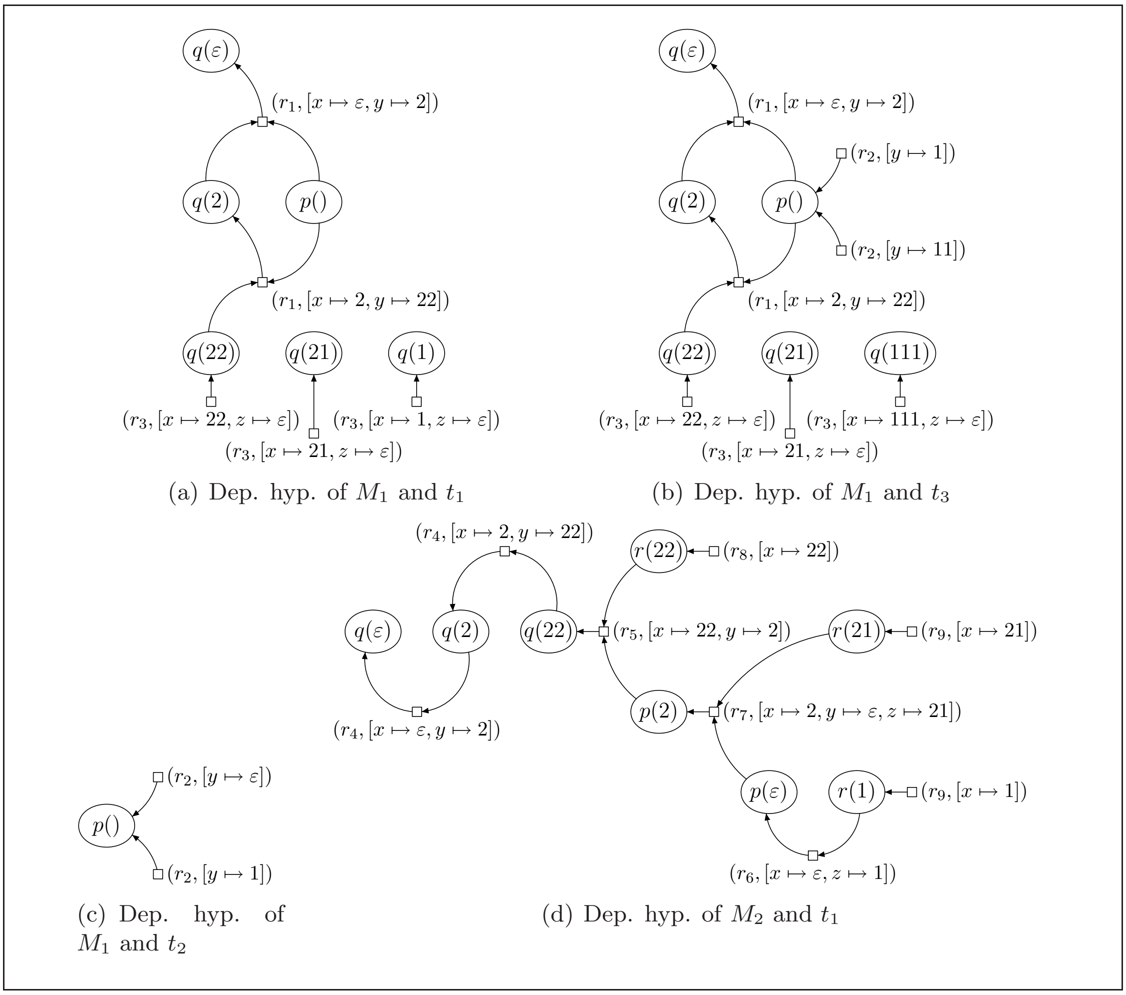


Figure 4.2: Dependency hypergraphs of Example 4.10.

Observation 4.11. *Let* $t \in T_\Sigma$ *and* $c \in P(\mathrm{pos}(t))$.

1. $\{e \in \Phi_{M,t} \mid \mathrm{out}(e) = c\} = \Phi_{M,t,c}$.

2. *Let* $e = (r, \rho) \in \Phi_{M,t}$ *and* $\mathrm{indyield}(\rho(r_\mathrm{b})) = c_1 \cdots c_k$ *for some* $k \in \mathbb{N}$, *and* $c_1, \ldots, c_k \in P(\mathrm{pos}(t))$. *Then* $k = \mathrm{rk}(e)$ *and* $\mathrm{in}_i(e) = c_i$ *for every* $i \in [k]$.

3. *Let* E *be the set of edges of the dependency graph of* M *and* t. *Then for every* $c' \in P(\mathrm{pos}(t))$, $(c', c) \in E$ *iff there is a* $(r, \rho) \in \Phi_{M,t,c}$ *such that* $c' \in \mathrm{ind}(\rho(r_\mathrm{b}))$.

4. *For every* $e = (r, \rho) \in \Phi_{M,t,c}$ *let* $k_e \in \mathbb{N}$ *and* $c^e_1, \ldots, c^e_{k_e} \in P(\mathrm{pos}(t))$ *such that* $c^e_1 \cdots c^e_{k_e} = \mathrm{indyield}(\rho(r_\mathrm{b}))$. *Let* $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. *Then*

$$\mathrm{H}^c_G = \{e(\eta_1, \ldots, \eta_{k_e}) \mid e \in \Phi_{M,t,c}, \eta_1 \in \mathrm{H}^{c^1_e}_G, \ldots, \eta_{k_e} \in \mathrm{H}^{c^e_{k_e}}_G)\} .$$

Consider the dependency hypergraph of Figure 4.2(a). The value of the atom instance $q(\varepsilon)$ depends on the values of $q(2)$ and $p()$, and the value of $q(2)$ depends on $q(22)$ and

$p()$. Intuitively, the values of these atom instances should be computed in the following order: first compute the values of $q(22)$ and $p()$, then of $q(2)$, and finally of $q(\varepsilon)$. However, such a topological ordering is not possible when the dependencies are circular, i.e., when some atom instance depends directly or indirectly on itself; then there might not be a meaningful way to associate values of the carrier set of the considered m-monoid with atom instances. Roughly speaking, in general the semantics are only definable for mwmd which do not exhibit such circularities. The notions of circularity and non-circularity are defined formally in the following definition; they are inspired by the definitions of circularity for attribute grammars [32, 50], attributed tree transducers [56, 60], and weighted monadic datalog [122].

Definition 4.12. The mwmd M is called ***circular*** if there is a $t \in T_\Sigma$ such that the dependency graph of M and t is cyclic. Otherwise M is called ***non-circular***.

The mwmd M is called ***weakly non-circular*** if, for every $t \in T_\Sigma$, $\mathrm{H}_G^{q(\varepsilon)}$ is finite, where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. □

In the previous definition we introduced another version of non-circularity, which we call weak non-circularity. We defined this notion for the following two reasons:

- The definition of non-circularity (according to the following definition) is unnecessarily strong because there are mwmd that are circular but nevertheless allow for a meaningful definition of the semantics. This case occurs if, for instance, the cycle has no impact on the value of the atom instance $p(\varepsilon)$ (where p is the query predicate; the value of $p(\varepsilon)$ is the value that we are interested in at the end of the computation of the semantics, i.e., it is the resulting value of the semantics); more precisely, if $p(\varepsilon)$ does not depend on the cycle in the dependency hypergraph.

- In Chapter 5 we will introduce syntactic normal forms of mwmd. In general these constructions do not preserve the property of non-circularity. For instance, consider the rule $p(x) \leftarrow \sigma(p(x), q(y))\ ;\{\mathrm{root}(x), \mathrm{leaf}(x), \mathrm{child}_1(y,z)\}$. This rule does not lead to cyclic dependencies because there is no valid variable assignment: if both $\mathrm{root}(x)$ and $\mathrm{leaf}(x)$ are satisfied, then the input tree has only one node, which is the root node; if, however, $\mathrm{child}_1(y,z)$ is satisfied, then the input tree must have at least two nodes. When constructing the connected normal form (see Lemma 5.33), then this rule is replaced by the two rules $p(x) \leftarrow \sigma(p(x), r())\ ;\{\mathrm{root}(x), \mathrm{leaf}(x)\}$ and $r() \leftarrow q(y)\ ;\{\mathrm{child}_1(y,z)\}$. Now the resulting mwmd is circular because for any input tree consisting of only one node the rule $p(x) \leftarrow \sigma(p(x), r());\{\mathrm{root}(x), \mathrm{leaf}(x)\}$ together with the variable assignment $[x \mapsto \varepsilon]$ will lead to cycles in the dependency graph. While non-circularity is not preserved by this construction, the property of weak non-circularity is preserved.

The following lemma shows that the name "weak non-circularity" is justified.

Lemma 4.13. *Let M be non-circular. Then M is weakly non-circular.*

Proof. Assume that there is a $t \in T_\Sigma$ such that $\mathrm{H}_G^{q(\varepsilon)}$ is infinite, where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. We will derive a contradiction. By Lemma 2.25 there is a $c \in P(\mathrm{pos}(t))$ such that $c \prec_G^+ c \prec_G^* q(\varepsilon)$. Then Lemma 2.31 implies $(c,c) \in E^+$, where E is the set of edges of the dependency graph G' of M and t; thus, G' is cyclic, a contradiction. ■

Example 4.14 (*Continuation of Example 4.10*)**.** Consider the mwmd M_1 from Example 4.10 and the input tree t_1. From Figure 4.2(a) it is easy to see that the dependency graph of M_1 and t_1 has precisely four edges, namely $(q(2), q(\varepsilon))$, $(p(), q(\varepsilon))$, $(q(22), q(2))$, and $(p(), q(2))$; thus, the dependency graph is not cyclic. Similarly, the dependency graphs of M_1 and t_2 as well as of M_1 and t_3 are not cyclic. It is easy to check that this holds for every input tree; hence, M_1 is non-circular. Likewise, Figure 4.2(d) shows that the dependency graph of M_2 and t_1 is not cyclic. Again, it is easy to check that M_2 is non-circular.

By Lemma 4.13 we conclude that both M_1 and M_2 are weakly non-circular. Let us establish this property for M_1 explicitely. First consider the dependency hypergraph of M_1 and t_1 (Figure 4.2(a)). Observe that the set $\mathrm{H}_G^{q(\varepsilon)}$ is empty, where $G = \mathrm{G}_{M_1,t_1}^{\mathrm{dep}}$; this is due to the fact that there is no derivation ending in $p()$ and, hence, no derivation ending in $q(\varepsilon)$. In particular we obtain that $\mathrm{H}_G^{q(\varepsilon)}$ is finite. Now consider the input tree t_3 and the according dependency hypergraph (Figure 4.2(b)). We define $e_1 = (r_1, [x \mapsto \varepsilon, y \mapsto 2])$, $e_2 = (r_1, [x \mapsto 2, y \mapsto 22])$, $e_3 = (r_3, [x \mapsto 22, z \mapsto \varepsilon])$, $e_4 = (r_2, [y \mapsto 1])$, and $e_5 = (r_2, [y \mapsto 11])$. There are two derivations ending in $p()$, namely e_4 and e_5; therefore, there are four derivations ending in $q(\varepsilon)$, namely $e_1(e_2(e_3, e_4), e_4)$, $e_1(e_2(e_3, e_4), e_5)$, $e_1(e_2(e_3, e_5), e_4)$, and $e_1(e_2(e_3, e_5), e_5)$; hence, $\mathrm{H}_G^{q(\varepsilon)}$ is finite, where $G = \mathrm{G}_{M_1,t_3}^{\mathrm{dep}}$. It is easy to check that for every input tree the number of derivations ending in $q(\varepsilon)$ is finite.

Now let us consider a more interesting example. Let $M_3 = (P_3, R_3, q)$ be an mwmd over Σ_{ex} and Δ_{ex}, where $P_3 = (q^{(1)}, p^{(0)}, r^{(0)})$ and R_3 consists of the following rules:

$$\begin{aligned} r_{10} &= q(x) \leftarrow p() \,;\, \{\mathrm{root}(y), \mathrm{label}_\alpha(y)\} \,, \\ r_{11} &= p() \leftarrow \sigma(p(), r()) \,;\, \emptyset \,, \\ r_{12} &= r() \leftarrow \alpha \,;\, \{\mathrm{root}(y), \mathrm{label}_\beta(y)\} \,, \\ r_{13} &= p() \leftarrow \beta \,;\, \emptyset \,. \end{aligned}$$

It is easy to see that M_3 is circular because for every input tree the rule r_{11} together with the empty variable assignment induces the edge $(p(), p())$ in the dependency graph. However, M_3 is weakly non-circular. Consider the dependency hypergraphs for the input trees $t_4 = \alpha$ and $t_5 = \beta$ in Figure 4.3. Although there is a cyclic dependency of $p()$ on itself, the set of derivations ending in $q(\varepsilon)$ is finite in both cases. Note that this holds for different reasons for each of the trees t_4 and t_5. For t_4 the cycle is "not active" because there is no derivation ending in $r()$; therefore, there is only one derivation ending in $q(\varepsilon)$, namely $(r_{10}, [y \mapsto \varepsilon])\big((r_{13}, [\ ])\big)$. For the input tree t_5 the cycle is active (the set of derivations ending in $p()$ is infinite; it is the set of all left-descending combs of the form $e_2(e_2(\cdots e_2(e_2(e_1, e_3), e_3)\cdots), e_3)$, where $e_1 = (r_{13}, [\ ])$ and $e_2 = (r_{11}, [\ ])$ and $e_3 = (r_{12}, [y \mapsto \varepsilon])$); however, this cycle cannot be reached from $q(\varepsilon)$; the set of derivations ending in $q(\varepsilon)$ is empty. It is easy to see that for every other input tree the set of derivations ending in $q(\varepsilon)$ is empty as well.

Let r'_{10} originate from r_{10} by replacing its guard by the empty set and let M_4 originate from M_3 by replacing r_{10} by r'_{10}. Then M_4 is not even weakly non-circular because for the input tree t_5 the set of derivations ending in $q(\varepsilon)$ is infinite (see Figure 4.3(c)). □

In Example 4.14 we argued that it is easy to check that the mwmd M_1 and M_2 are non-circular and weakly non-circular. In general this checking might be hard as there are infinitely many input trees that have to be analyzed. In Chapter 6 we will show rigor-

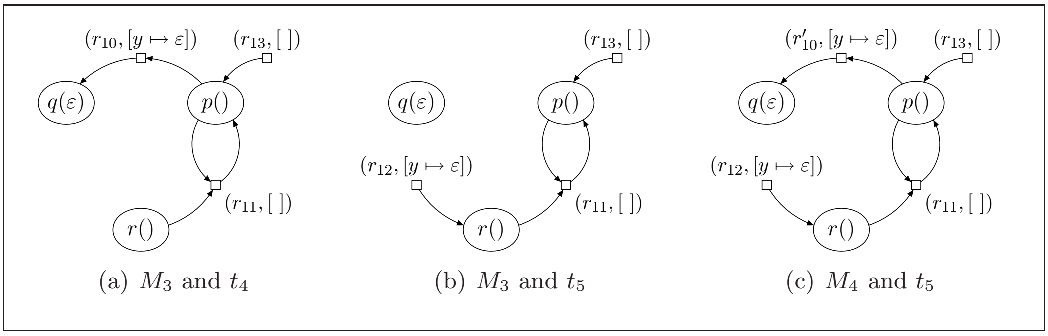


Figure 4.3: Dependency hypergraphs of M_3 and M_4 (Example 4.14).

ously that both the properties of non-circularity and weak non-circularity are effectively decidable.

Finitary and infinitary semantics

In the introduction to this chapter we explained that we are going to define two types of semantics: fixpoint semantics and hypergraph semantics. In the previous section we discussed that, in general, these semantics are only applicable if the considered mwmd is weakly non-circular. For each of these two types of semantics we will define an alternative version that can be applied to arbitrary mwmd, even to mwmd that are not weakly non-circular; let us call this version the infinitary semantics and the version that is only applicable to weakly non-circular mwmd the finitary semantics.

Clearly, the infinitary semantics are not applicable in general. Instead of requiring the considered mwmd to be weakly non-circular they require the considered m-monoid to be capable of carrying out "infinitary computations". More precisely, the infinitary fixpoint semantics is only defined for ω-continuous m-monoids and the infinitary hypergraph semantics is only defined for ω-complete m-monoids. An overview of all kinds of semantics that we define in this thesis and their respective restrictions on the mwmd and the m-monoid is given in Table 4.1.

	fixpoint semantics	hypergraph semantics
finitary	weakly non-circular mwmd absorptive m-monoid	weakly non-circular mwmd arbitrary m-monoid
	non-circular mwmd arbitrary m-monoid	
infinitary	arbitrary mwmd ω-continuous m-monoid	arbitrary mwmd ω-complete m-monoid

Table 4.1: In this thesis we define four kinds of semantics.

Note that the finitary fixpoint semantics is only defined if (i) the mwmd is weakly non-circular and the m-monoid is absorptive or (ii) the mwmd is non-circular and the m-monoid is arbitrary. We included Case (ii) only for completeness and we will show that, roughly speaking, Case (ii) is subsumed by Case (i) (see Proposition 4.22).

For the remainder of this section we fix an m-monoid $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ over Δ.

4.2.2 Fixpoint semantics

The fixpoint semantics, which is based on the application of an immediate consequence operator, is inspired by the definition of the semantics for the Horn calculus [102, 74, 75] and monadic datalog [68, 69].

The idea of the fixpoint semantics is as follows. Given an mwmd, an input tree, and an m-monoid, we associate every atom instance with an element of the carrier set of the m-monoid; such an association is called an interpretation. Among all interpretations is one designated interpretation, which is computed by the fixpoint semantics in a stepwise manner. Finally, the semantics returns the element of the m-monoid that is associated with the atom instance $q(\varepsilon)$ by the designated interpretation, i.e., the element that the designated interpretation associates with the query predicate at the root of the input tree.

The computation of the designated interpretation is based on the immediate consequence operator; this is a mapping from the set of interpretations into itself; it takes an interpretation I and returns an interpretation, called the consequence interpretation of I. The computation of the consequence interpretation is guided by the rule instances. As an example consider the following rule instance:

$$\big(p(x) \leftarrow \sigma(p(y), q(x))\ ; \{\mathrm{child}_1(x, y)\}\ ,\ [x \mapsto 2, y \mapsto 21]\big)\ .$$

Its meaning is that the value of $p(2)$ under the consequence interpretation of I is the result of applying the operation σ (evaluated in the considered m-monoid) to the values of the atom instances of $p(21)$ and $q(2)$ under I. If there are multiple rule instances for $p(2)$ (i.e., rule instances whose output vertex is $p(2)$ in the dependency hypergraph), then their resulting values should be added up by using the monoid operation of the m-monoid.

Definition 4.15. Let $t \in T_\Sigma$. An ***interpretation*** *over M, t, and $\mathcal{A}$* is a mapping $I : P(\mathrm{pos}(t)) \to A$. We denote the set of all interpretations over M, t, and $\mathcal{A}$ by $\mathcal{I}_{M,t,\mathcal{A}}$. For every $I \in \mathcal{I}_{M,t,\mathcal{A}}$ we denote by h_I the unique Δ-homomorphism from $T_\Delta\big(P(\mathrm{pos}(t))\big)$ to (A, θ) extending the interpretation I. The ***empty interpretation*** $I_\mathbf{0} \in \mathcal{I}_{M,t,\mathcal{A}}$ is defined by $I_\mathbf{0}(c) = \mathbf{0}$ for every $c \in P(\mathrm{pos}(t))$.

We define the ***immediate consequence operator*** $\mathcal{T}_{M,t,\mathcal{A}} : \mathcal{I}_{M,t,\mathcal{A}} \to \mathcal{I}_{M,t,\mathcal{A}}$ *over M, t, and $\mathcal{A}$* by letting for every $I \in \mathcal{I}_{M,t,\mathcal{A}}$ and $c \in P(\mathrm{pos}(t))$:

$$\mathcal{T}_{M,t,\mathcal{A}}(I)(c) = \sum\nolimits_{(r,\rho) \in \Phi_{M,t,c}} \mathrm{h}_I(\rho(r_\mathrm{b}))\ ;$$

we call $\mathcal{T}_{M,t,\mathcal{A}}(I)$ the ***consequence interpretation*** *of I*. Note that this sum is always finite. If M, t, and $\mathcal{A}$ are clear from the context, we simply write $\mathcal{I}$ and $\mathcal{T}$ instead of $\mathcal{I}_{M,t,\mathcal{A}}$ and $\mathcal{T}_{M,t,\mathcal{A}}$, respectively. □

Example 4.16 (*Continuation of Example 4.14***).** Let us consider the m-monoid $\mathcal{A}_1 = (\mathbb{R}_{\geq 0} \cup \{\infty\}, +, 0, \theta_1)$ over Δ_ex, where for every $a, b \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ we have

$$\theta_1(\alpha)() = 3\ , \qquad \theta_1(\beta)() = 2\ , \qquad \theta_1(\gamma)(a) = a$$

$$\theta_1(\sigma)(a, b) = \begin{cases} 0\ , & \text{if } 0 \in \{a, b\}\ , \\ \infty\ , & \text{if } a \neq 0 \neq b \text{ and } \infty \in \{a, b\}\ , \\ a/2 + b + 1\ , & \text{otherwise.} \end{cases}$$

Note that $\mathcal{A}_1$ is absorptive but not distributive; in fact, we obtain that $\theta_1(\sigma)(2, 1+3) = 1 + (1+3) + 1 = 6 \neq 8 = 3 + 5 = \theta_1(\sigma)(2,1) + \theta_1(\sigma)(2,3)$.

In the following table we give three examples of interpretations over M_1, t_1, and $\mathcal{A}_1$, and compute their respective consequence interpretations. As an example we compute the value of $\mathcal{T}(I_1)(q(\varepsilon))$ explicitely. Using Figure 4.2(a) it is easy to see that $\Phi_{M_1,t_1,q(\varepsilon)} = \{(r_1, [x \mapsto \varepsilon, y \mapsto 2])\}$; then $\mathcal{T}(I_1)(q(\varepsilon)) = \mathrm{h}_{I_1}(\sigma(q(2), p())) = \theta_1(\sigma)(I_1(q(2)), I_1(p())) = \theta_1(\sigma)(6,1) = 3 + 1 + 1 = 5$.

	I_1	$\mathcal{T}(I_1)$	I_2	$\mathcal{T}(I_2)$	I_3	$\mathcal{T}(I_3)$
$q(\varepsilon)$	4	5	2	6	0	0
$q(1)$	3	2	1	2	2	2
$q(2)$	6	4	2	0	0	0
$q(21)$	1	2	0	2	2	2
$q(22)$	4	2	0	2	2	2
$p()$	1	0	4	0	0	0

Observe that the immediate consequence operator leaves interpretation I_3 unchanged. We call such an interpretation a *fixpoint interpretation*. It is easy to check that I_3 is the only fixpoint interpretation over M_1, t_1, and $\mathcal{A}_1$. For the input tree t_3 we obtain that there is also a unique fixpoint interpretation, which looks as follows (this can easily be verified by using Figure 4.2(b)):

$q(\varepsilon)$	$q(1)$	$q(11)$	$q(111)$	$q(2)$	$q(21)$	$q(22)$	$p()$
11	0	0	2	8	2	2	6

□

In Example 4.16 we turned our attention to fixpoint interpretations. It is self-evident that we are interested in computing such a fixpoint interpretation when evaluating the semantics because, roughly speaking, fixpoint interpretations are consistent with the rule instances. However, we have to deal with two problems: (i) a fixpoint interpretation does not necessarily exist and (ii) if there are multiple fixpoint interpretations, which one should we choose? These two problems are dealt with differently in the finitary and the infinitary version of the fixpoint semantics.

We conclude this section with a technical lemma, which states the following. When we use an absorptive m-monoid and apply the immediate consequence operator to the empty interpretation an arbitrary number of times, then the resulting interpretation will associate the neutral element of the m-monoid with every atom instance that has no derivations ending in it.

Lemma 4.17. *Let $\mathcal{A}$ be absorptive, $t \in T_\Sigma$, and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. Moreover, let $n \in \mathbb{N}$.*

1. *Let $c \in P(\mathrm{pos}(t))$ such that $\mathrm{H}^c_G = \emptyset$. Then $\mathcal{T}^n(I_\mathbf{0})(c) = \mathbf{0}$.*
2. *Let $(r, \rho) \in \Phi_{M,t}$ and assume that $\mathrm{H}^c_G = \emptyset$ for some $c \in \mathrm{ind}(\rho(r_\mathrm{b}))$. Then we have $\mathrm{h}_{\mathcal{T}^n(I_\mathbf{0})}(\rho(r_\mathrm{b})) = \mathbf{0}$.*

Proof. First we prove Statement 1 by induction on n.

Induction base. Clearly, $\mathcal{T}^0(I_\mathbf{0})(c) = I_\mathbf{0}(c) = \mathbf{0}$ for every $c \in P(\mathrm{pos}(t))$.

Induction step. Let $n \in \mathbb{N}$ and $c \in P(\mathrm{pos}(t))$ such that $\mathrm{H}^c_G = \emptyset$. For every $d \in P(\mathrm{pos}(t))$ with $\mathrm{H}^d_G = \emptyset$ we assume $\mathcal{T}^n(I_\mathbf{0})(d) = \mathbf{0}$. We show $\mathcal{T}^{n+1}(I_\mathbf{0})(c) = \mathbf{0}$. Observation 4.11(4) yields that for every $e = (r, \rho) \in \Phi_{M,t,c}$ there is a $d \in \mathrm{ind}(\rho(r_\mathrm{b}))$ such that $\mathrm{H}_G^d = \emptyset$. Then Lemma 3.9(1) and the induction hypothesis imply $\mathcal{T}^{n+1}(I_{\mathbf{0}})(c) = \sum_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^n(I_{\mathbf{0}})}(\rho(r_\mathrm{b})) = \mathbf{0}$.

Statement 2 follows from Statement 1 and Lemma 3.9(1). ■

The finitary case

Throughout this section we fix a tree $t \in T_\Sigma$.

Recall that we will define the finitary semantics for the following two cases: (i) the mwmd is non-circular and the m-monoid is arbitrary or (ii) the mwmd is weakly non-circular and the m-monoid is absorptive. In Case (i) there is always precisely one fixpoint interpretation (cf. [122, Lemma 3.26]). As an example of this property consider the non-circular mwmd M_1 from Example 4.16; in this example we have already established that for the m-monoid $\mathcal{A}_1$ and the input trees t_1 and t_3 there is a unique fixpoint interpretation.

Unfortunately, in Case (ii) there can be zero, one, or many fixpoint interpretations; this is witnessed by the following example.

Example 4.18 (*Continuation of Example 4.16***).** Recall the weakly non-circular mwmd M_3 from Example 4.14. Let $\mathcal{A}_2 = (\mathbb{N}, +, 0, \theta_2)$ be the distributive m-monoid over Δ_ex such that for every $a, b \in \mathbb{N}$:

$$\theta_2(\alpha)() = \theta_2(\beta)() = 1\ , \qquad \theta_2(\gamma)(a) = a\ , \qquad \theta_1(\sigma)(a,b) = a \cdot b\ .$$

Consider the tree $t_5 = \beta$ (the dependency hypergraph of M_3 and t_5 is shown in Figure 4.3(b)). We prove that there is no fixpoint interpretation over M_3, t_5, and $\mathcal{A}_2$. Assume, contrary to our claim, that I is a fixpoint interpretation. Clearly, $\mathcal{T}(I)(r()) = \mathrm{h}_I(\alpha)$ and $\mathcal{T}(I)(p()) = \mathrm{h}_I(\beta) + \mathrm{h}_I\big(\sigma(p(), r())\big)$. Hence, $I(r()) = \mathcal{T}(I)(r()) = \theta_2(\alpha)() = 1$ and $I(p()) = \mathcal{T}(I)(p()) = \theta_2(\beta)() + \theta_2(\sigma)\big(I(p()), I(r())\big) = 1 + I(p()) \cdot 1$. There is no $I(p()) \in \mathbb{N}$ satisfying $I(p()) = 1 + I(p())$. □

Although there might not be a fixpoint interpretation, every weakly non-circular mwmd in conjunction with an absorptive m-monoid exhibits a "weak fixpoint behavior" in the following sense. When we start with the empty interpretation and repeatedly apply the immediate consequence operator to this interpretation, i.e., compute the sequence $\mathcal{T}^0(I_{\mathbf{0}}), \mathcal{T}^1(I_{\mathbf{0}}), \mathcal{T}^2(I_{\mathbf{0}}), \ldots$, then after n steps (where n is the number of atom instances) the value that the resulting interpretations assign to $q(\varepsilon)$ stays fixed. This is stated formally by the following lemma.

Lemma 4.19. *Let $n = |P(\mathrm{pos}(t))|$ and assume that at least one of the following conditions is satisfied:*

(i) M is non-circular, or

(ii) M is weakly non-circular and $\mathcal{A}$ is absorptive.

Then for every $m \in \mathbb{N}$ we have $\mathcal{T}^n(I_{\mathbf{0}})(q(\varepsilon)) = \mathcal{T}^{n+m}(I_{\mathbf{0}})(q(\varepsilon))$.

Proof. Throughout this proof we abbreviate $\mathrm{G}_{M,t}^{\mathrm{dep}}$ by G and we let E be the set of edges of the dependency graph of M and t. We define the relation $\prec$ on $P(\mathrm{pos}(t))$ as follows: (i) if $\mathcal{A}$ is absorptive, then $\prec\ =\ \prec_G$ (recall the definition of $\prec_G$ from Page 28) and (ii) otherwise we let $\prec\ = E$.

Let $C = \{c \in P(\text{pos}(t)) \mid c \prec^* q(\varepsilon)\}$ and $\sqsubset = \prec \cap (C \times C)$. We show that $\sqsubset^+$ is irreflexive. First consider the case that $\mathcal{A}$ is absorptive. Then $\prec = \prec_G$. By Lemma 4.13 either one of Conditions (i) and (ii) yields hat M is weakly non-circular; hence, $\mathrm{H}_G^{q(\varepsilon)}$ is finite. Then Lemma 2.26 implies that $\sqsubset^+$ is irreflexive. Now consider the case that $\mathcal{A}$ is not absorptive. Then Condition (i) must hold, i.e., M is non-circular; hence, $\prec^+ = E^+$ is irreflexive. Since $\sqsubset^+ = (\prec \cap (C \times C))^+ \subseteq \prec^+ \cap (C \times C)$, also $\sqsubset^+$ is irreflexive.

Since C is finite and $\sqsubset^+$ is an irreflexive relation on C, we obtain that $\sqsubset$ is well-founded on C. For every $c \in C$ we define the number $n_c \in \mathbb{N}$ by well-founded recursion on $\sqsubset$ by letting $n_c = 1 + \max\{n_{c'} \mid c' \in C, c' \sqsubset c\}$. It is easy to see that $n_c \leq |C|$ for every $c \in C$; hence, in particular $n_{q(\varepsilon)} \leq |C| \leq |P(\text{pos}(t)|$ because $q(\varepsilon) \in C$.

We claim that for every $c \in C$, $m \in \mathbb{N}$, and $n' \in \mathbb{N}$ with $n' \geq n_c$ we have that $\mathcal{T}^{n'}(I_\mathbf{0})(c) = \mathcal{T}^{n'+m}(I_\mathbf{0})(c)$. This claim trivially implies the lemma because $n_{q(\varepsilon)} \leq |P(\text{pos}(t)| = n$. We prove this claim by well-founded induction on $\sqsubset$.

Let $c \in C$ and assume that we have already proved the claim for every $c' \in C$ with $c' \sqsubset c$. Let $m, n' \in \mathbb{N}$ such that $n' \geq n_c$. We show $\mathcal{T}^{n'}(I_\mathbf{0})(c) = \mathcal{T}^{n'+m}(I_\mathbf{0})(c)$. By the definition of n_c, $n' \geq n_c \geq 1$. Then by the definition of $\mathcal{T}$ it suffices to show that for every $(r, \rho) \in \Phi_{M,t,c}$, $\mathrm{h}_{\mathcal{T}^{n'-1}(I_\mathbf{0})}(\rho(r_\mathrm{b})) = \mathrm{h}_{\mathcal{T}^{n'-1+m}(I_\mathbf{0})}(\rho(r_\mathrm{b}))$. Let $(r, \rho) \in \Phi_{M,t,c}$.

First we consider the case that for every $c' \in \text{ind}(\rho(r_\mathrm{b}))$ we have $c' \sqsubset c$; thus, $n_{c'} < n_c$ and the induction hypothesis yields $\mathcal{T}^{n'-1}(I_\mathbf{0})(c') = \mathcal{T}^{n'-1+m}(I_\mathbf{0})(c')$. We obtain $\mathrm{h}_{\mathcal{T}^{n'-1}(I_\mathbf{0})}(\rho(r_\mathrm{b})) = \mathrm{h}_{\mathcal{T}^{n'-1+m}(I_\mathbf{0})}(\rho(r_\mathrm{b}))$.

Next we consider the case that there is a $c' \in \text{ind}(\rho(r_\mathrm{b}))$ such that $c' \not\sqsubset c$. If $c' \prec c$, then $c' \in C$ and, thus, $c' \sqsubset c$, a contradiction. Hence, $c' \not\prec c$. Clearly, $(c', c) \in E$ by Observation 4.11(3). We obtain that $\mathcal{A}$ is absorptive because otherwise $\prec = E$, which implies the contradiction $c' \prec c$. Moreover, $\prec = \prec_G$ by the definition of $\prec$. Since $c' \not\prec_G c$, Lemma 2.20 and Observation 4.11(1,2) imply that there is a $c' \in \text{ind}(\rho(r_\mathrm{b}))$ such that $\mathrm{H}_G^{c'} = \emptyset$. Then by Lemma 4.17(2) $\mathrm{h}_{\mathcal{T}^{n'-1}(I_\mathbf{0})}(\rho(r_\mathrm{b})) = \mathbf{0} = \mathrm{h}_{\mathcal{T}^{n'-1+m}(I_\mathbf{0})}(\rho(r_\mathrm{b}))$. ■

Now we define the finitary fixpoint semantics. Our definition is justified by the previous lemma.

Definition 4.20. Suppose that one of the following conditions holds: (i) M is non-circular or (ii) M is weakly non-circular and $\mathcal{A}$ is absorptive. Then we define the ***tree series fixpoint-defined*** *by M and $\mathcal{A}$* as the tree series $[\![M]\!]^{\text{fix}}_{\mathcal{A}} \in \mathcal{A}\langle\!\langle T_\Sigma \rangle\!\rangle$ with $[\![M]\!]^{\text{fix}}_{\mathcal{A}}(t) = \mathcal{T}^{|P(\text{pos}(t))|}(I_\mathbf{0})(q(\varepsilon))$ for every $t \in T_\Sigma$. □

Example 4.21 (*Continuation of Example 4.18***).** First consider the mwmd M_1, input tree t_1, and m-monoid $\mathcal{A}_1$. The sequence of interpretations $\mathcal{T}^i(I_0)$ is shown in the following table.

	$q(\varepsilon)$	$q(1)$	$q(2)$	$q(21)$	$q(22)$	$p()$
$\mathcal{T}^0(I_0)$	0	0	0	0	0	0
$\mathcal{T}^1(I_0)$	0	2	0	2	2	0
$\mathcal{T}^2(I_0)$	0	2	0	2	2	0

For every $i \in \mathbb{N}$ with $i \geq 1$ we have $\mathcal{T}^i(I_0) = \mathcal{T}^1(I_0)$. Hence, $[\![M_1]\!]^{\text{fix}}_{\mathcal{A}_1}(t_1) = 0$. For the input tree t_3 we obtain the following table:

	$q(\varepsilon)$	$q(1)$	$q(11)$	$q(111)$	$q(2)$	$q(21)$	$q(22)$	$p()$
$\mathcal{T}^0(I_0)$	0	0	0	0	0	0	0	0
$\mathcal{T}^1(I_0)$	0	0	0	2	0	2	2	6
$\mathcal{T}^2(I_0)$	0	0	0	2	8	2	2	6
$\mathcal{T}^3(I_0)$	11	0	0	2	8	2	2	6

For every $i \in \mathbb{N}$ with $i \geq 3$ we have $\mathcal{T}^i(I_0) = \mathcal{T}^3(I_0)$; thus, $[\![M_1]\!]^{\text{fix}}_{\mathcal{A}_1}(t_3) = 11$. Observe how the values propagate through the dependency hypergraph from $p()$ over $q(2)$ to $q(\varepsilon)$. Note that for non-circular mwmd we do not only reach a fixed valued for $q(\varepsilon)$ after n steps (where n is the number of atom instances) but even a fixpoint interpretation (cf. [122, Lemma 3.26]).

Now consider the mwmd M_3, input tree t_5, and m-monoid $\mathcal{A}_2$. We obtain

	$q(\varepsilon)$	$r()$	$p()$
$\mathcal{T}^0(I_0)$	0	0	0
$\mathcal{T}^1(I_0)$	0	1	1
$\mathcal{T}^2(I_0)$	0	1	2
$\mathcal{T}^3(I_0)$	0	1	3

For every $i \in \mathbb{N}$ we have $\mathcal{T}^i(I_0)(p()) = i$. Although there is no fixpoint interpretation, the value that the interpretations in this sequence assign to $q(\varepsilon)$ stays fixed. We obtain $[\![M_3]\!]^{\text{fix}}_{\mathcal{A}_2}(t_5) = 0$. Moreover, for the input tree t_4 we obtain $[\![M_3]\!]^{\text{fix}}_{\mathcal{A}_2}(t_4) = 1$. □

In order to reduce the number of cases that we have to consider in the sequel, we restrict ourselves to employing the semantics of Definition 4.20 only in the case that M is weakly non-circular and $\mathcal{A}$ is absorptive. This is justified due to the following proposition.

Proposition 4.22. *Let $\mathcal{A}$ be an arbitrary m-monoid over Δ. Then there is a signature $\Delta_\perp$ and an absorptive m-monoid $\mathcal{A}_\perp$ over $\Delta_\perp$ such that the carrier set of $\mathcal{A}_\perp$ contains the carrier set of $\mathcal{A}$, and for every non-circular mwmd M over Σ and Δ there is a weakly non-circular mwmd $M_\perp$ over Σ and $\Delta_\perp$ such that $[\![M]\!]^{\text{fix}}_{\mathcal{A}} = [\![M_\perp]\!]^{\text{fix}}_{\mathcal{A}_\perp}$.*[1]

Proof. Let $\mathcal{A} = (A, +, \mathbf{0}, \theta)$. We let $\Delta_\perp = \Delta \uplus \{\text{null}^{(0)}\}$ and $\mathcal{A}_\perp = (A \uplus \{\perp\}, +', \perp, \theta')$, where $+'$ is the extension of $+$ to the set $A \cup \{\perp\}$ defined by $\perp +' a = a = a +' \perp$ for every $a \in A \cup \{\perp\}$, and for every $\delta \in \Delta$, $\theta'(\delta)$ is the extension of $\theta(\delta)$ to $A \cup \{\perp\}$ such that $\theta'(\delta)(\ldots, \perp, \ldots) = \perp$; moreover, we put $\theta'(\text{null})() = \mathbf{0}$. Obviously, $\mathcal{A}_\perp$ is absorptive.

Let $M = (P, R, q)$ be a non-circular mwmd over Σ and Δ. We define $M_\perp = (P, R \cup R', q)$ such that

$$R' = \{p(x) \leftarrow \text{null}()\,; \emptyset \mid p \in P^{(1)}\} \cup \{p() \leftarrow \text{null}()\,; \emptyset \mid p \in P^{(0)}\}\,,$$

where x is chosen arbitrarily in V. First we show that $M_\perp$ is weakly non-circular. Let $t \in T_\Sigma$ and let E be the set of edges of the dependency graph of M and t, and let $E_\perp$ be the set of edges of the dependency graph of $M_\perp$ and t. It is easy to check that $E = E_\perp$ because the rules in R' generate no edges in $E_\perp$. Then $M_\perp$ is non-circular because M is non-circular. We conclude that $M_\perp$ is weakly non-circular by Lemma 4.13.

Let $t \in T_\Sigma$. It remains to show that $[\![M]\!]^{\text{fix}}_{\mathcal{A}}(t) = [\![M_\perp]\!]^{\text{fix}}_{\mathcal{A}_\perp}(t)$. Since M is non-circular, the relation E^+ is irreflexive. Therefore, E is well-founded on $P(\text{pos}(t))$ because $P(\text{pos}(t))$

[1]Note that there is no "conflict of types" when writing $[\![M]\!]^{\text{fix}}_{\mathcal{A}} = [\![M_\perp]\!]^{\text{fix}}_{\mathcal{A}_\perp}$ because we defined mappings $f : A \to B$ as relations from A to B and not as triples (A, B, ρ), where ρ is a relation from A to B.

is finite. For every $c \in P(\mathrm{pos}(t))$ we define $n_c \in \mathbb{N}$ by well-founded recursion on E: $n_c = 1 + \max\{n_{c'} \mid c' \in P(\mathrm{pos}(t)), (c', c) \in E\}$.

We claim that for every $c \in P(\mathrm{pos}(t))$ and for every $n \in \mathbb{N}$ with $n \geq n_c$, $\mathcal{T}^n_{M,t,\mathcal{A}}(I_{\mathbf{0}})(c) = \mathcal{T}^n_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)(c)$. This claim implies $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}(t) = [\![M_\perp]\!]^{\mathrm{fix}}_{\mathcal{A}_\perp}(t)$ because for every atom instance $c \in P(\mathrm{pos}(t))$ it is easy to check that $n_c \leq |P(\mathrm{pos}(t))|$; in particular, $n_{q(\varepsilon)} \leq |P(\mathrm{pos}(t))|$.

We prove our claim by well-founded induction on E. Let $c \in P(\mathrm{pos}(t))$ and assume that the claim holds for every $c' \in P(\mathrm{pos}(t))$ with $(c', c) \in E$. Let $n \in \mathbb{N}$ with $n \geq n_c$; by the definition of n_c, $n \geq n_c \geq 1$. The induction hypothesis and Observation 4.11(3) yield that for every $(r, \rho) \in \Phi_{M,t,c}$:

$$\mathrm{h}_{\mathcal{T}^{n-1}_{M,t,\mathcal{A}}(I_{\mathbf{0}})}(\rho(r_{\mathrm{b}})) = \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho(r_{\mathrm{b}})) , \tag{4.1}$$

because $n - 1 \geq n_{c'}$ for every $c' \in \mathrm{ind}(\rho(r_{\mathrm{b}}))$ by the definition of n_c and Observation 4.11(3).

Observe that there is exactly one $(r_c, \rho_c) \in \Phi_{M_\perp,t,c}$ with $r_c \in R'$. More precisely, if $c = p()$ for some $p \in P^{(0)}$, then r_c is the rule $p() \leftarrow \mathrm{null}() \,; \emptyset$ and ρ_c is the empty mapping; if $c = p(w)$ for some $p \in P^{(1)}$ and $w \in \mathrm{pos}(t)$, then r_c is the rule $p(x) \leftarrow \mathrm{null}() \,; \emptyset$ and $\rho_c(x) = w$. Clearly, $\rho_c((r_c)_{\mathrm{b}}) = \mathrm{null}()$ and, thus, $\mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho_c((r_c)_{\mathrm{b}})) = \theta'(\mathrm{null})() = \mathbf{0}$. It is easy to see that $\Phi_{M_\perp,t,c} = \Phi_{M,t,c} \cup \{(r_c, \rho_c)\}$.

In the remainder of the proof we denote the extension of $+$ and $+'$ to finite families by $\sum$ and $\sum'$, respectively. Note that $\sum$ and $\sum'$ agree on all nonempty families over A; however, when applied to the empty family, then $\sum$ yields $\mathbf{0}$ and $\sum'$ yields $\perp$. It is easy to see that for every finite set J and family $(a_j \mid j \in J)$ over A we have $\sum_{j \in J} a_j = \mathbf{0} +' \sum'_{j \in J} a_j$. By putting all these facts together we obtain

$$\begin{aligned}
\mathcal{T}^n_{M,t,\mathcal{A}}(I_{\mathbf{0}})(c) &= \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^{n-1}_{M,t,\mathcal{A}}(I_{\mathbf{0}})}(\rho(r_{\mathrm{b}})) \\
&= \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho(r_{\mathrm{b}})) && \text{(by Equation (4.1))} \\
&= \mathbf{0} +' \sum\nolimits'_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho(r_{\mathrm{b}})) \\
&= \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho_c((r_c)_{\mathrm{b}})) +' \sum\nolimits'_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho(r_{\mathrm{b}})) \\
&= \sum\nolimits'_{(r,\rho)\in\Phi_{M_\perp,t,c}} \mathrm{h}_{\mathcal{T}^{n-1}_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)}(\rho(r_{\mathrm{b}})) = \mathcal{T}^n_{M_\perp,t,\mathcal{A}_\perp}(I_\perp)(c) \,. \qquad \blacksquare
\end{aligned}$$

Definition 4.23. Let $\mathcal{A}$ be absorptive. The set of all tree series fixpoint-defined by weakly non-circular mwmd over Σ and Δ, and $\mathcal{A}$ is denoted by $\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$. □

The infinitary case

We motivate the definition of the infinitary fixpoint semantics by means of the following example.

Example 4.24 (*Continuation of Example 4.21*)**.** Let us consider mwmd M_4, input tree t_5, and m-monoid $\mathcal{A}_2$ (the dependency hypergraph for this combination of mwmd and input tree is given in Figure 4.3(c)). Recall that M_4 is not weakly non-circular. The following table shows the according sequence of interpretations $\mathcal{T}^i(I_0)$.

	$q(\varepsilon)$	$r()$	$p()$
$\mathcal{T}^0(I_0)$	0	0	0
$\mathcal{T}^1(I_0)$	0	1	1
$\mathcal{T}^2(I_0)$	1	1	2
$\mathcal{T}^3(I_0)$	2	1	3

For every $i \in \mathbb{N}_+$ we obtain that $\mathcal{T}^i(I_0)(q(\varepsilon)) = i - 1$; hence, we will not reach a fixpoint for $q(\varepsilon)$ in the sequence $(\mathcal{T}^i(I_0) \mid i \in \mathbb{N})$ of interpretations. In fact, for every interpretation I we obtain that $\mathcal{T}^3(I)(q(\varepsilon)) = \mathcal{T}^2(I)(p()) = \theta_2(\beta)() + \theta_2(\sigma)\big(\mathcal{T}(I)(p()), \mathcal{T}(I)(r())\big) = 1 + \mathcal{T}(I)(p())\cdot\theta_2(\alpha)() = 1+\mathcal{T}^2(I)(q(\varepsilon))\cdot 1 = 1+\mathcal{T}^2(I)(q(\varepsilon))$; hence, $T^3(I)(q(\varepsilon)) \neq T^2(I)(q(\varepsilon))$. Thus, there is no interpretation I that assigns a consistent value to $q(\varepsilon)$, i.e., a value that will stay constant when repeatedly applying the immediate consequence operator to I. Roughly speaking, there is no meaningful way to define the semantics for M_4, t_5, and $\mathcal{A}_2$.

Now let us consider the m-monoid $\mathcal{A}_1$. Then we obtain:

	$q(\varepsilon)$	$r()$	$p()$
$\mathcal{T}^0(I_0)$	0	0	0
$\mathcal{T}^1(I_0)$	0	3	$2 + 0 = 2$
$\mathcal{T}^2(I_0)$	2	3	$2 + (2/2 + 3 + 1) = 7$
$\mathcal{T}^3(I_0)$	7	3	$2 + (7/2 + 3 + 1) = 19/2$
$\mathcal{T}^4(I_0)$	$19/2$	3	$2 + (19/4 + 3 + 1) = 43/4$
$\mathcal{T}^5(I_0)$	$43/4$	3	$2 + (43/8 + 3 + 1) = 91/8$

It is easy to show by induction that for every $i \in \mathbb{N}$ with $i \geq 2$ we have that $\mathcal{T}^i(I_0)(q(\varepsilon)) = 12 - 5 \cdot 2^{3-i}$. Hence, for the m-monoid $\mathcal{A}_1$ we will not reach a fixpoint for $q(\varepsilon)$ in the sequence $(\mathcal{T}^i(I_0) \mid i \in \mathbb{N})$, either. However, a fixpoint interpretation exists nevertheless: in fact, the interpretation I with $I(q(\varepsilon)) = I(p()) = 12$ and $I(r()) = 3$ is a fixpoint for the immediate consequence operator. Observe that, roughly speaking, the interpretation I is the limit of the sequence $\mathcal{T}^0(I_0), \mathcal{T}^1(I_0), \mathcal{T}^2(I_0), \ldots$ of interpretations. □

In Example 4.24 we have shown that for the mwmd M_4 and input tree t_5 the m-monoid $\mathcal{A}_2$ does not admit a fixpoint interpretation, whereas $\mathcal{A}_1$ does admit one. This is due to the fact that for $\mathcal{A}_1$ there is a partial order $\leq$ (namely the natural order on $\mathbb{R}_{\geq 0} \cup \{\infty\}$) such that $(\mathcal{A}_1, \leq)$ is an ω-continuous m-monoid.

Now we show that this property holds for arbitrary ω-continuous m-monoids. More precisely, we prove that for every ω-continuous m-monoid the set of interpretations is an ω-cpo, that the family $(\mathcal{T}^i(I_0) \mid i \in \mathbb{N})$ is an ω-chain and that the supremum of this ω-chain is a fixpoint interpretation. We will employ this fixpoint interpretation in the definition of the infinitary fixpoint semantics.

In this section we let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid.

Definition 4.25. We lift the order $\leq$ to the set $\mathcal{I}$ as follows for every $I_1, I_2 \in \mathcal{I}$: $I_1 \leq I_2$ iff $I_1(c) \leq I_2(c)$ for every $c \in P(\text{pos}(t))$. □

Lemma 4.26. *The poset $(\mathcal{I}, \leq)$ is an ω-cpo with least element $I_{\mathbf{0}}$. Moreover, we have $(\bigvee\{b(n) \mid n \in \mathbb{N}\})(c) = \bigvee\{b(n)(c) \mid n \in \mathbb{N}\}$ for every $c \in P(\text{pos}(t))$ and ω-chain $b : \mathbb{N} \to \mathcal{I}$.*

Proof. Clearly, $I_{\mathbf{0}}$ is the least element of $\mathcal{I}$. Let $b : \mathbb{N} \to \mathcal{I}$ be an ω-chain. Observe that the interpretation $I \in \mathcal{I}$ with $I(c) = \bigvee\{b(n)(c) \mid n \in \mathbb{N}\}$, for every $c \in P(\text{pos}(t))$, is the supremum of b. ■

The following lemma is crucial for the proof of the result that the supremum of the family $(\mathcal{T}^i(I_0) \mid i \in \mathbb{N})$ is a fixpoint interpretation.

Lemma 4.27. *The immediate consequence operator $\mathcal{T}$ is ω-continuous.*

Proof. Let $c \in P(\mathrm{pos}(t))$ and $b : \mathbb{N} \to \mathcal{I}$ be an ω-chain. Then we show that we have $\mathcal{T}(\vee\{b(n) \mid n \in \mathbb{N}\})(c) = \big(\vee\{\mathcal{T}(b(n)) \mid n \in \mathbb{N}\}\big)(c)$. Let $I = \vee\{b(n) \mid n \in \mathbb{N}\}$. Then

$$\begin{aligned}
\mathcal{T}(I)(c) &= \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_I(\rho(r_\mathrm{b})) \\
&= \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \vee\{\mathrm{h}_{b(n)}(\rho(r_\mathrm{b})) \mid n \in \mathbb{N}\} && (\star)\\
&= \vee\Big\{\sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{b(n)}(\rho(r_\mathrm{b})) \mid n \in \mathbb{N}\Big\} && (\star\star)\\
&= \vee\big\{\mathcal{T}(b(n))(c) \mid n \in \mathbb{N}\big\}\\
&= \big(\vee\{\mathcal{T}(b(n)) \mid n \in \mathbb{N}\}\big)(c)\ . && \text{(by Lemma 4.26)}
\end{aligned}$$

At Equation $(\star)$ we used Lemma 3.36(2) with the following instantiations: $C = P(\mathrm{pos}(t))$ and $f_n = b(n)$, $f = I$, $g_n = \mathrm{h}_{b(n)}$, $g = \mathrm{h}_I$ for every $n \in \mathbb{N}$. The instantiation $f = I$ is justified by the fact that, for every $c' \in P(\mathrm{pos}(t))$, $I(c') = \vee\{b(n)(c') \mid n \in \mathbb{N}\}$, which follows from Lemma 4.26. Equation $(\star\star)$ holds by Lemma 3.32 applied to $\nu = +$; here, for every $(r,\rho) \in \Phi_{M,t,c}$, $(\mathrm{h}_{b(n)}(\rho(r_\mathrm{b})) \mid n \in \mathbb{N})$ is an ω-chain due to Lemma 3.36(1). ∎

Lemma 4.28. *The family $(\mathcal{T}^n(I_\mathbf{0}) \mid n \in \mathbb{N})$ is an ω-chain. Furthermore, the supremum $\vee\{\mathcal{T}^n(I_\mathbf{0}) \mid n \in \mathbb{N}\}$ is the least fixpoint of $\mathcal{T}$.*

Proof. This follows from Lemmas 4.26 and 4.27, and Theorem 3.29. ∎

We denote the least fixpoint of $\mathcal{T}_{M,t,\mathcal{A}}$, whose existence is asserted by Lemma 4.28, by $\mathcal{T}^\omega_{M,t,\mathcal{A}}$. Again, if M, t, and $\mathcal{A}$ are clear from the context, then we simply write $\mathcal{T}^\omega$ instead of $\mathcal{T}^\omega_{M,t,\mathcal{A}}$.

Now we are prepared to define the infinitary fixpoint semantics.

Definition 4.29. The ***tree series fixpoint-defined*** *by M and* $(\mathcal{A}, \leq)$ is the tree series $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)} \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle$ with $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}(t) = \mathcal{T}^\omega(q(\varepsilon))$ for every $t \in T_\Sigma$. The set of all tree series fixpoint-defined by arbitrary mwmd over Σ and Δ, and $(\mathcal{A}, \leq)$ is denoted by $\mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big)$. □

Example 4.30 (*Continuation of Example 4.24***).** Let $\leq$ be the natural order on the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$ and observe that $(\mathcal{A}_1, \leq)$ is an ω-continuous m-monoid.

Let us consider the mwmd M_4, input tree t_5, and the ω-continuous m-monoid $(\mathcal{A}_1, \leq)$. We have already shown that for every $i \in \mathbb{N}_+$ we have

$$\mathcal{T}^i(I_0)(r()) = 3\ , \qquad \mathcal{T}^i(I_0)(p()) = 12 - 5 \cdot 2^{2-i}\ ,$$

$$\mathcal{T}^i(I_0)(q(\varepsilon)) = \begin{cases} 0\ , & \text{if } i = 1\ ,\\ 12 - 5 \cdot 2^{3-i}\ , & \text{otherwise.}\end{cases}$$

Lemmas 4.28 and 4.26 yield that we have $\mathcal{T}^\omega(q(\varepsilon)) = \big(\vee\{\mathcal{T}^n(I_0) \mid n \in \mathbb{N}\}\big)(q(\varepsilon)) = \vee\{\mathcal{T}^n(I_0)(q(\varepsilon)) \mid n \in \mathbb{N}\} = \vee\{12 - 5 \cdot 2^{3-n} \mid n \in \mathbb{N}\} = 12$. Likewise, we have

$\mathcal{T}^\omega(p()) = 12$ and $\mathcal{T}^\omega(r()) = 3$. Clearly, $\mathcal{T}^\omega$ is a fixpoint interpretation. Finally, we obtain $[\![M_4]\!]^{\text{fix}}_{(\mathcal{A}_1,\leq)}(t_5) = 12$.

Now consider the m-monoid $\mathcal{A}_3 = \big(\mathcal{P}(T_{\Delta_{\text{ex}}}), \cup, \emptyset, \theta_3\big)$ over Δ_{ex}, where for every $\delta \in \Delta_{\text{ex}}$, $\theta_3(\delta)$ is the δ-language top concatenation. Then $(\mathcal{A}_3, \subseteq)$ is an ω-continuous m-monoid (cf. Example 3.34(5)). For M_4 and t_5 we obtain

	$q(\varepsilon)$	$r()$	$p()$
$\mathcal{T}^0(I_0)$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathcal{T}^1(I_0)$	$\emptyset$	$\{\alpha\}$	$\{\beta\}$
$\mathcal{T}^2(I_0)$	$\{\beta\}$	$\{\alpha\}$	$\{\beta, \sigma(\beta,\alpha)\}$
$\mathcal{T}^3(I_0)$	$\{\beta, \sigma(\beta,\alpha)\}$	$\{\alpha\}$	$\{\beta, \sigma(\beta,\alpha), \sigma(\sigma(\beta,\alpha),\alpha)\}$

This implies that $[\![M_4]\!]^{\text{fix}}_{(\mathcal{A}_3,\subseteq)}(t_5) = \mathcal{T}^\omega(q(\varepsilon))$ is the set of all left-descending combs of the form $\sigma(\sigma(\cdots\sigma(\sigma(\beta,\alpha),\alpha)\cdots),\alpha)$. □

In Example 3.34(6) we have shown that there are m-monoids $\mathcal{A}$ and distinct partial orders $\leq$ and $\sqsubseteq$ such that both $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sqsubseteq)$ are ω-continuous m-monoids. This does not necessarily mean that the infinitary fixpoint semantics for $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sqsubseteq)$ are distinct. However, the following lemma shows that in general the semantics are distinct for different ω-continuous m-monoids over the same underlying m-monoid.

Lemma 4.31. *There are a ranked alphabet Σ, a signature Δ, a distributive m-monoid $\mathcal{A}$ over Δ, ω-continuous m-monoids $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sqsubseteq)$, and an mwmd M over Σ and Δ such that $[\![M]\!]^{\text{fix}}_{(\mathcal{A},\leq)} \neq [\![M]\!]^{\text{fix}}_{(\mathcal{A},\sqsubseteq)}$.*

The proof of Lemma 4.31 is given in Appendix A.3.

Now we show that under certain assumptions the least fixpoint interpretation can be reached by a finite number of applications of the immediate consequence operator when starting with the empty interpretation.

Definition 4.32. We call $\mathcal{A}$ ***operationally locally finite*** (for short: ***olf***), if for every finite $\Delta' \subseteq \Delta$ there is a finite $A' \subseteq A$ containing $\mathbf{0}$ such that A' is closed under $+$ and under $\theta(\delta)$, for every $\delta \in \Delta'$. □

Lemma 4.33. *Suppose that $\mathcal{A}$ is olf. Then there is an $n \in \mathbb{N}$ with $\mathcal{T}^\omega = \mathcal{T}^n(I_\mathbf{0})$. If A is finite, then there is such an n with $(|A| - 1) \cdot |P| \cdot |\text{pos}(t)| \geq n$.*

Proof. Let Δ' be the set of elements of Δ that occur in the bodies of rules of M. Clearly, Δ' is finite. Then there is a finite $A' \subseteq A$ containing $\mathbf{0}$ such that A' is closed under $+$ and under $\theta(\delta)$, for every $\delta \in \Delta'$. Before we proceed with the main argument, we show that for every $n \in \mathbb{N}$ and $c \in P(\text{pos}(t))$ we have $\mathcal{T}^n(I_\mathbf{0})(c) \in A'$. We give a proof by induction on n.

Induction base. Clearly, for every $c \in P(\text{pos}(t))$, $\mathcal{T}^0(I_\mathbf{0})(c) = I_\mathbf{0}(c) = \mathbf{0} \in A'$.

Induction step. Let $n \in \mathbb{N}$ and assume that $\mathcal{T}^n(I_\mathbf{0})(c') \in A'$ for every $c' \in P(\text{pos}(t))$. Let $c \in P(\text{pos}(t))$. Since A' is closed under $+$, it suffices to show that $\text{h}_{\mathcal{T}^n(I_\mathbf{0})}(\rho(r_\text{b})) \in A'$ for every $(r, \rho) \in \Phi_{M,t,c}$. This follows from Lemma 3.9(2).

We claim that there is an $n \in \mathbb{N}$ with $(|A'| - 1) \cdot |P(\text{pos}(t))| \geq n$ such that $\mathcal{T}^\omega = \mathcal{T}^n(I_\mathbf{0})$. This claim implies our lemma because $|A| \geq |A'|$ and $|P| \cdot |\text{pos}(t)| \geq |P(\text{pos}(t))|$. Now we prove the claim.

Since the strict part $<$ of $\leq$ is irreflexive and transitive, and since A' is finite, the relation $<_{A'} = < \cap (A' \times A')$ is well-founded. We define the mapping $m : A' \to \mathbb{N}$ by well-founded recursion on $<_{A'}$ by letting $m(a) = \max\{1 + m(a') \mid a' \in A', a' <_{A'} a\}$, for every element $a \in A'$. It is easy to see that for every $a \in A'$, $|A'| - 1 \geq m(a)$. We define $f : \mathbb{N} \to \mathbb{N}$ as follows for every $n \in \mathbb{N}$: $f(n) = \sum_{c \in P(\mathrm{pos}(t))} m(\mathcal{T}^n(I_\mathbf{0})(c))$. This is well-defined because $\mathcal{T}^n(I_\mathbf{0})(c) \in A'$ for every $n \in \mathbb{N}$ and $c \in P(\mathrm{pos}(t))$, as we have shown above. Now we present three facts about f.

Fact 1. $(|A'| - 1) \cdot |P(\mathrm{pos}(t))| \geq f(n)$ for every $n \in \mathbb{N}$. This is easy to see because $|A'| - 1 \geq m(a)$ for every $a \in A'$.

Fact 2. $f(n + 1) \geq f(n)$ for every $n \in \mathbb{N}$. By Lemma 4.28, $\mathcal{T}^n(I_\mathbf{0}) \leq \mathcal{T}^{n+1}(I_\mathbf{0})$. Hence, for every $c \in P(\mathrm{pos}(t))$, $\mathcal{T}^n(I_\mathbf{0})(c) \leq \mathcal{T}^{n+1}(I_\mathbf{0})(c)$ and, thus, $m(\mathcal{T}^{n+1}(I_\mathbf{0})(c)) \geq m(\mathcal{T}^n(I_\mathbf{0})(c))$ by the definition of m. We obtain $f(n + 1) \geq f(n)$.

Fact 3. $f(n + 1) = f(n)$ implies $\mathcal{T}^n(I_\mathbf{0}) = \mathcal{T}^{n+1}(I_\mathbf{0})$, for every $n \in \mathbb{N}$. Assume $f(n + 1) = f(n)$. Then $m(\mathcal{T}^{n+1}(I_\mathbf{0})(c)) = m(\mathcal{T}^n(I_\mathbf{0})(c))$ for every $c \in P(\mathrm{pos}(t))$ because $m(\mathcal{T}^{n+1}(I_\mathbf{0})(c)) \geq m(\mathcal{T}^n(I_\mathbf{0})(c))$ for every $c \in P(\mathrm{pos}(t))$ as shown in the proof of Fact 2. Then for every $c \in P(\mathrm{pos}(t))$ we obtain $\mathcal{T}^n(I_\mathbf{0})(c) = \mathcal{T}^{n+1}(I_\mathbf{0})(c)$ because $\mathcal{T}^n(I_\mathbf{0})(c) \leq \mathcal{T}^{n+1}(I_\mathbf{0})(c)$ and by the definition of m. Hence, $\mathcal{T}^n(I_\mathbf{0}) = \mathcal{T}^{n+1}(I_\mathbf{0})$.

By putting Facts 1 to 3 together we obtain that there is an $n \in \mathbb{N}$ such that $(|A'| - 1) \cdot |P(\mathrm{pos}(t))| \geq n$ and $\mathcal{T}^n(I_\mathbf{0}) = \mathcal{T}^{n+1}(I_\mathbf{0})$. Then $\mathcal{T}^n(I_\mathbf{0}) = \mathcal{T}^{n+m}(I_\mathbf{0})$ for every $m \in \mathbb{N}$ and, thus, $\mathcal{T}^n(I_\mathbf{0}) = \mathcal{T}^\omega$. ∎

Although the finitary and the infinitary version of the fixpoint semantics are defined in a different manner, they coincide if both of them are defined.

Lemma 4.34. *Let M be weakly non-circular, $\mathcal{A}$ be absorptive, and $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then* $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}} = [\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}$.

Proof. Let $t \in T_\Sigma$ and $n = |P(\mathrm{pos}(t))|$. By Definition 4.25 and Lemmas 4.26 and 4.28, $\big(\mathcal{T}^m(I_\mathbf{0})(q(\varepsilon)) \mid m \in \mathbb{N}\big)$ is an ω-chain. Then, Lemma 4.19 yields $\vee\{\mathcal{T}^m(I_\mathbf{0})(q(\varepsilon)) \mid m \in \mathbb{N}\} = \mathcal{T}^n(I_\mathbf{0})(q(\varepsilon))$. By Lemmas 4.26 and 4.28, $\mathcal{T}^\omega(q(\varepsilon)) = \vee\{\mathcal{T}^m(I_\mathbf{0}) \mid m \in \mathbb{N}\}(q(\varepsilon)) = \vee\{\mathcal{T}^m(I_\mathbf{0})(q(\varepsilon)) \mid m \in \mathbb{N}\}$ and thus, $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}(t) = \mathcal{T}^n(I_\mathbf{0})(q(\varepsilon)) = \vee\{\mathcal{T}^m(I_\mathbf{0})(q(\varepsilon)) \mid m \in \mathbb{N}\} = \mathcal{T}^\omega(q(\varepsilon)) = [\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}(t)$. ∎

Corollary 4.35. *Let $\mathcal{A}$ be absorptive and let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then*

$$\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) \subseteq \mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, (\mathcal{A}, \leq)) .$$

4.2.3 Hypergraph semantics

The idea of the hypergraph semantics is as follows. For a given mwmd and input tree we consider the set of derivations ending in $q(\varepsilon)$. For every such derivation we compute a weight, which is an element of the m-monoid, and sum up over the weights of all derivations ending in $q(\varepsilon)$. The semantics is defined as the resulting value of this summation.

Now let us explain how the weight of a derivation is defined. Every derivation is a tree whose labels are rule instances. Such a tree can be transformed into a tree over Δ by, roughly speaking, replacing every label (r, ρ) by the tree $(\rho(r_\mathrm{b}))$; we will define this transformation function formally in the next definition. After the transformation we evaluate the resulting tree over Δ by means of the Δ-algebra in the considered m-monoid; this evaluation yields an element of the m-monoid, which is the weight of the derivation.

Definition 4.36. Let $t \in T_\Sigma$. Recall that we consider $\Phi_{M,t}$ as a ranked alphabet (see Definition 4.9). We define the $\Phi_{M,t}$-algebra $\mathcal{G}_{M,t} = \big(T_\Delta(P(\mathrm{pos}(t))), \theta'\big)$ where for every $k \in \mathbb{N}$, $(r, \rho) \in (\Phi_{M,t})^{(k)}$, and $s_1, \ldots, s_k \in T_\Delta(P(\mathrm{pos}(t))$ we let $\theta'(r,\rho)(s_1,\ldots,s_k) = \rho(r_\mathrm{b}) \leftarrow s_1 \cdots s_k$; note that this is well-defined because $k = |\mathrm{indyield}(\rho(r_\mathrm{b}))|$.

We define $\mathrm{h}_{M,t} : T_{\Phi_{M,t}}(P(\mathrm{pos}(t))) \to T_\Delta(P(\mathrm{pos}(t)))$ as the unique $\Phi_{M,t}$-homomorphism from $\mathcal{T}_{\Phi_{M,t}}(P(\mathrm{pos}(t)))$ to $\mathcal{G}_{M,t}$ extending $\mathrm{id}_{P(\mathrm{pos}(t))}$. □

Example 4.37 (*Continuation of Example 4.14*)**.** Consider the mwmd M_1 and input tree t_3 from Example 4.10 (the according dependency hypergraph is shown in Figure 4.2(b)). Moreover, consider the derivation $\eta = e_1(e_2(e_3, e_4), e_5)$ ending in $q(\varepsilon)$. Let us compute $\mathrm{h}_{M_1,t_3}(\eta)$:

$$\begin{aligned}
&\mathrm{h}_{M_1,t_3}\big(e_1(e_2(e_3,e_4),e_5)\big) = [x \mapsto \varepsilon, y \mapsto 2]((r_1)_\mathrm{b}) \leftarrow \mathrm{h}_{M_1,t_3}\big(e_2(e_3,e_4)\big)\mathrm{h}_{M_1,t_3}\big(e_5\big)\\
&= \sigma(q(2),p()) \leftarrow \mathrm{h}_{M_1,t_3}\big(e_2(e_3,e_4)\big)\mathrm{h}_{M_1,t_3}\big(e_5\big) = \sigma\big(\mathrm{h}_{M_1,t_3}\big(e_2(e_3,e_4)\big),\mathrm{h}_{M_1,t_3}\big(e_5\big)\big)\\
&= \sigma\big([x \mapsto 2, y \mapsto 22]((r_1)_\mathrm{b}) \leftarrow \mathrm{h}_{M_1,t_3}(e_3)\mathrm{h}_{M_1,t_3}(e_4), [y \mapsto 11]((r_2)_\mathrm{b}) \leftarrow \varepsilon\big)\\
&= \sigma\big(\sigma(q(22),p()) \leftarrow \mathrm{h}_{M_1,t_3}(e_3)\mathrm{h}_{M_1,t_3}(e_4), \alpha \leftarrow \varepsilon\big)\\
&= \sigma\big(\sigma\big(\mathrm{h}_{M_1,t_3}(e_3),\mathrm{h}_{M_1,t_3}(e_4)\big),\alpha\big)\\
&= \sigma\big(\sigma\big([x \mapsto 22, z \mapsto \varepsilon]((r_3)_\mathrm{b}) \leftarrow \varepsilon, [y \mapsto 1]((r_2)_\mathrm{b}) \leftarrow \varepsilon\big),\alpha\big)\\
&= \sigma\big(\sigma(\beta \leftarrow \varepsilon, \alpha \leftarrow \varepsilon),\alpha\big) = \sigma(\sigma(\beta,\alpha),\alpha)\ .
\end{aligned}$$

By similar derivations we obtain $\mathrm{h}_{M_1,t_3}\big(e_1(e_2(e_3,e_4),e_4)\big) = \mathrm{h}_{M_1,t_3}\big(e_1(e_2(e_3,e_5),e_4)\big) = \mathrm{h}_{M_1,t_3}\big(e_1(e_2(e_3,e_5),e_5)\big) = \sigma(\sigma(\beta,\alpha),\alpha)$.

Now consider mwmd M_2 and input tree t_1 (see Figure 4.2(d)). Observe that there is precisely one derivation η' ending in $q(\varepsilon)$. We obtain $\mathrm{h}_{M_2,t_1}(\eta') = \sigma(\sigma(\beta,\beta),\alpha)$. □

Now we present two technical lemmas concerning basic properties of the transformation function $\mathrm{h}_{M,t}$.

Lemma 4.38. *Let $t \in T_\Sigma$ and $\eta \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t)))$. Then we have* $\mathrm{indyield}(\mathrm{h}_{M,t}(\eta)) = \mathrm{indyield}(\eta)$*. Moreover,* $\mathrm{h}_{M,t}(T_{\Phi_{M,t}}) \subseteq T_\Delta$*.*

Proof. The second part of the lemma follows from the first part, which we prove by induction on the structure of η.

Induction base. If $\eta \in P(\mathrm{pos}(t))$, then $\mathrm{h}_{M,t}(\eta) = \eta$ implies the statement.

Induction step. Let $k \in \mathbb{N}$, $(r,\rho) \in (\Phi_{M,t})^{(k)}$, and $\eta_1,\ldots,\eta_k \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t)))$ such that $\eta = (r,\rho)(\eta_1,\ldots,\eta_k)$. Then Observation 2.9 together with the induction hypothesis yields that we have $\mathrm{indyield}(\mathrm{h}_{M,t}(\eta)) = \mathrm{indyield}(\rho(r_\mathrm{b}) \leftarrow \mathrm{h}_{M,t}(\eta_1)\cdots\mathrm{h}_{M,t}(\eta_k)) = \mathrm{indyield}(\mathrm{h}_{M,t}(\eta_1))\cdots\mathrm{indyield}(\mathrm{h}_{M,t}(\eta_k)) = \mathrm{indyield}(\eta_1)\cdots\mathrm{indyield}(\eta_k)$, which is equal to $\mathrm{indyield}(\eta)$. ■

Lemma 4.39. *Let $t \in T_\Sigma$, $s \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t))$, $k = |\mathrm{indyield}(s)|$, and $s_1,\ldots,s_k \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t))$. Then $\mathrm{h}_{M,t}(s \leftarrow s_1\cdots s_k) = \mathrm{h}_{M,t}(s) \leftarrow \mathrm{h}_{M,t}(s_1)\cdots\mathrm{h}_{M,t}(s_k)$.*

Proof. We give a proof by induction on s.

Induction base. Let $s \in P(\mathrm{pos}(t))$. Then $k = 1$ and, thus, $\mathrm{h}_{M,t}(s \leftarrow s_1) = \mathrm{h}_{M,t}(s_1) = s \leftarrow \mathrm{h}_{M,t}(s_1) = \mathrm{h}_{M,t}(s) \leftarrow \mathrm{h}_{M,t}(s_1)$ because $\mathrm{h}_{M,t}$ extends $\mathrm{id}_{P(\mathrm{pos}(t))}$.

Induction step. Let $l \in \mathbb{N}$, $e = (r, \rho) \in (\Phi_{M,t})^{(l)}$, and $s'_1, \ldots, s'_l \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t))$ such that $s = e(s'_1, \ldots, s'_l)$. For every $i \in [l]$ there are $n_i \in \mathbb{N}$ and $s^i_1, \ldots, s^i_{n_i} \in T_{\Phi_{M,t}}(P(\mathrm{pos}(t))$ such that $|\mathrm{indyield}(s'_i)| = n_i$ and $s^1_1 \cdots s^1_{n_1} \cdots s^l_1 \cdots s^l_{n_l} = s_1 \cdots s_k$. We obtain

$$\begin{aligned}
&\mathrm{h}_{M,t}(s \leftarrow s_1 \cdots s_k) \\
&= \mathrm{h}_{M,t}\big(e(s'_1 \leftarrow s^1_1 \cdots s^1_{n_1}, \ldots, s'_l \leftarrow s^l_1 \cdots s^l_{n_l})\big) \\
&= \rho(r_\mathrm{b}) \leftarrow \mathrm{h}_{M,t}(s'_1 \leftarrow s^1_1 \cdots s^1_{n_1}) \cdots \mathrm{h}_{M,t}(s'_l \leftarrow s^l_1 \cdots s^l_{n_l}) \\
&= \rho(r_\mathrm{b}) \leftarrow \big(\mathrm{h}_{M,t}(s'_1) \leftarrow \mathrm{h}_{M,t}(s^1_1) \cdots \mathrm{h}_{M,t}(s^1_{n_1})\big) \cdots \\
&\qquad\qquad \big(\mathrm{h}_{M,t}(s'_l) \leftarrow \mathrm{h}_{M,t}(s^l_1) \cdots \mathrm{h}_{M,t}(s^l_{n_l})\big) \qquad \text{(by the induction hypothesis)} \\
&= \big(\rho(r_\mathrm{b}) \leftarrow \mathrm{h}_{M,t}(s'_1) \cdots \mathrm{h}_{M,t}(s'_l)\big) \leftarrow \mathrm{h}_{M,t}(s_1) \cdots \mathrm{h}_{M,t}(s_k) \qquad \text{(by Lemma 2.10)} \\
&= \mathrm{h}_{M,t}\big(e(s'_1, \ldots, s'_l)\big) \leftarrow \mathrm{h}_{M,t}(s_1) \cdots \mathrm{h}_{M,t}(s_k) \\
&= \mathrm{h}_{M,t}(s) \leftarrow \mathrm{h}_{M,t}(s_1) \cdots \mathrm{h}_{M,t}(s_k)\ .
\end{aligned}$$

■

Now we define the finitary hypergraph semantics. It is defined as the sum of the weights of all derivations ending in $q(\varepsilon)$, where the computation of the weight of a derivation is broken down into two steps: (i) first the derivation is transformed into a tree over Δ by means of the mapping $\mathrm{h}_{M,t}$, (ii) then the resulting tree is evaluated in the Δ-algebra of the considered m-monoid.

Definition 4.40. Let M be weakly non-circular. The ***tree series hypergraph-defined** by M and $\mathcal{A}$* is the tree series $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$ such that, for every $t \in T_\Sigma$,

$$[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) = \sum\nolimits_{\eta \in \mathrm{H}^{q(\varepsilon)}_G} h(\mathrm{h}_{M,t}(\eta))\ ,$$

where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$ and h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). Note that $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}$ is well-defined due to the definition of weakly non-circular mwmd and Lemma 4.38. The set of all tree series hypergraph-defined by weakly non-circular mwmd over Σ and Δ, and $\mathcal{A}$ is denoted by $\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A})$. □

Example 4.41 (*Continuation of Examples 4.37 and 4.30***).** Consider the mwmd M_1, input tree t_3, and m-monoid $\mathcal{A}_1$. In Example 4.14 we have shown that there are four derivations ending in $q(\varepsilon)$ and in Example 4.37 we have established that for every such derivation η we have $\mathrm{h}_{M_1,t_3}(\eta) = \sigma(\sigma(\beta, \alpha), \alpha)$. Thus, for $G = \mathrm{G}^{\mathrm{dep}}_{M_1,t_3}$ we obtain

$$\begin{aligned}
[\![M_1]\!]^{\mathrm{hyp}}_{\mathcal{A}_1}(t_3) &= \sum\nolimits_{\eta \in \mathrm{H}^{q(\varepsilon)}_G} h(\mathrm{h}_{M_1,t_3}(\eta)) = 4 \cdot h\big(\sigma(\sigma(\beta, \alpha), \alpha)\big) \\
&\qquad \text{(where h is the Δ_{ex}-homomorphism from $\mathcal{T}_{\Delta_{\mathrm{ex}}}$ to $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \theta_1)$)} \\
&= 4 \cdot \theta_1(\sigma)(\theta_1(\sigma)(\theta_1(\beta)(), \theta_1(\alpha)()), \theta_1(\alpha)()) \\
&= 4 \cdot \theta_1(\sigma)(\theta_1(\sigma)(2, 3), 3) = 4 \cdot \theta_1(\sigma)(2/2 + 3 + 1, 3) \\
&= 4 \cdot \theta_1(\sigma)(5, 3) = 4 \cdot (5/2 + 3 + 1) = 4 \cdot 13/2 = 26\ .
\end{aligned}$$

Now consider mwmd M_2 and input tree t_1 (see Figure 4.2(d)). In Example 4.37 we showed that there is precisely one derivation η' ending in $q(\varepsilon)$ and that $\mathrm{h}_{M_2,t_1}(\eta') = \sigma(\sigma(\beta, \beta), \alpha)$. Recall m-monoid $\mathcal{A}_3$ from Example 4.30. We obtain

$$[\![M_2]\!]^{\mathrm{hyp}}_{\mathcal{A}_3}(t_1) = [\![M_2]\!]^{\mathrm{hyp}}_{\mathcal{A}_3}\big(\sigma(\beta, \sigma(\beta, \alpha))\big) = h\big(\sigma(\sigma(\beta, \beta), \alpha)\big) = \big\{\sigma(\sigma(\beta, \beta), \alpha)\big\}\ ,$$

where h is the $\Delta_{\rm ex}$-homomorphism from $\mathcal{T}_{\Delta_{\rm ex}}$ to $(\mathcal{P}(T_{\Delta_{\rm ex}}), \theta_3)$.

Note that $[\![M_2]\!]^{\rm hyp}_{\mathcal{A}_3}$ is a mapping from trees over $\Sigma_{\rm ex}$ to sets of trees over $\Delta_{\rm ex} = \Sigma_{\rm ex}$. In particular, we obtain that if $[\![M_2]\!]^{\rm hyp}_{\mathcal{A}_3}$ takes a right-descending comb having the form $\sigma(\alpha_1, \sigma(\alpha_2, \cdots \sigma(\alpha_n, \alpha_{n+1}) \cdots))$, for some $n \in \mathbb{N}_+$ and $\alpha_1, \ldots, \alpha_{n+1} \in \{\alpha, \beta\}$, then it produces a singleton set with the related left-descending comb $\sigma(\sigma(\cdots \sigma(\alpha_1, \alpha_2) \cdots \alpha_n), \alpha_{n+1})$ (cf. [60, Example 5.4]; compare the six rules that are required to define the mwmd M_2 versus the 11 rules that are required to define the attributed tree transducer in [60, Example 5.4]). □

When the considered mwmd is weakly non-circular, then the set of derivations ending in $q(\varepsilon)$ is finite. The following lemma states that the number of derivations is even bounded from above.

Lemma 4.42. *Let M be weakly non-circular. Then there is a constant $n \in \mathbb{N}$ such that for every $t \in T_\Sigma$ we have $|\mathrm{H}^{q(\varepsilon)}_G| \leq 2^{(2^{n \cdot \mathrm{size}(t)})}$, where $G = \mathrm{G}^{\rm dep}_{M,t}$.*

This bound is tight in the following sense: If $\Delta^{(0)} \neq \emptyset$ and Δ is not monadic, then for every $n \in \mathbb{N}$ there is a weakly non-circular mwmd M_n over Σ and Δ such that for every $t \in T_\Sigma$ we have $|\mathrm{H}^{q(\varepsilon)}_{G_n}| = 2^{(2^{n \cdot \mathrm{size}(t)})}$, where $G_n = \mathrm{G}^{\rm dep}_{M_n,t}$.

Proof. Let $M = (P, R, q)$ and let $b = \max\{|\mathrm{indyield}(r_{\rm b})| \mid r \in R\}$. Then for every $t \in T_\Sigma$ and $e \in \Phi_{M,t}$ we have $\mathrm{rk}(e) \leq b$. Let $t \in T_\Sigma$ and $G = \mathrm{G}^{\rm dep}_{M,t}$. Let $\eta \in \mathrm{H}^{q(\varepsilon)}_G$. Due to Lemmas 2.24 and 2.25(1 ⇒ 2) and due to the fact that $\mathrm{H}^{q(\varepsilon)}_G$ is finite, we obtain $\mathrm{height}(\eta) < |P(\mathrm{pos}(t))|$. Then the sets $\mathrm{H}^{q(\varepsilon)}_G$ and $\{\eta \in \mathrm{H}^{q(\varepsilon)}_G \mid \mathrm{height}(\eta) \leq |P(\mathrm{pos}(t))| - 1\}$ coincide. Thus, Lemma 2.6 and the fact that $\mathrm{maxrk}(\Phi_{M,t}) \leq b$ imply that $|\mathrm{H}^{q(\varepsilon)}_G| \leq |\Phi_{M,t}|^{k_t}$, where $k_t = (b+1)^{|P(\mathrm{pos}(t))|-1}$.

Let us estimate the size of the set $\Phi_{M,t}$. Let $l = \max\{|\mathrm{var}(r)| \mid r \in R\}$. Clearly, for every $r \in R$ we have that there are at most $|\mathrm{pos}(t)|^l = \mathrm{size}(t)^l$ many r,t-variable assignments. Thus, $|\Phi_{M,t}| \leq |R| \cdot \mathrm{size}(t)^l$. Observe that $|P(\mathrm{pos}(t))| \leq |P| \cdot |\mathrm{pos}(t)| = |P| \cdot \mathrm{size}(t)$. We obtain

$$\begin{aligned}
|\mathrm{H}^{q(\varepsilon)}_G| &\leq |\Phi_{M,t}|^{k_t} \leq (|R| \cdot \mathrm{size}(t)^l)^{\left((b+1)^{|P(\mathrm{pos}(t))|-1}\right)} \leq (|R| \cdot \mathrm{size}(t)^l)^{\left((b+1)^{|P| \cdot \mathrm{size}(t)}\right)} \\
&\leq (k \cdot \mathrm{size}(t)^k)^{\left(k^{k \cdot \mathrm{size}(t)}\right)} && (\text{where } k = \max\{|R|, |P|, b+1, l\}) \\
&= k^{\left(k^{k \cdot \mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(k \cdot k^{k \cdot \mathrm{size}(t)}\right)} = k^{\left((k^k)^{\mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(k \cdot (k^k)^{\mathrm{size}(t)}\right)} \\
&\leq j^{\left(j^{\mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(j \cdot j^{\mathrm{size}(t)}\right)} && (\text{where } j = k^k) \\
&\leq j^{\left(j^{\mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(j^{\mathrm{size}(t)} \cdot j^{\mathrm{size}(t)}\right)} && (\text{because } \mathrm{size}(t) \geq 1) \\
&\leq i^{\left(i^{\mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(i^{\mathrm{size}(t)}\right)} && (\text{where } i = j \cdot j)
\end{aligned}$$

Observe that $i \leq 2^i$ and, thus, $i^{(i^{\mathrm{size}(t)})} \leq (2^i)^{(i^{\mathrm{size}(t)})} = 2^{(i \cdot i^{\mathrm{size}(t)})} \leq 2^{(i^{\mathrm{size}(t)} \cdot i^{\mathrm{size}(t)})} = 2^{((i^2)^{\mathrm{size}(t)})}$. Moreover, $\mathrm{size}(t) \leq 2^{2^{\mathrm{size}(t)}}$ implies that $\mathrm{size}(t)^{\left(i^{\mathrm{size}(t)}\right)} \leq \left(2^{2^{\mathrm{size}(t)}}\right)^{\left(i^{\mathrm{size}(t)}\right)} = 2^{\left(2^{\mathrm{size}(t)} \cdot i^{\mathrm{size}(t)}\right)} = 2^{((2i)^{\mathrm{size}(t)})}$. Thus,

$$\begin{aligned}
|\mathrm{H}^{q(\varepsilon)}_G| &\leq i^{\left(i^{\mathrm{size}(t)}\right)} \cdot \mathrm{size}(t)^{\left(i^{\mathrm{size}(t)}\right)} \leq 2^{((i^2)^{\mathrm{size}(t)})} \cdot 2^{((2i)^{\mathrm{size}(t)})} \\
&\leq 2^{(m^{\mathrm{size}(t)})} \cdot 2^{(m^{\mathrm{size}(t)})} = 2^{(2 \cdot m^{\mathrm{size}(t)})} && (\text{where } m = \max\{i^2, 2i\})
\end{aligned}$$

$$\leq 2^{(2^{\text{size}(t)} \cdot m^{\text{size}(t)})} = 2^{((2m)^{\text{size}(t)})} .$$

Let $n \in \mathbb{N}$ be minimal in the set $\{n' \in \mathbb{N} \mid 2m \leq 2^{n'}\}$. Then we obtain that $|\mathrm{H}_G^{q(\varepsilon)}| \leq 2^{((2\cdot m)^{\text{size}(t)})} \leq 2^{((2^n)^{\text{size}(t)})} = 2^{(2^{n\cdot\text{size}(t)})}$; this holds for every $t \in T_\Sigma$ because $|R|$, $|P|$, b, and l (and, hence, k, j, i, m, and n) do only depend on M and not on t.

Now we show that this bound is tight. Suppose that $\Delta^{(0)} \neq \emptyset$ and that Δ is not monadic; thus, there is a $k \in \mathbb{N}$ with $k > 1$ such that $\Delta^{(k)} \neq \emptyset$; choose $\alpha \in \Delta^{(0)}$ and $\delta \in \Delta^{(k)}$.

For reasons of simplicity we assume that Σ is a monadic ranked alphabet. The arguments in this proof can easily be extended to arbitrary ranked alphabets. Let $n \in \mathbb{N}$. We construct the mwmd $M_n = (P, R, p_n)$ over Σ and Δ as follows:

$$\begin{aligned} P &= \{p_0^{(1)}, p_1^{(1)}, \ldots, p_n^{(1)}\} , \\ R &= \{p_i(x) \leftarrow \delta(p_{i-1}(x), p_{i-1}(x), \alpha, \ldots, \alpha) \,;\, \emptyset \mid i \in [n]\} \\ &\quad \cup \big\{p_0(x) \leftarrow p_n(y) \,;\, \{\text{child}_1(x, y)\}\big\} \\ &\quad \cup \big\{p_0(x) \leftarrow \alpha \,;\, \{\text{leaf}(x)\}, \quad p_0(y) \leftarrow \alpha \,;\, \{\text{leaf}(y)\}\big\} , \end{aligned}$$

where the number of occurrences of α in the tree $\delta(p_{i-1}(x), p_{i-1}(x), \alpha, \ldots, \alpha)$ is $\text{rk}(\delta) - 2$. Note that we require the rule $p_0(x) \leftarrow \alpha \,;\, \{\text{leaf}(x)\}$ to occur twice in the set R. Since R is a set (and not a multiset) and, thus, allows every rule only to occur once, we cope with this problem by adding the rule $p_0(y) \leftarrow \alpha \,;\, \{\text{leaf}(y)\}$ instead (where y is chosen arbitrarily in $\mathrm{V} \setminus \{x\}$), which is essentially the same rule. It is easy to see that M_n is weakly non-circular (it is even non-circular).

Let $t \in T_\Sigma$ and $G_n = \mathrm{G}^{\text{dep}}_{M_n,t}$. We show that $|\mathrm{H}_{G_n}^{p_n(\varepsilon)}| = 2^{(2^{n\cdot\text{size}(t)})}$. By our assumption that Σ is monadic we obtain that there are $\beta \in \Sigma^{(0)}$, $m \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_m \in \Sigma^{(1)}$ such that $t = \gamma_1(\gamma_2(\cdots\gamma_m(\beta)\cdots))$. Then $\text{size}(t) = m + 1$ and $\text{pos}(t) = \{1^0, 1^1, 1^2, \ldots, 1^m\}$, where 1^i denotes the string of i successive occurrences of 1, for every $i \in \{0, \ldots, n\}$. We claim that for every $j \in \{0, \ldots, m\}$ and $i \in \{0, \ldots, n\}$ we have that $|\mathrm{H}_{G_n}^{p_i(w_j)}| = 2^{(2^{n\cdot j+i})}$, where $w_j = 1^{m-j}$. Clearly, this claim implies $|\mathrm{H}_{G_n}^{p_n(\varepsilon)}| = 2^{(2^{n\cdot m+n})} = 2^{(2^{n\cdot(m+1)})} = 2^{(2^{n\cdot\text{size}(t)})}$. We prove our claim by induction on j.

Induction base. Let $j = 0$. We prove that for every $i \in \{0, \ldots, n\}$ we have that $|\mathrm{H}_{G_n}^{p_i(w_0)}| = 2^{(2^i)}$. We give a proof by induction on i. Clearly, for $i = 0$, $|\mathrm{H}_{G_n}^{p_0(w_0)}| = 2 = 2^{(2^0)}$ because w_0 is a leaf position in t. Now let $i \in [n]$ and assume that $|\mathrm{H}_{G_n}^{p_{i-1}(w_0)}| = 2^{(2^{i-1})}$. Then Corollary 4.11(4) yields $|\mathrm{H}_{G_n}^{p_i(w_0)}| = \big(|\mathrm{H}_{G_n}^{p_{i-1}(w_0)}|\big)^2 = \big(2^{(2^{i-1})}\big)^2 = 2^{(2^i)}$.

Induction step. Let $j \in [m]$ and assume that for every $i \in \{0, \ldots, n\}$ we have that $|\mathrm{H}_{G_n}^{p_i(w_{j-1})}| = 2^{(2^{n\cdot(j-1)+i})}$. We show for every $i \in \{0, \ldots, n\}$ that $|\mathrm{H}_{G_n}^{p_i(w_j)}| = 2^{(2^{n\cdot j+i})}$. We give a proof by induction on i. For $i = 0$ we obtain by means of Corollary 4.11(4) and the induction hypothesis (for the outer inductive proof over j) that $|\mathrm{H}_{G_n}^{p_0(w_j)}| = |\mathrm{H}_{G_n}^{p_n(w_{j-1})}| = 2^{(2^{n\cdot(j-1)+n})} = 2^{(2^{n\cdot j})}$. Now let $i \in [n]$ and assume that $|\mathrm{H}_{G_n}^{p_{i-1}(w_j)}| = 2^{(2^{n\cdot j+i-1})}$. Then Corollary 4.11(4) yields $|\mathrm{H}_{G_n}^{p_i(w_j)}| = \big(|\mathrm{H}_{G_n}^{p_{i-1}(w_j)}|\big)^2 = \big(2^{(2^{n\cdot j+i-1})}\big)^2 = 2^{(2^{n\cdot j+i})}$. ∎

Now we define the infinitary hypergraph semantics, which is only defined for ω-complete m-monoids. The definition looks similar to Definition 4.40. However, the summation in Definition 4.40 is the extension of the monoid operation to finite families whereas the summation in the following definition is the ω-infinitary sum operation of the given ω-complete m-monoid.

Definition 4.43. Suppose that $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Then the ***tree series hypergraph-defined*** *by* M *and* $(\mathcal{A}, \sum)$ is the tree series $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$ such that, for every $t \in T_\Sigma$,

$$[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)}(t) = \sum\nolimits_{\eta \in \mathrm{H}_G^{q(\varepsilon)}} h(\mathrm{h}_{M,t}(\eta)) \ ,$$

where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$ and h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). The set of all tree series hypergraph-defined by arbitrary mwmd over Σ and Δ, and $(\mathcal{A}, \sum)$ is denoted by $\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)$. □

Example 4.44 (*Continuation of Example 4.30*)**.** Consider the mwmd M_4 and input tree t_5 (see Figure 4.3(c)). Let $e_6 = (r'_{10}, [y \mapsto \varepsilon])$, $e_7 = (r_{13}, [\ ])$, $e_8 = (r_{11}, [\ ])$, and $e_9 = (r_{12}, [y \mapsto \varepsilon])$. Note that the set of derivations ending in $q(\varepsilon)$ consists of all trees of the form $e_6\big(e_8(e_8(\cdots e_8(e_8(e_7, e_9), e_9) \cdots), e_9)\big)$.

Observe that for all derivations η, η' of M_4 and t_5 we have

$$\mathrm{h}_{M_4,t_5}(e_6(\eta)) = \mathrm{h}_{M_4,t_5}(\eta) \ , \qquad \mathrm{h}_{M_4,t_5}(e_7) = \beta \ ,$$
$$\mathrm{h}_{M_4,t_5}(e_8(\eta, \eta')) = \sigma\big(\mathrm{h}_{M_4,t_5}(\eta), \mathrm{h}_{M_4,t_5}(\eta')\big) \ , \qquad \mathrm{h}_{M_4,t_5}(e_9) = \alpha \ .$$

Consider the ω-complete m-monoid $(\mathcal{A}_3, \cup)$. Let h be the Δ_{ex}-homomorphism from $\mathcal{T}_{\Delta_{\mathrm{ex}}}$ to $(\mathcal{P}(T_{\Delta_{\mathrm{ex}}}), \theta_3)$. For every tree $t \in T_{\Delta_{\mathrm{ex}}}$ we have $h(t) = \{t\}$. Hence, for every derivation η of the form $e_6\big(e_8(e_8(\cdots e_8(e_8(e_7, e_9), e_9) \cdots), e_9)\big)$ (with n occurrences of e_9) we obtain $h(\mathrm{h}_{M_4,t_5}(\eta)) = \big\{\sigma(\sigma(\cdots \sigma(\sigma(\beta, \alpha), \alpha) \cdots), \alpha)\big\}$ (with n occurrences of α). Therefore, $[\![M_4]\!]^{\mathrm{hyp}}_{(\mathcal{A}_3,\cup)}(t_5)$ is the set of left combs of the form $\sigma(\sigma(\cdots \sigma(\sigma(\beta, \alpha), \alpha) \cdots), \alpha)$. □

Clearly, if both the finitary and the infinitary version of the hypergraph semantics are applicable, then they coincide.

Lemma 4.45. *Let M be weakly non-circular and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Then $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)}$.*

Proof. This is an immediate consequence of Proposition 3.16. ■

Corollary 4.46. *Let $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Then*

$$\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) \subseteq \mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)) \ .$$

4.2.4 Comparison of fixpoint and hypergraph semantics

The computation of the fixpoint semantics consists of interleaved applications of the operations of the Δ-algebra and the monoid operation of the given m-monoid, whereas the hyperpath semantics is computed by summing up values that are obtained by merely using operations from the Δ-algebra. Roughly speaking, the computation of the hyperpath semantics can be obtained from the computation of the fixpoint semantics by distributing out the monoid operation. In this sense the fixpoint semantics resembles the initial algebra semantics of bottom-up weighted tree automata [15, 63], whereas the hypergraph semantics resembles the run semantics.

Note that in general the fixpoint semantics and hypergraph semantics do not coincide. In Example 4.21 we have shown that $[\![M_1]\!]^{\mathrm{fix}}_{\mathcal{A}_1}(t_3) = 11$; however, in Example 4.41 we have established the fact that $[\![M_1]\!]^{\mathrm{hyp}}_{\mathcal{A}_1}(t_3) = 26$; hence, $[\![M_1]\!]^{\mathrm{fix}}_{\mathcal{A}_1} \neq [\![M_1]\!]^{\mathrm{hyp}}_{\mathcal{A}_1}$. This difference is due to the fact that $\mathcal{A}_1$ is not distributive.

In fact, in the remainder of this chapter we show that the fixpoint and the hypergraph semantics coincide if the considered m-monoid is distributive. To this end we introduce an auxiliary notion.

Definition 4.47. Let $t \in T_\Sigma$, $c \in P(\mathrm{pos}(t))$, and $n \in \mathbb{N}$. The *set of n-**bounded derivations** of M and t ending in c* is defined as the set

$$\mathrm{H}^{c,n}_G = \{\eta \in \mathrm{H}^c_G \mid \mathrm{height}(\eta) < n\} . \qquad \square$$

The following observation is a consequence of the definition of the height of trees and Observation 4.11(4).

Observation 4.48. *Let $t \in T_\Sigma$, $c \in P(\mathrm{pos}(t))$, and $n \in \mathbb{N}$. Moreover, for every $e = (r, \rho) \in \Phi_{M,t,c}$ let $k_e \in \mathbb{N}$ and $c^e_1, \dots, c^e_{k_e} \in P(\mathrm{pos}(t))$ such that $c^e_1 \cdots c^e_{k_e} = \mathrm{indyield}(\rho(r_\mathrm{b}))$. Let $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. Then*

$$\mathrm{H}^{c,n+1}_G = \{e(\eta_1, \dots, \eta_{k_e}) \mid e \in \Phi_{M,t,c}, \eta_1 \in \mathrm{H}^{c^e_1,n}_G, \dots, \eta_{k_e} \in \mathrm{H}^{c^e_{k_e},n}_G)\} .$$

Before we show that the semantics coincide for distributive m-monoids, we need to prove another technical lemma.

Lemma 4.49. *Let $\mathcal{A}$ be distributive, $t \in T_\Sigma$, and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. Moreover, let h be the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ), and $n \in \mathbb{N}$. Then for every $c \in P(\mathrm{pos}(t))$:*

$$\sum_{\eta \in \mathrm{H}^{c,n}_G} h(\mathrm{h}_{M,t}(\eta)) = \mathcal{T}^n(I_\mathbf{0})(c) . \tag{4.2}$$

Proof. We give a proof by induction on n.

Induction base. For $n = 0$ we have $\mathrm{H}^{c,0}_G = \emptyset$ for every $c \in P(\mathrm{pos}(t))$; hence, both sides of Equation (4.2) are equal to $\mathbf{0}$.

Induction step. Let $n \in \mathbb{N}$ and $c \in P(\mathrm{pos}(t))$. For every $e = (r, \rho) \in \Phi_{M,t,c}$ let $k_e \in \mathbb{N}$ and $c^e_1, \dots, c^e_{k_e} \in P(\mathrm{pos}(t))$ such that $c^e_1 \cdots c^e_{k_e} = \mathrm{indyield}(\rho(r_\mathrm{b}))$. Then

$$\begin{aligned}
&\sum_{\eta \in \mathrm{H}^{c,n+1}_G} h(\mathrm{h}_{M,t}(\eta)) \\
&= \sum_{e \in \Phi_{M,t,c}} \sum_{\eta_1 \in \mathrm{H}^{c^e_1,n}_G, \dots, \eta_{k_e} \in \mathrm{H}^{c^e_{k_e},n}_G} h\big(\mathrm{h}_{M,t}(e(\eta_1, \dots, \eta_{k_e}))\big) && \text{(By Observation 4.48)} \\
&= \sum_{e=(r,\rho) \in \Phi_{M,t,c}} \sum_{\eta_1 \in \mathrm{H}^{c^e_1,n}_G, \dots, \eta_{k_e} \in \mathrm{H}^{c^e_{k_e},n}_G} h\big(\rho(r_\mathrm{b}) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_{k_e})\big) \\
&= \sum_{e=(r,\rho) \in \Phi_{M,t,c}} h'_e(\rho(r_\mathrm{b})) , && \text{(by Lemma 3.14(1))}
\end{aligned}$$

where for every $e = (r, \rho) \in \Phi_{M,t,c}$ the mapping h'_e is the unique Δ-homomorphism from the $\mathcal{T}_\Delta\big(\mathrm{ind}(\rho(r_\mathrm{b}))\big)$ to (A, θ) such that for every $c' \in \mathrm{ind}(\rho(r_\mathrm{b}))$ we have $h'_e(c') = \sum_{\eta \in \mathrm{H}^{c',n}_G} h(\mathrm{h}_{M,t}(\eta))$. We used Lemma 3.14(1) with the following instantiations: for every $e = (r, \rho) \in \Phi_{M,t,c}$ and $c' \in \mathrm{ind}(\rho(r_\mathrm{b}))$ we let $I_{c'} = \mathrm{H}^{c',n}_G$ and, for every $i \in I_{c'}$, $s^{c'}_i = \mathrm{h}_{M,t}(i)$.

Let $e \in \Phi_{M,t,c}$ and $c' \in \mathrm{ind}(\rho(r_\mathrm{b}))$. The induction hypothesis yields

$$h'_e(c') = \sum_{\eta \in \mathrm{H}^{c',n}_G} h(\mathrm{h}_{M,t}(\eta)) = \mathcal{T}^n(I_\mathbf{0})(c') . \tag{4.3}$$

This yields $h'_e(\rho(r_\mathrm{b})) = \mathrm{h}_{\mathcal{T}^n(I_\mathbf{0})}(\rho(r_\mathrm{b}))$ due to Corollary 3.11. We obtain

$$\begin{aligned}&\sum\nolimits_{\eta \in \mathrm{H}_G^{c,n+1}} h(\mathrm{h}_{M,t}(\eta)) = \sum\nolimits_{e=(r,\rho)\in \Phi_{M,t,c}} h'_e(\rho(r_\mathrm{b})) && \text{(shown above)}\\ &= \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}^n(I_\mathbf{0})}(\rho(r_\mathrm{b})) = \mathcal{T}^{n+1}(I_\mathbf{0})(c) \ . && \blacksquare\end{aligned}$$

Now we show that the finitary versions of the fixpoint and the hypergraph semantics coincide for distributive m-monoids.

Lemma 4.50. *Let M be weakly non-circular and $\mathcal{A}$ be distributive. Then $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}$.*

Proof. Let $t \in T_\Sigma$ and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. The set $\mathrm{H}_G^{q(\varepsilon)}$ is finite because M is weakly non-circular. Hence, there is an $n \in \mathbb{N}$ such that $\mathrm{H}_G^{q(\varepsilon)} = \mathrm{H}_G^{q(\varepsilon),n}$. Let $m = \max\big(n, |P(\mathrm{pos}(t))|\big)$. Let h be the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). Then

$$\begin{aligned}[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) &= \sum\nolimits_{\eta\in\mathrm{H}_G^{q(\varepsilon)}} h(\mathrm{h}_{M,t}(\eta)) = \sum\nolimits_{\eta\in\mathrm{H}_G^{q(\varepsilon),m}} h(\mathrm{h}_{M,t}(\eta))\\ &= \mathcal{T}^m(I_\mathbf{0})(q(\varepsilon)) && \text{(by Lemma 4.49)}\\ &= \mathcal{T}^{|P(\mathrm{pos}(t))|}(I_\mathbf{0})(q(\varepsilon)) && \text{(by Lemma 4.19)}\\ &= [\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}(t) \ . && \blacksquare\end{aligned}$$

Now we show that the infinitary versions of the fixpoint and the hypergraph semantics coincide for distributive and related ω-complete and ω-continuous m-monoids.

Lemma 4.51. *Let $\mathcal{A}$ be distributive. Moreover, let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid such that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related. Then $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} = [\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}$.*

Proof. Let $t \in T_\Sigma$ and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. For every $n \in \mathbb{N}$ let $H_n = \mathrm{H}_G^{q(\varepsilon),n+1} \setminus \mathrm{H}_G^{q(\varepsilon),n}$ and $a_n = \sum_{\eta\in H_n} h(\mathrm{h}_{M,t}(\eta))$. Observe that $(H_n \mid n \in \mathbb{N})$ is a generalized partition of $\mathrm{H}_G^{q(\varepsilon)}$ because for every $\eta \in \mathrm{H}_G^{q(\varepsilon)}$ and $n \in \mathbb{N}$ we have $\eta \in H_n$ iff $n = \mathrm{height}(\eta)$. Thus,

$$\begin{aligned}&[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)}(t) = \sum\nolimits_{\eta\in\mathrm{H}_G^{q(\varepsilon)}} h(\mathrm{h}_{M,t}(\eta))\\ &= \sum\nolimits_{n\in\mathbb{N}} \sum\nolimits_{\eta\in H_n} h(\mathrm{h}_{M,t}(\eta)) = \sum\nolimits_{n\in\mathbb{N}} a_n\\ &= \bigvee\{a_{\pi(0)} + \cdots + a_{\pi(n)} \mid n \in \mathbb{N}\} \ ,\end{aligned}$$

where $\pi = \mathrm{id}_\mathbb{N}$; this statement holds by Lemma 3.39(3) and since $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related;

$$\begin{aligned}&= \bigvee\{a_0 + \cdots + a_n \mid n \in \mathbb{N}\}\\ &= \bigvee\Big\{\sum\nolimits_{\eta\in H_0} h(\mathrm{h}_{M,t}(\eta)) + \cdots + \sum\nolimits_{\eta\in H_n} h(\mathrm{h}_{M,t}(\eta)) \mid n \in \mathbb{N}\Big\}\\ &= \bigvee\Big\{\sum\nolimits_{\eta\in\mathrm{H}_G^{q(\varepsilon),n+1}} h(\mathrm{h}_{M,t}(\eta)) \mid n \in \mathbb{N}\Big\}\\ &\qquad \text{(because } (H_m \mid m \in \{0,\dots,n\}) \text{ is a generalized partition of } \mathrm{H}_G^{q(\varepsilon),n+1})\end{aligned}$$

$$= \bigvee\big\{\mathcal{T}^{n+1}(I_{\mathbf{0}})(q(\varepsilon)) \mid n \in \mathbb{N}\big\} \qquad \text{(by Lemma 4.49)}$$
$$= \bigvee\big\{\mathcal{T}^{n+1}(I_{\mathbf{0}}) \mid n \in \mathbb{N}\big\}(q(\varepsilon)) \qquad \text{(by Lemmas 4.26 and 4.28)}$$
$$= \mathcal{T}^{\omega}(q(\varepsilon)) = [\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}(t) \ . \qquad \blacksquare$$

Example 4.52 (*Continuation of Examples 4.30 and 4.44*)**.** Clearly, $(\mathcal{A}_3, \subseteq)$ and $(\mathcal{A}_3, \cup)$ are related. Since $\mathcal{A}_3$ is distributive, Lemma 4.51 yields $[\![M_4]\!]^{\mathrm{hyp}}_{(\mathcal{A}_3,\cup)} = [\![M_4]\!]^{\mathrm{fix}}_{(\mathcal{A}_3,\subseteq)}$. We have already shown in Examples 4.30 and 4.44 that both $[\![M_4]\!]^{\mathrm{hyp}}_{(\mathcal{A}_3,\cup)}$ and $[\![M_4]\!]^{\mathrm{fix}}_{(\mathcal{A}_3,\subseteq)}$ are the set of all left-descending combs of the form $\sigma(\sigma(\cdots\sigma(\sigma(\beta,\alpha),\alpha)\cdots),\alpha)$. □

Theorem 4.53. *Let $\mathcal{A}$ be a dm-monoid. Moreover, let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid such that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related. Then*

$$\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) \ ,$$
$$\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) = \mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big) \ .$$

Proof. This theorem is an immediate consequence of Lemma 4.50 and 4.51. ■

Normal forms

In this chapter we will study four syntactic subclasses of mwmd, called restricted, connected, local, and proper mwmd. Let us give an informal definition of these classes.

Restricted: the positions of the variables in the body of rules must obey a certain structure.

Connected: there are no rules having variables that form independent clusters.

Local: for every rule and every valid variable assignment for that rule the variables of the rule are assigned to directly neighboring nodes of the input tree; the rules resemble the rules in attribute grammars [32, 50] and attributed tree transducers [56, 60].

Proper: every user-defined predicate is unary.

For each of these subclasses (and intersections of them) we will study conditions that allow a general mwmd to be transformed into a semantically equivalent mwmd belonging to this subclass, i.e., we investigate what subclasses can be considered to be normal forms of mwmd. The notion 'semantically equivalent' that we used in the previous sentence is quite ambiguous because we did not specify which of the four variants of semantics, that we defined in the previous chapter, this relates to. In this chapter we aim for normal form constructions that allow for the strongest possible definition of semantic equivalence: these are constructions that preserve all four kinds of semantics simultaneously. Since equivalence proofs for such constructions are very laborious, we will, as a preliminary step, first prove a generic equivalence result that we will employ for (almost every) of the normal form constructions later in this chapter.

The connected normal form has been introduced by Gottlob and Koch [69, Theorem 4.2]; it has also been studied in [122, 28]. The remaining three syntactic classes have first been investigated in [28]. The constructions that we present in this section are based on the constructions in [28].

This chapter is organized as follows. In Section 5.1 we define our syntactic classes formally and state the normal form theorem that we prove in the remainder of this chapter. In Section 5.2 we develop and prove our auxiliary generic equivalence result that we use in the remaining three sections, where we deal with proper mwmd (Section 5.3), restricted and connected mwmd (Section 5.4), and local mwmd (Section 5.5).

5.1 Syntactic Subclasses

In this section we define the four syntactic subclasses that we deal with in this chapter formally. Afterwards we state the main theorem of this chapter.

Definition 5.1. Let $r \in R$. The ***variable connection relation*** $\sim_r$ *of* r is the transitive reflexive closure of $\{(x, y) \in \mathrm{var}(r) \times \mathrm{var}(r) \mid \exists\, b \in r_{\mathrm{G}} : \{x, y\} \subseteq \mathrm{var}(b)\}$. We call the rule r ***restricted*** if for every equivalence class $C \in \mathrm{var}(r)/{\sim_r}$ with $C \cap \mathrm{var}(r_{\mathrm{b}}) \neq \emptyset$ there is a $w_C \in \text{pos}(r_\text{b})$ such that the following statements are equivalent for every $w' \in \text{pos}(r_\text{b})$ with $r_\text{b}(w') \in P(\text{var}(r))$:

- $r_\text{b}(w') \in P(C)$ and
- w_C is a prefix of w'.

Moreover, we call r ***connected*** if $\sim_r = \text{var}(r) \times \text{var}(r)$. We call M ***restricted*** if every $r \in R$ is restricted and we say that M is ***connected*** if every $r \in R$ is connected. □

Example 5.2. Consider the following two rules

$$r = q(y) \leftarrow \sigma\big(q(z), \sigma(q(x), q(y))\big) \; ; \{\text{child}_1(x,y), \text{child}_2(x,x')\} \; ,$$
$$r' = q(z') \leftarrow \sigma\big(q(x), \sigma(q(z), q(y))\big) \; ; \{\text{child}_1(x,y), \text{child}_1(z,z'), \text{child}_2(z',z'')\} \; .$$

We have $\text{var}(r) = \{x, x', y, z\}$ and $\text{var}(r') = \{x, y, z, z', z''\}$. Moreover, the equivalence relation $\sim_r$ has precisely the two equivalence classes $\{x, y, x'\}$ and $\{z\}$, and $\sim_{r'}$ has the equivalence classes $\{x, y\}$ and $\{z, z', z''\}$. Therefore, neither of the rules r and r' is connected.

The rule r is restricted because for $C_1 = \{x, y, x'\}$ and $C_2 = \{z\}$ we obtain that $w_{C_1} = 2$ and $w_{C_2} = 1$ satisfy the required properties. The rule r' is not restricted because for $C_3 = \{x, y\}$ there is no position w_{C_3} having the required properties: (i) if $w_{C_3} = \varepsilon$ or $w_{C_3} = 2$, then for the position $w' = 21 \in \text{pos}(r'_\text{b})$ we have that: $r'_\text{b}(w') = q(z) \notin P(C_3)$ but w_{C_3} is a prefix of w'; (ii) if $w_{C_3} = 1$, then for the position $w' = 22 \in \text{pos}(r'_\text{b})$ we have that: $r'_\text{b}(w') = q(y) \in P(C_3)$ but w_{C_3} is not a prefix of w'; and (iii) if $w_{C_3} = 21$ or $w_{C_3} = 22$, then for the position $w' = 1 \in \text{pos}(r'_\text{b})$ we have that: $r'_\text{b}(w') = q(x) \in P(C_3)$ but w_{C_3} is not a prefix of w'.

Now consider the mwmd M_1 from Example 4.5. It is not connected because the rule r_3 is not connected ($x \not\sim_{r_3} z$). However, M_1 is restricted. It is easy to check that the mwmd M_2 is both connected and restricted.

The mwmd M_3 from Example 4.14 is not connected due to rule r_{10} (we have $x \not\sim_{r_{10}} y$) but it is restricted. The mwmd M_4 is both connected and restricted because $\text{var}(r'_{10}) = \{x\}$. □

Every connected mwmd that we considered in Example 5.2 is also restricted. The following lemma states that this holds in general.

Observation 5.3. *If $r \in R$ is connected, then it is also restricted.*

Proof. This is obvious if $\text{var}(r) = \emptyset$. Otherwise $\text{var}(r)/{\sim_r}$ contains a unique element C. Then it is easy to see that for $w_C = \varepsilon$ we have, for every $w' \in \text{pos}(r_\text{b})$ with $r_\text{b}(w') \in P(\text{var}(r))$, that $r_\text{b}(w') \in P(C)$ iff w_C is a prefix of w'. ■

Definition 5.4. We call M ***proper*** if $P^{(0)} = \emptyset$, i.e., all user-defined predicates are unary. □

Definition 5.5. Let $r \in R$. We say that r is ***local*** if both of the following conditions hold

- r is connected and
- $\text{var}(r) = \emptyset$ or there is an $x \in \text{var}(r)$ such that for every $b \in r_\text{G}$ and $y \in \text{var}(b) \setminus \{x\}$ we have that $b = \text{child}_i(x, y)$ for some $i \in [\text{maxrk}(\Sigma)]$.

We say that M is ***local*** if M is proper and every $r \in R$ is local. □

Example 5.6. In this example we consider the mwmd M_1 and M_2 from Example 4.5 and the mwmd M_3 and M_4 from Example 4.14. Clearly, M_2 is proper but neither M_1, M_3, nor M_4 are proper.

The rules r_1 and r_2 are local but r_3 is not because it is not connected. The rule r_4 is local but r_5 is not because the variable x occurs in the guard atom $\mathrm{leaf}(x)$ and y occurs at the first position in the guard atom $\mathrm{child}_2(y, x)$. The rule r_6 is local because x occurs in $\mathrm{root}(x)$ and in the first position in $\mathrm{child}_1(x, z)$. The rule r_7 is not local whereas r_8 and r_9 are local.

Observe that r_{10} is not local because it is not connected; however, r_{11}, r_{12}, and r_{13} are local. Therefore, M_3 is not local. Note that even M_4 is not local (although all rules in M_4 are local) because M_4 is not proper. □

Now we define semantic subclasses based on our syntactic subclasses.

Definition 5.7. Let $\mathcal{A}$ be an m-monoid over Δ. We define the following restrictions of the class $\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$. By $\mathrm{r\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$ ($\mathrm{c\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$, $\mathrm{l\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$, $\mathrm{p\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$, respectively) we denote the set of all tree series fixpoint-defined by *restricted* (*connected*, *local*, *proper* respectively), weakly non-circular mwmd over Σ, Δ, and $\mathcal{A}$.

Likewise, we let $\mathrm{pr\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$ ($\mathrm{pc\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$, respectively) be the set of all tree series fixpoint-defined by *restricted and proper* (*connected and proper*, respectively) weakly non-circular mwmd over Σ, Δ, and $\mathcal{A}$.

Similarly, we define such restrictions for $\mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big)$, $\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A})$, and $\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)$ (where $(\mathcal{A}, \leq)$ is an arbitrary ω-continuous m-monoid and $(\mathcal{A}, \sum)$ is an arbitrary ω-complete m-monoid); e.g., $\mathrm{pr\text{–}WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)$ denotes the set of all tree series hypergraph-defined by proper and restricted mwmd over Σ, Δ, and $(\mathcal{A}, \sum)$. □

In this chapter we will prove the following theorem.

Theorem 5.8. *Let $\mathcal{A}$ be an m-monoid over Δ, $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid, and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Let $\mathcal{C}$ be any of the classes* $\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A})$, $\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, (\mathcal{A}, \leq))$, $\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A})$, *or* $\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)$.

If (i) $\mathcal{C} \in \{\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}), \mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)\}$ *or (ii) $\mathcal{A}$ is absorptive and* $\mathcal{C} \in \{\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}), \mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big)\}$, *then*

$$\mathrm{l\text{–}}\mathcal{C} = \mathrm{c\text{–}}\mathcal{C} = \mathrm{pc\text{–}}\mathcal{C} \subseteq \mathrm{r\text{–}}\mathcal{C} = \mathrm{pr\text{–}}\mathcal{C} \subseteq \mathcal{C} = \mathrm{p\text{–}}\mathcal{C}\ . \tag{5.1}$$

If $\mathcal{A}$ is idempotent and distributive and at least one of the following two properties holds: (i) $\mathcal{C} \in \{\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}), \mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}), \mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big)\}$ *or (ii) $(\mathcal{A}, \sum)$ is ω-idempotent and ω-distributive and* $\mathcal{C} = \mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \sum)\big)$, *then*

$$\mathrm{l\text{–}}\mathcal{C} = \mathrm{c\text{–}}\mathcal{C} = \mathrm{pc\text{–}}\mathcal{C} = \mathrm{r\text{–}}\mathcal{C} = \mathrm{pr\text{–}}\mathcal{C}\ . \tag{5.2}$$

Proof. By definition every local mwmd is both proper and connected. By Observation 5.3 every connected mwmd is also restricted. Then Equation (5.1) follows from Corollaries 5.27 and 5.41. Moreover, Equation (5.2) follows from Corollary 5.34. ■

The proof of Theorem 5.8 is based on semantics preserving constructions, i.e., for a given mwmd we construct a semantically equivalent mwmd belong to a certain syntactic subclass. It turns out that for some of our syntactic subclasses we can only carry out constructions that preserve the hypergraph semantics but are not guaranteed to preserve the fixpoint semantics. Therefore we have to give two definitions of semantic equivalence of mwmd. The first one, called hyp-equivalence, only requires that the hypergraph semantics of two mwmd coincides. The second definition, called complete equivalence, requires that also the fixpoint semantics of two mwmd are equal.

When defining the semantic equivalence of two mwmd, we have to take another condition into account, namely that the finitary semantics should be applicable for the first mwmd iff it is applicable for the second mwmd or, equivalently, that the first mwmd should be weakly non-circular iff the second mwmd is so.

Definition 5.9. Let M and M' be mwmd over Σ and Δ. We say that M and M' are ***hyp-equivalent*** if all of the following conditions hold.

1. M is weakly non-circular iff M' is weakly non-circular.
2. If M and M' are weakly non-circular, then $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}$ for every m-monoid $\mathcal{A}$.
3. $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} = [\![M']\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)}$ for every ω-complete m-monoid $(\mathcal{A}, \sum)$.

Moreover, we call M and M' ***completely equivalent*** if all of the following conditions hold.

1. M and M' are hyp-equivalent.
2. If M and M' are weakly non-circular, then $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}$ for every absorptive m-monoid $\mathcal{A}$.
3. $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)} = [\![M']\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}$ for every ω-continuous m-monoid $(\mathcal{A}, \leq)$ such that $\mathcal{A}$ is an absorptive m-monoid. □

5.2 Relatedness

In the introduction to this chapter we mentioned that proving semantic equivalence of two mwmd wrt four kinds of semantics is a laborious task. In order to avoid the requirement to give such an equivalence proof for each of the normal form constructions that we carry out in this chapter, we follow the following approach: we define a property, called relatedness, for pairs of mwmd in such a way that

(i) whenever two mwmd are related, then they are also semantically equivalent and

(ii) for every normal form construction and every given mwmd M we have that M is related with the mwmd that is constructed from M.

Then we need to prove Condition (i) once, and for every construction we only need to prove Condition (ii).

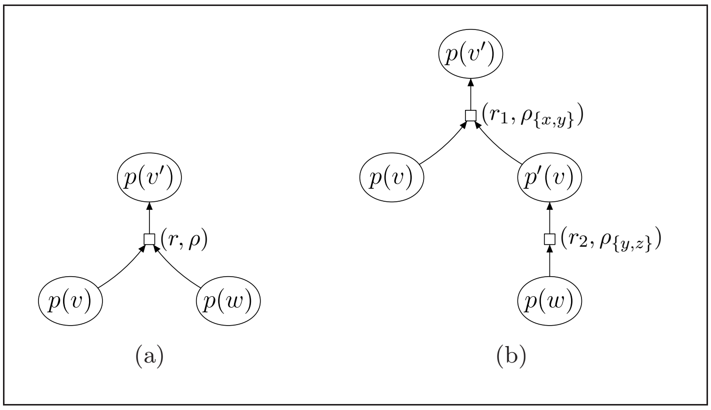


Figure 5.1: Fragments of the dependency hypergraph when using (a) rule r or (b) rules r_1 and r_2 instead.

5.2.1 Motivation

Let us first give an intuitive explanation of the definition of the notion of related mwmd. Roughly speaking, most of the normal form constructions that we will deal with later in this chapter are carried out by breaking down the information transport in an mwmd into smaller steps. Consider the rule $r = p(x) \leftarrow \sigma\big(p(y), \gamma(p(z))\big)$; $\{\text{child}_1(x, y), \text{child}_2(y, z)\}$. This rule is not local but can be replaced (while preserving semantics) by the two local rules $r_1 = p(x) \leftarrow \sigma(p(y), p'(y))$; $\{\text{child}_1(x, y)\}$ and $r_2 = p'(y) \leftarrow \gamma(p(z))$; $\{\text{child}_2(y, z)\}$, where p' is a new user-defined predicate. Then every rule instance of the original mwmd which is of the form (r, ρ) is simulated by the rule instances $(r_1, \rho|_{\{x,y\}})$ and $(r_2, \rho|_{\{y,z\}})$. This situation is represented in terms of dependency hypergraphs in Figure 5.1 (where $v' = \rho(x)$, $v = \rho(y)$, and $w = \rho(z)$); we can consider the *hyperedge* (r, ρ) to be simulated by the *hyperpath segment* $(r_1, \rho|_{\{x,y\}})\big(p(v), (r_2, \rho|_{\{y,z\}})(p(w)\big)$.

We have just given a simple example of related mwmd; its core concept is that hyperedges in dependency hypergraphs of the first mwmd are simulated by hypergraph segments in corresponding dependency hypergraphs of the second mwmd. More precisely, two mwmd M and M' are related if for every input tree t the dependency hypergraphs G and G' of M and t, and M' and t, respectively, are correlated in the following sense (for an example we refer to Figure 5.2).

- For every vertex (atom instance) in G there is a unique corresponding vertex in G' (this is represented by dotted lines in Figure 5.2). We will model this correspondence by an injective mapping. Let us refer to vertices of G' that do not correspond to vertices of G by *auxiliary vertices*; in Figure 5.2 they are represented by small circles. In our introductory example the atom instance $p'(v)$ is such an auxiliary vertex (see Figure 5.1(b)).
- There is a one-to-one correspondence between hyperedges of G and those hyperpath segments of G' whose inner vertices are auxiliary vertices and whose output and input vertices are not auxiliary vertices. This correspondence is required to satisfy the following two conditions:
  (A) it preserves input-output behavior, i.e., the input and output vertices of any hyperedge of G correspond to the input and output vertices of the corresponding

hyperpath segment (regarding the example in Figure 5.2: a possible correspondence that satisfies this input-output condition is given in Table 5.1), and

(B) the tree of operations that are encoded by any hyperedge must agree with the tree of operations that is encoded by its corresponding hyperpaths segment; for instance, in Figure 5.1 both the hyperedge (r, ρ) and the hyperpath segment $(r_1, \rho|_{\{x,y\}})\big(p(v), (r_2, \rho|_{\{y,z\}})(p(w)\big)$ encode the term $\sigma\big(p(v), \gamma(p(w))\big)$.

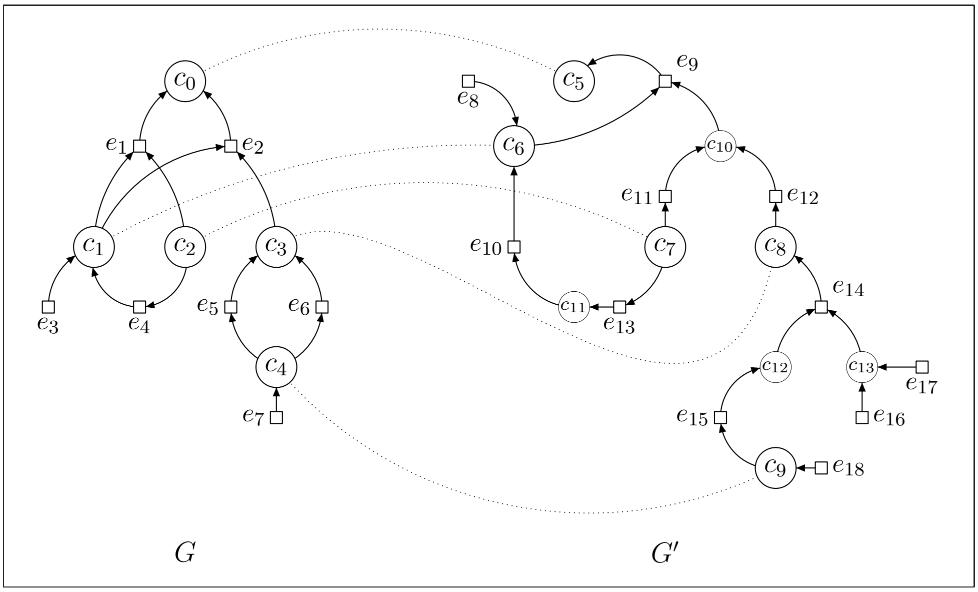


Figure 5.2: A sketch of the relationships between dependency hypergraphs of related mwmd.

hyperedge in G	hyperpath segment in G'
e_1	$e_9(c_6, e_{11}(c_7))$
e_2	$e_9(c_6, e_{12}(c_8))$
e_3	$e_8()$
e_4	$e_{10}(e_{13}(c_7))$
e_5	$e_{14}(e_{15}(c_9), e_{16}())$
e_6	$e_{14}(e_{15}(c_9), e_{17}())$
e_7	$e_{18}()$

Table 5.1: A hyperedge-hyperpath segment-correspondence for Figure 5.2.

5.2.2 Formal definition

For the remainder of Section 5.2 we fix two mwmd $M = (P, R, q)$ and $M' = (P', R', q')$ over Σ and Δ. In order to simplify notation we abbreviate, for every $t \in T_\Sigma$, $\mathrm{G}^{\mathrm{dep}}_{M,t}$ by G_t and $\mathrm{G}^{\mathrm{dep}}_{M',t}$ by G'_t. Furthermore, we fix the two families $\nu = (\nu_t \mid t \in T_\Sigma)$ and $\pi = \big(\pi_{t,c} \mid t \in T_\Sigma, c \in P(\mathrm{pos}(t))\big)$ such that for every $t \in T_\Sigma$ and $c \in P(\mathrm{pos}(t))$,

- $\nu_t : P(\text{pos}(t)) \to P'(\text{pos}(t))$ is an injective mapping,
- $\pi_{t,c} : \Phi_{M,t,c} \to \mathrm{H}_{G'_t}^{\nu_t(c),\text{ran}(\nu_t)}$ is a bijective mapping.

The mappings ν_t are the vertex-vertex correspondences and the mappings $\pi_{t,c}$ are the hyperedge-hyperpath segment correspondences that we mentioned in Section 5.2.1.

In order to reduce notational overhead in the following derivations let us introduce some more auxiliary definitions and abbreviations. Let $t \in T_\Sigma$, $c \in P(\text{pos}(t))$, $c' \in P'(\text{pos}(t))$, $e \in \Phi_{M',t}$, $k = \text{rk}(e)$, and $i \in [k]$. Then we let

- $H_t(c)$, $H'_t(c')$, and $H'_t(c', \text{ran}(\nu_t))$ be abbreviations for the three sets $\mathrm{H}^c_{G_t}$, $\mathrm{H}^{c'}_{G'_t}$, and $\mathrm{H}^{c',\text{ran}(\nu_t)}_{G'_t}$ respectively,
- $\lhd_t$ be an abbreviation for $\prec_{G'_t} \setminus \big(\text{ran}(\nu_t) \times P'(\text{pos}(t))\big)$
- h_{ν_t} denote the Δ-homomorphism from $\mathcal{T}_\Delta\big(P(\text{pos}(t))\big)$ to $\mathcal{T}_\Delta\big(P'(\text{pos}(t))\big)$ extending ν_t,
- $H^e_i = H'_t\big(\text{in}_i(e), \text{ran}(\nu_t)\big)$ if $\text{in}_i(e) \notin \text{ran}(\nu_t)$ and $H^e_i = \{\text{in}_i(e)\}$ otherwise.

Unfortunately, this is a large number of definitions but the reader may rest assured that without them notations in this section would be very cluttered. Now we define the notion of relatedness. For our purposes we need to define two versions of this notion, which we call (weak) relatedness and strong relatedness; the former version only guarantees that two related mwmd are hyp-equivalent, whereas the latter version implies that two related mwmd are even completely equivalent.

Definition 5.10. We call M and M' (*weakly*) ***related*** *via* ν *and* π if for every $t \in T_\Sigma$:

- $\nu_t(q(\varepsilon)) = q'(\varepsilon)$ and
- for every $c \in P(\text{pos}(t))$ and $(r, \rho) \in \Phi_{M,t,c}$, $\mathrm{h}_{\nu_t}(\rho(r_\mathrm{b})) = \mathrm{h}_{M',t}\big(\pi_{t,c}(r, \rho)\big)$.

M and M' are ***strongly related*** *via* ν *and* π if M and M' are related via ν and π, and for every $t \in T_\Sigma$, $c \in P(\text{pos}(t))$, $\eta, \eta' \in H'_t(\nu_t(c), \text{ran}(\nu_t))$, and $w \in \text{pos}(\eta) \cap \text{pos}(\eta')$ with $\eta|_w \notin \text{ran}(\nu_t)$ and $\eta'|_w \notin \text{ran}(\nu_t)$:

- if $\rho(r_\mathrm{b}) \notin P'(\text{pos}(t))$, where $(r, \rho) = \eta(w)$, and $\eta(w') = \eta'(w')$ for every prefix w' of w,
- then $\eta|_w = \eta'|_w$. □

Essentially the definition of weak relatedness states that the mappings $\pi_{t,c}$ (i.e., the hyperedge-hyperpath segment correspondences) must satisfy Conditions (A) and (B) from Section 5.2.1. Roughly speaking, the definition of strong relatedness states that every auxiliary vertex in the dependency hypergraph G'_t that has more than one hyperpath segment ending in it, may be the input vertex only of ε-rule instances. We will discuss strong relatedness more thoroughly in Section 5.2.4.

In the remainder of Section 5.2 we will prove the following theorem.

Theorem 5.11. *If M and M' are related via ν and π, then M and M' are hyp-equivalent. If M and M' are strongly related via ν and π, then M and M' are completely equivalent.*

Proof. This theorem follows from Lemmas 5.18 and 5.24. ■

5.2.3 Weak relatedness

In this section we will prove the first part of Theorem 5.11. First let us unearth some basic properties of related mwmd. The following observation is easy to prove by structural induction on s.

Observation 5.12. *Let $t \in T_\Sigma$, $s \in T_\Delta\big(P(\mathrm{pos}(t))\big)$, $l \in \mathbb{N}$, and $c_1, \dots, c_l \in P(\mathrm{pos}(t))$ such that* $\mathrm{indyield}(s) = c_1 \cdots c_l$.

Then $\mathrm{indyield}(\mathrm{h}_{\nu_t}(s)) = \nu_t(c_1) \cdots \nu_t(c_l)$ *and, thus,* $\mathrm{ind}(\mathrm{h}_{\nu_t}(s)) = \nu_t(\mathrm{ind}(s))$*. Moreover, for every $s_1, \dots, s_l \in T_\Delta\big(P(\mathrm{pos}(t))\big)$,* $\mathrm{h}_{\nu_t}(s) \leftarrow s_1 \cdots s_l = s \leftarrow s_1 \cdots s_l$.

Lemma 5.13. *Let $t \in T_\Sigma$ and $c' \in P'(\mathrm{pos}(t))$. Then*

$$H'_t(c', \mathrm{ran}(\nu_t)) = \{e(\eta_1, \dots, \eta_{\mathrm{rk}(e)}) \mid e \in \Phi_{M',t,c'}, \eta_1 \in H^e_1, \dots, \eta_{\mathrm{rk}(e)} \in H^e_{\mathrm{rk}(e)}\} .$$

Proof. This is a consequence of Lemma 2.14 and Observation 4.11(1). ∎

Lemma 5.14. *Let M and M' be related via ν and π and let $t \in T_\Sigma$.*

1. *Let $k \in \mathbb{N}$, $e = (r, \rho) \in \Phi_{M,t}$, and $c_1, \dots, c_k \in P(\mathrm{pos}(t))$ such that $c_1 \cdots c_k =$* $\mathrm{indyield}(\rho(r_\mathrm{b}))$*. Then* $\nu_t(c_1) \cdots \nu_t(c_k) = \mathrm{indyield}(\pi_{t,\mathrm{out}(e)}(e))$.

2. *Let $c \in P(\mathrm{pos}(t))$, $\eta \in H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$, $k \in \mathbb{N}$, and $c'_1, \dots, c'_k \in P'(\mathrm{pos}(t))$ with $c'_1 \cdots c'_k = \mathrm{indyield}(\eta)$. Let $(r, \rho) = \pi^{-1}_{t,c}(\eta)$. Then $\nu_t^{-1}(c'_1) \cdots \nu_t^{-1}(c'_k) =$* $\mathrm{indyield}(\rho(r_\mathrm{b}))$.

Proof. 1. By Observation 5.12, $\nu_t(c_1) \cdots \nu_t(c_k) = \mathrm{indyield}\big(\mathrm{h}_{\nu_t}(\rho(r_\mathrm{b}))\big)$. Since M and M' are related, $\mathrm{indyield}\big(\mathrm{h}_{\nu_t}(\rho(r_\mathrm{b}))\big) = \mathrm{indyield}\big(\mathrm{h}_{M',t}(\pi_{t,\mathrm{out}(e)}(r, \rho))\big)$. Moreover, Lemma 4.38 implies $\mathrm{indyield}\big(\mathrm{h}_{M',t}(\pi_{t,\mathrm{out}(e)}(r, \rho))\big) = \mathrm{indyield}(\pi_{t,\mathrm{out}(e)}(e))$.

2. Let $l \in \mathbb{N}$ and $c_1, \dots, c_l \in P(\mathrm{pos}(t))$ such that $c_1 \cdots c_l = \mathrm{indyield}(\rho(r_\mathrm{b}))$. By Statement 1, $\nu_t(c_1) \cdots \nu_t(c_l) = \mathrm{indyield}(\pi_{t,c}(e)) = \mathrm{indyield}(\eta) = c'_1 \cdots c'_k$. Thus, $l = k$ and $c_i = \nu_t^{-1}(c'_i)$ for every $i \in [k]$. ∎

Now we define a mapping from derivations of G_t to derivations of G'_t by lifting the mappings $\pi_{t,c}$ from hyperedges to hyperpaths. Roughly speaking, this mapping is defined as follows: every derivation of G_t is a tree of hyperedges; by applying the mappings $\pi_{t,c}$ to these hyperedges we obtain a tree that is labeled with hyperpath segments; if M and M' are related, then the hyperpath segments in the resulting tree can be ‘glued’ together; this yields a derivation of M'. For instance, consider the dependency hypergraphs in Figure 5.2. When we lift the hyperedge-hyperpath segment correspondence from Table 5.1 to hyperpaths, then, e.g., the derivation $e_2(e_3, e_5(e_7))$ in G corresponds to the derivation $e_9\big(e_8, e_{12}(e_{14}(e_{15}(e_{18}), e_{16})\big)$ in G'.

Definition 5.15. Let M and M' be related via ν and π. For every $t \in T_\Sigma$ we define the $\Phi_{M,t}$-algebra $\Pi_t = (T_{\Phi_{M',t}}, \theta')$ such that for every $j \in \mathbb{N}$, $e = (r, \rho) \in (\Phi_{M,t})^{(k)}$, and $\eta_1, \dots, \eta_k \in T_{\Phi_{M',t}}$ we let

$$\theta'(e)(\eta_1, \dots, \eta_k) = \pi_{t,\mathrm{out}(e)}(e) \leftarrow \eta_1 \cdots \eta_k ;$$

note that this is well-defined because by Lemma 5.14(1), we have that $k = \mathrm{rk}(e) = |\mathrm{indyield}(\rho(r_\mathrm{b}))| = |\mathrm{indyield}(\pi_{t,\mathrm{out}(e)}(e))|$. Moreover, by h_{π_t} we will denote the unique $\Phi_{M,t}$-homomorphism from $\mathcal{T}_{\Phi_{M,t}}$ to Π_t. For every $c \in P(\mathrm{pos}(t))$ let $\mathrm{h}^c_{\pi_t}$ be the restriction of h_{π_t} to $H_t(c)$ (which is clearly a subset of $T_{\Phi_{M,t}}$). □

Before we prove the main lemma of this section, let us first investigate some properties of the mapping h_{π_t}. If M and M' are related, then for every input tree t and atom instance c we obtain that the mapping $\mathrm{h}^c_{\pi_t}$ is a bijection between derivations of G_t ending in c and derivations of G'_t ending in $\nu_t(c)$. Moreover, this mapping preserves semantics in the following sense: for any derivation η we have that η and $\mathrm{h}_{\pi_t}(\eta)$ encode the same tree of operations. The third statement of the following lemma is a technical property that we will employ in the next section.

Lemma 5.16. *Let M and M' be related via ν and π. Moreover, let $t \in T_\Sigma$.*

1. $\mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta)) = \mathrm{h}_{M,t}(\eta)$ *for every* $\eta \in T_{\Phi_{M,t}}$.
2. *For every* $c \in P(\mathrm{pos}(t))$, $\mathrm{h}^c_{\pi_t}$ *is a bijection from* $H_t(c)$ *to* $H'_t(\nu_t(c))$.
3. *Let* $c, d \in P(\mathrm{pos}(t))$ *and* $c' \in P'(\mathrm{pos}(t))$ *with* $\nu_t(d) \prec_{G'_t} c'$ *and* $c' \triangleleft^*_t \nu_t(c)$. *Then* $d \prec_{G_t} c$.

Proof. 1. We give a proof by structural induction on η. Let $\eta \in T_{\Phi_{M,t}}$, $k \in \mathbb{N}$, $e = (r, \rho) \in (\Phi_{M,t})^{(k)}$, and $\eta_1, \dots, \eta_k \in T_{\Phi_{M,t}}$ such that $\eta = e(\eta_1, \dots, \eta_k)$. We assume that $\mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta_i)) = \mathrm{h}_{M,t}(\eta_i)$ for every $i \in [k]$ and show that $\mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta)) = \mathrm{h}_{M,t}(\eta)$. We derive

$$
\begin{aligned}
\mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta)) &= \mathrm{h}_{M',t}\big(\mathrm{h}_{\pi_t}(e(\eta_1, \dots, \eta_k))\big) = \mathrm{h}_{M',t}\big(\pi_{t,\mathrm{out}(e)}(e) \leftarrow \mathrm{h}_{\pi_t}(\eta_1) \cdots \mathrm{h}_{\pi_t}(\eta_k)\big) \\
&= \mathrm{h}_{M',t}(\pi_{t,\mathrm{out}(e)}(e)) \leftarrow \mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta_1)) \cdots \mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta_k)) && \text{(by Lemma 4.39)} \\
&= \mathrm{h}_{M',t}(\pi_{t,\mathrm{out}(e)}(e)) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k) && \text{(by the induction hypothesis)} \\
&= \mathrm{h}_{\nu_t}(\rho(r_\mathrm{b})) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k) && \text{(since } M \text{ and } M' \text{ are related)} \\
&= \rho(r_\mathrm{b}) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k) && \text{(by Observation 5.12)} \\
&= \mathrm{h}_{M,t}(e(\eta_1, \dots, \eta_k)) = \mathrm{h}_{M,t}(\eta) \ .
\end{aligned}
$$

2. First we show that for every $c \in P(\mathrm{pos}(t))$ we have $\mathrm{ran}(\mathrm{h}^c_{\pi_t}) \subseteq H'_t(\nu_t(c))$. To this end we show by structural induction that for every $\eta \in T_{\Phi_{M,t}}$ and $c \in P(\mathrm{pos}(t))$ such that $\eta \in H_t(c)$ we have $\mathrm{h}^c_{\pi_t}(\eta) \in H'_t(\nu_t(c))$. Let $\eta \in T_{\Phi_{M,t}}$ and $c \in P(\mathrm{pos}(t))$ such that $\eta \in H_t(c)$. By Observation 4.11(4) there are $k \in \mathbb{N}$, $e = (r, \rho) \in \Phi_{M,t,c}$, $c_1, \dots, c_k \in P(\mathrm{pos}(t))$, and $\eta_1 \in H_t(c_1), \dots, \eta_k \in H_t(c_k)$ such that $c_1 \cdots c_k = \mathrm{indyield}(\rho(r_\mathrm{b}))$ and $\eta = e(\eta_1, \dots, \eta_k)$. The induction hypothesis yields $\mathrm{h}^{c_i}_{\pi_t}(\eta_i) \in H'_t(\nu_t(c_i))$ for every $i \in [k]$. Moreover, Lemma 5.14(1) implies that $\mathrm{indyield}(\pi_{t,\mathrm{out}(e)}(e)) = \nu_t(c_1) \cdots \nu_t(c_k)$. Thus, we can apply Lemma 2.29 and the fact that $\pi_{t,\mathrm{out}(e)}(e) \in H'_t\big(\nu_t(\mathrm{out}(e)), \mathrm{ran}(\nu_t)\big) = H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$ in order to obtain

$$
\begin{aligned}
\mathrm{h}^c_{\pi_t}(\eta) &= \mathrm{h}^c_{\pi_t}(e(\eta_1, \dots, \eta_k)) = \pi_{t,\mathrm{out}(e)}(e) \leftarrow \mathrm{h}_{\pi_t}(\eta_1) \cdots \mathrm{h}_{\pi_t}(\eta_k) \\
&= \pi_{t,\mathrm{out}(e)}(e) \leftarrow \mathrm{h}^{c_1}_{\pi_t}(\eta_1) \cdots \mathrm{h}^{c_k}_{\pi_t}(\eta_k) \in H'_t(\nu_t(c)) \ .
\end{aligned}
$$

Next we show that for every $c \in P(\mathrm{pos}(t))$, $\mathrm{h}^c_{\pi_t} : H_t(c) \to H'_t(\nu_t(c))$ is a bijection. Before we proceed with the main proof let us first define the mapping $g_c : H'_t(\nu_t(c)) \to H_t(c)$, for every $c \in P(\mathrm{pos}(t))$, by structural recursion as follows. Let $c \in P(\mathrm{pos}(t))$ and $\eta \in H'_t(\nu_t(c))$. Moreover, let $\eta' = \mathrm{dec}{\uparrow}(\eta, \mathrm{ran}(\nu_t))$, $l \in \mathbb{N}$, and $c'_1, \dots, c'_l \in \mathrm{ran}(\nu_t)$ such that $c'_1 \cdots c'_l = \mathrm{indyield}(\eta')$. By Lemma 2.28 there are $\hat{\eta}_1 \in H'_t(c'_1), \dots, \hat{\eta}_l \in H'_t(c'_l)$ such that $\mathrm{dec}{\downarrow}(\eta, \mathrm{ran}(\nu_t)) = \hat{\eta}_1 \cdots \hat{\eta}_l$; then we define

$$
g_c(\eta) = \pi^{-1}_{t,c}(\eta')\big(g_{\nu_t^{-1}(c'_1)}(\hat{\eta}_1), \dots, g_{\nu_t^{-1}(c'_l)}(\hat{\eta}_l)\big) \ .
$$

Observe that $g_c(\eta)$ is well-defined for the following two reasons.

- $\hat{\eta}_i$ is a proper subtree of η, for every $i \in [l]$, due to Lemma 2.28.
- Let $e = (r, \rho) = \pi_{t,c}^{-1}(\eta') \in \Phi_{M,t,c}$ and for every $i \in [l]$ let $c_i = \nu_t^{-1}(c'_i)$. Then Lemma 5.14(2) yields $\text{indyield}(\rho(r_\text{b})) = c_1 \cdots c_l$. Moreover, for every $i \in [l]$, $g_{\nu_t^{-1}(c'_i)}(\hat{\eta}_i) \in H_t(c_i)$ due to the definition of $g_{\nu_t^{-1}(c'_i)} = g_{c_i}$. We obtain that $g_c(\eta) \in H_t(c)$ because of Observation 4.11(4).

Let $c \in P(\text{pos}(t))$. In order to show that $\text{h}_{\pi_t}^c : H_t(c) \to H'_t(\nu_t(c))$ is a bijection, it suffices to show that (i) $\text{h}_{\pi_t}^c \,;\, g_c$ is the identity relation on $H_t(c)$ (this implies that $\text{h}_{\pi_t}^c$ is injective) and that (ii) $g_c \,;\, \text{h}_{\pi_t}^c$ is the identity relation on $H'_t(\nu_t(c))$ (this implies that $\text{h}_{\pi_t}^c$ is surjective onto $H'_t(\nu_t(c))$).

First we prove Statement (i). We show by structural induction that for every $\eta \in T_{\Phi_{M,t}}$ and $c \in P(\text{pos}(t))$ with $\eta \in H_t(c)$ we have $g_c(\text{h}_{\pi_t}^c(\eta)) = \eta$. Let $\eta \in T_{\Phi_{M,t}}$ and $c \in P(\text{pos}(t))$ such that $\eta \in H_t(c)$. By Observation 4.11(4) there are $k \in \mathbb{N}$, $e = (r, \rho) \in \Phi_{M,t,c}$, $c_1, \ldots, c_k \in P(\text{pos}(t))$, and $\eta_1 \in H_t(c_1), \ldots, \eta_k \in H_t(c_k)$ such that $c_1 \cdots c_k = \text{indyield}(\rho(r_\text{b}))$ and $\eta = e(\eta_1, \ldots, \eta_k)$. Then

$$\begin{aligned}
&g_c(\text{h}_{\pi_t}^c(\eta)) = g_c(\text{h}_{\pi_t}(\eta)) = g_c\big(\text{h}_{\pi_t}(e(\eta_1, \ldots, \eta_k))\big)\\
&= g_c\big(\pi_{t,\text{out}(e)}(e) \leftarrow \text{h}_{\pi_t}(\eta_1) \cdots \text{h}_{\pi_t}(\eta_k)\big) = g_c\big(\pi_{t,c}(e) \leftarrow \text{h}_{\pi_t}(\eta_1) \cdots \text{h}_{\pi_t}(\eta_k)\big)\\
&= g_c\big(\pi_{t,c}(e) \leftarrow \text{h}_{\pi_t}^{c_1}(\eta_1) \cdots \text{h}_{\pi_t}^{c_k}(\eta_k)\big) \,. \qquad \text{(because } \eta_i \in H_t(c_i) \text{ for every } i \in [k])
\end{aligned}$$

By means of Lemma 2.29 we obtain $\text{dec}{\uparrow}\big(\pi_{t,c}(e) \leftarrow \text{h}_{\pi_t}^{c_1}(\eta_1) \cdots \text{h}_{\pi_t}^{c_k}(\eta_k), \text{ran}(\nu_t)\big) = \pi_{t,c}(e)$ and $\text{dec}{\downarrow}\big(\pi_{t,c}(e) \leftarrow \text{h}_{\pi_t}^{c_1}(\eta_1) \cdots \text{h}_{\pi_t}^{c_k}(\eta_k), \text{ran}(\nu_t)\big) = \text{h}_{\pi_t}^{c_1}(\eta_1) \cdots \text{h}_{\pi_t}^{c_k}(\eta_k)$. Together with the fact that $\text{h}_{\pi_t}^{c_i}(\eta_i) \in H'_t(\nu_t(c_i))$, for every $i \in [l]$, this implies

$$\begin{aligned}
&g_c\big(\pi_{t,c}(e) \leftarrow \text{h}_{\pi_t}^{c_1}(\eta_1) \cdots \text{h}_{\pi_t}^{c_k}(\eta_k)\big)\\
&= \pi_{t,c}^{-1}(\pi_{t,c}(e))\big(g_{\nu_t^{-1}(\nu_t(c_1))}(\text{h}_{\pi_t}^{c_1}(\eta_1)), \ldots, g_{\nu_t^{-1}(\nu_t(c_k))}(\text{h}_{\pi_t}^{c_k}(\eta_k))\big)\\
&= e\big(g_{c_1}(\text{h}_{\pi_t}^{c_1}(\eta_1)), \ldots, g_{c_k}(\text{h}_{\pi_t}^{c_k}(\eta_k))\big)\\
&= e(\eta_1, \ldots, \eta_k) = \eta \,. \qquad \text{(by the induction hypothesis)}
\end{aligned}$$

Next we prove Statement (ii). We show by structural induction that for every $\eta \in T_{\Phi_{M',t}}$ and $c \in P(\text{pos}(t))$ with $\eta \in H'_t(\nu_t(c))$ we have $\text{h}_{\pi_t}^c(g_c(\eta)) = \eta$. Let $\eta \in T_{\Phi_{M',t}}$ and $c \in P(\text{pos}(t))$ such that $\eta \in H'_t(\nu_t(c))$. Moreover, let $\eta' = \text{dec}{\uparrow}(\eta, \text{ran}(\nu_t))$, $l \in \mathbb{N}$, and $c'_1, \ldots, c'_l \in \text{ran}(\nu_t)$ such that $c'_1 \cdots c'_l = \text{indyield}(\eta')$. By Lemma 2.28 there are $\hat{\eta}_1 \in H'_t(c'_1), \ldots, \hat{\eta}_l \in H'_t(c'_l)$ such that $\text{dec}{\downarrow}(\eta, \text{ran}(\nu_t)) = \hat{\eta}_1 \cdots \hat{\eta}_l$. Let $c_i = \nu_t^{-1}(c'_i)$ for every $i \in [l]$. Then

$$\begin{aligned}
&\text{h}_{\pi_t}^c(g_c(\eta)) = \text{h}_{\pi_t}^c\big(\pi_{t,c}^{-1}(\eta')\big(g_{\nu_t^{-1}(c'_1)}(\hat{\eta}_1), \ldots, g_{\nu_t^{-1}(c'_l)}(\hat{\eta}_l)\big)\big)\\
&= \text{h}_{\pi_t}\big(\pi_{t,c}^{-1}(\eta')\big(g_{c_1}(\hat{\eta}_1), \ldots, g_{c_l}(\hat{\eta}_l)\big)\big)\\
&= \pi_{t,c}(\pi_{t,c}^{-1}(\eta')) \leftarrow \text{h}_{\pi_t}(g_{c_1}(\hat{\eta}_1)) \cdots \text{h}_{\pi_t}(g_{c_l}(\hat{\eta}_l)) \qquad \text{(because out}(\pi_{t,c}^{-1}(\eta')) = c)\\
&= \eta' \leftarrow \text{h}_{\pi_t}^{c_1}(g_{c_1}(\hat{\eta}_1)) \cdots \text{h}_{\pi_t}^{c_l}(g_{c_l}(\hat{\eta}_l)) \qquad \text{(because } g_{c_i}(\hat{\eta}_i) \in H_t(c_i) \text{ for every } i \in [l])\\
&= \eta' \leftarrow \hat{\eta}_1 \cdots \hat{\eta}_l \qquad \text{(by the induction hypothesis)}\\
&= \eta \,. \qquad \text{(by Lemma 2.28)}
\end{aligned}$$

3. We need to show that there is an $\tilde{\eta} \in H_t(c)$ and an $i \in [\text{rk}(\tilde{\eta}(\varepsilon))]$ such that $\tilde{\eta}|_i \in H_t(d)$.

Since $c' \lhd_t^* \nu_t(c)$, there is an $n \in \mathbb{N}_+$ and there are $c'_1, \ldots, c'_n \in P'(\text{pos}(t))$ such that $c'_1 = c'$, $c'_n = \nu_t(c)$, and $c'_{i-1} \lhd_t c'_i$ for every $i \in \{2, \ldots, n\}$. Thus, $c'_1, \ldots, c'_{n-1} \notin \text{ran}(\nu_t)$

and $c'_{i-1} \prec_{G'_t} c'_i$ for every $i \in \{2,\ldots,n\}$. Let $c'_0 = \nu_t(d)$. Then we have $c'_{i-1} \prec_{G'_t} c'_i$, for every $i \in [n]$, because $\nu_t(d) \prec_{G'_t} c'$ by our assumption. By Lemma 2.23(1 $\Rightarrow$ 2) there are $\eta \in H'_t(c'_n) = H'_t(\nu_t(c))$ and $w \in \mathrm{pos}(\eta)$ such that $|w| = n$ and $c'_{n-|w'|} = \mathrm{out}(\eta(w'))$ for every prefix w' of w. We put $\tilde{\eta} = g_c(\eta)$, where g_c is the mapping from the proof of Statement 2. Then $\tilde{\eta} \in H_t(c)$. It remains to show that there is an $i \in [\mathrm{rk}(\tilde{\eta}(\varepsilon))]$ such that $\tilde{\eta}|_i \in H_t(d)$.

Let $\eta' = \mathrm{dec}{\uparrow}(\eta, \mathrm{ran}(\nu_t))$. For every proper prefix position w' of w with $w' \neq \varepsilon$ we have $|w| - |w'| \in [n-1]$ and, hence, $\mathrm{out}(\eta(w')) = c'_{n-|w'|} = c'_{|w|-|w'|} \notin \mathrm{ran}(\nu_t)$. Therefore Lemma 2.18(2) yields that $w \in \mathrm{pos}(\eta')$.

Since $\mathrm{out}(\eta(w)) = c'_{n-|w|} = c'_0 = \nu_t(d) \in \mathrm{ran}(\nu_t)$ and $|w| = n > 0$, Lemma 2.18(1) yields that $\nu_t(d) = \mathrm{out}(\eta(w)) = \eta'|_w$. Thus, $\nu_t(d) \in \mathrm{ind}(\eta')$. Clearly, $\eta' \in H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$ because $\eta \in H'_t(\nu_t(c))$. Let $e = (r, \rho) = \pi^{-1}_{t,c}(\eta')$. By the fact that $\nu_t(d) \in \mathrm{ind}(\eta')$ and due to Lemma 5.14(2) we obtain $d \in \mathrm{ind}(\rho(r_{\mathrm{b}}))$; hence, by Observation 4.11(2) there is an $i \in \mathrm{rk}(e)$ such that $\mathrm{in}_i(e) = d$. The definition of g_c yields that $\tilde{\eta}(\varepsilon) = g_c(\eta)(\varepsilon) = \pi^{-1}_{t,c}(\eta') = e$. Then Observation 4.11(4) implies that $\tilde{\eta}|_i \in H_t(d)$. ∎

The following corollary is an immediate consequence of Lemma 5.16(2).

Corollary 5.17. *Let M and M' be related via ν and π. Moreover, let $t \in T_\Sigma$ and $c \in P(\mathrm{pos}(t))$. Then $H_t(c) = \emptyset$ iff $H'_t(\nu_t(c)) = \emptyset$.*

Now we are prepared to show that relatedness implies hyp-equivalence.

Lemma 5.18. *Let M and M' be related via ν and π. Then M and M' are hyp-equivalent.*

Proof. For every $t \in T_\Sigma$ we have that $H_t(q(\varepsilon))$ is infinite iff $H'_t(q'(\varepsilon))$ is infinite; this holds due to Lemma 5.16(2) and to the fact that $\nu_t(q(\varepsilon)) = q'(\varepsilon)$. Thus, M is weakly non-circular iff M' is weakly non-circular.

Assume that both M and M' are weakly non-circular and let $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an m-monoid. We show that $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}$. Let $t \in T_\Sigma$ and let h be the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). Then

$$\begin{aligned}
[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) &= \sum\nolimits_{\eta \in H_t(q(\varepsilon))} h(\mathrm{h}_{M,t}(\eta)) \\
&= \sum\nolimits_{\eta \in H_t(q(\varepsilon))} h(\mathrm{h}_{M',t}(\mathrm{h}_{\pi_t}(\eta))) && \text{(by Lemma 5.16(1))} \\
&= \sum\nolimits_{\eta \in H_t(q(\varepsilon))} h(\mathrm{h}_{M',t}(\mathrm{h}^{q(\varepsilon)}_{\pi_t}(\eta))) && \text{(because } \eta \in H_t(q(\varepsilon))\text{)} \\
&= \sum\nolimits_{\eta' \in H'_t(q'(\varepsilon))} h(\mathrm{h}_{M',t}(\eta')) && \text{(by Lemma 5.16(2))} \\
&= [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}(t)\ .
\end{aligned}$$

In a similar way one can show that $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} = [\![M']\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)}$ for every ω-complete m-monoid $(\mathcal{A}, \sum)$. ∎

5.2.4 Strong relatedness

In this section we prove the second part of Theorem 5.11. For two reasons this proof is more involved than the proof in the previous section.

Firstly, weak relatedness does generally not imply equivalence of fixpoint semantics. This is witnessed by the following example.

Example 5.19 (*Continuation of Example 4.5*)**.** Let $M = (P, R, q)$ and $M' = (P', R', q')$ be mwmd over Σ_{ex} and Δ_{ex} such that $P = \{q^{(1)}\}$, $P' = \{q'^{(1)}, p^{(0)}\}$, $R = \{r_\alpha, r_\beta\}$, and $R' = \{r, r'_\alpha, r'_\beta\}$ where

$$r_\alpha = q(x) \leftarrow \gamma(\alpha)\,;\emptyset\,, \qquad r_\beta = q(x) \leftarrow \gamma(\beta)\,;\emptyset\,,$$
$$r'_\alpha = p() \leftarrow \alpha\,;\emptyset\,, \qquad r'_\beta = p() \leftarrow \beta\,;\emptyset\,, \qquad r = q'(x) \leftarrow \gamma(p())\,;\emptyset\,.$$

For every $t \in T_{\Sigma_{\mathrm{ex}}}$ and $w \in \mathrm{pos}(t)$ we let $\nu_t(q(w)) = q'(w)$ and

$$\pi_{t,q(w)}(r_\alpha, [x \mapsto w]) = (r, [x \mapsto w])\big((r'_\alpha, [\,])()\big)\,,$$
$$\pi_{t,q(w)}(r_\beta, [x \mapsto w]) = (r, [x \mapsto w])\big((r'_\beta, [\,])()\big)\,.$$

It is easy to see that M and M' are related via the two families $(\nu_t \mid t \in T_{\Sigma_{\mathrm{ex}}})$ and $\big(\pi_{t,c} \mid t \in T_{\Sigma_{\mathrm{ex}}}, c \in P(\mathrm{pos}(t))\big)$. Thus, the hypergraph semantics of M and M' coincide due to Lemma 5.18. However, in general this does not hold for the fixpoint semantics. In order to prove this let $\mathcal{A} = (\mathbb{N}, +, 0, \theta)$ be the m-monoid over Δ_{ex} such that for every $n, n' \in \mathbb{N}$ we have $\theta(\alpha)() = \theta(\beta)() = 1$, $\theta(\gamma)(n) = (n+1) \cdot n$, and $\theta(\sigma)(n, n') = n \cdot n'$. Clearly, $\mathcal{A}$ is absorptive but not distributive.

For the fixpoint interpretation I of M we have $I(q(\varepsilon)) = \theta(\gamma)\big(\theta(\alpha)()\big) + \theta(\gamma)\big(\theta(\beta)()\big) = 2 + 2 = 4$. On the other hand we obtain for the fixpoint interpretation I' of M' that $I'(p()) = \theta(\alpha)() + \theta(\beta)() = 1 + 1 = 2$ and $I'(q'(\varepsilon)) = \theta(\gamma)(I(p()) = (2+1) \cdot 2 = 6$. Hence, $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}(t) = 4 \neq 6 = [\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(t)$.

The reason why the fixpoint semantics of M and M' do not coincide is that M and M' are not strongly related. In fact, we obtain for $t = \alpha$, $c = q(\varepsilon)$, $\eta = (r, [x \mapsto \varepsilon])\big((r'_\alpha, [\,])()\big)$, $\eta' = (r, [x \mapsto \varepsilon])\big((r'_\beta, [\,])()\big)$, and position $w = \varepsilon$ that $\eta, \eta' \in H'_t\big(q'(\varepsilon), \mathrm{ran}(\nu_t)\big) = H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$, $\eta|_w = \eta \notin \mathrm{ran}(\nu_t)$, $\eta'|_w = \eta' \notin \mathrm{ran}(\nu_t)$, $\eta(w) = \eta'(w) = (r, [x \mapsto \varepsilon])$, $[x \mapsto \varepsilon](r_{\mathrm{b}}) = \gamma(p()) \notin P'(\mathrm{pos}(t))$, and $\eta|_w = \eta \neq \eta' = \eta'|_w$. □

Before we explain the second reason why the proof of the second part of Theorem 5.11 is more involved than the proof for the first part, let us first state an important technical lemma. The first two statements are basic properties of the set of derivations of dependency hypergraphs of M'. The third property states that, roughly speaking, every auxiliary vertex of any dependency hypergraph of M' that is the input vertex of some hyperedge that is not an ε-rule instance, has at most one derivation ending in it.

Lemma 5.20. *Let $t \in T_\Sigma$, $c' \in P'(\mathrm{pos}(t))$, $e = (r, \rho) \in \Phi_{M',t,c'}$, and $k = \mathrm{rk}(e)$.*

1. *Let $H'_t(\mathrm{in}_i(e)) \neq \emptyset$ for every $i \in [k]$. Then we have that $H'_t(\mathrm{in}_i(e), \mathrm{ran}(\nu_t)) \neq \emptyset$ and $\mathrm{in}_i(e) \prec_{G'_t} c'$, for every $i \in [k]$.*

2. *Let $i \in [k]$ such that $H'_t(\mathrm{in}_i(e)) = \emptyset$. Then for every $\eta_i \in H^e_i$ there is a $d' \in \mathrm{ind}(\eta_i)$ with $H'_t(d') = \emptyset$.*

3. *Let $c \in P(\mathrm{pos}(t))$ such that $c' \lhd^*_t \nu_t(c)$. Suppose that M and M' are strongly related via ν and π, that $H'_t(\mathrm{in}_i(e)) \neq \emptyset$ for every $i \in [\mathrm{rk}(e)]$, and that $\rho(r_{\mathrm{b}}) \notin P'(\mathrm{pos}(t))$. Let $j \in [\mathrm{rk}(e)]$ and $\eta, \eta' \in H^e_j$ such that $H'_t(d') \neq \emptyset$ holds for every $d' \in \mathrm{ind}(\eta) \cup \mathrm{ind}(\eta')$. Then $\eta = \eta'$.*

Proof. 1. For every $i \in [k]$ choose an $\eta_i \in H'_t(\mathrm{in}_i(e))$; clearly, $\mathrm{dec}{\uparrow}(\eta_i, \mathrm{ran}(\nu_t)) \in H'_t(\mathrm{in}_i(e), \mathrm{ran}(\nu_t))$. Moreover, Corollary 2.16 and Observation 4.11(1) imply that $\eta =$

$e(\eta_1, \ldots, \eta_k) \in H'_t(c')$. For every $i \in [k]$ the fact that $\eta|_i = \eta_i \in H'_t(\mathrm{in}_i(e))$ yields $\mathrm{in}_i(e) \prec_{G'_t} c'$.

2. Let $\eta_i \in H^e_i$ and assume that $H'_t(d') \neq \emptyset$ for every $d' \in \mathrm{ind}(\eta_i)$. We will derive a contradiction. First we consider the case that $\mathrm{in}_i(e) \in \mathrm{ran}(\nu_t)$. Then $\eta_i = \mathrm{in}_i(e)$ and $\mathrm{ind}(\eta_i) = \{\mathrm{in}_i(e)\}$. Hence $H'_t(\mathrm{in}_i(e)) \neq \emptyset$, a contradiction.

Now we consider the case that $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$. Then $\eta_i \in H'_t\big(\mathrm{in}_i(e), \mathrm{ran}(\nu_t)\big)$ and Lemma 2.30 yields that $H'_t(\mathrm{in}_i(e)) \neq \emptyset$, a contradiction.

3. If $\mathrm{in}_j(e) \in \mathrm{ran}(\nu_t)$, then $H^e_j = \{\mathrm{in}_j(e)\}$ and, thus, $\eta = \mathrm{in}_j(e) = \eta'$ holds trivially.

For the remainder of the proof we assume that $\mathrm{in}_j(e) \notin \mathrm{ran}(\nu_t)$. In order to show that $\eta = \eta'$ we need to take a detour and construct a sequence of intermediate trees $\eta_0, \eta_1, \eta_2, \eta_3$ and $\eta'_0, \eta'_1, \eta'_2, \eta'_3$.

The assumption $\mathrm{in}_j(e) \notin \mathrm{ran}(\nu_t)$ implies $H^e_j = H'_t\big(\mathrm{in}_j(e), \mathrm{ran}(\nu_t)\big)$ and, hence, $\eta, \eta' \in H'_t\big(\mathrm{in}_j(e), \mathrm{ran}(\nu_t)\big)$. Since $H'_t(d') \neq \emptyset$ for every $d' \in \mathrm{ind}(\eta) \cup \mathrm{ind}(\eta')$, Lemma 2.30 yields that there are $\eta_0, \eta'_0 \in H'_t(\mathrm{in}_j(e))$ such that $\mathrm{dec}{\uparrow}(\eta_0, \mathrm{ran}(\nu_t)) = \eta$ and $\mathrm{dec}{\uparrow}(\eta'_0, \mathrm{ran}(\nu_t)) = \eta'$.

For every $i \in [\mathrm{rk}(e)]$ choose a $\tilde{\eta}_i \in H'_t(\mathrm{in}_i(e))$; such an $\tilde{\eta}_i$ exists by assumption. Let $\eta_1 = e(\tilde{\eta}_1, \ldots, \tilde{\eta}_{j-1}, \eta_0, \tilde{\eta}_{j+1}, \ldots, \tilde{\eta}_k)$ and $\eta'_1 = e(\tilde{\eta}_1, \ldots, \tilde{\eta}_{j-1}, \eta'_0, \tilde{\eta}_{j+1}, \ldots, \tilde{\eta}_k)$. By Corollary 2.16 and Observation 4.11(1) we have $\eta_1, \eta'_1 \in H'_t(c')$.

Since $c' \lhd^*_t \nu_t(c)$, there are $n \in \mathbb{N}$, $c'_0, \ldots, c'_{n-1} \in P'(\mathrm{pos}(t)) \setminus \mathrm{ran}(\nu_t)$, and $c'_n \in P'(\mathrm{pos}(t))$ such that $c'_0 = c'$, $c'_n = \nu_t(c)$, and $c'_{m-1} \prec_{G'_t} c'_m$ for every $m \in [n]$. Therefore, Corollary 2.22 yields that there are $w \in (\mathbb{N}_+)^n$ and $e_1, \ldots, e_n \in \Phi_{M',t}$ such that $\mathrm{out}(e_l) = c'_l$, for every $l \in [n]$, and for every $\tilde{\eta}' \in H'_t(c'_0) = H'_t(c')$ there is an $\tilde{\eta} \in H'_t(c'_n)$ with: (a) $w \in \mathrm{pos}(\tilde{\eta})$, (b) $\tilde{\eta}|_w = \tilde{\eta}'$, and (c) $\tilde{\eta}(w') = e_{n-|w'|}$ for every proper prefix w' of w. This holds in particular for $\tilde{\eta} = \eta_1$ and $\tilde{\eta} = \eta'_1$. More precisely, there are $\eta_2, \eta'_2 \in H'_t(c'_n) = H'_t(\nu_t(c))$ such that (a) $w \in \mathrm{pos}(\eta_2) \cap \mathrm{pos}(\eta'_2)$, (b) $\eta_2|_w = \eta_1$, $\eta'_2|_w = \eta'_1$, and (c) $\eta_2(w') = e_{n-|w'|} = \eta'_2(w')$ for every proper prefix w' of w. Observe that Condition (b) implies $\eta_2(w) = \eta_1(\varepsilon) = e = \eta'_1(\varepsilon) = \eta'_2(w)$. Hence, for every prefix w' of w we have

$$\eta_2(w') = \eta'_2(w') \;, \qquad\qquad \eta_2(w) = e = \eta'_2(w) \;. \tag{5.3}$$

Due to Condition (c), we obtain for every proper prefix w' of w that $\mathrm{out}(\eta_2(w')) = \mathrm{out}(e_{n-|w'|}) = c'_{n-|w'|}$. Moreover, Condition (b) together with the fact that $\eta_1 \in H'_t(c')$ yields $\mathrm{out}(\eta_2(w)) = \mathrm{out}(\eta_1(\varepsilon)) = c' = c'_0$. Therefore, we have for every prefix w' of w that

$$\mathrm{out}(\eta_2(w')) = \mathrm{out}(\eta'_2(w')) = c'_{n-|w'|} \;. \tag{5.4}$$

Finally, we let $\eta_3 = \mathrm{dec}{\uparrow}(\eta_2, \mathrm{ran}(\nu_t))$ and $\eta'_3 = \mathrm{dec}{\uparrow}(\eta'_2, \mathrm{ran}(\nu_t))$. Clearly, $\eta_3, \eta'_3 \in H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$. Observe that for every prefix w' of w with $w' \neq \varepsilon$ we have $n - |w'| = |w| - |w'| \in \{0, \ldots, n-1\}$ and, thus, $\mathrm{out}(\eta_2(w')) = \mathrm{out}(\eta'_2(w')) = c'_{n-|w'|} \notin \mathrm{ran}(\nu_t)$ by Equation (5.4). Moreover, $\mathrm{out}(\eta_2(wj)) = \mathrm{out}((\eta_2|_w)(j)) = \mathrm{out}(\eta_1(j)) = \mathrm{out}(\eta_0(\varepsilon)) = \mathrm{in}_j(e)$ because $\eta_0 \in H'_t(\mathrm{in}_j(e))$. Likewise, $\mathrm{out}(\eta'_2(wj)) = \mathrm{out}(\eta'_0(\varepsilon)) = \mathrm{in}_j(e)$. Hence, $\mathrm{out}(\eta_2(wj)) = \mathrm{out}(\eta'_2(wj)) = \mathrm{in}_j(e) \notin \mathrm{ran}(\nu_t)$ by assumption. Thus, we have for every prefix w' of wj with $w' \neq \varepsilon$ that

$$\mathrm{out}(\eta_2(w')) = \mathrm{out}(\eta'_2(w')) \notin \mathrm{ran}(\nu_t) \;. \tag{5.5}$$

Therefore we obtain the following facts.

(i) $w' \in \mathrm{pos}(\eta_3) \cap \mathrm{pos}(\eta'_3)$ for every prefix w' of w due to Lemma 2.18(2).

(ii) $\eta_3(w') = \eta_2(w') = \eta'_2(w') = \eta'_3(w')$ for every prefix w' of w due to Fact (i), Lemma 2.18(1), and Equations (5.3) and (5.5). In particular, $\eta_3(w) = e = \eta'_3(w)$.

(iii) Lemma 2.18(3) implies that $\eta_3|_{wj} = \mathrm{dec}{\uparrow}(\eta_2|_{wj}, \mathrm{ran}(\nu_t)) = \mathrm{dec}{\uparrow}(\eta_1|_j, \mathrm{ran}(\nu_t)) = \mathrm{dec}{\uparrow}(\eta_0, \mathrm{ran}(\nu_t)) = \eta$ and, likewise, $\eta'_3|_{wj} = \eta'$. This implies in particular that $\eta_3(w) \notin \mathrm{ran}(\nu_t)$ and $\eta'_3(w) \notin \mathrm{ran}(\nu_t)$, because $wj \in \mathrm{pos}(\eta_3) \cap \mathrm{pos}(\eta'_3)$.

The facts that M and M' are strongly related via ν and π, $e = (r, \rho)$, $\rho(r_\mathrm{b}) \notin P'(\mathrm{pos}(t))$, $\eta_3, \eta'_3 \in H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$, and Facts (ii) and (iii) imply that $\eta_3|_w = \eta'_3|_w$. Then Fact (iii) yields that $\eta = \eta'$. ■

Now let us discuss the second reason that makes the proof of the second part of Theorem 5.11 more difficult. Let us consider an example. Assume that M contains (possibly among others) the rule $r = p() \leftarrow \sigma(q(), q()); \emptyset$ and that M' is obtained from M by replacing r by the rules $r_1 = p() \leftarrow \sigma(q(), q'())\,; \emptyset$ and $r_2 = q'() \leftarrow q()\,; \emptyset$. Then it is easy to see that M and M' are strongly related. For the following discussion let us fix an m-monoid $\mathcal{A} = (A, +, \mathbf{0}, \theta)$. For every $n \in \mathbb{N}$ let use denote the value of $\mathcal{T}^n_M(I_\mathbf{0})(q())$ by a_n and assume that also $\mathcal{T}^n_{M'}(I_\mathbf{0})(q()) = a_n$. Then we obtain that for every $n \in \mathbb{N}$ with $n \geq 2$ we have $\mathcal{T}^n_M(I_\mathbf{0})(p()) = \theta(\sigma)(a_{n-1}, a_{n-1})$ and $\mathcal{T}^n_{M'}(I_\mathbf{0})(p()) = \theta(\sigma)(a_{n-1}, a_{n-2})$. Hence, the introduction of an intermediate step in the information transport of the mwmd M' (i.e., the atom instance $q'()$) implies that the input values of the operation $\theta(\sigma)$ are, roughly speaking, out of sync. Thus, the sequences $\big(\mathcal{T}^n_M(I_\mathbf{0})(p()) \mid n \in \mathbb{N}\big)$ and $\big(\mathcal{T}^n_{M'}(I_\mathbf{0})(p()) \mid n \in \mathbb{N}\big)$ have ostensibly nothing in common. The differences between these two sequences intensify if there are even more intermediate steps in M' or if the information is fed back, which happens if the value of $q()$ depends on the value of $p()$.

We will resolve this problem differently for the finitary and infinitary fixpoint semantics. In the case of the finitary semantics we will show that the sequence $a_0, a_1, a_2, \ldots$ will be ultimately constant, i.e., it will be constant for almost all indices; this implies that also the sequences $\big(\theta(\sigma)(a_{n-1}, a_{n-1}) \mid n \in \mathbb{N}\big)$ and $\big(\theta(\sigma)(a_{n-1}, a_{n-2}) \mid n \in \mathbb{N}\big)$ will coincide for almost all indices. In the case of the infinitary semantics we will show that the sequences $\big(\mathcal{T}^n_M(I_\mathbf{0})(p()) \mid n \in \mathbb{N}\big)$ and $\big(\mathcal{T}^n_{M'}(I_\mathbf{0})(p()) \mid n \in \mathbb{N}\big)$ are mutually cofinal and, thus, that their suprema are equal. We conclude that we have to prove the following three statements.

(i) For the finitary semantics: there are $N, M \in \mathbb{N}$ such that for every $n, m \in \mathbb{N}$ with $n \geq N$ and $m \geq M$ we have $T^n_M(I_\mathbf{0})(p()) = T^m_{M'}(I_\mathbf{0})(p())$.

(ii) For the infinitary semantics: for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ with $T^n_M(I_\mathbf{0})(p()) \leq T^m_{M'}(I_\mathbf{0})(p())$, where $\leq$ is the order of the considered ω-continuous m-monoid.

(iii) For the infinitary semantics: for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $T^m_{M'}(I_\mathbf{0})(p()) \leq T^n_M(I_\mathbf{0})(p())$.

In general we have to prove these three statements not only for the atom instance $p()$ but for every[1] atom instance in M and its corresponding atom instance in M'.

The proofs of Statements (i) to (iii) are rather technical and laborious. Fortunately, they share a common structure; this allows us to extract common parts of their proofs.

[1]Actually, Statement (i) does generally not hold for every atom instance. However, we are only required to prove it for a special subset of atom instances; these are, roughly speaking, those atom instances that $q(\varepsilon)$ depends on, where q is the query predicate.

In the following generic lemma we have aggregated the common part of Statements (i) to (iii). One of the parameters of this lemma is a reflexive relation, denoted by $\bowtie$. When we will apply this lemma later in this section, we will instantiate this relation with the following particular relations: the identity relation on the carrier set of the considered m-monoid (for the proof of Statement (i)), the partial order relation of the ω-continuous m-monoid (for the proof of Statement (ii)), and the inverse of the partial order relation (for the proof of Statement (iii)).

Let us give an informal description of the following lemma. For all natural numbers n and m and atom instance c let us call the triple (n, m, c) *proper* if $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \bowtie \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$ holds. Now the following lemma states that the triple $(n+1, m, c)$ is proper given that certain other triples (n', m', d) (with smaller indices, i.e., $n' < n+1$ and $m' < m$) are proper. This allows us to employ the following lemma in order to give inductive proofs of Statements (i) to (iii).

Lemma 5.21. *Let M and M' be strongly related via ν and π, $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an absorptive m-monoid over Δ, and $t \in T_\Sigma$. Moreover, let $\bowtie$ be a reflexive relation on A such that $+$ and $\theta(\delta)$, for every $\delta \in \Delta$, are monotone wrt $\bowtie$. Then for every $c \in P(\mathrm{pos}(t))$ there is an $l_c^{\bowtie} \in \mathbb{N}$ such that the following implication holds for every $n, m \in \mathbb{N}$:*

(i) if

- $l_c^{\bowtie} \leq m$ *or* $a \bowtie \mathbf{0}$ *for every* $a \in A$, *and*
- $\mathcal{T}_M^n(I_{\mathbf{0}})(d) \bowtie \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(\nu_t(d))$ *for every* $d \in P(\mathrm{pos}(t))$ *with* $d \prec_{G_t} c$ *and for every* $j \in [\min(m, l_c^{\bowtie})]$,

(ii) then $\mathcal{T}_M^{n+1}(I_{\mathbf{0}})(c) \bowtie \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$.

Proof. Let $c \in P(\mathrm{pos}(t))$. We define the set $C = \{c' \in P'(\mathrm{pos}(t)) \mid c' \lhd_t^* \nu_t(c)\}$ and the relation $\sqsubset = \lhd_t \cap (C \times C)$. Since $\Phi_{M,t,c}$ is finite, also $H_t'\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$ is finite due to the definition of $\pi_{t,c}$. Hence, Lemma 2.26 yields[2] that $\sqsubset^+$ is irreflexive. Since C is finite and $\sqsubset^+$ is an irreflexive relation on C, we obtain that $\sqsubset$ is well-founded on C. For every $c' \in C$ we define the number $k_{c'} \in \mathbb{N}$ by well-founded recursion on $\sqsubset$ as follows: $k_{c'} = 1 + \max\{k_{d'} \mid d' \in C, d' \sqsubset c'\}$. Clearly, $\nu_t(c) \in C$; we put $l_c^{\bowtie} = k_{\nu_t(c)}$.

Let $n, m \in \mathbb{N}$ and assume that Condition *(i)* holds. We need to show that Condition *(ii)* holds as well. Choose an $I \in \mathcal{I}_{M'}$ such that $I(\nu_t(d)) = \mathcal{T}_M^n(I_{\mathbf{0}})(d)$ for every $d \in P(\mathrm{pos}(t))$. Such an I exists because ν_t is injective. Before we proceed, we state an important fact concerning the interpretation I.

Fact A. *Let $\eta \in T_{\Phi_{M',t}}(\mathrm{ran}(\nu_t))$ such that there is a $d' \in \mathrm{ind}(\eta)$ with $H_t'(d') = \emptyset$. Then* $\mathrm{h}_I(\mathrm{h}_{M',t}(\eta)) = \mathbf{0}$.

Proof of Fact A. By Lemma 4.38 there is a $d' \in \mathrm{ind}(\mathrm{h}_{M',t}(\eta))$ with $H_t'(d') = \emptyset$. For this particular d' we have $d' \in \mathrm{ind}(\mathrm{h}_{M',t}(\eta)) = \mathrm{ind}(\eta) \subseteq \mathrm{ran}(\nu_t)$. We obtain by means of Corollary 5.17 that $H_t(\nu_t^{-1}(d')) = \emptyset$ and, hence, $\mathcal{T}_M^n(I_{\mathbf{0}})(\nu_t^{-1}(d')) = \mathbf{0}$ due to Lemma 4.17(1). The definition of I implies that $\mathrm{h}_I(d') = I(d') = \mathbf{0}$. Thus, $\mathrm{h}_I(\mathrm{h}_{M',t}(\eta)) = \mathbf{0}$ by Lemma 3.9(1).

Continuation of the main proof. Now we define a set $D \subseteq C$ as follows. For every $c' \in C$ we let $c' \in D$ iff the following equation holds for every $j \in \{0, \ldots, \min(m, l_c^{\bowtie} - k_{c'})\}$:

$$\sum\nolimits_{\eta \in H_t'(c', \mathrm{ran}(\nu_t))} \mathrm{h}_I(\mathrm{h}_{M',t}(\eta)) \bowtie \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(c') \,. \tag{5.6}$$

[2]Here we apply Lemma 2.26 with the following instantiations: $G = G_t' = \mathrm{G}_{M',t}^{\mathrm{dep}}$, $\lhd = \lhd_t$, $V = P'(\mathrm{pos}(t))$, $v = \nu_t(c)$, $U = \mathrm{ran}(\nu_t)$, and $V' = C$.

We claim that $\nu_t(c) \in D$. This claim implies Condition *(ii)* due to the following derivation:

$$\begin{aligned}
&\mathcal{T}_M^{n+1}(I_{\mathbf{0}})(c) = \sum\nolimits_{(r,\rho)\in\Phi_{M,t,c}} \mathrm{h}_{\mathcal{T}_M^n(I_{\mathbf{0}})}(\rho(r_\mathrm{b}))\\
&= \sum\nolimits_{\eta\in H'_t(\nu_t(c),\mathrm{ran}(\nu_t))} \mathrm{h}_{\mathcal{T}_M^n(I_{\mathbf{0}})}\big(\rho_\eta((r_\eta)_\mathrm{b})\big) && \text{(where } (r_\eta,\rho_\eta)=\pi_{t,c}^{-1}(\eta))\\
&= \sum\nolimits_{\eta\in H'_t(\nu_t(c),\mathrm{ran}(\nu_t))} \mathrm{h}_I\big(\mathrm{h}_{\nu_t}\big(\rho_\eta((r_\eta)_\mathrm{b})\big)\big)\\
&\qquad \text{(it is easy to check that } \mathrm{h}_{\nu_t}\,;\mathrm{h}_I = \mathrm{h}_{\mathcal{T}_M^n(I_{\mathbf{0}})} \text{ due to the definition of } I)\\
&= \sum\nolimits_{\eta\in H'_t(\nu_t(c),\mathrm{ran}(\nu_t))} \mathrm{h}_I(\mathrm{h}_{M',t}(\eta)) && \text{(because } M \text{ and } M' \text{ are related)}\\
&\bowtie \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))\ . && \text{(by our claim that } \nu_t(c)\in D, \text{ for the instance } j=0)
\end{aligned}$$

It remains to prove our claim, i.e., $\nu_t(c) \in D$. To this end we show by well-founded induction on $\sqsubset$ that for every $c' \in C$ we have $c' \in D$. Let $c' \in C$ and assume that $d' \in D$ for every $d' \in C$ with $d' \sqsubset c'$. We show that $c' \in D$. Let $j \in \{0,\dots,\min(m, l_c^{\bowtie} - k_{c'})\}$. We prove that Equation (5.6) holds. First we consider the case that $j = m$. Then $l_c^{\bowtie} - k_{c'} \geq m$, which implies $l_c^{\bowtie} \geq m + k_{c'} \geq m+1 > m$. By Condition *(i)* we conclude that $a \bowtie \mathbf{0}$ for every $a \in A$. Hence, $\sum_{\eta\in H'_t(c',\mathrm{ran}(\nu_t))} \mathrm{h}_I(\mathrm{h}_{M',t}(\eta)) \bowtie \mathbf{0} = \mathcal{T}_{M'}^0(I_{\mathbf{0}})(c') = \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(c')$, which proves Equation (5.6).

It remains to consider the case $j < m$. Let $j' = j + 1$. Equation (5.6) results from the following derivation:

$$\begin{aligned}
&\sum\nolimits_{\eta\in H'_t(c',\mathrm{ran}(\nu_t))} \mathrm{h}_I(\mathrm{h}_{M',t}(\eta))\\
&= \sum\nolimits_{e\in\Phi_{M',t,c'}} \sum\nolimits_{(\eta_1,\dots,\eta_{\mathrm{rk}(e)})\in H_1^e\times\cdots\times H^e_{\mathrm{rk}(e)}} \mathrm{h}_I(\mathrm{h}_{M',t}(e(\eta_1,\dots,\eta_{\mathrm{rk}(e)})) && \text{(by Lemma 5.13)}\\
&\bowtie \sum\nolimits_{(r,\rho)\in\Phi_{M',t,c'}} \mathrm{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\rho(r_\mathrm{b}))\ . && (\star)\\
&= \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(c')\ .
\end{aligned}$$

It remains to prove $(\star)$. Since $+$ is monotone wrt $\bowtie$, it suffices to show for every $e = (r,\rho) \in \Phi_{M',t,c'}$ that

$$\sum\nolimits_{(\eta_1,\dots,\eta_k)\in H_1^e\times\cdots\times H_k^e} \mathrm{h}_I(\mathrm{h}_{M',t}(e(\eta_1,\dots,\eta_k)) \bowtie \mathrm{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\rho(r_\mathrm{b}))\ , \tag{5.7}$$

where $k = \mathrm{rk}(e)$.

Let $e = (r,\rho) \in \Phi_{M',t,c'}$ and $k = \mathrm{rk}(e)$. We distinguish two cases in order to prove Equation (5.7).

Case 1: There is an $i \in [k]$ such that $H'_t(\mathrm{in}_i(e)) = \emptyset$. We show that the left- and the right-hand side of Equation (5.7) are equal to $\mathbf{0}$. For every $\eta_i \in H_i^e$ there is a $d' \in \mathrm{ind}(\eta_i)$ with $H'_t(d') = \emptyset$ because of Lemma 5.20(2). Thus, for every $(\eta_1,\dots,\eta_k) \in H_1^e \times\cdots\times H_k^e$ there is a $d' \in \mathrm{ind}(e(\eta_1,\dots,\eta_k))$ with $H'_t(d') = \emptyset$; then Fact A yields that $\mathrm{h}_I\big(\mathrm{h}_{M',t}(e(\eta_1,\dots,\eta_k))\big) = \mathbf{0}$. Since this holds for arbitrary $(\eta_1,\dots,\eta_k) \in H_1^e\times\cdots\times H_k^e$, we obtain that the left-hand side of Equation (5.7) is equal to $\mathbf{0}$.

Now consider the right-hand side of Equation (5.7). Since there is an $i \in [k]$ such that $H'_t(\mathrm{in}_i(e)) = \emptyset$, we conclude that there is a $d' \in \mathrm{ind}(\rho(r_\mathrm{b}))$ such that $H'_t(d') = \emptyset$ because of Observation 4.11(2). Lemma 4.17(2) yields that the right-hand side of Equation (5.7) is equal to $\mathbf{0}$. Since both the left- and the right-hand side of Equation (5.7) are equal to $\mathbf{0}$ and since $\bowtie$ is reflexive, we obtain that Equation (5.7) holds.

Case 2: $H'_t(\text{in}_i(e)) \neq \emptyset$ *for every* $i \in [k]$. Then Lemma 5.20(1) yields that $\text{in}_i(e) \prec_{G'_t} c'$ for every $i \in [k]$. Let us state and prove another fact before we proceed with the main proof of Case 2.

Fact B. *For every for every* $i \in [k]$ *we have*

$$\sum\nolimits_{\eta \in H_i^e} \text{h}_I(\text{h}_{M',t}(\eta)) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\text{in}_i(e)) \,.$$

Proof of Fact B. Let $i \in [k]$. First we consider the case that $\text{in}_i(e) \notin \text{ran}(\nu_t)$. Then $\text{in}_i(e) \lhd_t c'$ because $\text{in}_i(e) \prec_{G'_t} c'$. This implies $\text{in}_i(e) \in C$ because $c' \in C$. Hence, $\text{in}_i(e) \sqsubset c'$ and the induction hypothesis yields that $\text{in}_i(e) \in D$. Moreover, $\text{in}_i(e) \sqsubset c'$ implies $k_{\text{in}_i(e)} < k_{c'}$. Since also $j < m$ and $j \in \{0, \dots, \min(m, l_c^{\bowtie} - k_{c'})\}$, we conclude $j' = j+1 \in \{0, \dots, \min(m, l_c^{\bowtie} - k_{\text{in}_i(e)})\}$. Thus, $\text{in}_i(e) \in D$ implies $\sum_{\eta \in H'_t(\text{in}_i(e), \text{ran}(\nu_t))} \text{h}_I(\text{h}_{M',t}(\eta)) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\text{in}_i(e))$. Then Fact B follows from the fact that $H_i^e = H'_t(\text{in}_i(e), \text{ran}(\nu_t))$ whenever $\text{in}_i(e) \notin \text{ran}(\nu_t)$.

Now we consider the case that $\text{in}_i(e) \in \text{ran}(\nu_t)$. Let $d \in P(\text{pos}(t))$ such that $\nu_t(d) = \text{in}_i(e)$. Then $H_i^e = \{\text{in}_i(e)\} = \{\nu_t(d)\}$ and $\sum_{\eta \in H_i^e} \text{h}_I(\text{h}_{M',t}(\eta)) = \text{h}_I\big(\text{h}_{M',t}(\nu_t(d))\big) = \text{h}_I(\nu_t(d)) = I(\nu_t(d)) = \mathcal{T}_M^n(I_{\mathbf{0}})(d)$ by the definition of I. It remains to prove that $\mathcal{T}_M^n(I_{\mathbf{0}})(d) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\nu_t(d))$. Since $\nu_t(d) = \text{in}_i(e) \prec_{G'_t} c'$ and $c' \lhd_t^* \nu_t(c)$ (because $c' \in C$) we obtain that $d \prec_{G_t} c$ due to Lemma 5.16(3). Therefore Condition *(i)* of this Lemma yields that $\mathcal{T}_M^n(I_{\mathbf{0}})(d) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\nu_t(d))$ because $j' \in [\min(m, l_c^{\bowtie})]$ (which follows from the facts that $j < m$, $k_{c'} \geq 1$ and $j' - 1 = j \in \{0, \dots, \min(m, l_c^{\bowtie} - k_{c'})\}$).

Continuation of the main proof of Case 2. We distinguish two subcases.

Case 2.1. $\rho(r_\text{b}) \in P'(\text{pos}(t))$. Then $k = \text{rk}(e) = 1$ and $\text{indyield}(\rho(r_\text{b})) = \rho(r_\text{b}) = \text{in}_1(e)$. We prove Equation (5.7) as follows by means of Fact B:

$$\begin{aligned}
\sum\nolimits_{\eta_1 \in H_1^e} \text{h}_I(\text{h}_{M',t}(e(\eta_1)) &= \sum\nolimits_{\eta_1 \in H_1^e} \text{h}_I\big(\rho(r_\text{b}) \leftarrow \text{h}_{M',t}(\eta_1)\big) \\
&= \sum\nolimits_{\eta_1 \in H_1^e} \text{h}_I(\text{h}_{M',t}(\eta_1)) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\text{in}_1(e)) = \text{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\rho(r_\text{b})) \,.
\end{aligned}$$

Case 2.2. $\rho(r_\text{b}) \notin P'(\text{pos}(t))$. Let $i \in [k]$. First we show that $H_i^e \neq \emptyset$. If $\text{in}_i(e) \in \text{ran}(\nu_t)$, then $H_i^e = \{\text{in}_i(e)\} \neq \emptyset$. If $\text{in}_i(e) \notin \text{ran}(\nu_t)$, then $H_i^e = H'_t(\text{in}_i(e), \text{ran}(\nu_t))$ and Lemma 5.20(1) yields that $H_i^e \neq \emptyset$.

Let $\eta, \eta' \in H_i^e$. We make one observation.

- Assume that $H'_t(d') \neq \emptyset$ for every $d' \in \text{ind}(\eta) \cup \text{ind}(\eta')$. Since $c' \in C$, we have $c' \lhd_t^* \nu_t(c)$. Therefore, all premises of Lemma 5.20(3) are satisfied and we obtain that $\eta = \eta'$.

Hence, $\eta = \eta'$ or there is a $d' \in \text{ind}(\eta) \cup \text{ind}(\eta')$ such that $H'_t(d') = \emptyset$. Therefore and since $H_i^e \neq \emptyset$ we conclude that there is an $\tilde{\eta}_i \in H_i^e$ such that for every $\eta' \in H_i^e \setminus \{\tilde{\eta}_i\}$ there is a $d' \in \text{ind}(\eta')$ with $H'_t(d') = \emptyset$ (hence, $\text{h}_I(\text{h}_{M',t}(\eta')) = \mathbf{0}$ by Fact A). Thus, by means of Fact B we obtain that

$$\text{h}_I(\text{h}_{M',t}(\tilde{\eta}_i)) = \sum\nolimits_{\eta \in H_i^e} \text{h}_I(\text{h}_{M',t}(\eta)) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\text{in}_i(e)) \,. \tag{5.8}$$

Note that, if there is an $\eta \in H_i^e$ such that $H'_t(d') \neq \emptyset$ for every $d' \in \text{ind}(\eta)$, then $\tilde{\eta}_i$ is uniquely determined (because then $\tilde{\eta}_i$ must be this particular η). Otherwise, if for every

$\eta \in H_i^e$ there is a $d' \in \mathrm{ind}(\eta)$ with $H'_t(d') = \emptyset$, then $\tilde{\eta}_i$ can be chosen arbitrarily from H_i^e. We choose $\tilde{\eta}_1 \in H_1^e, \ldots, \tilde{\eta}_k \in H_k^e$ in such a way that for every $i, l \in [k]$ with $H_i^e = H_l^e$ we have $\tilde{\eta}_i = \tilde{\eta}_l$. Such a choice clearly exists.

For every $i, l \in [k]$ with $\mathrm{in}_i(e) = \mathrm{in}_l(e)$ we have that also $H_i^e = H_l^e$ and, thus, $\tilde{\eta}_i = \tilde{\eta}_l$. Hence, the mapping $g : \mathrm{ind}(\rho(r_{\mathrm{b}})) \to T_\Delta(P'(\mathrm{pos}(t)))$ with $g(\mathrm{in}_i(e)) = \mathrm{h}_{M',t}(\tilde{\eta}_i)$, for every $i \in [k]$, is well-defined. Let h' be the unique Δ-homomorphism from $\mathcal{T}_\Delta\big(\mathrm{ind}(\rho(r_{\mathrm{b}}))\big)$ to (A, θ) extending $g \,;\, \mathrm{h}_I$. Then for every $i \in [k]$ we obtain by means of Equation (5.8) that $h'(\mathrm{in}_i(e)) = \mathrm{h}_I(g(\mathrm{in}_i(e))) = \mathrm{h}_I(\mathrm{h}_{M',t}(\tilde{\eta}_i)) \bowtie \mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})(\mathrm{in}_i(e)) = \mathrm{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\mathrm{in}_i(e))$. Therefore we can apply Lemma 3.10 and obtain that $h'(\rho(r_{\mathrm{b}})) \bowtie \mathrm{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\rho(r_{\mathrm{b}}))$. We derive

$$
\begin{aligned}
&\textstyle\sum_{(\eta_1,\ldots,\eta_k) \in H_1^e \times \cdots \times H_k^e} \mathrm{h}_I(\mathrm{h}_{M',t}(e(\eta_1, \ldots, \eta_k))) \\
&= \mathrm{h}_I(\mathrm{h}_{M',t}(e(\tilde{\eta}_1, \ldots, \tilde{\eta}_k))) && \text{(by Fact A and the definition of } \tilde{\eta}_1, \ldots, \tilde{\eta}_k\text{)} \\
&= \mathrm{h}_I\big(\rho(r_{\mathrm{b}}) \leftarrow \mathrm{h}_{M',t}(\tilde{\eta}_1) \cdots \mathrm{h}_{M',t}(\tilde{\eta}_k)\big) \\
&= \mathrm{h}_I\big(\rho(r_{\mathrm{b}}) \leftarrow g(\mathrm{in}_1(e)) \cdots g(\mathrm{in}_k(e))\big) \\
&= h'(\rho(r_{\mathrm{b}})) && \text{(by Lemma 3.14(2))} \\
&\bowtie \mathrm{h}_{\mathcal{T}_{M'}^{m-j'}(I_{\mathbf{0}})}(\rho(r_{\mathrm{b}}))\ .
\end{aligned}
$$

This finishes the proof of Equation (5.7). ■

Now we will prove Statement (i), i.e., that the finitary fixpoint semantics coincide given that M and M' are strongly related.

Lemma 5.22. *Let M and M' be strongly related via ν and π, and $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an absorptive m-monoid over Δ. Suppose that M is weakly non-circular. Then $[\![M]\!]_{\mathcal{A}}^{\mathrm{fix}} = [\![M']\!]_{\mathcal{A}}^{\mathrm{fix}}$.*

Proof. Let $t \in T_\Sigma$. We define the set $C = \{c \in P(\mathrm{pos}(t)) \mid c \prec_{G_t}^* q(\varepsilon)\}$ and the relation $\sqsubset\ =\ \prec_{G_t} \cap\, (C \times C)$. Since M is weakly non-circular, $H_t(q(\varepsilon))$ is finite. Then $\sqsubset^+$ is irreflexive due to Lemma 2.26.[3] Thus, $\sqsubset$ is well-founded on C because C is finite. For every $c \in C$ we define the number $k_c \in \mathbb{N}$ by well-founded recursion as follows: $k_c = 1 + \max\{k_d \mid d \in C, d \sqsubset c\}$. We claim that for every $c \in C$ there is a number $j_c \in \mathbb{N}$ such that for every $n, m \in \mathbb{N}$ with $n \geq k_c$ and $m \geq j_c$ we have $\mathcal{T}_M^{n+1}(I_{\mathbf{0}})(c) = \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$. This claim together with Lemma 4.19 and the facts that $q(\varepsilon) \in C$ and $\nu_t(q(\varepsilon)) = q'(\varepsilon)$ yields $\mathcal{T}_M^{|P(\mathrm{pos}(t))|}(I_{\mathbf{0}})(q(\varepsilon)) = \mathcal{T}_{M'}^{|P'(\mathrm{pos}(t))|}(I_{\mathbf{0}})(q'(\varepsilon)))$; hence, $[\![M]\!]_{\mathcal{A}}^{\mathrm{fix}}(t) = [\![M']\!]_{\mathcal{A}}^{\mathrm{fix}}(t)$.

It remains to prove our claim. We give a proof by well-founded induction on the relation $\sqsubset$. Let $c \in C$ and let $l_c^{\mathrm{id}_A}$ be the number from Lemma 5.21 (note that id_A is a reflexive relation on A and that $+$ as well as $\theta(\delta)$, for every $\delta \in \Delta$, are monotone wrt id_A). We put $j_c = l_c^{\mathrm{id}_A} + \max\{j_d \mid d \in P(\mathrm{pos}(t)), d \sqsubset c\}$.

Let $n, m \in \mathbb{N}$ such that $n \geq k_c$ and $m \geq j_c$. We show that $\mathcal{T}_M^{n+1}(I_{\mathbf{0}})(c) = \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$. By Lemma 5.21 it suffices to show that

- $l_c^{\mathrm{id}_A} \leq m$ and
- $\mathcal{T}_M^n(I_{\mathbf{0}})(d) = \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(\nu_t(d))$ for every $d \in P(\mathrm{pos}(t))$ with $d \prec_{G_t} c$ and for every $j \in [\min(m, l_c^{\mathrm{id}_A})]$.

[3] Note that we applied Lemma 2.26 with the instantiations $G = G_t = \mathrm{G}_{M,t}^{\mathrm{dep}}$, $U = \emptyset$, $\lhd = \prec_{G_t}$, and $V' = C$.

The first item follows from $m \geq j_c \geq l_c^{\mathrm{id}_A}$. Now we prove that the second item holds as well. Let $d \in P(\mathrm{pos}(t))$ with $d \prec_{G_t} c$ and let $j \in [\min(m, l_c^{\mathrm{id}_A})]$. Since $c \in C$, we have $c \prec_{G_t}^* q(\varepsilon)$ and, thus, $d \prec_{G_t}^* q(\varepsilon)$. This implies $d \in C$ and $d \sqsubset c$. Hence, $k_c > k_d$ and $j_c \geq l_c^{\mathrm{id}_A} + j_d$. Then the facts that $n \geq k_c$, $m \geq j_c$, and $j \leq l_c^{\mathrm{id}_A}$ imply $n - 1 \geq k_d$ and $m - j \geq j_d$. Due to the induction hypothesis we obtain $\mathcal{T}_M^n(I_{\mathbf{0}})(d) = \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(\nu_t(d))$. ■

Now we will prove the equivalence of the infinitary fixpoint semantics; this involves the proofs of Statements (ii) and (iii).

Lemma 5.23. *Let M and M' be strongly related via ν and π, and $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an absorptive m-monoid over Δ. Moreover, let $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)} = [\![M']\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}$.*

Proof. Let $t \in T_\Sigma$. We claim that

(i) For every n there is an m such that for every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \leq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$.

(ii) For every m there is an n such that for every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c)) \leq \mathcal{T}_M^n(I_{\mathbf{0}})(c)$.

Due to these claims and the fact that $\nu_t(q(\varepsilon)) = q'(\varepsilon)$, the sets $\{\mathcal{T}_M^n(I_{\mathbf{0}})(q(\varepsilon)) \mid n \in \mathbb{N}\}$ and $\{\mathcal{T}_{M'}^n(I_{\mathbf{0}})(q'(\varepsilon)) \mid n \in \mathbb{N}\}$ are mutually cofinal. Lemmas 3.25(2), 4.26, and 4.28 imply that $\mathcal{T}_M^\omega(q(\varepsilon)) = \vee\{\mathcal{T}_M^n(I_{\mathbf{0}})(q(\varepsilon)) \mid n \in \mathbb{N}\} = \vee\{\mathcal{T}_{M'}^n(I_{\mathbf{0}})(q'(\varepsilon)) \mid n \in \mathbb{N}\} = \mathcal{T}_{M'}^\omega(q'(\varepsilon))$. Hence, $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}(t) = [\![M']\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}(t)$.

It remains to prove both claims. By Observation 3.28(2) we have that $+$ and $\theta(\delta)$, for every $\delta \in \Delta$, are monotone wrt $\leq$ because $(\mathcal{A}, \leq)$ is an ω-continuous m-monoid. Let $\geq$ be the inverse relation of $\leq$. Clearly, $+$ and $\theta(\delta)$, for every $\delta \in \Delta$, are also monotone wrt $\geq$.

First we prove Claim (i). To this end we show by induction on n that for every $n \in \mathbb{N}$ there is an $M_n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ with $m \geq M_n$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \leq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$.

Induction base. For $n = 0$ we put $M_n = 0$. Clearly, for every $m \in \mathbb{N}$ with $m \geq M_n$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^0(I_{\mathbf{0}})(c) = \mathbf{0} \leq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$.

Induction step. Let $n \in \mathbb{N}$ and assume that there is an $M_n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ with $m \geq M_n$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \leq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$. We let $M_{n+1} = M_n + \max\{l_c^{\leq} \mid c \in P(\mathrm{pos}(t))\}$, where $l_c^{\leq}$ is the number from Lemma 5.21, for every $c \in P(\mathrm{pos}(t))$. Let $m \in \mathbb{N}$ with $m \geq M_{n+1}$ and let $c \in P(\mathrm{pos}(t))$. We need to show that $\mathcal{T}_M^{n+1}(I_{\mathbf{0}})(c) \leq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$. In view of Lemma 5.21 it suffices to show that

- $l_c^{\leq} \leq m$ and
- $\mathcal{T}_M^n(I_{\mathbf{0}})(d) \leq \mathcal{T}_{M'}^{m-j}(I_{\mathbf{0}})(\nu_t(d))$ for every $d \in P(\mathrm{pos}(t))$ and $j \in [\min(m, l_c^{\leq})]$.

The first item follows from the fact that $m \geq M_{n+1} \geq l_c^{\leq}$ and the second item follows from the fact that $j \leq l_c^{\leq}$ implies $m - j \geq M_{n+1} - l_c^{\leq} \geq M_n$ and the induction hypothesis.

Next we prove Claim (ii). We show by induction on m that for every $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N_m$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \geq \mathcal{T}_{M'}^m(I_{\mathbf{0}})(\nu_t(c))$.

Induction base. For $m = 0$ we put $N_m = 0$. Clearly, for every $n \in \mathbb{N}$ with $n \geq N_m$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_{\mathbf{0}})(c) \geq \mathbf{0} = \mathcal{T}_{M'}^0(I_{\mathbf{0}})(\nu_t(c))$.

Induction step. Let $m \in \mathbb{N}$ and assume that there is an $N_m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq N_m$ and every $c \in P(\mathrm{pos}(t))$ we have $\mathcal{T}_M^n(I_\mathbf{0})(c) \geq \mathcal{T}_{M'}^m(I_\mathbf{0})(\nu_t(c))$. We let $N_{m+1} = N_m + 1$. Let $n \in \mathbb{N}$ with $n \geq N_{m+1}$ and let $c \in P(\mathrm{pos}(t))$. We need to show that $\mathcal{T}_M^n(I_\mathbf{0})(c) \geq \mathcal{T}_{M'}^{m+1}(I_\mathbf{0})(\nu_t(c))$.

Let $d \in P(\mathrm{pos}(t))$. Clearly, $n \geq N_m + 1$ and, hence $n - 1 \geq N_m$. Then the induction hypothesis yields $\mathcal{T}_M^{n-1}(I_\mathbf{0})(d) \geq \mathcal{T}_{M'}^m(I_\mathbf{0})(\nu_t(d))$. Since Lemma 4.28 implies $\mathcal{T}_{M'}^m(I_\mathbf{0})(\nu_t(d)) \geq \mathcal{T}_{M'}^{m'}(I_\mathbf{0})(\nu_t(d))$ for every $m' \in \mathbb{N}$ with $m' \leq m$, we have $\mathcal{T}_M^{n-1}(I_\mathbf{0})(d) \geq \mathcal{T}_{M'}^{m+1-j}(I_\mathbf{0})(\nu_t(d))$ for every $j \in [m]$.

Therefore Lemma 5.21 together with the fact that $a \geq \mathbf{0}$, for every $a \in A$, yields that $\mathcal{T}_M^n(I_\mathbf{0})(c) \geq \mathcal{T}_{M'}^{m+1}(I_\mathbf{0})(\nu_t(c))$. ∎

Lemma 5.24. *Let M and M' be strongly related via ν and π. Then M and M' are completely equivalent.*

Proof. Clearly, M and M' are related and therefore hyp-equivalent due to Lemma 5.18. The remainder of this proof follows from Lemmas 5.22 and 5.23. ∎

This finishes the proof of Theorem 5.11. We conclude this section with a lemma that provides a simple condition that guarantees that two mwmd are strongly related. This lemma will be useful later in this chapter because it simplifies proofs that two particular mwmd are strongly related.

Lemma 5.25. *Let M and M' be related via ν and π. Suppose that for every $t \in T_\Sigma$, $c' \in P'(\mathrm{pos}(t))$, $e = (r, \rho) \in \Phi_{M',t,c'}$, and $i \in [\mathrm{rk}(e)]$ with $\rho(r_\mathrm{b}) \notin P'(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ we have that $|H'_t(\mathrm{in}_i(e), \mathrm{ran}(\nu_t))| \leq 1$. Then M and M' are strongly related.*

Proof. Let $t \in T_\Sigma$, $c \in P(\mathrm{pos}(t))$, $\eta, \eta' \in H'_t\big(\nu_t(c), \mathrm{ran}(\nu_t)\big)$, $w \in \mathrm{pos}(\eta) \cap \mathrm{pos}(\eta')$, and $(r, \rho) \in \Phi_{M',t}$, such that such that $\eta|_w \notin \mathrm{ran}(\nu_t)$, $\eta'|_w \notin \mathrm{ran}(\nu_t)$, $(r, \rho) = \eta(w) = \eta'(w)$, $\rho(r_\mathrm{b}) \notin P'(\mathrm{pos}(t))$, and $\eta(w') = \eta'(w')$ for every prefix w' of w. We show that $\eta|_w = \eta'|_w$.

Let $e = (r, \rho)$, $c' = \mathrm{out}(e)$, and $k = \mathrm{rk}(e)$. By Lemma 2.15 we have that $\eta|_w, \eta'|_w \in H'_t(c', \mathrm{ran}(\nu_t))$. Then in order to show that $\eta|_w = \eta'|_w$ it suffices to show that, for every $i \in [k]$, we have $|H_i^e| \leq 1$ due to Lemma 5.13. Let $i \in [k]$. If $\mathrm{in}_i(e) \in \mathrm{ran}(\nu_t)$ we have $H_i^e = \{\mathrm{in}_i(e)\}$ and, hence $|H_i^e| = 1$. If $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$, then $H_i^e = H'_t(\mathrm{in}_i(e), \mathrm{ran}(\nu_t))$ and $|H_i^e| \leq 1$ follows by assumption. ∎

5.3 Proper

In this section we show that for every mwmd there is a completely equivalent proper mwmd. Moreover, we show that we can carry out this construction in such a way that restrictedness and connectedness are preserved; hence, for every restricted mwmd there is a completely equivalent proper and restricted mwmd, and likewise for connected mwmd.

Let us motivate our construction. Given an mwmd M having a nullary user-defined predicate $p^{(0)}$ we need to construct an mwmd M' that has a unary predicate $p'^{(1)}$ instead. The simplest way to construct M' is by replacing in every rule r every occurrence of $p()$ by $p'(x)$, where x is a new variable, and adding the atom $\mathrm{root}(x)$ to the guard of the rule. Hence, the atom instance $p'(\varepsilon)$ in M' does simulate the behavior of the atom instance $p()$ in M; $p()$ *corresponds to* $p'(\varepsilon)$. The other atom instances involving p' (e.g., $p'(211)$) are virtually inactive; they have no effect on the behavior of M'.

This naive construction is simple but it is easy to see that in general it does neither preserve restrictedness nor connectedness (the new variable x is not connected to the other variables is the rule). A more sophisticated approach is not to introduce a new variable x in the rule but use one of the variables y that is already present in the rule r; then replace $p()$ by $p'(y)$ instead of $p'(x)$. This construction does preserve connectedness (but not necessarily restrictedness; for the sake of simplicity let us disregard this problem in this informal motivation). However, now the transport of information may be disconnected: if $p()$ is the head of r, then the head of the resulting rule r' is $p'(y)$ and for any rule instance (r', ρ) of r' the output vertex is $p'(\rho(y))$ instead of $p'(\varepsilon)$, as it should be. Similarly, for every occurrence of $p()$ in the body of r the according input vertex of the rule instance (r', ρ) is $p'(\rho(y))$ instead of $p'(\varepsilon)$.

We can remedy this by introducing transport rules in M' that reconnect the information transport. These are rules for transporting information upwards to the root of the tree and rules for transporting information downwards from the root of the tree. The upward transport rules are of the form $p'(x) \leftarrow p'(y) ; \mathrm{child}_i(x, y)$ and ensure that, if $p()$ is the head of the rule r, then the output vertex $p'(\rho(y))$ of the rule instance (r', ρ) is connected to the atom instance $p'(\varepsilon)$. The downward transport rules are of the form $p'(y) \leftarrow p'(x) ;$ $\mathrm{child}_i(x, y)$ and ensure that, if $p()$ occurs in the body of the rule r, then the atom instance $p'(\varepsilon)$ is connected to any input vertex $p'(\rho(y))$ of the rule instance (r', ρ).

Unfortunately, this construction introduces loops in the information transport and, thus, the resulting mwmd M' may not be weakly non-circular anymore. This is due to the problem that upward and downward information transport are not separated, both of them are based on the predicate p'. We solve this problem as follows: instead of p' we introduce two variants of this predicate, namely p'_{in} and p'_{out}. We will use p'_{out} for upward information transport and will replace the atom $p()$ in the head of any rule by $p'_{\mathrm{out}}(y)$ (for an appropriate variable y). Likewise, p'_{in} is used for downward information transport and we will replace any occurrence of $p()$ in the body of any rule by $p'_{\mathrm{in}}(y)$. Additionally, we need to add a rule to M' that ensures that, roughly speaking, upward and downward transport are connected to each other at the root of the input tree, i.e., the rule $p'_{\mathrm{in}}(x) \leftarrow p'_{\mathrm{out}}(x) ; \mathrm{root}(x)$. Then the atom instance $p()$ in M corresponds both to $p'_{\mathrm{in}}(\varepsilon)$ and $p'_{\mathrm{out}}(\varepsilon)$ in M'.

Lemma 5.26 (cf. [28, Lemma 5]). *Let M be an mwmd over Σ and Δ. Then there is a proper mwmd M' over Σ and Δ such that*

1. *M and M' are completely equivalent, and*
2. *if M is restricted, then M' is restricted,*
3. *if M is connected, then M' is connected.*

Proof. The construction that we carry out in this proof depends on whether M is restricted or connected. First we assume that M is either connected or not restricted. We consider the case that M is restricted but not connected at the end of this proof.

Let $M = (P, R, q)$. For every rule $r \in R$ we define the variable x_r as follows: (i) if $\mathrm{var}(r) \neq \emptyset$, then we choose $x_r \in \mathrm{var}(r)$ arbitrarily and (ii) otherwise we choose $x_r \in \mathrm{V}$ arbitrarily.

We define $M' = (P', R', q)$ as follows:

$$P' = P^{(1)} \cup \{p^{(1)}_{\mathrm{in}} \mid p \in P^{(0)}\} \cup \{p^{(1)}_{\mathrm{out}} \mid p \in P^{(0)}\} ,$$

$$R' = \{\bar{r} \mid r \in R\} \cup \{r^{\text{up}}_{p,i} \mid p \in P^{(0)}, i \in [\text{maxrk}(\Sigma)]\}$$
$$\cup \{r^{\text{down}}_{p,i} \mid p \in P^{(0)}, i \in [\text{maxrk}(\Sigma)]\}$$
$$\cup \{r^{\text{trans}}_{p} \mid p \in P^{(0)}\} ,$$

where for every $r \in R$ the rule $\bar{r}$ is obtained from r as follows:

- (i) if $r_{\text{h}} = p(x)$ for some $p \in P^{(1)}$ and $x \in \text{V}$, then $\bar{r}_{\text{h}} = r_{\text{h}}$ and (ii) if $r_{\text{h}} = p()$ for some $p \in P^{(0)}$, then $\bar{r}_{\text{h}} = p_{\text{out}}(x_r)$,
- $\bar{r}_{\text{b}}$ originates from r_{b} by replacing for every $p \in P^{(0)}$ every occurrence of $p()$ by $p_{\text{in}}(x_r)$,
- $\bar{r}_{\text{G}} = r_{\text{G}}$ if $\text{var}(r) \neq \emptyset$ and $\bar{r}_{\text{G}} = \{\text{root}(x_r)\}$ otherwise.

Moreover, for every $p \in P^{(0)}$ and $i \in [\text{maxrk}(\Sigma)]$ we let

$$r^{\text{up}}_{p,i} = p_{\text{out}}(x_\varepsilon) \leftarrow p_{\text{out}}(x_i) \,;\, \text{child}_i(x_\varepsilon, x_i) ,$$
$$r^{\text{down}}_{p,i} = p_{\text{in}}(x_i) \leftarrow p_{\text{in}}(x_\varepsilon) \,;\, \text{child}_i(x_\varepsilon, x_i) ,$$
$$r^{\text{trans}}_{p} = p_{\text{in}}(x_\varepsilon) \leftarrow p_{\text{out}}(x_\varepsilon) \,;\, \text{root}(x_\varepsilon) .$$

We have to show that Conditions 1, 2, and 3 of this lemma hold. First let us prove Conditions 2 and 3. Since we assumed that M is either connected or not restricted, it suffices to show that M' is connected whenever M is connected because every connected mwmd is also restricted due to Observation 5.3. Assume that M is connected. For every $p \in P^{(0)}$ and $i \in [\text{maxrk}(\Sigma)]$ the rules $r^{\text{up}}_{p,i}$, $r^{\text{down}}_{p,i}$, and r^{trans}_{p} are obviously connected. It remains to show that $\bar{r}$ is connected for every $r \in R$; if $\text{var}(r) \neq \emptyset$, then obviously $\text{var}(r) = \text{var}(\bar{r})$ and $\sim_r = \sim_{\bar{r}}$; if $\text{var}(r) = \emptyset$, then $\text{var}(\bar{r}) = \{x_r\}$ and $\bar{r}$ is trivially connected.

Now we show Condition 1, i.e., that M and M' are completely equivalent. In view of Lemma 5.24 it suffices to show that there are families ν and π such that M and M' are strongly related via ν and π.

Let $t \in T_\Sigma$. We define the injective mapping $\nu_t : P(\text{pos}(t)) \to P'(\text{pos}(t))$ as follows for every $p \in P$ and $w \in \text{pos}(t)$: (i) if $p \in P^{(0)}$, then $\nu_t(p()) = p_{\text{out}}(\varepsilon)$, and (ii) if $p \in P^{(1)}$, then $\nu_t(p(w)) = p(w)$.

Let $c' \in P'(\text{pos}(t))$ and let us study the set $\Phi_{M',t,c'}$. First consider the case that $c' = p_{\text{in}}(\varepsilon)$ for some $p \in P^{(0)}$. Then $\Phi_{M',t,c'} = \{e^{\text{trans}}_p\}$, where e^{trans}_p is the hyperedge $(r^{\text{trans}}_p, [x_\varepsilon \mapsto w])$. Observe that $\text{rk}(e^{\text{trans}}_p) = 1$ and $\text{in}_1(e^{\text{trans}}_p) = p_{\text{out}}(\varepsilon)$.

Next we consider the case that $c' = p_{\text{in}}(wi)$ for some $p \in P^{(0)}$, $w \in \text{pos}(t)$, and $i \in \mathbb{N}_+$ such that $wi \in \text{pos}(t)$. Then we obtain that $\Phi_{M',t,c'} = \{e^{\text{down}}_{p,w,i}\}$, where $e^{\text{down}}_{p,w,i} = (r^{\text{down}}_{p,i}, [x_\varepsilon \mapsto w, x_i \mapsto wi])$. Clearly, $\text{rk}(e^{\text{down}}_{p,w,i}) = 1$ and $\text{in}_1(e^{\text{down}}_{p,w,i}) = p_{\text{in}}(w)$.

Now assume that $c' = p(w)$ for some $p \in P^{(1)}$ and $w \in \text{pos}(t)$. Then we have that also $c' \in P(\text{pos}(t))$. Observe that for every $(r, \rho) \in \Phi_{M',t,c'}$ the rule r must be of the form $\bar{r}_0$ for some $r_0 \in R$ such that $r_{\text{h}} = (\bar{r}_0)_{\text{h}} = (r_0)_{\text{h}} = p(x)$ for some $x \in \text{V}$; hence, $\text{var}(r_0) \neq \emptyset$ and, thus, $x_r \in \text{var}(r_0)$, $\text{var}(r) = \text{var}(r_0)$, and $r_{\text{G}} = (\bar{r}_0)_{\text{G}} = (r_0)_{\text{G}}$; we obtain that $(r_0, \rho) \in \Phi_{M,t,c'}$. The converse holds as well, i.e., for every $(r_0, \rho) \in \Phi_{M,t,c'}$ we have $(\bar{r}_0, \rho) \in \Phi_{M',t,c'}$. We conclude that $\Phi_{M',t,c'} = \{(\bar{r}, \rho) \mid (r, \rho) \in \Phi_{M,t,c'}\}$. Let $(r, \rho) \in \Phi_{M,t,c'}$. Then $\text{rk}((\bar{r}, \rho)) = \text{rk}((r, \rho))$ and for every $i \in [\text{rk}((r, \rho))]$ we have that $\text{in}_i((\bar{r}, \rho)) = \text{in}_i((r, \rho))$ if $\text{in}_i((r, \rho)) \in P^{(1)}(\text{pos}(t))$, and $\text{in}_i((\bar{r}, \rho)) = {p'}_{\text{in}}(\rho(x_r))$ if $\text{in}_i((r, \rho)) = p'()$ for some $p' \in P^{(0)}$.

In a similar fashion one can analyze the case that $c' = p_{\text{out}}(w)$ for some $p \in P^{(0)}$ and $w \in \text{pos}(t)$. For every $i \in \mathbb{N}_+$ let $e^{\text{up}}_{p,w,i} = (r^{\text{up}}_{p,i}, [x_\varepsilon \mapsto w, x_i \mapsto wi])$; then we obtain that

$$\begin{aligned}\Phi_{M',t,c'} = \{e^{\text{up}}_{p,w,i} \mid i \in \mathbb{N}_+, wi \in \text{pos}(t)\} \\ \cup \{(\bar{r}, \rho) \mid (r, \rho) \in \Phi_{M,t,p()}, \text{var}(r) \neq \emptyset, \rho(x_r) = w\} \\ \cup \{(\bar{r}, [x_\varepsilon \mapsto \varepsilon]) \mid (r, \emptyset) \in \Phi_{M,t,p()}, \text{var}(r) = \emptyset, w = \varepsilon\} .\end{aligned}$$

The last line results from the fact that for every $r \in R$ with $\text{var}(r) = \emptyset$ we have that $r_{\text{G}} = \emptyset$, $x_r = x_\varepsilon$, and $\bar{r}_{\text{G}} = \{\text{root}(x_\varepsilon)\}$. For every $i \in \mathbb{N}_+$ with $wi \in \text{pos}(t)$ we have that $\text{rk}(e^{\text{up}}_{p,w,i}) = 1$ and $\text{in}_1(e^{\text{up}}_{p,w,i}) = p_{\text{out}}(wi)$.

Now we are prepared to study the sets $H'_t(c', \text{ran}(\nu_t))$ for every $c' \in P'(\text{pos}(t))$, where $\text{ran}(\nu_t) = P^{(1)}(\text{pos}(t)) \cup \{p_{\text{out}}(\varepsilon) \mid p \in P^{(0)}\}$. Note that in the remainder of this proof we will make heavy use of Lemma 5.13 without referring to it explicitely. If $c' = p_{\text{in}}(w)$ for some $w \in \text{pos}(t)$, then there are $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}_+$ such that $w = i_1 \cdots i_n$. Then it is easy to see that $H'_t(p_{\text{in}}(w), \text{ran}(\nu_t)) = \{\eta_{\text{in},w}\}$, where the single element $\eta_{\text{in},p,w}$ is defined as

$$\eta_{\text{in},p,w} = e^{\text{down}}_{p,i_1\cdots i_{n-1},i_n}(e^{\text{down}}_{p,i_1\cdots i_{n-2},i_{n-1}}(\cdots(e^{\text{down}}_{p,\varepsilon,i_1}(e^{\text{trans}}_p(p_{\text{out}}(\varepsilon)))\cdots)) .$$

Clearly, $\text{h}_{M',t}(\eta_{\text{in},p,w}) = p_{\text{out}}(\varepsilon)$.

The other cases are slightly more complex. In order to deal with them in a succinct way let us introduce an auxiliary definition. Let $(r, \rho) \in \Phi_{M,t}$. We let $k_{r,\rho} = \text{rk}((r, \rho))$ and, for every $i \in [k_{r,\rho}]$, let $\eta^{r,\rho}_i = \text{in}_i((r, \rho))$ if $\text{in}_i((r, \rho)) \in P^{(1)}(\text{pos}(t))$ and let $\eta^{r,\rho}_i = \eta_{\text{in},p',\rho(x_r)}$ if $\text{in}_i((r, \rho)) = p'()$ for some $p' \in P^{(0)}$; observe that $\text{h}_{M',t}(\eta^{r,\rho}_i) = \nu_t(\text{in}_i((r, \rho)))$.

Now we consider the case that $c' = p(w)$ for some $p \in P^{(1)}$ and $w \in \text{pos}(t)$. It is easy to check that

$$H'_t(p(w), \text{ran}(\nu_t)) = \{(\bar{r}, \rho)(\eta^{r,\rho}_1, \ldots, \eta^{r,\rho}_{k_{r,\rho}}) \mid (r, \rho) \in \Phi_{M,t,c'}\} ,$$

and that $\text{h}_{M',t}(\bar{r}, \rho)(\eta^{r,\rho}_1, \ldots, \eta^{r,\rho}_{k_{r,\rho}}) = \text{h}_{\nu_t}(\rho(r_{\text{b}}))$ for every $(r, \rho) \in \Phi_{M,t,c'}$.

Finally, we consider the atom instances of the form $p_{\text{out}}(w)$ for some $p \in P^{(0)}$ and $w \in \text{pos}(t)$. Let $(r, \rho) \in \Phi_{M,t,p()}$. Let $\rho' = \rho$ if $\text{var}(r) \neq \emptyset$ and let $\rho = [x_\varepsilon \mapsto \varepsilon]$ otherwise. Moreover, let $w = \rho'(x_r)$. It is easy to see that the hyperpath segment $\eta^{\text{out}}_{r,\rho} = (\bar{r}, \rho')(\eta^{r,\rho}_1, \ldots, \eta^{r,\rho}_{k_{r,\rho}})$ is an element of the set $H'_t(p_{\text{out}}(w), \text{ran}(\nu_t))$ and that $\text{h}_{M',t}(\eta^{\text{out}}_{r,\rho}) = \text{h}_{\nu_t}(\rho(r_{\text{b}}))$. There are $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \mathbb{N}_+$ such that $w = i_1 \cdots i_n$. We let

$$\hat{\eta}^{\text{out}}_{r,\rho} = e^{\text{up}}_{p,\varepsilon,i_1}(e^{\text{up}}_{p,i_1\cdots i_1,i_2}(\cdots(e^{\text{up}}_{p,i_1\cdots i_{n-1},i_n}(\eta^{\text{out}}_{r,\rho}))\cdots)) .$$

Clearly, $\hat{\eta}^{\text{out}}_{r,\rho} \in H'_t(p_{\text{out}}(\varepsilon), \text{ran}(\nu_t))$ and $\text{h}_{M',t}(\hat{\eta}^{\text{out}}_{r,\rho}) = \text{h}_{M',t}(\eta^{\text{out}}_{r,\rho}) = \text{h}_{\nu_t}(\rho(r_{\text{b}}))$; we obtain that

$$H'_t(p_{\text{out}}(\varepsilon), \text{ran}(\nu_t)) = \{\hat{\eta}^{\text{out}}_{r,\rho} \mid (r, \rho) \in \Phi_{M,t,p()}\} .$$

For every $c \in P(\text{pos}(t))$ we define the mapping $\pi_{t,c} : \Phi_{M,t,c} \to H'_t\big(\nu_t(c), \text{ran}(\nu_t)\big)$ as follows: if $c \in P^{(1)}(\text{pos}(t))$, then, for every $(r, \rho) \in \Phi_{M,t,c}$, we let $\pi_{t,c}(r, \rho) = (\bar{r}, \rho)(\eta^{r,\rho}_1, \ldots, \eta^{r,\rho}_{k_{r,\rho}})$. Otherwise, if $c = p()$ for some $p \in P^{(0)}$, then for every $(r, \rho) \in \Phi_{M,t,c}$ we let $\pi_{t,c}(r, \rho) = \hat{\eta}^{\text{out}}_{r,\rho}$.

We have already shown that for every $c \in P(\text{pos}(t))$ and $(r, \rho) \in \Phi_{M,t,c}$ we have $\text{h}_{\nu_t}(\rho(r_{\text{b}})) = \text{h}_{M',t}(\pi_{t,c}(r, \rho))$. Therefore and due to the fact that $\nu_t(q(\varepsilon)) = q(\varepsilon)$ we conclude that M and M' are related via the families $\nu = (\nu_t \mid t \in T_\Sigma)$ and $\pi = \big(\pi_{t,c} \mid t \in T_\Sigma, c \in P(\mathrm{pos}(t))\big)$.

It remains to show that M and M' are also strongly related via ν and π. By Lemma 5.25 it suffices to show that for every $t \in T_\Sigma$, $c' \in P'(\mathrm{pos}(t))$, $e = (r', \rho') \in \Phi_{M',t,c'}$, and $i \in [\mathrm{rk}(e)]$ with $\rho'(r'_\mathrm{b}) \notin P'(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ we have that $|H'_t\big(\mathrm{in}_i(e), \mathrm{ran}(\nu_t)\big)| \leq 1$. It is easy to check that the condition $\rho'(r'_\mathrm{b}) \notin P'(\mathrm{pos}(t))$ yields that $r' = \bar{r}$ for some $r \in R$. Then the condition $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ implies that $\mathrm{in}_i(e) = p_\mathrm{in}(w)$ for some $w \in \mathrm{pos}(t)$. We have already shown that $H'_t\big(p_\mathrm{in}(w), \mathrm{ran}(\nu_t)\big) = \{\eta_{\mathrm{in},w}\}$; hence, $|H'_t\big(\mathrm{in}_i(e), \mathrm{ran}(\nu_t)\big)| \leq 1$.

This finishes the proof for the case that M is either connected or not restricted. Now let us consider the case that M is restricted but not connected. We will give an informal proof because this case can be handled similarly to the first one.

Due to Condition 2 the mwmd M' needs to be restricted, too. The construction that we carried out above (for the case that M is either connected or not restricted) will not satisfy Condition 2 in general. This is witnessed by the following example. Assume that R contains a rule r such that $r_\mathrm{b} = \delta\big(\sigma(p(x), r(), p(x)), \sigma(p(y), r(), p(y))\big)$, where $\sigma \in \Delta^{(3)}$, $\delta \in \Delta^{(2)}$, $p \in P^{(1)}$, and $r \in P^{(0)}$. Moreover, assume that $r_\mathrm{G} = \emptyset$; hence, $x \not\sim_r y$, i.e., the rule r is not connected, but it is easy to check that r is restricted. When applying the construction that we described above, we obtain for the rule $\bar{r}$ that $\bar{r}_\mathrm{b} = \delta\big(\sigma(p(x), r_\mathrm{in}(x_r), p(x)), \sigma(p(y), r_\mathrm{in}(x_r), p(y))\big)$ where x_r is some variable $x_r \in \mathrm{var}(r)$ (e.g., $x_r = x$ or $x_r = y$). Then we have that $x_r \not\sim_{\bar{r}} x$ or $x_r \not\sim_{\bar{r}} y$ must hold. This implies that $\bar{r}$ is not restricted.

The reason for this problem is that we used the same variable x_r at every occurrence of the predicate r_in. This is not necessary. Instead we could replace the first occurrence of x_r by the variable x and the second occurrence of x_r by the variable y, i.e., $\bar{r}_\mathrm{b} = \delta\big(\sigma(p(x), r_\mathrm{in}(x), p(x)), \sigma(p(y), r_\mathrm{in}(y), p(y))\big)$; this yields a restricted rule.

In general the mwmd M' is defined similarly to the case that M is either connected or not restricted. However, for every $r \in R$ the body $\bar{r}_\mathrm{b}$ of the rule $\bar{r}$ is obtained from r_b by replacing for every $p \in P^{(0)}$ and $w \in \mathrm{pos}(r_\mathrm{b})$ with $r_\mathrm{b}(w) = p()$ the occurrence of $p()$ in r_b at the position w by $p_\mathrm{in}(x)$, where the variable x is defined as follows:

(i) if there is an equivalence class $C \in \mathrm{var}(r)/{\sim_r}$ such that $C \cap \mathrm{var}(r_\mathrm{b}) \neq \emptyset$, and w_C is a prefix of w (where w_C is the position from Definition 5.1), then choose $x \in C$ arbitrary,

(ii) otherwise let $x = x_{p,w}$ be a new variable that does not occur in r and add $\mathrm{root}(x_{p,w})$ to $\bar{r}_\mathrm{G}$.

Then the rule $\bar{r}$ is restricted. The proof that M and M' are completely equivalent is similar to the case that M is either connected or not restricted. ■

The following corollary is an immediate consequence of Lemma 5.26 and the definition of completely equivalent mwmd.

Corollary 5.27. *Let $\mathcal{A}$ be an m-monoid over Δ, $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid, and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Then for every $x \in \{\varepsilon, \mathrm{r}, \mathrm{c}\}$*

$$x\text{–}\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{p}x\text{–}\mathrm{WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) ,$$
$$x\text{–}\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) = \mathrm{p}x\text{–}\mathrm{WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) ,$$

moreover, if $\mathcal{A}$ is absorptive, then

$$x\text{–WMD}^{\text{fix}}(\Sigma,\Delta,\mathcal{A}) = \text{p}x\text{–WMD}^{\text{fix}}(\Sigma,\Delta,\mathcal{A})\ ,$$
$$x\text{–WMD}^{\text{fix}}\big(\Sigma,\Delta,(\mathcal{A},\le)\big) = \text{p}x\text{–WMD}^{\text{fix}}\big(\Sigma,\Delta,(\mathcal{A},\le)\big)\ .$$

Note that if $x = \varepsilon$, then, e.g., $x\text{–WMD}^{\text{hyp}}(\Sigma,\Delta,\mathcal{A})$ denotes $\text{WMD}^{\text{hyp}}(\Sigma,\Delta,\mathcal{A})$.

5.4 Connected

In this section we deal with restricted and connected mwmd. Clearly, the class of restricted mwmd is a subclass of the class of all mwmd and the class of connected mwmd is a subclass of the class of all restricted mwmd (see Observation 5.3). We will study whether (i) every mwmd can be transformed into a (hyp- or completely) equivalent restricted one and (ii) whether such a transformation exists from the class of restricted mwmd to the class of connected mwmd. It turns out that the answer to both problems is negative. In fact, in this section we will prove the following three statements.

Statement C1: We will show that there is a restricted mwmd such that there is no hyp-equivalent connected mwmd (see Lemma 5.30).

Statement C2: However, we show that for every restricted mwmd there is a connected mwmd such that their hyperpath and fixpoint semantics coincide for a certain subclass of m-monoids, namely the class of idempotent and distributive m-monoids (see Corollary 5.34).

Statement C3: We will show that there is an mwmd such that there is no a hyp-equivalent restricted mwmd (see Lemma 5.35).

Proof of Statement C1

We show that there is a restricted mwmd such that there is no hyp-equivalent connected mwmd. First let us give two lemmas that state basic properties of connected mwmd.

For the remainder of this chapter we fix a ranked alphabet Σ and a signature Δ. Moreover we fix an mwmd $M = (P, R, q)$ over Σ and Δ.

Lemma 5.28. *Let $r \in R$ be connected and let $X \subseteq \text{var}(r)$, $x \in X$, and $y \in \text{var}(r) \setminus X$. Then there are $z \in X$, $z' \in \text{var}(r) \setminus X$, and $i \in [\text{maxrk}(\Sigma)]$ such that $\text{child}_i(z, z') \in r_{\text{G}}$ or $\text{child}_i(z', z) \in r_{\text{G}}$.*

Proof. Since r is connected, we have $x \sim_r y$. Therefore there are $n \in \mathbb{N}$, $x_0, \dots, x_n \in \text{var}(r)$, and $b_1, \dots, b_n \in r_{\text{G}}$ such that $x_0 = x$, $x_n = y$, and for every $j \in [n]$ we have that $\{x_{j-1}, x_j\} \in \text{var}(b_j)$. Since $x = x_0 \in X$ and $y = x_n \notin X$, there is a $j \in [n]$ such that $x_{j-1} \in X$ and $x_j \notin X$. Hence, $x_{j-1} \ne x_j$ and by using the fact that $x_{j-1} \in \text{var}(b_j)$ and $x_j \in \text{var}(b_j)$ we obtain that there is an $i \in [\text{maxrk}(\Sigma)]$ such that either $b_j = \text{child}_i(x_{j-1}, x_j)$ or $b_j = \text{child}_i(x_j, x_{j-1})$. ■

Lemma 5.29. *Let $M = (P, R, q)$ be a connected mwmd over Σ and Δ, and let $t \in T_\Sigma$.*

1. *Let* $(r,\rho),(r',\rho') \in \Phi_{M,t}$ *such that* $r = r'$. *If there is an* $x \in \mathrm{var}(r)$ *with* $\rho(x) = \rho'(x)$, *then* $\rho = \rho'$.

2. *Let* $r \in R$. *Then* $|\{e \in \Phi_{M,t} \mid \mathrm{pr}_1(e) = r\}| \leq |\mathrm{pos}(t)|$. *Moreover, we have* $|\Phi_{M,t}| \leq |R| \cdot |\mathrm{pos}(t)|$.

3. *Let* $c \in P(\mathrm{pos}(t))$. *If* $c \in P^{(1)}(\mathrm{pos}(t))$, *then* $|\Phi_{M,t,c}| \leq |R|$. *Moreover, if* $c \in P^{(0)}(\mathrm{pos}(t))$, *then* $|\Phi_{M,t,c}| \leq |R| \cdot |\mathrm{pos}(t)|$.

PROOF. 1. Suppose that there is an $x \in \mathrm{var}(r)$ such that $\rho(x) = \rho'(x)$. Assume, contrary to our claim, that $\rho \neq \rho'$. This assumption implies that there is a $y \in \mathrm{var}(r)$ with $\rho(y) \neq \rho'(y)$. Let $X = \{z \in \mathrm{var}(r) \mid \rho(z) = \rho'(z)\}$. Then $x \in X$ and $y \notin X$. Hence, Lemma 5.28 yields that there are $z \in X$, $z' \in \mathrm{var}(r) \setminus X$, and $i \in [\mathrm{maxrk}(\Sigma)]$ such that $\mathrm{child}_i(z,z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z',z) \in r_{\mathrm{G}}$. The fact that $(r,\rho),(r',\rho') \in \Phi_{M,t}$ implies $\rho(b),\rho'(b) \in \mathrm{B}_t$. This yields a contradiction to the facts that $\rho(z) = \rho'(z)$, $\rho(z') \neq \rho'(z')$, and $\mathrm{child}_i(z,z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z',z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$. Therefore our assumption $\rho \neq \rho'$ was false.

2. The first part is an immediate consequence of Statement 1 and the second part follows from the first part.

3. First we consider the case that $c \in P^{(1)}(\mathrm{pos}(t))$. Thus, there is a $p \in P^{(1)}$ and $w \in \mathrm{pos}(t)$ such that $p(w) = c$. Let $(r,\rho),(r',\rho') \in \Phi_{M,t,c}$ such that $r = r'$. Then $\rho(r_{\mathrm{h}}) = c = \rho'(r_{\mathrm{h}})$, i.e., there is an $x \in \mathrm{var}(r)$ such that $r_{\mathrm{h}} = p(x)$, $\rho(x) = w = \rho'(x)$. Statement 1 yields that $\rho = \rho'$. We conclude that $|\Phi_{M,t,c}| \leq |R|$.

Now consider the case that $c \in P^{(0)}(\mathrm{pos}(t))$. Clearly, $|\Phi_{M,t,c}| \leq |\Phi_{M,t}| \leq |R| \cdot |\mathrm{pos}(t)|$ by Statement 2. ∎

Now we are prepared to prove Statement C1.

Lemma 5.30. *There is a ranked alphabet* Σ, *a signature* Δ, *and a restricted mwmd* M *over* Σ *and* Δ *such that there is no connected mwmd* M' *over* Σ *and* Δ *that is hyp-equivalent to* M.

PROOF. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}\}$, $\Delta = \{\alpha^{(0)}\}$. Consider the mwmd $M = (P,R,p)$ over Σ and Δ such that $P = \{p^{(1)}\}$ and $R = \{r_1, r_2\}$, where r_1 and r_2 are defined as follows:

$$r_1 = p(x) \leftarrow p(y)\,;\{\mathrm{child}_1(x,y), \mathrm{label}_\gamma(z)\}\ ,$$
$$r_2 = p(x) \leftarrow \alpha\,;\{\mathrm{leaf}(x)\}\ .$$

Clearly, M is weakly non-circular (it is even non-circular) and M is not connected because $x \not\sim_{r_1} z$. However, M is restricted. We show that there is no connected mwmd M' over Σ and Δ that is hyp-equivalent to M.

It suffices to prove that there is an m-monoid $\mathcal{A}$ over Δ such that for every weakly non-circular and connected mwmd M' over Σ and Δ we have $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} \neq [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}$. We define the m-monoid $\mathcal{A}$ by letting $\mathcal{A} = (\mathbb{N}, +, 0, \theta)$ where $\theta(\alpha)() = 1$. Note that $\mathcal{A}$ is distributive. Now we show that $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} \neq [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}$ holds for every weakly non-circular and connected mwmd M' over Σ and Δ.

Let $n \in \mathbb{N}$ and $t = \gamma^n(\alpha) \in T_\Sigma$. We show by induction that for every index $i \in \{0,\ldots,n\}$ we have that $\mathcal{T}^{i+1}(I_0)(p(w_i)) = n^i$, where w_i is the unique position in $\mathrm{pos}(t)$ such that $|w_i| = n - i$.

Induction base. Suppose that $i = 0$. Clearly, w_0 is the leaf position in t; thus, $\mathcal{T}(I_0)(p(w_0)) = \sum_{(r,\rho)\in\Phi_{M,t,p(w_0)}} \mathrm{h}_{I_0}(\rho(r_{\mathrm{b}})) = \mathrm{h}_{I_0}(\alpha) = 1 = n^0$.

Induction step. Let $i \in [n]$. Assume $\mathcal{T}^i(I_0)(p(w_{i-1})) = n^{i-1}$. Then $\mathcal{T}^{i+1}(I_0)(p(w_i)) = \mathcal{T}(\mathcal{T}^i(I_0))(p(w_i)) = \sum_{(r,\rho)\in\Phi_{M,t,p(w_i)}} \mathrm{h}_{\mathcal{T}^i(I_0)}(\rho(r_\mathrm{b})) = \sum_{j\in[n]} \mathrm{h}_{\mathcal{T}^i(I_0)}\big(\rho_j(p(y))\big)$, where $\rho_j = [x \mapsto w_i, y \mapsto w_i1, z \mapsto w_j]$ for every $j \in [n]$; this holds because the label γ occurs at positions $\{w_1, \ldots, w_n\}$ in t. Since $w_i1 = w_{i-1}$, we derive $\sum_{j\in[n]} \mathrm{h}_{\mathcal{T}^i(I_0)}\big(\rho_j(p(y))\big) = \sum_{j\in[n]} T^i(I_0)(p(w_{i-1})) = n \cdot n^{i-1} = n^i$ due to the induction hypothesis. This finishes the inductive proof.

In particular, we have $\mathcal{T}^{n+1}(I_0)(p(w_n)) = n^n$. By means of the fact that $|P(\mathrm{pos}(t))| = \big|\{p(w) \mid w \in \mathrm{pos}(t)\}\big| = \mathrm{size}(t) = n + 1$ and the fact that $w_n = \varepsilon$ we obtain that $n^n = \mathcal{T}^{|P(\mathrm{pos}(t))|}(I_0)(p(\varepsilon)) = [\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha))$ by the definition of the fixpoint semantics.

Assume, contrary to our claim, that there is a weakly non-circular and connected mwmd $M' = (P', R', q)$ over Σ and Δ such that $[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}} = [\![M]\!]^{\mathrm{fix}}_{\mathcal{A}}$ (note that here it does not matter which kind of semantics we compare due to Lemma 4.50 and the facts that M and M' are weakly non-circular and $\mathcal{A}$ is distributive); hence, for every $n \in \mathbb{N}$ we have $[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha)) = n^n$. We will derive a contradiction. First let us present a claim.

Claim A. There is a constant $k \in \mathbb{N}_+$ such that $[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha)) \leq (n+1)^k \cdot k^{n+1}$ for every $n \in \mathbb{N}$.

Claim A contradicts the assumption that $[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha)) = n^n$, for every $n \in \mathbb{N}$, because for $n > 4^{2k}$ we have

$$\begin{aligned}
&[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha)) \leq (n+1)^k \cdot k^{n+1} \\
&\leq 2^{(n+1)k} \cdot 2^{k(n+1)} && (\forall\, i, j \in \mathbb{N}_+ : i \leq 2^i;\ \text{thus, } i^j \leq (2^i)^j = 2^{ij}) \\
&\leq 2^{(n+n)k} \cdot 2^{k(n+n)} = 2^{2kn} \cdot 2^{2kn} = 4^{2kn} = (4^{2k})^n < n^n = [\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(\gamma^n(\alpha))\ .
\end{aligned}$$

We conclude that there is no weakly non-circular and connected mwmd M' over Σ and Δ such that $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}$.

It remains to prove Claim A. We put $k = \max\{|P'|, |R'|^{|P'|}\}$. Let $n \in \mathbb{N}$, $t = \gamma^n(\alpha)$, and $G' = \mathrm{G}^{\mathrm{dep}}_{M',t}$. We show that $[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(t) \leq (n+1)^k \cdot k^{n+1}$. Since M' is weakly non-circular, the set $\mathrm{H}^{q(\varepsilon)}_{G'}$ is finite. Let $C = \{c \in P'(\mathrm{pos}(t)) \mid c \prec^*_{G'} q(\varepsilon)\}$ and $\sqsubset\, = \,\prec_{G'} \cap\, (C \times C)$. Then $\sqsubset^+$ is irreflexive because of Lemma 2.26. Together with the fact that C is finite this implies that $\sqsubset$ is well-founded on C. For every $c \in C$ we define the number $l_c \in \mathbb{N}$ and the number $j_c \in \mathbb{N}$ by well-founded recursion on $\sqsubset$ as follows:

$$\begin{aligned}
l_c &= 1 + \max\{l_d \mid d \in C, d \sqsubset c\}\ , \\
j_c &= \begin{cases} 1 + \max\{j_d \mid d \in C, d \sqsubset c\}\ , & \text{if } c \in P'^{(0)}(\mathrm{pos}(t)), \\ \max\{j_d \mid d \in C, d \sqsubset c\}\ , & \text{otherwise.} \end{cases}
\end{aligned}$$

Let us state another claim.

Claim B. For every $c \in C$ and $m \in \mathbb{N}$ with $m \geq l_c$ we have that $\mathcal{T}^m_{M'}(I_0)(c) \leq (n+1)^{j_c} \cdot |R'|^{l_c}$.

It is easy to show by well-founded induction on $\sqsubset$ that for every $c \in C$ we have $l_c \leq |P'(\mathrm{pos}(t))| \leq |P'| \cdot |\mathrm{pos}(t)| = |P'| \cdot (n+1)$ and $j_c \leq |P'^{(0)}(\mathrm{pos}(t))| = |P'^{(0)}| \leq |P'|$. Then Claim B together with the fact that $l_{q(\varepsilon)} \leq |P'(\mathrm{pos}(t))|$ implies for $c = q(\varepsilon)$ that

$$\begin{aligned}
&[\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}(t) = \mathcal{T}^{|P'(\mathrm{pos}(t))|}_{M'}(I_0)(q(\varepsilon)) \leq (n+1)^{|P'|} \cdot |R'|^{|P'|\cdot(n+1)} \\
&= (n+1)^{|P'|} \cdot (|R'|^{|P'|})^{n+1} \\
&\leq (n+1)^k \cdot k^{n+1}\ . && (\text{because } k = \max\{|P'|, |R'|^{|P'|}\})
\end{aligned}$$

This proves Claim A. It remains to show Claim B. We give a proof by well-founded induction on $\sqsubset$.

Let $c \in C$ and assume that for every $d \in C$ with $d \sqsubset c$ and every $m' \in \mathbb{N}$ with $m' \geq l_d$ we have $\mathcal{T}_{M'}^{m'}(I_0)(d) \leq (n+1)^{j_d} \cdot |R'|^{l_d}$. Let $m \in \mathbb{N}$ with $m \geq l_c$. We show that $\mathcal{T}_{M'}^{m}(I_0)(c) \leq (n+1)^{j_c} \cdot |R'|^{l_c}$.

This is trivial if $R' = \emptyset$. Now assume that $|R'| \geq 1$. Clearly, $m \geq l_c \geq 1$; we put $m' = m - 1 \in \mathbb{N}$. Let us state yet another claim.

Claim C. For every $(r, \rho) \in \Phi_{M',t,c}$ we have

$$\mathrm{h}_{\mathcal{T}_{M'}^{m'}(I_0)}(\rho(r_\mathrm{b})) \leq \begin{cases} (n+1)^{j_c} \cdot |R'|^{l_c - 1}\,, & \text{if } c \in P'^{(1)}(\mathrm{pos}(t)), \\ (n+1)^{j_c - 1} \cdot |R'|^{l_c - 1}\,, & \text{otherwise.} \end{cases}$$

Claim C together with Lemma 5.29(3) and the fact that $\mathrm{pos}(t) = n + 1$ implies that

$$\mathcal{T}_{M'}^{m}(I_0)(c) = \sum\nolimits_{(r,\rho) \in \Phi_{M',t,c}} \mathrm{h}_{\mathcal{T}_{M'}^{m'}(I_0)}(\rho(r_\mathrm{b})) \leq (n+1)^{j_c} \cdot |R'|^{l_c}\,.$$

This proves Claim B. It remains to show Claim C.

Let $(r, \rho) \in \Phi_{M',t,c}$. Since $\Delta = \{\alpha\}$ we obtain that either (i) $\rho(r_\mathrm{b}) = \alpha$ or (ii) $\rho(r_\mathrm{b}) \in P'(\mathrm{pos}(t))$. If Case (i) holds, then $\mathrm{h}_{\mathcal{T}_{M'}^{m'}(I_0)}(\rho(r_\mathrm{b})) = 1$ and, thus, Claim C holds because we assumed that $|R'| \geq 1$. Now let us consider Case (ii). Let $d = \rho(r_\mathrm{b}) \in P'(\mathrm{pos}(t))$. Then Lemma 2.20(3 $\Rightarrow$ 1) implies that (ii.i) $\mathrm{H}_{G'}^{d} = \emptyset$ or (ii.ii) $d \prec_{G'} c$. In Case (ii.i) Lemma 4.17(2) yields $\mathrm{h}_{\mathcal{T}_{M'}^{m'}(I_0)}(\rho(r_\mathrm{b})) = 0$ and therefore Claim C holds trivially.

It remains to analyze Case (ii.ii). It is easy to see that $d \sqsubset c$ and, thus, $l_d < l_c$, which implies $m' \geq l_d$. We obtain that $\mathrm{h}_{\mathcal{T}_{M'}^{m'}(I_0)}(\rho(r_\mathrm{b})) = \mathcal{T}_{M'}^{m'}(I_0)(d) \leq (n+1)^{j_d} \cdot |R'|^{l_d}$ by the induction hypothesis of the inductive proof of Claim B. Clearly, the fact $l_d < l_c$ implies $l_d \leq l_c - 1$. Moreover, if $c \in P'^{(1)}(\mathrm{pos}(t))$, then $j_d \leq j_c$ and if $c \in P'^{(0)}(\mathrm{pos}(t))$, then $j_d \leq j_c - 1$. We conclude that Claim C holds. ■

Remark 5.31. Recall that in Lemma 4.42 we determined an upper bound on the number of derivations of weakly non-circular mwmd and showed that this bound is tight; more precisely, for every $n \in \mathbb{N}$ we constructed a weakly non-circular mwmd M_n over Σ and Δ such that, for every input tree t, the number of derivations of M_n on the input tree t is $2^{(2^{n \cdot \mathrm{size}(t)})}$. Observe that the mwmd M_n we constructed in Lemma 4.42 is connected (it is even local). Does such a weakly non-circular and connected mwmd M_n exist for every signature Δ (recall that in Lemma 4.42 we required that Δ is not monadic)?

The answer is no! Let Δ and $\mathcal{A} = (A, +, 0, \theta)$ be as in the proof of Lemma 5.30. We have shown that for every weakly non-circular and connected mwmd $M' = (P', R', q)$ over Σ and Δ there is a $k \in \mathbb{N}_+$ such that $[\![M']\!]_{\mathcal{A}}^{\mathrm{fix}}(\gamma^m(\alpha)) \leq (m+1)^k \cdot k^{m+1}$ for every $m \in \mathbb{N}$. Since $\mathcal{A}$ is distributive, Lemma 4.50 yields that $[\![M']\!]_{\mathcal{A}}^{\mathrm{hyp}}(\gamma^m(\alpha)) \leq (m+1)^k \cdot k^{m+1}$ for every $m \in \mathbb{N}$. Let $m \in \mathbb{N}$, $t = \gamma^m(\alpha)$, and $G' = \mathrm{G}_{M',t}^{\mathrm{dep}}$. Since $\Delta = \{\alpha\}$, we have for every $\eta \in \mathrm{H}_{G'}^{q(\varepsilon)}$ that $\mathrm{h}_{M',t}(\eta) = \alpha$ and, hence $h(\mathrm{h}_{M',t}(\eta)) = 1$, where h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). Thus, the fact $[\![M']\!]_{\mathcal{A}}^{\mathrm{hyp}}(\gamma^m(\alpha)) \leq (m+1)^k \cdot k^{m+1}$ implies $|\mathrm{H}_{G'}^{q(\varepsilon)}| \leq (m+1)^k \cdot k^{m+1}$. It is easy to check that for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $(m+1)^k \cdot k^{m+1} < 2^{(2^{n \cdot (m+1)})} = 2^{(2^{n \cdot \mathrm{size}(\gamma^n(\alpha))})}$. Hence, there is no M_n with the required properties. □

Proof of Statement C2

Now we show that for every restricted mwmd there is a connected mwmd such that their hyperpath and fixpoint semantics coincide for a certain subclass of m-monoids. In the following, we motivate informally the constructions that are involved in this result. To this end, consider the rule

$$r = p(x) \leftarrow \delta(q(x), p(y)) \; ; \{\mathrm{label}_\gamma(z)\} \enspace .$$

Let $t \in T_\Sigma$. We can make two observations: First, the variable z is not connected (i.e., related by $\sim_r$) to any variable occurring in r_h or r_b. Thus, if t contains a node labeled γ, we can omit the guard, and otherwise, we can omit the whole rule, each time preserving semantics for t. This idea can be used for a construction which does not depend on t, but suffice it to say that this involves duplication of all the remaining rules. Therefore, in order to have a terminating procedure, our construction deals with all rules at once. Second, the variable y, while trivially connected to some variable in the body, is not connected to the variable in the head. In this case, we may replace $p(y)$ by $p'()$, where p' is a new nullary predicate, adding a rule $p'() \leftarrow p(y) \; ; \emptyset$.

This example suggests that there are two different ways of a rule not to be connected (first that a non-connected variable occurs only in the guard and second that a non-connected variable occurs in the head or body) and that both require different constructions. We will refer to a rule that is either connected or not connected only in the second sense as semiconnected.

Definition 5.32. Let $r \in R$. We define the set of ***independent guards*** *of* r as follows: $I(r) = \{b \in r_\mathrm{G} \mid \forall x \in \mathrm{var}(r) : \forall y \in \mathrm{var}(r_\mathrm{h}) \cup \mathrm{var}(r_\mathrm{b}) : x \not\sim_r y\}$ and for every $R' \subseteq R$ the set $I(R') = \bigcup_{r \in R'} I(r)$. We call M ***semiconnected*** iff r is connected or $I(r) = \emptyset$ for every $r \in R$. □

In order to simplify matters we split the construction of an "equivalent" connected mwmd from a given restricted mwmd M into two phases: (i) first we construct an "equivalent" semiconnected mwmd M_sc and (ii) then we construct a hyp-equivalent connected mwmd M_c from the semiconnected mwmd M_sc that we constructed in Phase (i) (we put the word equivalent into quotes because the semiconnected mwmd of Phase (i) is neither hyp- nor completely equivalent to M; in fact, M and M_sc will only exhibit equivalent behavior for idempotent and distributive m-monoids).

We postpone the construction of the "equivalent" semiconnected mwmd from a given restricted mwmd (i.e., Phase (i)) to Chapter 8 (see Corollary 8.18) where we will develop tools that allow us to give a simple correctness proof of the construction. In the current section we will instead deal with the problem to construct a hyp-equivalent connected mwmd from a given semiconnected one (i.e., Phase (ii)).

Lemma 5.33 (cf. [28, Lemma 3]). *Let M be a restricted and semiconnected mwmd over Σ and Δ. Then there is a connected mwmd M' over Σ and Δ such that M and M' are hyp-equivalent.*

Proof. Let $M = (P, R, q)$ and for every $r \in R$ let $n(r) = 0$ if $\mathrm{var}(r) = \emptyset$ and $n(r) = |\mathrm{var}(r)/{\sim_r}| - 1$ otherwise; clearly, r is connected iff $n(r) = 0$. Let $n(M) = \sum_{r \in R} n(r)$; then M is connected iff $n(M) = 0$. If M is already connected, then we put $M' = M$;

the assertion follows trivially. For the remainder of the proof assume that M is not yet connected, i.e., $n(M) > 0$.

In this proof we do not give a direct construction of the connected mwmd M', instead we construct a semiconnected mwmd $M_1 = (P_1, R_1, q_1)$ such that $n(M_1) = n(M) - 1$, M and M_1 are hyp-equivalent, and M_1 is restricted if M is so. Then it is obvious that we can perform this construction a finite number of times and generate a sequence of pairwise hyp-equivalent mwmd $M_1, M_2, \ldots, M_{n(M)} = M'$ in order to construct M'.

Since $n(M) > 0$, there is an $r \in R$ such that $|\mathrm{var}(r)/{\sim_r}| > 1$. Choose an equivalence class $C \in \mathrm{var}(r)/{\sim_r}$ such that $C \cap \mathrm{var}(r_\mathrm{h}) = \emptyset$. Observe that the fact that M is semiconnected implies $C \cap \mathrm{var}(r_\mathrm{b}) \neq \emptyset$. Since M is restricted we obtain that for every $w' \in \mathrm{pos}(r_\mathrm{b})$ with $r_\mathrm{b}(w) \in P(\mathrm{var}(r))$, we have that (i) w is a prefix of w' iff (ii) $r_\mathrm{b}(w') \in P(C)$.

Let $G = \{a \in r_\mathrm{G} \mid \mathrm{var}(a) \subseteq C\}$. We define $M_1 = (P', R', q)$ where $P' = P \cup \{p^{(0)}\}$ and $R' = (R \setminus \{r\}) \cup \{r_1, r_2\}$, where

$$r_1 = r_\mathrm{h} \leftarrow r_\mathrm{b}[p()]_w \,;\, r_\mathrm{G} \setminus G \;,$$
$$r_2 = p() \leftarrow r_\mathrm{b}|_w \,;\, G \;.$$

Clearly, this construction preserves semiconnectedness and restrictedness. It remains to show that M and M_1 are hyp-equivalent. In view of Lemma 5.18 it suffices to show that M and M_1 are related via some family $\nu = (\nu_t \mid t \in T_\Sigma)$ and $\pi = \big(\pi_{t,c} \mid t \in T_\Sigma, c \in P(\mathrm{pos}(t))\big)$. Let $t \in T_\Sigma$. For every $c \in P(\mathrm{pos}(t))$ we let $\nu_t(c) = \nu_t(c)$. Let $c \in P(\mathrm{pos}(t))$ and $(r', \rho) \in \Phi_{M,t,c}$. If $r' \neq r$, then we let $\pi_{t,c}(r', \rho) = (r', \rho)(c_1, \ldots, c_k)$, where $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in P(\mathrm{pos}(t))$ such that $c_1 \cdots c_k = \mathrm{indyield}(\rho(r_\mathrm{b}))$. Now assume that $r' = r$. Then there are $k, l, j \in \mathbb{N}$ and atom instances $c_1, \ldots, c_k, d_1, \ldots, d_l, e_1, \ldots, e_j \in P(\mathrm{pos}(t))$, such that $\mathrm{indyield}\big(\rho(r_\mathrm{b}[p()]_w)\big) = c_1 \cdots c_k p() d_1 \cdots d_l$ and $\mathrm{indyield}\big(\rho(r_\mathrm{b}|_w)\big) = e_1 \cdots e_j$; clearly, then we have that $\mathrm{indyield}(\rho(r_\mathrm{b})) = c_1 \cdots c_k e_1 \cdots e_j d_1 \cdots d_l$. We let $\pi_{t,c}(r', \rho) = (r_1, \rho|_{\mathrm{var}(r) \setminus C})(c_1, \ldots, c_k, (r_2, \rho|_C)(e_1, \ldots, e_j), d_1, \ldots, d_l)$.

It is easy to check that M and M_1 are related via ν and π. ∎

The following corollary follows from Corollary 8.18 and Lemma 5.33.

Corollary 5.34. *Let $\mathcal{A}$ be an idempotent dm-monoid over Δ, $(\mathcal{A}, \sum)$ be an ω-idempotent, ω-distributive ω-complete m-monoid, and $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid. Then*

$$\mathrm{r\text{–}WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{c\text{–}WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) \;,$$
$$\mathrm{r\text{–}WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) = \mathrm{c\text{–}WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) \;,$$
$$\mathrm{r\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{c\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) \;,$$
$$\mathrm{r\text{–}WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big) = \mathrm{c\text{–}WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big) \;.$$

Proof. Since every connected mwmd is also restricted, the right-hand sides of all four equations are trivially contained in their respective left-hand sides. We show that the left-hand sides are contained in their respective right-hand sides, too.

The first two equations follows from Corollary 8.18 and Lemma 5.33. The third equation follows from the first one together with Theorem 4.53. The last equation follows from the second one, Theorem 4.53, and the following two facts: (i) there is an ω-complete m-monoid $(\mathcal{A}, \sum')$ that is related to $(\mathcal{A}, \leq)$ (see Lemma 3.40) and (ii) $(\mathcal{A}, \sum')$ is ω-idempotent and ω-distributive due to Lemma 3.39(3). ∎

Proof of Statement C3

Now we show that there is an mwmd such that there is no hyp-equivalent restricted mwmd.

Lemma 5.35. *There is a ranked alphabet Σ, a signature Δ, and an mwmd M over Σ and Δ such that there is no restricted mwmd M' over Σ and Δ that is hyp-equivalent to M.*

Proof. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Delta = \Sigma \cup \{\delta^{(4)}\}$. Moreover, let $M = (P, R, q)$ be an mwmd over Σ and Δ such that $P = \{q^{(1)}, p^{(1)}\}$ and $R = \{r, r_\gamma, r_\alpha\}$ with

$$\begin{aligned} r &= q(x) \leftarrow \delta\big(p(y), p(z), p(y), p(z)\big) \,;\, \emptyset \,, \\ r_\gamma &= p(x) \leftarrow \gamma(p(y)) \,;\, \{\mathrm{label}_\gamma(x), \mathrm{child}_1(x,y)\} \,, \\ r_\alpha &= p(x) \leftarrow \alpha \,;\, \{\mathrm{label}_\alpha(x)\} \,. \end{aligned}$$

Observe that M is not restricted because the rule r is not restricted. Moreover, M is weakly non-circular (it is even non-circular). Assume, contrary to our claim, that there is a restricted mwmd M' over Σ and Δ that is hyp-equivalent to M. We will derive a contradiction.

Since M and M' are hyp-equivalent, the mwmd M' is weakly non-circular and for every m-monoid $\mathcal{A}$ over Δ we have that $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}}$. This holds in particular for the m-monoid $\mathcal{A}$ that is defined by letting $\mathcal{A} = (\mathcal{P}(T_\Delta), \cup, \emptyset, \theta)$ where for every $\delta' \in \Delta$, $\theta(\delta')$ is δ'-language top concatenation. Note that $\mathcal{A}$ is idempotent and distributive; thus, Corollary 5.34 yields that there is a connected mwmd $M_0 = (P_0, R_0, q_0)$ over Σ and Δ such that $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M_0]\!]^{\mathrm{hyp}}_{\mathcal{A}}$. It is easy to check that for every $t \in T_\Sigma$ we obtain

$$[\![M_0]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) = [\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) = \big\{\delta\big(t|_w, t|_v, t|_w, t|_v\big) \mid w, v \in \mathrm{pos}(t)\big\} \,. \tag{5.9}$$

Let $t \in T_\Sigma$ such that $|R_0| < |\mathrm{pos}(t)|$; such a tree exists. Moreover, let $G = \mathrm{G}^{\mathrm{dep}}_{M_0,t}$ and $w, v \in \mathrm{pos}(t)$. Due to the definition of the hypergraph semantics and Equation (5.9) we obtain that there is a derivation $\eta_{w,v} \in \mathrm{H}^{q_0(\varepsilon)}_G$ such that $\delta(t|_w, t|_v, t|_w, t|_v) \in h(\mathrm{h}_{M_0,t}(\eta_{w,v}))$, where h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). Observe that $h(s) = \{s\}$ for every $s \in T_\Delta$; thus, $\mathrm{h}_{M_0,t}(\eta_{w,v}) = \delta(t|_w, t|_v, t|_w, t|_v)$. This implies that there is a position $u_{w,v} \in \mathrm{pos}(\eta_{w,v})$ such that δ occurs in the tree $\rho_{w,v}((r_{w,v})_{\mathrm{b}})$, where $(r_{w,v}, \rho_{w,v}) = \eta_{w,v}(u_{w,v})$.

Note that $\rho_{w,v}((r_{w,v})_{\mathrm{b}})$ is of the form $\delta(s_1, s_2, s_3, s_4)$, where for every $i \in [4]$ either $s_i = \gamma^{n_i}(c_i)$ (with $c_i \in P_0(\mathrm{pos}(t))$) or $s_i = t|_w$ (for $i \in \{1,3\}$) or $s_i = t|_v$ (for $i \in \{2,4\}$); it is easy to check that if $s_i = \gamma^{n_i}(c_i)$, then for every $\eta \in \mathrm{H}^{c_i}_G$ we have that $\eta_{w,v}[\eta]_{u_{w,v}\cdot i} \in \mathrm{H}^{q_0(\varepsilon)}_G$ and, hence, $\mathrm{h}_{M_0,t}\big(\eta_{w,v}[\eta]_{u_{w,v}\cdot i}\big) = \delta(t|_{w'}, t|_{v'}, t|_{w'}, t|_{v'})$ for some $w', v' \in \mathrm{pos}(t)$.

For $i = 1$ we have $\mathrm{h}_{M_0,t}\big(\eta_{w,v}[\eta]_{u_{w,v}\cdot i}\big) = \delta\big(\gamma^{n_i}(\mathrm{h}_{M_0,t}(\eta)), t|_v, t|_w, t|_v\big)$. Thus, we have $\gamma^{n_i}(\mathrm{h}_{M_0,t}(\eta)) = t|_{w'} = t|_w$. Likewise, we obtain for $i = 3$ that $\gamma^{n_i}(\mathrm{h}_{M_0,t}(\eta)) = t|_w$ and for $i = 2$ or $i = 4$ that $\gamma^{n_i}(\mathrm{h}_{M_0,t}(\eta)) = t|_v$.

We conclude that for every derivation η' with $\eta'(\varepsilon) = (r_{w,v}, \rho_{w,v})$ we have $\mathrm{h}_{M_0,t}(\eta) = \delta(t|_w, t|_v, t|_w, t|_v)$. Thus, for every w', v' with $(w, v) \neq (w', v')$ we have that $(r_{w,v}, \rho_{w,v}) \neq (r_{w',v'}, \rho_{w',v'})$. Hence, $|\mathrm{pos}(t)|^2 \leq |\Phi_{M_0,t}|$. By Lemma 5.29(2) and the assumption $|R_0| < |\mathrm{pos}(t)|$ we obtain that $|\Phi_{M_0,t}| \leq |R_0| \cdot |\mathrm{pos}(t)| < |\mathrm{pos}(t)|^2$, a contradiction. ■

5.5 Local

In this section we show that for every connected and proper mwmd there is a completely equivalent local mwmd. First we give a brief explanation of our construction. Let r be the rule $q(\mathrm{x}_\varepsilon) \leftarrow p(\mathrm{x}_{21})\ ;\{\mathrm{child}_2(\mathrm{x}_\varepsilon,\mathrm{x}_2),\mathrm{child}_1(\mathrm{x}_2,\mathrm{x}_{21}),\mathrm{label}_\sigma(\mathrm{x}_2)\}$. In order to make r local we split it into local components while introducing an auxiliary predicate p'; hence, we obtain two rules

$$\begin{aligned} q(\mathrm{x}_\varepsilon) &\leftarrow p'(\mathrm{x}_2)\ ;\{\mathrm{child}_2(\mathrm{x}_\varepsilon,\mathrm{x}_2)\}\ , \\ p'(\mathrm{x}_\varepsilon) &\leftarrow p(\mathrm{x}_1)\ ;\{\mathrm{child}_1(\mathrm{x}_\varepsilon,\mathrm{x}_1),\mathrm{label}_\sigma(\mathrm{x}_\varepsilon)\}\ . \end{aligned}$$

Special care has to be taken if both the head and the body of the given rule belong entirely to one of the local components: consider the rule r' which originates from r by replacing the variable x_{21} in the body by x_ε. In this case we have to make a detour and construct three rules

$$\begin{aligned} q(\mathrm{x}_\varepsilon) &\leftarrow p'(\mathrm{x}_2)\ ;\{\mathrm{child}_2(\mathrm{x}_\varepsilon,\mathrm{x}_2)\}\ , \\ p'(\mathrm{x}_\varepsilon) &\leftarrow p''(\mathrm{x}_\varepsilon)\ ;\{\mathrm{child}_1(\mathrm{x}_\varepsilon,\mathrm{x}_1),\mathrm{label}_\sigma(\mathrm{x}_\varepsilon)\}\ , \\ p''(\mathrm{x}_2) &\leftarrow p(\mathrm{x}_\varepsilon)\ ;\{\mathrm{child}_2(\mathrm{x}_\varepsilon,\mathrm{x}_2)\}\ . \end{aligned}$$

Before we carry on with the construction of a local mwmd from a given connected and proper one, let us first make an observation concerning connected mwmd which makes it possible to simplify further considerations significantly. Consider the rule

$$r = p(x) \leftarrow \delta(q(x),p(y))\ ;\{\mathrm{child}_1(z,x),\mathrm{child}_2(z,y)\}\ .$$

For every $t \in T_\Sigma$ and valid r,t-variable assignment ρ, we obtain $\rho(x) = \rho(z)1$ and $\rho(y) = \rho(z)2$. Hence, we may reflect this fact in syntax by rephrasing r to

$$p(\mathrm{x}_1) \leftarrow \delta(q(\mathrm{x}_1),p(\mathrm{x}_2))\ ;\{\mathrm{child}_1(\mathrm{x}_\varepsilon,\mathrm{x}_1),\mathrm{child}_2(\mathrm{x}_\varepsilon,\mathrm{x}_2)\}\ .$$

Now we formalize this concept.

Definition 5.36. Let $r \in R$ be connected. A mapping $f : \mathrm{var}(r) \to \mathbb{N}^*$ is called r-***position mapping*** if

(i) $\mathrm{var}(r) = \emptyset$ or $\varepsilon \in \mathrm{ran}(f)$,

(ii) for every $x,y \in \mathrm{var}(r)$ and $i \in [\mathrm{maxrk}(\Sigma)]$ such that $\mathrm{child}_i(x,y) \in r_\mathrm{G}$ we have that $f(x)\cdot i = f(y)$. □

Lemma 5.37. *Let $r \in R$ be connected.*

1. *If there is no r-position mapping, then the set $\{e \in \Phi_{M,t} \mid \mathrm{pr}_1(e) = r\}$ is empty for every $t \in T_\Sigma$.*

2. *If f and f' are r-position mappings, then $f = f'$ and for every $t \in T_\Sigma$ and $(r,\rho) \in \Phi_{M,t}$ there is a $w \in \mathrm{pos}(t)$ such that $w \cdot f(x) = \rho(x)$ for every $x \in \mathrm{var}(r)$.*

PROOF. 1. Assume that there a $t \in T_\Sigma$ and $e \in \Phi_{M,t}$ such that $\mathrm{pr}_1(e) = r$. We show that there is an r-position mapping $f : \mathrm{var}(r) \to \mathbb{N}^*$. This is trivial if $\mathrm{var}(r) = \emptyset$. For the remainder of the proof of Statement 1 we assume that $\mathrm{var}(r) \neq \emptyset$. Let $\rho = \mathrm{pr}_2(e)$ and let w be the longest common prefix of $\mathrm{ran}(\rho)$. For every $x \in \mathrm{var}(r)$ let $f(x) = w_x$, where w_x is the unique string in $\mathbb{N}^*$ such that $w \cdot w_x = \rho(x)$.

We show that f is an r-position mapping. Condition (ii) follows from the fact that for every $x, y \in \mathrm{var}(r)$ and $i \in [\mathrm{maxrk}(\Sigma)]$ with $\mathrm{child}_i(x, y) \in r_{\mathrm{G}}$ we have that $\rho(x) \cdot i = \rho(y)$. It remains to prove Condition (i), i.e., that $\varepsilon \in \mathrm{ran}(f)$. It suffices to show that there is an $x \in \mathrm{var}(r)$ with $\rho(x) = w$. Clearly, there are $x, y \in \mathrm{var}(r)$ such that w is the longest common prefix of $\rho(x)$ and $\rho(y)$. If $\rho(x)$ is a prefix of $\rho(y)$ or the other way around, then $w \in \{\rho(x), \rho(y)\}$. Now assume that neither $\rho(x)$ is a prefix of $\rho(y)$ nor $\rho(y)$ a prefix of $\rho(x)$. Then there are $i_1, i_2 \in [\mathrm{maxrk}(\Sigma)]$ with $i_1 \neq i_2$ such that wi_1 is a prefix of $\rho(x)$ and wi_2 is a prefix of $\rho(y)$. Let $X = \{z \in \mathrm{var}(r) \mid wi_1 \text{ is a prefix of } \rho(z)\}$. Then $x \in X$ and $y \notin X$. By Lemma 5.28 there are $z \in X$ and $z' \in \mathrm{var}(r) \setminus X$ such that $\mathrm{child}_i(z, z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$; it is easy to see that this implies that $\mathrm{child}_{i_1}(z', z) \in r_{\mathrm{G}}$ and $\rho(z') = w$.

2. This statement is trivial if $\mathrm{var}(r) = \emptyset$. For the remainder of the proof we assume that $\mathrm{var}(r) \neq \emptyset$. Let f and f' be r-position mappings. First we show that $f = f'$. By the definition of r-position mappings there is an $x_0 \in \mathrm{var}(r)$ such that $f(x_0) = \varepsilon$. Let $w = f'(x_0)$.

Assume that there is a $y \in \mathrm{var}(r)$ such that $w \cdot f(y) \neq f'(y)$. We define the set $X = \{z \in \mathrm{var}(r) \mid w \cdot f(z) = f'(z)\}$. Clearly, $x_0 \in X$ and $y \notin X$. By Lemma 5.28 there are $z \in X$ and $z' \in \mathrm{var}(r) \setminus X$ such that $\mathrm{child}_i(z, z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$. If $\mathrm{child}_i(z, z') \in r_{\mathrm{G}}$, then the fact that $z \in X$ and that f and f' are r-position mappings implies $w \cdot f(z') = w \cdot f(z) \cdot i = f'(z) \cdot i = f'(z')$; hence, $z' \in X$, a contradiction. Likewise, the case $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ yields a contradiction. We conclude that $w \cdot f(y) = f'(y)$ holds for every $y \in \mathrm{var}(r)$. Together with the fact that $\varepsilon \in \mathrm{ran}(f')$ we obtain that $w = \varepsilon$. Hence, $f = f'$.

With similar arguments one can show that for every $t \in T_\Sigma$ and $(r, \rho) \in \Phi_{M,t}$ there is a $w \in \mathrm{pos}(t)$ such that $w \cdot f(x) = \rho(x)$ for every $x \in \mathrm{var}(r)$. ∎

Remark 5.38. Note that it is decidable whether a connected $r \in R$ admits an r-position mapping f and that f can be constructed effectively for the following reasons. It is easy to check that for every r-position mapping f we have that $\mathrm{ran}(f) \subseteq W$, where $W = \{w \in [\mathrm{maxrk}(\Sigma)]^* \mid |w| < |\mathrm{var}(r)|\}$. The set W is finite and, hence, there are only finitely many mappings from $\mathrm{var}(r)$ to W. Thus, it suffices to enumerate all mappings from $\mathrm{var}(r)$ to W and to check whether such a mapping is an r-position mapping in order to decide whether an r-position mapping exists and in order to construct one effectively. □

The description of the construction of a local mwmd M' from a given proper and connected mwmd M is much simpler if every rule of M admits an injective r-position mapping. Fortunately, we can assume that this is the case.

Lemma 5.39. *Let M be a connected and proper mwmd over Σ and Δ. Then there is a connected and proper mwmd M' over Σ and Δ such that M and M' are completely equivalent and every rule r of M' admits an injective r-position mapping.*

PROOF. Let $M = (P, R, q)$. We let $M' = (P, R', q)$ where

$$R' = \{\bar{r} \mid r \in R, \text{ there is an } r\text{-position mapping}\} ,$$

and for every $r \in R$ that admits an r-position mapping f the rule $\bar{r}$ is constructed as follows. The mapping f is uniquely determined due to Lemma 5.37(2). We define the equivalence relation $\sim$ on $\mathrm{var}(r)$ as follows for every $x, y \in \mathrm{var}(r)$: $x \sim y$ iff $f(x) = f(y)$. For every equivalence class C modulo $\sim$ choose a representative $x_C \in C$. Then $\bar{r}$ originates from r by replacing for every $y \in \mathrm{var}(r)$ every occurrence of y in r by $x_{[y]_\sim}$. It is easy to check that $\bar{r}$ admits an $\bar{r}$-position mapping and that this $\bar{r}$-position mapping is injective.

Clearly, M' is connected and proper. It remains to show that M and M' are completely equivalent. By Lemma 5.37(1), for every $r \in R \setminus R'$, the set $\{e \in \Phi_{M,t} \mid \mathrm{pr}_1(e) = r\}$ is empty for every $t \in T_\Sigma$. It is easy to check that by means of Lemma 5.37(2) we obtain that for every $t \in T_\Sigma$ and m-monoid $\mathcal{A}$ over Δ we have that the immediate consequence operators $\mathcal{T}_{M,t,\mathcal{A}}$ and $\mathcal{T}_{M',t,\mathcal{A}}$ coincide and that $\mathrm{H}_G^{q(\varepsilon)} = \mathrm{H}_{G'}^{q(\varepsilon)}$, where $G = \mathrm{G}_{M,t}^{\mathrm{dep}}$ and $G' = \mathrm{G}_{M',t}^{\mathrm{dep}}$. It is easy to see that this implies that M and M' are completely equivalent. ■

Now we are prepared to present the main lemma of this section.

Lemma 5.40 (cf. [28, Lemma 6]). *Let M be a connected and proper mwmd over Σ and Δ. Then there is a local mwmd M_{loc} over Σ and Δ such that M and M' are completely equivalent.*

Proof. Let $M = (P, R, q)$. In view of Lemma 5.39 we can assume that every rule $r \in R$ admits an injective r-position mapping; by Lemma 5.37(2) this r-position mapping is unique; we will denote this unique mapping by f_r in this proof.

First let us introduce an auxiliary notion. For every $b \in \mathrm{sp}_\Sigma(\mathrm{V})$ we define $\mathrm{var}_1(b) \in \mathrm{var}(b)$ as follows: (i) if $\mathrm{var}(b) = \{x\}$ for some $x \in \mathrm{V}$, then $\mathrm{var}_1(b) = x$, and (ii) if $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$ and $x, y \in \mathrm{V}$, then $\mathrm{var}_1(b) = x$. Let us state a fact.

Fact A. Let $r \in R$. Then the following two statements are equivalent: (i) r is local iff (ii) $\sum_{b \in r_\mathrm{G}} |f_r(\mathrm{var}_1(b))| = 0$.

Proof of Fact A. "$\Rightarrow$": By Statement (i) there is an $x \in \mathrm{var}(r)$ such that for every $b \in r_\mathrm{G}$ and $y \in \mathrm{var}(b) \setminus \{x\}$ we have that $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$. Hence, for every $b \in r_\mathrm{G}$, $\mathrm{var}_1(b) = x$. Therefore it suffices to show that $f_r(x) = \varepsilon$. Let $y \in \mathrm{var}(r) \setminus x$. Since r is connected, there is a $b \in r_\mathrm{G}$ such that $y \in \mathrm{var}(b)$. Then $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$; hence, $f_r(y) = f_r(x) \cdot i$. We obtain that $f_r(x) = \varepsilon$ because $\varepsilon \in \mathrm{ran}(f_r)$.

"$\Leftarrow$": If $\mathrm{var}(r) = \emptyset$, then r is local for trivial reasons. Now assume that $\mathrm{var}(r) \neq \emptyset$. Hence, there is an $x \in \mathrm{var}(r)$ such that $f_r(x) = \varepsilon$. We show that for every $b \in r_\mathrm{G}$ and $y \in \mathrm{var}(b) \setminus \{x\}$ we have that $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$. Let $b \in r_\mathrm{G}$ and $y \in \mathrm{var}(b) \setminus \{x\}$. By Statement (ii), $|f_r(\mathrm{var}_1(b))| = 0$; hence, $f_r(\mathrm{var}_1(b)) = \varepsilon$. Since f_r is injective, $\mathrm{var}_1(b) = x$. This together with the fact that $y \in \mathrm{var}(b) \setminus \{x\}$ implies that $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$.

Continuation of the main proof. For every $r \in R$ we abbreviate $\sum_{b \in r_\mathrm{G}} |f_r(\mathrm{var}_1(b))|$ by n_r. We put $n(M) = \sum_{r \in R} (2^{n_r} - 1)$. Clearly, M is local iff $n(M) = 0$ due to Fact A. If M is already local, then we put $M_{\mathrm{loc}} = M$; the assertion of the lemma follows trivially. For the remainder of the proof assume that M is not yet local, i.e., $n(M) > 0$.

In this proof we do not give a direct construction of the local mwmd M_{loc}, instead we construct a connected and proper mwmd $M_1 = (P_1, R_1, q)$ such that $n(M_1) < n(M)$, M and M_1 are completely equivalent, and every rule of M_1 admits an injective position mapping. Then it is obvious that we can perform this construction a finite number of times and generate a sequence of pairwise completely equivalent mwmd $M_1, M_2, \ldots, M_n = M_{\mathrm{loc}}$, for some $n \in \mathbb{N}_+$, in order to construct M_{loc}.

Since M is not local, there is a rule $r \in R$ that is not local. Select an $x \in \mathrm{var}(r)$ such that $x \in \{\mathrm{var}_1(b) \mid b \in r_{\mathrm{G}}\}$ and $|f_r(x)|$ is maximal in $\{|f_r(\mathrm{var}_1(b))| \mid b \in r_{\mathrm{G}}\}$. Then $f_r(x) \neq \varepsilon$ due to Fact A.

We let B be the set of all atoms $b \in \{r_{\mathrm{h}}\} \cup \mathrm{ind}(r_{\mathrm{b}})$ such that $f_r(x)$ is a proper prefix of $f_r(y)$, where y is the unique variable in $\mathrm{var}(b)$. Moreover, we let $G = \{b \in r_{\mathrm{G}} \mid \mathrm{var}_1(b) = x\}$. We define $M_1 = (P_1, R_1, q)$ by case distinction.

Case 1*:* $B = \emptyset$. Then we let $P_1 = P \cup \{p_1^{(1)}, p_2^{(1)}\}$ and $R_1 = (R \setminus \{r\}) \cup \{r_1, r_2, r_3\}$:

$$\begin{aligned} r_1 &= r_{\mathrm{h}} \leftarrow p_1(x) \,;\, r_{\mathrm{G}} \setminus G \,, \\ r_2 &= p_1(x) \leftarrow p_2(x) \,;\, G \,, \\ r_3 &= p_2(x) \leftarrow r_b \,;\, r_{\mathrm{G}} \setminus G \,. \end{aligned}$$

Now we show that M_1 has all required properties.

Proper. Clearly, M_1 is proper.

Connected. Now we show that M_1 is connected. It suffices to show that r_1, r_2, and r_3 are connected. First let us consider the rule r_1. Assume that there are $x, y \in \mathrm{var}(r_1)$ such that $x \not\sim_{r_1} y$.

First we show that $f_r(x)$ is not a proper prefix of $f_r(y)$. Assume, contrary to our claim, that $f_r(x)$ is a proper prefix of $f_r(y)$. Then $y \notin \mathrm{var}(r_{\mathrm{h}}) \cup \mathrm{var}(r_{\mathrm{b}})$ because $B = \emptyset$. Hence, $y \in \mathrm{var}(r_{\mathrm{G}} \setminus G)$; let $b \in r_{\mathrm{G}} \setminus G$ such that $y \in \mathrm{var}(b)$. Then $y \neq \mathrm{var}_1(b)$ because $|f_r(x)|$ is maximal in the set $\{|f_r(\mathrm{var}_1(b))| \mid b \in r_{\mathrm{G}}\}$. Then $b = \mathrm{child}_i(z, y)$ for some $z \in \mathrm{var}(r)$ and $i \in [\mathrm{maxrk}(\Sigma)]$. Since $|f_r(x)|$ is maximal in the set $\{|f_r(\mathrm{var}_1(b))| \mid b \in r_{\mathrm{G}}\}$, $f_r(x)$ is a proper prefix of $f_r(y)$, and f_r is injective, we obtain that $z = x$. But then the fact that $\mathrm{var}_1(b) = z = x$ implies that $b \in G$; this contradicts the statement $b \in r_{\mathrm{G}} \setminus G$. Therefore, the assumption that $f_r(x)$ is a proper prefix of $f_r(y)$ was wrong.

Consider the set $X = \{z \in \mathrm{var}(r) \mid x \sim_{r_1} z \vee f_r(x) \text{ is a proper prefix of } f_r(z)\}$. Clearly, $x \in X$ and $y \notin X$. By Lemma 5.28 (applied to the connected rule r, not r_1) there are $z \in X$ and $z' \in \mathrm{var}(r) \setminus X$ such that $\mathrm{child}_i(z, z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$.

Assume that $x \not\sim_{r_1} z$. Then $z \in X$ implies that $f_r(x)$ is a proper prefix of $f_r(z)$. Since $z' \in X$, $f_r(x)$ is not a proper prefix of $f_r(z')$. This implies that $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$ and that $x = z'$. But then $x \sim_{r_1} z' = x$, which contradicts the fact that $z' \notin X$. Hence, the assumption $x \not\sim_{r_1} z$ was wrong.

We have shown that $x \sim_{r_1} z$ and that $\mathrm{child}_i(z, z') \in r_{\mathrm{G}}$ or $\mathrm{child}_i(z', z) \in r_{\mathrm{G}}$ for some $i \in [\mathrm{maxrk}(\Sigma)]$; hence, $\mathrm{child}_i(z, z') \in (r_1)_{\mathrm{G}}$ or $\mathrm{child}_i(z', z) \in (r_1)_{\mathrm{G}}$ because $f_r(x)$ is not a proper prefix of $f_r(z')$ and $z' \neq x$, which follows from $z' \notin X$. This implies that $x \sim_{r_1} z'$, a contradiction to the fact that $z' \notin X$. Hence, the assumption that there are $x, y \in \mathrm{var}(r_1)$ with $x \not\sim_{r_1} y$ was wrong. We conclude that r_1 is connected. For similar reasons the rule r_3 is connected, too.

The rule r_2 is obviously connected, because $x \in \mathrm{var}(b)$ for every $b \in G$; hence, $x \sim_{r_2} y$ for every $y \in \mathrm{var}(r_2)$.

Existence of injective position mappings. It suffices to show that r_1, r_2, and r_3 admit injective position mappings. Observe that $f_r|_{\mathrm{var}(r_1)}$ and $f_r|_{\mathrm{var}(r_3)}$ are injective r_1- and r_3-position mappings, respectively. Moreover, there is a unique r_2-position mapping f_{r_2} such that $f_{r_2}(y) = f_r(x) \cdot f_r(y)$ for every $y \in \mathrm{var}(r_2)$; clearly, f_{r_2} is injective.

Proof that $n(M_1) < n(M)$. It suffices to show that $2^{n_{r_1}} - 1 + 2^{n_{r_2}} - 1 + 2^{n_{r_3}} - 1 < 2^{n_r} - 1$. Clearly, $n_{r_2} = 0$ because $f_{r_2}(x) = \varepsilon$. It is easy to check that $n_{r_1} < n_r$ and $n_{r_3} < n_r$ because $r_{\mathrm{G}} \setminus G$ is a proper subset of G (G is nonempty because $x \in \{\mathrm{var}_1(b) \mid b \in r_{\mathrm{G}}\}$) and because for every $b \in G$ we have $|f_r(\mathrm{var}_1(b))| = |f_r(x)| > 0$ (this follows from the fact that $f_r(x) \neq \varepsilon$). Hence, $2^{n_{r_1}} - 1 + 2^{n_{r_2}} - 1 + 2^{n_{r_3}} - 1 \leq 2^{n_r - 1} - 1 + 2^0 - 1 + 2^{n_r - 1} - 1 = 2^{n_r} - 2 < 2^{n_r} - 1$.

Completely equivalent. Due to Lemma 5.24 it suffices to show that there are families ν and π such that M and M' are strongly related via ν and π.

Let $t \in T_\Sigma$. We define the injective mapping $\nu_t : P(\mathrm{pos}(t)) \to P_1(\mathrm{pos}(t))$ by $\nu_t(c) = c$ for every $c \in P(\mathrm{pos}(t))$. For every $c \in P(\mathrm{pos}(t))$ we define the mapping $\pi_{t,c} : \Phi_{M,t,c} \to \mathrm{H}_{G_1}^{c,P(\mathrm{pos}(t))}$, where $G_1 = \mathrm{G}_{M_1,t}^{\mathrm{dep}}$, as follows for every $e = (r', \rho') \in \Phi_{M,t,c}$. If $r' \neq r$, then we let $\pi_{t,c}(r', \rho') = (r', \rho')(c_1, \ldots, c_k)$, where $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in P(\mathrm{pos}(t))$ such that $c_1 \cdots c_k = \mathrm{indyield}(\rho'(r'_{\mathrm{b}}))$. Otherwise, if $r' = r$, then we let $\pi_{t,c}(r', \rho') = (r_1, \rho'|_{\mathrm{var}(r_1)})\big((r_2, \rho'|_{\mathrm{var}(r_2)})\big((r_3, \rho'|_{\mathrm{var}(r_3)})(c_1, \ldots, c_k)\big)\big)$, for some $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in P(\mathrm{pos}(t))$ such that $c_1 \cdots c_k = \mathrm{indyield}(\rho'(r'_{\mathrm{b}}))$.

Let $\nu = (\nu_t \mid t \in T_\Sigma)$ and $\pi = (\pi_{t,c} \mid t \in T_\Sigma, c \in P(\mathrm{pos}(t)))$. It is easy to see that M and M' are related via ν and π. Now we show that M and M' are even strongly related via ν and π. By Lemma 5.25 it suffices to show that for every $t \in T_\Sigma$, $c' \in P_1(\mathrm{pos}(t))$, $e = (r', \rho') \in \Phi_{M_1,t,c'}$, and $i \in [\mathrm{rk}(e)]$ with $\rho'(r'_{\mathrm{b}}) \notin P_1(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ we have that $|\mathrm{H}_{G_1}^{\mathrm{in}_i(e),P(\mathrm{pos}(t))}| \leq 1$, where $G_1 = \mathrm{G}_{M_1,t}^{\mathrm{dep}}$. This is vacuously true because it is easy to check that there are no $c' \in P_1(\mathrm{pos}(t))$, $e = (r', \rho') \in \Phi_{M_1,t,c'}$, and $i \in [\mathrm{rk}(e)]$ such that $\rho'(r'_{\mathrm{b}}) \notin P_1(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$. **Case 2**: $B \neq \emptyset$. Then we let $P_1 = P \cup \{\bar{b} \mid b \in B\}$ where $\bar{b}$ is a new unary predicate for every $b \in B$, and $R_1 = (R \setminus \{r\}) \cup \{\bar{r}\} \cup R'$, where $\bar{r}$ is obtained from r by replacing every occurrence of every $b \in B$ by $\bar{b}(x)$ and replacing r_{G} by $r_{\mathrm{G}} \setminus G$, and R' is the smallest set such that for every $b \in B$:

- if $b = r_{\mathrm{h}}$, then R' contains the rule $b \leftarrow \bar{b}(x)\,; G$
- if $b \in \mathrm{ind}(r_{\mathrm{b}})$, then R' contains the rule $\bar{b}(x) \leftarrow b\,; G$

The proof that M_1 has all required properties is similar to the proof of Case 1, therefore we will only sketch this proof.

Connected and Existence of injective position mappings. The rule $\bar{r}$ is connected and admits an injective position mapping for the same reasons that the rules r_1 and r_3 of Case 1 are connected and admit injective position mappings. The rules in R' are connected and have injective position mappings for the same reasons that the rule r_2 of Case 1 is connected has an injective position mapping.

Proof that $n(M_1) < n(M)$. Clearly, $n_{\bar{r}} < n_r$ for the same reason that $n_{r_1} < n_r$ in Case 1. Moreover, for every rule $r' \in R'$ we have $n_{r'} = 0$ for the same reasons that $n_{r_2} = 0$ in Case 1.

Completely equivalent. For every $t \in T_\Sigma$, $c \in P(\mathrm{pos}(t))$, and $(r', \rho') \in \Phi_{M,t,c}$ we let $\nu_t(c) = c$, and $\pi_{t,c}(r', \rho')$ be defined similarly to Case 1 if $r' \neq r$, and if $r' = r$ we let

$$\pi_{t,c}(r', \rho') = \begin{cases} (\bar{r}, \rho'|_{\mathrm{var}(\bar{r})})(\eta_1, \ldots, \eta_k)\,, & \text{if } r_{\mathrm{h}} \notin B\,, \\ \big(r_{\mathrm{b}} \leftarrow \overline{r_{\mathrm{b}}}(x)\,; G, \rho'|_{\mathrm{var}(G)}\big)\big((\bar{r}, \rho'|_{\mathrm{var}(\bar{r})})(\eta_1, \ldots, \eta_k)\big)\,, & \text{otherwise,} \end{cases}$$

where $k \in \mathbb{N}$ and $b_1, \ldots, b_k \in P(\mathrm{pos}(t))$ such that $\mathrm{indyield}(r'_{\mathrm{b}}) = b_1 \cdots b_k$, and for every $i \in [k]$ we have $\eta_i = \rho'(b_i)$ if $b_i \notin B$ and $\eta_i = \big(\bar{b}_i(x) \leftarrow b_i\,; G, \rho'|_{\mathrm{var}(G)}\big)(\rho'(b_i))$ otherwise. It is easy to see that M and M' are related via ν and π. Now we show that M and M' are even strongly related via ν and π. By means of Lemma 5.25 it suffices to show that for every

$t \in T_\Sigma$, $c' \in P_1(\mathrm{pos}(t))$, $e = (r', \rho') \in \Phi_{M_1,t,c'}$, and $i \in [\mathrm{rk}(e)]$ with $\rho'(r'_\mathrm{b}) \notin P_1(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ we have that $|\mathrm{H}_{G_1}^{\mathrm{in}_i(e),P(\mathrm{pos}(t))}| \leq 1$, where $G_1 = \mathrm{G}^{\mathrm{dep}}_{M_1,t}$. It is easy to check that the conditions $\rho'(r'_\mathrm{b}) \notin P_1(\mathrm{pos}(t))$ and $\mathrm{in}_i(e) \notin \mathrm{ran}(\nu_t)$ imply that $r' = \bar{r}$ and $\mathrm{in}_i(e) = \rho'(\bar{b}(x))$ for some $b \in B \cap \mathrm{ind}(r_\mathrm{b})$. Note that for every $e' \in \Phi_{M_1,t,\rho(\bar{b}(x))}$ we have that $\mathrm{pr}_1(e') = \bar{b}(x) \leftarrow b\,; G$. therefore,

$$\mathrm{H}_{G_1}^{\rho(\bar{b}(x)),P(\mathrm{pos}(t))} = \{(r', \rho')(\rho'(b)) \mid (r', \rho') \in \Phi_{M_1,t,\rho(\bar{b}(x))}\} \ .$$

By Lemma 5.29(1) we obtain that $|\Phi_{M_1,t,\rho(\bar{b}(x))}| \leq 1$, hence $|\mathrm{H}_{G_1}^{\mathrm{in}_i(e),P(\mathrm{pos}(t))}| \leq 1$. ■

The following corollary is an immediate consequence of Lemma 5.40 and the definition of completely equivalent mwmd.

Corollary 5.41. *Let $\mathcal{A}$ be an m-monoid over Δ, $(\mathcal{A}, \leq)$ be an ω-continuous m-monoid, and $(\mathcal{A}, \sum)$ be an ω-complete m-monoid. Then*

$$\mathrm{l\text{–}WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{pc\text{–}WMD}^{\mathrm{hyp}}(\Sigma, \Delta, \mathcal{A}) \ ,$$
$$\mathrm{l\text{–}WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) = \mathrm{pc\text{–}WMD}^{\mathrm{hyp}}\big(\Sigma, \Delta, (\mathcal{A}, \textstyle\sum)\big) \ ,$$

moreover, if $\mathcal{A}$ is absorptive, then

$$\mathrm{l\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) = \mathrm{pc\text{–}WMD}^{\mathrm{fix}}(\Sigma, \Delta, \mathcal{A}) \ ,$$
$$\mathrm{l\text{–}WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big) = \mathrm{pc\text{–}WMD}^{\mathrm{fix}}\big(\Sigma, \Delta, (\mathcal{A}, \leq)\big) \ .$$

Deciding circularity

In Chapter 4, we laid out two types of semantics for mwmd, namely fixpoint and hypergraph semantics. For each of these semantics we defined a variant, called finitary semantics, that is only applicable to weakly non-circular mwmd but is defined for arbitrary (absorptive) m-monoids. Such a semantics is only then of practical value if it is decidable whether a given mwmd is weakly non-circular.

In this chapter we prove that there is an effective procedure that decides whether an mwmd is weakly non-circular. For the sake of completeness we will show that such a decision procedure for the property of non-circularity exists, too.

Theorem 6.1. *Let Σ be a ranked alphabet and Δ be a signature. Moreover, let M be an mwmd over Σ and Δ.*

1. *It is effectively decidable whether M is non-circular.*
2. *It is effectively decidable whether M is weakly non-circular.*

This theorem is a consequence of Theorem 6.2 and Corollaries 6.12 and 6.15.

The definitions of weak non-circularity and non-circularity of mwmd are inspired by and adapted from the definitions of non-circularity for attribute grammars [32, 50], attributed tree transducers [56, 60], and weighted monadic datalog [122]. It is self-evident that decision procedures for non-circularity of these devices are of central importance in their respective theories. A decision procedure for the non-circularity of attribute grammars, called a circularity test, has first been studied by Knuth [90, 89] (also see [3] and [94, Figure 3.6, Lemma 3.25]). A similar circularity test for attributed tree transducers has been proposed in [56] and investigated in [60, Figure 5.7, Lemma 5.17]. Both of these circularity tests are based on the inductive construction of a finite set of graphs, called is-graphs, that are checked for cycles. It is worth pointing out that the problem to decide whether a given attribute grammar or attributed tree transducer is non-circular, is inherently exponential [79, 78]. Since attributed tree transducers are special mwmd (see Chapter 8), every circularity test for mwmd will also have exponential time complexity.

In this thesis we will not follow the approach to develop a circularity test that is based on the construction of is-graphs, or any similar methods. This is due to the following two reasons.

- Attribute grammars and attributed tree transducers are similar to local mwmd. A circularity test that is based on is-graphs is easy to define for local mwmd but very hard to extend to mwmd that have an arbitrary structure.
- The property of non-circularity can nicely be captured with is-graphs. However, this does not hold for weak non-circularity. Though one can extend the notion of is-graphs such that they can be used for testing weak non-circularity, this extension

is not straightforward and the correctness proof of the decision procedure is fairly tedious.

Due to these reasons we will use the following approach instead. Let $M = (P, R, q)$ be an mwmd and let L_M be the set of input trees t such that the set of derivations of the dependency hypergraph of M and t that end in $q(\varepsilon)$ is infinite. Clearly, M is weakly non-circular iff L_M is empty. We will show that there effectively is an MSO-logic formula that defines L_M; this implies that L_M is a recognizable tree language. The decidability of weak non-circularity of mwmd follows from the fact that the emptiness problem of recognizable tree languages is decidable.

This chapter is organized as follows. In Section 6.1 we will recall the basic concepts of recognizable tree languages and MSO-logic. In Section 6.2 we will give a stepwise construction of the MSO-formula that defines the tree language L_M and prove its correctness along the way.

In this chapter we fix a ranked alphabet Σ and a signature Δ.

6.1 Recognizable tree languages

In this section we recall basic concepts of finite state tree automata [37, 38, 128], recognizable tree languages [65, 66], and monadic second order logic [131, 38].

Finite state tree automata

A (*bottom-up*) ***finite state tree automaton*** [37, 49, 66, 128] (for short: ***fta***) *over* Σ is a triple $\mathcal{M} = (Q, \delta, F)$, where Q is a finite non-empty set, $\delta = (\delta_k \mid k \in \mathbb{N})$ is a family of sets $\delta_k \subseteq Q^k \times \Sigma^{(k)} \times Q$, and $F \subseteq Q$. We refer to the elements in Q and F as ***states*** and ***final states***, respectively. In the sequel we will simply write δ instead of δ_k, for every $k \in \mathbb{N}$.

Let $\mathcal{M} = (Q, \delta, F)$ be an fta. We call $\mathcal{M}$ ***deterministic*** if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k \in Q$ there is at most one $p \in Q$ with $(q_1 \cdots q_k, \sigma, p) \in \delta$. Let $t \in T_\Sigma$. A ***successful run*** *of* $\mathcal{M}$ *on* t is a mapping $\kappa : \mathrm{pos}(t) \to Q$ such that $\kappa(\varepsilon) \in F$ and for every $w \in \mathrm{pos}(t)$ we have that $(\kappa(w1) \cdots \kappa(wk), t(w), \kappa(w)) \in \delta$, where $k = \mathrm{rk}(t(w))$. The ***tree language*** $\mathcal{L}(\mathcal{M}) \subseteq T_\Sigma$ *recognized by* $\mathcal{M}$ is the set of all trees $t \in T_\Sigma$ such that there is a successful run of $\mathcal{M}$ on t. If a tree language $L \subseteq T_\Sigma$ is recognized by an fta over Σ, then L is called ***recognizable***. The following theorem is well-known (see [38, Corollary 1.12(i)] or [131, Theorem 7]).

Theorem 6.2. *If $\mathcal{M}$ is an fta over Σ, then it is effectively decidable whether $\mathcal{L}(\mathcal{M}) = \emptyset$.*

Monadic second order logic

As usual in monadic second order logic [131, 38] (for short: MSO-logic), we use first-order variables, like $x, x_1, x_2, \ldots, y, z$ and second-order variables, like $X, X_1, X_2, \ldots, Y, Z$. We assume that the set V is contained in the set of all first-order variables. We define the set MSO(Σ) of ***MSO-logic formulas*** *over* Σ by the following EBNF with nonterminal φ (cf. [66]):

$$\varphi ::= \mathrm{label}_\sigma(x) \mid \mathrm{edge}_i(x, y) \mid x \in X \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi)$$

$$\forall x.\varphi \mid \forall X.\varphi \mid \exists x.\varphi \mid \exists X.\varphi \; ,$$

where $i \in [\mathrm{maxrk}(\Sigma)]$, $\sigma \in \Sigma$, x and y are first-order variables, and X is a second-order variable. We will drop parentheses if no confusions arise. Moreover, we will use the following abbreviations for every $\varphi, \psi \in \mathrm{MSO}(\Sigma)$ and all first-order variables x, y:

$$\begin{aligned}
(\varphi \to \psi) &= (\neg\varphi \vee \psi) \; , && (\varphi \text{ implies } \psi)\\
(\varphi \leftrightarrow \psi) &= (\varphi \to \psi) \wedge (\psi \to \varphi) \; , && (\varphi \text{ and } \psi \text{ are equivalent})\\
(x \equiv y) &= \forall X.(x \in X \leftrightarrow y \in X) \; , && (x \text{ and } y \text{ are at the same position})\\
\mathrm{true} &= \forall x.\forall X.((x \in X) \to (x \in X)) \; . && (\text{a tautology})\\
\mathrm{false} &= \neg\mathrm{true} \; . && (\text{a contradiction})
\end{aligned}$$

The set of free variables of a formula $\varphi \in \mathrm{MSO}(\Sigma)$, denoted by $\mathrm{Free}(\varphi)$, is defined as usual; if $\mathrm{Free}(\varphi) = \emptyset$, then φ is called a ***sentence***.

Let $\mathcal{V}$ be a finite set of first-order and second-order variables. Let $t \in T_\Sigma$. A ***$\mathcal{V}$-assignment*** *for t* is a function with domain $\mathcal{V}$ which maps first-order variables to elements of $\mathrm{pos}(t)$ and second-order variables to subsets of $\mathrm{pos}(t)$. By $\Psi_{\mathcal{V},t}$ we denote the set of all $\mathcal{V}$-assignments for t.

Let $\varphi \in \mathrm{MSO}(\Sigma)$ and $\mathcal{V}$ be a finite set of variables containing $\mathrm{Free}(\varphi)$. For every $t \in T_\Sigma$ and $\rho \in \Psi_{\mathcal{V},t}$ we define the relation "(t,ρ) satisfies φ", denoted by $(t,\rho) \models \varphi$ as usual; if $\mathcal{V} = \emptyset$, then we also write $t \models \varphi$ instead of $(t,\rho) \models \varphi$. We define the set $\mathcal{L}_{\mathcal{V}}(\varphi) = \{(t,\rho) \mid t \in T_\Sigma, \rho \in \Psi_{\mathcal{V},t}, (t,\rho) \models \varphi\}$ and abbreviate $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi)$ by $\mathcal{L}(\varphi)$. The following consistency lemma is folklore.

Lemma 6.3. *Let $\varphi \in \mathrm{MSO}(\Sigma)$ and $\mathcal{V}$ be a finite set of variables containing $\mathrm{Free}(\varphi)$. Then for every $t \in T_\Sigma$ and $\rho \in \Psi_{\mathcal{V},t}$ the following holds: $(t,\rho) \models \varphi$ iff $(t, \rho|_{\mathrm{Free}(\varphi)}) \models \varphi$.*

A tree language $L \subseteq T_\Sigma$ is called ***definable*** (*in MSO-logic*) if there is a sentence $\varphi \in \mathrm{MSO}(\Sigma)$ such that $L = \mathcal{L}(\varphi)$.

In the subsequent lemmas, whose purpose is to prove Theorem 6.1, we distinguish between "existence" and "effective existence" of an object (e.g., a formula or an fta). We give a brief explanation of this distinction. Let A and B be sets and τ be a relation from A to B. If for every $a \in A$ the set $\tau(\{a\})$ is nonempty, then we say that *for every $a \in A$ there is a $b \in B$ such that $(a,b) \in \tau$*. If we say instead that *for every $a \in A$ there effectively is a $b \in B$ such that $(a,b) \in \tau$*, then we imply that there is an algorithm (or a Turing machine [132], etc.) that, for every input $a \in A$, computes an output $b_a \in B$ with $(a, b_a) \in \tau$; we call this an *effective construction* of b_a from a. For the sake of brevity, we will not explicitly mention the set A; it will always be clear from the context. For example, in the first statement of the following well-known theorem the set A consists of all pairs (Σ', φ) such that Σ' is a ranked alphabet and $\varphi \in \mathrm{MSO}(\Sigma')$ (both encoded appropriately in some finite way such that A is a countable set and not a proper class).

Theorem 6.4 (cf. [38, Corollary 3.11], [131, Theorem 17]).

1. *Let $\varphi \in \mathrm{MSO}(\Sigma)$ be a sentence. Then there effectively is an fta $\mathcal{M}$ over Σ such that $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\varphi)$.*

2. *Let $L \subseteq T_\Sigma$. Then L is recognizable iff L is definable.*

In order to simplify notation we will adhere to the following convention with regard to free variables and assignments. Let $n \in \mathbb{N}$, $\mathcal{X}_1, \dots, \mathcal{X}_n$ be first- or second-order variables, and $\varphi \in \mathrm{MSO}(\Sigma)$. Then we write $\varphi(\mathcal{X}_1, \dots, \mathcal{X}_n)$ instead of φ in order to express that $\mathrm{Free}(\varphi) \subseteq \{\mathcal{X}_1, \dots, \mathcal{X}_n\}$. Moreover, for every $t \in T_\Sigma$ and $\{\mathcal{X}_1, \dots, \mathcal{X}_n\}$-assignment ρ for t we will sometimes write $t \models \varphi(W_1, \dots, W_n)$ instead of $(t, \rho) \models \varphi$, where $W_i = \rho(\mathcal{X}_i)$ for every $i \in [n]$.[1]

6.2 Defining circularity

In this section we show that for every mwmd $M = (P, R, q)$ there effectively exists an MSO-logic sentence such that for every input tree t the sentence holds for t iff the set of derivations of the dependency hypergraph of M and t that end in $q(\varepsilon)$ is infinite; thus the language defined by the sentence is empty iff M is weakly non-circular. For completeness we will also give an effective construction of a sentence that holds for an input tree t iff the dependency graph of M and t is cyclic; thus, the language defined by the sentence is empty iff M is non-circular.

For the sake of simplicity we will restrict ourselves to proper mwmd. This is no real restriction since for every mwmd M there effectively exists a proper mwmd that is weakly non-circular iff M is so (see Lemma 5.26). Apart from this it is easy to extend our construction to arbitrary mwmd.

The basic idea of the construction of the MSO-logic formula that expresses that the mwmd M is weakly non-circular is to employ Lemma 2.25: the set of derivations in the dependency hypergraph G of M and t that end in $q(\varepsilon)$ is infinite iff there is an atom instance c such that $c \prec_G^+ c \prec_G^* q(\varepsilon)$. Therefore, it suffices to show that (i) the relation $\prec_G$ can be defined in MSO-logic and that (ii) transitive and transitive reflexive closures of relations can be expressed in MSO-logic.

Let us first discuss Statement (ii). It is well-known that transitive closures of relations on the semantic domain can be defined in MSO-logic. However, the relation $\prec_G$ is not a relation on the semantic domain $\mathrm{pos}(t)$ but on the set of atom instances $P(\mathrm{pos}(t))$. Since we assume that M is proper, the set $P(\mathrm{pos}(t))$ can be considered to be $|P|$ copies of the set $\mathrm{pos}(t)$; therefore, we have to extend the concept of the definition of transitive closures of relations on the semantic domain to, roughly speaking, the definition of transitive closures of relations on a finite number of copies of the semantic domain (see Lemma 6.8).

Now let us analyze Statement (i). In order to define the relation $\prec_G$ we will make use of Lemma 2.20 and Observation 4.11(1,2): for two atom instances c and c' we have $c \prec_G c'$ iff there is a hyperedge in G such that c' is its output vertex and c one of its input vertices and every input vertex c'' has a nonempty set of derivations ending in c''. Hence, we are left with the task to show that the set of hyperedges of the dependency graph G and their respective input and output vertices are, roughly speaking, definable in MSO-logic (see Lemma 6.10); moreover, we need to show that the set of atom instances that have no derivations ending in them is definable in MSO-logic (see Lemma 6.13).

Throughout this section we fix a proper mwmd $M = (P, R, q)$ over Σ and Δ.

For the sake of simplicity let us assume that the set P is of the form $\{p_1, \dots, p_n\}$ for

[1] Note that W_i might be a single position or a set of positions in t, depending on whether $\mathcal{X}_i$ is a first-order or a second-order variable, respectively.

some $n \in \mathbb{N}$ and pairwise distinct predicates $p_1, \dots, p_n$ such that $q = p_1$. Moreover, we fix distinct first-order variables x, y and pairwise distinct second order variables $X_1, \dots, X_n$.

Definition 6.5. For every $t \in T_\Sigma$ let the partial order $\leq_t$ on $\big(\mathcal{P}(\mathrm{pos}(t))\big)^n$ be defined as follows for every $d, d' \in \big(\mathcal{P}(\mathrm{pos}(t))\big)^n$: $d \leq_t d'$ iff $\mathrm{pr}_i(d) \subseteq \mathrm{pr}_i(d')$ for every $i \in [n]$. In the sequel we will simply write $\mathcal{P}(\mathrm{pos}(t))^n$ instead of $\big(\mathcal{P}(\mathrm{pos}(t))\big)^n$. □

The following well-known fixpoint theorem is due to Knaster and Tarski [125, Theorem 1].

Theorem 6.6. *Let $(B, \leq)$ be a complete lattice and let $g : B \to B$ be monotone wrt $\leq$. Then $\wedge\{b \in B \mid g(b) \leq b\}$ is the least fixpoint of g.*

In what follows, let us call a mapping f from the power set of atom instances to the power set of atom instances a *closure operator* if it is monotone, i.e., for all sets C and D of atom instances we have that $C \subseteq D$ implies $f(C) \subseteq f(D)$. Moreover, let us call the set $\bigcup_{n \in \mathbb{N}} f^n(\emptyset)$ the *closure* of f. The following lemma states that for a given closure operator that is definable in MSO-logic, also the closure of this closure operator is definable in MSO-logic; the proof of this lemma makes use of Theorem 6.6. It is rather technical because the semantic domain of MSO-logic on trees is $\mathrm{pos}(t)$ and not the set of atom instances $P(\mathrm{pos}(t))$; therefore, MSO-definable closure operators are not expressed by a single MSO-logic formula but by a family of n MSO-logic formulas. We will use the following lemma in order to show that transitive closures of relations on atom instances are definable in MSO-logic, and that the set of atom instances that have no derivations ending in them is definable in MSO-logic.

Lemma 6.7. *Let $\varphi = \big(\varphi_j(x, y, X_1, \dots, X_n) \mid j \in [n]\big)$ be a family over $\mathrm{MSO}(\Sigma)$. Moreover, let $f = \big(f_{t,v} \mid t \in T_\Sigma, v \in \mathrm{pos}(t)\big)$ be a family such that for every $t \in T_\Sigma$, $v \in \mathrm{pos}(t)$, $W_1, \dots, W_n \subseteq \mathrm{pos}(t)$, and $j \in [n]$ we have $f_{t,v} : \mathcal{P}(\mathrm{pos}(t))^n \to \mathcal{P}(\mathrm{pos}(t))^n$ and*

$$\mathrm{pr}_j\big(f_{t,v}(W_1, \dots, W_n)\big) = \big\{w \in \mathrm{pos}(t) \mid t \models \varphi_j(v, w, W_1, \dots, W_n)\big\} . \tag{6.1}$$

Suppose that for every $t \in T_\Sigma$ and $v \in \mathrm{pos}(t)$, $f_{t,v}$ is monotone wrt $\leq_t$. Then for every $j \in [n]$ there effectively is a formula $\mathrm{close}_j^\varphi(x, y)$ in $\mathrm{MSO}(\Sigma)$ such that for every $t \in T_\Sigma$ and $v \in \mathrm{pos}(t)$:

$$\big\{w \in \mathrm{pos}(t) \mid t \models \mathrm{close}_j^\varphi(v, w)\big\} = \bigcup_{m \in \mathbb{N}} \mathrm{pr}_j\big((f_{t,v})^m(\emptyset, \dots, \emptyset)\big) . \tag{6.2}$$

Proof. We define the formula $\psi(x, X_1, \dots, X_n) \in \mathrm{MSO}(\Sigma)$ and for every $j \in [n]$ we define $\mathrm{close}_j^\varphi(x, y) \in \mathrm{MSO}(\Sigma)$ as follows:

$$\psi = \bigwedge_{k \in [n]} \forall y.(\varphi_k \to (y \in X_k)) ,$$
$$\mathrm{close}_j^\varphi = \forall X_1 \cdots \forall X_n.\psi \to (y \in X_j) .$$

It is easy to see that $\mathrm{Free}(\psi) \subseteq \{x, X_1, \dots, X_n)$ and $\mathrm{Free}(\mathrm{close}_j^\varphi) \subseteq \{x, y\}$, for every $j \in [n]$; moreover, the construction of close_j^φ is clearly effective.

Let $t \in T_\Sigma$ and $v \in \mathrm{pos}(t)$. Observe that $\big(\mathcal{P}(\mathrm{pos}(t))^n, \leq_t\big)$ is a complete lattice; in fact, we have for every $D \subseteq \mathcal{P}(\mathrm{pos}(t))^n$ and $j \in [n]$ that $\mathrm{pr}_j(\wedge_t D) = \bigcap_{d \in D} \mathrm{pr}_j(d)$ and $\mathrm{pr}_j(\vee_t D) = \bigcup_{d \in D} \mathrm{pr}_j(d)$, where $\wedge_t$ and $\vee_t$ denote the infimum and supremum wrt $\leq_t$, respectively. Moreover, $f_{t,v}$ is ω-continuous wrt $\leq_t$ because of Observation 3.28(2) and due to the facts that (i) $f_{t,v}$ is monotone wrt $\leq_t$ by assumption and (ii) every ω-chain wrt $\leq_t$ is ultimately constant because $\mathcal{P}(\mathrm{pos}(t))^n$ is finite. Thus, we obtain for every $j \in [n]$ that

$$
\begin{aligned}
&\big\{w \in \mathrm{pos}(t) \mid t \models \mathrm{close}_j^{\varphi}(v,w)\big\} \\
&= \{w \in \mathrm{pos}(t) \mid \forall (W_1,\dots,W_n) \in \mathcal{P}(\mathrm{pos}(t))^n : t \models \psi(v,W_1,\dots,W_n) \Rightarrow w \in W_j\} \\
&= \bigcap\nolimits_{d \in \{(W_1,\dots,W_n) \in \mathcal{P}(\mathrm{pos}(t))^n \mid t \models \psi(v,W_1,\dots,W_n)\}} \mathrm{pr}_j(d) \\
&= \mathrm{pr}_j\big(\wedge_t\{(W_1,\dots,W_n) \in \mathcal{P}(\mathrm{pos}(t))^n \mid t \models \psi(v,W_1,\dots,W_n)\}\big) \\
&= \mathrm{pr}_j\big(\wedge_t\{(W_1,\dots,W_n) \in \mathcal{P}(\mathrm{pos}(t))^n \mid f_{t,v}(W_1,\dots,W_n) \leq_t (W_1,\dots,W_n)\}\big) \qquad (\star) \\
&= \mathrm{pr}_j(\mathrm{lfp}(f_{t,v})) \qquad \text{(by Theorem 6.6, where } \mathrm{lfp}(f_{t,v}) \text{ is the least fixpoint of } f_{t,v}) \\
&= \mathrm{pr}_j\big(\vee_t\{(f_{t,v})^m(\emptyset,\dots,\emptyset) \mid m \in \mathbb{N}\}\big) \qquad \text{(by Theorem 3.29)} \\
&= \bigcup\nolimits_{m \in \mathbb{N}} \mathrm{pr}_j\big((f_{t,v})^m(\emptyset,\dots,\emptyset)\big) \; .
\end{aligned}
$$

It remains to prove Equation ($\star$). For every $(W_1,\dots,W_n) \in \mathcal{P}(\mathrm{pos}(t))^n$ we have

$$
\begin{aligned}
&t \models \psi(v,W_1,\dots,W_n) \\
&\text{iff } \forall k \in [n] : \forall w \in \mathrm{pos}(t) : t \models \varphi_k(v,w,W_1,\dots,W_n) \Rightarrow w \in W_k \\
&\text{iff } \forall k \in [n] : \forall w \in \mathrm{pos}(t) : w \in \mathrm{pr}_k\big(f_{t,v}(W_1,\dots,W_n)\big) \Rightarrow w \in W_k \qquad \text{(by Eq. (6.1))} \\
&\text{iff } \forall k \in [n] : \mathrm{pr}_k(f_{t,v}(W_1,\dots,W_n)) \subseteq W_k \\
&\text{iff } f_{t,v}(W_1,\dots,W_n) \leq_t (W_1,\dots,W_n) \; .
\end{aligned}
$$

■

The following lemma states that transitive closures of relations on atom instances are definable in MSO-logic.

Lemma 6.8. *Let* $\chi = (\chi_{i,j}(x,y) \mid i,j \in [n])$ *be a family over* $\mathrm{MSO}(\Sigma)$ *and for every* $t \in T_\Sigma$ *let* $\tau_t = \big\{(p_i(v),p_j(w)) \mid i,j \in [n], v,w \in \mathrm{pos}(t), t \models \chi_{i,j}(v,w)\big\}$. *Then for every* $i,j \in [n]$ *there effectively is a formula* $\mathrm{trans}_{i,j}^{\chi}(x,y) \in \mathrm{MSO}(\Sigma)$ *such that for every* $t \in T_\Sigma$ *and* $v,w \in \mathrm{pos}(t)$ *we have* $t \models \mathrm{trans}_{i,j}^{\chi}(v,w)$ *iff* $(p_i(v),p_j(w)) \in \tau_t^+$.

Proof. First let us define some families of auxiliary formulas. For every $i \in [n]$ we let the family $\varphi_i = (\varphi_{i,j}(x,y,X_1,\dots,X_n) \mid j \in [n])$ over $\mathrm{MSO}(\Sigma)$ be defined by letting for every $j \in [n]$:

$$\varphi_{i,j} = \chi_{i,j} \vee \big(\bigvee\nolimits_{k \in [n]} \exists x.(x \in X_k \wedge \chi_{k,j})\big) \; .$$

Observe that $\mathrm{Free}(\varphi_{i,j}) \subseteq \{x,y,X_1,\dots,X_n\}$ for every $i,j \in [n]$. For every $t \in T_\Sigma$, $i,j \in [n]$, $v,w \in \mathrm{pos}(t)$, and $W_1,\dots,W_n \subseteq \mathrm{pos}(t)$ we obtain

$$
\begin{aligned}
&t \models \varphi_{i,j}(v,w,W_1,\dots,W_n) \\
&\text{iff } t \models \chi_{i,j}(v,w) \text{ or } \exists\, k \in [n] : \exists\, v' \in W_k : t \models \chi_{k,j}(v',w) \\
&\text{iff } (p_i(v),p_j(w)) \in \tau_t \text{ or } \exists\, k \in [n] : \exists\, v' \in W_k : (p_k(v'),p_j(w)) \in \tau_t \; . \qquad (6.3)
\end{aligned}
$$

For every $i \in [n]$ we define the family $f_i = \big(f_{i,t,v} \mid t \in T_\Sigma, v \in \mathrm{pos}(t)\big)$ such that for every $t \in T_\Sigma$, $v \in \mathrm{pos}(t)$, $W_1,\dots,W_n \subseteq \mathrm{pos}(t)$, and $j \in [n]$ we have $f_{i,t,v} : \mathcal{P}(\mathrm{pos}(t))^n \to \mathcal{P}(\mathrm{pos}(t))^n$ and

$$\mathrm{pr}_j\big(f_{i,t,v}(W_1,\dots,W_n)\big) = \big\{w \in \mathrm{pos}(t) \mid t \models \varphi_{i,j}(v,w,W_1,\dots,W_n)\big\} \; .$$

By means of Equivalence (6.3) it is easy to check that for every $i \in [n]$, $t \in T_\Sigma$, and $v \in \mathrm{pos}(t)$ the mapping $f_{i,t,v}$ is monotone wrt $\leq_t$. Thus, we can employ Lemma 6.7 as follows. For every $i, j \in [n]$ we define $\mathrm{trans}^{\chi}_{i,j}(x, y) \in \mathrm{MSO}(\Sigma)$ by letting $\mathrm{trans}^{\chi}_{i,j} = \mathrm{close}^{f_i}_j$ (this construction is effective due to Lemma 6.7); then we obtain for every $t \in T_\Sigma$ and $v, w \in \mathrm{pos}(t)$ that

$$
\begin{aligned}
& t \models \mathrm{trans}^{\chi}_{i,j}(v,w) \ \text{ iff } \ t \models \mathrm{close}^{f_i}_j(v,w) \\
& \text{iff } w \in \textstyle\bigcup_{m\in\mathbb{N}} \mathrm{pr}_j\big((f_{i,t,v})^m(\emptyset,\dots,\emptyset)\big) && \text{(by Lemma 6.7)} \\
& \text{iff } (p_i(v), p_j(w)) \in \tau_t^+ \ . && (\star)
\end{aligned}
$$

It remains to prove Equivalence ($\star$). To this end we introduce an auxiliary notion. For the remainder of the proof we fix a $t \in T_\Sigma$; we let $\tau_t^{(0)} = \emptyset$ and, for every $m \in \mathbb{N}$, we let $\tau_t^{(m+1)} = \tau_t \cup (\tau_t^{(m)} \,;\, \tau_t)$. Clearly, for every $m \in \mathbb{N}$, $\tau_t^{(m)}$ is a relation on $P(\mathrm{pos}(t))$. Furthermore, it is easy to check that $\bigcup_{m\in\mathbb{N}} \tau_t^{(m)} = \tau_t^+$. Thus, in order to prove Equivalence ($\star$) it suffices to show that for every $m \in \mathbb{N}$, $i, j \in [n]$, and $v, w \in \mathrm{pos}(t)$ we have: $w \in \mathrm{pr}_j\big((f_{i,t,v})^m(\emptyset,\dots,\emptyset)\big)$ iff $(p_i(v), p_j(w)) \in \tau_t^{(m)}$. We give a proof by induction on m.

Induction base. This is trivial because $\mathrm{pr}_j\big((f_{i,t,v})^0(\emptyset,\dots,\emptyset)\big) = \emptyset = \tau_t^{(0)}$.

Induction step. Let $m \in \mathbb{N}$. We derive

$$
\begin{aligned}
& w \in \mathrm{pr}_j\big((f_{i,t,v})^{m+1}(\emptyset,\dots,\emptyset)\big) \\
& \text{iff } w \in \mathrm{pr}_j\big(f_{i,t,v}((f_{i,t,v})^m(\emptyset,\dots,\emptyset))\big) \\
& \text{iff } t \models \varphi_{i,j}\big(v, w, \mathrm{pr}_1((f_{i,t,v})^m(\emptyset,\dots,\emptyset)), \dots, \mathrm{pr}_n((f_{i,t,v})^m(\emptyset,\dots,\emptyset))\big) \\
& && \text{(by the definition of } f_{i,t,v}) \\
& \text{iff } (p_i(v), p_j(w)) \in \tau_t \text{ or } \exists\, k \in [n] : \exists\, v' \in \mathrm{pr}_k((f_{i,t,v})^m(\emptyset,\dots,\emptyset)) : \\
& \qquad (p_k(v'), p_j(w)) \in \tau_t \ . && \text{(by Equivalence (6.3))} \\
& \text{iff } (p_i(v), p_j(w)) \in \tau_t \text{ or } \exists\, k \in [n] : \exists\, v' \in \mathrm{pos}(t) : (p_i(v), p_k(v')) \in \tau_t^{(m)} \wedge \\
& \qquad (p_k(v'), p_j(w)) \in \tau_t \ . && \text{(by the induction hypothesis)} \\
& \text{iff } (p_i(v), p_j(w)) \in \tau_t \text{ or } (p_i(v), p_j(w)) \in (\tau_t^{(m)} \,;\, \tau_t) \\
& \text{iff } (p_i(v), p_j(w)) \in \tau_t^{(m+1)} \ .
\end{aligned}
$$

■

In what follows, we need to show that it is definable in MSO-logic whether, for a given rule r and input tree t, a particular r,t-variable assignment ρ is valid. To this end we will not distinguish between r,t-variable assignments and $\mathcal{V}$-assignments for t if $\mathcal{V} = \mathrm{var}(r)$.

The following lemma states that it is definable in MSO-logic whether a guard (i.e., a finite set of structural atoms) is satisfied by a given variable assignment.

Lemma 6.9. *Let* $G \subseteq \mathrm{sp}_\Sigma(\mathrm{V})$ *be finite. Then there effectively is a formula* $\mathrm{guard}_G \in \mathrm{MSO}(\Sigma)$ *with* $\mathrm{Free}(\mathrm{guard}_G) = \mathrm{var}(G)$ *such that for every finite set* $\mathcal{V} \subseteq \mathrm{V}$ *containing* $\mathrm{Free}(\mathrm{guard}_G)$*, every* $t \in T_\Sigma$*, and every* $\rho \in \Psi_{\mathcal{V},t}$ *we have* $(t,\rho) \models \mathrm{guard}_G$ *iff* $\rho(G) \subseteq \mathrm{B}_t$*.*

Proof. First we show that for every $g \in \mathrm{sp}_\Sigma(\mathrm{V})$ there is a $\psi_g \in \mathrm{MSO}(\Sigma)$ such that (i) $\mathrm{Free}(\psi_g) = \mathrm{var}(g)$ and (ii) for every finite set $\mathcal{V} \subseteq \mathrm{V}$ containing $\mathrm{Free}(\psi_g)$, every $t \in T_\Sigma$, and every $\rho \in \Psi_{\mathcal{V},t}$ we have $(t,\rho) \models \psi_g$ iff $\rho(g) \in \mathrm{B}_t$. We distinguish four cases.

- $g = \mathrm{root}(z)$ for some $z \in \mathrm{V}$. Then we let $\psi_g = \forall z'. \bigwedge_{i\in[\mathrm{maxrk}(\Sigma)]} \neg\mathrm{edge}_i(z', z)$.

- $g = \mathrm{leaf}(z)$ for some $z \in \mathrm{V}$. Then we let $\psi_g = \forall z'. \bigwedge_{i \in [\mathrm{maxrk}(\Sigma)]} \neg \mathrm{edge}_i(z, z')$.
- $g = \mathrm{label}_\sigma(z)$ for some $\sigma \in \Sigma$ and $z \in \mathrm{V}$. Then we let $\psi_g = \mathrm{label}_\sigma(z')$.
- $g = \mathrm{child}_i(z, z')$ for some $i \in [\mathrm{maxrk}(\Sigma)]$ and $z, z' \in \mathrm{V}$. Then we let $\psi_g = \mathrm{edge}_i(z, z')$.

It is easy to check that Properties (i) and (ii) are satisfied in each case.

Now let $\mathrm{guard}_G = \bigwedge_{g \in G} \psi_g$. Observe that this construction is effective. Clearly, $\mathrm{Free}(\mathrm{guard}_G) = \bigcup_{g \in G} \mathrm{Free}(\psi_g) = \bigcup_{g \in G} \mathrm{var}(g) = \mathrm{var}(G)$. Let $\mathcal{V} \subseteq \mathrm{V}$ be a finite set containing $\mathrm{Free}(\mathrm{guard}_G)$, let $t \in T_\Sigma$ and let $\rho \in \Psi_{\mathcal{V},t}$. Then $(t, \rho) \models \mathrm{guard}_G$ iff $\forall\, g \in G$: $(t, \rho) \models \psi_g$ iff $\forall\, g \in G : \rho(g) \in B_t$ iff $\{\rho(g) \mid g \in G\} \subseteq \mathrm{B}_t$ iff $\rho(G) \subseteq \mathrm{B}_t$. ∎

The following lemma states that it is definable in MSO-logic whether the dependency hypergraph of M and an input tree t contains a hyperedge satisfying a certain MSO-logic definable property. First we need to fix some notation. For every $r \in R$ let $k_r \in \mathbb{N}$ and pairwise distinct $x_1^r, \ldots, x_{k_r}^r \in \mathrm{V}$ such that $\mathrm{var}(r) = \{x_1^r, \ldots, x_{k_r}^r\}$ and x_1^r is the unique variable that occurs in the head of r; without loss of generality we assume that $\{x_1^r, \ldots, x_{k_r}^r\}$ and $\{x, y\}$ are disjoint. Moreover, for every $j \in [n]$ let R_j be the set of all $r \in R$ such that p_j is the unique predicate that occurs in the head of r.

Lemma 6.10. *Let $\psi = (\psi_k(y, X_k) \mid k \in [n])$ be a family over* $\mathrm{MSO}(\Sigma)$*, $j \in [n]$, and $i \in \{0, \ldots, n\}$. Then there is a formula* $\mathrm{hyperedge}_{i,j}^\psi(x, y, X_1, \ldots, X_n)$ *in* $\mathrm{MSO}(\Sigma)$ *that can be constructed effectively such that for every $t \in T_\Sigma$, $v, w \in \mathrm{pos}(t)$, and $W_1, \ldots, W_n \subseteq \mathrm{pos}(t)$ the following statements are equivalent:*

1. $t \models \mathrm{hyperedge}_{i,j}^\psi(v, w, W_1, \ldots, W_n)$,
2. *there is a $(r, \rho) \in \Phi_{M,t,p_j(w)}$ such that*
   - *$i \in [n]$ implies $p_i(v) \in \mathrm{ind}(\rho(r_\mathrm{b}))$ and*
   - *$t \models \psi_k(v', W_k)$ for every $p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b}))$.*

Proof. First we assume that $i \in [n]$. We define $\mathrm{hyperedge}_{i,j}^\psi(x, y, X_1, \ldots, X_n)$ and for every $r \in R$ we define $\varphi_{\exists,r}(x, x_1^r, \ldots, x_{k_r}^r)$, $\varphi_{\forall,r}(X_1, \ldots, X_n, x_1^r, \ldots, x_{k_r}^r)$ in $\mathrm{MSO}(\Sigma)$ as follows:

$$\mathrm{hyperedge}_{i,j}^\psi = \bigvee_{r \in R_j} \exists x_1^r \cdots \exists x_{k_r}^r . \big((y \equiv x_1^r) \wedge \mathrm{guard}_{r_\mathrm{G}} \wedge \varphi_{\exists,r} \wedge \varphi_{\forall,r}\big)\ ,$$

$$\varphi_{\exists,r} = \bigvee_{\substack{l \in [k_r] \\ p_i(x_l^r) \in \mathrm{ind}(r_\mathrm{b})}} (x \equiv x_l^r)\ ,$$

$$\varphi_{\forall,r} = \bigwedge_{k \in [n]} \bigwedge_{\substack{l \in [k_r] \\ p_k(x_l^r) \in \mathrm{ind}(r_\mathrm{b})}} \forall y.((y \equiv x_l^r) \rightarrow \psi_k)\ .$$

It is easy to check that, for every $r \in R$, $\mathrm{Free}(\varphi_{\exists,r}) \subseteq \{x, x_1^r, \ldots, x_{k_r}^r\}$, $\mathrm{Free}(\varphi_{\forall,r}) \subseteq \{X_1, \ldots, X_n, x_1^r, \ldots, x_{k_r}^r\}$, and $\mathrm{Free}(\mathrm{hyperedge}_{i,j}^\psi) \subseteq \{x, y, X_1, \ldots, X_n\}$ (by using the fact that $\mathrm{Free}(\mathrm{guard}_{r_\mathrm{G}}) = \mathrm{var}(r_\mathrm{G}) \subseteq \mathrm{var}(r)$ due to Lemma 6.9). The construction of the formula $\mathrm{hyperedge}_{i,j}^\psi$ is effective due to Lemma 6.9.

Let $t \in T_\Sigma$, $v, w \in \mathrm{pos}(t)$, and $W_1, \ldots, W_n \subseteq \mathrm{pos}(t)$. Moreover, let the assignment $\rho_0 \in \Psi_{\{x,y,X_1,\ldots,X_n\},t}$ be defined by $\rho_0(x) = v$, $\rho_0(y) = w$, and $\rho_0(X_l) = W_l$ for every $l \in [n]$. Then

$$t \models \mathrm{hyperedge}_{i,j}^\psi(v, w, W_1, \ldots, W_n) \quad \text{iff} \quad (t, \rho_0) \models \mathrm{hyperedge}_{i,j}^\psi$$

$$\begin{aligned}
&\text{iff } \exists\, r \in R_j : \exists\, \rho \in \Psi_{\mathrm{var}(r),t} : w = \rho(x_1^r), (t, \rho_0 \cup \rho) \models (\mathrm{guard}_{r_\mathrm{G}} \wedge \varphi_{\exists,r} \wedge \varphi_{\forall,r}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{because } \mathrm{var}(r) \text{ and } \{x, y\} \text{ are disjoint by assumption}) \\
&\text{iff } \exists r \in R_j : \exists \rho \in \Psi_{\mathrm{var}(r),t} : w = \rho(x_1^r), (t, \rho) \models \mathrm{guard}_{r_\mathrm{G}}, (t, \rho_0 \cup \rho) \models (\varphi_{\exists,r} \wedge \varphi_{\forall,r}) \\
&\qquad\qquad (\text{by Lemma 6.3 and because } \mathrm{Free}(\mathrm{guard}_{r_\mathrm{G}}) \subseteq \mathrm{var}(r) \text{ by Lemma 6.9}) \\
&\text{iff } \exists\, r \in R_j : \exists\, \rho \in \Psi_{\mathrm{var}(r),t} : w = \rho(x_1^r), \rho(r_\mathrm{G}) \subseteq \mathrm{B}_t, (t, \rho_0 \cup \rho) \models (\varphi_{\exists,r} \wedge \varphi_{\forall,r}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{by Lemma 6.9}) \\
&\text{iff } \exists\, r \in R : \exists\, \rho \in \Psi_{\mathrm{var}(r),t} : \rho(r_\mathrm{h}) = p_j(w), \rho(r_\mathrm{G}) \subseteq \mathrm{B}_t, (t, \rho_0 \cup \rho) \models (\varphi_{\exists,r} \wedge \varphi_{\forall,r}) \\
&\qquad (\text{because } r_\mathrm{h} = p_j(x_1^r) \text{ and } w = \rho(x_1^r) \text{ together are equivalent to } \rho(r_\mathrm{h}) = p_j(w)) \\
&\text{iff } \exists\, (r, \rho) \in \Phi_{M,t,p_j(w)} : (t, \rho_0 \cup \rho) \models \varphi_{\exists,r}, (t, \rho_0 \cup \rho) \models \varphi_{\forall,r} \;.
\end{aligned}$$

The last equivalence follows from the definition of $\Phi_{M,t,p_j(w)}$. Let $(r, \rho) \in \Phi_{M,t,p_j(w)}$. It remains to show that:

(i) $(t, \rho_0 \cup \rho) \models \varphi_{\exists,r}$ iff $p_i(v) \in \mathrm{ind}(\rho(r_\mathrm{b}))$ and

(ii) $(t, \rho_0 \cup \rho) \models \varphi_{\forall,r}$ iff $t \models \psi_k(v', W_k)$ for every $p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b}))$.

First we prove Statement (i):

$$\begin{aligned}
&(t, \rho_0 \cup \rho) \models \varphi_{\exists,r} \\
&\text{iff } \exists\, l \in [k_r] : p_i(x_l^r) \in \mathrm{ind}(r_\mathrm{b}) \text{ and } \rho_0(x) = \rho(x_l^r) \\
&\text{iff } \exists\, l \in [k_r] : p_i(x_l^r) \in \mathrm{ind}(r_\mathrm{b}) \text{ and } v = \rho(x_l^r) \\
&\text{iff } p_i(v) \in \mathrm{ind}(\rho(r_\mathrm{b})) \;.
\end{aligned}$$

Next we prove Statement (ii):

$$\begin{aligned}
&(t, \rho_0 \cup \rho) \models \varphi_{\forall,r} \\
&\text{iff } \forall\, k \in [n] : \forall\, l \in [k_r] : p_k(x_l^r) \in \mathrm{ind}(r_\mathrm{b}) \text{ implies} \\
&\qquad\qquad\qquad \big(\forall\, w' \in \mathrm{pos}(t) : w' = \rho(x_l^r) \Rightarrow (t, \rho_0[y \mapsto w'] \cup \rho) \models \psi_k\big) \\
&\text{iff } \forall\, k \in [n] : \forall\, l \in [k_r] : p_k(x_l^r) \in \mathrm{ind}(r_\mathrm{b}) \text{ implies } (t, \rho_0[y \mapsto \rho(x_l^r)] \cup \rho) \models \psi_k \\
&\text{iff } \forall\, k \in [n] : \forall\, l \in [k_r] : p_k(x_l^r) \in \mathrm{ind}(r_\mathrm{b}) \text{ implies } t \models \psi_k(\rho(x_l^r), W_k) \\
&\text{iff } \forall\, k \in [n] : \forall\, v' \in \mathrm{pos}(t) : p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b})) \text{ implies } t \models \psi_k(v', W_k) \\
&\text{iff } \forall\, p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b})) : t \models \psi_k(v', W_k) \;.
\end{aligned}$$

This finishes the proof of the case that $i \in [n]$. Now assume that $i = 0$. Then we let

$$\mathrm{hyperedge}^{\psi}_{i,j} = \bigvee_{r \in R_j} \exists x_1^r \cdots \exists x_{k_r}^r . \big((y \equiv x_1^r) \wedge \mathrm{guard}_{r_\mathrm{G}} \wedge \varphi_{\forall,r}\big) \;.$$

The remainder of the proof is similar to the proof of the case that $i \in [n]$. ■

Now we show that non-circularity can be expressed in MSO-logic.

Lemma 6.11. *There effectively is a sentence $\varphi^{\mathrm{nc}} \in \mathrm{MSO}(\Sigma)$ such that $\mathcal{L}(\varphi^{\mathrm{nc}})$ is the set of all trees $t \in T_\Sigma$ such that the dependency graph of M and t is cyclic.*

Proof. We define the sentence $\varphi^{\mathrm{nc}} \in \mathrm{MSO}(\Sigma)$ and for every $i, j, k \in [n]$ we define formulas $\psi_k(y, X_k)$ and $\chi_{i,j}(x, y)$ in $\mathrm{MSO}(\Sigma)$ as follows:

$$\psi_k = \mathrm{true} \;,$$

$$\chi_{i,j} = \forall X_1 \cdots \forall X_n.\text{hyperedge}^{\psi}_{i,j} \;,$$
$$\varphi^{\text{nc}} = \bigvee_{i\in[n]} \exists x.\exists y.((x \equiv y) \wedge \text{trans}^{\chi}_{i,i}) \;,$$

where $\psi = (\psi_k \mid k \in [n])$ and $\chi = (\chi_{i,j} \mid i,j \in [n])$. It is easy to see that, for every $i,j,k \in [n]$, we have $\text{Free}(\psi_k) \subseteq \{y, X_k\}$, $\text{Free}(\chi_{i,j}) \subseteq \{x,y\}$ (by using Lemma 6.10) and that φ^{nc} is a sentence (by Lemma 6.8). The construction of φ^{nc} is effective due to Lemmas 6.8 and 6.10.

Let $t \in T_\Sigma$ and let E be the set of edges of the dependency graph of M and t. It remains to prove that $t \models \varphi^{\text{nc}}$ iff $(c,c) \in E^+$ for some $c \in P(\text{pos}(t))$. We claim that for every $i,j \in [n]$ and $v,w \in \text{pos}(t)$, $t \models \chi_{i,j}(v,w)$ iff $(p_i(v), p_j(w)) \in E$. This claim implies

$$\begin{aligned}
& t \models \varphi^{\text{nc}} \\
& \text{iff } \exists\, i \in [n] : \exists\, v \in \text{pos}(t) : t \models \text{trans}^{\chi}_{i,i}(v,v) \\
& \text{iff } \exists\, i \in [n] : \exists\, v \in \text{pos}(t) : (p_i(v), p_i(v)) \in E^+ \qquad \text{(by Lemma 6.8 and the claim)} \\
& \text{iff } \exists\, c \in P(\text{pos}(t)) : (c,c) \in E^+ \;.
\end{aligned}$$

It remains to prove the claim. For every $i,j \in [n]$ and $v,w \in \text{pos}(t)$:

$$\begin{aligned}
& t \models \chi_{i,j}(v,w) \\
& \text{iff } \forall\, W_1,\dots,W_n \subseteq \text{pos}(t) : t \models \text{hyperedge}^{\psi}_{i,j}(v,w,W_1,\dots,W_n) \\
& \text{iff } \forall\, W_1,\dots,W_n \subseteq \text{pos}(t) : \exists\,(r,\rho) \in \Phi_{M,t,p_j(w)} : \big(p_i(v) \in \text{ind}(\rho(r_{\text{b}})) \text{ and} \\
& \qquad \big(\forall\, p_k(v') \in \text{ind}(\rho(r_{\text{b}})) : t \models \psi_k(v', W_k)\big)\big) \qquad \text{(by Lemma 6.10)} \\
& \text{iff } \forall\, W_1,\dots,W_n \subseteq \text{pos}(t) : \exists\,(r,\rho) \in \Phi_{M,t,p_j(w)} : p_i(v) \in \text{ind}(\rho(r_{\text{b}})) \\
& \qquad \text{(since } \psi_k = \text{true for every } k \in [n]) \\
& \text{iff } \exists\,(r,\rho) \in \Phi_{M,t,p_j(w)} : p_i(v) \in \text{ind}(\rho(r_{\text{b}})) \\
& \qquad \text{(because this statement is independent from } W_1,\dots,W_n) \\
& \text{iff } (p_i(v), p_j(w)) \in E \;. \qquad \text{(by Observation 4.11(3))}
\end{aligned}$$

■

The following corollary is an immediate consequence of Theorem 6.4(1) and Lemma 6.11.

Corollary 6.12. *There effectively is an fta $\mathcal{M}^{\text{nc}}$ such that $\mathcal{L}(\mathcal{M}^{\text{nc}}) = \emptyset$ iff M is non-circular.*

Before we show that also weak non-circularity can be expressed in MSO-logic, we prove that the set of atom instances that have no derivations ending in them is definable in MSO-logic.

Lemma 6.13. *For every $k \in [n]$ there effectively is a formula* $\text{nonempty}_k(y)$ *in* $\text{MSO}(\Sigma)$ *such that for every $t \in T_\Sigma$ and $w \in \text{pos}(t)$: $t \models \text{nonempty}_k(w)$ iff $\text{H}_G^{p_k(w)} \neq \emptyset$, where $G = \text{G}^{\text{dep}}_{M,t}$.*

Proof. First let us introduce some auxiliary formulas. For every $j,k \in [n]$ we define $\psi_k(y, X_k)$ and $\varphi_j(x,y,X_1,\dots,X_n)$ in $\text{MSO}(\Sigma)$ as follows:

$$\psi_k = y \in X_k \;, \qquad\qquad \varphi_j = \text{hyperedge}^{\psi}_{0,j} \;,$$

where $\psi = (\psi_k \mid k \in [n])$. Obviously, $\mathrm{Free}(\psi_k) \subseteq \{y, X_k\}$ for every $k \in [n]$. By Lemma 6.10, for every $j \in [n]$, $\mathrm{Free}(\varphi_j) \subseteq \{x, y, X_1, \ldots, X_n\}$ and the construction of φ_j is effective. For every $t \in T_\Sigma$, $j \in [n]$, $v, w \in \mathrm{pos}(t)$, and $W_1, \ldots, W_n \subseteq \mathrm{pos}(t)$ we obtain, by using Lemma 6.10, that

$$\begin{aligned}
&t \models \varphi_j(v, w, W_1, \ldots, W_n) \text{ iff } t \models \mathrm{hyperedge}^{\psi}_{0,j}(v, w, W_1, \ldots, W_n)\\
&\text{iff } \exists\, (r, \rho) \in \Phi_{M,t,p_j(w)} : \forall\, p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b})) : t \models \psi_k(v', W_k)\\
&\text{iff } \exists\, (r, \rho) \in \Phi_{M,t,p_j(w)} : \forall\, p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b})) : v' \in W_k\ . \qquad (6.4)
\end{aligned}$$

We define the family $f = \big(f_{t,v} \mid t \in T_\Sigma, v \in \mathrm{pos}(t)\big)$ such that for every $t \in T_\Sigma$, $v \in \mathrm{pos}(t)$, $W_1, \ldots, W_n \subseteq \mathrm{pos}(t)$, and $j \in [n]$ we have $f_{t,v} : \mathcal{P}(\mathrm{pos}(t))^n \to \mathcal{P}(\mathrm{pos}(t))^n$ and

$$\mathrm{pr}_j\big(f_{t,v}(W_1, \ldots, W_n)\big) = \big\{w \in \mathrm{pos}(t) \mid t \models \varphi_j(v, w, W_1, \ldots, W_n)\big\}\ .$$

By means of Equivalence (6.4) it is easy to check that for every $t \in T_\Sigma$ and $v \in \mathrm{pos}(t)$ the mapping $f_{t,v}$ is monotone wrt $\leq_t$. Thus, we can employ Lemma 6.7 to the family $\varphi = (\varphi_j \mid j \in [n])$ as follows. Let $k \in [n]$. We define the formula $\mathrm{nonempty}_k(y) \in \mathrm{MSO}(\Sigma)$ by letting $\mathrm{nonempty}_k = \forall x.\mathrm{close}^{\varphi}_k$. Lemma 6.7 implies that $\mathrm{Free}(\mathrm{nonempty}_k) \subseteq \{y\}$ and that the construction of $\mathrm{nonempty}_k$ is effective. Let $t \in T_\Sigma$, $w \in \mathrm{pos}(t)$, and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. It remains to prove that $t \models \mathrm{nonempty}_k(w)$ iff $\mathrm{H}^{p_k(w)}_G \neq \emptyset$. We obtain

$$\begin{aligned}
&t \models \mathrm{nonempty}_k(w)\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : t \models \mathrm{close}^{\varphi}_k(v, w)\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : w \in \textstyle\bigcup_{m\in\mathbb{N}} \mathrm{pr}_k\big((f_{t,v})^m(\emptyset, \ldots, \emptyset)\big) \qquad \text{(by Lemma 6.7)}\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : \mathrm{H}^{p_k(w)}_G \neq \emptyset \qquad (\star)\\
&\text{iff } \mathrm{H}^{p_k(w)}_G \neq \emptyset\ . \qquad \text{(because this statement is independent from } v)
\end{aligned}$$

It remains to prove Equivalence $(\star)$. We claim that for every $m \in \mathbb{N}$, $k' \in [n]$, and $v, v' \in \mathrm{pos}(t)$ we have $v' \in \mathrm{pr}_{k'}\big((f_{t,v})^m(\emptyset, \ldots, \emptyset)\big)$ iff $\mathrm{H}^{p_{k'}(v'),m}_G \neq \emptyset$ (recall the definition of $\mathrm{H}^{p_k(w),m}_G$ from Definition 4.47). In particular, for $k' = k$ and $v' = w$, this claim implies

$$\begin{aligned}
&\forall\, v \in \mathrm{pos}(t) : w \in \textstyle\bigcup_{m\in\mathbb{N}} \mathrm{pr}_k\big((f_{t,v})^m(\emptyset, \ldots, \emptyset)\big)\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : \exists\, m \in \mathbb{N} : w \in \mathrm{pr}_k\big((f_{t,v})^m(\emptyset, \ldots, \emptyset)\big)\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : \exists\, m \in \mathbb{N} : \mathrm{H}^{p_k(w),m}_G \neq \emptyset\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : \textstyle\bigcup_{m\in\mathbb{N}} \mathrm{H}^{p_k(w),m}_G \neq \emptyset\\
&\text{iff } \forall\, v \in \mathrm{pos}(t) : \mathrm{H}^{p_k(w)}_G \neq \emptyset\ . \qquad \big(\text{because } \textstyle\bigcup_{m\in\mathbb{N}} \mathrm{H}^{p_k(w),m}_G = \mathrm{H}^{p_k(w)}_G\big)
\end{aligned}$$

This proves Equivalence $(\star)$. Now we show that the claim holds. We give a proof by induction on m.

Induction base. $v' \in \mathrm{pr}_{k'}\big((f_{t,v})^0(\emptyset, \ldots, \emptyset)\big)$ iff $v' \in \emptyset$ iff $\emptyset \neq \emptyset$ iff $\mathrm{H}^{p_{k'}(v'),0}_G \neq \emptyset$, for every $k' \in [n]$ and $v, v' \in \mathrm{pos}(t)$.

Induction step. Let $m \in \mathbb{N}$. We derive

$$v' \in \mathrm{pr}_{k'}\big((f_{t,v})^{m+1}(\emptyset, \ldots, \emptyset)\big)$$

$$\begin{aligned}
&\text{iff } v' \in \mathrm{pr}_{k'}\big(f_{t,v}((f_{t,v})^m(\emptyset,\dots,\emptyset))\big)\\
&\text{iff } t \models \varphi_{k'}\big(v, v', \mathrm{pr}_1((f_{t,v})^m(\emptyset,\dots,\emptyset)),\dots,\mathrm{pr}_n((f_{t,v})^m(\emptyset,\dots,\emptyset))\big)\\
&\hfill\text{(by the definition of } f_{t,v})\\
&\text{iff } \exists\,(r,\rho)\in\Phi_{M,t,p_{k'}(v')} : \forall\, p_{k''}(v'')\in \mathrm{ind}(\rho(r_{\mathrm{b}})) : v''\in \mathrm{pr}_{k''}((f_{t,v})^m(\emptyset,\dots,\emptyset))\\
&\hfill\text{(by Equivalence 6.4)}\\
&\text{iff } \exists\,(r,\rho)\in\Phi_{M,t,p_{k'}(v')} : \forall\, p_{k''}(v'')\in \mathrm{ind}(\rho(r_{\mathrm{b}})) : \mathrm{H}_G^{p_{k''}(v''),m} \neq \emptyset\\
&\hfill\text{(by the induction hypothesis)}\\
&\text{iff } \mathrm{H}_G^{p_{k'}(v'),m+1} \neq \emptyset\ . \qquad \text{(by Observation 4.48)}
\end{aligned}$$

■

Now we show that weak non-circularity can be expressed in MSO-logic.

Lemma 6.14. *There effectively is a sentence $\varphi^{\mathrm{wnc}} \in \mathrm{MSO}(\Sigma)$ such that $\mathcal{L}(\varphi^{\mathrm{wnc}})$ is the set of all trees $t \in T_\Sigma$ such that $\mathrm{H}_G^{q(\varepsilon)}$ is infinite, where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$.*

Proof. First let us define some auxiliary formulas. For every $i,j,k \in [n]$ we define $\psi_k(y, X_k)$, $\chi_{i,j}(x,y)$, and $\varphi_{i,j}(x,y)$ in $\mathrm{MSO}(\Sigma)$ as follows:

$$\begin{aligned}
\psi_k &= \mathrm{nonempty}_k\ ,\\
\chi_{i,j} &= \forall X_1\cdots\forall X_n.\mathrm{hyperedge}^{\psi}_{i,j}\ ,\\
\varphi_{i,j} &= \mathrm{trans}^{\chi}_{i,j}\ ,
\end{aligned}$$

where $\psi = (\psi_k \mid k \in [n])$ and $\chi = (\chi_{i,j} \mid i,j \in [n])$. It is easy to see that, for every $i,j,k \in [n]$, we have $\mathrm{Free}(\psi_k) \subseteq \{y, X_k\}$ (by Lemma 6.13), $\mathrm{Free}(\chi_{i,j}) \subseteq \{x,y\}$ (by Lemma 6.10), and $\mathrm{Free}(\varphi_{i,j}) \subseteq \{x,y\}$ (by Lemma 6.8); the construction of $\varphi_{i,j}$ is effective due to Lemmas 6.8, 6.10, and 6.13. For every $t \in T_\Sigma$, $i,j \in [n]$, and $v,w \in \mathrm{pos}(t)$ we obtain for $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$ that

$$\begin{aligned}
&t \models \chi_{i,j}(v,w)\\
&\text{iff } \forall\, W_1,\dots,W_n \subseteq \mathrm{pos}(t) : t \models \mathrm{hyperedge}^{\psi}_{i,j}(v,w,W_1,\dots,W_n)\\
&\text{iff } \forall\, W_1,\dots,W_n \subseteq \mathrm{pos}(t) : \exists\,(r,\rho)\in\Phi_{M,t,p_j(w)} : \big(p_i(v)\in\mathrm{ind}(\rho(r_{\mathrm{b}})) \text{ and}\\
&\qquad \big(\forall\, p_k(v')\in\mathrm{ind}(\rho(r_{\mathrm{b}})) : t \models \psi_k(v', W_k)\big)\big) \qquad \text{(by Lemma 6.10)}\\
&\text{iff } \forall\, W_1,\dots,W_n \subseteq \mathrm{pos}(t) : \exists\,(r,\rho)\in\Phi_{M,t,p_j(w)} : \big(p_i(v)\in\mathrm{ind}(\rho(r_{\mathrm{b}})) \text{ and}\\
&\qquad \big(\forall\, p_k(v')\in\mathrm{ind}(\rho(r_{\mathrm{b}})) : \mathrm{H}_G^{p_k(v')} \neq \emptyset\big)\big) \qquad \text{(by Lemma 6.13)}\\
&\text{iff } \exists\,(r,\rho)\in\Phi_{M,t,p_j(w)} : p_i(v)\in\mathrm{ind}(\rho(r_{\mathrm{b}})), \forall\, p_k(v')\in\mathrm{ind}(\rho(r_{\mathrm{b}})) : \mathrm{H}_G^{p_k(v')} \neq \emptyset\\
&\hfill\text{(because this statement is independent from } W_1,\dots,W_n)\\
&\text{iff } (p_i(v), p_j(w)) \in \prec_G\ . \qquad \text{(by Lemma 2.20 and Observation 4.11(1,2))}
\end{aligned}$$

Thus, by Lemma 6.8:

$$\begin{aligned}
&t \models \varphi_{i,j}(v,w) \ \text{ iff } \ t \models \mathrm{trans}^{\chi}_{i,j}(v,w)\\
&\text{iff } (p_i(v), p_j(w)) \in \prec_G^+ \ \text{ iff } \ p_i(v) \prec_G^+ p_j(w)\ .
\end{aligned} \tag{6.5}$$

Now we define the sentence $\varphi^{\mathrm{wnc}} \in \mathrm{MSO}(\Sigma)$ as follows:

$$\varphi^{\mathrm{wnc}} = \big(\exists x.\exists y.(\mathrm{root}(x) \wedge \mathrm{root}(y) \wedge \varphi_{1,1})\big) \vee$$

$$\bigvee_{i\in[n]}\big(\exists x.\exists y.\big((x\equiv y)\wedge\varphi_{i,i}\wedge\exists y.(\mathrm{root}(y)\wedge\varphi_{i,1})\big)\big)\ ,$$

where $\mathrm{root}(z)$ is the macro $\forall z'.\bigwedge_{i\in[\mathrm{maxrk}(\Sigma)]}\neg\mathrm{edge}_i(z',z)$, for every $z\in\{x,y\}$. Observe that φ^{wnc} is a sentence and that its construction is effective. Let $t\in T_\Sigma$ and $G=\mathrm{G}^{\mathrm{dep}}_{M,t}$. Then

$$\begin{aligned}
& t\models\varphi^{\mathrm{wnc}} \\
\text{iff } & t\models\varphi_{1,1}(\varepsilon,\varepsilon) \text{ or } \exists\, i\in[n]:\exists\, v\in\mathrm{pos}(t): t\models\varphi_{i,i}(v,v), t\models\varphi_{i,1}(v,\varepsilon) \\
\text{iff } & p_1(\varepsilon)\prec^+_G p_1(\varepsilon) \text{ or } \exists\, p_i(v)\in P(\mathrm{pos}(t)): \\
& \qquad p_i(v)\prec^+_G p_i(v), p_i(v)\prec^+_G p_1(\varepsilon) && \text{(by Equivalence (6.5))} \\
\text{iff } & \exists\, c\in P(\mathrm{pos}(t)): c\prec^+_G c, c\prec^*_G q(\varepsilon) && (q=p_1 \text{ by definition}) \\
\text{iff } & \mathrm{H}^{q(\varepsilon),\emptyset}_G \text{ is infinite.} && \text{(by Lemma 2.25)}
\end{aligned}$$

■

The following corollary is an immediate consequence of Theorem 6.4(1) and Lemma 6.14.

Corollary 6.15. *There effectively is an fta $\mathcal{M}^{\mathrm{wnc}}$ such that $\mathcal{L}(\mathcal{M}^{\mathrm{wnc}})=\emptyset$ iff M is weakly non-circular.*

Weighted monadic datalog

In this chapter we study the semantics of mwmd for a certain class of m-monoids, viz. the class of m-monoids that behave like strong bimonoids [44] or semirings [67, 72]. Roughly speaking, we investigate semantics of mwmd that are evaluated in strong bimonoids or semirings instead of m-monoids. In this context we will refer to m-weighted monadic datalog programs simply as weighted monadic datalog programs. This chapter is a revised and extended version of [122]; note that the scope of investigation in [122] is weighted monadic datalog over semirings and unranked trees [127], whereas we study weighted monadic datalog over strong bimonoids and ranked trees in this thesis.

Weighted monadic datalog (for short: wmd) is an extension of the concept of monadic datalog [68, 69]. In fact, monadic datalog is obtained from wmd when employing the Boolean semiring. The method of extending monadic datalog to wmd by introducing weights of some strong bimonoid is similar to the extension of horn calculus to semiring-based constraint logic programming [17, 18, 19].

This chapter is organized as follows. In Section 7.1 we will recall basic concepts concerning strong bimonoids and semirings. In Section 7.2 we study properties of the syntax and semantics of mwmd over strong bimonoids. Section 7.3 deals with the expressive power of weighted monadic datalog and compares it with the class of recognizable tree series. In Section 7.4 we will investigate the evaluation complexity of weighted monadic datalog and provide sufficient conditions that allow for an efficient computation of the semantics of wmd.

7.1 Strong bimonoids and semirings

First let us recall the notions of strong bimonoids and semirings. Moreover, let us introduce the class of m-monoids that behave like strong bimonoids or semirings. A strong bimonoid is an algebra that consists of two monoids, called the additive and multiplicative monoid, where the additive monoid is commutative and its neutral element is absorbing wrt the multiplicative operation. A semiring is a strong bimonoid such that multplication distributes over addition.

Definition 7.1. A ***strong bimonoid*** [44] is a tuple $\mathcal{S} = (S, +, \cdot, \mathbf{0}, \mathbf{1})$ where

- S is a set,
- $(S, +, \mathbf{0})$ is a commutative monoid,
- $(S, \cdot, \mathbf{1})$ is a monoid,
- $\mathbf{0}$ is absorbing wrt $\cdot$.

The strong bimonoid $\mathcal{S}$ is called a ***semiring*** [67, 72] if the operation $\cdot$ distributes over $+$, i.e., for every $a, b, c \in S$ we have

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \qquad \text{and} \qquad (a + b) \cdot c = (a \cdot c) + (b \cdot c) \ .$$

We call $\mathcal{S}$ ***commutative*** if $\cdot$ is commutative and ***idempotent*** if $+$ is idempotent. Moreover, $\mathcal{S}$ is ***locally finite*** if for every finite subset $S' \subseteq S$ also $\langle S' \rangle_{+,\cdot}$ is finite (where $\langle S' \rangle_{+,\cdot}$ is the smallest set $S'' \subseteq S$ containing $S' \cup \{\mathbf{0}, \mathbf{1}\}$ that is closed under $+$ and $\cdot$) and S is ***additively locally finite*** if for every finite subset $S' \subseteq S$ also $\langle S' \rangle_{+}$ is finite (where $\langle S' \rangle_{+}$ is defined accordingly). □

Example 7.2 (cf. [44, Example 1]). Now we present some example strong bimonoids and semirings.

1. The ***tropical bimonoid*** is the strong bimonoid $(\mathbb{N} \cup \{\infty\}, +, \min, 0, \infty)$ with $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ and the usual extensions of $+$ and min from $\mathbb{N}$ to $\mathbb{N}_\infty$. We note that it is not a semiring, because there are $a, b, c \in \mathbb{N}_\infty$ with $\min\{a, b+c\} \neq \min\{a, b\} + \min\{a, c\}$ (e.g., take $a = b = c \neq 0$).

2. The ***tropical semiring*** is the semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$.

3. The algebra $([0, 1], \oplus, \cdot, 0, 1)$ with the usual multiplication $\cdot$ of real numbers is a strong bimonoid for, e.g., each of the following two definitions of $\oplus$ for every $a, b \in [0, 1]$:
   - $a \oplus b = a + b - a \cdot b$ (called *algebraic sum* in [85]) and
   - $a \oplus b = \min\{a + b, 1\}$ (called *bounded sum* in [85]).

   In neither of the two cases $([0, 1], \oplus, \cdot, 0, 1)$ is a semiring.

4. Let $(C, +, 0)$ be a commutative monoid and let A be the set of all mappings from C into itself with pointwise addition, composition of mappings, constant mapping zero, and the identity mapping. Then A constitutes a strong bimonoid satisfying only one distributivity law (which depends on the order used for defining the composition). Such structures are also called near semirings [133, 92].

5. Let Σ be an alphabet. Consider the strong bimonoid $(\Sigma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$ where $\wedge$ is the longest common prefix operation, $\cdot$ is the usual concatenation of words, and ∞ is a new element such that $w \wedge \infty = \infty \wedge w = w$ and $w \cdot \infty = \infty \cdot w = \infty$ for every $w \in \Sigma^* \cup \{\infty\}$. This bimonoid occurs in investigations for natural language processing, see [108]. It is clear that $(\Sigma^* \cup \{\infty\}, \wedge, \cdot, \infty, \varepsilon)$ is left distributive but not right distributive.

6. The ***Boolean semiring*** is the semiring $(\mathbb{B}, \vee, \wedge, 0, 1)$ with $\mathbb{B}$ consisting of the truth values 0 and 1, and $\vee$ and $\wedge$ are disjunction and conjunction, respectively.

7. Bounded lattices (lattices containing a greatest element 1 and a smallest element 0) are strong bimonoids. As is well known, there are large classes of lattices that are not distributive [71].

8. Moreover, bounded distributive lattices, semiring-reducts of semi-lattice ordered monoids and of complete residuated lattices, and Brouwerian lattices are semirings. □

Every strong bimonoid induces a signature, its associated signature, and an m-monoid, its associated m-monoid, over its associated signature. The construction of the associated m-monoid is taken from [58, Definition 8.5] and [59]. Now let us define these concepts formally.

Definition 7.3. Let $\mathcal{S} = (S, +, \cdot, \mathbf{0}, \mathbf{1})$ be a strong bimonoid. The ***associated signature*** $\Delta_\mathcal{S}$ *of* $\mathcal{S}$ is the signature $\Delta_\mathcal{S} = \{\mathrm{mul}_a^k \mid a \in S, k \in \mathbb{N}\}$, where for every $a \in S$ and $k \in \mathbb{N}$, mul_a^k has the rank k. The ***associated m-monoid*** *of* $\mathcal{S}$ is the m-monoid $\mathcal{A}_\mathcal{S} = (S, +, \mathbf{0}, \theta_\mathcal{S})$ over $\Delta_\mathcal{S}$ such that for every $a \in S$, $k \in \mathbb{N}$, and $a_1, \ldots, a_k \in S$ we have $\theta_\mathcal{S}(\mathrm{mul}_a^k)(a_1, \ldots, a_k) = a_1 \cdot \ldots \cdot a_k \cdot a$.

Let $\leq$ be a partial order on S. We call $(\mathcal{S}, \leq)$ an ***ω-continuous strong bimonoid*** if $(\mathcal{A}_\mathcal{S}, \leq)$ is an ω-continuous m-monoid. We define ***ω-continuous semirings*** likewise. □

Note that in [122] ω-continuous semirings are called ω-cpo semirings. In this thesis we use the name ω-continuous semiring because it is more consistent with the previous definitions (e.g., ω-continuous m-monoids).

Observation 7.4. *Let $\mathcal{S} = (S, +, \cdot, \mathbf{0}, \mathbf{1})$ be a strong bimonoid.*

1. *The associated m-monoid $\mathcal{A}_\mathcal{S}$ of $\mathcal{S}$ is absorptive. Moreover, if $\mathcal{S}$ is a semiring, then $\mathcal{S}_\mathcal{S}$ is a dm-monoid.*

2. *Let $\leq$ be a partial over on S. Then $(\mathcal{S}, \leq)$ is an ω-continuous strong bimonoid iff $(S, \leq)$ is an ω-cpo, and*
   - *$\mathbf{0}$ is the least element of A wrt $\leq$,*
   - *$+$ and $\cdot$ are monotone wrt $\leq$, i.e., $a \leq b$ implies $s + a \leq s + b$, $s \cdot a \leq s \cdot b$, and $a \cdot s \leq b \cdot s$ for every $a, b, s \in S$,*
   - *$+$ and $\cdot$ are continuous wrt $\leq$, i.e., for every $s \in S$ and ω-chain c we have*
     - $\bigvee\{c(i) \mid i \in \mathbb{N}\} + s = \bigvee\{c(i) + s \mid i \in \mathbb{N}\}$,
     - $\bigvee\{c(i) \mid i \in \mathbb{N}\} \cdot s = \bigvee\{c(i) \cdot s \mid i \in \mathbb{N}\}$,
     - $s \cdot \bigvee\{c(i) \mid i \in \mathbb{N}\} = \bigvee\{s \cdot c(i) \mid i \in \mathbb{N}\}$.

Example 7.5 ([122]). Now we give some examples of ω-continuous semirings.

1. The Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$ with the natural order $0 \leq 1$ is an ω-continuous semiring.

2. The tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ together with the reverse natural order of natural numbers is an ω-continuous semiring.

   Note that the arctic semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ together with the natural order on natural numbers is not an ω-continuous semiring because it lacks the element ∞ and is, thus, not an ω-cpo.

3. Every complete semiring $(\mathcal{S}, \sum)$ (cf. [67, Chapter 22]) that is ω-idempotent, i.e., that satisfies $\sum_{i \in I} s = s$ for every non-empty family $(s \mid i \in I)$ over the carrier set S of $\mathcal{S}$, together with the partial order $\leq$, that is defined for every $a, b \in S$ as $a \leq b$ iff $a + b = b$, is an ω-continuous semiring.

4. The semiring of nonnegative real numbers with infinity $(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, \cdot, 0, 1)$ together with the natural order on nonnegative real numbers with infinity is an ω-continuous semiring.

5. A *c-semiring* [17] is a complete[1] semiring $(\mathcal{S}, \sum)$ with $\mathcal{S} = (S, +, \cdot, \mathbf{0}, \mathbf{1})$ such that $\sum$ is ω-idempotent, $\cdot$ is commutative, and $\mathbf{1}$ is absorbing wrt $+$. It is easy to see that the semiring $\mathcal{S}$ together with the binary relation $\leq$ on S defined for every $s_1, s_2 \in S$ by $s_1 \leq s_2$ iff $s_1 + s_2 = s_2$, is an ω-continuous semiring; in fact, $(\mathcal{S}, \leq)$ forms a complete lattice (see [17, Theorem 9]). On the other hand, there are ω-continuous semirings that cannot be represented as c-semirings in this manner (e.g., the ω-continuous semiring from Item 4 of this example is such an ω-continuous semiring because 1 is not absorbing wrt $+$). □

7.2 Weighted monadic datalog programs

In this section we will study mwmd over a given ranked alphabet Σ and the associated signature of a given strong bimonoid $\mathcal{S}$. Moreover, we will investigate properties of their intended semantics, viz. semantics that are evaluated in the associated m-monoid of $\mathcal{S}$.

Throughout this chapter let Σ be a ranked alphabet and $\mathcal{S} = (S, +, \cdot, \mathbf{0}, \mathbf{1})$ be a strong bimonoid.

Definition 7.6. Let M be an mwmd over Σ and $\Delta_{\mathcal{S}}$. Then we also call M a ***weighted monadic datalog program*** (for short: ***wmd***) *over* Σ *and* $\mathcal{S}$.

If M is weakly non-circular, then we will simply write $[\![M]\!]$ instead of $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}_{\mathcal{S}}}$. Moreover, for every partial order $\leq$ on S such that $(\mathcal{S}, \leq)$ is an ω-continuous strong bimonoid, we write $[\![M]\!]_{\leq}$ instead of $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A}_{\mathcal{S}}, \leq)}$. We call $[\![M]\!]$ and $[\![M]\!]_{\leq}$ the ***tree series defined by*** M *and* $\mathcal{S}$ and the ***tree series defined by*** M *and* $(\mathcal{S}, \leq)$, respectively.

Likewise, we will abbreviate the class $\mathrm{WMD}^{\mathrm{fix}}(\Sigma, \Delta_{\mathcal{S}}, \mathcal{A}_{\mathcal{S}})$ by $\mathrm{WMD}(\Sigma, \mathcal{S})$ and the class $\mathrm{WMD}^{\mathrm{fix}}\big(\Sigma, \Delta_{\mathcal{S}}, (\mathcal{A}_{\mathcal{S}}, \leq)\big)$ by $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big)$. For convenience we abbreviate $\mathcal{A}_{\mathcal{S}}\langle\!\langle T_\Sigma \rangle\!\rangle$ by $\mathcal{S}\langle\!\langle T_\Sigma \rangle\!\rangle$. Hence, $\mathrm{WMD}(\Sigma, \mathcal{S}) \subseteq \mathcal{S}\langle\!\langle T_\Sigma \rangle\!\rangle$ and $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big) \subseteq \mathcal{S}\langle\!\langle T_\Sigma \rangle\!\rangle$.

Let M' be a wmd over Σ and $\mathcal{S}$. We say that M and M' are ***semantically equivalent*** if

- M is weakly non-circular iff M' is weakly non-circular,
- if M is weakly non-circular, then $[\![M]\!] = [\![M']\!]$, and
- for every partial order $\leq$ on S such that $(\mathcal{S}, \leq)$ is an ω-continuous strong bimonoid, we have $[\![M]\!]_{\leq} = [\![M']\!]_{\leq}$. □

Note that the notions in the previous definition are well-defined because $\mathcal{A}_{\mathcal{S}}$ is absorptive for every strong bimonoid $\mathcal{S}$ (see Observation 7.4(1)). Furthermore, note that in [122] the query predicate of a weighted monadic datalog program has been defined to be a nullary predicate. For reasons of consistency with the definitions in the previous chapters we require the query predicates of wmd to be unary in this thesis.

As the previous definition suggests, we restrict ourselves to the fixpoint semantics when studying wmd. The following observation (that follows from Theorem 4.53 and Observation 7.4(1)) states that when employing semirings, then this is no real restriction.

[1] i.e., sums are defined for arbitrary families of semiring elements; note that in [18] c-semirings have been defined without the requirement that the sum of an infinite number of elements exists

Observation 7.7. *Let $\mathcal{S}$ be a semiring and M be an wmd over Σ and $\mathcal{S}$. Then* $[\![M]\!] = [\![M]\!]^{\text{fix}}_{\mathcal{A}_\mathcal{S}} = [\![M]\!]^{\text{hyp}}_{\mathcal{A}_\mathcal{S}}$.

Let $\leq$ be a partial order on S such that $(\mathcal{S}, \leq)$ is an ω-continuous strong bimonoid. Then $[\![M]\!]_\leq = [\![M]\!]^{\text{fix}}_{(\mathcal{A}_\mathcal{S},\leq)} = [\![M]\!]^{\text{hyp}}_{(\mathcal{A}_\mathcal{S},\sum)}$, where $(\mathcal{A}_\mathcal{S}, \sum)$ is the ω-complete m-monoid that is related with $(\mathcal{A}_\mathcal{S}, \leq)$ (see Lemma 3.40).

The following example is inspired by [122, Example 3.7]; it is adapted from the setting of unranked trees to ranked trees.

Example 7.8. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}, \sigma^{(2)}\}$ be a ranked alphabet and consider the arctic semiring $\mathcal{S} = (\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$. Consider the wmd $M = (P, R, q)$ over Σ and $\mathcal{S}$ such that $P = \{q^{(1)}, p^{(1)}\}$ and $R = \{r_1, r_2, r_3, r_4, r_5\}$ with

$$\begin{aligned}
r_1 &= q(x) \leftarrow p(y)\,;\{\text{label}_\sigma(x), \text{child}_1(x,y)\}\ ,\\
r_2 &= q(x) \leftarrow q(z)\,;\{\text{label}_\sigma(x), \text{child}_2(x,z)\}\ ,\\
r_3 &= q(x) \leftarrow \text{mul}^0_0\,;\{\text{label}_\alpha(x)\}\ ,\\
r_4 &= p(x) \leftarrow \text{mul}^0_1\,;\{\text{label}_\alpha(x)\}\ ,\\
r_5 &= p(x) \leftarrow \text{mul}^1_1(p(y))\,;\{\text{label}_\gamma(x), \text{child}_1(x,y)\}\ .
\end{aligned}$$

Clearly, the wmd M is weakly non-circular. Now let us compute the semantics $[\![M]\!]$ of M. First consider the tree $t_1 = \alpha$. Then $P(\text{pos}(t_1)) = \{q(\varepsilon), p(\varepsilon)\}$. The sequence of interpretations $\mathcal{T}^i(I_{-\infty})$ is shown in the following table. Let us compute the value of $\mathcal{T}^1(I_{-\infty})(q(\varepsilon))$ explicitely. It is easy to see that $\Phi_{M,t_1,q(\varepsilon)} = \{(r_3, [x \mapsto \varepsilon])\}$. Then $\mathcal{T}^1(I_{-\infty})(q(\varepsilon)) = \text{h}_{I_{-\infty}}\big([x \mapsto \varepsilon]((r_3)_\text{b})\big) = \text{h}_{I_{-\infty}}(\text{mul}^0_0) = \theta_\mathcal{S}(\text{mul}^0_0)() = 0$.

	$\mathcal{T}^0(I_{-\infty})$	$\mathcal{T}^1(I_{-\infty})$	$\mathcal{T}^2(I_{-\infty})$
$q(\varepsilon)$	$-\infty$	0	0
$p(\varepsilon)$	$-\infty$	1	1

Thus, $[\![M]\!](t_1) = 0$. Now consider the input tree $t_2 = \sigma(\alpha, \sigma(\gamma(\alpha), \alpha))$. In the following table we have compiled the sequence of interpretations $\mathcal{T}^i(I_{-\infty})$ for all the relevant atom instances. Let us compute the values of $\mathcal{T}^2(I_{-\infty})(p(21))$ and $\mathcal{T}^3(I_{-\infty})(q(2))$ explicitely. It is easy to check that we have $\Phi_{M,t_2,p(21)} = \{(r_5, [x \mapsto 21, y \mapsto 211])\}$ and $\Phi_{M,t_2,q(2)} = \{(r_1, [x \mapsto 2, y \mapsto 21]), (r_2, [x \mapsto 2, z \mapsto 22])\}$. Therefore, we have $\mathcal{T}^2(I_{-\infty})(p(21)) = \theta_\mathcal{S}(\text{mul}^1_1)\big(\mathcal{T}^1(I_{-\infty})(p(211))\big) = 1 + 1 = 2$ and $\mathcal{T}^3(I_{-\infty})(q(2)) = \max\big\{\mathcal{T}^2(I_{-\infty})(p(21)), \mathcal{T}^2(I_{-\infty})(q(22))\big\} = \max\{2, 0\} = 2$.

	$\mathcal{T}^0(I_{-\infty})$	$\mathcal{T}^1(I_{-\infty})$	$\mathcal{T}^2(I_{-\infty})$	$\mathcal{T}^3(I_{-\infty})$	$\mathcal{T}^4(I_{-\infty})$	$\mathcal{T}^5(I_{-\infty})$
$q(\varepsilon)$	$-\infty$	$-\infty$	1	1	2	2
$q(2)$	$-\infty$	$-\infty$	0	2	2	2
$q(22)$	$-\infty$	0	0	0	0	0
$p(1)$	$-\infty$	1	1	1	1	1
$p(21)$	$-\infty$	$-\infty$	2	2	2	2
$p(211)$	$-\infty$	1	1	1	1	1

We obtain $[\![M]\!](t_2) = 2$. For every $n \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{N}$ we have $[\![M]\!](t) = 1 + \max\{m_i \mid i \in [n]\}$, where $t = \sigma(\gamma^{m_1}(\alpha), \sigma(\gamma^{m_2}(\alpha), \cdots \sigma(\gamma^{m_n}(\alpha), \alpha)\cdots))$, i.e., t is the right-descending comb where the i-the prong is of the form $\gamma^{m_i}(\alpha)$, for every $i \in [n]$. □

For the remainder of this chapter let $M = (P, R, q)$ be a wmd over Σ and $\mathcal{S}$.

The notation we have developed so far is fairly cumbersome. Consider the rules r_3 and r_4 in the previous example. Essentially, their bodies mul_0^0 and mul_1^0 represent the constant semiring elements 0 and 1, respectively. Therefore, we will follow the convention to denote these bodies by 0 and 1 instead, if no confusion arises. We can extend this idea to operations of arity greater than 0 as follows: for every body of the form $\mathrm{mul}_a^k(b_1, \ldots, b_k)$ (for $a \in S$, $k \in \mathbb{N}$, and atoms $b_1, \ldots, b_k$), we will write $b_1, \ldots, b_k, a$ instead. This notation is quite intuitive because when evaluating this body in the m-monoid $\mathcal{A}_\mathcal{S}$, then one computes the product of the values of the instances of the atoms b_1 to b_k and the strong bimonoid element a in this order.

In this manner we can extend this scheme of notation to arbitrary bodies consisting of multiple occurrences of operations of the form mul_a^k, and, hence, denote every body in a flattened form consisting of a finite sequence of atoms and strong bimonoid elements, e.g., we will denote the body $\mathrm{mul}_a^2(\mathrm{mul}_b^1(q()), \mathrm{mul}_c^2(p(), r()))$ by $q(), b, p(), r(), c, a$; this process is reminiscent of transforming a mathematical term into reverse Polish notation [29] or of performing the post-order walk of a tree. We will now formally define this alternative notation and will exhibit its practical use for the computation of the semantics of wmd in the following lemma.

Definition 7.9. We define the mapping $\xi_M : T_{\Delta_\mathcal{S}}(P(\mathrm{V})) \to (P(\mathrm{V}) \cup S)^*$, called ***flattening** of M*, by structural recursion as follows for every $s \in T_{\Delta_\mathcal{S}}(P(\mathrm{V}))$:

- if $s \in P(\mathrm{V})$, then $\xi_M(s) = s$, and
- if $s = \mathrm{mul}_a^k(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}$, $a \in S$, and $s_1, \ldots, s_k \in T_{\Delta_\mathcal{S}}(P(\mathrm{V}))$, then $\xi_M(s) = \xi_M(s_1) \cdots \xi_M(s_k)\ a$.

For better readability, we will separate the symbols in the string $\xi_M(s)$ by commas, e.g., we write $p(), a, q(x), b$ instead of $p()\, a\, q(x)\, b$.

Let $r \in R$, $k \in \mathbb{N}$, and $b_1, \ldots, b_k \in P(\mathrm{V}) \cup S$ such that $\xi(r_\mathrm{b}) = b_1, \ldots, b_k$. We put $\mathrm{size}(r) = k + |r_\mathrm{G}|$ and $\mathrm{size}(M) = \sum_{r \in R} \mathrm{size}(r)$. □

Lemma 7.10. *For every $r \in R$ let $k_r \in \mathbb{N}$ and $b_1^r, \ldots, b_{k_r}^r \in P(\mathrm{V}) \cup S$ such that $\xi_M(r_\mathrm{b}) = b_1^r, \ldots, b_{k_r}^r$. Moreover, let $t \in T_\Sigma$ and $I \in \mathcal{I}$. Then for every $c \in P(\mathrm{pos}(t))$ we have*

$$\mathcal{T}(I)(c) = \sum\nolimits_{(r,\rho) \in \Phi_{M,t,c}} I(\rho(b_1^r)) \cdot \ldots \cdot I(\rho(b_{k_r}^r)) ,$$

where we put $\rho(a) = a$ and $I(a) = a$ for every $a \in S$.

Proof. Since $\mathcal{T}(I)(c) = \sum_{(r,\rho) \in \Phi_{M,t,c}} \mathrm{h}_I(\rho(r_\mathrm{b}))$ for every $c \in P(\mathrm{pos}(t))$, it suffices to show for every $s \in T_{\Delta_\mathcal{S}}(P(\mathrm{V}))$, $\rho : \mathrm{var}(s) \to \mathrm{pos}(t)$, $l \in \mathbb{N}$, and $b_1, \ldots, b_l \in P(\mathrm{V}) \cup S$ with $\xi_M(s) = b_1, \ldots, b_l$ that $\mathrm{h}_I(\rho(s)) = I(\rho(b_1)) \cdot \ldots \cdot I(\rho(b_l))$. We give a proof by structural induction.

Induction base. If $s \in P(\mathrm{V})$, then $l = 1$ and $b_1 = s$; thus, $\mathrm{h}_I(\rho(s)) = I(\rho(s)) = I(\rho(b_1))$.

Induction step. Suppose that $s = \mathrm{mul}_a^k(s_1, \ldots, s_k)$ for some $k \in \mathbb{N}$, $a \in S$, and $s_1, \ldots, s_k \in T_{\Delta_\mathcal{S}}(P(\mathrm{V}))$. For every $i \in [k]$ let $l_i \in \mathbb{N}$ and $b_1^i, \ldots, b_{l_i}^i \in P(\mathrm{V}) \cup S$ such that $\xi_M(s_i) = b_1^i, \ldots, b_{l_i}^i$. Then $b_1^1, \ldots, b_{l_1}^1, \ldots, b_1^k, \ldots, b_{l_k}^k, a = b_1, \ldots, b_l$ and we obtain

$$\mathrm{h}_I(\rho(s)) = \theta_\mathcal{S}(\mathrm{mul}_a^k)\big(\mathrm{h}_I(\rho(s_1)), \ldots, \mathrm{h}_I(\rho(s_k))\big) = \mathrm{h}_I(\rho(s_1)) \cdot \ldots \cdot \mathrm{h}_I(\rho(s_k)) \cdot a$$

$$= I(\rho(b_1^1)) \cdot \ldots \cdot I(\rho(b_{l_1}^1)) \cdot \ldots \cdot I(\rho(b_1^k)) \cdot \ldots \cdot I(\rho(b_{l_k}^k)) \cdot a \qquad \text{(by ind. hyp.)}$$
$$= I(\rho(b_1^1)) \cdot \ldots \cdot I(\rho(b_{l_1}^1)) \cdot \ldots \cdot I(\rho(b_1^k)) \cdot \ldots \cdot I(\rho(b_{l_k}^k)) \cdot I(\rho(a))$$
$$= I(\rho(b_1)) \cdot \ldots \cdot I(\rho(b_l)) \;. \qquad ■$$

Remark 7.11. In the sequel we will denote every rule r of a weighted monadic datalog program M by $r_{\mathrm{h}} \leftarrow \xi_M(r_{\mathrm{b}})\,; r_{\mathrm{G}}$.

Observe that this notation is ambiguous. For example, when denoting the body of a rule by the sequence $p(), q(), b, a$, then it is not clear whether the body is actually the tree $b = \mathrm{mul}_a^2\big(p(), \mathrm{mul}_b^1(q())\big)$ or the tree $b' = \mathrm{mul}_a^1\big(\mathrm{mul}_b^2(p(), q())\big)$. However, this is no problem because b and b' are, roughly speaking, semantically equivalent (clearly, the semantics of wmd does not depend on the actual tree representation r_{b} of the sequence $\xi_M(r_{\mathrm{b}})$ due to Lemma 7.10).

Moreover, observe that there are sequences $\bar{b}$ of atoms and strong bimonoid elements that do not correspond to trees over $\Delta_{\mathcal{S}}$ indexed by $P(\mathrm{V})$, i.e., there is no $s \in T_{\Delta_{\mathcal{S}}}(P(\mathrm{V}))$ with $\xi_M(s) = \bar{b}$. An example of such a sequence is $a, p(), q()$. In this case we will silently assume that the sequence is succeeded by the strong bimonoid element $\mathbf{1}$; e.g., $a, p(), q()$ is the result of flattening the tree $\mathrm{mul}_{\mathbf{1}}^3(a, p(), q())$.

Note that in [122] the syntax of wmd rules has been defined in such a way that the body and the guard of a rule are denoted together by one sequence of user-defined atoms, semiring elements, and structural atoms. However, in this thesis we denote the body and the guard of a wmd rule separately; this is due to reasons of consistency with the definition of the syntax of mwmd. □

Let us conclude this section with one observation and one lemma, which is an adaptation of Lemma 4.33 to the setting of wmd.

Observation 7.12 (cf. [122, Obs. 3.28]). *Let M be weakly non-circular. Then for every $t \in T_\Sigma$ we have $[\![M]\!](t) \in \langle S' \rangle_{+,\cdot}$, where S' is the set of strong bimonoid elements occurring in the rules of M.*

Lemma 7.13. *Suppose that $\mathcal{S}$ is a locally finite strong bimonoid and that $(\mathcal{S}, \leq)$ is an ω-complete strong bimonoid. Then there is an $n \in \mathbb{N}$ with $\mathcal{T}^\omega = \mathcal{T}^n(I_{\mathbf{0}})$. If $\mathcal{S}$ is finite, then there is such an n with $n \leq (|S| - 1) \cdot |P| \cdot |\mathrm{pos}(t)|$.*

Proof. In view of Lemma 4.33 it suffices to show that $\mathcal{A}_{\mathcal{S}}$ is olf. Let $\Delta' \subseteq \Delta_{\mathcal{S}}$ be finite. Then there is a finite set $S' \subseteq S$ such that for every $\delta \in \Delta'$ there is an $a \in S'$ and $k \in \mathbb{N}$ with $\delta = \mathrm{mul}_a^k$. Clearly, then also $S'' = \langle S' \rangle_{+,\cdot}$ is finite. It is easy to see that S'' contains $\mathbf{0}$ and is closed under $+$ and under $\theta_{\mathcal{S}}(\mathrm{mul}_a^k)$, for every $a \in S'$ and $k \in \mathbb{N}$ and, thus, under $\theta_{\mathcal{S}}(\delta)$ for every $\delta \in \Delta'$. Hence, $\mathcal{A}_{\mathcal{S}}$ is olf. ■

7.3 Expressiveness of wmd

In this section we study the expressive power of wmd, i.e., we compare the classes $\mathrm{WMD}(\Sigma, \mathcal{S})$ and $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big)$ with classes of tree series over strong bimonoids. We will investigate the relationships between (i) the class of tree series that are defined by weakly non-circular wmd and the class of tree series defined by arbitrary wmd, (ii) wmd over the Boolean semiring and monadic datalog, and (iii) the class of tree series that are defined by non-circular wmd and the class of recognizable tree series.

7.3.1 Comparison of finitary with infinitary semantics

First let us study how the two classes $\mathrm{WMD}(\Sigma, \mathcal{S})$ and $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big)$ for a given ranked alphabet Σ and an ω-continuous strong bimonoid $(\mathcal{S}, \leq)$ relate. Due to Corollary 4.35 we immediately obtain that the class $\mathrm{WMD}(\Sigma, \mathcal{S})$ is contained in $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big)$.

Corollary 7.14 (cf. [122, Corollary 3.30]). *Let $(\mathcal{S}, \leq)$ be an ω-continuous strong bimonoid. Then $\mathrm{WMD}(\Sigma, \mathcal{S}) \subseteq \mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big)$.*

The following lemma shows that arbitrary wmd have stronger expressiveness than weakly non-circular wmd.

Lemma 7.15 (cf. [122, Lemma 3.31]). *There is a ranked alphabet Σ and a commutative ω-complete semiring $(\mathcal{S}, \leq)$ with $\mathrm{WMD}\big(\Sigma, (\mathcal{S}, \leq)\big) \setminus \mathrm{WMD}(\Sigma, \mathcal{S}) \neq \emptyset$.*

Proof. Let $\mathcal{S} = (\mathcal{P}(\mathbb{N}), \cup, \circ, \emptyset, \{0\})$ where $N_1 \circ N_2 = \{n_1 + n_2 \mid n_1 \in N_1, n_2 \in N_2\}$ for every $N_1, N_2 \subseteq \mathbb{N}$. Observe that $(\mathcal{S}, \subseteq)$ is a commutative ω-complete semiring. Let $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}\}$ and consider the wmd $M = (P, R, q)$ such that $P = \{c^{(1)}, q^{(1)}\}$ and R contains the following rules:

$$c(x) \leftarrow \{0\} \,;\{\mathrm{leaf}(x)\}\,, \qquad c(x) \leftarrow c(y), \{1\} \,;\{\mathrm{child}_1(x, y)\}\,,$$
$$q(x) \leftarrow \{0\} \,;\emptyset\,, \qquad q(x) \leftarrow q(x), c(x), \{0\} \,;\{\mathrm{root}(x)\}\,.$$

Observe that, due to the last rule, M is not weakly non-circular. Let $k \in \mathbb{N}$ and $t = \gamma^k(\alpha)$. It is easy to verify that the following equivalences hold for every $i \in \{0, \ldots, k\}$ and $n \in \mathbb{N}$:

$$\mathcal{T}^n(I_\emptyset)(c(1^i)) = \begin{cases} \emptyset\,, & \text{if } n < k - i + 1, \\ \{k - i\}, & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^n(I_\emptyset)(q(\varepsilon)) = \begin{cases} \emptyset\,, & \text{if } n = 0, \\ \{0\}, & \text{if } 0 < n \leq k + 1, \\ \{k \cdot j \mid 0 \leq j < n - k\}, & \text{if } k + 1 < n. \end{cases}$$

Then $\{k \cdot i \mid i \in \mathbb{N}\} = \mathcal{T}^\omega(q(\varepsilon)) = [\![M]\!]_\subseteq(t)$.

Now we assume that there is a weakly non-circular wmd $M' = (P', R', q')$ over Σ and $\mathcal{S}$ such that $[\![M']\!] = [\![M]\!]_\subseteq$. Let S' be the finite set of semiring elements of $\mathcal{S}$ that occur in the rules of M'. Then $[\![M']\!](t) \in \langle S' \rangle_{\cup,\circ}$, for every $t \in T_\Sigma$, by Observation 7.12. It suffices to show that there is a $k \in \mathbb{N}$ such that $\{k \cdot i \mid i \in \mathbb{N}\} \notin \langle S' \rangle_{\cup,\circ}$.

Let $N \subseteq \mathbb{N}$ be infinite. We define the least element distance of N, denoted by $l(N)$, as $l(N) = \min\{i \in \mathbb{N} \mid \text{there is a } j \in N \text{ with } j + i \in N\}$. Now let m be the maximal of such distances in S', i.e.,

$$m = \max\{i \in \mathbb{N} \mid \text{there is an infinite } N \in S' \text{ with } i = l(N)\}\,.$$

The number m is well-defined because S' is finite. Now observe that $\langle S' \rangle_{\cup,\circ}$ cannot contain an infinite set N with $l(N) > m$ because:

- For every $N_1, N_2 \subseteq \mathbb{N}$ we have that if $N_1 \cup N_2$ is infinite or $N_1 \circ N_2$ is infinite, then N_1 or N_2 must be infinite.
- If $N_1, N_2 \subseteq \mathbb{N}$ such that N_1 is infinite, then we have $l(N_1 \cup N_2) \leq l(N_1)$ and $l(N_1 \circ N_2) \leq l(N_1)$ (if $N_1 \circ N_2$ is infinite).

Thus, there is a $k \in \mathbb{N}$ (viz. $k = m + 1$) such that $\{k \cdot i \mid i \in \mathbb{N}\} \notin \langle S' \rangle_{\cup,\circ}$, because $l(\{k \cdot i \mid i \in \mathbb{N}\}) = m + 1$. ∎

7.3.2 Comparison with monadic datalog

The class of tree language that can be defined in monadic datalog is the class of MSO-definable tree languages (see [69, Corollary 4.7]) and, thus, the class of recognizable tree languages (see Theorem 6.4)(2). Here we show that the class of languages that are the support of wmd-definable tree series over the Boolean semiring is also the class of recognizable tree languages.

In analogy to the definition of the set $\mathrm{WMD}(\Sigma, \mathcal{S})$ we define $\mathrm{MD}(\Sigma)$ to comprise of all tree languages definable by nullary monadic datalog queries over Σ (cf. [69]). The following lemma states the correspondence between monadic datalog and weighted monadic datalog over the Boolean semiring $\mathbb{B}$ (where for every class of tree series Ψ we define $\mathrm{supp}(\Psi) = \{\mathrm{supp}(\lambda) \mid \lambda \in \Psi\}$).

Lemma 7.16 (cf. [122, Lemma 3.22]). *Let $\mathbb{B}$ be the Boolean semiring and $\leq$ be the natural order on $\mathbb{B}$. Then* $\mathrm{supp}\big(\mathrm{WMD}(\Sigma, (\mathbb{B}, \leq))\big) = \mathrm{MD}(\Sigma)$.

Proof. Here we only give a proof idea. The inclusion $\mathrm{supp}\big(\mathrm{WMD}(\Sigma, (\mathbb{B}, \leq))\big) \supseteq \mathrm{MD}(\Sigma)$ is easy to see because every monadic datalog query (R, q) over Σ can be considered as a wmd M over Σ and $\mathbb{B}$, and the support of the tree series defined by M and $(\mathbb{B}, \leq)$ coincides with the semantics of (R, q) intended in [68, 69].

For the inclusion $\mathrm{supp}\big(\mathrm{WMD}(\Sigma, (\mathbb{B}, \leq))\big) \subseteq \mathrm{MD}(\Sigma)$ let M be a wmd over Σ to $\mathbb{B}$. It is easy to see that M can easily be transformed into an equivalent monadic datalog program as follows: (i) remove every occurrence of the semiring element 1 in any of the rules in M, and (ii) drop every rule that contains the semiring element 0. ■

7.3.3 Comparison with recognizable tree series

Suppose that $\mathcal{S}$ is a semiring. In this section we compare the class $\mathrm{WMD}(\Sigma, \mathcal{S})$ with the class $\mathrm{Rec}(\Sigma, S)$ of recognizable tree series over Σ and $\mathcal{S}$. This is a robust and important subclass of $\mathcal{S}\langle\langle T_\Sigma \rangle\rangle$ and is characterized by, e.g., weighted tree automata [15, 63], semiring weighted MSO-logic [41, 45], rational tree series expressions [22, 43, 113], and weighted regular grammars [4]. For a thorough introduction into the theory of recognizable tree series we refer to [63]. We will show that the class of tree series that are defined by non-circular wmd contains the class of recognizable tree series and that this inclusion is, in general, proper (see Theorem 7.18).

Here we will define the class of recognizable tree series in terms of weighted bottom-up tree automata. First we recall this concept.

Definition 7.17. Suppose that $\mathcal{S}$ is a semiring. A ***weighted tree automaton*** (for short: ***wta***) *over Σ and $\mathcal{S}$* is a triple $\mathcal{M} = (Q, \mu, F)$, where Q is a finite, non-empty set, $\mu = (\mu_k \mid k \in \mathbb{N})$ is a family of mappings $\mu_k : Q^k \times \Sigma^{(k)} \times Q \to S$, and $F \subseteq Q$.

Let $t \in T_\Sigma$. The set $R_{\mathcal{M}}(t)$ of ***successful runs*** *over $\mathcal{M}$ and t* is defined to be the set $\{\kappa \mid \kappa : \mathrm{pos}(t) \to Q, \kappa(\varepsilon) \in F\}$. Every $\kappa \in R_{\mathcal{M}}(t)$ induces a mapping $\mathrm{wt}_{\mathcal{M},t}(\kappa) : \mathrm{pos}(t) \to S$ which is defined as follows for every $w \in \mathrm{pos}(t)$:

$$\mathrm{wt}_{\mathcal{M},t}(\kappa)(w) = \mathrm{wt}_{\mathcal{M},t}(\kappa)(w1) \cdot \ldots \cdot \mathrm{wt}_{\mathcal{M},t}(\kappa)(wk) \cdot \mu_k\big(\kappa(w1) \cdots \kappa(wk), t(w), \kappa(w)\big) \ ,$$

where $k = \mathrm{rk}(t(w))$. If $\mathcal{M}$ and t are clear from the context, then we also write $\mathrm{wt}(\kappa)$ instead of $\mathrm{wt}_{\mathcal{M},t}(\kappa)$.

The ***tree series recognized*** *by* $\mathcal{M}$, denoted by $[\![\mathcal{M}]\!] \in \mathcal{S}\langle\langle T_\Sigma\rangle\rangle$, is defined for every $t \in T_\Sigma$ by $[\![\mathcal{M}]\!](t) = \sum_{\kappa \in R_{\mathcal{M}}(t)} \mathrm{wt}(\kappa)(\varepsilon)$. A tree series $\lambda \in \mathcal{S}\langle\langle T_\Sigma\rangle\rangle$ is called ***recognizable*** over Σ and $\mathcal{S}$ if there is a wta $\mathcal{M}$ over Σ and $\mathcal{S}$ such that $[\![\mathcal{M}]\!] = \lambda$. The set of all recognizable tree series over Σ and $\mathcal{S}$ is denoted by $\mathrm{Rec}(\Sigma, \mathcal{S})$. □

We point out that sometimes in the literature the mapping μ_k has the type $\Sigma^{(k)} \to S^{Q^k \times Q}$; clearly this is equivalent to the type we introduce here. Note that in [114] the definition of wta has been extended to arbitrary strong bimonoids; here we restrict ourselves to wta over semirings.

Now we present the result comparing the expressiveness of wta and wmd. The following theorem is an adaptation of [122, Theorem 4.4] from unranked to ranked trees.

Theorem 7.18. $\mathrm{Rec}(\Sigma, \mathcal{S}) \subseteq \mathrm{WMD}(\Sigma, \mathcal{S})$ *for every commutative semiring* $\mathcal{S}$*. There is a ranked alphabet* Σ *and a commutative semiring* $\mathcal{S}$ *such that* $\mathrm{Rec}(\Sigma, \mathcal{S})$ *is a proper subset of* $\mathrm{WMD}(\Sigma, \mathcal{S})$*.*

Proof. First we show that $\mathrm{Rec}(\Sigma, \mathcal{S}) \subseteq \mathrm{WMD}(\Sigma, \mathcal{S})$ for every commutative semiring $\mathcal{S}$. To this end let $\mathcal{M} = (Q, \mu, F)$ be a wta over Σ and $\mathcal{S}$. We construct a non-circular wmd M over Σ and $\mathcal{S}$ such that $[\![M]\!] = [\![\mathcal{M}]\!]$ as follows: $M = (P, R, q)$ with $P^{(0)} = \emptyset$ and $P^{(1)} = Q \cup \{q\}$, where $q \notin Q$, and

$$R = \{r_{k,\sigma,p,p_1,\ldots,p_k} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, p, p_1, \ldots, p_k \in Q\} \cup \{r_p \mid p \in F\} ,$$

such that for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, p, p_1, \ldots, p_k \in Q$ we have

$$r_{k,\sigma,p,p_1,\ldots,p_k} = p(\mathrm{x}_\varepsilon) \leftarrow \mathrm{mul}^k_{\mu_k(p_1\cdots p_k,\sigma,p)}\big(p_1(\mathrm{x}_1), \ldots, p_k(\mathrm{x}_k)\big);$$
$$\{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \ldots, \mathrm{child}_k(\mathrm{x}_\varepsilon, \mathrm{x}_k)\} ,$$

and for every $p \in F$ we have

$$r_p = q(\mathrm{x}_\varepsilon) \leftarrow p(\mathrm{x}_\varepsilon)\, ; \{\mathrm{root}(\mathrm{x}_\varepsilon)\} .$$

Note that we have denoted the bodies of the rules of M in tree form instead of the flattened sequence form for simplifying the following proof. It is easy to see that M is non-circular.

Now we prove that $[\![M]\!] = [\![\mathcal{M}]\!]$. Let $t \in T_\Sigma$ and define the mapping $\pi : R_{\mathcal{M}}(t) \to \mathrm{H}_G^{q(\varepsilon)}$ as follows (where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$). Let $\kappa \in R_{\mathcal{M}}(t)$. Then $\pi(\kappa)$ is defined as the derivation $\eta \in \mathrm{H}_G^{q(\varepsilon)}$ such that $\mathrm{pos}(\eta) = \{\varepsilon\} \cup \{1 \cdot w \mid w \in \mathrm{pos}(t)\}$, $\eta(\varepsilon) = (r_{\kappa(\varepsilon)}, [\mathrm{x}_\varepsilon \mapsto \varepsilon])$ and for every $w \in \mathrm{pos}(t)$ we have

$$\eta(1 \cdot w) = (r_{k,t(w),\kappa(w),\kappa(w1),\ldots,\kappa(wk)}, [\mathrm{x}_\varepsilon \mapsto w, \mathrm{x}_1 \mapsto w1, \ldots, \mathrm{x}_k \mapsto wk]) ,$$

where $k = \mathrm{rk}(t(w))$. Let us prove that this definition is correct, i.e., that $\eta = \pi(\kappa) \in \mathrm{H}_G^{q(\varepsilon)}$.

- First we show $\eta \in T_{\Phi_{M,t}}$, i.e., that for every $w \in \mathrm{pos}(\eta)$, $w \cdot (\mathrm{rk}(\eta(w)) + 1) \notin \mathrm{pos}(\eta)$ and, if $\mathrm{rk}(\eta(w)) \neq 0$, then $w \cdot \mathrm{rk}(\eta(w)) \in \mathrm{pos}(\eta)$. This follows immediately from the the definition of $\mathrm{pos}(\eta)$ and the facts that $\mathrm{rk}(\eta(\varepsilon)) = \mathrm{rk}\big((r_{\kappa(\varepsilon)}, [\mathrm{x}_\varepsilon \mapsto \varepsilon])\big) = 1$ and that, for every $w \in \mathrm{pos}(t)$, $\mathrm{rk}(\eta(1 \cdot w)) = \mathrm{rk}(t(w))$.
- Now we show that $\mathrm{out}(\eta(\varepsilon)) = q(\varepsilon)$. Clearly, $\mathrm{out}(\eta(\varepsilon)) = [\mathrm{x}_\varepsilon \mapsto \varepsilon]((r_{\kappa(\varepsilon)})_{\mathrm{h}}) = q(\varepsilon)$.

- Finally, we show that for every $w \in \mathrm{pos}(\eta)$ and $i \in [\mathrm{rk}(\eta(w))]$ we have that $\mathrm{out}(\eta(wi)) = \mathrm{in}_i(\eta(w))$. If $w = \varepsilon$, then $i = 1$ and $\mathrm{out}(\eta(wi)) = \mathrm{out}(\eta(1 \cdot \varepsilon)) = \big(\kappa(\varepsilon)\big)(\varepsilon) = \mathrm{in}_1(\eta(\varepsilon)) = \mathrm{in}_i(\eta(w))$. If $w = 1 \cdot v$ for some $v \in \mathrm{pos}(t)$, then $\mathrm{out}(\eta(wi)) = \mathrm{out}(\eta(1vi)) = \big(\kappa(vi)\big)(vi) = \mathrm{in}_i(\eta(1v)) = \mathrm{in}_i(\eta(w))$.

Hence, $\pi : R_{\mathcal{M}}(t) \to \mathrm{H}_G^{q(\varepsilon)}$. We show that π is a bijection. First we prove that π is injective. Let $\kappa, \kappa' \in R_{\mathcal{M}}(t)$ be distinct. Then there is a $w \in \mathrm{pos}(t)$ with $\kappa(w) \neq \kappa'(w)$. It is easy to see that this implies $\pi(\kappa)(1 \cdot w) \neq \pi(\kappa')(1 \cdot w)$; hence, $\pi(\kappa) \neq \pi(\kappa')$. Now we show that π is surjective onto $\mathrm{H}_G^{q(\varepsilon)}$. Let $\eta \in \mathrm{H}_G^{q(\varepsilon)}$.

- We prove that $\mathrm{pos}(\eta) = \{\varepsilon\} \cup \{1 \cdot w \mid w \in \mathrm{pos}(t)\}$. To this end we show by induction on the length of w that for every $w \in \mathrm{pos}(t)$ we have that $1 \cdot w \in \mathrm{pos}(\eta)$ and that there is a $p_w \in Q$ such that $\mathrm{out}(\eta(1 \cdot w)) = p_w(w)$. First let us consider the base case, i.e., $w = \varepsilon$. Since $\mathrm{out}(\eta(\varepsilon)) = q(\varepsilon)$, we obtain that $\mathrm{rk}(\eta(\varepsilon)) = 1$ and that there is a $p_\varepsilon \in Q$ such that $\mathrm{out}(\eta(1)) = \mathrm{in}_1(\eta(\varepsilon)) = p_\varepsilon(\varepsilon)$; note that p_ε is even in F. Now let us assume that there is a $v \in \mathrm{pos}(t)$ and $i \in [\mathrm{rk}(t(v))]$ such that $w = vi$. The induction hypothesis yields that $1 \cdot v \in \mathrm{pos}(\eta)$ and that there is a $p_v \in Q$ such that $\mathrm{out}(\eta(1 \cdot v)) = p_v(v)$. We obtain that $\mathrm{rk}(\eta(1 \cdot v)) = \mathrm{rk}(t(v))$ due to the definition of R. Hence, $i \in [\mathrm{rk}(\eta(1 \cdot v))]$, i.e., $1 \cdot w = 1 \cdot v \cdot i \in \mathrm{pos}(\eta)$. Moreover, the definition of R implies that there is a $p_w \in Q$ such that $\mathrm{out}(\eta(1 \cdot w)) = \mathrm{out}(\eta(1 \cdot v \cdot i)) = \mathrm{in}_i(\eta(1 \cdot v)) = p_w(vi) = p_w(w)$. This finishes the inductive proof.

  We have shown that $\mathrm{rk}(\eta(\varepsilon)) = 1$ and that for every $w \in \mathrm{pos}(t)$, $\mathrm{rk}(\eta(1 \cdot w)) = \mathrm{rk}(t(w))$. This implies that $\mathrm{pos}(\eta) = \{\varepsilon\} \cup \{1 \cdot w \mid w \in \mathrm{pos}(t)\}$.

- Let $\kappa \in R_{\mathcal{M}}(t)$ be defined by $\kappa(w) = p_w$ for every $w \in \mathrm{pos}(t)$. Clearly, $\kappa \in R_{\mathcal{M}}(t)$ because $p_\varepsilon \in F$, as we have shown in the previous item. It is easy to check that $\pi(\kappa) = \eta$.

Now we show that, for every $\kappa \in R_{\mathcal{M}}(t)$, we have $\mathrm{wt}(\kappa)(\varepsilon) = h\big(\mathrm{h}_{M,t}(\pi(\kappa))\big)$, where h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to $(S, \theta_{\mathcal{S}})$. To this end we show by (reverse) induction that for every $w \in \mathrm{pos}(t)$ we have $\mathrm{wt}(\kappa)(w) = h\big(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w})\big)$. This implies $\mathrm{wt}(\kappa)(\varepsilon) = h\big(\mathrm{h}_{M,t}(\pi(\kappa))\big)$ because $\mathrm{h}_{M,t}(\pi(\kappa)) = \mathrm{h}_{M,t}(\pi(\kappa)|_1)$ by the definition of π and R.

Let $w \in \mathrm{pos}(t)$ and $k = \mathrm{rk}(t(w))$ and assume that for every $i \in [k]$ we have $\mathrm{wt}(\kappa)(w \cdot i) = h\big(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w \cdot i})\big)$. Then

$$\begin{aligned}
&\mathrm{wt}(\kappa)(w) = \mathrm{wt}(\kappa)(w1) \cdot \ldots \cdot \mathrm{wt}(\kappa)(wk) \cdot \mu_k(\kappa(w1) \cdots \kappa(wk), t(w), \kappa(w)) \\
&= h(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w \cdot 1})) \cdot \ldots \cdot h(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w \cdot k})) \cdot \mu_k(\kappa(w1) \cdots \kappa(wk), t(w), \kappa(w)) \\
&\hfill \text{(by the induction hypothesis)} \\
&= h\big(\mathrm{mul}^k_{\mu_k(\kappa(w1)\cdots\kappa(wk), t(w), \kappa(w))}\big(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w \cdot 1}), \ldots, \mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w \cdot k})\big)\big) \\
&\hfill \text{(by the definition of } h \text{ and } \theta_{\mathcal{S}}) \\
&= h\big(\mathrm{h}_{M,t}(\pi(\kappa)|_{1 \cdot w})\big) \;. \qquad \text{(by the definition of } \eta(1 \cdot w) \text{ and } \mathrm{h}_{M,t})
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
[\![M]\!](t) &= [\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}_{\mathcal{S}}}(t) && \text{(by Observation 7.7)} \\
&= \sum\nolimits_{\eta \in \mathrm{H}_G^{q(\varepsilon)}} h(\mathrm{h}_{M,t}(\eta))
\end{aligned}$$

$$= \sum_{\kappa \in R_{\mathcal{M}}(t)} h(\mathrm{h}_{M,t}(\pi(\kappa))) \qquad \text{(because } \pi \text{ is a bijection)}$$
$$= \sum_{\kappa \in R_{\mathcal{M}}(t)} \mathrm{wt}(\kappa)(\varepsilon) \qquad \text{(shown above)}$$
$$= [\![\mathcal{M}]\!](t) \; .$$

This finishes the proof of the first part of this theorem.

Next we show that $\mathrm{Rec}(\Sigma, \mathcal{S}) = \mathrm{WMD}(\Sigma, \mathcal{S})$ does not hold in general. Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\mathcal{S} = (\mathbb{N}, +, \cdot, 0, 1)$ be the semiring of natural numbers. Consider the non-circular wmd $M = (P, R, q)$ over Σ and $\mathcal{S}$ where $P = \{q^{(1)}\}$ and R contains the rules

$$q(x) \leftarrow 2\,; \{\mathrm{leaf}(x)\}\;, \qquad q(x) \leftarrow q(y), q(y)\;; \{\mathrm{child}_1(x,y)\}\;.$$

Then M defines the tree series $\lambda \in \mathcal{S}\langle\!\langle T_\Sigma \rangle\!\rangle$ with $(\lambda, \gamma^n(\alpha)) = 2^{2^n}$ for every $n \in \mathbb{N}$. Assume that λ is recognized by a wta $\mathcal{M} = (Q, \mu, F)$ over Σ and $\mathcal{S}$. But then it is easy to check that $[\![\mathcal{M}]\!](\gamma^n(\alpha)) \leq |Q|^{n+1} \cdot c^{n+1}$ where $c = \max(\bigcup_{k \in \mathbb{N}} \mathrm{ran}(\mu_k))$. So there is an $n \in \mathbb{N}$ with $[\![\mathcal{M}]\!](\gamma^n(\alpha)) < \lambda(\gamma^n(\alpha))$, which contradicts the assumption that λ is recognized by $\mathcal{M}$. Thus, λ cannot be recognized by any wta. ■

7.4 Combined Complexity

In this section we prove that the semantics of a weakly non-circular wmd over a commutative semiring can be computed efficiently. As an auxiliary tool, we first show that for every wmd over a commutative semiring there is a connected wmd which is semantically equivalent to the original one. Both results are based on investigations in [69], which have been extended to the setting of wmd in [122]; the results that we present in this section are stronger versions of the propositions in [122] in the following sense: (i) we relax the requirement that the considered mwmd is non-circular to the requirement that it is weakly non-circular and (ii) we generalize some of these propositions from semirings to strong bimonoids.

First let us motivate this construction. In Section 5.4 we have shown that for an arbitrary mwmd M and an arbitrary m-monoid $\mathcal{A}$ there is in general no connected mwmd that behaves equivalently to M when evaluated in $\mathcal{A}$. However, we have proved that if the mwmd M is restricted and the m-monoid $\mathcal{A}$ is idempotent and distributive, then such a construction can be carried out. We will show that in the setting of wmd we can drop the conditions that M is restricted and that the m-monoid $\mathcal{A}_\mathcal{S}$ is idempotent (note that $\mathcal{A}_\mathcal{S}$ is distributive because $\mathcal{S}$ is a semiring). Unlike the construction we presented in Section 5.4, which consists of two phases (where in the first phase a semiconnected mwmd is constructed from a given restricted mwmd and in the second phase a connected mwmd is constructed from the semiconstructed mwmd that results from the first phase), we will instead provide a direct construction of connected wmd from arbitrary wmd.

Let us consider an example. Let r be the following rule:

$$r_\mathrm{h} = q(x_1)\;,$$
$$r_\mathrm{b} = p(y_1), r(x_2), p(x_3), q(y_2)\;,$$
$$r_\mathrm{G} = \{\mathrm{child}_1(x_1, y_1), \mathrm{child}_3(x_2, y_2), \mathrm{child}_2(x_3, y_3), \mathrm{leaf}(y_3)\}\;.$$

Clearly, r is not connected, because $\{x_1, y_1\}$, $\{x_2, y_2\}$, and $\{x_3, y_3\}$ are the equivalence classes of $\sim_r$. We construct three new rules r_β, r_α^1, and r_α^2 as follows:

$$r_\beta = q(x_1) \leftarrow p(y_1), p_1(), p_2()\;; \{\mathrm{child}_2(x_1, y_1)\}\;,$$

$$r_\alpha^1 = p_1() \leftarrow r(x_2), q(y_2) \, ; \{\text{child}_3(x_2, y_2)\} \ ,$$
$$r_\alpha^2 = p_2() \leftarrow p(x_3) \, ; \{\text{child}_2(x_3, y_3), \text{leaf}(y_3)\} \ ,$$

where $p_1()$ and $p_2()$ are new nullary predicates. Observe that r_β, r_α^1, and r_α^2 are connected and that they, roughly speaking, behave similarly to r for commutative semirings. By replacing r by these three rules in a wmd M over a commutative semiring we obtain a semantically equivalent wmd that has less non-connected rules than M. Now let us prove the correctness of this construction formally.

Definition 7.19. We define the mapping $\text{weight}_M : T_{\Delta_\mathcal{S}}(P(\text{V})) \to S$ by structural recursion as follows: (i) for every $c \in P(\text{V})$ let $\text{weight}_M(c) = \mathbf{1}$ and (ii) for every $k \in \mathbb{N}$, $a \in S$, and $s_1, \dots, s_k \in T_{\Delta_\mathcal{S}}(P(\text{V}))$ let $\text{weight}_M\big(\text{mul}_a^k(s_1, \dots, s_k)\big) = \text{weight}_M(s_1) \cdot \ldots \cdot \text{weight}_M(s_k) \cdot a$. □

Lemma 7.20. *Suppose that $\mathcal{S}$ is a commutative semiring. Moreover, let h be the unique $\Delta_\mathcal{S}$-homomorphism from $\mathcal{T}_{\Delta_\mathcal{S}}$ to $(S, \theta_\mathcal{S})$.*

1. *Let $s \in T_{\Delta_\mathcal{S}}(P(\text{V}))$ and $k = |\text{indyield}(s)|$. Moreover, let $s_1, \dots, s_k \in T_{\Delta_\mathcal{S}}$. Then $h(s \leftarrow s_1 \cdots s_k) = h(s_1) \cdot \ldots \cdot h(s_k) \cdot \text{weight}_M(s)$.*
2. *For every $t \in T_\Sigma$ and $\eta \in \text{H}_G^{q(\varepsilon)}$ (where $G = \text{G}_{M,t}^{\text{dep}}$) we have*
$$h(\text{h}_{M,t}(\eta)) = \prod\nolimits_{w \in \text{pos}(\eta)} \text{weight}_M\big((\text{pr}_1(\eta(w)))_\text{b}\big) \ .$$
3. *Let $r \in R$ and $\xi_M(r_\text{b}) = b_1, \dots, b_k$ for some $k \in \mathbb{N}$ and $b_1, \dots, b_k \in P(\text{V}) \cup S$. Then $\text{weight}_M(r_\text{b}) = \prod_{i \in I} b_i$, where $I = \{i \in [k] \mid b_i \in S\}$.*
4. *We can construct a semantically equivalent connected wmd M' over Σ and $\mathcal{S}$ in time $\mathcal{O}(size(M))$ such that $size(M') = \mathcal{O}(size(M))$ (see [122, Lemma 5.6]).*

Proof. 1. We give a proof by structural induction.

Induction base. If $s \in P(\text{V})$, then $k = 1$ and $h(s \leftarrow s_1) = h(s_1) \cdot \mathbf{1} = h(s_1) \cdot \text{weight}_M(s)$.

Induction step. Suppose that $s = \text{mul}_a^l(s'_1, \dots, s'_l)$ for some $l \in \mathbb{N}$, $a \in S$, and $s'_1, \dots, s'_l \in T_{\Delta_\mathcal{S}}(P(\text{V}))$. For every $i \in [l]$ let $k_i = |\text{indyield}(s'_i)|$. Then for every $i \in [l]$ there are $s_1^i, \dots, s_{k_i}^i \in T_{\Delta_\mathcal{S}}$ such that $s_1^1, \dots, s_{k_1}^1, \dots, s_1^l, \dots, s_{k_l}^l = s_1, \dots, s_k$. Using the fact that $\mathcal{S}$ is commutative we derive

$$\begin{aligned}
&h(s \leftarrow s_1 \cdots s_k) = h\big(\text{mul}_a^l(s'_1 \leftarrow s_1^1 \cdots s_{k_1}^1, \dots, s'_l \leftarrow s_1^l \cdots s_{k_l}^l)\big) \\
&= h(s'_1 \leftarrow s_1^1 \cdots s_{k_1}^1) \cdot \ldots \cdot h(s'_l \leftarrow s_1^l \cdots s_{k_l}^l) \cdot a \\
&= h(s_1^1) \cdot \ldots \cdot h(s_{k_1}^1) \cdot \text{weight}_M(s'_1) \cdot \ldots \cdot h(s_1^l) \cdot \ldots \cdot h(s_{k_l}^l) \cdot \text{weight}_M(s'_l) \cdot a \\
&\hfill \text{(by the induction hypothesis)} \\
&= h(s_1^1) \cdot \ldots \cdot h(s_{k_1}^1) \cdot \ldots \cdot h(s_1^l) \cdot \ldots \cdot h(s_{k_l}^l) \cdot \text{weight}_M(s'_1) \cdot \ldots \cdot \text{weight}_M(s'_l) \cdot a \\
&= h(s_1) \cdot \ldots \cdot h(s_k) \cdot \text{weight}_M(s) \ .
\end{aligned}$$

2. Let $t \in T_\Sigma$. We show by structural induction that for every $\eta \in T_{\Phi_{M,t}}$ we have $h(\text{h}_{M,t}(\eta)) = \prod_{w \in \text{pos}(\eta)} \text{weight}_M\big((\text{pr}_1(\eta(w)))_\text{b}\big)$. This proves the assertion because $\text{H}_G^{q(\varepsilon)} \subseteq$

$T_{\Phi_{M,t}}$. Let $k \in \mathbb{N}$, $e = (r, \rho) \in (\Phi_{M,t})^{(k)}$, and $\eta_1, \dots, \eta_k \in T_{\Phi_{M,t}}$ such that $e(\eta_1, \dots, \eta_k) = \eta$. Then $(\mathrm{pr}_1(\eta(\varepsilon)))_{\mathrm{b}} = (\mathrm{pr}_1(r, \rho))_{\mathrm{b}} = r_{\mathrm{b}}$ and

$$\begin{aligned}
&h(\mathrm{h}_{M,t}(\eta)) = h\big(\rho(r_{\mathrm{b}}) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k)\big) \\
&= h\big(r_{\mathrm{b}} \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k)\big) && (\star) \\
&= h(\mathrm{h}_{M,t}(\eta_1)) \cdot \ldots \cdot h(\mathrm{h}_{M,t}(\eta_k)) \cdot \mathrm{weight}_M(r_{\mathrm{b}}) && \text{(by Statement 1)} \\
&= \prod_{i\in[k]} \prod_{w\in \mathrm{pos}(\eta_i)} \mathrm{weight}_M\big((\mathrm{pr}_1(\eta_i(w)))_{\mathrm{b}}\big) \cdot \mathrm{weight}_M(r_{\mathrm{b}}) && \text{(by ind. hyp.)} \\
&= \prod_{i\in[k]} \prod_{w\in \mathrm{pos}(\eta_i)} \mathrm{weight}_M\big((\mathrm{pr}_1(\eta(i\cdot w)))_{\mathrm{b}}\big) \cdot \mathrm{weight}_M\big((\mathrm{pr}_1(\eta(\varepsilon)))_{\mathrm{b}}\big) \\
&= \prod_{w\in \mathrm{pos}(\eta)} \mathrm{weight}_M\big((\mathrm{pr}_1(\eta(w)))_{\mathrm{b}}\big) \ .
\end{aligned}$$

Equation $(\star)$ follows from $\rho(r_{\mathrm{b}}) \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k) = r_{\mathrm{b}} \leftarrow \mathrm{h}_{M,t}(\eta_1) \cdots \mathrm{h}_{M,t}(\eta_k)$, which is easy to prove by structural induction.

3. This statement is obvious.

4. Assume that M is not yet connected. Then there is an $r \in R$ that is not connected. Moreover, there is an $n \in \mathbb{N}_+$ and there are pairwise disjoint sets $C_0, \dots, C_n \subseteq \mathrm{var}(r)$ such that $\{C_0, \dots, C_n\} = \mathrm{var}(r)/\!\sim_r$ and $\mathrm{var}(r_{\mathrm{h}}) \subseteq C_0$.

Let $k \in \mathbb{N}$ and $b_1, \dots, b_k \in P(\mathrm{V}) \cup S$ such that $r_{\mathrm{b}} = b_1, \dots, b_k$. Let $B = \{b_1, \dots, b_k\}$ and for every $i \in [n]$ let $B_i = P^{(1)}(C_i) \cap B$. Moreover, let $B_0 = P(C_0) \cap B$; clearly, $(B_i \mid i \in \{0, \dots, n\})$ is a generalized partition of $P(\mathrm{V}) \cap B$. For every $i \in \{0, \dots, n\}$ let $k_i \in \mathbb{N}$ and $b^i_1, \dots, b^i_{k_i} \in P(\mathrm{V}) \cup S$ be pairwise distinct such that $B_i = \{b^i_1, \dots, b^i_{k_i}\}$.

Now we define the wmd $M' = (P', R', q)$ over Σ and $\mathcal{S}$ with $P' = P \cup \{p^{(0)}_1, \dots, p^{(0)}_n\}$ (where $p_1, \dots, p_n$ are new predicates not occurring in P) and $R' = R \setminus \{r\} \cup \{r_\beta, r^1_\alpha, \dots, r^n_\alpha\}$ such that, for every $i \in [n]$,

$$\begin{aligned}
r_\beta &= r_{\mathrm{h}} \leftarrow b^0_1, \dots, b^0_{k_0}, \mathrm{weight}_M(r_{\mathrm{b}}), p_1(), \dots, p_n() \,;\, \{g \in r_{\mathrm{G}} \mid \mathrm{var}(g) \subseteq C_0\} \ , \\
r^i_\alpha &= r_{\mathrm{h}} \leftarrow b^i_1, \dots, b^i_{k_i} \,;\, \{g \in r_{\mathrm{G}} \mid \mathrm{var}(g) \subseteq C_i\} \ .
\end{aligned}$$

Observe that $r_\beta, r^1_\alpha, \dots, r^n_\alpha$ can be constructed in time $\mathcal{O}(\mathrm{size}(r))$, that $\mathrm{size}(r_\beta) + \mathrm{size}(r^1_\alpha) + \cdots + \mathrm{size}(r^n_\alpha)$ are of order $\mathcal{O}(\mathrm{size}(r))$, and that the rules $r_\beta, r^1_\alpha, \dots, r^n_\alpha$ are connected. Thus, if M and M' are semantically equivalent, then by repeatedly applying this construction to every non-connected rule in R we obtain a connected wmd M'' in time $\mathcal{O}(\mathrm{size}(M))$ such that M and M'' are semantically equivalent and $\mathrm{size}(M'') = \mathcal{O}(\mathrm{size}(M))$.

It remains to show that M and M' are semantically equivalent; we only sketch this proof because it uses proof techniques that are similar to those that we developed in Chapter 5.

Due to the definition of weak non-circularity, hypergraph semantics, and Observation 7.7 we obtain that it suffices to show that for every $t \in T_\Sigma$, there is a mapping $f : \mathrm{H}^{q(\varepsilon)}_G \to \mathrm{H}^{q(\varepsilon)}_{G'}$ (where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$, and $G' = \mathrm{G}^{\mathrm{dep}}_{M',t}$) such that

(E1) $f : \mathrm{H}^{q(\varepsilon)}_G \to \mathrm{H}^{q(\varepsilon)}_{G'}$ is a bijection and

(E2) for every $\eta \in \mathrm{H}^{q(\varepsilon)}_G$ we have $h(\mathrm{h}_{M,t}(\eta)) = h\big(\mathrm{h}_{M',t}(f(\eta))\big)$, where h is the unique homomorphism from $\mathcal{T}_{\Delta_\mathcal{S}}$ to $(S, \theta_\mathcal{S})$.

Let $t \in T_\Sigma$ and $c \in P(\mathrm{pos}(t))$. We define the mapping $\pi_c : \Phi_{M,t,c} \to \mathrm{H}^{c,P(\mathrm{pos}(t))}_{G'}$ as follows for every $e = (r', \rho) \in \Phi_{M,t,c}$:

- if $r' \neq r$, then $\pi_c(e) = e(\mathrm{in}_1(e), \dots, \mathrm{in}_l(e))$, where $l = \mathrm{rk}(e)$, and
- if $r' = r$, then $\pi_c(e) = (r_\beta, \rho|_{C_0})\big(\rho(c_1^0), \dots, \rho(c_{l_0}^0), \eta_1, \dots, \eta_n\big)$, where, for every $i \in [n]$, $\eta_i = (r_\alpha^i, \rho|_{C_i})\big(\rho(c_1^i), \dots, \rho(c_{l_i}^i)\big)$.

It is easy to check that $\pi_c : \Phi_{M,t,c} \to \mathrm{H}_{G'}^{c,P(\mathrm{pos}(t))}$ is a bijection. Before we proceed, we show that for every $e = (r', \rho) \in \Phi_{M,t,c}$ we have

$$\mathrm{weight}_M(r'_{\mathrm{b}}) = \prod_{\substack{w \in \mathrm{pos}(\pi_c(e)) \\ \pi_c(e)|_w \notin P(\mathrm{pos}(t))}} \mathrm{weight}_{M'}\big((\mathrm{pr}_1(\pi_c(e)(w)))_{\mathrm{b}}\big)\ . \tag{7.1}$$

Proof of Equation (7.1): Let $e = (r', \rho) \in \Phi_{M,t,c}$. Let us consider the case that $r' \neq r$; then we have $\{w \in \mathrm{pos}(\pi_c(e)) \mid \pi_c(e)|_w \notin P(\mathrm{pos}(t))\} = \{\varepsilon\}$ and, hence, the right-hand side of Equation (7.1) is equal to $\mathrm{weight}_{M'}\big((\mathrm{pr}_1(\pi_c(e)(\varepsilon)))_{\mathrm{b}}\big) = \mathrm{weight}_{M'}\big((\mathrm{pr}_1(e))_{\mathrm{b}}\big) = \mathrm{weight}_{M'}(r'_{\mathrm{b}})$. If $r' = r$, then it is easy to see that the right-hand side of Equation (7.1) is equal to $\mathrm{weight}_{M'}\big((r_\beta)_{\mathrm{b}}\big) \cdot \mathrm{weight}_{M'}\big((r_\alpha^1)_{\mathrm{b}}\big) \cdot \ldots \cdot \mathrm{weight}_{M'}\big((r_\alpha^n)_{\mathrm{b}}\big) = \mathrm{weight}_{M'}\big(r_{\mathrm{b}}\big) \cdot \mathbf{1} \cdot \ldots \cdot \mathbf{1}$ due to Statement 3.

Now we lift the family $(\pi_c \mid c \in P(\mathrm{pos}(t)))$ of mappings to the mapping $f : \mathrm{H}_G^{q(\varepsilon)} \to \mathrm{H}_{G'}^{q(\varepsilon)}$. We leave out the details of this definition; it is similar to the lifting of the mapping of the form $\pi_{t,c}$ to the mapping h_{π_t} in Definition 5.15. Then it is easy to show that the fact that, for every $c \in P(\mathrm{pos}(t))$, π_c is a bijection, implies that $f : \mathrm{H}_G^{q(\varepsilon)} \to \mathrm{H}_{G'}^{q(\varepsilon)}$ is a bijection; thus Condition (E1) is satisfied. Moreover Equation (7.1) implies that for every for every $\eta \in \mathrm{H}_G^{q(\varepsilon)}$ we have

$$\prod_{w \in \mathrm{pos}(\eta)} \mathrm{weight}_M\big((\mathrm{pr}_1(\eta(w)))_{\mathrm{b}}\big) = \prod_{w \in \mathrm{pos}(f(\eta))} \mathrm{weight}_{M'}\big((\mathrm{pr}_1(f(\eta)(w)))_{\mathrm{b}}\big)\ .$$

Then Statement 2 implies Condition (E2). ■

A fact that makes (unweighted) monadic datalog remarkably useful is that a monadic datalog program R' can be evaluated in time $\mathcal{O}(size(R') \cdot |\mathrm{pos}(t)|)$ for every input tree t (cf. Theorem 4.2 in [69]), i.e., if we consider a fixed input tree, then evaluating a monadic datalog program can be done in time linear in the size of the query. On the other hand processing a fixed monadic datalog program on multiple input trees can be done in time linear in the size of each tree. So monadic datalog is said to have *linear combined complexity*. This nice complexity result generalizes to the weighted case as follows.

Theorem 7.21 (cf. [122, Theorem 5.7]). *Let M be a wmd and $t \in T_\Sigma$. The following three statements hold, provided that the strong bimonoid operations (i.e., $+$ and $\cdot$) can be evaluated in one computation step.*

1. *If M is weakly non-circular and connected, then we can compute $[\![M]\!](t)$ in time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$ (i.e., linear on the size of M).*
2. *If $\mathcal{S}$ is a commutative semiring and M is weakly non-circular, then $[\![M]\!](t)$ can be computed in time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$ (i.e., linear on the size of M).*
3. *If $\mathcal{S}$ is a finite commutative semiring and $(\mathcal{S}, \leq)$ is an ω-continuous semiring, then $[\![M]\!]_\leq(t)$ can be computed in time $\mathcal{O}(|S| \cdot (\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)^2)$ (i.e., quadratic in the size of M).*

PROOF. First we prove Statement 1. As the first computation step we construct the dependency hypergraph $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$ of M and t. This can be done as follows: as the set V of vertices we take the set $P(\mathrm{pos}(t))$; thus we have $|V| = |P(\mathrm{pos}(t))| \leq |P| \cdot |\mathrm{pos}(t)| \leq \mathrm{size}(M) \cdot |\mathrm{pos}(t)|$ different vertices (assuming that every predicate of P is used in M); for every $r \in R$ and valid r,t-variable assignment ρ we add the edge (r, ρ) to the set $\Phi_{M,t}$ of hyperedges. Clearly, $|\Phi_{M,t}| \leq |R| \cdot |\mathrm{pos}(t)|$ by Lemma 5.29(2) and the fact that M is connected. Hence, G can be constructed in time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$ and $|V| + |\Phi_{M,t}| = \mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$.

Second, we compute the set $C_\emptyset = \{c \in P(\mathrm{pos}(t)) \mid \mathrm{H}^c_G = \emptyset\}$. This can be done as follows: we compute the sequence $C_0, C_1, C_2, C_3, \ldots$ by recursion, where, for every $n \in \mathbb{N}$, $C_n = \{c \in P(\mathrm{pos}(t)) \mid \exists\ \eta \in \mathrm{H}^c_G : \mathrm{height}(c) \leq n\}$; then there is an $m \in \mathbb{N}$ with $m \leq |P(\mathrm{pos}(t))|$ such that $C_m = \bigcup_{n\in\mathbb{N}} C_n$. Finally, let $C_\emptyset = P(\mathrm{pos}(t)) \setminus C_m$.

Third, we compute the direct dependence relation $\prec_G$ of G by means of the set $C_\emptyset$ and Lemma 2.20($1 \Leftrightarrow 3$).

Fourth, we compute the set $C = \{c \in P(\mathrm{pos}(t)) \mid c \prec^*_G q(\varepsilon)\}$ and compute the relation $\sqsubset\, = \,\prec \cap\, (C \times C)$ on C. Then $\sqsubset^+$ is irreflexive due to Lemma 2.26 and because M is weakly non-circular (i.e., $\mathrm{H}^{q(\varepsilon)}_G$ is finite). Hence, the directed graph $(C, \sqsubset)$ is acyclic.

Fifth, we sort the directed graph $(C, \sqsubset)$ topologically, i.e., we compute some sequence $c_1, \ldots, c_k$ such that $k = |C|$, $c_1, \ldots, c_k \in C$ are pairwise distinct, and for every $i, j \in [k]$ with $i < j$ we have $c_i \sqsubset^+ c_j$. This step is possible because $(C, \sqsubset)$ is acyclic.

Each of these five steps can be computed in time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$. Finally, we compute the sequence $I_0, I_1, \ldots, I_{|C|} \in \mathcal{I}$ as follows: $I_0 = \boldsymbol{I_0}$ and for every $i \in [\,|C|\,]$ the interpretation I_i originates from I_{i-1} by replacing $I_{i-1}(c_i)$ with $\mathcal{T}(I_{i-1})(c_i)$. Clearly, for every $i \in [\,|C|\,]$ the computation of I_i from I_{i-1} can be done in time $\mathcal{O}(\sum_{(r,\rho)\in\Phi_{M,t,c_i}} \mathrm{size}(r))$, because for every $(r, \rho) \in \Phi_{M,t,c_i}$ at most $\mathrm{size}(r)$ multiplications have to be carried out. Hence, we conclude that the computation of the entire sequence $I_0, I_1, \ldots, I_{|C|}$ can be accomplished in time $\mathcal{O}\big(\sum_{i\in[\,|C|\,]} \sum_{(r,\rho)\in\Phi_{M,t,c_i}} \mathrm{size}(r)\big) = \mathcal{O}\big(\sum_{(r,\rho)\in\Phi_{M,t}} \mathrm{size}(r)\big) = \mathcal{O}\big(\sum_{r\in R} \sum_{e\in\{e'\in\Phi_{M,t} \mid \mathrm{pr}_1(e')=r\}} \mathrm{size}(r)\big)$; Lemma 5.29(2) implies that the last term is equal to $\mathcal{O}\big(\sum_{r\in R} |\mathrm{pos}(t)| \cdot \mathrm{size}(r)\big) = \mathcal{O}\big(|\mathrm{pos}(t)| \cdot \mathrm{size}(M)\big)$.

For every $i, j \in [\,|C|\,]$ with $i \leq j$ we have that $I_j(c_i) = \mathcal{T}^{|P(\mathrm{pos}(t))|}(\boldsymbol{I_0})(c_i)$. This can be shown by induction on i. We omit this proof here because it is similar to the proof of Lemma 4.19. Hence, we obtain that $[\![M]\!](t) = I_{|C|}(q(\varepsilon))$.

Statement 2 is a consequence of Statement 1 and Lemma 7.20(4).

Next we prove Statement 3. Using Lemma 7.20(4) we transform M into a semantically equivalent, connected wmd M' in time $\mathcal{O}(\mathrm{size}(M))$. Observe that for every input tree $t \in T_\Sigma$ a single application of the immediate consequence operator $\mathcal{T}$ for a connected wmd M' takes time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)$ (this can be shown by means of Lemma 5.29(2)). By Lemma 7.13 this has to be done at most $(|S|-1) \cdot |P| \cdot |\mathrm{pos}(t)|$ times. Thus, $\mathcal{T}^\omega$ can be computed in time $\mathcal{O}(\mathrm{size}(M) \cdot |\mathrm{pos}(t)| \cdot (|S|-1) \cdot |P| \cdot |\mathrm{pos}(t)|) = \mathcal{O}(|S| \cdot (\mathrm{size}(M) \cdot |\mathrm{pos}(t)|)^2)$. ■

7.5 Open problems

In Section 7.3.3 we explained that in general wmd are strictly more expressive than wta, even when restricting to non-circular wmd. It is well-known that semiring weighted MSO-logic over trees is also strictly more expressive than wta (see [63]). This suggests that non-circular wmd and semiring weighted MSO-logic might be equally expressive.

Monadic datalog tree transducers

A tree transducer [65, 66, 48] is a formal model that defines a tree transformation, which is a mapping from input trees to sets of output trees. In the introduction of [60] the notion of tree transducers has been described to emerge from formal models that compute tree series by abstracting from the semantic domain and using the term algebra instead.

In this chapter we will study monadic datalog tree transducers (for short: mdtt), which are, roughly speaking, obtained from mwmd by abstracting from the semantic domain. Hence, we will not evaluate the semantics of such mwmd in an arbitrary given m-monoid but will use an m-monoid instead, that behaves like the term algebra. This chapter is a revised and extended version of [28], where monadic datalog tree transducers have first been investigated.

Mdtt and (nondeterministic) attributed tree transducers [56, 11, 12] (for short:att) share conceptual ideas. We will prove that the class of tree transformations that are definable by attributed tree transducers coincides with the class of tree transformations that are computed by restricted mdtt (see Theorem 8.21).

This chapter is organized as follows. In Section 8.1 we will study classes of m-monoids that, roughly speaking, abstract from particular semantic domains. In Section 8.2 we will study mwmd that employ the m-monoids that we introduced in Section 8.1; we will refer to such mwmd as mdtt. In Section 8.3 we study normal forms of mdtt and compare the concepts of att and mdtt.

8.1 Free m-monoids

Every mwmd M is defined on a signature Δ, the elements of which may be used in the body of the rules of M. When evaluating the hypergraph semantics of M for a given input tree t and a given m-monoid $(A, +, \mathbf{0}, \theta)$, then (i) one computes a set of derivations, which are trees labeled with rule instances; (ii) afterwards, these derivations are transformed into trees over Δ by means of the homomorphism $\mathrm{h}_{M,t}$; (iii) finally, for every resulting tree over Δ one computes an element in A by means of the evaluation homomorphism of the Δ-algebra (A, θ) and adds the resulting valued for every tree. Only the last step depends on the given m-monoid. Thus, the set of trees over Δ that one computes in the second step is, roughly speaking, an abstract respresentation of the semantics of M and t. This abstract representation turns out to be very useful; unfortunately, it is 'destroyed' in the last step of the computation of the semantics.

In this section we will investigate m-monoids that do behave, roughly speaking, like the identity in the last of the above three steps; hence, they do not destroy the abstract representation. The carrier set of these m-monoids is the set of tree languages over Δ and they interpret every symbol $\delta \in \Delta$ by the δ-language top concatenation. When evaluating the semantics of M in such an m-monoid, one obtains the abstract representation of the semantics of M.

It turns out that such m-monoids are free in a particular subclass of all m-monoids. First let us recall algebra theoretic concepts like free algebras and homomorphisms and explain how these concepts carry over to m-monoids.

Throughout this chapter we fix a ranked alphabet Σ and a signature Δ.

Let $\Delta_{\mathrm{mon}} = \Delta \dot\cup \Sigma_{\mathrm{mon}}$ (recall the definition of Σ_{mon} from Example 2.1). Every m-monoid $(A, +, \mathbf{0}, \theta)$ over Δ can be considered as a Δ_{mon}-algebra (A, θ') as follows:

- $\theta'(e) = \mathbf{0}$,
- $\theta'(\circ) = +$ and
- $\theta'(\delta) = \theta(\delta)$ for every $\delta \in \Delta$.

Therefore, the class of m-monoids over Δ can be considered as a particular class of Δ_{mon}-algebras. We carry over notions for Δ_{mon}-algebras to m-monoids over Δ in an obvious manner. In particular, we will refer to Δ_{mon}-homomorphisms as ***m-monoid homomorphisms***. Now we extend these concepts to ω-complete m-monoids.

Definition 8.1. Let $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an m-monoid over Δ and let $(\mathcal{A}, \sum^A)$ be an ω-complete m-monoid. Moreover, let $A' \subseteq A$. We say that $(\mathcal{A}, \sum^A)$ is ***generated*** by A' if for every set A'' satisfying the following properties:

- $A' \subseteq A'' \subseteq A$,
- A'' is closed under $+$ and under $\theta(\delta)$ for every $\delta \in \Delta$, and
- A'' is closed under $\sum^A$, i.e., for every countable set I and family $(a_i \mid i \in I)$ over A'' we have that $\sum^A_{i \in I} a_i \in A''$,

we have that $A'' = A$.

Let $(\mathcal{B}, \sum^B)$ be an ω-complete m-monoid an let h be an m-monoid homomorphism from $\mathcal{A}$ to $\mathcal{B}$. Then we say that h is a ***complete m-monoid homomorphism*** *from* $(\mathcal{A}, \sum^A)$ *to* $(\mathcal{B}, \sum^B)$ if for every countable set I and family $(a_i \mid i \in I)$ over A we have that $h(\sum^A_{i \in I} a_i) = \sum^B_{i \in I} h(a_i)$.

We extend notions like free m-monoids to ω-complete m-monoids in an obvious fashion. □

Now we are prepared to study free m-monoids and free ω-complete m-monoids. We will first give some auxiliary definitions and then prove the existence of particular free m-monoids and free ω-complete m-monoids.

Let D be a set. Recall that by $\mathcal{P}_{\mathrm{fin}}(T_\Delta(D))$ we denote the set of finite subsets of $T_\Delta(D)$. Moreover, in this chapter we will denote by $\mathcal{P}_{\aleph_0}(T_\Delta(D))$ the set of countable subsets of $T_\Delta(D)$. Then both $\big(\mathcal{P}_{\mathrm{fin}}(T_\Delta(D)), \cup, \emptyset\big)$ and $\big(\mathcal{P}_{\aleph_0}(T_\Delta(D)), \cup, \emptyset\big)$ are commutative monoids, where in the former monoid the operation $\cup$ is the union of finite subsets of $T_\Delta(D)$ and in the latter monoid the operation $\cup$ is the union of countable subsets of $T_\Delta(D)$.

We lift these two commutative monoids to the idempotent and distributive m-monoids $\mathcal{P}^{\Delta,D}_{\mathrm{fin}} = \big(\mathcal{P}_{\mathrm{fin}}(T_\Delta(D)), \cup, \emptyset, \theta_{\mathcal{P}}\big)$ and $\mathcal{P}^{\Delta,D}_{\aleph_0} = \big(\mathcal{P}_{\aleph_0}(T_\Delta(D)), \cup, \emptyset, \theta_{\mathcal{P}}\big)$, where for every $\delta \in \Delta$ the operation $\theta_{\mathcal{P}}(\delta)$ is the δ-language top concatenation (restricted to finite languages in the dm-monoid $\mathcal{P}^{\Delta,D}_{\mathrm{fin}}$ and to countable languages in the dm-monoid $\mathcal{P}^{\Delta,D}_{\aleph_0}$; we will not distinguish between these two versions of the operation $\theta_{\mathcal{P}}(\delta)$).

Note that $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ is an ω-complete m-monoid; this follows from the fact that a countable union of countable sets is a countable set (see Footnote 5 on page 45; this result requires the Axiom of Choice). Moreover, it is easy to check that $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ is ω-idempotent and ω-distributive.

Lemma 8.2. *Let D be a set.*

1. *The dm-monoid $\mathcal{P}_{\mathrm{fin}}^{\Delta,D}$ is freely generated by the set $\{\{d\} \mid d \in D\}$ for the class of all idempotent dm-monoids over Δ.*
2. *The ω-complete dm-monoid $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ is freely generated by $\{\{d\} \mid d \in D\}$ for the class of all ω-idempotent, ω-distributive, ω-complete m-monoids over Δ.*

Proof. The proof of Statement 1 is similar to the slightly more involved proof of Statement 2. For this reason we restrict ourselves to the proof of Statement 2.

We start this proof with showing that $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ is generated by $\{\{d\} \mid d \in D\}$. Clearly, for every subset S of $\mathcal{P}_{\aleph_0}(T_\Delta(D))$ containing $\{\{d\} \mid d \in D\}$ which is closed under $\theta_{\mathcal{P}}(\delta)$ for every $\delta \in \Delta$, we obtain that S contains $\{t\}$ for every $t \in T_\Delta(D)$; hence, if S is also closed under $\bigcup$, then $S = \mathcal{P}_{\aleph_0}(T_\Delta(D))$.

Let $\mathcal{A} = (A, +, \mathbf{0}, \tau)$ be a dm-monoid and $(\mathcal{A}, \sum)$ be an ω-idempotent, ω-distributive, ω-complete m-monoid. Moreover, let $f : \{\{d\} \mid d \in D\} \to A$. We show that f can be extended to a complete m-monoid homomorphism f' from $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ to $(\mathcal{A}, \sum)$. Let $g : D \to A$ be defined by letting $g(d) = f(\{d\})$ for every $d \in D$, and let $g' : T_\Delta(D) \to A$ be the unique Δ-homomorphism from $\mathcal{T}_\Delta(D)$ to (A, τ) extending g. We define f' as follows for every countable $S \subseteq T_\Delta(D)$:

$$f'(S) = \sum\nolimits_{s \in S} g'(s) \ ,$$

note that this is well-defined because S is a countable set. Clearly, $f'(\{d\}) = g'(d) = f(\{d\})$ for every $d \in D$; hence, f' extends f. We show that f' is a complete m-monoid homomorphism from $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$ to $(\mathcal{A}, \sum)$. It is easy to see that $f'(\emptyset) = \mathbf{0}$. Let I be a countable index set and let $(S_i \mid i \in I)$ be a family over $\mathcal{P}_{\aleph_0}(T_\Delta(D))$. We show that $f'(\bigcup_{i \in I} S_i) = \sum_{i \in I} f'(S_i)$. To this end we define, for every $i \in I$, the set $\bar{S}_i = \{i\} \times S_i$ and the set $\bar{S} = \bigcup_{i \in I} \bar{S}_i$. Clearly, $(\bar{S}_i \mid i \in I)$ is a generalized partition of $\bar{S}$ because $\bar{S}_i$ and $\bar{S}_j$ are disjoint for every $i, j \in I$ with $i \neq j$. Then

$$\begin{aligned}
f'(\bigcup\nolimits_{i \in I} S_i) &= \sum\nolimits_{s \in \bigcup_{i \in I} S_i} g'(s) \\
&= \sum\nolimits_{s \in \{s' \mid \exists i \in I : s' \in S_i\}} g'(s) = \sum\nolimits_{s \in \{s' \mid \exists i \in I : (i,s') \in \bar{S}\}} g'(s) \\
&= \sum\nolimits_{p \in \bar{S}} g'(\mathrm{pr}_2(p)) \qquad \text{(by Observation 3.21 and since } (\mathcal{A}, \textstyle\sum) \text{ is } \omega\text{-idempotent)} \\
&= \sum\nolimits_{i \in I} \sum\nolimits_{p \in \bar{S}_i} g'(\mathrm{pr}_2(p)) \qquad \text{(by Equation (2.5))} \\
&= \sum\nolimits_{i \in I} \sum\nolimits_{s \in S_i} g'(s) = \sum\nolimits_{i \in I} f'(S_i) \ .
\end{aligned}$$

In particular, we obtain that $f'(S_1 \cup S_2) = f'(S_1) + f'(S_2)$ for every countable $S_1, S_2 \subseteq T_\Delta(D)$. Now let $k \in \mathbb{N}$, $\delta \in \Delta^{(k)}$, and $S_1, \ldots, S_k \subseteq T_\Delta(D)$ be countable. Then

$$f'\big(\theta_{\mathcal{P}}(\delta)(S_1, \ldots, S_k)\big) = \sum\nolimits_{s \in \theta_{\mathcal{P}}(\delta)(S_1, \ldots, S_k)} g'(s)$$

$$
\begin{aligned}
&= \sum\nolimits_{s_1 \in S_1, \ldots, s_k \in S_k} g'(\delta(s_1, \ldots, s_k)) \\
&= \sum\nolimits_{s_1 \in S_1, \ldots, s_k \in S_k} \tau(\delta)(g'(s_1), \ldots, g'(s_k)) \\
&= \tau(\delta)\Big(\sum\nolimits_{s_1 \in S_1} g'(s_1), \ldots, \sum\nolimits_{s_k \in S_k} g'(s_k)\Big) && (\omega\text{-distributivity of } \mathcal{A}) \\
&= \tau(\delta)(f'(S_1), \ldots, f'(S_k)) \; . && \blacksquare
\end{aligned}
$$

Remark 8.3. In Lemma 8.2(2) we considered an m-monoid whose carrier set is the set of countable tree languages instead of all tree languages for the following reason. We may occasionally want to use signatures Δ that are not countable (e.g., in the previous section we considered signatures that are associated with a given semiring; many familiar semirings are uncountable, for instance the semiring of real numbers, and, thus, their associated signature is, too). Clearly, the set of trees over an uncountable signature is uncountable, too. However, in this case an ω-complete m-monoid that is defined similarly to $(\mathcal{P}_{\aleph_0}^{\Delta,D}, \bigcup)$, but whose carrier set is the set of all tree languages, is not freely generated in the sense of Lemma 8.2(2), because the operations of the ω-complete m-monoid, in particular the ω-infinitary sum operation $\bigcup$, are not capable of generating uncountable languages. $\square$

We will denote $\mathcal{P}_{\text{fin}}^{\Delta,\emptyset}$ and $\mathcal{P}_{\aleph_0}^{\Delta,\emptyset}$ by $\mathcal{P}_{\text{fin}}^{\Delta}$ and $\mathcal{P}_{\aleph_0}^{\Delta}$, respectively. According to Lemma 8.2, the m-monoid $\mathcal{P}_{\text{fin}}^{\Delta}$ is initial for the class of all idempotent dm-monoids over Δ, and $(\mathcal{P}_{\aleph_0}^{\Delta}, \bigcup)$ is initial for the class of all ω-idempotent, ω-distributive, ω-complete m-monoids over Δ.

The following lemma shows that the semantics of a given mwmd M for idempotent and distributive m-monoids (or ω-idempotent, ω-distributive, ω-complete m-monoids) can be expressed by the semantics of M for the ω-complete m-monoid $(\mathcal{P}_{\aleph_0}^{\Delta}, \bigcup)$ followed by a homomorphism.

Lemma 8.4. *Let M be an mwmd over Σ and Δ.*

1. *Suppose that M is weakly non-circular. Let $\mathcal{A}$ be an idempotent dm-monoid and let h be the unique m-monoid homomorphism from $\mathcal{P}_{\text{fin}}^{\Delta}$ to $\mathcal{A}$. Then for every $t \in T_\Sigma$ we have*
$$[\![M]\!]_{\mathcal{A}}^{\text{hyp}}(t) = h\big([\![M]\!]_{\mathcal{P}_{\text{fin}}^{\Delta}}^{\text{hyp}}(t)\big) = h\big([\![M]\!]_{(\mathcal{P}_{\aleph_0}^{\Delta},\bigcup)}^{\text{hyp}}(t)\big) \; .$$

2. *Let $(\mathcal{A}, \sum)$ be an ω-idempotent, ω-distributive, ω-complete m-monoid over Δ, and let h be the unique complete m-monoid homomorphism from $(\mathcal{P}_{\aleph_0}^{\Delta}, \bigcup)$ to $(\mathcal{A}, \sum)$. Then for every $t \in T_\Sigma$ we have*
$$[\![M]\!]_{(\mathcal{A},\sum)}^{\text{hyp}}(t) = h\big([\![M]\!]_{(\mathcal{P}_{\aleph_0}^{\Delta},\bigcup)}^{\text{hyp}}(t)\big) \; .$$

Proof. First we prove Statement 1. Let $\mathcal{A} = (A, +, \mathbf{0}, \theta)$, $t \in T_\Sigma$, and $G = \mathrm{G}_{M,t}^{\text{dep}}$. Let f be the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ) and let g be the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to $(\mathcal{P}_{\text{fin}}(T_\Delta), \theta_{\mathcal{P}})$. It is easy to show by structural induction that for every $s \in T_\Delta$ we have $g(s) = \{s\}$ and $h(\{s\}) = f(s)$. Therefore,

$$[\![M]\!]_{\mathcal{A}}^{\text{hyp}}(t) = \sum\nolimits_{\eta \in \mathrm{H}_G^{q(\varepsilon)}} f(\mathrm{h}_{M,t}(\eta)) = \sum\nolimits_{\eta \in \mathrm{H}_G^{q(\varepsilon)}} h\big(g(\mathrm{h}_{M,t}(\eta))\big)$$

$$= h\big(\bigcup_{\eta\in \mathrm{H}_G^{q(\varepsilon)}} g(\mathrm{h}_{M,t}(\eta))\big) = h\big([\![M]\!]^{\mathrm{hyp}}_{\mathcal{P}^{\Delta}_{\mathrm{fin}}}(t)\big)$$
$$= h\big([\![M]\!]^{\mathrm{hyp}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)}(t)\big) \ .$$

The last equation holds because $[\![M]\!]^{\mathrm{hyp}}_{\mathcal{P}^{\Delta}_{\mathrm{fin}}}(t) = [\![M]\!]^{\mathrm{hyp}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)}(t)$; this equality is easy to prove by using the fact that M is weakly non-circular and, thus, $\mathrm{H}_G^{q(\varepsilon)}$ is finite.

We omit the proof of Statement 2 because it is almost identical to the proof of Statement 1. ■

Lemma 8.4 demonstrates that the study of the semantics of mwmd for idempotent and distributive m-monoids can be reduced to the study of the semantics of mwmd for the ω-complete m-monoid $(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)$.

8.2 Mwmd over free m-monoids

In the remainder of this chapter we will deal with m-weighted monadic datalog programs over the ω-idempotent and ω-distributive ω-complete m-monoid $(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)$. Note that $(\mathcal{P}^{\Delta}_{\aleph_0},\subseteq)$ is an ω-continuous m-monoid that is related to $(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)$. Hence, the hypergraph semantics for $(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)$ and the fixpoint semantics for $(\mathcal{P}^{\Delta}_{\aleph_0},\subseteq)$ coincide for every mwmd over Σ and Δ due to Lemma 4.51. Thus, there is no need to distinguish between these semantics when using the m-monoid $\mathcal{P}^{\Delta}_{\aleph_0}$ in conjunction with the ω-infinitary sum operation $\bigcup$ and the partial order $\subseteq$; more precisely, $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)} = [\![M]\!]^{\mathrm{hyp}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\subseteq)}$ for every mwmd M over Σ and Δ. Therefore, in this chapter we will simply write $[\![M]\!]$ instead of $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)}$ and instead of $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\subseteq)}$.

Definition 8.5. A mapping $\tau : T_\Sigma \to \mathcal{P}(T_\Delta)$ is called a ***tree transformation*** *from* Σ *to* Δ; the tree transformation τ is called ***finite*** if $\mathrm{ran}(\tau) \subseteq \mathcal{P}_{\mathrm{fin}}(T_\Delta)$.

In this chapter we will refer to mwmd over Σ and Δ as ***monadic datalog tree transducers*** (for short: ***mdtt***). For every mdtt M over Σ and Δ we call $[\![M]\!]$ the ***tree transformation*** *computed by* M. For every mdtt M' over Σ and Δ we say that M and M' are ***equivalent*** if $[\![M]\!] = [\![M']\!]$. The set of tree transformations computed by restricted mdtt over Σ and Δ is denoted by $\mathrm{r\text{–}MDTT}(\Sigma,\Delta)$. □

For the remainder of this section we fix an mdtt $M = (P, R, q)$ *over* Σ *and* Δ.

The following lemma relates the definition of the fixpoint semantics with the definition of the hypergraph semantics of mdtt.

Lemma 8.6. *Let* $t \in T_\Sigma$, $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$, $c \in P(\mathrm{pos}(t))$, *and* $n \in \mathbb{N}$. *Then we have*

$$\mathcal{T}^n(I_\emptyset)(c) = \bigcup_{\eta\in \mathrm{H}_G^{c,n}} \{\mathrm{h}_{M,t}(\eta)\} \quad \text{and}$$
$$\bigcup_{m\in\mathbb{N}} \mathcal{T}^m(I_\emptyset)(q(\varepsilon)) = \bigcup_{\eta\in \mathrm{H}_G^{q(\varepsilon)}} \{\mathrm{h}_{M,t}(\eta)\} = [\![M]\!](t) \ ,$$

where $\mathrm{H}_G^{c,n}$ *is the set of* n*-bounded derivations of* M *and* t *starting in* c *(see Definition 4.47).*

PROOF. The first line follows from Lemma 4.49 and the fact that for the unique Δ-homomorphism h from $\mathcal{T}_\Delta$ to $(\mathcal{P}_{\aleph_0}(T_\Delta), \theta_{\mathcal{P}})$ we have $h(s) = \{s\}$ for every $s \in T_\Delta$. The second line follows from the first line, the fact that $\bigcup_{m\in\mathbb{N}} \mathrm{H}_G^{q(\varepsilon),m} = \mathrm{H}_G^{q(\varepsilon)}$, and the definition of the hypergraph-defined tree series $[\![M]\!]^{\mathrm{hyp}}_{(\mathcal{P}^{\Delta}_{\aleph_0},\bigcup)}$. ■

The following example is taken from the proof of Lemma 5.35.

Example 8.7. Now we consider a simple example mdtt $M_{\mathrm{ex}} = (P, R, q)$ over $\Sigma = \{\alpha^{(0)}, \gamma^{(1)}\}$ and $\Delta = \Sigma \cup \{\delta^{(4)}\}$, where $P = \{q^{(1)}, r^{(1)}\}$ and R contains the rules

$$\begin{aligned} &q(x) \leftarrow \delta(r(y), r(z), r(y), r(z))\,;\emptyset\,,\\ &r(x) \leftarrow \alpha\,;\{\mathrm{label}_\alpha(x)\}\,,\\ &r(x) \leftarrow \gamma(x_1)\,;\{\mathrm{label}_\gamma(x), \mathrm{child}_1(x, x_1)\}\,. \end{aligned}$$

Then $[\![M_{\mathrm{ex}}]\!](t) = \{\delta(t|_w, t|_v, t|_w, t|_v) \mid w, v \in \mathrm{pos}(t)\}$ for every $t \in T_\Sigma$. □

If there is an $n \in \mathbb{N}$ with $\mathcal{T}^n(I_\emptyset)(q(\varepsilon)) = [\![M]\!](t)$, then $[\![M]\!](t)$ is obviously finite. It is easy to see that the converse holds as well because $\mathcal{T}$ is monotone. We show that in order to determine whether $[\![M]\!](t)$ is finite, it suffices to consider the interpretation $\mathcal{T}^n(I_\emptyset)$, where $n = |P(\mathrm{pos}(t))|$. Before we state and prove this fact formally, we first need to introduce an auxiliary notion.

Definition 8.8. Let $t \in T_\Sigma$ and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. We define the relation $\prec^{\mathrm{p}}_G$ on $P(\mathrm{pos}(t))$, called ***pumping dependence relation*** *of* G, as follows for every $c_1, c_2 \in P(\mathrm{pos}(t))$: $c_1 \prec^{\mathrm{p}}_G c_2$ iff there are $e = (r, \rho) \in \Phi_{M,t}$ and $i \in [\mathrm{rk}(e)]$ such that $\rho(r_{\mathrm{b}}) \notin P(\mathrm{pos}(t))$, $\mathrm{out}(e) = c_2$, $\mathrm{in}_i(e) = c_1$, and $\mathrm{H}_G^{\mathrm{in}_l(e)} \neq \emptyset$ for every $l \in [\mathrm{rk}(e)]$. □

Note that due to Lemma 2.20(3 ⇒ 1), $c_1 \prec^{\mathrm{p}}_G c_2$ implies $c_1 \prec_G c_2$ for every $c_1, c_2 \in P(\mathrm{pos}(t))$.

Theorem 8.9. *Let $t \in T_\Sigma$, $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$, and $n = |P(\mathrm{pos}(t))|$. Then the following statements are equivalent.*

1. $[\![M]\!](t) = \mathcal{T}^n(I_\emptyset)(q(\varepsilon))$.
2. $[\![M]\!](t)$ *is finite.*
3. *There are no $c, c_1, c_2 \in P(\mathrm{pos}(t))$ such that $c \prec^*_G c_1 \prec^{\mathrm{p}}_G c_2 \prec^*_G c \prec^*_G q(\varepsilon)$.*

PROOF. "1 ⇒ 2": Clearly, for every $I \in \mathcal{I}$ such that $\mathrm{ran}(I) \subseteq \mathcal{P}_{\mathrm{fin}}(T_\Delta)$ we have that $\mathrm{ran}(\mathcal{T}(I)) \subseteq \mathcal{P}_{\mathrm{fin}}(T_\Delta)$. Then it is easy to show by induction that $\mathcal{T}^n(I_\emptyset)(q(\varepsilon))$ is finite.

"2 ⇒ 3": Suppose that $[\![M]\!](t)$ is finite. Assume, contrary to our claim, that there are $c, c_1, c_2 \in P(\mathrm{pos}(t))$ such that $c \prec^*_G c_1 \prec^{\mathrm{p}}_G c_2 \prec^*_G c \prec^*_G q(\varepsilon)$. We will derive a contradiction. Clearly, there are $n, j, l \in \mathbb{N}_+$ and $c'_0, \ldots, c'_n \in P(\mathrm{pos}(t))$ such that $j \in [n]$, $l \in [j]$, $c'_0 = c'_j = c$, $c'_{l-1} = c_1$, $c'_l = c_2$, $c'_n = q(\varepsilon)$, $c'_{k-1} \prec_G c'_k$ for every $k \in [n]$, and $c'_{l-1} \prec^{\mathrm{p}}_G c'_l$. Then Lemma 2.20(1 ⇒ 3) yields that for every $k \in [n]$ there are $e_k = (r_k, \rho_k) \in \Phi_{M,t}$ and $i_k \in [\mathrm{rk}(e_k)]$ such that $\mathrm{out}(e_k) = c'_k$, $\mathrm{in}_{i_k}(e_k) = c'_{k-1}$, and $\mathrm{H}_G^{\mathrm{in}_m(e_k)} \neq \emptyset$ for every $m \in [\mathrm{rk}(e_k)]$; the definition of $\prec^{\mathrm{p}}_G$ implies that there is such an $e_l = (r_l, \rho_l)$ such that $\rho_l((r_l)_{\mathrm{b}}) \notin P(\mathrm{pos}(t))$.

Let $m = \max\{\text{height}(s) \mid s \in [\![M]\!](t)\} + 2$. By Lemma 2.21(2) and the fact that $c'_0 = c'_j$ we obtain that there are $\eta \in \text{H}_G^{c'_n} = \text{H}_G^{q(\varepsilon)}$ and $w \in \text{pos}(\eta)$ such that $|w| = j(m-1)+n$ and $\eta(w') = e_{f(|w|-|w'|)}$ for every proper prefix w' of w, where the mapping $f : [j(m-1)+n] \to [n]$ is defined as in Lemma 2.21(2).

Let $n' = j(m-1)+n$. For every $i \in [m]$ let $w_i \in \text{pos}(\eta)$ be the unique proper prefix of w of length $n'-l-j(i-1)$; observe that this is well-defined, i.e., that $0 \leq n'-l-j(i-1) < n' = |w|$ because $l \in [j]$ and $j \in [n]$. It is easy to check that for every $i \in [m]$ we have that $f(j(i-1)+l) = l$ and, hence, $\eta(w_i) = e_{f(|w|-|w_i|)} = e_{f(n'-(n'-l-j(i-1)))} = e_{f(j(i-1)+l)} = e_l$. Moreover, observe that the positions $w_1, \ldots, w_m$ are pairwise distinct. Since $w_1, \ldots, w_m$ are prefixes of w and since $\rho_l((r_l)_\text{b}) \notin P(\text{pos}(t))$ and $\eta(w_1) = \cdots = \eta(w_m) = e_l$ it is easy to show by induction that for every $i \in [m]$ we have $\text{height}(\text{h}_{M,t}(\eta|_{w_i})) \geq i-1$; in particular we have $\text{height}(\text{h}_{M,t}(\eta)) \geq \text{height}(\text{h}_{M,t}(\eta|_{w_m})) \geq m-1$. Since $\eta \in \text{H}_G^{q(\varepsilon)}$, Lemma 8.6 yields that $\max\{\text{height}(s) \mid s \in [\![M]\!](t)\} \geq \text{height}(\text{h}_{M,t}(\eta)) \geq m-1$; a contradiction to the fact that $m = \max\{\text{height}(s) \mid s \in [\![M]\!](t)\} + 2$.

"3 ⇒ 1": We give an indirect proof. Assume that $[\![M]\!](t) \neq \mathcal{T}^n(I_\emptyset)(q(\varepsilon))$. We show that there are $c, c_1, c_2 \in P(\text{pos}(t))$ such that $c \prec_G^* c_1 \prec_G^\text{p} c_2 \prec_G^* c \prec_G^* q(\varepsilon)$. By Lemma 8.6 we have $[\![M]\!](t) = \bigcup_{n \in \mathbb{N}} \mathcal{T}^n(I_\emptyset)(q(\varepsilon))$; thus, there is a $j \geq n$ with $\mathcal{T}^j(I_\emptyset)(q(\varepsilon)) \subset \mathcal{T}^{j+1}(I_\emptyset)(q(\varepsilon))$. This implies that there is an $s \in \mathcal{T}^{j+1}(I_\emptyset)(q(\varepsilon)) \setminus \mathcal{T}^j(I_\emptyset)(q(\varepsilon))$. By Lemma 8.6, there is an $\eta \in \text{H}_G^{q(\varepsilon),j+1}$ with $s = \text{h}_{M,t}(\eta)$ and for every $\eta' \in \text{H}_G^{q(\varepsilon),j}$ we have $s \neq \text{h}_{M,t}(\eta')$.

We define the family $w = (w_k \mid k \in \{0, \ldots, j\})$ over $\text{pos}(\eta)$ such that for every $k \in \{0, \ldots, j\}$ the following three properties are satisfied: (i) $|w_k| = k$, (ii) there is no $\eta' \in \text{H}_G^{\text{out}(\eta(w_k)),j-k}$ such that $\text{h}_{M,t}(\eta|_{w_k}) = \text{h}_{M,t}(\eta')$, and (iii) if $k \geq 1$, then w_{k-1} is a prefix of w_k. We define the family w by recursion.

Recursion base. Let $w_0 = \varepsilon$. Clearly, $|w_0| = 0$ and there is no $\eta' \in \text{H}_G^{\text{out}(\eta(\varepsilon)),j} = \text{H}_G^{q(\varepsilon),j}$ such that $\text{h}_{M,t}(\eta) = \text{h}_{M,t}(\eta')$, because $\text{h}_{M,t}(\eta) = s$.

Recursion step. Let $k \in [j]$ and assume that w_{k-1} has already been defined such that Conditions (i), (ii), and (iii) are satisfied. Let $e = (r, \rho) = \eta(w_{k-1})$, $l = \text{rk}(e)$, and for every $i \in [l]$ let $c_i = \text{out}(\eta(w_{k-1} \cdot i)) = \text{in}_i(\eta(w_{k-1})) = \text{in}_i(e)$. Assume that for every $i \in [l]$ there is an $\eta_i \in \text{H}_G^{c_i,j-k}$ such that $\text{h}_{M,t}(\eta|_{w_{k-1} \cdot i}) = \text{h}_{M,t}(\eta_i)$. Let $\eta' = e(\eta_1, \ldots, \eta_l)$. By Observation 4.48 we obtain that $\eta' \in \text{H}_G^{c,j-(k-1)}$, where $c = \text{out}(e)$. Moreover, $\text{h}_{M,t}(\eta|_{w_{k-1}}) = \rho(r_\text{b}) \leftarrow \text{h}_{M,t}(\eta|_{w_{k-1} \cdot 1}) \cdots \text{h}_{M,t}(\eta|_{w_{k-1} \cdot l}) = \rho(r_\text{b}) \leftarrow \text{h}_{M,t}(\eta_1) \cdots \text{h}_{M,t}(\eta_l) = \text{h}_{M,t}(\eta')$. Hence, $\eta' \in \text{H}_G^{c,j-(k-1)}$ and $\text{h}_{M,t}(\eta|_{w_{k-1}}) = \text{h}_{M,t}(\eta')$, a contradiction to the fact that w_{k-1} satisfies Condition (ii). Thus, our assumption that for every $i \in [l]$ there is an $\eta_i \in \text{H}_G^{c_i,j-k}$ with $\text{h}_{M,t}(\eta|_{w_{k-1} \cdot i}) = \text{h}_{M,t}(\eta_i)$ was false. We conclude that there is an $i \in [l]$ such that for every $\eta_i \in \text{H}_G^{c_i,j-k}$ we have $\text{h}_{M,t}(\eta|_{w_{k-1} \cdot i}) \neq \text{h}_{M,t}(\eta_i)$. We put $w_k = w_{k-1} \cdot i$. Obviously w_k satisfies Conditions (i), (ii), and (iii).

For every $k \in \{0, \ldots, j\}$ let $e_k = (r_k, \rho_k) = \eta(w_k)$ and $b_k = \rho_k((r_k)_\text{b})$. There are $k, k' \in \{0, \ldots, j\}$ with $k < k'$ and $\text{out}(e_k) = \text{out}(e_{k'})$ because $j \geq n = |P(\text{pos}(t))|$.

Assume that $b_l \in P(\text{pos}(t))$ for every $l \in \{k, \ldots, k'-1\}$. We show by (reverse) induction that for every $l \in \{k, \ldots, k'\}$ we have that $\text{h}_{M,t}(\eta|_{w_l}) = \text{h}_{M,t}(\eta|_{w_{k'}})$. This is trivial for the base case that $l = k'$. Now let $l \in \{k, \ldots, k'-1\}$ and assume that $\text{h}_{M,t}(\eta|_{w_{l+1}}) = \text{h}_{M,t}(\eta|_{w_{k'}})$. Since $b_l \in P(\text{pos}(t))$ implies that $\text{rk}(e_l) = 1$ and, thus, $w_{l+1} = w_l \cdot 1$, we obtain that $\text{h}_{M,t}(\eta|_{w_l}) = b_l \leftarrow \text{h}_{M,t}(\eta|_{w_l \cdot 1}) = \text{h}_{M,t}(\eta|_{w_l \cdot 1}) = \text{h}_{M,t}(\eta|_{w_{l+1}}) = \text{h}_{M,t}(\eta|_{w_{k'}})$. This finishes the inductive proof. In particular, we obtain that $\text{h}_{M,t}(\eta|_{w_k}) = \text{h}_{M,t}(\eta|_{w_{k'}})$. Lemma 2.15 together with the fact that $\eta \in \text{H}_G^{q(\varepsilon)}$ yields that $\eta|_{w_{k'}} \in \text{H}_G^{\text{out}(\eta(w_{k'}))} =$

$\mathrm{H}_G^{\mathrm{out}(e_{k'})} = \mathrm{H}_G^{\mathrm{out}(e_k)}$. The fact $\eta \in \mathrm{H}_G^{q(\varepsilon),j+1}$ implies that $\mathrm{height}(\eta) < j+1$; thus, $\mathrm{height}(\eta|_{w_{k'}}) < j+1-|w_{k'}| = j+1-k' \leq j+1-k-1 = j-k$ because $k < k'$. We obtain that $\eta|_{w_{k'}} \in \mathrm{H}_G^{\mathrm{out}(e_k),j-k}$. This together with the fact that $\mathrm{h}_{M,t}(\eta|_{w_k}) = \mathrm{h}_{M,t}(\eta|_{w_{k'}})$ contradicts Condition (ii) for w_k. Hence, our assumption that $b_l \in P(\mathrm{pos}(t))$ holds for every $l \in \{k,\dots,k'-1\}$ was wrong.

We conclude that there is an $l \in \{k,\dots,k'-1\}$ such that $b_l \notin P(\mathrm{pos}(t))$. Since $\mathrm{out}(e_k) = \mathrm{out}(e_{k'})$ and $\mathrm{out}(e_0) = \mathrm{out}(\eta(w_0)) = \mathrm{out}(\eta(\varepsilon)) = q(\varepsilon)$ it suffices to show that $\mathrm{out}(e_i) \prec_G \mathrm{out}(e_{i-1})$ holds for every $i \in [j]$ and that $\mathrm{out}(e_{l+1}) \prec_G^{\mathrm{p}} \mathrm{out}(e_l)$, in order to finish our indirect proof.

For every $i \in [j]$, $\mathrm{out}(e_i) \prec_G \mathrm{out}(e_{i-1})$ follows from Lemma 2.20(2 ⇒ 1) and the facts that w_{i-1} is a prefix of w_i, $|w_i| - |w_{i-1}| = 1$, $\mathrm{out}(\eta(w_i)) = \mathrm{out}(e_i)$, and $\mathrm{out}(\eta(w_{i-1})) = \mathrm{out}(e_{i-1})$.

Since $|w_l| = l$, $|w_{l+1}| = l+1$, and w_l is a prefix of w_{l+1}, we have that there is an $i \in [\mathrm{rk}(\eta(w_l))] = [\mathrm{rk}(e_l)]$ such that $w_l i = w_{l+1}$. Hence, $\mathrm{in}_i(e_l) = \mathrm{in}_i(\eta(w_l)) = \mathrm{out}(\eta(w_l i)) = \mathrm{out}(\eta(w_{l+1})) = \mathrm{out}(e_{l+1})$. Observe Lemma 2.15 implies that for every $m \in [\mathrm{rk}(e_l)]$, $\eta|_{w_l m} \in \mathrm{H}_G^{\mathrm{out}(\eta(w_l m))} = \mathrm{H}_G^{\mathrm{in}_m(\eta(w_l))} = \mathrm{H}_G^{\mathrm{in}_m(e_l)}$, i.e., $\mathrm{H}_G^{\mathrm{in}_m(e_l)} \neq \emptyset$. We conclude that $\mathrm{out}(e_{l+1}) \prec_G^{\mathrm{p}} \mathrm{out}(e_l)$ because $b_l \notin P(\mathrm{pos}(t))$. ■

Corollary 8.10. *Let $h_{\mathrm{m}} = \max\{\mathrm{height}(r_{\mathrm{b}}) \mid r \in R\}$. Then for every $t \in T_\Sigma$ either the height of trees in $[\![M]\!](t)$ is unbounded or bounded by $h_{\mathrm{m}} \cdot |P(\mathrm{pos}(t))|$.*

Theorem 8.9 shows that for every mdtt and input tree the result of the semantics can either be computed in n iterations of the immediate consequence operator, where n is linear in the size of the input tree, or it yields an infinite set of output trees and, roughly speaking, cannot be computed in finite time. Clearly, only those mdtt that, for every input tree, guarantee a semantics that is computable in finite time, are useful for practical purposes. Let us define this concept formally.

Definition 8.11. We call M ***executable*** iff $[\![M]\!](t)$ is finite for every $t \in T_\Sigma$. □

Observe that the class of executable mdtt contains the class of weakly non-circular mdtt. We will now show that the class of executable mdtt is decidable.

Lemma 8.12. *Let M be an mdtt over Σ and Δ. Then it is effectively decidable whether M is executable.*

Proof. This proof is based on and similar to the proofs in Chapter 6. Therefore we will only sketch this proof.

Similarly to Chapter 6 assume that $M = (P, R, q)$ is proper and that $P = \{p_1,\dots,p_n\}$ for some $n \in \mathbb{N}$ and pairwise distinct predicates $p_1,\dots,p_n$. Moreover suppose that $q = p_1$. For every $j \in [n]$ let R_j^{p} be the set of all $r \in R$ such that $r_{\mathrm{b}} \notin P(\mathrm{V})$ and p_j is the unique predicate that occurs in the head of r.

Let $\psi = (\psi_k(y, X_k) \mid k \in [n])$ be a family over $\mathrm{MSO}(\Sigma)$, $j \in [n]$, and $i \in [n]$. We define the formula $\mathrm{hyperedge}_{i,j}^{\mathrm{p},\psi}(x, y, X_1,\dots,X_n)$ in $\mathrm{MSO}(\Sigma)$ to originate from the formula $\mathrm{hyperedge}_{i,j}^{\psi}(x, y, X_1,\dots,X_n)$ (see Lemma 6.10) by replacing $\bigvee_{r \in R_j}$ with $\bigvee_{r \in R_j^{\mathrm{p}}}$. Then we obtain that for every $t \in T_\Sigma$, $v, w \in \mathrm{pos}(t)$, and $W_1,\dots,W_n \subseteq \mathrm{pos}(t)$ the following statements are equivalent:

- $t \models \mathrm{hyperedge}_{i,j}^{\mathrm{p},\psi}(v, w, W_1,\dots,W_n)$,

- there is a $(r, \rho) \in \Phi_{M,t,p_j(w)}$ such that
  - $\rho(r_\mathrm{b}) \notin P(\mathrm{pos}(t))$,
  - $p_i(v) \in \mathrm{ind}(\rho(r_\mathrm{b}))$, and
  - $t \models \psi_k(v', W_k)$ for every $p_k(v') \in \mathrm{ind}(\rho(r_\mathrm{b}))$.

Note that we defined $\mathrm{hyperedge}^{\mathrm{p},\psi}_{i,j}$ only for $i \in [n]$ (and not for $i \in \{0, \dots, n\}$ as we did for the formula $\mathrm{hyperedge}^{\psi}_{i,j}$).

The following constructions are along the lines of those carried out in the proof of Lemma 6.14. For every $i, j, k \in [n]$ we define $\psi_k(y, X_k)$, $\chi_{i,j}(x, y)$, $\chi^{\mathrm{p}}_{i,j}(x, y)$, and $\varphi_{i,j}(x, y)$ in $\mathrm{MSO}(\Sigma)$ as follows:

$$\begin{aligned}
\psi_k &= \mathrm{nonempty}_k \ , \\
\chi_{i,j} &= \forall X_1 \cdots \forall X_n.\mathrm{hyperedge}^{\psi}_{i,j} \ , \\
\chi^{\mathrm{p}}_{i,j} &= \forall X_1 \cdots \forall X_n.\mathrm{hyperedge}^{\mathrm{p},\psi}_{i,j} \ , \\
\varphi_{i,j} &= \begin{cases} \mathrm{trans}^{\chi}_{i,j} \vee x \equiv y \ , & \text{if } i = j, \\ \mathrm{trans}^{\chi}_{i,j} \ , & \text{otherwise.} \end{cases}
\end{aligned}$$

For every $t \in T_\Sigma$, $i, j \in [n]$, and $v, w \in \mathrm{pos}(t)$ we obtain for $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$ that

$$\begin{aligned}
t \models \chi^{\mathrm{p}}_{i,j}(v, w) \ &\text{ iff } \ p_i(v) \prec^{\mathrm{p}}_G p_j(w) \ , \\
t \models \varphi_{i,j}(v, w) \ &\text{ iff } \ p_i(v) \prec^{*}_G p_j(w) \ .
\end{aligned}$$

Now we define the sentence $\varphi^{\mathrm{exe}} \in \mathrm{MSO}(\Sigma)$ as follows

$$\begin{aligned}
\varphi^{\mathrm{exe}} = \bigvee_{i \in [n]} \bigvee_{i_1 \in [n]} \bigvee_{i_2 \in [n]} & \exists z.\exists z_1.\exists z_2. \\
& \big(\exists x.\exists y.(x \equiv z \wedge y \equiv z_1 \wedge \varphi_{i,i_1}) \wedge \exists x.\exists y.(x \equiv z_1 \wedge y \equiv z_2 \wedge \chi^{\mathrm{p}}_{i_1,i_2}) \\
& \wedge \exists x.\exists y.(x \equiv z_2 \wedge y \equiv z \wedge \varphi_{i_2,i}) \wedge \exists x.\exists y.(x \equiv z \wedge \mathrm{root}(y) \wedge \varphi_{i,1})\big) \ .
\end{aligned}$$

Clearly, we have for every $t \in T_\Sigma$ that (i) $t \models \varphi^{\mathrm{exe}}$ iff (ii) there are $c, c_1, c_2 \in P(\mathrm{pos}(t))$ with $c \prec^{*}_G c_1 \prec^{\mathrm{p}}_G c_2 \prec^{*}_G c \prec^{*}_G q(\varepsilon)$ iff (iii) $[\![M]\!](t)$ is infinite due to Theorem 8.9. Thus, M is executable iff $\mathcal{L}(\varphi^{\mathrm{exe}})$ is empty. Then by Theorems 6.4(1) and 6.2 it is decidable whether M is executable. ∎

8.3 Normal forms

In this section we study normal forms of mdtt. In Chapter 5 we have already defined five syntactic classes of mwmd: the classes of restricted, semiconnected, connected, proper, and local mwmd. These classes and the results from Chapter 5 carry over straightforwardly to the setting of mdtt.

This section is divided into two parts. In Section 8.3.1 we will deal with the class of semiconnected mdtt (recall the definition of semiconnectedness from Definition 5.32). We will prove that for every mdtt there is an equivalent semiconnected mdtt; we provide a construction that preserves restrictedness and weak non-circularity (see Lemma 8.17). We will use this result in order to show Corollary 8.18; we already used this corollary in Section 5.4 but postponed its proof to the present section.

In Section 8.3.2 we will introduce another syntactic subclass, which is a subclass of local mdtt. We will refer to the mdtt in this class as attributed tree transducer mdtt (for short: att mdtt), because they are a syntactic redefinition of attributed tree transducers [56, 60] in terms of mdtt; moreover, they behave exactly like attributed tree transducers. We will show that for every local mdtt there is an equivalent att mdtt (see Lemma 8.20).

8.3.1 Semiconnected

In this section we prove that for every mdtt there is an equivalent semiconnected mdtt. The detailed proof that we present in this section has been sketched in [28]. First let us give an informal description of this construction. Assume that in the given mdtt M there is the following rule: $r = p(x) \leftarrow q(x)\ ;\ \{\text{label}_\gamma(z)\}$. This rule is apparently not semiconnected because the variable z is not connected to the variable x. Thus, for every $t \in T_\Sigma$ we have that if t contains a node labeled γ, we can omit the guard, and otherwise, we can omit the whole rule, each time preserving semantics for t.

Our construction is based on the idea that we have just laid out. We will construct two copies of M, where in the first copy we will omit the rule r and in the second copy we will keep r but replace its guard by the empty set; moreover, we ensure, by adding additional rules, that the second copy of M will only be "active" for input trees that contain a node labeled γ. Then for every input tree that has no γ-labeled node only the first copy will be active; this copy will then behave precisely like M. On the other hand, for every input tree that has a γ-labeled node, both copies will be active; the second copy will behave like M and the first copy will behave like M without the rule r; we will show that, since $\mathcal{P}^{\Delta}_{\aleph_0}$ is an idempotent m-monoid, the first copy will not interfere with the second one, i.e., these two copies together behave like M.

We have broken down our construction into four lemmas in order to make it more accessible. The first lemma states that for two mdtt M and M' and an input tree t such that for every rule instance in M there is a rule instance M' that encodes the same tree of operations, the semantics of M for t is a subset of the semantics of M' for t.

Lemma 8.13. *Let $M = (P, R, q)$ and $M' = (P', R', q')$ be mdtt over Σ and Δ such that $P = P'$ and $q = q'$. Let $t \in T_\Sigma$. Assume that for every $c \in P(\text{pos}(t))$ and $(r, \rho) \in \Phi_{M,t,c}$ there is a $(r', \rho') \in \Phi_{M',t,c}$ such that $\rho(r_\text{b}) = \rho'(r'_\text{b})$. Then $[\![M]\!](t) \subseteq [\![M']\!](t)$ and for every $c, c' \in P(\text{pos}(t))$ with $c \prec_G c'$ we have that also $c \prec_{G'} c'$, where $G = \text{G}^{\text{dep}}_{M,t}$ and $G' = \text{G}^{\text{dep}}_{M',t}$.*

Proof. For every $c \in P(\text{pos}(t))$ and $(r, \rho) \in \Phi_{M,t,c}$ choose a $(r', \rho') \in \Phi_{M',t,c}$ such that $\rho(r_\text{b}) = \rho'(r'_\text{b})$ and denote this (r', ρ') by $\lambda(r, \rho)$. Clearly, for every $(r, \rho) \in \Phi_{M,t,c}$ we have that $\text{out}((r, \rho)) = \text{out}(\lambda(r, \rho))$, $\text{rk}((r, \rho)) = \text{rk}(\lambda(r, \rho))$, and $\text{in}_i((r, \rho)) = \text{in}_i(\lambda(r, \rho))$ for every $i \in [\text{rk}((r, \rho))]$.

We define the mapping $\text{h}_\lambda : T_{\Phi_{M,t}} \to T_{\Phi_{M',t}}$ by recursion as follows for every $k \in \mathbb{N}$, $(r, \rho) \in (\Phi_{M,t})^{(k)}$, $\eta_1, \ldots, \eta_k \in T_{\Phi_{M,t}}$:

$$\text{h}_\lambda((r, \rho)(\eta_1, \ldots, \eta_k)) = \lambda(r, \rho)(\text{h}_\lambda(\eta_1), \ldots, \text{h}_\lambda(\eta_k)) .$$

Let $c \in P(\text{pos}(t))$ and $\eta \in \text{H}^c_G$. It is easy to prove by structural induction that $\text{h}_\lambda(\eta) \in \text{H}^c_{G'}$ and that $\text{h}_{M,t}(\eta) = \text{h}_{M',t}(\text{h}_\lambda(\eta))$. The fact that, for every $c, c' \in P(\text{pos}(t))$, $c \prec_G c'$ implies $c \prec_{G'} c'$, follows trivially. Moreover,

$$[\![M]\!](t) = \bigcup\nolimits_{\eta \in \text{H}^{q(\varepsilon)}_G} \{\text{h}_{M,t}(\eta)\} \qquad \text{(by Lemma 8.6)}$$

$$= \bigcup_{\eta \in \mathrm{H}_G^{q(\varepsilon)}} \{\mathrm{h}_{M',t}(\mathrm{h}_\lambda(\eta))\} \subseteq \bigcup_{\eta \in \mathrm{H}_{G'}^{q(\varepsilon)}} \{\mathrm{h}_{M',t}(\eta)\}$$
$$= [\![M']\!](t) \ . \qquad \blacksquare$$

Before we proceed, let us introduce an auxiliary notion.

Definition 8.14. For every $G \subseteq \mathrm{sp}_\Sigma(\mathrm{V})$ we define the ***language accepted*** *by* G as follows: $L(G) = \{t \in T_\Sigma \mid \exists\, \rho : \mathrm{var}(G) \to \mathrm{pos}(t) : \rho(G) \subseteq \mathrm{B}_t\}$. □

Observe that $\mathrm{var}(G_1) \cap \mathrm{var}(G_2) = \emptyset$ implies $L(G_1 \cup G_2) = L(G_1) \cap L(G_2)$ for every $G_1, G_2 \subseteq \mathrm{sp}_\Sigma(V)$.

Lemma 8.15. *Let $M = (P, R, q)$ be an mdtt over Σ and Δ, and let $R' \subseteq R$. Then there is an mdtt $\tau_1^{R'}(M) = (P, R_1, q)$ over Σ and Δ such that the following conditions hold.*

1. *$\tau_1^{R'}(M)$ is semiconnected and if M is restricted, then $\tau_1^{R'}(M)$ is restricted.*
2. *Let $t \in T_\Sigma$, $\tau(G) = \mathrm{G}^{\mathrm{dep}}_{\tau_1^{R'}(M),t}$, and $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$.*
   a) *If $t \in L(I(R'))$, then $[\![\tau_1^{R'}(M)]\!](t) \subseteq [\![M]\!](t)$ and for every $c, c' \in P(\mathrm{pos}(t))$ we have that $c \prec_{\tau(G)} c'$ implies $c \prec_G c'$.*
   b) *If $R' = \{r \in R \mid t \in L(I(r))\}$, then $[\![\tau_1^{R'}(M)]\!](t) = [\![M]\!](t)$ and for every $c, c' \in P(\mathrm{pos}(t))$ we have that $c \prec_{\tau(G)} c'$ iff $c \prec_G c'$.*

Proof. Without loss of generality, we assume that $\mathrm{var}(r_1) \cap \mathrm{var}(r_2) \neq \emptyset$ implies $r_1 = r_2$ for every $r_1, r_2 \in R$.

We construct the mdtt $\tau_1^{R'}(M) = (P, R_1, q)$ such that $R_1 = \{\bar{r} \mid r \in R'\}$, where, for every $r \in R'$, $\bar{r} = r_\mathrm{h} \leftarrow r_\mathrm{b} \,;\, r_\mathrm{G} \setminus I(r)$. It is easy to check that $\tau_1^{R'}(M)$ is semiconnected and that the construction preserves restrictedness. It remains to prove Statement 2.

First we prove Statement (a). Let $t \in L(I(R'))$. Then for every $r \in R'$ there is a $\rho_r : \mathrm{var}(I(r)) \to \mathrm{pos}(t)$ such that $\rho_r(I(r)) \subseteq \mathrm{B}_t$. Observe that for every $c \in P(\mathrm{pos}(t))$, $r \in R'$, and $(\bar{r}, \rho) \in \Phi_{\tau_1^{R'}(M),t,c}$ we have $(r, \rho \cup \rho_r) \in \Phi_{M,t,c}$; clearly, $\rho(\bar{r}_\mathrm{b}) = (\rho \cup \rho_r)(r_\mathrm{b})$. Hence, Lemma 8.13 yields that $[\![\tau_1^{R'}(M)]\!](t) \subseteq [\![M]\!](t)$ and that for every $c, c' \in P(\mathrm{pos}(t))$, $c \prec_{\tau(G)} c'$ implies $c \prec_G c'$.

Next we prove Statement (b). Suppose that $R' = \{r \in R \mid t \in L(I(r))\}$. Then for every $r \in R'$, $t \in L(I(r))$, i.e., $t \in L(I(R'))$. Therefore the first part of Fact (a) yields $[\![\tau_1^{R'}(M)]\!](t) \subseteq [\![M]\!](t)$ and, for every $c, c' \in P(\mathrm{pos}(t))$, $c \prec_{\tau(G)} c'$ implies $c \prec_G c'$. It remains to show that $[\![\tau_1^{R'}(M)]\!](t) \supseteq [\![M]\!](t)$ and, for every $c, c' \in P(\mathrm{pos}(t))$, $c \prec_G c'$ implies $c \prec_{\tau(G)} c'$. Let $c \in P(\mathrm{pos}(t))$ and $(r, \rho) \in \Phi_{M,t,c}$. Then $\rho|_{\mathrm{var}(I(r))} : \mathrm{var}(I(r)) \to \mathrm{pos}(t)$ with $\rho|_{\mathrm{var}(I(r))}(I(r)) \subseteq \mathrm{B}_t$; hence, $t \in L(I(r))$ and we conclude that $r \in R'$. Let $(r', \rho') = (\bar{r}, \rho|_{\mathrm{var}(\bar{r})})$ Thus, $(r', \rho') \in \Phi_{\tau_1^{R'}(M),t,c}$. Clearly, $\rho(r_\mathrm{b}) = \rho'(r'_\mathrm{b})$. Hence, Lemma 8.13 yields that $[\![\tau_1^{R'}(M)]\!](t) \supseteq [\![M]\!](t)$ and that for every $c, c' \in P(\mathrm{pos}(t))$, $c \prec_G c'$ implies $c \prec_{\tau(G)} c'$. ■

Lemma 8.16. *Let $M = (P, R, q)$ be a semiconnected mdtt over Σ and Δ, and let $G \subseteq \mathrm{sp}_\Sigma(\mathrm{V})$ be finite. Then there is an mdtt $\tau_2^G(M) = (P_0, R_0, q_0)$ over Σ and Δ such that the following conditions hold.*

1. *$P \subseteq P_0$.*

2. $\tau_2^{R'}(M)$ *is semiconnected and if* M *is restricted, then* $\tau_2^{R'}(M)$ *is restricted.*

3. *Let* $t \in T_\Sigma$, $\tau(G) = \mathrm{G}^{\mathrm{dep}}_{\tau_2^G(M),t}$, *and* $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$.

   a) *For every* $c, c' \in P(\mathrm{pos}(t))$ *we have* $c \prec_{\tau(G)} c'$ *iff* $c \prec_G c'$.

   b) *If* $t \in L(G)$, *then* $[\![\tau_2^G(M)]\!](t) = [\![M]\!](t)$ *and the following statements are equivalent: (i) there is a* $c_0 \in P_0(\mathrm{pos}(t))$ *such that* $c_0 \prec^+_{\tau(G)} c_0$ *and* $c_0 \prec^*_{\tau(G)} q_0(\varepsilon)$ *and (ii) there is a* $c \in P(\mathrm{pos}(t))$ *such that* $c \prec^+_G c$ *and* $c \prec^*_G q(\varepsilon)$.

   c) *If* $t \notin L(G)$, *then* $[\![\tau_2^G(M)]\!](t) = \emptyset$ *and there is no* $c_0 \in P_0(\mathrm{pos}(t))$ *such that* $c_0 \prec^+_{\tau(G)} c_0$ *and* $c_0 \prec^*_{\tau(G)} q_0(\varepsilon)$.

Proof. Without loss of generality, we assume that $\mathrm{var}(r_1) \cap \mathrm{var}(r_2) \neq \emptyset$ implies $r_1 = r_2$ for every $r_1, r_2 \in R$. We define $\sim_G$ as the transitive reflexive closure of the relation $\{(a_1, a_2) \in G \times G \mid \mathrm{var}(a_1) \cap \mathrm{var}(a_2) \neq \emptyset\}$. Clearly, $\sim_G$ is an equivalence relation on G. Let $k \in \mathbb{N}$ and $G_1, \ldots, G_k \subseteq G$ be pairwise disjoint such that $\{G_1, \ldots, G_k\} = G/{\sim_G}$.

We construct $\tau_2^G(M) = (P_0, R_0, q_0)$ where $P_0 = P \cup P'$, $R_0 = R \cup R'$ and

$$\begin{aligned} P' &= \{q_0^{(1)}\} \cup \{q_1^{(0)}, \ldots, q_{k+1}^{(0)}\} \text{ is disjoint from } P\,, \\ R' &= \{q_0(x) \leftarrow q_1()\,;\emptyset\} \cup \{q_i() \leftarrow q_{i+1}()\,; G_i \mid i \in [k]\} \\ &\quad \cup \big\{q_{k+1}() \leftarrow q(x)\,; \{\mathrm{root}(x)\}\big\}\,. \end{aligned}$$

Note that this construction preserves semiconnectedness and restrictedness. It remains to prove Statement 3. Let $t \in T_\Sigma$. It is easy to check that for every $c \in P(\mathrm{pos}(t))$ we have $\mathrm{H}^c_G = \mathrm{H}^c_{\tau(G)}$; hence Statement (a) holds. Observe that the following two statements are equivalent: (i) $t \in L(G)$ and (ii) for every $i \in [k]$ the set $\Phi_{\tau_2^G(M),t,q_i()}$ is nonempty. Moreover, for every $i \in [k]$ and $(r, \rho) \in \Phi_{\tau_2^G(M),t,q_i()}$ we have $\rho(r_{\mathrm{b}}) = q_{i+1}()$. Using these facts it is easy to check that Statements (b) and (c) hold. ∎

Now we state the main lemma of Section 8.3.1. In the proof of this lemma we present the main construction of a semiconnected mdtt from a given arbitrary mdtt.

Lemma 8.17. *Let* M *be an mdtt over* Σ *and* Δ. *Then there is a semiconnected mdtt* M' *over* Σ *and* Δ *such that*

- M *is weakly non-circular iff* M' *is weakly non-circular,*
- $[\![M]\!] = [\![M']\!]$, *and*
- *if* M *is restricted, then* M' *is restricted.*

Proof. Let $M = (P, R, q)$. First let us introduce a family $(M_{R'} \mid R' \subseteq R)$ of auxiliary mdtt over Σ and Δ. Let $R' \subseteq R$. We define $M_{R'} = (P_{R'}, R_{R'}, q_{R'}) = \tau_2^{I(R')}(\tau_1^{R'}(M))$. Let $t \in T_\Sigma$ and $G_{R'} = \mathrm{G}^{\mathrm{dep}}_{M_{R'},t}$. We distinguish two cases.

Case 1. $t \notin L(I(R'))$. By Lemma 8.16, $[\![M_{R'}]\!](t) = \emptyset$ and there is no $c \in P_{R'}(\mathrm{pos}(t))$ such that $c \prec^+_{G_{R'}} c$ and $c \prec^*_{G_{R'}} q_{R'}(\varepsilon)$.

Case 2. $t \in L(I(R'))$. By Lemmas 8.15 and 8.16, $[\![M_{R'}]\!](t) \subseteq [\![M]\!](t)$ and whenever there is a $c \in P_{R'}(\mathrm{pos}(t))$ with $c \prec^+_{G_{R'}} c$ and $c \prec^*_{G_{R'}} q_{R'}(\varepsilon)$, then there is a $c \in P(\mathrm{pos}(t))$ with $c \prec^+_G c$ and $c \prec^*_G q(\varepsilon)$, where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$. Moreover, if we even have the equality $R' = \{r \in R \mid t \in L(I(r))\}$, then $[\![M_{R'}]\!](t) = [\![M]\!](t)$ and the following statements are equivalent: (i) there is a $c \in P_{R'}(\mathrm{pos}(t))$ with $c \prec^+_{G_{R'}} c$ and $c \prec^*_{G_{R'}} q_{R'}(\varepsilon)$ and (ii) there is a $c \in P(\mathrm{pos}(t))$ with $c \prec^+_G c$ and $c \prec^*_G q(\varepsilon)$, where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$.

Now we need to aggregate the family $(M_{R'} \mid R' \subseteq R)$ of mdtt into one mdtt M'. To this end we define $M' = (P_0, R_0, q_0)$ where $P_0 = \{q_0\} \cup \bigcup_{R' \subseteq R}(P_{R'} \times \{R'\})$ and $R_0 = \{q_0(x) \leftarrow (q_{R'}, R')(x) \, ; \emptyset \mid R' \subseteq R\} \cup \bigcup_{R' \subseteq R} \tilde{R}_{R'}$ where the set $\tilde{R}_{R'}$ is obtained from $R_{R'}$ by replacing every occurrence of every $p \in P_{R'}$ by (p, R').

Let $t \in T_\Sigma$. Clearly, $[\![M']\!](t) = \bigcup_{R' \subseteq R}[\![M_{R'}]\!](t) = [\![M]\!](t)$, which is an immediate consequence of Cases 1 and 2 and the fact that there is a subset $R' \subseteq R$ such that $R' = \{r \in R \mid t \in L(I(r))\}$.

Moreover, it is easy to see that the following statements are equivalent: (i) there is an $R' \subseteq R$ and $c \in P_{R'}(\mathrm{pos}(t))$ with $c \prec^+_{G_{R'}} c$ and $c \prec^*_{G_{R'}} q_{R'}(\varepsilon)$ and (ii) there is a $c \in P_0(\mathrm{pos}(t))$ with $c \prec^+_{G'} c$ and $c \prec^*_{G'} q_0(\varepsilon)$, where $G' = \mathrm{G}^{\mathrm{dep}}_{M',t}$. Then Cases 1 and 2 yield that the following two statements are equivalent: (i) there is a $c \in P(\mathrm{pos}(t))$ with $c \prec^+_G c$ and $c \prec^*_G q(\varepsilon)$ and (ii) there is a $c \in P_0(\mathrm{pos}(t))$ with $c \prec^+_{G'} c$ and $c \prec^*_{G'} q_0(\varepsilon)$.

Thus $[\![M]\!] = [\![M']\!]$ and M is weakly non-circular iff M' is weakly-noncircular. ■

The following corollary is an immediate consequence of Lemmas 4.50, 4.51, 8.4, and 8.17. It is used in Section 5.4.

Corollary 8.18. *Let M be an mdtt over Σ and Δ. Then there is a semiconnected mdtt M' over Σ and Δ such that*

- *M is weakly non-circular iff M' is weakly non-circular,*
- *if M is weakly non-circular, then for every idempotent dm-monoid $\mathcal{A}$ over Δ we have $[\![M]\!]^{\mathrm{fix}}_{\mathcal{A}} = [\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{hyp}}_{\mathcal{A}} = [\![M']\!]^{\mathrm{fix}}_{\mathcal{A}}$,*
- *for every ω-idempotent, ω-distributive, and ω-complete m-monoid $(\mathcal{A}, \sum)$ over Δ and every related ω-continuous m-monoid $(\mathcal{A}, \leq)$ we have $[\![M]\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)} = [\![M]\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} = [\![M']\!]^{\mathrm{hyp}}_{(\mathcal{A},\sum)} = [\![M']\!]^{\mathrm{fix}}_{(\mathcal{A},\leq)}$, and*
- *if M is restricted, then M' is restricted.*

8.3.2 Attributed Tree Transducers

In this section we introduce the syntactic class of attributed tree transducer mdtt (for short: att mdtt) and show that for every local mdtt there is an equivalent att mdtt.

(Nondeterministic) attributed tree transducers [56, 11, 12] (for short: att), an abstract form of attribute grammars [90, 35], are introduced as a formal model of syntax-directed semantics [60], that is, a model for specifying tree transformations. The semantics of att are defined to be tree transformations. We will now give a definition of att in terms of mdtt. In the literature there are varying definitions of att. We will adopt the definition from [20, Sect. 2.3], this choice seems to be best adapted to our theory.

For the remainder of this chapter, let $M = (P, R, q)$ be an mdtt over Σ and Δ.

As a prerequisite, we fix pairwise distinct variables $\mathrm{x}_\varepsilon, \mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots \in \mathrm{V}$. Moreover we denote $\{\mathrm{x}_1, \ldots \mathrm{x}_k\}$ by X_k, for every $k \in \mathbb{N}$.

Definition 8.19. The mdtt M is an ***attributed tree transducer mdtt*** (for short: ***att mdtt***) *over* Σ *and* Δ if there are disjoint sets A_{syn} and A_{inh} such that $P = P^{(1)} = A_{\mathrm{syn}} \cup A_{\mathrm{inh}}$, $q \in A_{\mathrm{syn}}$, and the following holds for every $r \in R$:

- either there are $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $a \in A_{\mathrm{syn}}(\{\mathrm{x}_\varepsilon\}) \cup A_{\mathrm{inh}}(\mathrm{X}_k)$ and $b \in T_\Delta(A_{\mathrm{syn}}(\mathrm{X}_k) \cup A_{\mathrm{inh}}(\{\mathrm{x}_\varepsilon\}))$ such that

$$r = a \leftarrow b\,;\{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \ldots, \mathrm{child}_k(\mathrm{x}_\varepsilon, \mathrm{x}_k)\}$$

- or there are $a \in A_{\mathrm{inh}}(\{\mathrm{x}_\varepsilon\})$ and $b \in T_\Delta(A_{\mathrm{syn}}(\{\mathrm{x}_\varepsilon\}))$ such that

$$r = a \leftarrow b\,;\{\mathrm{root}(\mathrm{x}_\varepsilon)\}\ .$$

The class of all tree transformations computed by att mdtt over Σ and Δ is denoted by $\mathrm{ATT}(\Sigma, \Delta)$. □

The class $\mathrm{ATT}(\Sigma, \Delta)$ defined here can be best compared with the class $\mathcal{TA}$ from [56], which differs from $\mathrm{ATT}(\Sigma, \Delta)$ in that it only takes *non-circular* atts into account, a decidable syntactic subclass which guarantees that the semantics can be evaluated in finite time in their framework.

Now we show that we can transform every local mdtt M into an equivalent attributed tree transducer.

Lemma 8.20 (cf. [28, Lemma 7]). *Let M be local. Then there is an att mdtt M' equivalent to M.*

Proof (Sketch). First let us list the syntactic differences between local mdtt and att mdtt:

1. the variables in rules do not need to be of the form x_ε or x_i for some $i \in \mathbb{N}_+$,
2. for att mdtt certain atoms are mandatory in guards, e.g., the atom $\mathrm{label}_\sigma(\mathrm{x}_\varepsilon)$ or $\mathrm{root}(\mathrm{x}_\varepsilon)$, and certain atoms are not allowed to occur, e.g., $\mathrm{leaf}(\mathrm{x}_\varepsilon)$,
3. the set of user-defined predicates is partitioned into inherited and synthesized attributes,
4. rules whose guard contains $\mathrm{root}(\mathrm{x}_\varepsilon)$ have to be of a very restricted form.

Our construction of M' is divided into four phases, each of which dealing with one of the syntactic differences in the order listed above.

Phase 1. Since M is local, we have that for every $r \in R$ with $\mathrm{var}(r) \neq \emptyset$ there is an $x \in \mathrm{var}(r)$ such that for every $b \in r_{\mathrm{G}}$ and $y \in \mathrm{var}(b) \setminus \{x\}$ we have that $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$. We rename every occurrence of x in r to x_ε and for every $y \in \mathrm{var}(r) \setminus \{x\}$ we rename every occurrence of y as follows: since r is connected, there is a $b \in r_{\mathrm{G}}$ with $y \in \mathrm{var}(b)$; then $b = \mathrm{child}_i(x, y)$ for some $i \in [\mathrm{maxrk}(\Sigma)]$; we rename y to x_i. If this renaming cannot be done consistently, i.e., there is an $i' \in [\mathrm{maxrk}(\Sigma)]$ with $i \neq i'$ such that $\mathrm{child}_{i'}(x, y) \in r_{\mathrm{G}}$, then the rule is inconsistent and can be dropped.

Phase 2. Suppose that M has already passed Phase 1. This construction is split into five steps. In the first step we drop all rules r such that there are distinct $\sigma, \sigma' \in \Sigma$ with $\{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{label}_{\sigma'}(\mathrm{x}_\varepsilon)\} \subseteq r_\mathrm{G}$, since these rules are obviously inconsistent. In the second step we take care of all rules whose guard does not yet contain $\mathrm{label}_\sigma(\mathrm{x}_\varepsilon)$ for any $\sigma \in \Sigma$. For such a rule r we add for every $\sigma \in \Sigma$ a copy of r that additionally contains $\mathrm{label}_\sigma(\mathrm{x}_\varepsilon)$ in the guard. Afterwards we remove r. In the resulting mdtt we have that for every rule r' there is a unique $\sigma \in \Sigma$ with $\mathrm{label}_\sigma(\mathrm{x}_\varepsilon) \in r'_\mathrm{G}$; we denote this σ by $\sigma_{r'}$. In the third step we add to the guard of every rule r the atoms $\mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \ldots, \mathrm{child}_k(\mathrm{x}_\varepsilon, \mathrm{x}_k)$ where $k = \mathrm{rk}(\sigma_r)$.

In the fourth step we remove all rules that are obviously inconsistent; these are all rules r such that (i) $\mathrm{rk}(\sigma_r) > 0$ and $\mathrm{leaf}(\mathrm{x}_\varepsilon) \in r_\mathrm{G}$ or (ii) $\mathrm{child}_i(\mathrm{x}_\varepsilon, \mathrm{x}_i) \in r_\mathrm{G}$ for some $i > \mathrm{rk}(\sigma_r)$. In the fifth step we consider all rules r whose guard contains $\mathrm{leaf}(\mathrm{x}_\varepsilon)$: by the construction in the fourth step it is obvious that $\mathrm{rk}(\sigma_r) = 0$. Therefore $\mathrm{leaf}(\mathrm{x}_\varepsilon)$ is redundant in r_G and we can simply drop it. The constructions that are carried out in each step yield an mdtt equivalent to the original one. Note that we deal with all rules that contain $\mathrm{root}(\mathrm{x}_\varepsilon)$ in Phase 4.

We illustrate this construction with an example. Assume that R contains the rules $q(\mathrm{x}_\varepsilon) \leftarrow \alpha\,;\{\mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1)\}$ and $p(\mathrm{x}_\varepsilon) \leftarrow \alpha\,;\{\mathrm{leaf}(\mathrm{x}_\varepsilon)\}$ and that $\Sigma = \{\alpha^{(0)}, \sigma^{(2)}\}$. Then after the third step of our construction we obtain four rules, while two of these rules are removed in the fourth step. The fifth step yields

$$\begin{aligned} q(\mathrm{x}_\varepsilon) &\leftarrow \alpha\,;\{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \mathrm{child}_2(\mathrm{x}_\varepsilon, \mathrm{x}_2)\}\ , \\ p(\mathrm{x}_\varepsilon) &\leftarrow \alpha\,;\{\mathrm{label}_\alpha(\mathrm{x}_\varepsilon)\}\ . \end{aligned}$$

Phase 3. Suppose that M has already passed Phases 1 and 2. We give an intuitive description of our construction. Assume that $P = \{p, q\}$ and consider the input tree $\gamma(\gamma(\alpha))$. Furthermore assume that R determines data transport from $p(\varepsilon)$ to $p(1)$, $p(11)$ to $p(1)$, $p(1)$ to $q(\varepsilon)$, and $p(1)$ to $q(11)$. This situation is depicted on the left-hand side of Fig. 8.1, where the data transport is represented by dashed arrows. Obviously, p behaves both like a synthesized and an inherited attribute. Thus, we have to replace p by a predicate (p, syn) (simulating its synthesizing behavior, i.e., transporting data bottom-up) and a predicate (p, inh) (simulating its inheriting behavior, i.e., transporting data top-down). Since (p, syn) should also receive all the data that p receives from the top part of the tree, we need to introduce a rule which transports data from (p, inh) to (p, syn). The same holds for (p, inh): here we need to add a rule for transporting data from (p, syn) to (p, inh) (note that we need to take special care for the root of the tree). This is illustrated on the right-hand side of Fig. 8.1.

Phase 4. Suppose that M has already passed Phases 1, 2 and 3. If M is not already an att mdtt, then there is a rule r containing $\mathrm{root}(\mathrm{x}_\varepsilon)$. We remove this rule from M, add two new user-defined predicates (r, syn) and (r, inh), and add the rules

$$\begin{aligned} r_\mathrm{h} &\leftarrow (r, \mathrm{inh})(\mathrm{x}_\varepsilon)\,; r_\mathrm{G} \setminus \{\mathrm{root}(\mathrm{x}_\varepsilon)\}\ , \\ (r, \mathrm{inh})(\mathrm{x}_\varepsilon) &\leftarrow (r, \mathrm{syn})(\mathrm{x}_\varepsilon)\,;\{\mathrm{root}(\mathrm{x}_\varepsilon)\}\ , \\ (r, \mathrm{syn})(\mathrm{x}_\varepsilon) &\leftarrow r_\mathrm{b}\,; r_\mathrm{G} \setminus \{\mathrm{root}(\mathrm{x}_\varepsilon)\}\ . \end{aligned}$$

Obviously, these three new rules comply with the definition of att mdtt. Thus, by doing this construction for every non-compliant rule we will eventually obtain an att mdtt equivalent to M.

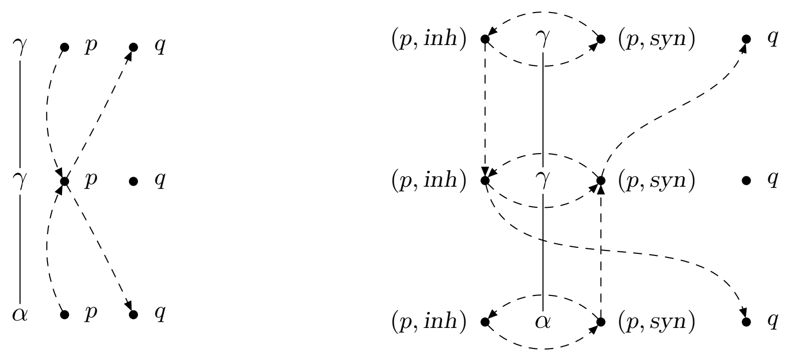


Figure 8.1: Illustration of the construction of Phase 3.

Theorem 8.21 (cf. [28, Theorem 3]). $\text{r–MDTT}(\Sigma, \Delta) = \text{ATT}(\Sigma, \Delta)$.

Proof. This theorem follows immediately from Theorem 5.8 and Lemma 8.20. ■

8.4 Open problems

One major open problem is the question whether executable mdtt and (weakly) non-circular mdtt have the same expressive power, i.e., whether it is always possible to eliminate cycles in an mdtt that do not generate new output trees (i.e., cycles such that every involved hyperedge is an ε-rule instance; these are rule instances (r, ρ) with $r_\text{b} \in P(\text{V})$).

Another open problem is to find a characterization of tree transformations that are computed by mdtt but not by restricted mdtt. Is it decidable whether a given tree transformation is not computable by restricted mdtt?

CHAPTER 9

Weighted multioperator tree automata

Every mwmd can be evaluated in a variety of appropriate m-monoids; every such pair of mwmd and m-monoid determines one fixpoint- and one hypergraph-defined tree series. Thus, every syntactic class of mwmd and every class of m-monoids determines a class of (fixpoint- or hypergraph-defined) tree series; let us call such a class a semantic class. In the previous two chapters we considered semantic classes that we obtained by restricting our attention to particular classes of m-monoids.

In this chapter we will study a semantic class that results from restricting the class of mwmd (whereas allowing the full diversity of possible m-monoids). Since the information transport of mwmd in this class resembles the bottom-up information transport of weighted bottom-up tree automata [15, 63], we will refer to the mwmd in this class as weighted multioperator tree automata mwmd (for short: wmta mwmd).

We will show that wmta mwmd are essentially equivalent to the concept of weighted multioperator tree automata [96, 103, 58, 123] (for short: wmta). Roughly speaking, a wmta is a finite state tree automaton [66] in which every transition is equipped with an operation from the considered m-monoid (where the rank of the operation has to agree with the rank of the transition).

This chapter is a revised version of the most important results of [123, 59]. We will restrict ourselves to proving the following two main results.

1. We will show that, for a given absorptive m-monoid satisfying some additional condition, the class of tree series recognized by wmta over $\mathcal{A}$ can be decomposed into the class of relabeling tree transformations, followed by the class of characteristic tree transformations of recognizable tree languages, and followed by the class of tree series recognized by homomorphism wmta over $\mathcal{A}$, where a homomorphism wmta is a wmta having precisely one state (see Theorem 9.17).

2. We will give an alternative characterization of the class of tree series recognized by wmta. This characterization is based on m-expressions, which form a new kind of weighted MSO-logic. This characterization is a Büchi-like theorem [26, 46] for the class of tree series recognized by wmta (see Theorem 9.26).

This chapter is organized as follows. In Section 9.1 we will study the syntactic class of wmta mwmd and investigate their relationship to wmta. In Section 9.2 we will prove the composition and decomposition results of wmta. In Section 9.3 we will introduce the concept of m-expressions and in Section 9.4 we will prove that the class of tree series definable by m-expressions and the class of tree series recognized by wmta coincides. Finally, we briefly mention further implications of the results that we present in this chapter in Section 9.5.

9.1 Syntax and semantics of weighted multioperator tree automata

Now we will define the syntactic class of wmta mwmd. This class is a syntactic subclass of local mwmd. It is obtained by introducing the following restrictions.

- There are two types of rules: *final rules* and *computation rules*.
- Every final rule is of the form $q(\mathrm{x}_\varepsilon) \leftarrow p(\mathrm{x}_\varepsilon)\ ;\ \{\mathrm{root}(\mathrm{x}_\varepsilon)\}$, where q is the query predicate and $q \neq p$.
- For every $k \in \mathbb{N}$, k-ary input symbol σ, and user-defined predicates $p, p_1, \ldots, p_k$, which are no query predicates, there is precisely one computation rule r; this rule has the form $p(\mathrm{x}_\varepsilon) \leftarrow \delta(p_1(\mathrm{x}_1), \ldots, p_k(\mathrm{x}_k)); \{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \ldots, \mathrm{child}_k(\mathrm{x}_\varepsilon, \mathrm{x}_k)\}$ for some k-ary symbol δ. It is easy to see that in computation rules information transport takes place only from the bottom of the tree (its leafs) to the top (its root).

Due to this strictly restricted syntax, wmta mwmd can be represented in a very concise way. A wmta mwmd is completely specified by the following objects: (i) its set of user-defined predicates (since every user-defined predicate in a local mwmd needs to be unary, it is not required to specify the rank of each user-defined predicate), (ii) for every $k \in \mathbb{N}$, k-ary input symbol σ, and user-defined predicates $p, p_1, \ldots, p_k$, which are no query predicates, the according computation rule is specified by the operation δ that is applied in its body, and (iii) the set of user-defined predicates that occur in the body of final rules. Therefore, every wmta mwmd can compactly be specified by a triple; we will call such a triple a weighted multioperator tree automaton. Now let us define these concepts formally.

> *In this chapter we let Σ be a ranked alphabet, Δ be a signature, and $\mathcal{A} = (A, +, \mathbf{0}, \theta)$ be an m-monoid over Δ.*

Definition 9.1. A ***weighted multioperator tree automaton*** (abbreviated by ***wmta***) *over* Σ *and* Δ is a triple $\mathcal{M} = (Q, \mu, F)$, where

- Q is a finite, non-empty set,
- $\mu = (\mu_k \mid k \in \mathbb{N})$ is a family of mappings $\mu_k : Q^k \times \Sigma^{(k)} \times Q \to \Delta^{(k)}$, and
- $F \subseteq Q$.

Let $M = (P, R, q)$ be an mwmd over Σ and Δ. We say that M is a ***weighted multioperator tree automaton mwmd*** (for short: ***wmta mwmd***) if there is a wmta (Q, μ, F) over Σ and Δ such that

- $P^{(0)} = \emptyset$ and $P^{(1)} = \{q\} \cup Q$, where $q \notin Q$.
- $R = \{r_{k,\sigma,p,p_1,\ldots,p_k} \mid k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, p, p_1, \ldots, p_k \in Q\} \cup \{r_p \mid p \in F\}$ where for every $k \in \mathbb{N}, \sigma \in \Sigma^{(k)}, p, p_1, \ldots, p_k \in Q$ we have

$$r_{k,\sigma,p,p_1,\ldots,p_k} = p(\mathrm{x}_\varepsilon) \leftarrow \mu_k(p_1 \cdots p_k, \sigma, p)(p_1(\mathrm{x}_1), \ldots, p_k(\mathrm{x}_k));$$
$$\{\mathrm{label}_\sigma(\mathrm{x}_\varepsilon), \mathrm{child}_1(\mathrm{x}_\varepsilon, \mathrm{x}_1), \ldots, \mathrm{child}_k(\mathrm{x}_\varepsilon, \mathrm{x}_k)\}\ ,$$

and for every $p \in F$ we have

$$r_p = q(\mathrm{x}_\varepsilon) \leftarrow p(\mathrm{x}_\varepsilon)\,; \{\mathrm{root}(\mathrm{x}_\varepsilon)\}\ .$$

We say that (Q, μ, F) ***represents*** M. □

Since the wmta representation of a wmta mwmd is more compact and easier to handle, we prefer to deal with wmta instead of wmta mwmd in the sequel. Clearly, one can carry over the definition of the semantics of mwmd to wmta naturally; i.e., the semantics of a wmta is the semantics of the mwmd wmta it represents. In this chapter we will restrict ourselves to the hypergraph semantics. Since the syntax of wmta mwmd is strictly restricted, it turns out that the hypergraph semantics of wmta can be expressed in simplified terms. This is stated formally by the following definition and lemma.

Definition 9.2. Let $t \in T_\Sigma$ an $\mathcal{M} = (Q, \mu, F)$ be a wmta over Σ and Δ. We define the set $R_{\mathcal{M}}(t)$ of ***successful runs*** *over* $\mathcal{M}$ *and* t as the set $\{\kappa \mid \kappa : \mathrm{pos}(t) \to Q, \kappa(\varepsilon) \in F\}$. A successful run κ over $\mathcal{M}$ and t is called ***supportive*** *for* $\mathcal{A}$ if for every $w \in \mathrm{pos}(t)$ we have that $\theta\big(\mu_k(\kappa(w \cdot 1) \cdots \kappa(w \cdot k), t(w), \kappa(w))\big)$ is supportive, where $k = \mathrm{rk}_t(w)$; the set of all runs over $\mathcal{M}$ and t that are supportive for $\mathcal{A}$ is denoted by $R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t)$.

Every $\kappa \in R_{\mathcal{M}}(t)$ induces a mapping $\mathrm{wt}_{\mathcal{M},t,\mathcal{A}}(\kappa) : \mathrm{pos}(t) \to A$ which is defined as follows for every $w \in \mathrm{pos}(t)$:

$$\mathrm{wt}_{\mathcal{M},t,\mathcal{A}}(\kappa)(w) = \omega\big(\mathrm{wt}_{\mathcal{M},t,\mathcal{A}}(\kappa)(w1), \ldots, \mathrm{wt}_{\mathcal{M},t,\mathcal{A}}(\kappa)(wk)\big)\ ,$$

where $k = \mathrm{rk}(t(w))$ and $\omega = \mu_k\big(\kappa(w1) \cdots \kappa(wk), t(w), \kappa(w)\big)$. If $\mathcal{M}$, t, and $\mathcal{A}$ are clear from the context, then we also write $\mathrm{wt}(\kappa)$ instead of $\mathrm{wt}_{\mathcal{M},t,\mathcal{A}}(\kappa)$. □

Lemma 9.3. *Let $\mathcal{M} = (Q, \mu, F)$ be a wmta over Σ and Δ and let M be the wmta mwmd over Σ and Δ that is represented by $\mathcal{M}$. Then for every $t \in T_\Sigma$ we have*

$$[\![M]\!]^{\mathrm{hyp}}_{\mathcal{A}}(t) = \sum\nolimits_{\kappa \in R_{\mathcal{M}}(t)} \mathrm{wt}(\kappa)(\varepsilon)\ .$$

Proof. We only sketch this proof because it is similar to the proof of Theorem 7.18. Let $M = (P, R, q)$ and the rules in R be denoted as in Definition 9.1. Moreover, let $t \in T_\Sigma$ and define the mapping $\pi : R_{\mathcal{M}}(t) \to \mathrm{H}^{q(\varepsilon)}_G$ as follows (where $G = \mathrm{G}^{\mathrm{dep}}_{M,t}$). Let $\kappa \in R_{\mathcal{M}}(t)$. Then $\pi(\kappa)$ is the derivation $\eta \in \mathrm{H}^{q(\varepsilon)}_G$ such that $\mathrm{pos}(\eta) = \{\varepsilon\} \cup \{1 \cdot w \mid w \in \mathrm{pos}(t)\}$, $\eta(\varepsilon) = (r_{\kappa(\varepsilon)}, [\mathrm{x}_\varepsilon \mapsto \varepsilon])$ and for every $w \in \mathrm{pos}(t)$ we have

$$\eta(1 \cdot w) = (r_{k,t(w),\kappa(w),\kappa(w1),\ldots,\kappa(wk)}, [\mathrm{x}_\varepsilon \mapsto w, \mathrm{x}_1 \mapsto w1, \ldots, \mathrm{x}_k \mapsto wk])\ ,$$

where $k = \mathrm{rk}(t(w))$. Then $\pi : R_{\mathcal{M}}(t) \to \mathrm{H}^{q(\varepsilon)}_G$ is a bijection and for every $\kappa \in R_{\mathcal{M}}(t)$ we have $\mathrm{wt}(\kappa)(\varepsilon) = h\big(\mathrm{h}_{M,t}(\pi(\kappa))\big)$, where h is the unique Δ-homomorphism from $\mathcal{T}_\Delta$ to (A, θ). This implies the assertion. ■

We use the previous lemma to define the hypergraph semantics for wmta directly (without referring to the wmta mwmd it represents).

Definition 9.4. Let $\mathcal{M} = (Q, \mu, F)$ be a wmta over Σ and Δ. The ***tree series recognized*** *by* $\mathcal{M}$ *(and* $\mathcal{A}$*)*, denoted by $[\![\mathcal{M}]\!]_{\mathcal{A}} \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$, is defined for every $t \in T_\Sigma$ by

$$[\![\mathcal{M}]\!]_{\mathcal{A}}(t) = \sum\nolimits_{\kappa \in R_{\mathcal{M}}(t)} \mathrm{wt}(\kappa)(\varepsilon) \ .$$

If $\mathcal{A}$ is clear from the context, we simply write $[\![\mathcal{M}]\!]$ instead of $[\![\mathcal{M}]\!]_{\mathcal{A}}$. We will refer to $[\![\mathcal{M}]\!]$ as the ***run semantics*** *of* M.

A tree series $\lambda \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$ is called ***recognizable*** *over* Σ *and* $\mathcal{A}$ if there is a wmta $\mathcal{M}$ over Σ and Δ with $[\![\mathcal{M}]\!]_{\mathcal{A}} = \lambda$. □

The following lemma states that if a run of a wmta $\mathcal{M}$ over an absorptive m-monoid is not supportive, then it can be disregarded. This can easily be shown by well-founded induction on tree positions.

Lemma 9.5. *Suppose that* $\mathcal{A}$ *is absorptive. Let* $M = (Q, \mu, F)$ *be a wmta over* Σ *and* Δ*. Then the following two statements hold for every* $t \in T_\Sigma$*.*

1. *For every* $\kappa \in R_{\mathcal{M}}(t) \setminus R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t)$ *we have* $\mathrm{wt}(\kappa)(\varepsilon) = \mathbf{0}$ *.*
2. $[\![\mathcal{M}]\!](t) = \sum_{\kappa \in R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t)} \mathrm{wt}(\kappa)(\varepsilon)$ *.*

Now let us consider an example wmta.

Example 9.6. We consider trees over the ranked alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$. Let $t \in T_\Sigma$ and $w \in \mathrm{pos}(t)$. The *unbalancedness* $\mathrm{ubal}(t, w) \in \mathbb{N}$ *of* t *at* w is defined as

$$\mathrm{ubal}(t, w) = \begin{cases} |\mathrm{height}(t|_{w \cdot 1}) - \mathrm{height}(t|_{w \cdot 2})| \ , & \text{if } t(w) = \sigma, \\ 0 \ , & \text{if } t(w) = \alpha. \end{cases}$$

Furthermore, we define the *unbalancedness* $\mathrm{ubal}(t)$ *of* t as $\mathrm{ubal}(t) = \max_{w \in \mathrm{pos}(t)} \mathrm{ubal}(t, w)$. For example, the unbalancedness of every balanced binary tree (e.g., $\sigma(\sigma(\alpha,\alpha),\sigma(\alpha,\alpha))$) is 0, and every right comb $\sigma(\alpha, \sigma(\alpha, \ldots \sigma(\alpha, \alpha) \ldots))$ with n occurrences of σ has unbalancedness $n - 1$.

Algorithms that operate on binary trees (as, e.g., insertion sort into a search tree) are often the less efficient the more unbalanced the input tree is. Thus, when using such algorithms it is worthwhile to provide an automaton that computes the unbalancedness of an input tree so that the tree can be restructured if it turns out to be highly unbalanced. We construct a wmta M that accomplishes this task. First we define the signature $\Delta = \{\mathrm{nil}_0^{(0)}, \mathrm{nil}_2^{(2)}, \mathrm{zero}^{(0)}, \mathrm{incmax}^{(2)}, \mathrm{diff}^{(2)}, \mathrm{proj}_1^{(2)}, \mathrm{proj}_2^{(2)}\}$. Let $\mathcal{M} = (Q, \mu, F)$ be a wmta over Σ and Δ with $Q = \{h, u\}$, $F = \{u\}$, and μ is defined as follows:

$$\mu_0(\varepsilon, \alpha, h) = \mu_0(\varepsilon, \alpha, u) = \mathrm{zero} \ ,$$
$$\mu_2(hh, \sigma, h) = \mathrm{incmax} \ , \qquad \mu_2(hh, \sigma, u) = \mathrm{diff} \ ,$$
$$\mu_2(uh, \sigma, u) = \mathrm{proj}_1 \ , \qquad \mu_2(hu, \sigma, u) = \mathrm{proj}_2 \ ,$$
$$\mu_2(q_1 q_2, \sigma, p) = \mathrm{nil}_2 \text{ for every other combination of states } q_1,\ q_2, \text{ and } p.$$

Moreover, let $\mathcal{A} = (\mathbb{N} \cup \{-\infty\}, \max, -\infty, \theta)$ be the m-monoid over Δ, where θ is defined as follows for every $a, b \in \mathbb{N}$:

$$\theta(\mathrm{nil}_0)() = -\infty \ , \qquad \theta(\mathrm{zero})() = 0 \ ,$$

$$\theta(\text{incmax})(a,b) = 1 + \max(a,b) \ , \qquad \theta(\text{diff})(a,b) = |a-b| \ ,$$
$$\theta(\text{proj}_1)(a,b) = a \ , \qquad \theta(\text{proj}_2)(a,b) = b \ ,$$
$$\theta(\text{nil}_2)(a,b) = -\infty \ .$$

and for every $\delta \in \Delta^{(2)}$ and $a, b \in \mathbb{N} \cup \{-\infty\}$ with $-\infty \in \{a,b\}$ we let $\theta(\delta)(a,b) = -\infty$.

Now we give an intuition of how $\mathcal{M}$ processes a tree $t \in T_\Sigma$. First observe that $\mathcal{A}$ is absorptive and, thus, we have that $[\![\mathcal{M}]\!](t) = \max\{\text{wt}(\kappa)(\varepsilon) \mid \kappa \in R^{\text{u},\mathcal{A}}_{\mathcal{M}}(t)\}$ by Lemma 9.5(2). Consider a supportive run $\kappa \in R^{\text{u},\mathcal{A}}_{\mathcal{M}}(t)$. By the definition of μ there is a unique position $w \in \text{pos}(t)$ such that for every $w' \in \text{pos}(t)$ we have that $\kappa(w') = u$ if w' is a prefix of w and $\kappa(w') = h$ otherwise, i.e., every position on the unique path from the root of t to w is mapped to u under κ and every other position is mapped to h. In particular, we have for every $w'' \in \text{pos}(t)$ such that w is a proper prefix of w'' that $\kappa(w'') = h$ and, thus, by the definition of $\theta(\mu_0(\varepsilon,\alpha,h))$ and $\theta(\mu_2(hh,\sigma,h))$ we obtain $\text{wt}(\kappa)(w'') = \text{height}(t|_{w''})$. But then $\text{wt}(\kappa)(w) = \text{ubal}(t,w)$ due to the definition of $\theta(\mu_0(\varepsilon,\alpha,u))$ and $\theta(\mu_2(hh,\sigma,u))$. This value is propagated to the root of t, i.e., $\text{wt}(\kappa)(\varepsilon) = \text{ubal}(t,w)$ because of the definition of $\theta(\mu_2(uh,\sigma,u))$ and $\theta(\mu_2(hu,\sigma,u))$.

So every supportive run κ determines a $w \in \text{pos}(t)$ with $\text{wt}(\kappa)(\varepsilon) = \text{ubal}(t,w)$. Conversely, let $w \in \text{pos}(t)$ and consider the run $\kappa \in R_{\mathcal{M}}(t)$ such that for every $w' \in \text{pos}(t)$ we have $\kappa(w') = u$ if w' is a prefix of w and $\kappa(w') = h$ otherwise. Then κ is supportive and $\text{wt}(\kappa)(\varepsilon) = \text{ubal}(t,w)$. Hence, we have $[\![\mathcal{M}]\!](t) = \max_{\kappa \in R^{\text{u},\mathcal{A}}_{\mathcal{M}}(t)} \text{wt}(\kappa)(\varepsilon) = \max_{w \in \text{pos}(t)} \text{ubal}(t,w) = \text{ubal}(t)$. □

Let us conclude this section with a definition of syntactic subclasses of wmta.

Definition 9.7. Let $\mathcal{M} = (Q,\mu,F)$ be a wmta over Σ and Δ. The wmta $\mathcal{M}$ is called a ***homomorphism wmta*** (for short: ***hom wmta***) if $|Q| = |F| = 1$. It is called ***total*** *for* $\mathcal{A}$ if for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1,\ldots,q_k \in Q$ there is at least one $p \in Q$ such that the operation $\theta(\mu_k(q_1 \cdots q_k,\sigma,p))$ is supportive.

We define the classes $\text{Rec}(\Sigma,\mathcal{A})$, h–$\text{Rec}(\Sigma,\mathcal{A})$, and th–$\text{Rec}(\Sigma,\mathcal{A})$ as follows:

- $\text{Rec}(\Sigma,\mathcal{A}) = \{\lambda \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle \mid \lambda$ is recognizable over Σ and $\mathcal{A}.\}$,
- h–$\text{Rec}(\Sigma,\mathcal{A}) = \{\lambda \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle \mid \lambda$ is recognizable over Σ and $\mathcal{A}$ by a hom wmta$\}$,
- th–$\text{Rec}(\Sigma,\mathcal{A}) = \{\lambda \in \mathcal{A}\langle\langle T_\Sigma\rangle\rangle \mid \lambda$ is recognizable over Σ and $\mathcal{A}$ by a hom wmta that is total for $\mathcal{A}\}$.

By Π we denote the class of all ranked alphabets. Then we let $\text{Rec}(\mathcal{A}) = \bigcup_{\Sigma' \in \Pi} \text{Rec}(\Sigma',\mathcal{A})$ and we define the classes h–$\text{Rec}(\mathcal{A})$ and th–$\text{Rec}(\mathcal{A})$ likewise. □

We note that for every hom wmta $\mathcal{M} = (\{*\},\mu,\{*\})$ over Σ and Δ there is a Σ-algebra (A,θ') such that $[\![\mathcal{M}]\!]$ is the unique Σ-homomomorphism from $\mathcal{T}_\Sigma$ to (A,θ'). In fact, we have $\theta'(\sigma) = \theta(\mu_k(* \cdots *,\sigma,*))$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$.

9.2 Decomposition and composition

In this section we prove a decomposition and composition result of wmta. The idea of the decomposition is taken from the following classical result of formal language theory: for every generalized sequential machine mapping (for short: gsm mapping) $\tau : \Sigma^* \to \Delta^*$

there are an alphabet Γ, homomorphisms $h_1 : \Gamma^* \to \Sigma^*$ and $h_2 : \Gamma^* \to \Delta^*$, and a recognizable language $R \subseteq \Gamma^*$ such that $\tau(L) = h_2(h_1^{-1}(L) \cap R)$ for every input language $L \subseteq \Sigma^*$ [111], also see [14, Theorem 4.1].

This decomposition of gsm has been generalized in [48] to bottom-up tree transducers [116, 129]: for every bottom-up tree transducer M there is a relabeling tree transformation R, an fta tree transformation L, and a homomorphism tree transformation H such that $[\![M]\!] = R; L; H$ where $[\![M]\!]$ is the tree transformation computed by M. In fact, R generalizes the inverse of h_1, and L simulates the intersection with a recognizable tree language.

In [48] even the converse result and the following characterization on the level of classes of tree transformations have been proved (cf. Theorem 3.5, Lemmas 4.1 and 4.2 of [48]): $\mathrm{BOT} = \mathrm{REL}; \mathrm{FTA}; \mathrm{HOM}$ where BOT and HOM are the classes of bottom-up tree transformations and homomorphism tree transformations, respectively.

Before we carry over these results to the setting of wmta, we need to recall and introduce notions that are related to tree transformations.

9.2.1 Tree transformations

Recall the notion of finite tree transformation from Definition 8.5. The set of all finite tree transformations from Σ to Δ is denoted by $\mathrm{FIN}(\Sigma, \Delta)$. We call $\lambda \in \mathrm{FIN}(\Sigma, \Delta)$ ***non-overlapping*** if $\lambda(t) \cap \lambda(t') = \emptyset$ for every $t, t' \in T_\Sigma$ with $t \neq t'$. Moreover, λ is called ***shape preserving*** if for every $t \in T_\Sigma$ and $s \in \lambda(t)$ we have $\mathrm{pos}(s) = \mathrm{pos}(t)$. Let $L \subseteq T_\Sigma$. The ***characteristic tree transformation*** *induced by* L, denoted by $\chi_L^{\mathrm{tt}} \in \mathrm{FIN}(\Sigma, \Sigma)$, is defined by $\chi_L^{\mathrm{tt}}(t) = L \cap \{t\}$ for every $t \in T_\Sigma$. Note that every characteristic tree transformation is shape preserving and non-overlapping.

The set $\mathrm{fork}(\Sigma) = \{(\sigma, \sigma_1, \ldots, \sigma_k) \mid k \in \mathbb{N}_+, \sigma \in \Sigma^{(k)}, \sigma_1, \ldots, \sigma_k \in \Sigma\}$ is the set of Σ-***forks***. Let $G \subseteq \mathrm{fork}(\Sigma)$ and $H \subseteq \Sigma$. The ***tree language defined*** *by G and H*, denoted by $[\![G, H]\!] \subseteq T_\Sigma$, is the set of all trees $t \in T_\Sigma$ such that $t(\varepsilon) \in H$ and, for every $w \in \mathrm{pos}(t)$ with $\mathrm{rk}(t(w)) > 0$, $\big(t(w), t(w\cdot 1), \ldots, t(w\cdot \mathrm{rk}(t(w)))\big) \in G$. Observe that $[\![\mathrm{fork}(\Sigma), \Sigma]\!] = T_\Sigma$. A tree language L over Σ is called ***local*** [66, Section 8], if there is a $G \subseteq \mathrm{fork}(\Sigma)$ and $H \subseteq \Sigma$ with $L = [\![G, H]\!]$. By $\mathscr{L}_{\mathrm{LOC}}(\Sigma)$ we denote the set of all local tree languages over Σ and by $\mathrm{LOC}(\Sigma)$ the set of all characteristic tree transformations induced by local languages over Σ. It is obvious that $\mathscr{L}_{\mathrm{LOC}}(\Sigma)$ is closed under intersection.

Recall the definition of recognizable tree languages from Section 6.1. By $\mathrm{FTA}(\Sigma)$ we denote the set of all characteristic tree transformations induced by recognizable tree languages $L \subseteq T_\Sigma$. It is obvious that $\mathrm{LOC}(\Sigma) \subseteq \mathrm{FTA}(\Sigma)$ ([66]).

A ***relabeling*** *from Σ to Δ* is a mapping $\rho : \Sigma \to \mathcal{P}(\Delta)$ with $\rho(\sigma) \subseteq \Delta^{(\mathrm{rk}(\sigma))}$ for every $\sigma \in \Sigma$. The ***tree transformation defined*** *by ρ*, denoted by $[\![\rho]\!] \in \mathrm{FIN}(\Sigma, \Delta)$, is for every $t \in T_\Sigma$ given by

$$[\![\rho]\!](t) = \{s \in T_\Delta \mid \mathrm{pos}(s) = \mathrm{pos}(t) \text{ and } s(w) \in \rho(t(w)) \text{ for every } w \in \mathrm{pos}(s)\} \, .$$

We define

- $\mathrm{REL}(\Sigma, \Delta) = \{[\![\rho]\!] \mid \rho \text{ is a relabeling from } \Sigma \text{ to } \Delta\}$ and
- $i\text{–}\mathrm{REL}(\Sigma, \Delta) = \{\lambda \in \mathrm{REL}(\Sigma, \Delta) \mid \lambda \text{ is non-overlapping}\}$.

Every tree transformation in $\mathrm{REL}(\Sigma, \Delta)$ is finite and shape preserving. Now we define the classes $\mathscr{L}_{\mathrm{LOC}}$, LOC, FTA, REL, and i–REL as follows: for every $\mathcal{C} \in \{\mathscr{L}_{\mathrm{LOC}}, \mathrm{LOC}, \mathrm{FTA}\}$

let $\mathcal{C}$ be defined as $\bigcup_{\Sigma' \in \Pi} \mathcal{C}(\Sigma')$, and for every $\mathcal{D} \in \{\mathrm{REL}, i\text{–REL}\}$ let $\mathcal{D}$ be defined as $\bigcup_{\Sigma', \Delta' \in \Pi} \mathcal{D}(\Sigma', \Delta')$.

Let $\lambda \in \mathrm{FIN}(\Sigma, \Delta)$ be a finite tree transformation and $\psi \in \mathcal{A}\langle\langle T_\Delta \rangle\rangle$ a tree series. The ***composition*** $\lambda; \psi \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$ *of* λ *and* ψ is defined as follows for every $t \in T_\Sigma$:

$$(\lambda; \psi)(t) = \sum\nolimits_{s \in \lambda(t)} \psi(s) \ .$$

Observe that this sum is finite. We lift this operation to classes as follows: let Φ be a class of finite tree transformations and let Ψ be a class of tree series. Then

$$\begin{aligned}\Phi; \Psi = \{\lambda; \psi \mid & \text{ there are ranked alphabets } \Sigma, \Delta \text{ and an m-monoid} \\ & \mathcal{A} \text{ such that } \lambda \in \Phi \cap \mathrm{FIN}(\Sigma, \Delta) \text{ and } \psi \in \Psi \cap \mathcal{A}\langle\langle T_\Delta \rangle\rangle\}.\end{aligned}$$

The following observation shows a weak kind of associativity law of the composition.

Observation 9.8. *1. Let* $\lambda \in \mathrm{FIN}(\Sigma, \Delta)$, $L \subseteq T_\Delta$, *and* $\psi \in \mathcal{A}\langle\langle T_\Delta \rangle\rangle$. *Then we have* $(\lambda; \chi_L^{\mathrm{tt}}); \psi = \lambda; (\chi_L^{\mathrm{tt}}; \psi)$.

2. Let Φ *be a class of finite tree transformations,* $\mathcal{L}$ *be a class of tree languages, and* Ψ *be a class of tree series. Then* $(\Phi; \chi_{\mathcal{L}}^{\mathrm{tt}}); \Psi = \Phi; (\chi_{\mathcal{L}}^{\mathrm{tt}}; \Psi)$, *where* $\chi_{\mathcal{L}}^{\mathrm{tt}} = \{\chi_L^{\mathrm{tt}} \mid L \in \mathcal{L}\}$.

In view of Observation 9.8 we will drop parentheses in expressions of the form $(\lambda; \chi_L^{\mathrm{tt}}); \psi$ or $(\Phi; \chi_{\mathcal{L}}^{\mathrm{tt}}); \Psi$.

Overview

Here we will present the first main result of this chapter: for every absorptive m-monoid $\mathcal{A}$, we have that $\mathrm{Rec}(\mathcal{A}) = \mathrm{REL}; \mathrm{FTA}; h\text{–}\mathrm{Rec}(\mathcal{A})$ (cf. Theorem 9.17). Actually, we will prove a whole variety of characterizations $\mathrm{Rec}(\mathcal{A}) = \mathcal{R}; \mathcal{L}; \mathcal{H}$ where $\mathcal{R} \in \{i\text{–REL}, \mathrm{REL}\}$, $\mathcal{L} \in \{\mathrm{LOC}, \mathrm{FTA}\}$, $\mathcal{H} \in \{th\text{–}\mathrm{Rec}(\mathcal{A}), h\text{–}\mathrm{Rec}(\mathcal{A})\}$.

It has turned out to be useful for both, the underlying decomposition and composition results, to have the following technical tool available. Let $\mathcal{M}$ be a wmta over Σ and Δ, $\lambda \in \mathrm{FIN}(\Sigma, \Gamma)$ be a shape preserving finite tree transformation from Σ to Γ, and $\mathcal{M}_{\mathrm{hom}}$ a hom wmta over Γ and Δ. Roughly speaking, we call $\mathcal{M}$ and $(\lambda, \mathcal{M}_{\mathrm{hom}})$ related if for every input tree $t \in T_\Sigma$, the finite set $\lambda(t)$ of images of t under λ is bijective (via some mapping b_t) to the set $R_{\mathcal{M}}(t)$ of all successful runs over $\mathcal{M}$ and t, and for every run $\kappa \in R_{\mathcal{M}}(t)$, the wmta $\mathcal{M}$ produces on t the same operations as $\mathcal{M}_{\mathrm{hom}}$ produces on $b_t(\kappa)$, more formally, $\mathrm{wt}(\kappa)(\varepsilon) = [\![\mathcal{M}_{\mathrm{hom}}]\!](b_t(\kappa))$. In fact, this idea is taken from the way in which gsm and bottom-up tree transducers have been decomposed, cf. [111] and [48], respectively.

For later purposes, it is convenient not to relate just the set $R_{\mathcal{M}}(t)$ with $\lambda(t)$, but an arbitrary subset $R_t \subseteq R_{\mathcal{M}}(t)$. Then, we will instantiate R_t either to $R_{\mathcal{M}}(t)$ or to $R_{\mathcal{M}}^{\mathrm{u},\mathcal{A}}(t)$.

Definition 9.9. (cf. Figure 9.1) Let $\mathcal{M}$ be a wmta over Σ and Δ. Moreover, let $\lambda \in \mathrm{FIN}(\Sigma, \Gamma)$ be a shape preserving finite tree transformation from Σ to Γ and let $\mathcal{M}_{\mathrm{hom}}$ be a hom wmta over Γ and Δ.

We call $\mathcal{M}$ ***related*** *with* $(\lambda, \mathcal{M}_{\mathrm{hom}})$ if for every $t \in T_\Sigma$ there is a set $R_t \subseteq R_{\mathcal{M}}(t)$ and a bijection $b_t : R_t \to \lambda(t)$ such that

(i) $\mathrm{wt}(\kappa)(\varepsilon) = [\![\mathcal{M}_{\mathrm{hom}}]\!](b_t(\kappa))$ for every $\kappa \in R_t$, and

(ii) for every $\kappa \in R_{\mathcal{M}}(t) \setminus R_t$ we have $\mathrm{wt}(\kappa)(\varepsilon) = \mathbf{0}$. □

Lemma 9.10 (cf. [123, Lemma 2]). *Let $\mathcal{M}$, $\lambda \in \mathrm{FIN}(\Sigma,\Gamma)$, $\mathcal{M}_{\mathrm{hom}}$ be as in Definition 9.9 and suppose that $\mathcal{M}$ is related with $(\lambda, \mathcal{M}_{\mathrm{hom}})$. Then $[\![\mathcal{M}]\!] = \lambda; [\![\mathcal{M}_{\mathrm{hom}}]\!]$.*

Proof. For every $t \in T_\Sigma$,

$$\begin{aligned}
[\![\mathcal{M}]\!](t) &= \sum\nolimits_{\kappa \in R_t} \mathrm{wt}(\kappa)(\varepsilon) + \sum\nolimits_{\kappa \in R_{\mathcal{M}}(t)\setminus R_t} \mathrm{wt}(\kappa)(\varepsilon) \\
&= \sum\nolimits_{\kappa \in R_t} \mathrm{wt}(\kappa)(\varepsilon) && \text{((ii) of Definition 9.9)} \\
&= \sum\nolimits_{\kappa \in R_t} [\![\mathcal{M}_{\mathrm{hom}}]\!]\big(b_t(\kappa)\big) && \text{((i) of Definition 9.9)} \\
&= \sum\nolimits_{s \in \lambda(t)} [\![\mathcal{M}_{\mathrm{hom}}]\!]\big(s\big) && \text{(because } b_t \text{ is a bijection)} \\
&= (\lambda; [\![\mathcal{M}_{\mathrm{hom}}]\!])(t) \; .
\end{aligned}$$

■

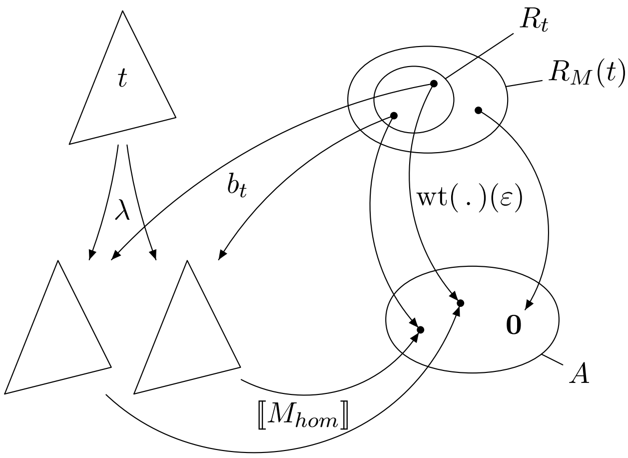


Figure 9.1: Illustration of the Definition 9.9 for a tree t with $|\lambda(t)| = 2$.

The following lemma states a syntactic criterion which guarantees Condition (i) of Definition 9.9. It can easily be shown by well-founded induction on tree positions.

Lemma 9.11 (cf. [123, Lemma 3]). *Let $\mathcal{M} = (Q, \mu, F)$ be a wmta over Σ and Δ and $\mathcal{M}_{\mathrm{hom}} = (\{*\}, \mu_{\mathrm{hom}}, \{*\})$ be a hom wmta over Γ and Δ. Moreover, let $t \in T_\Sigma$, $\kappa \in R_{\mathcal{M}}(t)$, and $s \in T_\Gamma$ be such that $\mathrm{pos}(t) = \mathrm{pos}(s)$. If for every $w \in \mathrm{pos}(t)$ we have $\mu_k(\kappa(w \cdot 1) \cdots \kappa(w \cdot k), t(w), \kappa(w)) = (\mu_{\mathrm{hom}})_k(* \cdots *, s(w), *)$, where $k = \mathrm{rk}(t(w))$, then $\mathrm{wt}(\kappa)(\varepsilon) = [\![\mathcal{M}_{\mathrm{hom}}]\!](s)$.*

9.2.2 Decomposition

Now we can prove the decomposition of wmta.

Lemma 9.12 (cf. [123, Lemma 4]). *1.* $\mathrm{Rec}(\mathcal{A}) \subseteq i\text{–}\mathrm{REL}; \mathrm{LOC}; h\text{–}\mathrm{Rec}(\mathcal{A})$.
2. $\mathrm{Rec}(\mathcal{A}) \subseteq i\text{–}\mathrm{REL}; \mathrm{LOC}; th\text{–}\mathrm{Rec}(\mathcal{A})$, *if $\mathcal{A}$ is absorptive.*

Proof. First we prove Statement 1. Let $\lambda \in \mathrm{Rec}(\mathcal{A})$. Hence, there is a ranked alphabet Σ and a wmta $\mathcal{M} = (Q, \mu, F)$ over Σ and Δ such that $\lambda = [\![\mathcal{M}]\!]$. We define the ranked alphabet

$$\Gamma = \{(q_1, \ldots, q_k, \sigma, p)^{(k)} \mid k \in \mathbb{N},\, \sigma \in \Sigma^{(k)},\, q_1, \ldots, q_k, p \in Q\} \; .$$

In the sequel we will, for every $\gamma = (q_1, \ldots, q_k, \sigma, p) \in \Gamma$ access the components of γ by means of projections functions pr_i (e.g., $\mathrm{pr}_{k+1}(\gamma) = \sigma$); we will denote the projection to the last of component of γ by $\mathrm{pr}_{\mathrm{last}}$, (e.g., $\mathrm{pr}_{\mathrm{last}}(\gamma) = p$).

Moreover, we define a relabeling ρ from Σ to Γ, a set $G \subseteq \mathrm{fork}(\Gamma)$ of Γ-forks, a set $H \subseteq \Gamma$, and a hom wmta $\mathcal{M}_{\mathrm{hom}}$ over Γ and Δ as follows:

- $\rho(\sigma) = \{\gamma \in \Gamma^{(k)} \mid \mathrm{pr}_{k+1}(\gamma) = \sigma\}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$,
- $G = \{(\gamma, \gamma_1, \ldots, \gamma_k) \in \mathrm{fork}(\Gamma) \mid \mathrm{pr}_i(\gamma) = \mathrm{pr}_{\mathrm{last}}(\gamma_i)$ for every $i \in [k]\}$,
- $H = \{\gamma \in \Gamma \mid \mathrm{pr}_{\mathrm{last}}(\gamma) \in F\}$,
- $\mathcal{M}_{\mathrm{hom}} = (\{*\}, \mu_{\mathrm{hom}}, \{*\})$, where for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, p \in Q$ with $(q_1, \ldots, q_k, \sigma, p) \in \Gamma$ we have

$$(\mu_{\mathrm{hom}})_k\big(* \cdots *, (q_1, \ldots, q_k, \sigma, p), *\big) = \mu_k(q_1 \cdots q_k, \sigma, p)\ .$$

Observe that $[\![\rho]\!]$ is non-overlapping and, hence, $[\![\rho]\!] \in i$–REL. Furthermore, observe that $[\![\rho]\!]; \chi^{\mathrm{tt}}_{[\![G,H]\!]}$ is shape preserving because every finite tree transformation in i–REL and LOC is shape preserving. In view of Lemma 9.10 it suffices to prove that the wmta $\mathcal{M}$ is related with $(\lambda, \mathcal{M}_{\mathrm{hom}})$ where $\lambda = [\![\rho]\!]; \chi^{\mathrm{tt}}_{[\![G,H]\!]}$.

Let $t \in T_\Sigma$. We define the set $R_t = R_{\mathcal{M}}(t)$ and the mapping $b_t : R_t \to \lambda(t)$ for every $\kappa \in R_t$ as the tree $b_t(\kappa) \in T_\Gamma$ with $\mathrm{pos}(b_t(\kappa)) = \mathrm{pos}(t)$ and

$$b_t(\kappa)(w) = \big(\kappa(w \cdot 1), \ldots, \kappa(w \cdot \mathrm{rk}(t(w))), t(w), \kappa(w)\big)$$

for every $w \in \mathrm{pos}(t)$. We need to show that this construction is well-defined. It is obvious that $b_t(\kappa) \in T_\Gamma$ due to the definition of R_t. It is also easy to see that $b_t(\kappa) \in [\![\rho]\!](t)$, because $\mathrm{pos}(b_t(\kappa)) = \mathrm{pos}(t)$ and because for every $w \in \mathrm{pos}(t)$ and $k = \mathrm{rk}(t(w))$ we have $\mathrm{pr}_{k+1}(b_t(\kappa)(w)) = t(w)$, which implies $b_t(\kappa)(w) \in \rho(t(w))$. In order to prove that $b_t(\kappa) \in [\![G, H]\!]$ we show that (i) $\big(b_t(\kappa)(w), b_t(\kappa)(w{\cdot}1), \ldots, b_t(\kappa)(w{\cdot}\mathrm{rk}(t(w)))\big) \in G$ for every $w \in \mathrm{pos}(t)$ and that (ii) $b_t(\kappa)(\varepsilon) \in H$. For every $w \in \mathrm{pos}(t)$ and $j \in [\mathrm{rk}(t(w))]$ we have $\mathrm{pr}_j(b_t(\kappa)(w)) = \kappa(w \cdot j) = \mathrm{pr}_{\mathrm{last}}(b_t(\kappa)(w \cdot j))$ which establishes (i). Since $\kappa \in R_t \subseteq R_{\mathcal{M}}(t)$ we have $\mathrm{pr}_{\mathrm{last}}(b_t(\kappa)(\varepsilon)) = \kappa(\varepsilon) \in F$ which implies (ii).

We show that b_t is a bijection. In fact, b_t is injective, because for every $w \in \mathrm{pos}(t)$ the last component of the tuple $b_t(\kappa)(w)$ is $\kappa(w)$ and, thus, b_t maps different runs κ to different trees $b_t(\kappa)$. In order to show that b_t is surjective let $s \in \lambda(t)$. This implies that $s \in [\![\rho]\!](t)$ and $s \in [\![G, H]\!]$. By $s \in [\![\rho]\!](t)$ we get $\mathrm{pos}(s) = \mathrm{pos}(t)$ and, thus, we can construct a run κ over $\mathcal{M}$ and t with $\kappa(w) = \mathrm{pr}_{\mathrm{last}}(s(w))$ for every $w \in \mathrm{pos}(t)$. Since $s \in [\![G, H]\!]$ we have $s(\varepsilon) \in H$ and, thus, $\kappa(\varepsilon) = \mathrm{pr}_{\mathrm{last}}(s(\varepsilon)) \in F$. Therefore, $\kappa \in R_{\mathcal{M}}(t)$. Let $w \in \mathrm{pos}(t)$ and $k = \mathrm{rk}(t(w))$. Then $s \in [\![G, H]\!]$ yields $(s(w), s(w \cdot 1), \ldots, s(w \cdot k)) \in G$ which gives $s(w) \in \Gamma$ and $\mathrm{pr}_i(s(w)) = \mathrm{pr}_{\mathrm{last}}(s(w \cdot i)) = \kappa(w \cdot i)$ for every $i \in [k]$. Together with the fact that $\mathrm{pr}_{k+1}(s(w)) = t(w)$, which is implied by $s \in [\![\rho]\!](t)$, we obtain $\big(\kappa(w \cdot 1), \ldots, \kappa(w \cdot k), t(w), \kappa(w)\big) = s(w) \in \Gamma$ and, thus, $\kappa \in R_t$ and $b_t(\kappa) = s$. Hence, b_t is a bijection.

Now we prove that Conditions (i) and (ii) of Definition 9.9 are satisfied. Let $\kappa \in R_t$, $w \in \mathrm{pos}(t)$, and $k = \mathrm{rk}(t(w))$. Then we obtain that $(\mu_{\mathrm{hom}})_k(* \cdots *, b_t(\kappa)(w), *) = (\mu_{\mathrm{hom}})_k\big(* \cdots *, (\kappa(w \cdot 1), \ldots, \kappa(w \cdot k), t(w), \kappa(w)), *\big) = \mu_k(\kappa(w \cdot 1) \cdots \kappa(w \cdot k), t(w), \kappa(w))$. Then Lemma 9.11 yields Condition (i) of Definition 9.9.
Finally, the fact that $R_{\mathcal{M}}(t) \setminus R_t = \emptyset$ implies Condition (ii) of Definition 9.9.

Now we show the second statement of the lemma and assume that $\mathcal{A}$ is absorptive. The proof is the same as the proof of the first statement with the following two exceptions. Exception 1: we define

$$\Gamma = \{(q_1, \ldots, q_k, \sigma, p)^{(k)} \mid k \in \mathbb{N},\ \sigma \in \Sigma^{(k)},\ q_1, \ldots, q_k, p \in Q \\ \text{such that } \theta(\mu_k(q_1 \cdots q_k, \sigma, p)) \text{ is supportive for } \mathcal{A}\}\ .$$

Note that it is possible that $\Gamma = \emptyset$. Also note that the definitions of ρ, G, H, and $\mathcal{M}_{\text{hom}}$ depend on the choice of Γ, thus, ρ, G, H, and $\mathcal{M}_{\text{hom}}$ are different from the ρ, G, H, and $\mathcal{M}_{\text{hom}}$ in the proof of the first statement of the lemma. Then $\mathcal{M}_{\text{hom}}$ is total for $\mathcal{A}$ because for every $(q_1, \ldots, q_k, \sigma, p) \in \Gamma$ the operation $\theta\big((\mu_{\text{hom}})_k\big(* \cdots *, (q_1, \ldots, q_k, \sigma, p), *\big)\big) = \theta\big(\mu_k(q_1 \cdots q_k, \sigma, p)\big)$ is supportive. Thus, $[\![\mathcal{M}_{\text{hom}}]\!] \in th\text{–Rec}(\mathcal{A})$.

Exception 2: for every $t \in T_\Sigma$ we define R_t to be the set $R^{\text{u},\mathcal{A}}_{\mathcal{M}}(t)$. Then Condition (ii) of Definition 9.9 is implied by Lemma 9.5(1). ■

Example 9.13 (Continuation of Example 9.6). Now we will decompose the wmta $\mathcal{M}$ of Example 9.6 that computes the unbalancedness of trees over the alphabet $\Sigma = \{\sigma^{(2)}, \alpha^{(0)}\}$ and show that $[\![\mathcal{M}]\!] \in i\text{–REL}; \text{LOC}; th\text{–Rec}(\mathcal{A})$.

By using the constructions that occur in the proof of Lemma 9.12(2) we obtain a ranked alphabet Γ, a relabeling ρ from Σ to Γ, a set $G \subseteq \text{fork}(\Gamma)$, a set $H \subseteq \Gamma$, and a total hom wmta $\mathcal{M}_{\text{hom}}$ over Γ and Δ with $[\![\mathcal{M}]\!] = [\![\rho]\!]; \chi^{\text{tt}}_{[\![G,H]\!]}; [\![\mathcal{M}_{\text{hom}}]\!]$ as follows:

$$\begin{aligned}
\Gamma &= \{(h,h,\sigma,h)^{(2)}, (h,h,\sigma,u)^{(2)}, (u,h,\sigma,u)^{(2)}, (h,u,\sigma,u)^{(2)}, \\
&\qquad (\alpha,h)^{(0)}, (\alpha,u)^{(0)}\}\ , \\
\rho(\sigma) &= \{(h,h,\sigma,h), (h,h,\sigma,u), (u,h,\sigma,u), (h,u,\sigma,u)\}\ , \\
\rho(\alpha) &= \{(\alpha,h), (\alpha,u)\}\ , \\
G &= \{(\gamma, \gamma_1, \ldots, \gamma_k) \in \text{fork}(\Gamma) \mid \text{pr}_i(\gamma) = \text{pr}_{\text{last}}(\gamma_i) \text{ for every } i \in [k]\}\ , \\
H &= \{(h,h,\sigma,u), (u,h,\sigma,u), (h,u,\sigma,u), (\alpha,u)\}\ ,
\end{aligned}$$

and $\mathcal{M}_{\text{hom}} = (\{*\}, \mu_{\text{hom}}, \{*\})$ such that for every $a, b \in \mathbb{N}$,

$$\begin{aligned}
(\mu_{\text{hom}})_0(\varepsilon, (\alpha,h), *) = (\mu_{\text{hom}})_0(\varepsilon, (\alpha,u), *) &= \text{zero}\ , \\
(\mu_{\text{hom}})_2\big(**, (h,h,\sigma,h), *\big) &= \text{incmax}\ , \\
(\mu_{\text{hom}})_2\big(**, (h,h,\sigma,u), *\big) &= \text{diff}\ , \\
(\mu_{\text{hom}})_2\big(**, (u,h,\sigma,u), *\big) &= \text{proj}_1\ , \\
(\mu_{\text{hom}})_2\big(**, (h,u,\sigma,u), *\big) &= \text{proj}_2\ .
\end{aligned}$$

Consider the tree $t = \sigma(\alpha, \sigma(\alpha, \alpha)) \in T_\Sigma$. Then $[\![\mathcal{M}]\!](t) = \text{ubal}(t) = 1$. Now we show that also $([\![\rho]\!]; \chi^{\text{tt}}_{[\![G,H]\!]}; [\![\mathcal{M}_{\text{hom}}]\!])(t) = 1$. Let $\lambda = [\![\rho]\!]; \chi^{\text{tt}}_{[\![G,H]\!]} \in \text{FIN}(\Sigma, \Gamma)$. We have

$$\begin{aligned}
(\lambda; [\![\mathcal{M}_{\text{hom}}]\!])(t) &= \max\{[\![\mathcal{M}_{\text{hom}}]\!](s) \mid s \in \lambda(t)\} \\
&= \max\{[\![\mathcal{M}_{\text{hom}}]\!](s) \mid s \in [\![\rho]\!](t) \text{ and } s \in [\![G,H]\!]\}\ .
\end{aligned}$$

The definitions of ρ, G, and H yield that there are exactly five trees s_1 to s_5 with $s_i \in [\![\rho]\!](t)$ and $s_i \in [\![G,H]\!]$ for every $1 \le i \le 5$. (see Figure 9.2). These trees correspond to the five runs over $\mathcal{M}$ and t that are supportive for $\mathcal{A}$. Clearly, $[\![\mathcal{M}_{\text{hom}}]\!](s_1) = 1$ and $[\![\mathcal{M}_{\text{hom}}]\!](s_i) = 0$ for every $2 \le i \le 5$, hence, $(\lambda; [\![\mathcal{M}_{\text{hom}}]\!])(t) = \max\{1, 0\} = 1$. □

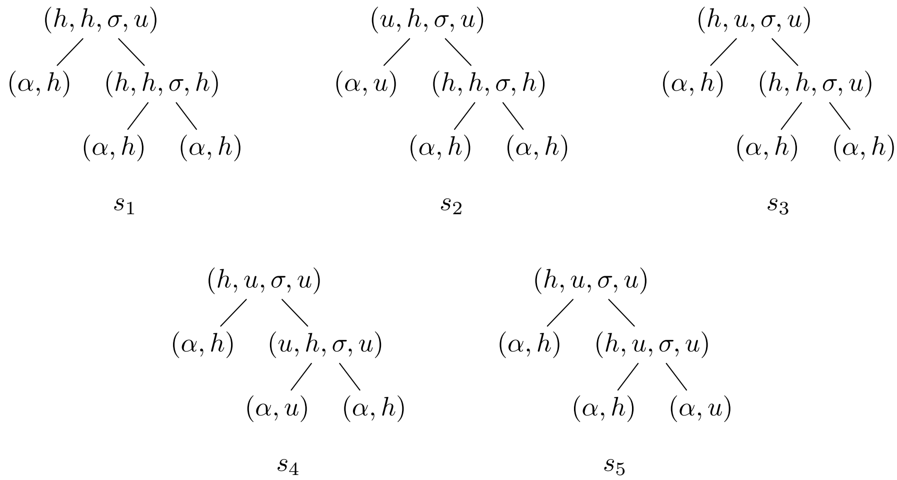


Figure 9.2: The five trees s_1 to s_5 of Example 9.13.

9.2.3 Composition

Now we prove that also the converse inclusion holds; in fact, we will even prove that $\mathrm{REL};\mathrm{FTA};h\text{–}\mathrm{Rec}(\mathcal{A}) \subseteq \mathrm{Rec}(\mathcal{A})$. For this we will first show that $\mathrm{FTA};h\text{–}\mathrm{Rec}(\mathcal{A}) \subseteq \mathrm{Rec}(\mathcal{A})$, and second that $\mathrm{REL};\mathrm{Rec}(\mathcal{A}) \subseteq \mathrm{Rec}(\mathcal{A})$.

For the first inclusion, let us consider a deterministic fta $\mathcal{M}_{\mathrm{fta}}$ over Γ and a hom wmta $\mathcal{M}_{\mathrm{hom}}$ over Γ and Δ. Then we construct a wmta $\mathcal{M}$ over Γ and Δ by simply combining the state behavior of $\mathcal{M}_{\mathrm{fta}}$ with the operation produced by $\mathcal{M}_{\mathrm{hom}}$. If $\mathcal{M}_{\mathrm{fta}}$ does not contain a transition on a k-ary input symbol for a particular state behavior, then $\mathcal{M}$ produces the operation $\mathbf{0}^{(k)}$, where $\mathbf{0}^{(k)}$ is the k-ary operation that is not supportive; thus, we have to assume that the m-monoid $\mathcal{A}$ provides such operations. Let us define this formally.

Definition 9.14. We say that $\mathcal{A}$ is ***strongly absorptive*** if it is absorptive and, for every $k \in \mathbb{N}$, there is a $\delta \in \Delta^{(k)}$ such that $\theta(\delta)$ is not supportive, i.e., $\mathrm{ran}(\theta(\delta)) = \{\mathbf{0}\}$; if $\mathcal{A}$ is clear from the context, we will denote this ranked symbol δ by $\mathbf{0}^{(k)}$. □

Lemma 9.15 (cf. [123, Lemma 5]). *Suppose that $\mathcal{A}$ is strongly absorptive. Then we have that* $\mathrm{FTA};h\text{–}\mathrm{Rec}(\mathcal{A}) \subseteq \mathrm{Rec}(\mathcal{A})$.

PROOF. Let $\lambda \in \mathrm{FTA};h\text{–}\mathrm{Rec}(\mathcal{A})$. Then there is a ranked alphabet Γ, an fta $\mathcal{M}_{\mathrm{fta}} = (Q, \delta_{\mathrm{fta}}, F)$ over Γ, and a hom wmta $\mathcal{M}_{\mathrm{hom}} = (\{*\}, \mu_{\mathrm{hom}}, \{*\})$ over Γ and Δ such that $\lambda = \chi^{\mathrm{tt}}_{\mathcal{L}(\mathcal{M}_{\mathrm{fta}})}; [\![\mathcal{M}_{\mathrm{hom}}]\!]$. We may assume that $\mathcal{M}_{\mathrm{fta}}$ is deterministic (cf. [66]).

We construct the wmta $\mathcal{M} = (Q, \mu, F)$ over Γ and Δ such that for every $k \in \mathbb{N}$, $\gamma \in \Gamma^{(k)}$, $q_1, \ldots, q_k, p \in Q$ we have

$$\mu_k(q_1 \cdots q_k, \gamma, p) = \begin{cases} (\mu_{\mathrm{hom}})_k(* \cdots *, \gamma, *)\ , & \text{if } (q_1, \ldots, q_k, \gamma, p) \in (\delta_{\mathrm{fta}})_k\ ; \\ \mathbf{0}^{(k)}\ , & \text{otherwise}\ . \end{cases}$$

Note that μ is well-defined because $\mathcal{A}$ is strongly absorptive.

The proof is completed by showing that $[\![\mathcal{M}]\!] = \lambda$. By Lemma 9.10 it suffices to show that $\mathcal{M}$ is related with $\big(\chi^{\mathrm{tt}}_{\mathcal{L}(\mathcal{M}_{\mathrm{fta}})}, \mathcal{M}_{\mathrm{hom}}\big)$. Let $t \in T_\Sigma$. We define the set $R_t \subseteq R_{\mathcal{M}}(t)$ and the bijection $b_t : R_t \to \chi^{\mathrm{tt}}_{\mathcal{L}(\mathcal{M}_{\mathrm{fta}})}(t)$ as

$$R_t = \{\kappa \in R_{\mathcal{M}}(t) \mid \kappa \text{ is a successful run over } \mathcal{M}_{\mathrm{fta}} \text{ and } t\} ,$$
$$b_t(\kappa) = t \text{ for every } \kappa \in R_t .$$

Since $\mathcal{M}_{\mathrm{fta}}$ is deterministic, either $R_t = \emptyset$ or it contains the only successful run over $\mathcal{M}_{\mathrm{fta}}$ and t, depending on whether $\mathcal{M}_{\mathrm{fta}}$ accepts t or not. Thus, b_t is well-defined and a bijection.

If $\kappa \in R_t$, $w \in \mathrm{pos}(t)$, and $k = \mathrm{rk}(t(w))$, then $\mu_k(\kappa(w \cdot 1) \cdots \kappa(w \cdot k), t(w), \kappa(w)) = (\mu_{\mathrm{hom}})_k(* \cdots *, t(w), *) = (\mu_{\mathrm{hom}})_k(* \cdots *, b_t(\kappa)(w), *)$ by the definition of R_t. Then we obtain Condition (i) of Definition 9.9 due to Lemma 9.11.

Let $\kappa \in R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t)$. Then for every $w \in \mathrm{pos}(t)$ and $k = \mathrm{rk}(t(w))$ the definition of μ yields $(\kappa(w \cdot 1), \ldots, \kappa(w \cdot k), t(w), \kappa(w)) \in (\delta_{\mathrm{fta}})_k$. Thus, $\kappa \in R_t$. Hence, $R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t) \subseteq R_t$ and, thus, $R_{\mathcal{M}}(t) \setminus R_t \subseteq R_{\mathcal{M}}(t) \setminus R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}}(t)$. Then Lemma 9.5(1) yields Condition (ii) of Definition 9.9. ■

For the proof of the second inclusion, let us consider a relabeling ρ from Σ to Γ and a wmta $\mathcal{M}$ over Γ and Δ; let Q be the set of states of $\mathcal{M}$. Roughly speaking, we construct a wmta $\mathcal{M}'$ with state set $Q \times \Gamma$; for a given input tree t, $\mathcal{M}$ guesses (in the second component of its states) at every position w of t a relabeling of $t(w)$ and then simulates the state behavior of $\mathcal{M}$ on this relabeling.

Lemma 9.16 (cf. [123, Lemma 6]). *If $\mathcal{A}$ is strongly absorptive, then* $\mathrm{REL};\mathrm{Rec}(\mathcal{A}) \subseteq \mathrm{Rec}(\mathcal{A})$.

Proof. Let $\lambda \in \mathrm{REL};\mathrm{Rec}(\mathcal{A})$. Then there are ranked alphabets Σ, Γ, a relabeling ρ from Σ to Γ, and a wmta $\mathcal{M} = (Q, \mu, F)$ over Γ and Δ, such that $\lambda = [\![\rho]\!];[\![\mathcal{M}]\!]$. If $\Gamma = \emptyset$, then $[\![\rho]\!](t) = \emptyset$ and $\lambda(t) = \mathbf{0}$ for every $t \in T_\Sigma$, i.e., $\mathrm{supp}(\lambda) = \emptyset$; thus, $\lambda \in \mathrm{Rec}(\mathcal{A})$ follows immediately. For the remainder of the proof we assume that $\Gamma \neq \emptyset$. We construct the wmta $\mathcal{M}' = (Q \times \Gamma, \mu', F \times \Gamma)$ over Σ and Δ such that for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $q_1, \ldots, q_k, p \in Q \times \Gamma$ we have

$$\mu'_k(q_1 \cdots q_k, \sigma, p) = \begin{cases} \mu_k\big(\mathrm{pr}_1(q_1) \cdots \mathrm{pr}_1(q_k), \mathrm{pr}_2(p), \mathrm{pr}_1(p)\big) , & \text{if } \mathrm{pr}_2(p) \in \rho(\sigma) ; \\ \mathbf{0}^{(k)} , & \text{otherwise} . \end{cases}$$

Observe that μ' is well-defined because $\mathcal{A}$ is strongly absorptive. The proof is completed by showing that $[\![\mathcal{M}']\!] = \lambda$.

Let $t \in T_\Sigma$. We let $P_t = \{\kappa \in R_{\mathcal{M}'}(t) \mid \mathrm{pr}_2(\kappa(w)) \in \rho(t(w)) \text{ for every } w \in \mathrm{pos}(t)\}$. The definition of μ' yields $R^{\mathrm{u},\mathcal{A}}_{\mathcal{M}'}(t) \subseteq P_t$. Hence, by Lemma 9.5(1) we have $\mathrm{wt}(\kappa)_{\mathcal{M}',t,\mathcal{A}}(\varepsilon) = \mathbf{0}$ for every $\kappa \in R_{\mathcal{M}'}(t) \setminus P_t$. Furthermore, we let D be the set of mappings $\kappa : \mathrm{pos}(t) \to Q$ with $\kappa(\varepsilon) \in F$; we define the mapping $c_t : D \times [\![\rho]\!](t) \to P_t$ as follows: for every $\kappa \in D$ and $s \in [\![\rho]\!](t)$ we let $c_t(\kappa, s)(w) = (\kappa(w), s(w))$ for every $w \in \mathrm{pos}(t)$. Then c_t is well-defined because $[\![\rho]\!]$ is shape preserving; furthermore, c_t is a bijection from $D \times [\![\rho]\!](t)$ to P_t. For every $\kappa \in D$, $s \in [\![\rho]\!](t)$, and $w \in \mathrm{pos}(t)$, the following identity can easily be verified by well-founded induction over $\mathrm{pos}(t)$,

$$\mathrm{wt}(c_t(\kappa, s))_{\mathcal{M}',t,\mathcal{A}}(w) = \mathrm{wt}(\kappa)_{\mathcal{M},s,\mathcal{A}}(w) . \tag{9.1}$$

Now we derive

$$\begin{aligned}
[\![\mathcal{M}']\!](t) &= \sum\nolimits_{\kappa' \in R_{\mathcal{M}'}(t)} \mathrm{wt}(\kappa')_{\mathcal{M}',t,\mathcal{A}}(\varepsilon) \\
&= \sum\nolimits_{\kappa' \in P_t} \mathrm{wt}(\kappa')_{\mathcal{M}',t,\mathcal{A}}(\varepsilon) \quad \text{(because } \mathrm{wt}(\kappa')_{\mathcal{M}',t,\mathcal{A}}(\varepsilon) = \mathbf{0} \text{ for every } \kappa' \in R_{\mathcal{M}'}(t) \setminus P_t) \\
&= \sum\nolimits_{s \in [\![\rho]\!](t)} \sum\nolimits_{\kappa \in D} \mathrm{wt}(c_t(\kappa, s))_{\mathcal{M}',t,\mathcal{A}}(\varepsilon) \quad \text{(since } c_t \text{ is a bijection)} \\
&= \sum\nolimits_{s \in [\![\rho]\!](t)} \sum\nolimits_{\kappa \in D} \mathrm{wt}(\kappa)_{\mathcal{M},s,\mathcal{A}}(\varepsilon) \quad \text{(by Equation (9.1))} \\
&= \sum\nolimits_{s \in [\![\rho]\!](t)} \sum\nolimits_{\kappa \in R_{\mathcal{M}}(s)} \mathrm{wt}(\kappa)_{\mathcal{M},s,\mathcal{A}}(\varepsilon) \\
&\qquad \text{(since } D = R_{\mathcal{M}}(s) \text{ because } [\![\rho]\!] \text{ is shape preserving)} \\
&= \sum\nolimits_{s \in [\![\rho]\!](t)} [\![\mathcal{M}]\!](s) = ([\![\rho]\!]; [\![\mathcal{M}]\!])(t) \ .
\end{aligned}$$

■

Theorem 9.17 (cf. [123, Theorem 1]). *If $\mathcal{A}$ is a strongly absorptive m-monoid, then for every $\mathcal{R} \in \{i\text{–REL}, \mathrm{REL}\}$, $\mathcal{L} \in \{\mathrm{LOC}, \mathrm{FTA}\}$, and $\mathcal{H} \in \{th\text{–Rec}(\mathcal{A}), h\text{–Rec}(\mathcal{A})\}$ we have $\mathrm{Rec}(\mathcal{A}) = \mathcal{R}; \mathcal{L}; \mathcal{H}$.*

PROOF. The inclusion $\mathrm{Rec}(\mathcal{A}) \subseteq \mathcal{R}; \mathcal{L}; \mathcal{H}$ follows from Lemma 9.12 and the facts i–REL $\subseteq$ REL, LOC $\subseteq$ FTA (see Sect. 9.2.1), and $th\text{–Rec}(\mathcal{A}) \subseteq h\text{–Rec}(\mathcal{A})$. The inclusion $\mathcal{R}; \mathcal{L}; \mathcal{H} \subseteq \mathrm{Rec}(\mathcal{A})$ is due to $\mathcal{R}; \mathcal{L}; \mathcal{H} \subseteq \mathrm{REL}; \mathrm{FTA}; h\text{–Rec}(\mathcal{A})$ and by Lemmas 9.15 and 9.16. ■

9.3 M-definable tree series

In this section we introduce multioperator expressions over Σ and Δ, define the concept of m-definable tree series, and illustrate these new concepts by means of three extended examples.

Intuitively, when evaluated in trees and an m-monoid $\mathcal{A}$, multioperator expressions describe calculations in $\mathcal{A}$ where parts of these calculations can be guarded by formulas of Boolean-valued MSO-logics.

Syntax

First let us introduce the syntax of m-expression, which is reminiscent to the syntax of monadic second order logic. Similarly to MSO-logic (see Section 6.1) the m-expressions involve first- and second-order variables.

Definition 9.18. A Σ-***family of operations*** *in* Δ is a family $\omega = (\omega_\sigma \mid \sigma \in \Sigma)$ such that $\omega_\sigma \in \Delta^{(k)}$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$.

For every finite set $\mathcal{V}$ of first- and second-order variables we define the ranked alphabet $\Sigma_{\mathcal{V}}$ by letting $\Sigma_{\mathcal{V}}^{(k)} = \Sigma^{(k)} \times \mathcal{P}(\mathcal{V})$ for every $k \in \mathbb{N}$; as a convention, we identify the sets Σ and $\Sigma_\emptyset$.

Now we define the set $\mathrm{MExp}(\Sigma, \Delta)$ of ***multioperator expressions*** (for short: ***m-expressions***) *over* Σ *and* Δ by the following EBNF with nonterminal e:

$$e \ ::= \mathrm{H}(\omega) \mid (e + e) \mid \sum\nolimits_x e \mid \sum\nolimits_X e \mid (\varphi \rhd e) \ ,$$

where ω is a $\Sigma_{\mathcal{U}}$-family of operations in Δ for some finite set $\mathcal{U}$ of first-order and second-order variables, x is a first-order variable, X is a second-order variable, and $\varphi \in \mathrm{MSO}(\Sigma)$. We will drop parentheses whenever no confusions arise. □

Since we will define the semantics of m-expressions inductively, we need the concept of free variables (as in logics).

Definition 9.19. For every $e \in \mathrm{MExp}(\Sigma, \Delta)$ we define the set of ***free variables*** *of* e, denoted by $\mathrm{Free}(e)$, by recursion on the structure of m-expressions as follows:

- if $\mathcal{U}$ is a finite set of variables and ω is a $\Sigma_{\mathcal{U}}$-family of operations in Δ, then $\mathrm{Free}(\mathrm{H}(\omega)) = \mathcal{U}$,
- $\mathrm{Free}(e_1 + e_2) = \mathrm{Free}(e_1) \cup \mathrm{Free}(e_2)$,
- $\mathrm{Free}(\sum_x e) = \mathrm{Free}(e) \setminus \{x\}$ and $\mathrm{Free}(\sum_X e) = \mathrm{Free}(e) \setminus \{X\}$,
- $\mathrm{Free}(\varphi \rhd e) = \mathrm{Free}(\varphi) \cup \mathrm{Free}(e)$.

We call e a ***sentence*** if $\mathrm{Free}(e) = \emptyset$. □

Semantics

Now we define the semantics of m-expressions by induction on their structure. We note that the semantics of $e_1 + e_2$, $\sum_x e$, and $\sum_X e$ correspond to the semantics of the formulas $\varphi_1 \vee \varphi_2$, $\exists x.\varphi$, and $\exists X.\varphi$, respectively, of weighted MSO-logic. Before we describe the semantics formally, we need to introduce some more auxiliary notions.

Definition 9.20. Let ω be a Σ-family of operations in Δ and observe that $(A, (\omega\,;\theta))$ is a Σ-algebra. We denote the unique Σ-homomorphism from the $\mathcal{T}_\Sigma$ to the Σ-algebra $(A, (\omega\,;\theta))$ by h_ω and call it the ***homomorphism induced*** *by* ω.

As a technical tool we have to extend the index set of a family of operations. Let $\mathcal{U}$ and $\mathcal{V}$ be finite sets of first- and second-order variables with $\mathcal{U} \subseteq \mathcal{V}$. Moreover, let ω be a $\Sigma_{\mathcal{U}}$-family of operations in Δ. We define $\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]$ to be the $\Sigma_{\mathcal{V}}$-family of operations in Δ defined for every $\sigma \in \Sigma$ and $V \subseteq \mathcal{V}$ by $\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]_{(\sigma,V)} = \omega_{(\sigma,\mathcal{U}\cap V)}$.

Let $t \in T_\Sigma$. In the usual way, we can encode a pair (t, ρ), where ρ is a $\mathcal{V}$-assignment for t, as a tree over the ranked alphabet $\Sigma_{\mathcal{V}}$ (recall the definition of $\mathcal{V}$-assignments from Section 6.1). A tree $s \in T_{\Sigma_{\mathcal{V}}}$ is called ***valid*** if for every first-order variable $x \in \mathcal{V}$ there is a unique $w \in \mathrm{pos}(s)$ such that x occurs in the second component of $s(w)$. We denote the set of all valid trees in $T_{\Sigma_{\mathcal{V}}}$ by $T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$.

There is a bijection between the two sets $\{(t, \rho) \mid t \in T_\Sigma, \rho \in \Phi_{\mathcal{V},t}\}$ and $T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$ via the correspondence $(t, \rho) \mapsto s$, where $\mathrm{pos}(s) = \mathrm{pos}(t)$ and for every $w \in \mathrm{pos}(s)$,

$$s(w) = \big(t(w), \{x \in \mathcal{V}^{(1)} \mid w = \rho(x)\} \cup \{X \in \mathcal{V}^{(2)} \mid w \in \rho(X)\}\big) ,$$

where $\mathcal{V}^{(1)}$ is the set of first-order variables occurring in $\mathcal{V}$ and $\mathcal{V}^{(2)}$ the set of second-order variables in $\mathcal{V}$; therefore we will no further distinguish between the sets $T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$ and $\{(t, \rho) \mid t \in T_\Sigma, \rho \in \Phi_{\mathcal{V},t}\}$.

Let $(t, \rho) \in T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$, x be a first-order variable, and $w \in \mathrm{pos}(t)$. By $(t, \rho)[x \mapsto w]$ we denote the valid tree $(t, \rho[x \mapsto w])$ over $\Sigma_{\mathcal{V}\cup\{x\}}$. Similarly, if X is a second-order variable and $W \subseteq \mathrm{pos}(s)$, then $(t, \rho)[X \mapsto W]$ denotes the valid tree $(t, \rho[X \mapsto W])$ over $\Sigma_{\mathcal{V}\cup\{X\}}$. □

Now we are prepared to define the semantics of m-expressions.

Definition 9.21. Let $e \in \mathrm{MExp}(\Sigma, \Delta)$ and $\mathcal{V}$ be a finite set of variables containing $\mathrm{Free}(e)$. The ***semantics*** *of* e *with respect to* $\mathcal{V}$ *and* $\mathcal{A}$ is the tree series $[\![e]\!]_{\mathcal{V},\mathcal{A}} \in \mathcal{A}\langle\langle T_{\Sigma_\mathcal{V}} \rangle\rangle$ such that $\mathrm{supp}([\![e]\!]_{\mathcal{V},\mathcal{A}}) \subseteq T^{\mathrm{v}}_{\Sigma_\mathcal{V}}$ and for every $s \in T^{\mathrm{v}}_{\Sigma_\mathcal{V}}$ we define $[\![e]\!]_{\mathcal{V},\mathcal{A}}(s)$ inductively as follows:

- for every $\mathcal{U} \subseteq \mathcal{V}$ and $\Sigma_\mathcal{U}$-family ω of operations in Δ:
  $[\![\mathrm{H}(\omega)]\!]_{\mathcal{V},\mathcal{A}}(s) = \mathrm{h}_{\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]}(s)$,
- for every $e_1, e_2 \in \mathrm{MExp}(\Sigma, \Delta)$:
  $[\![e_1 + e_2]\!]_{\mathcal{V},\mathcal{A}}(s) = [\![e_1]\!]_{\mathcal{V},\mathcal{A}}(s) + [\![e_2]\!]_{\mathcal{V},\mathcal{A}}(s)$,
- for every first-order variable x and $e \in \mathrm{MExp}(\Sigma, \Delta)$:
  $[\![\sum_x e]\!]_{\mathcal{V},\mathcal{A}}(s) = \sum_{w \in \mathrm{pos}(s)} [\![e]\!]_{\mathcal{V} \cup \{x\},\mathcal{A}}(s[x \mapsto w])$,
- for every second-order variable X and $e \in \mathrm{MExp}(\Sigma, \Delta)$:
  $[\![\sum_X e]\!]_{\mathcal{V},\mathcal{A}}(s) = \sum_{W \subseteq \mathrm{pos}(s)} [\![e]\!]_{\mathcal{V} \cup \{X\},\mathcal{A}}(s[X \mapsto W])$,
- for every $\varphi \in \mathrm{MSO}(\Sigma)$ and $e \in \mathrm{MExp}(\Sigma, \Delta)$:
  $$[\![\varphi \rhd e]\!]_{\mathcal{V},\mathcal{A}}(s) = \begin{cases} [\![e]\!]_{\mathcal{V},\mathcal{A}}(s) , & \text{if } s \in \mathcal{L}_\mathcal{V}(\varphi) \\ 0 , & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ is clear from the context, then we write $[\![e]\!]_\mathcal{V}$ instead of $[\![e]\!]_{\mathcal{V},\mathcal{A}}$. Moreover, we write $[\![e]\!]$ rather than $[\![e]\!]_{\mathrm{Free}(e)}$. We say that a tree series $s \in \mathcal{A}\langle\langle T_\Sigma \rangle\rangle$ is ***definable by m-expressions*** *over* Σ *and* $\mathcal{A}$ (or: ***m-definable***) if there is a sentence $e \in \mathrm{MExp}(\Sigma, \Delta)$ with $[\![e]\!]_\mathcal{A} = s$. By $\mathrm{M}(\Sigma, \mathcal{A})$ we denote the set of tree series that are definable by m-expressions over Σ and $\mathcal{A}$. □

Example 9.22 (cf. [58, Example 3.10]). Let Σ be some ranked alphabet, let $\Delta = \Sigma$, and consider the m-expression $e = \mathrm{H}(\sigma \mid \sigma \in \Sigma)$ over Σ and Δ.

Let $\mathcal{A}_{\mathrm{height}} = (\mathbb{N}, +, 0, \theta_{\mathrm{height}})$ be the m-monoid over Δ, where $\theta_{\mathrm{height}}(\sigma)(n_1, \ldots, n_k) = \max\{1 + n_1, \ldots, 1 + n_k\}$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $n_1, \ldots, n_k \in \mathbb{N}$. Then $[\![e]\!]_{\mathcal{A}_{\mathrm{height}}}$ is the tree series $\mathrm{height} \in \mathcal{A}_{\mathrm{height}}\langle\langle T_\Sigma \rangle\rangle$, which associates every tree in T_Σ with its height. Hence, this tree series is m-definable over Σ and $\mathcal{A}_{\mathrm{height}}$.

Similarly, for the m-monoid $\mathcal{A}_{\mathrm{size}} = (\mathbb{N}, +, 0, \theta_{\mathrm{size}})$ over Δ, where $\theta_{\mathrm{size}}(\sigma)(n_1, \ldots, n_k) = 1 + n_1 + \cdots + n_k$ for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $n_1, \ldots, n_k \in \mathbb{N}$, we have that $[\![e]\!]_{\mathcal{A}_{\mathrm{height}}}$ is the tree series $\mathrm{size} \in \mathcal{A}_{\mathrm{size}}\langle\langle T_\Sigma \rangle\rangle$. Hence, size is m-definable over Σ and $\mathcal{A}_{\mathrm{size}}$.

In general, for every Σ-algebra (A, θ), the unique Σ-homomorphism from $\mathcal{T}_\Sigma$ to (A, θ) is the semantics of the m-expression e wrt the m-monoid $(A, +, \mathbf{0}, \theta)$ over $\Sigma = \Delta$. □

Example 9.23 (*Continuation of Example 9.6***).** The tree series ubal of Example 9.6 is m-definable over Σ and $\mathcal{A}$. In order to prove this, we construct the following m-expression e over Σ and Δ:

$$e = \sum\nolimits_x \sum\nolimits_Y \sum\nolimits_{Z_1} \sum\nolimits_{Z_2} \varphi(x, Y, Z_1, Z_2) \rhd \mathrm{H}\big(\omega_\delta \mid \delta \in \Sigma_\mathcal{U}\big) ,$$

where $\mathcal{U} = \{x, Y, Z_1, Z_2\}$ and

$$\omega_{(\alpha,\{Y\})} = \omega_{(\alpha,\{x\})} = \mathrm{zero} , \qquad \omega_{(\sigma,\{Y\})} = \mathrm{incmax} , \qquad \omega_{(\sigma,\{x\})} = \mathrm{diff} ,$$
$$\omega_{(\sigma,\{Z_1\})} = \mathrm{proj}_1 , \qquad \omega_{(\sigma,\{Z_2\})} = \mathrm{proj}_2 ,$$

and $\omega_\delta = \mathrm{nil}_k$ for every $k \in \{0, 2\}$ and every other $\delta \in \Sigma^{(k)}_\mathcal{U}$. It is straightforward to define the formula $\varphi(x, Y, Z_1, Z_2)$ in $\mathrm{MSO}(\Sigma)$ such that it is true for a given tree $s = (t, \rho) \in T^{\mathrm{v}}_{\Sigma_\mathcal{U}}$ iff

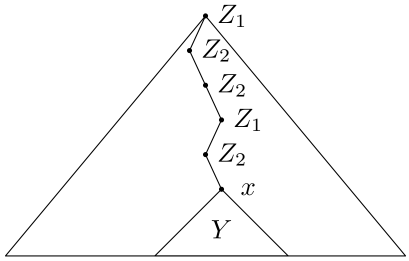


Figure 9.3: A variable assignment for Example 9.23 satisfying $\varphi(x, Y, Z_1, Z_2)$.

- $\rho(Y)$ comprises exactly all positions below $\rho(x)$,
- $\rho(Z_1)$ comprises exactly all positions w such that $\rho(x)$ occurs in $t|_{w1}$, and
- $\rho(Z_2)$ comprises exactly all positions w such that $\rho(x)$ occurs in $t|_{w2}$ (see Figure 9.3).

Clearly, for every $t \in T_\Sigma$ and position $w \in \text{pos}(t)$ there is exactly one combination of sets W, W_1, and W_2 of positions such that $\varphi(w, W, W_1, W_2)$ is true. For such sets W, W_1, and W_2 it is easy to see that

$$[\![\text{H}(\omega)]\!](t[x \mapsto w][Y \mapsto W][Z_1 \mapsto W_1][Z_2 \mapsto W_2]) = \text{ubal}(t, w) \ .$$

Thus, $[\![e]\!] = \text{ubal}$. □

Example 9.24 (cf. [59, Example 3.7]). Consider the alphabet $\Sigma = \{\sigma^{(2)}, \gamma^{(1)}, \alpha^{(0)}\}$ and the pattern $p = \sigma(\cdot, \alpha)$. Let $\tau \in \text{FIN}(\Sigma, \Sigma)$ be the finite tree transformation defined as follows for every $t \in T_\Sigma$ (see [63, Example 5.4]): if p does not occur in t, then $\tau(t) = \emptyset$; if p occurs in t, then $\tau(t) = \{t\} \cup \{t_w \mid w \in \text{pos}(t), p \text{ occurs at } w\}$, where t_w is the tree obtained from t by deleting p at occurrence w in t, i.e., $t_w = t[t|_{w1}]_w$.

Let $\Delta = \{\text{top}_\delta^{(k)} \mid k \in \mathbb{N}, \delta \in \Sigma^{(k)}\} \cup \{\text{proj}^{(2)}\}$ be a signature and consider the m-monoid $\mathcal{A} = (\mathcal{P}(T_\Sigma), \cup, \emptyset, \theta)$ over Δ, where for every $\delta \in \Sigma$ we let $\theta(\text{top}_\delta)$ be the δ-language top concatenation and $\theta(\text{proj})(L_1, L_2) = L_1$ for every $L_1, L_2 \subseteq T_\Sigma$.

Now we show that τ is m-definable over Σ and $\mathcal{A}$. To this end we define the m-expression e_τ over Σ and Δ as follows:

$$e_\tau = \varphi \rhd \Big(\text{H}(\omega) + \sum\nolimits_x \psi(x) \rhd \text{H}(\omega')\Big) \ ,$$

where

- $\varphi \in \text{MSO}(\Sigma)$ is a sentence that is true for a given tree $t \in T_\Sigma$ iff p occurs in t, e.g., $\varphi = \exists x.\psi(x)$,
- $\psi(x) \in \text{MSO}(\Sigma)$ is true for a tree $(t, \rho) \in T^{\text{v}}_{\Sigma_{\{x\}}}$ iff p occurs in t at position $\rho(x)$, e.g., $\psi(x) = \text{label}_\sigma(x) \wedge \big(\exists y.\text{edge}_2(x, y) \wedge \text{label}_\alpha(y)\big)$,
- $\omega = (\text{top}_\delta \mid \delta \in \Sigma)$,
- $\omega' = (\omega_{(\delta,V)} \mid (\delta, V) \in \Sigma_{\{x\}})$ and $\omega_{(\delta,V)} = \text{top}_\delta$ if $(\delta, V) \neq (\sigma, \{x\})$, and $\omega_{(\delta,V)} = \text{proj}$ otherwise.

Let us compute $[\![e_\tau]\!](t)$ for the tree $t = \sigma(\alpha, \sigma(\alpha, \alpha))$. Since $t \in \mathcal{L}(\varphi)$, we have $[\![e_\tau]\!](t) = [\![\mathrm{H}(\omega)]\!](t) \cup [\![e]\!](t)$, where $e = \sum_x \psi(x) \rhd \mathrm{H}(\omega')$. Since $\mathrm{Free}(\omega) = \emptyset$ we have $[\![\mathrm{H}(\omega)]\!](t) = \mathrm{h}_{\omega[\emptyset \rightsquigarrow \emptyset]}(t) = \mathrm{h}_\omega(t) = \{t\}$. Now we can evaluate the second subexpression as follows:

$$\begin{aligned}
[\![e]\!](t) &= \bigcup\nolimits_{w \in \mathrm{pos}(t)} [\![\psi(x) \rhd \mathrm{H}(\omega')]\!]_{\{x\}}(t[x \mapsto w]) \\
&= [\![\mathrm{H}(\omega')]\!]_{\{x\}}(t[x \mapsto 2]) \\
&= [\![\mathrm{H}(\omega')]\!]_{\{x\}}\Big((\sigma, \emptyset)\big((\alpha, \emptyset), \big((\sigma, \{x\})\big((\alpha, \emptyset), (\alpha, \emptyset)\big)\big)\big)\Big) ,
\end{aligned}$$

because $t[x \mapsto 2] \in \mathcal{L}_{\{x\}}(\psi(x))$ and $t[x \mapsto w] \notin \mathcal{L}_{\{x\}}(\psi(x))$ for every $w \in \mathrm{pos}(t) \setminus \{2\}$. Since $\mathrm{Free}(\omega') = \{x\}$, we derive as follows: $[\![\mathrm{H}(\omega')]\!]_{\{x\}}(t[x \mapsto 2]) = \mathrm{h}_{\omega'}(t[x \mapsto 2]) = \mathrm{top}_\sigma(\mathrm{top}_\alpha, \mathrm{proj}(\mathrm{top}_\alpha, \mathrm{top}_\alpha)) = \{\sigma(\alpha, \alpha)\}$. Hence $([\![e_\tau]\!], t) = \{t\} \cup \{\sigma(\alpha, \alpha)\} = \tau(t)$. □

Next we prove a consistency lemma for m-expressions (compare Lemma 6.3 for the analogous lemma on MSO-logic formulas).

Lemma 9.25 (cf. [59, Lemma 3.8]). *Let* $e \in \mathrm{MExp}(\Sigma, \Delta)$ *and let* $\mathcal{V}$ *and* $\mathcal{W}$ *be finite sets of variables with* $\mathrm{Free}(e) \subseteq \mathcal{W} \subseteq \mathcal{V}$. *Then for every* $(t, \rho) \in T^{\mathrm{v}}_{\Sigma_\mathcal{V}}$, *we have* $[\![e]\!]_\mathcal{V}(t, \rho) = [\![e]\!]_\mathcal{W}(t, \rho|_\mathcal{W})$.

Proof. Throughout this proof we abbreviate $\rho|_\mathcal{W}$ by ρ'. We prove this lemma by structural induction on $e \in \mathrm{MExp}(\Sigma, \Delta)$.

Case $e = \mathrm{H}(\omega)$. Let $\mathcal{U}$ be a finite set of variables and ω be a $\Sigma_\mathcal{U}$-family of operations in Δ. We need to show that $\mathrm{h}_{\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]}(t, \rho) = \mathrm{h}_{\omega[\mathcal{U} \rightsquigarrow \mathcal{W}]}(t, \rho')$. Since $\mathrm{pos}(t, \rho) = \mathrm{pos}(t) = \mathrm{pos}(t, \rho')$, it suffices to show for every $w \in \mathrm{pos}(t)$ that

$$\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]_{(t,\rho)(w)} = \omega[\mathcal{U} \rightsquigarrow \mathcal{W}]_{(t,\rho')(w)} . \tag{9.2}$$

It is easy to check that Equation (9.2) holds.

Case $e = e_1 + e_2$. By the fact that $\mathrm{Free}(e_i) \subseteq \mathrm{Free}(e) \subseteq \mathcal{W}$ for every $i \in \{1, 2\}$, we obtain:

$$\begin{aligned}
[\![e]\!]_\mathcal{V}(t, \rho) &= [\![e_1]\!]_\mathcal{V}(t, \rho) + [\![e_2]\!]_\mathcal{V}(t, \rho) \\
&= [\![e_1]\!]_\mathcal{W}(t, \rho') + [\![e_2]\!]_\mathcal{W}(t, \rho') && \text{(by ind. hyp.)} \\
&= [\![e]\!]_\mathcal{W}(t, \rho') .
\end{aligned}$$

Case $e = \sum_x e'$. Since $\mathrm{Free}(e') \subseteq \mathrm{Free}(e) \cup \{x\} \subseteq \mathcal{W} \cup \{x\}$, we obtain

$$\begin{aligned}
&\sum\nolimits_{w \in \mathrm{pos}(t)} [\![e']\!]_{\mathcal{V} \cup \{x\}}(t, \rho[x \mapsto w]) \\
&= \sum\nolimits_{w \in \mathrm{pos}(t)} [\![e']\!]_{\mathcal{W} \cup \{x\}}(t, (\rho[x \mapsto w])|_{\mathcal{W} \cup \{x\}}) && \text{(by ind. hyp.)} \\
&= \sum\nolimits_{w \in \mathrm{pos}(t)} [\![e']\!]_{\mathcal{W} \cup \{x\}}(t, (\rho|_\mathcal{W})[x \mapsto w]) .
\end{aligned}$$

Case $e = \sum_X e'$: This case can be shown in the same way as $e = \sum_x e'$.

Case $e = \varphi \triangleright e'$: Let $\varphi \in \mathrm{MSO}(\Sigma)$ and $e' \in \mathrm{MExp}(\Sigma, \Delta)$. By Lemma 6.3 we have $(t, \rho) \in \mathcal{L}_{\mathcal{V}}(\varphi)$ iff $(t, \rho') \in \mathcal{L}_{\mathcal{W}}(\varphi)$. Hence, if $(t, \rho) \notin \mathcal{L}_{\mathcal{V}}(\varphi)$, the statement obviously holds. If $(t, \rho) \in \mathcal{L}_{\mathcal{V}}(\varphi)$, then the induction hypothesis yields

$$[\![e]\!]_{\mathcal{V}}(t, \rho) = [\![e']\!]_{\mathcal{V}}(t, \rho) = [\![e']\!]_{\mathcal{W}}(t, \rho') = [\![e]\!]_{\mathcal{W}}(t, \rho') . \qquad \blacksquare$$

9.4 A Büchi-like theorem

Here we prove that a tree series is recognizable by a wmta iff it is definable by an m-expression whenever the considered m-monoid is strongly absorptive. We recall that every m-monoid can be extended easily to a strongly absorptive m-monoid (see Remark 3.12).

Theorem 9.26 (cf. [59, Theorem 4.1]). *If $\mathcal{A}$ is strongly absorptive, then* $\mathrm{Rec}(\Sigma, \mathcal{A}) = \mathrm{M}(\Sigma, \mathcal{A})$.

This theorem follows from Lemmas 9.27 and 9.35, which will be proved in the next two subsections, respectively.

9.4.1 From automata to m-expressions

For an arbitrary wmta $\mathcal{M}$ we will construct an equivalent m-expression e. The idea of the construction is the same as in the proof of the fact that recognizability implies MSO-definability, and e has the form:

$$e = \sum\nolimits_{X_1} \cdots \sum\nolimits_{X_n} (\varphi \triangleright \mathrm{H}(\omega)) .$$

Intuitively, using a sequence $X_1, \ldots, X_n$ of second-order variables, an MSO-formula φ checks, for a given input tree t, whether the associated sets $\rho(X_1), \ldots, \rho(X_n) \subseteq \mathrm{pos}(t)$ represent a successful run κ over $\mathcal{M}$ on t. If so, then the homomorphism h_ω on (t, ρ) maps every transition $(q_1 \cdots q_k, \sigma, q)$ of $\mathcal{M}$ to the operation $\mu_k(q_1 \cdots q_k, \sigma, q)$, thereby computing the weight of κ.

We note that the form of the m-expression e nicely resembles the fact that each wmta $\mathcal{M}$ can be decomposed into a relabeling (reflected by $\sum_{X_1} \cdots \sum_{X_n}$), followed by a partial identity on a recognizable tree language (reflected by φ), followed by a homomorphism (reflected by $\mathrm{H}(\omega)$); see Section 9.2.

Lemma 9.27 (cf. [59, Lemma 4.2]). *Let $\mathcal{A}$ be strongly absorptive and let $\mathcal{M}$ be a wmta over Σ and Δ. Then there effectively exists a sentence $e \in \mathrm{MExp}(\Sigma, \Delta)$ such that $[\![\mathcal{M}]\!] = [\![e]\!]$.*

Proof. Let $\mathcal{M} = (Q, \mu, F)$. We define the set $\mathcal{V} = (\bigcup_{\sigma \in \Sigma} Q^{\mathrm{rk}(\sigma)}) \times Q$ and consider every element of $\mathcal{V}$ to be a second-order variable. For every $t \in T_\Sigma$ and successful run $\kappa \in R_{\mathcal{M}}(t)$ we define the $\mathcal{V}$-assignment $\rho_{t,\kappa} \in \Phi_{\mathcal{V},t}$ as follows for every $(q_1 \cdots q_k, q) \in \mathcal{V}$:

$$\rho_{t,\kappa}((q_1 \cdots q_k, q)) = \{w \in \mathrm{pos}(t) \mid \kappa(w) = q, \mathrm{rk}(t(w)) = k, \forall\, i \in [k] : \kappa(wi) = q_i\} .$$

We will now define a formula $\varphi \in \mathrm{MSO}(\Sigma)$ such that $\mathrm{Free}(\varphi) = \mathcal{V}$ and for every $(t, \rho) \in T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$ we have $(t, \rho) \in \mathcal{L}(\varphi)$ iff there is a successful run $\kappa \in R_{\mathcal{M}}(t)$ with $\rho = \rho_{t,\kappa}$. We let $\varphi = \varphi_{\mathrm{part}} \wedge \varphi_{\mathrm{run}} \wedge \varphi_{\mathrm{suc}}$, where

$$\varphi_{\mathrm{part}} = \forall x. \Big(\bigvee\nolimits_{X \in \mathcal{V}} \big(x \in X \wedge \bigwedge\nolimits_{\substack{Y \in \mathcal{V} \\ Y \neq X}} \neg(x \in Y) \big) \Big) ,$$

$$\varphi_{\text{run}} = \forall x. \bigwedge_{(q_1\cdots q_k,q)\in\mathcal{V}} \Big(x \in (q_1\cdots q_k,q) \to (\bigvee_{\sigma\in\Sigma^{(k)}} \text{label}_\sigma(x) \wedge \bigwedge_{i\in[k]} \forall y.\text{edge}_i(x,y) \to (\bigvee_{\substack{(q'_1\cdots q'_{k'},q')\in\mathcal{V}\\ q'=q_i}} y \in (q'_1\cdots q'_{k'},q')))\Big) ,$$

$$\varphi_{\text{suc}} = \bigvee_{\substack{(q_1\cdots q_k,q)\in\mathcal{V}\\ q\in F}} \forall x.(\text{root}(x) \to x \in (q_1\cdots q_k,q)) ,$$

where $\text{root}(x) = \neg\exists y.(\text{edge}(y,x))$ and $\text{edge}(x,y) = \bigvee_{1\le i\le \text{maxrk}(\Sigma)} \text{edge}_i(x,y)$. It is easy to check that $\text{Free}(\varphi) = \mathcal{V}$ and for every $(t,\rho) \in T^{\text{v}}_{\Sigma_\mathcal{V}}$ we have $(t,\rho) \in \mathcal{L}(\varphi)$ iff there is a successful run $\kappa \in R_\mathcal{M}(t)$ with $\rho = \rho_{t,\kappa}$.

Now we choose a $\Sigma_\mathcal{V}$-family $\omega = (\omega_{(\sigma,V)} \mid (\sigma,V) \in \Sigma_\mathcal{V})$ of operations in Δ by letting for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $V \subseteq \mathcal{V}$:

$$\omega_{(\sigma,V)} = \begin{cases} \mu_k(q_1\cdots q_k,\sigma,q) & \text{if } V = \{(q_1\cdots q_k,q)\} , \\ \mathbf{0}^{(k)} & \text{otherwise.} \end{cases}$$

Let $t \in T_\Sigma$ and $\kappa \in R_\mathcal{M}(t)$. We show by induction over w that for every $w \in \text{pos}(t)$,

$$\text{h}_\omega((t,\rho_{t,\kappa})|_w) = \text{wt}(\kappa)(w) . \tag{9.3}$$

Let $w \in \text{pos}(t)$, $\sigma = t(w)$, and $k = \text{rk}(\sigma)$. Then we obtain the following equalities: $(t,\rho_{t,\kappa})(w) = \big(\sigma, \{X \in \mathcal{V} \mid w \in \rho_{t,\kappa}(X)\}\big) = \big(\sigma, \{(\kappa(w1)\cdots\kappa(wk),\kappa(w))\}\big)$. Hence, using the abbreviation $t = (\kappa(w1)\cdots\kappa(wk),\sigma,\kappa(w))$, we have

$$\begin{aligned} \text{h}_\omega((t,\rho_{t,\kappa})|_w) &= \mu_k(t)\big(\text{h}_\omega((t,\rho_{t,\kappa})|_{w1}),\ldots,\text{h}_\omega((t,\rho_{t,\kappa})|_{wk})\big) \\ &= \mu_k(t)\big(\text{wt}(\kappa)(w1),\ldots,\text{wt}(\kappa)(wk)\big) && \text{(by ind. hyp.)} \\ &= \text{wt}(\kappa)(w) . \end{aligned}$$

Let $\mathcal{V} = \{X_1,\ldots,X_n\}$. We put $e = \sum_{X_1}\cdots\sum_{X_n}(\varphi \rhd \text{H}(\omega))$. Then we obtain $\text{Free}(e) = \emptyset$ and for every $t \in T_\Sigma$:

$$\begin{aligned} [\![e]\!](t) &= \sum_{W_1,\ldots,W_n\subseteq \text{pos}(t)} [\![\varphi \rhd \text{H}(\omega)]\!]_\mathcal{V} t[X_1 \mapsto W_1,\ldots,X_n \mapsto W_n] \\ &= \sum_{\rho\in\Phi_{\mathcal{V},t}} [\![\varphi \rhd \text{H}(\omega)]\!]_\mathcal{V}(t,\rho) = \sum_{\substack{\rho\in\Phi_{\mathcal{V},t}\\ (t,\rho)\in\mathcal{L}_\mathcal{V}(\varphi)}} [\![\text{H}(\omega)]\!]_\mathcal{V}(t,\rho) \\ &= \sum_{\kappa\in R_\mathcal{M}(t)} \sum_{\substack{\rho\in\Phi_{\mathcal{V},t}\\ \rho=\rho_{t,\kappa}}} \text{h}_\omega((t,\rho)) = \sum_{\kappa\in R_\mathcal{M}(t)} \text{h}_\omega((t,\rho_{t,\kappa})) \\ &= \sum_{\kappa\in R_\mathcal{M}(t)} \text{wt}(\kappa)(\varepsilon) && \text{(by (9.3) for } w=\varepsilon) \\ &= [\![\mathcal{M}]\!](t) . \end{aligned}$$

Hence, $[\![\mathcal{M}]\!] = [\![e]\!]$. ■

9.4.2 From m-expressions to automata

In this section we will prove that the semantics of every m-expression is recognizable. As usual, the proof is by induction on the structure of the m-expression. We start with atomic m-expressions and first prove that the homomorphism induced by a family of operations is recognizable.

Lemma 9.28 (cf. [59, Lemma 4.3]). *Let ω be a Σ-family of operations in Δ. Then* $\mathrm{h}_\omega \in \mathrm{Rec}(\Sigma, \mathcal{A})$.

Proof. Let $\omega = (\omega_\sigma \mid \sigma \in \Sigma)$. We construct the wmta $\mathcal{M} = (\{*\}, \mu, \{*\})$ over Σ and $\mathcal{A}$ with $\mu_k(* \cdots *, \sigma, *) = \omega_\sigma$ for every $k \in \mathbb{N}$ and $\sigma \in \Sigma^{(k)}$. It is straightforward to prove that $[\![\mathcal{M}]\!] = \mathrm{h}_\omega$. ■

As the first item of Definition 9.21 shows, $[\![\mathrm{H}(\omega)]\!]_\mathcal{V}$ coincides with $\mathrm{h}_{\omega[U \rightsquigarrow \mathcal{V}]}$ on valid trees over $\Sigma_\mathcal{V}$, and on non-valid trees it yields $\mathbf{0}$. Thus, we can express $[\![\mathrm{H}(\omega)]\!]_\mathcal{V}$ as the composition of (1) filtering out from $T_{\Sigma_\mathcal{V}}$ the valid trees, followed by (2) the execution of $\mathrm{h}_{\omega[U \rightsquigarrow \mathcal{V}]}$. We model the filtering in the first step as the characteristic tree transformation χ_L^{tt} with $L = T_{\Sigma_\mathcal{V}}^{\mathrm{v}} \subseteq T_{\Sigma_\mathcal{V}}$.

In order to show that $[\![\mathrm{H}(\omega)]\!]_\mathcal{V}$ is recognizable it remains to prove that the class of recognizable tree series is closed under pre-composition with $\mathrm{FTA}(\Sigma)$.

Lemma 9.29 (cf. [59, Lemma 4.4]). *Suppose that $\mathcal{A}$ is strongly absorptive. Then we have that* $\mathrm{FTA}(\Sigma); \mathrm{Rec}(\Sigma, \mathcal{A}) \subseteq \mathrm{Rec}(\Sigma, \mathcal{A})$.

Proof. Let $\lambda \in \mathrm{FTA}(\Sigma); \mathrm{Rec}(\Sigma, \mathcal{A})$. Then there is an fta $\mathcal{M}_{\mathrm{fta}} = (Q_{\mathrm{fta}}, \delta_{\mathrm{fta}}, F_{\mathrm{fta}})$ over Σ and a wmta $\mathcal{M} = (Q, \mu, F)$ over Σ and Δ such that $\lambda = \chi^{\mathrm{tt}}_{\mathcal{L}(\mathcal{M}_{\mathrm{fta}})}; [\![\mathcal{M}]\!]$. We may assume that $\mathcal{M}_{\mathrm{fta}}$ is deterministic (cf. [66]).

We construct the wmta $\mathcal{M}' = (Q', \mu', F')$ over Σ and Δ by letting $Q' = Q_{\mathrm{fta}} \times Q$, $F' = F_{\mathrm{fta}} \times F$ and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, $q_1, \ldots, q_k, p \in Q_{\mathrm{fta}}$, and $q'_1, \ldots, q'_k, p' \in Q$:

$$\mu'_k((q_1, q'_1) \cdots (q_k, q'_k), \sigma, (p, p')) = \begin{cases} \mu_k(q'_1 \cdots q'_k, \sigma, p') & \text{if } (q_1 \cdots q_k, \sigma, p) \in (\delta_{\mathrm{fta}})_k \ , \\ \mathbf{0}^{(k)} & \text{otherwise.} \end{cases}$$

The proof is completed by showing that $[\![\mathcal{M}']\!] = \lambda$, i.e., that for every $t \in T_\Sigma$, $[\![\mathcal{M}']\!](t) = [\![\mathcal{M}]\!](t)$ if $t \in \mathcal{L}(\mathcal{M}_{\mathrm{fta}})$, and $[\![\mathcal{M}']\!](t) = \mathbf{0}$ otherwise. Let $t \in T_\Sigma$. We denote the set of mappings $\kappa : \mathrm{pos}(t) \to Q_{\mathrm{fta}}$ with $\kappa(\varepsilon) \in F_{\mathrm{fta}}$ by $K_{\mathcal{M}_{\mathrm{fta}}}(t)$. Note that the set $K_{\mathcal{M}_{\mathrm{fta}}}(t)$ includes the set of successful runs of $\mathcal{M}_{\mathrm{fta}}$ on t. Obviously, there is a bijection π between $K_{\mathcal{M}_{\mathrm{fta}}}(t) \times R_\mathcal{M}(t)$ and $R_{\mathcal{M}'}(t)$ given by $\pi(\kappa_{\mathrm{fta}}, \kappa)(w) = (\kappa_{\mathrm{fta}}(w), \kappa(w))$ for every $(\kappa_{\mathrm{fta}}, \kappa) \in K_{\mathcal{M}_{\mathrm{fta}}}(t) \times R_\mathcal{M}(t)$ and $w \in \mathrm{pos}(t)$. Let $\kappa' \in R_{\mathcal{M}'}(t)$ and $(\kappa_{\mathrm{fta}}, \kappa) = \pi^{-1}(\kappa')$. It is easy to check that $\mathrm{wt}_{\mathcal{M}',t}(\kappa')(\varepsilon) = \mathrm{wt}_{\mathcal{M},t}(\kappa)(\varepsilon)$ if κ_{fta} is a successful run of $\mathcal{M}_{\mathrm{fta}}$ on t, and otherwise $\mathrm{wt}_{\mathcal{M}',t}(\kappa')(\varepsilon) = \mathbf{0}$ due to Lemma 9.5 (which is applicable because $\mathcal{A}$ is absorptive).

First assume that $t \notin \mathcal{L}(\mathcal{M}_{\mathrm{fta}})$. Then for every $\kappa' \in R_{\mathcal{M}'}(t)$ the mapping in the first component of $\pi^{-1}(\kappa')$ is not a successful run and thus, $\mathrm{wt}_{\mathcal{M}',t}(\kappa')(\varepsilon) = \mathbf{0}$. Hence, $[\![\mathcal{M}']\!](t) = \mathbf{0}$.

Now assume that $t \in \mathcal{L}(\mathcal{M}_{\mathrm{fta}})$. Since $\mathcal{M}_{\mathrm{fta}}$ is deterministic, there is a unique successful run κ_{suc} of $\mathcal{M}_{\mathrm{fta}}$ on t. We obtain

$$\begin{aligned} [\![\mathcal{M}']\!](t) &= \sum\nolimits_{\kappa' \in R_{\mathcal{M}'}(t)} \mathrm{wt}_{\mathcal{M}',t}(\kappa')(\varepsilon) \\ &= \sum\nolimits_{\kappa_{\mathrm{fta}} \in K_{\mathcal{M}_{\mathrm{fta}}}(t)} \sum\nolimits_{\kappa \in R_\mathcal{M}(t)} \mathrm{wt}_{\mathcal{M}',t}\big(\pi(\kappa_{\mathrm{fta}}, \kappa)\big)(\varepsilon) \\ &= \sum\nolimits_{\kappa \in R_\mathcal{M}(t)} \mathrm{wt}_{\mathcal{M}',t}\big(\pi(\kappa_{\mathrm{suc}}, \kappa)\big)(\varepsilon) = \sum\nolimits_{\kappa \in R_\mathcal{M}(t)} \mathrm{wt}_{\mathcal{M},t}(\kappa)(\varepsilon) = [\![\mathcal{M}]\!](t) \ . \end{aligned}$$

■

Lemma 9.30 (cf. [59, Lemma 4.5]). *Let $\mathcal{A}$ be absorptive and ω be a $\Sigma_{\mathcal{U}}$-family of operations in Δ for some finite set $\mathcal{U}$ of variables. Moreover, let $\mathcal{V}$ be a finite set of variables with $\mathcal{U} \subseteq \mathcal{V}$. Then we have $[\![\mathrm{H}(\omega)]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$.*

PROOF. Let $T = T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}} \subseteq T_{\Sigma_{\mathcal{V}}}$. Clearly, T is recognizable. Then $[\![\mathrm{H}(\omega)]\!]_{\mathcal{V}} = \chi^{\mathrm{tt}}_T ; \mathrm{h}_{\omega[\mathcal{U} \rightsquigarrow \mathcal{V}]}$ which is a tree series in $\mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$ by Lemmas 9.28 and 9.29. ∎

Next we prove that $\mathrm{Rec}(\Sigma, \mathcal{A})$ is closed under summation. It turns out that the m-monoid $\mathcal{A}$ has to be absorptive (see Example 9.32).

Lemma 9.31 (cf. [58, Lemma 7.5] and [59, Lemma 4.6]). *If $\mathcal{A}$ is strongly absorptive, then for every $\lambda, \lambda' \in \mathrm{Rec}(\Sigma, \mathcal{A})$ also $\lambda + \lambda' \in \mathrm{Rec}(\Sigma, \mathcal{A})$.*

PROOF. Let $\mathcal{M} = (Q, \mu, F)$ and $\mathcal{M}' = (Q', \mu', F')$ be wmta over Σ and Δ such that $[\![\mathcal{M}]\!] = \lambda$ and $[\![\mathcal{M}']\!] = \lambda'$. We assume that Q and Q' are disjoint. We construct the wmta $\mathcal{M}^+ = (Q^+, \mu^+, F^+)$ over Σ and Δ by letting $Q^+ = Q \cup Q'$, $F^+ = F \cup F'$ and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $q_1, \ldots, q_k, p \in Q^+$ we define

$$\mu^+_k(q_1 \cdots q_k, \sigma, p) = \begin{cases} \mu_k(q_1 \cdots q_k, \sigma, p)\ , & \text{if } \{q_1, \ldots, q_k, p\} \subseteq Q\ , \\ \mu'_k(q_1 \cdots q_k, \sigma, p)\ , & \text{if } \{q_1, \ldots, q_k, p\} \subseteq Q'\ , \\ \mathbf{0}^{(k)}\ , & \text{otherwise.} \end{cases}$$

We show that $[\![\mathcal{M}^+]\!] = \lambda + \lambda'$. Let $t \in T_\Sigma$ and $\kappa \in R_{\mathcal{M}^+}(t)$. If $\mathrm{ran}(\kappa) \subseteq Q$ (i.e., $\kappa \in R_{\mathcal{M}}(t)$), then $\mathrm{wt}_{\mathcal{M}^+,t}(\kappa)(\varepsilon) = \mathrm{wt}_{\mathcal{M},t}(\kappa)(\varepsilon)$. Likewise, if $\kappa \in R_{\mathcal{M}'}(t)$, then $\mathrm{wt}_{\mathcal{M}^+,t}(\kappa)(\varepsilon) = \mathrm{wt}_{\mathcal{M}',t}(\kappa)(\varepsilon)$. If neither $\kappa \in R_{\mathcal{M}}(t)$ nor $\kappa \in R_{\mathcal{M}'}(t)$, then $\mathrm{wt}_{\mathcal{M}^+,t}(\kappa)(\varepsilon) = \mathbf{0}$ by Lemma 9.5. Thus,

$$\begin{aligned} [\![\mathcal{M}^+]\!](t) &= \sum\nolimits_{\kappa \in R_{\mathcal{M}^+}(t)} \mathrm{wt}_{\mathcal{M}^+,t}(\kappa)(\varepsilon) \\ &= \sum\nolimits_{\kappa \in R_{\mathcal{M}}(t)} \mathrm{wt}_{\mathcal{M},t}(\kappa)(\varepsilon) + \sum\nolimits_{\kappa \in R_{\mathcal{M}'}(t)} \mathrm{wt}_{\mathcal{M}',t}(\kappa)(\varepsilon) \\ &= [\![\mathcal{M}]\!](t) + [\![\mathcal{M}']\!](t) = (\lambda + \lambda')(t)\ . \end{aligned}$$ ∎

Note that there is an m-monoid $\mathcal{A}$ that is not absorptive such that $\mathrm{Rec}(\Sigma, \mathcal{A})$ is not closed under summation; this is witnessed by the following example.

Example 9.32 (cf. [59, Example 4.7]). Let $\Sigma = \{\gamma^{(1)}, \alpha^{(0)}\}$ and $\Delta = \{\mathrm{one}^{(1)}\} \cup \{\mathrm{zero}^{(k)}_k \mid k \in \mathbb{N}\}$. Moreover, let $\mathcal{A} = (\mathbb{N}, +, 0, \theta)$ be the m-monoid over Δ such that

- $\theta(\mathrm{one})$ is the unary mapping with $\theta(\mathrm{one})(n) = 1$ for every $n \in \mathbb{N}$,
- for every $k \in \mathbb{N}$, $\theta(\mathrm{zero}_k)$ is the k-ary operation with $\mathrm{ran}(\theta(\mathrm{zero}_k)) = \{\mathbf{0}\}$.

Obviously, $\mathcal{A}$ is not absorptive. Consider the wmta $\mathcal{M} = (Q, \mu, Q)$ and $\mathcal{M}' = (Q', \mu', Q')$ over Σ and Δ, where Q has one state, say $*$, Q' has two states, $\mu_1(*, \gamma, *) = \mathrm{one}$, and $\mu'_1(p', \gamma, q') = \mathrm{one}$ for every $p', q' \in Q'$ (the other values in μ and μ' are not relevant for our considerations). It is easy to see that $[\![\mathcal{M}]\!](\gamma^n(\alpha)) = 1$ and $[\![\mathcal{M}']\!](\gamma^n(\alpha)) = 2^{n+1}$ for every $n \in \mathbb{N}_+$.

Now assume that there is a wmta $\mathcal{M}^+ = (Q^+, \mu^+, F^+)$ over Σ and Δ with $[\![\mathcal{M}^+]\!] = [\![\mathcal{M}]\!] + [\![\mathcal{M}']\!]$. Let $a = \big|\{(p, q) \mid p, q \in Q^+, \mu^+_1(p, \gamma, q) = \mathrm{one}, q \in F^+\}\big|$. By the definition of $\mathcal{A}$ we obtain for every $n \in \mathbb{N}_+$:

$$[\![\mathcal{M}^+]\!](\gamma^n(\alpha)) = \sum\nolimits_{\kappa \in R_{\mathcal{M}^+}(\gamma^n(\alpha))} \mu^+_1(\kappa(1), \gamma, \kappa(\varepsilon))\big(\mathrm{wt}(\kappa)(1)\big)$$

$$= \sum_{\substack{\kappa \in R_{\mathcal{M}^+}(\gamma^n(\alpha)) \\ \mu_1^+(\kappa(1),\gamma,\kappa(\varepsilon))=1^{(1)}}} 1 = \left|\{\kappa \mid \kappa \in R_{\mathcal{M}^+}(\gamma^n(\alpha)), \mu_1^+(\kappa(1),\gamma,\kappa(\varepsilon)) = 1^{(1)}\}\right|$$
$$= \left|\{\kappa \mid \kappa : \mathrm{pos}(\gamma^n(\alpha)) \to Q^+, \kappa(\varepsilon) \in F^+, \mu_1^+(\kappa(1),\gamma,\kappa(\varepsilon)) = 1^{(1)}\}\right|$$
$$= |Q^+|^{n-1} \cdot a \ .$$

However, it is easy to see that there are no integers $|Q^+|$ and a such that $1 + 2^{n+1} = |Q^+|^{n-1} \cdot a$ for every $n \in \mathbb{N}_+$. □

Now we prove that $[\![\sum_x e]\!]_{\mathcal{V}}$ and $[\![\sum_X e]\!]_{\mathcal{V}}$ are recognizable tree series provided that $[\![e]\!]_{\mathcal{V}\cup\{x\}}$ and $[\![e]\!]_{\mathcal{V}\cup\{X\}}$, respectively, are so. As known from MSO-logic for the existential quantification, the operators $\sum_x$ and $\sum_X$ induce a relabeling on the given tree.

Lemma 9.33 (cf. [59, Lemma 4.9]). *Suppose that $\mathcal{A}$ is strongly absorptive and let $e \in \mathrm{MExp}(\Sigma, \Delta)$. Moreover, let $\mathcal{V}$ be a finite set of variables.*

1. *If* $\mathrm{Free}(\sum_x e) \subseteq \mathcal{V}$ *and* $[\![e]\!]_{\mathcal{V}\cup\{x\}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}\cup\{x\}}, \mathcal{A})$*, then* $[\![\sum_x e]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$.
2. *If* $\mathrm{Free}(\sum_X e) \subseteq \mathcal{V}$ *and* $[\![e]\!]_{\mathcal{V}\cup\{X\}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}\cup\{X\}}, \mathcal{A})$*, then* $[\![\sum_X e]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$.

Proof. Let $T = T^{\mathrm{v}}_{\Sigma_{\mathcal{V}}}$. Clearly T is recognizable.

First we prove Statement 1. Let $e' = \sum_x e$ and assume that $\mathrm{Free}(\sum_x e) \subseteq \mathcal{V}$ and $[\![e]\!]_{\mathcal{V}\cup\{x\}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}\cup\{x\}}, \mathcal{A})$. Let ρ be the relabeling from $\Sigma_{\mathcal{V}}$ to $\Sigma_{\mathcal{V}\cup\{x\}}$ which is defined by $\rho((\sigma, V)) = \{(\sigma, V \setminus \{x\}), (\sigma, V \cup \{x\})\}$ for every $\sigma \in \Sigma$ and $V \subseteq \mathcal{V}$. We define the tree language $T' = T^{\mathrm{v}}_{\Sigma_{\mathcal{V}\cup\{x\}}}$. Then

$$\chi^{\mathrm{tt}}_T; ([\![\rho]\!]; (\chi^{\mathrm{tt}}_{T'}; [\![e]\!]_{\mathcal{V}\cup\{x\}})) \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$$

by the hypothesis and due to Lemmas 9.29 and 9.16. Therefore it suffices to show that $[\![e']\!]_{\mathcal{V}}(t) = \big(\chi^{\mathrm{tt}}_T; ([\![\rho]\!]; (\chi^{\mathrm{tt}}_{T'}; [\![e]\!]_{\mathcal{V}\cup\{x\}}))\big)(t)$ for every $t \in T_{\Sigma_{\mathcal{V}}}$. This equation does obviously hold if $t \notin T$. Now assume that $t \in T$. Then

$$\big(\chi^{\mathrm{tt}}_T; ([\![\rho]\!]; (\chi^{\mathrm{tt}}_{T'}; [\![e]\!]_{\mathcal{V}\cup\{x\}}))\big)(t) = \big([\![\rho]\!]; (\chi^{\mathrm{tt}}_{T'}; [\![e]\!]_{\mathcal{V}\cup\{x\}})\big)(t)$$
$$= \sum_{s \in [\![\rho]\!](t)} \sum_{s' \in \chi^{\mathrm{tt}}_{T'}(s)} [\![e]\!]_{\mathcal{V}\cup\{x\}}(s') = \sum_{s \in [\![\rho]\!](t) \cap T'} [\![e]\!]_{\mathcal{V}\cup\{x\}}(s)$$
$$= \sum_{s \in \{t[x \mapsto w] \mid w \in \mathrm{pos}(t)\}} ([\![e]\!]_{\mathcal{V}\cup\{x\}}(s) \qquad (\star)$$
$$= \sum_{w \in \mathrm{pos}(t)} [\![e]\!]_{\mathcal{V}\cup\{x\}}(t[x \mapsto w]) \qquad \text{(since } t[x \mapsto w] \neq t[x \mapsto w'] \text{ for } w \neq w'\text{)}$$
$$= [\![e']\!]_{\mathcal{V}}(t) \ .$$

At $(\star)$ we used the fact that $[\![\rho]\!](t) \cap T' = \{t[x \mapsto w] \mid w \in \mathrm{pos}(t)\}$, which is easy to check.

Statement 2 can be shown by a similar argument. ■

Finally we prove that $[\![\varphi \rhd e]\!]_{\mathcal{V}}$ is recognizable if $[\![e]\!]_{\mathcal{V}}$ is recognizable.

Lemma 9.34 (cf. [59, Lemma 4.10]). *Let $\mathcal{A}$ be strongly absorptive, $\varphi \in \mathrm{MSO}(\Sigma)$, and $e \in \mathrm{MExp}(\Sigma, \Delta)$. Moreover, let $\mathcal{V}$ be a finite set of variables containing* $\mathrm{Free}(\varphi \rhd e)$*. Then $[\![e]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$ implies $[\![\varphi \rhd e]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$.*

Proof. By Theorem 6.4(2) the language $\mathcal{L}_{\mathcal{V}}(\varphi)$ is recognizable. Thus, $[\![\varphi \rhd e]\!]_{\mathcal{V}} = \chi^{\mathrm{tt}}_{\mathcal{L}_{\mathcal{V}}(\varphi)}; [\![e]\!]_{\mathcal{V}}$ is a tree series in $\mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$ by Lemma 9.29. ■

Now we can prove the fact that m-definable tree series are recognizable by wmta.

Lemma 9.35 (cf. [59, Lemma 4.11]). *Let $\mathcal{A}$ be absorptive,* $e \in \mathrm{MExp}(\Sigma, \Delta)$*, and let* $\mathcal{V}$ *be a finite set of variables containing* $\mathrm{Free}(e)$*. Then* $[\![e]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, \mathcal{A})$*.*

PROOF. This follows by induction on the structure of e as follows.

Case $e = H(\omega)$. Let ω be a $\Sigma_{\mathcal{U}}$-family of operations in Δ for some finite set $\mathcal{U}$ of variables. Then $\mathcal{U} = \mathrm{Free}(e) \subseteq \mathcal{V}$ and the statement follows from Lemma 9.30.

Case $e = e_1 + e_2$. Since $\mathrm{Free}(e) = \mathrm{Free}(e_1) \cup \mathrm{Free}(e_2)$, we have $\mathrm{Free}(e_1) \subseteq \mathcal{V}$ and $\mathrm{Free}(e_2) \subseteq \mathcal{V}$. By the induction hypothesis, $[\![e_1]\!]_{\mathcal{V}}, [\![e_2]\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, A)$. Then the statement follows from Lemma 9.31.

Case $e = \sum_x e'$. Clearly, the fact that $\mathrm{Free}(e) = \mathrm{Free}(e') \setminus \{x\}$ implies $\mathrm{Free}(e') \subseteq \mathcal{V} \cup \{x\}$. The induction hypothesis yields $[\![e']\!]_{\mathcal{V}\cup\{x\}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}\cup\{x\}}, A)$. Then the statement follows from Lemma 9.33.

Case $e = \sum_X e'$. This proof of this case is similar to the proof of $e = \sum_x e'$.

Case $e = \varphi \rhd e'$. We have $\mathrm{Free}(e') \subseteq \mathcal{V}$. The induction hypothesis yields $[\![e']\!]_{\mathcal{V}} \in \mathrm{Rec}(\Sigma_{\mathcal{V}}, A)$. Then the statement follows from Lemma 9.34. ■

9.5 Further results

In this chapter we presented only the most essential results from [123, 59]. Let us briefly mention important consequences of the two theorems that we presented in this chapter.

Implications of Theorem 9.17 are studied in [123].

- One consequence of Theorem 9.17 is the following characterization of the class p–$\mathrm{BOT}(\mathcal{S})$ of tree series transformations computed by polynomial bottom-up tree series transducers (for short: polynomial bu-tst) over some semiring $\mathcal{S}$: $p\text{-}\mathrm{BOT}(\mathcal{S}) = \mathrm{REL}; \mathrm{FTA}; \mathrm{HOM}(\mathcal{S})$ (see [123, Theorem 2]), where $\mathrm{HOM}(\mathcal{S})$ is the class of tree series transformations computed by homomorphism bu-tst, and a tree series transformation over $\mathcal{S}$ is a mapping $\varphi : T_\Sigma \to \mathcal{S}\langle\langle T_\Delta \rangle\rangle$. Polynomial bu-tst were investigated in, e.g., [51, 104, 63].

- Another consequence of Theorem 9.17 is a characterization of the class $\mathrm{Rec}(\Sigma, \mathcal{S})$ of tree series which are recognizable by weighted tree automata over some semiring $\mathcal{S}$: $\mathrm{Rec}(\Sigma, \mathcal{S}) = \mathrm{PROJ}(\Sigma, \mathcal{S})(\mathscr{L}_{\mathrm{LOC}})$ (see [123, Theorem 3]), where $\mathrm{PROJ}(\Sigma, \mathcal{S})$ is the class of tree series transformations which are computed by projection bu-tst. Recognizable tree series over $\mathcal{S}$ were investigated in, e.g., [4, 52, 43].

A corollary of Theorem 9.26 is the Büchi-like result $\mathrm{Rec}(\Sigma, \mathcal{S}) = \mathrm{srMSO}(\Sigma, \mathcal{S})$ (see [63, Theorem 3.49]), where $\mathrm{srMSO}(\Sigma, \mathcal{S})$ denotes the class of tree series definable by syntactically restricted weighted MSO-formulas over Σ and $\mathcal{S}$ (cf. [59, Corollary 5.15]).

Additional proofs

A.1 Proper classes in formal language theory

The class $\mathscr{L}_{\mathrm{rec}}$ of recognizable string languages is a proper class. Let us briefly explain why $\mathscr{L}_{\mathrm{rec}}$ is not a set.

Let S be a set. Then $\{S\}$ is a set, too. In the setting of formal language theory, $\{S\}$ is called an alphabet because it is a finite nonempty set. Then the set $\{S\}^*$ of strings over the alphabet $\{S\}$ (i.e., $\{S\}^* = \{\varepsilon, S, SS, SSS, \ldots\}$) is obviously a recognizable language over the alphabet $\{S\}$; therefore, $\{S\}^* \in \mathscr{L}_{\mathrm{rec}}$.

If we assume that $\mathscr{L}_{\mathrm{rec}}$ is a set, then $\{\mathscr{L}_{\mathrm{rec}}\}^* \in \mathscr{L}_{\mathrm{rec}}$. This contradicts the Axiom of Regularity [124, 112] of ZF (also known as the Axiom of Foundation), that asserts that no set may contain itself directly or indirectly.

However, the assumption that $\mathscr{L}_{\mathrm{rec}}$ is a set entails a more profound contradiction even when using an axiomatization of set theory without the Axiom of Regularity. Let us explain why such a problem arises. For every set A we define the property $\varphi(A)$ as follows: $\varphi(A)$ iff $\{A\}^* \notin A$. Assume that $\mathscr{L}_{\mathrm{rec}}$ is a set. Then the Axiom of Separation yields that $L = \{S \in \mathscr{L}_{\mathrm{rec}} \mid \exists\, A : S = \{A\}^* \wedge \varphi(A)\}$ is a set, too. Therefore, $\{L\}^* \in \mathscr{L}_{\mathrm{rec}}$ as shown above. Let us determine whether the language $\{L\}^*$ is contained in the set L.

$$
\begin{aligned}
& \{L\}^* \in L \\
& \text{iff } \{L\}^* \in \mathscr{L}_{\mathrm{rec}} \wedge \exists\, A : \big(\{L\}^* = \{A\}^* \wedge \varphi(A)\big) \\
& \text{iff } \exists\, A : \big(\{L\}^* = \{A\}^* \wedge \varphi(A)\big) && \text{(because } \{L\}^* \in \mathscr{L}_{\mathrm{rec}}) \\
& \text{iff } \varphi(L) && \text{(because } \{L\}^* = \{A\}^* \text{ implies } L = A) \\
& \text{iff } \{L\}^* \notin L\ . && \text{(by the definition of } \varphi(L))
\end{aligned}
$$

Since "$\{L\}^* \in L$ iff $\{L\}^* \notin L$" is a contradiction, the assumption that $\mathscr{L}_{\mathrm{rec}}$ is a set was wrong. Thus, $\mathscr{L}_{\mathrm{rec}}$ is a proper class. The class L is similar to the Russel class [124, Theorem 4.14] (i.e., the class of all sets that do not contain themselves), a class that demonstrates that the class of all sets is not a set.

A.2 Counterexample for (R1) and (R2)

Example A.1. Consider the semigroups $\mathcal{S}_1 = (S_1, \uplus)$ and $\mathcal{S}_2 = (S_2, \sqcup)$, where

- $S_1 = \big(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}\big) \cup \{\top\}$, $\top$ is absorbing wrt $\uplus$ and for every $N, N' \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$, $N \uplus N' = N \cup N'$ if N and N' are disjoint and $N \uplus N' = \top$ otherwise,

- $S_2 = \{1, 1', 2, 2', \top\}$, $\sqcup$ is commutative and idempotent, the element $\top$ is absorbing wrt $\sqcup$, $1 \sqcup 1' = 1'$, $2 \sqcup 2' = 2'$ and $a \sqcup b = \top$ for every $a \in \{1, 1'\}$ and $b \in \{2, 2'\}$, i.e., $\mathcal{S}_2$

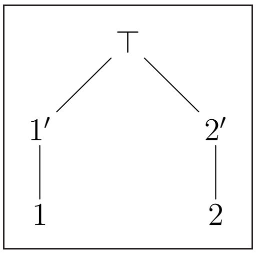


Figure A.1: The join-semilattice S_2.

is the join-semilattice [71] given in Figure A.1 (it is even a complete join-semilattice[1] because it is finite).

By Lemma 3.2, $\mathcal{S}_1 \times \mathcal{S}_2$ is a semigroup. Let $\equiv$ be the equivalence relation on $S_1 \times S_2$ such that

$$\begin{aligned}(S_1 \times S_2)/{\equiv} = \big\{&\{(\mathbb{N},1),(\mathbb{N},2')\},\{(\mathbb{N},2),(\mathbb{N},1')\},\\ &\{(x,y) \in S_1 \times S_2 \mid x = \top \vee y = \top\}\big\}\\ &\cup \big\{\{(N,y)\} \mid N \subset \mathbb{N}, N \neq \emptyset, y \in \{1,1',2,2'\}\big\}\ ,\end{aligned}$$

i.e., $\equiv$ identifies $(\mathbb{N},1)$ with $(\mathbb{N},2')$, $(\mathbb{N},2)$ with $(\mathbb{N},1')$, and all elements in $S_1 \times S_2$ containing $\top$. It is easy to check that $\equiv$ is a semigroup congruence on $\mathcal{S}_1 \times \mathcal{S}_2$; hence, also $(\mathcal{S}_1 \times \mathcal{S}_2)/{\equiv}$ is a semigroup by Lemma 3.2.

Let $\Delta = \emptyset$ and let $\mathcal{A} = (A,+,\mathbf{0},\theta)$ be the m-monoid over Δ such that $(A,+,\mathbf{0})$ is the monoid that is obtained from the semigroup $(\mathcal{S}_1 \times \mathcal{S}_2)/{\equiv}$ by adding a neutral element $\mathbf{0}$ (see Remark 3.4; in particular, $A = \big((S_1 \times S_2)/{\equiv}\big) \cup \{\mathbf{0}\}$).

Now we define an ω-infinitary sum operation $\sum$ for $\mathcal{A}$ as follows. Let I be a countable set and $(a_i \mid i \in I)$ be a family over A. Let $I' = \{i \in I \mid a_i \neq \mathbf{0}\}$ and for every $i \in I'$ let $(b_i, c_i) \in S_1 \times S_2$ such that $[(b_i,c_i)]_{\equiv} = a_i$. We let

$$\sum_{i \in I} a_i = \begin{cases} \mathbf{0}\ , & \text{if } I' = \emptyset\ ,\\ [(\top,\top)]_{\equiv}\ , & \text{if } \exists\, i \in I' : b_i = \top \vee c_i = \top \\ & \text{or } \exists\, i,j \in I' : (b_i, b_j \neq \top) \wedge (b_i \cap b_j \neq \emptyset)\ ,\\ [(\bigcup_{i \in I'} b_i, \bigsqcup_{i \in I'} c_i)]_{\equiv}\ , & \text{otherwise,}\end{cases}$$

where the operation $\bigsqcup$ is the supremum operation in the complete join-semilattice $\mathcal{S}_2$. We omit the rather technical proof that $\sum$ is well-defined and an ω-infinitary sum operation for $\mathcal{A}$.

It is easy to see that $(\mathcal{A},\sum)$ has property (R1)[2]. It is also easy to check that $(\mathcal{A},\sum)$ has property (R2).

Now we show that the ω-complete m-monoid $(\mathcal{A},\sum)$ does not admit a related ω-continuous m-monoid. Suppose, contrary to our claim, that there is an ω-continuous

[1] A complete join-semilattice is a poset [71] such that every *nonempty* set of elements of the lattice has a supremum.

[2] In fact, this is the reason why we did not (i) add the empty set to the carrier set of $\mathcal{S}_1$ and did not (ii) use union instead of disjoint union for the binary operation of $\mathcal{S}_1$; otherwise, for every $j \in \{1,2\}$, we would have $[(\mathbb{N},j)]_{\equiv} + [(\emptyset,j')]_{\equiv} = [(\mathbb{N},j')]_{\equiv}$ in case (i) and $[(\mathbb{N},j)]_{\equiv} + [(\mathbb{N},j')]_{\equiv} = [(\mathbb{N},j')]_{\equiv}$ in case (ii).

m-monoid $(\mathcal{A}, \leq)$ such that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related. Let $j \in \{1, 2\}$. For every $n \in \mathbb{N}$ let $a_n^j = [(\{n\}, j)]_\equiv$. We obtain $\sum_{n \in N} a_n^j = [(N, j)]_\equiv$ for every nonempty $N \subseteq \mathbb{N}$. Thus, for every nonempty $N \in \mathcal{P}_{\text{fin}}(\mathbb{N})$, $\big(\sum_{n \in N} a_n^j\big) + [(\mathbb{N} \setminus N, j')] = [(\mathbb{N}, j')]_\equiv$; hence, $\big(\sum_{n \in N} a_n^j\big) \leq [(\mathbb{N}, j')]_\equiv$. Since also $\sum_{n \in \emptyset} a_n^j = \mathbf{0} \leq [(\mathbb{N}, j')]_\equiv$, we obtain

$$[(\mathbb{N}, j)]_\equiv = \sum\nolimits_{n \in \mathbb{N}} a_n^j = \vee \big\{ \sum\nolimits_{n \in N} a_n^j \mid N \in \mathcal{P}_{\text{fin}}(\mathbb{N}) \big\} \leq [(\mathbb{N}, j')]_\equiv \ ,$$

because $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sum)$ are related. Hence, $[(\mathbb{N}, 1)]_\equiv \leq [(\mathbb{N}, 1')]_\equiv = [(\mathbb{N}, 2)]_\equiv$ and, likewise, $[(\mathbb{N}, 2)]_\equiv \leq [(\mathbb{N}, 1)]_\equiv$. However, $[(\mathbb{N}, 1)]_\equiv \neq [(\mathbb{N}, 2)]_\equiv$, which means that $\leq$ is not a partial order, a contradiction. Thus, $\big(\mathcal{A}, \sum\big) \notin \mathfrak{A}_{\text{s}\rightsquigarrow\text{c}}$. □

A.3 Proof of Lemma 4.31

Proof. We let $\Sigma = \{\alpha^{(0)}\}$, $\Delta = \{\text{null}^{(0)}, \text{suc}^{(1)}\}$, and $\mathcal{A} = (A, \circ, (\emptyset, 1), \theta)$ such that

- $A = \big((\mathcal{P}(\mathbb{N}) \times \{1, 2\}) \setminus \{(\emptyset, 2)\}\big) \cup \{\top\}$,
- for every $a_1, a_2 \in A$ we let

$$a_1 \circ a_2 = \begin{cases} \big(\text{pr}_1(a_1) \cup \text{pr}_1(a_2), \max(\text{pr}_2(a_1), \text{pr}_2(a_2))\big) & \text{if } a_1, a_2 \in A \setminus \{\top\} \text{ and} \\ & \ \text{pr}_1(a_1) \cap \text{pr}_1(a_2) = \emptyset, \\ \top & \text{otherwise,} \end{cases}$$

- $\theta(\text{null})() = \big(\{0\}, 1\big)$
- for every $a \in A$, $\theta(\text{suc})(a) = \big(\{n + 1 \mid n \in \text{pr}_1(a)\}, \text{pr}_2(a)\big)$ if $a \in A \setminus \{\top\}$, and $\theta(\text{suc})(a) = \top$ otherwise.

It is easy to check that $\mathcal{A}$ is a distributive m-monoid over Δ. Next we define the relations $\leq$ and $\sqsubseteq$. To this end we define three auxiliary relations $\prec_0$, $\prec_1$, and $\prec_2$ on $A \setminus \{\top\}$ as follows:

$$\begin{aligned} \prec_0 &= \big\{(a, b) \mid a, b \in A \setminus \{\top\}, \text{pr}_1(a) \subset \text{pr}_1(b), \text{pr}_2(b) \geq \text{pr}_2(a)\big\} \ , \\ \prec_1 &= \big\{\big((N, 1), (N, 2)\big) \mid N \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}\big\} \ , \\ \prec_2 &= \big\{\big((N, 2), (N', 1)\big) \mid N, N' \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, N \subseteq N'\big\} \ . \end{aligned}$$

Now we let

$$\begin{aligned} \leq &= \text{id}_A \cup (A \times \{\top\}) \cup \prec_0 \cup \prec_1 \ , \\ \sqsubseteq &= \text{id}_A \cup (A \times \{\top\}) \cup \prec_0 \cup \prec_2 \ . \end{aligned}$$

We prove that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sqsubseteq)$ are ω-continuous m-monoids.

First we show that $\leq$ and $\sqsubseteq$ are partial orders on A. Clearly, $\leq$ and $\sqsubseteq$ are reflexive, because $\text{id}_A \subseteq \leq$ and $\text{id}_A \subseteq \sqsubseteq$. Next we prove transitivity. Let $a, b, c \in A$ such that $a \leq b \leq c$. The only interesting case is that $(a, b) \in \prec_0 \cup \prec_1$ and $(b, c) \in \prec_0 \cup \prec_1$. Clearly, $a \prec_0 b \prec_0 c$ implies $a \prec_0 c$. It is easy to check that $a \prec_1 b \prec_0 c$ or $a \prec_0 b \prec_1 c$ implies $a \prec_0 c$. The case $a \prec_1 b \prec_1 c$ is not possible. The proof that $\sqsubseteq$ is transitive is slightly more complicated but similar to the proof that $\leq$ is transitive. Next we prove antisymmetry. Let $a, b \in A$ such that $a \leq b \leq a$. The only interesting case is that $(a, b) \in \prec_0 \cup \prec_1$ and

$(b, a) \in \prec_0 \cup \prec_1$. Then $\mathrm{pr}_1(a) \subseteq \mathrm{pr}_1(b) \subseteq \mathrm{pr}_1(a)$ and, since $a \prec_1 b \prec_1 a$ is not possible, we obtain that one of the inclusions is proper, i.e., $\mathrm{pr}_1(a) \subset \mathrm{pr}_1(a)$, a contradiction. The proof that $\sqsubseteq$ is antisymmetric is similar to the proof that $\leq$ is antisymmetric.

Next we prove that $(A, \leq)$ and $(A, \sqsubseteq)$ are ω-cpos. Let us denote the supremum wrt $\leq$ by $\vee$ and the supremum wrt $\sqsubseteq$ by $\sqcup$. Observe that $(\emptyset, 1)$ is the least element of A wrt $\leq$ and the least element of A wrt $\sqsubseteq$. Let $b : \mathbb{N} \to A$ be an ω-chain wrt $\leq$. We show that b has a supremum wrt $\leq$. This is trivial if b is ultimately constant. If b is not ultimately constant, then $\mathrm{ran}(b) \subseteq A \setminus \{\top\}$ and for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $\mathrm{pr}_1(b(n)) \subset \mathrm{pr}_1(b(n+m))$; it is easy to check that

$$\vee\{b(n) \mid n \in \mathbb{N}\} = \Big(\bigcup_{n\in\mathbb{N}} \mathrm{pr}_1(b(n)), \max\{\mathrm{pr}_2(b(n)) \mid n \in \mathbb{N}\}\Big) . \tag{A.1}$$

Let $b' : \mathbb{N} \to A$ be an ω-chain wrt $\sqsubseteq$. We show that b' has a supremum wrt $\sqsubseteq$. This is trivial if b' is ultimately constant. If b' is not ultimately constant, then it is easy to check that

$$\sqcup\{b'(n) \mid n \in \mathbb{N}\} = \Big(\bigcup_{n\in\mathbb{N}} \mathrm{pr}_1(b'(n)), 2\Big) . \tag{A.2}$$

It remains to prove that the operations $\circ$, $\theta(\mathrm{null})$, and $\theta(\mathrm{suc})$ are ω-continuous wrt the partial orders $\leq$ and $\sqsubseteq$. By Observation 3.28(2) it suffices to show monotonicity and b-continuity for ω-chains b that are not ultimately constant.

"$\circ$ wrt $\leq$": First we show that $\circ$ is monotone wrt $\leq$. Since $\circ$ is commutative, it suffices to show that $a \leq a'$ implies $a \circ b \leq a' \circ b$ for every a, a', b. This is trivial if $a' \circ b = \top$; therefore, assume $a' \circ b \in A \setminus \{\top\}$. The only interesting case is that $(a, a') \in \prec_0 \cup \prec_1$. Then it is easy to check that $(a \circ b, a' \circ b) \in \prec_0 \cup \prec_1$.

Let b be an ω-chain wrt $\leq$ that is not ultimately constant. We prove that $\circ$ is b-continuous. Since $\circ$ is commutative, it suffices to show that, for every $a \in A$, $a \circ \vee\{b(n) \mid n \in \mathbb{N}\} = \vee\{a \circ b(n) \mid n \in \mathbb{N}\}$. This is trivial if $a = \top$. If $a \neq \top$ and $\mathrm{pr}_1(a) \cap \mathrm{pr}_1(b(n)) \neq \emptyset$ for some $n \in \mathbb{N}$, then obviously $a \circ \vee\{b(n) \mid n \in \mathbb{N}\} = \vee\{a \circ b(n) \mid n \in \mathbb{N}\}$ is an immediate consequence of Equation (A.1). Otherwise the ω-chain $(a \circ b(n) \mid n \in \mathbb{N})$ is not ultimately constant, either; then it is easy to check that $a \circ \vee\{b(n) \mid n \in \mathbb{N}\} = \vee\{a \circ b(n) \mid n \in \mathbb{N}\}$ by using Equation (A.1).

"$\circ$ wrt $\sqsubseteq$": This can be shown similarly to the proof that $\circ$ is ω-continuous wrt $\leq$ while using Equation (A.2) instead of Equation (A.1).

"$\theta(\mathrm{null})$ wrt $\leq$ and $\sqsubseteq$": This is trivial because $\theta(\mathrm{null})$ is nullary.

"$\theta(\mathrm{suc})$ wrt $\leq$": First we show that $\theta(\mathrm{suc})$ is monotone wrt $\leq$. Let $a, a' \in A$ with $a \leq a'$. The only interesting case is that $(a, a') \in \prec_0 \cup \prec_1$; then it is easy to check that $\big(\theta(\mathrm{suc})(a), \theta(\mathrm{suc})(a')\big) \in \prec_0 \cup \prec_1$.

Let b be an ω-chain wrt $\leq$ that is not ultimately constant. We prove that $\theta(\mathrm{suc})$ is b-continuous. Clearly, the ω-chain $\big(\theta(\mathrm{suc})(b(n)) \mid n \in \mathbb{N}\big)$ is not ultimately constant, either; thus, Equation (A.1) yields $\theta(\mathrm{suc})\big(\vee\{b(n) \mid n \in \mathbb{N}\}\big) = \vee\{\theta(\mathrm{suc})(b(n)) \mid n \in \mathbb{N}\}$.

"$\theta(\mathrm{suc})$ wrt $\sqsubseteq$": This can be shown similarly to the proof that $\theta(\mathrm{suc})$ is ω-continuous wrt $\leq$ while using Equation (A.2) instead of Equation (A.1).

This finishes the proof that $(\mathcal{A}, \leq)$ and $(\mathcal{A}, \sqsubseteq)$ are ω-continuous m-monoids. It remains to define the mwmd M over Σ and Δ. We let $M = (P, R, q)$ such that $P = \{q^{(1)}\}$ and R contains the following two rules:

$$q(x) \leftarrow \mathrm{null}() \,;\, \emptyset \,, \qquad\qquad q(x) \leftarrow \mathrm{suc}(q(x)) \,;\, \emptyset \,.$$

Let $t = \alpha()$. It is easy to show by induction on n that for every $n \in \mathbb{N}$ we obtain $\mathcal{T}^n(I_{(\emptyset,1)})(q(\varepsilon)) = \big(\{m \in \mathbb{N} \mid n > m\}, 1\big)$. Equations (A.1) and (A.2) yield

$$\vee\{\mathcal{T}^n(I_{(\emptyset,1)})(q(\varepsilon)) \mid n \in \mathbb{N}\} = (\mathbb{N}, 1) \ ,$$
$$\sqcup\{\mathcal{T}^n(I_{(\emptyset,1)})(q(\varepsilon)) \mid n \in \mathbb{N}\} = (\mathbb{N}, 2) \ .$$

Hence, $[\![M]\!]^{\text{fix}}_{(\mathcal{A},\leq)}(t) = (\mathbb{N}, 1)$ and $[\![M]\!]^{\text{fix}}_{(\mathcal{A},\sqsubseteq)}(t) = (\mathbb{N}, 2)$. This finished the proof that $[\![M]\!]^{\text{fix}}_{(\mathcal{A},\leq)} \neq [\![M]\!]^{\text{fix}}_{(\mathcal{A},\sqsubseteq)}$. ∎

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List of Symbols

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