Levenberg-Marquardt Algorithms for Nonlinear Equations, Multi-objective Optimization, and Complementarity Problems

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Abstract

The Levenberg-Marquardt algorithm is a classical method for solving nonlinear systems of equations that can come from various applications in engineering and economics.

Recently, Levenberg-Marquardt methods turned out to be a valuable principle for obtaining fast convergence to a solution of the nonlinear system if the classical nonsingularity assumption is replaced by a weaker error bound condition. In this way also problems with nonisolated solutions can be treated successfully. Such problems increasingly arise in engineering applications and in mathematical programming.

In this thesis we use Levenberg-Marquardt algorithms to deal with nonlinear equations, multi-objective optimization and complementarity problems. We develop new algorithms for solving these problems and investigate their convergence properties.

For sufficiently smooth nonlinear equations we provide convergence results for inexact Levenberg-Marquardt type algorithms. In particular, a sharp bound on the maximal level of inexactness that is sufficient for a quadratic (or a superlinear) rate of convergence is derived. Moreover, the theory developed is used to show quadratic convergence of a robust projected Levenberg-Marquardt algorithm.

The use of Levenberg-Marquardt type algorithms for unconstrained multi-objective optimization problems is investigated in detail. In particular, two globally and locally quadratically convergent algorithms for these problems are developed. Moreover, assumptions under which the error bound condition for a Pareto-critical system is fulfilled are derived.

We also treat nonsmooth equations arising from reformulating complementarity problems by means of NCP functions. For these reformulations, we show that existing smoothness conditions are not satisfied at degenerate solutions. Moreover, we derive new results for positively homogeneous functions. The latter results are used to show that appropriate weaker smoothness conditions (enabling a local Q-quadratic rate of convergence) hold for certain reformulations.

Zusammenfassung

Der Levenberg-Marquardt-Algorithmus ist ein klassisches Verfahren zur Lösung nichtlinearer Gleichungssysteme. Diese findet man u.a. in vielen Anwendungen der Ingenieur-und Wirtschaftswissenschaften.

Kürzlich erwiesen sich Levenberg-Marquardt-Methoden als wichtiges Prinzip zur Erreichung schneller Konvergenz gegen eine Lösung des nichtlinearen Systems, wenn die klassische Regularitätsbedingung durch eine schwächere Fehlerschranke ersetzt wird. So lassen sich auch Probleme mit nicht isolierten Lösungen erfolgreich behandeln. Solche Probleme treten zunehmend in ingenieurwissenschaftlichen Anwendungen und in der mathematischen Optimierung auf.

In dieser Arbeit werden Levenberg-Marquardt-Methoden für nichtlineare Gleichungen, multikriterielle Optimierung und Komplementaritätsprobleme verwendet. Neue Algorithmen werden entwickelt und ihre Konvergenzeigenschaften untersucht.

Für nichtlineare Gleichungssysteme mit hinreichend glatten Funktionen werden Konvergenzergebnisse für inexakte Levenberg-Marquardt-Algorithmen gezeigt. Insbesondere wird eine verbesserte scharfe obere Schranke für das Maß der Inexaktheit hergeleitet, die noch quadratische (oder superlineare) Konvergenz erlaubt. Außerdem wird die entwickelte Theorie benutzt, um die quadratische Konvergenz eines robusten projizierten Levenberg- Marquardt-Algorithmus zu zeigen.

Die Verwendung von Levenberg-Marquardt-Algorithmen für unrestringierte multikriterielle Optimierungsprobleme wird im Detail untersucht. Insbesondere wurden zwei global und lokal quadratisch konvergente Algorithmen für diese Optimierungsprobleme entwickelt. Außerdem konnten Bedingungen hergeleitet werden, unter denen die Fehlerschranke für ein Pareto-kritisches System gilt.

Die Arbeit behandelt auch nichtglatte Gleichungssysteme, die aus der Umformulierung von Komplementaritätsproblemen durch NCP-Funktionen entstehen. Dafür wird gezeigt, dass übliche Glattheitsvoraussetzungen im Falle degenerierter Lösungen nicht erfüllt sind. Außerdem werden neue Ergebnisse für positiv homogene Funktionen hergeleitet. Diese Ergebnisse werden verwendet um zu zeigen, dass für einige Umformulierungen bestimmte (für die lokal schnelle Konvergenz ausreichende) schwächere Glattheitsvoraussetzungen gelten.

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Basic Notation

$\mathbb{R}$	Real numbers.
$\mathbb{R}^n$	$=\{x = (x_1, \ldots, x_n)^\top \mid x_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n\}$.
$\mathbb{R}^n_+$	non-negative orthant i.e., $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0, \text{for all } i = 1, 2, \ldots, n\}$.
$\mathbb{N}$	Natural numbers i.e., $\mathbb{N} = \{1, 2, \ldots\}$.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	set of integers i.e., $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
0_n	null vector in $\mathbb{R}^n$.
1_n	$(1, 1, \ldots, 1)^\top \in \mathbb{R}^n$.
$\mathrm{bd}(S)$	boundary of a set S.
$\mathrm{int}(S)$	interior of a set S.
$\|x\|$	Euclidean norm of a vector x i.e., $\|x\| := \sqrt{x_1^2 + \ldots + x_n^2}$.
$\|x\|_q$	ℓ_q norm of a vector x, $q \in (1, \infty)$ i.e., $\|x\|_q := (\lvert x_1\rvert^q + \ldots + \lvert x_n\rvert^q)^{\frac{1}{q}}$.
$\|x\|_\infty$	max norm of a vector x i.e., $\|x\|_\infty := \max\{\lvert x_1\rvert, \ldots, \lvert x_n\rvert\}$.
$\mathcal{B}(x, \delta)$	the ball around x with radius δ i.e., $\mathcal{B}(x, \delta) := \{y \mid \|y - x\| \leq \delta\}$.
$\mathrm{dist}\,[x, S]$	distance of x to a set S i.e., $\mathrm{dist}\,[x, S] := \inf_{y \in S} \|y - x\|$.
f	a scalar valued function from $\mathbb{R}^n$ to $\mathbb{R}$.
F	a vector valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or to $\mathbb{R}^n$).
$\mathcal{D}_f$	set of all points at which the function f is Fréchet differentiable.
$\nabla f(x)$	gradient of f at x.
$\nabla^2 f(x)$	Hessian of f at x.
$JF(x)$	Jacobian matrix of F at x.
$\nabla F(x)$	$=JF(x)^\top$.
X_p	set of all **P**areto-optimal points.
X_w	set of all **w**eakly Pareto-optimal points.
X_{pp}	set of all **p**roperly **P**areto-optimal points.
X_{pc}	set of all **P**areto-**c**ritical points.
$\mathrm{Null}\,(M)$	Null space of matrix M.
$r(M)$	Rank of matrix M.
$\mathbb{R}^{n \times p}$	Set of $(n \times p)$-matrices with elements in $\mathbb{R}$.

Chapter 1

Introduction

1.1 Overview

In this thesis we deal with *nonlinear equations*, *multi-objective optimization* and *complementarity problems*. We develop new algorithms for solving these problems and investigate their convergence properties.

The new algorithms developed are based on the Levenberg-Marquardt algorithm. This algorithm is a classical method for solving nonlinear systems of equations and least squares problems that come from various applications in engineering, physics and economics. The algorithm can be regarded as a regularized Gauss-Newton method.

Recently, Levenberg-Marquardt algorithms turned out to be a valuable means for ensuring fast convergence to a solution of the nonlinear system if the classical nonsingularity assumption is replaced by a weaker error bound condition so that problems with nonisolated solutions can be treated successfully. Computing a nonisolated solution is a difficult task and the algorithms in this area are much less developed than say Newton's method. Nonisolatedness is a kind of degeneracy coming from several sources.

A first source for problems with nonisolated solutions are Karush-Kuhn-Tucker (KKT) systems that belong to an optimization problem with constraints. If no suitable constraint qualification holds the Lagrange multipliers can be nonunique and nonisolated. For this type of difficulties, a number of techniques have been developed during the last years, see [19; 25; 35; 68].

Sources for more difficult problems with nonisolated solutions are underdetermined nonlinear equations, optimization problems with nonisolated primal solutions (often caused by redundant variables), multi-objective optimization problems or complementarity problems that do not arise from optimization problems (like reformulations of generalized Nash equilibria, see [20]).

This thesis aims at the design and analysis of new and improved Levenberg-

Marquardt type algorithms for solving nonlinear equations, multi-objective optimization and complementarity problems. These algorithms share basic advantages of existing Levenberg-Marquardt methods like fast convergence under weak conditions and the promising practical robustness.

In the remaining sections of this chapter we will briefly discuss these problems and some basic tools that we will use to tackle them. We will also describe the Levenberg-Marquardt subproblem and discuss the relations between the classical nonsingularity assumption versus an error bound.

For nonlinear equations Levenberg-Marquardt type algorithms were shown to have a quadratic rate of convergence if an appropriate error bound condition holds, see [24; 26; 69; 71]. These results are valid if the map H defining the nonlinear equation is sufficiently smooth (basically H has to be differentiable with locally Lipschitz continuous derivative). The results differ in the possible range for the regularization parameter and in the proof techniques. The result in [69] allow this parameter to be chosen proportional to the square of $\|H\|$. In contrast to this, according to approaches in [26] and [24] the regularization parameter can be chosen proportional to $\|H\|$ without destroying the convergence rate. This could lead to more stable subproblems.

Since the subproblems of a Levenberg-Marquardt type algorithm will usually be solved only inexactly it is important to know the level of accuracy required to preserve the convergence rate. Corresponding results require that the accuracy is proportional to the power of 4 [10] or to the power of 3 [23] of $\|H\|$. However, this shows a significant gap to the accuracy level (power of 2) that holds for the classical inexact Newton method. This is investigated in detail Chapter 2 where it is shown that an accuracy, proportional to the power of 2, is sufficient to preserve the Q-quadratic convergence rate. In the same chapter, we also show Q-quadratic convergence of a projected Levenberg-Marquardt algorithm with a large regularization parameter.

The use of Levenberg-Marquardt type algorithms for multi-objective optimization problems are investigated in Chapters 3 and 4. Many methods have been proposed in the literature to find a Pareto-critical point (which satisfies the first order KKT optimality conditions for an unconstrained multi-objective problem). The most notable among them are the steepest descent based methods in [16; 31; 32; 33; 51; 52; 55]. These methods use gradient information of all the objective functions to find a search direction. However, all of these methods converge only quite slowly (comparable to gradient methods for programs with a single objective). Multi-objective optimization problems usually possess a set of nonisolated Pareto critical points. The nonisolatedness of Pareto-critical points has never been looked at in the past. This view gives fresh insights into developing new algorithms with a local Q-quadratic convergence rate. In particular, two quadratically and globally convergent algorithms are developed. Moreover, assumptions are derived under which the error bound for a Pareto-critical system is fulfilled.

In Chapter 5, we discuss nonsmooth nonlinear equations arising from reformulating nonlinear complementarity problems. This reformulation is done by so called NCP functions. In [43], a smoothness condition on the reformulated nonlinear equation near a solution is employed for showing local Q-quadratic convergence of the constrained Levenberg-Marquardt method. In Chapter 5, we provide results in detail that this smoothness condition does not hold near some special types of solutions (known as degenerate solutions). Recently [27], this smoothness condition has been weakened so that the constrained Levenberg-Marquardt method can have local Q-quadratic convergence if the NCP function is defined as the min function. All the smoothness conditions are discussed in detail in Chapter 5. In Chapter 5 we use positively homogeneous NCP functions to investigate the weaker smoothness properties. In particular, we examine for what class of NCP functions the weaker smoothness conditions in [27] are satisfied. For this, we extend some fundamental identities known for differentiable homogeneous functions, like Euler's and derivative identity, to nonsmooth homogeneous functions.

Finally, conclusions and possible extensions will be presented in the last chapter of our work.

1.2 Problems

Nonlinear Equations

In many applications one needs to find values of variables that satisfy a number of conditions in the form of equations. This can be described by

$$H(z) = 0, \tag{1.1}$$

where $z \in \mathbb{R}^n$ is the unknown variable and $H : \mathbb{R}^n \to \mathbb{R}^m$ is a given function.

If the equations are all linear then (1.1) is a linear system. There are many exact methods for solving a linear system, for example, Gaussian elimination. However, most real-world problems, especially those encountered in engineering, mathematics and physics are inherently nonlinear in nature. Systems of nonlinear equations are much more difficult to solve than linear ones. Numerical solution techniques almost exclusively rely on iterative procedures. We will mostly assume H to be continuously differentiable and, will discuss solution procedures for general functions H.

Many times one cannot model a real-world problem in the form given by (1.1). For example, one has to consider additional constraints. Time, for example, cannot be negative (except possibly in a relativistic setting). In chemical equilibrium problems, the concentration of a substrate must always be non-negative. Or, the mapping H might not be defined everywhere. A constrained system of nonlinear equations can be described by

$$H(z) = 0, \text{ s.t. } z \in \Omega, \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is the constraint set. Usually, for simplicity we will restrict ourselves to the case when Ω is convex.

There are many sources of nonlinear (or constrained) nonlinear equations. In this thesis will discuss in detail two such sources: Multi-objective optimization problems and complementarity problems.

Multi-objective Optimization Problems

There are usually multiple conflicting objectives in engineering design problems (see for example [12; 62]) and they all shall be *minimized* (or *maximized*, as the case may be) in some sense. For example, while buying a car we would like to pay as less as possible (minimize the cost) and also would like to have a car with minimal fuel consumption (minimize fuel consumption). However, usually a point that is a simultaneous minimizer of all the objective functions usually does not exists (in our example, the cheapest car might not be fuel efficient).

In this thesis we will be concerned with unconstrained multi-objective optimization problems. The concept of Pareto-optimality (defined in Chapter 3) is often used to characterize a desirable point of such a problem (see [18]). It will be shown in Chapter 3 that finding a Pareto-optimal point amounts to solving a constrained system of equations, i.e., solving (1.2) with particular choices of H and Ω. Thus, we can adapt some classical techniques to solve (1.2), depending upon the choice of H and Ω.

Complementarity Problems

For a given function $F : \mathbb{R}^n \to \mathbb{R}^n$, solving the nonlinear complementarity problem (NCP(F) in short) is to find a vector $x \in \mathbb{R}^n$ so that

$$x \geq 0, \quad F(x) \geq 0, \quad x^\top F(x) = 0. \tag{1.3}$$

Complementarity problems arise in a variety of engineering applications, especially in equilibrium problems from economics, transportation sciences and frictional contact problems (see [22] for further details). Another very common source of complementarity problems are constrained optimization problems (both single objective and multi-objective). It will be shown in Chapter 5 that solving the nonlinear complementarity problem amounts to solving a constrained system of equations, i.e., solving (1.2) with particular choices of H and Ω. Hence, as in the case of unconstrained multi-objective problems, we can adapt some classical techniques to solve (1.2), depending upon the choice of H and Ω.

1.3 Algorithmic Principles and Preliminaries

The solution set of (1.2) is denoted by

$$Z := \{z \in \Omega | H(z) = 0\}. \tag{1.4}$$

Let us first consider the case when H is continuously differentiable, $\Omega := \mathbb{R}^n$ and $n = m$. For this case, among the many iterative methods for solving (1.2), *Newton's method* is the most popular. In this method, if z^k is the point at iteration k the next iterate is $z^{k+1} := z^k + d^k$, where

$$d^k := -(\nabla H(z^k))^{-1} H(z^k). \tag{1.5}$$

Obviously, Newton's method requires that $\nabla H(z^k)$ is nonsingular (invertible). Now let us relax the condition $m = n$ by $m \neq n$. For such a case, a popular method for solving (1.2) is the *Gauss-Newton method*. In this method, if z^k is the point at iteration k the next iterate is $z^{k+1} := z^k + d^k$, where

$$d^k := -(\nabla H(z^k) \nabla H(z^k)^\top)^{-1} \nabla H(z^k) H(z^k). \tag{1.6}$$

Obviously, the Gauss-Newton method requires that $\nabla H(z^k)\nabla H(z^k)^\top$ is nonsingular. The classical *Levenberg-Marquardt method* on the other hand, does not require that $\nabla H(z^k)\nabla H(z^k)^\top$ is nonsingular. The Levenberg-Marquardt method goes back to Levenberg [45] and Marquardt [49]. In this method, if z^k is the point at iteration k the next iterate is $z^{k+1} := z^k + d^k$, where

$$d^k := -\left(\nabla H(z^k) \nabla H(z^k)^\top + \alpha(z^k) I\right)^{-1} \nabla H(z^k) H(z^k), \tag{1.7}$$

with $\alpha(z^k) > 0$. Obviously, the matrix $\nabla H(z^k)\nabla H(z^k)^\top + \alpha(z^k)I$ is positive definite and d^k from (1.7) is well-defined. This method can also be regarded as a regularized Gauss-Newton (or regularized Newton) method and does not require that $m = n$ or invertibility of any matrix. Rather, the following assumption is crucial for the convergence analysis of this method.

Error Bound *For some $z^* \in Z$, there are $C > 0$ and $\delta > 0$ so that*

$$\|H(z)\| \geq C \text{dist}\,[z, Z], \qquad \textit{for all } z \in \mathcal{B}(z^*, \delta), \tag{1.8}$$

holds.

Let us define the concept of isolated and nonisolated solutions of (1.2).

Definition *A solution $z^* \in Z$ is called an* isolated solution *if there exists a $\delta > 0$ so that $\mathcal{B}(z^*, \delta) \cap Z = \{z^*\}$. If this condition does not holds then we call z^* a* nonisolated solution*.*

Taylor's theorem implies the following.

Lemma *Let $z^* \in Z$, $m = n$, H be continuously differentiable and let $\nabla H(z^*)$ be nonsingular. Then, z^* is an isolated solution. Moreover, the Error Bound holds.*

Proof: By Taylor's theorem we obtain

$$H(z^* + h) = H(z^*) + \nabla H(z^*)^\top h + o(h).$$

As $z^* \in Z$, this simplifies to

$$H(z^* + h) = \nabla H(z^*)^\top h + o(h). \tag{1.9}$$

The nonsingularity of $\nabla H(z^*)$, shows that $\nabla H(z^*)^\top h = 0$ holds if and only if $h = 0$. Hence, from (1.9) we obtain that z^* is an isolated solution. Moreover, we have that

$$\text{dist}\,[z^* + h, Z] = \|h\| \leq \left\|\nabla H(z^*)^{-1}\right\| \|H(z^* + h)\| + \left\|\nabla H(z^*)^{-1}\right\| \|o(h)\|,$$

for all h with $\|h\|$ sufficiently small. This simplifies to

$$\frac{1}{2}\text{dist}\,[z^* + h, Z] \leq \|h\| - \left\|\nabla H(z^*)^{-1}\right\| \|o(h)\| \leq \left\|\nabla H(z^*)^{-1}\right\| \|H(z^* + h)\|,$$

for all h with $\|h\|$ sufficiently small. This obviously implies that the Error Bound holds. △

Hence, if H is continuously differentiable, the Error Bound is a weaker condition than nonsingularity of $\nabla H(z^*)$. It is strictly weaker since for $H : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H(x, y) := \left(x, x(1 + y^2)\right)^\top,$$

it is easy to see that the Error Bound holds but $\nabla H(z^*)$ is singular for any $z^* \in Z$.

Chapter 2

Robust Levenberg-Marquardt Algorithms for Nonlinear Equations

2.1 Introduction

For a continuously differentiable mapping $H : \mathbb{R}^n \to \mathbb{R}^m$, let us consider the equation

$$H(z) = 0. \tag{2.1}$$

Its solution set is denoted by

$$Z := \{z \in \mathbb{R}^n | H(z) = 0\} \tag{2.2}$$

Levenberg-Marquardt methods for solving Equation (2.1) are known ([69]) to possess superlinear convergence properties, if the starting point is sufficiently close to some $z^* \in Z$, and if a certain local error bound condition holds. This condition requires that $C > 0$ and $\delta > 0$ exist so that

$$\|H(z)\| \leq C \mathrm{dist}\,[z, Z] \tag{2.3}$$

for all z in a ball of radius δ around z^*. It is well known that this condition can hold even for problems (2.1) which have nonisolated solutions. Such problems arise in different fields such as optimization, variational inequalities, and game theory, for examples see [21; 43].

In this chapter, we will improve the inexactness level for robust Levenberg-Marquardt methods. The Levenberg-Marquardt method is called robust, if its regularization parameter is chosen as large as possible without destroying a desired convergence rate. The issues of inexactness and robustness will be detailed in the following text. We first consider a subproblem of the Levenberg-Marquardt method. Given an iterate $s \in \mathbb{R}^n$ and a parameter $\alpha(s) > 0$, such a subproblem consists of the following linear equation

$$\nabla H(s)H(s) + \left(\nabla H(s)\nabla H(s)^\top + \alpha(s)I\right)(z - s) = 0. \tag{2.4}$$

Obviously, the matrix $\nabla H(s)\nabla H(s)^\top + \alpha(s)I$ is positive definite and Equation (2.4) has a unique solution that provides the next iterate. The resulting method is thus well defined for any starting point and goes back to Levenberg [45] and Marquardt [49]. This method can also be regarded as a regularized Gauss-Newton method. The inexact (perturbed) Levenberg-Marquardt subproblem is given by

$$\nabla H(s)H(s) + \left(\nabla H(s)\nabla H(s)^\top + \alpha(s)I\right)(z - s) = p(s), \tag{2.5}$$

where $p(s)$ denotes the value of a perturbation function $p : \mathbb{R}^n \to \mathbb{R}^n$. Again, Equation (2.5) has a unique solution. This solution will be denoted by $z(s)$.

For a given starting vector $z^0 \in \mathbb{R}^n$, the inexact Levenberg-Marquardt method generates a sequence z^k of iterates defined by,

$$z^{k+1} := z(z^k), \quad k \in \mathbb{N}_0. \tag{2.6}$$

For later use, the solution of the unperturbed subproblem (3) is denoted by $z_0(s)$. Obviously, $z_0(s)$ is the unique solution of Equation (2.5) with $p(s) = 0$. The inexact subproblem (2.5) can be equivalently written as the unconstrained quadratic program

$$\min \psi(z),$$

where $\psi : \mathbb{R}^n \to [0, \infty)$ is defined by

$$\psi(s) := \frac{1}{2}\left\|H(s) + \nabla H(s)^\top(z - s)\right\|^2 + \frac{1}{2}\alpha(s)\|z - s\|^2 - p(s)^\top(z - s). \tag{2.7}$$

This is easily seen using the necessary optimality condition $\nabla\psi(z) = 0$ which, due to the (strong) convexity of ψ, is also sufficient.

Based on the error bound condition (2.3) convergence rate results for the inexact LevenbergMarquardt method have been developed by Dan et al. [10] and by Fan and Pan [23]. Regardless slightly different smoothness assumptions, they obtained a quadratic rate of the inexact Levenberg-Marquardt method under different assumptions on the magnitudes of $\alpha(s)$ and $\|p(s)\|$ in terms of $\|H(z)\|$, see Table 2.1. The result in [10] is based on [69], the first paper showing a superlinear rate for the Levenberg-Marquardt method under the error bound condition (2.3). The improved result in [23] exploits a technique developed in [24], that makes use of singular value decompositions of Jacobians of H at certain points. Obviously, the result in [23] allows a larger perturbation term $p(s)$. Moreover, it guarantees a numerically more robust solution of the subproblems, since the regularization term $\alpha(s)I$ within the inexact Levenberg-Marquardt subproblems is locally much larger, if $\alpha(s)$ is proportional to $\|H(s)\|$ instead of being proportional to $\|H(s)\|^2$.

If we compare the results in [10] and [23] with the magnitude of the perturbation term in inexact Newton methods for regular solutions of quadratic systems of equations, the question arises whether it is possible to show a quadratic rate of convergence for the inexact Levenberg-Marquardt method with the more robust choice of $\alpha(s) = O(\|H(s)\|)$ (see [23] in Table 2.1), but with a significantly larger perturbation term $p(s)$.

Table 2.1: Results from literature

	$\alpha(s)$	$\|p(s)\|$
[10]	$O(\|H(s)\|^{\mathbf{2}})$	$O(\|H(s)\|^{\mathbf{4}})$
[23]	$O(\|H(s)\|^{\mathbf{1}})$	$O(\|H(s)\|^{\mathbf{3}})$

In more detail, let us consider the case when $m = n$ and the Jacobian ∇H is locally Lipschitz continuous. Then, if $H(z^*) = 0$ and $\nabla H(z^*)$ is nonsingular (z^* is said to be a regular solution), it is well known by the work of Dembo et al. [14] that inexact Newton methods with perturbations not larger than $O(\|H(s)\|^2)$ converge Q-quadratically to the isolated solution z^*. It is also known ([14]) that the quadratic rate is lost (decreases), if perturbations with

$$\|p(s)\| = O\left(\|H(s)\|^{1+t}\right) \qquad \text{and } t \in (0,1) \tag{2.8}$$

are allowed. Under the conditions stated in this paragraph the Levenberg-Marquardt subproblem (2.4) can be rewritten as the following inexact Newton subproblem

$$H(s) + \nabla H(s)^\top (z - s) = p_\alpha(s) \tag{2.9}$$

with

$$\begin{aligned} p_\alpha(s) &:= \left[I - \nabla H(s)^\top \left(\nabla H(s) \nabla H(s)^\top + \alpha(s) I\right)^{-1} \nabla H(s)\right] H(s) \\ &= \left[I - \left(I + \alpha(s)(\nabla H(s)^\top \nabla H(s))^{-1}\right)^{-1}\right] H(s), \end{aligned}$$

if s belongs to a sufficiently small neighborhood of z^*. This can be verified by checking that the solution of Equation (2.4) solves (2.9). Moreover, without giving details here, it can be further shown that

$$\|p_\alpha(s)\| = O(\alpha(s)\|H(s)\|)$$

is valid for all s in some neighborhood of z^*. This means that according to ([14]), the quadratic rate of the Levenberg-Marquardt method is lost in general if $\alpha(s) = O(\|H(s)\|^t)$ with $t \in (0, 1)$. Since $t = 1$ is the smallest value for which the Q-quadratic rate is possible (see above), the Levenberg-Marquardt method is called *robust* if α with $\alpha(s) = \|H(s)\|^1$ is used. A more general notion of robustness that depends on a desired convergence rate will be given in Section 2.5.

The first purpose of this chapter (see Section 2.2) is to improve the level of inexactness that can be allowed in a robust Levenberg-Marquardt method without destroying quadratic convergence. It will be shown that

$$p(s) = O(\|H(s)\|^2)$$

is sufficient. To this end we use an approach that is different from [10] and [23]. Rather, we exploit a result given by Fischer [26]. Roughly speaking, it says that the error bound condition (2.3) provides an error bound for the equation

$$F(z) := \nabla H(z)H(z) = 0, \tag{2.10}$$

see Lemma 2.2.1.

The second purpose of this chapter is to apply the improved inexactness level to a projected Levenberg-Marquardt method suggested by Kanzow et al. [69]. The projections guarantee the feasibility of the iterates with respect to a given convex set. It will turn out that the projected Levenberg-Marquardt method can be regarded as an inexact Levenberg-Marquardt method. Based on this and on the results on inexact robust Levenberg-Marquardt methods we will show in Section 2.3 that a projected robust Levenberg-Marquardt method also converges Q-quadratically. Section 2.4 presents some numerical experiments, whereas the concluding remarks at the end of the chapter show possible extensions.

Most of the results in this chapter can be found in [29].

2.2 Local Convergence Analysis

Throughout this section the assumptions given below are required to be fulfilled. For the sake of clarity we also repeat assumptions that were already used in Section 2.1

Assumption 2.2.1 *The function* $H : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable and* $\nabla H : \mathbb{R}^n \to$

$\mathbb{R}^{n\times m}$ *is locally Lipschitz continuous.*

Assumption 2.2.2 *For some $z^* \in Z$, there are $C > 0$ and $\delta > 0$ so that*

$$\|H(s)\| \geq C\text{dist}\,[z, Z], \qquad \textit{for all } z \in \mathcal{B}(z^*, \delta). \tag{2.11}$$

Obviously, Assumption 2.2.2 implicitly assumes that the solution set Z is nonempty. Now, a condition on the level of inexactness within the subproblems (2.5) of the Levenberg-Marquardt method (2.6) is stated.

Assumption 2.2.3 *There is $c_p > 0$ so that*

$$\|p(s)\| \leq c_p\|H(s)\|^2 \qquad \textit{for all } s \in \mathcal{B}(z^*, \delta). \tag{2.12}$$

with $z^* \in Z$ and δ from Assumption 2.2.2.

The remaining assumption guarantees not only the robustness of the subproblems but also that for technical reasons, the subproblem (2.5) has a unique solution even if $s \in Z$.

Assumption 2.2.4 *Let $\alpha : \mathbb{R}^n \to (0, \infty)$ be defined by*

$$\alpha(s) = \begin{cases} \|H(s)\| & \textit{if } s \in \mathbb{R}^n \setminus Z; \\ 1 & \textit{if } s \in Z. \end{cases} \tag{2.13}$$

Blanket Assumption for Section 2.2: *Assumptions 2.2.1, 2.2.2, 2.2.3 and 2.2.4 hold.*

Replacing the definition of $\alpha(s)$ for $s \in \mathbb{R}^n \setminus Z$ by the more general condition that

$$\alpha_0\|H(s)\| \leq \alpha(s) \leq \alpha_1\|H(s)\| \tag{2.14}$$

(with some $\alpha_1 \geq \alpha_0 > 0$) would lead to the same results but is avoided here for the sake of simplicity. In particular, under Assumptions 2.2.1 and 2.2.2,

$$\alpha(s) := \|F(s)\| = \|\nabla H(s)H(s)\| \tag{2.15}$$

satisfies condition (2.14) for all s in an arbitrary ball around z^* with suitably chosen $\alpha_0, \alpha_1 > 0$. To verify this claim the following Lemma 2.2.1 can be employed. Lemma 2.2.1 is an immediate consequence of [26, Corollary 2] and plays an important role in deriving the convergence results in this section.

Lemma 2.2.1 *With $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by (2.10), there are $C_F > 0$ and $\delta_F > 0$ so that*

$$\|F(z)\| \geq C_F \text{dist}\,[z, Z] \qquad \textit{for all } z \in \mathcal{B}(z^*, \delta_F). \tag{2.16}$$

The next lemma provides smoothness properties following from the Blanket Assumption for this section.

Lemma 2.2.2 *There is $L > 0$ so that the inequalities*

$$\begin{aligned} \|\nabla H(z) - \nabla H(s)\| &\leq L\|z - s\|, & (2.17)\\ \|H(z) - H(s) - \nabla H(s)^\top (z - s)\| &\leq L\|z - s\|^2, & (2.18)\\ \|H(z) - H(s)\| &\leq L\|z - s\|, & (2.19)\\ \|\nabla H(z)\| &\leq L, & (2.20) \end{aligned}$$

are satisfied for all $z, s \in \mathcal{B}(z^*, 2\delta)$.

Proof: Since $\mathcal{B}(z^*, 2\delta)$ is compact, the inequalities (2.17)-(2.20) follow easily from Assumption 2.2.1. △

Lemma 2.2.3 *There is $\kappa > 0$ so that*

$$\|z(s) - s\| \leq \kappa \text{dist}\,[s, Z] \qquad \textit{for all } s \in \mathcal{B}(z^*, \delta). \tag{2.21}$$

Proof: Let s denote an arbitrary but fixed element of $\mathcal{B}(z^*, \delta) \setminus Z$. By Assumptions 2.2.1 and 2.2.2, the set Z is closed and nonempty. Thus, a point $s^\perp \in Z$ exists with

$$\|s - s^\perp\| = \text{dist}\,[s, Z]. \tag{2.22}$$

Moreover, $s^\perp \in \mathcal{B}(z^*, 2\delta)$ since

$$\|s^\perp - z^*\| \leq \|s^\perp - s\| + \|s - z^*\| \leq 2\delta. \tag{2.23}$$

Thus, $s^\perp$ can be used in Lemma 2.2.2 as a possible instance for z.

The function ψ defined in (2.7) has a unique minimizer. For $p(s) := 0$ the minimizer was denoted by $z_0(s)$ (see Section 2.1). This immediately leads to

$$\alpha(s)\|z_0(s) - s\|^2 \leq 2\psi(z_0(s)) \leq 2\psi(s^\perp). \tag{2.24}$$

From (2.7) with $p(s) = 0$ and the definition of α in (2.13) we obtain

$$2\psi(s^\perp) = \|H(s) + \nabla H(s)^\top (s^\perp - s)\|^2 + \|H(s)\|\|s - s^\perp\|^2. \tag{2.25}$$

Taking into account inequalities (2.18) and (2.19) for $z := s^\perp$ and $H(s^\perp) = 0$, we have

$$2\psi(s^\perp) \le L^2\|s - s^\perp\|^4 + L\|s - s^\perp\|^3. \tag{2.26}$$

Together with Equations (2.24), (2.22) and the definition of α this yields

$$\|z_0(s) - s\|^2 \le 2\alpha(s)^{-1}\psi(s^\perp) \le \|H(s)\|^{-1}(L^2\delta + L)\text{dist}\,[s, Z]^3 \tag{2.27}$$

and, with Assumption 2.2.2,

$$\|z_0(s) - s\| \le \sqrt{C^{-1}(L^2\delta + L)}\,\text{dist}\,[s, Z] \tag{2.28}$$

follows. According to the definition of $z_0(s)$ and $z(s)$ as solution of Equations (2.4) and (2.5), respectively, we have

$$F(s) + A(s)(z_0(s) - s) = 0, \qquad F(s) + A(s)(z(s) - s) = p(s), \tag{2.29}$$

where $A(s) := \nabla H(s)\nabla H(s)^\top + \alpha(s)I$ and $F(s) = \nabla H(s)H(s)$ (see Equation (2.10)). Subtracting both equations leads to

$$z(s) - z_0(s) = A(s)^{-1}p(s). \tag{2.30}$$

Obviously, the matrix $A(s)$ is symmetric and positive definite. Its smallest eigenvalue is bounded below by $\alpha(s) = \|H(s)\|$. Thus, the largest eigenvalue of $A(s)^{-1}$ is bounded above by $\|H(s)\|^{-1}$ and

$$\|A(s)\|^{-1} \le \|H(s)\|^{-1}. \tag{2.31}$$

Due to Assumption 2.2.3, (2.19) and (2.22) it follows that

$$\begin{aligned} \|z(s) - z_0(s)\| &\le \|A(s)\|^{-1} \le \|p(s)\| \\ &\le c_p\|H(s)\| \\ &= c_p\|H(s) - H(s^\perp)\| \\ &\le c_p L\text{dist}\,[s, Z]. \end{aligned}$$

With (2.28) we therefore have

$$\|z(s) - s\| \le \|z(s) - z_0(s)\| + \|z_0(s) - s\| \le \kappa\text{dist}\,[s, Z], \tag{2.32}$$

where

$$\kappa := c_pL + \sqrt{C^{-1}(L^2\delta + L)}. \tag{2.33}$$

For $s \in Z$, Assumption 2.2.3 implies $p(s) = 0$. Then, with Equation (2.13), $z(s) = s$ follows so that the assertion of the lemma is shown for all $s \in \mathcal{B}(z^*, \delta)$. $\triangle$

Lemma 2.2.4 *There are $\hat{C} > 0$ and $\hat{\delta} > 0$ so that*

$$\operatorname{dist}[z(s), Z] \leq \hat{C} \operatorname{dist}[s, Z]^2 \qquad \textit{for all } s \in \mathcal{B}(z^*, \hat{\delta}). \tag{2.34}$$

Proof: With δ_F from Lemma 2.2.1 and κ from Lemma 2.2.3 we define

$$\hat{\delta} := \frac{1}{\kappa + 1} \delta_F. \tag{2.35}$$

Note that $0 < \hat{\delta} < \delta_F \leq \delta$ and let $s \in \mathcal{B}(z^*, \hat{\delta}) \setminus Z$ and $z \in \mathcal{B}(z^*, \delta_F)$ be arbitrarily chosen. By the definitions of ψ in (2.7) and of F in (2.10) we have

$$\nabla \psi(z) = \nabla H(s) \left(H(s) + \nabla H(s)^\top (z - s) \right) + \alpha(s)(z - s) - p(s) \tag{2.36}$$

and

$$\begin{aligned} F(z) - \nabla \psi(z) = \quad & \nabla H(s) \left(H(z) - H(s) - \nabla H(s)^\top (z - s) \right) - \alpha(s)(z - s) + p(s) \\ & + (\nabla H(z) - \nabla H(s)) H(z). \end{aligned}$$

Therefore, with formulas (2.20), (2.18), (2.13), Assumption 2.2.3 and the inequality (2.17) it follows that

$$\|F(z) - \nabla \psi(z)\| \leq L^2 \|z - s\|^2 + \|H(s)\| \|z - s\| + c_p \|H(s)\|^2 + L \|z - s\| \|H(z)\|. \tag{2.37}$$

Since Lemma 2.2.3 implies

$$\|z(s) - z^*\| \leq \|z(s) - s\| + \|s - z^*\| \leq \kappa \operatorname{dist}[s, Z] + \hat{\delta} \leq (\kappa + 1)\hat{\delta} \leq \delta_F,$$

we have

$$z(s) \in \mathcal{B}(z^*, \delta_F) \tag{2.38}$$

so that $z \in \mathcal{B}(z^*, \delta_F)$ in (2.37) can be replaced by $z(s)$. Taking into account Lemma 2.2.1 and $\nabla \psi(z(s)) = 0$ we get

$$\|F(z(s)) - \nabla \psi(z(s))\| = \|F(z(s))\| \geq C_F \operatorname{dist}[z(s), Z]. \tag{2.39}$$

Defining $s^\perp \in Z$ according to (2.22) then, similarly to (2.23),

$$\|s^\perp - z^*\| \leq 2\hat{\delta} \leq 2\delta \tag{2.40}$$

follows. Therefore, (2.19) yields

$$\|H(s)\| = \|H(s^{\perp}) - H(s)\| \leq L\text{dist}\,[s, Z]. \tag{2.41}$$

Furthermore, by Equations (2.38), (2.40), (2.18) and Lemma 2.2.3, we obtain

$$\begin{aligned} \|H(z(s))\| &= \|H(z(s)) - H(s^{\perp})\| \\ &\leq L\|z(s) - s^{\perp}\| \\ &\leq L\|z(s) - s\| + L\|s - s^{\perp}\| \\ &\leq (\kappa + 1)L\text{dist}\,[s, Z]. \end{aligned}$$

This together with (2.41), Lemma 2.2.3 and (2.37) yields

$$\|F(z(s)) - \nabla\psi(z(s))\| \leq \text{dist}\,[s, Z]^2 L\left(\kappa^2 L + \kappa + c_p L + \kappa(\kappa + 1)L\right).$$

This inequality and Equation (2.39) provide the desired assertion for all $s \in \mathcal{B}(z^*, \hat{\delta}) \setminus Z$, where

$$\hat{C} := C_F^{-1} L\left(\kappa^2 L + \kappa + c_p L + \kappa(\kappa + 1)L\right). \tag{2.42}$$

For $s \in Z$, the assertion is obviously true, just apply the arguments at the end of the proof of Lemma 2.2.3. △

The next result does not depend on a particular algorithm and seems also useful for other applications. It gives sufficient conditions for the convergence of a general sequence to a limit and for a certain Q-order of this convergence.

Lemma 2.2.5 *Let $\{w^k\} \subset \mathbb{R}^n$, $\{r_k\} \subset [0, 1)$ be sequences, and $r \in [0, 1)$, $R > 0$ numbers so that, for $k \in \mathbb{N}_0$,*

$$\|w^k - w^0\| \leq r_0 \frac{R}{1 - r} \tag{2.43}$$

implies

$$r_{k+1} \leq r r_k \text{ and } \|w^{k+1} - w^k\| \leq R r_k. \tag{2.44}$$

Then, the following assertions hold:

(a) *$\{r_k\}$ converges to 0 and $\{w^k\}$ converges to some $\hat{w} \in \mathbb{R}^n$.*

(a) *If, for some $\tau > 1$ and $c > 0$,*

$$r_{k+1} \leq c r_k^{\tau} \text{ and } \|\hat{w} - w^k\| \geq r_k \tag{2.45}$$

is satisfied for $k \in \mathbb{N}_0$*, then* $\{w^k\}$ *converges to* $\hat{w}$ *with the Q-order of* τ*. In particular,*

$$\|w^{k+1} - \hat{w}\| \geq \frac{cR}{1-r}\|w^{k+1} - \hat{w}\|^{\tau}$$

is valid for $k \in \mathbb{N}_0$*.*

Proof: (a) We first show by induction that Equations (2.43) and (2.44) hold for all $k \geq 0$. Obviously, (2.43) is valid for $k = 0$. Let us now assume that for some k, inequality (2.43) is satisfied for $\nu = 1, \ldots, k$. Then, by assumption, the inequalities in (2.44) are valid for $\nu = 0, \ldots, k$ and we have that for ν, ℓ, with $k + 1 \geq \nu > \ell \geq 0$

$$r_\nu \leq r r_{\nu-1} \leq r_\ell r^{\nu-\ell} \tag{2.46}$$

and

$$\|w^\nu - w^\ell\| \leq \sum_{i=0}^{\nu-\ell-1} \|w^{\ell+i+1} - w^{\ell+i}\| \leq R \sum_{i=0}^{\nu-\ell-1} r_{\ell+i} \leq R r_\ell \sum_{i=0}^{\nu-\ell-1} r^i. \tag{2.47}$$

By $0 \leq r < 1$, (2.47) implies

$$\|w^\nu - w^\ell\| < r_\ell \frac{R}{1-r} \tag{2.48}$$

for ν, ℓ with $k + 1 \geq \nu \geq \ell \geq 0$. Taking $\nu = k + 1$ and $\ell = 0$ in (2.48) we see that (2.43) is valid for $\nu = 0, \ldots, k + 1$. By induction it follows that (2.43) and (2.44) hold for all $k \geq 0$, and that (2.46) , (2.47) and (2.48) hold for arbitrary integers ν, ℓ, with $\nu \geq \ell \geq 0$. Hence, by $0 \leq r < 1$ and by (2.46) , the sequence $\{r_k\}$ must converge to 0. Furthermore, with (2.48) we then conclude that $\{w^k\}$ is a Cauchy sequence and, thus, converges to some $\hat{w} \in \mathbb{R}^n$.

(b) For $\nu > \ell := k + 1$ we obtain from (2.48) and the first inequality in (2.45) that

$$\|w^\nu - w^{k+1}\| < r_{k+1} \frac{R}{1-r} \leq r_k^\tau \frac{cR}{1-r}. \tag{2.49}$$

Passing to the limit for $\nu \to \infty$ and using the second inequality in (2.45) yields

$$\frac{\|w^\nu - w^{k+1}\|}{\|w^\nu - w^k\|^\tau} \leq \frac{cR}{1-r} < \infty \tag{2.50}$$

for $k \in \mathbb{N}_0$. Hence, the sequence $\{w^k\}$ converges to $\hat{w}$ with the Q-order of τ. △

Theorem 2.2.1 *Let* $\{z^k\}$ *be a sequence generated by the inexact Levenberg Marquardt method (2.6) . Then, there are* $\tilde{\delta} > 0$ *and* $\tilde{C} > 0$ *so that* $z^0 \in \mathcal{B}(z^*, \tilde{\delta})$

implies that $\{z^k\}$ *belongs to* $\mathcal{B}(z^*, \tilde{\delta})$ *and converges to some* $\hat{z} \in Z$ *with*

$$\|z^{k+1} - \hat{z}\| \leq \tilde{C}\|z^k - \hat{z}\|^2 \qquad \text{for } k \in \mathbb{N}_0, \tag{2.51}$$

i.e. the inexact LevenbergMarquardt method converges Q-quadratically to a solution of (2.1).

Proof: As noted in Section 2.1 the inexact Levenberg-Marquardt method is well defined for any starting vector z^0. Moreover, it generates an infinite sequence $\{z^k\}$. Let $\tilde{\delta} > 0$ be chosen so that

$$\tilde{\delta}(2\kappa + 1) \leq \hat{\delta} \quad \text{and } \tilde{\delta}\hat{C}(2\kappa + 1) \leq \frac{1}{2} \tag{2.52}$$

with κ from Lemma 2.2.3 and $\hat{C}, \hat{\delta}$ from Lemma 2.2.4. To apply Lemma 2.2.5 we first set

$$w^k := z^k, \quad r_k := \text{dist}\,[z^k, Z], \quad R := \kappa, \quad r := \frac{1}{2}, \quad c := \hat{C}, \quad \tau := 2 \tag{2.53}$$

for $k \in \mathbb{N}_0$. Then, if we suppose that (2.43) is valid for some k, $z^0 \in \mathcal{B}(z^*, \tilde{\delta})$ implies

$$\begin{aligned} \|z^k - z^*\| &\leq \|z^k - z^0\| + \|z^0 - z^*\| \\ &\leq r_0 R(1-r)^{-1} + \tilde{\delta} \\ &= 2\kappa \text{dist}\,[z^0, Z] + \tilde{\delta} \\ &\leq (2\kappa + 1)\tilde{\delta} \\ &\leq \hat{\delta} \end{aligned} \tag{2.54}$$

and

$$\hat{C}\text{dist}\,[z^k, Z] \leq \hat{C}\|z^k - z^*\| \leq \hat{C}(2\kappa + 1)\tilde{\delta} \leq \frac{1}{2} = r.$$

Now, we see that Lemmas 2.2.3 and 2.2.4 with $s := z^k$ lead to

$$\|w^{k+1} - w^k\| = \|z^{k+1} - z^k\| \leq \kappa \text{dist}\,[z^k, Z] = R r_k \tag{2.55}$$

and

$$r_{k+1} = \text{dist}\,[z^{k+1}, Z] \leq \hat{C}\text{dist}\,[z^k, Z]^2 = c r_k^2 \leq r r_k. \tag{2.56}$$

Thus, the assumption in Lemma 2.2.5 that (2.43) implies (2.44) is satisfied for $k \in \mathbb{N}_0$. Therefore, according to part (a) of Lemma 2.2.5, the sequence $\{\text{dist}\,[z^k, Z]\}$ converges to 0 and $\{z^k\}$ converges to some $\hat{z} := \hat{w}$. As a consequence,

$\hat{z}$ belongs to Z. Moreover, due to (2.56), the first inequality in (2.45) is fulfilled. The second inequality in (2.45) is valid as well since

$$\|\hat{w} - w^k\| = \|\hat{z} - z^k\| \geq \text{dist}\,[z^k, Z] = r_k. \tag{2.57}$$

Hence, the assumptions used in Lemma 2.2.5 are fulfilled and with (2.53) the assertions of the theorem (with $\tilde{C} := 2\kappa\hat{C}$) follow except that $\{z^k\} \subset \mathcal{B}(z^*, \tilde{\delta})$. The latter is valid due to (2.54). △

2.3 A Projected Robust Levenberg-Marquardt Algorithm

Let us consider the problem of solving

$$H(z) = 0, \quad z \in Y, \tag{2.58}$$

where $Y \subset \mathbb{R}^n$ is a nonempty closed convex set. The solution set of problem (2.58) is denoted by Y^*. In [43] a projected Levenberg-Marquardt method for computing a solution of (2.58) is suggested. To detail this method let for any $z \in \mathbb{R}^n$, and any closed nonempty set $S \subseteq \mathbb{R}^n$, $\pi_S(z)$ denote the the Euclidean projection of $z \in \mathbb{R}^n$ onto the set S i.e., $\|\pi_S(z) - z\| = \text{dist}\,[z, S]$ holds. Recall that $z_0(s)$ denotes the solution of the unperturbed Levenberg-Marquardt subproblem (2.4), see Section 2.1. Then, for any starting vector $y^0 \in \mathbb{R}^n$, the projected Levenberg-Marquardt method generates a sequence $\{y^k\}$ defined by

$$y^{k+1} := \pi_Y(z_0(y^k)), \qquad k = 0, 1, 2, ... \tag{2.59}$$

Obviously, all iterates (possibly except y^0) belong to Y . Under certain assumptions the local Q-quadratic convergence of $\{y^k\}$ to a solution of (2.58) is shown in [43] if the regularization parameter in the subproblem (2.5) is defined by $\alpha(s) := \|H(s)\|^2$. By means of the results in the previous section we will now show that the Q-quadratic convergence is not destroyed if the robust version of the subproblems (2.5) is used, i.e., if $\alpha(s)$ is defined according to (2.13).

Instead of Assumption 2.2.2 the following error bound condition is required in [43].

Assumption 2.3.1 *For some $y^* \in Y^*$, there are $\omega_Y > 0$ and $\delta_Y > 0$ so that*

$$\|H(z)\| \geq \omega_Y \text{dist}\,[z, Y^*] \quad \textit{for all } z \in \mathcal{B}(y^*, \delta_Y).$$

Blanket Assumption for Section 2.3: *Assumptions 2.2.1, 2.2.4 and 2.3.1 hold.*

As a consequence of Assumption 2.3.1, we have the following.

Lemma 2.3.1 *It holds that y^* belongs to Z,*

$$Y^* \cap \mathcal{B}(y^*, \delta_Y) = Z \cap \mathcal{B}(y^*, \delta_Y), \tag{2.60}$$

and

$$\operatorname{dist}[z, Z] = \operatorname{dist}[z, Y^*] \quad \textit{for all } z \in \mathcal{B}\left(y^*, \frac{1}{2}\delta_Y\right).$$

Proof: Obviously, $Y^* \subseteq Z$. This implies $Y^* \cap \mathcal{B}(y^*, \delta_Y) \subseteq Z \cap \mathcal{B}(y^*, \delta_Y)$. Now, consider any $z \in Z \cap \mathcal{B}(y^*, \delta_Y)$. Then, $H(z) = 0$. From Assumption 2.3.1 it follows that $\operatorname{dist}[z, Y^*] = 0$. By Assumption 2.2.1, Y^* is closed so that z belongs to $Y^* \cap \mathcal{B}(y^*, \delta_Y)$. Thus, (2.60) is valid. Now, with the definitions of π_Z and π_{Y^*} in Section 2.1, $z \in \mathcal{B}\left(y^*, \frac{1}{2}\delta_Y\right)$ implies

$$\|y^* - \pi_Z(z)\| \leq \|y^* - z\| + \|z - \pi_Z(z)\| \leq \frac{1}{2}\delta_Y + \frac{1}{2}\delta_Y = \delta_Y$$

and, similarly, $\|y^* - \pi_{Y^*}(z)\| \leq \delta_Y$. So, we have $\pi_Z(z) \in \mathcal{B}(y^*, \delta_Y)$ and $\pi_{Y^*}(z) \in \mathcal{B}(y^*, \delta_Y)$. Therefore, with (2.60), $\operatorname{dist}[z, Z] = \operatorname{dist}[z, Y^*]$ follows for all $z \in \mathcal{B}\left(y^*, \frac{1}{2}\delta_Y\right)$. $\triangle$

We now show that the projected Levenberg-Marquardt method (2.59) is a particular inexact Levenberg-Marquardt method that locally satisfies Assumption 2.2.3 on the level of inexactness. To this end define

$$\hat{p}(s) := \left(\nabla H(s)\nabla H(s)^\top + \alpha(s)I\right)(\pi_Y(z_0(s)) - z_0(s))$$

for all $s \in \mathbb{R}^n$. Then,

$$\nabla H(s)H(s) + \left(\nabla H(s)\nabla H(s)^\top + \alpha(s)I\right)(\pi_Y(z_0(s)) - s) = \hat{p}(s), \tag{2.61}$$

i.e., $\pi_Y(z_0(s))$ can be regarded as the solution of the robust Levenberg-Marquardt subproblem (5) with the particular perturbation $p(s) := \hat{p}(s)$.

Lemma 2.3.2 *There are $\epsilon > 0$ and $c_{\hat{p}} > 0$ so that*

$$\|\hat{p}(s)\| \leq c_{\hat{p}}\|H(s)\|^2 \quad \textit{for all } s \in \mathcal{B}(y^*, \epsilon). \tag{2.62}$$

Proof: First note that, due to Assumption 2.3.1 and Lemma 2.3.1, Assumption 2.2.2 is satisfied for $z^* := y^*$, $C := \omega_Y$ and $\delta := \frac{1}{2}\delta_Y$. As defined at the beginning of this section, $z_0(s)$ denotes the solution of the Levenberg-Marquardt subproblem (2.6) with perturbation $p(s) := 0$. Thus, for such subproblems Assumption 2.2.3 holds with $c_p = 0$. Therefore, Lemmas 2.2.1-2.2.4 from Section 2.2 are valid and can be exploited later within this proof. In particular, the constants κ, $\hat{C}$ and $\hat{\delta}$ are given by Lemmas 2.2.3 and 2.2.4. Now, let $\epsilon > 0$ be given by

$$\epsilon := \min\left\{\hat{\delta}, \frac{1}{4(\kappa_0 + 1)}\delta_Y\right\}. \tag{2.63}$$

Since $Y^* \subseteq Y$ and with π_Y according to the definition at the beginning of this section, we have

$$\|\pi_Y(z_0(s)) - z_0(s)\| \leq \text{dist}\,[z_0(s), Y^*]. \tag{2.64}$$

For any $s \in \mathcal{B}(y^*, \epsilon)$, Lemma 2.2.3 and (2.63) ensure

$$\|z_0(s) - y^*\| \leq \|z_0(s) - s\| + \|s - y^*\| \leq \kappa_0 \text{dist}\,[s, Z] + \frac{1}{4}\delta_Y \leq \frac{1}{2}\delta_Y. \tag{2.65}$$

This together with Lemma 2.3.1 and (2.64) yields

$$\|\pi_Y(z_0(s)) - z_0(s)\| \leq \text{dist}\,[z_0(s), Y^*] = \text{dist}\,[z_0(s), Z].$$

Taking into account Lemma 2.2.4, (2.63), and Assumption 2.2.2 (which is fulfilled according to the first lines of this proof) it follows that

$$\|\pi_Y(z_0(s)) - z_0(s)\| \leq \text{dist}\,[z_0(s), Z] \leq \hat{C}\text{dist}\,[s, Z]^2 \leq \frac{\hat{C}}{\omega_Y^2}\|H(s)\|^2$$

is satisfied for all $s \in \mathcal{B}(y^*, \epsilon)$. Combining this with (2.19) and (2.20) of Lemma 2.2.2 and with the definition (2.13) of α, we obtain

$$\|\hat{p}(s)\| \leq \left(\|\nabla H(s)\nabla H(s)^\top\| + \nabla H(s)\|\right)\frac{\hat{C}}{\omega_Y^2}\|H(s)\|^2 \leq c_{\hat{p}}\|H(s)\|^2$$

for all $s \in \mathcal{B}(y^*, \epsilon)$ with

$$c_{\hat{p}} := \frac{\hat{C}}{\omega_Y^2}(L^2 + L\epsilon).$$

△

Now, the convergence theorem for the projected robust Levenberg-Marquardt method can be given.

Theorem 2.3.1 *Let $\{y^k\}$ be a sequence generated by the projected Levenberg-Marquardt method (2.59). Then, there is $\tilde{\delta} > 0$ so that $y^0 \in \mathcal{B}(y^*, \tilde{\delta})$ implies that $\{y^k\}$ converges Q-quadratically to some $\hat{y} \in Y^*$.*

Proof: With $z^* := y^*, \delta := \epsilon$ (according to (2.63)), $\omega_H := \frac{1}{2}\omega_Y$ (from Assumption 2.3.1) it can easily be verified that Assumptions 2.2.2 and 2.2.3 are fulfilled, just use Assumption 2.3.1, Lemmas 2.3.1 and 2.3.2, and the definition of ϵ in (2.63). Therefore, the theorem is an immediate consequence of Theorem 2.2.1. $\triangle$

2.4 Computational Results

We have applied an inexact version of the Levenberg-Marquardt method to the four test problems given in [10]. It will turn out that a larger level of inexactness is numerically more efficient. This underlines the value of the theory presented in the chapter.

The subproblems of the Levenberg-Marquardt method are solved by the conjugate gradient (CG) method up to a certain accuracy level. Given an iterate $s := z^0$ the CG method is stopped if a vector z^1 is computed that solves the inexact LM subproblem (2.5), where

$$\|p(s)\| \leq \|H(s)\|^{\zeta} \|F(s)\|^1 \tag{2.66}$$

is required with some positive integer ζ. As detailed in Section 2.2 it is known that, under Assumptions 2.4 and 2.5, $\|F(s)\| = O(\|H(s)\|)$ holds in any ball around z^*. Therefore, (2.66) means

$$\|p(s)\| \leq O\left(\|H(s)\|^{\zeta+1}\right). \tag{2.67}$$

For the numerical tests in [10] the stopping rule (2.67) was used for $\zeta = 2$ together with $\alpha(s) = \|H(s)\|$ (regardless of modifications to deal with iterates that are farther from a solution which we will not encounter in our local setting). Interestingly, the convergence analysis in [10] does not guarantee a quadratic convergence for this choice of ζ and $\alpha(s)$.

In our numerical tests we always use $\alpha(s)$ according to (2.13) and choose ζ from $\{1, 2, 3\}$. Thus, one can easily estimate the level of inexactness according to (2.67). In particular, for $\zeta = 1$ we have $\|p(s)\| \leq O(\|H(s)\|^2)$ which due to the results in Section 2.2 provides Q-quadratic convergence. For $\zeta = 2$ we have $\|p(s)\| \leq O(\|H(s)\|^3)$ which meets the theoretical setting for quadratic convergence analyzed in [23] and the test setting in [10]. Finally, $\zeta = 3$ yields

$\|p(s)\| \le O(\|H(s)\|^4)$ and corresponds to the conditions for quadratic convergence in [10] except the choice of $\alpha(s)$.

We only report results on Problem 1 of the four test problems in [10] since the results for the remaining three are similar. The mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ defining Problem 1 (for n even) is given by

$$H_i(z) = \begin{cases} \sqrt{i}\exp\left(z_i + z_{i+1}\right)/n - \sqrt{i} & \text{if} \quad \bmod (i,2) = 1; \\ \sqrt{i}(z_{i-1} + z_i)(z_{i-1} + z_i - 1) & \text{otherwise}, \end{cases}$$

As in [10] we set n equal to 1000. The starting points z^0 for the inexact LM method were generated as

$$z^0 := z^* + \Delta \frac{u}{\|u\|}, \tag{2.68}$$

where $z^* := (2, -2, 2, -2, \ldots, 2, -2)^\top$ is a nonisolated solution of $H(z) = 0$ and $u \in \mathbb{R}^n$ is chosen randomly from an n-dimensional uniform distribution in the cube $\{z \in \mathbb{R}^n | -1 \le z_i \le 1, i = 1, \ldots, n\}$. Several values of Δ (like $10^{-3}, \ldots, 10^{-6}$ in Table 2.2) are used to explore the behavior of the inexact Levenberg-Marquardt method for starting points in larger and smaller neighborhoods of z^*. For each Δ we generated 10 starting points according to (2.68). Table 2.2 shows the average number of Conjugate Gradient iterations (#CG) required for computing z^1 by one step of the inexact Levenberg-Marquardt method for different levels of accuracy within the stopping rule (2.66). In addition, for each Δ and each accuracy level, Table 2.2 shows the average of the experimental convergence rates E defined as solution of

$$\|H(z^1)\| = \|H(z^0)\|^E.$$

A reasonable comparison of the efficiency of the inexact Levenberg-Marquardt method for different accuracy levels is now possible by comparing the average ratios #CG/ln(E) in the last three columns of Table 2.2. These ratios tell us how many CG iterations were required in average to obtain one more digit of accuracy. A related efficiency measure was introduced by Ostrowski, see [54, Chapter 3, §11]. The last three columns of Table 2.2 show that the largest level of inexactness ($\|p(s)\| \le O(\|H(s)\|^2)$ for $\zeta = 1$) requires less conjugate gradient iterations per digit of accuracy than the smaller levels ($\zeta = 2, 3$) do. Our numerical results for the other problems in [10] show a similar behavior. Moreover, the behavior did not change much if the dimension of z is varied (at least up to $n = 5000$). It is also noted that using $\alpha(s) = \|H(s)\|^2$ (or even lesser values) instead of $\alpha(s) = \|H(s)\|$ did not worsen the behavior. It might be interesting to figure out the reason for this observation. In contrast to this, if a direct method is used for solving the Levenberg-Marquardt subproblem the magnitude of $\alpha(s)$ turns out to be crucial for achieving a reasonable accuracy;

the smaller the value of $\alpha(s)$ is the earlier the linear algebra of the direct method breaks down and the earlier the Levenberg-Marquardt method stops.

Table 2.2: Numerical results for one step of the inexact Levenberg-Marquardt method

Δ	#CG averaged			E averaged			#CG/$ln(E)$		
	$\zeta = 1$	$\zeta = 2$	$\zeta = 3$	$\zeta = 1$	$\zeta = 2$	$\zeta = 3$	$\zeta = 1$	$\zeta = 2$	$\zeta = 3$
10^{-3}	6.0	30.4	70.7	1.89	2.70	3.47	9.4	30.6	56.9
10^{-4}	39.4	91.6	121.1	2.20	2.92	2.92	49.9	85.4	112.9
10^{-5}	82.6	124.4	154.2	2.45	2.67	2.67	92.0	126.6	157.0
10^{-6}	106.8	149.0	176.1	2.51	2.53	2.53	115.8	160.8	190.0

Using a level of inexactness $\|p(s)\|$ larger than $O(\|H(s)\|^2)$ will destroy the local quadratic convergence, see (2.8) in Section 2.1. The same will happen if the regularization parameter $\alpha(s)$ is larger than $O(\|H(s)\|)$. However, allowing such a larger value of $\|p(s)\|$ might be computationally favorable since less CG iterations are needed per Levenberg-Marquardt subproblem. To explore this we applied the inexact Levenberg-Marquardt method under the same circumstances as before but with $\zeta = 0.5$ in the stopping rule (2.66) and with $\alpha(s) = \|H(s)\|^{0.5}$. Again, for each value of Δ, 10 starting points were randomly generated. The results can be found in Table 2.3, their meaning is the same as in Table 2.2. On the one hand, the experimental convergence rate E is now less than 2 and corresponds to the theoretical rate of 1.5 predicted by Theorem 2.5.1, see Section 2.5. On the other hand, the computational expense per digit of accuracy (i.e., the ratio #CG/$\ln(E)$) is even smaller than in Table 2.2. Similar results were obtained for the other test problems in [10].

2.5 Discussion

The local smoothness assumptions we use and those in [23] are the same but we were able to provide a significantly larger inexactness level. Slightly weaker smoothness assumptions are employed in [10]. However, the inexactness level for quadratic convergence is even smaller than in [23]. In general, it might be a fruitful and challenging task to weaken smoothness conditions needed for analyzing the local behavior of Levenberg-Marquardt methods. We discuss some weaker smoothness condition in Chapter 5.

We can extend the notion of robust Levenberg-Marquardt methods if, instead of a quadratic convergence rate, a desired rate $\tau \in (1, 2]$ is fixed. Then, we speak of a robust Levenberg-Marquardt method for the rate τ if the exponent $a = \tau - 1$

Table 2.3: Numerical results for convergence rate of 1.5

Δ	#CG (averaged)	E (averaged)	#CG/$ln(E)$ (averaged)
10^{-3}	2.0	1.53	5.2
10^{-4}	11.8	1.86	21.2
10^{-5}	43.6	1.91	66.7
10^{-6}	77.9	2.03	108.7
10^{-7}	97.9	1.98	142.3
10^{-8}	114.1	1.90	177.0
10^{-9}	129.4	1.69	245.8
10^{-10}	141.8	1.49	351.4

is used in the definition of the regularization parameter

$$\alpha(s) = \begin{cases} \|H(s)\|^a & \text{if } s \in \mathbb{R}^n \setminus Z; \\ 1 & \text{if } s \in Z. \end{cases} \tag{2.69}$$

since for values of a smaller than $\tau - 1$ the convergence rate of an (unperturbed) Levenberg-Marquardt method becomes less than τ . Then, following the lines in Section 2.2, Theorem 2.2.1 can be extended accordingly to show the maximal possible level of inexactness. For the sake of completeness we now present this result but omit its proof. In Section 2.4 the practical value of the following theorem is exemplified.

Theorem 2.5.1 *Let Assumptions 2.2.1 and 2.2.2 be satisfied. For some* $\tau \in (1, 2]$ *suppose that* $\alpha : \mathbb{R}^n \to (0, \infty)$ *is given by (2.69) with* $a := \tau - 1$*. Moreover, let the sequence* $\{z^k\}$ *be generated by the inexact Levenberg-Marquardt method (2.6). Then* $\{z^k\}$ *converges to some* $\hat{z} \in Z$ *with the Q-order* τ *if there are* $\tilde{\delta} > 0$ *and* $c_p > 0$ *so that* $z^0 \in \mathcal{B}(z^*, \tilde{\delta})$ *and*

$$\|p(s)\| \leq c_p \|H(s)\|^\tau \quad \textit{for all } s \in \mathcal{B}(z^*, \delta).$$

With reference to Section 2.3 it is easily seen that for inexact projected robust Levenberg-Marquardt methods a convergence result similar to that of Theorem 2.3.1 must hold. This might be of particular interest for projections onto general convex sets where inexactness can be caused by numerical errors in the projection.

Chapter 3

A Levenberg-Marquardt Algorithm for Multi-objective Optimization

3.1 Introduction

A multi-objective optimization problem (MOP) is characterized by multiple and (usually) conflicting objective functions $F_1, \ldots, F_m : \mathbb{R}^n \to \mathbb{R}$, all of them need to be minimized. However usually there does not exists a point that is a simultaneous minimizer of all the objective functions. Such problems commonly occur in engineering design, management sciences among others (see for example [62]). In this chapter we will be concerned with unconstrained MOP's. To characterize a desirable point of an unconstrained MOP the concept of Pareto-optimality is often used.

Definition 3.1.1 *A point $x^* \in \mathbb{R}^n$ is called Pareto-optimal if no $y \in \mathbb{R}^n$ exists so that $F_i(y) \leq F_i(x^*)$ for all $i = 1, \ldots, m$ with strict inequality for at least one index i. If, as a slightly weaker requirement, there is no $y \in \mathbb{R}^n$ so that $F_i(y) < F_i(x^*)$ for all $i = 1, \ldots, m$, the point x^* is called weakly Pareto-optimal.*

Another well known optimality notion in multi-objective optimization is that of a properly Pareto-optimal solution. It is a slightly restricted definition of Pareto-optimality that eliminates solutions of anomalous types (in a certain sense). One such widely used definition of a properly Pareto-optimal solution is as follows (Geoffrion [34]).

Definition 3.1.2 *A point $x^* \in \mathbb{R}^n$ is called properly Pareto-optimal if x^* is Pareto-optimal and if there exists an $M > 0$ such that for each $(x, i) \in \mathbb{R}^n \times$*

$\{1, 2, \dots, m\}$ *satisfying* $F_i(x) < F_i(x^*)$ *there exists an index* $j \neq i$ *with* $F_j(x) > F_j(x^*)$ *and*

$$\frac{F_i(x^*) - F_i(x)}{F_j(x) - F_j(x^*)} \leq M.$$

Let X_p, X_w and X_{pp} denote the set of all Pareto-optimal, weakly Pareto-optimal and properly Pareto-optimal points of (MOP), respectively. It is well known that (see [18])

$$X_{pp} \subseteq X_p \subseteq X_w. \tag{3.1}$$

We call the set $\{(F_1(x), F_2(x), \dots, F_m(x))^\top \in \mathbb{R}^m | x \in X_p\}$ the *efficient front*.

Under appropriate differentiability assumptions on $F := (F_1, \dots, F_m)^\top$ any (weakly, properly) Pareto-optimal point can be characterized by necessary optimality conditions. More in detail, let $\mathcal{I}$ denote the index set $\{1, 2, \cdots, m\}$ and for any $\varepsilon \in [0, \frac{1}{m}]$, let

$$\Lambda_\varepsilon := \left\{ \lambda \in \mathbb{R}^m | \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq \varepsilon, i \in \mathcal{I} \right\}. \tag{3.2}$$

denote the set of (nonempty) weight vectors. Then, x^* is called Pareto-critical if a weight vector $\lambda^* \in \Lambda_0$ exists so that $\nabla F(x^*)\lambda^* = 0$. The set of all Pareto-critical points will be denoted by X_{pc}. It is well known that each weakly Pareto-optimal point is Pareto-critical, i.e.,

$$X_w \subseteq X_{pc}. \tag{3.3}$$

From (3.1) and (3.3) we obtain that each Pareto-optimal or properly Pareto-optimal point is also Pareto-critical. Moreover, if $F_1, \dots, F_m$ are convex functions any Pareto-critical point is also weakly Pareto-optimal. If under this convexity assumption, there exists a weight vector $\lambda^* \in \Lambda_\varepsilon$ with $\varepsilon > 0$ so that $\nabla F(x^*)\lambda^* = 0$, then x^* is properly Pareto-optimal (and by (3.1) is also Pareto-optimal). Details about these statements can be found in [18, Section 3.3], for example.

To determine Pareto-critical points, we suggest in this chapter to directly solve the following constraint system of equations

$$H(z) := \nabla F(x)\lambda = 0, \quad \lambda \in \Lambda_0, \tag{3.4}$$

where $z = (x, \lambda)$. Usually system (3.4) has infinitely many and nonisolated solutions. Therefore, for solving (3.4) a Levenberg-Marquardt type method is chosen. Under appropriate assumptions such methods are known to possess a fast local convergence even in cases where nonisolated solutions occur. Regarding these assumptions the most crucial question is under which conditions $\|H\|$ is (in some local sense) an error bound for the solution set

$$Z := \{z \in \mathbb{R}^{n+m} \,|\, H(z) = 0,\, \lambda \in \Lambda_0\}$$

of system (3.4). We will derive such conditions in Section 3.2. Then, in Section 3.3, the Levenberg-Marquardt method applied to the constraint system (3.4) is described and its global and local convergence behavior is analyzed. The globalization is done by means of an Armijo type line-search and solely uses the search directions generated by the Levenberg-Marquardt subproblems. This technique and corresponding convergence results can also be applied to the solution of other constraint systems of equations. In Section 3.4 we derive stronger results under additional convexity assumptions. Some numerical results are presented in Section 3.5. Finally, globalization techniques and desirable modifications are discussed in Section 3.6.

For ease of use we define the sets

$$\Omega_\varepsilon := \{z = (x, \lambda) \in \mathbb{R}^{n+m} \,|\, \lambda \in \Lambda_\varepsilon\} \quad \text{and} \quad X(\lambda) := \{x \in \mathbb{R}^n \,|\, (x, \lambda) \in Z\}.$$

Most of the results in this chapter can be found in [28].

3.2 Existence of a Local Error bound

First of all, we state the following assumption.

Blanket Assumption for Chapter 3: *For all $i \in \mathcal{I}$, the function $F_i : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable. Moreover, the function $\alpha : \mathbb{R}^{n+m} \to (0, \infty)$ is given by*

$$\alpha(z) = \begin{cases} \|H(z)\|^2 & \text{if } z \in \mathbb{R}^{n+m} \setminus Z; \\ 1 & \text{if } z \in Z. \end{cases} \tag{3.5}$$

As a basic ingredient for the Levenberg-Marquardt method and its fast convergence we first provide conditions under which a certain local error bound for the solution set Z of system (3.4) holds. To this end let $z^* := (x^*, \lambda^*) \in Z$ be fixed throughout this section.

Definition 3.2.1 *If there are $\delta > 0$ and $C > 0$ so that*

$$\|H(z)\| \geq C \text{dist}\,[z, Z] \qquad \text{for all } z \in \mathcal{B}(z^*, \delta) \cap \Omega_0, \tag{3.6}$$

then we say that $\|H\|$ has the error bound property around z^*.

The following two lemmas provide sufficient conditions for this.

Lemma 3.2.1 *If, for some* $\delta_0 > 0$*, there is a continuous function* $\xi : \mathcal{B}(\lambda^*, \delta_0) \cap \Lambda_0 \to \mathbb{R}^n$ *so that*

(i) $\xi(\lambda) \in X(\lambda)$ *for all* $\lambda \in \mathcal{B}(\lambda^*, \delta_0)$*,*

(ii) $\xi(\lambda^*) = x^*$*, and,*

(iii) for some $\delta_1 > 0$ *and* $C > 0$*, it holds that*

$$\|H(x, \lambda)\| \geq C\mathrm{dist}\,[x, X(\lambda)]$$

for all (x, λ) *satisfying* $\lambda \in \mathcal{B}(\lambda^*, \delta_0) \cap \Lambda_0$ *and* $x \in \mathcal{B}(\xi(\lambda), \delta_1)$

then $\|H\|$ *has the error bound property around* z^**.*

Proof: Choose any $x \in \mathcal{B}(x^*, \delta_1/2)$. Then, by the continuity of ξ and condition (ii) in Lemma 3.2.1, there is $\tilde{\delta} \in (0, \delta_0]$ such that

$$\|\xi(\lambda) - x^*\| \leq \delta_1/2 \quad \text{for all } \lambda \in \mathcal{B}(\lambda^*, \tilde{\delta}) \cap \Lambda_0.$$

Using the triangle inequality,

$$\|x - \xi(\lambda)\| \leq \|x - x^*\| + \|x^* - \xi(\lambda)\| \leq \delta_1$$

follows for all $\lambda \in \mathcal{B}(\lambda^*, \tilde{\delta}) \cap \Lambda_0$.

Now, for $\delta \in (0, \min\{\tilde{\delta}, \delta_1/2\}]$, condition (iii) in Lemma 3.2.1 yields

$$\|H(z)\| = \|H(x, \lambda)\| \geq C\mathrm{dist}\,[x, X(\lambda)] = C\mathrm{dist}\,[z, (X(\lambda), \lambda)] \geq C\mathrm{dist}\,[z, Z]$$

for all $z \in \mathcal{B}(z^*, \delta) \cap \Omega$, where, for any $\lambda \in \Lambda_0$, $(X(\lambda), \lambda)$ denotes the set $\{(x, \lambda) \,|\, x \in X(\lambda)\}$ which is obviously contained in Z. △

Remark 3.2.1 *Conditions (i) and (ii) in Lemma 3.2.1 seem quite weak for a sufficiently smooth MOP. Although condition (i) is not explicitly needed within the proof of Lemma 3.2.1 it is required if condition (iii) comes into play. Moreover, we can easily verify that most of the continuous problems in [12; 39] satisfy these conditions. Let us fix any* λ *in* $\mathcal{B}(\lambda^*, \delta_0) \cap \Lambda_0$ *then condition (iii) requires that* $\|H(\cdot, \lambda)\|$ *is a local error bound for the set* $X(\lambda)$ *around* $\xi(\lambda)$*. This is only reasonable if* $\xi(\lambda)$ *belongs to* $X(\lambda)$ *which is guaranteed by condition (i). For any*

$\lambda \in \Lambda_0$ we have that $X(\lambda) = \{x \in \mathbb{R}^n \,|\, \nabla F(x)\lambda = 0\}$ is exactly the set of Karush-Kuhn-Tucker points of the scalarized (single objective) optimization problem

$$\min f_\lambda(x) := \sum_{i=1}^{m} \lambda_i F_i(x).$$

For single objective optimization problems a local error bound condition for the set of KKT points is among the weakest regularity conditions used for designing algorithms with fast local convergence properties. Therefore, condition (iii) in Lemma 3.2.1 does not seem too strong.

Lemma 3.2.2 *Suppose that $\nabla^2 F_1, \ldots, \nabla^2 F_m$ are Lipschitz continuous in a neighborhood of x^*. If $\nabla^2 f_{\lambda^*}(x^*)$ is nonsingular then $\|H\|$ has the error bound property around z^*.*

Proof: By assumption we have that $\nabla_x H(z^*) = \nabla^2 f_{\lambda^*}(x^*)$ is nonsingular. Therefore, the result directly follows from Lemma 2 in [20]. △

3.3 Convergence of a Constrained Levenberg-Marquardt Method

In [43] a Levenberg-Marquardt type method is applied to a constraint system of equations the first time. There, to obtain global convergence, a hybrid technique is suggested that combines Levenberg-Marquardt steps with projected (damped) gradient steps, where the gradient of a merit function is meant. Here, instead of this technique, we show the same global convergence result for an Armijo type line-search applied to the Levenberg-Marquardt steps itself. Thus, only Levenberg-Marquardt steps are required. Lemma 3.3.1 and the Theorems 3.3.1 and 3.3.2 do not rely on the particular definitions of H and Ω_ε so that they can be easily extended to a more general setting. The merit function $\phi : \mathbb{R}^{n+m} \to [0, \infty)$ with

$$\phi(z) := \frac{1}{2}\|H(z)\|^2$$

will be used within the Armijo stepsize procedure.

The algorithm is formally stated on page 30.

Lemma 3.3.1 *Algorithm 3.1 generates a well defined sequence $\{z^k\} \subset \Omega_0$.*

Proof: Obviously, step (S1) is well defined since Ω_0 is nonempty and closed. Assume that Algorithm 3.1 has determined an iterate $z^k \in \Omega_0$ with $H(z^k) \neq 0$.

Algorithm 3.1

(S1) Choose $z^0 = (x^0, \lambda^0) \in \Omega_0$, $\beta \in (0,1), \kappa \in (0,1/2)$, and set $k = 0$.

(S2) Determine d^k as the solution of

$$\min \theta_k(d) := \|H(z^k) + \nabla H(z^k)^\top d\|^2 + \alpha(z^k)\|d\|^2 \quad \text{s.t.} \quad z^k + d \in \Omega_0.$$

(S3) Determine t_k as the largest $t \in \{1, \beta, \beta^2, \ldots\}$ satisfying

$$\phi(z^k + td^k) - \phi(z^k) \le -\kappa t \alpha(z^k)\|d^k\|^2$$

(S4) Set $z^{k+1} := z^k + t_k d^k$, $k := k+1$, and go to (S2)

Then, the minimization problem in step (S2) has a unique solution d^k since θ is a strongly convex quadratic function and Ω_0 is nonempty, convex and closed. This implies $\theta_k(d^k) \le \theta_k(0)$. With the definition of θ_k and $\nabla\phi(z) = \nabla H(z)H(z)$ we see that this is equivalent to

$$2\phi(z^k) + 2\nabla\phi(z^k)^\top d^k + (d^k)^\top(\nabla H(z^k)\nabla H(z^k)^\top + \alpha_k I)d^k \le 2\phi(z^k).$$

Hence,

$$\nabla\phi(z^k)^\top d^k \le -\frac{1}{2}\alpha(z^k)\|d^k\|^2 \tag{3.7}$$

follows. Therefore, a Taylor expansion of ϕ at z^k shows the existence of $t_k > 0$ according to step (S3). Note that the merit function ϕ is continuously differentiable because F is a twice continuously differentiable function. Since Ω_0 is convex z^{k+1} belongs Ω_0. Thus, by induction, it can be verified that $\{z^k\}$ is well defined and lies in Ω_0. △

Theorem 3.3.1 *Any accumulation point generated by Algorithm 3.1 is a stationary point of the minimization problem*

$$\min \phi(z) \qquad \text{s.t. } z \in \Omega_0. \tag{3.8}$$

Proof: Suppose that there is a subsequence $\{z^k\}_{k\in K}$ converging to a non-stationary point $\bar{z}$ of (3.8). Then, $H(\bar{z}) \neq 0$ and, by the definition of $\alpha(z^k)$ in (S2), $\bar{\alpha} > 0$ exists so that $\alpha(z^k) \ge \bar{\alpha}$ for all $k \in K$ sufficiently large. With (3.7) and the continuity of $\nabla\phi$ this implies the boundedness of the sequence $\{d^k\}_K$. By the non-stationarity of $\bar{z}$ there is some $\tilde{z} \in \Omega_0$ with $\nabla\phi(\bar{z})^\top(\tilde{z} - \bar{z}) < 0$. Thus, by continuity arguments, $\varkappa > 0$ exists so that

$$-\nabla\phi(z^k)^\top(\tilde{d}^k) \geq \varkappa > 0$$

for all $k \in K$ sufficiently large, where $\tilde{d}^k := \tilde{z} - z^k$. By $\tilde{z} \in \Omega_0$ it follows that $\theta_k(d^k) \leq \theta_k(\tau\tilde{d}^k)$ for all $\tau \in [0,1]$. This inequality gives

$$2\nabla\phi(z^k)^\top d^k \leq -2\tau\varkappa + \tau^2(\tilde{d}^k)^\top(\nabla H(z^k)\nabla H(z^k)^\top + \alpha(z^k)I)\tilde{d}^k$$

for all $\tau \in [0,1]$ and $k \in K$. Due to the boundedness of $\{\|\nabla H(z^k)\nabla H(z^k)^\top + \alpha(z^k)I\|\}_K$ there is some $\bar{\varkappa} > 0$ so that $\nabla\phi(z^k)^\top d^k \leq -\bar{\varkappa}$ for all $k \in K$. Therefore, $\liminf_{k\in K, k\to\infty} \|d^k\| > 0$ follows from the boundedness of $\{\nabla\phi(z^k)\}_K$. Since $\{\phi(z^k)\}$ is monotonically decreasing we obtain from (3.7) by standard arguments that $\lim_{k\to\infty}\phi(z^k) = -\infty$. This is a contradiction. Hence $\bar{z}$ is a stationary point of (3.8). △

Lemma 3.3.2 *Let $z^* = (x^*, \lambda^*)$ be a stationary point of (3.8). If $\nabla^2 f_{\lambda^*}(x^*)$ is nonsingular then x^* is Pareto-critical. Moreover, if $x^* \in X_{pc}$ then $z^* \in Z$.*

Proof: Due to the stationarity of z^* and the convexity of Ω_0 we have that $\nabla\phi(z^*)^\top(z - z^*) \geq 0$ for all $z \in \Omega_0$.

Since

$$\begin{aligned}\nabla\phi(z^*)^\top(z - z^*) &= H(z^*)^\top\nabla H(z^*)^\top(z - z^*)\\ &= H(z^*)^\top\left(\nabla^2 f_{\lambda^*}(x^*)(x - x^*) + \nabla F(x^*)(\lambda - \lambda^*)\right)\end{aligned} \tag{3.9}$$

we obtain $H(z^*)^\top\nabla^2 f_{\lambda^*}(x^*)(x-x^*) \geq 0$ for all $x \in \mathbb{R}^n$ because $z = (x, \lambda^*)$ belongs to Ω_0 for all $x \in \mathbb{R}^n$ and can be used in (3.9). Hence, $H(z^*)^\top\nabla^2 f_{\lambda^*}(x^*) = 0$ and, under the nonsingularity condition on $\nabla^2 f_{\lambda^*}(x^*)$, $H(z^*) = 0$ follows.

In a similar way, from (3.9) we obtain

$$H(z^*)^\top(H(x^*, \lambda) - H(z^*)) \geq 0 \tag{3.10}$$

for all $\lambda \in \Lambda_0$ because $z = (x^*, \lambda)$ belongs to Ω_0 for all $\lambda \in \Lambda_0$ and can be used in (3.9). Now if $x^* \in X_{pc}$ then, by definition, there exists a $\tilde{\lambda} \in \Lambda_0$ so that $H(x^*, \tilde{\lambda}) = 0$. Hence using (3.10) we obtain

$$-\|H(z^*)\|^2 = H(z^*)^\top(H(x^*, \tilde{\lambda}) - H(z^*)) \geq 0.$$

This gives $\|H(z^*)\| = 0$ and hence $z^* \in Z$. △

Theorem 3.3.2 *Suppose that $\nabla^2 F_1, \ldots, \nabla^2 F_m$ are Lipschitz continuous in a neighborhood of x^*, that $\|H\|$ has the error bound property around $z^* = (x^*, \lambda^*)$, and that Algorithm 3.1 generates the sequence $\{z^k\}$. Then there is $\hat{\delta} \in (0, \delta)$ so that $z^\ell \in \mathcal{B}(z^*, \hat{\delta})$ implies the Q-quadratic convergence of $\{z^k\}$ to some $\hat{z} \in Z$.*

Proof: In [43] the local Q-quadratic convergence of a constrained Levenberg-Marquardt method is shown in a general setting. If we apply these results to our more special case we find that the assumptions required in [43] are satisfied. Therefore, for $\hat{\delta} > 0$ sufficiently small, $\|z^\ell - z^*\| \leq \hat{\delta}$ implies $\|d^\ell\| = O(\|H(z^\ell)\|)$, $\|H(z^\ell + d^\ell)\| = O(\|H(z^\ell)\|^2)$ and, thus $\phi(z^\ell + d^\ell) = O(\|H(z^\ell)\|^4)$. Combining this with the definitions of $\phi(z^\ell)$ and α_l (see step (S2) of Algorithm 3.1) we obtain, for $\hat{\delta} > 0$ sufficiently small,

$$\phi(z^\ell + d^\ell) - \phi(z^\ell) \leq -\frac{1}{4}\|H(z^\ell)\|^2 \leq -\kappa \alpha_\ell \|d^\ell\|^2.$$

Hence, a step length is accepted in step (S3) of Algorithm 3.1 for any $z^\ell \in \mathcal{B}(z^*, \hat{\delta})$ with $\hat{\delta} > 0$ sufficiently small. The Q-quadratic convergence now follows with Theorem 2.11 in [43]. △

3.4 Results under Convexity Assumptions

Theorem 3.4.1 *Let $z^* = (x^*, \lambda^*)$ be a stationary point of (3.8). Then the following assertions hold:*

a) *If the functions $F_1, \ldots, F_m$ are strongly convex then x^* is weakly Pareto-optimal.*

b) *If the functions $F_1, \ldots, F_m$ are convex and if, for one i, the function F_i is strongly convex and $\lambda_i^* > 0$, then x^* is weakly Pareto-optimal.*

Proof: In both cases a) and b) we have that $\nabla^2 f_{\lambda^*}(x^*) = \sum_{i=1}^m \lambda_i^* \nabla^2 F_i(x^*)$ is nonsingular. Thus, by Lemma 3.3.2, x^* is Pareto-critical. The convexity of $F_1, \ldots, F_m$ then implies that x^* is weakly Pareto-optimal (see [18, Chapter 3.3], for example). △

Lemma 3.4.1 *Suppose that, for some $z^* \in Z$, $\nabla^2 F_1, \ldots, \nabla^2 F_m$ are Lipschitz continuous in a neighborhood of x^*. If the functions $F_1, \ldots, F_m$ are convex and if there is $i \in \{1, \ldots, m\}$ so that F_i is strongly convex and $\lambda_i^* > 0$ then $\|H\|$ has the error bound property around z^*.*

Proof: According to the assumptions the matrix $\nabla^2 f_{\lambda^*}(x^*)$ is positive definite. Thus, Lemma 3.2.2 yields the result. △

Theorem 3.4.2 *If the functions $F_1, \ldots, F_m$ are strongly convex then any sequence $\{z^k\}$ generated by Algorithm 3.1 converges Q-quadratically to a solution $(x^*, \lambda^*) \in Z$ such that x^* is weakly Pareto-optimal.*

Proof: Let $z^0 \in \Omega_0$ be arbitrarily chosen. We first verify that the level set

$$\mathcal{L}_\phi(z^0) := \{z \in \Omega_0 \,|\, \phi(z) \leq \phi(z^0)\}$$

is bounded. To this end suppose that a sequence $\{z^\nu\} \subset \mathcal{L}_\phi(z^0)$ exists with $\lim_{\nu\to\infty} \|z^\nu\| = \infty$. By the definition of Ω_0 and with $z^\nu = (x^\nu, \lambda^\nu)$ this implies $\lim_{\nu\to\infty} \|x^\nu\| = \infty$. Moreover, we can find a subsequence $\{z^\nu\}_K$, an index $i \in \{1, \ldots, m\}$, and $c > 0$ with $\lambda_i^\nu \geq c$ for all $\nu \in K$. The strong convexity of $F_1, \ldots, F_m$ implies $\lim_{\nu\to\infty, \nu\in K} \|H(z^\nu)\| = \infty$. This contradicts to $\{z^\nu\} \subset \mathcal{L}_\phi(z^0)$. Hence, $\mathcal{L}_\phi(z^0)$ is bounded.

The sequence $\{z^k\}$ generated by Algorithm 3.1 belongs to $\mathcal{L}_\phi(z^0)$ and, thus, has at least one accumulation point z^*. Taking into account Theorems 3.3.1 and 3.4.1 a) it follows that $z^* = (x^*, \lambda^*) \in Z$ such that x^* is weakly Pareto-optimal. By Lemma 3.4.1, $\|H\|$ has the error bound property around z^*. Then, Theorem 3.3.2 yields the Q-quadratic convergence of $\{z^k\}$ to z^*. △

Remark 3.4.1 *To further weaken the convexity assumptions used in this section it is possible to replace the sets Λ_0 and Ω_0 throughout the paper by Λ_ε and Ω_ε respectively, for some $\varepsilon \in (0, \frac{1}{m}]$. Then, the results in Sections 3.2, 3.3, and 3.4 remain valid. Moreover, in Theorems 3.4.1 and 3.4.2 the point x^* turns out to be properly Pareto-optimal which (by 3.1) is also Pareto-optimal.*
In addition, by the use of Λ_ε and Ω_ε the condition that $\lambda_i^* > 0$ in part b) of Theorem 3.4.1 is no longer needed (since it is automatically satisfied). The same applies for Lemma 3.4.1. Finally, the strong convexity of all functions $F_1, \ldots, F_m$ in Theorem 3.4.2 can be replaced by their convexity plus the strong convexity of only one *of the functions $F_1, \ldots, F_m$.*

3.5 Computational Results

In this section we solve two unconstrained test problems using Algorithm 3.1. Both of these problems have nonisolated solutions. The first of these is taken from [41] and is known as JOS. It is bi-objective and F is given by:

$$F_1(x) := \frac{1}{50}\sum_{i=1}^{50} x_i^2$$

$$F_2(x) := \frac{1}{50}\sum_{i=1}^{50} (x_i - 2)^2.$$

Both of the objective functions of JOS are strongly convex. The Pareto-optimal set of JOS is the set $\{(a, a, \ldots, a) \in \mathbb{R}^{50} | a \in [0, 2]\}$. We choose 100 starting points from a uniform random distribution from the set

$$\Omega := \{(x, \lambda) \in \mathbb{R}^{52} | \lambda \in \Lambda_0, x \in [-5, 5]^{50}\}.$$

The algorithm parameters β and κ are chosen to be 0.5 and 0.1 respectively. The maximum number of iterations is set to be 100. The results (in the objective space) form the simulation are shown in Figure 3.1. We see that Algorithm 3.1 is able to find multiple Pareto-optimal solutions.

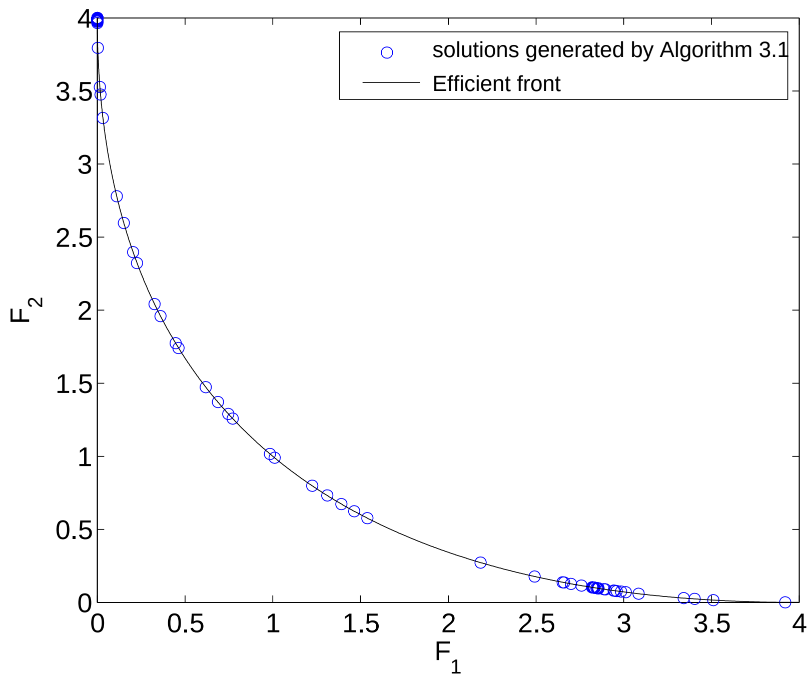

Figure 3.1: Performance of Algorithm 3.1 on the problem JOS.

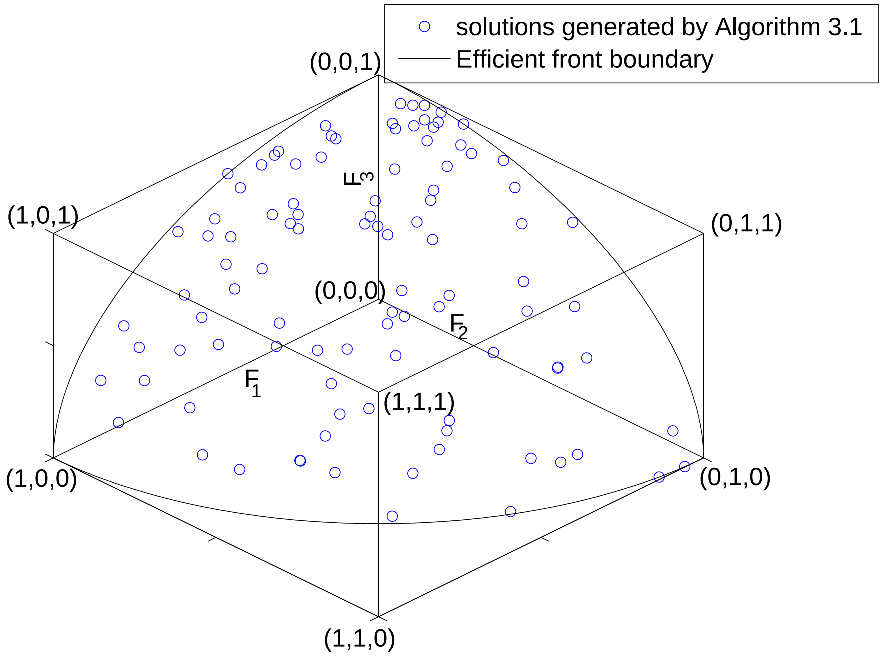

Figure 3.2: Performance of Algorithm 3.1 on the problem DTLZ2.

As a second test problem we have chosen the non-convex, 12 dimensional, 3-objective DTLZ2 problem from [13]. Its objective functions are given as:

$$\begin{aligned} F_1(x) &:= (1+g(x))\cos\left(\frac{x_1\pi}{2}\right)\cos\left(\frac{x_2\pi}{2}\right), \\ F_2(x) &:= (1+g(x))\cos\left(\frac{x_1\pi}{2}\right)\sin\left(\frac{x_2\pi}{2}\right), \\ F_3(x) &:= (1+g(x))\sin\left(\frac{x_1\pi}{2}\right), \end{aligned}$$

with $g(x) := \sum_{i=3}^{12}(x_i - 0.5)^2$.

Its Pareto-optimal set is $\{x \in \mathbb{R}^{12} | x_3 = x_4 = \ldots = x_{12} = 0.5\}$. In the objective space the Pareto-optimal solutions satisfy $F_1^2 + F_2^2 + F_3^2 = 1$. We choose 100 starting points from a uniform random distribution from the set

$$\Omega := \{(x, \lambda) \in \mathbb{R}^{15} | \lambda \in \Lambda_0, x \in [0,1]^{12}\}.$$

The other parameters are chosen in the same way as for the JOS problem. The results (in the objective space) form the simulation of Algorithm 3.1 are shown in Figure 3.2. We see that the algorithm is able to find multiple Pareto-optimal solutions for this non-convex problem.

3.6 Discussion

An interesting aim is to ensure that an accumulation point of Algorithm 3.1 yields a Pareto-optimal point of (MOP) under weaker conditions than the convexity assumptions used in the present chapter. From single objective minimization several techniques are known which avoid or reduce the cases where an accumulation point (generated by some algorithm) is not a local minimizer. Such globalization techniques take into account the objective function in some way. This raises the question whether similar techniques can be designed for Algorithm 3.1. A simple idea to tackle this question is a hybrid method that combines Levenberg-Marquardt subproblems with subproblems that generate directions of simultaneous decrease for all the objective functions $F_1, \ldots, F_m$. Algorithms that generate directions for simultaneous decrease have been suggested in [31; 52]. In a local phase of such a hybrid technique projected Levenberg-Marquardt subproblems (see Chapter 2, [43] and [29]) could be used. This would lead to less expensive subproblems. Another idea of globalizing is the use of (more sophisticated) Levenberg-Marquardt type subproblems that itself lead to a simultaneous decrease. The last topic is thoroughly discussed in the next chapter.

Chapter 4

A Simultaneous Descent Levenberg-Marquardt Algorithm for Multi-objective Optimization

4.1 Introduction

To determine Pareto-critical or (weakly) Pareto-optimal points of an MOP we suggested in Chapter 3 to solve the following constrained system of equations

$$H(z) := \nabla F(x)\lambda = 0, \quad \lambda \in \Lambda_0, \tag{4.1}$$

where $z := (x, \lambda)$ and Λ_0 and F are defined in Chapter 3. In the same chapter (and also in an earlier work [28]), for globalizing the Levenberg-Marquardt method we used a line search technique with $\|H\|^2$ as the merit function. As already discussed in Section 3.6, using such a globalization scheme has some limitations. For example, such an algorithm may converge to a non Pareto-critical point if appropriate assumptions do not hold. To alleviate this behavior, in this chapter objective function values are used for globalization. This might be of use in MOPs having non-convex objective functions.

Here we present a globally convergent Levenberg-Marquardt method for finding a Pareto-critical point. The error bound condition and other assumptions required for local convergence analysis of the method are discussed. In this method, each iteration provides a decrease in all the objective function values. This is achieved locally by using a suitably modified form of a constrained Levenberg-Marquardt method (presented in [43]) where now the constraint set changes at each iteration. For globalization our method combines the Levenberg-Marquardt direction and a simultaneous descent direction ([31; 52]). Global convergence to Pareto-optimal points is shown under convexity assumptions. An important feature of our method is that for non-convex problems, convergence to Pareto-critical points can still be shown.

Many methods have been proposed in the literature to find a Pareto-critical point. The most notable among them are the steepest descent based methods in [16; 31; 32; 33; 51; 52; 55]. These methods use gradient information of all the objective functions to find a search direction. However, all of these methods converge only quite slowly (comparable to gradient methods for programs with a single objective). The methods in [28; 30] are the first to present an algorithm with a local Q-quadratic convergence rate. These methods use the Hessian information of all the objective functions. In this chapter we will present another algorithm with a local Q-quadratic convergence rate. It has similarities to the Newton based approach in the recent work [30]. Advantages and disadvantages of both approaches are discussed later in detail.

For the sake of brevity, if the usage is clear, we sometimes use row vectors instead of column vectors. For an $(n+m)$-dimensional vector, say $u \in \mathbb{R}^{n+m}$, we denote by u_x the vector of the first n components and by u_λ the vector of the last m components, respectively. Instead of u we also write (u_x, u_λ). Inequality (also strict) or equality signs between vectors are understood componentwise. The other notations in this chapter are the same as defined in Chapter 3.

Although throughout the rest of this chapter we use Λ_0 and get convergence results for weakly Pareto-optimal points, the corresponding results can be obtained also for properly Pareto-optimal points by replacing the set Λ_0 by Λ_ε throughout with some $\varepsilon \in \left(0, \frac{1}{m}\right]$.

The chapter is divided into six sections, of which this is the first. The next section presents the Levenberg-Marquardt algorithm with simultaneous descent, while its convergence is discussed in Section 4.3. Section 4.4 describes a duality based approach for solving the Levenberg-Marquardt subproblems. Section 4.5 analyzes the algorithm under convexity assumptions. Concluding remarks, comparisons with other methods and extensions to MOPs with a polyhedral ordering cone are presented in the last section.

4.2 The Levenberg-Marquardt Algorithm with Simultaneous Descent

In the following we shall investigate a constrained Levenberg-Marquardt algorithm with simultaneous descent for finding a Pareto-critical point of an unconstrained MOP. Before we do so, we state the following smoothness condition on the objective functions.

Assumption 4.2.1 *For all $i \in \mathcal{I}$, the function $F_i : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and $\nabla^2 F_i : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is locally Lipschitz continuous.*

Next, we state a condition on the regularization parameter to be used in the Levenberg-Marquardt method.

Assumption 4.2.2 *The function $\alpha : \mathbb{R}^{n+m} \to (0, \infty)$ is given by*

$$\alpha(z) = \begin{cases} \|H(z)\|^2 & \text{if } z \in \mathbb{R}^{n+m} \setminus Z; \\ 1 & \text{if } z \in Z. \end{cases} \tag{4.2}$$

For each $x \in \mathbb{R}^n$, let $\nu(x) \in \mathbb{R}^m$ denote a global minimizer of the convex quadratic optimization problem $(QP(x))$

$$\min_{\nu} \|H(x,\nu)\|^2 \quad \text{s.t. } \nu \in \Lambda_0, \tag{4.3}$$

where we recall that Λ_0 is defined in (3.2). As discussed later, we need the optimal value of $(QP(x))$ to bound the search directions of our algorithm. It is shown in [59, Theorem 2.1] that the function $q : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$q(x) := -H(x, \nu(x)) \tag{4.4}$$

is locally Lipschitz continuous. Moreover, the direction $q(x) \in \mathbb{R}^n$ is a descent direction for all the objective functions $F_i, i \in \mathcal{I}$, at x. More in detail,

$$\nabla F_i(x)^\top q(x) < 0, \quad \text{for all } i \in \mathcal{I}. \tag{4.5}$$

A direction which is a descent direction for all the objective functions will be called a simultaneous descent direction.

Remark 4.2.1 *We note that instead of solving $(QP(x))$, the value $\|H(x,\nu(x))\|$ (or $\nu(x)$) can also be found out by solving the following equivalent problem $(\overline{QP}(x))$*

$$\min_{\mathsf{v}} \left\| H\left(x, \left(\mathsf{v}, 1 - \sum_{i=1}^{m-1} \mathsf{v}_i\right)\right)\right\|^2 \quad \textit{s.t. } \mathsf{v} \geq 0, \sum_{i=1}^{m-1} \mathsf{v}_i \leq 1.$$

If $\mathsf{v}(x) \in \mathbb{R}^{m-1}$ is a solution of $(\overline{QP}(x))$ then a solution of $(QP(x))$ can be easily computed by

$$\nu(x) := \left(\mathsf{v}(x), 1 - \sum_{i=1}^{m-1} \mathsf{v}_i(x)\right).$$

For a bi-objective MOP, the objective function of $(\overline{QP}(x))$ *is a quadratic function in* one *variable, and hence an explicit formula can be given for obtaining* $\mathsf{v}(x)$ *and hence,* $\nu(x)$.

Remark 4.2.2 *An analytic expression for the set of all simultaneous descent direction can be found in [6].*

For any $z \in \mathbb{R}^{n+m}$, let $\theta(\cdot, z) : \mathbb{R}^{n+m} \to \mathbb{R}$ be the function defined by

$$\theta(d, z) := \|H(z) + \nabla H(z)^\top d\|^2 + \alpha(z)\|d\|^2. \tag{4.6}$$

The function $\theta(\cdot, z)$ is the quadratic function that is minimized in standard Levenberg-Marquardt methods, see [29; 43; 69], where z is the current iterate. However, as we need to guarantee a simultaneous decrease in all the objective functions we need additional constraints within the Levenberg-Marquardt subproblems. The constraint set $\Omega(z)$ that we choose depends on $z = (x, \lambda)$ and is given by

$$\Omega(z) := \{d := (d_x, d_\lambda) \in \mathbb{R}^{n+m} | \lambda + d_\lambda \in \Lambda_0, \|d_x\| \le \|H(x, \nu(x))\|^{0.9},$$
$$\nabla F_i(x)^\top d_x + \frac{1}{2} d_x^\top \nabla^2 F_i(x) d_x \le -\|H(x, \nu(x))\|^{2.5} \quad \text{for all } i \in \mathcal{I}\}.$$

For any $z \in \mathbb{R}^{n+m}$, let us define the constrained Levenberg-Marquardt subproblem $(P(z))$ as

$$\min_d \theta(d, z) \qquad \text{s.t. } d \in \Omega(z).$$

$(P(z))$ is a *Quadratically Constrained Quadratic Programming problem* (QCQP). Note that $\theta(\cdot, z)$ is a strongly convex function and $\Omega(z)$ is compact. Hence, if $\Omega(z)$ is nonempty and all the objective functions are convex then $\Omega(z)$ is convex and $(P(z))$ has a unique solution. On the other hand if $\Omega(z)$ is nonempty and nonconvex, by the theorem of Weierstrass the minimum value of $\theta(\cdot, z)$ is attained at some point in $\Omega(z)$. Later we use duality results to obtain a sufficient condition for infeasibility of $\Omega(z)$.

Let $d(z)$ denote a global solution of $(P(z))$ for each $z \in \mathbb{R}^{n+m}$ such that $\Omega(z) \neq \emptyset$. Let $\Omega_0 := \mathbb{R}^n \times \Lambda_0$ and $\mathcal{Y} := \{F(x) \in \mathbb{R}^m | x \in \mathbb{R}^n\}$ be the feasible set in the objective space. Moreover, for each $z = (x, \lambda) \in \mathbb{R}^{m+n}$, we define the sets $S(z) \subseteq Z$ and $\mathcal{Y}(x) \subseteq R^m$ (shown in Figure 4.1) by

$$\begin{aligned} S(z) &:= Z \cap (z + \Omega(z)), \text{ and} \\ \mathcal{Y}(x) &:= (F(x) - \mathbb{R}^m_+) \cap \mathcal{Y}. \end{aligned}$$

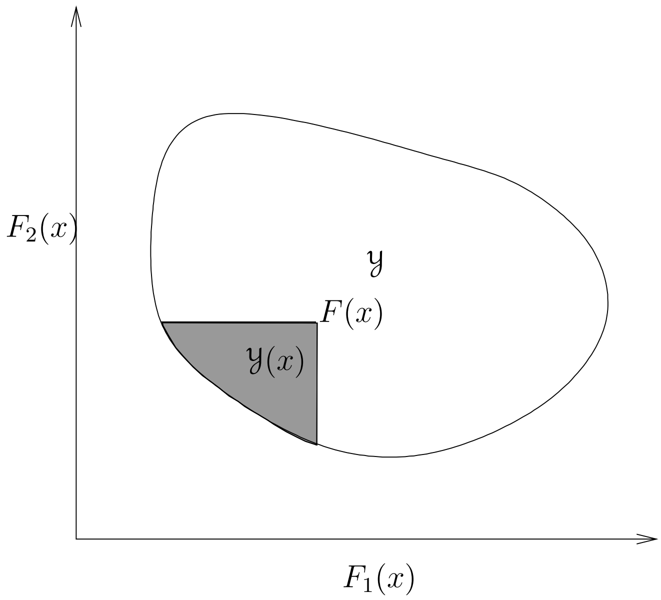

Figure 4.1: Illustration of the set $\mathcal{Y}(x)$ in the bi-objective space.

In the objective space, a simultaneous decrease in all the objective functions implies that we move the point $F(x)$ into the region $\text{int}(\mathcal{Y}(x))$.

The algorithm is formally stated on page 42. A short description of the various steps in the algorithm is as follows. Step (S1) of Algorithm 4.1 is the initialization step while step (S2) is the stopping criterion. Step (S3) finds a Levenberg-Marquardt based local search direction while step (S4) is an Armijo-type line search based on a simultaneous descent direction. Finally, step (S5) updates the current iterate. The parameter ϱ is used to make sure that $d(z^k)$ gives a sufficient decrease in $\|H\|$ if step (S3) is employed. This is needed for global convergence of Algorithm 4.1 and is discussed later in detail.

Lemma 4.2.1 *Algorithm 4.1 generates a well defined sequence* $\{z^{k+1}\} \subset \mathbb{R}^{n+m}$ *with* $F(x^{k+1}) \in \text{int}(\mathcal{Y}(x^k))$ *and* $\lambda^k \in \Lambda_0$ *for all* $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

Proof: Obviously, $\lambda^0 \in \Lambda_0$ and step (S1) is well defined since Ω_0 is nonempty. The algorithm terminates if the current iterate is a solution of (4.1), which is checked in step (S2). For any $k \in \mathbb{N}_0$, if $(P(z^k))$ is infeasible then the algorithm continues with step (S4). On the other hand, if $\Omega(z^k) \neq \emptyset$ then the subproblem $(P(z^k))$ has a solution. Then, if (4.7) or (4.8) is violated the algorithm continues with (S4), else d^k is obtained from step (S3). Thus step (S3) is well defined. The Armijo-type line search in step (S4) is well defined as the direction $q(x^k)$ is a simultaneous descent direction for all the objective functions (see (4.5)

Algorithm 4.1

(S1) Choose $\beta, \kappa \in (0,1)$ and $z^0 = (x^0, \lambda^0) \in \Omega_0$. Set $\varrho := \|H(z^0)\|$ and $k := 0$.

(S2) If $H(x^k, \nu(x^k)) = 0$ then STOP.

(S3) If $(P(z^k))$ is infeasible go to (S4). If

$$\begin{aligned} \|H(z^k + d(z^k))\| &\leq \kappa\varrho \text{ and} && (4.7)\\ F(x^k + d(z^k)_x) &< F(x^k) && (4.8) \end{aligned}$$

then set $d^k := d(z^k)$, $\varrho := \|H(z^k + d^k)\|$ and go to (S5).

(S4) Determine t_k as the largest $t \in \{1, \beta, \beta^2, \ldots\}$ satisfying

$$F\left(x^k + tq(x^k)\right) - F\left(x^k\right) \leq \kappa t \nabla F(x^k)^\top q(x^k).$$

Let $\tilde{x} := x^k + t_k q(x^k)$ and set $d^k := (\tilde{x}, \nu(\tilde{x})) - z^k$.

(S5) Set $z^{k+1} := z^k + d^k$, $k := k+1$ and go to (S2)

and [52; 59] for details). If $\|H(x^k, \nu(x^k))\| \neq 0$ then $q(x^k)$ is also non-zero and $F\left(x^k + t_k q(x^k)\right) - F\left(x^k\right) \in \operatorname{int}(\mathcal{Y}(x^k))$, where the t_k is obtained from Armijo line search in step (S3). Hence, for all $k \in \mathbb{N}_0$, the direction d^k in step (S5) is always such that $F(x^{k+1}) \in \operatorname{int}(\mathcal{Y}(x^k))$ and $\lambda^{k+1} \in \Lambda_0$. This is irrespective of whether d^k is obtained from step (S3) or (S4). △

An efficient way to solve the quadratically constrained quadratic programming problem $(P(z))$ is by using a dual method discussed in Section 4.4. The duality based method also gives information about the infeasibility of $(P(z))$ which is needed in step (S3) of Algorithm 4.1. We note that if all the objective functions are convex then the infeasibility can also be detected (in a computationally efficient way) using techniques from [53].

Lemma 4.2.2 *If $z \in Z$ then $d(z) = 0$. Moreover, if $x \in X_{pc}$ then $d(z)_x = 0$ and vice versa.*

Proof: Let $z = (x, \lambda) \in Z$. Hence, we have that $H(z) = H(x, \nu(x)) = 0$. We also see that $0 \in \Omega(z)$ and that $\theta(0, z) = 0$. Hence 0 is a global minimizer of $(P(z))$. Moreover, any $d \neq 0$ cannot be a global minimizer of $(P(z))$ as then $\theta(d, z) > 0$. Hence $d(z) = 0$ is the only global minimizer of $(P(z))$.

If $d(z)_x = 0$, then from the definition of $\Omega(z)$, x is Pareto-critical. On the other hand if $x \in X_{pc}$, then the constraint

$$\|d_x\| \leq \|H(x, \nu(x))\|^{0.9}$$

implies that $d(z)_x = 0$. △

Remark 4.2.3 *In the approach [30] at each iteration the following subproblem $(P'(x))$ has to be solved*

$$\begin{array}{ll} \min\limits_{(t,s)\in\mathbb{R}\times\mathbb{R}^n} & t \\ s.t. & \nabla F_j(x)^\top s + \frac{1}{2} s^\top \nabla^2 F_j(x) s - t \leq 0 \text{ for all } i \in \mathcal{I}. \end{array}$$

A solution of $(P'(x))$ exists if at least one of the functions F_i is assumed to be strongly convex. However, as the next two examples will show, the above subproblem can be unbounded for both convex and non-convex objective functions. A simple way to alleviate this difficulty is to modify $(P'(x))$ by adding constraints of the form $\|s\|^2 \leq \hat{C}$. However, for such a modification the convergence analysis presented in [30] fails. In contrast to this, in our method $(P(z))$ is a kind of a regularized subproblem. A further discussion of assumptions employed for Q-quadratic convergence rate in our method and in method of [30] is given in Section 4.6.

Example 4.2.1 *Consider the bi-objective, non-convex MOP where the objective functions $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are given by*

$$\begin{array}{rcl} F_1(x) & := & x^3 - x \\ F_2(x) & := & \begin{cases} x & \text{if } x \leq -2; \\ (x+2)^4 + x & \text{otherwise.} \end{cases} \end{array}$$

Both of the objective functions are three times continuously differentiable. Figure 4.2 shows the plot of both objective functions.

Now, at a non Pareto-critical point $x = -1$, we can easily see that $(P'(-1))$ is

$$\begin{array}{ll} \min\limits_{(t,s)\in\mathbb{R}\times\mathbb{R}} & t \\ s.t. & 2s - 3s^2 \leq t \\ & s \leq t, \end{array}$$

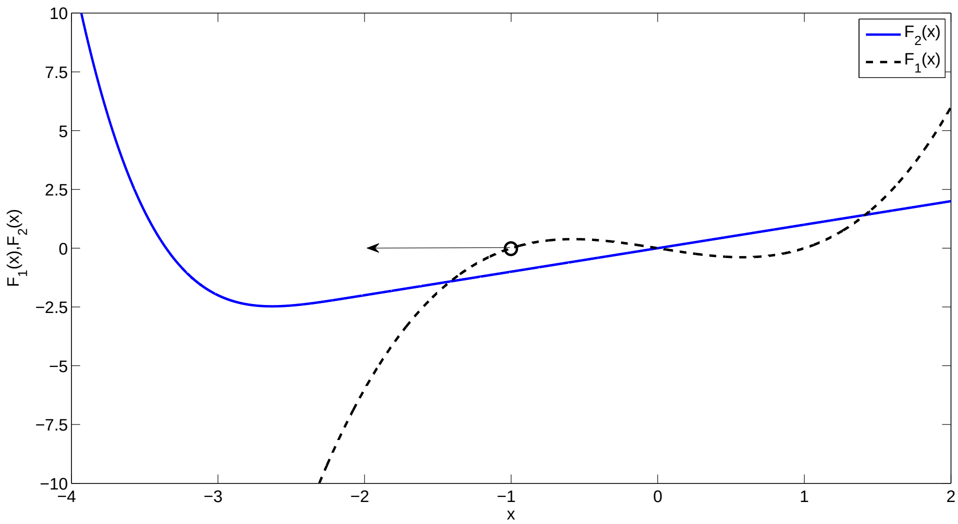

Figure 4.2: Plot of the objective functions F_1 and F_2 in Example 4.2.1. The arrow at $x = -1$ shows the simultaneous descent direction (-1) found out by solving $P(-1, \lambda)$.

and that it is unbounded.

On the other hand, let us take an arbitrary but fixed $\lambda \in \Lambda_0$ *and consider the problem* $P(-1, \lambda)$*. It is easy to see that*

$$\|H(-1, \nu(-1))\| = 1.$$

Hence, $P(-1, \lambda)$ *can be written as*

$$\begin{aligned} \min_{d} \quad & \|H(-1,\lambda) + \nabla H(-1,\lambda)^\top d\|^2 + \alpha(-1,\lambda)\|d\|^2 \\ s.t. \quad & \lambda + d_\lambda \in \Lambda_0 \\ & |d_x| \leq 1 \\ & 2d_x - 3(d_x)^2 \leq -1 \\ & d_x \leq -1 \end{aligned}$$

and has a unique *solution* $d(-1, \lambda)$ *with* $d(-1, \lambda)_x = -1$*. Moreover,* $d(-1, \lambda)_x$ *is a simultaneous descent direction.*

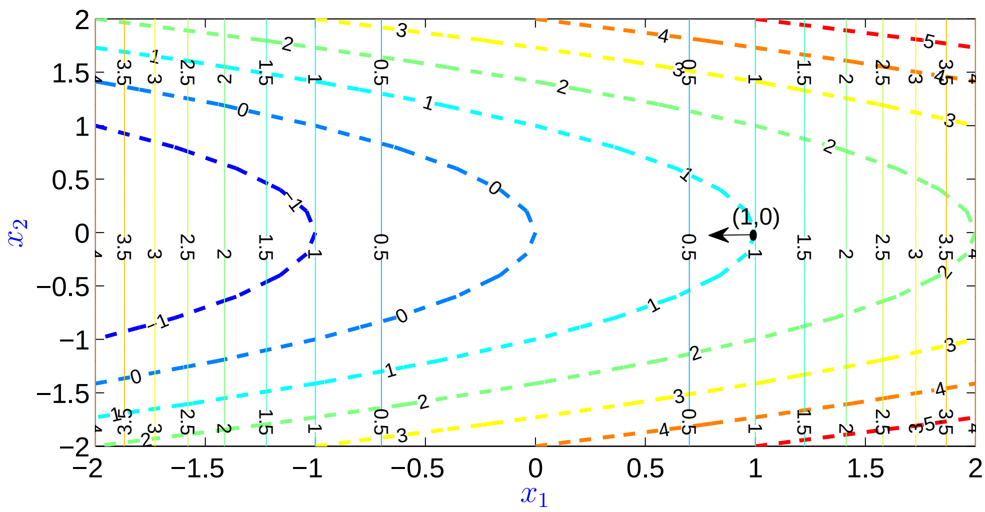

Figure 4.3: Contours of the objective functions F_1 and F_2 in Example 4.2.2. Curved lines are contours of F_2 while straight lines are contours of F_1. The arrow at $(1, 0)$ shows a simultaneous descent direction $(-1, 0)$ found out by solving $P(1, 0, \lambda)$.

Example 4.2.2 *Consider the bi-objective, convex MOP where the objective functions $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ are given by*

$$\begin{aligned} F_1(x) &:= x_1^2 \\ F_2(x) &:= x_1 + x_2^2. \end{aligned}$$

Both of the objective functions are analytic. Figure 4.3 shows the contour plot of both of the objective functions.

Now, at the non Pareto-critical point $(x_1, x_2) = (1, 0)$, we can easily see that $(P'(1, 0))$ is

$$\begin{aligned} \min_{(t,s) \in \mathbb{R} \times \mathbb{R}^2} \quad & t \\ s.t. \quad & 2s_1 \leq t \\ & s_1 + s_2^2 \leq t, \end{aligned}$$

and that it is unbounded.

On the other hand, let us take an arbitrary but fixed $\lambda \in \Lambda_0$ *and consider the problem* $P(1,0,\lambda)$*. It is easy to see that*

$$\|H(1,0,\nu(1,0))\| = 1.$$

Hence, $P(1,0,\lambda)$ *can be written as*

$$\begin{array}{ll} \min\limits_{d} & \|H(1,0,\lambda) + \nabla H(1,0,\lambda)^\top d\|^2 + \alpha(1,0,\lambda)\|d\|^2 \\ s.t. & \lambda + d_\lambda \in \Lambda_0 \\ & \|d_x\| \leq 1 \\ & 2(d_x)_1 \leq -1 \\ & (d_x)_1 + (d_x)_2^2 \leq -1 \end{array}$$

and has a unique *solution* $d(1,0,\lambda)$ *with* $d(1,0,\lambda)_x = (-1,0)$*. Moreover,* $d(1,0,\lambda)_x$ *is a simultaneous descent direction.*

4.3 Convergence of Algorithm 4.1

In this section we provide a convergence analysis for Algorithm 4.1. In step (S3) of Algorithm 4.1 a constrained Levenberg-Marquardt subproblem is solved where the constraint set is changing with each iteration. This is in contrast to all the existing Levenberg-Marquardt methods where the constraint set (if any) is fixed throughout the iterations (see for example [43]). Hence, the existing technique for local convergence analysis cannot be applied to our algorithm.

We now state the assumptions that we will require to hold in this section. The first is an error bound property for the solution set Z. Let $z^* := (x^*, \lambda^*) \in Z$ be fixed. This implicitly means that Z is nonempty.

Assumption 4.3.1 *There are constants* $C, \hat{\delta} > 0$ *so that*

$$\|H(z)\| \geq C \operatorname{dist}[z, Z] \quad \textit{for all } z \in \mathcal{B}(z^*, \hat{\delta}) \cap \Omega_0. \tag{4.9}$$

Sufficient conditions for Assumption 4.3.1 to hold are discussed in Chapter 3. The next assumption is needed in the local convergence analysis of the constrained Levenberg-Marquardt method with constraints changing at each iteration.

Assumption 4.3.2 *There are constants* $c^\triangleright, \delta > 0$ *so that*

$$\operatorname{dist}[z, S(z)] \leq c^\triangleright \operatorname{dist}[z, Z] \quad \textit{for all } z \in \mathcal{B}(z^*, \delta) \cap \Omega_0. \tag{4.10}$$

If Assumption 4.3.2 holds then, since Z is non-empty, it is clear that

$$S(z) \neq \emptyset \text{ and } \Omega(z) \neq \emptyset \quad \text{for all } z \in \mathcal{B}(z^*, \delta) \cap \Omega_0.$$

Assumption 4.3.2 is discussed in detail in Section 4.5 where a sufficient condition for it to hold is also given. Here, we just note that convexity of the objective functions is in general not necessary for either Assumption 4.3.1 or Assumption 4.3.2 to hold.

Assumption 4.3.3 *There exists an $x^{\triangleright} \in \mathbb{R}^n$ so that the level set*

$$\mathcal{L}_F(x^{\triangleright}) := \{x \in \mathbb{R}^n | F(x) \leq F(x^{\triangleright})\}$$

is bounded.

Assumption 4.3.2 is used for local convergence analysis while we use Assumption 4.3.3 for global convergence analysis (to show existence of accumulation points). Assumption 4.3.3 is a standard sufficient condition assumed in the literature for existence of weakly Pareto-optimal points (see for example in [31; 52]).

Blanket Assumption for Sections 4.3 and 4.4: *Assumptions 4.2.1, 4.2.2, 4.3.1, 4.3.2 and 4.3.3 hold.*

Let $\delta := \min\{\hat{\delta}, \delta\}$ and $c := \dfrac{c^{\triangleright}}{C}$. It is easy to see that Assumptions 4.3.1 and 4.3.2 imply that

$$\text{dist}\,[z, S(z)] \leq c^{\triangleright}\text{dist}\,[z, Z] \leq c\|H(z)\| \quad \text{for all } z \in \mathcal{B}(z^*, \delta) \cap \Omega_0, \tag{4.11}$$

and vice versa.

Definition 4.3.1 *If, for some $z^{\triangleright} \in Z$, (4.11) holds with some $c^{\triangleright}, c, \delta > 0$ then we say, in short, that $\|H\|$ has the constrained error bound property around $z^{\triangleright}$.*

Lemma 4.3.1 *For all $z \in \mathcal{B}(z^*, \delta) \cap \Omega_0$, the set Z is non-empty and closed and, the sets $\Omega(z)$ and $S(z)$ are nonempty and compact. Moreover, $d(z)$ exists for all $z \in \mathcal{B}(z^*, \delta) \cap \Omega_0$.*

Proof: Take an arbitrary but fixed $z \in \mathcal{B}(z^*, \delta) \cap \Omega_0$. Obviously, under Assumptions 4.3.1 and 4.3.2, Z and $S(z)$ are nonempty. From the structure of $\Omega(z)$, it is easy to deduce that $\Omega(z)$ is compact. Moreover, under Assumption 4.2.1, Z is closed. Since both the sets Z and $\Omega(z)$ are closed, the set $S(z) = Z \cap (z + \Omega(z))$ is also closed and bounded and hence compact. As noted earlier, $\theta(\cdot, z)$ is a strongly convex function. By the compactness and nonemptiness of $\Omega(z)$ the subproblem $(P(z))$ has at least one solution $d(z)$. $\triangle$

Lemma 4.3.2 *There is* $L > 0$ *so that the inequalities*

$$\begin{aligned} \|H(z) - H(s)\| &\leq L\|z - s\|, && (4.12)\\ \|\nabla H(z) - \nabla H(s)\| &\leq L\|z - s\|, && (4.13)\\ \|H(z) - H(s) - \nabla H(s)^\top (z - s)\| &\leq L\|z - s\|^2, && (4.14)\\ \|H(z)\| &\leq L, && (4.15) \end{aligned}$$

are satisfied for all $z, s \in \mathcal{B}(z^*, \delta)$*, and*

$$\begin{aligned} \|F(x) - F(y)\| &\leq L\|x - y\| && (4.16)\\ \|\nabla^2 F_i(x) - \nabla^2 F_i(y)\| &\leq L\|x - y\|, \ \textit{for all } i \in \mathcal{I} && (4.17)\\ \|H(x, \nu(x)) - H(y, \nu(y))\| &\leq L\|x - y\| && (4.18) \end{aligned}$$

are satisfied for all $x, y \in \mathcal{B}(x^*, \delta)$.

Proof: Inequalities (4.12), (4.13), (4.16) and (4.17) follow directly from Assumption 4.2.1. From [59, Theorem 2.1] we obtain that the function $q : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and hence (4.18) also follows. The compactness of $\mathcal{B}(z^*, \delta)$ and the continuity of $\|H\|$ ensures (4.15). Finally, Taylor's formula together with Assumption 4.2.1 yields (4.14). △

In the following, for a given $z \in \mathcal{B}(z^*, \delta) \cap \Omega_0$, we denote by $z^{\llcorner}$ and $z^{\perp}$ vectors in $S(z)$ and Z, respectively, so that

$$\begin{aligned} \|z - z^{\llcorner}\| &= \text{dist}\,[z, S(z)], \ \text{and} && (4.19)\\ \|z - z^{\perp}\| &= \text{dist}\,[z, Z] && (4.20) \end{aligned}$$

hold. Note that as $S(z) = (z + \Omega(z)) \cap Z$, we have that

$$z^{\llcorner} - z \in \Omega(z). \tag{4.21}$$

Recall that for any $z \in \mathbb{R}^{n+m}$ such that $\Omega(z) \neq \emptyset$, $d(z)$ denotes an arbitrary global minimizer of $(P(z))$.

Lemma 4.3.3 *There is* $C_1 > 0$ *so that the inequalities*

$$\begin{aligned} \|d(z)\| &\leq C_1 \text{dist}\,[z, S(z)],\\ \|H(z) + \nabla H(z)^\top d(z)\| &\leq C_1 \text{dist}\,[z, S(z)]^2 \end{aligned}$$

are satisfied for all $z \in \mathcal{B}\left(z^*, \dfrac{\delta}{1 + c^{\triangleright}}\right) \cap \Omega_0$.

Proof: As $\mathcal{B}\left(z^*, \dfrac{\delta}{1+c^{\triangleright}}\right) \subset \mathcal{B}(z^*, \delta)$, Lemma 4.3.1 shows that $S(z)$ is non-empty and that $d(z)$ exists. From Lemma 4.2.2 it easy to see that the statement of the lemma holds if $z \in Z$. Hence, we consider an arbitrary but fixed $z \in \mathcal{B}\left(z^*, \dfrac{\delta}{1+c^{\triangleright}}\right) \setminus Z$. Using the triangle inequality, Assumption 4.3.2 and that $z \in \mathcal{B}\left(z^*, \dfrac{\delta}{1+c^{\triangleright}}\right)$, we obtain

$$\|z^{\llcorner} - z^*\| \;\leq\; \|z^{\llcorner} - z\| + \|z - z^*\| \leq c^{\triangleright}\|z - z^*\| + \|z - z^*\| \leq \delta,$$

which means that $z^{\llcorner} \in \mathcal{B}(z^*, \delta)$. Hence it follows from (4.11), (4.2) and (4.12) that $\alpha(z)$ satisfies

$$\frac{1}{c^2}\|z^{\llcorner} - z\|^2 \leq \|H(z)\|^2 = \alpha(z) \leq L^2\|z^{\llcorner} - z\|^2. \tag{4.22}$$

Since $d(z)$ is a global minimizer of $\theta(\cdot, z)$, using (4.21), (4.6), (4.14) and (4.22) we obtain

$$\begin{aligned}
\|d(z)\|^2 &\leq \frac{\theta(d(z), z)}{\alpha(z)} \\
&\leq \frac{\theta(z^{\llcorner} - z, z)}{\alpha(z)} \\
&= \frac{\|H(z) + \nabla H(z)^\top (z^{\llcorner} - z)\|^2 + \alpha(z)\|(z^{\llcorner} - z)\|^2}{\alpha(z)} \\
&\leq L^2 c^2 \|z^{\llcorner} - z\|^2 + \|z^{\llcorner} - z\|^2 \\
&\leq (L^2 c^2 + 1)\|z^{\llcorner} - z\|^2.
\end{aligned}$$

The above inequality implies that

$$\|d(z)\| \leq \sqrt{L^2c^2 + 1}\,\mathrm{dist}\,[z, S(z)]. \tag{4.23}$$

We also observe that,

$$\begin{aligned}
\|H(z) + \nabla H(z)^\top d(z)\| &\leq \sqrt{\theta(d, z)} \\
&\leq \sqrt{\theta(z^{\llcorner} - z, z)} \\
&= \sqrt{\|H(z) + \nabla H(z)^\top (z^{\llcorner} - z)\|^2 + \alpha(z)\|z^{\llcorner} - z\|^2}.
\end{aligned}$$

This together with (4.14) and the last part of (4.22) yields,

$$\begin{aligned}
\|H(z) + \nabla H(z)^\top d(z)\| &\leq \sqrt{L^2\|z^{\llcorner} - z\|^4 + L^2\|z^{\llcorner} - z\|^4} \\
&\leq \sqrt{2}L\mathrm{dist}\,[z, S(z)]^2.
\end{aligned}$$

This inequality together with (4.23) shows that the result of the lemma follows by setting

$$C_1 := \max\{\sqrt{L^2c^2+1}\,,\ \sqrt{2}L\}.$$

$\triangle$

Lemma 4.3.4 *Let* $z \in \mathcal{B}\left(z^*, \frac{\delta}{1+c^{\triangleright}}\right)$ *be such that* $z + d(z) \in \mathcal{B}\left(z^*, \frac{\delta}{1+c^{\triangleright}}\right)$. *Then, there is* $C_2 > 0$ *so that*

$$\operatorname{dist}[z+d(z), S(z+d(z))] \le C_2 \operatorname{dist}[z, S(z)]^2.$$

Proof: Using the triangle inequality we obtain

$$\|H(z+d(z))\| \le \|H(z) + \nabla H(z)^\top d(z)\| + \|H(z+d(z)) - H(z) - \nabla H(z)^\top d(z)\|.$$

Since both $z + d(z)$ and z belong to $\mathcal{B}\left(z^*, \frac{\delta}{1+c^{\triangleright}}\right)$ and by (4.14), we have

$$\|H(z+d(z))\| \le \|H(z) + \nabla H(z)^\top d(z)\| + L\|d(z)\|^2. \tag{4.24}$$

With Lemma 4.3.3, (4.24) further simplifies to

$$\|H(z+d(z))\| \le C_1\|z - z^{\llcorner}\|^2 + LC_1^2\|z - z^{\llcorner}\|^2. \tag{4.25}$$

Using Lemma 4.3.1 we obtain that $S(z+d(z))$ is non-empty. Now, (4.11) together with (4.25) gives

$$\begin{aligned} \operatorname{dist}[z+d(z), S(z+d(z))] &\le c\|H(z+d(z))\| \\ &\le C_2 \operatorname{dist}[z, S(z)]^2, \end{aligned}$$

where the last inequality follows by setting $C_2 := cC_1(1 + LC_1)$. $\triangle$

Lemma 4.3.5 *For any* $\bar{\delta} \in (0, \delta]$, *there is an* $\epsilon\left(\bar{\delta}\right) \in \left(0, \frac{\bar{\delta}}{1+c^{\triangleright}}\right]$ *so that, if* $\tilde{z}^0 \in \mathcal{B}\left(z^*, \epsilon\left(\bar{\delta}\right)\right)$ *then, for all* $k \in \mathbb{N}_0$, $d(\tilde{z}^k)$ *is well defined and*

$$\tilde{z}^{k+1} := \tilde{z}^k + d(\tilde{z}^k) \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1+c^{\triangleright}}\right)$$

.

Proof: Take an arbitrary but fixed $\bar{\delta} \in (0, \delta]$ and let

$$\epsilon\left(\bar{\delta}\right) := \min\left\{\frac{\bar{\delta}}{(1+c^{\triangleright})(1+3C_1c^{\triangleright})}, \frac{1}{2C_2c^{\triangleright}}\right\} \tag{4.26}$$

with C_1 from Lemma 4.3.3, C_2 from Lemma 4.3.4 and $c^{\triangleright}$ from Assumption 4.3.2. We use induction to show the desired result. To this end suppose $\tilde{z}^0 \in \mathcal{B}\left(z^*, \epsilon(\bar{\delta})\right)$. Obviously, $\tilde{z}^0 \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1+c^{\triangleright}}\right)$ from the definition of $\epsilon(\bar{\delta})$. Now, from the triangle inequality, Lemma 4.3.1, Lemma 4.3.3, Assumption 4.3.2 and the definition of $\epsilon\left(\bar{\delta}\right)$ if follows that

$$\begin{aligned}
\|\tilde{z}^1 - z^*\| &= \|\tilde{z}^0 + d(\tilde{z}^0) - z^*\| \\
&\leq \|\tilde{z}^0 - z^*\| + \|d(\tilde{z}^0)\| \\
&\leq \|\tilde{z}^0 - z^*\| + C_1 \text{dist}\left[\tilde{z}^0, S(\tilde{z}^0)\right] \\
&\leq \|\tilde{z}^0 - z^*\| + C_1c^{\triangleright}\|\tilde{z}^0 - z^*\| \\
&\leq (1+C_1c^{\triangleright})\epsilon\left(\bar{\delta}\right) \\
&\leq \frac{\bar{\delta}}{1+c^{\triangleright}},
\end{aligned}$$

which means that $\tilde{z}^1 \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1+c^{\triangleright}}\right)$.

Suppose that $\tilde{z}^i \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1+c^{\triangleright}}\right)$ for $i = 1, 2, \ldots, k$. Then $d(\tilde{z}^i)$ exists for all $i = 1, 2, \ldots, k$. Now, for an arbitrary but fixed $i \in \{1, 2, \ldots, k\}$, using Lemma 4.3.4 we obtain that

$$\begin{aligned}
\|\tilde{z}^i - (\tilde{z}^i)^{\llcorner}\| &= \|\tilde{z}^{i-1} + d(\tilde{z}^{i-1}) - (\tilde{z}^i)^{\llcorner}\| \\
&= \|\tilde{z}^{i-1} + d(\tilde{z}^{i-1}) - (\tilde{z}^{i-1} + d(\tilde{z}^{i-1}))^{\llcorner}\| \\
&= \text{dist}\left[\tilde{z}^{i-1} + d(\tilde{z}^{i-1}), S(\tilde{z}^{i-1} + d(\tilde{z}^{i-1}))\right] \\
&\leq C_2 \text{dist}\left[\tilde{z}^{i-1}, S(\tilde{z}^{i-1})\right]^2 \\
&= C_2\|\tilde{z}^{i-1} - (\tilde{z}^{i-1})^{\llcorner}\|^2
\end{aligned}$$

holds. Hence, using Lemma 4.3.4 repeatedly and the definition of $\epsilon\left(\bar{\delta}\right)$ we further obtain that

$$\begin{aligned}
\|\tilde{z}^i - (\tilde{z}^i)^{\llcorner}\| &\leq C_2\|\tilde{z}^{i-1} - (\tilde{z}^{i-1})^{\llcorner}\|^2 \\
&\leq C_2\left(C_2\|\tilde{z}^{i-2} - (\tilde{z}^{i-2})^{\llcorner}\|^2\right)^2 \\
&\leq \ldots \\
&\leq C_2^{(2^i-1)}\|\tilde{z}^0 - (\tilde{z}^0)^{\llcorner}\|^{2^i}.
\end{aligned} \tag{4.27}$$

From the definition of $\epsilon(\bar{\delta})$ and Assumption 4.3.2 we obtain that

$$\begin{aligned} \|\tilde{z}^0 - (\tilde{z}^0)^{\llcorner}\| &\leq c^{\triangleright}\text{dist}\left[\tilde{z}^0, Z\right] \\ &\leq c^{\triangleright}\|\tilde{z}^0 - z^*\| \\ &\leq c^{\triangleright}\epsilon(\bar{\delta}), \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} C_2^{2^i-1}\|\tilde{z}^0 - (\tilde{z}^0)^{\llcorner}\|^{2^i-1} &\leq C_2^{2^i-1}\left(c^{\triangleright}\frac{1}{2C_2c^{\triangleright}}\right)^{2^i-1} \\ &= \left(\frac{1}{2}\right)^{2^i-1} \end{aligned} \tag{4.29}$$

hold. Using (4.28) and (4.29) in (4.27) yields

$$\begin{aligned} \|\tilde{z}^i - (\tilde{z}^i)^{\llcorner}\| &\leq c^{\triangleright}\epsilon(\bar{\delta})\left(\frac{1}{2}\right)^{2^i-1} \\ &= 2c^{\triangleright}\epsilon(\bar{\delta})\left(\frac{1}{2}\right)^{2^i}. \end{aligned}$$

Hence, it follows from Lemma 4.3.3 and the definition of $\epsilon\left(\bar{\delta}\right)$ that

$$\begin{aligned} \|\tilde{z}^{k+1} - z^*\| &\leq \|\tilde{z}^0 - z^*\| + \sum_{i=0}^{k}\|d(\tilde{z}^k)\| \\ &\leq \epsilon\left(\bar{\delta}\right) + C_1\sum_{i=0}^{k}\|\tilde{z}^i - (\tilde{z}^i)^{\llcorner}\| \\ &\leq \epsilon\left(\bar{\delta}\right) + 2\epsilon\left(\bar{\delta}\right)C_1c^{\triangleright}\sum_{i=0}^{k}\left(\frac{1}{2}\right)^{2^i} \\ &\leq (1 + C_1c^{\triangleright})\epsilon\left(\bar{\delta}\right) + 2\epsilon\left(\bar{\delta}\right)C_1c^{\triangleright}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i} \\ &\leq (1 + 3C_1c^{\triangleright})\epsilon\left(\bar{\delta}\right) \\ &\leq \frac{\bar{\delta}}{1 + c^{\triangleright}}. \end{aligned}$$

Hence, we see that $\tilde{z}^{k+1} \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1 + c^{\triangleright}}\right)$ and, by Lemma 4.3.1, $d(\tilde{z}^{k+1})$ is well defined. The statement of the lemma follows as the choice of $\bar{\delta} \in (0, \delta]$ was arbitrary. $\triangle$

Lemma 4.3.6 *There are* $\varepsilon, \bar{\delta} \in (0, \delta)$ *so that* $\tilde{z}^0 \in \mathcal{B}(z^*, \varepsilon)$ *implies* $\tilde{z}^k \in \mathcal{B}(z^*, \bar{\delta})$ *and*

$$\begin{aligned} \|H(\tilde{z}^{k+1})\| &\leq \kappa \|H(\tilde{z}^k)\|, &(4.30)\\ F(\tilde{x}^{k+1}) &\leq F(\tilde{x}^k) &(4.31)\end{aligned}$$

for all $k \in \mathbb{N}_0$*, where* κ *is from Algorithm 4.1 and the sequence* $\{\tilde{z}^k\}$ *is defined in Lemma 4.3.5. Moreover, equality in (4.31) occurs if and only if* $\tilde{x}^k$ *is Pareto-critical.*

Proof: Let $\bar{\delta} := \min\left\{\frac{\delta}{1+\bar{c}}, \frac{1}{L^{4.5}}, \frac{\kappa}{c^2L^2C_2}\right\}$ and let $\epsilon := \epsilon(\bar{\delta})$ be given by (4.26). From Lemma 4.3.5 we obtain that $\tilde{z}^k \in \mathcal{B}\left(z^*, \frac{\bar{\delta}}{1+c^{\triangleright}}\right)$ for all $k \in \mathbb{N}_0$.

Let us assume that for some $k \in \mathbb{N}_0$, $\tilde{x}^k$ is Pareto-critical. Then, $\|H(\tilde{x}^k, \nu(\tilde{x}^k))\| = 0$ and it is easy to see that $\Omega(\tilde{z}^k)$ is non-empty. Moreover, $\tilde{x}^{k+1} = \tilde{x}^k$ and hence (4.31) holds with equality. On the other hand, if $\tilde{z}^k \in Z$ for some $k \in \mathbb{N}_0$ then, applying Lemma 4.2.2, (4.30) and (4.31) follows. In the remaining part of the proof we assume that $\tilde{z}^k \notin Z$ and $\tilde{x}^k$ is not Pareto-critical for any $k \in \mathbb{N}_0$.

For an arbitrary but fixed $k \in \mathbb{N}_0$, (4.12) together with the definition of $\bar{\delta}$ gives

$$\|H(\tilde{z}^k)\| = \|H(\tilde{z}^k) - H(z^*)\| \leq L\|\tilde{z}^k - z^*\| \leq \frac{L\bar{\delta}}{1+c^{\triangleright}} \leq \left(\frac{\kappa}{c^2LC_2}\right). \tag{4.32}$$

Using (4.12), (4.20) and Lemma 4.3.4 we obtain

$$\|H(\tilde{z}^{k+1})\| \leq L\|\tilde{z}^{k+1} - (\tilde{z}^{k+1})^{\perp}\| \leq L\|\tilde{z}^{k+1} - (\tilde{z}^{k+1})^{\llcorner}\| \leq LC_2\|\tilde{z}^k - (\tilde{z}^k)^{\llcorner}\|^2. \tag{4.33}$$

Furthermore, (4.33), (4.11), (4.32) and that $\tilde{z}^k \notin Z$ give

$$\begin{aligned} \frac{\|H(\tilde{z}^{k+1})\|}{\|H(\tilde{z}^k)\|} &\leq \frac{cLC_2\|\tilde{z}^k - (\tilde{z}^k)^{\llcorner}\|^2}{\|\tilde{z}^k - (\tilde{z}^k)^{\llcorner}\|} \\ &= cLC_2\|\tilde{z}^k - (\tilde{z}^k)^{\llcorner}\| \\ &\leq c^2LC_2\|H(\tilde{z}^k)\| \\ &\leq \kappa. \end{aligned}$$

Hence, (4.30) is satisfied. This proves the first part of the lemma.

For showing (4.31), observe that by Lemma 4.3.1, $d(\tilde{z}^k)$ is well defined. Since $d(\tilde{z}^k)$ is optimal to $(P(z^k))$, it must also be feasible to $\Omega(z^k)$ i.e., $d(\tilde{z}^k) \in \Omega(\tilde{z}^k)$

and we obtain

$$\nabla F_i(\tilde{x}^k)^\top d(\tilde{z}^k)_x + \frac{1}{2} d(\tilde{z}^k)_x^\top \nabla^2 F_i(\tilde{x}^k) d(\tilde{x}^k)_x \leq -\left\| H\left(\tilde{x}^k, \nu(\tilde{z}^k)\right)\right\|^{2.5} \tag{4.34}$$

for all $i \in \mathcal{I}$. As $\tilde{x}^k$ is not Pareto-critical, from (4.34) we easily see that

$$\|d(\tilde{z}^k)_x\| \neq 0. \tag{4.35}$$

We further obtain

$$\|d(\tilde{z}^k)_x\| \leq \left\| H\left(\tilde{x}^k, \nu(\tilde{x}^k)\right)\right\|^{0.9}, \tag{4.36}$$

and hence (4.34) yields

$$\nabla F_i(\tilde{x}^k)^\top d(\tilde{z}^k)_x + \frac{1}{2} d(\tilde{z}^k)_x^\top \nabla^2 F_i(\tilde{x}^k) d(\tilde{z}^k)_x \leq -\|d(\tilde{z}^k)_x\|^{\frac{2.5}{0.9}} \tag{4.37}$$

for all $i \in \mathcal{I}$. Now, for any $i \in \mathcal{I}$, using Assumption 4.2.1 and Taylor's theorem we obtain (see [15, Lemma 4.1.14] for example)

$$F_i(\tilde{x}^{k+1}) - F_i(\tilde{x}^k) - \nabla F_i(\tilde{x}^k)^\top d(\tilde{z}^k)_x - \frac{1}{2} d(\tilde{z}^k)_x^\top \nabla^2 F_i(\tilde{z}^k) d(\tilde{z}^k)_x \leq \frac{L}{6}\|d(\tilde{z}^k)_x\|^3.$$

This together with (4.37), the definition of ϵ and (4.35) implies

$$\begin{aligned} F_i(\tilde{x}^{k+1}) - F_i(\tilde{x}^k) &\leq -\|d(\tilde{z}^k)_x\|^{\frac{2.5}{0.9}} + \frac{L}{6}\|d(\tilde{z}^k)_x\|^3 \\ &= \|d(\tilde{z}^k)_x\|^{\frac{2.5}{0.9}} \left(-1 + \frac{L}{6}\|d(\tilde{z}^k)_x\|^{\frac{2}{9}}\right) \\ &\leq -\left(\frac{5}{6}\|d(\tilde{z}^k)_x\|^{\frac{2.5}{0.9}}\right) \\ &< 0. \end{aligned}$$

Since the choice of index i was arbitrary we obtain a simultaneous descent, i.e., $F(\tilde{x}^{k+1}) < F(\tilde{x}^k)$. The statement of lemma holds since the choice of $k \in \mathbb{N}_0$ was arbitrary. △

Remark 4.3.1 *Let us define the constraint set $\Omega(z,p)$ depending on $z = (x, \lambda)$ and $p := (p_1, p_2) \in \mathbb{R}^2$ by*

$$\begin{aligned} \Omega(z,p) := \{ d := (d_x, d_\lambda) \in \mathbb{R}^{n+m} | \lambda + d_\lambda \in \Lambda_0, \|d_x\| \leq \|H(x, \nu(x))\|^{p_1}, \\ \nabla F_i(x)^\top d_x + \frac{1}{2} d_x^\top \nabla^2 F_i(x) d_x \leq -\|H(x,\nu(x))\|^{p_2} \quad \textit{for all } i \in \mathcal{I} \}. \end{aligned}$$

Following the proof of Lemma 4.3.6 we easily observe that it goes through even if instead of $\Omega(z)$ we take $\Omega(z,p)$ with any

$$p \in \mathcal{P} := \left\{(p_1, p_2) \in \mathbb{R}^2 | 0 < p_1 < 1,\ 2 < p_2 < 3,\ 2p_1 < p_2 < 3p_1\right\}. \tag{4.38}$$

Definition 4.3.2 *A sequence $\{z^k\} := \{(x^k, \lambda^k)\} \subset \mathbb{R}^{n+m}$ is called a* Pareto-decreasing sequence *if*

- $F(x^{k+1}) < F(x^k)$ *for all* $k \in \mathbb{N}_0$, *and*
- *there is a sub-sequence of $\{z^k\}$ converging to some $\bar{z} := (\bar{x}, \bar{\lambda}) \in Z$.*

Lemma 4.3.7 *Let $\{z^k\}$ be a Pareto-decreasing sequence. Then, the sequence $\{F(x^k)\}$ converges to $F(\bar{x})$.*

Proof: From the definition of the Pareto-deceasing sequence we observe that the sequence $\{F(x^k)\}$ is component-wise monotonically decreasing and bounded below by $F(\bar{x})$, where $F(\bar{x})$ is from Definition 4.3.2. Hence, it is easy to see that $\{F(x^k)\}$ converges to $F(\bar{x})$. $\triangle$

The next theorems present the convergence results for Algorithm 4.1. Note that if the algorithm stops after a finite number of iterations then from (S2) it stops at a Pareto-critical point. Moreover, from step (S2) of Algorithm 4.1, the algorithms stops if a Pareto-critical point is found. Hence in the remaining part of this section we assume that an infinite sequence $\{z^k\}$ is generated such that x^k is not Pareto-critical for all $k \in \mathbb{N}$.

Theorem 4.3.1 *Let $\{z^k\}$ be an infinite sequence generated by Algorithm 4.1 so that $z^0 \in \mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$ (see Assumption 4.3.3). Then, this sequence has at least one accumulation point. Let $\hat{z} := (\hat{x}, \hat{\lambda})$ be any of these accumulation points. Then $\hat{z}$ belongs to Z and $\hat{x}$ is Pareto-critical. Moreover, the entire sequence $\{F(x^k)\}$ converges to $F(\hat{x})$.*

Proof: Using Assumption 4.3.3 and the definition of Λ_0 we see that the set $\mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$ is bounded. From Lemma 4.2.1 the entire sequence $\{z^k\}$ lies in the set $\mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$. Hence, by the Bolzano-Weierstrass theorem, the sequence $\{z^k\}$ has at least one accumulation point. Let us take a *convergent* subsequence $\{z^k\}_{\mathbb{U}}$ of $\{z^k\}$. There are three cases:

1. d^{k-1} is obtained from (S3) for all $k \in \mathbb{U}$ sufficiently large. From (4.7) the sequence $\{\|H(z^k)\|\}_{\mathbb{U}}$ converges to zero. By continuity of H we easily obtain that $\{z^k\}_{\mathbb{U}}$ converges to some $\hat{z} \in Z$.

2. d^{k-1} is obtained from (S4) for all $k \in \mathbb{U}$ sufficiently large. Now, as $\{x^k\}$ is convergent to $\hat{x}$, we have that $\hat{x}$ is Pareto-critical (see [31, Theorem 1] or [52, Theorem 2]). By (4.18) $\|H(\cdot, \nu(\cdot))\|$ is continuous and we easily obtain that the sequence $\{\|H(z^k)\|\}_{\mathbb{U}}$ converges to zero and hence, $\{z^k\}_{\mathbb{U}}$ converges to some $\hat{z} \in Z$.

3. d^{k-1} is obtained from both (S3) and (S4) for all $k \in \mathbb{U}$ sufficiently large. Then, using the analysis of case 1., we see that $\{z^k\}_{\mathbb{U}}$ has a convergent sub-sequence that converges to $\hat{z} \in Z$. Hence the sequence $\{z^k\}_{\mathbb{U}}$ itself converges to $\hat{z} \in Z$.

Thus, in all the cases we showed that any accumulation point $\hat{z} = (\hat{x}, \hat{\lambda})$ of $\{z^k\}$ is a solution of (4.1). Now invoking Lemma 4.3.7, we obtain that the sequence $\{F(x^k)\}$ converges to $F(\hat{x})$. △

Theorem 4.3.2 *Let $\{z^k\}$ be a sequence generated by Algorithm 4.1 so that $z^0 \in \mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$. Suppose that $\{z^k\}$ converges to some $\hat{z} \in \mathcal{B}\left(z^*, \frac{\epsilon}{2}\right)$, with ϵ from Lemma 4.3.6. Then, there exists $\tilde{k} \in \mathbb{N}$ so that $d^k = d(z^k)$ for all $k \geq \tilde{k}$. Moreover, the sequence $\{z^k\}$ converges Q-quadratically to $\hat{z}$ and $\hat{z}$ belongs to Z.*

Proof: As $z^0 \in \mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$, by Theorem 4.3.1 any convergent subsequence converges to a point in Z. Hence, by the assumption that $\{z^k\}$ is convergent to some $\hat{z}$, we have that $\hat{z} \in Z$.

We first show that the sequence $\{d^k\}$ is obtained from (S3) infinitely many times. Assume on the contrary, that there exists $k_1 \in \mathbb{N}$ so that d^k is obtained from (S4) for all $k \geq k_1$. Let ρ be the value of the parameter ϱ at iteration k_1. Since the sequence $\{z^k\}$ converges to $\hat{z} \in Z$, the sequence $\{\|H(z^k)\|\}$ goes to zero. Hence, there exists a $k_2 > k_1$ so that for all $k \geq k_2$, $\|H(z^k)\| < \rho$ and

$$\begin{aligned}\|z^k - z^*\| &\leq \|z^* - \hat{z}\| + \|\hat{z} - z^k\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

This means that $z^k \in \mathcal{B}(z^*, \epsilon)$ for all $k \geq k_2$. Note that x^k is not Pareto-critical (else the algorithm would stop at step (S2)). Hence, by Lemma 4.3.6, if we set $\tilde{z}^0 := z^k$, we obtain that

$$\begin{aligned}\|H(z^k + d(z^k))\| &\leq \kappa\|H(z^k)\| < \kappa\rho, \text{ and} \\ F((x^k + d(z^k)_x) &< F(x^k)\end{aligned}$$

hold. Thus, both (4.7) and (4.8) are satisfied and d^{k+1} is obtained from (S3), which is a contradiction. This means that d^k is obtained from (S3) infinitely many times.

Let $\tilde{k} \in \mathbb{N}$ be sufficiently large so that $z^{\tilde{k}} \in \mathcal{B}(z^*, \epsilon)$ and that $d^{\tilde{k}-1} = d(z^{\tilde{k}-1})$, i.e.,

$$z^{\tilde{k}} = z^{\tilde{k}-1} + d(z^{\tilde{k}-1}).$$

Now, by Lemma 4.3.6, if we set $\tilde{z}^0 := z^{\tilde{k}}$, we obtain that both (4.7) and (4.8) are satisfied and thus, d^k is obtained from (S3) for all $k \geq \tilde{k}$.

We next show that $\{z^k\}$ converges Q-quadratically to $\hat{z}$. From Lemma 4.3.5, Lemma 4.3.4 and (4.26) we obtain that $z^k \in \mathcal{B}(z^*, \bar{\delta})$ for all $k \geq \tilde{k}$ and that

$$\text{dist}\,[z^{k+1}, S(z^{k+1})] \leq C_2 \text{dist}\,[z^k, S(z^k)]^2 \leq \frac{1}{2}\text{dist}\,[z^k, S(z^k)] \tag{4.39}$$

holds. Now for any $l_1 \geq l_2 \geq \tilde{k}$ with $l_1, l_2 \in \mathbb{N}$, using Lemma 4.3.3 and (4.39) we obtain

$$\begin{aligned}
\|z^{l_1} - z^{l_2}\| &\leq \sum_{i=l_2}^{l_1-1} \|z^{i+1} - z^i\| \\
&= \sum_{i=l_2}^{l_1-1} \|d^i\| \\
&\leq C_1 \sum_{i=l_2}^{l_1-1} \text{dist}\,[z^i, S(z^i)] \\
&\leq C_1 \text{dist}\,[z^{l_2}, S(z^{l_2})] \sum_{i=0}^{l_1-l_2-1} 2^{-i} \\
&\leq 2C_1 \text{dist}\,[z^{l_2}, S(z^{l_2})].
\end{aligned}$$

In the above estimation using Lemma 4.3.3, Assumption 4.3.2 and l_2+1, l_2+j instead of l_2, l_1 respectively leads to

$$\begin{aligned}
\|z^{l_2+j} - z^{l_2+1}\| &\leq 2C_1 \text{dist}\,[z^{l_2+1}, S(z^{l_2+1})] \\
&\leq 2C_1 C_2 \text{dist}\,[z^{l_2}, S(z^{l_2})]^2 \leq 2c^{\triangleright} C_1 C_2 \text{dist}\,[z^{l_2}, Z]^2
\end{aligned}$$

for any $l_2, j \in \mathbb{N}$. Passing to the the limit as $j \to \infty$ we obtain

$$\|\hat{z} - z^{l_2+1}\| \leq 2c^{\triangleright} C_1 C_2 \text{dist}\,[z^{l_2}, Z]^2$$

and, by $\|\hat{z} - z^{l_2}\| \geq \operatorname{dist}[z^{l_2}, Z]$, we obtain that

$$\lim_{l_2 \to \infty} \frac{\|\hat{z} - z^{l_2+1}\|}{\|\hat{z} - z^{l_2}\|^2} \leq 2c^{\flat} C_1 C_2.$$

Hence the statement of the theorem follows. △

Remark 4.3.2 *From Theorem 4.3.1, we obtain that if z^1 and z^2 are two accumulation points of the sequence $\{z^k\}$ generated by Algorithm 4.1 then, $F(x^1) = F(x^2)$. Thus, the assumption that the sequence $\{z^k\}$ converges to some $\hat{z}$ seems reasonable. Later in Section 4.5, Lemma 4.5.3 gives a condition under which $F(x^1) = F(x^2)$ implies that $x^1 = x^2$. Moreover, we can avoid this assumption by requiring that for any $\hat{z}$ from Theorem 4.3.1, a $\tilde{z} \in \mathcal{B}\left(\hat{z}, \frac{\epsilon}{2}\right) \cap Z$ exists so that $\|H\|$ has the constrained error bound property around $\tilde{z}$. The proof of this is involved and is not presented here.*

Remark 4.3.3 *We observe that in Theorem 4.3.2 the local rate of convergence is $\frac{\rho+2}{2}$ for $\rho \in [1, 2)$, if $\alpha(z) := \|H(z)\|^\rho$ is chosen instead of $\alpha(z) := \|H(z)\|^2$.*

4.4 A Duality Based Method for Solving $(P(z))$

In the dual method instead of solving $(P(z))$ (which we call primal problem), we solve its dual instead. This is one of the standard approaches for solving quadratically constrained quadratic problems [1]. If strong duality holds we can obtain the solution of the primal problem as well. Convexity of the primal problem and Slater's constraint qualification guarantee strong duality for a general nonlinear programming problem. Details about these statements can be found in [7, Chapter 5].

Recall the Blanket Assumption in the beginning of Section 4.3. Hence, throughout this section Assumptions 4.2.1, 4.2.2, 4.3.1, 4.3.2 and 4.3.3 hold. By $0_k \in \mathbb{R}^k$ and by $1_k \in \mathbb{R}^k$ we denote the vectors having zero everywhere and one everywhere, respectively.

Let us define the matrices $Q^i(z) \in \mathbb{R}^{(n+m)\times(n+m)}$ for $i = 0, \ldots, m+1$ by

$$\begin{aligned} Q^0(z) &:= 2\left(\nabla H(z) \nabla H(z)^\top + \alpha(z) I\right) \\ Q^i(z) &:= 2\begin{pmatrix} \nabla^2 F_i(x) & 0 \\ 0 & 0 \end{pmatrix}, \text{ for all } i \in \mathcal{I} \text{ and,} \\ Q^{m+1}(z) &:= 2\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

The vectors $b^i(z) \in \mathbb{R}^{n+m}$ for $i = 0, \ldots, 2m+1$ are as follows.

$$\begin{aligned}
b^0(z) &:= \nabla H(z)H(z) \\
b^i(z) &:= \begin{pmatrix} \nabla F_i(x) \\ 0_m \end{pmatrix}, \text{ for all } i \in \mathcal{I}, \\
b^{m+1}(z) &:= 0_{n+m}, \\
b^{2m+2} &:= \begin{pmatrix} 0_n \\ 1_n \end{pmatrix}.
\end{aligned}$$

Moreover, for all $i \in \mathcal{I}$, $b^{m+i+1}(z) \in \mathbb{R}^{n+m}$ is the vector with -1 in the $(n+i)^{th}$ place and 0 elsewhere. The scalars $c^i(z)$ and $c^{m+i+1}(z)$ for $i = 0, \ldots, 2m+2$ are defined by

$$\begin{aligned}
c^0(z) &:= \|H(z)\|^2, \\
c^i(z) &:= -\|H(x, \nu(x))\|^{2.5}, \text{ for all } i \in \mathcal{I}, \\
c^{m+1}(z) &:= -\|H(x, \nu(x))\|^{1.8}, \\
c^{m+i+1}(z) &:= -z_{m+i} \text{ for all } i \in \mathcal{I} \text{ and}, \\
c^{2m+2}(z) &:= 0.
\end{aligned}$$

Using the above definitions, we can write $(P(z))$ in a simplified way as

$$\begin{aligned}
\min_d \quad & \frac{1}{2} d^\top Q^0(z) d + b^0(z)^\top d + c^0(z) \\
\text{s.t.} \quad & \frac{1}{2} d^\top Q^i(z) d + b^i(z)^\top d + c^i(z) \leq 0, \quad \forall i = 1, 2, \ldots, m+1 \\
& b^{m+1+i}(z)^\top d + c^{m+1+i}(z) \leq 0, \quad \forall i = 1, 2, \ldots, m \\
& b^{2m+2}(z)^\top d + c^{2m+2}(z) = 0.
\end{aligned}$$

The Lagrangian $L : \mathbb{R}^{n+m} \times \mathbb{R}^{2m+2} \to \mathbb{R}$ associated with the problem $(P(z))$ is given by

$$\begin{aligned}
L(d, \upsilon) := \; & \frac{1}{2} d^\top \left(Q^0(z) + \sum_{i=1}^{m+1} \upsilon_i Q^i(z) \right) d + \left(b^0(z) + \sum_{i=1}^{2m+2} \upsilon_i b^i(z) \right)^\top d \\
& + \left(c^0(z) + \sum_{i=1}^{2m+2} \upsilon_i c^i(z) \right).
\end{aligned}$$

We will construct the dual of $(P(z))$ following the results in [7, Section 5.2.4].

Let $\vartheta(\cdot, z) : \mathbb{R}^{2m+2} \to \mathbb{R}$ be the concave function defined as

$$\vartheta(\upsilon, z) := -\left(b^0(z) + \sum_{i=1}^{2m+2} \upsilon_i b^i(z)\right)^\top \left(Q^0(z) + \sum_{i=1}^{m+1} \upsilon_i Q^i(z)\right)^{-1} \left(b^0(z) + \sum_{i=1}^{2m+2} \upsilon_i b^i(z)\right) - \sum_{i=1}^{2m+2} \upsilon_i c^i(z)$$

The concave dual problem $(DP(z))$ of $(P(z))$ is given by (see [7, Section 5.2.4]):

$$\begin{array}{ll} \max\limits_{\upsilon} & \vartheta(\upsilon, z) \\ \text{s.t.} & \upsilon \in \Upsilon := \left\{x \in \mathbb{R}^{2m+2} | x_i \geq 0, \text{for all } i = 1, 2, \ldots, 2m+1\right\}. \end{array}$$

Let $\bar{f} : \mathbb{R}^{m+n} \to \mathbb{R}$ be the function defined by

$$\begin{aligned} \bar{f}(z) := \chi_{\max}(Q_0(z)) \left(1 + \|H(x, \nu(x))\|^{0.9}\right)^2 + \|H(z)\|^2 \\ + \|2b_0(z)\| \left(1 + \|H(x, \nu(x))\|^{0.9}\right) \end{aligned}$$

where we recall that $\chi_{\max}(Q_0(z))$ is the maximal eigenvalue of the matrix $Q_0(z)$. Using the dual method, we replace step (S3) of Algorithm 4.1 by the following steps (S3a) and (S3b).

(S3a) If, while solving $(DP(z^k))$, a $\bar{\upsilon} \in \Upsilon$ is found so that $\vartheta(\bar{\upsilon}, z) > \bar{f}(z)$ then go to (S4).

(S3b) Let υ^* be optimal to $(DP(z^k))$. If the linear system

$$\nabla_d L(d, \upsilon^*) = 0. \tag{4.40}$$

has no solution then go to (S4). Otherwise, let d^* be a solution of (4.40). If

$$\|H(z^k + d^*)\| \leq \kappa \varrho \text{ and} \tag{4.41}$$

$$F(x^k + d_x^*) < F(x^k) \tag{4.42}$$

then set $d^k := d^*$, $\varrho := \|H(z^k + d^k)\|$ and go to (S5).

The new algorithm having the above changes in step (S3) of Algorithm 4.1 is called Dual Algorithm 4.1. Step (S3a) of the Dual Algorithm 4.1 detects infeasibility of the primal problem $(P(z))$ as shown by the following lemma.

Lemma 4.4.1 *If for some $z \in \mathbb{R}^{n+m}$ there exists $\bar{\upsilon} \in \Upsilon$ such that $\vartheta(\bar{\upsilon}, z) > \bar{f}(z)$ then $\Omega(z)$ is empty.*

Proof: Let, for some $z \in \mathbb{R}^{n+m}$, $\bar{\upsilon} \in \Upsilon$ be so that $\vartheta(\bar{\upsilon}, z) > \bar{f}(z)$. Assume on the contrary that $\Omega(z)$ is nonempty. Then, since

$$\max_{\lambda \in \Lambda_0} \|\lambda\| = 1,$$

any d that is feasible to $(P(z))$ must satisfy

$$\|d\| \leq \|d_x\| + \|d_\lambda\| \leq \|H(x, \nu(x))\|^{0.9} + 1. \tag{4.43}$$

Consider the problem $\bar{P}(z)$ as

$$\begin{array}{ll} \max\limits_{d} & \theta(d, z) := \frac{1}{2} d^\top Q^0(z) d + b^0(z)^\top d + \|H(z)\|^2 \\ \text{s.t.} & d \in \Omega(z). \end{array}$$

The objective function of $\bar{P}(z)$ is a continuous function of d and the constraint set $\Omega(z)$ is non-empty and compact. Hence, by the theorem of Weierstrass there is a $\bar{d}$ where the maximum of $\bar{P}(z)$ is attained. From (4.43) we obtain that

$$\begin{array}{rcl} \theta(\bar{d}, z) & \leq & \chi_{\max}\left(Q^0(z)\right)\left(1 + \|H(x, \nu(x))\|^{0.9}\right)^2 + \|H(z)\|^2 \\ & & + \|b^0(z)\| \left(1 + \|H(x, \nu(x))\|^{0.9}\right) \\ & = & \bar{f}(z). \end{array} \tag{4.44}$$

Let d^* be optimal to $(P(z))$. From weak duality we obtain

$$\vartheta(\upsilon, z) \leq \theta(d^*, z) \quad \text{for all } \upsilon \in \Upsilon.$$

Therefore, we have

$$\begin{array}{rcl} \vartheta(\bar{\upsilon}, z) & \leq & \theta(d^*, z) \\ & \leq & \theta(\bar{d}, z) \quad (\text{as } \bar{d} \text{ is optimal to } \bar{P}(z)) \\ & \leq & \bar{f}(z) \quad (\text{from (4.44)}). \end{array}$$

This is a contradiction to $\vartheta(\bar{\upsilon}, z) > \bar{f}(z)$. Hence the statement of the lemma follows. △

Remark 4.4.1 *In the general case infeasibility of the primal problem* cannot *be determined by the dual problem even if the problem is convex. See a convex example in [3] where the primal feasible set is empty whereas the dual optimal value can take any value on the extended real line* $\bar{R} = [-\infty, +\infty]$*. However we were able to provide a condition (see Lemma 4.4.1) for the infeasibility of* $(P(z))$*.*

Remark 4.4.2 *The dual based approach for solving a quadratically constrained quadratic program is one of the standard methods. If all the objective functions are strongly convex, the dual problem $(DP(z))$ can be solved using an efficient gradient projection method presented in [1]. This method consists only of matrix vector multiplications and avoids computing the inverse of the matrix $\left(Q^0(z) + \sum_{i=1}^{m+1} \upsilon_i Q^i(z)\right)$ at each iteration of a gradient projection method used to solve $(DP(z))$.*

The next lemma discusses strong duality for $(P(z))$.

Lemma 4.4.2 *Let all the objective functions F_i, $i \in \mathcal{I}$ be convex and suppose that $\Omega(z)$ is nonempty for some $z \in \mathbb{R}^{n+m}$. Then, strong duality holds for $(P(z))$.*

Proof: If, for some $z \in \mathbb{R}^{n+m}$, $\Omega(z)$ is nonempty then, obviously, $(P(z))$ has a solution and

$$-\infty < \theta(d(z), z) < \infty$$

holds. Furthermore, since all the objective functions F_i are assumed to be convex, $(P(z))$ is a convex quadratically constrained quadratic programming problem. Then, we obtain from [4, Proposition 6.5.6] that strong duality holds for $(P(z))$. $\triangle$

Lemma 4.4.3 *Let the same conditions as in Lemma 4.4.2 hold. Then, the linear system (4.40) has a unique solution $d^* \in \Omega(z)$ and d^* is optimal to $(P(z))$.*

Proof: The result easily follows by [7, Section 5.5.5]. $\triangle$

The next theorem presents the convergence results for Dual Algorithm 4.1

Theorem 4.4.1 *Let us consider Dual Algorithm 4.1. For this:*

1. *Theorem 4.3.1 holds.*

2. *Let there exist an $\bar{\epsilon} > 0$ so that for all $z \in \mathcal{B}(z^*, \bar{\epsilon})$ strong duality holds for $(P(z))$ and (4.40) has a unique solution. Then, Theorem 4.3.2 holds.*

Proof:

1. If, in step (S3b), d^* exists so that (4.41) and (4.42) hold, it can be used instead of $d(z^k)$ in step (S3) of Algorithm 4.1 and satisfies (4.7) and (4.8) in step (S3). This is all what is needed in Theorem 4.3.1.

2. If strong duality holds and the solution of (4.40) is unique, then this is also a solution of the primal problem $(P(z))$. In such a case we obtain $d(z^k)$ for all $z \in \mathcal{B}(z^*, \bar{\epsilon})$. Hence, Theorem 4.3.1 holds. $\triangle$

4.5 Results under Convexity/ Non-singularity Assumptions

In this section we will give sufficient conditions for some assumptions used in this chapter. The next lemma presents a sufficient condition for Assumption 4.3.3.

Lemma 4.5.1 *If, for some $i \in \mathfrak{I}$, the function F_i is strongly convex then Assumption 4.3.3 holds.*

Proof: Take any $x^{\triangleright} \in \mathbb{R}^n$. Now, strong convexity of F_i implies that the level set

$$\mathcal{L}_{F_i}(x^{\triangleright}) := \{x \in \mathbb{R}^n | F_i(x) \leq F_i(x^{\triangleright})\}$$

is bounded (see for example [7, Section 9.1.2]). Hence the statement of the lemma follows by noting that $\mathcal{L}_F(x^{\triangleright}) \subseteq \mathcal{L}_{F_i}(x^{\triangleright})$. △

Some of the sufficient conditions for Assumption 4.3.1 to hold are given in Chapter 3 (see Lemma 3.2.1, 3.2.2 and 3.4.1). In the rest of this section we discuss Assumption 4.3.2.

Blanket Assumption for Section 4.5: *Assumptions 4.2.1 and 4.3.3 hold. Moreover, all the objective functions F_i, $i \in \mathfrak{I}$ are convex.*

As discussed at the beginning of Chapter 3, we then have that the set of Pareto-critical points is equal to the set of weak Pareto-optimal points, i.e.

$$X_{pc} = X_w.$$

We first discuss a condition under which $S(z)$ is nonempty in a neighborhood of z^*. Since $S(z) \subseteq z + \Omega(z)$, $S(z) \neq \emptyset$ implies $\Omega(z) \neq \emptyset$.

Lemma 4.5.2 *Let $\alpha \in \left(0, \frac{1}{m}\right]$ and $(x, \lambda) \in \mathcal{L}_F(x^{\triangleright}) \times \Lambda_\alpha$ with $x^{\triangleright}$ from Assumption 4.3.3 be given. Then, the single-objective problem $(SP(x, \lambda))$*

$$\begin{aligned} \max_{(y,t)\in\mathbb{R}^n\times\mathbb{R}} \phi(y,t) &:= t \\ s.t.\ F(y) &= F(x) - t\lambda \\ t &\geq 0, \end{aligned}$$

has at least one solution. Moreover, let $(\bar{x}, \bar{t})$ be a solution of $(SP(x, \lambda))$, then

$$0 \leq F_i(x) - F_i(\bar{x}) \leq \frac{1}{\alpha}\|H(x, \nu(x))\|\|x - \bar{x}\| \text{ for all } i \in \mathfrak{I}. \tag{4.45}$$

Proof: Let us first consider the case when x is weakly Pareto-optimal. Then, from the definition of weak Pareto-optimality, it easily follows that $(\bar{x}, \bar{t}) := (x, 0)$ is a solution of $(SP(x, \lambda))$. Moreover, for any solution $(\bar{x}, \bar{t}) := (x, 0)$ of $(SP(x, \lambda))$ it holds that $F(\bar{x}) = F(x)$ and $\bar{t} = 0$. Weak Pareto-optimality of x also gives that $H(x, \nu(x)) = 0$. Thus, in this case, (4.45) holds.

In the remaining part of the proof we assume that $x \notin X_w$. Let

$$\Psi := \{(y, t) \in \mathbb{R}^n \times \mathbb{R} | F(y) = F(x) - t\lambda, t \geq 0\}$$

be the constraint set for $(SP(x, \lambda))$. It is easy to see that $(x, 0) \in \Psi$ and hence Ψ is nonempty. Let $(y, t) \in \Psi$ be an arbitrary point. As

$$F(y) \leq F(x) \leq F(x^{\triangleright}),$$

we have that $y \in \mathcal{L}_F(x^{\triangleright})$. Hence, from Assumption 4.3.3, the set

$$\{y \in \mathbb{R}^n | \exists t \geq 0, (y, t) \in \Psi\}$$

is bounded. With this, we obtain the boundedness of $\{F(y) | \exists t \geq 0, (y, t) \in \Psi\}$. Thus,

$$\{t \in \mathbb{R} | \exists y \in \mathbb{R}^n, (y, t) \in \Psi\}$$

is bounded. Together, we have that Ψ is bounded. Since Ψ is also closed, it Ψ is compact. Furthermore, note that ϕ is a continuous function. Hence, by the theorem of Weierstrass, a point $(\bar{x}, \bar{t}) \in \Psi$ exists where the maximum of ϕ on Ψ is attained.

For any solution $(\bar{x}, \bar{t})$ of $(SP(x, \lambda))$, we obviously have

$$0 \leq F_i(x) - F_i(\bar{x}) \quad \text{for all } i \in \mathfrak{I}. \tag{4.46}$$

Since $\nu(x)$ is a minimizer of $QP(x)$ (see 4.3), for any $\lambda \in \Lambda_0$ we obtain

$$\left\| \sum_{i=1}^{m} \lambda_i \nabla F_i(x) \right\| = \|H(x, \lambda)\| \geq \|H(x, \nu(x))\|.$$

As $x \notin X_w$ we have $\|H(x, \nu(x))\| \neq 0$ and we can define the unit vector

$$\theta := \frac{H(x, \nu(x))}{\|H(x, \nu(x))\|}. \tag{4.47}$$

Now, for any $y \in \mathbb{R}^n$, using (4.47) and convexity of F_i for all $i \in \mathcal{I}$ (see Blanket Assumption for this section) we obtain

$$\nu(x)^\top F(y) - \nu(x)^\top F(x) \geq \left(\sum_{i=1}^{m} \nu(x)_i \nabla F_i(x) \right)^\top (y-x) = \|H(x,\nu(x))\| \theta^\top (y-x).$$

Since $\|\theta\| = 1$, the Cauchy-Schwartz inequality gives

$$|\theta^\top (y-x)| \leq \|y-x\| \text{ for all } y \in \mathbb{R}^n.$$

Hence for all $y \in \mathbb{R}^n$ we obtain

$$\nu(x)^\top F(y) + \|H(x,\nu(x))\| \|y-x\| \geq \nu(x)^\top F(x). \tag{4.48}$$

Claim:

$$\bar{t} \leq \frac{1}{\alpha} \|H(x,\nu(x))\| \|\bar{x}-x\|. \tag{4.49}$$

To show this, assume on the contrary that

$$\bar{t} > \frac{1}{\alpha} \|H(x,\nu(x))\| \|\bar{x}-x\|.$$

Hence, since $F(x) - F(\bar{x}) = \bar{t}\lambda$ we obtain

$$F(\bar{x}) + \frac{1}{\alpha} \|H(x,\nu(x))\| \|\bar{x}-x\| \lambda < F(x). \tag{4.50}$$

Since $\lambda \in \Lambda_\alpha$, we have that $\lambda_i \geq \alpha$ for all $i \in \mathcal{I}$. This together with (4.50) yields

$$F(\bar{x}) + \|H(x,\nu(x))\| \|\bar{x}-x\| 1_m < F(x), \tag{4.51}$$

where by 1_m we denote the vector in $\mathbb{R}^m$ having 1 in all its components. Multiplying this by $\nu(x)^\top$ further gives

$$\nu(x)^\top F(\bar{x}) + \|H(x,\nu(x))\| \|\bar{x}-x\| < \nu(x)^\top F(x), \tag{4.52}$$

since $\nu(x) \in \Lambda_0$ implies $\nu(x)^\top 1_m = 1$. This contradicts to (4.48). Thus, (4.49) is proved and we have

$$F_i(x) - F_i(\bar{x}) = \lambda_i \bar{t} \leq \bar{t} \leq \frac{1}{\alpha} \|H(x,\nu(x))\| \|\bar{x}-x\|,$$

for all $i \in \mathcal{I}$. $\triangle$

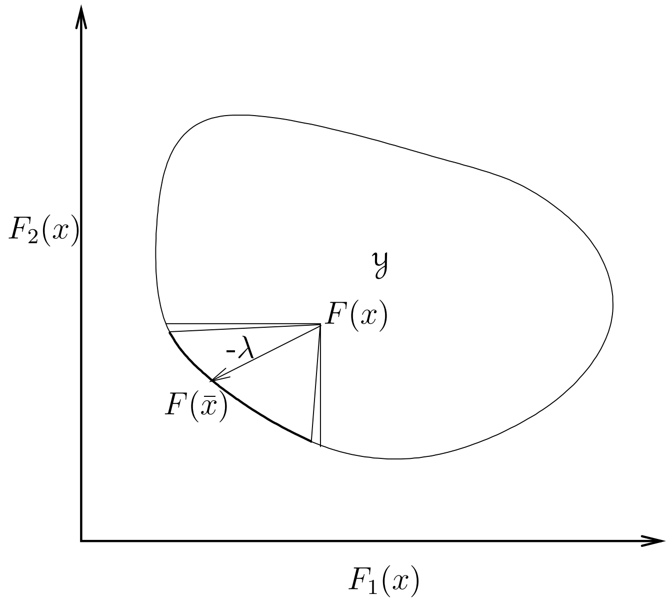

Figure 4.4: Illustration of the point $F(\bar{x})$ obtained by solving $(SP(x,\lambda))$ in the bi-objective space. The dark portion of the efficient front is obtainable by solving $(SP(x,\lambda))$ with $\lambda \in \Lambda_\alpha$ for some $\alpha > 0$

.

We next describe some characterization of the efficient front. We recall that X_p and X_w denote the set of all Pareto-optimal and weakly Pareto-optimal points, respectively.

Lemma 4.5.3 *If F_i is strictly convex for at least $i \in \mathfrak{I}$, then the mapping F restricted on the domain X_p is injective.*

Proof: Assume on the contrary that the restriction of mapping F on X_p is not injective. Hence

$$F(x^*) = F(y^*),$$

for some distinct $x^*, y^* \in X_p$. This together with the convexity of all the objective functions and strict convexity of F_i gives

$$F_j\left(\frac{x^* + y^*}{2}\right) \leq \frac{1}{2}F_j(x^*) + \frac{1}{2}F_j(y^*) = F_j(x^*),$$

for all $j \in \mathfrak{I}$ with strict inequality for at least one $j = i$. This gives a contradiction to the Pareto-optimality of x^*. $\triangle$

Remark 4.5.1 *If we assume that all the functions F_i, $i \in \mathcal{I}$ are strictly convex then, in a similar way, we obtain that the restriction of mapping F on X_w is injective.*

Property 4.5.1 *Let $\alpha \in \left(0, \frac{1}{m}\right]$ be given. Then, there is $\tilde{\delta} \in (0, \delta]$ so that $\mathcal{B}\left(x^*, \tilde{\delta}\right) \subseteq \mathcal{L}_F(x^{\triangleright})$. Moreover, for any $x \in \mathcal{B}\left(x^*, \tilde{\delta}\right)$ there is $\lambda^x \in \Lambda_\alpha$ so that $x^s \in X_p$, where (x^s, t^s) is a solution of $(SP(x, \lambda^x))$.*

For an arbitrary but fixed $\alpha \in \left(0, \frac{1}{m}\right]$ and $(x, \lambda) \in \mathcal{L}_F(x^{\triangleright}) \times \Lambda_0$, solving $(SP(x, \lambda))$ and finding the corresponding $\bar{x}$ amounts to shooting a ray from the point $F(x)$ in the objective space in the direction $-\lambda$ and obtaining the corresponding point $F(\bar{x})$ on the *boundary* of $\mathcal{Y}$ (see Figure 4.4 for an illustration for the case of two objective functions). Since all the efficient points are a subset of the boundary of $\mathcal{Y}$, it is reasonable that $\bar{x}$ belongs to X_p, especially if x is taken from a sufficiently small neighborhood of a Pareto-optimal point. Thus Property 4.5.1 seems reasonable and quite weak. The direction of the ray $-\lambda \in -\mathbb{R}^m_+$ and the fact that $(SP(x, \lambda))$ is a maximization problem, makes sure that $F(x^s) \in F(x) - \mathbb{R}^m_+$. Such direction based approaches for solving an (MOP) have been emphasized in the last four decades [2; 11; 44; 50; 57; 58; 61].

In our next remark we present a sufficient condition for Property 4.5.1 to hold. Before this, we recall from Chapter 3 that corresponding to any $\lambda \in \Lambda$, $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is the weighted objective function defined as

$$f_\lambda(x) := \sum_{i=1}^{m} \lambda_i F_i(x).$$

Remark 4.5.2 *If $\lambda^* > 0$, the matrix $\nabla^2 f_{\lambda^*}(x^*)$ is non-singular, $\nabla F(x^*)$ is of rank $m-1$ and $\nabla F(x)$ is of rank m for some $x \in \mathcal{B}(x^*, \delta)$ then, Property 4.5.1 holds. This statement can be shown using concepts from differential geometry ([67]) and from [36, Theorem 2.2]. The proof is involved and is not presented here.*

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, consider the following unconstrained nonlinear programming problem (NLP)

$$\min f(x).$$

For this problem, the notion of a strict local minimizer of order p is defined as follows (see for example [9; 65]).

Definition 4.5.1 *A point $x^\diamond \in \mathbb{R}^n$ is a strict local minimizer of order $p \geq 1$ for the (NLP) if there exist constants $\alpha^\diamond, \delta^\diamond > 0$ so that*

$$f(x) > f(x^\diamond) + \alpha^\diamond \|x - x^\diamond\|^p \quad \text{for all } x \in \mathcal{B}(x^\diamond, \delta^\diamond) \setminus \{x^\diamond\}. \tag{4.53}$$

This classical notion has been extended to multi-objective problems in the following way (see [40]).

Definition 4.5.2 *A point $x^\diamond \in \mathbb{R}^n$ is a strict local minimizer of order $p \geq 1$ for (MOP) if there exist constants $\alpha^\diamond, \delta^\diamond > 0$ so that*

$$(F(x) + \mathbb{R}^m_+) \cap \mathcal{B}(F(x^\diamond), \alpha^\diamond \|x - x^\diamond\|^p) = \emptyset \quad \text{for all } x \in \mathcal{B}(x^\diamond, \delta^\diamond) \setminus \{x^\diamond\}. \tag{4.54}$$

Note that for an (MOP) any strict local minimizer of any order $p \geq 1$ is Pareto-optimal (see [40, Proposition 3.3]). Our next lemma relates a strict local minimizer of order p of a scalarized problem to a strict local minimizer of order p for an (MOP).

Lemma 4.5.4 *Let, for some $\lambda \in \Lambda_0$, $x^\diamond$ be a strict local minimizer of order p (with constants $\alpha^\diamond$ and $\delta^\diamond$) for the following unconstrained problem*

$$\min f_\lambda(x). \tag{4.55}$$

Then, $x^\diamond$ is a strict local minimizer of order p for (MOP) with the constants $\alpha^\diamond$ and $\delta^\diamond$.

Proof: Let $x^\diamond$ be a strict local minimizer of order p for problem (4.55) with some constants $\alpha^\diamond, \delta^\diamond > 0$. Using Definition 4.5.1 this translates to

$$f_\lambda(x) > f_\lambda(x^\diamond) + \alpha^\diamond \|x - x^\diamond\|^p \quad \text{for all } x \in \mathcal{B}(x^\diamond, \delta^\diamond) \setminus \{x^\diamond\}. \tag{4.56}$$

Now assume that the assertion of the lemma does not hold. Hence, from Definition 4.5.2 we obtain that there exists an $\hat{x} \in \mathcal{B}(x^\diamond, \delta^\diamond) \setminus \{x^\diamond\}$ such that

$$(F(\hat{x}) + \mathbb{R}^m_+) \cap \mathcal{B}(F(x^\diamond), \alpha^\diamond \|\hat{x} - x^\diamond\|^p) \neq \emptyset. \tag{4.57}$$

This further implies that there exist $d \in \mathbb{R}^m_+$ and $b \in \mathcal{B}(0_n, \alpha^\diamond \|\hat{x} - x^\diamond\|^p)$ such that

$$F(\hat{x}) + d = F(x^\diamond) + b.$$

The inner product of both sides with λ yields

$$f_\lambda(\hat{x}) - f_\lambda(x^\diamond) - \lambda^\top b = -\lambda^\top d \leq 0.$$

Further simplification gives

$$\begin{aligned}
f_\lambda(\hat{x}) &\leq f_\lambda(x^\diamond) + \langle \lambda, b \rangle \\
&\leq f_\lambda(x^\diamond) + \|\lambda\| \|b\| \quad \text{(using the Cauchy-Schwartz inequality)} \\
&\leq f_\lambda(x^\diamond) + \|b\| \quad (\text{since } \|\lambda\| \leq 1) \\
&\leq f_\lambda(x^\diamond) + \alpha^\diamond \|x - x^\diamond\|^p \quad (\text{since } b \in \mathcal{B}(0_n, \alpha^\diamond \|\hat{x} - x^\diamond\|^p)).
\end{aligned}$$

This is a contradiction to (4.56). Hence the statement of the lemma follows. △

Remark 4.5.3 *A similar result can also be found in [47, Proposition 5.1]. The main difference to [47] is that Lemma 4.5.4 shows that the constants $\alpha^\diamond$ and $\delta^\diamond$ are the same for the scalarized problem (4.55) and the (MOP). In [47, Proposition 5.1] just the existence of these constants for the (MOP) is shown. Hence the result of Lemma 4.5.4 is more general than that of [47]. Moreover, we will explicitly make use of these constants in the next lemma.*

Lemma 4.5.5 *Let the matrix $\nabla^2 f_{\lambda^*}(x^*)$ be positive definite. Then, there exist constants $\alpha^\diamond, \delta^\diamond, \delta_1 > 0$ so that*

$$(F(x) + \mathbb{R}^m_+) \cap \mathcal{B}(F(x^\diamond), \alpha^\diamond \|x - x^\diamond\|^2) = \emptyset$$

for all $x^\diamond \in \mathcal{B}(x^*, \delta_1) \cap X_w$ and for all $x \in \mathcal{B}(x^\diamond, \delta^\diamond) \setminus \{x^\diamond\}$.

Proof: Let $M(z) := \nabla^2 f_\lambda(x)$. By Assumption 4.2.1, $M(z)$ is a continuous function of z. Since the matrix $\nabla^2 f_{\lambda^*}(x^*)$ is assumed to be positive definite we obtain that $u^\top M(z^*) u \geq \chi \|u\|^2$ for all $u \in \mathbb{R}^n$ with some $\chi > 0$. By continuity of M there exists a $\bar{\delta}_1 > 0$ so that

$$u^\top M(z) u \geq \frac{\chi}{2} \|d\|^2 \quad \text{for all } u \in \mathbb{R}^n \text{ and all } z \in \mathcal{B}(z^*, \bar{\delta}_1). \tag{4.58}$$

Let $\delta_1 := \min\{\bar{\delta}, \delta\}$. Take some arbitrary but fixed $z^\diamond \in \mathcal{B}(z^*, \delta_1) \cap Z$. A Taylor's expansion of $f_{\lambda^\diamond}$ around $x^\diamond$ together with [15, Lemma 4.1.14] shows that

$$f_{\lambda^\diamond}(x^\diamond + d_{x^\diamond}) = f_{\lambda^\diamond}(x^\diamond) + \nabla f_{\lambda^\diamond}(x^\diamond)^\top d_{x^\diamond} + \frac{1}{2} d_{x^\diamond}^\top \nabla^2 f_{\lambda^\diamond}(x^\diamond) d_{x^\diamond} + O(\|d_{x^\diamond}\|^3), \tag{4.59}$$

and

$$|O(\|d_{x^\diamond}\|^3)| \leq \frac{L}{6} \|d_{x^\diamond}\|^3 \tag{4.60}$$

hold. Since $(x^\diamond, \lambda^\diamond) \in Z$ we obtain

$$\nabla f_{\lambda^\diamond}(x^\diamond) = 0. \tag{4.61}$$

Moreover, using (4.58) we obtain

$$\frac{1}{2}d_{x^\diamond}^\top \nabla^2 f_{\lambda^\diamond}(x^\diamond) d_{x^\diamond} \geq \frac{1}{4}\chi \|d_{x^\diamond}\|^2. \tag{4.62}$$

It is easy to see that (4.17) is also valid if F_i is replaced by $f_{\lambda^\diamond}$. Now, in (4.59), using (4.61), (4.62) and (4.17) for $f_{\lambda^\diamond}$ we easily obtain

$$f_{\lambda^\diamond}(x^\diamond + d_{x^\diamond}) > f_{\lambda^\diamond}(x^\diamond) + \frac{1}{8}\chi\|d_{x^\diamond}\|^2 \quad \forall d_{x^\diamond} \in \mathcal{B}\left(0_n, \frac{3\chi}{8L}\right) \setminus \{0\}. \tag{4.63}$$

Hence $x^\diamond$ is a strict local minimizer of order 2 for the unconstrained problem

$$\min f_{\lambda^\diamond}(x).$$

A simple application of Lemma 4.5.4 and setting $\alpha^\diamond := \frac{1}{8}\chi$, $\delta^\diamond := \frac{3\chi}{8L}$ gives the desired result. △

We next state a continuity type assumption on the objective function.

Property 4.5.2 *For any $\bar{\delta} \in (0, \delta]$, there is a $\tau(\bar{\delta}) \in (0, \bar{\delta}]$ so that for all $u \in \mathcal{B}\left(F(x^*), \tau(\bar{\delta})\right) \cap \mathcal{Y}$ there is $x^u \in \mathcal{B}(x^*, \bar{\delta})$ with $F(x^u) = u$.*

This property relates points in the objective space close to $F(x^*)$ to points in the variable space close to x^* and seems quite weak. We can easily verify that most of the problems in [12; 39] satisfy Property 4.5.2. Moreover, the next lemma presents a sufficient condition for Property 4.5.2 to hold.

Lemma 4.5.6 *Let $\lambda^* > 0$, $F(x^*) \in \operatorname{int}\left(F(x^\triangleright) - \mathbb{R}^m_+\right)$ and let for some $i \in \mathcal{I}$, F_i be strictly convex. Then, Property 4.5.2 holds.*

Proof: Since $\lambda^* > 0$, we have that $x^* \in X_p$ (see Section 3.1). Under these assumptions by Lemma 4.5.3 we obtain that x^* is the only solution of the equation

$$F(x) - F(x^*) = 0. \tag{4.64}$$

Although Lemma 4.5.3 says that F is injective only on the set X_p, we do not need to put the $x \in X_p$ constraint to (4.64) because any $x \in \mathbb{R}^n$ with $F(x) = F(x^*)$ belongs to X_p by definition.

Now, corresponding to a parameter $u \in \mathbb{R}^m$, let us define the following parametric problem $(PM(u))$

$$\begin{array}{ll} \min\limits_{x} & \|F(x) - u\| \\ \text{s.t.} & x \in \mathcal{L}_F(x^\triangleright). \end{array}$$

Let $\mathcal{S}(u)$ denote the set of global minimizers of $(PM(u))$. Note that since $F(x^*) \in \operatorname{int}\left(F(x^\triangleright) - \mathbb{R}^m_+\right)$ is assumed, we have that $x^* \in \mathcal{L}_F(x^\triangleright)$. This together with (4.64) yields

$$\mathcal{S}(F(x^*)) = \{x^*\}. \tag{4.65}$$

Moreover, we easily see that

$$y \in \mathcal{S}(F(y)) \quad \text{for all } y \in \mathcal{L}_F(x^\triangleright). \tag{4.66}$$

Now, as the set $\mathcal{L}_F(x^\triangleright)$ is bounded (from Assumption 4.3.3), this parametric problem satisfies both the *Local Compactness* and *Constraint Qualification* conditions described in [63] for $u^* := F(x^*)$. Hence [63, Lemma 6] gives that the (set-valued) mapping $\mathcal{S}$ is outer semi-continuous (osc) at u^*.

From the definition of outer semi-continuity (see [63, Definition 2b]) we therefore obtain that, for *any* sequences $\{\tilde{u}^l\} \subset \mathbb{R}^m$, $\{\tilde{x}^l\} \subset \mathbb{R}^n$, $l \in \mathbb{N}$ with $\tilde{u}^l \to u^*$, $\tilde{x}^l \in \mathcal{S}(\tilde{u}^l)$, it holds that

$$\|\tilde{x}^l - x^*\| \to 0 \tag{4.67}$$

as $l \to \infty$. Since $F(x^*) \in \operatorname{int}\left(F(x^\triangleright) - \mathbb{R}^m_+\right)$ there is a $\delta^\square > 0$ so that

$$\left(\mathcal{B}(F(x^*), \delta^\square) \cap \mathcal{Y}\right) \subset \operatorname{int}\left(F(x^\triangleright) - \mathbb{R}^m_+\right). \tag{4.68}$$

Let us assume on the contrary that Property 4.5.2 does not hold. Then, a $\bar{\delta} \in (0, \delta]$ exists so that, for any $l \in \mathbb{N}$, there is $u^l \in \left(\mathcal{B}(F(x^*), \frac{\delta^\square}{l}\right) \cap \mathcal{Y}$ so that

$$\left\{x \in \mathbb{R}^n | F(x) = u^l\right\} \cap \mathcal{B}(x^*, \bar{\delta}) = \emptyset. \tag{4.69}$$

As $u^l \in \left(\mathcal{B}(F(x^*), \frac{\delta^\square}{l}\right) \cap \mathcal{Y}$ and $l \in \mathbb{N}$, (4.68) shows that

$$u^l \in \operatorname{int}\left(F(x^\triangleright) - \mathbb{R}^m_+\right).$$

Hence, corresponding to the sequence $\{u^l\}$, there is a sequence $\{x^l\} \subset \mathcal{L}_F(x^\triangleright)$ so that $F(x^l) = u^l$.

Setting $y := x^l$ in (4.66) yields

$$x^l \in \mathcal{S}(F(x^l)) \quad \text{for all } l \in \mathbb{N}. \tag{4.70}$$

Moreover, we easily see that, for all $l \in \mathbb{N}$,

$$\mathcal{S}(u^l) = \mathcal{S}(F(x^l)) \subseteq \left\{x \in \mathbb{R}^n | F(x) = u^l\right\}$$

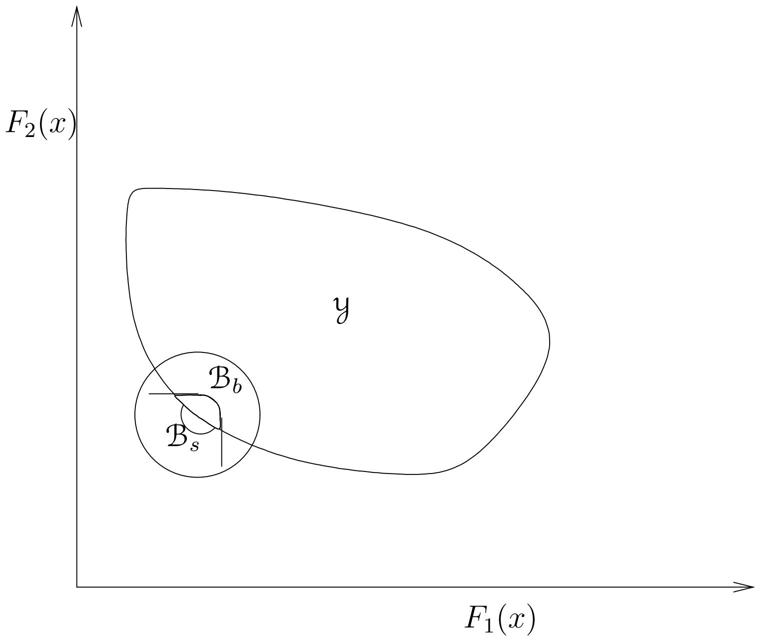

Figure 4.5: Illustration of Property 4.5.3. $\mathcal{B}_s := \mathcal{B}\left(F(x^*), \frac{\bar{\delta}}{w}\right)$ and $B_b := \mathcal{B}(F(x^*), \bar{\delta})$.

holds. With this (4.69) gives

$$\mathcal{S}(F(x^l)) \cap \mathcal{B}(x^*, \bar{\delta}) = \emptyset,$$

and, using (4.70) we further obtain that

$$\{x^l\} \cap \mathcal{B}(x^*, \bar{\delta}) = \emptyset \quad \text{for all } l \in \mathbb{N}. \tag{4.71}$$

Setting $\tilde{u}^l := u^l$, $\tilde{x}^l := x^l$ and taking into account (4.70), (4.67) yields

$$\|x^l - x^*\| \to 0,$$

as $l \to \infty$. This is a contradiction to (4.71) and hence the assertion of the lemma follows. $\triangle$

Remark 4.5.4 *As F is locally Lipschitz continuous, from Property 4.5.2 we easily see that $\tau(\bar{\delta}) \to 0$ as $\bar{\delta} \to 0$.*

Property 4.5.3 *There is a $w \geq \frac{1}{L}$, so that for any $\bar{\delta} \in (0, \delta]$*

$$\mathcal{Y} \cap \left(\mathcal{B} \left(F(x^*), \frac{\bar{\delta}}{w} \right) - \mathbb{R}^m_+ \right) \subseteq \mathcal{B}(F(x^*), \bar{\delta}) \cap \mathcal{Y}. \tag{4.72}$$

Property 4.5.3 is illustrated in Figure 4.5.

Remark 4.5.5 *If $\lambda^* > 0$ and the matrix $\nabla^2 f_{\lambda^*}(x^*)$ is non-singular then we can show that Property 4.5.3 holds. This constant w depends upon the ratios $\frac{\lambda_i}{\lambda_j}$, $i \neq j$ for λ vectors in a ball around λ^* of sufficiently small radius. The idea of the proof is geometrical and is not presented here.*

Lemma 4.5.7 *Let Properties 4.5.1, 4.5.2 and 4.5.3 hold, let the matrix $\nabla^2 f_{\lambda^*}(x^*)$ be positive definite and, for one $i \in \mathcal{I}$, let F_i be strictly convex. Then, there exist constants $\alpha^\diamond, \delta^\diamond > 0$ so that*

$$F_i(x) > F_i(x^s) + \alpha^\diamond \|x - x^s\|^2 \quad \textit{for all } i \in \mathcal{I},\ x \in \mathcal{B}(x^*, \delta^\diamond) \setminus \{x^s\}, \tag{4.73}$$

where $x^s \in X_p$ comes from Property 4.5.1.

Proof: As $\nabla^2 f_{\lambda^*}(x^*)$ is assumed to be positive definite, from Lemma 4.5.5 we obtain the constants $\alpha^\diamond > 0$, $\delta_1 > 0$ and $\delta^\diamond > 0$. Let

$$\hat{\delta}_1 := \frac{\min\{\delta_1, \delta^\diamond, \tilde{\delta}\}}{2} > 0 \text{ and } \delta^\diamond := \frac{\tau(\hat{\delta}_1)}{wL} > 0, \tag{4.74}$$

where $\tilde{\delta}$ is from Property 4.5.1, $\tau(\hat{\delta}_1) \in (0, \hat{\delta}_1]$ is from Property 4.5.2 and $w > 0$ is from Property 4.5.3. From (4.16) we obtain that

$$F\left(\mathcal{B}(x^*, \delta^\diamond)\right) \subseteq \mathcal{B}\left(F(x^*), \frac{\tau(\hat{\delta}_1)}{w} \right). \tag{4.75}$$

Now take an arbitrary but fixed

$$x \in \mathcal{B}(x^*, \delta^\diamond) \setminus \{x^s\}, \tag{4.76}$$

where $x^s \in X_p$ comes from Property 4.5.1 (note that $x^s \in X_p$ is well-defined as $\delta^\diamond \leq \tilde{\delta}$). From (4.75) we obtain that

$$F(x) \in \mathcal{B}\left(F(x^*), \frac{\tau(\hat{\delta}_1)}{w} \right). \tag{4.77}$$

Moreover, from Lemma 4.5.2, it is clear that

$$F(x^s) \in F(x) - \mathbb{R}^m_+. \tag{4.78}$$

This together with (4.77) yields

$$F(x^s) \in \mathcal{B}\left(F(x^*), \frac{\tau(\hat{\delta}_1)}{w}\right) - \mathbb{R}^m_+. \tag{4.79}$$

Using Property 4.5.3 with $\bar{\delta} := \tau(\hat{\delta}_1)$, we further obtain

$$u := F(x^s) \in \mathcal{B}(F(x^*), \tau(\hat{\delta}_1)) \cap \mathcal{Y}. \tag{4.80}$$

From (4.80) and using Property 4.5.2 we obtain $x^u \in \mathcal{B}(x^*, \hat{\delta}_1)$ such that $F(x^u) = F(x^s)$. As for some $i \in \mathcal{I}$, F_i is assumed to be strictly convex and $x^s \in X_p$, invoking Lemma 4.5.3,

$$x^u := x^s$$

follows. Hence, we have that

$$x^s \in \mathcal{B}(x^*, \hat{\delta}_1). \tag{4.81}$$

From the triangle inequality, (4.81), (4.76), (4.74) and from Property 4.5.2 we obtain

$$\|x - x^s\| \leq \|x^s - x^*\| + \|x^* - x\| \leq \hat{\delta}_1 + \delta^\diamond \leq 2\hat{\delta}_1 \leq \delta^\diamond, \tag{4.82}$$

and hence, $x \in \mathcal{B}(x^s, \delta^\diamond)$ follows. This together with (4.81) and Lemma 4.5.5 (with $x^\diamond := x^s$) yields

$$(F(x) + \mathbb{R}^m_+) \cap \mathcal{B}(F(x^s), \alpha^\diamond \|x - x^s\|^2) = \emptyset.$$

This translates to

$$\|F(x) - F(x^s)\| > \alpha^\diamond \|x - x^s\|^2. \tag{4.83}$$

Now, recall from Property 4.5.1 that (x^s, t^s) is a solution of $(SP(x, \lambda^x))$. This implies that

$$F(x) - F(x^s) = t^s \lambda^x, \tag{4.84}$$

with $t^s > 0$ and $\lambda^x \in \Lambda_\alpha$. This together with (4.83) shows that

$$t^s > \frac{\alpha^\diamond \|x - x^s\|^2}{\|\lambda^x\|} \geq \alpha^\diamond \|x - x^s\|^2, \tag{4.85}$$

where the last inequality follows as $\max \|\lambda\| = 1$.

From (4.85) and (4.83) we obtain that

$$F_i(x) - F_i(x^s) = t^s \lambda_i^x \geq t^s \alpha > \alpha\alpha^\diamond \|x - x^s\|^2.$$

The statement of the lemma follows by setting $\alpha^\diamond := \alpha\alpha^\diamond$. △

Next we provide a new error bound result.

Lemma 4.5.8 *Let Properties 4.5.1, 4.5.2 and 4.5.3 hold, let the matrix $\nabla^2 f_{\lambda^*}(x^*)$ be positive definite and, for one $i \in \mathcal{I}$, let F_i be strictly convex. Then, there exists $\bar{C} > 0$ so that*

$$\|x - x^s\| \leq \bar{C}\|H(x, \nu(x))\| \qquad \text{for all } x \in \mathcal{B}(x^*, \delta^\diamond), \tag{4.86}$$

where x^s is from Property 4.5.1 and $\delta^\diamond$ is from Lemma 4.5.7.

Proof: If $x \in X_w$ then, by (4.45), (4.86) holds. The statement of this lemma, for $x \notin X_w$ follows by an immediate application of Lemma 4.5.2 (set $\bar{x} := x^s$), Lemma 4.5.7 and by setting $\bar{C} := \frac{1}{\alpha\alpha^\diamond}$. △

Lemma 4.5.9 *Let Properties 4.5.1, 4.5.2 and 4.5.3 hold, the matrix $\nabla^2 f_{\lambda^*}(x^*)$ be positive definite and for one $i \in \mathcal{I}$ let F_i be strictly convex. Then there exists $\delta^\star > 0$, so that $(x^s, \nu(x^s)) \in S(z)$ for all $z \in \mathcal{B}(z^*, \delta^\star)$.*

Proof: Since $S(z) := Z \cap \{z + d | d \in \Omega(z)\}$ we have to equivalently show that $(x^s, \nu(x^s)) \in Z$ and that $d := (x^s - x, \nu(x^s) - \lambda) \in \Omega(z)$ for all $z \in \mathcal{B}(z^*, \delta^\star)$ with $\delta^\star > 0$ sufficiently small. Obviously $(x^s, \nu(x^s)) \in Z$ since x^s is weakly Pareto-optimal under the assumptions (and hence $H(x^s, \nu(x^s)) = 0$).

By Lemma 4.5.8,

$$\|x^s - x\| \leq \|H(x, \nu(x))\|^{0.9}$$

holds for all points in a sufficiently small neighborhood around x^*. Let us take an arbitrary but fixed $i \in \mathcal{I}$. Then, for all x in a sufficiently small neighborhood around x^*, Taylor's theorem together with (4.73) and (4.18) yields,

$$\begin{aligned}
\nabla F_i(x)^\top (x^s - x) + \frac{1}{2}(x^s - x)^\top \nabla^2 F_i(x)(x^s - x) &= F_i(x^s) - F_i(x) + O(\|x^s - x\|^3) \\
&\leq -\alpha^\diamond \|x - x^s\|^2 + O(\|x^s - x\|^3) \\
&\leq -\|x^s - x\|^{2.4} \\
&\leq -\|H(x^s, \nu(x^s) - H(x, \nu(x))\|^{2.5} \\
&= -\|H(x, \nu(x))\|^{2.5}.
\end{aligned}$$

Hence, we see that $d \in \Omega(z)$ and the result of the lemma follows. △

Remark 4.5.6 *In a similar way, one can show that Lemma 4.5.9 holds also if we take $\Omega(z,p)$ instead of $\Omega(z)$ (see (4.38)).*

We present a Lipschitz property of ν as a function of x.

Lemma 4.5.10 *Let $\lambda^* > 0$ and the rank of the matrix $\nabla F(x^*)$ be $m-1$. Then, $\breve{L}, \breve{\delta} > 0$ exist so that for all $x \in \mathcal{B}(x^*, \breve{\delta}) \cap X_p$ problem $(QP(x))$ has a unique solution and*

$$\|\nu(x) - \nu(y)\| \leq \breve{L}\|x - y\| \qquad \text{for all } x, y \in \mathcal{B}(x^*, \breve{\delta}) \cap X_p. \tag{4.87}$$

Proof: Consider the convex problem $(\overline{QP}(x))$

$$\min_{\mathsf{v}} \left\| H\left(x, \left(\mathsf{v}, 1 - \sum_{i=1}^{m-1} \mathsf{v}_i\right)\right)\right\|^2 \quad \text{s.t. } \mathsf{v} \geq 0, \sum_{i=1}^{m-1} \mathsf{v}_i \leq 1 \tag{4.88}$$

as a parametric optimization problem with parameter x and let $\mathsf{v}(x) \in \mathbb{R}^{m-1}$ be a solution of $(\overline{QP}(x))$. Consider the linear system

$$H(x^*, \lambda) := \nabla F(x^*)\lambda = 0. \tag{4.89}$$

As the rank of $\nabla F(x^*)$ is $m-1$, the rank of the nullity of $\nabla F(x^*)$ equals one. Imposing the additional constraint $\sum_{i=1}^{m} \lambda_i = 1$ will thus give an unique solution of (4.89). This solution is given by $\nu(x^*) = \lambda^*$ as $z^* := (x^*, \lambda^*) \in Z$. Thus, we have that the set of minimizers of $(QP(x^*))$ is a singleton with

$$\left(\mathsf{v}(x^*), 1 - \sum_{i=1}^{m-1} \mathsf{v}_i\right) = \nu(x^*) = \lambda^*.$$

Moreover, we see that

$$\left(\nabla^2_{\bar{\lambda}} \left\| H\left(x, \left(\bar{\lambda}, 1 - \sum_{i=1}^{m-1} \bar{\lambda}_i\right)\right)\right\|^2\right)_{(x^*)} = A^\top A, \tag{4.90}$$

where A is the $n \times (m-1)$-matrix having $\nabla F_i(x^*) - \nabla F_m(x^*)$ as columns for all $i = 1, 2, \ldots, m-1$. From elementary linear algebra we have that ranks of $(A^\top A)$ and A are same and equal to $m-1$. Hence the square matrix $A^\top A$ is of full rank and nonsingular. Moreover, note that as $\lambda^* > 0$, we have that $\mathsf{v}(x^*)$ is an unconstrained minimizer of $(\overline{QP}(x^*))$. Now the desired result follows by using [63, Theorems 2,4]. △

Lemma 4.5.11 *Let Properties 4.5.1, 4.5.2 and 4.5.3 hold, the matrix* $\nabla^2 f_{\lambda^*}(x^*)$ *be positive definite and for one* $i \in \mathcal{I}$ *let* F_i *be strongly convex. If* $\lambda^* > 0$ *and* $\nabla F(x^*)$ *is of rank* $m-1$ *then, there exist* $c^{\triangleright}, \delta^{\triangle} > 0$ *such that*

$$\text{dist}\,[(x,\lambda), S(x,\lambda)] \leq \|(x,\lambda) - (x^s, \nu(x^s))\| \leq c^{\triangleright}\text{dist}\,[(x,\lambda), Z], \tag{4.91}$$

for all $z := (x,\lambda) \in \mathcal{B}(z^*, \delta^{\triangle}) \cap \Omega_0$ *and hence Assumption 4.3.2 is satisfied.*

Proof: The first part of (4.91) holds since by Lemma 4.5.9 $(x^s, \nu(x^s)) \in S(x,\lambda)$.

Recall from Chapter 3 that $X(\lambda)$ is the set defined as $X(\lambda) := \{x \in \mathbb{R}^n \,|\, (x,\lambda) \in Z\}$ and $\xi : \mathbb{R}^m \to \mathbb{R}^n$ is a function so that Conditions (i) and (ii) in Lemma 3.2.1 are satisfied. In particular, $\xi(\lambda) \in X(\lambda)$ holds.

As $\lambda^* > 0$ and one F_i, $i \in \mathcal{I}$ is strongly convex we obtain that the function f_λ, for all $\lambda > 0$ is strongly convex. The matrix $\nabla^2 f_\lambda(x)$ is nonsingular for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Hence, from Lemma 3.2.2 in Chapter 1, using $H = \nabla f_\lambda$, Error Bound for scalarized problems hold. Thus, there exist $c_1, r_1 > 0$ so that

$$\|H(x,\lambda)\| \geq c_1 \text{dist}\,[x, X(\lambda)] \text{ for all } z \in \mathcal{B}(z^*, r_1). \tag{4.92}$$

Let

$$\delta^{\triangle} \leq \min\left\{r_1, \check{\delta}, \delta^{\diamond}\right\},$$

be sufficiently small so that any $\tilde{\lambda} \in \mathcal{B}(\lambda^*, \delta^{\triangle})$ satisfy $\tilde{\lambda} > 0$. Consider an arbitrary but fixed $z \in \mathcal{B}(z^*, \delta^{\triangle})$. Now from (4.12) we obtain that

$$\text{dist}\,[(x,\lambda), Z] \geq L^{-1}\|H(x,\lambda)\|. \tag{4.93}$$

Moreover, the triangle inequality, (4.86) and (4.87) yield

$$\begin{aligned}
\|(x,\lambda) - (x^s, \nu(x^s))\| &\leq \|x - x^s\| + \|\lambda - \nu(x^s)\| \\
&\leq \bar{C}\|H(x,\nu(x))\| + \|\lambda - \nu(x^s)\| \\
&\leq \bar{C}\|H(x,\nu(x))\| + \check{L}\|\xi(\lambda) - x^s\| \\
&\leq \bar{C}\|H(x,\nu(x))\| + \check{L}(\|\xi(\lambda) - x\| + \|x - x^s\|).
\end{aligned}$$

Note that as $\lambda > 0$, we have that $\xi(\lambda) \in X_p$. Using (4.92) and (4.86) we can further simplify the last inequality as

$$\|(x,\lambda) - (x^s, \nu(x^s))\| \leq \bar{C}\|H(x,\nu(x))\| + \check{L}\left(\frac{\|H(x,\lambda)\|}{c_1} + \bar{C}\|H(x,\nu(x))\|\right).$$

This together with (4.93) gives

$$\frac{\|(x,\lambda)-(x^s,\nu(x^s))\|}{\operatorname{dist}\left[(x,\lambda),Z\right]} \leq \frac{\bar{C}\|H(x,\nu(x))\| + \breve{L}\left(\frac{\|H(x,\lambda)\|}{c_1} + \bar{C}\|H(x,\nu(x))\|\right)}{L^{-1}\|H(x,\lambda)\|}.$$

Hence the result of the Lemma follows by noting that $\|H(x),\nu(x)\| \leq \|H(x,\lambda)\|$ and defining

$$c^{\triangleright} := L\left(\bar{C} + \frac{\breve{L}}{c_1} + \breve{L}\bar{C}\right).$$

$\triangle$

4.6 Discussion

Let us consider the following convex bi-criteria problem (from [32]):

$$\min_{x\in\mathbb{R}} F(x) := \left(x, \sqrt{x^2+1}\right)^{\top}$$

For this problem $X_w = \{x|x \leq 0\}$. The weighted sum method is one of the most popular methods for solving multi-objective optimization problems [18]. The application of this method requires the decision maker to specify some a-priori weights for the objectives (not all zero) which are used to convert the multi-objective optimization problem into a single-criteria optimization problem. In general, the choice of the weight vector is of prime importance. For example, it can be easily checked that the scalarized optimization problem

$$\min_{x\in\mathbb{R}} f_\lambda(x) := \lambda_1 x + \lambda_2\sqrt{x^2+1}$$

has no solutions if $\lambda_1 \geq \lambda_2$ since then $\inf_{x\in\mathbb{R}} f_\lambda(x)$ does not exist. Hence, the weighted sum method fails for this example for all $\lambda \in \Lambda_0 \cap \{(\lambda_1,\lambda_2)|\lambda_1 \geq \lambda_2\}$. However, Algorithm 4.1 is well defined for this problem for any starting point and it can be easily verified that Assumption 4.3.1, 4.2.1, 4.3.2 and 4.3.3 hold. Moreover, it can be verified that Step (S2) is used locally. Thus, Algorithm 4.1 converges locally Q-quadratically to a weak-Pareto optimal point from any starting point.

To recapitulate, the algorithm presented in this chapter is a viable way to obtain Pareto-critical points with a local fast convergence. The solvability of subproblems of our algorithm and their relations to the subproblems in [30] has already been discussed in Section 4.2. Instead of requiring that all the objective functions are strongly convex as in [30], we need that only *one* objective functions is strongly convex. The convergence analysis of the method in [30] fails if one of the objective functions is not strongly convex ([30, Lemma 4.3] does not hold). Even for non-convex problems our method is able to converge to Pareto-critical points. However, the subproblems for a non-convex MOP are also non-convex and in practise these are difficult problems to solve.

As a final remark we mention that using results from [66], Algorithm 4.1 can be easily generalized when an arbitrary polyhedral ordering cone is used instead of $\mathbb{R}^m_+$.

Chapter 5

Levenberg-Marquardt Algorithms for Nonlinear Complementarity Problems

5.1 Introduction

For a given function $F : \mathbb{R}^n \to \mathbb{R}^n$, solving the nonlinear complementarity problem (NCP(F) in short) is to find a vector $x \in \mathbb{R}^n$ so that

$$x \geq 0, \quad F(x) \geq 0, \quad x^\top F(x) = 0. \tag{5.1}$$

When the function F is linear i.e., $F(x) := Mx + q$ (M being an $n \times n$ matrix and q an n-dimensional vector), NCP(F) is commonly known as the linear complementarity problem (LCP(M, q)). Complementarity problems (both linear and nonlinear) arise in a variety of engineering applications (see [22] for further details).

The solution set of NCP(F) is given by

$$S := \{x \in \mathbb{R}^n | x \geq 0, F(x) \geq 0, x^\top F(x) = 0\}.$$

A well-known approach for solving nonlinear complementarity problems is to reformulate them as an equivalent systems of nonlinear equations. This is achieved by means of a so called NCP function $\psi : \mathbb{R}^2 \to \mathbb{R}$ having the property that

$$\psi(a, b) = 0 \iff a \geq 0, \quad b \geq 0, \quad ab = 0 \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

A well-known NCP function is the min function, i.e., $\psi_{\min}(a, b) := \min\{a, b\}$. Over the years many more NCP functions have been proposed [22].

Using a given NCP function ψ, one possible way to reformulate NCP(F) is as follows:

$$H_\psi(z) = 0, \tag{5.2}$$

where

$$H_\psi(z) := \begin{pmatrix} \psi(x_1, y_1) \\ \vdots \\ \psi(x_n, y_n) \\ F_1(x) - y_1 \\ \vdots \\ F_n(x) - y_n \end{pmatrix}, \tag{5.3}$$

$z := (x^\top, y^\top)^\top \in \mathbb{R}^{2n}$ and $x, y \in \mathbb{R}^n$. For the i^{th} component $H_\psi(z)_i$ of $H_\psi(z)$ we then have

$$\psi(z_i, z_{n+i}) = \psi(x_i, y_i) \text{ for all } i \in \mathcal{I} := \{1, 2, \dots, n\}.$$

The solution set of (5.2) is denoted by Z, i.e.,

$$Z := \{z \in \mathbb{R}^{2n} | H_\psi(z) = 0\}.$$

It is easy to see that if $\hat{z} = (\hat{x}^\top, \hat{y}^\top)^\top \in Z$ then $\hat{x} \in S$ and if $\hat{x} \in S$ then $(\hat{x}^\top, F(\hat{x})^\top)^\top \in Z$. Hence we can obtain a solution of NCP(F) by solving (5.2) and vice versa.

In this chapter we investigate the use of a constrained Levenberg-Marquardt method (discussed in [43]) for solving (5.2). In [43] this method is suggested for solving the following constrained nonlinear equation

$$H(z) = 0, \quad \text{s.t. } z \in \Omega, \tag{5.4}$$

where $\Omega \subseteq \mathbb{R}^{2n}$ is some nonempty convex set. We restrict Ω to ensure that $Z \subseteq \Omega$. In this way the solution set of (5.2) and of (5.4) coincide when $H := H_\psi$. In [43], a smoothness condition on H_ψ near a solution of (5.4) is employed for showing local Q-quadratic convergence of the constrained Levenberg-Marquardt method. However, as we discuss later in detail, this smoothness condition does not hold for $H = H_\psi$ near some special types of solutions of (5.2) (known as degenerate solutions). Recently ([27]), this smoothness condition has been weakened so that the constrained Levenberg-Marquardt method has local Q-quadratic convergence if the NCP function ψ is defined as the min function. All the smoothness conditions are discussed later in detail.

In this chapter we use positively homogeneous NCP functions to investigate the smoothness properties of H_ψ. In particular, we examine for what class of

NCP functions the new smoothness conditions in [27] are satisfied. For the sake of brevity, if the usage is clear, we sometimes write (a, b) instead of $(a, b)^\top$ and (x, y) instead of $(x^\top, y^\top)^\top$, where $a, b \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

This chapter is divided into five sections of which this is the first. Some assumptions and preliminary results required for local convergence analysis of the constrained Levenberg-Marquardt method are discussed in the next section. Section 5.3 analyzes the smoothness assumptions in [43]. In Section 5.4 we extend some fundamental identities known for differentiable homogeneous functions to nonsmooth homogeneous functions. Using these identities, in Section 5.5 we discuss the new weaker smoothness condition of [27] for several choices of the NCP function ψ.

At the end of every section and sub-section we recapitulate and highlight the important results therein.

5.2 Preliminaries

First we state some assumptions required for local convergence analysis of the constrained Levenberg-Marquardt method (see [43] for the description of this method).

Assumption 5.2.1 *The function $F : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and $\nabla F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is locally Lipschitz continuous.*

Assumption 5.2.2 *The NCP function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous.*

Assumptions 5.2.1 and 5.2.2 are smoothness assumptions on F and ψ, respectively. The smoothness assumptions discussed in Section 5.3 and 5.5 are conditions on H_ψ. A basic assumption for local convergence analysis of the Levenberg-Marquardt method is given next. To this end let $z^* := (x^*, y^*) \in Z$ be fixed throughout this chapter. This implicitly means that Z is nonempty.

Assumption 5.2.3 *There are constants $C, \delta > 0$ so that*

$$C\,\mathrm{dist}\,[z, Z] \leq \|H_\psi(z)\| \quad \text{for all } z \in \mathcal{B}(z^*, \delta) \cap \Omega. \tag{5.5}$$

This assumption has been introduced in ([43]) for H instead of H_ψ. For $\Omega = \mathbb{R}^{2n}$, Assumption 5.2.3 is commonly known in the literature as *error bound* condition, see [26; 43; 69]. This condition may hold also in cases where the Jacobian of H_ψ (if it exists at all) is singular at z^*. It is easy to see that Assumptions 5.2.1 and

5.2.2 imply that H_ψ is locally Lipschitz continuous. Hence there is an $L > 0$ so that

$$\|H_\psi(z^1) - H_\psi(z^2)\| \le L\|z^1 - z^2\| \text{ for all } z^1, z^2 \in \mathcal{B}(z^*, \delta). \tag{5.6}$$

Blanket Assumption Chapter 5: *Assumptions 5.2.1, 5.2.2 and 5.2.3 hold. Moreover, Ω is a set of the form*

$$\Omega := \mathcal{S} \times \ldots \times \mathcal{S} \supseteq \mathbb{R}^{2n}_+, \tag{5.7}$$

where $\mathcal{S}$ is a closed convex cone in $\mathbb{R}^2$.

The convex cone $\mathcal{S}$ can always be described by

1. $\{(a,b) \in \mathbb{R}^2 | c^a a + c^b b \ge 0\}$, or
2. $\{(a,b) \in \mathbb{R}^2 | c_1^a a + c_1^b b \ge 0, c_2^a a + c_2^b b \ge 0\}$, or
3. $\mathbb{R}^2$,

with some constants $c^a, c^b, c_1^a, c_1^b, c_2^a, c_2^b \ge 0$. Under the blanket assumption it is clear that Ω is a polyhedral cone since $\mathcal{S}$ is polyhedral.

Definition 5.2.1 *Let $\mathcal{J} := \{i \in \mathcal{I} | x_i^* = y_i^* = 0\}$. The solution z^* is said to be degenerate if the index set $\mathcal{J}$ is nonempty. Otherwise, we call z^* non-degenerate. $\mathcal{J}$ is called the set of degenerate indices.*

For any $\hat{z} := (\hat{x}, \hat{y}) \in Z$, let the set $D(\hat{z}) \subseteq \mathbb{R}^{2n}$ be defined as

$$D(\hat{z}) := \{d | \exists (i, \hat{t}) \in \mathcal{I} \times (0, \infty), (\hat{x}_i, \hat{y}_i) + t(d_i, d_{n+i}) \neq (z_i, z_{n+i}), \forall (z, t) \in Z \times (0, \hat{t}], \\ \hat{z} + \hat{t}d \in \Omega\}.$$

$D(\hat{z})$ is the set of all directions $d \in \mathbb{R}^{2n}$ such that, for at least one index $i \in \mathcal{I}$,

$$(\hat{x}_i, \hat{y}_i) + t(d_i, d_{n+i}) \neq (z_i, z_{n+i}) \text{ for all } z \in Z,$$

holds for all $t > 0$ sufficiently small. Hence, $\hat{z} + td \notin Z$ for all $t > 0$ sufficiently small. This means that

$$(\hat{z} + D(\hat{z})) \cap Z = \emptyset.$$

However, there can be a direction $d \in \mathbb{R}^{2n}$ with $d \notin D(\hat{z})$ and $\hat{z} + td \notin Z$. Obviously, $D(\hat{z}) \neq \emptyset$ for all $\hat{z} \in Z$, as any $d \in \mathbb{R}^{2n}$ such that $d_1 = d_{n+1} = 1$ always belongs to $D(\hat{z})$. It is clear that $0 \notin D(\hat{z})$ and, if $\hat{z} \in \text{int}(\Omega)$ and $\hat{z}$ is an isolated solution of (5.2), then $D(\hat{z}) = \mathbb{R}^{2n}$. As Ω is a cone, $z + \Omega \subseteq \Omega$ holds for any $z \in \Omega$, and thus we further have that if $\hat{z} \in \mathrm{bd}(\Omega)$ and $\hat{z}$ is an isolated solution of (5.2), then $D(\hat{z}) \supseteq \Omega$. Moreover, since Ω is convex and $\hat{z}, \hat{z} + \hat{t}d \in \Omega$ we have that $\hat{z} + td \in \Omega$ for all $t \in [0, \hat{t}\,]$.

Lemma 5.2.1 *For any $d \in D(z^*)$, there exist $c_1(d), c_2(d), t(d) > 0$ so that*

$$c_1(d)t \leq \|H_\psi(z^* + td)\| \leq c_2(d)t \quad \textit{for all } t \in [0, t(d)]. \tag{5.8}$$

Proof: First of all, let us define $\bar{Z}$ as

$$\bar{Z} := \{(x, y) \in \mathbb{R}^{2n} | \psi(x_i, y_i) = 0 \text{ for all } i \in \mathcal{I}\}. \tag{5.9}$$

Obviously, we have that $z^* \in \bar{Z}$ and $Z \subseteq \bar{Z}$.

Let $D_1(z^*) \subseteq D(z^*)$ be defined as

$$D_1(z^*) := \{d \in D(z^*) | \exists (i, \hat{t}\,) \in \mathcal{I} \times (0, \infty), \psi(x_i^* + td_i, y_i^* + td_{n+i}) \neq 0, \forall t \in (0, \hat{t}\,]\}.$$

$D_1(z^*)$ is the set of all directions $d \in \mathbb{R}^{2n}$ such that $z^* + td \notin \bar{Z}$ for all $t > 0$ sufficiently small. We easily see that the set $D_1(z^*)$ is nonempty as any $d \in \mathbb{R}^{2n}$ such that $d_1 = d_{n+1} = 1$ always belongs to $D_1(\hat{z})$. In contrast, the set $D(z^*) \setminus D_1(z^*)$ could be empty (for example, if $Z = \bar{Z}$ then $D(z^*) \setminus D_1(z^*)$ is empty).

Now, take an arbitrary but fixed $d \in D(z^*)$. For any $t \geq 0$ let $\bar{z}(t) := (\bar{x}(t), \bar{y}(t)) \in \bar{Z}$ be so that

$$\|z^* + td - \bar{z}(t)\| = \mathrm{dist}\,[z^* + td, \bar{Z}]. \tag{5.10}$$

Such a $\bar{z}(t) \in \bar{Z}$ exists for all $t \geq 0$ as $\bar{Z}$ is closed and nonempty. We have the following two cases.

Case A: $d \in D(z^*) \setminus D_1(z^*)$**.** Let $(i, \tilde{t}\,) \in \mathcal{I} \times (0, \infty)$ be so that

$$\begin{aligned} (x_i^*, y_i^*) + t(d_i, d_{n+i}) &\neq (z_i, z_{n+i}), \quad \text{for all } (z, t) \in Z \times (0, \tilde{t}\,], \text{ and} \\ z^* + td &\in \Omega, \quad \text{for all } t \in (0, \tilde{t}\,]. \end{aligned}$$

This together with $d \notin D_1(z^*)$ easily gives

$$\begin{aligned} \mathrm{dist}\,[z^* + td, Z] &= \|z^* + td - \bar{z}(t)\| \\ &\geq \|(x_i^* + td_i, y_i^* + td_{n+i}) - (\bar{x}(t)_i, \bar{y}(t)_i)\| \\ &\geq t\,\|(d_i, d_{n+i})\|, \end{aligned}$$

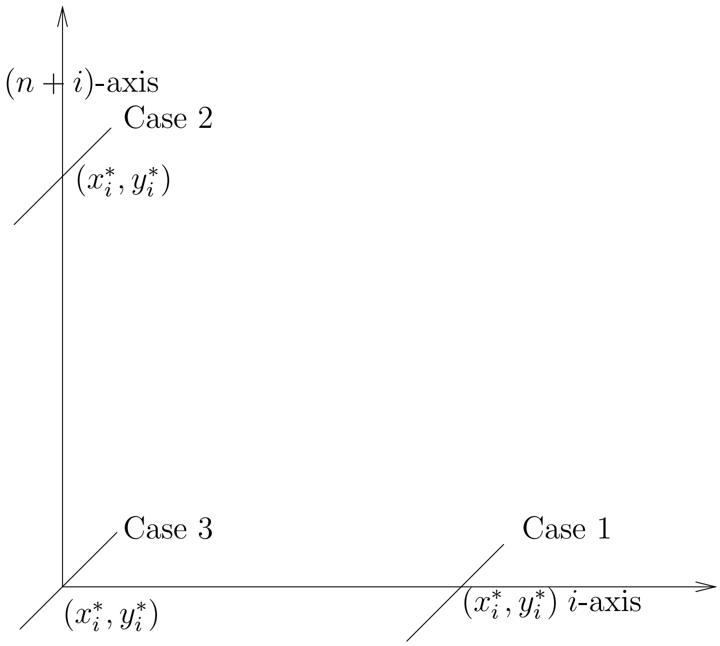

Figure 5.1: Illustration of the three cases in the proof of Lemma 5.2.1.

for all $t \in \left[0, \frac{\tilde{t}}{2}\right]$. Using (5.5) and (5.6) we further obtain

$$tC\,\|(d_i, d_{n+i})\| \leq \|H_\psi(z^* + td)\| \leq tL\|d\| \quad \text{for all } t \in \left[0, \min\left\{\delta, \frac{\tilde{t}}{2}\right\}\right].$$

Hence, we set $c_1(d) := C\,\|(d_i, d_{n+i})\|$, $c_2(d) := L\|d\|$ and $t(d) := \min\left\{\delta, \frac{\tilde{t}}{2}\right\}$ and obtain the desired result.

Case B: $d \in D_1(z^*)$. For such a d, let $(i, \tilde{t}\,) \in \mathcal{I} \times (0, \infty)$ be so that

$$\psi(x_i^* + td_i, y_i^* + td_{n+i}) \neq 0 \quad \text{for all } t \in (0, \tilde{t}\,].$$

We observe that $z^* + td \notin Z$ for all $t \in (0, \tilde{t}\,]$. Moreover, $z^* + td \in \Omega$ for all $t \in (0, \tilde{t}\,]$. For any $t \geq 0$, since $(\bar{x}(t), \bar{y}(t)) \in \bar{Z}$, we obtain that $\psi(\bar{x}(t)_i, \bar{y}(t)_i) = 0$.

We now analyze three subcases (illustrated in Figure 5.1)

Case 1: $x_i^* \neq 0$, $y_i^* = 0$. Due to $d \in D_1(z^*)$, $d_{n+i} \neq 0$. Hence,

$$\|z^* + td - \bar{z}(t)\| \geq \|(x_i^* + td_i, y_i^* + td_{n+i}) - (\bar{x}(t)_i, \bar{y}(t)_i)\| = t|d_{n+i}|,$$

holds for all $t \in \left[0, \min\left(\tilde{t}, \frac{x_i^*}{4}\right)\right]$.

Case 2: $x_i^* = 0$**,** $y_i^* \neq 0$**.** Due to $d \in D_1(z^*)$, $d_i \neq 0$. Hence,

$$\|z^* + td - \bar{z}(t)\| \geq \|(x_i^* + td_i, y_i^* + td_{n+i}) - (\bar{x}(t)_i, \bar{y}(t)_i)\| = t|d_i|,$$

holds for all $t \in \left[0, \min\left(\tilde{t}, \frac{y_i^*}{4}\right)\right]$.

Case 3: $x_i^* = y_i^* = 0$**.** In this case as $d \in D_1(z^*)$, $(d_i, d_{n+i}) \notin \{s_1, s_2 \geq 0, s_1 s_2 = 0\}$. Let the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be defined as

$$\begin{array}{rcl} \mathcal{A} & := & \{s \in \mathbb{R}^2 | s_1 < 0, s_2 > 0\} \cup \{s \in \mathbb{R}^2 | s_1 > 0, s_1 \leq s_2\} \\ \mathcal{B} & := & \{s \in \mathbb{R}^2 | s_2 < 0, s_1 > 0\} \cup \{s \in \mathbb{R}^2 | s_2 > 0, s_2 \leq s_1\} \\ \mathcal{C} & := & \mathbb{R}^2 \setminus (\mathcal{A} \cup \mathcal{B} \cup \{s_1, s_2 \geq 0, s_1 s_2 = 0\}). \end{array}$$

Note that $\mathcal{C}$ is equal to the set $\{s \in \mathbb{R}^2 | s_1, s_2 < 0\}$. From the structure of the above three sets, we easily obtain

$$\begin{array}{rcl} d_i & \neq & 0 \quad \text{if } (d_i, d_{n+i}) \in \mathcal{A} \\ d_{n+i} & \neq & 0 \quad \text{if } (d_i, d_{n+i}) \in \mathcal{B} \\ d_i, d_{n+i} & \neq & 0 \quad \text{if } (d_i, d_{n+i}) \in \mathcal{C}, \end{array}$$

and,

$$\begin{array}{rcl} \|z^* + td - \bar{z}(t)\| & \geq & \|(x_i^* + td_i, y_i^* + td_{n+i}) - (\bar{x}(t)_i, \bar{y}(t)_i)\| \\ & = & \begin{cases} t|d_i| & \text{if } (d_i, d_{n+i}) \in \mathcal{A}; \\ t|d_{n+i}| & \text{if } (d_i, d_{n+i}) \in \mathcal{B}; \\ t\|(d_i, d_{n+i})\| & \text{if } (d_i, d_{n+i}) \in \mathcal{C}, \end{cases} \end{array}$$

for all $t \in [0, \tilde{t}\,]$.

Hence, taking results from Cases 1, 2 and 3, there is a $c(d) > 0$ so that

$$\text{dist}\,[z^* + td, \bar{Z}] = \|z^* + td - \bar{z}(t)\| \geq tc(d),$$

for all $t \geq 0$ sufficiently small.

Since $Z \subseteq \bar{Z}$,

$$\text{dist}\,[z^* + td, Z] \geq \text{dist}\,[z^* + td, \bar{Z}] \geq tc(d), \tag{5.11}$$

holds for all $t \geq 0$ sufficiently small. Since $z^* + td \in \mathcal{B}(z^*, \delta) \cap \Omega$, there exists a $t(d) > 0$ sufficiently small so that for all $t \in [0, t(d)]$, Assumption 5.2.3

together with (5.6) gives

$$C\mathrm{dist}\,[z^* + td, Z] \leq \|H_\psi(z^* + td)\| = \|H_\psi(z^* + td) - H_\psi(z^*)\| \leq Lt\|d\|.$$

Using (5.11), we further obtain,

$$tc(d)C \leq \|H_\psi(z^* + td)\| \leq Lt\|d\| \quad \text{for all } t \in [0, t(d)]. \tag{5.12}$$

Hence, in Case B, setting $c_1(d) := c(d)C$, $c_2(d) := L\|d\|$ in (5.12) gives the desired result.

The proof of the lemma follows by noting that both in Case A and in Case B, we can find $c_1(d), c_2(d), t(d) > 0$ so that (5.8) holds and that these constants can be chosen independent of $i \in \mathcal{I}$. △

Lemma 5.2.2 *Let $z^* = (x^*, y^*) \in \mathbb{R}^{2n}$ be a degenerate solution of $H_\psi(z) = 0$. Then, exactly one of the following statements is true.*

(i) ψ is not differentiable everywhere.

(ii) If $d \in \mathrm{Null}\,(JH_\psi(z^*))$ then $d \notin D(z^*)$.

Proof: Let us assume that (i) does not hold. Then, ψ is differentiable everywhere. In particular, ψ is differentiable at $(0, 0)$. Let $(\check{c}_1, \check{c}_2)^\top := \nabla\psi(0, 0)$. Differentiability of ψ at $(0, 0)$ implies that

$$\lim_{r \to (0,0)} \frac{|\psi(r) - \psi(0, 0) - (\check{c}_1, \check{c}_2)(r - (0, 0))^\top|}{\|r - (0, 0)\|} = 0.$$

Equivalently,

$$\lim_{r \to (0,0)} \frac{|\psi(r) - (\check{c}_1, \check{c}_2)r^\top|}{\|r\|} = 0. \tag{5.13}$$

Taking the limit in (5.13) along the positive a-axis and the positive b-axis respectively we easily obtain that $\check{c}_1 = \check{c}_2 = 0$. Hence,

$$\nabla\psi(0, 0) = (0, 0)^\top. \tag{5.14}$$

Since both ψ and H are differentiable everywhere (from Assumption 5.2.1) we obtain that H_ψ is differentiable everywhere, in particular at z^*. Due to assuming that z^* is degenerate, we have $\mathcal{J} \neq \emptyset$. Using the definition of H_ψ (see (5.3)) together with (5.14) we obtain that the j^{th}-row of $JH_\psi(z^*)$, for all $j \in \mathcal{J}$, is zero.

Thus, the dimension of the null space of $JH_\psi(z^*)$ is greater than zero and there exists a non-zero $d \in \text{Null}\,(JH_\psi(z^*))$.

Assume further that $d \in D(z^*)$. Then, Lemma 5.2.1 gives the existence of $c_1(d), c_2(d), t(d) > 0$ so that

$$c_1(d)t \leq \|H_\psi(z^* + td)\| \leq c_2(d)t \quad \text{for all } t \in [0, t_1(d)]. \tag{5.15}$$

Differentiability of H_ψ at z^* gives

$$\lim_{z \to z^*} \frac{\|H_\psi(z) - H_\psi(z^*) - JH_\psi(z^*)(z - z^*)\|}{\|z - z^*\|} = 0. \tag{5.16}$$

In particular, taking the limit in (5.16) along the direction d and using $z^* \in Z$ we obtain

$$\lim_{t \to 0} \frac{\|H_\psi(z^* + td) - JH_\psi(z^*)(td)\|}{t\|d\|} = 0.$$

Since $d \in \text{Null}\,(JH_\psi(z^*))$, $JH_\psi(z^*)d = 0$, we further get

$$\lim_{t \to 0} \frac{\|H_\psi(z^* + td)\|}{t} = 0.$$

This is clearly a contradiction to (5.15). Hence, the result of the lemma follows. $\triangle$

Remark 5.2.1 *A similar result can be found in [22, Proposition 9.1.1]. There, it is shown that continuous differentiability of ψ and F together with z^* being degenerate implies that $JH_\psi(z^*)$ is singular. The result of Lemma 5.2.2 is stronger than that of [22, Proposition 9.1.1] as it shows that Assumption 5.2.3 (which is weaker than non-singularity of $JH_\psi(z^*)$) does not hold if ψ is assumed differentiable everywhere and there exists a $d \in \text{Null}\,(JH_\psi(z^*)) \cap D(z^*)$.*

Lemma 5.2.3 *Let $\Omega = \mathbb{R}^{2n}$ and $z^* = (x^*, y^*) \in \mathbb{R}^{2n}$ be a solution of the linear complementarity problem $LCP(M, q)$ with $M \neq 0$. Then, there exist $c_1', c_2', t' > 0$, $v \in \mathbb{R}^{2n}$ so that*

$$c_1' t \leq \left\| \begin{pmatrix} \psi(x_1^* + tv_1, y_1^* + tv_{n+1}) \\ \vdots \\ \psi(x_n^* + tv_n, y_n^* + tv_{2n}) \end{pmatrix} \right\| \leq c_2' t \quad \textit{for all } t \in [0, t']. \tag{5.17}$$

Proof: Let $\mathbf{e}^1, \mathbf{e}^1, \ldots, \mathbf{e}^n \in \mathbb{R}^n$ denote the canonical basis vectors of $\mathbb{R}^n$. Since $M \neq 0$ by assumption, without loss of generality, we assume that the first row of M is non-zero.

Let $\hat{\mathbf{e}} \in R^n$ denote the vector

$$\hat{\mathbf{e}} = \begin{cases} \mathbf{e}^1 & \text{if } M_{11} \neq 0; \\ \mathbf{e}^1 + \mathbf{e}^\ell & \text{otherwise,} \end{cases} \tag{5.18}$$

where $\ell \in \mathcal{I}$ is any index satisfying $M_{1\ell} \neq 0$. The vector $\hat{\mathbf{e}}$ is well defined as the first row of M is non-zero. Let $v \in R^{2n}$ be defined as

$$v := \begin{pmatrix} \hat{\mathbf{e}} \\ M\hat{\mathbf{e}} \end{pmatrix}. \tag{5.19}$$

We observe that $z^* + tv \notin \bar{Z}$ for all $t > 0$ sufficiently small, since $v_1 = \hat{\mathbf{e}}_1 \neq 0$ and $v_{n+1} = (M\hat{\mathbf{e}})_1 \neq 0$ holds. Note that $\bar{Z}$ is defined by (5.9).

It is easy to see that $H_\psi(z^* + tv)_{n+i} = 0$ for all $i \in \mathcal{I}$. By $\Omega = \mathbb{R}^{2n}$, v belonging to $D(z^*)$ and

$$H_\psi(z^* + tv) = \begin{pmatrix} \psi(x_1^* + tv_1, y_1^* + tv_{n+1}) \\ \vdots \\ \psi(x_n^* + tv_n, y_n^* + tv_{2n}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{5.20}$$

Lemma 5.2.1 gives constants $c_1(v), c_2(v), t(v) > 0$ so that

$$tc_1(v) \leq \left\| \begin{pmatrix} \psi(x_1^* + tv_1, y_1^* + tv_{n+1}) \\ \vdots \\ \psi(x_n^* + tv_n, y_n^* + tv_{2n}) \end{pmatrix} \right\| \leq tc_2(v) \quad \text{for all } t \in [0, t(v)]. \tag{5.21}$$

Hence, the lemma follows by setting $c_1' := c_1(v)$, $c_2' := c_2(v)$ and $t' := t(v)$. △

Definition 5.2.2 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous of degree $k \in \mathbb{Z}$ if, for all $y \in \mathbb{R}^n$ and all $t \in \mathbb{R} \setminus \{0\}$,*

$$f(ty) = t^k f(y). \tag{5.22}$$

Lemma 5.2.4 *An NCP function ψ cannot be homogeneous of any degree.*

Proof: Let ψ be a homogeneous function of degree $k \in \mathbb{Z}$. We obtain a contradiction to the definition of an NCP function by observing that

$$0 \neq \psi(-1, 0) = (-1)^k \psi(1, 0) = 0.$$

$\triangle$

Definition 5.2.3 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be positively homogeneous of degree $k \in \mathbb{R}$ if, for all $y \in \mathbb{R}^n$ and all $t > 0$,*

$$f(ty) = t^k f(y). \tag{5.23}$$

Table 5.1 shows some of the commonly used NCP functions (see for example [22; 38; 42]) where $(a, b) \in \mathbb{R}^2$, $a_+ := \max\{0, a\}$, $b_+ := \max\{0, b\}$, $\|\cdot\|_q$ denotes

Table 5.1: Commonly used ψ-functions used in (5.2)

$\psi(a,b)$	Parameter	Type
$\psi_{\min}(a,b) := \min(a,b);$		I
$\psi_{FB}(a,b) := \sqrt{a^2+b^2} - (a+b);$		I
$\psi_{LT}(a,b) := \|(a,b)\|_q - (a+b);$	$q \in (1, \infty],$	I
$\psi_{\text{poly}}(a,b) := \|(a,b)\|_{\text{poly}} - (a+b);$		I
$\psi_{KK}(a,b) := \frac{\sqrt{(a-b)^2+2qab}-(a+b)}{2-q};$	$q \in [0, 2),$	I
$\psi_{CCK}(a,b) := \psi_{FB}(a,b) - qa_+b_+;$	$q > 0,$	II
$\psi_{KP}(a,b) := \begin{pmatrix} \lambda\psi_{FB}(a,b) \\ (1-\lambda)a_+b_+ \end{pmatrix};$	$\lambda \in (0, 1),$	III

the L^q-norm of vectors and $\|\cdot\|_{\text{poly}}$ denotes those polyhedral (or block) vector norms (see [56; 60]) such that

$$\{(a,b) \in \mathbb{R}^2 : \|(a,b)\|_{\text{poly}} = 1\} \cap \{(a,b) \in \mathbb{R}^2 : a + b = 1\} = \{(0,1), (1,0)\}$$

is satisfied. Note that, similar to the L^q-norm of $(a,b) \in \mathbb{R}^2$, any polyhedral (or block) vector norm of (a,b) satisfies

$$\|(a,b)\|_{\text{poly}} = \|(|a|, |b|)\|_{\text{poly}} \text{ for all } (a,b) \in \mathbb{R}^2, \tag{5.24}$$

see [60]. Some of the above ψ-functions are equivalent for some parameter values.

Definition 5.2.4 *An NCP function will be called a* ψ-function of Type-I, *if it is positively homogeneous of degree one. An NCP function will be called a* ψ-function of Type-II, *if it can be written as a sum of a* ψ*-function of Type-*I *and of a positively homogeneous function* $\psi_k : \mathbb{R}^2 \to \mathbb{R}$ *of degree* $k > 1$*, such that* $\psi_k \not\equiv 0$*. A function* $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ *will be called a* ψ-function of Type-III, *if*

$$\psi(a,b) := \begin{pmatrix} \lambda\psi_1(a,b) \\ (1-\lambda)\psi_k(a,b) \end{pmatrix},$$

where $\lambda \in (0,1)$*,* ψ_1 *is a* ψ*-function of Type-*I *and* ψ_k *is a positively homogeneous function of degree* $k > 1$*, such that* $\psi_k \not\equiv 0$*.*

All the NCP functions we have found in the literature such that Assumption 5.2.3 with $\Omega = \mathbb{R}^{2n}$ can be satisfied, are either ψ-function of Type-I or of Type-II. Apart from the reformulation (5.2), other reformulations of NCP(H) are also possible. For example, such a reformulation of NCP(H) (see [42]) could be

$$H_{\psi_{KP}}(z) = 0,$$

where ψ_{KP} comes from Table 5.1 and is a ψ-function of Type-III. Assumption 5.2.3 with $\Omega = \mathbb{R}^{2n}$ can also be satisfied for such kind of reformulations. A way to construct an NCP function from positively homogeneous functions of degree one can be found in [38].

We next present two basic results about differentiable homogeneous functions.

Theorem 5.2.1 (Euler's homogeneous function theorem) *Let a function* $f : \mathbb{R}^n \to \mathbb{R}$ *be differentiable at some* $\tilde{x} \in \mathbb{R}^n$*. Moreover, let* f *be homogeneous of degree* $k \in \mathbb{Z}$ *or positively homogeneous of some degree* $k \in \mathbb{R}$*. Then,*

$$\nabla f(\tilde{x})^\top \tilde{x} = k f(\tilde{x}). \tag{5.25}$$

Proof: Let f be homogeneous of degree $k \in \mathbb{Z}$. The proof when f is positively homogeneous of some degree $k \in \mathbb{R}$ is very similar. Let us consider two cases:

Case 1: $\tilde{x} \neq 0$**.** Since f is differentiable at $\tilde{x}$, we obtain

$$\lim_{t \to 1} \frac{f(t\tilde{x}) - f(\tilde{x}) - \nabla f(\tilde{x})^\top (t\tilde{x} - \tilde{x})}{\|t\tilde{x} - \tilde{x}\|} = 0.$$

This together with (5.22) and L'Hôpital's rule gives

$$\begin{aligned} 0 &= \lim_{t\to 1} \frac{t^k f(\tilde{x}) - f(\tilde{x}) - \nabla f(\tilde{x})^\top \tilde{x}(t-1)}{\|\tilde{x}\|(t-1)} \\ &= \lim_{t\to 1} \frac{(t^k-1) f(\tilde{x}) - (t-1)(\nabla f(\tilde{x})^\top \tilde{x})}{(t-1)\|\tilde{x}\|} \\ &= \lim_{t\to 1} \frac{k t^{k-1} f(\tilde{x}) - \nabla f(\tilde{x})^\top \tilde{x}}{\|\tilde{x}\|} \\ &= \frac{k f(\tilde{x}) - \nabla f(\tilde{x})^\top \tilde{x}}{\|\tilde{x}\|}. \end{aligned}$$

Since $\|\tilde{x}\| \neq 0$ this is equivalent to $\nabla f(\tilde{x})^\top \tilde{x} = k f(\tilde{x})$.

Case 2: $\tilde{x} = 0$**.** In this case (5.22) with $y := 0$ gives

$$f(0) = t^k f(0) \quad \text{for all } t \in \mathbb{R} \setminus \{0\}. \tag{5.26}$$

This is possible only if $f(0) = 0$ or $k = 0$ and in both these situations $\nabla f(\tilde{x})^\top \tilde{x} = k f(\tilde{x})$ holds.

Hence, the theorem follows as in both Case 1 and 2, (5.25) is satisfied. △

Theorem 5.2.2 (Derivative identity) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at some $\tilde{x} \in \mathbb{R}^n$. Moreover, let f be homogeneous of degree $k \in \mathbb{Z}$ (or positively homogeneous of some degree $k \in \mathbb{R}$). Then, f is differentiable at $t\tilde{x}$ for all $t \in \mathbb{R} \setminus \{0\}$ (for all $t > 0$) and*

$$\nabla f(t\tilde{x}) = t^{k-1} \nabla f(\tilde{x}). \tag{5.27}$$

Proof: Let f be homogeneous of degree $k \in \mathbb{Z}$. The proof when f is positively homogeneous of some degree $k \in \mathbb{R}$ is very similar.

Let $t \in \mathbb{R} \setminus \{0\}$ be fixed. Differentiability of f at $\tilde{x}$ gives

$$\lim_{\frac{y}{t}\to\tilde{x}} \frac{\left| f(\frac{y}{t}) - f(\tilde{x}) - \nabla f(\tilde{x})^\top (\frac{y}{t} - \tilde{x}) \right|}{\|\frac{y}{t} - \tilde{x}\|} = 0.$$

Hence

$$\lim_{\frac{y}{t}\to\tilde{x}} \frac{t^k \left| f(\frac{y}{t}) - f(\tilde{x}) - \nabla f(\tilde{x})^\top (\frac{y}{t} - \tilde{x}) \right|}{t\|\frac{y}{t} - \tilde{x}\|} = 0,$$

and, applying (5.22) we obtain that

$$0 \quad = \quad \lim_{y \to t\tilde{x}} \frac{\left|f(y) - f(t\tilde{x}) - t^{k-1}\nabla f(\tilde{x})^\top (y - t\tilde{x})\right|}{\|y - t\tilde{x}\|}$$

Thus, we obtain that f is differentiable at $t\tilde{x}$ with $\nabla f(t\tilde{x}) = t^{k-1}\nabla f(\tilde{x})$. △

Remark 5.2.2 *Although Theorems 5.2.1 and 5.2.2 are basic results about differentiable homogeneous functions ([48, Page 138]), we did not find many references for them. In the few references ([5; 48]), the proof of these results assumes that f is differentiable everywhere. However, this is not needed as Theorems 5.2.1 and 5.2.2 show.*

Remark 5.2.3 *In Theorem 5.2.2 we showed that if a positively homogeneous function is differentiable at some point $\tilde{x}$ then it is differentiable along the entire ray tx, with $t > 0$. This is an improvement over the usual derivative identity that we find in existing texts. We need this improvement in later sections for proving some results.*

Remark 5.2.1 together with Theorems 5.2.1 and 5.2.2 are the main results of this section.

5.3 Existing Smoothness Assumption

Besides Assumptions 5.2.1, 5.2.2 and 5.2.3 additional smoothness conditions are needed to prove local Q-quadratic convergence of a Levenberg-Marquardt method. We are now going to discuss such conditions/ properties and their relations. One such assumption used in the literature (see for example [43]) is given next.

Assumption 5.3.1 *There exist $c, \varepsilon > 0$ and a function $G : \mathcal{B}(z^*, \varepsilon) \cap \Omega \to \mathbb{R}^{2n\times 2n}$ so that*

$$\|H_\psi(z) - H_\psi(\hat{z}) - G(\hat{z})(z - \hat{z})\| \leq c\|z - \hat{z}\|^2 \tag{5.28}$$

holds for all pairs $(z, \hat{z})$ with $z, \hat{z} \in \mathcal{B}(z^*, \varepsilon) \cap \Omega$.

In (5.28), $\hat{z}$ and z can be regarded as the current and the next iterate, respectively. However, the Levenberg-Marquardt method stops if $\hat{z}$ is the solution of (5.2). Hence, the following weaker form of the assumption will be used.

Assumption 5.3.2 *There exist* $c, \varepsilon > 0$ *and a function* $G : \mathcal{B}(z^*, \varepsilon) \cap \Omega \to \mathbb{R}^{2n \times 2n}$ *so that*

$$\|H_\psi(z) - H_\psi(\hat{z}) - G(\hat{z})(z - \hat{z})\| \leq c\|z - \hat{z}\|^2 \tag{5.29}$$

holds for all pairs $(z, \hat{z})$ *with* $z \in \mathcal{B}(z^*, \varepsilon) \cap \Omega$, $\hat{z} \in (\mathcal{B}(z^*, \varepsilon) \cap \Omega) \setminus Z$.

If H_ψ is differentiable and ∇H_ψ locally Lipschitz continuous then Assumptions 5.3.1 and 5.3.2 hold if $G(\hat{z})$ is chosen to be $\nabla H_\psi(\hat{z})^\top$. For the last n components of H_ψ this might be regarded as a reasonable condition. However, for the first n components of H_ψ whether or not Assumption 5.3.2 holds depends upon the NCP function ψ and upon z^*. The next lemma shows that Assumption 5.3.2 implies differentiability of H_ψ at some points.

Lemma 5.3.1 *Let Assumption 5.3.2 hold. Then,* H_ψ *is differentiable at all points* $\hat{z} \in \operatorname{int}((\mathcal{B}(z^*, \varepsilon) \cap \Omega) \setminus Z)$. *Moreover, for all such points* $\hat{z}$,

$$G(\hat{z}) := \nabla H_\psi(\hat{z})^\top$$

holds.

Proof: The proof follows easily by noting that Assumption 5.3.2 together with $\hat{z} \in \operatorname{int}(\mathcal{B}(z^*, \varepsilon) \cap \Omega)$ gives

$$\lim_{z \to \hat{z}} \frac{\|H_\psi(z) - H_\psi(\hat{z}) - G(\hat{z})(z - \hat{z})\|}{\|z - \hat{z}\|} = 0. \tag{5.30}$$

Hence, H_ψ is differentiable at $\hat{z}$ and $G(\hat{z}) := \nabla H_\psi(\hat{z})^\top$ is the unique matrix satisfying (5.30). △

Lemma 5.3.2 *Let Assumption 5.3.2 hold and, for* $k \in \mathcal{I}$, *let the set* $T_k \subseteq \mathbb{R}^{2n}_+$ *be defined as*

$$T_k := \left\{ z \in \mathbb{R}^{2n}_+ \cap \mathcal{B}\left(z^*, \frac{\varepsilon}{2}\right) | z_k, z_{n+k} > 0 \right\}. \tag{5.31}$$

Then, for any $k \in \mathcal{I}$ *and* $\hat{z} \in T_k$, ψ *is differentiable at* $(\hat{z}_k, \hat{z}_{n+k})$ *and*

$$(G(\hat{z})_{k,k}, G(\hat{z})_{k,n+k}) = \nabla\psi(\hat{z}_k, \hat{z}_{n+k})^\top \tag{5.32}$$

holds.

Proof: For an arbitrary but fixed $k \in \mathcal{I}$ take any $\hat{z} \in T_k$. Observe that $\hat{z}_k, \hat{z}_{n+k} > 0$. Consider an arbitrary but fixed sequence $\{a^l, b^l\} \subset \mathcal{B}\left((\hat{z}_k, \hat{z}_{n+k}), \frac{\varepsilon}{2}\right)$, $l \in \mathbb{N}$, converging to $(\hat{z}_k, \hat{z}_{n+k})$. For $l \in \mathbb{N}$, let the vector $v(l) \in \mathbb{R}^{2n}$ be defined as

$$v(l)_j = \begin{cases} a^l & \text{if } j = k; \\ b^l & \text{if } j = n+k; \\ \hat{z}_j & \text{otherwise.} \end{cases}$$

We can easily see that $v(l) \notin Z$ for all $l \in \mathbb{N}$ sufficiently large since both a^l and b^l become positive.

Now, Assumption 5.3.2 gives for $z = v(l)$,

$$\left| \psi(v(l)_k, v(l)_{n+k}) - \psi(\hat{z}_k, \hat{z}_{n+k}) - (G(\hat{z})_{k,k}, G(\hat{z})_{k,n+k}) \begin{pmatrix} a^l - \hat{z}_k \\ b^l - \hat{z}_{n+k} \end{pmatrix} \right| \leq c\|v(l) - \hat{z}\|^2$$

for all $l \in \mathbb{N}$ sufficiently large.

This implies

$$\lim_{l \to \infty} \frac{\left| \psi(a^l, b^l) - \psi(\hat{z}_k, \hat{z}_{n+k}) - (G(\hat{z})_{k,k}, G(\hat{z})_{k,n+k}) \begin{pmatrix} a^l - \hat{z}_k \\ b^l - \hat{z}_{n+k} \end{pmatrix} \right|}{\|(a^l, b^l) - (\hat{z}_k, \hat{z}_{n+k})\|} = 0. \quad (5.33)$$

Hence, ψ is differentiable at $(\hat{z}_k, \hat{z}_{n+k})$ and

$$(G(\hat{z})_{k,k}, G(\hat{z})_{k,n+k}) = \nabla \psi(\hat{z}_k, \hat{z}_{n+k})^\top.$$

$\triangle$

Remark 5.3.1 *Lemma 5.3.2 shows that if z^* is degenerate Assumption 5.3.2 excludes all such NCP functions which have points of non-differentiability in* $\operatorname{int}(\mathbb{R}^2_+) \cap \mathcal{B}((0,0), \varepsilon)$ *for all* $\varepsilon > 0$. *For example, Assumption 5.3.2 does not hold if z^* is degenerate and $\psi = \psi_{\min}$ or $\psi = \psi_{\text{poly}}$ are used. This result is not based on positive homogeneity of ψ. Later, we show that Assumption 5.3.2 also does not hold for other NCP functions from Table 5.1.*

Remark 5.3.2 *For any $k \in \mathcal{I}$, there are points $z^k \in T_k$ with $z^k \notin \operatorname{int}(\mathcal{B}(z^*, \varepsilon) \cap \Omega)$. Hence, although it might seem, it is not possible to derive Lemma 5.3.2 from Lemma 5.3.1.*

Lemma 5.3.3 *An NCP function ψ that is positively homogeneous of degree one cannot be continuously differentiable everywhere.*

Proof: Let us assume on the contrary that ψ is continuously differentiable everywhere and let

$$(\tilde{a}, \tilde{b})^\top := \nabla\psi(0,0).$$

For any $r \in \mathbb{R}^2$, using the Lemma 5.2.2 we obtain

$$\nabla\psi(tr) = \nabla\psi(r) \quad \text{for all } t > 0. \tag{5.34}$$

Since ψ is continuously differentiable at $(0,0)$ using (5.34) we obtain

$$(\tilde{a}, \tilde{b})^\top = \nabla\psi(0,0) = \lim_{t\to 0} \nabla\psi(tr) = \lim_{t\to 0} \nabla\psi(r) = \nabla\psi(r), \tag{5.35}$$

for all $r \in \mathbb{R}^2$. Hence

$$\frac{\partial\psi(r)}{\partial a} = \tilde{a} \quad \text{and} \quad \frac{\partial\psi(r)}{\partial b} = \tilde{b} \quad \text{for all } r \in \mathbb{R}^2.$$

By integration, we easily obtain

$$\psi(a,b) := \tilde{a}a + \tilde{b}b + \tilde{c} \quad \text{for all } (a,b) \in \mathbb{R}^2,$$

with some constants $\tilde{a}, \tilde{b}$ and $\tilde{c}$. This gives a contradiction as this is not an NCP function for any values of $\tilde{a}, \tilde{b}$ and $\tilde{c}$. △

Lemma 5.3.4 *Let ψ be a given NCP function and $\Lambda : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2_+$ be a set-valued map defined as*

$$\Lambda(\gamma) := \left\{(a,b)^\top \in \mathbb{R}^2 | a + b = \gamma, a, b \geq 0\right\}. \tag{5.36}$$

Then, for any $g \in \mathbb{R}^2$ and $\gamma > 0$, there exists a point $(\tilde{a}, \tilde{b})^\top \in \Lambda(\gamma)$ so that

$$\psi(\tilde{a}, \tilde{b}) - g^\top \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \neq 0. \tag{5.37}$$

Proof: Let us assume the contrary. For any fixed $g \in \mathbb{R}^2$ and any fixed $\gamma > 0$ this leads to

$$\psi(a,b) = g^\top \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{for all } (a,b)^\top \in \Lambda(\gamma). \tag{5.38}$$

By taking $(a, b) = (\gamma, 0)$ and $(0, \gamma)$ in (5.38) (both these points are in $\Lambda(\gamma)$) we obtain

$$\begin{array}{rcl} 0 & = & \psi(\gamma, 0) = g_1\gamma \text{ and,} \\ 0 & = & \psi(0, \gamma) = g_2\gamma, \end{array}$$

and hence $g_1 = g_2 = 0$. Thus,

$$\psi(a, b) = 0 \quad \text{for all } (a, b)^\top \in \Lambda(\gamma).$$

This is clearly a contradiction to the definition of an NCP function. Hence the result follows. $\triangle$

The next theorem is the main result of this section.

Theorem 5.3.1 *Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP function. Moreover, let ψ be given by*

$$\psi(a, b) := \psi_1(a, b) + \psi_k(a, b) \quad \textit{for all } (a, b) \in \mathbb{R}^2,$$

where ψ_1 and ψ_k are positively homogeneous functions of degree one and of degree $k > 1$ respectively, and ψ_1 is a locally Lipschitz NCP function. If z^* is a degenerate solution of $H_\psi(z) = 0$ then Assumption 5.3.2 does not hold.

Proof: Let us assume that Assumption 5.3.2 holds. With $\varepsilon > 0$ from Assumption 5.3.2 there is a $\hat{r} := (\hat{r}_1, \hat{r}_2)^\top \in \text{int}\left(\mathcal{B}((0, 0)^\top, \frac{\varepsilon}{2}) \cap \mathbb{R}^2_+\right)$ at which ψ_1 is differentiable. This follows by Rademacher's theorem since ψ_1 is locally Lipschitz. From Lemma 5.3.2 we obtain that ψ is differentiable at $\hat{r}$. Since both ψ and ψ_1 are differentiable at $\hat{r}$ it follows that $\psi_k = \psi - \psi_1$ is differentiable at $\hat{r}$ as well. From Theorem 5.2.2 both ψ_1 and ψ_k (and hence ψ) are differentiable at $t\hat{r}$ for all $t > 0$.

Let $\hat{C}$ be the optimal value of the following maximization problem $(P1)$

$$\begin{array}{ll} \max & \left| \psi_1(a, b) - \nabla\psi_1(\hat{r})^\top \begin{pmatrix} a \\ b \end{pmatrix} \right| \\ \text{s.t.} & (a, b) \in \Lambda(\hat{r}_1 + \hat{r}_2), \end{array}$$

where the constraint set $\Lambda(\hat{r}_1 + \hat{r}_2)$ is defined by (5.36). The objective function of $(P1)$ is continuous and the feasible set is compact. Hence, by the theorem of Weierstrass the maximal value $\hat{C}$ is attained at some point $\hat{s} := (\hat{a}, \hat{b}) \in \Lambda(\hat{r}_1 + \hat{r}_2)$. From Lemma 5.3.4 we obtain that $\hat{C} > 0$. Moreover, using Theorem 5.2.1, we observe that $\hat{s} \neq \hat{r}$, otherwise the objective function of $(P1)$ would vanish at $\hat{s}$.

Since z^* is degenerate by assumption, the index set $\mathcal{J}$ is nonempty. For some $j \in \mathcal{J}$, let $d^1, d^2 \in \mathbb{R}^{2n}_+$ be defined as follows

$$d^1_i = \begin{cases} \hat{r}_1 & \text{if } i = j; \\ \hat{r}_2 & \text{if } i = n+j; \\ 0 & \text{otherwise,} \end{cases} \text{ and } d^2_i = \begin{cases} \hat{a} & \text{if } i = j; \\ \hat{b} & \text{if } i = n+j; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that there exists $t_1 > 0$ so that $z^* + td^1 \in \left(\mathcal{B}\left(z^*, \frac{\varepsilon}{2}\right) \cap \Omega\right) \setminus Z$ for all $t \in (0, t_1]$ and $z^* + td^2 \in \mathcal{B}\left(z^*, \frac{\varepsilon}{2}\right) \cap \Omega$ for all $t \in [0, t_1]$. Hence, for all $t \in (0, t_1]$, $z^* + td^1$ and $z^* + td^2$ can be used for $\hat{z}$ and z, respectively, in Assumption 5.3.2. Moreover we see that $z^* + td^1 \neq z^* + td^2$ for all $t > 0$.

Now, using the definition of a positive homogeneous function, and applying Theorems 5.2.1 and 5.2.2 we will simplify the left hand side of (5.29). Using Lemma 5.3.2 and $j \in \mathcal{J}$ we obtain

$$(G(z^* + td^1)_{j,j}, G(z^* + td^1)_{j,n+j}) = \nabla\psi(t\hat{r})^\top \quad \text{for all } t \in (0, t_1). \tag{5.39}$$

For all $t \in (0, t_1)$, and in view of (5.29) let $E(t)$ be defined as

$$E(t) := [H_\psi(z^* + td^2) - H_\psi(z^* + td^1) - G(z^* + td^1)\left(t(d^2 - d^1)\right)]_j. \tag{5.40}$$

Now,

$$[G(z^* + td^1)\left(t(d^2 - d^1)\right)]_j = \left(G(z^* + td^1)_{j,j}, G(z^* + td^1)_{j,n+j}\right) t(\hat{s} - \hat{r}). \tag{5.41}$$

Applying (5.39) and (5.41) together with $x^*_j = y^*_j = 0$ (since $j \in \mathcal{J}$), $E(t)$ can be simplified as

$$E(t) = \psi(t\hat{s}) - \psi(t\hat{r}) - t\nabla\psi(t\hat{r})^\top(\hat{s} - \hat{r}). \tag{5.42}$$

Theorems 5.2.1 5.2.2 further give,

$$\begin{aligned} t\nabla\psi(t\hat{r})^\top\hat{r} &= t\left(\nabla\psi_1(\hat{r}) + t^{k-1}\nabla\psi_k(\hat{r})\right)^\top\hat{r} \\ &= t\psi_1(\hat{r}) + kt^k\psi_k(\hat{r}). \end{aligned}$$

This together with (5.42) and the homogeneity properties of ψ_1 and ψ_k show

$$\begin{aligned} E(t) &= t\psi_1(\hat{s}) + t^k\psi_k(\hat{s}) + (k-1)t^k\psi_k(\hat{r}) - \left(\nabla\psi_1(\hat{r}) + t^{k-1}\nabla\psi_k(\hat{r})\right)^\top t\hat{s} \\ &= t\left(\psi_1(\hat{s}) - \nabla\psi_1(\hat{r})^\top\hat{s}\right) + t^k\left(\psi_k(\hat{s}) + (k-1)\psi_k(\hat{r}) - \nabla\psi_k(\hat{r})^\top\hat{s}\right). \end{aligned}$$

Since $\hat{s}$ is optimal to $(P1)$,

$$|\psi_1(\hat{s}) - \nabla\psi_1(\hat{r})^\top\hat{s}| = \hat{C} > 0. \tag{5.43}$$

For $t \in (0, t_1)$, using the triangle inequality and (5.43) we obtain

$$\begin{aligned} \frac{|E(t)|}{t} &\geq \left|\psi_1(\hat{s}) - \nabla\psi_1(\hat{r})^\top \hat{s}\right| - \left|t^{k-1}\left(\psi_k(\hat{s}) + (k-1)\psi_k(\hat{r}) - \nabla\psi_1(\hat{r})^\top \hat{s}\right)\right| \\ &= \hat{C} - \left|t^{k-1}\left(\psi_k(\hat{s}) + (k-1)\psi_k(\hat{r}) - \nabla\psi_1(\hat{r})^\top \hat{s}\right)\right|. \end{aligned}$$

Since $k > 1$, for $t \in (0, t_1)$ sufficiently small, we have

$$|E(t)| \geq \frac{t}{2}\hat{C} > t^2 c\|\hat{s} - \hat{r}\|^2. \tag{5.44}$$

However, on the other hand, from Assumption 5.3.2

$$\begin{aligned} |E(t)| &\leq c\|z + td^1 - (z + td^2)\|^2 \\ &= t^2 c\|\hat{s} - \hat{r}\|^2. \end{aligned}$$

This is a contradiction to (5.44) and hence the result of the theorem follows. △

Remark 5.3.3 *Due to the appearance of the constraint set Ω in (5.29), Assumption 5.3.2 is weaker than assuming the differentiability of H_ψ together with local Lipschitz continuity of ∇H_ψ. However, as the above theorem shows, even this weaker smoothness Assumption 5.3.2 is not satisfied at degenerate solutions. This holds for any choice of the convex set Ω such that $Z \subset \Omega$.*

Remark 5.3.4 *If we choose $\psi_k \equiv 0$ then ψ is a positively homogeneous NCP function of degree one. Thus, any reformulation $H_\psi(z) = 0$ of NCP(H) with a positively homogeneous NCP function ψ of degree one does not satisfy Assumption 5.3.2 if z^* is degenerate. Moreover, obviously Theorem 5.3.1 holds also for reformulations based on Type III-ψ functions (see Definition 5.2.4). This shows that Theorem 5.3.1 is applicable for all ψ-functions of Type I, II and III and, in particular, for all the ψ-functions listed in Table 5.1.*

Remark 5.3.5 *If $\psi_k \equiv 0$, then $\psi = \psi_1$ and the local Lipschitzness of ψ_1 is not required. Local Lipschitzness of ψ_1 is only needed to ensure that both ψ_1 and ψ_k ($= \psi - \psi_1$) are differentiable at $\hat{r}$.*

Remark 5.3.6 *It is interesting to note that Theorem 5.3.1 only requires $z^* \in Z$ being degenerate and that ψ is locally Lipschitz continuous and positively homogeneous. In particular no condition on H is assumed. For example, even if H is linear then Assumption 5.3.2 cannot hold.*

Remark 5.3.1, Lemma 5.3.3, Theorem 5.3.1 and Remarks 5.3.3, 5.3.4, 5.3.5 and 5.3.6 are the main results of this section. If we restrict ourselves to the ψ-functions from Table 5.1, then, Table 5.2 gives summarizes the satisfiability of Assumption 5.3.2 on these ψ-functions.

Table 5.2: Satisfiability of Assumption 5.3.2 on the ψ-functions from Table 5.1

ψ-functions	Conditions	Assumption 5.3.2
$\psi_{\min}$, ψ_{FB}, ψ_{LT}, ψ_{poly}, ψ_{KK}, ψ_{CCK}, ψ_{KP}	z^* degenerate	Does not hold

5.4 Fundamental Identities for Nonsmooth Homogeneous Functions

Here we present a nonsmooth version of Euler's theorem and the derivative identity for homogeneous (or positively homogeneous) functions. These results are based on Clarke's subdifferential (see, for example, [8]). Although the results of this section are used for NCP functions later, we present the results in a general setting.

For a given function $f : \mathbb{R}^n \to \mathbb{R}$, let $\mathcal{D}_f \subseteq \mathbb{R}^n$ be the set of points at which f is differentiable. Similarly, for a vector valued function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$, let $\mathcal{D}_{\mathcal{F}} \subseteq \mathbb{R}^n$ be the set of points at which $\mathcal{F}$ is differentiable. The B-subdifferential and Clarke's subdifferential play a major role in the study of convergence analysis of nonsmooth Newton methods. The B-subdifferential of a locally Lipschitz continuous function is given as follows (see [17])

Definition 5.4.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Then, the B-subdifferential is given by*

$$\partial_B f(x) := \left\{ \lim_{k\to\infty} \nabla f(x^k)^\top | x^k \to x, x^k \in \mathcal{D}_f \right\}. \tag{5.45}$$

If f is a locally Lipschitz continuous function, using Rademacher's theorem we obtain that the set $\mathbb{R}^n \setminus \mathcal{D}_f$ is of measure zero (in the sense of Lebesgue measure). Thus, if f is locally Lipschitz continuous then $\partial_B f(x) \neq \emptyset$. A characterization of Clarke's subdifferential is (see [8, Theorem 2.5.1]) as follows.

Definition 5.4.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Then, Clarke's subdifferential is given by*

$$\partial f(x) := \text{co}\, \{\partial_B f(x)\}, \tag{5.46}$$

where co $(\cdot)$ *denotes the convex hull of a set.*

From Proposition 2.1.2 in [8], we know that $\partial f(x)$ is nonempty, convex and compact for each $x \in \mathbb{R}^n$ if f is locally Lipschitz continuous. Moreover, it is clear that $\partial_B f(x) \subseteq \partial f(x)$.

Extending 5.4.2 to vector valued functions, see [8], the notion of Clarke's generalized Jacobian is defined as follows.

Definition 5.4.3 *Let $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function. Then, Clarke's generalized Jacobian is given by*

$$\partial \mathcal{F}(x) := \text{co}\left\{ \lim_{k\to\infty} J\mathcal{F}(x_k) | x^k \to x, x^k \in \mathcal{D}_{\mathcal{F}} \right\}. \tag{5.47}$$

Moreover, let the set $\bar{\partial}\mathcal{F}(x)$ be defined by

$$\bar{\partial}\mathcal{F}(x) := \partial\mathcal{F}_1(x) \times \partial\mathcal{F}_2(x) \times \ldots \times \partial\mathcal{F}_m(x), \tag{5.48}$$

where the latter denotes the set of all matrices whose i^{th} row belongs to $\partial\mathcal{F}_i(x)^\top$ for each $i = 1, 2, \ldots, m$.

From [8, Proposition 2.6.2], we obtain that $\partial\mathcal{F}(x) \subseteq \bar{\partial}\mathcal{F}(x)$.

Next we state and prove Euler's identity for locally Lipschitz continuous homogeneous (or positively homogeneous) functions.

Theorem 5.4.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and homogeneous of degree $k \in \mathbb{Z}$ or positively homogeneous of some degree $k \in \mathbb{R}$. Then, for each $x \in \mathbb{R}^n$ and each g with $g^\top \in \partial f(x)$,*

$$g^\top x = kf(x) \tag{5.49}$$

holds.

Proof: Let $\tilde{f} : \mathbb{R}^{2n} \to \mathbb{R}$ be the function defined as

$$\tilde{f}(x, y) := y^\top x - kf(x). \tag{5.50}$$

Since f is locally Lipschitz continuous we obtain that $\tilde{f}$ is also locally Lipschitz continuous. Now take $x \in \mathbb{R}^n$ and $g \in \partial f(x)$. Now, since $\partial f(x)$ is the convex hull of $\partial_B f(x)$, Carathéodory's theorem gives a subset of $\partial_B f(x)$ consisting of at most $n+1$ points such that g lies in the convex hull of the subset. Equivalently,

$$g = \sum_{i=1}^{n+1} \lambda_j g^j, \quad \lambda_j \geq 0 \quad \text{and } \textstyle\sum_{j=1}^{n+1} \lambda_j = 1 \tag{5.51}$$

where every g^j is in $\partial_B f(x)$. For all $j = 1, 2, \ldots, n+1$, let $\{\nabla f(x^{jl})\}$ be sequences so that $\nabla f(x^{jl}) \to g^j$, $x^{jl} \to x$ (such sequences exist by definition of g^j). Now as $(x^{jl}, \nabla f(x^{jl})) \to (x, g^j)$, continuity of $\tilde{f}$ gives

$$\lim_{l\to\infty} \tilde{f}(x^{jl}, \nabla f(x^{jl})) = \tilde{f}(x, g^j), \quad \text{for all } j = 1, 2, \ldots, n+1. \tag{5.52}$$

Using (5.50), this further simplifies to

$$\lim_{l\to\infty} \left(\nabla f(x^{jl})^\top x^{jl} - kf(x^{jl})\right) = (g^j)^\top x - kf(x). \tag{5.53}$$

Now, by Theorem 5.2.1, we obtain that $\nabla f(x^{jl})^\top x^{jl} - kf(x^{jl}) = 0$ for all $l \in \mathbb{N}$ and $j = 1, 2, \ldots, n+1$. Hence, (5.53) gives

$$(g^j)^\top x - kf(x) = 0. \tag{5.54}$$

This together with (5.51) and $\sum_{j=1}^{n+1} \lambda_j = 1$ shows that

$$g^\top x - kf(x) = \left(\sum_{i=1}^{n+1} \lambda_j g^j\right)^\top x - kf(x) = \sum_{i=1}^{n+1} \lambda_j \left((g^j)^\top x - kf(x)\right) = 0.$$

Hence, $g^\top x = kf(x)$ and the statement of the theorem follows. $\triangle$

Remark 5.4.1 *In [70], Euler's identity is extended to non-smooth functions using the β-subdifferential defined on a β-smooth Banach space.*

Next, we generalize Theorem 5.2.2 for locally Lipschitz continuous homogeneous (or positively homogeneous) functions.

Theorem 5.4.2 *Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous and homogeneous of degree $k \in \mathbb{Z}$ (positively homogeneous of some degree $k \in \mathbb{R}$). Then, for any $t \in \mathbb{R} \setminus \{0\}$ ($t > 0$),*

$$\partial f(tx) = t^{k-1} \partial f(x) \tag{5.55}$$

holds.

Proof: Let f be homogeneous of degree $k \in \mathbb{Z}$. Using (5.46) and Lemma 5.2.2 we obtain, for any $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}
g^\top \in \partial f(tx) \quad &\Longleftrightarrow \quad g^\top \in \mathrm{co}\left\{\lim_{j\to\infty} \nabla f(x_j)^\top | x_j \to tx, x_j \in \mathcal{D}_f\right\} \\
&\Longleftrightarrow \quad g^\top \in \mathrm{co}\left\{\lim_{j\to\infty} \nabla f(x_j)^\top | \frac{x_j}{t} \to x, \frac{x_j}{t} \in \mathcal{D}_f\right\} \\
&\Longleftrightarrow \quad g^\top \in \mathrm{co}\left\{\lim_{j\to\infty} t^{k-1} \nabla f\left(\frac{x_j}{t}\right)^\top | \frac{x_j}{t} \to x, \frac{x_j}{t} \in \mathcal{D}_f\right\} \\
&\Longleftrightarrow \quad g^\top \in t^{k-1}\mathrm{co}\left\{\lim_{j\to\infty} \nabla f\left(\frac{x_j}{t}\right)^\top | \frac{x_j}{t} \to x, \frac{x_j}{t} \in \mathcal{D}_f\right\} \\
&\Longleftrightarrow \quad g^\top \in t^{k-1} \partial f(x).
\end{aligned}$$

The case when f is positively homogeneous of some degree $k \in \mathbb{R}$ can be proved similarly. Hence (5.55) follows. △

Remark 5.4.2 *From the proof of Theorem 5.4.2 it is clear that (5.55) holds even when ∂ is replaced by ∂_B. This means, if f is locally Lipschitz continuous and homogeneous of degree $k \in \mathbb{Z}$ (positively homogeneous of some degree $k \in \mathbb{R}$) then for any $t \in \mathbb{R} \setminus \{0\}$ ($t > 0$)*

$$\partial_B f(tx) = t^{k-1} \partial_B f(x)$$

holds.

As the next theorem shows, Theorems 5.4.1 and 5.4.2 are also valid if the function $f : \mathbb{R}^n \to \mathbb{R}$ is replaced by $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$. We note that, if the function f is replaced by $\mathcal{F}$ then Definitions 5.2.2 and 5.2.3 can be correspondingly modified.

Theorem 5.4.3 *Suppose that the function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz continuous and homogeneous of degree $k \in \mathbb{Z}$ (positively homogeneous of some degree $k \in \mathbb{R}$). Then, for each $x \in \mathbb{R}^n$, $G \in \bar{\partial}\mathcal{F}(x)$ and $t \in \mathbb{R} \setminus \{0\}$ ($t > 0$),*

$$\begin{aligned}
Gx &= k\mathcal{F}(x), && (5.56)\\
\partial\mathcal{F}(tx) &= t^{k-1}\partial\mathcal{F}(x) \text{ and} && (5.57)\\
\bar{\partial}\mathcal{F}(tx) &= t^{k-1}\bar{\partial}\mathcal{F}(x) && (5.58)
\end{aligned}$$

hold.

Proof: The proof is omitted as it is very similar to that of Theorems 5.4.1 and 5.2.2. △

Theorems 5.4.1 and 5.4.2 are the main results of this section.

5.5 New Smoothness Assumption

Recently (see [27]), Assumption 5.3.2 has been weakened so that the constrained Levenberg-Marquardt method applied to nonsmooth equation based reformulation of NCP by the min function can be shown to have quadratic rate of convergence. The weaker smoothness assumption consists of the following two conditions, where $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ is some function.

Condition 5.5.1 *There are* $\omega_1 > 0$ *and* $\delta_1 > 0$ *so that for any* $z \in \mathcal{B}(z^*, \delta_1) \cap \Omega \setminus Z$ *there is* $z^\diamond \in Z$ *with*

$$\begin{aligned} \|z - z^\diamond\| &\le \omega_1 \text{dist}\,[z, Z], \textit{ and} \\ \|H_\psi(z) + G(z)(z^\diamond - z)\| &\le \omega_1 \text{dist}\,[z, Z]^2. \end{aligned}$$

Condition 5.5.2 *There are* $\omega_2 > 0$ *and* δ_2 *so that*

$$w \in \left\{ w \in \Omega | \|w - z\| \le \alpha, \|H_\psi(z) + G(z)(w - z)\| \le \alpha^2 \right\}$$

implies

$$\|H_\psi(w)\| \le \omega_2 \alpha^2$$

for all $z \in \mathcal{B}(z^*, \delta_2) \cap \Omega \backslash Z$ *and all* $\alpha \in [0, \delta_2]$.

In this section we discuss in detail Conditions 5.5.1 and 5.5.2 for several choices of the NCP function ψ.

It is shown in [27] that Assumption 5.3.2 implies Conditions 5.5.1 and 5.5.2. Assumption 5.3.2 as well as Conditions 5.5.1 and 5.5.2 require a suitable choice of a function G which maps $z \in \mathbb{R}^{2n}$ to a matrix $G(z) \in \mathbb{R}^{2n\times 2n}$. If H_ψ is differentiable at $z := (x, y) \in \mathbb{R}^{2n}$ then we take $G(z) := \nabla H_\psi(z)^\top$. However, if H_ψ is non-differentiable at z, an element from $\partial H_\psi(z)$ (or $\bar{\partial} H_\psi(z)$) seems a natural choice for the matrix $G(z)$.

Blanket Assumption for Section 5.5: *The functions* $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ *and* $g : \mathbb{R}^2 \to \mathbb{R}^2$ *are such that* $G(z) \in \bar{\partial} H_\psi(z)$ *and* $g(r)^\top \in \partial\psi(r)$ *for any* $z \in \mathbb{R}^{2n}$ *and* $r \in \mathbb{R}^2$*, respectively.*

For all $i \in \mathcal{I}$, the blanket assumptions for Section 5.5 gives that the i^{th} and $(n+i)^{th}$ rows of $G(z)$ are given by

$$\left(0_{i-1}^\top, g(x_i, y_i)_1, 0_{n-i}^\top, g(x_i, y_i)_2, 0_{n-i}^\top\right)$$

and

$$\left(\nabla_x H_i(\hat{x})^\top, 0_{i-1}^\top, -1, 0_{n-i}^\top\right),$$

respectively, where 0_{i-1} and 0_{n-i} are $(i-1)$ and $(n-i)$ dimensional null vectors. For some vector $w \in \mathbb{R}^{2n}$ and an index $i \in \mathcal{I}$, the i^{th} and the $(n+i)^{th}$ components of $G(z)w \in \mathbb{R}^{2n}$ are given by

$$(G(z)w)_i = g(x_i, y_i)^\top \begin{pmatrix} w_i \\ w_{n+i} \end{pmatrix} \tag{5.59}$$

and

$$(G(z)w)_{n+i} = \nabla_x H_i(x)^\top \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} - w_{n+i}, \tag{5.60}$$

respectively. Note that, for all $i \in \mathcal{I}$, the i^{th} row of $H_\psi(z)$ is only a function of (x_i, y_i). Hence, $G(z)_{i,j} = G(z)_{i,n+j} = 0$ for all $j \neq i$, $i, j \in \mathcal{I}$.

Lemma 5.5.1 *Let the NCP function ψ be positively homogeneous of degree one. Then, for any $w, z \in \mathbb{R}^{2n}$ and $i \in \mathcal{I}$,*

$$\left(H_\psi(z) + G(z)(w - z)\right)_i = (G(z)w)_i. \tag{5.61}$$

Proof: For any $w, z := (x, y) \in \mathbb{R}^{2n}$ and $i \in \mathcal{I}$, using (5.3), (5.59) and Theorem 5.4.1 we obtain that

$$\begin{aligned} (H_\psi(z) + G(z)(w - z))_i &= \psi(x_i, y_i) + (G(z)(w - z))_i \\ &= \psi(x_i, y_i) + (G(z)w)_i - (G(z)z)_i \\ &= \psi(x_i, y_i) + (G(z)w)_i - (x_i, y_i)g(x_i, y_i) \\ &= (G(z)w)_i \end{aligned}$$

follows. △

We can assume without loss of generality that $\delta \leq \min\{\delta_1, \delta_2\}$ (if not just replace δ in Assumption 5.2.3 by $\min\{\delta_1, \delta_2\}$). Recall the blanket assumption for Chapter 5 and the definition of H_ψ. As a result, the last n components of H_ψ

are differentiable and their derivatives are locally Lipschitz continuous. Thus, for the last n components, Assumption 5.3.1 holds and it is easy to see that the next two lemma's hold.

Lemma 5.5.2 *For any $\tilde{C} \geq 1$ there exits $\delta_{\tilde{C}} \in [0, \varepsilon]$ so that for $z \in \mathcal{B}(z^*, \delta_{\tilde{C}})$, $z^\diamond \in Z$ and $\|z - z^\diamond\| \leq \tilde{C}\mathrm{dist}[z, Z]$ implies*

$$|(H_\psi(z) + G(z)(z^\diamond - z))_{n+i}| \quad \leq \quad c\tilde{C}^2\mathrm{dist}[z, Z]^2 \text{ for all } i \in \mathcal{I},$$

where $c, \varepsilon > 0$ are from Assumption 5.3.1.

Proof: Let $\delta_{\tilde{C}} := \frac{\varepsilon}{2\tilde{C}}$. As $z \in \mathcal{B}(z^*, \delta_{\tilde{C}})$, we have that $z \in \mathcal{B}(z^*, \varepsilon)$. Now, we note that

$$\|z^\diamond - z^*\| \leq \|z^\diamond - z\| + \|z - z^*\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2\tilde{C}} \leq \varepsilon.$$

Hence $z^\diamond \in \mathcal{B}(z^*, \varepsilon)$ and we can use $z, z^\diamond$ as instances of $\hat{z}, z$, respectively, in Assumption 5.3.1. Then, it is easy to see the assertion of the lemma holds.

Lemma 5.5.3 *There are $\beta, \delta > 0$ so that*

$$w \in \left\{w \in \Omega | \|w - z\| \leq \alpha, |(H_\psi(z) + G(z)(w - z))_{n+i}| \leq \alpha^2\right\} \tag{5.62}$$

implies

$$|H_\psi(w)_{n+i}| \leq \beta\alpha^2$$

for all $i \in \mathcal{I}$, $z \in \mathcal{B}(z^*, \delta) \cap \Omega \backslash Z$ and all $\alpha \in [0, \delta]$.

5.5.1 Discussion of Condition 5.5.1

Taking into account Lemma 5.5.2, we will consider only the first n components of H_ψ and, accordingly, present a sufficient condition for Condition 5.5.1 to hold.

Condition 5.5.3 *The NCP function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is positively homogeneous of degree one. Moreover, there are $c^\diamond, \delta_S > 0$ so that for any $z \in \mathcal{B}(z^*, \delta_S) \cap \Omega \setminus Z$ there is $z^\diamond \in Z$ with*

$$\|z - z^\diamond\| \quad \leq \quad c^\diamond \mathrm{dist}[z, Z] \tag{5.63}$$

$$\left| g(x_i, y_i) \begin{pmatrix} z_i^\diamond \\ z_{n+i}^\diamond \end{pmatrix} \right| \quad \leq \quad c^\diamond \mathrm{dist}[z, Z]^2 \quad \text{for all } i \in \mathcal{I}. \tag{5.64}$$

Lemma 5.5.4 *Condition 5.5.3 implies Condition 5.5.1.*

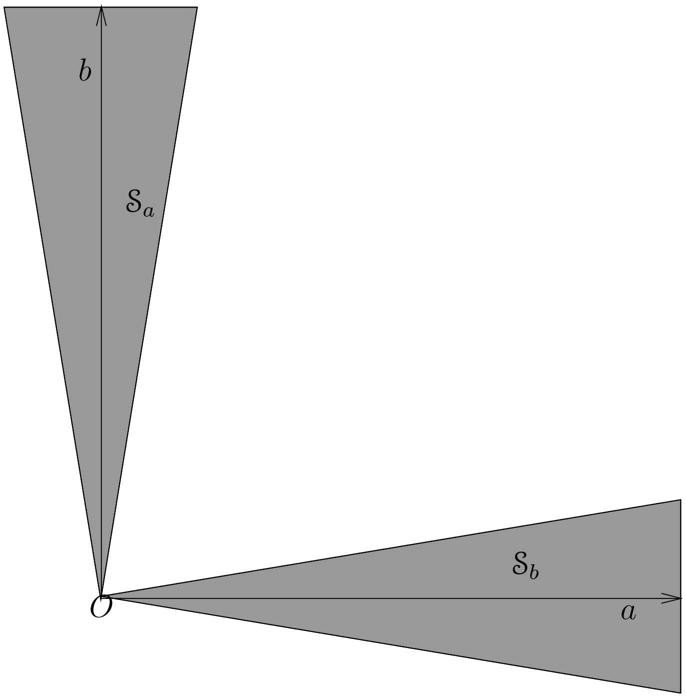

Figure 5.2: Illustration of the cones $\mathcal{S}_a$ and $\mathcal{S}_b$.

Proof: Let Condition 5.5.3 hold. It is clear that $c^\diamond \geq 1$. Using $\tilde{C} := c^\diamond$ in Lemma 5.5.2 we obtain a $\delta_1 := \delta_{c^\diamond} > 0$. Take an arbitrary but fixed $z \in \mathcal{B}(z^*, \min\{\delta_1, \delta_S\}) \setminus Z$. Using Lemma 5.5.1 and Condition 5.5.3, we obtain that there is a $z^\diamond$ so that

$$\begin{aligned} \|z - z^\diamond\| &\leq c^\diamond \text{dist}\,[z, Z], \text{and} \\ \left|(H_\psi(z) + G(z)(z^\diamond - z))_i\right| &\leq c^\diamond \text{dist}\,[z, Z]^2 \quad \text{for all } i \in \mathcal{I}, \end{aligned}$$

holds.

Now, applying Lemma 5.5.2, we further obtain that

$$|(H_\psi(z) + G(z)(z^\diamond - z))_{n+i}| \leq c(c^\diamond)^2 \text{dist}\,[z, Z]^2 \text{ for all } i \in \mathcal{I},$$

Thus we see that Condition 5.5.1 is satisfied with $\omega_1 := \max\{c^\diamond, c(c^\diamond)^2\}\sqrt{2n}$ and $\delta_1 := \min\{\delta_1, \delta_S\}$. △

Lemma 5.5.5 *Let $\psi_{\text{poly}} : \mathbb{R}^2 \to \mathbb{R}$ be any NCP function as described in Table 5.1. Then, there are $c_1, c_2 > 0$ and closed and convex cones $\mathcal{S}_a \subset \mathbb{R}^2$ and $\mathcal{S}_b \subset \mathbb{R}^2$ such* *that* $\{(a,b) \in \mathbb{R}^2_+ | a = 0, b > 0\} \in \text{int}(S_a)$ *and* $\{(a,b) \in \mathbb{R}^2_+ | b = 0, a > 0\} \in \text{int}(S_b)$, *and*

$$\psi_{\text{poly}}(a,b) = c_1 a \quad \textit{for all } (a,b) \in \mathcal{S}_a, \tag{5.65}$$

$$\psi_{\text{poly}}(a,b) = c_2 b \quad \textit{for all } (a,b) \in \mathcal{S}_b \tag{5.66}$$

hold.

Proof: Since $\psi_{\text{poly}}(a,b) := \|(a,b)\|_{\text{poly}} - (a+b)$ and $\|\cdot\|_{\text{poly}}$ is a polyhedral (block) norm [60], we obtain that ψ_{poly} is positively homogeneous of degree one and piecewise-linear. Hence, by (5.24), there exists a closed and convex cone $\mathcal{S}_a$ such that $\{(a,b) \in \mathbb{R}^2_+ | a = 0, b > 0\} \in \text{int}(\mathcal{S}_a)$ and

$$\psi_{\text{poly}}(a,b) = c_1 a + \tilde{c}_1 b \quad \text{for all } (a,b) \in \mathcal{S}_a.$$

Now, as $\psi_{\text{poly}}(a,b)$ is an NCP function we further obtain

$$0 = \psi_{\text{poly}}(0,1) = \tilde{c}_1.$$

Hence (5.65) holds. The proof of (5.66) is very similar. Both these cones are illustrated in Figure 5.2. △

Lemma 5.5.6 *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be locally Lipschitz continuous and homogeneous of degree* $k \in \mathbb{Z}$ *or positively homogeneous of some degree* $k \in \mathbb{R}$. *Moreover, let* $\mathfrak{L} \subset \mathbb{R}^2$ *be a line* not *passing through the origin and* $\bar{\mathfrak{L}}$ *be a segment of* $\mathfrak{L}$ *with distinct end points. Then, there exists a point in* $\bar{\mathfrak{L}}$ *at which* f *is differentiable.*

Proof: Let us assume the contrary, i.e.,

$$\bar{\mathfrak{L}} \cap \mathcal{D}_f = \emptyset.$$

Now, applying Theorem 5.2.2 and that $\mathfrak{L}$ is not passing through the origin, we obtain that f is not differentiable at all the points in the triangle formed by the origin and the (distinct) end points of $\bar{\mathfrak{L}}$. This triangle has a non-zero measure in $\mathbb{R}^2$. Since f is not differentiable on the whole triangle this is a contradiction to Rademacher's theorem. Hence the result of the lemma follows. △

Property 5.5.1 *There is* $c^\circ > 0$ *so that*

$$|\nabla\psi(1,t)_1| \leq c^\circ t^2, \quad \textit{for all } t \in [-2,1] \textit{ such that } (1,t) \in \mathcal{D}_\psi, \tag{5.67}$$

and

$$|\nabla\psi(t,1)_2| \le c^\circ t^2, \quad \textit{for all } t \in [-2,1] \textit{ such that } (t,1) \in \mathcal{D}_\psi. \tag{5.68}$$

Lemma 5.5.7 *Property 5.5.1 holds for $\psi_{\text{poly}}(a,b)$.*

Proof: As noted earlier, ψ_{poly} is positively homogeneous of degree one and piecewise-linear. Hence, from Theorem 5.2.2 there exists $k \in \mathbb{N}$ and constants c_{1j}, c_{2j}, for $j = 1, 2, \ldots, k$, such that

$$\nabla\psi(r) \in \{(c_{1i}, c_{1j})^\top \in \mathbb{R}^2 | \text{ for some } i, j = 1, 2, \ldots, k\}, \quad \forall r \in \mathcal{D}_\psi. \tag{5.69}$$

Let c_1 be the maximum among the constants $|c_{1j}|, |c_{2j}|$, for all $j = 1, 2, \ldots, k$. Without loss of generality, let the angle of cones $\mathcal{S}_a$ and $\mathcal{S}_b$ be $2\phi \in \left(0, \frac{\pi}{4}\right]$. Using Lemma 5.5.5 we obtain that

$$|\nabla\psi(1,t)_1| = 0, \quad \text{for all } t \in [-\tan\phi, \tan\phi] \text{ such that } (1,t) \in \mathcal{D}_\psi, \tag{5.70}$$

and

$$|\nabla\psi(t,1)_2| = 0, \quad \text{for all } t \in [-\tan\phi, \tan\phi] \text{ such that } (t,1) \in \mathcal{D}_\psi. \tag{5.71}$$

Hence, we see that (5.67) and (5.68) hold with $c^\circ := \dfrac{c_1}{\tan^2\phi}$. △

Remark 5.5.1 *We can easily verify that Property 5.5.1 is satisfied for all positively homogeneous NCP functions of degree one which are discussed in [22; 64] or given in Table 5.1. Property 5.5.1 requires checking only at differentiable points of two line segments.*

Lemma 5.5.8 *Let the NCP function ψ be of Type I (see Definition 5.2.4) and satisfy Property 5.5.1. Then for any $t \in [-2,2]$, we have the following.*

(a) *For any $g^\top \in \partial\psi(1,t)$ it holds that*

$$|g_1| \le c^\circ t^2 \textit{ and,} \tag{5.72}$$

(b) *for any $g^\top \in \partial\psi(t,1)$ it holds that*

$$|g_2| \le c^\circ t^2. \tag{5.73}$$

Proof: Let

$$\mathcal{U} := \{(a,b) \in \mathbb{R}^2 | a+b \geq 0, \|(a,b)\|_\infty = 1\},$$

where we recall that $\|\cdot\|_\infty$ is the infinity norm. This set $\mathcal{U}$ can be equivalently described by

$$\mathcal{U} := \{(1,t) \in \mathbb{R}^2 | t \in [-1,1]\} \cup \{(t,1) \in \mathbb{R}^2 | t \in [-1,1]\}.$$

Take an arbitrary but fixed point $\hat{s} \in \mathcal{U}$. We claim that

$$\partial\psi(\hat{s}) = \text{co}\left\{\lim_{j\to\infty} \nabla\psi(\xi_j)^\top | \xi_j \to \hat{s}, \|\xi_j\|_\infty = 1, \xi_j \in \mathcal{D}_\psi\right\}. \tag{5.74}$$

Since ψ is of Type-I, it is positively homogeneous of degree one. Hence, (5.46), Theorem 5.2.2 and $\hat{s} \in \mathcal{U}$ gives

$$\begin{aligned}
\hat{g}^\top \in \partial\psi(\hat{s}) \quad &\Longleftrightarrow \quad \hat{g}^\top \in \text{co}\left\{\lim_{j\to\infty} \nabla\psi(s_j)^\top | s_j \to \hat{s}, s_j \in \mathcal{D}_\psi\right\} \\
&\Longleftrightarrow \quad \hat{g}^\top \in \text{co}\left\{\lim_{j\to\infty} \nabla\psi\left(\frac{s_j}{\|s_j\|_\infty}\right)^\top \Bigg| \frac{s_j}{\|s_j\|_\infty} \to \hat{s}, \frac{s_j}{\|s_j\|_\infty} \in \mathcal{D}_\psi\right\} \\
&\Longleftrightarrow \quad \hat{g}^\top \in \text{co}\left\{\lim_{j\to\infty} \nabla\psi(\xi_j)^\top | \xi_j \to \hat{s}, \|\xi_j\|_\infty = 1, \xi_j \in \mathcal{D}_\psi\right\}.
\end{aligned}$$

Hence (5.74) holds.

Now, due to (5.74), we easily see that any element from $\partial\psi(\hat{s})$ satisfies conditions (5.67) and (5.68). Hence, (5.72) and (5.73) hold. Hence the statement of the lemma holds. △

Definition 5.5.1 *The solution z^* will be called* isolated in the degenerate components (IDC)*, if, either $\mathcal{J}$ is empty (z^* is non-degenerate), or, if there exists $\delta_1 > 0$ so that*

$$\hat{z}_i = \hat{z}_{n+i} = 0 \textit{ for all } i \in \mathcal{J} \textit{ and all } \hat{z} \in (\mathcal{B}(z^*, \delta_1) \cap Z) \setminus \{z^*\}. \tag{5.75}$$

It is clear that without loss of generality we can also assume that $\delta_1 \leq \delta$.

Convergence analysis of the constrained Levenberg-Marquardt method applied to solving NCPs requires less conditions if z^* is IDC. On the other hand, the case when z^* is not IDC is difficult to handle. These are shown in the next theorems.

Theorem 5.5.1 *Let the NCP function ψ be of Type I and satisfy Property 5.5.1. Moreover, let $\mathcal{S} \subseteq \{(a,b) \in \mathbb{R}^2 | a+b \geq 0\}$. Then, Condition 5.5.1 is satisfied if z^* is IDC.*

Proof: Let z^* be IDC and let

$$\mathcal{S} \subseteq \{(a,b) \in \mathbb{R}^2 | a+b \geq 0\}. \tag{5.76}$$

Hence, from Definition 5.5.1, z^* is either non-degenerate or there exist $\delta_1, \delta_2 > 0$ so that (5.75) holds. Let $\hat{\delta}$ be defined as

$$\hat{\delta} := \begin{cases} \min\left\{\max\limits_{i\in\mathcal{I}}\left\{\dfrac{x_i^*}{4}, \dfrac{y_i^*}{4}\right\}, \dfrac{\delta}{2}\right\} & \text{if } z^* \text{ is non-degenerate, and} \\ \min\left\{\max\limits_{i\in(\mathcal{I}\setminus\mathcal{J})}\left\{\dfrac{x_i^*}{4}, \dfrac{y_i^*}{4}\right\}, \dfrac{\delta_1}{2}\right\} & \text{otherwise.} \end{cases}$$

It is easy to see that $\hat{\delta} > 0$ and that for any $z \in \mathcal{B}(z^*, \hat{\delta})$, $\text{dist}\,[z, Z]$ is realized at a point in the ball $\mathcal{B}(z^*, \delta)$.

From Lemma 5.5.4, we know that Condition 5.5.3 suffices Condition 5.5.1. Let us consider a point $z := (x,y) \in \mathcal{B}(z^*, \hat{\delta}) \cap \Omega$. Let $z^\diamond := (x^\diamond, y^\diamond)$ be a projection of z onto Z. Such a $z^\diamond$ always exists as Z is closed and nonempty.

In the first part of the remaining proof we deal with the case that z^* is non-degenerate. Now for an arbitrary but fixed $i \in \mathcal{I}$ consider the point $(x_i, y_i) \in \mathcal{B}((x_i^*, y_i^*), \hat{\delta}) \cap \mathcal{S}$. From the definition of $\hat{\delta}$, we obtain that

$$(x_i, y_i) \neq (0,0) \quad \text{for all } i \in \mathcal{I}, \tag{5.77}$$

and that either $x_i < y_i$ or $x_i > y_i$ holds. We discuss these two cases in detail.

Case A: $x_i < y_i$. Here, obviously $x_i^\diamond = 0$ and thus,

$$\begin{aligned} g(x_i, y_i)^\top \begin{pmatrix} x_i^\diamond \\ y_i^\diamond \end{pmatrix} &= g(x_i, y_i)^\top \begin{pmatrix} 0 \\ y_i^\diamond \end{pmatrix} \\ &= g(x_i, y_i)_2 y_i^\diamond. \end{aligned}$$

Moreover, as $(x_i, y_i) \in \mathcal{S}$ and $x_i < y_i$, using (5.76), we see that $y_i > 0$. From Theorem 5.4.2, we obtain that there exists $\tilde{g}$ such that $\tilde{g}^\top \in \partial\psi\left(\frac{x_i}{y_i}, 1\right)$ and

$$g(x_i, y_i) = \tilde{g}. \tag{5.78}$$

As $y_i > 0$ and $x_i^\diamond = 0$, from the definition of $\hat{\delta}$ we have that

$$\begin{aligned}\left|\frac{x_i}{y_i}\right| &= \frac{|x_i|}{y_i} \\ &\leq \frac{\hat{\delta}}{y_i^* - \hat{\delta}} \\ &\leq 1.\end{aligned}$$

Now using (5.78) and Lemma 5.5.8 we obtain

$$\begin{aligned}|g(x_i, y_i)_2 y_i^\diamond| &= |\tilde{g}_2|\, y_i^\diamond \\ &\leq c^\circ \left(\frac{x_i}{y_i}\right)^2 y_i^\diamond \\ &= \left(\frac{c^\circ y_i^\diamond}{y_i^2}\right) x_i^2 \\ &\leq \left(\frac{c^\circ y_i^\diamond}{y_i^2}\right) \|(x_i, y_i) - (0, y_i^\diamond)\|^2 \\ &\leq \left(\frac{c^\circ y_i^\diamond}{y_i^2}\right) \text{dist}\,[z, Z]^2.\end{aligned} \tag{5.79}$$

Case B: $y_i < x_i$. In this case, proceeding along the lines of Case A, we obtain

$$\begin{aligned}g(x_i, y_i)^\top \begin{pmatrix} x_i^\diamond \\ y_i^\diamond \end{pmatrix} &= g(x_i, y_i)^\top \begin{pmatrix} x_i^\diamond \\ 0 \end{pmatrix} \\ &= g(x_i, y_i)_1 x_i^\diamond,\end{aligned}$$

and

$$|g(x_i, y_i)_1 x_i^\diamond| \leq \left(\frac{c^\circ x_i^\diamond}{x_i^2}\right) \text{dist}\,[z, Z]^2. \tag{5.80}$$

Now, we easily see that

$$\max_{\substack{i \in \mathcal{I} \\ z, \hat{z} \in \mathcal{B}\left(z^*, \hat{\delta}\right) \cap \Omega}} \left\{\frac{c^\circ y_i^\diamond}{\hat{y}_i^2}, \frac{c^\circ x_i^\diamond}{\hat{x}_i^2}\right\} \leq c^\circ \frac{\max_{i \in \mathcal{I}}\{x_i^*, y_i^*\} + \hat{\delta}}{(\min_{i \in \mathcal{I}}\{x_i^*, y_i^*\} - \hat{\delta})^2} =: c_1. \tag{5.81}$$

From the definition of $\hat{\delta}$ we observe that $c_1 > 0$. Hence from (5.79) and (5.80), it is easy to see that Condition 5.5.3 is satisfied for $c^\diamond$ defined as

$$c^\diamond := \max\{1, c_1\}.$$

In the second part we consider the case when z^* is degenerate and IDC. Then, for all $i \in \mathcal{I} \setminus \mathcal{J}$, (5.79) and (5.80) still hold. The constant c_1 is now given by

$$\max_{\substack{i\in\mathcal{I}\setminus\mathcal{J}\\ z,\hat{z}\in\mathcal{B}(z^*,\hat{\delta})\cap\Omega}} \left\{ \frac{c^\circ y_i^\diamond}{\hat{y}_i^2}, \frac{c^\circ x_i^\diamond}{\hat{x}_i^2} \right\} \leq c^\circ \frac{\max_{i\in\mathcal{J}}\{x_i^*, y_i^*\} + \hat{\delta}}{(\min_{i\in\mathcal{I}\setminus\mathcal{J}}\{x_i^*, y_i^*\} - \hat{\delta})^2} =: c_1. \tag{5.82}$$

(5.64) holds for all $i \in \mathcal{I} \setminus \mathcal{J}$. For all $i \in \mathcal{J}$, $(x_i^\diamond, y_i^\diamond) = (z_i^\diamond, z_{n+i}^\diamond) = (0,0)$ and hence (5.64) holds trivially. Thus, Condition 5.5.3 is also satisfied if z^* is degenerate and IDC.

Hence, the statement of the theorem follows. △

Theorem 5.5.2 *Let the NCP function ψ be of Type II, i.e.,*

$$\psi(a,b) := \psi_1(a,b) + \psi_k(a,b) \quad \textit{for all } (a,b) \in \mathbb{R}^2,$$

or let ψ be of Type III, i.e.,

$$\psi(a,b) := \begin{pmatrix} \psi_1(a,b) \\ \psi_k(a,b) \end{pmatrix} \quad \textit{for all } (a,b) \in \mathbb{R}^2,$$

(see Definition 5.2.4). Let the NCP function $\psi_1(a,b)$ satisfy Property 5.5.1. Moreover, let Assumption 5.3.1 hold with $H_\psi := H_{\psi_k}$ for some $\mathcal{S} \subseteq \{(a,b) \in \mathbb{R}^2 | a+b \geq 0\}$. Then, Condition 5.5.1 is satisfied if z^* is IDC.

Proof: Let us assume the NCP function ψ to be of Type II. Observe that by [8, Proposition 2.3.3],

$$\partial\psi(r) \subseteq \partial\psi_1(r) + \partial\psi_k(r), \text{ for all } r \in \mathbb{R}^2.$$

Since ψ_1 is positive homogeneous of degree one, applying Theorem 5.2.1, we obtain that

$$(H_\psi(z) + G(z)(z^\diamond - z))_i = g_1(x_i,y_i)^\top \begin{pmatrix} x_i^\diamond \\ y_i^\diamond \end{pmatrix} + \psi_k(x_i,y_i) + g_k(x_i,y_i)^\top \begin{pmatrix} x_i^\diamond - x_i \\ y_i^\diamond - y_i \end{pmatrix},$$

for all $z := (x,y), z^\diamond := (x^\diamond, y^\diamond) \in \mathbb{R}^{2n}$ and $i \in \mathcal{I}$, where $g_1, g_k : \mathbb{R}^2 \to \mathbb{R}^2$ are functions such that $g_1(r) \in \partial\psi_1(r)$ and $g_k(r) \in \partial\psi_k(r)$ for all $r \in \mathbb{R}^2$. Using the triangle inequality in the above expression further shows that

$$|(H_\psi(z) + G(z)(z^\diamond - z))_i| \le \left|\psi_k(x_i, y_i) + g_k(x_i, y_i)^\top \begin{pmatrix} x_i^\diamond - x_i \\ y_i^\diamond - y_i \end{pmatrix}\right| + \left|g_1(x_i, y_i)^\top \begin{pmatrix} x_i^\diamond \\ y_i^\diamond \end{pmatrix}\right|.$$

Now, Assumption 5.3.1 is satisfied with $H_\psi := H_{\psi_k}$, hence the ψ_k term in the above inequality satisfies Assumption 5.3.2 and also Condition 5.5.1. Moreover, from Theorem 5.5.1, the ψ_1 term also satisfies Condition 5.5.1 (as ψ_1 is an NCP function of Type-I).

For showing the other part, let ψ be of Type III. Again, the statement of the theorem follows by noting that from Theorem 5.5.1 and Assumption 5.3.1, both ψ_1 and ψ_k terms satisfy Condition 5.5.1. △

Remark 5.5.2 *If $\psi_k(a,b) := ab$ then it is easy to see that Assumption 5.3.1 holds if $\Omega := \mathbb{R}_+^{2m}$. Note that $\psi_k(a,b) = ab = a_+b_+$ for all $(a,b) \in \mathbb{R}_+^2$ and then, ψ_{KP} and ψ_{CCK} satisfy Condition 5.5.1.*

We investigate next what happens if the solution z^* is *not* IDC. To discuss Condition 5.5.1 in this case, we first state the following property.

Property 5.5.2 *There is a $\bar{t} \in \left(0, \frac{1}{2}\right]$ so that the NCP function ψ satisfies*

(i) $\nabla\psi(1,t)_1 \neq 0$, *for all $t \in (0, \bar{t}\,]$ such that $(1,t) \in \mathcal{D}_\psi$, and*

(ii) $\nabla\psi(t,1)_2 \neq 0$, *for all $t \in (0, \bar{t}\,]$ such that $(t,1) \in \mathcal{D}_\psi$.*

It is easy to verify that all the NCP functions from Table 5.1 except $\psi_{\min}$ and ψ_{poly} satisfy Property. 5.5.2. Let us define a weaker smoothness condition than Condition 5.5.1.

Condition 5.5.4 *There are $\omega_1 > 0$, $\delta_1 > 0$ and $\rho > 0$ so that for any $z \in \mathcal{B}(z^*, \delta_1) \cap \Omega \setminus Z$ there is $z^\diamond \in Z$ with*

$$\begin{aligned} \|z - z^\diamond\| &\le \omega_1 \text{dist}\,[z, Z], \textit{ and} \\ \|H_\psi(z) + G(z)(z^\diamond - z)\| &\le \omega_1 \text{dist}\,[z, Z]^{1+\rho}. \end{aligned}$$

Conditions 5.5.4 and 5.5.2 together with the blanket assumption for Chapter 5 are sufficient for a local convergence rate of $(1+\rho)$ of the constrained Levenberg-Marquardt method for

$$H_\psi(z) = 0, \quad \text{s.t. } z \in \Omega,$$

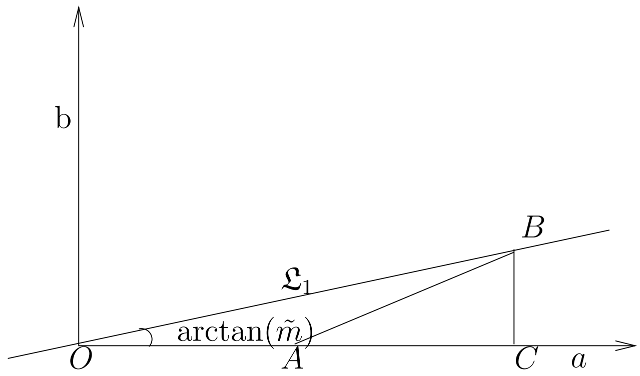

Figure 5.3: Illustration of the line $\mathfrak{L}_1$.

(see [27]).

Theorem 5.5.3 *Let the NCP function ψ be of Type I and satisfy Property 5.5.2. If z^* is not IDC then Condition 5.5.4 and hence, Condition 5.5.1 is* not *satisfied.*

Proof: As z^* is not IDC, without loss of generality, we can assume that $1 \in \mathcal{J}$ and that

$$\forall \zeta > 0 \, \exists \hat{z} \in \mathcal{B}(z^*, \zeta) \cap Z : \hat{z}_1 \in \mathcal{B}(0, \zeta) \setminus \{0\}. \tag{5.83}$$

Hence, there exists a sequence of positive reals $\{\bar{\zeta}_k\}$ and $\{\zeta_k\}$ converging to zero, such that there exists $\hat{z}^k \in \mathcal{B}(z^*, \bar{\zeta}_k) \cap Z$ with $\hat{z}_1^k = \zeta_k$, for all $k \in \mathbb{N}$.

Let the function $m : [\sqrt{2}, \infty) \to \left(0, \frac{1}{2}\right]$ be defined by

$$m(c) := \frac{1}{2\sqrt{c^2 - 1}}. \tag{5.84}$$

It is clear that m is strictly monotonically decreasing and it holds that

$$\lim_{c \to \infty} m(c) = 0. \tag{5.85}$$

Now let us assume that Condition 5.5.4 holds. Therefore, it is clear that $\omega_1 \geq 1$. Let $\bar{\omega}_1 \geq 1$ be so that

$$\bar{m} := m(\bar{\omega}_1) = \bar{t} \leq \frac{1}{2}, \tag{5.86}$$

where $\bar{t}$ comes from Property 5.5.2. Moreover, let

$$\omega := \max\{\omega_1, \bar{\omega}_1\}, \quad \text{and } \tilde{m} := m(\omega). \tag{5.87}$$

From (5.87), it is clear that Condition 5.5.4 holds if ω_1 is replaced by ω. As m is monotonically decreasing, from (5.86) we have that

$$\tilde{m} \leq \bar{m} \leq \frac{1}{2} \tag{5.88}$$

and

$$\omega \geq \sqrt{2}. \tag{5.89}$$

We can assume that ψ is differentiable at the point $(1, \tilde{m})$. Otherwise, from Lemma 5.5.6, there exists a point $(1, \tilde{m}_1)$ in the line segment $\{(1,t) | t \in [\frac{\tilde{m}}{2}, \tilde{m}]\}$ at which ψ is differentiable. With this, we take $\tilde{m}_1$ and the corresponding $\bar{\omega} > \omega$ such that $m(\bar{\omega}) = \bar{m}_1$ and redefine appropriately.

Since ψ is differentiable at $(1, \tilde{m})$, from Theorem 5.2.2, it is differentiable at $(t, t\tilde{m})$ for all $t > 0$.

Let us consider the line $\mathfrak{L}_1 \subset \mathbb{R}^2$ (illustrated in Figure 5.3) with $\mathfrak{L}_1 := \{(a,b)^\top \in \mathbb{R}^2 | b = \tilde{m}a\}$ and consider the sequence $\{z^k\} \subset \mathbb{R}^{2n}$ given by

$$z_i^k = \begin{cases} \hat{z}_i^k & \text{if } i \neq 1 \text{ or } i \neq n+1; \\ \zeta_k \, (= \hat{z}_1^k) & \text{if } i = 1; \\ \tilde{m}\zeta_k & \text{if } i = n+1. \end{cases}$$

For all $k \in \mathbb{N}$ sufficiently large, $z^k \in \mathcal{B}(z^*, \delta_1) \cap \Omega \setminus Z$ with δ_1 from Condition 5.5.4. Hence, z^k can be used as z in Condition 5.5.4 for $k \in \mathbb{N}$ sufficiently large.

Claim A:

$$\text{dist}\,[z^k, Z] = \tilde{m}\zeta_k \text{ for all } k \in \mathbb{N}. \tag{5.90}$$

To show this, consider an arbitrary but fixed $k \in \mathbb{N}$ and let $z^\perp \in Z$ be such that $\text{dist}\,[z^k, Z] = \|z^k - z^\perp\|$. Then, $z^\perp \in Z$ together with (5.88) gives that

$$\begin{aligned} \text{dist}\,[z^k, Z] &= \|z^k - z^\perp\| \\ &\geq \|(z_1^k, z_{n+1}^k) - (z_1^\perp, z_{n+1}^\perp)\| \\ &\geq \min\{\zeta_k, \tilde{m}\zeta_k\} \\ &= \tilde{m}\zeta_k. \end{aligned}$$

This together with $\hat{z}_k \in Z$ yields

$$\text{dist}\,[z^k, Z] \leq \left\|z^k - \hat{z}_k\right\| = \tilde{m}\zeta_k,$$

which proves Claim A.

Now, as the NCP function ψ is of Type-I, it is positively homogeneous of degree one. Using Property 5.5.2 and Theorem 5.2.2, we note that

$$0 < |\nabla\psi(1, \tilde{m})_1| = |\nabla\psi(t, t\tilde{m})_1| = |\nabla\psi(z_1^k, z_{n+1}^k)_1| \tag{5.91}$$

for all $t > 0$ and all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, let us consider Figure 5.3 and let $B := (z_1^k, z_{n+1}^k)$ and let $C := (z_1^k, 0)$. From Claim A, we obtain that

$$\text{dist}\,[z^k, Z] = |\overline{BC}|. \tag{5.92}$$

Claim B: For any $k \in \mathbb{N}$ and $z^\diamond \in Z$ such that

$$\|z^k - z^\diamond\| \leq \omega \text{dist}\,[z^k, Z],$$

we have that

$$z_{n+1}^\diamond = 0 \tag{5.93}$$

and

$$z_1^\diamond \geq \frac{1 - \frac{1}{\sqrt{2}}}{2} \cdot \frac{\text{dist}\,[z^k, Z]}{\tilde{m}} > 0. \tag{5.94}$$

To show this, consider an arbitrary but fixed $k \in \mathbb{N}$ and take an arbitrary but fixed $z^\diamond \in Z$ such that $\|z^k - z^\diamond\| \leq \omega \text{dist}\,[z^k, Z]$. Now, (5.93) follows, as from the definitions of z^k and $z^\diamond$, we see that

$$z_1^\diamond \geq z_1^k - \omega \text{dist}\,[z^k, Z] = \zeta_k - \omega\tilde{m}\zeta_k = \zeta_k \left(1 - \frac{\omega}{2\sqrt{\omega^2 - 1}}\right) > \frac{1 - \frac{1}{\sqrt{2}}}{2}\zeta_k, \tag{5.95}$$

where the last inequality follows as the function $m_1 : (\sqrt{2}, \infty) \to \left(1, \frac{1}{2}\right)$ defined by

$$m_1(c) := 1 - m(c)c$$

is monotonically increasing. Using Claim A, (5.95) can be further simplified as

$$z_1^\diamond > \frac{1 - \frac{1}{\sqrt{2}}}{2} \cdot \frac{\text{dist}\,[z^k, Z]}{\tilde{m}} > 0, \tag{5.96}$$

and hence (5.94) also holds. This proves Claim B.

Now applying Lemma 5.5.1 and taking into account Equations (5.59), (5.93) and (5.91) we obtain

$$|(H_\psi(z^k) + G(z^k)(z^\diamond - z^k))_1| = |(G(z^k)z^\diamond)_1| = |G(z^k)_{11} z_1^\diamond| = |\nabla\psi(1, \tilde{m})_1| z_1^\diamond. \tag{5.97}$$

This together with (5.94) yields,

$$|(H_\psi(z^k) + G(z^k)(z^\diamond - z^k))_1| \geq |\nabla\psi(1, \tilde{m})_1| \frac{1 - \frac{1}{\sqrt{2}}}{2} \cdot \frac{\operatorname{dist}[z^k, Z]}{\tilde{m}}. \tag{5.98}$$

For $k \in \mathbb{N}$ sufficiently large we obtain a contradiction to (5.98) by noting that Condition 5.5.4 implies

$$|(H_\psi(z^k) + G(z^k)(z^\diamond - z^k))_1| = |G(z^k)z_1^\diamond| \leq \omega \operatorname{dist}[z^k, Z]^{1+\rho}.$$

$\triangle$

Remark 5.5.3 *Theorem 5.5.3 holds also for Type-II and Type-III ψ functions if the NCP function $\psi_1(a,b)$ satisfies Property 5.5.2.*

Remark 5.5.4 *Theorem 5.5.3 gives a negative answer for obtaining a superlinear rate of convergence if a certain class of ψ function is employed. It might be interesting to investigate if superlinear convergence without any particular rate is possible or not.*

Property 5.5.2 is satisfied by many *non piecewise-linear* NCP functions like ψ_{FB}, ψ_{LT} (for any $q \in (1, \infty)$), ψ_{CCK} and ψ_{KK}. With no hope to obtain a superlinear rate of convergence using such NCP functions if z^* is not IDC, we next turn to *piecewise-linear* NCP functions like $\psi_{\min}$ or ψ_{poly}. For this, we need an additional error bound condition. Before this is presented, let us introduce some new notation.

Let $\mathcal{K}$ be a nonempty subset of the index set $\mathcal{I}$. For a given $z \in \mathbb{R}^{2n}$, let $z^{\mathcal{K}} := (x^{\mathcal{K}}, y^{\mathcal{K}}) \in \mathbb{R}^{2n}$ denote the vector defined by

$$z_i^{\mathcal{K}} := \begin{cases} 0 & \text{if } i \in \mathcal{K} \text{ or } i \in n + \mathcal{K}; \\ z_i & \text{otherwise.} \end{cases}$$

Moreover, let $Z^{\mathcal{K}} \subseteq Z$ be defined as

$$Z^{\mathcal{K}} := \{z^{\mathcal{K}} \in Z | \exists z \in \Omega : H_\psi(z^{\mathcal{K}}) = 0\}.$$

Note that for any $\mathcal{K} \subseteq \mathcal{J}$, $z^* = (z^*)^{\mathcal{K}} \in Z^{\mathcal{K}}$.

Now we are in a position to state an error bound condition which we will use to analyze the case when z^* is not IDC.

Condition 5.5.5 *There are $c^{\mathcal{J}}, \delta^{\mathcal{J}} > 0$ so that, for any $\mathcal{K} \subseteq \mathcal{J}$*

$$c^{\mathcal{J}} \operatorname{dist}[\hat{z}^{\mathcal{K}}, Z^{\mathcal{K}}] \leq \left\| F(\hat{x}^{\mathcal{K}}) - \hat{y}^{\mathcal{K}} \right\| \quad \textit{for all } \hat{z} \in \mathcal{B}(z^*, \delta^{\mathcal{J}}) \cap Z. \tag{5.99}$$

Remark 5.5.5 *It is important to note that Condition 5.5.5 is independent of the choice of the NCP function ψ. This is in contrast to most of the other conditions in this chapter.*

We can easily see that Condition 5.5.5 is equivalent to a number of error bound conditions, that is exponential in the number of degenerate indices. Moreover, note that by $\mathcal{K} \subseteq \mathcal{J}$ in Condition 5.5.5,

$$\left\|F(\hat{x}^{\mathcal{K}}) - \hat{y}^{\mathcal{K}}\right\| = \left\|H_\psi(\hat{z}^{\mathcal{K}})\right\| \quad \text{for all } \hat{z} \in \mathcal{B}(z^*, \delta^{\mathcal{J}}) \cap Z \tag{5.100}$$

holds. As shown by the next lemma, Condition 5.5.5 does not seem too strong.

Lemma 5.5.9 *Condition 5.5.5 holds for LCP(M, q).*

Proof: Let us assume that the set $\mathcal{J}$ (corresponding to the solution z^*) is nonempty, since otherwise, there is nothing to prove. Consider an arbitrary but fixed, nonempty subset $\mathcal{K}$ of $\mathcal{J}$. Recall that $\bar{Z}$ is defined in (5.9). Let, for any $\bar{z} := (\bar{x}, \bar{y}) \in \bar{Z}$, the index sets $\mathcal{I}_1(\bar{z}), \mathcal{I}_2(\bar{z}), \mathcal{I}_3(\bar{z}) \subseteq \mathcal{I}$ be defined as

$$\begin{aligned} \mathcal{I}_1(\bar{z}) &:= \{i \in \mathcal{I} | \bar{x}_i > 0\} \\ \mathcal{I}_2(\bar{z}) &:= \{i \in \mathcal{I} | \bar{y}_i > 0\} \\ \mathcal{I}_3(\bar{z}) &:= \{i \in \mathcal{I} | \bar{x}_i = \bar{y}_i = 0\}, \end{aligned}$$

and let the set $\mathcal{S}(\bar{z}) \subseteq \mathbb{R}^{2n}$ be defined as

$$\begin{aligned} \mathcal{S}(\bar{z}) := \{z := (x, y) \in \mathbb{R}^{2n} | x_i \geq 0, y_i = 0 \, \forall i \in \mathcal{I}_1(\bar{z}), x_i = 0, y_i \geq 0 \, \forall i \in \mathcal{I}_2(\bar{z}), \\ x_i = y_i = 0 \, \forall i \in \mathcal{I}_3(\bar{z})\}. \end{aligned}$$

We can easily see that $\mathcal{I}_1(\bar{z}) \cup \mathcal{I}_2(\bar{z}) \cup \mathcal{I}_3(\bar{z}) = \mathcal{I}$, $\mathcal{S}(\bar{z})$ is polyhedral, and that

$$\bar{z} \in \mathcal{S}(\bar{z}) \text{ and,} \tag{5.101}$$

$$\mathcal{S}(\bar{z}) \subset \bar{Z} \tag{5.102}$$

hold.

Let $\hat{\delta}$ be defined as

$$\hat{\delta} := \min \left\{ \max_{i \in (\mathcal{I} \setminus \mathcal{J})} \left\{ \frac{x_i^*}{4}, \frac{y_i^*}{4} \right\}, \delta \right\}.$$

For an arbitrary but fixed $\hat{z} \in \mathcal{B}(z^*, \hat{\delta}) \cap Z$ consider the following (linearly) constrained system of linear equations

$$Mx + q - y = 0, \qquad \text{s.t. } (x, y) \in \mathcal{S}(\hat{z}^{\mathcal{K}}). \tag{5.103}$$

From the definitions of $\hat{\delta}$ and $\mathcal{S}(\hat{z}^{\mathcal{K}})$ and by $\mathcal{K} \subseteq \mathcal{J}$ we can easily see that $z^* \in \mathcal{S}(\hat{z}^{\mathcal{K}})$ and hence z^* is a solution of (5.103). It is also clear that the solution set of (5.103) is polyhedral and nonempty.

Claim: *The solution set of (5.103) is equal to the set $Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}})$.*

To show this, let us take an arbitrary but fixed $\tilde{z} := (\tilde{x}, \tilde{y})$ that solves (5.103). Hence,

$$M\tilde{x} + q - \tilde{y} = 0 \qquad \tilde{z} \in \mathcal{S}(\hat{z}^{\mathcal{K}}). \tag{5.104}$$

From (5.102) and by $\hat{z} \in \mathcal{K} \in \bar{Z}$ we obtain that

$$\psi(\tilde{x}_i^{\mathcal{K}}, \tilde{y}_i^{\mathcal{K}}) = 0 \text{ for all } i \in \mathcal{I}$$

Hence, $H_\psi(\tilde{z}) = 0$ and, $\tilde{z} \in Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}})$. In a similar way one can show that any element from $Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}})$ is a solution of (5.103). This proves our claim.

Since $\hat{z}^{\mathcal{K}} \in \mathcal{S}(\hat{z}^{\mathcal{K}})$ from (5.101), applying Hoffman's error bound result [37] yields that there exists $\tau > 0$ so that

$$\tau \text{dist}\,[\hat{z}^{\mathcal{K}}, Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}})] \leq \|M\hat{x}^{\mathcal{K}} + q - \hat{y}^{\mathcal{K}}\|. \tag{5.105}$$

As $Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}}) \subseteq Z^{\mathcal{K}}$,

$$\tau \text{dist}\,[\hat{z}^{\mathcal{K}}, Z^{\mathcal{K}}] \leq \|M\hat{x}^{\mathcal{K}} + q - \hat{y}^{\mathcal{K}}\| \tag{5.106}$$

follows. The constant τ depends on the index sets $\mathcal{I}_1(\hat{z}), \mathcal{I}_2(\hat{z})$ and $\mathcal{I}_3(\hat{z})$. However as $\mathcal{I}_1(\hat{z}), \mathcal{I}_2(\hat{z}), \mathcal{I}_3(\hat{z}) \subseteq \mathcal{I}$, the number of such subsets is finite. Hence, Condition 5.5.5 holds with $c^{\mathcal{J}}$ defined as the minimum of τ's for all possible subsets $\mathcal{I}_1(\hat{z}), \mathcal{I}_2(\hat{z})$ and $\mathcal{I}_3(\hat{z})$ for all $\hat{z} \in \mathcal{B}(z^*, \hat{\delta}) \cap Z$ and, $\delta^{\mathcal{J}} := \hat{\delta}$. △

Remark 5.5.6 *As the proof of the above lemma shows, a condition stronger than Condition 5.5.5 holds for LCP(M, q) (when the set $Z^{\mathcal{K}}$ is changed to a* smaller *set $Z \cap \mathcal{S}(\hat{z}^{\mathcal{K}})$)*

Without loss of generality we assume that

$$C \geq c^{\mathcal{J}} \text{and } \delta \leq \delta^{\mathcal{J}}, \tag{5.107}$$

where we recall that C, δ are from (5.5) (else just replace C by $c^{\mathcal{J}}$ and δ by $\delta^{\mathcal{J}}$ everywhere).

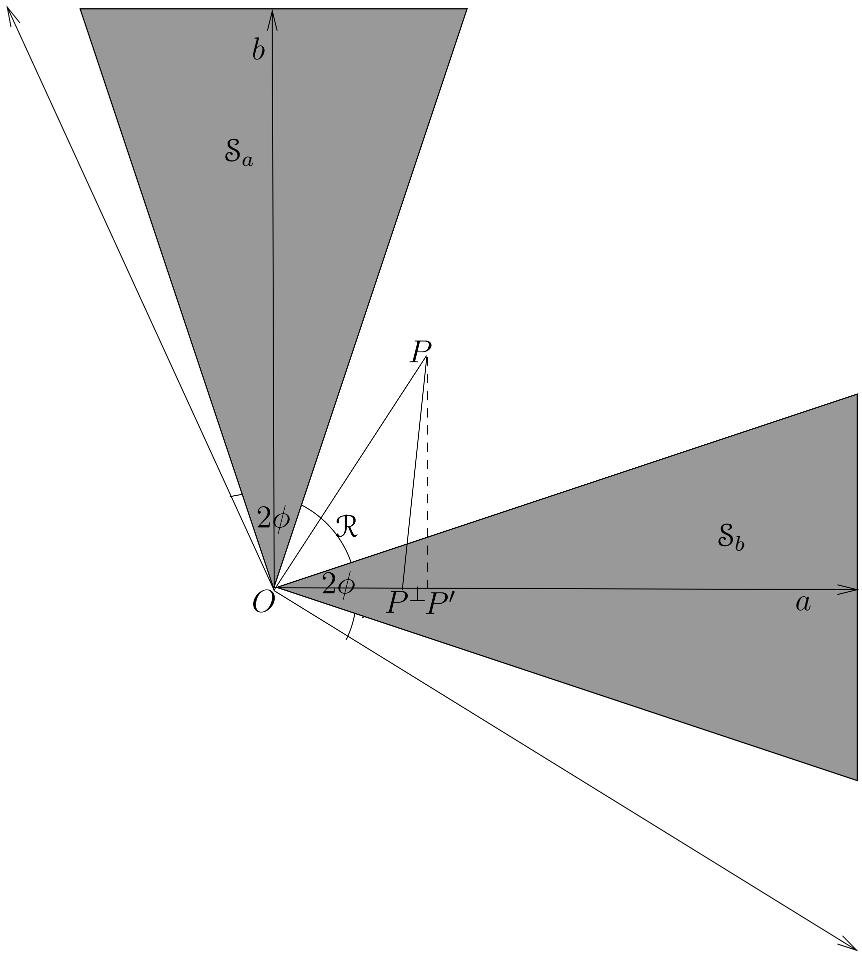

Figure 5.4: Illustration of the sets $\mathcal{S}_a$, $\mathcal{S}_b$ and $\mathcal{R}$.

Theorem 5.5.4 *Let Condition 5.5.5 hold and* $\psi := \psi_{\text{poly}}$. *Then, Condition 5.5.1 is satisfied.*

Proof: By Lemma 5.5.4, we have to show that Condition 5.5.3 holds. The function ψ_{poly} is (always) a positively homogeneous function of degree one , see [56; 60]. The proof that the remaining parts of Condition 5.5.3 are valid will be done by finding a solution $z^\diamond$ satisfying (5.63) such that the left side of the inequality in (5.64) is always zero.

Recall that $\mathcal{J}$ is the set of degenerate indices corresponding to the solution $z^* = (x^*, y^*)$. Let $\hat{\delta}$ be defined by

$$\hat{\delta} := \min\left\{ \max_{i \in (\mathcal{I} \setminus \mathcal{J})} \left\{ \frac{x_i^*}{4}, \frac{y_i^*}{4} \right\}, \frac{\delta}{2} \right\}.$$

It is easy to see that $\hat{\delta} > 0$ and that for all $z \in \mathcal{B}(z^*, \hat{\delta})$, $\operatorname{dist}[z, Z]$ is realized at a point in the ball $\mathcal{B}(z^*, \delta)$.

Without loss of generality, we assume that the cones $\mathcal{S}_a$ and $\mathcal{S}_b$ from Lemma 5.5.5 are symmetric with respect to the b-axis and the a-axis, respectively. Moreover, let the angle of each cone be 2ϕ and let $\phi \in \left(0, \frac{\pi}{4}\right)$. Let us define $\mathcal{R}$ by

$$\mathcal{R} := \mathcal{S} \setminus (\operatorname{int}(\mathcal{S}_a) \cup \operatorname{int}(\mathcal{S}_b)).$$

The situation is illustrated in Figure 5.4. From Lemma 5.5.5, we can easily see that

$$\partial\psi_{\text{poly}}(a, b) = \begin{cases} \{(c_1, 0)\} & \text{if } (a, b) \in \operatorname{int}(\mathcal{S}_a); \\ \{(0, c_2)\} & \text{if } (a, b) \in \operatorname{int}(\mathcal{S}_b), \end{cases}$$

where the constants $c_1, c_2 > 0$ are from Lemma 5.5.5. For any $\tilde{z} := (\tilde{x}, \tilde{y}) \in Z$ and $z := (x, y) \in \Omega$, let the index set $\mathcal{I}(z, \tilde{z})$ be defined by

$$\mathcal{I}(z, \tilde{z}) := \{i \in \mathcal{J} | (x_i, y_i) \in \operatorname{int}(\mathcal{S}_a), (\tilde{x}_i, \tilde{y}_i) \notin \mathcal{S}_a \text{ or } (x_i, y_i) \in \operatorname{int}(\mathcal{S}_b), (\tilde{x}_i, \tilde{y}_i) \notin \mathcal{S}_b \\ \text{or } (x_i, y_i) \in \mathcal{R}, (\tilde{x}_i, \tilde{y}_i) \neq (0, 0)\}$$

Let us take an arbitrary but fixed $z := (x, y) \in \mathcal{B}(z^*, \hat{\delta}) \cap \Omega \setminus Z$. Moreover, note that $G(z) \in \partial H_\psi(z)$ from the blanket assumption for Section 5.5. For this z let $z^\perp := (x^\perp, y^\perp)$ be a point in Z so that

$$\|z - z^\perp\| = \operatorname{dist}[z, Z]. \tag{5.108}$$

If $\mathcal{I}(z, z^\perp)$ is empty then we use $z^\diamond := z^\perp$ and then, with $c^\diamond = 1$, (5.63) holds due to (5.108). Moreover, (5.64) also holds as the left side of the inequality in (5.64) is zero. In the remaining part of the proof we assume that $\mathcal{I}(z, z^\perp)$ is nonempty.

In Figure 5.4, for an arbitrary but fixed $i \in \mathcal{I}(z)$, let $P := (x_i, y_i)$, $P^\perp := (x_i^\perp, y_i^\perp)$, and $P' := (x_i, 0)$. A simple geometrical analysis gives

$$\frac{|\overline{PO}|}{|\overline{PP^\perp}|} \leq \frac{|\overline{PO}|}{|\overline{PP'}|} = \frac{1}{\sin(\measuredangle POP')} = \csc(\measuredangle POP') \leq \csc(\phi).$$

Hence,

$$|\overline{PO}| = \|(x_i, y_i)\| \leq \csc(\phi)|\overline{PP^\perp}| \leq \csc(\phi)\operatorname{dist}[z, Z]. \tag{5.109}$$

Although, for simplicity, we assumed that $(x_i, y_i) \in \mathcal{R}$, it is easy to deduce that (5.109) holds for any $i \in \mathcal{I}(z)$ and any $z \in \mathcal{B}(z^*, \hat{\delta}) \cap \Omega \setminus Z$. Let $\mathcal{K} := \mathcal{I}(z, z^\perp)$.

Now, using (5.109) we obtain

$$\begin{aligned} \|z - (z^\perp)^{\mathcal{K}}\| &\leq \sum_{i\in\mathcal{K}} \|(x_i, y_i)\| + \sum_{i\in\mathcal{I}\setminus\mathcal{K}} \big\|(x_i, y_i) - \big((z^\perp)_i, (z^\perp)_{n+i}\big)\big\| \\ &\leq |\mathcal{K}| \csc(\phi) \mathrm{dist}\,[z, Z] + \mathrm{dist}\,[z, Z] \\ &\leq (|\mathcal{J}| \csc(\phi) + 1) \mathrm{dist}\,[z, Z], \end{aligned} \tag{5.110}$$

where the last inequality follows as $\mathcal{K} \subseteq \mathcal{J}$.

As $\mathcal{K} = \mathcal{I}(z)$ and $\delta^{\mathcal{J}} \geq \delta$ by (5.107), Condition 5.5.5 with $\hat{z} := z^\perp$ together with (5.100) give

$$c^{\mathcal{J}} \mathrm{dist}\,[(z^\perp)^{\mathcal{K}}, Z^{\mathcal{K}}] \leq \|H_\psi((z^\perp)^{\mathcal{K}})\|. \tag{5.111}$$

Let $\bar{z} := (\bar{x}, \bar{y}) \in Z^{\mathcal{K}}$ be such that

$$\|\bar{z} - (z^\perp)^{\mathcal{K}}\| = \mathrm{dist}\,[(z^\perp)^{\mathcal{K}}, Z^{\mathcal{K}}],$$

where we recall that $Z^{\mathcal{K}}$ is nonempty (as $z^* \in Z^{\mathcal{K}}$). From the definition of $Z^{\mathcal{K}}$, $\bar{x}_i = \bar{y}_i = 0$ for all $i \in \mathcal{K}$ follows. Repeated application of the triangle inequality together with $z^\perp \in Z$, (5.6) and (5.110) gives

$$\begin{aligned} c^{\mathcal{J}}\|\bar{z} - (z^\perp)^{\mathcal{K}}\| &\leq \|H_\psi((z^\perp)^{\mathcal{K}}) - H_\psi(z)\| + \|H_\psi(z)\| \\ &= \|H_\psi((z^\perp)^{\mathcal{K}}) - H_\psi(z)\| + \|H_\psi(z) - H_\psi(z^\perp)\| \\ &\leq L\|(z^\perp)^{\mathcal{K}} - z\| + L\mathrm{dist}\,[z, Z] \\ &\leq (|\mathcal{J}| \csc(\phi) + 2)L\mathrm{dist}\,[z, Z]. \end{aligned} \tag{5.112}$$

Again using the triangle inequality and (5.109) we obtain

$$\begin{aligned} \|z - \bar{z}\| &\leq \|z - (z^\perp)^{\mathcal{K}}\| + \|(z^\perp)^{\mathcal{K}} - \bar{z}\| \\ &\leq \sum_{i\in\mathcal{K}} \|(x_i, y_i)\| + \|z - z^\perp\| + \|(z^\perp)^{\mathcal{K}} - \bar{z}\| \\ &\leq c_1^\diamond \mathrm{dist}\,[z, Z], \end{aligned} \tag{5.113}$$

where

$$c_1^\diamond := |\mathcal{J}| \csc(\phi) + 1 + \frac{(|\mathcal{J}| \csc(\phi) + 2)L}{c^{\mathcal{J}}}. \tag{5.114}$$

If $\mathcal{I}(z, \bar{z})$ is empty then we use $z^\diamond := \bar{z}$ and then, with $c^\diamond := c_1^\diamond$, (5.63) holds due to (5.113). Moreover, (5.64) also holds as the left side of the inequality in (5.64) is zero. In the remaining part of the proof we assume that $\mathcal{I}(z, \bar{z})$ is nonempty.

We will now use Condition 5.5.5 with $\mathcal{K} := \mathcal{I}(z, z^\perp) \cup \mathcal{I}(z, \bar{z})$ to get $\bar{z}^1 \in Z$ such that (5.63) is satisfied. For this, consider the vector $\bar{z}^{\mathcal{K}}$. Now, with (5.109) and (5.113) we obtain that

$$\begin{aligned} \|z-\bar{z}^{\mathcal{K}}\| &\leq \sum_{i\in\mathcal{K}}\|(x_i,y_i)\|+\|z-\bar{z}\| \\ &\leq |\mathcal{J}|\csc(\phi)\mathrm{dist}\,[z,Z]+c_1^{\diamond}\mathrm{dist}\,[z,Z] \\ &= (|\mathcal{J}|+c_1^{\diamond})\,\mathrm{dist}\,[z,Z]. \end{aligned} \tag{5.115}$$

Let

$$\hat{\delta}_1 := \min\left\{\hat{\delta},\frac{\delta}{|\mathcal{J}|+c_1^{\diamond}}\right\}. \tag{5.116}$$

Now, take any arbitrary but fixed $z\in\mathcal{B}(z^*,\hat{\delta}_1)$. If still, both the sets $\mathcal{I}(z,z^{\perp})$ and $\mathcal{I}(z,\bar{z})$ are nonempty then we proceed further. The fact that $z\in\mathcal{B}(z^*,\hat{\delta}_1)$ together with (5.115) and (5.116) ensures that

$$\|z-\bar{z}^{\mathcal{K}}\|\leq\delta\leq\delta^{\mathcal{J}}. \tag{5.117}$$

Hence, Condition 5.5.5 with $\hat{z}:=\bar{z}$ together with (5.100) gives

$$c^{\mathcal{J}}\mathrm{dist}\,[\bar{z}^{\mathcal{K}},Z^{\mathcal{K}}]\leq\|H_\psi(\bar{z}^{\mathcal{K}})\|. \tag{5.118}$$

Let $\bar{z}^1\in Z^{\mathcal{K}}$ be such that

$$\|\bar{z}^1-\bar{z}^{\mathcal{K}}\|=\mathrm{dist}\,[\bar{z}^{\mathcal{K}},Z^{\mathcal{K}}]. \tag{5.119}$$

Repeated application of the triangle inequality together with (5.118), (5.119), $z^{\perp}\in Z$, (5.6) and (5.115) gives

$$\begin{aligned} c^{\mathcal{J}}\|\bar{z}^1-\bar{z}^{\mathcal{K}}\| &\leq \|H_\psi(\bar{z}^{\mathcal{K}})-H_\psi(z)\|+\|H_\psi(z)\| \\ &= \|H_\psi(\bar{z}^{\mathcal{K}})-H_\psi(z)\|+\|H_\psi(z)-H_\psi(z^{\perp})\| \\ &\leq L\|\bar{z}^{\mathcal{K}}-z\|+L\mathrm{dist}\,[z,Z] \\ &\leq (|\mathcal{J}|+c_1^{\diamond}+1)L\mathrm{dist}\,[z,Z]. \end{aligned}$$

Again using the triangle inequality and (5.115), we obtain

$$\|z-\bar{z}^1\|\leq\|z-\bar{z}^{\mathcal{K}}\|+\|\bar{z}^{\mathcal{K}}-\bar{z}^1\|\leq c_2^{\diamond}\mathrm{dist}\,[z,Z], \tag{5.120}$$

where

$$c_2^{\diamond}:=|\mathcal{J}|+c_1^{\diamond}+\frac{(|\mathcal{J}|+c_1^{\diamond}+1)L}{c^{\mathcal{J}}}. \tag{5.121}$$

If $\mathcal{I}(z,\bar{z}^1)$ is empty then we use $z^{\diamond}:=\bar{z}^1$ and then, with $c^{\diamond}:=c_2^{\diamond}$, (5.63) holds due to (5.120). Moreover, (5.64) also holds as the left side of the inequality in (5.64) is zero.

On the other hand, if $\mathcal{I}(z, \bar{z}^1) \neq \emptyset$ we repeat the procedure again, i.e., with $\mathcal{K} := \mathcal{I}(z, z^{\perp}) \cup \mathcal{I}(z, \bar{z}) \cup \mathcal{I}(z, \bar{z}^1)$ and replace $\bar{z}$ by $\bar{z}^1$. We note that the procedure stops since $|\mathcal{J}|$ is finite. Hence the statement of the theorem follows. $\triangle$

Remark 5.5.7 *In a similar way one can show that Theorem 5.5.4 also holds for*

$$\psi(a, b) := \psi'(a, b) := \begin{pmatrix} \lambda\psi_{\text{poly}}(a, b) \\ (1 - \lambda)a_+ b_+ \end{pmatrix}$$

with $\lambda \in (0, 1)$.

Remark 5.5.8 *From the proof of the above theorem we observe that this theorem holds regardless of the behavior of the NCP function* ψ *in the region* $\mathcal{R}$.

Theorems 5.5.1, 5.5.2, 5.5.3 and 5.5.4 are the main results of this section. If we restrict ourselves to the ψ-functions from Table 5.1, then, Tables 5.3 and 5.4 summarize the satisfiability of Condition 5.5.1 on these ψ-functions.

Table 5.3: Satisfiability of Condition 5.5.1 on the ψ-functions from Table 5.1, if z^* is IDC

ψ-functions	Conditions	Condition 5.5.1
$\psi_{\min}$, ψ_{FB}, ψ_{LT}, ψ_{poly}, ψ_{KK}, ψ_{CCK}	Blanket Assumptions for Chapter 5 and Section 5.5, $\mathcal{S}$ from (5.76), z^* is IDC	Satisfied
ψ_{KP}	Blanket Assumptions for Chapter 5 and Section 5.5, $\mathcal{S} := \mathbb{R}^2_+$, z^* is IDC	Satisfied

Table 5.4: Satisfiability of Conditions 5.5.4 and 5.5.1 on the ψ-functions from Table 5.1, if z^* is *not* IDC

ψ-functions	Conditions	Conditions 5.5.4 and 5.5.1
ψ_{FB}, ψ_{LT}, ψ_{KK}, ψ_{KP}, ψ_{CCK}	Blanket Assumption for Section 5.5	Not satisfied
$\psi_{\min}$, ψ_{poly}	Blanket Assumptions for Chapter 5 and Section 5.5, LCP(M, q)	Satisfied
$\psi_{\min}$, ψ_{poly}	Blanket Assumptions for Chapter 5 and Section 5.5, Condition 5.5.5	Satisfied

5.5.2 Discussion of Condition 5.5.2

We first present a sufficient condition for Condition 5.5.2 to hold.

Condition 5.5.6 *The NCP function $\psi : \mathbb{R}^2 \to \mathbb{R}$ is positively homogeneous of degree one. Moreover, there is $\beta > 0$ so that*

$$s \in \mathbb{L}(\tilde{s}, \alpha) := \left\{ s \in \mathcal{S} \middle| |g(\tilde{s})^\top s| \leq \alpha^2 \right\} \Rightarrow |\psi(s)| \leq \beta\alpha^2 \tag{5.122}$$

for all $\tilde{s} \in \mathcal{S}$ and all $\alpha > 0$.

Lemma 5.5.10 *Condition 5.5.6 implies Condition 5.5.2.*

Proof: Recall that β and δ are given by Lemma 5.5.3. Take an arbitrary but fixed $z \in \mathcal{B}(z^*, \delta) \cap \Omega \backslash Z$, $\alpha \in (0, \delta]$ and let $\hat{w}$ be an arbitrary but fixed element from the set

$$\overline{\mathbb{L}} := \left\{ w \in \Omega \middle| \|w - z\| \leq \alpha, \|H_\psi(z) + G(z)(w - z)\| \leq \alpha^2 \right\}.$$

Looking at (5.62) and (5.122), it is easy to see that

$$\hat{w} \in \left\{ w \in \Omega \middle| \|w - z\| \leq \alpha, |(H_\psi(z) + G(z)(w - z))_{n+i}| \leq \alpha^2 \right\} \tag{5.123}$$

and by Lemma 5.5.1,

$$(\hat{w}_i, \hat{w}_{n+i}) \in \mathbb{L}((z_i, z_{n+i}), \alpha) \tag{5.124}$$

hold for all $i \in \mathcal{I}$.

Now, applying Lemma 5.5.3 and Condition 5.5.6 we obtain that

$$\begin{aligned}
\|H_\psi(\hat{w})\| &\leq \|(H_\psi(\hat{w})_1, \ldots, H_\psi(\hat{w})_n)^\top\| + \|(H_\psi(\hat{w})_{n+1}, \ldots, H_\psi(\hat{w})_{2n})^\top\| \\
&= \|(\psi(\hat{w}_1, \hat{w}_{n+1}), \ldots, \psi(\hat{w}_n, \hat{w}_{2n}))^\top\| + \|(H_\psi(\hat{w})_{n+1}, \ldots, H_\psi(\hat{w})_{2n})^\top\| \\
&\leq \sqrt{n}\beta\alpha^2 + \sqrt{n}\beta\alpha^2 \\
&= 2\sqrt{n}\beta\alpha^2,
\end{aligned}$$

holds. Hence, if $\alpha \in (0, \delta]$, we see that Condition 5.5.2 hold with $\omega_2 := 2\sqrt{n}\beta$ and $\delta_2 := \delta$. Moreover, if $\alpha = 0$, then Condition 5.5.2 holds trivially. Hence the statement of the lemma follows. △

We discuss how to choose the constraint set $\mathcal{S}$ so that Condition 5.5.6 can be satisfied. Before this, we present some results to be used later.

Lemma 5.5.11 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function of degree one. Then, for any $\alpha > 0$ and $t > 0$ the level set $\mathcal{L}_f(\alpha) := \{x \in \mathbb{R}^n | f(x) \leq \alpha\}$ satisfies*

$$\mathcal{L}_f(t\alpha) = t\mathcal{L}_f(\alpha). \tag{5.125}$$

Proof: By definition,

$$\begin{aligned} x \in \mathcal{L}_f(t\alpha) &\iff f(x) \le t\alpha \\ &\iff \frac{f(x)}{t} \le \alpha \\ &\iff f\left(\frac{x}{t}\right) \le \alpha \\ &\iff x \in t\mathcal{L}_f(\alpha). \end{aligned}$$

$\triangle$

The following lemma also follows by definition of positively homogeneous functions.

Lemma 5.5.12 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positively homogeneous function of degree k. Then $|f|$ is also positively homogeneous of degree k.*

Let us define $\mathcal{T} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ as the following set-valued map

$$\mathcal{T}(\kappa) := \left(\mathbb{R}^2_+ - (\kappa, \kappa)\right) \setminus \left(\mathbb{R}^2_+ + (\kappa, \kappa)\right).$$

Figure 5.5 illustrates the set $\mathcal{T}(\kappa)$.

Lemma 5.5.13 *Let ψ be a positively homogeneous NCP function of degree one. Then, there exists $\varkappa > 0$ so that*

$$\mathcal{T}(\varkappa) \subseteq \mathcal{L}_{|\psi|}(1).$$

Proof: Since ψ is locally Lipschitz by Assumption 5.2.2, we obtain that $|\psi|$ is also locally Lipschitz. Let $\tilde{L} > 0$ be so that

$$||\psi(\tilde{s})| - |\psi(\tilde{r})|| \le \tilde{L}\|\tilde{r} - \tilde{s}\|,$$

for all $\tilde{r}, \tilde{s} \in \mathcal{B}((0,0),1)$. For any $s, r \in \mathbb{R}^2$, there exists $\tilde{k} > 0$ sufficiently large so that $\left(\frac{s}{\tilde{k}}, \frac{r}{\tilde{k}}\right) \in \mathcal{B}((0,0)^\top, 1)$. Hence using positive homogeneity of degree one

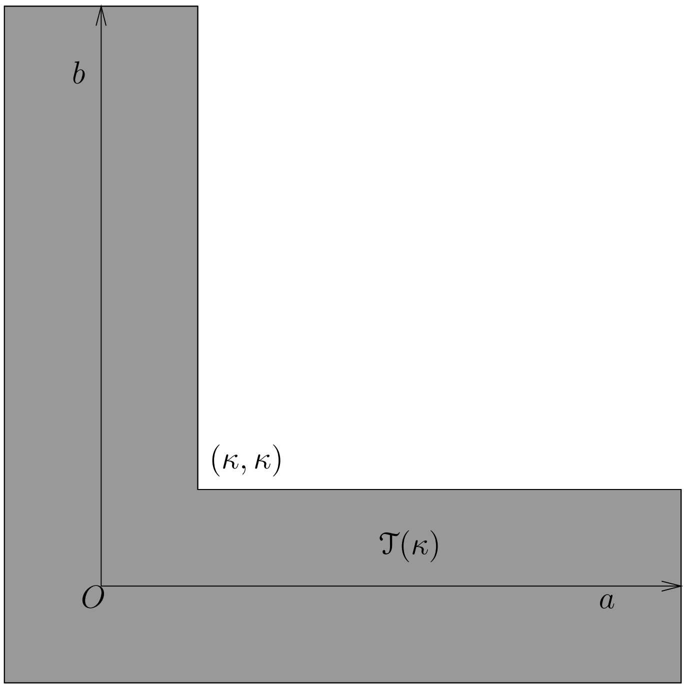

Figure 5.5: Illustration of the set $\mathcal{T}(\kappa)$, for some $\kappa > 0$. The shaded non-convex region is the set $\mathcal{T}(\kappa)$.

of ψ, we obtain

$$\begin{aligned} ||\psi(s)| - |\psi(r)|| &= \tilde{k} \left| \left| \psi\left(\frac{s}{\tilde{k}}\right) \right| - \left| \psi\left(\frac{r}{\tilde{k}}\right) \right| \right| \\ &\leq \tilde{L}\tilde{k} \left\| \frac{s}{\tilde{k}} - \frac{r}{\tilde{k}} \right\| \\ &= \tilde{L}\|s - r\|. \end{aligned}$$

This shows that $|\psi|$ is *globally* Lipschitz continuous.

Let $\varkappa := \frac{1}{\sqrt{2}\tilde{L}}$. For an arbitrary but fixed $r \in \mathcal{T}(\varkappa)$, let $r^{\perp}$ be a point such that

$$\|r - r^{\perp}\| := \operatorname{dist}\left[r, \{(a, b) \in \mathbb{R}^2 | a, b \geq 0, ab = 0\}\right].$$

By the definition of $\mathcal{T}(\varkappa)$, we have that

$$\|r - r^{\perp}\| \leq \sqrt{2}\varkappa.$$

Using global Lipschitz continuity of $|\psi|$ we further obtain that

$$|\psi(r)| = ||\psi(r)| - |\psi(r^{\perp})|| \leq L\|r - r^{\perp}\| \leq L\sqrt{2}\varkappa \leq 1.$$

Hence, $r \in \mathcal{L}_{|\psi|}(1)$ and the result of the lemma follows. $\triangle$

Corollary 5.5.1 *Let ψ be a positively homogeneous NCP function of degree one. Then, for any $t > 0$*

$$\mathcal{T}(t\varkappa) \subseteq \mathcal{L}_{|\psi|}(t), \tag{5.126}$$

where $\varkappa$ is from Lemma 5.5.13.

Proof: The statement of the lemma easily follows using Lemma 5.5.11 and by observing that $\mathcal{T}(t\varkappa) = t\mathcal{T}(\varkappa)$ for $t > 0$. $\triangle$

Remark 5.5.9 *Homogeneity of degree one is crucial both for Lemma 5.5.13 and for Corollary 5.5.1 to hold. Basically (5.126) holds since for positively homogeneous functions of degree one level curves satisfy (5.125).*

Assumption 5.5.1 *Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be an NCP function. There is $\varrho > 0$ so that, for all*

$$r \in \{(a,b)^{\top} \in \mathbb{R}^2 | a + b = 1, (a,b)^{\top} \in \mathcal{S} \cap \mathcal{D}_{\psi}\}, \tag{5.127}$$

it holds that

(i) $\|\nabla\psi(r)\| \geq \varrho$ *and*

(ii) $\frac{\partial\psi(r)}{\partial a} \cdot \frac{\partial\psi(r)}{\partial b} \geq 0.$

All NCP functions of Types I and II that we have found in the literature (for example that are discussed in [22; 64]) satisfy Assumption 5.5.1. Part (ii) of Assumption 5.5.1 has been useful for showing that every stationary point of the merit function (used for globalization of algorithms) is a solution of NCP(F) (see [46], for example). Lemma 5.5.15 gives a sufficient condition for Assumption 5.5.1 to hold.

Lemma 5.5.14 *Let the NCP function ψ be positively homogeneous of degree one and satisfy Assumption 5.5.1. Then for any $r \in \text{int}(\mathcal{S})$ and any $g(r)$*

(i) $\|g(r)\| \geq \varrho$, *and*

(ii) $g(r)_1 \cdot g(r)_2 \geq 0.$

Moreover, for any $r \in \mathrm{bd}(\mathcal{S})$*, (i) and (ii) hold for some* g*.*

Proof: The proof easily follows from the definition of $\partial\psi(\hat{a}, \hat{b})$. △

Lemma 5.5.15 *Let the NCP function* ψ *be positively homogeneous of degree one and be either convex or concave. Then, Assumption 5.5.1 holds.*

Proof: Suppose that the NCP function ψ is positively homogeneous of degree one and convex. The case when ψ is concave is handled in the same manner by taking $-\psi$ instead of ψ (note that if ψ is an NCP function then so is $-\psi$). Assume on the contrary that part (i) of Assumption 5.5.1 is not satisfied. Hence, for a sequence $\{\epsilon_k\}$ converging to zero, we can find a sequence $\{r_k\} \subset \{(a,b)^\top \in \mathbb{R}^2 | a+b=1, (a,b)^\top \in \mathcal{S} \cap \mathcal{D}_\psi\}$ so that

$$\|\nabla\psi(r_k)\| \leq \epsilon_k.$$

Corresponding to $\{r_k\}$, let us consider the sequence $\{\tilde{r}_k\}$ converging to $(0,0)^\top$ and defined by

$$\tilde{r}_k := \epsilon_k \frac{r_k}{\|r_k\|}. \tag{5.128}$$

This together with Lemma 5.2.2 gives

$$\|\nabla\psi(\tilde{r}_k)\| = \|\nabla\psi(r_k)\| \leq \epsilon_k. \tag{5.129}$$

From this, by the definition of Clarke's subdifferential we obtain that

$$(0,0)^\top \in \partial\psi(0,0),$$

which further shows that $(0,0)^\top$ is a minimizer of the function ψ. Convexity of ψ further gives that

$$0 \neq \psi(1,1) \leq \frac{1}{2}\psi(2,0) + \frac{1}{2}\psi(0,2) = 0.$$

Hence $\psi(1,1) < 0 = \psi(0,0)$ contradicting that $(0,0)^\top$ is a minimizer of ψ. Hence, part (i) of Assumption 5.5.1 holds.

To prove part (ii) of Assumption 5.5.1 let us assume the contrary, i.e., there exists a point $r \in \{(a,b)^\top \in \mathbb{R}^2 | a+b=1, (a,b)^\top \in \mathcal{S} \cap \mathcal{D}_\psi\}$ so that

$$\frac{\partial\psi(r)}{\partial a} \cdot \frac{\partial\psi(r)}{\partial b} < 0. \tag{5.130}$$

Without loss of generality, let us assume that $\frac{\partial\psi(r)}{\partial b} < 0$, $\frac{\partial\psi(r)}{\partial a} > 0$. For such an r, it is easy to see that there exists $s \in \text{int}(\mathbb{R}^2_+)$ such that $\nabla\psi(r)^\top s > 0$. Now consider the one dimensional convex function ψ_s defined as

$$\psi_s(t) := \psi(r + ts).$$

From the chain rule, $\psi_s'(0) = \nabla\psi(r)^\top s > 0$. Hence, ψ_s is a strictly increasing function and at some $\hat{t} > 0$, $\psi_r(\hat{t}) > 0$ and $r + \hat{t}s \in \text{int}(\mathbb{R}^2_+)$. This is a contradiction as convexity of ψ easily gives that $\psi(a, b) < 0$ for all $(a, b) \in \text{int}(\mathbb{R}^2_+)$. △

Remark 5.5.10 *It can be seen that many of the functions in Table 5.1 (for example, $\psi_{\min}, \psi_{FB}$) are either convex or concave.*

Next, we go on to show how to choose Ω to satisfy Condition 5.5.6 and, thus, Condition 5.5.2 for a large class of locally Lipschitz continuous positively homogeneous NCP function ψ of degree one (Type-I).

Theorem 5.5.5 *Let $\mathcal{S} := \mathbb{R}^2_+$ and let the NCP function ψ be of Type-I and satisfy Assumption 5.5.1. Then, Condition 5.5.2 holds.*

Proof: Using Lemma 5.5.10, it is sufficient to show that Condition 5.5.6 holds. Take any arbitrary but fixed $\alpha > 0$, $\tilde{s} \in \mathcal{S}$, $s := (s_1, s_2) \in \mathbb{L}(\tilde{s}, \alpha)$ and $g(\tilde{s})^\top \in \partial\psi(\tilde{s})$. Let $g(\tilde{s})^\top =: (c_1, c_2)$. Taking into account Assumption 5.5.1 and that $\tilde{s} \in \mathbb{R}^2_+$ we have the following three cases.

Case 1: $|c_1| > 0, c_2 = 0$**.** Here, $|g(\tilde{s})^\top s| = |c_1|s_1$. Since $s \in \mathbb{L}(\tilde{s}, \alpha)$,

$$\begin{aligned}
|s_1| \quad &\leq \quad \frac{\alpha^2}{|c_1|} \leq \frac{\alpha^2}{\varrho} \quad \text{with } (s_1, s_2) \in \mathbb{R}^2_+ \\
&\Rightarrow \quad (s_1, s_2) \in \mathcal{T}\left(\frac{\alpha^2}{\varrho}\right) \\
&\Rightarrow \quad (s_1, s_2) \in \mathcal{L}_{|\psi|}\left(\frac{\alpha^2}{\varkappa\varrho}\right) \quad \text{(from Corollary 5.5.1)} \\
&\Rightarrow \quad |\psi(s)| \leq \frac{\alpha^2}{\varkappa\varrho}.
\end{aligned}$$

Hence Condition 5.5.6 holds with $\beta := \frac{1}{\varkappa\varrho}$.

Case 2: $|c_1| > 0, |c_2| > 0$**.** Then, $|g(\tilde{s})^\top| = |c_1|s_1 + |c_2|s_2$. Without loss of generality we assume that $s_1 \leq s_2$. This together with $(s_1, s_2) \in \mathbb{L}(\tilde{s}, \alpha)$ gives

$$s_1 \leq \frac{|c_1|s_1}{|c_1| + |c_2|} + \frac{|c_2|s_2}{|c_1| + |c_2|} \leq \frac{\alpha^2}{|c_1| + |c_2|}.$$

Hence,

$$\begin{aligned} (s_1, s_2) &\in \mathcal{T}\left(\frac{\alpha^2}{\varrho}\right) \\ \Rightarrow (s_1, s_2) &\in \mathcal{L}_{|\psi|}\left(\frac{\alpha^2}{\varkappa\varrho}\right) \quad \text{(from Corollary 5.5.1).} \end{aligned}$$

Hence Condition 5.5.6 holds with $\beta := \frac{1}{\varkappa\varrho}$.

Case 3: $|c_2| > 0, c_1 = 0$**.** Then, $|g(\tilde{s})^\top s| = |c_2|s_2$ and the rest analysis is similar to that in Case 1.

Thus, in all cases Condition 5.5.6 holds with $\beta := \frac{1}{\varkappa\varrho}$. Hence the result of the theorem follows. △

Theorem 5.5.6 *Let $\mathcal{S} := \{(a, b) \in \mathbb{R}^2 | a + b \geq 0\}$ and let the NCP function ψ be of Type-I and satisfy Assumption 5.5.1. Moreover, assume that the function g is such that $g_1(r)g_2(r) = 0$ for all $r \in \mathcal{S}$. Then, Condition 5.5.2 holds.*

Proof: Using Lemma 5.5.10, it is sufficient to show that Condition 5.5.6 holds. Take any arbitrary but fixed $\alpha > 0$, $\tilde{s} \in \mathcal{S}$, $s := (s_1, s_2) \in \mathbb{L}(\tilde{s}, \alpha)$ and $g(\tilde{s})^\top \in \partial\psi(\tilde{s})$. Let us consider the case $g(\tilde{s})^\top =: (c_1, 0)$. Then, $|g(\tilde{s})^\top s| = |c_1 s_1|$ and then taking into account Assumption 5.5.1 we obtain

$$\begin{aligned} |s_1| &\leq \frac{\alpha^2}{|c_1|} \leq \frac{\alpha^2}{\varrho} \quad \text{with } s_1 + s_2 \geq 0 \\ &\Rightarrow (s_1, s_2) \in \mathcal{T}\left(\frac{\alpha^2}{\varrho}\right) \\ &\Rightarrow (s_1, s_2) \in \mathcal{L}_{|\psi|}\left(\frac{\alpha^2}{\varkappa\varrho}\right) \quad \text{(from Corollary 5.5.1)} \\ &\Rightarrow |\psi(s)| \leq \frac{\alpha^2}{\varkappa\varrho}. \end{aligned}$$

Hence Condition 5.5.6 holds with $\beta := \frac{1}{\varkappa\varrho}$.

The proof of the other case is similar and is not presented here. △

Theorem 5.5.6 is useful for $\psi_{\min}(a,b)$ NCP function and other similar NCP functions where it is always possible to find a subdifferential element having one of its components as zero (by taking an element from the B-subdifferential).

Theorem 5.5.7 *Let the NCP function ψ be of Type-I and satisfy Assumption 5.5.1. Moreover, assume that for some $k \in \mathbb{N}$ and some $(c_{11}, c_{21}), (c_{12}, c_{22}), \dots, (c_{1k}, c_{2k}) \in \mathbb{R}^2$, the function $g(r)$ can always be chosen from the set $\mathfrak{S} := \{(c_{1j}, c_{2j}) | j = 1, 2, \dots, k\}$. Then there exist constants $\tilde{c}_1, \tilde{c}_2 \geq 0$ so that with*

$$\mathcal{S} := \{(a,b) \in \mathbb{R}^2 | \tilde{c}_1 a + b \geq 0, \tilde{c}_2 a + b \geq 0\}$$

Condition 5.5.2 holds.

Proof: Let $\mathfrak{S}_1 := \{(c_{1j}, c_{2j}) : |c_{1j}|, |c_{2j}| > 0, j = 1, 2, \dots, k\}$ and $\bar{\mathfrak{S}}_1 := \mathfrak{S} \setminus \mathfrak{S}_1$. Let

$$n := \min \left\{ \frac{c_{1j}}{c_{2j}} \in \mathfrak{S}_1, j = 1, 2, \dots, k \right\} \text{ and } M := \max \left\{ \frac{c_{1j}}{c_{2j}} \in \mathfrak{S}_1, j = 1, 2, \dots, k \right\}.$$

From the definition of $\mathfrak{S}_1$ we have that $n, M \in (0, \infty)$.

We claim that the statement of the theorem holds with $\tilde{c}_1 := \frac{n}{2}$ and $\tilde{c}_2 := 2M$. Using Lemma 5.5.10, it is sufficient to show that Condition 5.5.6 holds. Take any arbitrary but fixed $\alpha > 0$, $\tilde{s} \in \mathcal{S}$, $s := (s_1, s_2) \in \mathbb{L}(\tilde{s}, \alpha)$ and $g(\tilde{s})^\top \in \partial\psi(\tilde{s})$. We have the following two cases.

Case 1: $g(\tilde{s}) \in \mathfrak{S}_1$**.** Let $\tilde{c}$ denote the minimum among $\{c_{2j} | j = 1, 2, \dots, k\}$. For some $\beta > 0$ the lines $s_1 + 2Mt_2 = 0$ and $s_1 + Mt_2 = \beta$ intersect at $(\frac{-\beta}{M}, 2\beta)$. Similarly the lines $s_1 + \frac{n}{2} s_2 = 0$ and $s_1 + mt_2 = \beta$ intersect at $(\frac{2\beta}{n}, -\beta)$. Moreover, let $\mathfrak{T}_1(\beta)$ be the triangle formed by the vertices $(\frac{-\beta}{M}, 2\beta), (0,0)$ and $(\frac{2\beta}{n}, -\beta)$. It is easy to see that

$$\mathbb{L}(\tilde{s}, \alpha) \subseteq \mathfrak{T}_1 \left(\frac{\alpha^2}{\tilde{c}} \right). \tag{5.131}$$

This implies that $s = (s_1, s_2) \in \mathcal{L}_{|\psi|} \left(\frac{2\alpha^2}{\tilde{c}} \right)$. Hence

$$|\psi(s)| \leq \frac{2\alpha^2}{\tilde{c}},$$

and the statement of the theorem follows.

Case 2: $g_\psi(r) \in \bar{\mathfrak{S}}_1$**.** In this case the proof follows along the lines of Theorem 5.5.6. In this case we obtain

$$|\psi(s)| \leq \frac{\alpha^2}{\varkappa\varrho}.$$

Hence in all cases Condition 5.5.6 holds with $\beta := \max\{\frac{2}{\tilde{c}}, \frac{1}{\varkappa\varrho}\}$. Hence the result follows. △

Theorem 5.5.7 is useful for

$$\begin{aligned} \psi(a,b) &:= \|(a,b)\|_\infty - (a+b) = \max(|a|,|b|) - (a+b), \quad \text{or} \\ \psi(a,b) &:= \begin{cases} |a| + |b|/2 - a - b & \text{if } a \geq b; \\ |a|/2 + |b| - a - b & \text{otherwise} \end{cases} \end{aligned}$$

NCP functions and other similar NCP functions like $\psi_{\text{poly}}(a,b)$. In these kind of NCP functions it is always possible to find a subdifferential element from the finite set $g_\psi(r) \in \{(c_{1j}, c_{2j}) | j = 1, 2, \ldots, k, k \in \mathbb{N}\}$.

Remark 5.5.11 *Theorems 5.5.5, 5.5.6 and 5.5.7 (with their choice of Ω's) can also be used for ψ-functions of Type-III.*

An interesting topic currently under investigation is whether it is possible to apply the results of last three theorems, for NCP functions of Type-II. For such functions, although Assumption 5.5.1 holds, it is not clear how to bound their level sets and hence Condition 5.5.2 might not hold.

Theorems 5.5.5, 5.5.6, and 5.5.7 are the main results of this section. If we restrict ourselves to the ψ-functions from Table 5.1, then, Table 5.5 summarizes the satisfiability of Condition 5.5.2 on these ψ-functions.

Table 5.5: Satisfiability of Condition 5.5.2 on the ψ-functions from Table 5.1

ψ-functions	Conditions	Condition 5.5.2
$\psi_{\min}$, ψ_{FB}, ψ_{LT}, ψ_{poly}, ψ_{KK}, ψ_{KP}	Blanket Assumptions for Chapter 5 and Section 5.5, $\mathcal{S}$ from Theorems 5.5.5, 5.5.6 or 5.5.7	Satisfied
ψ_{CCK}	Blanket Assumptions for Chapter 5 and Section 5.5	Not known

Chapter 6

Conclusions and Outlook

The final chapter presents a summary of the mathematical contributions made in this thesis. Moreover, some possible directions of future research are presented.

6.1 Conclusions

The thesis is primarily devoted to solving nonlinear equations using Levenberg-Marquardt algorithms. In Chapter 2 we improved the level of inexactness of Levenberg-Marquardt methods without destroying their Q-quadratic rate of convergence. The bound on the level of inexactness is shown to be a tight one. We called a Levenberg-Marqurdt algorithm robust, if its regularization parameter is chosen as large as possible without destroying a desired convergence rate. Numerical experiments showed the efficiency of the robust algorithm over existing inexact Levenberg-Marquardt methods. The theory is also used to show that a projected robust Levenberg-Marquardt method also converges Q-quadratically.

In Chapters 3-4, we reformulated multi-objective optimization problems as a constrained system of nonlinear equations. In Chapter 3, we developed a descent based Q-quadratically convergent algorithm for unconstrained multi-objective optimization. Conditions for the error bound property to hold were derived. Global convergence to weakly Pareto-optimal points was shown under appropriate convexity assumptions.

Chapter 4 presents another Levenberg-Marquardt type algorithm for unconstrained multi-objective optimization. In this algorithm, each iteration provides a decrease in all the objective function values. This is achieved locally by using a suitably modified form of a constrained Levenberg-Marquardt method [43] where, in contrast to [43], the constraint set changes from iteration to iteration. As changing constraint sets have never been earlier used in Levenberg-Marquardt methods, we presented a detailed local convergence analysis. This analysis is based on a new error bound property which was shown to hold for multi-objective optimization problems under suitable conditions. Global convergence to weakly Pareto-optimal points was shown under appropriate convexity assumptions.

Chapter 5 deals with equation based reformulations of nonlinear complementarity problems. These reformulations are done by means of so called NCP functions. It was shown in detail that an existing smoothness conditions does not holds for these equation based reformulations near degenerate solutions. Recently ([27]), this smoothness condition has been weakened so that the constrained Levenberg-Marquardt method can have local Q-quadratic convergence if the NCP function is defined as the min function. In this chapter we provided a general framework for analyzing the weaker smoothness conditions for positively homogeneous NCP functions. Using this, we also analyzed various cases where the weaker smoothness conditions are satisfied and where they do not hold.

6.2 Outlook

This thesis has opened various possible directions for future research. Some important ones are discussed now.

It may be useful to develop a robust constrained Levenberg-Marquardt algorithm, with a robust choice of the regularization parameter. This should be done without destroying the Q-quadratic rate of convergence. Moreover, inexact versions of the (robust) constrained Levenberg-Marquardt can be developed, so that both the maximal level of inexactness and the Q-quadratic convergence are retained. In Section 2.4, from the numerical experiments we observed that using the regularization parameter $\alpha(s)$ to be smaller than $O(\|H(s)\|^2)$ did not worsen the results, if a CG method is used. It would be interesting to figure out a theoretical reason for this behavior. Moreover, an inexact version of the robust projected Levenberg-Marquardt method can be investigated for projections onto general convex sets

The results from Chapter 2 could be applied to design various new algorithms. We envisage the following three.

- Algorithm 4.1 could be modified using a hybrid technique. In a local phase of such a technique, projected Levenberg-Marquardt subproblems (see Chapter 2, [43] and [29]) could be used. This would lead to less expensive subproblems. For globalization, simultaneous descent directions could be employed.

- For multi-objective problems with a general domination structure Algorithm 4.1 could be easily modified to take care of this domination structure. This is possible by including additional constraints for the λ variable (as the multipliers belong to the dual of the domination cone).

- It would be interesting to use Algorithm 4.1 to find a good representation of the set of efficient points. For this techniques from [11; 61] are useful. For example, the equality constraints from [11] could be appended to the $H(x, \lambda)$ vector.

As the subproblems of Chapter 4 are quadratically constrained, it might be useful to have simpler subproblems. This could be tackled by incorporating a quadratic penalty term in the objective function and by using a projected Levenberg-Marquardt.

As the new weaker smoothness conditions seem too strong for some NCP functions, it would be interesting to weaken these conditions further. In general weakening of smoothness conditions is a challenging task.

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in Germany nor in any other country.

I have written this dissertation at Dresden University of Technology under the scientific supervision of Prof. Dr. Andreas Fischer.

There have been no prior attempts to obtain a PhD at any university.

I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU Dresden, issued March 20, 2000 with the changes in effect since April 16, 2003 and October 01, 2008.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prfungsbehörde vorgelegt.

Die vorliegende Dissertation habe ich an der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. Andreas Fischer angefertigt.

Es wurden zuvor keine Promotionsvorhaben unternommen.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20. Mrz 2000, in der geänderten Fassung mit Gültig-keit vom 16. April 2003 und vom 01.10.2008 an.

Datum: 25.02.2010

Unterschrift