Lyapunov Exponents for Random Dynamical Systems

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Introduction

A dynamical system is a concept in mathematics where a specified rule describes the time dependence of a point in a state space. The mathematical models used to describe the motion of planets, the swinging of a clock pendulum in Newton's mechanics, the flow of water in a pipe, chemical reactions, or even the number of fish each spring in a lake are examples of dynamical systems. A dynamical system is determined by a state space and a fixed evolution rule which describes how future states follow from the current state.

The history of dynamical system began with the foundational work of Poincare [117] and Lyapunov [90] on the qualitative behavior of ordinary differential equations. The concept of dynamical systems was then introduced by Birkhoff [18], followed by important contributions by Markov [94], Nemytskii and Stefanov [107], Bhatia and Szegoe [16], Smale [127], among others. The main goal of this theory is to study the qualitative behavior of systems from geometrical and topological view points.

The concept of random dynamical systems is a comparatively recent development combining ideas and methods from the well developed areas of probability theory and dynamical systems. Due to our inaccurate knowledge of the particular model or due to computational or theoretical limitations (lack of sufficient computational power, inefficient algorithms or insufficiently developed mathematical and physical theory, for example), the mathematical models never correspond exactly to the phenomenon they are meant to model. Moreover, when considering practical systems we cannot avoid either external noise or inaccuracy errors in measurements, so every realistic mathematical model should allow for small errors along orbits. To be able to cope with unavoidable uncertainty about the "correct" parameter values, observed initial states and even the specific mathematical formulation involved, we let randomness be embedded within the model. Therefore, random dynamical systems arise naturally in the modeling of many phenomena in physics, biology, economics, climatology, etc.

The concept of random dynamical systems was mainly developed by Arnold [3] and his "Bremen group", based on the research of Baxendale [13], Bismut [19], Elworthy [52], Gihman and Skorohod [65], Ikeda and Watanabe [72] and Kunita [83] on two-parameter stochastic flows generated by stochastic differential equations. Three main classes of random dynamical systems are:

- Products of random maps: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta : \Omega \to \Omega$ an ergodic transformation preserving the probability $\mathbb{P}$. Let $(X, \mathcal{B})$ be a measurable space. For a given measurable function $\psi : \Omega \times X \to X$ we can define the corresponding random dynamical system $\varphi : \mathbb{N} \times \Omega \times X \to X$ by

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & \text{if } n \geq 1, \\ \mathrm{id}_X, & \text{otherwise.} \end{cases} \tag{1}$$

The random dynamical system φ is said to be generated by the random mapping ψ. Conversely, every one-sided discrete time random dynamical system has the form (1), i.e. is a *product of random mappings*, or an *iterated function system*, or a *system in a random environment* (see Arnold [3, pp. 50]).

- Random differential equations: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\theta_t)_{t\in\mathbb{R}} : \Omega \to \Omega$ an ergodic flow preserving the probability $\mathbb{P}$. Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function satisfying that for a fixed $\omega \in \Omega$ the function $(t, x) \mapsto f(\theta_t\omega, x)$ is locally Lipschitz in x, integrable in t and

$$\|f(\omega, x)\| \leq \alpha(\omega)\|x\| + \beta(\omega),$$

where $t \mapsto \alpha(\theta_t\omega)$ and $t \mapsto \beta(\theta_t\omega)$ are locally integrable. Then the random differential equation

$$\dot{x} = f(\theta_t\omega, x)$$

uniquely generates a continuous random dynamical system $\varphi : \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\varphi(t, \omega)x = x + \int_0^t f(\theta_s\omega, \varphi(s, \omega)x)\, ds,$$

(we refer to Arnold [3, pp. 57–63] for more details).

- Stochastic differential equations: The classical Stratonovic stochastic differential equation

$$dx_t = f_0(x_t)dt + \sum_{j=1}^{m} f_j(x_t) \circ dW_t^j, \quad t \in \mathbb{R},$$

where $f_0, \dots, f_m$ are smooth vector fields, and W is standard Brownian motion of $\mathbb{R}^m$ generates a unique (up to indistinguishability) smooth random dynamical system φ over the filtered dynamical system describing Brownian motion (we refer to Arnold [3, pp. 68-107] for more details).

For the gap between random dynamical systems and continuous skew products we refer to the paper by Berger and Siegmund [15].

Lyapunov exponent or Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. The concept was introduced by Lyapunov when studying the stability of non-stationary solutions of ordinary differential equations, and has been widely employed in studying dynamical systems since then.

The fundamental results on Lyapunov exponents for random dynamical systems on finite dimensional systems were first obtained by Oseledets [109] in 1968, which is now called the Oseledets Multiplicative Ergodic Theorem. Originally formulated for products of random matrices, it has been reformulated and reproved several times during the past thirty years. Basically, there are two classes of proofs. One makes use of the Kingman's Subadditive Ergodic Theorem together with the polar decomposition of square matrices (see Arnold [3] and the references therein). The other one relies on the triangularization of a linear cocycle and the classical ergodic theorem for the triangular cocycles. This technique was also used in the contemporaneous paper of Millionščikov [97] who independently derived a portion of the multiplicative ergodic theorem, and then taken up again by Johnson, Palmer and Sell [74] (assuming a topological setting for the metric dynamical system).

Thanks to the multiplicative ergodic theorem of Oseledets [109], the Lyapunov spectrum of products of random matrices is well defined (under some integrability conditions) and it is a generalization of the Lyapunov spectrum in the deterministic case and the Oseledets subspaces are generalizations of the eigenspaces.

The study of the Lyapunov spectrum of linear cocycles is one of the central tasks of the theory of random dynamical systems (see Arnold [3]). In various situations it is of theoretical and practical importance to know when the Lyapunov spectrum is simple and the Oseledets splitting is exponentially separated. Recently, Arbieto and Bochi [2], Bochi [21], Bochi and Viana [22, 23], Bonatti and Viana [24] and Cong [36] have derived some new results on genericity of hyperbolicity of several classes of dynamical systems including smooth dynamical systems and linear cocycles. Let us mention here a result of Cong [36] stating that the set of cocycles with integral separation is open and dense in the space of all bounded $Gl(d, \mathbb{R})$-cocycles equipped with the uniform topology. As a consequence, a generic bounded linear cocycle has simple Lyapunov spectrum and exponentially separated Oseledets splitting. In Chapter 2, we show that this result cannot be extended to the case of unbounded cocycles. In particular, we construct an open set of cocycles with simple Lyapunov spectrum but no exponentially separated splitting.

Generic properties of Lyapunov exponents have recently been extended to continuous time. Basse [14] has shown that for a C^0-generic subset of all the 2-dimensional conservative nonautonomous linear differential equations, either Lyapunov exponents are zero or there is a dominated splitting. Dai [42] investigated the generic properties of continuous linear skew-product systems. His results ensured that based on a uniquely ergodic equi-continuous endomorphism the set of linear hyperbolic skew-product systems is open and dense in the set of all skew product systems. In Chapter 3, using the approach in Arnold and Cong [4] we obtain several generic properties of Lyapunov exponents of linear random differential equations. Precisely, we are able to show that the Lyapunov exponents are of the second Baire class. As a consequence, there exists a residual set on which the Lyapunov exponents are continuous. In other words, generically the Lyapunov exponents of linear random differential equations depend continuously on the coefficient.

Multiplicative ergodic theory becomes much more difficult when considering infinite dimensional random dynamical systems, i.e. random dynamical systems on infinite dimensional Banach spaces. Recall that for a finite dimensional linear deterministic system the Lyapunov exponents are precisely the real parts of the eigenvalues of A (for continuous time, $\dot{x} = Ax$) or the logarithms of the eigenvalues of A (for discrete time, $x_{n+1} = Ax_n$), respectively. Thus, the Lyapunov exponents are determined by the spectrum. Since the spectra of infinite dimensional operators in general have a considerably more complicated structure than finite dimensional ones, it is clear that much less can be expected for infinite dimensional random dynamical systems.

In his remarkable paper [122], Ruelle extended the classical multiplicative ergodic theorem to compact linear random operators in a separable Hilbert space with a base measurable metric dynamical system in a probability space. A typical example of these maps is the time-one map of the solution operator of a stochastic or random parabolic partial differential equation. In this case, one has to face the difficulties arising from the fact that the phase space is not locally compact and the dynamical system may not be invertible over the phase space. Ruelle's results have been applied to study certain stochastic partial differential equations and delay differential equations (see, e.g., Mohammed and Scheutzow [102]).

Later, Mañé [93] extended the multiplicative ergodic theorem to compact operators in a Banach space, where the base metric dynamical system is a homeomorphism over a compact topological space. A drawback of Mañé's results is that they can not be applied to random dynamical systems generated by stochastic partial differential equations. Besides the obstacles Ruelle encountered in a Hilbert space, one also needs to overcome the problem that there is no inner product.

Thieullen [136] further extended Mañé's results on Lyapunov exponents to bounded linear operators in a Banach space, where the base metric dynamical system is a homeomorphism over a topological space which is homeomorphic to a Borel subset of a separable metric space.

In [54], Flandoli and Schaumlöffel obtained a multiplicative ergodic theorem for random isomorphisms on a separable Hilbert space with a measurable metric dynamical system over a probability space. This result is used to study hyperbolic stochastic partial differential equations. Schaumlöffel [124] extended the multiplicative ergodic theorem to a class of bounded random linear operators which map a closed linear subspace onto a closed linear subspace in a Banach space with certain convexity.

Recently, Lian and Lu [89] extended the multiplicative ergodic theorem to a general setting, products of random bounded operators on a Banach space with a measurable metric dynamical system over a probability space. Crauel, Doan and Siegmund [41] used this result to study scalar difference equations with random delay. After that, based on the multiplicative ergodic theorem results by Lian and Lu [89], differential equations with random delay are investigated by Doan and Siegmund [45]. The work of Crauel, Doan and Siegmund [41] and Doan and Siegmund [45] can be considered as the first step towards a general theory of difference and differential equations incorporating unbounded random delays. In Chapter 4 we extend the results in Crauel, Doan and Siegmund [41] on Lyapunovs exponents of difference equations with random delay to arbitrary dimension. Moreover, the coefficients are also random. In particular, we show that the number of Lyapunov exponents of difference equations is always finite. Using the materials in Doan and Siegmund [45], differential equations with random delay are investigated in Chapter 5.

Computational methods are a basic tool in investigation of dynamical systems, both to explore what may happen and to approximate specific dynamical features such as limit cycles and attractors and, more generally, invariant measures, see e.g., Stuart and Humphries [134].

Invariant measures is a central concept in the theory of dynamical system, both deterministic and random, and their investigation has been closely related to developments in ergodic theory, see e.g., Katok and Hasselblatt [76]. A variety of methods have been proposed and implemented for computing invariant measures, see e.g., Dellnitz, Froyland and Junge [43], Dellnitz and Junge [44], Diamond, Kloeden and Pokrovskii [46] and Guder, Dellnitz and Kreuzer [68].

By discretizing the state space and replacing the action of generator by the transition mechanism of a Markov chain, Imkeller and Kloeden [73] provided a method for computing invariant measures of dynamical system generated by difference equations. For iterated functions systems, which are important examples of random dynamical system, Perrugia [114] introduced a general method of discretisation as a way of approximating the attracting set and invariant measure. Using an extension of this construction, Froyland [56] and Froyland and Aihara [57] present a computational method for rigorously approximating the unique invariant measure of an iterated function system which is contractive on average. The advantage of this method is that it provides quantitative bounds on the accuracy of the approximation. Using the same idea, Cong, Doan and Siegmund [38] extended this method to infinite iterated systems which are contractive on average. In Chapter 6, we go one further step to provide a computational method for computing the invariant measure of a contractive on average iterated functions system with place-dependent probabilities and an infinite iterated functions system which is l-contractive on average - a notion which is more general than contractive on average. With a rigorous method for computing invariant measures at hand, we also provide a method for computing the Lyapunov exponents of products of random matrices which serve as a main generator of random dynamical systems.

Invariant manifold theory for RDS based on the MET is an important part of *smooth ergodic theory*. It was started in 1976 with the pioneering work of Pesin [115, 116]. He constructed the classical stable and unstable manifolds of a deterministic diffeomorphism on a compact Riemannian manifolds preserving a measure which is absolutely continuous with respect to the Riemannian volume. His technique is to cope with the non-uniformity of the MET (random norms, ε-slowly varying functions). This technique is also used in Wanner [139] and Arnold [3] to construct invariant manifolds for RDS on finite dimensional space. In chapter 7, we provide the Lyapunov norm corresponding to a linear equation with random delay. This can be considered as the first technical step toward the nonlinear theory of equations with random delay.

To conclude the introduction let us outline the structure of the thesis. Chapter 1 is devoted to provide some fundamental aspects of random dynamical systems. We first start with the notion of metric dynamical system. Based on a metric dynamical system the notion of random dynamical systems is defined. Three important classes of linear random dynamical systems, namely products of random matrices, random differential equations and stochastic differential equations, are discussed. In the remaining part of this chapter, one of the most important theorems for random dynamical systems, the multiplicative ergodic theorem, is presented.

In Chapter 2, we deal with the generic properties of random dynamical systems having dominated splitting. The notion of dominated splitting is discussed carefully in the first part of this chapter. More precisely, we point out that in the definition of dominated splitting the condition that the angle between the invariant subspaces are uniformly bounded from zero plays an important role in deciding the robustness of this notion. In the remaining part of this chapter, we construct an explicit open set of linear random dynamical systems with simple Lyapunov spectrum but no dominated splitting. Consequently, the set of all random dynamical systems having dominated splitting is not generic. Moreover, unlike the case of bounded linear random dynamical systems the continuity of Lyapunov exponents is not equivalent to the existence of a dominated splitting.

The generic properties of Lyapunov exponents for random differential equations is the main topic in Chapter 3. In this chapter, we first introduce the space of all random differential equations satisfying the integrability condition of the multiplicative ergodic theorem. On this space, we show that the top Lyapunov exponent is upper semi-continuous. Consequently, the repeated Lyapunov exponents are of the first Baire class. However, the Lyapunov exponents are only of the second Baire class.

Difference equations with random delay is the topic of Chapter 4. By introducing an appropriate initial value space, we obtain random dynamical systems corresponding to difference equations with random delay in infinite dimension. Under natural assumptions on the random delay and coefficients we show that the generated random dynamical system satisfies the integrability condition of the multiplicative ergodic theorem by Lian and Lu [89]. The Kuratowski measures of the generated random dynamical systems are explicitly computed. Consequently, the Lyapunov exponents for difference equations with random delay are provided. It is also worth emphasizing that the number of Lyapunov exponents for difference equations with random delay is finite. Difference equations with constant delay and bounded random delay are also investigated in order to see the link between classical results and our new results about infinite dimensional random dynamical systems.

In Chapter 5, we extend the results of Chapter 4 to differential equations with random delay. We first introduce the space of initial values. Second, we prove the existence and uniqueness of solutions of differential equations with random delay. Based on these results, the corresponding random dynamical system is defined. Checking the integrability condition and computing the Kuratowski mueasure of the random dynamical system leads to a multiplicative ergodic theorem for random differential equations with random delay.

In Chapter 6, we provide a method to compute invariant measures for iterated function systems with place-dependent probabilities and infinite iterated function systems. We start this chapter by introducing the notion of iterated function systems, iterated function systems with place-dependent probabilities, and infinite iterated function systems. A short proof of an ergodic theorem for infinite iterated function systems which are l-contractive on average is given. We then construct an approximating sequence of finite iterated function systems. Using the method for computing the invariant measure of a finite iterated function system we obtain a numerical method to compute the invariant measures of iterated function systems with place-dependent probabilities and infinite iterated function systems. In the last section of the chapter, we apply the above procedure to compute numerically the Lyapunov exponents for a special class of random dynamical systems, products of random matrices. Several examples are provided to illustrate the method.

Finally, in the first part of Chapter 7 we state and prove the MET for one-sided RDS on Banach space. In the last part of Chapter 7, we construct the Lyapunov norm corresponding to a linear equation with random delay. This work is the first attempt to establish the nonlinear theory of equations with random delay.

This thesis contains new results, some are published with multiple authors in Cong and Doan [37], Cong, Doan and Siegmund [38], Crauel, Doan and Siegmund [41] and Doan and Siegmund [45]. Not all of the results of these papers are repeated here, but only those to which I actively and critically made contributions.

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Chapter 1

Background on Random Dynamical Systems

This foundation chapter is devoted to recall some basic definitions and facts about random dynamical systems. For a more detailed discussion of the theory and applications of random dynamical systems we refer to the monograph Arnold [3]. We pay particular attention to the notion of the generator and Lyapunov exponent for random dynamical systems.
Throughout the thesis we will be concerned with a probability space by which we mean a triple $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a set, $\mathcal{F}$ is a σ-algebra of sets in Ω, and $\mathbb{P}$ is a nonnegative σ-additive measure on $\mathcal{F}$ with $\mathbb{P}(\Omega) = 1$. The time $\mathbb{T}$ always stands for the following semigroups or groups :

- $\mathbb{T} = \mathbb{R}$: *Two-sided continuous* time.
- $\mathbb{T} = \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$: *Two-sided discrete* time.

1.1 Definition of Random Dynamical System

A random dynamical system is an object consisting of a metric dynamical system and a cocycle over this system. We need a metric dynamical system for modeling of random perturbations. We begin with a definition of a metric dynamical system.

Definition 1.1.1 (Metric Dynamical System). A *metric dynamical system*[1] with time $\mathbb{T}$, $\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$, is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a family of transformations $\theta_t : \Omega \to \Omega, t \in \mathbb{T}$ such that

(i) it is an one-parameter group, i.e.

$$\theta_0 = \mathrm{id}_\Omega, \quad \theta_t \circ \theta_s = \theta_{t+s} \qquad \text{for all } t, s \in \mathbb{T},$$

where id_Ω is the identical map on Ω,

[1]"Metric Dynamical System(s)" is henceforth often abbreviated as "MDS".

(ii) The mapping $(t,\omega) \mapsto \theta_t\omega$ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F}$ measurable,

(iii) $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{T}$, i.e. $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for all $B \in \mathcal{F}$ and all $t \in \mathbb{T}$.

A set $B \in \mathcal{F}$ is called *θ-invariant* (for short invariant) if $\theta_t B = B$ for all $t \in \mathbb{T}$. A metric dynamical system θ is said to be *ergodic under* $\mathbb{P}$ if for any invariant set $B \in \mathcal{F}$ we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

In the case that $\mathbb{T}$ is discrete, i.e. $\mathbb{T} = \mathbb{Z}$ we use the notation $(\Omega, \mathcal{F}, \mathbb{P}, (\theta^n)_{n\in\mathbb{Z}})$ instead of the notation $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ which is usually used in the continuous time case $\mathbb{T} = \mathbb{R}$ to denote an MDS with time $\mathbb{T}$. We refer to Cornfeld, Fomin and Sinai [39], Walters [138] for the references and presentation of MDS and ergodic theorem. Now we give several important examples of MDS.

Example 1.1.2 (Periodic Case). Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a circle of unit circumference, $\mathcal{F}$ is the σ-algebra of Borel sets and $\mathbb{P}$ is the Lebesgue measure on Ω. Let $(\theta_t)_{t\in\mathbb{R}}$ be the group of rotations of the circle. It is easy to see that we obtain an ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with continuous time.

Example 1.1.3 (Quasi-Periodic Case). Let Ω be a d-dimensional torus, $\Omega = \text{Tor}^d$. Assume that its points are written as $x = (x_1, x_2, \ldots, x_d)$ with $x_i \in [0, 1)$. Let $\mathcal{F}$ be the σ-algebra of Borel sets of Tor^d and $\mathbb{P}$ the Lebesgue measure on Tor^d. We define transformations $(\theta_t)_{t\in\mathbb{T}}$ by

$$\theta_t x = \big(x_1 + ta_1(mod\ 1), x_2 + ta_2(mod\ 1), \ldots, x_d + ta_d(mod\ 1)\big), \quad t \in \mathbb{T},$$

for a given $a = (a_1, a_2, \ldots, a_d)$. Thus we obtain an MDS. If the numbers $a_1, a_2, \ldots, a_d$ are rationally independent, then this MDS is ergodic (see, e.g., Rudolph [119]).

Example 1.1.4 (Almost Periodic Case). Let $f(x)$ be a Bohr almost periodic function on $\mathbb{R}$. We define the hull $H(f)$ of the function f as the closure of the set $\{f(x+t), t \in \mathbb{R}\}$ in the norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. The hull $H(f)$ is a compact metric space and it has a natural commutative group structure. Therefore, it processes a Haar measure which, if normalized to unity, makes $H(f)$ into a probability space. If we define transformations $(\theta_t)_{t\in\mathbb{R}}$ as shifts

$$\theta_t g(x) = g(x+t) \qquad \text{for all } g \in H(f),$$

then we obtain an ergodic MDS with continuous time. For details we refer to Ellis [49] and Leviton and Zhikov [88].

Example 1.1.5 (Ordinary Differential Equations). Let us consider a system of ordinary differential equation in $\mathbb{R}^d$:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_d), \quad i = 1, 2, \ldots, d. \tag{1.1}$$

Assume that the Cauchy problem for this system is well-posed. We define transformations $(\theta_t)_{t\in\mathbb{R}} : \mathbb{R}^d \to \mathbb{R}^d$ by $\theta_t x = x(t)$, where $x(t)$ is the solution of (1.1) with $x(0) = x$.

Assume that a nonnegative smooth function $\rho(x_1, x_2, \ldots, x_d)$ satisfies the stationary Liouville equation

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \big(\rho(x_1, x_2, \ldots, x_d) f_i(x_1, x_2, \ldots, x_d)\big) = 0 \tag{1.2}$$

and possesses the property $\int_{\mathbb{R}^d} \rho(x)\ dx = 1$. Then $\rho(x)$ is a density of a probability measure on $\mathbb{R}^d$. By Liouville's theorem we have

$$\int_{\mathbb{R}^d} f(\theta_t x)\rho(x)\ dx = \int_{\mathbb{R}^d} f(x)\rho(x)\ dx$$

for all bounded continuous functions $f(x)$ on $\mathbb{R}^d$. Therefore in this situation an MDS is generated with $\Omega = \mathbb{R}^d$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^d)$ the Borel σ-algebra of sets in $\mathbb{R}^d$ and $\mathbb{P}(dx) = \rho(x)\,dx$. Sometimes it is also possible to construct an MDS connected with (1.1), when the solution ρ of (1.2) possesses a first integral (e.g., if (1.1) is a Hamiltonian system) with appropriate properties (see, e.g., Sinai [126] for more details).

Example 1.1.6 (Bernoulli Shifts). Let $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ be a probability space and $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space of infinite sequences $\omega = (\omega_i)_{i\in\mathbb{Z}}$, where $\omega_i \in \Omega_0, i \in \mathbb{Z}$. Here $\mathcal{F}$ is the σ-algebra generated by finite dimensional cylinders

$$C_{i_1,i_2,\ldots,i_m} = \{\omega \,|\, \omega_{i_k} \in C_k, k = 1, 2, \ldots, m\},$$

where $C_k \in \mathcal{F}_0$ and $i_1, i_2, \ldots, i_m \in \mathbb{Z}$. The probability measure $\mathbb{P}$ is defined such that $\mathbb{P}(C_{i_1,i_2,\ldots,i_m}) = \mathbb{P}_0(C_1)\mathbb{P}_0(C_2)\ldots\mathbb{P}_0(C_m)$. We define transformations $(\theta_t)_{t\in\mathbb{Z}}$ by $(\theta_t\omega)_i = \omega_{t+i}$ for all $i \in \mathbb{Z}, \omega \in \Omega$. Since

$$\theta_t C_{i_1,i_2,\ldots,i_m} = \{\omega \,|\, \omega_{i_k - t} \in C_k, k = 1, 2, \ldots, m\},$$

the probability measure $\mathbb{P}$ is invariant under θ_t. Thus we obtain an MDS. In the particular case when $\Omega_0 = \{0, 1\}$ is a two-point set and $\mathbb{P}_0(\{0\}) = \mathbb{P}_0(\{1\}) = 1/2$, we have the standard Bernoulli shift. In the general case we can interpret this MDS as one generated by an infinite sequence of independent identically distributed random variables. We refer the reader to Walter [138] for more details.

Example 1.1.7 (Stationary Random Process). Let $\xi = (\xi(t))_{t\in\mathbb{T}}$ be a stationary random process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the σ-algebra generated by ξ. Assume that in the continuous time case ($\mathbb{T} = \mathbb{R}$) the process ξ possesses the following property: all trajectories are right-continuous and have limits from the left. Then the shift $\xi(t) \mapsto (\theta_\tau \xi)(t) = \xi(t + \tau)$ generate an MDS. See Arnold [3] and the references therein for details.

In the framework of stochastic equations the following example of an MDS is of importance.

Example 1.1.8 (Wiener Process). Let $W_t = (W_t^1, W_t^2, \ldots, W_t^d)$ be a Wiener process with values in $\mathbb{R}^d$ and two-sided time $\mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the corresponding canonical Wiener space. More precisely, let $C_0(\mathbb{R}, \mathbb{R}^d)$ be the space of continuous functions ω from $\mathbb{R}$ into $\mathbb{R}^d$ such that $\omega(0) = 0$, endowed with the compact-open topology, i.e. with the topology generated by the metric

$$\rho(\omega, \omega^*) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(\omega, \omega^*)}{1 + \rho_n(\omega, \omega^*)}, \quad \rho_n(\omega, \omega^*) = \max_{t \in [-n,n]} |\omega(t) - \omega^*(t)|.$$

Let $\widetilde{\mathcal{F}}$ be the corresponding Borel σ-algebra of $C_0(\mathbb{R}, \mathbb{R}^d)$, and let $\mathbb{P}$ be the Wiener measure on $\widetilde{\mathcal{F}}$. We suppose that Ω is the subset in $C_0(\mathbb{R}, \mathbb{R}^d)$ consisting of the functions that have a growth rate less than linear when $t \to \pm\infty$ and $\mathcal{F}$ is the restriction of $\widetilde{\mathcal{F}}$ to Ω. In this realization $W_t(\omega) = \omega(t)$, where $\omega \in \Omega$, i.e. the elements of Ω are identified with the paths of the Wiener process. We define an MDS θ by

$$\theta_t \omega(\cdot) := \omega(t + \cdot) - \omega(t) \qquad \text{for all } \omega \in \Omega.$$

These transformations preserve the Wiener measure and are ergodic. Thus we have an ergodic MDS. The flow $(\theta_t)_{t \in \mathbb{R}}$ is called the Wiener shift (for more details we refer to Arnold [3, pp. 544-548]).

With the notion of MDS at hand, we are in a position to state the notion of random dynamical system.

Definition 1.1.9 (Random Dynamical System [3]). A *measurable random dynamical system*[2] on the measurable space $(X, \mathcal{B})$ over an MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ with time $\mathbb{T}$ is a mapping

$$\varphi : \mathbb{T} \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

with the following properties:

(i) *Measurablity*: φ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}, \mathcal{B}$-measurable.

(ii) *Cocycle property*: The mappings $\varphi(t, \omega) := \varphi(t, \omega, \cdot) : X \to X$ form a *cocycle* over $(\theta_t)_{t \in \mathbb{T}}$, i.e. they satisfy

$$\varphi(0, \omega) = \mathrm{id}_X \qquad \text{for all } \omega \in \Omega \quad (\text{if } 0 \in \mathbb{T}),$$

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \qquad \text{for all } s, t \in \mathbb{T}, \omega \in \Omega,$$

where id_X is the identical map on X.

Here " $\circ$ " means composition, which canonically defines an action on the left of the semigroup of self-mappings of X on the space X, i.e. $(f \circ g)(x) = f(g(x))$.

It is very useful to imagine an RDS as fiber maps on the bundle $\Omega \times X$. Figure 1.1 can be explained as follows: While ω is shifted by the dynamical system θ in time s to the point $\theta_s \omega$ on the base space Ω, the cocycle $\varphi(s, \omega)$ moves the point x in the fiber $\{\omega\} \times X$ over ω to the point $\varphi(s, \omega)x$ in the fiber $\{\theta_s \omega\} \times X$ over $\theta_s \omega$. The cocycle property can be clearly visualized on this bundle.

[2]"Random Dynamical System(s)" is henceforth often abbreviated as "RDS".

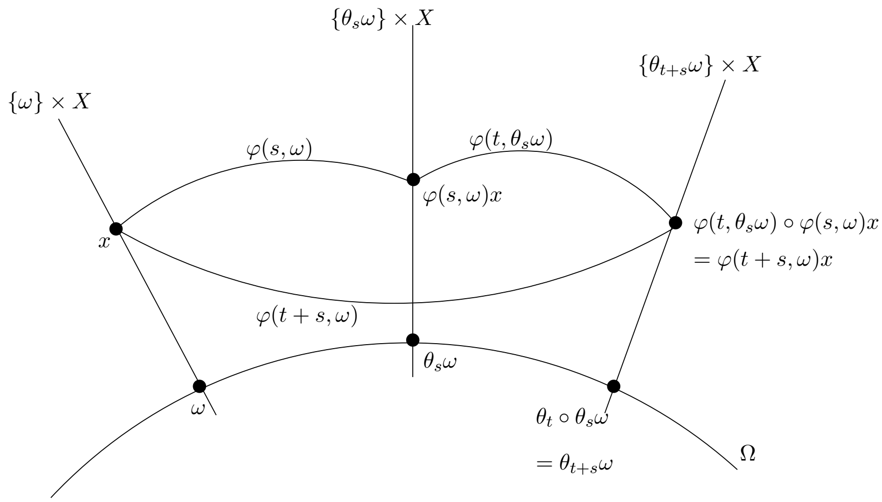

Figure 1.1: A random dynamical system is an action on the bundle $\Omega \times X$

Definition 1.1.10 (Continuous RDS [3])**.** A *continuous* or *topological RDS* on the topological space X over the MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ is a measurable RDS which satisfies in addition the following property: For each $\omega \in \Omega$ the mapping

$$\varphi(\cdot, \omega, \cdot) : \mathbb{T} \times X \to X, \quad (t, x) \mapsto \varphi(t, \omega, x)$$

is continuous.

Definition 1.1.11 (Smooth RDS [3])**.** A *smooth RDS of class C^k*, or a *C^k RDS*, where $1 \leq k \leq \infty$, on a d-dimensional (C^∞) manifold X is a topological RDS which in addition satisfies the following property: For each $(t, \omega) \in \mathbb{T} \times \Omega$ the mapping

$$\varphi(t, \omega) = \varphi(t, \omega, \cdot) : X \to X, \quad x \mapsto \varphi(t, \omega, x)$$

is C^k (i.e. k times differentiable with respect to x, and the derivatives are continuous with respect to (t, x)).

Definition 1.1.12 (Linear RDS [3])**.** A continuous RDS on a Banach space X is called a *linear RDS*, if $\varphi(t, \omega) \in \mathcal{L}(X)$ for each $t \in \mathbb{T}$, $\omega \in \Omega$, where $\mathcal{L}(X)$ is the space of bounded linear operators of X.

A mapping $\varphi : \mathbb{T} \times \Omega \to \mathcal{L}(X)$ is said to be *strongly measurable* if for a fixed $x \in X$ the mapping $\mathbb{T} \times \Omega \to X$ defined by

$$(t, \omega) \mapsto \varphi(t, \omega)x$$

is measurable.

Lemma 1.1.1. *Let $\varphi : \mathbb{T} \times \Omega \to \mathcal{L}(X)$ be a map satisfying the cocycle property, i.e. $\varphi(0, \omega) = id_X$ for all $\omega \in \Omega$ and*

$$\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega) \qquad \text{for all } s, t \in \mathbb{T}, \omega \in \Omega,$$

where X is a separable Banach space. Assume that φ is strongly measurable. Then φ is $\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B}(\mathcal{L}(X))$-measurable. In particular, if $X=\mathbb{R}^d$ then the mapping defined by

$$\mathbb{T}\times\Omega\times X \ni (t,\omega,x)\mapsto \varphi(t,\omega)x\in X$$

is also $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}\otimes\mathcal{B}(X),\mathcal{B}(X)$-measurable and is therefore a linear RDS.

Proof. Since X is a separable Banach space it follows that there exists a countable set $\{x_i\}_{i=1}^\infty$ which is dense in X. For a fixed $T\in\mathcal{L}(X)$ and $\varepsilon>0$, we define

$$\Omega_T := \{(t,\omega)\in\mathbb{T}\times\Omega : \|\varphi(t,\omega)-T\|\le\varepsilon\}.$$

This implies together with the fact that $\{x_i\}_{i=1}^\infty$ is dense in X that

$$\Omega_T=\bigcap_{i=1}^\infty\{(t,\omega) : \|\varphi(t,\omega)x_i-Tx_i\|\le\varepsilon\|x_i\|\}.$$

Using strong measurability of φ, the set

$$\{(t,\omega) : \|\varphi(t,\omega)x_i-Tx_i\|\le\varepsilon\|x_i\|\}\qquad\text{is measurable for all } i=1,2,\dots,$$

which proves that Ω_T is a measurable set. Hence, φ is $\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B}(\mathcal{L}(X))$-measurable. For the remaining part of the proof, we deal with the case that $X=\mathbb{R}^d$. Choose and fix $x\in\mathbb{R}^d$ and $\varepsilon>0$. Define

$$\Omega_x := \{(t,\omega,y)\in\mathbb{T}\times\Omega\times\mathbb{R}^d : \|\varphi(t,\omega)y-x\|<\varepsilon\}.$$

Our aim is to show that Ω_x is measurable. Since φ is $\mathcal{B}(\mathbb{T})\otimes\mathcal{F},\mathcal{B}(\mathcal{L}(\mathbb{R}^d))$-measurable, there exists a sequence of mappings $\varphi_n:\mathbb{T}\times\Omega\to\mathcal{L}(\mathbb{R}^d)$ of the form

$$\varphi_n=\sum_{i=1}^n \chi_{\Omega_i}T_i, \tag{1.3}$$

where $\Omega_i\subset\mathbb{T}\times\Omega$ are disjoint measurable sets and $T_i\in\mathcal{L}(\mathbb{R}^d)$ for $i=1,\dots,n$, such that

$$\lim_{n\to\infty}\|\varphi_n(t,\omega)-\varphi(t,\omega)\|_{\mathcal{L}(\mathbb{R}^d)}=0\qquad\text{for all } (t,\omega)\in\mathbb{T}\times\Omega. \tag{1.4}$$

For each $k\ge1$, we define

$$\Omega_{n,x}^k := \left\{(t,\omega,y)\in\mathbb{T}\times\Omega\times X : \|\varphi_n(t,\omega)y-x\|\le\varepsilon-\frac{1}{k}\right\}.$$

Clearly, we have $\Omega_{n,x}^1\subseteq\Omega_{n,x}^2\subseteq\dots$. We show that

$$\Omega_x=\bigcup_{k=1}^\infty\bigcap_{i=1}^\infty\bigcup_{n=i}^\infty\Omega_{n,x}^k. \tag{1.5}$$

By the definition of Ω_x, for each $(t,\omega,y)\in\Omega_x$ we have $\|\varphi(t,\omega)y-x\|<\varepsilon-\frac{1}{k}$ for some $k\in\mathbb{N}$. Due to (1.4) there exists $N\in\mathbb{N}$ such that

$$\|\varphi_n(t,\omega)y-x\|<\varepsilon-\frac{1}{k}\qquad \text{for all } n\geq N,$$

which implies that $(t,\omega,y)\in\bigcap_{i=1}^\infty\bigcup_{n=i}^\infty\Omega_{n,x}^k$ and hence $\Omega_x\subset\bigcup_{k=1}^\infty\bigcap_{i=1}^\infty\bigcup_{n=i}^\infty\Omega_{n,x}^k$. For $(t,\omega,y)\in\bigcup_{k=1}^\infty\bigcap_{i=1}^\infty\bigcup_{n=i}^\infty\Omega_{n,x}^k$, by the definition of the set $\Omega_{n,x}^k$ there exist $k\in\mathbb{N}$ and a sequence $\{k_n\}_{n=1}^\infty$ with $\lim_{n\to\infty}k_n=\infty$ such that $\|\varphi_{k_n}(t,\omega)y-x\|\leq\varepsilon-\frac{1}{k}$. This implies together with (1.4) that $(t,\omega,y)\in\Omega_x$ and therefore (1.5) is proved. As a consequence, to prove the measurability of Ω_x, it is therefore sufficient to show the measurability of $\Omega_{n,x}^k$ for all $n,k\in\mathbb{N}$. From expression (1.3), we derive

$$\begin{aligned}\Omega_{n,x}^k &= \{(t,\omega,y)\in\mathbb{T}\times\Omega\times X:\|\sum_{i=1}^n\chi_{\Omega_i}(t,\omega)T_i(y)-x\|\leq\varepsilon-\frac{1}{k}\}\\ &= \bigcup_{i=1}^n\Omega_i\times T_i^{-1}(\overline{B_{\varepsilon-\frac{1}{k}}(x)}),\end{aligned}$$

where $\overline{B_{\varepsilon-\frac{1}{k}}(x)}:=\{y\in\mathbb{R}^d:\|x-y\|\leq\varepsilon-\frac{1}{k}\}$, which leads to the measurability of $\Omega_{n,x}^k$ and the proof is completed. ☐

Remark 1.1.2. According to Lemma 1.1.1, throughout this thesis a strongly measurable mapping $\varphi:\mathbb{T}\times\Omega\to\mathcal{L}(X)$ satisfying the cocycle property as in Definition 1.1.9 is also called a linear RDS.

In the following lemma, some fundamental properties of RDS with two-sided time are provided. The proof can be found in Arnold [3, pp. 7].

Theorem 1.1.3 (Basic Properties of RDS with Two-Sided Time, [3])**.** *Suppose that $\mathbb{T}$ is two-sided (i.e. $\mathbb{T}=\mathbb{R}$ or $\mathbb{Z}$). Let φ be a measurable RDS on a measurable space $(X,\mathcal{B})$ over an MDS $(\Omega,\mathcal{F},\mathbb{P},\theta)$. Then for all $(t,\omega)\in\mathbb{T}\times\Omega$, $\varphi(t,\omega)$ is a bimeasurable bijection of $(X,\mathcal{B})$ and*

$$\varphi(t,\omega)^{-1}=\varphi(-t,\theta_t\omega)\qquad \textit{for all } (t,\omega)\in\mathbb{T}\times\Omega,$$

or, equivalently,

$$\varphi(-t,\omega)=\varphi(t,\theta_{-t}\omega)^{-1}\qquad \textit{for all } (t,\omega)\in\mathbb{T}\times\Omega.$$

Moreover, the mapping

$$(t,\omega,x)\to\varphi(t,\omega)^{-1}x$$

is measurable.

Remark 1.1.4 (RDS as a skew product)**.** Given an RDS φ. Then the mapping

$$(\omega,x)\mapsto(\theta_t\omega,\varphi(t,\omega)x):=\Theta(t)(\omega,x),\qquad t\in\mathbb{T},$$

is a measurable dynamical system on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$, which is called the *skew product* of the MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ and the cocycle $\varphi(t, \omega)$ on X. Conversely, every such measurable skew product dynamical system Θ of the MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ and the cocycle $\varphi(t, \omega)$ on X defines a cocycle φ on its x component, thus a measurable RDS. We can consequently use "RDS φ", "cocycle φ" and "skew product Θ", synonymously.

1.2 Generation

1.2.1 Discrete Time: Products of Random Mappings

Let φ be an RDS on X over θ with time $\mathbb{T} = \mathbb{Z}$. Introduce the *time-one mapping*

$$\psi(\omega) := \varphi(1, \omega) : X \to X.$$

By the cocycle property, the mapping $\psi(\omega)$ and the time-minus-one mapping $\varphi(-1, \omega)$ are related by

$$\varphi(-1, \omega) = \varphi(1, \theta^{-1}\omega)^{-1} = \psi(\theta^{-1}\omega)^{-1},$$

so the mapping $\psi(\omega) : X \to X$ is invertible for all ω. The repeated application of the cocycle property forwards and backwards in time gives

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \mathrm{id}_X, & n = 0, \\ \psi(\theta^n\omega)^{-1} \circ \cdots \circ \psi(\theta^{-1}\omega)^{-1}, & n \leq -1. \end{cases} \tag{1.6}$$

This defines an RDS φ if and only if the mappings

$$(\omega, x) \mapsto \psi(\omega)x \quad \text{and} \quad (\omega, x) \mapsto \psi(\omega)^{-1}x, \tag{1.7}$$

are measurable. Moreover, the RDS φ is continuous or C^k if and only if $\psi(\omega) \in \mathrm{Homeo}(X)$ or $\mathrm{Diff}^k(X)$, respectively.

Conversely, let for each ω an invertible mapping $\psi(\omega) : X \to X$ be given such that the two mappings in (1.7) are measurable. Then φ defines via (1.6) an RDS. We say that φ is *generated* by ψ.

Hence every two-sided discrete RDS has the form (1.6), i.e. is a *product of* (a stationary sequence of) *random mappings*, or an *iterated function system*, or a *system in a random environment*.

To emphasize the dynamical perspective, we can write the discrete time cocycle $\varphi(n, \omega)$ as the "solution" of an initial value problem for a *random difference equation*

$$x_{n+1} = \psi(\theta^n\omega)x_n, \quad n \in \mathbb{Z}, \quad x_0 \in X.$$

The sequence of random points $(\varphi(n, \omega)x)_{n\in\mathbb{Z}}$ in the state space X is the *orbit* of the point x under the RDS φ.

Example 1.2.1. The cases $X = \mathbb{R}^d$, $\psi(\omega)$ an invertible matrix, or $\psi(\omega)$ an invertible affine mapping, are of particular importance.

(i) *Linear RDS, products of random matrices*: Let $Gl(d)$ be the group of all nonsingular matrices in $\mathbb{R}^{d\times d}$, with matrix multiplication as composition. A linear RDS has thus the form

$$\Phi(n,\omega) := \begin{cases} A(\theta^{n-1}\omega)\circ\cdots\circ A(\omega), & n>0,\\ I_d, & n=0,\\ A(\theta^{n}\omega)^{-1}\circ\cdots\circ A(\theta^{-1}\omega)^{-1}, & n<0,\end{cases}$$

where I_d is the identical matrix of dimension d and $A:\Omega\to Gl(d)$ is measurable. The theory of products of random matrices together with the multiplicative ergodic theorem (see Section 1.3) is the core of the theory of RDS, with many fundamental papers such as Furstenberg and Kesten [60], Furstenberg [61], Oseledets [109], Ruelle [120].

(iii) *Affine RDS*: Let $\psi(\omega) = A(\omega)x + b(\omega)$ be the time-one mapping of the affine cocycle φ. We have

$$\varphi(1,\omega)x = A(\omega)x + b(\omega), \quad \varphi(-1,\omega) = A(\theta^{-1}\omega)^{-1}\big(x - b(\theta^{-1}\omega)\big),$$

where $A:\Omega\to Gl(d)$ and $b:\Omega\to\mathbb{R}^d$ are measurable. By induction,

$$\varphi(n,\omega)x = \begin{cases} \Phi(n,\omega)\left(x+\sum_{i=0}^{n-1}\Phi(i+1,\omega)^{-1}b(\theta^i\omega)\right), & n>0,\\ x, & n=0,\\ \Phi(n,\omega)\left(x-\sum_{i=n}^{-1}\Phi(i+1,\omega)^{-1}b(\theta^i\omega)\right), & n<0,\end{cases}$$

where Φ is the linear cocycle generated by A. Affine RDS are *iterated function systems* in the classical sense. They are important for encoding and visualizing fractals (see Chapter 6 for more details).

1.2.2 Continuous Time 1: Random Differential Equations

Let $\mathbb{T} = \mathbb{R}, X = \mathbb{R}^d$, and θ be an MDS. We establish a one-to-one correspondence between RDS over θ which are absolutely continuous with respect to t and *random differential equations*[3] *driven by* θ

$$\dot{x}_t = f(\theta_t\omega, x_t). \tag{1.8}$$

The integral form of (1.8) is given by

$$\varphi(t,\omega)x = x + \int_0^t f(\theta_s\omega, \varphi(s,\omega)x)\, ds, \tag{1.9}$$

which is valid in global, i.e. for all $t\in\mathbb{R}$. If (1.9) holds, we say that $t\mapsto\varphi(t,\omega)x$ is a *solution* of the RDE (1.8), or that the RDS *generates* φ. The following theorem

[3]Random Differential Equations is henceforth often abbreviated as "RDE".

provides a sufficient condition for the generation of RDS by RDE. The proof can be found in Arnold [3, Remark 2.2.3]. We first recall the following notions: Let $\mathcal{C}^{0,1}$ denote the Fréchet space of locally Lipschitz continuous functions $f : \mathbb{R}^d \to \mathbb{R}^d$ with seminorms

$$\|f\|_{0,1;K} := \sup_{x\in K} |f(x)| + \sup_{x,y\in K, x\neq y} \frac{|f(x)| - f(y)}{|x-y|},$$

where K is a compact convex subset of $\mathbb{R}^d$. Let $L_{\text{loc}}(\mathbb{R}, \mathcal{C}^{0,1})$ be the set of measurable functions $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ for which

- $f(t, \cdot) \in \mathcal{C}^{0,1}$ for Lebesgue-almost all $t \in \mathbb{R}$,
- for every compact set $K \subset \mathbb{R}^d$ and every bounded interval $[a, b] \subset \mathbb{R}$

$$\int_a^b \|f(t,\cdot)\|_{0,1;K}\, dt < \infty.$$

Theorem 1.2.1 (RDS from RDE, [3])**.** *Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable, consider the pathwise RDE*

$$\dot{x}_t = f(\theta_t\omega, x_t), \tag{1.10}$$

and for fixed ω let $f_\omega(t, x) := f(\theta_t\omega, x)$. Assume that $f_\omega \in L_{\text{loc}}(\mathbb{R}, \mathcal{C}^{0,1})$ and

$$\|f(\omega, x)\| \leq \alpha(\omega)\|x\| + \beta(\omega),$$

where $t \mapsto \alpha(\theta_t\omega)$ and $t \mapsto \beta(\theta_t\omega)$ are locally integrable. Then (1.10) *generates uniquely a continuous RDS φ over θ.*

Example 1.2.2 (Linear and Affine RDE). (i) *Linear RDE*: Let the measurable function $A : \Omega \to \mathbb{R}^{d\times d}$ satisfy $A \in \mathcal{L}^1(\mathbb{P})$. Then $f_\omega(t, x) := A(\theta_t\omega)x$ satisfies the conditions in Theorem 1.2.1. Hence the linear RDE

$$\dot{x}_t = A(\theta_t\omega)x_t,$$

generates a unique RDS Φ satisfying

$$\Phi(t, \omega) = I_d + \int_0^t A(\theta_s\omega)\Phi(s, \omega)\, ds$$

and

$$\det \Phi(t, \omega) = \exp \int_0^t \text{trace} A(\theta_s\omega)\, ds.$$

Moreover, differentiating $\Phi(t, \omega)\Phi(t, \omega)^{-1} = I_d$ yields

$$\Phi(t, \omega)^{-1} = I_d - \int_0^t \Phi(s, \omega)^{-1} A(\theta_s\omega)\, ds.$$

(ii) *Affine RDE*: Similarly, the equation

$$\dot{x}_t = A(\theta_t\omega)x_t + b(\theta_t\omega), \qquad A, b \in \mathcal{L}^1(\mathbb{P}),$$

generates a unique RDS. The variation of constants formula yields

$$\begin{aligned}\varphi(t,\omega)x &= \Phi(t,\omega)x + \int_0^t \Phi(t,\omega)\Phi(u,\omega)^{-1}b(\theta_u\omega)\,du \\ &= \Phi(t,\omega)x + \int_0^t \Phi(t-u,\theta_u\omega)b(\theta_u\omega)\,du,\end{aligned}$$

where Φ is the matrix cocycle generated by $\dot{x}_t = A(\theta_t\omega)x_t$. Consequently, the RDS φ consists of affine mappings.

In the next example, we compute explicitly the RDS generated from a nonlinear RDE.

Example 1.2.3. Consider a scalar RDE of the following form

$$\dot{x}_t = (1+\xi(\theta_t\omega))x_t - x_t^3, \tag{1.11}$$

where $\xi : \Omega \to \mathbb{R}$ is a random variable with $\xi \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $E\xi = 0$. Equation (1.11) can be solved explicitly and the generated RDS is

$$\varphi(t,\omega)x := \frac{x^{t+S_t(\omega)}}{\left(1+2x^2\int_0^t e^{2(s+S_s(\omega))}\,ds\right)^{\frac{1}{2}}},$$

where $S_t(\omega) := \int_0^t \xi(\theta_s\omega)\,ds$.

We now deal with the inverse problem of when for a given RDS φ on $\mathbb{R}^d$ over θ with time $\mathbb{T} = \mathbb{R}$ there exists an RDE $\dot{x}_t = f(\theta_t\omega, x_t)$ which generates φ.

Theorem 1.2.2 (RDE from RDS)**.** *Let φ be a continuous RDS for which $t \mapsto \varphi(t,\omega)x$ is absolutely continuous for all $t \in \mathbb{R}$ and (ω, x). Then there exists a measurable function $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ for which for all (ω, x)*

$$\varphi(t,\omega)x = x + \int_0^t f(\theta_s\omega, \varphi(s,\omega))\,ds,$$

i.e. φ is a solution of $\dot{x}_t = f(\theta_t\omega, x_t)$. The function f is unique in the sense that if $\widetilde{f}$ is another generator then for all (ω, x), $f(\theta_t\omega, x) = \widetilde{f}(\theta_t\omega, x)$ for Lebesgue-almost $t \in \mathbb{R}$.

Proof. See Arnold [3, Theorem 2.2.13]. □

1.2.3 Continuous Time 2: Stochastic Differential Equations

An RDS can be also generated by a stochastic differential equation. We emphasize here that in this situation, to obtain the cocycle we have to construct the probability space, the dynamical system θ (which is usually the shift operator). Because of the complexity of the stochastic case, we aim in this section only to discuss how a precise affine stochastic differential equation generates an RDS and for more details we refer to Arnold [3]. Let $(\Omega, \mathcal{F}^0, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be the canonical MDS describing $\mathbb{R}^m$-valued Brownian motion $W_t(\omega) = \omega(t)$. Then the following equation

$$dx_t = \sum_{j=0}^{m} (A_j x_t + b_j) \circ dW_t^j, \quad A_j \in \mathbb{R}^{d\times d}, \quad b_j \in \mathbb{R}^d$$

uniquely generates a global C^∞ RDS, which consists of affine mappings given by the variation of constants formula

$$\varphi(t)x = \Phi(t)\left(x + \sum_{j=0}^{m} \int_0^t \Phi(s)^{-1} b_j \circ dW_s^j \right),$$

where Φ is the fundamental matrix of the corresponding linear stochastic dynamical system

$$dx_t = \sum_{j=0}^{m} A_j x_t \circ dW_t^j,$$

which is a linear RDS over θ.

1.3 Multiplicative Ergodic Theorem in $\mathbb{R}^d$

It is well-known that the dynamics of the autonomous linear system $\dot{x} = Ax, x \in \mathbb{R}^d$, is completely described by linear algebra, more precisely, by the spectral theory of A. It might be surprise that an important class of nonautonomous linear systems, namely those driven by an MDS has a spectral theory, with probability one. This is the content of the celebrated *multiplicative ergodic theorem*[4] of Oseledets [109]. Our aim in this Section is to state the MET for RDS on finite dimensional spaces. The version of the MET for RDS on an arbitrary Banach space will be provided in the next section. In order to obtain MET, we first start with some preparatory tools, singular values and exterior powers.

1.3.1 Singular Values

Let $\mathbb{R}^d$ be endowed with the standard scalar product and $(e_i)_{i=1}^d$ be the standard basis. Define

$$O(d, \mathbb{R}) := \{U \in Gl(d, \mathbb{R}) : U^* U = I_d\},$$

where U^* denotes the transpose of U, the orthogonal group. We say for $A \in \mathbb{R}^{d\times d}$ that

$$A = VDU$$

[4]"Multiplicative ergodic theorem" is henceforth abbreviated as "MET"

is a *singular value decomposition* of A if $U, V \in O(d, \mathbb{R})$ and $D = \operatorname{diag}(\delta_1, \ldots, \delta_d)$ with $0 \le \delta_d \le \cdots \le \delta_1$. Then $\delta_i, i = 1, \ldots, d$, are called the *singular values* of A. The following lemma gives some fundamental properties of the singular values for a matrix. Its proof can be seen easily in standard books about linear algebra (see e.g. Gantmacher [64]).

Lemma 1.3.1 (Singular Value Decomposition)**.** *Any $d \times d$ matrix A has a singular value decomposition. Moreover, $0 \le \delta_d \le \cdots \le \delta_1$ are necessarily the eigenvalues of $\sqrt{A^*A}$, and the columns of U^* are corresponding eigenvectors of $\sqrt{A^*A}$. In particularly, $\|A\| = \delta_1$, where $\|\cdot\|$ is the operator norm associated with the standard Euclidean norm in $\mathbb{R}^d$, and $|\det A| = \delta_1 \ldots \delta_d$.*

1.3.2 Exterior Powers

Let E be a real vector space of dimension d and for $1 \le k \le d$, let $\wedge^k E$, the *k-fold exterior power* of E, be the vector space of alternating k-linear forms on the dual space E^* (see e.g. Temam [135, Chap.V]). The space $\wedge^k E$ can be identified with the set of formal expressions

$$\sum_{i=1}^{m} c_i (u_1^{(i)} \wedge \cdots \wedge u_k^{(i)}) \quad \text{with } m \in \mathbb{N}, c_1, \ldots, c_m \in \mathbb{R}, \text{ and } u_1^{(i)}, \ldots, u_k^{(i)} \in E$$

if we compute with the following conventions:

1. $u_1 \wedge \cdots \wedge (u_j + u_j') \wedge \cdots \wedge u_k = (u_1 \wedge \cdots \wedge u_j \wedge \cdots \wedge u_k) + $
$+ (u_1 \wedge \cdots \wedge u_j' \wedge \cdots \wedge u_k)$,
2. $u_1 \wedge \cdots \wedge c u_j \wedge \cdots \wedge u_k = c\,(u_1 \wedge \cdots \wedge u_j \wedge \cdots \wedge u_k)$,
3. for any permutation π of $\{1, \ldots, k\}$

$$u_{\pi(1)} \wedge \cdots \wedge u_{\pi(k)} = \operatorname{sign}(\pi)\, u_1 \wedge \cdots \wedge u_k.$$

The elements in $\wedge^k E$ of the form $u_1 \wedge \cdots \wedge u_k$ are called *decomposable k-vectors* and the set of decomposable k-vectors is denoted by $\wedge_0^k E$. Clearly, $\wedge^k E = \operatorname{span}(\wedge_0^k E)$. The next proposition provides the fundamental properties of singular values of exterior power.

Proposition 1.3.2 (Singular Values of Exterior Power)**.** *Let A be a $d \times d$ matrix, let $A = VDU$ be a singular value decomposition and let $0 \le \delta_d \le \cdots \le \delta_1$ be the singular values of A. Then*

(i) $\wedge^k A = (\wedge^k V)(\wedge^k D)(\wedge^k U)$ *is a singular value decomposition of* $\wedge^k A$.

(ii) $\wedge^k D = \operatorname{diag}(\delta_{i_1} \ldots \delta_{i_k} : 1 \le i_1 < \cdots < i_k \le d)$. *In particular, the top singular value of $\wedge^k A$ is $\delta_1 \ldots \delta_k$, and the smallest is $\delta_{d-k+1} \ldots \delta_d$.*

(iii) $\|\wedge^k A\| = \delta_1 \ldots \delta_k$ *and* $\|\wedge^{k+m} A\| \le \|\wedge^k A\| \|\wedge^m A\|$, $1 \le k, m \le d$ *with* $k + m \le d$. *Here $\|\cdot\|$ is the corresponding operator norm associated with the standard Euclidean norm in $\mathbb{R}^d$.*

Proof. See Arnold [3, Proposition 3.2.7]. □

1.3.3 The Furstenberg-Kesten Theorem

We now present a theorem of Furstenberg and Kesten [60] which now bears their names. Based on this theorem, the MET is a direct consequence. However, we first introduce some new notations. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we denote by $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ the space of all integral measurable functions. For each $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ the number

$$\mathbb{E}f := \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$$

is called the expectation of the random variable f. For a real-valued function $f : X \to \mathbb{R}$, where X is an arbitrary space, we define the function $f^+ : X \to \mathbb{R}$ by

$$f^+(x) = \max\big\{0, f(x)\big\} \qquad \text{for all } x \in X.$$

Theorem 1.3.3 (Furstenberg-Kesten Theorem)**.** *Let Φ be a linear cocycle with two-sided time over the MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$.*

(A) Discrete time case $\mathbb{T} = \mathbb{Z}$: Assume that the generator $A : \Omega \to Gl(d, \mathbb{R})$ of Φ satisfies

$$\log^+ \|A\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \textit{and} \quad \log^+ \|A^{-1}\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Then the following statements hold:

(i) For each $k = 1, \dots, d$ the sequence

$$f_n^{(k)}(\omega) := \log \| \wedge^k \Phi(n, \omega)\|, \quad n \in \mathbb{N},$$

is subadditive and $f_1^{(k)+} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

(ii) There is an invariant set $\widetilde{\Omega}$ of full measure and measurable functions $\gamma^{(k)} : \Omega \to \mathbb{R}$ with $\gamma^{(k)+} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that on $\widetilde{\Omega}$

$$\lim_{n\to\infty} \log \| \wedge^k \Phi(n, \omega)\| = \gamma^{(k)}(\omega),$$

and

$$\gamma^{(k)}(\theta\omega) = \gamma^{(k)}(\omega), \quad \gamma^{(k+m)}(\omega) \le \gamma^{(k)}(\omega) + \gamma^{(m)}(\omega),$$

$\gamma^{(k)}(\omega) = \mathbb{E}\gamma^k$ in the ergodic case. Further,

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} \log \| \wedge^k \Phi(n, \cdot)\| = \mathbb{E}\gamma^{(k)} = \inf_{n\in\mathbb{N}} \frac{1}{n} \mathbb{E} \log \| \wedge^k \Phi(n, \cdot)\|.$$

(iii) The measurable functions Λ_k successively defined by

$$\Lambda_1(\omega) + \cdots + \Lambda_k(\omega) := \gamma^k(\omega), k = 1, \dots, d,$$

have the following properties on $\widetilde{\Omega}$:

$$\Lambda_k(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \delta_k(\Phi(n, \omega),$$

where $\delta_k(\Phi(n,\omega))$ *are the singular values of* $\Phi(n,\omega)$, *and*

$$\Lambda_k(\theta\omega) = \Lambda_k(\omega), \quad \Lambda_d(\omega) \le \cdots \le \Lambda_1(\omega),$$

$\Lambda_k(\omega) = \mathbb{E}\Lambda_k$ *in the ergodic case. Further,*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\log \delta_k(\Phi(n,\cdot)) = \mathbb{E}\Lambda_k.$$

(iv) Define $\Psi(n,\omega) := \Phi(-n,\omega)$. *Then* Ψ *is a cocycle over* θ^{-1} *generated by* $A^{-1}\circ\theta^{-1}$, *and on* $\widetilde{\Omega}$ *we have for* $k = 1,\dots,d$

$$\gamma^{(k)-}(\omega) := \lim_{n\to\infty}\frac{1}{n}\log\|\wedge^k \Phi(-n,\omega)\| = \gamma^{(d-k)}(\omega) - \gamma^{(d)}(\omega)$$

and

$$\Lambda_k^-(\omega) := \lim_{n\to\infty}\frac{1}{n}\log \delta_k(\Phi(-n,\omega)) = -\Lambda_{d+1-k}(\omega).$$

(B) Continuous time case $\mathbb{T} = \mathbb{R}$: *Assume that* $\alpha^\pm \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$, *where*

$$\alpha^\pm(\omega) := \sup_{0\le t\le 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\| \qquad \textit{for all } \omega\in\Omega.$$

Then all statements of part (A) hold with n *and* $\mathbb{N}$ *replaced by* t *and* $\mathbb{R}^+$, *respectively.*

Using the Furstenberg-Kesten theorem, the spectrum of a linear cocycle can be well-defined. It can be considered as an extension of the notion of spectrum of a constant matrix.

Definition 1.3.1 (Lyapunov Spectrum)**.** Suppose that Φ is a linear cocycle over an ergodic MDS θ for which Theorem 1.3.3 holds. Then the functions $\Lambda_i(\cdot), i = 1,2,\dots,d$, are constant on the invariant set $\widetilde{\Omega}$ of full measure. Denote on $\widetilde{\Omega}$ by

$$\lambda_p < \lambda_{p-1} < \cdots < \lambda_1,$$

the different numbers in the sequence $\Lambda_d \le \Lambda_{d-1} \le \cdots \le \Lambda_1$. Denote by d_i the multiplicities of appearance of λ_i in this sequence. The numbers λ_i are called the *Lyapunov exponents* of Φ, and d_i their *multiplicities*. The set

$$S(\theta,\Phi) := \{(\lambda_i, d_i) : i = 1,\dots,p\}$$

is called the *Lyapunov spectrum* of Φ.

Remark 1.3.4. Assume that Φ is a linear cocycle with two-sided time over an ergodic MDS $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{T}})$. Then $\Phi(t,\omega) := \Phi(-t,\omega)$ is a coycle over θ^{-1} and

$$S(\theta^{-1},\Phi(-\cdot)) = -S(\theta,\Phi) := \{(-\lambda_i,d_i) : i = 1,\dots,p\}.$$

We now give some explicit formulas of Lyapunov exponents of the products of triangular matrices.

Example 1.3.2 (Products of 2×2 Triangular Matrices). Let $A : \Omega \to Gl(2, \mathbb{R})$, where

$$A(\omega) = \begin{pmatrix} a(\omega) & c(\omega) \\ 0 & b(\omega) \end{pmatrix}, \quad a(\omega) \neq 0, b(\omega) \neq 0.$$

These matrices form a subgroup of $Gl(2, \mathbb{R})$ and the cocycle on $\mathbb{T} = \mathbb{N}$ over θ generated by A is

$$\Phi(n, \cdot) = A_{n-1} \dots A_0 = \begin{pmatrix} a_{n-1} \dots a_0 & \sum_{k=0}^{n-1} a_{n-1} \dots a_{k+1} c_k b_{k-1} \dots b_0 \\ 0 & b_{n-1} \dots b_0 \end{pmatrix},$$

where $a_k := a(\theta^k \omega), b_k := b(\theta^k \omega)$ and $c_k := c(\theta^k \omega)$. The following facts are easily verified

(i) $\log^+ \|A^{\pm 1}\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ if and only if $\log |a|, \log |b|$ and $\log^+ |c| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, which we assume from now on in this example.

(ii) By the above assumptions and using the Birkhoff Theorem (see Appendix A), we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |a_k| = \mathbb{E} \log |a| =: \alpha,$$
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |b_k| = \mathbb{E} \log |b| =: \beta,$$

hence

$$\lim_{n\to\infty} \log |\det \Phi(n, \cdot)| = \gamma^{(2)} = \Lambda_1 + \Lambda_2 =: 2\lambda_\Sigma = \alpha + \beta.$$

(iii) The Lyapunov exponent of $\Phi(n, \omega)_{11}$ is α, that of $\Phi(n, \omega)_{22}$ is β, and that of $\Phi(n, \omega)_{12}$ is less than or equal to $\max(\alpha, \beta)$. Therefore, by using Euclidean norm we obtain

$$\lim_{n\to\infty} \log \|\Phi(n, \omega)\| = \gamma^{(1)} = \Lambda_1 = \max\{\alpha, \beta\},$$

hence for $\alpha \neq \beta$

$$\lambda_1 = \max\{\alpha, \beta\} > \lambda_\Sigma = \frac{1}{2}(\alpha + \beta) > \lambda_2 = \min\{\alpha, \beta\}.$$

For $\alpha = \beta, \lambda_1 = \lambda_\Sigma = \alpha = \beta$ with multiplicity $d_1 = 2$.

1.3.4 Multiplicative Ergodic Theorem

Theorem 1.3.5 (Multiplicative Ergodic Theorem)**.** *Let Φ be a linear cocycle with two-sided time over an ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$.*

(A) Discrete time $\mathbb{T} = \mathbb{Z}$: Let

$$\Phi(n,\omega) = \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & n > 0, \\ I_d, & n = 0, \\ A(\theta^{n}\omega)^{-1} \circ \cdots \circ A(\theta^{-1}\omega)^{-1}, & n < 0, \end{cases}$$

be generated by $A : \Omega \to Gl(d, \mathbb{R})$ and assume

$$\log^+ \|A(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \quad \text{and} \quad \log^+ \|A^{-1}(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{1.12}$$

Then there exists an invariant set $\widetilde{\Omega}$ of full measure such that for each $\omega \in \Omega$ the following assertions hold:

(i) The limit $\lim_{n\to\infty}(\Phi(n,\omega)^*\Phi(n,\omega))^{1/2n} =: \Psi(\omega)$ exists. Furthermore, the different eigenvalues of $\Psi(\omega)$, denoted by $e^{\lambda_p} < \cdots < e^{\lambda_1}$, are almost surely constant.

(ii) There exists a splitting

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_p(\omega)$$

of $\mathbb{R}^d$ into random subspaces $E_i(\omega)$ depending measurably on ω with constant dimension $\dim E_i(\omega) = d_i$ with the following properties: For $i \in \{1, \ldots, p\}$

- *if $P_i(\omega) : \mathbb{R}^d \to E_i(\omega)$ denotes the projection onto $E_i(\omega)$ along $F_i(\omega) := \oplus_{j\neq i} E_j(\omega)$, then*
$$A(\omega)P_i(\omega) = P_i(\theta\omega)A(\omega),$$
equivalently
$$A(\omega)E_i(\omega) = E_i(\theta\omega).$$
- $$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)x\| = \lambda_i \Leftrightarrow x \in E_i(\omega) \setminus \{0\}.$$

(B) Continuous time $\mathbb{T} = \mathbb{R}$: Assume that $\alpha^{\pm} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\alpha^{\pm}(\omega) := \sup_{0\le t\le 1} \log^+ \|\Phi(t,\omega)^{\pm 1}\|.$$

Then all statements of part (A) hold with n, θ and $A(\omega)$ replaced by t, θ_t and $\Phi(t,\omega)$, respectively.

1.4 Multiplicative Ergodic Theorem in Banach Spaces

In order to state the MET in Banach spaces we recall a measure of noncompactness of an operator and its properties. Let $(X, \|\cdot\|)$ be a Banach space and B a subset of

X. Assume that $A : \Omega \to \mathcal{L}(X)$ is strongly measurable and define the corresponding one-sided linear RDS $\Phi : \mathbb{N}_0 \times \Omega \to \mathcal{L}(X)$ by

$$\Phi(n,\omega) = \begin{cases} \mathrm{id}_X, & \text{if } n = 0, \\ A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & \text{otherwise.} \end{cases} \tag{1.13}$$

The *Kuratowski measure* α of noncompactness is defined by

$$\alpha(B) := \inf\{d : B \text{ has a finite cover by sets of diameter } d\}. \tag{1.14}$$

For each $L \in \mathcal{L}(X)$ we define

$$\|L\|_\alpha = \alpha(L(B_1(0))),$$

where $B_1(0)$ is the unit ball of X with center at 0. Furthermore, $\|\cdot\|_\alpha$ is a multiplicative semi-norm, i.e. for all $L_1, L_2 \in \mathcal{L}(X)$ we have

$$\|L_1 + L_2\|_\alpha \le \|L_1\|_\alpha + \|L_2\|_\alpha, \quad \|L_1 \circ L_2\|_\alpha \le \|L_1\|_\alpha \|L_2\|_\alpha.$$

We introduce the following quantities

$$l_\alpha(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|_\alpha$$

and

$$\kappa(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|$$

and note that they are constant $\mathbb{P}$-a.s. due to the ergodicity of θ and the Kingman subadditive ergodic theorem (see Appendix B).

Remark 1.4.1. (i) If $\Phi(\omega) : X \to X$ is a compact operator for $\mathbb{P}$-a.e. $\omega \in \Omega$ then $l_\alpha = -\infty$.

(ii) Since $\|L\|_\alpha \le \|L\|$ for all linear operators $L \in \mathcal{L}(X)$ it follows that $l_\alpha(\Phi) \le \kappa(\Phi)$.

Now, we cite a short version of the MET for RDS on a separable Banach space from Lian and Lu [89] in the ergodic case.

Theorem 1.4.2 (MET in Banach Spaces, [89])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic MDS and X a separable Banach space. Assume that $A : \Omega \to \mathcal{L}(X)$ is strongly measurable, injective almost everywhere, and the integrability condition*

$$\log^+ \|A(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$$

holds. Let $\Phi : \mathbb{N}_0 \times X \to X$ denote the one-sided RDS generated by A as in (1.13)*. Then there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that exactly one of the following alternatives holds:*

(I) $\kappa(\Phi) = l_\alpha(\Phi)$.

(II) There exists $k \in \mathbb{N}$*,* Lyapunov exponents $\lambda_1 > \cdots > \lambda_k > l_\alpha(\Phi)$ *and a splitting into measurable* Oseledets spaces

$$X = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

with finite dimensional linear subspaces $E_j(\omega)$ *and an infinite dimensional linear subspace* $F(\omega)$ *such that the following properties hold:*

(i) Invariance: $\Phi(\omega)E_j(\omega) = E_j(\theta\omega)$ *and* $\Phi(\omega)F(\omega) \subset F(\theta\omega)$*.*

(ii) Lyapunov exponents:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j \quad \text{for all } v \in E_j(\omega) \setminus \{0\} \text{ and } j = 1, \ldots, k \ .$$

(iii) Exponential Decay Rate on $F(\omega)$*:*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F(\omega)}\| \leq l_\alpha(\Phi)$$

and if $v \in F(\omega) \setminus \{0\}$ *and* $(\Phi(n,\theta^{-n}\omega))^{-1}v$ *exists for all* $n \geq 0$*, which is denoted by* $\Phi(-n,\omega)v$*, then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -l_\alpha(\Phi) \ .$$

(III) There exist infinitely many finite dimensional measurable subspaces $E_j(\omega)$*, infinitely many infinite dimensional measurable subspaces* $F_j(\omega)$ *and infinitely many* Lyapunov exponents

$$\lambda_1 > \lambda_2 > \cdots > l_\alpha(\Phi) \qquad \text{with} \lim_{j\to+\infty} \lambda_j = l_\alpha(\Phi)$$

such that the following properties hold:

(i) Invariance: $\Phi(\omega)E_j(\omega) = E_j(\theta\omega)$ *and* $\Phi(\omega)F_j(\omega) \subset F_j(\theta\omega)$*.*

(ii) Invariant Splitting:

$$X = E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) \quad \text{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega) \ .$$

(iii) Lyapunov exponents:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j \quad \text{for all } v \in E_j(\omega) \setminus \{0\} \ .$$

(iv) Exponential Decay Rate on $F_j(\omega)$*:*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F_j(\omega)}\| = \lambda_{j+1}$$

and if $v \in F_j(\omega) \setminus \{0\}$ *such that* $\Phi(-n,\omega)v$ *exists for all* $n \in \mathbb{N}$ *then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\| \geq -\lambda_{j+1} \ .$$

The next theorem is the MET for continuous-time RDS in Banach spaces.

Theorem 1.4.3 (MET for Continuous Time Linear RDS in Banach Spaces, [89])**.** *Let $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(X)$ be a continuous-time linear RDS and X be a separable Banach space. Assume that $\Phi(1, \cdot) : \Omega \to \mathcal{L}(X)$ is strongly measurable and $\Phi(1, \omega)$ is injective almost everywhere, and*

$$\sup_{0\leq s\leq 1} \log^+ \|\Phi(s, \cdot)\|, \quad \sup_{0\leq s\leq 1} \log^+ \|\Phi(1 - s, \theta_s \cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{1.15}$$

Define

$$l_\alpha(\Phi) := \lim_{s\to\infty} \frac{1}{s} \log \|\Phi(s, \omega)\|_\alpha$$

and

$$\kappa(\Phi) := \lim_{s\to\infty} \frac{1}{s} \log \|\Phi(s, \omega)\|.$$

Then there exists a θ_t-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that all statements of Theorem 1.4.2 hold with $n \in \mathbb{N}$ replaced by $t \in \mathbb{R}_+$.

Chapter 2

Generic Properties of Lyapunov Exponents of Discrete Random Dynamical Systems

2.1 The Space of Linear Cocycles

Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic MDS. Throughout this chapter, we assume additionally that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a non-atomic Lebesgue space, i.e. any measurable set of positive probability in Ω includes a measurable subset of a less but nonzero probability. A measurable mapping A from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the topological space $Gl(d, \mathbb{R})$ equipped with its Borel σ-algebra is called a *random linear map*. A generates a linear cocycle (see also Section 1.2.1) over the dynamical system θ via

$$\Phi_A(n, \omega) := \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & n > 0, \\ I_d, & n = 0, \\ A(\theta^n\omega)^{-1} \circ \cdots \circ A(\theta^{-1}\omega)^{-1}, & n < 0. \end{cases}$$

Conversely, if we are given a linear cocycle over θ, then its time-one map is a linear random map. Therefore, we usually speak of a linear cocycle A, meaning the cocycle Φ_A generated by A. The above construction applies to any topological group G in place of $Gl(d, \mathbb{R})$ (in particular, G can be a Lie subgroup of $Gl(d, \mathbb{R})$, for instance $Sl(d, \mathbb{R})$). For simplicity of notation we denote by $\|\cdot\|$ both the standard Euclidean norm of $\mathbb{R}^d$ and the operator norm of linear operators of $\mathbb{R}^d$. We shall look at linear cocycles as linear operators of $\mathbb{R}^d$ and identify them with their matrix representations in the standard Euclidean basis of $\mathbb{R}^d$.

The MET of Oseledets [109] (see also Theorem 1.3.5) states that if $A(\cdot)$ satisfies the integrability conditions

$$\log^+ \|A(\cdot)^{\pm 1}\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \tag{2.1}$$

then the cocycle Φ_A has Lyapunov exponents $\lambda_p < \cdots < \lambda_1$ with multiplicities $d_p, \ldots, d_1$, which are independent of ω due to the ergodicity of θ, and the phase space $\mathbb{R}^d$ is decomposed into the direct sum of subspaces $E_i(\omega)$ of dimensions d_i corresponding to the Lyapunov exponents λ_i, $i = 1, \dots, p$, i.e.

$$\lim_{n\to\pm\infty} n^{-1} \log \|\Phi_A(n,\omega)x\| = \lambda_i \Longleftrightarrow x \in E_i(\omega)\backslash\{0\}.$$

The above splitting is called *Oseledets splitting* of Φ_A, and the subspaces $E_i(\omega)$ are called *Oseledets subspaces* of Φ_A, they are measurable and *invariant with respect to* A, i.e., $A(\omega)E_i(\omega) = E_i(\theta\omega)$. The Lyapunov spectrum of A, $\{(\lambda_i, d_i), i = 1, \dots, p\}$, is said to be *simple* if $p = d$. The cocycle A is called *hyperbolic* if none of its Lyapunov exponents vanishes. We note that the statements of the MET hold on an invariant set of full $\mathbb{P}$-measure. Since we deal with discrete-time cocycles we can always neglect sets of null measure, and we shall identify the random mappings which coincide $\mathbb{P}$-almost surely, and when needed we assume w.l.o.g. that the assertions of the Oseledets theorem hold on the whole of Ω.
Denote by $\mathcal{G}(d)$ the set of all $Gl(d,\mathbb{R})$-valued random maps. Let $\mathcal{G}_{IC}(d) \subset \mathcal{G}(d)$ denote the subset of those random maps which satisfy the integrability conditions (2.1) and $\mathcal{G}_\infty(d) \subset \mathcal{G}(d)$ the subset of those random maps which are essentially bounded together with their inverses. Clearly, $\mathcal{G}_\infty(d) \subset \mathcal{G}_{IC}(d)$. We define a metric $\rho_p, 1 \le p \le \infty$, on $\mathcal{G}(d)$ such that $(\mathcal{G}(d), \rho_p)$ can be considered as a version of the L_p-norm on $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$. For $A, B \in \mathcal{G}(d)$ set

$$\delta_p(A,B) := \begin{cases} \left(\int_\Omega \|A(\omega) - B(\omega)\|^p \, d\mathbb{P}(\omega) + \int_\Omega \|A(\omega)^{-1} - B(\omega)^{-1}\|^p \, d\mathbb{P}(\omega)\right)^{\frac{1}{p}}, \\ \infty, \quad \text{in case at least one of the above integrals does not exist,} \end{cases}$$

for $1 \le p < \infty$, and for $p = \infty$ put

$$\delta_\infty(A,B) := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| + \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)^{-1} - B(\omega)^{-1}\|.$$

Set

$$\rho_p(A,B) := \begin{cases} \delta_p(A,B)(1 + \delta_p(A,B))^{-1} & \text{if} \quad \delta_p(A,B) < \infty, \\ 1 & \text{if} \quad \delta_p(A,B) = \infty. \end{cases}$$

The following lemma ensures that $(\mathcal{G}(d), \rho_p)$ is a metric and provides some fundamental properties of this metric.

Lemma 2.1.1 (Arnold and Cong [4, 5])**.** *Let $1 \le p \le \infty$ and $\rho_p : \mathcal{G}(d) \times \mathcal{G}(d) \to \mathbb{R}$ be the function defined as in above. Then the following statements hold:*

(i) ρ_p is a metric on $\mathcal{G}(d)$, hence on $\mathcal{G}_{IC}(d)$ and $\mathcal{G}_\infty(d)$.

(ii) If $A \in \mathcal{G}_{IC}(d)$ and $B \in \mathcal{G}(d)$ with $\rho_p(A,B) < 1$, then $B \in \mathcal{G}_{IC}(d)$. In particular, $\mathcal{G}_{IC}(d)$ are both ρ_p-closed and ρ_p-open in $\mathcal{G}(d)$.

(iii) $(\mathcal{G}(d), \rho_p)$, hence $(\mathcal{G}_{IC}(d), \rho_p)$ is complete.

Remark 2.1.2. (i) We note that for $A, B \in \mathcal{G}(d)$ and $1 \leq p \leq p' \leq \infty$ we have $\delta_p(A,B) \leq \delta_{p'}(A,B)$, hence $\rho_p(A,B) \leq \rho_{p'}(A,B)$. Therefore, the topology generated by $\rho_{p'}$ is finer than the topology generated by ρ_p.
(ii) Being complete spaces, $(\mathcal{G}(d), \rho_p)$, $(\mathcal{G}_{IC}(d), \rho_p)$ and $(\mathcal{G}_\infty(d), \rho_p)$ are Baire spaces (see Theorem C.0.9).

The angle between two non-vanishing vectors $x, y \in \mathbb{R}^d$ is defined by

$$\measuredangle(x,y) := \arccos \frac{\langle x, y\rangle}{\|x\|\|y\|} \in [0,\pi]. \tag{2.2}$$

The (minimal) angle between two subspaces $E_1, E_2 \subset \mathbb{R}^d$ is defined by

$$\measuredangle(E_1,E_2) := \inf\{\measuredangle(x,y) \mid 0 \neq x \in E_1, 0 \neq y \in E_2\} \in [0, \frac{\pi}{2}]. \tag{2.3}$$

Throughout this chapter, we are only interested in ρ_∞ norm and for a better presentation we use the notation ρ to indicate ρ_∞.

2.2 Uniformly Hyperbolic Linear Cocycles

2.2.1 Exponential Dichotomy

Definition 2.2.1 (Exponential Dichotomy)**.** A linear cocycle $A \in \mathcal{G}(d)$ is said to admit an *exponential dichotomy* if there exist positive numbers $K > 0, \alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$ depending measurably on $\omega \in \Omega$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ the following inequalities hold:

$$\begin{aligned} \|\Phi_A(n,\omega)P_\omega\Phi_A(m,\omega)^{-1}\| &\leq K\,e^{-\alpha(n-m)} \quad \text{for all } n \geq m, \\ \|\Phi_A(n,\omega)(I_d - P_\omega)\Phi_A(m,\omega)^{-1}\| &\leq K\,e^{\alpha(n-m)} \quad \text{for all } n \leq m. \end{aligned}$$

Remark 2.2.1. (i) If $A \in \mathcal{G}(d)$ has an exponential dichotomy with positive constants K, α and a family of projections P_ω of $\mathbb{R}^d$, then the angle between the subspaces $P_\omega\mathbb{R}^d$ and $(I_d - P_\omega)\mathbb{R}^d$ is bounded away from zero by a positive constant which is independent of $\omega \in \Omega$.

(ii) The random subspaces $E_1(\omega) := P_\omega\mathbb{R}^d$ and $E_2(\omega) := (I_d - P_\omega)\mathbb{R}^d$ are invariant with respect to A, i.e., $\Phi_A(n,\omega)E_i(\omega) = E_i(\theta^n\omega)$ for all $n \in \mathbb{Z}$, $\omega \in \Omega$ and $i = 1, 2$.

(iii) Exponential dichotomy is also called *uniform hyperbolicity.*

Now we turn to the notion of cohomology of linear cocycles which is the notion of random basis change for the linear cocycles.

Definition 2.2.2. Two linear cocycles $A, B \in \mathcal{G}(d)$ are called *cohomologous* if there exists a measurable map $L \in \mathcal{G}(d)$ such that for almost all $\omega \in \Omega$

$$A(\omega) = L(\theta\omega)^{-1} \circ B(\omega) \circ L(\omega).$$

In the following lemmas, we study the relation between cohomology and exponential dichotomy of linear cocycles.

Lemma 2.2.2. *Suppose that $A \in \mathcal{G}(d)$ admits an exponential dichotomy with positive constants K, α and a family of projections P_ω of $\mathbb{R}^d$, and A is cohomologous to $B \in \mathcal{G}(d)$ by a bounded cohomology L which has bounded inverse, i.e. $A(\omega) = L(\theta\omega)^{-1}B(\omega)L(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then B also admits an exponential dichotomy. Furthermore, we have*

$$\begin{aligned} \|\Phi_B(n,\omega)Q_\omega\Phi_B(m,\omega)^{-1}\| &\leq KM_1M_2\, e^{-\alpha(n-m)} \quad \text{for all } n \geq m, \\ \|\Phi_B(n,\omega)(I_d - Q_\omega)\Phi_B(m,\omega)^{-1}\| &\leq KM_1M_2\, e^{\alpha(n-m)} \quad \text{for all } n \leq m, \end{aligned}$$

where

$$Q_\omega = L(\omega)P_\omega L(\omega)^{-1}, \quad M_1 := \operatorname*{ess\,sup}_{\omega\in\Omega} \|L(\omega)\|, \quad M_2 := \operatorname*{ess\,sup}_{\omega\in\Omega} \|L(\omega)^{-1}\|.$$

Proof. A direct computation yields that

$$\Phi_B(n,\omega)Q_\omega\Phi_B(m,\omega)^{-1} = L(\theta^n\omega)\Phi_A(n,\omega)P_\omega\Phi_A(m,\omega)^{-1}L(\theta^m\omega)^{-1}.$$

Therefore,

$$\|\Phi_B(n,\omega)Q_\omega\Phi_B(m,\omega)^{-1}\| \leq KM_1M_2e^{-\alpha(n-m)} \qquad \text{for all } n \geq m.$$

Similarly, we also have

$$\|\Phi_B(n,\omega)(I_d - Q_\omega)\Phi_B(m,\omega)^{-1}\| \leq KM_1M_2\, e^{\alpha(n-m)} \qquad \text{for all } n \leq m,$$

which completes the proof. □

Lemma 2.2.3. *Suppose that $A \in \mathcal{G}(d)$ admits an exponential dichotomy with positive constants K, α and a family of projections P_ω of $\mathbb{R}^d$. Then A is cohomologous to a block-diagonal cocycle*

$$\widetilde{A}(\omega) = \begin{pmatrix} \widetilde{A}_1(\omega) & 0 \\ 0 & \widetilde{A}_2(\omega) \end{pmatrix}, \quad \widetilde{A}_i \in \mathcal{G}(d_i), i = 1, 2,$$

by a cohomology $L \in \mathcal{G}(d)$ satisfying that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|L(\omega)\| \leq K\sqrt{2}, \quad \operatorname*{ess\,sup}_{\omega\in\Omega} \|L(\omega)^{-1}\| \leq \sqrt{2}.$$

Moreover,

$$\begin{aligned} \|\Phi_{\widetilde{A}_1}(n-m,\theta^m\omega)\| &\leq 2K^2e^{-\alpha(n-m)} \qquad \text{for all } n \geq m, \\ \|\Phi_{\widetilde{A}_2}(m-n,\theta^n\omega)^{-1}\| &\leq 2K^2e^{\alpha(n-m)} \qquad \text{for all } n \leq m. \end{aligned}$$

Proof. Choose orthonormal bases in the random subspaces $\mathrm{im}P_\omega$ and $\ker P_\omega$, and compose from them a random basis $\{f_1(\omega),\dots,f_d(\omega)\}$ of $\mathbb{R}^d$. Define a random linear mapping $L:\Omega\to Gl(d)$ by the formula

$$L(\omega)f_i(\omega)=e_i,\quad i=1,2,\dots,d,$$

where $\{e_1,\dots,e_d\}$ is the standard Euclidean basis of $\mathbb{R}^d$. Put

$$\widetilde{A}(\omega):=L(\theta\omega)A(\omega)L(\omega)^{-1}.$$

Clearly, $\widetilde{A}$ has the block-diagonal form as stated in the lemma, where $d_1=\dim(\mathrm{im}P_\omega)$ and $d_2=\dim(\ker P_\omega)$. We now estimate $\|L(\omega)\|$ and $\|L(\omega)^{-1}\|$. By the definition of $L(\omega)$ we have

$$\|L(\omega)x\|^2=\|P_\omega x\|^2+\|(I_d-P_\omega)x\|^2\qquad\text{for all } x\in\mathbb{R}^d,\tag{2.4}$$

which together with the fact that $\|P_\omega\|,\|I_d-P_\omega\|\le K$ implies that

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|L(\omega)\|\le K\sqrt{2}.$$

From equality (2.4) we derive

$$\frac{\|L(\omega)^{-1}x\|^2}{\|x\|^2}=\frac{\|L(\omega)^{-1}x\|^2}{\|P_\omega L(\omega)^{-1}x\|^2+\|(I_d-P_\omega)L(\omega)^{-1}x\|^2}\le 2,$$

which proves that $\|L(\omega)^{-1}\|\le\sqrt{2}$ almost surely. By the construction of $L(\omega)$ we obtain that

$$P:=L(\omega)P_\omega L(\omega)^{-1}=\begin{pmatrix} I_{d_1} & 0\\ 0 & 0\end{pmatrix}.$$

Note that the matrix $\widetilde{A}(\omega)$ commutes to P and we thus obtain

$$\begin{aligned}\|\Phi_{\widetilde{A}}(n,\omega)P\Phi_{\widetilde{A}}(m,\omega)^{-1}\| &= \|\Phi_{\widetilde{A}_1}(n-m,\theta^m\omega)\| \qquad\text{for all } n\ge m,\\ \|\Phi_{\widetilde{A}}(n,\omega)(I_d-P)\Phi_{\widetilde{A}}(m,\omega)^{-1}\| &= \|\Phi_{\widetilde{A}_2}(m-n,\theta^n\omega)^{-1}\| \qquad\text{for all } n\le m.\end{aligned}$$

Hence, the remaining part of the proof is a direct consequence of Lemma 2.2.2. □

The property of exponential dichotomy is robust under small perturbations. The proof of this statement for deterministic dynamical system can be found e.g. in Coppel [33], Ju and Wiggins [75]. Although the statement for RDS is used by many authors, as far as we know there is so far no proof in the literature. For sake of completeness, we provide a proof of the theorem on the robustness of exponential dichotomy for RDS. We follow the proof in Coppel [33, Chapter 5].

Theorem 2.2.4 (Robustness of Exponential Dichotomy)**.** *Let $A\in\mathcal{G}(d)$ be a linear cocycle exhibiting an exponential dichotomy with positive constants K,α and a family of projections P_ω of $\mathbb{R}^d$. Set*

$$\delta^*:=\min\left\{\frac{e^{2\alpha}-1}{72K^5e^\alpha},\frac{\alpha}{6\sqrt{2}K^3e^\alpha},\frac{\alpha}{6\sqrt{2}K^3e^{-\alpha}\alpha+24\sqrt{2}K^7e^{-\alpha}}\right\}.\tag{2.5}$$

Then any cocycle $B \in \mathcal{G}(d)$ satisfying that $\delta := \operatorname{esssup}_{\omega\in\Omega}\|B(\omega) - A(\omega)\| < \delta^*$ also exhibits an exponential dichotomy with the exponential rate β determined by

$$\beta := \min\left\{\alpha - 6\sqrt{2}K^3 e^{\alpha}\delta, \alpha - \frac{24\sqrt{2}K^7 e^{-\alpha}\delta}{1 - 6\sqrt{2}K^3 e^{-\alpha}\delta}\right\}.$$

Furthermore, the projections Q_ω of the exponential dichotomy of B satisfy

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|Q_\omega - P_\omega\| \le \frac{72K^6 e^{\alpha}\delta}{e^{2\alpha} - 1 - 36K^5 e^{\alpha}\delta}.$$

Proof. Let $B \in \mathcal{G}(d)$ satisfy that $\delta = \operatorname{esssup}_{\omega\in\Omega}\|B(\omega) - A(\omega)\| < \delta^*$. To simplify the formulas throughout the proof let us introduce the following constants

$$\eta := K^5\delta\frac{e^{\alpha}}{e^{2\alpha} - 1}, \quad \gamma := \frac{12\sqrt{2}K^5 e^{-2\alpha}\delta}{1 - 6\sqrt{2}K^3 e^{-\alpha}\delta}.$$

For convenience, we divide the proof into several steps.
Step 1: Transfer the linear cocycle A to a block-triangular form: According to Lemma 2.2.3, A is cohomologeous to a block-diagonal cocycle

$$\widetilde{A}(\omega) := \begin{pmatrix} \widetilde{A}_1(\omega) & 0 \\ 0 & \widetilde{A}_2(\omega) \end{pmatrix}, \quad \widetilde{A}_i \in \mathcal{G}(d_i), i = 1, 2,$$

by a cohomology $L \in \mathcal{G}(d)$, i.e. $\widetilde{A}(\omega) = L(\theta\omega)A(\omega)L(\omega)^{-1}$, satisfying that

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|L(\omega)\| \le \sqrt{2}K, \quad \operatorname*{ess\,sup}_{\omega\in\Omega}\|L(\omega)^{-1}\| \le \sqrt{2}, \tag{2.6}$$

and for $\mathbb{P}$-a.e. $\omega \in \Omega$ the following inequalities hold

$$\|\Phi_{\widetilde{A}_1}(n - m, \theta^m\omega)\| \le 2K^2\, e^{-\alpha(n-m)} \quad \text{for all } n \ge m, \tag{2.7}$$

$$\|\Phi_{\widetilde{A}_2}(m - n, \theta^n\omega)^{-1}\| \le 2K^2\, e^{\alpha(n-m)} \quad \text{for all } n \le m. \tag{2.8}$$

In other words, the linear cocycle $\widetilde{A}$ admits an exponential dichotomy with the exponential rate α and the projection $P := \begin{pmatrix} I_{d_1} & 0 \\ 0 & 0 \end{pmatrix}$. We define

$$\widetilde{B}(\omega) = L(\theta\omega)B(\omega)L(\omega)^{-1}, \quad \widetilde{\Delta}(\omega) = \widetilde{B}(\omega) - \widetilde{A}(\omega).$$

From inequality (2.6), we derive

$$\operatorname*{ess\,sup}_{\omega\in\Omega}\|\widetilde{\Delta}(\omega)\| \le 2K\operatorname*{ess\,sup}_{\omega\in\Omega}\|B(\omega) - A(\omega)\| = 2K\delta. \tag{2.9}$$

Step 2: Transfer the perturbed linear cocycle $\widetilde{B}$: For any matrix $E \in \mathbb{R}^{d\times d}$, we put

$$\begin{aligned} \{E\}_1 &:= PEP + (I_d - P)E(I_d - P), \\ \{E\}_2 &:= PE(I_d - P) + (I_d - P)EP, \end{aligned}$$

so that $\{E\}_1 + \{E\}_2 = E$. Obviously, the matrix $\{E\}_1$ commutes with P, i.e. $\{E\}_1 P = P\{E\}_1$, and $\|\{E\}_1\| \le \sqrt{2}\|E\|$. We look for a bounded cohomology $S \in \mathcal{G}(d)$ which has bounded inverse such that

$$S(\theta\omega)\widetilde{B}(\omega)S(\omega)^{-1} = \widetilde{A}(\omega) + \left\{\widetilde{\Delta}(\omega)S(\omega)^{-1}\right\}_1 := \widehat{B}(\omega). \tag{2.10}$$

In other words, we show that $\widetilde{B}$ and $\widehat{B}$ are cohomologeous by a cohomology which is bounded together with its inverse. For this purpose, let $\mathbf{B}$ denote the Banach space of matrix-valued functions $f : \mathbb{Z} \to \mathbb{R}^{d\times d}$ with the sup norm

$$\|f\|_{\mathbf{B}} := \sup_{n\in\mathbb{Z}} \|f(n)\|.$$

Let $\mathbf{B}_{\frac{1}{2}}(0)$ denote the closed ball with radius $\frac{1}{2}$ centered at 0. For each $\omega \in \Omega$ we define a mapping $T_\omega : \mathbf{B}_{\frac{1}{2}}(0) \to \mathbf{B}_{\frac{1}{2}}(0)$ by

$$\begin{aligned} T_\omega f(n) \;=\; & \sum_{k=-\infty}^{n} \Phi_{\widetilde{A}}(n,\omega)P\Phi_{\widetilde{A}}(k,\omega)^{-1}(I_d - f(k))\widetilde{\Delta}(\theta^{k-1}\omega)(I_d + f(k-1)) \cdot \\ & \qquad\qquad \cdot\Phi_{\widetilde{A}}(k-1,\omega)(I_d - P)\Phi_{\widetilde{A}}(n,\omega)^{-1} - \\ & - \sum_{k=n+1}^{\infty} \Phi_{\widetilde{A}}(n,\omega)(I_d - P)\Phi_{\widetilde{A}}(k,\omega)^{-1}(I_d - f(k))\widetilde{\Delta}(\theta^{k-1}\omega)(I_d + f(k-1)) \cdot \\ & \qquad\qquad \cdot\Phi_{\widetilde{A}}(k-1,\omega)P\Phi_{\widetilde{A}}(n,\omega)^{-1}. \end{aligned}$$

From the definition of η, inequalities (2.7), (2.8) and (2.9), we get

$$\|T_\omega f\| \le 16\eta(1 + \|f\|)^2 \le \frac{1}{2} \qquad \text{for all } f \in \mathbf{B}_{\frac{1}{2}}(0),$$

which implies that the mapping T_ω is well-defined. We now show that T_ω is a contraction. Consider $f, g \in \mathbf{B}_{\frac{1}{2}}(0)$. It is easy to prove that the following identity

$$(I_d-F)M(I_d+F)-(I_d-G)M(I_d+G) = (G-F)M+M(F-G)+(G-F)MG+FM(G-F)$$

holds for all $M, G, F \in \mathbb{R}^{d\times d}$. As a consequence, the following estimate

$$\|(I_d - f(k))\widetilde{\Delta}(\theta^{k-1}\omega)(I_d + f(k-1)) - (I_d - g(k))\widetilde{\Delta}(\theta^{k-1}\omega)(I_d + g(k-1))\| \le 6K\delta\|f - g\|$$

holds for all $f, g \in \mathbf{B}_{\frac{1}{2}}(0)$. Therefore, a direct estimate yields that

$$\|T_\omega f - T_\omega g\| \le 48\eta\|f - g\| \qquad \text{for all } f, g \in \mathbf{B}_{\frac{1}{2}}(0).$$

Hence, T_ω is a contraction on the closed subset $\mathbf{B}_{\frac{1}{2}}(0)$ of a Banach space. Consequently, by the contraction principle, the equation $T_\omega f = f$ has a unique solution denoted by f_ω. Obviously, the function f_ω depends measurably on ω and satisfies

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|f_\omega\| \le 36\eta. \tag{2.11}$$

Note that for all $f \in \mathbf{B}_{\frac{1}{2}}(0)$ we have

$$T_{\theta^k\omega} f(n) = T_\omega g(n+k) \qquad \text{for all } n,k \in \mathbb{Z},$$

where $g : \mathbb{Z} \to \mathbb{R}^{d\times d}$ is defined by $g(n) = f(n-k)$. As a consequence, we get

$$f_{\theta^k\omega}(n) = f_\omega(n+k) \qquad \text{for all } n,k \in \mathbb{Z} \tag{2.12}$$

We define a random linear mapping $S : \Omega \to Gl(d)$ by

$$S(\omega) = (I_d + f_\omega(0))^{-1} \qquad \text{for all } \omega \in \Omega. \tag{2.13}$$

From the fact that $\|f_\omega(0)\| \le \frac{1}{2}$, we derive that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|S(\omega)\| \le 2, \quad \operatorname*{ess\,sup}_{\omega\in\Omega} \|S(\omega)^{-1}\| \le \frac{3}{2}, \tag{2.14}$$

which together with inequality (2.9) and relation (2.10) implies that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|\widehat{B}(\omega) - \widetilde{A}(\omega)\| = \operatorname*{ess\,sup}_{\omega\in\Omega} \|\{\widetilde{\Delta}(\omega)S(\omega)^{-1}\}_1\| \le 3\sqrt{2}K\delta. \tag{2.15}$$

Since f_ω is the fixed point of T_ω it follows that

$$f_\omega(n+1)\widetilde{A}(\theta^n\omega) - \widetilde{A}(\theta^n\omega)f_\omega(n) = \Big\{(I_d - f_\omega(n+1))\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_2. \tag{2.16}$$

Moreover, f_ω satisfies

$$Pf_\omega(n)P = 0, \quad (I_d - P)f_\omega(n)(I_d - P) = 0.$$

Hence, $f_\omega = Pf_\omega + f_\omega P$. Therefore, we get

$$\begin{aligned}\Big\{(I_d - f_\omega(n+1))\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_2 &= \Big\{\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_2 - \\ &\quad - f_\omega(n+1)\Big\{\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_1.\end{aligned}$$

Together with (2.16) this implies that

$$\begin{aligned}\widetilde{A}(\theta^n\omega)f_\omega(n) + \Big\{\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_2 &= f_\omega(n+1)\widetilde{A}(\theta^n\omega) + \\ &\quad + f_\omega(n+1)\Big\{\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_1,\end{aligned}$$

which leads that

$$(I_d + f_\omega(n+1))\Big(\widetilde{A}(\theta^n\omega) + \Big\{\widetilde{\Delta}(\theta^n\omega)(I_d + f_\omega(n))\Big\}_1\Big) = \widetilde{B}(\theta^n\omega)(I_d + f_\omega(n)).$$

This together with (2.12) and (2.13) proves that the cohomology $S(\omega)$ satisfies relation (2.10). By combining Step 1 and Step 2, we obtain that two linear coycles B and $\widehat{B}$ are cohomologeous by the cohomology $S \circ L$. Furthermore, by using (2.6) and (2.14), we get

$$\|S(\omega) \circ L(\omega)\| \le 2\sqrt{2}K, \quad \|L(\omega)^{-1} \circ S(\omega)^{-1}\| \le \frac{3\sqrt{2}}{2}. \tag{2.17}$$

Step 3: Show that $\widehat{B}$ admits an exponential dichotomy: Since the matrix $\widehat{B}(\omega)$ commute with P it follows that $\widehat{B}(\omega)$ is of the form

$$\widehat{B}(\omega) = \begin{pmatrix} \widehat{B}_1(\omega) & 0 \\ 0 & \widehat{B}_2(\omega) \end{pmatrix}.$$

Hence, we get

$$\begin{aligned} \|\Phi_{\widehat{B}}(n,\omega) P \Phi_{\widehat{B}}(m,\omega)^{-1}\| &= \|\Phi_{\widehat{B}_1}(n-m, \theta^m\omega)\| \qquad \text{for all } n \ge m, \\ \|\Phi_{\widehat{B}}(n,\omega)(I_d - P)\Phi_{\widehat{B}}(m,\omega)^{-1}\| &= \|\Phi_{\widehat{B}_2}(m-n, \theta^n\omega)^{-1}\| \qquad \text{for all } n \le m. \end{aligned}$$

We now estimate $\|\Phi_{\widehat{B}_1}(n-m, \theta^m\omega)\|$. W.o.l.g. we assume that $m = 0$ and consider $n \ge 0$. Since

$$\Phi_{\widehat{B}_1}(n,\omega) = \Phi_{\widetilde{A}_1}(n,\omega) + \sum_{k=0}^{n-1} \Phi_{\widetilde{A}_1}(n-k-1, \theta^{k+1}\omega)(\widehat{B}_1(\theta^k\omega) - \widetilde{A}_1(\theta^k\omega))\Phi_{\widehat{B}_1}(k,\omega)$$

it follows together with(2.7) and (2.15) that

$$s_n \le 2K^2 e^{-\alpha n} + 6\sqrt{2}K^3\delta \sum_{k=0}^{n-1} e^{-\alpha(n-k-1)} s_k,$$

where $s_k := \|\Phi_{\widehat{B}_1}(k,\omega)\|$. Using the discrete Gronwall lemma (see e.g. Popenda [118]), we obtain

$$e^{\alpha n} s_n \le (2K^2 + 6\sqrt{2}K^3 e^{\alpha}\delta)(1 + 6\sqrt{2}K^3 e^{\alpha}\delta)^{n-1},$$

which implies together with the inequality $e^x \ge 1 + x$ for all $x \ge 0$ that the following relations

$$\begin{aligned} \|\Phi_{\widehat{B}_1}(n-m, \theta^m\omega)\| &\le 2K^2 e^{(-\alpha + 6\sqrt{2}K^3 e^{\alpha}\delta)(n-m)} \\ &\le 2K^2 e^{-\beta(n-m)} \end{aligned}$$

hold for all $n \ge m$. To prove the fact that $\widehat{B}$ admits an exponential dichotomy we need to estimate additionally the term $\|\Phi_{\widehat{B}_2}(m-n, \theta^n\omega)^{-1}\|$ for all $m \ge n$. W.l.o.g. we assume that $n = 0$ and consider $m \ge 0$. From inequality (2.8), we derive that $\|\widetilde{A}_2(\omega)^{-1}\| \le 2K^2 e^{-\alpha}$. This together with (2.15) implies that

$$\|\widehat{B}_2(\omega)^{-1} - \widetilde{A}_2(\omega)^{-1}\| \le \frac{12\sqrt{2}K^5 e^{-2\alpha}\delta}{1 - 6\sqrt{2}K^3 e^{-\alpha}\delta} = \gamma.$$

Since

$$\Phi_{\widehat{B}_2}(m,\omega)^{-1} = \Phi_{\widetilde{A}_2}(m,\omega)^{-1} + \sum_{k=0}^{m-1} \Phi_{\widehat{B}_2}(k,\omega)^{-1}(\widehat{B}_2(\theta^k\omega)^{-1} - \widetilde{A}_2(\theta^k\omega)^{-1}) \cdot \cdot\Phi_{\widetilde{A}_2}(m-k-1,\theta^{k+1}\omega)^{-1}$$

it follows together with (2.8) that

$$t_m \leq 2K^2 e^{-\alpha m} + 2K^2\gamma \sum_{k=0}^{m-1} e^{-\alpha(m-k-1)} t_k,$$

where $t_k := \|\Phi_{\widehat{B}_2}(k,\omega)^{-1}\|$. Using the discrete Gronwall lemma (see e.g. Popenda [118]), we obtain

$$e^{\alpha m} t_m \leq (2K^2 + 2K^2 e^{\alpha}\gamma)(1 + 2K^2 e^{\alpha}\gamma)^{m-1},$$

which implies together with the inequality $e^x \geq 1 + x$ for all $x \geq 0$ that the following relations

$$\begin{aligned} \|\Phi_{\widehat{B}_2}(m-n,\theta^m\omega)^{-1}\| &\leq 2K^2 e^{(-\alpha+2K^2e^{\alpha}\gamma)(m-n)} \\ &\leq 2K^2 e^{-\beta(m-n)} \end{aligned}$$

hold for all $n \leq m$. So far we have proved that the linear cocycle $\widehat{B}$ admits an exponential dichotomy with positive constants $2K^2, \beta$ and the projection P.
Step 4: Note that B and $\widehat{B}$ are cohomologous. By virtue of Lemma 2.2.2, the linear cocycle B admits an exponential dichotomy with the exponential rate β and the corresponding projections Q_ω are determined by

$$Q_\omega = S(\omega)L(\omega)PL(\omega)^{-1}S(\omega)^{-1} = S(\omega)P_\omega S(\omega)^{-1}.$$

Therefore, by (2.13) and the fact that $\|P_\omega\| \leq K$ we have

$$\begin{aligned} \|Q_\omega - P_\omega\| &\leq 2K(\|f_\omega(0)\| + \|f_\omega(0)\|^2 + \dots) \\ &\leq \frac{72K\eta}{1-36\eta}, \end{aligned}$$

where we use (2.11) to obtain the last estimate and the proof is complete. □

Remark 2.2.5. (i) Let $A \in \mathcal{G}(d)$ be a linear cocycle exhibiting an exponential dichotomy with the exponential rate α and the projections P_ω. Since

$$\lim_{\delta\to 0} \alpha - 6\sqrt{2}K^3 e^{\alpha}\delta = \lim_{\delta\to 0} \alpha - \frac{24\sqrt{2}K^7 e^{-\alpha}\delta}{1 - 6\sqrt{2}K^3 e^{-\alpha}\delta} = \alpha, \quad \lim_{\delta\to 0} \frac{72K^6 e^{\alpha}\delta}{e^{2\alpha} - 1 - 36K^5 e^{\alpha}\delta} = 0$$

it follows together with Theorem 2.2.4 that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $B \in \mathcal{G}(d)$ and $\operatorname{esssup}_{\omega\in\Omega}\|A(\omega) - B(\omega)\| < \delta$ then B admits an exponential dichotomy with the exponential rate β and the projections Q_ω satisfying

$$|\alpha - \beta| \leq \varepsilon, \quad \operatorname*{ess\,sup}_{\omega\in\Omega} \|Q_\omega - P_\omega\| \leq \varepsilon.$$

(ii) Let $A \in \mathcal{G}(d)$ be a linear cocycle exhibiting an exponential dichotomy. Since

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| \le \rho(A, B)$$

it follows together with Theorem 2.2.4 that there exists $\delta > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \delta$ also exhibits an exponential dichotomy.
(iii) Let $A \in \mathcal{G}(d)$ be a linear cocycle exhibiting an exponential dichotomy. We want to investigate and compute the following quantity

$$r_{\text{dich}}(A) := \inf\left\{\|\Delta\| : A + \Delta \text{ does not admit an exponential dichotomy}\right\}.$$

Theorem 2.2.4 ensures not only that $r_{\text{dich}}(A) > 0$ but also provides an explicit lower estimate on $r_{\text{dich}}(A)$.

2.2.2 Exponential Separation of Bounded Cocycles

Definition 2.2.3. Let $A \in \mathcal{G}_\infty(d)$ and

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad k \ge 2, \tag{2.18}$$

be an invariant splitting of A, i.e. for almost all $\omega \in \Omega$ and any $i = 1, \dots, k$ we have $A(\omega)E_i(\omega) = E_i(\theta\omega)$. Splitting (2.18) is called *exponentially separated* if there exist positive numbers $\alpha, K > 0$ such that for any $n \in \mathbb{N}$, $\omega \in \Omega$, and any $i = 1, \dots, k-1$ the inequality

$$\frac{\|\Phi_A(n,\omega)x\|}{\|x\|} \le Ke^{-n\alpha} \cdot \frac{\|\Phi_A(n,\omega)y\|}{\|y\|}$$

holds for all $0 \ne x \in E_1(\omega) \oplus \cdots \oplus E_i(\omega)$ and $0 \ne y \in E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$.

The notion of *exponential separation* given in Definition 2.2.3 (for discrete-time bounded cocycles) is equivalent to the notion of *domination* introduced by Viana and his co-workers [22, 23, 24] for classical dynamical systems on compact manifolds and cocycles over them. It is also a random version of the notion of exponential separation of ordinary differential equations which originated from the works of Bylov, Coppel, Sacker and Sell, Palmer and Siegmund (see [27, 33, 123, 111, 113]). For linear cocycles there is also a notion of *integral separateness* introduced by Cong [36] which is a random version of the notion of integral separateness of linear systems of differential equations and is equivalent to the notion of exponential separation (see Bylov, Vinograd, Grobman and Nemytskii [27] for the case of ordinary differential equations). Although these terms are equivalent in a sense, for linear cocycles we prefer the term *exponential separation* which has a longer history and emphasizes the "separation" of invariant subspaces.
Note that, like exponential dichotomy, in case of bounded cocycle, exponential separation implies boundedness away zero of the angle between invariant subspaces of the splitting (see also Bochi and Viana [22]).

Proposition 2.2.6. *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega),$$

then for any $i = 1, \ldots, k-1$ the angle between $E_i(\omega)$ and $E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$ is bounded away zero by a positive constant independent of $\omega \in \Omega$.

Proof. Put

$$F_i(\omega) := E_1(\omega) \oplus \cdots \oplus E_i(\omega) \quad \text{and} \quad G_{i+1}(\omega) := E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega).$$

Further,

$$M := \operatorname*{ess\,sup}_{\omega \in \Omega} \|A(\omega)\| + \operatorname*{ess\,sup}_{\omega \in \Omega} \|A^{-1}(\omega)\| < \infty. \tag{2.19}$$

Let m be a positive integer such that $2Ke^{-m\alpha} < 1$, where K, α are the constants provided by the definition of exponential separation of A. Let $x \in F_i(\omega)$ and $y \in G_{i+1}(\omega)$ be arbitrary unit vectors. Since

$$\|\Phi_A(m,\omega)x\| \leqslant Ke^{-\alpha m}\|\Phi_A(m,\omega)y\|$$

it follows that

$$2\|\Phi_A(m,\omega)x\| \leqslant \|\Phi_A(m,\omega)y\|.$$

Hence,

$$2\|\Phi_A(m,\omega)x\| \leqslant \|\Phi_A(m,\omega)x\| + \|\Phi_A(m,\omega)(y-x)\|,$$

which implies that

$$\|\Phi_A(m,\omega)x\| \leqslant \|\Phi_A(m,\omega)\|\|y-x\|.$$

Consequently, together with (2.19) we obtain

$$\begin{aligned} \|y-x\| &\geq \|\Phi_A(m,\omega)\|^{-1}\|\Phi_A(m,\omega)^{-1}\|^{-1} \\ &\geq M^{-2m}, \end{aligned}$$

which implies $\measuredangle(x,y) \geq M^{-2m}$. In other words, we have

$$\measuredangle(F_{i-1}(\omega), G_i(\omega)) \geq M^{-2m}. \tag{2.20}$$

Now, let $z \in E_i(\omega)$ and $h \in F_{i-1}(\omega) \oplus G_{i+1}(\omega)$ be arbitrary unit vectors. Then there exist $u \in F_{i-1}(\omega)$ and $v \in G_{i+1}(\omega)$ with $\|u\| = \|v\| = 1$ such that

$$h = a_1 u + a_2 v \qquad \text{for some } a_1, a_2 \in \mathbb{R}.$$

Clearly, $\max\big\{|a_1|, |a_2|\big\} \geq 1/2$. W.l.o.g. we can assume that $|a_1| \geq 1/2$. A direct computation yields that

$$\begin{aligned} \|h-z\| &= |a_1|\|u + \frac{a_2}{a_1}v - \frac{1}{a_1}z\| \\ &= |a_1|\|u+r\|, \end{aligned} \tag{2.21}$$

where $r := \left(\frac{a_2}{a_1}v - \frac{1}{a_1}z\right) \in G_i(\omega)$. Since $u \in F_{i-1}(\omega)$, $\|u\| = 1$ and (2.20), we have

$$|\langle u, r\rangle| \leq \cos\left(M^{-2m}\right)\|r\|.$$

Therefore, we get

$$\begin{aligned} \|u + r\|^2 &= 1 + 2\langle u, r\rangle + \|r\|^2 \\ &\geq 1 - 2\cos\left(M^{-2m}\right)\|r\| + \|r\|^2 \\ &\geq \sin\left(M^{-2m}\right)^2, \end{aligned}$$

which together with (2.21) and the inequality $\sin x \geq \frac{x}{2}$ for small x implies that $\|h - z\| \geq M^{-2m}/4$. Thus, $\measuredangle(E_i(\omega), F_{i-1}(\omega) \oplus G_{i+1}(\omega)) \geq M^{-2m}/4$ and it completes the proof. □

The property of having an exponentially separated splitting is also robust.

Proposition 2.2.7 (Robustness of Exponential Separation of Bounded Linear RDS)**.** *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| < \varepsilon$ also has an exponentially separated splitting. Moreover, for a small $\varepsilon > 0$ the exponentially separated rate and the splitting of the exponential separation B are close to those of A.*

The proof of this proposition is analogous to that of robustness of exponential dichotomy in Theorem 2.2.4 above (see also Bochi and Viana [22]).

Corollary 2.2.8. *If $A \in \mathcal{G}_\infty(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also has an exponentially separated splitting.*

Remark 2.2.9. Corollary 2.2.8 is in fact equivalent to Proposition 2.2.7 because in $\mathcal{G}_\infty(d)$ the distance defined by $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\|$ is equivalent to ρ. To see this, use the formula $B(\omega)^{-1} - A(\omega)^{-1} = B(\omega)^{-1}\big(A(\omega) - B(\omega)\big)A(\omega)^{-1}$.

2.2.3 Exponential Dichotomy is Strictly Stronger than Exponential Separation

Theorem 2.2.10 (Exponential Dichotomy Implies Exponential Separation)**.** *Suppose that $A \in \mathcal{G}(d)$ admits an exponential dichotomy with constants $K > 0, \alpha > 0$ and a family of projections P_ω of $\mathbb{R}^d$. Then the invariant splitting*

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega),$$

where $E_1(\omega) := P_\omega\mathbb{R}^d$ and $E_2(\omega) := (I_d - P_\omega)\mathbb{R}^d$, is exponentially separated.

Proof. As noted in Remark 2.2.1, the spaces $E_1(\omega)$ and $E_2(\omega)$ are invariant with respect to A. For any nonvanishing vectors $x \in E_1(\omega)$ and $y \in E_2(\omega)$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \frac{\|\Phi_A(n,\omega)x\|}{\|x\|} &\leq \|\Phi_A(n,\omega)P_\omega\| \\ &\leq Ke^{-\alpha n} \end{aligned}$$

and

$$\begin{aligned} \frac{\|\Phi_A(n,\omega)y\|}{\|y\|} &\geq \frac{1}{\|(I_d - P_\omega)\Phi_A(n,\omega)^{-1}\|} \\ &\geq \frac{e^{\alpha n}}{K}. \end{aligned}$$

Therefore, the splitting $\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega)$ is exponentially separated. □

The following proposition (based on a one-dimensional example by Cong [35, Proposition 3.2]) provides a bounded linear two-dimensional cocycle which has an exponentially separated splitting but does not exhibit an exponential dichotomy. First we recall from [35] a technical lemma which will be needed later.

Lemma 2.2.11 ([35]). *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic MDS. Suppose that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a non-atomic Lebesgue space. Then there exists a measurable set $U \subset \Omega$ which can be represented in the form*

$$U = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{3k-1} \theta^j U_k, \tag{2.22}$$

where the sets $\theta^j U_k$, $k = 1, \ldots,$ $j = 0, \ldots, 3k-1$, are pairwise disjoint and are all of positive $\mathbb{P}$-measure.

To better understand the above Lemma, let us describe its geometrical meaning in Fig 2.1. The figure points out that in order to construct a desired cocycle for some purposes on the whole space Ω one needs to construct it on two disjoint sets U and $\Omega \setminus U$. An advantage of the structure of the set U is that it is the union of infinitely many sets $\bigcup_{j=0}^{3k-1} \theta^j U_k$ for $k \in \mathbb{N}$ consisting of $3k$ disjoint sets $U_k, \theta U_k, \ldots, \theta^{3k-1} U_k$.

Proposition 2.2.12. *There exists $A \in \mathcal{G}_\infty(2)$ and $\varepsilon \in (0,1)$ such that any cocycle $B \in \mathcal{G}_\infty(2)$ satisfying $\rho(B, A) < \varepsilon$ is hyperbolic and the Oseledets splitting of B is exponentially separated but B exhibits no exponential dichotomy.*

Proof. Let $U \subset \Omega$ be a set with representation (2.22) provided by Lemma 2.2.11. It is easily seen that we may assume $\mathbb{P}(U) < \frac{1}{4}$. We construct a cocycle

$$A(\omega) := \begin{pmatrix} a_1(\omega) & 0 \\ 0 & a_2(\omega) \end{pmatrix} \in \mathcal{G}_\infty(2)$$

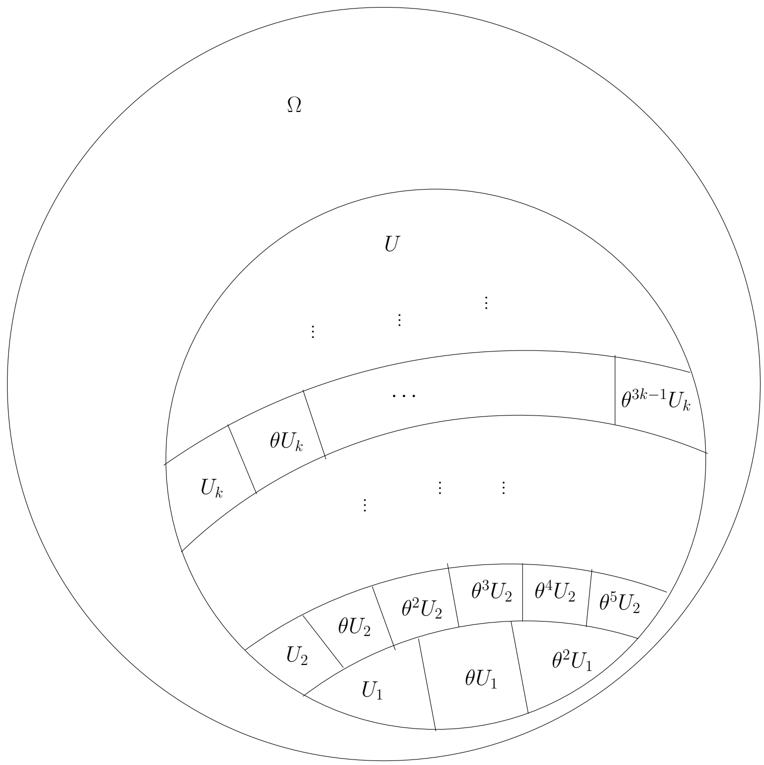

Figure 2.1: Structure of the sets Ω and U

by setting

$$a_1(\omega) = \begin{cases} \frac{1}{4} & \text{for } \omega \in U, \\ 1 & \text{for } \omega \in \Omega \setminus U, \end{cases}$$

$$a_2(\omega) = \begin{cases} \frac{1}{2} & \text{for } \omega \in U, \\ 2 & \text{for } \omega \in \Omega \setminus U. \end{cases}$$

Clearly,

$$\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \tag{2.23}$$

is an invariant and exponentially separated splitting of A. The Lyapunov exponents of A are easily computed and since $\mathbb{P}(U) < 1/4$ we have

$$\lambda_1(A) = \int_\Omega \log |a_1(\omega)| \, d\mathbb{P}(\omega) < 0,$$

$$\lambda_2(A) = \int_\Omega \log |a_2(\omega)| \, d\mathbb{P}(\omega) > 0,$$

hence A is hyperbolic. By Proposition 2.2.7, there is $\varepsilon > 0$ such that any $B \in \mathcal{G}_\infty(2)$ satisfying $\rho(B, A) < \varepsilon$ has an exponentially separated splitting which is close to the splitting (2.23). By making $\varepsilon > 0$ smaller if necessary we can show that B is also hyperbolic. Note that along the orbit segments on U the norm of $A(\omega)$ equals $1/2$, hence, for $\varepsilon < 1/4$, we have $\|B(\omega)\| < 3/4 < 1$. The set U contains arbitrarily long orbit segments and on these segments there are no expanding directions for B. Hence B has no uniformly expanding directions, and thus it cannot exhibit an exponential dichotomy. □

Remark 2.2.13. A higher dimensional example is easily constructed in a similar way. Thus the converse of Theorem 2.2.10 is false, hence exponential dichotomy is strictly stronger than exponential separation.

2.2.4 Exponential Separation of Unbounded Cocycles

For the general case of unbounded cocycles the definition of exponentially separated splitting is more subtle. Subsection 2.2.2 contains the definition of an exponentially separated splitting for bounded cocycles and some important properties concerning with robustness and boundedness away zero of the angles between subspaces. These two properties are no longer automatically satisfied in the unbounded case as will be shown in Proposition 2.2.14. Besides, it is not difficult to construct an unbounded cocycle which has an invariant splitting satisfying the properties stated in Definition 2.2.3 but the angles between subspaces are not bounded away from zero. Therefore, it is reasonable to require additionally an angle condition in the definition of exponential separation for unbounded cocycles. Thus we arrive at the following definition.

Definition 2.2.4 (Exponential Separation)**.** Let $A \in \mathcal{G}(d)$ and

$$\mathbb{R}^d = E_1(\omega) \oplus \cdots \oplus E_k(\omega), \quad k \geq 2, \tag{2.24}$$

be an invariant splitting of A, i.e. for almost all $\omega \in \Omega$ and any $i = 1, \ldots, k$ we have $A(\omega)E_i(\omega) = E_i(\theta\omega)$. The splitting (2.24) is called *exponentially separated* if the following two conditions are satisfied:

(i) there exist numbers $K, \alpha > 0$ such that for each $n \in \mathbb{N}$, $\omega \in \Omega$ and each $i = 1, \ldots, k-1$ the inequality

$$\frac{\|\Phi_A(n,\omega)x\|}{\|x\|} \leq Ke^{-n\alpha} \cdot \frac{\|\Phi_A(n,\omega)y\|}{\|y\|}$$

holds for all $0 \neq x \in E_1(\omega) \oplus \cdots \oplus E_i(\omega)$ and $0 \neq y \in E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$;

(ii) for each $i = 1, \ldots, k-1$, the angle between $E_i(\omega)$ and $E_1(\omega) \oplus \cdots \oplus E_{i-1}(\omega) \oplus E_{i+1}(\omega) \oplus \cdots \oplus E_k(\omega)$ is bounded away zero by a positive constant which is independent of $\omega \in \Omega$.

Note that due to Proposition 2.2.6, Definition 2.2.4 is equivalent to Definition 2.2.3 if $A \in \mathcal{G}_\infty(d)$. If $A \in \mathcal{G}_{IC}(d)$ has an exponentially separated splitting then it has at least two different Lyapunov exponents and its Oseledets splitting is nontrivial. The following proposition shows that the condition (ii) of Definition 2.2.4 is crucial for the robustness of an exponentially separated splitting.

Proposition 2.2.14. *If in Definition 2.2.4 we drop the angle condition (ii), then there exists $A \in \mathcal{G}_{IC}(2)$ such that A has an exponentially separated splitting but in any neighborhood of A there is a cocycle B which has no exponentially separated splitting.*

Proof. By Lemma 2.2.11 we can find a measurable set $U \subset \Omega$ such that

$$U = \bigcup_{k=4}^{\infty} \bigcup_{j=0}^{2k} \theta^j U_k, \tag{2.25}$$

where the sets $\theta^j U_k$, $j = 0, 1, \ldots, 2k$, $k \geq 4$, are measurable, disjoint, have positive measure and $\mathbb{P}(U_k) \leq \frac{1}{4^k}$ for all $k \geq 4$. Moreover, we can choose U such that for any $k \geq 4$ the sets U_k are not coboundaries, i.e. they cannot be represented in the form $U_k = V_k \triangle \theta V_k$ with $V_k \in \mathcal{F}$ (see Knill [79, Corollary 3.5]). We construct a cocycle A satisfying the assertion of the proposition together with the Oseledets splitting of A. Let $\{e_1, e_2\}$ denote the standard Euclidean basis of $\mathbb{R}^2$. We construct a random basis $\{f_1(\omega), f_2(\omega)\}$ of $\mathbb{R}^2$ by setting $f_1(\omega) \equiv e_1$, and for any $k \geq 4$

$$f_2(\omega) = \begin{cases} \cos(\frac{\pi}{2^{i+1}})e_1 + \sin(\frac{\pi}{2^{i+1}})e_2 & \text{if} \quad \omega \in \theta^i U_k,\ i = 0, 1, \ldots k-1, \\ \cos(\frac{\pi}{2^k})e_1 + \sin(\frac{\pi}{2^k})e_2 & \text{if} \quad \omega \in \theta^k U_k, \\ \cos(\frac{\pi}{2^{2k-i}})e_1 + \sin(\frac{\pi}{2^{2k-i}})e_2 & \text{if} \quad \omega \in \theta^i U_k,\ i = k+1, k+2, \ldots 2k-1, \\ e_2 & \text{if} \quad \omega = \theta^{2k}\omega_0,\ \omega_0 \in U_k, \end{cases}$$

and $f_2(\omega) = e_2$ for $\omega \in \Omega \setminus U$. See also Fig 2.2 for the geometrical meaning of the construction of $\{f_1(\omega), f_2(\omega)\}$ on the set $\bigcup_{j=0}^{2k} \theta^j U_k$ for $k \geq 4$.
For the definition of $A \in \mathcal{G}(2)$ we set $A(\omega) = I_2$ on $\bigcup_{k=4}^{\infty} \theta^{k-1} U_k \bigcup \theta^{2k-1} U_k$ and

$$A(\omega) f_1(\omega) = f_1(\theta\omega), \quad A(\omega) f_2(\omega) = \frac{1}{2} f_2(\theta\omega)$$

on $\Omega \setminus \bigcup_{k=4}^{\infty} \theta^{k-1} U_k \cup \theta^{2k-1} U_k$. By construction, $A \in \mathcal{G}_{IC}(2)$ and

$$\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega), \tag{2.26}$$

where $E_1(\omega), E_2(\omega)$ are the subspaces spanned by vectors $f_1(\omega), f_2(\omega)$ respectively, is an exponentially separated splitting of A. Now, let $\varepsilon \in (0, 1)$ be arbitrary. We will show that there exists $B \in \mathcal{G}_{IC}(2)$ such that $\rho(A, B) < \varepsilon$ and B has no exponentially separated splitting. Choose and fix $n \in \mathbb{N}$ such that $2^{n-3}\varepsilon > 1$. We define B by setting

(i) $B(\omega) = \begin{pmatrix} \cos(\frac{\pi}{2^n}) & -\sin(\frac{\pi}{2^n}) \\ \sin(\frac{\pi}{2^n}) & \cos(\frac{\pi}{2^n}) \end{pmatrix} A(\omega) \qquad \text{for } \omega \in \theta^{n-1} U_n,$

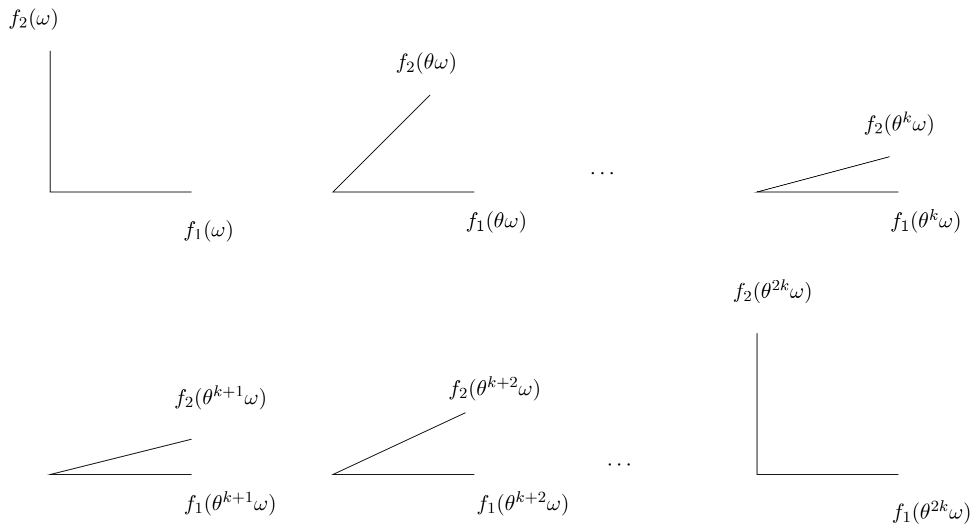

Figure 2.2: The construction of $\{f_1(\omega), f_2(\omega)\}$ on $\bigcup_{j=0}^{2k} \theta^j U_k$

(ii) $B(\omega) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} A(\omega)$ for $\omega \in \theta^{2n-1}U_n$, where $b = \frac{\cos\frac{\pi}{2^n}}{2^{n-1}}$,

(iii) $B(\omega) = A(\omega)$ for $\omega \in \Omega \setminus \big(\theta^{n-1}U_n \cup \theta^{2n-1}U_n\big)$.

By construction, $\rho(A, B) < \varepsilon$ and for any $\omega \in U_n$ we have

$$\Phi_B(2n,\omega)f_1(\omega) \quad \text{is collinear with } e_2,$$
$$\Phi_B(2n,\omega)f_2(\omega) \quad \text{is collinear with } e_1.$$

The set U_n is not a coboundary, $\mathbb{P}(U_n) > 0$, and the sets $U_n, \theta U_n, \dots, \theta^{2n}U_n$ are disjoint. Furthermore, for all $\omega \in U_n$ we have

$$\begin{aligned} \Phi_B(2n,\omega)E_1(\omega) &= E_2(\theta^{2n}\omega), \\ \Phi_B(2n,\omega)E_2(\omega) &= E_1(\theta^{2n}\omega). \end{aligned}$$

and on $\Omega \setminus \big(\bigcup_{i=0}^{2n} \theta^i U_n\big)$ we have $B(\omega) = A(\omega)$. Therefore, by a version of Lemma 4.4 of Knill [79] B has one Lyapunov exponent with multiplicity 2 (see also Bochi [20]), hence the Oseledets splitting of B is trivial and so B has no exponentially separated splitting. □

The following theorem shows that for unbounded cocycles the exponential separation property is also robust, thus gives another justification for the angle condition in the Definition 2.2.4.

Theorem 2.2.15 (Robustness of Exponentially Separated Splitting)**.** *If $A \in \mathcal{G}(d)$ has an exponentially separated splitting then there exists $\varepsilon > 0$ such that any cocycle $B \in \mathcal{G}(d)$ satisfying $\rho(A, B) < \varepsilon$ also has an exponentially separated splitting.*

Proof. By Lemma 2.11 of Cong [36], if A has an exponentially separated splitting then it is cohomologous to a block-diagonal cocycle by a cohomology which is bounded together with its inverse. Therefore, we may assume that A has a block diagonal form. First, we give a proof for the two-dimensional case. Suppose that we have a two-dimensional cocycle

$$A(\omega) = \begin{pmatrix} a_1(\omega) & 0 \\ 0 & a_2(\omega) \end{pmatrix}$$

with the exponentially separated splitting $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. Then there are positive constants K, α such that

$$\prod_{k=0}^{n-1} \|a_2(\theta^k\omega)\| \geq Ke^{\alpha n} \prod_{k=0}^{n-1} \|a_1(\theta^k\omega)\| \qquad \text{for all } \omega \in \Omega, n \in \mathbb{N}. \tag{2.27}$$

We can assume w.l.o.g. that $K < 1$. Put $\beta := \frac{\alpha}{2}$. We construct a diagonal cocycle $\widetilde{A}(\omega) = \begin{pmatrix} \widetilde{a}_1(\omega) & 0 \\ 0 & \widetilde{a}_2(\omega) \end{pmatrix}$ by setting $\widetilde{A}(\omega) = A(\omega)$ in case $|a_2(\omega)| \geq \frac{1}{K}e^\beta$ and $|a_1(\omega)| \leq Ke^{-\beta}$, and $\widetilde{A}(\omega) = \frac{e^\beta}{|a_2(\omega)|}A(\omega)$, i.e.

$$\widetilde{a}_1(\omega) = \frac{e^\beta a_1(\omega)}{|a_2(\omega)|}, \quad \widetilde{a}_2(\omega) = \frac{e^\beta a_2(\omega)}{|a_2(\omega)|}, \tag{2.28}$$

otherwise. We show that $\widetilde{A}$ exhibits an exponential dichotomy. Let $n \in \mathbb{N}$ and $\omega \in \Omega$ be arbitrary. We estimate the product

$$\|\Phi_{\widetilde{A}}(n,\omega)e_1\| = \prod_{k=0}^{n-1} \|\widetilde{a}_1(\theta^k\omega)\| \tag{2.29}$$

in three cases:
Case 1: $\widetilde{a}_1(\omega) = a_1(\omega)$ and $\widetilde{a}_1(\theta^i\omega) = \frac{e^\beta a_1(\theta^i\omega)}{|a_2(\theta^i\omega)|}$ for all $i = 1, \ldots, n-1$: In this case, by (2.27) and (2.28) we have $\widetilde{a}_1(\omega) = a_1(\omega) \leq Ke^{-\beta}$ and

$$\begin{aligned} \|\Phi_{\widetilde{A}}(n,\omega)e_1\| &= \prod_{k=0}^{n-1} \|\widetilde{a}_1(\theta^k\omega)\| \\ &\leq Ke^{-\beta}\frac{1}{Ke^{(n-1)\beta}} \\ &\leq e^{-n\beta}; \end{aligned}$$

Case 2: $\widetilde{a}_1(\theta^i\omega) = a_1(\theta^i\omega)$ for all $i = 0, \ldots, n-1$: In this case, by (2.28) we have $a_1(\theta^i\omega) \leq Ke^{-\beta}$ for all $i = 0, \ldots, n-1$, hence

$$\begin{aligned} \|\Phi_{\widetilde{A}}(n,\omega)e_1\| &= \prod_{k=0}^{n-1} \|\widetilde{a}_1(\theta^k\omega)\| \\ &\leq K^n e^{-n\beta} \\ &\leq e^{-n\beta}; \end{aligned}$$

Case 3: $\widetilde{a}_1(\theta^i\omega) = \frac{e^\beta a_1(\theta^i\omega)}{|a_2(\theta^i\omega)|}$ for all $i = 0, \ldots, n-1$: In this case, by (2.27) and (2.28) we have

$$\begin{aligned}\|\Phi_{\widetilde{A}}(n,\omega)e_1\| &= \prod_{k=0}^{n-1} \|\tilde{a}_1(\theta^k\omega)\| \\ &\leq \frac{1}{K}e^{-n\beta}.\end{aligned}$$

Note that for arbitrary $n \in \mathbb{N}$ and $\omega \in \Omega$ the product (2.29) can be decomposed into product of terms of the three basic types above. Furthermore, Case 3 can occurs possibly only once. Thus, we always have

$$\begin{aligned}\|\Phi_{\widetilde{A}}(n,\omega)e_1\| &= \prod_{k=0}^{n-1} \|\widetilde{a}_1(\theta^k\omega)\| \\ &\leq \frac{1}{K}e^{-n\beta}.\end{aligned}$$

By construction and by (2.28) we always have $|\widetilde{a}_2(\omega)| \geq e^\beta$, hence

$$\prod_{k=0}^{n-1} \|\Phi_{\widetilde{A}}(n,\omega)e_2\| \geq e^{\beta n} \qquad \text{for all } \omega \in \Omega, n \in \mathbb{N}.$$

Therefore, $\widetilde{A}$ exhibits an exponential dichotomy. Consequently, by Theorem 2.2.4 there exists $\delta_1 > 0$ such that any cocycle A' satisfying $\operatorname{ess\,sup}_{\omega\in\Omega} \|A'(\omega) - \widetilde{A}(\omega)\| < \delta_1$ also exhibits an exponential dichotomy. Choose and fix a number $\delta > 0$ which satisfies the following inequalities

$$\delta < \delta_1, \quad \delta < \frac{K^2}{e^\beta}, \quad \text{and } \delta < \frac{K^3\delta_1}{e^{2\beta} + K\delta_1 e^\beta}. \tag{2.30}$$

We show that any cocycle B satisfying $\rho(A, B) < \delta$ has an exponentially separated splitting. Thereto let us construct a cocycle $\widetilde{B}$ by setting $\widetilde{B}(\omega) = B(\omega)$ in case $|a_2(\omega)| \geq \frac{1}{K}e^\beta$ and $|a_1(\omega)| \leq Ke^{-\beta}$, and $\widetilde{B}(\omega) = \frac{e^\beta B(\omega)}{|a_2(\omega)|}$, i.e.

$$\widetilde{b}_1(\omega) = \frac{e^\beta b_1(\omega)}{|a_2(\omega)|}, \quad \widetilde{b}_2(\omega) = \frac{e^\beta b_2(\omega)}{|a_2(\omega)|},$$

otherwise. We will estimate $\|\widetilde{B}(\omega) - \widetilde{A}(\omega)\|$. There are two cases:

Case 1: $|a_2(\omega)| \geq \frac{1}{K}e^\beta$: If $|a_1(\omega)| \leq Ke^{-\beta}$ we have

$$\|\widetilde{B}(\omega) - \widetilde{A}(\omega)\| = \|B(\omega) - A(\omega)\| \leq \delta < \delta_1.$$

If $|a_1(\omega)| \geq Ke^{-\beta}$ then from the definition of $\widetilde{A}$ and $\widetilde{B}$ we have

$$\|\widetilde{B}(\omega) - \widetilde{A}(\omega)\| = \frac{e^\beta}{|a_2(\omega)|}\|B(\omega) - A(\omega)\| \leq K\delta < \delta_1.$$

Case 2: $|a_2(\omega)| < \frac{1}{K}e^{\beta}$: From (2.27) we get $|a_1(\omega)| \le \frac{1}{K}|a_2(\omega)|$, hence $\|A(\omega)\| \le \frac{1}{K}|a_2(\omega)| < \frac{e^{\beta}}{K^2}$. From the definition of $\widetilde{B}$ and $\widetilde{A}$ we have

$$\|\widetilde{B}(\omega) - \widetilde{A}(\omega)\| = \frac{e^{\beta}}{|a_2(\omega)|}\|B(\omega) - A(\omega)\|. \tag{2.31}$$

Setting $C(\omega) = A^{-1}(\omega) - B^{-1}(\omega)$, we see that $\|C(\omega)\| \le \rho(A, B) \le \delta$, and

$$B^{-1}(\omega) = A^{-1}(\omega)\big(I_d - A(\omega)C(\omega)\big).$$

Since $\|A(\omega)C(\omega)\| < \frac{e^{\beta}}{K^2} \cdot \delta := \delta_2 < 1$, the matrix $I_d - A(\omega)C(\omega)$ is invertible and $B(\omega) = \big(I_d - A(\omega)C(\omega)\big)^{-1}A(\omega)$. Put $D(\omega) := A(\omega)C(\omega)$. We have

$$B(\omega) = \big(I_d + D(\omega) + D(\omega)^2 \cdots\big)A(\omega),$$

which implies that

$$B(\omega) - A(\omega) = \big(D(\omega) + D(\omega)^2 + \cdots\big)A(\omega).$$

Therefore, we get

$$\begin{aligned} \|B(\omega) - A(\omega)\| &\le (\delta_2 + \delta_2^2 + \cdots)\|A(\omega)\| \\ &\le \frac{\delta_2}{1-\delta_2} \cdot \frac{1}{K}|a_2(\omega)|, \end{aligned}$$

which together with (2.31) implies that

$$\begin{aligned} \|\widetilde{B}(\omega) - \widetilde{A}(\omega)\| &\le \frac{e^{\beta}}{K} \cdot \frac{\delta_2}{1-\delta_2} \\ &\le \delta_1, \end{aligned}$$

where we use (2.30) to obtain the last inequality. Thus, in any case, by the choice of δ_1, the cocycle $\widetilde{B}$ exhibits an exponential dichotomy, hence has an exponentially separated splitting. Since B differs from $\widetilde{B}$ only by a scalar multiplier it follows that B has the same exponentially separated splitting as $\widetilde{B}$. The theorem is proved in this two-dimensional case.

The general d-dimensional case is similar to the two-dimensional case treated above. We list here the changes necessary for transition from the two-dimensional to the d-dimensional case with the splitting consisting of two subspaces: instead of scalars (one-dimensional matrices) $a_1(\omega)$, $a_2(\omega)$ we have to deal with matrices $a_1(\omega)$, $a_2(\omega)$ (of higher order, in general); the absolute values $|a_1(\omega)|$ should be changed to the matrix norm $\|a_1(\omega)\|$ and the absolute value $|a_2(\omega)|$ should be changed to the matrix co-norm $m(a_2(\omega)) := \|a_2(\omega)^{-1}\|^{-1}$ (cf. Bochi and Viana [23]); the product $\prod_{j=0}^{n-1}|a_1(\theta^j\omega)|$ should be changed to the norm $\|\prod_{j=0}^{n-1}a_1(\theta^j\omega)\|$ and the product $\prod_{j=0}^{n-1}|a_2(\theta^j\omega)|$ should be changed to the co-norm $m(\prod_{j=0}^{n-1}a_2(\theta^j\omega))$. The case of the splitting consisting of more than two subspaces can be easily deduced from the case of two subspaces. □

From the proof of Theorem 2.2.15 above we can see that for small $\varepsilon > 0$ the exponentially separated splitting of B is close to that of A (exponentially separated splitting varies continuously in $(\mathcal{G}(d), \rho)$). Although in $\mathcal{G}_\infty(d)$ the exponential separation is robust in the sup-norm as stated in Proposition 2.2.7, in the unbounded case this is no longer true (this is already indicated by the essential use of the smallness of $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega)^{-1} - B(\omega)^{-1}\|$ in the proof of Theorem 2.2.15).

Proposition 2.2.16. *There exists $A \in \mathcal{G}_{IC}(2)$ with exponentially separated splitting such that for any $\varepsilon > 0$ one can find a cocycle $B \in \mathcal{G}(2)$ which has no exponentially separated splitting and satisfies $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\| < \varepsilon$.*

Proof. By Lemma 2.2.11 we can find a measurable set F which can be represented in the form

$$F = \bigcup_{k=4}^{\infty} \bigcup_{j=0}^{k-1} \theta^j U_k,$$

where the sets $\theta^j U_k$, $k \geq 4$, $0 \leqslant j \leqslant k-1$, are pairwise disjoint and are all of positive measure. We can assume additionally that the sets U_k satisfy the inequality

$$\sum_{k=4}^{\infty} k^2 \mathbb{P}(U_k) \leqslant 1. \tag{2.32}$$

Define a cocycle $A \in \mathcal{G}(2)$ by

$$A(\omega) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{2k} \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=0}^{k-1} \theta^j U_k, \quad k \geq 4. \end{cases}$$

It is easily seen that $A \in \mathcal{G}_{IC}(2)$. For arbitrary $\varepsilon > 0$ we choose and fix $n \in \mathbb{N}$ such that $n \geq 4\varepsilon^{-1}$. Define $B \in \mathcal{G}_{IC}(2)$ by setting

$$B(\omega) = \begin{pmatrix} \frac{1}{2k} & 0 \\ 0 & \frac{1}{2k} \end{pmatrix}$$

for $\omega \in \cup_{j=0}^{k-1} \theta^j U_k$, $k \geq n$, and $B(\omega) = A(\omega)$ for other $\omega \in \Omega$. It is easily seen that B furnishes the assertions of the proposition. □

2.3 An Open Set of Cocycles with Simple Lyapunov Spectrum but no Exponentially Separated Splitting

In this section we will construct an open set of cocycles such that each cocycle in this set has simple Lyapunov spectrum but has no exponentially separated splitting.

Moreover, the Lyapunov exponents considered as function of cocycles are continuous in this set. This is a distinguished feature of unbounded cocycles since in the bounded case continuity of all Lyapunov exponents implies exponential separation of the Oseledets splitting (see Bochi [21] and Bochi and Viana [22, 23]). We will construct a cocycle $A_0 \in \mathcal{G}_{IC}(2)$ such that a neighborhood of it will have the properties claimed in the title of the section. First, by Lemma 2.2.11 we can find a measurable set $U = \bigcup_{k=4}^{\infty} \bigcup_{j=0}^{3k-1} \theta^j U_k$ such that the sets $\theta^j U_k$, $0 \le j \le 3k-1$, $k \ge 4$, are pairwise disjoint, measurable and of positive measure. Denote the probability of U_k by x_k for all $k \ge 4$. We can assume additionally that the sets U_k satisfy also the condition

$$\sum_{k=4}^{\infty} k^2 x_k \le \frac{1}{4}. \tag{2.33}$$

Set $F := \bigcup_{k=4}^{\infty} \bigcup_{j=k}^{2k-1} \theta^j U_k$, we get $\mathbb{P}(F) = \sum_{k=4}^{\infty} k x_k \le \frac{1}{16}$. Now, the cocycle $A_0 \in \mathcal{G}(2)$ is constructed as follows:

$$A_0(\omega) = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \begin{pmatrix} k+1 & 0 \\ 0 & k \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \ge 4. \end{cases}$$

From (2.33) it follows that $A_0 \in \mathcal{G}_{IC}(2)$. Clearly, $A_0 \notin \mathcal{G}_{\infty}(2)$. Denote by $\{e_1, e_2\}$ the standard Euclidean basis of $\mathbb{R}^2$. It is easy to see that

$$\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \tag{2.34}$$

is the Oseledets splitting of the cocycle A_0. Using the Birkhoff theorem (see Appendix A) we can compute and estimate the Lyapunov exponents of A_0 as follows

$$\begin{aligned} \lambda_1(A_0) &= \int_{\Omega} \log \|A_0(\omega)e_1\| \, d\mathbb{P}(\omega) \\ &= \sum_{k=4}^{\infty} k x_k \log(k+1) + (1 - \mathbb{P}(F)) \log 2 \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} \lambda_2(A_0) &= \int_{\Omega} \log \|A_0(\omega)e_2\| \, d\mathbb{P}(\omega) \\ &= \sum_{k=4}^{\infty} k x_k \log k - (1 - \mathbb{P}(F)) \log 2 \\ &< \frac{1}{4} - \frac{3\log 2}{4} \\ &< 0. \end{aligned}$$

Hence the cocycle A_0 is hyperbolic and has simple Lyapunov spectrum. However, A_0 has no exponentially separated splitting because otherwise the exponentially separated splitting must be the Oseledets splitting (2.34) but for fixed positive numbers α, K we can find $n > 4$ such that $(n+1)^n < Ke^{\alpha n} n^n$. We also define a cocycle $\widehat{A}$ by setting

$$\widehat{A}(\omega) := \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \text{for} \quad \omega \in \Omega \setminus F, \\ \begin{pmatrix} 1+\frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geq 4. \end{cases}$$

Proposition 2.3.1. *There exists a positive number δ such that any cocycle $B \in \mathcal{G}_{IC}(2)$ satisfying $\rho(A_0, B) < \delta$ has simple Lyapunov spectrum. Moreover, the functions $\lambda_i(\cdot) : \mathcal{G}_{IC}(2) \to \mathbb{R}$, $B \mapsto \lambda_i(B)$, $i = 1, 2$, are continuous on the ball centered at A_0 with radius δ in $(\mathcal{G}_{IC}(2), \rho)$.*

Proof. We choose and fix a positive number δ satisfying

$$\delta < \frac{1}{40}, \quad \delta < \frac{1}{3} \sum_{k=4}^{\infty} k x_k \log(1 + \frac{1}{k}).$$

Let $B \in \mathcal{G}_{IC}(2)$ be an arbitrary cocycle satisfying $\rho(A_0, B) < \delta/2$. Setting

$$\widehat{B}(\omega) := \begin{cases} B(\omega) & \text{for} \quad \omega \in \Omega \setminus F, \\ \frac{1}{k} B(\omega) & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geq 4, \end{cases}$$

and $\widehat{C}(\omega) := \widehat{B}(\omega) - \widehat{A}(\omega)$, we have

$$\begin{aligned} \|\widehat{C}(\omega)\| &\leq \delta \quad \text{if} \quad \omega \in \Omega \setminus F, \\ \|\widehat{C}(\omega)\| &\leq \frac{1}{k} \delta \quad \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geq 4. \end{aligned}$$

Define a random projector P_ω of $\mathbb{R}^2$ by setting $P_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for all $\omega \in \Omega$. We state the following claim.

Claim. *For any $\omega \in \Omega$ and any $n \in \mathbb{N}$ the following inequalities hold*

$$\sum_{k=0}^{n-1} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\| \leq (7 + \sqrt{2})\delta, \tag{2.35}$$

$$\sum_{k=n}^{\infty} \|\Phi_{\widehat{A}}^{-1}(k+1-n, \theta^n \omega)(I_d - P_{\theta^{k+1}\omega}) \widehat{C}(\theta^k \omega)\| \leq (7 + \sqrt{2})\delta. \tag{2.36}$$

To prove the claim we set

$$E_m := U_m \cup \theta U_m \cdots \cup \theta^{3m-2} U_m, \quad E := \bigcup_{m=4}^{\infty} E_m.$$

Note that $\|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega}\| = \prod_{j=k+1}^{n-1} |\widehat{a}_2(\theta^j \omega)|$, where

$$\widehat{a}_2(\omega) := \begin{cases} \frac{1}{2} & \text{for} \quad \omega \in \Omega \setminus F, \\ 1 & \text{for} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geq 4. \end{cases}$$

Therefore, by the construction of U and E it follows that if $\theta^{n-1}\omega \notin E$ then

$$\|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega}\| \leq \left(\frac{1}{\sqrt{2}}\right)^{n-k-1} \tag{2.37}$$

for all integers $0 \leq k \leq n-1$. Now, back to (2.35) we see that there are two cases: either $\theta^{n-1}\omega \in E$ or not. If $\theta^{n-1}\omega \notin E$, then by (2.37) we have

$$\begin{aligned} \sum_{k=0}^{n-1} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \hat{C}(\theta^k \omega)\| &\leq \delta \cdot \sum_{k=0}^{n-1} \left(\frac{1}{\sqrt{2}}\right)^{n-k-1} \\ &\leq (2+\sqrt{2})\delta, \end{aligned} \tag{2.38}$$

which proves (2.35). If $\theta^{n-1}\omega \in E$ then $\theta^{n-1}\omega \in \theta^h U_m$ for some $m \geq 4$ and $0 \leq h < 3m-1$. In this case $\theta^{n-1-h-1}\omega \notin E$, hence using (2.38) and the fact that $\|\Phi_{\widehat{A}}(r, \omega) P_\omega\| \leq 1$ for all $r \geq 0$ and all $\omega \in \Omega$, we have

$$\begin{aligned} \sum_{k=0}^{n-h-2} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\| &\leq \sum_{k=0}^{n-h-2} \|\Phi_{\widehat{A}}(n-h-k-2, \theta^{k+1}\omega) \cdot \\ &\qquad P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\| \\ &\leq (2+\sqrt{2})\delta. \end{aligned} \tag{2.39}$$

To estimate the term $\sum_{k=n-h-1}^{n-1} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\|$ we notice that

$$\sum_{k=n-h-1}^{n-1} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\| \leq \sum_{k=n-h-1}^{n-1} \|\widehat{C}(\theta^k \omega)\| \prod_{j=k+1}^{n-1} |\widehat{a}_2(\theta^j \omega)|.$$

By considering three possible cases $0 \leq h \leq m-1$, $m \leq h \leq 2m-1$ and $2m \leq h < 3m-1$, and using the definition of $\widehat{A}$ and $\widehat{C}$ (remember that on $\theta^h U_m$ with $m \leq h \leq 2m-1$ we have $\|\widehat{C}(\omega)\| \leq \delta/m$), one can show that

$$\sum_{k=n-h-1}^{n-1} \|\Phi_{\widehat{A}}(n-k-1, \theta^{k+1}\omega) P_{\theta^{k+1}\omega} \widehat{C}(\theta^k \omega)\| \leq 5\delta, \tag{2.40}$$

which together with (2.39) implies inequality (2.35). Inequality (2.36) can be proved in a similar way. Thus, the claim is proved.
Now, let **B** denote the Banach space of all bounded matrix-valued functions $f : \mathbb{N} \longrightarrow \mathbb{R}^{2\times 2}$, where $\mathbb{R}^{2\times 2}$ is the space of all two-by-two matrices with matrix norm, with the norm

$$\|f\|_{\mathbf{B}} = \sup_{n\in\mathbb{N}} \|f(n)\|.$$

For every $\omega \in \Omega$ we define a mapping $T_\omega : \mathbf{B} \to \mathbf{B}$ by

$$\begin{aligned} (T_\omega f)(n) &= \Phi_{\widehat{A}}(n,\omega)P_\omega + \sum_{k=0}^{n-1} \Phi_{\widehat{A}}(n,\omega)P_\omega \Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)f(k) \\ &\quad - \sum_{k=n}^{\infty} \Phi_{\widehat{A}}(n,\omega)(I_d - P_\omega)\Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)f(k). \end{aligned}$$

By the definition of $\widehat{A}$ and P_ω we have $\|\Phi_{\widehat{A}}(n,\omega)P_\omega\| \le 1$ for all $n \in \mathbb{N}, \omega \in \Omega$. Therefore, due to (2.35)–(2.36) the mapping T_ω is well-defined and depends measurably on $\omega \in \Omega$. Moreover, for every $f_1, f_2 \in \mathbf{B}$ we have

$$\begin{aligned} \|T_\omega f_1 - T_\omega f_2\|_{\mathbf{B}} &\le (14 + 2\sqrt{2})\delta \|f_1 - f_2\|_{\mathbf{B}} \\ &< \frac{1}{2}\|f_1 - f_2\|_{\mathbf{B}}, \end{aligned}$$

hence T_ω is a contraction mapping for all $\omega \in \Omega$. By the contraction principle, the mapping T_ω has a unique fixed point which depends measurably on $\omega \in \Omega$, too. Denoting this point by Y_ω, we have

$$\begin{aligned} Y_\omega(n) &= \Phi_{\widehat{A}}(n,\omega)P_\omega + \sum_{k=0}^{n-1} \Phi_{\widehat{A}}(n,\omega)P_\omega \Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)Y_\omega(k) - \\ &\quad - \sum_{k=n}^{\infty} \Phi_{\widehat{A}}(n,\omega)(I_d - P_\omega)\Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)Y_\omega(k). \end{aligned} \tag{2.41}$$

From this formula we derive $Y_\omega(n+1) = \widehat{B}(\theta^n\omega)Y_\omega(n)$. Since $Y_\omega(n)P_\omega$ is also a fixed point of T_ω we get $Y_\omega(n)P_\omega = Y_\omega(n)$. Put $Q_\omega := Y_\omega(0)$ then $Q_\omega P_\omega = Q_\omega$. Letting $n = 0$ in equality (2.41), we obtain

$$Q_\omega = Y_\omega(0) = P_\omega - \sum_{k=0}^{\infty} (I_d - P_\omega)\Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)Y_\omega(k), \tag{2.42}$$

which gives that $P_\omega Q_\omega = P_\omega$. Thus, $Y_\omega Q_\omega$ satisfies (2.41), hence it is also a fixed point of T_ω. Consequently, $Y_\omega Q_\omega = Y_\omega$, so $Q_\omega^2 = Q_\omega$ and Q_ω is a random projector. Set

$$M_\omega := \|Y_\omega\|_{\mathbf{B}} = \sup_{n\in\mathbb{N}} \|Y_\omega(n)\| = \sup_{n\in\mathbb{N}} \|\Phi_{\widehat{B}}(n,\omega)Q_\omega\|.$$

Using (2.41), for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\|Y_\omega(n)\| &\le \|\Phi_{\widehat{A}}(n,\omega)P_\omega\| + \sum_{k=0}^{n-1}\|\Phi_{\widehat{A}}(n,\omega)P_\omega\Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)Y_\omega(k)\| + \\ &\qquad + \sum_{k=n}^{\infty}\|\Phi_{\widehat{A}}(n,\omega)(I_d-P_\omega)\Phi_{\widehat{A}}(k+1,\omega)^{-1}\widehat{C}(\theta^k\omega)Y_\omega(k)\| \\ &\le 1+(14+2\sqrt{2})\delta M_\omega,\end{aligned}$$

which leads that $M_\omega \le 1+(14+2\sqrt{2})\delta M_\omega$. Equivalently, we have

$$M_\omega \le (1-(14+2\sqrt{2})\delta)^{-1} < 2. \tag{2.43}$$

Now we will show that the cocycle $\widehat{B}$ has simple Lyapunov spectrum. For this purpose, let f be a unit vector in the space $\mathrm{Im}Q_\omega$ (it exists because $Q_\omega \neq 0$), then for all $n \in \mathbb{N}$ we get

$$\frac{1}{n}\log\|\Phi_{\widehat{B}}(n,\omega)f\| = \frac{1}{n}\log\|\Phi_{\widehat{B}}(n,\omega)Q_\omega f\| = \frac{1}{n}\log\|Y_\omega(n)f\| \le \frac{\log M_\omega}{n} \le \frac{\log 2}{n},$$

where we use (2.43) to obtain the last inequality. As a consequence, the linear cocycle $\widehat{B}$ has a negative Lyapunov exponent. On the other hand, by the construction of the cocycle $\widehat{A}$ we have

$$\det\widehat{A}(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in \Omega\setminus F, \\ 1+\frac{1}{k} & \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1}\theta^j U_k, \quad k \ge 4.\end{cases}$$

Since $\|\widehat{B}(\omega)-\widehat{A}(\omega)\| \le \delta$ it follows that

$$|\log\det\widehat{B}(\omega) - \log\det\widehat{A}(\omega)| \le 3\delta.$$

Hence, an elementary computation yields that

$$\begin{aligned}\int_\Omega \log\det\widehat{B}(\omega)\,\mathbb{P}d\omega &\ge \sum_{k=4}^{\infty} kx_k\log(1+\frac{1}{k}) - 3\delta \\ &> 0,\end{aligned}$$

which together with an application of Theorem 1.3.3 implies that the linear cocycle $\widehat{B}$ has a positive Lyapunov exponent. Thus, the linear cocycle $\widehat{B}$ has simple Lyapunov spectrum and $\mathrm{im}Q_\omega$ is the Oseledets subspace corresponding to the negative Lyapunov exponent of $\widehat{B}$. Next we estimate the difference $\|Q_\omega - P_\omega\|$. Combining (2.36) and (2.42), we obtain

$$\begin{aligned}\|Q_\omega - P_\omega\| &\le M_\omega\sum_{k=0}^{\infty}\|(I-P_\omega)\Phi_{\hat{A}}(k+1,\omega)\hat{C}(\theta^k\omega)\| \\ &\le M_\omega(7+\sqrt{2})\delta \\ &< 17\delta.\end{aligned}$$

Since B differs from $\widehat{B}$ only by a scalar multiplier (scalar function), the linear cocycle B also has simple Lyapunov spectrum and has the same Oseledets splitting as $\widehat{B}$. Denote by $\lambda_1(B) > \lambda_2(B)$ the Lyapunov exponents and by $\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega)$ the Oseledets splitting of B. As is proved above, we get $E_2(\omega) = \text{im}Q_\omega$. Choose measurably a unit vector $f_2(\omega)$ in the space $E_2(\omega)$ then

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_B(n,\omega) f_2(\omega)\| = \lambda_2(B).$$

The unit measurable vector $f_2(\omega)$ can be given of the form $f_2(\omega) = \alpha(\omega)e_1 + \beta(\omega)e_2$, where $\alpha, \beta : \Omega \to \mathbb{R}$ are measurable and $\alpha(\omega)^2 + \beta(\omega)^2 = 1$. Set $x(\omega) := \|A_0(\omega)e_2\|$ and $y(\omega) := \|B(\omega)f_2(\omega)\|$. Using the Birkhoff theorem (see Appendix A), we have

$$\lambda_2(A_0) = \int_\Omega \log x(\omega)\, d\mathbb{P}(\omega), \quad \lambda_2(B) = \int_\Omega \log y(\omega)\, d\mathbb{P}(\omega).$$

By the construction of A_0 we have

$$x(\omega) = \begin{cases} \frac{1}{2} & \text{if} \quad \omega \in \Omega \setminus F, \\ k & \text{if} \quad \omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k, \quad k \geq 4. \end{cases}$$

Therefore,

$$\lambda_2(A_0) = \sum_{k=4}^{\infty} k x_k \log k - (1 - \mathbb{P}(F)) \log 2.$$

Note that $P_\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\|Q_\omega - P_\omega\| \leq 17\delta$ it follows that

$$|\alpha(\omega)| \leq \|P_\omega - Q_\omega\| \leq 17\delta, \qquad \beta(\omega) \geq \sqrt{1 - (17\delta)^2},$$

which implies together with $\delta < \frac{1}{40}$ that $\|f_2(\omega) - e_2\| < 30\delta$. Since $|x(\omega) - y(\omega)| \leq \|A_0(\omega)e_2 - B(\omega)f_2(\omega)\|$, for every $\omega \in \Omega$ we have

$$\begin{aligned} |x(\omega) - y(\omega)| &\leq \|A_0(\omega)e_2 - A_0(\omega)f_2(\omega)\| + \|A_0(\omega)f_2(\omega) - B(\omega)f_2(\omega)\| \\ &\leq \|A_0(\omega)\| \|e_2 - f_2(\omega)\| + \delta \\ &\leq (30\|A_0(\omega)\| + 1)\delta. \end{aligned}$$

Consequently,

1. For $\omega \in \Omega \setminus F$ we have $\|A_0(\omega)\| = 2$, $|x(\omega)| = \frac{1}{2}$, and

$$\frac{1}{2} - 60\delta - \delta \leq y(\omega) \leq \frac{1}{2} + 60\delta + \delta.$$

2. For $\omega \in \bigcup_{j=k}^{2k-1} \theta^j U_k$, $k \geq 4$, we have $\|A_0(\omega)\| = k + 1$, $|x(\omega)| = k$, and

$$k - 30(k+1)\delta - \delta \leq y(\omega) \leq k + 30(k+1)\delta + \delta.$$

Therefore, we obtain

$$\lambda_2(B) \geq (1-\mathbb{P}(F))\log(\frac{1}{2}-61\delta)+\sum_{k=4}^{\infty} kx_k \log(k-(30k+31)\delta),$$

and

$$\lambda_2(B) \leq (1-\mathbb{P}(F))\log(\frac{1}{2}+61\delta)+\sum_{k=4}^{\infty} kx_k \log(k+(30k+31)\delta).$$

From these inequalities, using the fact that for any $a>0$, $0<x<a/4$ the inequalities $\log(a+x)<\log a+x/a$ and $\log(a-x)>\log a-2x/a$ hold, we get for any $\delta<1/500$

$$\lambda_2(A_0)-2(122+3)\delta \leq \lambda_2(B) \leq \lambda_2(A_0)+(122+3)\delta.$$

It implies that the Lyapunov exponent $\lambda_2(\cdot)$ is continuous at A_0. Now, note that if we have another cocycle B' which also satisfies $\rho(A_0,B')<\delta$, and B' is close to B, then B' also has simple Lyapunov spectrum and the corresponding random projector Q'_ω of B' (onto its stable subspace) is close to the above random projector Q_ω of B. By the same arguments as that for proving $\lambda_2(B)$ is close to $\lambda_2(A_0)$ above, we can show that $\lambda_2(B')$ is close to $\lambda_2(B)$, hence $\lambda_2(\cdot)$ is continuous at B. The continuity of $\lambda_2(\cdot)$ implies the continuity of $\lambda_1(\cdot)$ because they add up to the exponent of the determinant, but the exponent of the determinant is easily seen to be continuous in $(\mathcal{G}_{IC}(2),\rho)$. □

Theorem 2.3.2. *There exist $A\in\mathcal{G}_{IC}(2)$ and $\varepsilon>0$ such that every cocycle $B\in\mathcal{G}_{IC}(2)$ satisfying $\rho(A,B)<\varepsilon$ has simple Lyapunov spectrum but has no exponentially separated splitting. Moreover, the functions $\lambda_i(\cdot):\mathcal{G}_{IC}(2)\to\mathbb{R}$, $B\mapsto\lambda_i(B)$, $i=1,2$, are continuous on the ball centered at A with radius ε in $(\mathcal{G}_{IC}(2),\rho)$.*

Proof. Take $A=A_0$ and $\varepsilon=\delta$ with A_0 and δ provided by Proposition 2.3.1. Due to Proposition 2.3.1, it remains only to prove that any $B\in\mathcal{G}_{IC}(2)$ satisfying $\rho(A,B)<\varepsilon$ has no exponentially separated splitting. Indeed, for any fixed positive numbers α,K we can find $n>4$ such that $(n+2)^n<Ke^{\alpha n}(n-1)^n$ and, as in the proof of no exponential separation of A_0 above, we see that B has no exponentially separated splitting. □

Note that Theorem 2.3.2 can be easily generalized to the d-dimensional case.

Chapter 3

Generic Properties of Lyapunov Exponents of Linear Random Differential Equations

In the early eighties, Millionshchikov has investigated some generic properties of the Lyapunov exponents of linear deterministic systems. He has proved that Lyapunov exponents in such a set-up are Baire functions of the second class (see Millionshchikov [99], [100] and [101]).

Mañé [92] has studied the deterministic multiplicative ergodic theorem from a generic viewpoint, i.e. for a generic C^1 diffeomorphism on a compact Riemannian manifold, and a generic invariant measure.

In the paper [4] by Arnold and Cong, the generic properties of Lyapunov exponents of linear cocycles generated by products of random matrices are investigated. More precisely, the authors show that the top Lyapunov exponent is upper semi-continuous and the smallest Lyapunov exponent is lower semi-continuous. Furthermore, on the one hand all the repeated Lyapunov exponents are Baire functions of the first class and on the other hand all Lyapunov exponents are of the second Baire class.

This chapter is devoted to the study of the Baire class functions of Lyapunov exponents of linear RDE which is an important generator of RDS on $\mathbb{R}$ (see Subsection 1.2.2). Note that in the conclusion of Arnold and Cong [4] the authors also mentioned that their results about the generic properties of Lyapuonv exponents can be extended to continuous time. However, our situation here is different. Instead of investigating the dependence of Lyapunov exponents of RDS with $\mathbb{T} = \mathbb{R}$ on some perturbations of the RDS (see Arnold and Cong [4]), we start from a linear RDE and thus deal with the dependence of Lyapunov exponents on some perturbations of the "vector field". The structure of this chapter is as follows: In Section 3.1, we introduce the space of linear RDE and the generic properties of Lyapunov exponents of linear RDS are discussed in Section 3.2.

3.1 Spaces of Linear Random Differential Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic Lebesgue probability space, and $(\theta_t)_{t\in\mathbb{R}}$ an ergodic flow of $(\Omega, \mathcal{F}, \mathbb{P})$ preserving the probability measure $\mathbb{P}$. We consider a linear pathwise RDE

$$\dot{x}_t = A(\theta_t\omega)x_t, \tag{3.1}$$

where $A \in \mathcal{L}^1(\mathbb{P})$. By virtue of Theorem 1.2.1, equation (3.1) generates a linear RDS Φ_A satisfying

$$\Phi_A(t,\omega) = I_d + \int_0^t A(\theta_s\omega)\Phi_A(s,\omega)\, ds. \tag{3.2}$$

Also, differentiating $\Phi(t,\omega)\Phi(t,\omega)^{-1} = I_d$ yields that

$$\Phi_A(t,\omega)^{-1} = I_d - \int_0^t \Phi_A(s,\omega)^{-1}A(\theta_s\omega)\, ds. \tag{3.3}$$

The following lemma ensures that the integrability of A implies the integrability condition for the linear RDS Φ_A. A proof can be found in Arnold [3, pp. 159]. However, for sake of completeness we present a short proof of the result.

Lemma 3.1.1. *If $A \in \mathcal{L}^1(\mathbb{P})$ then the corresponding linear RDS Φ_A satisfies the integrability condition in Theorem 1.3.5, i.e. $\alpha^+ \in \mathcal{L}^1(\mathbb{P})$ and $\alpha^- \in \mathcal{L}^1(\mathbb{P})$, where*

$$\alpha^{\pm}(\omega) := \sup_{0\le t\le 1} \log^+ \|\Phi_A(t,\omega)^{\pm 1}\|.$$

Proof. Since the linear RDS Φ_A satisfies (3.2) and (3.3) it follows that

$$\|\Phi_A(t,\omega)^{\pm 1}\| \le 1 + \int_0^t \|A(\theta_s\omega)\|\|\Phi_A(s,\omega)^{\pm 1}\|\, ds \qquad \text{for all } t \ge 0.$$

Using the Gronwall inequality (see Aulbach and Wanner [8]), we obtain

$$\|\Phi_A(t,\omega)^{\pm 1}\| \le \exp \int_0^t \|A(\theta_s\omega)\|\, ds.$$

Consequently,

$$\alpha^{\pm}(\omega) \le \int_0^1 \|A(\theta_s\omega)\|\, ds. \tag{3.4}$$

On the other hand, by the Fubini theorem, noting that $\mathbb{E}\|A(\theta_t\cdot)\| = m < \infty$, we get

$$\int_0^1 \mathbb{E}\|A(\theta_t\cdot)\|\, dt = m = \mathbb{E}\int_0^1 \|A(\theta_t\cdot)\|\, dt < \infty.$$

Together with (3.4) this concludes the proof. ∎

3.2 Generic Properties of Lyapunov Exponents of Linear Random Differential Equations

In this section, we consider the space of all linear pathwise RDE

$$\dot{x}_t = A(\theta_t\omega)x_t, \tag{3.5}$$

where $A \in \mathcal{L}^\infty(\mathbb{P})$, i.e. $\operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega)\| < \infty$. For each $A \in \mathcal{L}^\infty(\mathbb{P})$, Lemma 3.1.1 implies that the RDS Φ_A generated by (3.1) satisfies the integrability condition of Theorem 1.3.5. Let $\{(\lambda_i(A), d_i(A)) : i = 1, \dots, p(A)\}$ denote the Lyapunov spectrum and

$$\mathbb{R}^d = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_p(\omega),$$

the Oseledet splitting of Φ_A. We rewrite the Lyapunov exponents of Φ_A in the following sequence counting also their multiplicities

$$\chi_d(A) \le \chi_{d-1}(A) \le \cdots \le \chi_1(A),$$

where the Lyapunov exponent $\lambda_i(A)$ appears $d_i(A)$ times, $i = 1, 2, \dots, p(A)$. We call $\chi_1(A), \chi_2(A), \dots, \chi_d(A)$ the *repeated Lyapunov exponents* of Φ_A. For each $A \in \mathcal{G}_{IC}(d)$, there exists a one-to-one maping between the Lyapunov spectrum of Φ_A, $\{(\lambda_i(A), d_i(A)) : i = 1, \dots, p(A)\}$, and the repeated Lyapunov exponents of Φ_A, $\{\chi_1(A), \dots, \chi_d(A)\}$, but the dependence of Lyapunov exponents $\lambda_1(A), \dots, \lambda_{p(A)}(A)$ and repeated Lyapunov exponents $\chi_1(A), \chi_2(A), \dots, \chi_d(A)$ on A can be quite different as the results of this chapter show. Notice that $\chi_1(A) = \lambda_1(A)$ and $\chi_d(A) = \lambda_{p(A)}(A)$.
We define

$$\delta(A, B) = \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\|$$

and endow the space $\mathcal{L}^\infty(\mathbb{P})$ with the following metric

$$\rho(A, B) := \begin{cases} 1, & \text{if } \delta(A, B) = \infty, \\ \frac{\delta(A,B)}{1+\delta(A,B)}, & \text{otherwise.} \end{cases}$$

It is easy to see that $(\mathcal{L}^\infty(\mathbb{P}), \rho)$ is a complete metric space. In what follows, we will investigate the properties of the top Lyapunov exponent function $\lambda_1 : \mathcal{L}^\infty(\mathbb{P}) \to \mathbb{R}$, $A \mapsto \lambda_1(A)$.

Theorem 3.2.1. *The top Lyapunov exponent function $\lambda_1(\cdot) : (\mathcal{L}^\infty(\mathbb{P}), \rho) \to \mathbb{R}$ is upper-semi continuous.*

Proof. By virtue of the Furstenberg-Kesten theorem (see Theorem 1.3.3), we get

$$\lambda_1(A) = \lim_{t\to\infty} \frac{1}{t} \int_\Omega \log \|\Phi_A(t, \omega)\| \, d\mathbb{P}(\omega).$$

Fix $\varepsilon > 0$. Then there exists $T = T(\varepsilon) > 0$ such that

$$\frac{1}{T} \int_\Omega \log \|\Phi_A(T, \omega)\| \, d\mathbb{P}(\omega) \le \lambda_1(A) + \frac{\varepsilon}{2}. \tag{3.6}$$

Define $M(\omega) := \exp\int_0^T \|A(\theta_s\omega)\|\ ds$. Since A is essentially bounded it follows that $M(\cdot) \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Hence, there exists $\delta > 0$ such that

$$\int_\Omega \log\left(1 + \delta T M(\omega)^2 e^{\delta T M(\omega)}\right)\ d\mathbb{P}(\omega) < \frac{\varepsilon}{2}. \tag{3.7}$$

We now show that

$$\lambda_1(B) < \lambda_1(A) + \varepsilon \qquad \text{for all } B \in \mathcal{L}^\infty(d) \text{ with } \rho(A, B) < \frac{\delta}{1+\delta}. \tag{3.8}$$

By the variation of constants formula we obtain

$$\Phi_B(t,\omega) = \Phi_A(t,\omega) + \int_0^t \Phi_A(t-s, \theta_s\omega) C(\theta_s\omega) \Phi_B(s,\omega)\ ds,$$

for all $t \geq 0$, where $C(\omega) := B(\omega) - A(\omega)$. Together with

$$\|\Phi_A(t-s, \theta_s\omega)\| \leq \exp\int_0^t \|A(\theta_u\omega)\|\ du \qquad \text{for all } s \leq t,$$

we thus get

$$\|\Phi_B(t,\omega)\| \leq \|\Phi_A(t,\omega)\| + \delta M(\omega) \int_0^t \|\Phi_B(s,\omega)\|\ ds \qquad \text{for all } 0 \leq t \leq T.$$

Applying the Gronwall inequality (see Aulbach and Wanner [8]), we obtain

$$\|\Phi_B(T,\omega)\| \leq \|\Phi_A(T,\omega)\| + \delta M(\omega) \int_0^T \|\Phi_A(s,\omega)\| e^{\delta M(\omega)(T-s)}\ ds. \tag{3.9}$$

On the other hand, we have

$$\Phi_A(s,\omega) = \left(\Phi_A(T-s, \theta_s\omega)\right)^{-1} \circ \Phi_A(T,\omega) \qquad \text{for all } 0 < s < T.$$

This implies with the inequality $\|\left(\Phi_A(T-s,\theta_s\omega)\right)^{-1}\| \leq \exp\int_0^T \|A(\theta_u\omega)\|\ du$ that

$$\|\Phi_A(s,\omega)\| \leq M(\omega)\|\Phi_A(T,\omega)\| \qquad \text{for all } 0 < s < T. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\|\Phi_B(T,\omega)\| \leq \|\Phi_A(T,\omega)\| \left(1 + \delta T M(\omega)^2 e^{\delta T M(\omega)}\right).$$

Therefore,

$$\begin{aligned}\int_\Omega \log\|\Phi_B(T,\omega)\|\ d\mathbb{P}(\omega) \quad &\leq \quad \int_\Omega \log\|\Phi_A(T,\omega)\|\ d\mathbb{P}(\omega) \\ &\qquad + \int_\Omega \log\left(1 + \delta T M(\omega)^2 e^{\delta T M(\omega)}\right)\ d\mathbb{P}(\omega),\end{aligned}$$

which implies with (3.6), (3.7) that

$$\frac{1}{T}\int_\Omega \log \|\Phi_B(T,\omega)\|\, d\mathbb{P}(\omega) < \lambda_1(A) + \varepsilon.$$

Consequently, to prove (3.8) it suffices to show that

$$\lambda_1(B) \leq \frac{1}{T}\int_\Omega \log \|\Phi_B(T,\omega)\|\, d\mathbb{P}(\omega). \tag{3.11}$$

Since

$$\Phi_B(nT,\omega) = \Phi_B(T,\theta_{(n-1)T}\omega) \circ \Phi_B(T,\theta_{(n-2)T}\omega) \circ \cdots \circ \Phi_B(T,\omega) \qquad \text{for all } n \in \mathbb{N},$$

it follows that

$$\log \|\Phi_B(nT,\omega)\| \leq \log \|\Phi_B(T,\theta_{(n-1)T}\omega)\| + \log \|\Phi_B(T,\theta_{(n-2)T\omega})\| + \cdots + \log \|\Phi_B(T,\omega)\|.$$

Integrating both sides of this inequality implies together with the ergodicity of θ that

$$\frac{1}{nT}\int_\Omega \log \|\Phi_B(nT,\omega)\|\, d\mathbb{P}(\omega) \leq \frac{1}{T}\int_\Omega \log \|\Phi_B(T,\omega)\|\, d\mathbb{P}(\omega) \qquad \text{for all } n \in \mathbb{N}.$$

Letting n tend to infinity, we get (3.11). As a consequence, $\lambda_1(\cdot)$ is upper semi-continuous. □

Remark 3.2.2. Analogously, by considering the inverse direction of time the smallest Lyapunov exponent $\lambda_{p(A)}(A) = \chi_d(A)$ depends lower-semicontinously on A.

In the next theorem, we deal with the Baire class of the repeated Lyapunov exponents functions $\chi_i(\cdot), i = 1, \ldots, d$.

Theorem 3.2.3. *The functions $\chi_i(\cdot) : (\mathcal{L}^\infty(\mathbb{P}), \rho) \to \mathbb{R}, i = 1, \ldots, d$, are of the first Baire class.*

Proof. Let $\Lambda^k\Phi_A$ denote the RDS on the k-fold exterior power $\Lambda^k(\mathbb{R}^d), k = 1, \ldots, d$, induced by Φ_A. In view of the Furstenberg and Kesten theorem (see Theorem 1.3.3), we have

$$\Lambda^k(A) = \lim_{t\to\infty} \frac{1}{t}\int_\Omega \log \|\Lambda^k\Phi_A(t,\omega)\|\, d\mathbb{P}(\omega).$$

Let $\varepsilon > 0$ be arbitrary but fixed. Then there exists $T > 0$ such that

$$\Lambda^k(A) \leq \frac{\varepsilon}{2} + \frac{1}{T}\int_\Omega \log \|\Lambda^k\Phi_A(T,\omega)\|\, d\mathbb{P}(\omega).$$

By the same argument as in Theorem 3.2.1, there exists $\delta > 0$ such that the following inequality

$$\frac{1}{T}\int_\Omega \log \|\Lambda^k\Phi_B(T,\omega)\|\, d\mathbb{P}(\omega) \leq \frac{\varepsilon}{2} + \frac{1}{T}\int_\Omega \log \|\Lambda^k\Phi_A(T,\omega)\|\, d\mathbb{P}(\omega)$$

holds for all $B \in \mathcal{L}^\infty(\mathbb{P})$ with $\rho(A, B) < \delta$. This together with the inequality

$$\Lambda^k(B) \leq \frac{1}{T} \int_\Omega \log \|\Lambda^k \Phi_B(T, \omega)\| \, d\mathbb{P}(\omega)$$

implies the upper-semi continuity of $\Lambda^k(\cdot)$. Hence, the function $\Lambda^k(\cdot)$ is of the first Baire class (see e.g. Stromberg [133]). On the other hand, we have $\chi_i = \Lambda^i - \Lambda^{i-1}$ for all $i = 1, \ldots, d$. Therefore, the functions $\chi_i, i = 1, \ldots, d$, are of the first Baire class and the proof is completed. □

Using the definition of Lyapunov exponents and repeated Lyapunov exponents for an RDS, we obtain the following corollary (we refer to Arnold and Cong [4] for a proof).

Corollary 3.2.4. *The functions $\lambda_i(\cdot) : (\mathcal{L}^\infty(\mathbb{P}), \rho) \to \mathbb{R}$ are of the second Baire class.*

For completeness, let us quote a brief discussion about some generic properties of Baire functions from Arnold and Cong [4, Remark 2.8].

Remark 3.2.5. (i) Because all Baire functions are measurable with respect to the Borel σ-algebra of $\mathcal{L}^\infty(\mathbb{P})$ (see Goffman [67, Theorem 1]), the Lusin theorem (see Goffman [67, Theorem 5]) is applicable: Let μ be any probability measure on the measurable space $\mathcal{L}^\infty(\mathbb{P})$ with the Borel σ-algebra. For any $\varepsilon > 0$ there exists a set S_1, whose complement has μ-measure less than ε, such that the restrictions of $\lambda_1(\cdot), \ldots, \lambda_{p(\cdot)}(\cdot)$ to S_1 are continuous on S_1.

(ii) Due to the properties of the Baire functions (see Goffman [67, Theorem 2]), there exists a residual set S_2 such that the restrictions of $\lambda_1(\cdot), \ldots, \lambda_{p(\cdot)}(\cdot)$ to S_2 are continuous on S_2.

(iii) By virtue of Theorem C.0.10, the functions $\chi_1(\cdot), \ldots, \chi_d(\cdot)$ are generically continuous on the whole space $\mathcal{L}^\infty(\mathbb{P})$.

Chapter 4

Difference Equations with Random Delay

The work presented here is a first step towards a general theory of differential and difference equations incorporating random delays which are not assumed to be bounded. The main technical tool relies on recent work of Lian and Lu [89], which gives a generalization of the MET, going back to V. I. Oseledets [109], to Banach spaces.
We present a linear delay random difference equation with a random delay, which is not assumed to be bounded. Whereas the case of a bounded delay may be modeled using a finite dimensional state space, resulting in a product of random matrices, this does not work in the case of a delay for which the probability that it exceeds a value M is positive for every M. The first step is therefore the introduction of a suitable state space. Here we use a subspace of the linear space of all real valued sequences. With a suitable choice of a norm this gives a Banach space. Already here considerably more attention than in the finite dimensional case, where any two norms are equivalent, is needed.
For bounded delay, resulting in the finite dimensional case, the standard one-sided MET applies. The integrability condition is satisfied provided that the coefficients are integrable. On the other hand, one necessarily has a non-invertible and therefore only one-sided system as soon as the delay is not deterministic.
Using the model proposed here for the case of unbounded random delays, the integrability condition of the MET is satisfied provided the delay and the coefficients are integrable. Furthermore, the system is invertible.
We then show that modeling a bounded delay using the infinite dimensional setup, and then applying the MET in a Banach space, yields the same result as the finite-dimensional model with the standard MET, provided the norm on the infinite-dimensional space is chosen appropriately.
Delays in difference and in differential equations are used for mathematical modeling in many applications for the description of evolutions which incorporate influence of events from the past.
In particular, there is a vast literature on delays in biological systems. We briefly discuss some aspects here, but it should be emphasized that we do not claim to give an

exhaustive presentation, and we refer to Kot [80], Kuang [82], and MacDonald [91] as well as to recent work by Forde [58], also for further references.
Delays in biological applications may be caused by quite different sources. In population dynamics there are gestation and incubation times, which often can be assumed to be deterministic (see e. g. Finerty [53] and Flowerdew [55]) in first approximation. For a more detailed investigation it may also then become necessary to model these as random. For a discussion of human pregnancy see e. g. Forde [58, pp. 89]. However, there are populations with structurally non-deterministic maturation times. These may be caused by changes of external influences and living conditions due to weather or to climate changes, or also due to human influences such as fertilisation or forestial clearing. For insect populations where the end of the larval stage is governed by external conditions, for instance given by climatic or nutrimental parameters, the incorporation of a random delay with large and possibly even unbounded variance is justified, if not necessary.
Models for immune response are another situation where the incorporation of delays is reasonable (see e. g. Cooke et al. [40]). The incorporation of a random delay into models for immune response seems particularly appropriate, since the time of the outbreak of a disease caused by an infection is influenced by complicated biological and environmental processes. This is another case where the assumption of an a priori bound for the (random) delay, thereby discarding events which may occur with a small, albeit positive probability, may yield an erroneous model.
Recent investigations of discrete epidemiological models allow the period of infectivity to be of arbitrary length, see Lara-Sagahón, Kharchenko, and José [86]. This is another situation where the inclusion of randomness into the delay forbids the assumption of boundedness of the delay.
Also genetic changes which have taken place under circumstances which are long gone may get virulent, activating information which had long been hidden.
Another field of application of random delays is the stock market. The usual Black-Scholes-Merton-Samuelson (1965, 1973) model is based on the explicit assumption that information is uniformly held by everyone and that it does not play a role in stock prices. More recent models incorporate the evident nonuniformity of information in the market and the evident time delay until new information becomes generally known, see Shepp [125]. Clearly it would not be appropriate to model this information delay as deterministic, and also it seems questionable whether the assumption of a bounded random delay would yield a realistic model. In the economy also events such like corruption or other proceedings taking place in concealment come into operational conscience only after discovery. Often this is caused by 'lucky' circumstances, and may therefore well be modelled by a random time, acting as a delay.
In economic order quantity models it is often assumed that payments are made on the receipt of items. This is usually not fulfilled in practice. It is, on the contrary, quite common in the market that suppliers offer a credit period to retailers in order to stimulate demand, see Chang and Dye [31]. This is another situation where random delays may be used in order to obtain an appropriate model. Here the random delay, used to model the credit return time, should be assumed to be not bounded in order also to cover the credit loss risk.

Also mathematical modelling of other situations where an unexpected discovery or an event going back to a possibly far remote past are well modelled taking unbounded random delays into account. While in several situations it may be appropriate to consider these events as non-delayed random events, occuring actually and without delay, this will not be appropriate if the effects of a past event become virulent for several parts of vegetation or animal population which are otherwise independent.
For the investigation of a model incorporating fixed delays in a Black-Scholes-Merton-Samuelson model see Arriojas et al. [7] and Kazmerchuk et al. [77].
A major property of the model we are going to consider here is the fact that we allow for an unbounded distribution for the random delay time. As already mentioned, this makes it necessary to use an infinite dimensional state space. One may argue that the assumption of an unbounded distribution is not realistic for certain applications such as animal populations with backbreed evolution, and that there the standard finite-dimensional models would suffice. This being true in certain situations, it may not be appropriate to assume a bounded distribution founding on the argument that an unbounded delay time has never been observed yet. The probability of very large delays must be small anyway. Often the assumption of a bound for the possible delay times may be justified, as soon as only finite evolution times are of interest. However, for assertions about asymptotic properties it is relevant if the delay can be arbitrarily large with positive, albeit very small, probability. The Poincaré recurrence theorem as well as the ergodic theorem imply probability one for exceeding every value. Therefore the assumption of a bounded delay may be appropriate for investigations with a finite time horizon, but it may yield wrong conclusions for questions concerning asymptotic properties.
Further recent contributions on the theory of stochastic differential equations with delay, which may also be allowed to be random itself, are Caraballo et al. [28, 29], for unbounded delay see Caraballo et al. [30].

4.1 A Setting for Difference Equations with Random Delay

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \Omega \to \Omega$ an invertible ergodic map which preserves the probability measure $\mathbb{P}$ and let $r : \Omega \to \mathbb{N}$ be a measurable map which we call *random delay map*. Let $A, B : \Omega \to \mathbb{R}^{d\times d}$ be measurable functions. We consider a linear difference equation with random delay of the form

$$x_{n+1} = A(\theta^n\omega)x_n + B(\theta^n\omega)x_{n-r(\theta^n\omega)}. \tag{4.1}$$

In order to introduce an RDS generated by (4.1) we first need to construct an appropriate state space. Since the delay map r is in general unbounded, an initial value for (4.1) is an infinite sequence $(\dots, x_{-2}, x_{-1}, x_0)$ and for an arbitrary norm $\|\cdot\|$ we denote the normed linear space of all those sequences by

$$\mathbf{X} = \{\mathbf{x} = (\dots, x_{-2}, x_{-1}, x_0) : \quad \|\mathbf{x}\| < \infty\}.$$

The *time-1-map* $\Phi(1,\omega)=\Phi(\omega):\mathbf{X}\to\mathbf{X}$ generated by (4.1) is defined to be the map

$$\Phi:\Omega\times\mathbf{X}\to\mathbf{X},\qquad (\omega,\mathbf{x})\mapsto\Phi(\omega)\mathbf{x}=\big(\dots,x_{-2},x_{-1},x_0,A(\omega)x_0+B(\omega)x_{-r(\omega)}\big),\quad (4.2)$$

which gives rise to a linear cocycle $\Phi:\mathbb{N}_0\times\Omega\to\mathcal{L}(\mathbf{X})$ defined by

$$\Phi(n,\omega)=\begin{cases}\mathrm{id}_{\mathbf{X}}, & \text{if } n=0,\\ \Phi(1,\theta^{n-1}\omega)\circ\cdots\circ\Phi(1,\omega), & \text{otherwise.}\end{cases}\quad (4.3)$$

Remark 4.1.1. Note that $\Phi(\omega)$, and hence $\Phi(n,\omega)$ for every $n\in\mathbb{N}$, are injective, since $\ker\Phi(\omega)=\{0\}$. For $\omega\in\Omega, n\in\mathbb{N}$ and $\mathbf{x}\in\operatorname{im}\Phi(n,\theta^{-n}\omega)$ due to the injectivity of $\Phi(n,\theta^{-n}\omega)$ there is a unique $\mathbf{y}\in\mathbf{X}$, which is also denoted by $\Phi(-n,\omega)\mathbf{x}$, such that

$$\Phi(n,\theta^{-n}\omega)\mathbf{y}=\mathbf{x}.$$

In order to complete the definition of Φ we need to fix a norm on $\mathbf{X}$. Since in the unbounded delay case, the initial data is always a part of the solution. Hence, some kind of regularity must be imposed from the beginning, see e.g. [30, 63]. From now on we deal with a special class of norms on $\mathbf{X}$ which is appropriate for (4.1). For $\gamma>0$ fixed we define

$$\begin{aligned}\mathbf{X}_\gamma &:= \Big\{\mathbf{x}=(\cdots,x_{-1},x_0):\lim_{n\to\infty}e^{-\gamma n}x_{-n}\quad\text{exists}\Big\},\\ \|\mathbf{x}\|_\gamma &:= \sup_{n\in\mathbb{N}_0}e^{-\gamma n}|x_{-n}|=\sup_{n\in\mathbb{N}_0^-}e^{\gamma n}|x_n|\qquad\text{for all }\mathbf{x}\in\mathbf{X}_\gamma,\end{aligned}$$

where $|\cdot|$ is an arbitrary norm on $\mathbb{R}^d$. It is easy to see that $(\mathbf{X}_\gamma,\|\cdot\|_\gamma)$ is a Banach space. The following lemma provides the separability of the space $(\mathbf{X}_\gamma,\|\cdot\|_\gamma)$.

Lemma 4.1.2. *For $\gamma>0$ the space $(\mathbf{X}_\gamma,\|\cdot\|_\gamma)$ is separable.*

Proof. Let $\mathbb{Q}^d$ denote the set of all vectors in $\mathbb{R}^d$ whose components are rational. Clearly, $\mathbb{Q}^d$ is countable dense set in $\mathbb{R}^d$. For each $N\in\mathbb{N}$, we define

$$\mathbf{X}_N:=\Big\{f:\{-N,\dots,-1,0\}\to\mathbb{R}^d\Big\}.$$

We endow the space $\mathbf{X}_N$ with the sup norm $\|\cdot\|_\infty$, i.e.

$$\|f\|_\infty=\sup_{k\in\{-N,\dots,-1,0\}}|f(k)|\qquad\text{for all } f\in\mathbf{X}_N.$$

Obviously, $(\mathbf{X}_N,\|\cdot\|_\infty)$ is a separable Banach space. As a consequence, there exists a countable set

$$A_N:=\{f_1^{(N)},f_2^{(N)},\dots\},\qquad\text{where } f_1^{(N)},f_2^{(N)},\dots\in\mathbf{X}_N,$$

which is dense in $\mathbf{X}_N$. For each function $f_k^{(N)}\in A_N$ and $v\in\mathbb{Q}^d$, we define the extended function $\widetilde{f}_{k,v}^{(N)}:\mathbb{N}_0^-\to\mathbb{R}^d$ by

$$\widetilde{f}_{k,v}^{(N)}(n):=\begin{cases}f_k^{(N)}(n), & \text{for all } n\in\{-N,\dots,-1,0\},\\ e^{-\gamma n}v, & \text{otherwise.}\end{cases}\quad (4.4)$$

Since $\lim_{n\to\infty} e^{-\gamma n}\widetilde{f}_{k,v}^{(N)}(-n) = v$ it follows that $\widetilde{f}_{k,v}^{(N)} \in \mathbf{X}_\gamma$. Define

$$\widetilde{A}_N := \bigcup_{v\in\mathbb{Q}^d} \left\{\widetilde{f}_{1,v}^{(N)}, \widetilde{f}_{2,v}^{(N)}, \dots\right\} \qquad \text{for all } N \in \mathbb{N}.$$

To prove the separability of the Banach space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ we will show that

$$\widetilde{A} := \bigcup_{N\in\mathbb{N}} \widetilde{A}_N \qquad \text{is dense in } \mathbf{X}_\gamma.$$

For a given $\mathbf{x} = (\dots, x_{-1}, x_0) \in \mathbf{X}_\gamma$, set $u := \lim_{n\to\infty} e^{-\gamma n} x_{-n} \in \mathbb{R}^d$. Hence, for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|e^{-\gamma n} x_{-n} - u| \leq \frac{\varepsilon}{3} \qquad \text{for all } n \geq N. \tag{4.5}$$

Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$ it follows that there exists $v \in \mathbb{Q}^d$ such that $|v - u| \leq \frac{\varepsilon}{3}$. On the other hand, due to the denseness of A_N in $\mathbf{X}_N$ there exists $k \in \mathbb{N}$ such that

$$\sup_{n\in\{-N,\dots,-1,0\}} \left|f_k^{(N)}(n) - x_n\right| < \varepsilon. \tag{4.6}$$

We now estimate $\|\widetilde{f}_{k,v}^{(N)} - \mathbf{x}\|_\gamma$. By (4.4) and (4.6), the relation

$$\begin{aligned} e^{\gamma n}\left|\widetilde{f}_{k,v}^{(N)}(n) - x_n\right| &= e^{\gamma n}\left|f_k^{(N)}(n) - x_n\right| \\ &\leq \left|f_k^{(N)}(n) - x_n\right| \\ &\leq \varepsilon \end{aligned}$$

holds for all $n \in \{-N, \dots, -1, 0\}$. On the other hand, for all $n \in \{-N-1, -N-2, \dots\}$ by (4.4) we have

$$\begin{aligned} e^{\gamma n}\left|\widetilde{f}_{k,v}^{(N)}(n) - x_n\right| &= e^{\gamma n}|e^{-\gamma n} v - x_n| \\ &\leq |u - v| + |u - e^{\gamma n} x_n| \\ &\leq \frac{2}{3}\varepsilon, \end{aligned}$$

where we use (4.5) to obtain the last inequality. Therefore, we have

$$\|\widetilde{f}_{k,v}^{(N)} - \mathbf{x}\|_\gamma = \sup_{n\in\mathbb{N}_0^-} e^{\gamma n}\left|\widetilde{f}_{k,v}^{(N)}(n) - x_n\right| \leq \varepsilon,$$

which proves that $\widetilde{A}$ is dense in $\mathbf{X}_\gamma$ and this completes the proof. □

Throughout this chapter we assume that $\gamma > 0$ and consider equation (4.1) on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. In the following lemma, we provide a sufficient and necessary condition for which the solution of (4.1) tends to 0 when the time tends to infinity.

Lemma 4.1.3. *Let $\mathbf{x} \in \mathbf{X}_\gamma$. The following two statements are equivalent:*

(i) $\lim_{n\to\infty} \|\Phi(n,\omega)\mathbf{x}\|_\gamma = 0$.

(ii) $\lim_{n\to\infty} \left|\left(\Phi(n,\omega)\mathbf{x}\right)_0\right| = 0$, *where* $\left(\Phi(n,\omega)\mathbf{x}\right)_0$ *denotes the* 0*-th entry of* $\Phi(n,\omega)\mathbf{x}$.

Proof. ($\Rightarrow$) Suppose that $\lim_{n\to\infty} \|\Phi(n,\omega)\mathbf{x}\|_\gamma = 0$. Since $\|\Phi(n,\omega)\mathbf{x}\|_\gamma \geq \left|\left(\Phi(n,\omega)\mathbf{x}\right)_0\right|$ it follows that $\lim_{n\to\infty} \left|\left(\Phi(n,\omega)\mathbf{x}\right)_0\right| = 0$.
($\Leftarrow$) Conversely, we assume that $\lim_{n\to\infty} \left|\left(\Phi(n,\omega)\mathbf{x}\right)_0\right| = 0$. Thus for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|\left(\Phi(n,\omega)\mathbf{x}\right)_0\right| \leq \varepsilon$ for all $n \geq N$. Choose $k \in \mathbb{N}$ large enough such that $e^{-\gamma k}\|\Phi(N,\omega)\mathbf{x}\|_\gamma \leq \varepsilon$. Now we show that

$$\|\Phi(n,\omega)\mathbf{x}\|_\gamma \leq \varepsilon \qquad \text{for all } n \geq N+k. \tag{4.7}$$

From (4.2) and (4.3) we have for all $n \geq N+k$

$$\left(\Phi(n,\omega)\mathbf{x}\right)_{-j} = \begin{cases} \left(\Phi(n-j,\omega)\mathbf{x}\right)_0, & \text{if } 0 \leq j \leq n-N\,, \\ \left(\Phi(N,\omega)\mathbf{x}\right)_{n-N-j}, & \text{if } n-N+1 \leq j\,. \end{cases}$$

Hence, for all $n \geq N+k$ we get

$$\begin{aligned} \|\Phi(n,\omega)\mathbf{x}\|_\gamma &= \sup_{j\in\mathbb{N}_0} e^{-\gamma j}\left|\left(\Phi(n,\omega)\mathbf{x}\right)_{-j}\right| \\ &= \max\left\{\sup_{0\leq j\leq n-N} e^{-\gamma j}\left|\left(\Phi(n-j,\omega)\mathbf{x}\right)_0\right|, \sup_{n-N+1\leq j} e^{-\gamma j}\left|\left(\Phi(N,\omega)\mathbf{x}\right)_{n-N-j}\right|\right\} \\ &\leq \max\left\{\varepsilon, e^{-\gamma k}\|\Phi(N,\omega)\mathbf{x}\|_\gamma\right\}, \end{aligned}$$

proving (4.7) and this completes the proof. □

Example 4.1.1. We now discuss the case that $B = 0$ and A is a scalar number

$$x_{n+1} = Ax_n \qquad \text{for } A \neq 0.$$

It generates the deterministic cocycle

$$\Phi(n,\omega)\mathbf{x} = (\dots, x_{-1}, x_0, Ax_0, A^2x_0, \dots, A^n x_0)\,.$$

Since

$$\Phi(\omega)\left(\dots, \frac{1}{A^2}, \frac{1}{A}, 1\right) = A\left(\dots, \frac{1}{A^2}, \frac{1}{A}, 1\right)$$

it follows that A is an eigenvalue of $\Phi(\omega)$ with the corresponding eigenvector $\mathbf{x} = \left(\dots, \frac{1}{A^2}, \frac{1}{A}, 1\right)$. The eigenvector $\mathbf{x}$ is an element of $\mathbf{X}_\gamma$, i.e. $\lim_{n\to\infty} e^{-\gamma n}A^{-n}$ exists, if and only if $A \in (-\infty, -e^{-\gamma}) \cup [e^{-\gamma}, \infty)$. In this case we have

$$\Phi(n,\omega)\mathbf{x} = A^n\mathbf{x} \qquad \text{for all } n \in \mathbb{N},$$

which implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma = \log|A| \tag{4.8}$$

is a Lyapunov exponent of Φ. Let E_1 denote the subspace which realizes this exponent, i.e.

$$E_1 := \left\{ \mathbf{x} \in \mathbf{X}_\gamma : \lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma = \log|A| \right\}$$

and define

$$E := \big\{ \mathbf{x} \in \mathbf{X}_\gamma : \Phi(-n,\omega)\mathbf{x} \text{ exists for all } n \in \mathbb{N} \big\}.$$

It is easy to obtain that $E = \operatorname{span}\{\mathbf{x}\}$, where $\mathbf{x} = \big(\dots, \frac{1}{A^2}, \frac{1}{A}, 1\big)$. This implies together with (4.8) that $E_1 = E = \operatorname{span}\big\{\big(\dots, \frac{1}{A^2}, \frac{1}{A}, 1\big)\big\}$ and $\log|A|$ is the unique Lyapunov exponent of Φ. Define

$$F := \big\{ (\dots, x_{-1}, x_0) = \mathbf{x} \in \mathbf{X}_\gamma : x_0 = 0 \big\}.$$

It is clear to see that F is a complementary subspace of E_1 and invariant under $\Phi(\omega)$, i.e. $\Phi(\omega)F \subset F$. Moreover, we have

$$\|\Phi(n+1,\omega)\mathbf{x}\|_\gamma = e^{-\gamma}\|\Phi(n,\omega)\mathbf{x}\|_\gamma \qquad \text{for all } n \in \mathbb{N}, \mathbf{x} \in F.$$

As a consequence,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_F\|_\gamma = -\gamma$$

and we conclude that $\mathbf{X}_\gamma = E_1 \oplus F$ is the Oseledets splitting of Φ.

4.2 MET for Difference Equations with Random Delay

In this section we consider a difference equation with random delay of the form

$$x_{n+1} = A(\theta^n\omega)x_n + B(\theta^n\omega)x_{n-r(\theta^n\omega)}.$$

In the following lemma, we provide a sufficient condition for which the generated RDS satisfies the integrability condition of the MET.

Lemma 4.2.1 (Sufficient Integrability Condition). *If $r \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ and $A, B : \Omega \to \mathbb{R}^{d\times d}$ are measurable functions satisfying that*

$$\log^+|A(\cdot)|, \log^+|B(\cdot)| \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}),$$

then the linear cocycle Φ, defined as in (4.2)*, satisfies the integrability condition of the MET (see Theorem 1.4.2), i.e.*

$$\log^+\|\Phi(\cdot)\|_\gamma \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

Proof. For each $\mathbf{x} = (\ldots, x_{-1}, x_0) \in \mathbf{X}_\gamma$, by using (4.2), we obtain

$$\begin{aligned}\|\Phi(\omega)\mathbf{x}\|_\gamma &= \max\left\{\sup_{n\in\mathbb{N}_0} e^{-\gamma(n+1)}|x_{-n}|, |A(\omega)x_0 + B(\omega)x_{-r(\omega)}|\right\}\\ &= \max\left\{e^{-\gamma}\|\mathbf{x}\|_\gamma, |A(\omega)x_0 + B(\omega)x_{-r(\omega)}|\right\}\\ &\le \max\left\{e^{-\gamma}\|\mathbf{x}\|_\gamma, \left(|A(\omega)| + e^{\gamma r(\omega)}|B(\omega)|\right)\|\mathbf{x}\|_\gamma\right\}.\end{aligned}$$

Consequently, for all $\omega \in \Omega$ we have

$$\begin{aligned}\log^+\|\Phi(\omega)\|_\gamma &= \max\left\{0, \log\|\Phi(\omega)\|_\gamma\right\}\\ &\le \max\left\{0, \log e^{-\gamma}, \log\left(|A(\omega)| + e^{\gamma r(\omega)}|B(\omega)|\right)\right\}\\ &\le \log^+\left(|A(\omega)| + e^{\gamma r(\omega)}|B(\omega)|\right).\end{aligned}$$

This implies together with the inequality $\log^+(x+y) \le \log^+ x + \log^+ y + \log 2$ for all $x, y \in \mathbb{R}^+$ that

$$\log^+\|\Phi(\omega)\|_\gamma \le \log^+|A(\omega)| + \log^+\left(e^{\gamma r(\omega)}|B(\omega)|\right) + \log 2.$$

Since $\log^+\|A(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ it is thus sufficient to show that

$$\log^+\left(e^{\gamma r(\cdot)}|B(\cdot)|\right) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{4.9}$$

Indeed, using the inequality $\log^+(xy) \le \log^+ x + \log^+ y$, the following inequality

$$\begin{aligned}\log^+\left(e^{\gamma r(\omega)}|B(\omega)|\right) &\le \log^+\left(e^{\gamma r(\omega)}\right) + \log^+|B(\omega)|\\ &\le \gamma r(\omega) + \log^+|B(\omega)|\end{aligned}$$

holds for all $\omega \in \Omega$. This proves together with $\log^+|B(\cdot)|$, $r(\cdot) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ conclusion (4.9) and the proof is completed.

□

Recall that for a bounded linear map $L : \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ we define

$$\|L\|_\alpha = \alpha(L(B_1(0))),$$

where $\alpha(L(B_1(0)))$ is the Kuratowski measure of noncompactness of $B_1(0)$ which is defined as in (1.14). Let Φ be the linear cocycle defined as in (4.3). We define

$$l_\alpha(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log\|\Phi(n,\omega)\|_\alpha.$$

If $\Phi(n,\omega)$ would be a compact operator then $\|\Phi(n,\omega)\|_\alpha = 0$ and hence $l_\alpha(\Phi) = -\infty$. In the following lemma, we show that $\Phi(n,\omega)$ is not compact for all $\omega \in \Omega, n \in \mathbb{N}$ and we compute $l_\alpha(\Phi)$.

Lemma 4.2.2. *For each $\omega \in \Omega$ and $n \in \mathbb{N}$ the operator $\Phi(n,\omega) : \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ is not compact. Furthermore, we have $l_\alpha(\Phi) = -\gamma$.*

Proof. W.l.o.g. we assume that $|\cdot|$ is the max norm on $\mathbb{R}^d$, i.e.

$$|x| = \max_{1\le i\le d} |x_i| \qquad \text{for all } x=(x_1,\dots,x_d)\in\mathbb{R}^d.$$

We first show that the operator $\Phi(n,\omega)$ is not compact for all $\omega\in\Omega$ and $n\in\mathbb{N}$. Choose and fix $n\in\mathbb{N}$. For each $i\in\mathbb{N}$ we define a function $\mathbf{x}^i:\mathbb{N}_0^-\to\mathbb{R}^d$ by

$$\mathbf{x}^i(-k) = \begin{cases} 0, & \text{if } k\neq i,\\ (e^{\gamma i},\dots,e^{\gamma i}), & \text{if } k=i. \end{cases}$$

Since $\lim_{k\to\infty} e^{-\gamma k}\mathbf{x}^i(-k)=0$ it follows that $\mathbf{x}^i\in\mathbf{X}_\gamma$. A direct computation yields that

$$\|\mathbf{x}^i\|_\gamma = \sup_{k\in\mathbb{N}_0} e^{-\gamma k}|\mathbf{x}^i(-k)| = 1 \qquad \text{for all } i\in\mathbb{N}.$$

For an arbitrary $i,j\in\mathbb{N}$ with $i\neq j$ we now estimate $\|\Phi(n,\omega)\mathbf{x}^i-\Phi(n,\omega)\mathbf{x}^j\|_\gamma$. Since

$$\big(\Phi(n,\omega)(\mathbf{x}^i-\mathbf{x}^j)\big)(-k) = \begin{cases} (e^{\gamma i},\dots,e^{\gamma i}), & \text{if } k=i+n,\\ (-e^{\gamma j},\dots,-e^{\gamma j}), & \text{if } k=j+n, \end{cases}$$

it follows that

$$\|\Phi(n,\omega)\mathbf{x}^i-\Phi(n,\omega)\mathbf{x}^j\|_\gamma \ge e^{-\gamma n} \qquad \text{for all } i,j\in\mathbb{N}, i\neq j.$$

Consequently, there is no convergent subsequences of $\{\Phi(n,\omega)\mathbf{x}^i\}_{i\in\mathbb{N}}$ and hence $\Phi(n,\omega)$ is not a compact operator. Moreover, one has

$$\alpha\big(\Phi(n,\omega)B_1(0)\big) \ge e^{-\gamma n} \qquad \text{for all } n\in\mathbb{N},\omega\in\Omega.$$

Thus, $\|\Phi(n,\omega)\|_\alpha \ge e^{-\gamma n}$ and as a consequence we get

$$l_\alpha(\Phi) = \lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\|_\alpha \ge -\gamma. \tag{4.10}$$

Let ε be an arbitrary positive number. Choose and fix $n\in\mathbb{N}_0$ such that $n\ge\frac{\log 2}{\varepsilon}$. We show that $l_\alpha(\Phi)\le-\gamma+\varepsilon$. Since $\|\cdot\|_\alpha$ is multiplicative it follows that

$$\frac{1}{kn}\log\|\Phi(kn,\omega)\|_\alpha \le \frac{1}{kn}\left(\log\|\Phi(n,\theta^{(k-1)n}\omega)\|_\alpha+\cdots+\log\|\Phi(n,\omega)\|_\alpha\right).$$

Hence, it is sufficient to show that $\log\|\Phi(n,\omega)\|_\alpha\le n(-\gamma+\varepsilon)$ for all $\omega\in\Omega$. Equivalently,

$$\alpha\left(\Phi(n,\omega)B_1(0)\right) \le e^{n(-\gamma+\varepsilon)} \qquad \text{for all } \omega\in\Omega. \tag{4.11}$$

Fix $\omega\in\Omega$ and define $D:=\Phi(n,\omega)B_1(0)$. By the definition of $\Phi(n,\omega)$, see (4.3), we obtain that for any $\mathbf{x}\in D$ one has

$$|\mathbf{x}(-k-n)| \le e^{\gamma k} \qquad \text{for all } k\in\mathbb{N}_0, \tag{4.12}$$

and there exists $M > 0$ such that

$$|\mathbf{x}(-k)| \leq M \qquad \text{for all } k = 0, 1, \dots, n-1.$$

Since the set $[-M, M]^n$ is compact in $\mathbb{R}^n$ it follows that there exist $\widetilde{D}_1, \dots, \widetilde{D}_N \subset \mathbb{R}^n$ with

$$d(\widetilde{D}_i) \leq e^{n(-\gamma+\varepsilon)} \quad \text{for all } i = 1, \dots, N, \text{ and } [-M, M]^n \subset \bigcup_{i=1}^{N} \widetilde{D}_i. \tag{4.13}$$

For each $i \in \{1, \dots, N\}$ we define the following set

$$D_i := \Big\{\mathbf{x} \in \mathbf{X}_\gamma : |\mathbf{x}(-k-n)| \leq e^{\gamma k} \text{ for all } k \in \mathbb{N}_0, \text{ and } \big(\mathbf{x}(-n+1), \dots, \mathbf{x}(0)\big) \in \widetilde{D}_i\Big\}.$$

Combining (4.12) and (4.13), we get

$$\Phi(n, \omega) B_1(0) \subset \bigcup_{i=1}^{N} D_i, \text{ and } d(D_i) \leq \max\left\{e^{n(-\gamma+\varepsilon)}, 2e^{-\gamma n}\right\} \quad \text{for all } i = 1, \dots, N,$$

which implies that

$$\alpha(\Phi(n, \omega) B_1(0)) \leq \max\left\{e^{n(-\gamma+\varepsilon)}, 2e^{-\gamma n}\right\}.$$

This together with $n \geq \frac{\log 2}{\varepsilon}$ proves (4.11). Consequently, we obtain $l_\alpha(\Phi) \leq -\gamma + \varepsilon$. Since ε can be chosen arbitrarily small it follows that $l_\alpha(\Phi) \leq -\gamma$ and together with (4.10) completes the proof. □

We are now in a position to state our main result as an application of the MET by Lian and Lu [89] (see also Theorem 1.4.2).

Theorem 4.2.3 (MET for Difference Equations with Random Delay)**.** *Consider the difference equation* (4.1) *with a measurable random delay map r. Fix $\gamma > 0$ and let Φ denote the corresponding cocycle on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. Assume that the integrability condition*

$$\log^+ |A(\cdot)|, \log^+ |B(\cdot)|, r \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \tag{4.14}$$

holds. Then there exists a θ-invariant subset $\widetilde{\Omega} \subset \Omega$ of full measure such that exactly one of the following alternatives holds:

(I) $\kappa(\Phi) = -\gamma$.

(II) There exists $k \in \mathbb{N}$, Lyapunov exponents $\lambda_1 > \dots > \lambda_k > -\gamma$ *and a splitting into measurable* Oseledets spaces

$$\mathbf{X}_\gamma = E_1(\omega) \oplus \dots \oplus E_k(\omega) \oplus F(\omega)$$

with finite dimensional linear subspaces $E_j(\omega)$ and an infinite dimensional linear subspace $F(\omega)$ such that the following properties hold:

(i) *Invariance:* $\Phi(\omega)E_j(\omega) = E_j(\theta\omega)$ *and* $\Phi(\omega)F(\omega) \subset F(\theta\omega)$.

(ii) *Lyapunov exponents:*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\|_\gamma = \lambda_j \quad \text{for all } v \in E_j(\omega) \setminus \{0\} \text{ and } j = 1, \dots, k \ .$$

(iii) *Exponential Decay Rate on* $F(\omega)$*:*

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{F(\omega)}\|_\gamma \leq -\gamma$$

and if $v \in F(\omega) \setminus \{0\}$ *and* $(\Phi(-n,\omega))v$ *exists for all* $n \geq 0$ *then*

$$\liminf_{n\to+\infty} \frac{1}{n} \log \|\Phi(-n,\omega)v\|_\gamma \geq \gamma \ .$$

Proof. We first show the strong measurability of Φ. Fix $\mathbf{x} \in \mathbf{X}_\gamma$ and define

$$\Omega_n = \{\omega \in \Omega : r(\omega) = n\} \qquad \text{for each } n \in \mathbb{N}.$$

Since $A, B : \Omega \to \mathbb{R}^d$ are measurable, there exist sequences $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$, where $A_k, B_k : \Omega \to \mathbb{R}^d, k \in \mathbb{N}$, are simple functions with the properties

$$\lim_{k\to\infty} A_k(\omega) = A(\omega) \quad \text{and} \quad \lim_{k\to\infty} B_k(\omega) = B(\omega) \qquad \text{for all } \ \omega \in \Omega. \tag{4.15}$$

Define $f_k : \Omega \to \mathbf{X}_\gamma$ by

$$f_k(\omega)(n) = \begin{cases} \mathbf{x}(n+1) & \text{if } n \leq -1, \\ A_k(\omega)\mathbf{x}(0) + B_k(\omega)\mathbf{x}(-r(\omega)) & \text{if } n = 0. \end{cases} \tag{4.16}$$

Due to (4.15), we thus obtain

$$\lim_{k\to\infty} f_k(\omega) = \Phi(\omega)\mathbf{x} \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.17}$$

Measurability of r implies that Ω_n is a measurable set and from (4.16) we get for $\omega \in \Omega_n$

$$f_k(\omega)(n) = \begin{cases} \mathbf{x}(n+1) & \text{if } n \leq -1, \\ A_k(\omega)\mathbf{x}(0) + B_k(\omega)\mathbf{x}(-n) & \text{if } n = 0. \end{cases}$$

Together with the fact that A_k, B_k are simple functions, the map $\Omega \to \mathbf{X}_\gamma$, $\omega \mapsto f_k(\omega)$, is a simple function, i.e. it takes constant values on a measurable partition of Ω. This, together with (4.17), implies the strong measurability of Φ.

By virtue of Lemma 4.2.1, the linear RDS Φ satisfies the integrability condition, i.e. $\log^+ \|\Phi(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, the linear cocycle Φ fulfills all conditions of Theorem 1.4.2. It remains to show that Φ cannot have infinitely many Lyapunov exponents. To prove this, for each $n \in \mathbb{N}$ we define

$$\widetilde{\Omega}_n := \bigcup_{k=1}^{n} \Omega_k = \{\omega \in \Omega : r(\omega) \leq n\}.$$

Set $p_n = \mathbb{P}(\Omega_n)$. A straightforward computation yields that

$$\mathbb{P}(\widetilde{\Omega}_n) = p_1 + \cdots + p_n \qquad \text{for all } n \in \mathbb{N} \tag{4.18}$$

and

$$\sum_{n=1}^{\infty} np_n = \mathbb{E}r := \int_\Omega r(\omega)\, d\mathbb{P}(\omega) < \infty.$$

As a consequence, there exists $k \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} np_n < \frac{1}{2}. \tag{4.19}$$

Define

$$\widehat{\Omega} := \bigcap_{n=k}^{\infty} \theta^{k-n}\widetilde{\Omega}_n.$$

Using the inequality $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$ for all measurable sets $A, B \in \mathcal{F}$, we easily obtain together with (4.19) the following estimate

$$\mathbb{P}(\widehat{\Omega}) \geq 1 - \sum_{n=k}^{\infty}(n-k)p_n \geq \frac{1}{2}. \tag{4.20}$$

Hence, $\widehat{\Omega}$ is a measurable set with positive probability. Define

$$\mathbf{X}_k := \{\mathbf{x} \in \mathbf{X}_\gamma : \mathbf{x}(-n) = 0 \qquad \text{for all } 0 \leq n \leq k\}.$$

Obviously, $\mathbf{X}_k$ is an infinite dimensional subspace of $\mathbf{X}_\gamma$. Furthermore, for each $\omega \in \widehat{\Omega}$ and $\mathbf{x} \in \mathbf{X}_k$ we have

$$\Phi(n,\omega)\mathbf{x}(-m) = \begin{cases} 0 & \text{if } m \leq n+k, \\ \mathbf{x}(n-m) & \text{if } m \geq n+k+1. \end{cases}$$

Thus

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)|_{\mathbf{X}_k}\|_\gamma = -\gamma. \tag{4.21}$$

Define

$$\widetilde{\mathbf{X}}_k := \{\mathbf{x} \in \mathbf{X}_\gamma : \mathbf{x}(-n) = 0 \qquad \text{for all } n \geq k+1\}.$$

Obviously, we have $\dim \widetilde{\mathbf{X}}_k = k+1$ and

$$\mathbf{X}_\gamma = \mathbf{X}_k \oplus \widetilde{\mathbf{X}}_k.$$

Let π_k denote the projection of $\mathbf{X}_\gamma$ on $\widetilde{\mathbf{X}}_k$ along $\mathbf{X}_k$, i.e. $\operatorname{im} \pi_k = \widetilde{\mathbf{X}}_k$ and $\ker \pi_k = \mathbf{X}_k$. Fix $\omega \in \widehat{\Omega}$ and let $\lambda > -\gamma$ be a Lyapunov exponent of Φ and $\mathbf{x} \in \mathbf{X}_\gamma$ a vector corresponding to the Lyapunov exponent λ at the fiber ω, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma = \lambda,$$

which yields that

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\pi_k\mathbf{x}+\Phi(n,\omega)(I-\pi_k)\mathbf{x}\|_\gamma=\lambda.$$

Applying (4.21) with $(I-\pi_k)\mathbf{x}\in\mathbf{X}_k$, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)(I-\pi_k)\mathbf{x}\|_\gamma=-\gamma<\lambda.$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\pi_k\mathbf{x}\|_\gamma=\lambda.$$

Together with the fact that $\dim\widetilde{\mathbf{X}}_k=k+1$, we obtain that Φ has at most $k+1$ different Lyapunov exponents. This completes the proof. □

Theorem 4.2.4 (Lyapunov Exponents are Independent of Exponential Weight Factor)**.** *Let $\gamma>0$ and consider* (4.1) *on the state space $(\mathbf{X}_\gamma,\|\cdot\|_\gamma)$. Assume that $\lambda>-\gamma$ be a Lyapunov exponent of* (4.1)*, i.e. for $\mathbb{P}$-a.e. $\omega\in\Omega$ there exists $\mathbf{x}(\omega)\in\mathbf{X}_\gamma$ such that*

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\gamma=\lambda.$$

Then for every $\zeta>\gamma$ we have $\mathbf{x}(\omega)\in\mathbf{X}_\zeta$ and the number λ is also a Lyapunov exponent of (4.1) *on the state space $(\mathbf{X}_\zeta,\|\cdot\|_\zeta)$. In particular,*

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta=\lambda\qquad\textit{for } \mathbb{P}\textit{-a.e. } \omega\in\Omega. \tag{4.22}$$

Proof. Let $\mathbf{y}\in\mathbf{X}_\gamma$. From the definition of $\mathbf{X}_\gamma$ we obtain that $\lim_{n\to\infty}e^{-\gamma n}\mathbf{y}(-n)$ exists. For $\zeta>\gamma$ it is easy to see that

$$\begin{aligned}\lim_{n\to\infty}e^{-\zeta n}\mathbf{y}(-n) &= \lim_{n\to\infty}e^{(\gamma-\zeta)n}e^{-\gamma n}\mathbf{y}(-n)\\ &= 0,\end{aligned}$$

which implies that $\mathbf{y}\in\mathbf{X}_\zeta$. Furthermore, for any $\mathbf{y}\in\mathbf{X}_\gamma$ we have

$$\begin{aligned}\|\mathbf{y}\|_\zeta &= \sup_{n\in\mathbb{N}_0}e^{-\zeta n}|\mathbf{y}(-n)|\\ &\le \sup_{n\in\mathbb{N}_0}e^{-\gamma n}|\mathbf{y}(-n)|\\ &\le \|\mathbf{y}\|_\gamma.\end{aligned}$$

As a consequence, we get

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta\le\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\gamma, \tag{4.23}$$

and

$$\liminf_{n\to-\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta\ge\lim_{n\to-\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\gamma. \tag{4.24}$$

Let $-\zeta < \lambda_k < \lambda_{k-1} < \cdots < \lambda_1$ be the Lyapunov exponents of the linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ and

$$\mathbf{X}_\zeta = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

the corresponding Oseledets splitting of Φ. We write $\mathbf{x}(\omega)$ in the following form

$$\mathbf{x}(\omega) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k + \mathbf{x}_F,$$

where $\mathbf{x}_i \in E_i(\omega)$ and $\mathbf{x}_F \in F(\omega)$. For convenience, we divide the proof into several steps.

Step 1: We first show that $\mathbf{x}_F = 0$ by contradiction, i.e. we assume that $\mathbf{x}_F \neq 0$. In view of Theorem 4.2.3, we have

$$\limsup_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta,$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lim_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}_i\|_\zeta = \lambda_i.$$

Therefore, for any $\varepsilon \in \left(0, \frac{\lambda_k+\zeta}{4}\right)$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta + \varepsilon \qquad \text{for all } n \leq -N(\varepsilon),$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lambda_i - \varepsilon \leq \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}_i\|_\zeta \leq \lambda_i + \varepsilon \qquad \text{for all } n \leq -N(\varepsilon).$$

Hence, for all $n \leq -N(\varepsilon)$ we have

$$\begin{aligned}
\|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta &= \Big\|\Phi(n,\omega)\mathbf{x}_F + \sum_{i\in\{1,\dots,k\},\mathbf{x}_i\neq 0} \Phi(n,\omega)\mathbf{x}_i\Big\|_\zeta \\
&\geq \|\Phi(n,\omega)\mathbf{x}_F\|_\zeta - \Big\|\sum_{i\in\{1,\dots,k\},\mathbf{x}_i\neq 0} \Phi(n,\omega)\mathbf{x}_i\Big\|_\zeta \\
&\geq e^{n(-\zeta+\varepsilon)} - \sum_{i=1}^k e^{n(\lambda_i-\varepsilon)}.
\end{aligned}$$

Consequently,

$$\frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta \leq \frac{1}{n} \log \left(e^{n(-\zeta+\varepsilon)} - \sum_{i=1}^k e^{n(\lambda_i-\varepsilon)} \right) \qquad \text{for all } n \leq -N(\varepsilon),$$

which implies that

$$\limsup_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta \le -\zeta + \varepsilon,$$

where we use the fact that

$$\lim_{n\to-\infty} \frac{1}{n} \log \left(e^{na} - e^{nb}\right) = a \qquad \text{provided that } a < b,$$

to obtain the last inequality. Since ε can be chosen arbitrarily small it follows together with (4.24) that

$$-\zeta \ge \lim_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\gamma,$$

which contradicts to the fact that

$$-\zeta < -\gamma < \lambda = \lim_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\gamma.$$

Step 2: Define

$$i_{min} := \min_{\mathbf{x}_i \ne 0} i, \qquad i_{max} := \max_{\mathbf{x}_i \ne 0} i.$$

By the same arguments as in Step 1, we obtain that

$$\begin{aligned} \lambda_{i_{min}} &= \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta, \\ \lambda_{i_{max}} &= \lim_{n\to-\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta, \end{aligned}$$

which implies together with (4.23), (4.24), and the fact that $\lambda_{i_{min}} \ge \lambda_{i_{max}}$ that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}(\omega)\|_\zeta = \lambda,$$

proving (4.22) and the proof is completed. □

4.3 Some Examples

It is easy to see that alternative (I) of Theorem 4.2.3 occurs in the trivial case $A(\omega) = B(\omega) = 0$. We now present examples for alternative (II). For convenience, we prefer to consider the scalar difference equation with random delay of the following form

$$x_{n+1} = A(\theta^n\omega)x_n + B(\theta^n\omega)x_{n-r(\theta^n\omega)}, \tag{4.25}$$

where $A, B, r : \Omega \to \mathbb{R}$ are measurable functions. Suppose that $\log^+ |A(\cdot)|, \log^+ |B(\cdot)| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. For a fixed $\gamma > 0$, let $\Phi : \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ denote the RDS generated by (4.25).

4.3.1 Bounded Random Delay

Assume that the random delay map r is bounded, i.e. for a fixed $r^* \in \mathbb{N}$

$$r(\omega) \in \{1, \ldots, r^*\} \qquad \text{for all } \omega \in \Omega\,. \tag{4.26}$$

We define a random map $M : \Omega \to \mathbb{R}^{(r^*+1)\times(r^*+1)}$ by $M(\omega) = \big(m_{i,j}(\omega)\big)_{(r^*+1)\times(r^*+1)}$ with

$$m_{i,j}(\omega) := \begin{cases} A(\omega), & \text{if } i = j = r^*+1; \\ B(\omega), & \text{if } i = r^*+1, j = r^*+1-r(\omega); \\ 1, & \text{if } j = i+1; \\ 0, & \text{otherwise}; \end{cases}$$

i.e.

$$M(\omega) = \begin{pmatrix} 0 & 1 & & & & \\ & & & & & \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & & 0 & 1 \\ 0 & \ldots & B(\omega) & \ldots & 0 & A(\omega) \end{pmatrix}. \tag{4.27}$$

$$\uparrow$$
$$(r^*+1-r(\omega))\text{-th column}$$

Let $\pi : \mathbf{X}_\gamma \to \mathbb{R}^{r^*+1}$ be the projection defined by

$$\pi\mathbf{x} = (x_{-r^*}, \ldots, x_{-1}, x_0)^\top \qquad \text{for all } \mathbf{x} \in \mathbf{X}_\gamma.$$

Using (4.2), a straightforward computation shows the identity

$$\pi\Phi(\omega)\mathbf{x} = M(\omega)\pi\mathbf{x} \qquad \text{for all } \omega \in \Omega, \mathbf{x} \in \mathbf{X}_\gamma. \tag{4.28}$$

On the other hand, the random map M also generates an RDS over the dynamical systems θ via

$$\Phi_M(n,\omega) := \begin{cases} M(\theta^{n-1}\omega) \circ \cdots \circ M(\omega), & n > 0, \\ I_{r^*+1}, & n = 0. \end{cases}$$

Note that on the linear space $\mathbb{R}^{r^*+1}$ all norms are equivalent but for convenience let us fix the following norm

$$\|\mathbf{v}\| = \max_{0\le n\le r^*} e^{-\gamma n}|v_{-n}| \qquad \text{for all } \mathbf{v} = (v_{-r^*}, \ldots, v_{-1}, v_0)^\top \in \mathbb{R}^{r^*+1},$$

and

$$\|Q\| = \sup_{0\neq\mathbf{v}\in\mathbb{R}^{r^*+1}} \frac{\|Q\mathbf{v}\|}{\|\mathbf{v}\|} \qquad \text{for all } Q \in \mathbb{R}^{(r^*+1)\times(r^*+1)}.$$

From the definition of $M(\cdot)$ it is clear that $\log^+\|M(\cdot)\| \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$. Hence, the Multiplicative Ergodic Theorem (see Arnold [3, pp. 134]) ensures that the limit

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi_M(n,\omega)\mathbf{v}\|, \qquad \text{where } \mathbf{v}\in\mathbb{R}^{r^*+1}, \tag{4.29}$$

exists on a subset $\widehat{\Omega}\subset\Omega$ of full measure and takes on finitely many (non-random) values

$$-\infty \le \beta_k < \cdots < \beta_1 \qquad \text{for some } k\le r^*+1$$

as $\mathbf{v}$ varied over $\mathbb{R}^{r^*+1}$. Moreover, there exists a filtration of $\mathbb{R}^{r^*+1}$

$$\{0\} := V_{k+1}(\omega)\subset V_k(\omega)\subset\cdots\subset V_1(\omega)=\mathbb{R}^{r^*+1}$$

such that the limit (4.29) equals β_i, for $i=1,\dots,k$, if and only if $\mathbf{v}\in V_i(\omega)\setminus V_{i+1}(\omega)$. In the following theorem, we provide a relation between Lyapunov exponents of the RDS Φ_M and Φ.

Theorem 4.3.1 (Lyapunov Exponents for Bounded Delay)**.** *Consider the difference equation*

$$x_{n+1} = A(\theta^n\omega)x_n + B(\theta^n\omega)x_{n-r(\theta^n\omega)}$$

with bounded delay (4.26) *on the state space* $\mathbf{X}_\gamma$*. Define*

$$k^* := \min\{i : \beta_i > -\gamma\}.$$

Then there exist exactly k^* *Lyapunov exponents* $\lambda_{k^*}<\cdots<\lambda_1$ *of* Φ *and*

$$\{\beta_1,\dots,\beta_{k^*}\} = \{\lambda_1,\dots,\lambda_{k^*}\}\ .$$

Proof. Firstly, we show that $\lambda_i\in\{\beta_1,\beta_2,\dots,\beta_{k^*}\}$ for any $i\in\mathbb{N}$. Fix $i\in\mathbb{N}$ and let $0\neq\mathbf{x}\in\mathbf{X}_\gamma$ be a vector realizing the Lyapunov exponent λ_i, i.e.

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\mathbf{x}\|_\gamma = \lambda_i. \tag{4.30}$$

Now we show that

$$\beta := \lim_{n\to\infty}\frac{1}{n}\log\|\Phi_M(n,\omega)\pi\mathbf{x}\| = \lambda_i. \tag{4.31}$$

From (4.28), we derive

$$\Phi_M(n,\omega)\pi\mathbf{x} = \pi\Phi(n,\omega)\mathbf{x} \qquad \text{for all } n\in\mathbb{N},\omega\in\Omega, \tag{4.32}$$

which implies with $\|\pi\Phi(n,\omega)\mathbf{x}\|\le\|\Phi(n,\omega)\mathbf{x}\|_\gamma$ that $\beta\le\lambda_i$. Let us assume that $\beta<\lambda_i$ and derive a contradiction. Since $\beta<\lambda_i$ it follows that there exists $\varepsilon>0$ such that $\beta+2\varepsilon<\lambda_i$. From (4.30) and (4.31), $N\in\mathbb{N}$ can be chosen large enough such that

$$\|\Phi_M(n,\omega)\pi\mathbf{x}\| \le e^{(\beta+\varepsilon)n} \le \|\Phi(n,\omega)\mathbf{x}\|_\gamma \qquad \text{for all } n\ge N. \tag{4.33}$$

This implies with the definition of π and (4.32) that

$$\left|\big(\Phi(n,\omega)\mathbf{x}\big)_{-j}\right| \le e^{\gamma j} e^{(\beta+\varepsilon)n} \qquad \text{for all } 0 \le j \le r^*, n \ge N.$$

On the other hand, we have $\big(\Phi(n+1,\omega)\mathbf{x}\big)_{-j} = \big(\Phi(n,\omega)\mathbf{x}\big)_{-j+1}$ for all $j \ge 1$. Thus, we obtain

$$\begin{aligned}\|\Phi(n+1,\omega)\mathbf{x}\|_\gamma &\le \max\left\{e^{-\gamma}\|\Phi(n,\omega)\mathbf{x}\|_\gamma, \left|\big(\Phi(n+1,\omega)\mathbf{x}\big)_0\right|\right\}\\ &\le \max\left\{e^{-\gamma}\|\Phi(n,\omega)\mathbf{x}\|_\gamma, e^{(\beta+\varepsilon)(n+1)}\right\}.\end{aligned}$$

Together with (4.33) we derive

$$\|\Phi(n+1,\omega)\mathbf{x}\|_\gamma \le e^{-\gamma}\|\Phi(n,\omega)\mathbf{x}\|_\gamma \qquad \text{for all } n \ge N.$$

Therefore,

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma \le -\gamma$$

which contradicts to the fact that $\lambda_i > -\gamma$ and proves (4.31). Consequently, we get $\lambda_i \in \{\beta_1, \beta_2, \ldots, \beta_{k^*}\}$ for all $i \in \mathbb{N}$. For the remaining part of the proof, let $\beta_i > -\gamma$ be a Lyapunov exponent of Φ_M and $0 \ne \mathbf{v} = (v_{-r^*}, \ldots, v_{-1}, v_0) \in \mathbb{R}^{r^*+1}$ a vector realizing this Lyapunov exponent, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_M(n,\omega)\mathbf{v}\| = \beta_i.$$

Define $\mathbf{x} = (\ldots, x_{-1}, x_0) \in \mathbf{X}_\gamma$ by

$$x_{-j} = \begin{cases} v_{-j}, & \text{if } 0 \le j \le r^*,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\pi\mathbf{x} = \mathbf{v}$ and from (4.32), together with the fact that $\|\pi\Phi(n,\omega)\mathbf{x}\| \le \|\Phi(n,\omega)\mathbf{x}\|_\gamma$, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma \ge \beta_i > -\gamma.$$

This implies with (4.31) that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\mathbf{x}\|_\gamma = \beta_i,$$

which proves that $\beta_i \in \{\lambda_1, \ldots, \lambda_{k^*}\}$. and the proof is complete. □

Remark 4.3.2. Note that the random map M in (4.27) takes only finitely many values and thus the RDS Φ_M can be considered as the iteration of finitely many matrices with some specific chosen probability at each step. By approximating the invariant measure for such an iterated function system we can numerically compute the Lyapunov exponents of Φ_M (see Cong, Doan and Siegmund [38] and also in Chapter 6).

4.3.2 Deterministic Delay

Now we deal with a special case of bounded delay, namely a fixed deterministic delay, see Elaydi [48] for a more comprehensive treatment. The technical advantage in this special situation is that we can construct an invertible finite dimensional matrix, based on which we can represent both Lyapunov exponents as well as the Oseledets splitting of Φ. Assume that the random delay map $r : \Omega \to \mathbb{N}$ takes a constant value $r \in \mathbb{N}$. We consider the scalar difference equation with fixed delay time r

$$x_{n+1} = Ax_n + Bx_{n-r}, \qquad \text{where } B \neq 0. \tag{4.34}$$

It can be rewritten as a system

$$\big(x_{n-r-1}, \dots, x_{n+1}\big)^\top = M\big(x_{n-r}, \dots, x_n\big)^\top$$

with the matrix

$$M = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ B & 0 & \dots & 0 & A \end{pmatrix}$$

of dimension $r+1$. Its characteristic polynomial is

$$\sigma^{r+1} - A\sigma^r - B = 0\ .$$

Every root σ gives rise to a Lyapunov exponent $\log|\sigma|$ of (4.34). Let σ_1 denote the root with largest absolute value $|\sigma_1|$. Then (4.34) is asymptotically stable if $|\sigma_1| < 1$ and unstable if $|\sigma_1| > 1$. E.g. for $r = 2$ these regions are shown in Figure 6.2. Obviously, the generated RDS $\Phi(\omega)$ is independent of ω and we define

$$E := \big\{\mathbf{x} \in \mathbf{X}_\gamma : \Phi^{-n}\mathbf{x} \text{ exists for all } n \in \mathbb{N}\big\}\ . \tag{4.35}$$

It is easy to show that E is an invariant subspace of Φ in both backward and forward time, i.e. $\Phi E = E$. Let π be the projection map from $\mathbf{X}_\gamma$ into $\mathbb{R}^{r+1}$ defined by

$$\pi\mathbf{x} = (x_{-r}, \dots, x_{-1}, x_0) \qquad \text{for all } \mathbf{x} \in \mathbf{X}_\gamma.$$

A direct computation yields that

$$\pi\Phi^n\mathbf{x} = M^n\pi\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbf{X}_\gamma, n \in \mathbb{N}\ .$$

Consequently, $\pi_E : E \to \mathbb{R}^{r+1}$ is a linear bijective map, where π_E is the restriction of π on the linear subspace E. Let $\{\beta_i\}_{i=1}^k$ be the set of the logarithms of the moduli of the eigenvalues of M and $\{W_i\}_{i=1}^k$ the subspaces of $\mathbb{R}^{r+1}$ realizing $\{\beta_i\}_{i=1}^k$, respectively. Hence,

$$\mathbb{R}^{r+1} = W_1 \oplus \cdots \oplus W_{k-1} \oplus W_k\ .$$

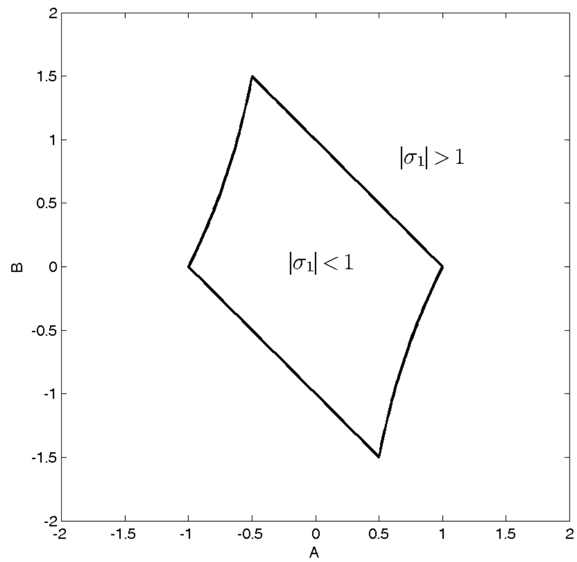

Figure 4.1: System (4.34) for $r = 2$ is stable if $|\sigma_1| < 1$ and unstable if $|\sigma_1| > 1$.

For an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^{r+1}$ then one has

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|M^n \mathbf{v}\| = \beta_i \quad \text{if and only if } \mathbf{v} \in W_i \setminus \{0\}.$$

The following theorem gives a relation between the Lyapunov exponents of M and Φ as well as the Oseledets splitting realizing these exponents.

Theorem 4.3.3 (Lyapunov Exponents for Deterministic Delay)**.** *Let γ be a positive number such that $-\gamma < \min_{1\le i\le k} \beta_i$. Then there exist exactly k Lyapunov exponents*

$$-\gamma < \lambda_k < \lambda_{k-1} < \cdots < \lambda_1$$

of Φ and

$$\{\beta_1, \beta_2, \ldots, \beta_k\} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}.$$

Moreover, let E_i denote the subspace of $\mathbf{X}_\gamma$ corresponding to λ_i in Theorem 4.2.3 then

$$E_i = \pi_E^{-1} W_i \qquad \text{for all } i = 1, \ldots, k,$$

and the infinite dimensional part F in the Oseledets splitting of Φ in Theorem 4.2.3 is determined as follows

$$F = \{\mathbf{x} = (\ldots, x_{-1}, x_0) \in \mathbf{X}_\gamma : x_0 = x_{-1} = \cdots = x_{-r} = 0\}.$$

Proof. By virtue of Theorem 4.3.1, we get $\lambda_i = \beta_i$ for all $1 \leq i \leq k$. Note that in Lian and Lu [89] (see also Theorem 1.4.2) the subspace E_i of $\mathbf{X}_\gamma$ realizing the Lyapunov exponent λ_i can be determined as follows

$$E_i := \Big\{\mathbf{x} \in \mathbf{X}_\gamma : \Phi^{-n}\mathbf{x} \text{ exists for all } n \in \mathbb{N} \text{ and } \lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi^n \mathbf{x}\| = \lambda_i\Big\}.$$

From the definition of E, see (4.35), one has $E_i \subset E$. Hence, we get

$$E_i = \Big\{\mathbf{x} \in E : \lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi^n \mathbf{x}\| = \lambda_i\Big\}.$$

Now we show that $E_i = \pi_E^{-1} W_i$ for all $i = 1, \dots, k$. Equivalently, we show that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|\Phi^n \mathbf{x}\|_\gamma = \lambda_i \text{ if and only if } \lim_{n\to\pm\infty} \frac{1}{n} \log \|M^n \pi_E \mathbf{x}\| = \lambda_i. \tag{4.36}$$

Using Theorem 4.3.1, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi^n \mathbf{x}\|_\gamma = \lim_{n\to\infty} \frac{1}{n} \log \|M^n \pi_E \mathbf{x}\| \qquad \text{for all } \mathbf{x} \in E. \tag{4.37}$$

Replacing Φ by Φ^{-1} and M by M^{-1}, we obtain

$$\lim_{n\to-\infty} \frac{1}{n} \log \|\Phi^n \mathbf{x}\|_\gamma = \lim_{n\to-\infty} \frac{1}{n} \log \|M^n \pi_E \mathbf{x}\| \qquad \text{for all } \mathbf{x} \in E,$$

this together with (4.37) implies statement (4.36). It thus remains to determine the infinite dimensional part of the Oseledets splitting of Φ. Define

$$F := \Big\{\mathbf{x} = (\dots, x_{-1}, x_0) \in \mathbf{X}_\gamma : x_0 = x_{-1} = \cdots = x_{-r} = 0\Big\}.$$

Obviously, F is invariant under Φ, i.e. $\Phi F \subset F$, and

$$\mathbf{X}_\gamma = E \oplus F = E_1 \oplus E_2 \oplus \cdots \oplus E_k \oplus F.$$

In order to prove that F is the infinite dimensional part as described in Theorem 4.2.3 (ii) it is sufficient to show that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi^n|_F\|_\gamma \leq -\gamma.$$

Indeed, from the definition of F it is easy to obtain that

$$\|\Phi^n \mathbf{x}\|_\gamma = e^{-\gamma n} \|\mathbf{x}\|_\gamma \qquad \text{for all } \mathbf{x} \in F, n \in \mathbb{N}.$$

Therefore, $\|\Phi^n|_F\|_\gamma = e^{-\gamma n}$ and this completes the proof. ◻

Chapter 5

Differential Equations with Random Delay

Delays in difference and in differential equations are used for mathematical modeling in many applications for the description of evolutions which incorporate influences of events from the past. Delays appear quite often in biological models when traditional pointwise modeling assumptions are replaced by more realistic distributed assumptions.

In contrast to ordinary differential equations the set of initial values of a differential equation with delay is an infinite dimensional space. As a consequence, a lot of technical problems arise when we deal with delay equations. Based on the recent work of Lian and Lu [89] the first step towards a general theory of difference equations incorporating random delays which are not assumed to be bounded is established in Chapter 4. In this chapter, we extend this work to differential equations with random delays.

5.1 Differential Equations with Random Delay

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\theta_t)_{t\in\mathbb{R}} : \Omega \to \Omega$ an ergodic flow which preserves the probability measure $\mathbb{P}$ and which has measurable inverse, and let $r : \Omega \to \mathbb{R}^+$ be a measurable map. We consider a random linear differential equation with random delay

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)) \qquad \text{for } t \geq 0, \tag{5.1}$$

where $A, B \in \mathcal{L}^1(\mathbb{P})$. In order to introduce an RDS generated by (5.1) we first need to construct an appropriate state space. Since the delay map r is in general unbounded, an initial value for (5.1) is a continuous function $\mathbf{x} : (-\infty, 0] \to \mathbb{R}^d$. A corresponding form of (5.1) is given by

$$\varphi(t,\omega)\mathbf{x} = \mathbf{x}(0) + \int_0^t A(\theta_s\omega)\varphi(s,\omega)\mathbf{x} + B(\theta_s\omega)\varphi(s - r(\theta_s\omega),\omega)\mathbf{x}\, ds, \tag{5.2}$$

with the convention that $\varphi(s,\omega)\mathbf{x} = \mathbf{x}(s)$ for all $s \leq 0$, which is valid for all $t \geq 0$. If (5.2) holds, we say that $t \mapsto \varphi(t,\omega)\mathbf{x} =: \varphi_\omega(t,\mathbf{x})$ *solves*, or is a *solution* of, equation (5.1) starting at 0 in $\mathbf{x}$.

Since in the unbounded delay case the initial data is always a part of the solution, some kind of regularity must be imposed from the beginning. The continuous time setting is discussed e.g. in Hale and Kato [63] and Hino, Murakami and Naito [69]. This leads us to work with a canonical phase space

$$\begin{aligned}\mathbf{X}_\gamma &:= \left\{\mathbf{x} \in \mathrm{C}\big((-\infty,0],\mathbb{R}^d\big) : \lim_{t\to-\infty} e^{\gamma t}\mathbf{x}(t) \quad \text{exists}\right\},\\ \|\mathbf{x}\|_\gamma &:= \sup_{t\in(-\infty,0]} e^{\gamma t}|\mathbf{x}(t)|.\end{aligned}$$

Throughout this chapter we assume that $\gamma > 0$ and consider system (5.1) on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. It is easy to see that $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ is a Banach space. The following lemma ensures the separability of the space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$.

Lemma 5.1.1. *For $\gamma > 0$ the space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ is separable.*

Proof. Let $\mathbb{Q}^d$ denote the set of all vectors in $\mathbb{R}^d$ whose components are rational. Clearly, $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$. For each $N \in \mathbb{N}$ we consider the Banach space $\mathrm{C}([-N,0],\mathbb{R}^d)$ together with the sup norm $\|\cdot\|_\infty$, i.e.

$$\|f\|_\infty = \sup_{t\in[-N,0]} |f(t)| \qquad \text{for all } f \in \mathrm{C}([-N,0],\mathbb{R}^d).$$

It is well known that $\big(\mathrm{C}([-N,0],\mathbb{R}^d), \|\cdot\|_\infty\big)$ is a separable Banach space (see e.g. Willard [141]). Consequently, there exists a countable set

$$A_N := \{f_1^{(N)}, f_2^{(N)}, \dots\}, \qquad f_1^{(N)}, f_2^{(N)}, \dots \in \mathrm{C}([-N,0],\mathbb{R}^d),$$

which is dense in $\big(\mathrm{C}([-N,0],\mathbb{R}^d), \|\cdot\|_\infty\big)$. For each function $f_k^{(N)}$, $v \in \mathbb{Q}^d$ and $p \in \mathbb{Q}^+$ we defined the extended function $\widetilde{f}_{k,v,p}^{(N)} : (-\infty,0] \to \mathbb{R}^d$ by

$$\widetilde{f}_{k,v,p}^{(N)}(t) := \begin{cases} f_k^{(N)}(t), & t \in [-N,0],\\ \left(\frac{N+t}{p}+1\right) f_k^{(N)}(-N) - \frac{N+t}{p}\, e^{(N+p)\gamma} v, & t \in [-N-p,-N),\\ e^{-\gamma t} v & t \in (-\infty, -N-p). \end{cases} \tag{5.3}$$

Obviously, for all $k \in \mathbb{N}, v \in \mathbb{Q}^d$ and $p \in \mathbb{Q}^+$ the function $\widetilde{f}_{k,v,p}^{(N)}$ is continuous and

$$\lim_{t\to-\infty} e^{\gamma t} \widetilde{f}_{k,v,p}^{(N)}(t) = v,$$

which implies that $\widetilde{f}_{k,v,p}^{(N)} \in \mathbf{X}_\gamma$. Define

$$\widetilde{A}_N = \bigcup_{v,p\in\mathbb{Q}^d\times\mathbb{Q}^+} \left\{\widetilde{f}_{1,v,p}^{(N)}, \widetilde{f}_{2,v,p}^{(N)}, \dots\right\} \qquad \text{for all } N \in \mathbb{N}.$$

To prove the separability of the Banach space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$, it is sufficient to show that

$$\bigcup_{N\in\mathbb{N}} \widetilde{A}_N \qquad \text{is dense in } (\mathbf{X}_\gamma, \|\cdot\|_\gamma). \tag{5.4}$$

For a given $\mathbf{x} \in \mathbf{X}_\gamma$, set $u := \lim_{t\to-\infty} e^{\gamma t}\mathbf{x}(t)$. Hence, for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|e^{\gamma t}\mathbf{x}(t) - u| \leq \frac{\varepsilon}{8} \qquad \text{for all } t \leq -N. \tag{5.5}$$

Since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$ it follows that there exists $v \in \mathbb{Q}^d$ such that $|v-u| \leq \frac{\varepsilon}{8}$. On the other hand, due to the denseness of A_N in the space $\mathrm{C}([-N,0],\mathbb{R}^d)$ there exists $k \in \mathbb{N}$ such that

$$\sup_{t\in[-N,0]} |f_k^{(N)}(t) - \mathbf{x}(t)| < \frac{\varepsilon}{8}. \tag{5.6}$$

Direct estimations yield that

$$\begin{aligned}
\lim_{p\to 0} \sup_{t\in[-N-p,-N]} e^{\gamma t}\left|f_k^{(N)}(-N) - \mathbf{x}(t)\right| &= e^{-\gamma N}\left|f_k^{(N)}(-N) - \mathbf{x}(-N)\right| \\
&\leq e^{-\gamma N}\frac{\varepsilon}{8}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{p\to 0} \sup_{t\in[-N-p,-N]} e^{\gamma t}\left|f_k^{(N)}(-N) - e^{(N+p)\gamma}v\right| &= \left|e^{-\gamma N} f_k^{(N)}(-N) - v\right| \\
&\leq e^{-\gamma N}\frac{\varepsilon}{8} + \left|e^{-\gamma N}\mathbf{x}(-N) - u\right| + |u-v| \\
&\leq \frac{3\varepsilon}{8}.
\end{aligned}$$

As a consequence, there exists $p \in \mathbb{Q}^+$ such that for all $t \in [-N-p,-N]$ we have

$$e^{\gamma t}\left|f_k^{(N)}(-N) - \mathbf{x}(t)\right| \leq \frac{\varepsilon}{3}, \quad e^{\gamma t}\left|f_k^{(N)}(-N) - e^{(N+p)\gamma}v\right| \leq \frac{\varepsilon}{2}. \tag{5.7}$$

We now estimate $\|\widetilde{f}_{k,v,p}^{(N)} - \mathbf{x}\|_\gamma$. By (5.3) and (5.6), the relation

$$\begin{aligned}
e^{\gamma t}\left|\widetilde{f}_{k,v,p}^{(N)}(t) - \mathbf{x}(t)\right| &= e^{\gamma t}|f_k^{(N)}(t) - \mathbf{x}(t)| \\
&\leq |f_k^{(N)}(t) - \mathbf{x}(t)| \\
&\leq \frac{\varepsilon}{8}
\end{aligned}$$

holds for all $t \in [-N, 0]$. For all $t \in (-\infty, -N-p]$, by (5.3) we have

$$\begin{aligned}
e^{\gamma t}\left|\widetilde{f}_{k,v,p}^{(N)}(t) - \mathbf{x}(t))\right| &= e^{\gamma t}|e^{-\gamma t}v - \mathbf{x}(t)| \\
&\leq |u-v| + |u - e^{\gamma t}\mathbf{x}(t)| \\
&\leq \frac{\varepsilon}{4},
\end{aligned}$$

where we use (5.5) to obtain the last inequality. On the other hand, for all $t \in [-N-p, -N]$ by (5.3) we have

$$\begin{aligned} e^{\gamma t}\left|\widetilde{f}^{(N)}_{k,v,p}(t) - \mathbf{x}(t))\right| &= e^{\gamma t}\left|\left(\frac{N+t}{p}+1\right) f_k^{(N)}(-N) - \frac{N+t}{p} e^{(N+p)\gamma} v - \mathbf{x}(t)\right| \\ &\leq e^{\gamma t}\left|f_k^{(N)}(-N) - \mathbf{x}(t)\right| + e^{\gamma t}\left|f_k^{(N)}(-N) - e^{(N+p)\gamma} v\right| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{2}, \end{aligned}$$

where we use (5.7) to obtain the last inequality. Therefore, we have

$$\|\widetilde{f}^{(N)}_{k,v,p} - \mathbf{x}\|_\gamma = \sup_{t\in(-\infty,0]} e^{\gamma t}|\widetilde{f}^{(N)}_{k,v,p}(t) - \mathbf{x}(t)| \leq \varepsilon,$$

which proves that $\widetilde{A}$ is dense in $\mathbf{X}_\gamma$ and the proof is completed. □

In the following theorem, we give a sufficient condition for the existence and uniqueness of solution of (5.1) on the state space $\mathbf{X}_\gamma$.

Theorem 5.1.2 (Existence of Solutions)**.** *Suppose that $A(\cdot), B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Then there exists a measurable set $\widetilde{\Omega}$ of full measure such that for every $\omega \in \widetilde{\Omega}$ the following pathwise random delay differential equation*

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)), \tag{5.8}$$

with the initial condition, $x(t) = \mathbf{x}(t)$ for all $t \in (-\infty, 0]$ for some $\mathbf{x} \in \mathbf{X}_\gamma$, has a unique solution on $\mathbb{R}$, denoted by $\varphi_\omega(\cdot, \mathbf{x})$. Furthermore, for a fixed $\mathbf{x} \in \mathbf{X}_\gamma$ and $T > 0$ the map $\widetilde{\Omega} \to \mathbb{R}^d$, defined by

$$\omega \mapsto \varphi_\omega(T, \mathbf{x}),$$

is measurable.

Proof. For convenience, we divide the proof into the several steps.

Step 1: We define

$$\widetilde{\Omega} := \{\omega \in \Omega : t \mapsto \|A(\theta_t\omega)\| + \|B(\theta_t\omega)\|e^{\gamma r(\theta_t\omega)} \quad \text{is locally integrable}\}. \tag{5.9}$$

It is easy to see that $\widetilde{\Omega}$ is a θ-invariant measurable set and $\mathbb{P}(\widetilde{\Omega}) = 1$ (see e.g. [3, Lemma 2.2.5]). We finish this step by showing that for all $0 < a < b$ and measurable functions $f : \widetilde{\Omega} \to \mathbb{R}^d$ the following function

$$\omega \mapsto \int_a^b A(\theta_s\omega)f(\omega)\, ds \qquad \text{is measurable for all } t \geq 0. \tag{5.10}$$

Since

$$\int_a^b |A(\theta_s\omega)v|\, ds \leq |v| \int_a^b \|A(\theta_s\omega)\|\, ds < \infty \quad \text{for all } v \in \mathbb{R}^d,$$

it follows that the map

$$\omega \mapsto \int_a^b A(\theta_s\omega)v\, ds \qquad \text{is measurable for all } v \in \mathbb{R}^d.$$

By approximating f by a sequence of simple functions, (5.10) is proved.

Step 2: For a fixed $\omega \in \widetilde{\Omega}$ and $T \in \mathbb{R}^+$ we show that equation (5.8) has a unique solution on $[0,T]$ with the initial value $\mathbf{x} \in \mathbf{X}_\gamma$. Define

$$\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d) := \{\mathbf{f} \in \mathrm{C}([0,T],\mathbb{R}^d): \quad \mathbf{f}(0) = \mathbf{x}(0)\}.$$

Obviously, $\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ is a closed subset of $\mathrm{C}([0,T],\mathbb{R}^d)$. Corresponding to each function $\mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$, we define the function $\widetilde{\mathbf{f}} : (-\infty,T] \to \mathbb{R}^d$ by

$$\widetilde{\mathbf{f}}(t) = \begin{cases} \mathbf{f}(t), & \text{if } t \geq 0, \\ \mathbf{x}(t), & \text{if } t \leq 0. \end{cases}$$

By the definition of $\mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$, $\widetilde{\mathbf{f}}$ is a continuous function from $(-\infty,T]$ to $\mathbb{R}^d$. Furthermore, by the definition of $\mathbf{X}_\gamma$ we have

$$\left|\widetilde{\mathbf{f}}(s - r(\theta_s\omega))\right| \leq \max\left\{\sup_{0\leq t\leq T} |\mathbf{f}(t)|, \|\mathbf{x}\|_\gamma e^{\gamma r(\theta_s\omega)}\right\} \qquad \text{for all } s \in [0,T].$$

Hence, by (5.9) we obtain

$$\int_0^t |A(\theta_s\omega)\mathbf{f}(s)|\, ds, \int_0^t |B(\theta_s\omega)\widetilde{\mathbf{f}}(s - r(\theta_s\omega))|\, ds < \infty \qquad \text{for all } t \in [0,T].$$

Therefore, to solve equation (5.8) we define the following operator $\mathbf{T}_\omega : \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d) \to \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ by

$$\mathbf{T}_\omega\mathbf{f}(t) := \mathbf{f}(0) + \int_0^t A(\theta_s\omega)\mathbf{f}(s)\, ds + \int_0^t B(\theta_s\omega)\widetilde{\mathbf{f}}(s - r(\theta_s\omega))\, ds \quad \text{for all } t \in [0,T]. \tag{5.11}$$

Clearly, $\mathbf{T}_\omega\mathbf{f}$ is a continuous function and $\mathbf{T}_\omega\mathbf{f}(0) = \mathbf{f}(0) = \mathbf{x}(0)$. Hence, $\mathbf{T}_\omega$ is well-defined. Let $\mathbf{f},\mathbf{g} \in \mathrm{C}_{\mathbf{x}}([0,T],\mathbb{R}^d)$ we show that

$$|\mathbf{T}_\omega^n\mathbf{f}(t) - \mathbf{T}_\omega^n\mathbf{g}(t)| \leq \frac{1}{n!}\left|\int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, ds\right|^n \|\mathbf{f} - \mathbf{g}\| \tag{5.12}$$

for all $t \in [0,T], n \in \mathbb{N}$. Indeed, due to (5.11) we obtain

$$\begin{aligned} |\mathbf{T}_\omega\mathbf{f}(t) - \mathbf{T}_\omega\mathbf{g}(t)| \;=\; & \int_0^t A(\theta_s\omega)(\mathbf{f}(s) - \mathbf{g}(s))\, ds + \\ & \int_0^t B(\theta_s\omega)\big(\widetilde{\mathbf{f}}(s - r(\theta_s\omega)) - \widetilde{\mathbf{g}}(s - r(\theta_s\omega))\big)\, ds. \end{aligned}$$

Together with the fact that $\widetilde{\mathbf{f}}(t) = \widetilde{\mathbf{g}}(t)$ for all $t \leq 0$ we have

$$|\mathbf{T}_\omega \mathbf{f}(t) - \mathbf{T}_\omega \mathbf{g}(t)| \leq \int_0^t \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\| \, ds \cdot \|\mathbf{f} - \mathbf{g}\|,$$

which proves that inequality (5.12) holds for $n = 1$. Now assume that inequality (5.12) is proven for some $n \in \mathbb{N}$. For $n+1$, using the proof for $n = 1$, we have

$$\begin{aligned}
\|\mathbf{T}_\omega^{n+1}\mathbf{f}(t) - \mathbf{T}_\omega^{n+1}\mathbf{g}(t)\| &\leq \int_0^t \|A(\theta_s\omega)\| |\mathbf{T}_\omega^n \mathbf{f}(s) - \mathbf{T}_\omega^n \mathbf{g}(s)| \, ds + \\
&\qquad \int_0^t \|B(\theta_s\omega)\| |\widetilde{\mathbf{T}}_\omega^n \mathbf{f}(s) - \widetilde{\mathbf{T}}_\omega^n \mathbf{g}(s)| \, ds \\
&\leq \int_0^t l(s) \cdot \frac{1}{n!} \left(\int_0^s l(u) \, du \right)^n ds \cdot \|\mathbf{f} - \mathbf{g}\|,
\end{aligned}$$

where $l(s) := \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|$. Together with the equality

$$\int_0^t l(s) \cdot \frac{1}{n!} \left(\int_0^s l(u) \, du \right)^n ds = \frac{1}{(n+1)!} \left(\int_0^t l(s) \, ds \right)^{n+1}$$

this proves (5.12) for $n+1$. Due to Step 1 we know that

$$\int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\| \, ds < \infty.$$

Therefore, there exists $N \in \mathbb{N}$ such that

$$K_N := \frac{1}{N!} \left| \int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\| \, ds \right|^N < 1,$$

which together with (5.12) implies that $\mathbf{T}_\omega^N$ is a contractive map from $\mathrm{C}_{\mathbf{x}}([0,T], \mathbb{R}^d)$ into itself. As an application of the Banach fixed point theorem, there exists a unique fixed point in $\mathrm{C}_{\mathbf{x}}([0,T], \mathbb{R}^d)$ for $\mathbf{T}_\omega$ denoted by $\mathbf{f}_\omega$. Since T can be chosen arbitrarily we thus can extend $\mathbf{f}_\omega$ to achieve the unique continuous function $\mathbf{f}_\omega : [0,\infty) \to \mathbb{R}^d$ such that $\mathbf{f}_\omega(0) = \mathbf{x}(0)$ and

$$\mathbf{f}_\omega(t) := \mathbf{f}_\omega(0) + \int_0^t A(\theta_s\omega)\mathbf{f}_\omega(s) \, ds + \int_0^t B(\theta_s\omega)\widetilde{\mathbf{f}}_\omega(s - r(\theta_s\omega)) \, ds \quad \text{for all } t \in \mathbb{R}_+.$$

In other words, equation (5.8) has a unique solution for each $\omega \in \widetilde{\Omega}$.

Step 3: It remains to show the measurability of the map $\widetilde{\Omega} \to \mathbb{R}^d$ defined by

$$\omega \mapsto \varphi_\omega(T, \mathbf{x}),$$

where $\mathbf{x} \in \mathbf{X}_\gamma$ and $T > 0$ are fixed and $\varphi_\omega(\cdot, \mathbf{x})$ is the solution of (5.8) with the initial value $\mathbf{x}$. Choose and fix $\mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,T], \mathbb{R}^d)$. Define a sequence of functions $g_n : [0,T] \times \Omega \to \mathbb{R}^d$ by

$$g_n(t,\omega) = \mathbf{T}_\omega^n \mathbf{f}(t) \qquad \text{for all } (t,\omega) \in [0,T] \times \widetilde{\Omega}.$$

By (5.11), we have

$$g_{n+1}(t,\omega) = \mathbf{f}(0) + \int_0^t A(\theta_s\omega) g_n(s,\omega)\, ds + \int_0^t B(\theta_s\omega)\widetilde{g}_n(s - r(\theta_s\omega),\omega)\, ds. \tag{5.13}$$

On the other hand, as is proved in Step 2 we have

$$\varphi_\omega(T,\mathbf{x}) = \lim_{n\to\infty} \mathbf{T}_\omega^n \mathbf{f}(T) = \lim_{n\to\infty} g_n(T,\omega) \quad \text{for all } \omega \in \widetilde{\Omega}.$$

Therefore, it is sufficient to show the measurability of the mappings $g_n(t,\cdot) : \widetilde{\Omega} \to \mathbb{R}^d$ for all $t \in [0,T], n \in \mathbb{N}$. We will prove this fact by induction. Clearly, the statement holds with $n = 0$. Suppose that for some $n \in \mathbb{N}$ the function $g_n(t,\cdot) : \Omega \to \mathbb{R}^d$ is measurable for all $t \in [0,T]$. Choose and fix $t \in [0,T]$. For each $k \in \mathbb{N}$, define $g_n^k : [0,t) \times \widetilde{\Omega} \to \mathbb{R}^d$ by

$$g_n^k(s,\omega) = \sum_{i=0}^{k-1} \chi_{[\frac{it}{k},\frac{(i+1)t}{k})}(s) g_n\left(\frac{it}{k},\omega\right) \qquad \text{for all } (s,\omega) \in [0,t) \times \widetilde{\Omega}.$$

Together with the fact that $g_n(\cdot,\omega) : [0,t] \to \mathbb{R}^d$ is a continuous function we derive that

$$\lim_{k\to\infty} \int_0^t A(\theta_s\omega) g_n^k(s,\omega)\, ds = \int_0^t A(\theta_s\omega) g_n(s,\omega)\, ds \qquad \text{for all } \omega \in \widetilde{\Omega}.$$

As a consequence, by using Step 1 the mapping

$$\omega \mapsto \int_0^t A(\theta_s\omega) g_n(s,\omega)\, ds$$

is $\mathcal{F},\mathcal{B}(\mathbb{R}^d)$-measurable for all $t \in [0,T]$. On the other hand, due to the measurability of the mapping $(s,\omega) \mapsto r(\theta_s\omega)$ there is a sequence of simple functions from $[0,t] \times \Omega$ to $\mathbb{R}$ converging pointwise to r. Using similar arguments as above, we also obtain the measurability of the map

$$\omega \mapsto \int_0^t B(\theta_s\omega)\widetilde{g}_n(s - r(\theta_s\omega),\omega)\, ds.$$

Hence, the mapping $g_n(t,\cdot)$ is measurable for all $t \in [0,T]$ and $n \in \mathbb{N}$. This completes the proof. □

Remark 5.1.3. Since we can choose a θ-invariant set $\widetilde{\Omega}$ with full measure, we can assume w.l.o.g. from now on that the statements in Theorem 5.1.2 hold on Ω.

Now we are at a position to define the random dynamical system on $\mathbf{X}_\gamma$ generated by (5.1) as follows.

Definition 5.1.1. Let $A, B \in \mathcal{L}^1(\mathbb{P})$ and $r : \Omega \to \mathbb{R}^+$ be a random delay satisfying that $B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Consider a random differential equation with random delay

$$\dot{x}(t) = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)) \qquad \text{for } t \geq 0. \tag{5.14}$$

The random dynamical system $\Phi : \mathbb{R}^+ \times \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ defined by

$$\Phi(t,\omega)\mathbf{x}(s) := \begin{cases} \mathbf{x}(t+s), & \text{if } t+s \le 0, \\ \varphi_\omega(t+s,\mathbf{x}), & \text{if } t+s \ge 0, \end{cases}$$

for all $s \in \mathbb{R}^-$, where $\varphi_\omega(\cdot,\mathbf{x})$ is the unique solution of (5.14) with the initial value $\mathbf{x}$, is called the *random dynamical system* generated by (5.14).

Remark 5.1.4. From the unique existence of solution of (5.14) we derive that $\Phi(t,\omega)$ is injective for all $t \in \mathbb{R}^+$. For $\omega \in \Omega, t \in \mathbb{R}^+$ and $\mathbf{x} \in \operatorname{im}\big(\Phi(t,\theta^{-t}\omega)\mathbf{X}_\gamma\big)$ due to the injectivity of $\Phi(t,\theta^{-t}\omega)$ there is a unique $\mathbf{y} \in \mathbf{X}_\gamma$, which is also denoted by $\Phi(-t,\omega)\mathbf{x}$, such that

$$\Phi(t,\theta^{-t}\omega)\mathbf{y} = \mathbf{x}.$$

Lemma 5.1.5 (Strong Measurability of Φ)**.** *Let Φ be the random dynamical system generated by* (5.14)*. Then the mapping $\Phi(1,\cdot) : \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ is strongly measurable, i.e., $\Phi(1,\cdot)\mathbf{x} : \Omega \to \mathbf{X}_\gamma$ is measurable for each $\mathbf{x} \in \mathbf{X}_\gamma$.*

Proof. It is sufficient to show that the set

$$A := \{\omega \in \Omega : \|\Phi(1,\omega)\mathbf{x} - \mathbf{y}\|_\gamma \le \varepsilon\}$$

is measurable for all $\mathbf{y} \in \mathbf{X}_\gamma$ and $\varepsilon > 0$. By Definition 5.1.1 we can rewrite the set A as follows

$$\begin{aligned} A \;:=\; \Big\{&\omega \in \Omega : e^{\gamma s}|\varphi_\omega(s+1,\mathbf{x}) - \mathbf{y}(s)| \le \varepsilon \text{ for all } s \in [-1,0], \\ & e^{\gamma s}|\mathbf{x}(s+1) - \mathbf{y}(s)| \le \varepsilon \text{ for all } s \in (-\infty,-1)\Big\}. \end{aligned}$$

Clearly, if the estimate

$$e^{\gamma s}|\mathbf{x}(s+1) - \mathbf{y}(s)| \le \varepsilon \qquad \text{for all } s \in (-\infty,-1) \tag{5.15}$$

does not hold then $A = \emptyset$ and hence A is measurable. Therefore, it remains to deal with the case that inequality (5.15) holds. Using continuity of $\mathbf{y}$ and $\varphi_\omega(\cdot,\mathbf{x})$, we obtain

$$A = \bigcap_{s \in \mathbb{Q}\cap[-1,0]} \{\omega \in \Omega : e^{\gamma s}|\varphi_\omega(s+1,\mathbf{x}) - \mathbf{y}(s)| \le \varepsilon\}.$$

According to Theorem 5.1.2, the set

$$\{\omega \in \Omega : e^{\gamma s}|\varphi_\omega(s+1,\mathbf{x}) - \mathbf{y}(s)| \le \varepsilon\}$$

is measurable for each $s \in [-1,0]$. Consequently, A is measurable and the proof is completed. □

5.2 MET for differential equations with random delay

So far we have proved the existence of the random dynamical system Φ generated by a random differential equation with random delay

$$\dot{x} = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)), \tag{5.16}$$

where $A(\cdot), B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Recall that Φ is said to satisfy the *integrability condition* provided that

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)\|_\gamma \quad \text{and} \quad \sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\cdot)\|_\gamma \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}),$$

(see Lian and Lu [89] and also Theorem 1.4.3).

5.2.1 Integrability

The aim of this subsection is to show the integrability of the random dynamical system Φ generated by equation (5.16).

Lemma 5.2.1 (Sufficient Integrability Condition)**.** *Let $A \in \mathcal{L}^1(\mathbb{P})$ and $r : \Omega \to \mathbb{R}^+$ be a random map such that $B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P})$. Denote by $\Phi : \mathbb{R}\times\Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ the random dynamical system generated* (5.16)*. Then Φ satisfies the integrability condition, i.e.*

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)\|_\gamma \quad \textit{and} \quad \sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\cdot)\|_\gamma \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

Proof. For each $\omega \in \Omega$, let $\varphi_\omega(\cdot,\mathbf{x})$ be the solution of (5.16) starting at $t = 0$ with the initial value $\mathbf{x} \in \mathbf{X}_\gamma$. By Definition 5.1.1, we obtain

$$\begin{aligned} \|\Phi(t,\omega)\mathbf{x}\|_\gamma &= \max\Big\{\sup_{s\in(-\infty,-t]} e^{\gamma s}|\mathbf{x}(t+s)|, \sup_{s\in(-t,0]} e^{\gamma s}|\varphi_\omega(t+s,\mathbf{x})|\Big\} \\ &= \max\Big\{e^{-\gamma t}\|\mathbf{x}\|_\gamma, \sup_{s\in(0,t]} e^{\gamma(s-t)}|\varphi_\omega(s,\mathbf{x})|\Big\}. \end{aligned}$$

Therefore, the following inequalities

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\omega)\|_\gamma \le \sup_{0\le t\le 1} \log^+ \|\varphi_\omega(t,\cdot)\| \tag{5.17}$$

and

$$\sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\omega)\|_\gamma \le \sup_{0\le t\le 1, 0\le s\le 1-t} \log^+ \|\varphi_{\theta_t\omega}(s,\cdot)\| \tag{5.18}$$

hold for all $\omega \in \Omega$. In what follows, we estimate $|\varphi_\omega(t,\mathbf{x})|$ for all $0 \le t \le 1$. To simplify the notation, we define a set

$$M_\omega := \big\{s \in \mathbb{R}_+ : s \ge r(\theta_s\omega)\big\}.$$

and an operator $\mathbf{T}_\omega : \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d) \to \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d)$ by

$$\begin{aligned}\mathbf{T}_\omega \mathbf{f}(t) &= \mathbf{x}(0) + \int_0^t A(\theta_s\omega)\mathbf{f}(s)\,ds + \int_{[0,t]\cap M_\omega} B(\theta_s\omega)\mathbf{f}(s - r(\theta_s\omega))\,ds + \\ &\quad + \int_{[0,t]\cap M_\omega^c} B(\theta_s\omega)\mathbf{x}(s - r(\theta_s\omega))\,ds.\end{aligned}$$

By (5.2), the function $\varphi_\omega(\cdot,\mathbf{x})$ is the unique fixed point of $\mathbf{T}_\omega$. Moreover, due to the contractiveness of $\mathbf{T}_\omega^N$ for some $N \in \mathbb{N}$ we have

$$\varphi_\omega(t,\mathbf{x}) = \lim_{n\to\infty} \mathbf{T}_\omega^n \mathbf{f}(t) \qquad \text{for all } \mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d), t \in [0,1]. \tag{5.19}$$

From the definition of $\mathbf{T}_\omega$, we derive that

$$\begin{aligned}|\mathbf{T}_\omega \mathbf{f}(t)| &\le |\mathbf{x}(0)| + \int_0^t \|A(\theta_s\omega)\||\mathbf{f}(s)|\,ds + \int_{[0,t]\cap M_\omega} \|B(\theta_s\omega)\||\mathbf{f}(s - r(\theta_s\omega))|\,ds + \\ &\quad + \int_{[0,t]\cap M_\omega^c} e^{\gamma(r(\theta_s\omega)-s)}\|B(\theta_s\omega)\|\|\mathbf{x}\|_\gamma\,ds.\end{aligned}$$

which implies that for all $0 \le t \le 1$

$$\begin{aligned}|\mathbf{T}_\omega \mathbf{f}(t)| &\le k(\omega)\|\mathbf{x}\|_\gamma + \int_{[0,t]\cap M_\omega} \|B(\theta_s\omega)\||\mathbf{f}(s - r(\theta_s\omega))|\,ds + \\ &\quad + \int_0^t \|A(\theta_s\omega)\||\mathbf{f}(s)|\,ds,\end{aligned}$$

where $k(\omega) := 1 + \int_0^1 \|B(\theta_s\omega)\|e^{\gamma(r(\theta_s\omega)-s)}\,ds$. A direct computation yields that the non-empty closed set

$$B_\omega := \{\mathbf{f} \in \mathrm{C}_{\mathbf{x}}([0,1],\mathbb{R}^d) : \mathbf{f}(t) \le k(\omega)\|\mathbf{x}\|_\gamma\, e^{\int_0^t \|A(\theta_s\omega)\|+\|B(\theta_s\omega)\|\,ds} \quad \text{for all } 0 \le t \le 1\}$$

is invariant under $\mathbf{T}_\omega$. Therefore, together with (5.19) we get

$$\|\varphi_\omega(t,\cdot)\| \le k(\omega)e^{\int_0^t \|A(\theta_s\omega)\|+\|B(\theta_s\omega)\|\,ds} \qquad \text{for all } 0 \le t \le 1,$$

which gives

$$\sup_{0\le t\le 1} \log^+ \|\varphi_\omega(t,\cdot)\| \le \log k(\omega) + \int_0^1 \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\,ds \tag{5.20}$$

and

$$\sup_{0\le t\le 1, 0\le s\le 1-t} \log^+ \|\varphi_{\theta_t\omega}(s,\cdot)\| \le \sup_{0\le t\le 1} \log k(\theta_t\omega) + \int_0^1 \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\,ds. \tag{5.21}$$

Using the inequality $\log(1+x) \le 1 + \log^+ x$ for $x \in \mathbb{R}_+$, we have

$$\sup_{0\le t\le 1} \log k(\theta_t\omega) \le 1 + \log^+ \int_0^2 \|B(\theta_s\omega)\|e^{\gamma r(\theta_s\omega)}\,ds. \tag{5.22}$$

By the Fubini theorem, we get

$$\int_\Omega \int_0^2 \|B(\theta_s\omega)\| e^{\gamma r(\theta_s\omega)}\, ds\, d\mathbb{P} = \int_0^2 \int_\Omega \|B(\theta_s\omega)\| e^{\gamma r(\theta_s\omega)}\, d\mathbb{P}\, ds.$$

On the other hand, for all $s \in [0,2]$

$$\begin{aligned}\int_\Omega \|B(\theta_s\omega)\| e^{\gamma r(\theta_s\omega)}\, d\mathbb{P}(\omega) &= \int_\Omega \|B(\omega)\| e^{\gamma r(\omega)}\, d\mathbb{P}(\omega)\\ &< \infty.\end{aligned}$$

Hence,

$$\int_0^2 \|B(\theta_s\cdot)\| e^{\gamma r(\theta_s\cdot)}\, ds \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}),$$

which together with (5.22) proves that $\sup_{0\le t\le 1}\log k(\theta_t\cdot) \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$. Therefore, by (5.17), (5.18) and (5.20), (5.21) we obtain

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)\|_\gamma,\ \sup_{0\le t\le 1} \log^+ \|\Phi(1-t,\theta_t\cdot)\|_\gamma \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

This completes the proof. □

5.2.2 Kuratowski Measure

Recall that for a subset $A \subset \mathbf{X}_\gamma$, the Kuratowski measure of noncompactness of A is defined by

$$\alpha(A) := \inf\{d : A \text{ has a finite cover by sets of diameter } d\}.$$

For a bounded linear map $L : \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ we define

$$\|L\|_\alpha = \alpha(L(B_1(0))).$$

Let Φ be the linear cocycle defined as in Definition 5.1.1. We recall the following quantity (see Subsection 1.4)

$$l_\alpha(\Phi) := \lim_{t\to\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\|_\alpha,$$

and note that it is constant $\mathbb{P}$-a.s. due to the ergodicity of θ and the Kingman subadditive ergodic theorem (see e.g. Arnold [3, pp. 122], Ruelle [120, Appendix A]). To compute the quantity $l_\alpha(\Phi)$, we first prove the following preparatory lemma.

Lemma 5.2.2. *(i) Let $T > 0$ and $a : [0,T] \to \mathbb{R}$ be an integrable function. Then for any $\varepsilon > 0$ there exists a partition $0 = t_0 < t_1 < \cdots < t_K = T$ such that*

$$\int_{t_i}^{t_{i+1}} |a(s)|\, ds \le \varepsilon \qquad \textit{for all } i = 0,\dots,K-1.$$

(ii) Let $T \in \mathbb{R}^+$ and $\omega \in \Omega$ satisfy that the function $t \mapsto \|A(\theta_t\omega)\| + \|B(\theta_t\omega)\| e^{\gamma(\theta_t\omega)}$ is locally integrable. Define

$$A \ := \ \{\varphi_\omega(\cdot,\mathbf{x}) : [0,T] \to \mathbb{R}^d : \mathbf{x} \in B_1(0)\}.$$

Then $l_\alpha(A) = 0$, where A is considered as a subset of $\mathrm{C}([0,T],\mathbb{R}^d)$.

Proof. (i) The proof is straightforward by using the fact that the function from $\mathbb{R}_+$ into itself defined by

$$t \mapsto \int_0^t |a(s)|\, ds$$

is continuous.
(ii) By the same arguments in the proof of Lemma 5.2.1, the following inequality

$$|\varphi_\omega(t,\mathbf{x})| \le \left(1 + \int_0^T \|B(\theta_s\omega)\| e^{\gamma r(\theta_s\omega)}\, ds\right) e^{\int_0^T \|A(\theta_s\omega)\| + \|B(\theta_s\omega)\|\, ds}$$

holds for all $\mathbf{x} \in B_1(0)$ and $t \in [0,T]$. Then there exists a positive number M which depends only on ω and T such that

$$|\varphi_\omega(t,\mathbf{x})| \le M \qquad \text{for all } t \in [0,T], \mathbf{x} \in B_1(0). \tag{5.23}$$

Using equation (5.2), we obtain that for any $t, s \in [0,T]$ with $t > s$ and $\mathbf{x} \in B_1(0)$

$$\begin{aligned}\varphi_\omega(t,\mathbf{x}) - \varphi_\omega(s,\mathbf{x}) &= \int_s^t A(\theta_u\omega)\varphi_\omega(u,\mathbf{x}) + B(\theta_u\omega)\varphi_\omega(u - r(\theta_u\omega),\mathbf{x})\, du \\ &= \int_s^t A(\theta_u\omega)\varphi_\omega(u,\mathbf{x})\, du + \int_{[s,t]\cap M_\omega} B(\theta_u\omega)\varphi_\omega(u - r(\theta_u\omega),\mathbf{x})\, du + \\ &\qquad + \int_{[s,t]\cap M_\omega^c} B(\theta_u\omega)\mathbf{x}(u - r(\theta_u\omega))\, du,\end{aligned}$$

where $M_\omega := \{s \in \mathbb{R}_+, s \ge r(\theta_s\omega)\}$. Together with estimate (5.23) this implies that

$$|\varphi_\omega(t,\mathbf{x}) - \varphi_\omega(s,\mathbf{x})| \le M \int_s^t \|A(\theta_u\omega)\| + \|B(\theta_u\omega)\|\, du + \int_s^t \|B(\theta_u\omega)\| e^{\gamma r(\theta_u\omega)}\, du$$

holds for all $\varphi_\omega(\cdot,\mathbf{x}) \in A$. Applying part (i) to the right hand side of the estimate, we get for an arbitrary $\varepsilon > 0$ a partition $0 = t_0 < t_1 < \cdots < t_K = T$ such that

$$|f(t) - f(s)| \le \frac{\varepsilon}{3} \qquad \text{for all } f \in A, t_k \le t, s \le t_{k+1}, k = 0,\dots,K-1. \tag{5.24}$$

In the following, we first give a proof in the scalar case, i.e. $d = 1$. Choose and fix $N \in \mathbb{N}$ such that $\frac{M}{N} \le \frac{\varepsilon}{3}$. For each index $(i_1,\dots,i_K) \in \{-N, -N+1, \dots, N-1, N\}^K$, by writing each $t \in [0,T]$ uniquely as $t = \alpha t_k + \beta t_{k+1}$ for $k \in \{0,\dots,K-1\}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, we define a continuous function $f_{i_1,\dots,i_K} \in \mathrm{C}([0,T],\mathbb{R})$ by

$$f_{i_1,\dots,i_K}(\alpha t_k + \beta t_{k+1}) = \alpha \frac{i_k M}{N} + \beta \frac{i_{k+1} M}{N}.$$

Now we show that

$$A \subset \bigcup_{-N \le i_1,\dots,i_K \le N} B_\varepsilon(f_{i_1,\dots,i_K}). \tag{5.25}$$

By the definition of A and inequality (5.23) we have

$$-M \leq f(t_k) \leq M \qquad \text{for all } f \in A, k = 0, \ldots, K-1.$$

which implies together with the inequality $\frac{M}{N} \leq \frac{\varepsilon}{3}$ that for any $f \in A$ there exists an index $(i_1, \ldots, i_K) \in \{-N, -N+1, \ldots, N-1, N\}^K$ such that

$$\left| f(t_k) - \frac{i_k M}{N} \right| \leq \frac{M}{N} \leq \frac{\varepsilon}{3} \qquad \text{for all } k = 0, \ldots, K-1.$$

Equivalently,

$$|f(t_k) - f_{i_1,\ldots,i_K}(t_k)| \leq \frac{\varepsilon}{3} \qquad \text{for all } k = 0, \ldots, K-1. \tag{5.26}$$

For any $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ we get

$$\begin{aligned} |f(\alpha t_k + \beta t_{k+1}) - \alpha f(t_k) - \beta f(t_{k+1})| &\leq \alpha |f(\alpha t_k + \beta t_{k+1}) - f(t_k)| \\ &\qquad + \beta |f(t_{k+1}) - f(\alpha t_k + \beta t_{k+1})| \\ &\leq \frac{\varepsilon}{3}, \end{aligned}$$

where we use (5.24) to obtain the last estimate. This implies with (5.26) that

$$\begin{aligned} |f(\alpha t_k + \beta t_{k+1}) - f_{i_1,\ldots,i_K}(\alpha t_k + \beta t_{k+1})| &\leq \frac{\varepsilon}{3} + \alpha |f(t_k) - f_{i_1,\ldots,i_K}(t_k)| \\ &\qquad + \beta |f(t_{k+1}) - f_{i_1,\ldots,i_K}(t_{k+1})| \\ &\leq \frac{2\varepsilon}{3} \end{aligned}$$

for all $k = 0, \ldots, K-1$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. This proves (5.25) and since ε can be chosen arbitrarily small it follows that $l_\alpha(A) = 0$ in the case that $d = 1$. Since each continuous function $f \in \mathrm{C}([0,T], \mathbb{R}^d)$ can be written of the form $f = (f_1, \ldots, f_d)$, where $f_1, \ldots, f_d$ are scalar continuous functions, the high dimension case can be reduced to the scalar case and therefore we also obtain the desired conclusion in the general case. This completes the proof. □

Proposition 5.2.3. *Let $\Phi : \mathbb{R} \times \Omega \times \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ be the random dynamical system generated by* (5.16). *Then*

$$l_\alpha = \lim_{t \to \infty} \frac{1}{t} \log \|\Phi(t, \omega)\|_\alpha = -\gamma.$$

Proof. For convenience, throughout the proof we only deal with the max norm on $\mathbb{R}^d$, i.e. $|x| = \max_{1 \leq i \leq d} |x_i|$ for all $x = (x_1, \ldots, x_d)^{\mathrm{T}} \in \mathbb{R}^d$. We first obtain the inequality $l_\alpha \geq -\gamma$ by sufficiently showing that

$$\alpha(\Phi(T, \omega) B_1(0)) \geq e^{-\gamma(T+1)} \qquad \text{for all } T > 0. \tag{5.27}$$

For this purpose, we define a sequence of functions $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ by

$$\mathbf{x}_n(t) = \begin{cases} 0, & \text{if } t \in (-n+1, 0], \\ e^{\gamma(n-1)}(-n+1-t)(1,\dots,1)^{\mathrm{T}}, & \text{if } t \in (-n, -n+1], \\ e^{\gamma(n-1)}(1,\dots,1)^{\mathrm{T}}, & \text{if } t \in (-\infty, -n]. \end{cases}$$

Obviously, the function $\mathbf{x}_n : (-\infty, 0] \to \mathbb{R}^d$ is continuous, and the relations

$$\lim_{t\to-\infty} e^{\gamma t}\mathbf{x}_n(t) = 0, \qquad \sup_{t\in(-\infty,0]} e^{\gamma t}|\mathbf{x}_n(t)| \leq 1$$

lead that $\mathbf{x}_n \in B_1(0)$ for all $n \in \mathbb{N}$. A straightforward computation yields that for all $m > n$ the following equality holds

$$\begin{aligned} \Phi(T,\omega)\mathbf{x}_m(-n-T) - \Phi(T,\omega)\mathbf{x}_n(-n-T) &= \mathbf{x}_m(-n) - \mathbf{x}_n(-n) \\ &= -e^{\gamma(n-1)}(1,\dots,1)^{\mathrm{T}}. \end{aligned}$$

Thus,

$$\|\Phi(T,\omega)\mathbf{x}_m - \Phi(T,\omega)\mathbf{x}_n\|_\gamma \geq e^{-\gamma(T+1)},$$

which proves (5.27). Hence,

$$l_\alpha(\Phi) = \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\|_\alpha \geq -\gamma.$$

Therefore, it remains to show that

$$l_\alpha(\Phi) \leq -\gamma + \varepsilon \qquad \text{for all } 0 < \varepsilon < \gamma. \tag{5.28}$$

Choose and fix $T \geq \frac{\log 3}{\varepsilon}$. By definition of $\Phi(T,\omega)$ (see Definition 5.1.1), we have

$$\Phi(T,\omega)\mathbf{x}(t) = \begin{cases} \mathbf{x}(t+T), & \text{for all } t \in (-\infty, -T), \\ \varphi_\omega(t+T, \mathbf{x}), & \text{for all } t \in [-T, 0], \end{cases}$$

for all $\mathbf{x} \in B_1(0)$. Therefore, we get

$$\Phi(T,\omega)\mathbf{x}(\cdot) \equiv \varphi_\omega(\cdot + T, \mathbf{x}) \qquad \text{on } [-T, 0]. \tag{5.29}$$

According to Lemma 5.2.2 (ii), there exist $\widetilde{f}_1, \dots, \widetilde{f}_n \in \mathrm{C}([-T,0], \mathbb{R}^d)$ such that

$$\{\varphi_\omega(\cdot, \mathbf{x}) : [0,T] \to \mathbb{R}^d, \mathbf{x} \in B_1(0)\} \subset \bigcup_{k=1}^{n} B_{e^{(-\gamma+\varepsilon)T}}(f_k),$$

which implies with (5.29) that

$$\left\{\Phi(T,\omega)\mathbf{x}|_{[-T,0]}, \mathbf{x} \in B_1(0)\right\} \subset \bigcup_{k=1}^{n} B_{e^{(-\gamma+\varepsilon)T}}(f_k), \tag{5.30}$$

where $f_k : [-T, 0] \to \mathbb{R}^d$ is defined by $f_k(t) = \widetilde{f}_k(t+T)$. Define $\widehat{f}_k : (-\infty, 0] \to \mathbb{R}^d$ by

$$\widehat{f}_k(t) := \begin{cases} f_k(t), & \text{if } t \in [-T, 0], \\ f_k(-T), & \text{if } t \in (-\infty, -T). \end{cases}$$

We show that

$$\Phi(T, \omega) B_1(0) \subset \bigcup_{k=1}^{n} B_{e^{(-\gamma+\varepsilon)T}}(\widehat{f}_k). \tag{5.31}$$

To prove this statement let $\mathbf{x} \in B_1(0)$. Using (5.30), there exists $k \in \{1, \ldots, n\}$ such that

$$|\Phi(T, \omega)\mathbf{x}(t) - f_k(t)| \leq e^{(-\gamma+\varepsilon)T} \qquad \text{for all } t \in [-T, 0].$$

In particular, $f_k(-T) \leq 1 + e^{(-\gamma+\varepsilon)T}$. On the other hand, for all $t \in (-\infty, -T]$ we get

$$\begin{aligned} e^{\gamma t}|\Phi(T, \omega)\mathbf{x}(t) - \widehat{f}_k(t)| &= e^{\gamma t}|\mathbf{x}(t+T) - f_k(-T)| \\ &\leq e^{\gamma t}\left(e^{-\gamma(t+T)} + 1 + e^{(-\gamma+\varepsilon)T}\right) \\ &\leq 3e^{-\gamma T}, \end{aligned}$$

which together with $T \geq \frac{\log 3}{\varepsilon}$ proves (5.31). Consequently, we have

$$\|\Phi(T, \omega)\|_\alpha \leq e^{(-\gamma+\varepsilon)T} \qquad \text{for all } T \geq \frac{\log 3}{\varepsilon},$$

which implies that

$$l_\alpha(\Phi) = \lim_{t \to \infty} \frac{1}{t} \log \|\Phi(t, \omega)\|_\alpha \leq -\gamma + \varepsilon,$$

proving (5.28) and the proof is completed. □

5.2.3 Multiplicative Ergodic Theorem

We have just proved in the above sections that the random dynamical system generated by a differential equation with random delay fulfills all assumptions of the multiplicative ergodic theorem on Banach space (see Lian and Lu [89]). Therefore, we are now at a position to state the multiplicative ergodic theorem for differential equations with random delay.

Theorem 5.2.4 (Multiplicative Ergodic Theorem for Differential Equations with Random Delay). *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be an ergodic MDS and $A, B : \Omega \to \mathbb{R}^{d\times d}$ and $r : \Omega \to \mathbb{R}_+$ be measurable functions satisfying that*

$$A(\cdot), B(\cdot)e^{\gamma r(\cdot)} \in \mathcal{L}^1(\mathbb{P}).$$

Denote by $\Phi : \mathbb{R}^+ \times \mathbf{X}_\gamma \to X_\gamma$ the random dynamical system generated by the differential equation with random delay

$$\dot{x} = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)).$$

Then, there exists a θ-invariant subset $\widetilde{\Omega} \subset \Omega$ of full measure such that for each $\omega \in \widetilde{\Omega}$ exactly one of the following statements holds

(I) $\kappa(\Phi) = -\gamma$

(II) There exists $k \in \mathbb{N}$, Lyapunov exponents $\lambda_1 > \cdots > \lambda_k > -\gamma$ *and a splitting into measurable* Oseledets spaces

$$\mathbf{X}_\gamma = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

with finite dimensional linear subspaces $E_j(\omega)$ *and an infinite dimensional linear subspace* $F(\omega)$ *such that the following properties hold:*

(i) Invariance: $\Phi(t,\omega)E_j(\omega) = E_j(\theta_t\omega)$ *and* $\Phi(t,\omega)F(\omega) \subset F(\theta_t\omega)$.

(ii) Lyapunov exponents:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \lambda_j \quad \textit{for all } \mathbf{x} \in E_j(\omega) \setminus 0 \textit{ and } j = 1,\dots,k.$$

(iii) Exponential Decay Rate on $F(\omega)$*:*

$$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t,\omega)|_{F(\omega)}\|_\gamma \le -\gamma.$$

Moreover, for $\mathbf{x} \in F(\omega) \setminus 0$ *such that* $\Phi(t,\theta_{-t}\omega)^{-1}\mathbf{x} := \Phi(-t,\omega)\mathbf{x}$ *exists for all* $t \in \mathbb{R}^+$ *we get*

$$\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(-t,\omega)\mathbf{x}\|_\gamma \ge \gamma.$$

(III) There exist infinitely many finite dimensional measurable subspaces $E_j(\omega)$*, infinitely many infinite dimensional subspaces* $F_j(\omega)$ *and infinitely many* Lyapunov exponents

$$\lambda_1 > \lambda_2 > \cdots > -\gamma \quad \textit{with} \quad \lim_{j\to+\infty} \lambda_j = -\gamma$$

such that the following properties hold:

(i) Invariance: $\Phi(t,\omega)E_j(\omega) = E_j(\theta_t\omega)$ *and* $\Phi(t,\omega)F_j(\omega) \subset F_j(\theta_t\omega)$.

(ii) Invariant Splitting

$$\mathbf{X}_\gamma = E_1(\omega) \oplus \cdots \oplus E_j(\omega) \oplus F_j(\omega) \quad \textit{and} \quad F_j(\omega) = E_{j+1}(\omega) \oplus F_{j+1}(\omega).$$

(iii) Lyapunov exponents:

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}\|_\gamma = \lambda_j \quad \textit{for all } \mathbf{x} \in E_j(\omega) \setminus 0 \textit{ and } j = 1,\dots,k.$$

(iv) Exponential Decay Rate on $F_j(\omega)$*:*

$$\limsup_{t\to+\infty} \frac{1}{t} \log \|\Phi(t,\omega)|_{F_j(\omega)}\|_\gamma = \lambda_{j+1}.$$

Moreover, for $\mathbf{x} \in F_j(\omega) \setminus 0$ *such that* $\Phi(t,\theta_{-t}\omega)^{-1}\mathbf{x} := \Phi(-t,\omega)\mathbf{x}$ *exists for all* $t \in \mathbb{R}^+$ *we get*

$$\liminf_{t\to+\infty} \frac{1}{t} \log \|\Phi(-t,\omega)\mathbf{x}\|_\gamma \ge -\lambda_{j+1}.$$

Theorem 5.2.5 (Lyapunov Exponents are Independent of Exponential Weight Factor)**.** *Let $\gamma > 0$ and consider (5.14) on the state space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$. Assume that $\lambda > -\gamma$ is a Lyapunov exponent of (5.14), i.e. for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists $\mathbf{x}(\omega) \in \mathbf{X}_\gamma$ such that*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma = \lambda \, .$$

Then for every $\zeta > \gamma$ satisfying that $e^{\zeta r(\cdot)}B(\cdot) \in \mathcal{L}^1(\mathbb{P})$ we have $\mathbf{x}(\omega) \in \mathbf{X}_\zeta$ and the number λ is also a Lyapunov exponent of (5.14) on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$. In particular,

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta = \lambda \, . \tag{5.32}$$

Proof. Let $\mathbf{y} \in \mathbf{X}_\gamma$. From the definition of $\mathbf{X}_\gamma$ we obtain that $\lim_{t\to\infty} e^{-\gamma t}\mathbf{y}(-t)$ exists. For $\zeta > \gamma$ it is easy to see that

$$\begin{aligned} \lim_{t\to\infty} e^{-\zeta t}\mathbf{y}(-t) &= \lim_{t\to\infty} e^{(\gamma-\zeta)t} e^{-\gamma t}\mathbf{y}(-t) \\ &= 0, \end{aligned}$$

which implies that $\mathbf{y} \in \mathbf{X}_\zeta$. Furthermore, for any $\mathbf{y} \in \mathbf{X}_\gamma$ we have

$$\begin{aligned} \|\mathbf{y}\|_\zeta &= \sup_{t\in[0,\,\infty)} e^{-\zeta t}|\mathbf{y}(-t)| \\ &\leq \sup_{t\in[0,\,\infty)} e^{-\gamma t}|\mathbf{y}(-t)| \\ &\leq \|\mathbf{y}\|_\gamma. \end{aligned}$$

As a consequence, we get

$$\limsup_{t\to\infty} \frac{1}{t} \, \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq \lim_{t\to\infty} \frac{1}{t} \, \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma, \tag{5.33}$$

and

$$\liminf_{t\to-\infty} \frac{1}{t} \, \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \geq \lim_{t\to-\infty} \frac{1}{t} \, \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma. \tag{5.34}$$

In view of Theorem 5.2.4 we divide the proof into several cases.

Case 1: The linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ has finitely many Lyapunov exponents. Let $-\zeta < \lambda_k < \lambda_{k-1} < \cdots < \lambda_1$ be the Lyapunov exponents of the linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ and

$$\mathbf{X}_\zeta = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F(\omega)$$

the corresponding Oseledets splitting of Φ. We write $\mathbf{x}(\omega)$ in the following form

$$\mathbf{x}(\omega) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k + \mathbf{x}_F,$$

where $\mathbf{x}_i \in E_i(\omega)$ and $\mathbf{x}_F \in F(\omega)$. For convenience, we divide the proof into several steps.

Step 1: We first show that $\mathbf{x}_F = 0$ by contradiction, i.e. we assume that $\mathbf{x}_F \neq 0$. In view of Theorem 5.2.4, we have

$$\limsup_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta,$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lim_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}_i\|_\zeta = \lambda_i.$$

Therefore, for any $\varepsilon \in \left(0, \frac{\lambda_k+\zeta}{4}\right)$ there exists $T(\varepsilon) \in \mathbb{R}^+$ such that

$$\frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta \leq -\zeta + \varepsilon \qquad \text{for all } t \leq -T(\varepsilon),$$

and for all $i \in \{1,\dots,k\}$ with $\mathbf{x}_i \neq 0$

$$\lambda_i - \frac{\varepsilon}{2} \leq \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}_i\|_\zeta \leq \lambda_i + \frac{\varepsilon}{2} \qquad \text{for all } t \leq -T(\varepsilon).$$

Hence, for all $t \leq -T(\varepsilon)$ we have

$$\begin{aligned}
\|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta &= \Big\|\Phi(t,\omega)\mathbf{x}_F + \sum_{i\in\{1,\dots,k\},\,\mathbf{x}_i\neq 0} \Phi(t,\omega)\mathbf{x}_i\Big\|_\zeta \\
&\geq \|\Phi(t,\omega)\mathbf{x}_F\|_\zeta - \Big\|\sum_{i\in\{1,\dots,k\},\,\mathbf{x}_i\neq 0} \Phi(t,\omega)\mathbf{x}_i\Big\|_\zeta \\
&\geq e^{t(-\zeta+\varepsilon)} - \sum_{i=1}^k e^{t(\lambda_i-\varepsilon)}.
\end{aligned}$$

Consequently,

$$\frac{1}{t}\log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq \frac{1}{t}\log\left(e^{t(-\zeta+\varepsilon)} - \sum_{i=1}^k e^{t(\lambda_i-\varepsilon)}\right) \qquad \text{for all } t \leq -T(\varepsilon),$$

which implies that

$$\limsup_{t\to-\infty} \frac{1}{t}\log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \leq -\zeta + \varepsilon,$$

where we use the fact that

$$\lim_{t\to-\infty} \frac{1}{t}\log\left(e^{ta} - e^{tb}\right) = a \qquad \text{provided that } a < b,$$

to obtain the last inequality. Since ε can be chosen arbitrarily small it follows together with (5.34) that

$$\lim_{t\to-\infty} \frac{1}{t}\ \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma \leq -\zeta,$$

which contradicts to the fact that

$$-\zeta < -\gamma < \lambda = \lim_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\gamma.$$

Step 2: Define

$$i_{min} := \min_{\mathbf{x}_i \neq 0} i, \qquad i_{max} := \max_{\mathbf{x}_i \neq 0} i.$$

By the same argument as in Step 1, we obtain that

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta &\geq \lambda_{i_{min}}, \\ \lim_{t\to-\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta &\leq \lambda_{i_{max}}, \end{aligned}$$

which implies together with (5.33), (5.34), and the fact that $\lambda_{i_{min}} \geq \lambda_{i_{max}}$ that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta = \lambda,$$

proving (5.32) and the proof in this case is completed.

Case 2: The linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ has infinitely many Lyapunov exponents. Let $-\zeta < \cdots < \lambda_2 < \lambda_1$ with $\lim_{k\to\infty} \lambda_k = -\zeta$ be the Lyapunov exponents of the linear cocycle Φ on the state space $(\mathbf{X}_\zeta, \|\cdot\|_\zeta)$ and

$$\mathbf{X}_\zeta = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_k(\omega)$$

the corresponding invariant splittings. We prove the fact that $\lambda \in \{\lambda_1, \lambda_2, \dots\}$ by contradiction, i.e. we assume that $\lambda_k \neq \lambda$ for all $k \in \mathbb{N}$. Since $\lim_{k\to\infty} \lambda_k = -\zeta$ it follows that there exists $k \in \mathbb{N}$ such that $\lambda_k < \lambda$. Set $k^* := \min\{k : \lambda_k < \lambda\}$. By using (5.34) and in view of Theorem 5.2.4, we obtain that $k^* > 1$. Hence, $\lambda_{k^*} < \lambda < \lambda_{k^*-1}$. We write $\mathbf{x}(\omega)$ in the following form

$$\mathbf{x}(\omega) = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{k^*-1} + \mathbf{x}_F,$$

where $\mathbf{x}_i \in E_i(\omega), i = 1, \dots, k^* - 1$ and $\mathbf{x}_F \in F_{k^*-1}(\omega)$. Using a similar proof as in Step 1 of Case 1, we also have $\mathbf{x}_F = 0$ and

$$\liminf_{t\to\infty} \frac{1}{t} \log \|\Phi(t,\omega)\mathbf{x}(\omega)\|_\zeta \geq \lambda_{k^*-1},$$

which together with (5.33) contradicts to the fact that $\lambda < \lambda_{k^*-1}$ and the proof is completed. □

5.3 Differential equations with bounded delay

The aim of this section is to investigate differential equations with bounded delay. We can easily observe that if the delay is bounded we do not need all information for $t \in (-\infty, 0]$ in order to know the value of solutions in the future. As a consequence, there are several options to define a dynamical system generated by such an equation. Naturally, we can ask the question whether there are any relations between the Lyapunov exponents of these dynamical systems. Throughout this section we consider the following system

$$\dot{x} = A(\theta_t\omega)x(t) + B(\theta_t\omega)x(t - r(\theta_t\omega)). \tag{5.35}$$

Assume that the random delay map r is bounded, i.e. there exists $M > 0$ such that

$$r(\omega) \le M \qquad \text{for all } \omega \in \Omega$$

and $A, B \in \mathcal{L}^1(\mathbb{P})$. Due to the boundedness of the delay the initial values of (5.35) can be either in $\mathbf{X}_\gamma$ or in $\mathrm{C}([-M, 0], \mathbb{R}^d)$. Using the same procedure to introduce and investigate the RDS in $\mathbf{X}_\gamma$ generated by (5.35) we also obtain an RDS in $\mathrm{C}([-M, 0], \mathbb{R}^d)$ generated by (5.35) as follows:

Random Dynamical System on $\mathrm{C}([-M, 0], \mathbb{R}^d)$: For each $\omega \in \Omega$ and an initial value $x \in \mathrm{C}([-M, 0], \mathbb{R}^d)$, equation (5.35) has a unique solution denoted by $\psi_\omega(\cdot, x)$, i.e. the equality

$$\psi_\omega(t, x) = \int_0^t A(\theta_s\omega)\psi_\omega(s, x) + \int_0^t B(\theta_s\omega)\widetilde{\psi}_\omega(s - r(\theta_s\omega), x)\, ds$$

holds for all $t \in \mathbb{R}_+$, where

$$\widetilde{\psi}_\omega(s - r(\theta_s\omega), x) = \begin{cases} x(s - r(\theta_s\omega)), & \text{if } s \le r(\theta_s\omega), \\ \psi_\omega(s - r(\theta_s\omega), x), & \text{otherwise.} \end{cases}$$

Based on the existence and uniqueness solution of (5.35) we can define an RDS $\Psi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(\mathrm{C}([-T, 0], \mathbb{R}^d))$, where $\mathcal{L}(\mathrm{C}([-T, 0], \mathbb{R}^d))$ denotes the space of all bounded linear operators from $\mathrm{C}([-T, 0], \mathbb{R}^d)$ into itself, by

$$\Psi(t, \omega)x(s) = \begin{cases} \psi_\omega(t + s, x), & \text{if } t + s \ge 0, \\ x(t + s), & \text{otherwise,} \end{cases}$$

for all $s \in [-M, 0]$.

Properties of Ψ: Along the lines of the proof of Theorem 5.1.2, Lemma 5.2.1 and Proposition 5.2.3 we have:

- Ψ is strongly measurable.

- Ψ satisfies the integrability condition, i.e.

$$\sup_{0\le t\le 1}\log^+\|\Psi(t,\cdot)\| \quad \text{and} \quad \sup_{0\le t\le 1}\log^+\|\Psi(1-t,\theta_t\cdot)\| \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

- $l_\alpha(\Psi)=-\infty$.

Theorem 5.3.1. *Let $\alpha_1>\alpha_2>\dots$ be the Lyapunov exponents of Φ and $\beta_1>\beta_2>\dots$ be the Lyapunov exponent of Ψ. Then*

$$\{\alpha_i\}=\{\beta_i:\beta_i>-\gamma\}.$$

Proof. ($\Rightarrow$) Suppose that λ is a Lyapunov exponent of Φ. Fix $\omega\in\Omega$ and let $\mathbf{x}\in\mathbf{X}_\gamma$ be a vector corresponding to this Lyapunov exponent, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma=\lambda.$$

Define $x\in \mathrm{C}([-M,0],\mathbb{R}^d)$ by

$$x(s)=\mathbf{x}(s) \qquad \text{for all } s\in[-M,0].$$

A direct computation yields that

$$\varphi_\omega(t,\mathbf{x})=\psi_\omega(t,x) \qquad \text{for all } t\ge 0,$$

which leads

$$\Phi(t,\omega)\mathbf{x}(s)=\Psi(t,\omega)x(s) \qquad \text{for all } s\in[-M,0], t\ge M.$$

Consequently, for all $t\ge M$ we have

$$\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma\ge\frac{1}{t}\log(e^{-\gamma M}\|\Psi(t,\omega)x\|),$$

proving that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Psi(t,\omega)x\|\le\lambda.$$

To prove λ is a Lyapunov exponent of Ψ, by virtue of Theorem 5.2.4 it is sufficient to show that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Psi(t,\omega)x\|=\lambda.$$

Thereto, we assume a contradiction, i.e. there exists $\varepsilon\in(0,\lambda+\gamma)$ and $T>0$ such that

$$\|\Psi(t,\omega)x\|\le e^{(\lambda-\varepsilon)t} \qquad \text{for all } t\ge T.$$

Therefore,

$$|\varphi_\omega(t,\mathbf{x})|=|\psi_\omega(t,x)|\le e^{(\lambda-\varepsilon)t} \qquad \text{for all } t\ge T.$$

As a consequence, for all $t \geq T$ we have

$$\begin{aligned}
\|\Phi(t,\omega)\mathbf{x}\|_\gamma &= \max\left\{\sup_{-\infty<s\leq -t} e^{\gamma s}|\Phi(t,\omega)\mathbf{x}(s)|, \sup_{-t\leq s\leq 0} e^{\gamma s}|\Phi(t,\omega)\mathbf{x}(s)|\right\}\\
&\leq \max\left\{e^{-\gamma t}\|\mathbf{x}\|_\gamma, \sup_{-t\leq s\leq 0} e^{\gamma s}|\varphi_\omega(t+s,\mathbf{x})|\right\}\\
&\leq \max\left\{e^{-\gamma t}\|\mathbf{x}\|_\gamma, e^{-\gamma t}\sup_{0\leq s\leq T} e^{\gamma s}|\varphi_\omega(s,\mathbf{x})|, e^{(\lambda-\varepsilon)t}\right\}.
\end{aligned}$$

This implies together with $-\gamma \leq \lambda - \varepsilon$ that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma \leq \lambda-\varepsilon.$$

This is a contradiction and we get the desired conclusion.
($\Leftarrow$) Assume that $\beta > -\gamma$ is a Lyapunov exponent of Ψ and let $x \in \mathrm{C}([-M,0],\mathbb{R}^d)$ be a vector corresponding to β, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\log\|\Psi(t,\omega)x\| = \beta.$$

Define $\mathbf{x} : (-\infty,0] \to \mathbb{R}^d$ by

$$\mathbf{x}(s) = \begin{cases} x(s), & \text{if } s\in[-M,0],\\ x(-M), & \text{otherwise.}\end{cases}$$

Using similar arguments as in the first part of the proof, we also have

$$\lim_{t\to\infty}\frac{1}{t}\log\|\Phi(t,\omega)\mathbf{x}\|_\gamma = \beta.$$

Therefore, β is a Lyapunov exponent of Φ and the proof is completed. □

Chapter 6

Computational Ergodic Theorem

Iterated function systems (IFS) consisting of finitely many affine transformations became popular as a method for constructing fractals like, e.g. the Sierpinski Gasket or the Barnsley Fern in the plane, Barnsley [10]. A common algorithm consists of picking a random point in the plane, then iteratively applying one of the functions chosen at random from the function system and drawing the point. Iterated function systems are examples of RDS. For IFS which are uniformly contractive Peruggia [114] introduces a general method of discretisation as a way of approximating the attracting sets and the invariant measure. Using an extension of this construction, Froyland [56] and Froyland and Aihara [57] present a computational method for rigorously approximating the unique invariant measure of an IFS which is contractive on average – a notion which is more general than uniformly contractive. An advantage of this method is that it provides quantitative bounds on the accuracy of the approximation. For the same class of IFS Elton [50] proved an ergodic theorem which states that the time average along almost every random iterate of any starting point converges to a constant number, the space average. This theorem is extended in a number of directions, e.g. to recurrent IFS by Barnsley, Elton and Hardin [12], to systems with time dependent probabilities by Stenflo [130], to systems with place-dependent probabilities by Barnsley, Demko, Elton and Geronimo [11], to contractive Markov systems by Werner [140]. Using the Banach limit technique, Forte and Mendivil [59] give a simple proof of the ergodic theorem for an IFS which is uniformly contractive. Based on the same method, Hyong-chol et al. [71] extend the ergodic theorem to infinite iterated function systems (IIFS) which are uniformly contractive. Combining the Kingman subadditive ergodic theorem and the Birkhoff ergodic theorem, Cong, Doan and Siegmund [38] provide a simple proof of the ergodic theorem for IIFS which are contractive on average.

In this chapter, we extend the result in Cong, Doan and Siegmund [38] to IIFS which are l-contractive on average, a notion which is weaker than contraction on average. We also construct an approximating sequence of finite IFS such that the corresponding sequence of invariant measures converges to the unique invariant measure of the approximated IIFS. One of our main results is a computational version of the ergodic theorem which allows to approximate the time average of an IIFS together with explicit error bounds.

At the same time, we also use the method in Froyland and Aihara [57] to establish an algorithm to compute the time average of a contractive on average place-dependent IFS. Having a rigorous method to compute the invariant measure, we apply this method to compute Lyapunov exponents of products of random matrices. Several examples are also provided to illustrate the theoretical results.

Given an RDS φ as in Definition 1.1.9 or its corresponding skew product Θ as in Remark 1.1.4 we define its time-one map $\tau : \Omega \times X \to \Omega \times X$ by

$$\tau(\omega, x) = \Theta(1)(\omega, x).$$

Then it is easy to see that $\Theta(n) = \tau^n$. Conversely, let $\tau : \Omega \times X \to \Omega \times X$ be a map satisfying $\tau(\omega, x) = (\theta\omega, \varphi(1, \omega)x)$ for an RDS φ. Then $\Theta(n) := \tau^n$ is the skew product corresponding to the RDS φ. Hence we can identify a skew product with its time-one map. Next we will recall the notion of invariant measure which is a central concept for RDS.

Definition 6.0.1 (Invariant Measure)**.** Let $\tau : \Omega \times X \to \Omega \times X$ be the skew product corresponding to an RDS φ over an MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. A probability measure μ on $(X, \mathcal{B})$ is said to be *invariant* under τ if the probability measure $\mathbb{P} \times \mu$ on $\Omega \times X$ is invariant with respect to the skew product τ, i.e. satisfies

$$\mathbb{P} \times \mu(B) = \mathbb{P} \times \mu(\tau^{-1}B) \qquad \text{for all } B \in \mathcal{F} \otimes \mathcal{B}.$$

We recall some elementary properties of Lipschitz functions which will be used in the next sections. Let (X_1, d_1) and (X_2, d_2) be metric spaces and $f : X_1 \to X_2$ a Lipschitz function. Denote by $\mathrm{Lip}(f)$ the Lipschitz constant of f, i.e.,

$$\mathrm{Lip}(f) = \sup_{x,y \in X_1} \frac{d_2(f(x), f(y))}{d_1(x, y)} < +\infty.$$

For a compact metric space (X, d), we define

$$\mathrm{Lip}_1(X) := \{h : X \to \mathbb{R} \mid \quad \mathrm{Lip}(h) \le 1\}.$$

Remark 6.0.2. (i) Let (X_i, d_i), with $i = 1, 2, 3$, be metric spaces and suppose that $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are Lipschitz functions. Then the composition function $g \circ f : X_1 \to X_3$ is also a Lipschitz function and $\mathrm{Lip}(g \circ f) \le \mathrm{Lip}(f)\mathrm{Lip}(g)$.

(ii) Let (X, d) be a metric space and $f : X \to \mathbb{R}$ a Lipschitz function. Then for any $\alpha \in \mathbb{R}$ the function αf is Lipschitz and $\mathrm{Lip}(\alpha f) = |\alpha|\mathrm{Lip}(f)$.

From now on, suppose that (X, d) is a compact metric space. Denote by $C(X)$, $\mathcal{M}(X)$ the space of real-valued continuous maps on X and probability measures on $(X, \mathcal{B}(X))$, respectively. In order to estimate the distance between two probability measures we introduce a metric d_H on $\mathcal{M}(X)$ which is known as the *Hutchinson metric* (see Hutchinson [70]) as follows

$$d_H(\nu_1, \nu_2) := \sup_{h \in \mathrm{Lip}_1(X)} \left| \int_X h \, d\nu_1 - \int_X h \, d\nu_2 \right| \qquad \text{for all } \nu_1, \nu_2 \in \mathcal{M}(X). \tag{6.1}$$

In the following remark, we collect some well-known properties of this metric space which are used later.

Remark 6.0.3. (i) $(\mathcal{M}(X), d_H)$ is a complete metric space.

(ii) Let $x_0 \in X$ be fixed. Then the Hutchinson metric satisfies

$$d_H(\nu_1, \nu_2) = \sup_{h \in \mathrm{Lip}_1(X), h(x_0)=0} \left| \int_X h \, d\nu_1 - \int_X h \, d\nu_2 \right| \qquad \text{for all } \nu_1, \nu_2 \in \mathcal{M}(X) .$$

(iii) If $\nu_1, \nu_2 \in \mathcal{M}(X)$ satisfy $d_H(\nu_1, \nu_2) = 0$ then $\int_X G(z) \, d\nu_1(z) = \int_X G(z) \, d\nu_2(z)$ for all continuous functions $G \in C(X)$.

(iv) If $\lim_{n \to \infty} \nu_n = \nu$ then

$$\lim_{n \to \infty} \int_X f \, d\nu_n = \int_X f \, d\nu \qquad \text{for all } f \in C(X).$$

(v) $(\mathcal{M}(X), d_H)$ is a compact metric space.

6.1 Iterated Function Systems

6.1.1 Finite Iterated Function Systems

Let (X, d) be a compact metric space, $k \in \mathbb{N}$ and $\mathbf{f} = \{f_n\}_{n=1}^k$ a sequence of k Lipschitz maps from X into itself. Let $\mathbf{p} = \{p_n\}_{n=1}^k$ be a collection of k positive probabilities $p_n > 0$, $\sum_{n=1}^k p_n = 1$. The pair $(\mathbf{f}, \mathbf{p})$ is called *(finite) iterated function system (IFS)*, see [9]. In order to explain how the IFS $(\mathbf{f}, \mathbf{p})$ generates an RDS on X we introduce the *space of addresses* containing k symbols

$$\Omega^{(k)} = \prod_{n=0}^{\infty} \{1, \dots, k\} = \{1, \dots, k\}^{\infty}$$

together with the σ-algebra $\mathcal{F}^{(k)}$ generated by the cylinders in $\Omega^{(k)}$ and define a product probability measure $\mathbb{P}^{(k)}$ on $(\Omega^{(k)}, \mathcal{F}^{(k)})$ by

$$\mathbb{P}^{(k)} = \prod_{n=0}^{\infty} \rho^{(k)} \qquad \text{with } \rho^{(k)}(\{n\}) = p_n .$$

Let $\theta : \Omega^{(k)} \to \Omega^{(k)}$ denote the left shift, i.e., $(\theta\omega)_j = \omega_{j+1}$ for all $\omega \in \Omega^{(k)}$ and $j \in \mathbb{N}_0$. It is well known that θ is an ergodic transformation preserving the probability $\mathbb{P}^{(k)}$ (see e.g. Walter [138]).

Remark 6.1.1 (RDS generated by IFS)**.** An IFS $(\mathbf{f}, \mathbf{p})$ generates a random dynamical system $\varphi^{(k)}$ over the MDS $(\Omega^{(k)}, \mathcal{F}^{(k)}, \mathbb{P}^{(k)}, \theta)$ by setting

$$\varphi^{(k)}(n, \omega)x = f_{\omega_{n-1}} \circ f_{\omega_{n-2}} \circ \cdots \circ f_{\omega_0}(x) \qquad \text{for } n \in \mathbb{Z}^+ .$$

The corresponding skew product $\tau^{(k)} : \Omega^{(k)} \times X \to \Omega^{(k)} \times X$ is given by

$$\tau^{(k)}(\omega, x) = (\theta\omega, f_{\omega_0}x) .$$

Definition 6.1.1 (l-Average Expansion Rate and Contraction)**.** Let $(\mathbf{f}, \mathbf{p})$ be an IFS. The number

$$\mathrm{Lip}_l(\mathbf{f}) := \sum_{n_1,n_2,\ldots,n_l=1}^{k} p_{n_1} p_{n_2} \ldots p_{n_l} \,\mathrm{Lip}(f_{n_1} \circ f_{n_2} \circ \cdots \circ f_{n_l})$$

is called the *l-average expansion rate* of $(\mathbf{f}, \mathbf{p})$. The IFS $(\mathbf{f}, \mathbf{p})$ is said to be *l-contractive on average* if $\mathrm{Lip}_l(\mathbf{f}) < 1$. The IFS $(\mathbf{f}, \mathbf{p})$ is said to be *contractive on average* if $\mathrm{Lip}_1(\mathbf{f}) < 1$.

Remark 6.1.2. *For an IFS $(\mathbf{f}, \mathbf{p})$ and $l \in \mathbb{N}$, we define an IFS $(\widetilde{\mathbf{f}}, \widetilde{\mathbf{p}})$ by*

$$\widetilde{f}_{n_1,\ldots,n_l} = f_{n_1} \circ f_{n_2} \circ \cdots \circ f_{n_l}, \quad \widetilde{p}_{n_1,\ldots,n_l} = p_{n_1} p_{n_2} \ldots p_{n_l},$$

where $n_1, \ldots, n_l = 1, \ldots, k$. Then

$$\mathrm{Lip}_l(\mathbf{f}) = \mathrm{Lip}_1(\widetilde{\mathbf{f}}).$$

The following lemma ensures that contraction on average implies l-contraction on average for all $l \in \mathbb{N}$.

Lemma 6.1.3. *Let $(\mathbf{f}, \mathbf{p})$ be an iterated function system. Then*

$$\mathrm{Lip}_l(\mathbf{f}) \le (\mathrm{Lip}_1(\mathbf{f}))^l \qquad \text{for all } l \in \mathbb{N}.$$

Consequently, contraction on average implies l-contraction on average for all $l \in \mathbb{N}$. However, the converse implication does not in general hold.

Proof. Using Remark 6.0.2 (i), we obtain

$$\begin{aligned}
\mathrm{Lip}_l(\mathbf{f}) &= \sum_{n_1,n_2,\ldots,n_l=1}^{k} p_{n_1} p_{n_2} \ldots p_{n_l} \,\mathrm{Lip}(f_{n_1} \circ f_{n_2} \circ \cdots \circ f_{n_l}) \\
&\le \sum_{n_1,n_2,\ldots,n_l=1}^{k} p_{n_1} p_{n_2} \ldots p_{n_l} \,\mathrm{Lip}(f_{n_1})\mathrm{Lip}(f_{n_2}) \ldots \mathrm{Lip}(f_{n_l}) \\
&= \sum_{n_1=1}^{k} p_{n_1} \,\mathrm{Lip}(f_{n_1}) \sum_{n_2=1}^{k} p_{n_2} \,\mathrm{Lip}(f_{n_2}) \cdots \sum_{n_l=1}^{k} p_{n_l} \,\mathrm{Lip}(f_{n_l}) \\
&= (\mathrm{Lip}_1(\mathbf{f}))^l,
\end{aligned}$$

proving that contraction on average implies l-contraction on average. For the remaining part of the proof, we need to construct an IFS $(\mathbf{f}, \mathbf{p})$ which is l-contractive on average but not contractive on average. Set $X = [0, 1], k = 2, p_1 = p_2 = 1/2$ and we define two functions $f_1, f_2 : X \to X$ by

$$f_1(x) := \frac{x^2}{2}, \qquad f_2(x) := 1 - \frac{x^2}{2}.$$

Note that the Lipschitz constant of a C^1 map $f : [0,1] \to [0,1]$ can be determined by

$$\mathrm{Lip}(f) = \max_{x\in[0,1]} |f'(x)|. \tag{6.2}$$

Hence,

$$\mathrm{Lip}(f_1) = \mathrm{Lip}(f_2) = 1,$$

which implies that the IFS $(\mathbf{f}, \mathbf{p})$ is not contractive on average. On the other hand, a direct computation yields that

$$f_1^2(x) = \frac{x^4}{8}, f_2^2(x) = \frac{1}{2} + \frac{x^2}{2} - \frac{x^4}{8}, f_1 \circ f_2(x) = \frac{1}{2} - \frac{x^2}{2} + \frac{x^4}{8}, f_2 \circ f_1 = 1 - \frac{x^4}{8}.$$

Therefore, by using statement (6.2), we have

$$\mathrm{Lip}(f_1^2) = \mathrm{Lip}(f_2 \circ f_1) = \frac{1}{2}, \quad \mathrm{Lip}(f_2^2) = \mathrm{Lip}(f_1 \circ f_2) = \frac{2\sqrt{2}}{3\sqrt{3}},$$

which implies that

$$\mathrm{Lip}_2(\mathbf{f}) = \frac{1}{4} + \frac{\sqrt{2}}{3\sqrt{3}} < 1.$$

As a consequence, $(\mathbf{f}, \mathbf{p})$ is 2-contractive on average. This completes the proof. □

Remark 6.1.4 (Uniform Contraction is Stronger than Contraction on Average)**.** An IFS $(\mathbf{f}, \mathbf{p})$ is said to be *uniformly contractive* if $\mathrm{Lip}(f_n) < 1$ for all $1 \le n \le k$. A uniformly contractive IFS is also contractive on average, since $\mathrm{Lip}(f_n) < 1$ together with $\sum_{n=1}^{k} p_n = 1$ implies that $\sum_{n=1}^{k} p_n \,\mathrm{Lip}(f_n) < 1$.

For an IFS $(\mathbf{f}, \mathbf{p})$, we call the operator $\mathcal{P}_{(k)} : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by

$$\mathcal{P}_{(k)}\nu = \sum_{n=1}^{k} p_n \nu \circ f_n^{-1} \qquad \text{for all } \nu \in \mathcal{M}(X) \tag{6.3}$$

the associated *Markov operator*. By the definition of $\mathcal{P}_{(k)}$ we have

$$\int_X h \, d\mathcal{P}_{(k)}^l \nu = \sum_{n_1,\dots,n_l=1}^{k} p_{n_1} \dots p_{n_l} \int_X h \circ f_{n_1} \circ \dots \circ f_{n_l} \, d\nu \qquad \text{for all } \nu \in \mathcal{M}(X). \tag{6.4}$$

The following lemma from Foryland [56] characterizes invariant measures of IFS as fixed points of an associated Markov operator.

Lemma 6.1.5 (Froyland [56])**.** *Let* $(\mathbf{f}, \mathbf{p})$ *be an IFS on a compact metric space* (X, d)*. Then a probability measure* $\mu \in \mathcal{M}(X)$ *is invariant under the IFS* $(\mathbf{f}, \mathbf{p})$*, i.e.* $\mu \times \mathbb{P}^{(k)}$ *is invariant under the skew product* $\tau^{(k)}$ *associated with* $(\mathbf{f}, \mathbf{p})$*, if and only if* μ *is a fixed point of the associated* Markov operator $\mathcal{P}_{(k)}$ *defined as in* (6.3)*.*

Moreover, in Froyland [56] it is shown that if the IFS $(\mathbf{f}, \mathbf{p})$ is contractive on average then the operator $\mathcal{P}_{(k)}$ is contractive with respect to the Hutchinson metric. More precisely, we have

$$\mathrm{Lip}(\mathcal{P}_{(k)}) \leq \sum_{n=1}^{k} p_n \mathrm{Lip}(f_n).$$

As a consequence, the contractive on average IFS $(\mathbf{f}, \mathbf{p})$ has a unique invariant probability measure. In the following lemma, we investigate a contractivity property of the Markov operator for an l-contractive IFS.

Lemma 6.1.6. *Let $(\mathbf{f}, \mathbf{p})$ be an IFS on a compact metric space (X, d). Then*

$$\mathrm{Lip}(\mathcal{P}_{(k)}^l) \leq \mathrm{Lip}_l(\mathbf{f}) \qquad \text{for all } l \in \mathbb{N}.$$

As a consequence, if the IFS $(\mathbf{f}, \mathbf{p})$ is l-contractive on average then $\mathcal{P}_{(k)}^l$ is a contractive operator and therefore the system $(\mathbf{f}, \mathbf{p})$ has a unique invariant probability measure.

Proof. It is equivalent to show that

$$d_H(\mathcal{P}_{(k)}^l \nu_1, \mathcal{P}_{(k)}^l \nu_2) \leq \mathrm{Lip}_l(\mathbf{f}) d_H(\nu_1, \nu_2) \qquad \text{for all } \nu_1, \nu_2 \in \mathcal{M}(X).$$

By the definition of Hutchinson metric, see (6.1), we have

$$d_H(\mathcal{P}_{(k)}^l \nu_1, \mathcal{P}_{(k)}^l \nu_2) = \sup_{h \in \mathrm{Lip}_1(X)} \left| \int_X h \, d\mathcal{P}_{(k)}^l \nu_1 - \int_X h \, d\mathcal{P}_{(k)}^l \nu_2 \right|,$$

which together with (6.4) implies that

$$\begin{aligned} d_H(\mathcal{P}_{(k)}^l \nu_1, \mathcal{P}_{(k)}^l \nu_2) \quad &\leq \quad \sup_{h \in \mathrm{Lip}_1(X)} \sum_{n_1,\dots,n_l=1}^{k} p_{n_1} \dots p_{n_l} \cdot \\ &\cdot \left| \int_X h \circ f_{n_1} \circ \dots \circ f_{n_l} \, d\nu_1 - \int_X h \circ f_{n_1} \circ \dots \circ f_{n_l} \, d\nu_2 \right|. \end{aligned} \tag{6.5}$$

On the other hand, by using Remark 6.0.2 (ii), we get

$$\mathrm{Lip}(h \circ f_{n_1} \circ \dots \circ f_{n_l}) \leq \mathrm{Lip}(f_{n_1} \circ \dots \circ f_{n_l}) \qquad \text{for all } h \in \mathrm{Lip}_1(X),$$

which gives together with estimate (6.5) that

$$\begin{aligned} d_H(\mathcal{P}_{(k)}^l \nu_1, \mathcal{P}_{(k)}^l \nu_2) \quad &\leq \quad \sum_{n_1,\dots,n_l=1}^{k} p_{n_1} \dots p_{n_l} \mathrm{Lip}(f_{n_1} \circ \dots \circ f_{n_l}) \, d_H(\nu_1, \nu_2) \\ &= \quad \mathrm{Lip}_l(\mathbf{f}) \, d_H(\nu_1, \nu_2). \end{aligned}$$

This completes the proof. □

Barnsley, Elton and Hardin [12] proved the following ergodic theorem for IFS which are l-contractive on average. For a simple proof in the more restrictive case of uniformly contractive IFS we refer to Forte and Mendivil [59].

Theorem 6.1.7 (Ergodic Theorem for l-Contractive on Average IFS, [12])**.** *Let* $(\mathbf{f}, \mathbf{p})$ *be an IFS which is* l*-contractive on average. Then for any continuous function* $G : X \to \mathbb{R}$ *and any* $x \in X$ *we have*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi^{(k)}(n,\omega)x) = \int_X G(z)\, d\mu(z) \qquad \text{for } \mathbb{P}^{(k)}\text{-a.e. } \omega \in \Omega^{(k)},$$

where μ *is the invariant probability measure of the IFS* $(\mathbf{f}, \mathbf{p})$*.*

6.1.2 Finite Iterated Function Systems with Place-dependent Probabilities

Let $\{f_n\}_{n=1}^k$ be Lipschitz continuous maps from a compact metric space (X, d) into itself. Associated to each map are given continuous probability weights $p_n : X \to (0, 1)$, $n = 1, \dots, k$, and

$$\sum_{n=1}^{k} p_n(x) = 1 \qquad \text{for all } x \in X.$$

To simplify the notation and to emphasize the fact that p_n depends on the state space, we define $\mathbf{f} = \{f_n\}_{n=1}^k$ and $\mathbf{p}(\cdot) = \{p_n(\cdot)\}_{n=1}^k$. We call the set $(\mathbf{f}, \mathbf{p}(\cdot))$ an *iterated function system with place-dependent probabilities* (place-dependent IFS), see e.g. Barnsley, Demko, Elton and Geronimo [11]. The operator $\mathcal{T} : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by

$$\mathcal{T}\nu(B) = \sum_{n=1}^{k} \int_{f_n^{-1}(B)} p_n(x)\, d\nu(x) \qquad \text{for all } \nu \in \mathcal{M}(X)$$

is called the *Markov operator* associated to the place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$. A probability measure $\mu \in \mathcal{M}(X)$ is said to be *invariant* or *stationary* if $\mathcal{T}\mu = \mu$. It is said to be *attractive* if for all $\nu \in \mathcal{M}(X)$,

$$\lim_{n\to\infty} \int_X h\, d\mathcal{T}^n\nu = \int_X h\, d\mu \qquad \text{for all } h \in C(X).$$

Remark 6.1.8. Suppose that $(\mathbf{f}, \mathbf{p})$ is an IFS which is l-contractive on average. Then according to Lemma 6.1.6, the system $(\mathbf{f}, \mathbf{p})$ has a unique attractive invariant measure.

We call a place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ *contractive on average* if there exists $r \in (0, 1)$ such that

$$\sum_{n=1}^{k} p_n(x) d(f_n(x), f_n(y)) \leq r d(x, y) \qquad \text{for all } x, y \in X,$$

(see e.g. Werner [140]). Before giving a sufficient condition for the existence of an attractive invariant measure we recall the following notion. A real-valued continuous function $h : X \to \mathbb{R}$ is called *Dini-continuous* if for some $c > 0$

$$\int_0^c \frac{\varphi_h(t)}{t}\, dt < \infty,$$

where φ_h is the *modulus of uniform continuity* of h, i.e.

$$\varphi_h(t) := \sup\{|h(x) - h(y)| : d(x, y) \le t, x, y \in X\} \qquad \text{for all } t \ge 0.$$

The following theorem is a consequence of a theorem proved in Barnsley, Demko, Elton and Geronimo [11]. However, we first note that since $p_1, \dots, p_k$ are continuous functions and the metric space (X, d) is compact, the functions $p_1, \dots, p_k$ are bounded away from 0, i.e. there exists $\delta > 0$ such that

$$p_n(x) \ge \delta \qquad \text{for all } x \in X, n = 1, \dots, k.$$

Theorem 6.1.9 (Existence of Attractive Invariant Measure for Contractive on Average Place-dependent IFS, [11]). *Let $(\mathbf{f}, \mathbf{p}(\cdot))$ be a contractive on average place-dependent IFS with all $f_1, \dots, f_k$ being Lipschitz-continuous and all $p_1, \dots, p_k$ are Dini-continuous. Then the place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ has an attractive (and thus necessarily unique) invariant measure.*

In Stenflo [132], an example is constructed to show that the content of the above theorem is no longer true under the weaker assumption that all $p_1, \dots, p_k$ are continuous. Other sufficient conditions for the existence of an invariant measure and an attractive measure can be found among others in Elton and Piccioni [51], Burton and Keller [26], Lasota and Yorke [87], etc. We refer to Stenflo [131] for a survey of results on the question of uniqueness of invariant measures for place-dependent IFS.
Now we follow the construction in Kwiecińska and Słomczyński [84] to define the RDS generated by a contractive on average place-dependent IFS. We consider a place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ which fulfills all assumptions of Theorem 6.1.9 and let μ denote the unique invariant measure of $(\mathbf{f}, \mathbf{p}(\cdot))$. To obtain an RDS generated by $(\mathbf{f}, \mathbf{p}(\cdot))$, we define the corresponding probability measure on the space of addresses $(\Omega^{(k)}, \mathcal{F}^{(k)})$ by first setting

$$\mathbb{P}((i_0, \dots, i_n)_{0,\dots,n}) = \int_X p_{i_0}(x) p_{i_1}(f_{i_0}(x)) \dots p_{i_n}(f_{i_{n-1}} \circ \dots \circ f_{i_0}(x))\, d\mu(x), \qquad (6.6)$$

for all $n \in \mathbb{N}$, $i_0, \dots, i_n \in \{1, \dots, k\}$, and then extend $\mathbb{P}$ to the whole $\mathcal{F}^{(k)}$. Having constructed the probability space, we consider the left shift on it, i.e. the mapping $\theta : \Omega^{(k)} \to \Omega^{(k)}$ defined by $(\theta\omega)_j = \omega_{j+1}$. Since μ is an attractive invariant measure of $(\mathbf{f}, \mathbf{p}(\cdot))$, the mapping θ is an ergodic transformation preserving the probability $\mathbb{P}$ (see e.g. Kwiecińska and Słomczyński [84, Proposition 1], Werner [140, Proposition 2.1]).

Remark 6.1.10 (RDS generated by place-dependent IFS). Suppose that $(\mathbf{f}, \mathbf{p}(\cdot))$ is a contractive on average place-dependent IFS and $p_1, \dots, p_k$ are Dini-continuous. Then

$(\mathbf{f}, \mathbf{p}(\cdot))$ generates a random dynamical system φ over the MDS $(\Omega^{(k)}, \mathcal{F}^{(k)}, \mathbb{P}, \theta)$ by setting

$$\varphi(n, \omega)x = f_{\omega_{n-1}} \circ f_{\omega_{n-2}} \circ \cdots \circ f_{\omega_0}(x).$$

The corresponding skew product $\tau : \Omega^{(k)} \times X \to \Omega^{(k)} \times X$ is given by

$$\tau(\omega, x) = (\theta\omega, f_{\omega_0}(x)).$$

So far we have discussed and explained why an RDS can be generated by a contractive on average place-dependent IFS. In the following theorem, an ergodic theorem for place-dependent IFS is stated and we refer to Barnsley, Demko, Elton and Geronimo [11] and Werner [140] for a proof.

Theorem 6.1.11 (Ergodic Theorem for Place-dependent IFS, [11])**.** *Let $(\mathbf{f}, \mathbf{p}(\cdot))$ be a contractive on average place-dependent IFS satisfying that $p_1, \dots, p_k$ are Dini-continuous. Let μ denote the unique measure of $(\mathbf{f}, \mathbf{p}(\cdot))$. Denote by $\mathbb{P}$ the generated measure on the address space $(\Omega^{(k)}, \mathcal{F}^{(k)})$ which is defined as in (6.6). Then for any continuous function $G : X \to \mathbb{R}$ and any $x \in X$ the following limit exists and equality holds*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi(n, \omega)x) = \int_X G(z)\, d\mu(z) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

6.1.3 Infinite Iterated Function Systems

In this subsection, we introduce a generalization of finite iterated function systems to systems of infinitely many functions. Let (X, d) be a compact metric space and $\mathbf{f} = \{f_n\}_{n=1}^{\infty}$ a sequence of Lipschitz maps on X. Let $\mathbf{p} = \{p_n\}_{n=1}^{\infty}$ be a sequence of probabilities $p_n > 0$ with $\sum_{n=1}^{\infty} p_n = 1$. Then the pair $(\mathbf{f}, \mathbf{p})$ is called an *infinite iterated function system (IIFS)*, see [95]. Similarly as in the case of finite iterated function systems where we had k symbols $\{1, \dots, k\}$ we now define the *space of addresses* but with infinitely many symbols $\mathbb{N}$

$$\Omega = \prod_{n=0}^{\infty} \mathbb{N} = \mathbb{N}^{\infty}$$

together with the σ-algebra $\mathcal{F}$ generated by the cylinders in Ω. For convenience, we define

$$(i_0, i_1, \dots, i_k)_{p_0, p_1 \dots, p_k} := \left\{\omega \in \Omega : \omega_{p_j} = i_j \text{ for all } j = 0, 1, \dots, k\right\}.$$

A probability measure on $(\Omega, \mathcal{F})$ is defined by

$$\mathbb{P} = \prod_{n=0}^{\infty} \rho \qquad \text{with } \rho(\{n\}) = p_n\ .$$

The left shift $\theta : \Omega \to \Omega$ with $(\theta\omega)_j = \omega_{j+1}$ for all $\omega \in \Omega$ and $j \in \mathbb{N}_0$ is ergodic and preserves the probability $\mathbb{P}$, see [138]. Moreover, using the same arguments as in the proof of Walter [138, Theorem 1.12], we also obtain that θ^l is an ergodic transformation for all $l \in \mathbb{N}$.

Remark 6.1.12 (RDS generated by IIFS)**.** An IIFS $(\mathbf{f},\mathbf{p})$ generates a random dynamical system φ over the MDS $(\Omega,\mathcal{F},\mathbb{P},\theta)$ by setting

$$\varphi(n,\omega)x = f_{\omega_{n-1}} \circ f_{\omega_{n-2}} \circ \cdots \circ f_{\omega_0}(x)\,.$$

The corresponding skew product $\tau : \Omega \times X \to \Omega \times X$ is given by

$$\tau(\omega,x) = (\theta\omega, f_{\omega_0}x)\,.$$

Similarly as for IFS we define the l-average rate of expansion for IIFS as follows.

Definition 6.1.2 (l-Average Expansion Rate and Contraction for IIFS)**.** Let $(\mathbf{f},\mathbf{p})$ be an IIFS. The number

$$\mathrm{Lip}_l(\mathbf{f}) := \sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1}\dots p_{n_l}\,\mathrm{Lip}(f_{n_1}\circ\cdots\circ f_{n_l})$$

is called the *l-average expansion rate* of $(\mathbf{f},\mathbf{p})$. The IIFS $(\mathbf{f},\mathbf{p})$ is said to be *l-contractive on average* if $\mathrm{Lip}_l(\mathbf{f}) < 1$.

For the remainder of this chapter we mainly deal with IIFS which are l-contractive on average. Next we extend Lemmas 6.1.5 and 6.1.6 to IIFS. However, we first recall a criteria to check whether a transformation is measure-preserving. In [138, Theorem 1.1], it is proved that a transformation T from a probability $(\Omega,\mathcal{F},\mathbb{P})$ into itself is measure-preserving if and only if $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{S}$, where $\mathcal{S}$ is a semi-algebra generating the σ-algebra $\mathcal{F}$. Recall that a collection $\mathcal{S}$ of subsets of Ω is called a *sigma-algebra* if the following three conditions hold:

(i) $\emptyset \in \mathcal{S}$,

(ii) if $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$,

(iii) if $A \in \mathcal{S}$ then $A^c = \bigcup_{i=1}^n E_i$, where each $E_i \in \mathcal{S}$ and $E_1,\dots,E_n$ are pairwise disjoint subsets of Ω.

Lemma 6.1.13. *Let $(\mathbf{f},\mathbf{p})$ be an IIFS on a compact metric space (X,d). Then a probability measure $\mu \in \mathcal{M}(X)$ is invariant under the IIFS $(\mathbf{f},\mathbf{p})$, i.e. $\mathbb{P}\times\mu$ is invariant under the skew product τ associated with $(\mathbf{f},\mathbf{p})$, if and only if μ is a fixed point of the* Markov operator $\mathcal{P} : \mathcal{M}(X) \to \mathcal{M}(X)$ *defined by*

$$\mathcal{P}\nu = \sum_{n=1}^{\infty} p_n \nu \circ f_n^{-1} \qquad \text{for } \nu \in \mathcal{M}(X)\,. \tag{6.7}$$

Moreover, if the IIFS $(\mathbf{f},\mathbf{p})$ is l-contractive on average then the operator $\mathcal{P}^l$ is contractive with respect to the Hutchinson metric. More precisely, $\mathrm{Lip}(\mathcal{P}^l) \le \mathrm{Lip}_l(\mathbf{f})$ *and as a consequence, the IIFS $(\mathbf{f},\mathbf{p})$ has a unique invariant probability measure.*

Proof. Let τ denote the skew product associated to the IIFS $(\mathbf{f}, \mathbf{p})$ as in Remark 6.1.12. We first prove that the invariance of $\mu \in \mathcal{M}(X)$ is equivalent to $\mathcal{P}\mu = \mu$.

($\Rightarrow$) Suppose that $\mu \in \mathcal{M}(X)$ is an invariant probability measure of the system $(\mathbf{f}, \mathbf{p})$. For any measurable set $B \in \mathcal{B}(X)$, we have

$$\begin{aligned}
\mathbb{P} \times \mu \left(\tau^{-1}(\Omega \times B)\right) &= \mathbb{P} \times \mu \left(\{(\omega, x) \in \Omega \times X : \theta\omega \in \Omega, f_{\omega_0}(x) \in B\}\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P} \times \mu \left(\left\{(\omega, x) \in \Omega \times X : \omega_0 = n, x \in f_n^{-1}(B)\right\}\right) \\
&= \sum_{n=1}^{\infty} p_n \mu \left(f_n^{-1}(B)\right) = \mathcal{P}\mu(B),
\end{aligned}$$

which together with $\mathbb{P} \times \mu(\Omega \times B) = \mu(B)$ proves that $\mathcal{P}\mu = \mu$.

($\Leftarrow$) Let μ be a probability measure on X satisfying that $\mu = \sum_{n=1}^{\infty} p_n \mu \circ f_n^{-1}$. To show that $\mathbb{P} \times \mu$ is invariant under the skew product τ we first prove the following claim:

Claim: The collection of sets

$$\mathcal{S} := \{\mathbb{N} \times \cdots \times \mathbb{N} \times B_p \times \cdots \times B_{p+k} \times \mathbb{N} \times \cdots \subset \Omega |\, p, k \in \mathbb{N}_0, B_p, \dots, B_{p+k} \subset \mathbb{N}\}$$

is a semi-algebra on Ω generating the sigma-algebra $\mathcal{F}$. Indeed, it is easy to see that $\mathcal{F}$ is the sigma-algebra generated by $\mathcal{S}$. For $A_1, A_2 \in \mathcal{S}$ with the form

$$\begin{aligned}
A_1 &= \mathbb{N} \times \cdots \times \mathbb{N} \times B_p \times \cdots \times B_{p+k} \times \mathbb{N} \times \dots, \\
A_2 &= \mathbb{N} \times \cdots \times \mathbb{N} \times C_{p'} \times \cdots \times C_{p'+k'} \times \mathbb{N} \times \dots,
\end{aligned}$$

a direct computation yields that $A_1 \cap A_2 \in \mathcal{S}$. Finally, for $A \in \mathcal{S}$ with the form

$$A = \mathbb{N} \times \cdots \times \mathbb{N} \times B_p \times \cdots \times B_{p+k} \times \mathbb{N} \times \dots,$$

we have $A^c = \bigcup_{i=0}^{k} E_i$, where

$$E_i = \mathbb{N} \times \cdots \times \mathbb{N} \times B_p \times \cdots \times B_{p+i-1} \times B_{p+i}^c \times \mathbb{N} \times \dots$$

Obviously, $E_0, \dots, E_k$ are pairwise disjoint and $E_0, \dots, E_k \in \mathcal{S}$. This proves the claim. Returning to the proof, by using the above claim, we obtain that the collection of sets

$$\widetilde{\mathcal{S}} := \{A \times B | A \in \mathcal{S}, B \in \mathcal{B}(X)\}$$

is a semi-algebra on $\Omega \times X$ generating the sigma-algebra $\mathcal{F} \otimes \mathcal{B}(X)$. On the other hand, each set in $\widetilde{\mathcal{S}}$ can be represented as a countably disjoint union of sets which have the form $(i_0, \dots, i_k)_{p,\dots,p+k} \times B$. Therefore, it is sufficient to prove that

$$\mathbb{P} \times \mu \left((i_0, \dots, i_k)_{p,\dots,p+k} \times B\right) = \mathbb{P} \times \mu \left(\tau^{-1}\left((i_0, \dots, i_k)_{p,\dots,p+k} \times B\right)\right) \tag{6.8}$$

holds for all $B \in \mathcal{B}(X)$, $p, k \in \mathbb{N}_0$, and $i_0, \dots, i_k \in \mathbb{N}$. The set $\tau^{-1}\left((i_0, \dots, i_k)_{p,\dots,p+k} \times B\right)$ can be represented as the following disjoint union

$$\tau^{-1}\left((i_0, \dots, i_k)_{p,\dots,p+k} \times B\right) = \bigcup_{n=1}^{\infty} (n, i_0, \dots, i_k)_{0,p+1,\dots,p+k+1} \times f_n^{-1}(B).$$

Together with the fact that $\mu(B) = \sum_{n=1}^{\infty} p_n \mu\left(f_n^{-1}(B)\right)$ this implies

$$\begin{aligned}\mathbb{P} \times \mu\left(\tau^{-1}\left((i_0, \dots, i_k)_{p,\dots,p+k} \times B\right)\right) &= p_{i_0} \dots p_{i_k} \sum_{n=1}^{\infty} p_n \mu\left(f_n^{-1}(B)\right)\\ &= p_{i_0} \dots p_{i_k} \mu(B),\end{aligned}$$

proving (6.8). For the remaining part of the proof, we suppose that the system $(\mathbf{f}, \mathbf{p})$ is l-contractive on average and we need to show that the following inequality

$$\left| \int_X h\, d\mathcal{P}^l \nu_1 - \int_X h\, d\mathcal{P}^l \nu_2 \right| \le \mathrm{Lip}_l(\mathbf{f})\, d_H(\nu_1, \nu_2) \tag{6.9}$$

holds for all $\nu_1, \nu_2 \in \mathcal{M}(X)$ and $h \in \mathrm{Lip}_1(X)$. A direct computation yields that

$$\int_X h\, d\mathcal{P}^l \nu = \sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1} \dots p_{n_l} \int_X h \circ f_{n_1} \circ \dots \circ f_{n_l}\, d\nu \qquad \text{for all } l \in \mathbb{N}, \nu \in \mathcal{M}(X).$$

Hence, we get

$$\begin{aligned}\left| \int_X h\, d\mathcal{P}^l \nu_1 - \int_X h\, d\mathcal{P}^l \nu_2 \right| \le \sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1} \dots p_{n_l} \Bigg| &\int_X h \circ f_{n_1} \circ \dots \circ f_{n_l}\, d\nu_1 \\ &- \int_X h \circ f_{n_1} \circ \dots \circ f_{n_l}\, d\nu_2 \Bigg|.\end{aligned}$$

Using Remark 6.0.2, we have $\mathrm{Lip}(h \circ f_{n_1} \circ \dots \circ f_{n_l}) \le \mathrm{Lip}(f_{n_1} \circ \dots \circ f_{n_l})$ and this together with the above inequality equality proves (6.9) and the proof is complete. □

The Barnsley ergodic theorem 6.1.7 for IFS was extended in many ways, e.g. to general IIFS with time-dependent probabilities by Stenflo [130], for an extension to uniformly contractive IIFS see Hyong-chol et al. [71]. Cong, Doan and Siegmund [38] extend this result to IIFS which are contractive on average. In the following, we present a short proof of an extension of Theorem 6.1.7 to IIFS which are l-contractive on average. Firstly, we extend a result from Furstenberg and Kesten [60] (see also Krengel [81, p.40]).

Lemma 6.1.14. *Let $(\mathbf{f}, \mathbf{p})$ be an IIFS which is l-contractive on average. Then there exists $\alpha < 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Lip}\left(\varphi(n, \omega)\right) = \alpha \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega\,. \tag{6.10}$$

Proof. For each $n \in \mathbb{N}$ we define a measurable function $g_n : \Omega \to \mathbb{R}$ by

$$g_n(\omega) = \log \mathrm{Lip}\left(\varphi(n, \omega)\right) \qquad \text{for all } \omega \in \Omega.$$

Since $\varphi(n + m, \omega) = \varphi(m, \theta^n \omega) \circ \varphi(n, \omega)$ and using Remark 6.0.2 (i), we get

$$g_{n+m}(\omega) \le g_m(\theta^n \omega) + g_n(\omega) \qquad \text{for all } n, m \in \mathbb{N}. \tag{6.11}$$

Therefore, $\{g_n\}_{n=1}^\infty$ is a subadditive sequence of random variables over the ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Now we show that $g_l^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. By the definition of g_l, we get

$$\int_\Omega g_l^+(\omega)\, d\mathbb{P}(\omega) = \sum_{(n_1,\dots,n_l)\in A} p_{n_1}\dots p_{n_l} \log \mathrm{Lip}(f_{n_1}\circ\dots\circ f_{n_l}),$$

where $A := \{(n_1,\dots,n_l)\in\mathbb{N}^l : \mathrm{Lip}(f_{n_1}\circ\dots\circ f_{n_l}) > 1\}$. Since $\log(\cdot)$ is a concave function it follows with $\mathrm{Lip}_l(\mathbf{f}) < 1$ that

$$\sum_{(n_1,\dots,n_l)\in A} p_{n_1}\dots p_{n_l} \log \mathrm{Lip}(f_{n_1}\circ\dots\circ f_{n_l}) \;\le\; \left(\sum_{(n_1,\dots,n_l)\in A} p_{n_1}\dots p_{n_l}\right)\cdot$$
$$\cdot \log \frac{1}{\sum_{(n_1,\dots,n_l)\in A} p_{n_1}\dots p_{n_l}}.$$

This implies with the inequality $x\log\frac{1}{x} \le 1$ for all $0 < x \le 1$ that $g_l^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Thus the subsequence $\{g_{nl}\}_{n=1}^\infty$ fulfills all assumptions of the Kingman subadditive ergodic theorem B.0.8 over the ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta^l)$. Consequently, there exists $\widehat{\Omega} \subset \Omega$ which is θ^l forward invariant with $\mathbb{P}(\widehat{\Omega}) = 1$ and

$$\lim_{n\to\infty} \frac{1}{n} g_{nl}(\omega) = \beta := \lim_{n\to\infty}\frac{1}{n}\int_\Omega g_{nl}(\omega)\, d\mathbb{P}(\omega) \qquad \text{for all } \omega\in\widehat{\Omega}. \tag{6.12}$$

We now show that $\beta < 0$. Integrating both sides of the inequality $g_{nl+ml}(\omega) \le g_{ml}(\theta^{nl}\omega) + g_{nl}(\omega)$ and using the fact that θ is an ergodic transformation preserving the probability $\mathbb{P}$, we obtain

$$\int_\Omega g_{nl+ml}(\omega)\, d\mathbb{P}(\omega) \le \int_\Omega g_{nl}(\omega)\, d\mathbb{P}(\omega) + \int_\Omega g_{ml}(\omega)\, d\mathbb{P}(\omega).$$

In particular,

$$\beta = \lim_{n\to\infty}\frac{1}{n}\int_\Omega g_{nl}(\omega)\, d\mathbb{P}(\omega) \le \int_\Omega g_l(\omega)\, d\mathbb{P}(\omega). \tag{6.13}$$

From the definition of g_l, we derive that

$$\int_\Omega g_l(\omega)\, d\mathbb{P}(\omega) = \sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1}\dots p_{n_l} \log \mathrm{Lip}(f_{n_1}\circ\dots\circ f_{n_l}).$$

Using the fact that $\log(\cdot)$ is a concave function together with $\mathrm{Lip}_l(\mathbf{f}) < 1$ implies

$$\int_\Omega g_l(\omega)\, d\mathbb{P}(\omega) \le \log\left(\sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1}\dots p_{n_l} \mathrm{Lip}(f_{n_1}\circ\dots\circ f_{n_l})\right) < 0,$$

which together with (6.13) shows that $\beta < 0$. Define $\alpha := \frac{\beta}{l}$ and to complete the proof we show that

$$\lim_{n\to\infty}\frac{1}{n} g_n(\omega) = \alpha \qquad \text{for all } \omega \in \bigcap_{i=0}^{l-1}\theta^i\widehat{\Omega}.$$

Using (6.11), for all $i = 0, 1, \ldots, l-1$ and $\omega \in \bigcap_{i=0}^{l-1} \theta^i \widehat{\Omega}$, we obtain the following inequality

$$\frac{g_{nl+l}(\theta^{i-l}\omega)}{nl+i} - \frac{g_{l-i}(\theta^{i-l}\omega)}{nl+i} \leq \frac{g_{nl+i}(\omega)}{nl+i} \leq \frac{g_{nl}(\theta^i\omega)}{nl+i} + \frac{g_i(\omega)}{nl+i},$$

where we use the fact that $\omega \in \theta^{l-i}\widehat{\Omega}$ to ensure the existence of $\theta^{i-l}\omega \in \widehat{\Omega}$. Note that since $\omega \in \theta^{l-i}\widehat{\Omega}$, we have $\theta^i\omega \in \theta^l\widehat{\Omega} \subset \widehat{\Omega}$. Taking $n \to \infty$ and using (6.12), we have

$$\lim_{n\to\infty} \frac{1}{nl+i}\, g_{nl+i}(\omega) = \alpha \qquad \text{for } i = 0, 1, \ldots, l-1 \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

This completes the proof. □

Remark 6.1.15. If the IIFS $(\mathbf{f}, \mathbf{p})$ is uniformly contractive, i.e., there exists $\beta < 1$ such that $\mathrm{Lip}(f_n) \leq \beta$ for all $n \in \mathbb{N}$ then the following inequality holds

$$\frac{1}{n} \log \mathrm{Lip}\,(\varphi(n,\omega)) \leq \log \beta < 0 \qquad \text{for all } n \in \mathbb{N}, \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The following theorem extends the Barnsley ergodic theorem 6.1.7 to IIFS which are l-contractive on average.

Theorem 6.1.16 (Ergodic Theorem for l-Contractive on Average IIFS)**.** *Suppose that the IIFS $(\mathbf{f}, \mathbf{p})$ is l-contractive on average. Then for any continuous function $G : X \to \mathbb{R}$ and any $x \in X$ the following limit exists and equality holds*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi(n,\omega)x) = \int_X G(z)\, d\mu(z) \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega,$$

where μ is the invariant probability measure of $(\mathbf{f}, \mathbf{p})$.

Proof. By using Lemma 6.1.14 together with the fact that θ is an ergodic transformation preserving the probability $\mathbb{P}$, there exists $\alpha < 0$ such that the following limits

$$\lim_{n\to\infty} \frac{1}{n} \log \mathrm{Lip}\,(\varphi(n,\omega)) = \alpha \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Lip}\,(\varphi(n,\theta\omega)) = \alpha \tag{6.14}$$

hold for $\mathbb{P}$ a.e. $\omega \in \Omega$. We define a real valued function $\widetilde{G} : \Omega \times X \to \mathbb{R}$ by $\widetilde{G}(\omega, x) = G(x)$. By Remark 6.1.12, we have $G(\varphi(n,\omega)x) = \widetilde{G}(\tau^n(\omega,x))$ and hence

$$\frac{1}{N+1} \sum_{n=0}^{N} G(\varphi(n,\omega)x) = \frac{1}{N+1} \sum_{n=0}^{N} \widetilde{G}(\tau^n(\omega,x)) \qquad \text{for all } (\omega,x) \in \Omega \times X.$$

By virtue of Lemma 6.1.13, the measure $\mathbb{P} \times \mu$ is invariant under τ and since $G \in C(X)$ it follows that $\widetilde{G}$ is a bounded measurable function from $\Omega \times X$ to $\mathbb{R}$. By applying the Birkhoff ergodic theorem (see Appendix A), there exists a measurable function $G^* : \Omega \times X \to \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \widetilde{G}(\tau^n(\omega, x)) = G^*(\omega, x) \tag{6.15}$$

holds for $\mathbb{P} \times \mu$ -a.e. $(\omega, x) \in \Omega \times X$ and

$$\int_{\Omega\times X} G^*(\omega, x)\; d(\mathbb{P}\times\mu)(\omega, x) = \int_{\Omega\times X} \widetilde{G}(\omega, x)\; d(\mathbb{P}\times\mu)(\omega, x). \tag{6.16}$$

Define

$$\widetilde{\Omega} = \{\omega \in \Omega : (6.14) \text{ holds and there exists } x \in X \text{ such that } (6.15) \text{ holds}\}.$$

It is easy to obtain that $\mathbb{P}(\widetilde{\Omega}) = 1$. We will show that the function G^* is constant on the set $\widetilde{\Omega} \times X$. For this purpose, choose and fix $\omega \in \widetilde{\Omega}$. According to (6.14), there exists $N_1 \in \mathbb{N}$ such that

$$\mathrm{Lip}\,(\varphi(n, \omega)) \le e^{\frac{\alpha n}{2}} \qquad \text{for all } n \ge N_1,$$

which gives that

$$d\,(\varphi(n, \omega)x, \varphi(n, \omega)y) \le e^{\frac{\alpha n}{2}} \mathrm{diam}(X) \qquad \text{for all } x, y \in X \text{ and } n \ge N_1.$$

On the other hand, we have

$$\widetilde{G}(\tau^n(\omega, x)) - \widetilde{G}(\tau^n(\omega, y)) = G(\varphi(n, \omega)x) - G(\varphi(n, \omega)y).$$

Together with (6.15) and the fact that G is a uniformly continuous function from X to $\mathbb{R}$ we conclude that the function G^* is independent of x on the set $\widetilde{\Omega} \times X$. Now we choose and fix $x \in X, \omega \in \widetilde{\Omega}$. According to (6.14) there exists $N_2 \in \mathbb{N}$ such that

$$\mathrm{Lip}\,(\varphi(n, \theta\omega)) \le e^{\frac{\alpha n}{2}} \qquad \text{for all } n \ge N_2.$$

Therefore,

$$d(\varphi(n, \theta\omega)x, \varphi(n+1, \omega)x) \le e^{\frac{\alpha n}{2}} \mathrm{diam}(X) \qquad \text{for all } n \ge N_2.$$

On the other hand,

$$\widetilde{G}(\tau^n(\theta\omega, x)) - \widetilde{G}(\tau^{n+1}(\omega, x)) = G(\varphi(n, \theta\omega)x) - G(\varphi(n+1, \omega)x).$$

This implies with (6.15) that $G^*(\theta\omega, x) = G(\omega, x)$. Since θ is ergodic it follows that $G^*(\omega, x)$ is independent of ω on $\widetilde{\Omega} \times X$ and the claim is proved. By using (6.16) and the fact that $\int_{\Omega\times X} \widetilde{G}(\omega, x)\; d(\mathbb{P}\times\mu)(\omega, x) = \int_X G(x)\; d\mu(x)$, we get

$$G^*(\omega, x) = \int_X G(x)\, d\mu(x) \qquad \text{for all } (\omega, x) \in \widetilde{\Omega} \times X.$$

This completes the proof. □

6.2 Computational Ergodic Theorem for Place-dependent IFS

In this section, we consider a place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ defined on a compact subset $X \subset \mathbb{R}^d$, where $\mathbf{f} = \{f_n\}_{n=1}^k, \mathbf{p} = \{p_n\}_{n=1}^k$, which is contractive on average, i.e. there exists $r \in (0,1)$ such that

$$\sum_{n=1}^{k} p_n(x) d\big(f_n(x), f_n(y)\big) \leq r d(x,y) \qquad \text{for all } x, y \in X.$$

We assume additionally that the functions $p_1, \ldots, p_k : X \to (0,1)$ are Dini-continuous. According to Lemma 6.1.9, the associated Markov operator $\mathcal{T} : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by

$$\mathcal{T}\nu(B) = \sum_{n=1}^{k} \int_{f_n^{-1}(B)} p_n(x)\, d\nu(x) \qquad \text{for all } \nu \in \mathcal{M}(X), B \in \mathcal{B}(X), \tag{6.17}$$

has a unique attractive invariant measure denoted by μ. Let $\mathbb{P}$ be the probability on the space of addresses $(\Omega^{(k)}, \mathcal{F}^{(k)})$ which is defined as in (6.6). For a given continuous map $G : X \to \mathbb{R}$, we define the time average of $(\mathbf{f}, \mathbf{p}(\cdot))$ with respect to G by

$$\lambda := \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi(n,\omega)x) = \int_X G(z)\, d\mu(z).$$

Theorem 6.1.11 implies that the limit almost surely equals the space average and therefore λ is well-defined. Our aim is to establish an algorithm to approximate λ.

An Approximation and Convergence Result

Construct a partition of X into K connected sets $\{X_1, X_2, \ldots, X_K\}$. From each set, choose a single point x_n, $n = 1, \ldots, K$, and for each mapping f_n, where $n = 1, \ldots, k$, define a $K \times K$ stochastic matrix $S_n = (S_{n,ij})_{i,j=1,\ldots,K}$ by setting

$$S_{n,ij} = \begin{cases} 1, & \text{if } f_n(x_i) \in X_j\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{6.18}$$

We use the matrices S_n to define a family of Markov operators that will approximate the Markov operator $\mathcal{T}$. Precisely, we define the operator $\mathcal{T}_K : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$\mathcal{T}_K\nu = \sum_{j=1}^{K} \left(\sum_{n=1}^{k} \sum_{i=1}^{K} \int_{X_i} p_n(x) S_{n,ij}\, d\nu(x) \right) \delta_{x_j} \qquad \text{for all } \nu \in \mathcal{M}(X), \tag{6.19}$$

where δ_{x_j} is the Dirac measure centered at the point x_j. The following lemma provides an estimate between the operator $\mathcal{T}_K$ and the operator $\mathcal{T}$.

Lemma 6.2.1. *Define* $\varepsilon_K := \max_{1\le n\le K} diam(X_n)$. *Then*

$$\sup_{\nu\in\mathcal{M}(X)} d_H(\mathcal{T}_K\nu, \mathcal{T}\nu) \le 2\varepsilon_K.$$

Proof. By the definition of the Hutchison metric as in (6.1), it is equivalent to show that

$$\left|\int_X h\ d\mathcal{T}\nu - \int_X h\ d\mathcal{T}_K\nu\right| \le 2\varepsilon_K \qquad \text{for all } h\in \mathrm{Lip}_1(X). \tag{6.20}$$

A direct computation yields that

$$\begin{aligned}\left|\int_X h\ d\mathcal{T}\nu - \int_X h\ d\mathcal{T}_K\nu\right| &= \Bigg|\sum_{n=1}^{k}\sum_{i=1}^{K}\int_{X_i} p_n(x) h\circ f_n(x)\ d\nu(x) - \\ &\qquad -\sum_{j=1}^{K}\left(\sum_{n=1}^{k}\sum_{i=1}^{K}\int_{X_i} p_n(x)S_{n,ij}\ d\nu(x)\right)h(x_j)\Bigg| \\ &\le S_1+S_2,\end{aligned}$$

where

$$\begin{aligned} S_1 &:= \left|\sum_{n=1}^{k}\sum_{i=1}^{K} h\circ f_n(x_i)\int_{X_i} p_n(x)\ d\nu(x) - \sum_{j=1}^{K}\left(\sum_{n=1}^{k}\sum_{i=1}^{K}\int_{X_i} p_n(x)S_{n,ij}\ d\nu(x)\right)h(x_j)\right|, \\ S_2 &:= \left|\sum_{n=1}^{k}\sum_{i=1}^{K}\int_{X_i} p_n(x)\big(h\circ f_n(x) - h\circ f_n(x_i)\big)\ d\nu(x)\right|.\end{aligned}$$

Since $h\in \mathrm{Lip}_1(X)$ and the system $(\mathbf{f},\mathbf{p}(\cdot))$ is contractive on average, we have

$$\begin{aligned} S_2 &\le \sum_{n=1}^{k}\sum_{i=1}^{K}\int_{X_i} p_n(x) d\big(f_n(x), f_n(x_i)\big)\ d\nu(x) \\ &= \sum_{i=1}^{K}\int_{X_i}\sum_{n=1}^{k} p_n(x) d\big(f_n(x), f_n(x_i)\big)\ d\nu(x) \\ &\le \varepsilon_K. \end{aligned} \tag{6.21}$$

On the other hand, using the equality $h\circ f_n(x_i) = \sum_{j=1}^K h\circ f_n(x_i)S_{n,ij}$ for all $i=1,\ldots,K$ and $n=1,\ldots,k$, we obtain

$$\begin{aligned} S_1 &= \left|\sum_{n=1}^{k}\sum_{i,j=1}^{K}\int_{X_i} S_{n,ij}\big(h\circ f_n(x_i) - h(x_j)\big)p_n(x)\ d\nu(x)\right| \\ &\le \sum_{n=1}^{k}\sum_{i,j=1}^{K}\int_{X_i}\varepsilon_K S_{n,ij}p_n(x)\ d\nu(x) \\ &= \varepsilon_K, \end{aligned}$$

which together with estimate (6.21) proves (6.20) and the proof is complete. □

To construct an invariant measure of $\mathcal{T}_K$, we use the following $K \times K$ stochastic matrix $S^{(K)} = (S_{ij}^{(K)})_{i,j=1,\ldots,K}$ defined by

$$S_{ij}^{(K)} = \sum_{n=1}^{k} p_n(x_i) S_{n,ij}, \tag{6.22}$$

where the stochastic matrices $S_1, \ldots, S_k$ are defined as in (6.18).

Lemma 6.2.2. *Let $s^{(K)} = (s_1, \ldots, s_K)$ be an arbitrary fixed left eigenvector of $S^{(K)}$. Then the probability measure*

$$\nu_K := \sum_{j=1}^{K} s_j \delta_{x_j} \tag{6.23}$$

is an invariant measure of $\mathcal{T}_K$

Proof. From (6.19), we derive that

$$\mathcal{T}_K \delta_{x_m} = \sum_{j=1}^{K} \left(\sum_{n=1}^{k} p_n(x_m) S_{n,mj} \right) \delta_{x_j},$$

which implies that

$$\begin{aligned} \mathcal{T}_K \sum_{m=1}^{K} s_m \delta_{x_m} &= \sum_{m=1}^{K} s_m \sum_{j=1}^{K} S_{mj}^{(K)} \delta_{x_j} \\ &= \sum_{j=1}^{k} s_j \delta_{x_j}, \end{aligned}$$

where we use the assumption that $(s_1, \ldots, s_K)$ is a fixed left eigenvector of $S^{(K)}$ to obtain the last equality. This completes the proof. □

Theorem 6.2.3 (Computational Ergodic Theorem for Place-dependent IFS)**.** *Let $(\mathbf{f}, \mathbf{p}(\cdot))$ be a contractive on average place-dependent IFS on a compact subset $X \subset \mathbb{R}^d$ satisfying that $p_1, \ldots, p_k$ are Dini-continuous. For a given continuous function $G : X \to \mathbb{R}$, we have*

$$\lim_{K\to\infty} \sum_{j=1}^{K} s_j G(x_j) = \lambda = \int_X G(z)\, d\mu(z), \qquad \textit{provided that } \lim_{K\to\infty} \varepsilon_K = 0,$$

where $s^{(K)} = (s_1, \ldots, s_K)$ is a fixed left eigenvector of the stochastic matrix $S^{(K)}$ defined as in (6.22) *and μ is the invariant measure of $(\mathbf{f}, \mathbf{p}(\cdot))$.*

Proof. According to Lemma 6.2.1 and Lemma 6.2.2, we have

$$d_H(\mathcal{T}_K \nu_K, \mathcal{T} \nu_K) = d_H(\nu_K, \mathcal{T} \nu_K) \le 2\varepsilon_K. \tag{6.24}$$

Let ν be an arbitrary limit point of the sequence $\{\nu_K\}_{K\in\mathbb{N}}$. Then there exists a subsequence $\{n_K\}_{K\in\mathbb{N}}$ with $\lim_{K\to\infty} n_K = \infty$ such that $\nu = \lim_{K\to\infty} \nu_{n_K}$. By the definition of $\mathcal{T}$ and using the fact that $p_1, \ldots, p_k$ and $f_1, \ldots, f_k$ are continuous function, we obtain that $\lim_{K\to\infty} \mathcal{T}\nu_{n_K} = \mathcal{T}\nu$. Together with (6.24) this implies that $\mathcal{T}\nu = \nu$. Hence, $\nu = \mu$ and therefore μ is the unique limit point of the sequence $\{\nu_K\}_{K\in\mathbb{N}}$. On the other hand, since the metric space $(\mathcal{M}(X), d_H)$ is compact (see e.g. Barnsley [10, pp. 355]) it follows that $\lim_{K\to\infty} \nu_K = \mu$. As a consequence, we have

$$\lim_{K\to\infty} \int_X G \, d\nu_K = \lim_{K\to\infty} \sum_{j=1}^{K} s_j G(x_j) = \int_X G(z) \, d\mu(z) = \lambda,$$

which completes the proof. □

6.3 Computational Ergodic Theorem for IIFS

Throughout this section we consider an IIFS $(\mathbf{f}, \mathbf{p})$ which is l-contractive on average and is defined on a compact subset $X \subset \mathbb{R}^d$. Let $G : X \to \mathbb{R}$ be a given Lipschitz function. Define the *time average* of $(\mathbf{f}, \mathbf{p})$ with respect to G by

$$\lambda = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi(n, \omega)x) \; . \tag{6.25}$$

Theorem 6.1.16 implies that the limit almost surely equals the space average and therefore λ is well-defined. Our aim is to establish an algorithm to approximate λ. The algorithm contains two steps: In the first step, we approximate the IIFS by a sequence of IFS. In the last step, we follows the idea in Froyland [56] to compute the invariant measure of the approximating IFS. Combining these steps, we obtain a method to compute the time-average of an l-contractive IIFS with respect to a Lipschitz function.

6.3.1 Approximating IIFS through a Sequence of IFS

We introduce a sequence of "approximating IFS" $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ consisting of k functions. More precisely, for each $k \in \mathbb{N}$ we consider the IFS $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ defined by $\mathbf{f}^{(k)} = \{f_1, f_2, \ldots, f_k\}$ and $\mathbf{p}^{(k)} = \{p_1^{(k)}, p_2^{(k)}, \ldots, p_k^{(k)}\}$ with

$$p_n^{(k)} = \frac{p_n}{p_1 + p_2 + \cdots + p_k} \qquad \text{for all } 1 \le n \le k. \tag{6.26}$$

Denote by $\varphi^{(k)}$ the RDS generated by the IFS $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ as in Remark 6.1.1. A direct computation yields that

$$\mathrm{Lip}_l(\mathbf{f}^{(k)}) = \frac{1}{(p_1 + \cdots + p_k)^l} \sum_{n_1, \ldots, n_l = 1}^{k} p_{n_1} \ldots p_{n_l} \mathrm{Lip}(f_{n_1} \circ \cdots \circ f_{n_l}).$$

Consequently,

$$\lim_{k\to\infty} \mathrm{Lip}_l(\mathbf{f}^{(k)}) = \mathrm{Lip}_l(\mathbf{f}),$$

which implies that there exists $K \in \mathbb{N}$ such that system $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ is l-contractive on average for all $k \geq K$. Because we are only interested in the difference between $(\mathbf{f}, \mathbf{p})$ and $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ when k tends to infinity we can therefore assume w.l.o.g. that $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ is l-contractive on average for all $k \in \mathbb{N}$. Hence, Theorem 6.1.7 enables us to define the *time average* of $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ with respect to the function G as follows:

$$\lambda^{(k)} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} G(\varphi^{(k)}(n,\omega)x)\ . \tag{6.27}$$

The Markov operators associated with these systems are determined as follows:

$$\mathcal{P}_{(k)}\nu(B) = \sum_{n=1}^{k} p_n^{(k)} \nu(f_n^{-1}(B)) \qquad \text{for all } B \in \mathcal{B}(X). \tag{6.28}$$

The relation between the sequence of Markov operators $\{\mathcal{P}_{(k)}\}_{k=1}^{\infty}$ for $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ and the Markov operator $\mathcal{P}$ for $(\mathbf{f}, \mathbf{p})$ is the content of the following lemma.

Lemma 6.3.1. *Let $\mathcal{P}_{(k)}$ denote the Markov operator defined in* (6.28)*. Then the following inequality holds*

$$\sup_{\nu\in\mathcal{M}(X)} d_H(\mathcal{P}_{(k)}^l \nu, \mathcal{P}^l \nu) \leq 2\,\mathrm{diam}(X)\Big[1 - (p_1 + \cdots + p_k)^l\Big].$$

Proof. Choose and fix $x_0 \in X$. Using Remark 6.0.3 (ii), it is sufficient to show that the following inequality

$$\left| \int_X h\, d\mathcal{P}_{(k)}^l \nu - \int_X h\, d\mathcal{P}^l \nu \right| \leq 2\,\mathrm{diam}(X)\Big[1 - (p_1 + \cdots + p_k)^l\Big]$$

holds for all $\nu \in \mathcal{M}(X)$ and $h \in \mathrm{Lip}_1(X)$ with $h(x_0) = 0$. For this purpose, we proceed

$$\begin{aligned}
\left| \int_X h\, d\mathcal{P}^l \nu - \int_X h\, d\mathcal{P}_{(k)}^l \nu \right| &= \Bigg| \sum_{n_1,\dots,n_l=1}^{\infty} p_{n_1} \dots p_{n_l} \int_X h \circ f_{n_1} \circ \cdots \circ f_{n_l}\, d\nu - \\
&\qquad - \sum_{n_1,\dots,n_l=1}^{k} p_{n_1}^{(k)} \dots p_{n_l}^{(k)} \int_X h \circ f_{n_1} \circ \cdots \circ f_{n_l}\, d\nu \Bigg| \\
&\leq \sum_{n_1,\dots,n_l \in A_k} p_{n_1} \dots p_{n_l} \left| \int_X h \circ f_{n_1} \circ \cdots \circ f_{n_l}\, d\nu \right| + \\
&\quad + \sum_{n_1,\dots,n_l=1}^{k} \Big(p_{n_1}^{(k)} \dots p_{n_l}^{(k)} - p_{n_1} \dots p_{n_l} \Big) \left| \int_X h \circ f_{n_1} \circ \cdots \circ f_{n_l}\, d\nu \right|,
\end{aligned}$$

where $A_k := \mathbb{N}^l \setminus \{1,\dots,k\}^l$. Since $h \in \mathrm{Lip}_1(X)$ and $h(x_0)=0$ it follows that $|h(x)| \leq \mathrm{diam}(X)$ for all $x \in X$ and we conclude that $\left|\int_X h \circ f_{n_1} \circ \cdots \circ f_{n_l}\, d\nu\right| \leq \mathrm{diam}(X)$ for all $n_1,\dots,n_l \in \mathbb{N}$. Therefore, for all $\nu \in \mathcal{M}(X)$ and $h \in \mathrm{Lip}_1(X)$ we have

$$\begin{aligned}\left|\int_X h\, d\mathcal{P}^l\nu - \int_X h\, d\mathcal{P}^l_{(k)}\nu\right| &\leq \mathrm{diam}(X)\Bigg[\sum_{n_1,\dots,n_l=1}^{k} p_{n_1}\cdots p_{n_l}\left(\frac{1}{(p_1+\dots p_k)^l}-1\right)+ \\ &\qquad + \sum_{n_1,\dots,n_l\in A_k} p_{n_1}\cdots p_{n_l}\Bigg] \\ &= 2\,\mathrm{diam}(X)\left[1-(p_1+\cdots+p_k)^l\right],\end{aligned}$$

which completes the proof. □

Now we are in a position to state the main result in this subsection on the approximation of the invariant measure of a l-contractive on average IIFS by a sequence of invariant measures of approximating IFS.

Theorem 6.3.2 (Approximation of Invariant Measure for IIFS)**.** *Let* $(\mathbf{f},\mathbf{p})$ *be an IIFS which is* l*-contractive on average and* $(\mathbf{f}^{(k)},\mathbf{p}^{(k)})$ *the corresponding sequence of approximating IFS. Let* $\mu^{(k)}$ *be the fixed point of* $\mathcal{P}^{(k)}$ *and* μ *the fixed point of* $\mathcal{P}$*. Then*

$$d_H(\mu,\mu^{(k)}) \leq 2\,\mathrm{diam}(X)\frac{1-(p_1+\cdots+p_k)^l}{1-\mathrm{Lip}_l(\mathbf{f})}.$$

Moreover, for $k \in \mathbb{N}$ *let* $\lambda^{(k)}$ *denote the time average of* $(\mathbf{f}^{(k)},\mathbf{p}^{(k)})$ *with respect to the Lipschitz function* G *as in* (6.27) *then*

$$|\lambda^{(k)}-\lambda| \leq 2\,\mathrm{diam}(X)\frac{1-(p_1+\cdots+p_k)^l}{1-\mathrm{Lip}_l(\mathbf{f})}\,\mathrm{Lip}(G).$$

Proof. Choose and fix $k \in \mathbb{N}$. Since μ is the fixed point of $\mathcal{P}$ and by using Lemma 6.1.13, we have

$$d_H(\mu,\mathcal{P}^l\mu^{(k)}) = d_H(\mathcal{P}^l\mu,\mathcal{P}^l\mu^{(k)}) \leq \mathrm{Lip}_l(\mathbf{f})d_H(\mu,\mu^{(k)}). \tag{6.29}$$

On the other hand, by using the triangle inequality, we have

$$\begin{aligned}d_H(\mu,\mu^{(k)}) &= d_H(\mathcal{P}^l\mu,\mathcal{P}^l_{(k)}\mu^{(k)}) \\ &\leq d_H(\mathcal{P}^l\mu,\mathcal{P}^l\mu^{(k)}) + d_H(\mathcal{P}^l\mu^{(k)},\mathcal{P}^l_{(k)}\mu^{(k)}),\end{aligned}$$

which implies with Lemma 6.3.1 and (6.29) that

$$d_H(\mu,\mu^{(k)}) \leq \mathrm{Lip}_l(\mathbf{f})d_H(\mu,\mu^{(k)}) + 2\mathrm{diam}(X)\left[1-(p_1+\cdots+p_k)^l\right].$$

Therefore,

$$d_H(\mu,\mu^{(k)}) \leq 2\,\mathrm{diam}(X)\frac{1-(p_1+\cdots+p_k)^l}{1-\mathrm{Lip}_l(\mathbf{f})}\ . \tag{6.30}$$

Now we will prove the relation between $\lambda^{(k)}$ and λ. Using the ergodic theory for IFS (see Theorem 6.1.7) and for IIFS (see Theorem 6.1.16), we have

$$\lambda = \int_X G(z)\, d\mu(z) \quad \text{and} \quad \lambda^{(k)} = \int_X G(z)\, d\mu^{(k)}(z)\,.$$

Hence

$$|\lambda^{(k)} - \lambda| = \left| \int_X G(z)\, d\mu(z) - \int_X G(z)\, d\mu^{(k)}(z) \right|.$$

By Remark 6.0.2 (ii) we get $\frac{G}{\mathrm{Lip}(G)} \in \mathrm{Lip}_1(X)$ and together with (6.30) this completes the proof. □

6.3.2 An Approximation and Convergence Result

As in Section 6.3.1, we approximate the system $(\mathbf{f}, \mathbf{p})$ by the sequence of "approximating IFS" $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$, where $\mathbf{f}^{(k)} = \{f_1, f_2, \ldots, f_k\}$ and $\mathbf{p}^{(k)} = \{p_1^{(k)}, p_2^{(k)}, \ldots, p_k^{(k)}\}$ defined as in (6.26). Now we compute the invariant measure of the IFS $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$. To do it, we first construct a partition of X into k connected sets $\{X_1, X_2, \ldots, X_k\}$. From each set, choose a single point x_n, $n = 1, \ldots, k$, and for each mapping $f_{n_1} \circ \cdots \circ f_{n_l}$, where $n_1, \ldots, n_l \in \{1, \ldots, k\}$, define a $k \times k$ stochastic matrix $S^{(k)}_{n_1,\ldots,n_l} = \left(S^{(k)}_{n_1,\ldots,n_l ij}\right)_{i,j=1,\ldots,k}$ by setting

$$S^{(k)}_{n_1,\ldots,n_l,\, ij} = \begin{cases} 1, & \text{if } f_{n_1} \circ \cdots \circ f_{n_l}(x_i) \in X_j\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{6.31}$$

We use the matrices $S^{(k)}_{n_1,\ldots,n_l}$ to define a family of Markov operators that will approximate the Markov operator $\mathcal{P}$. Let $\nu \in \mathcal{M}(X)$ and define the operator $\mathcal{S}_{(k)} : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$\mathcal{S}_{(k)}\nu = \sum_{j=1}^{k} \left(\sum_{n_1,\ldots,n_l=1}^{k} p^{(k)}_{n_1} \ldots p^{(k)}_{n_l} \sum_{i=1}^{k} \nu(X_i) S^{(k)}_{n_1,\ldots,n_l,\, ij} \right) \delta_{x_j}\,, \tag{6.32}$$

where δ_{x_j} is the Dirac measure centered at the point x_j. Recall that Lemma 3.11 in Froyland [56] provides an estimate between $\mathcal{S}_{(k)}$ and $\mathcal{P}_{(k)}$ in the case that $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ is contractive on average. We now go one step further to extend this result to the case that $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$ is l-contractive on average.

Lemma 6.3.3. *Define* $\varepsilon_k = \max_{1 \le n \le k} \mathrm{diam}(X_n)$. *Then*

$$\sup_{\nu \in \mathcal{M}(X)} d_H(\mathcal{S}_{(k)}\nu, \mathcal{P}^l_{(k)}\nu) \le \big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big)\varepsilon_k.$$

Proof. Choose and fix $h \in \mathrm{Lip}_1(X)$, $\nu \in \mathcal{M}(X)$. By the definition of the Hutchinson metric as in (6.1), we therefore need to show that

$$\left| \int_X h(x)\; d\mathcal{P}^l_{(k)}\nu(x) - \int_X h(x)\; d\mathcal{S}_{(k)}\nu(x) \right| \le \big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big)\varepsilon_k. \tag{6.33}$$

By (6.32) and (6.4), we obtain

$$
\begin{aligned}
& \left|\int_X h(x)\, d\mathcal{P}^l_{(k)}\nu(x) - \int_X h(x)\, d\mathcal{S}_{(k)}\nu(x)\right| \\
= & \left| \sum_{n_1,\dots,n_l=1}^{k} p^{(k)}_{n_1}\dots p^{(k)}_{n_l} \int_X h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x)\, d\nu(x) - \right. \\
& \left. - \sum_{j=1}^{k}\left(\sum_{n_1,\dots,n_l=1}^{k} p^{(k)}_{n_1}\dots p^{(k)}_{n_l}\sum_{i=1}^{k}\nu(X_i)S^{(k)}_{n_1,\dots,n_l,ij}\right) h(x_j)\right| \\
\leq & \sum_{n_1,\dots,n_l=1}^{k} p^{(k)}_{n_1}\dots p^{(k)}_{n_l}(S_1+S_2), \qquad (6.34)
\end{aligned}
$$

where for each $n_1,\dots,n_l \in \{1,\dots,k\}$

$$S_1 := \left|\int_X h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x)\, d\nu(x) - \sum_{i=1}^{k}\nu(X_i) h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x_i)\right|, \quad (6.35)$$

$$S_2 := \left|\sum_{i=1}^{k}\nu(X_i) h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x_i) - \sum_{i,j=1}^{k}\nu(X_i)S^{(k)}_{n_1,\dots,n_l,ij}h(x_j)\right|. \quad (6.36)$$

By (6.35), we obtain

$$
\begin{aligned}
S_1 & \leq \sum_{i=1}^{k}\nu(X_i)\left(\sup_{x\in X_i} h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x) - \inf_{x\in X_i} h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x)\right) \\
& \leq \varepsilon_k \operatorname{Lip}(f_{n_1}\circ\cdots\circ f_{n_l}),
\end{aligned}
$$

which gives that

$$\sum_{n_1,\dots,n_l=1}^{k} p^{(k)}_{n_1}\dots p^{(k)}_{n_l} S_1 \leq \varepsilon_k \mathrm{Lip}_l(\mathbf{f}^{(k)}). \quad (6.37)$$

By (6.36), we have

$$S_2 \leq \sum_{i=1}^{k}\nu(X_i)\left|h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x_i) - \sum_{j=1}^{k} S^{(k)}_{n_1,\dots,n_l,ij}h(x_j)\right|,$$

which implies that

$$S_2 \leq \sum_{i=1}^{k}\nu(X_i)\left|h\circ f_{n_1}\circ\cdots\circ f_{n_l}(x_i) - h(x_{j(i)})\right|,$$

where $j(i)$ denotes the unique number in $\{1,\dots,k\}$ such that

$$f_{n_1,\dots,n_l}(x_i) \in X_{j(i)} \qquad \text{for all } i\in\{1,\dots,k\}.$$

Hence,

$$S_2 \leq \sum_{i=1}^{k} \nu(X_i)\varepsilon_k = \varepsilon_k,$$

which together with (6.37) and (6.34) gives

$$\left| \int_X h(x)\, d\mathcal{P}^l_{(k)}\nu(x) - \int_X h(x)\, d\mathcal{S}_{(k)}\nu(x) \right| \leq \varepsilon_k\big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big),$$

proving (6.33). This completes the proof. □

The following proposition provides an estimate of the distance between the operators $\mathcal{S}_{(k)}$ and $\mathcal{P}^l$.

Proposition 6.3.4. *Define* $\varepsilon_k = \max_{1\leq n\leq k} \mathrm{diam}(X_n)$. *Then*

$$\sup_{\nu\in\mathcal{M}(X)} d_H(\mathcal{S}_{(k)}\nu, \mathcal{P}^l\nu) \leq 2\,\mathrm{diam}(X)\Big[1 - (p_1 + \cdots + p_k)^l\Big] + \varepsilon_k\big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big).$$

Proof. Let $\mathcal{P}_{(k)}$ denote the Markov operator associated with the system $(\mathbf{f}^{(k)}, \mathbf{p}^{(k)})$. Choose and fix $\nu \in \mathcal{M}(X)$. In view of Lemma 6.3.3, we have

$$d_H(\mathcal{P}^l_{(k)}\nu, \mathcal{S}_{(k)}\nu) \leq \varepsilon_k\big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big).$$

This implies with Lemma 6.3.1 that

$$\begin{aligned} d_H(\mathcal{S}_{(k)}\nu, \mathcal{P}^l\nu) &\leq d_H(\mathcal{P}^l_{(k)}\nu, \mathcal{S}_{(k)}\nu) + d_H(\mathcal{P}^l_{(k)}\nu, \mathcal{P}^l\nu) \\ &\leq \varepsilon_k\big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big) + 2\,\mathrm{diam}(X)\Big[1 - (p_1 + \cdots + p_k)^l\Big]. \end{aligned}$$

This completes the proof. □

For the operator $\mathcal{S}_{(k)}$, we can easily construct its fixed point. The following lemma provides a simple way to construct a fixed point of the operator $\mathcal{S}_{(k)}$.

Lemma 6.3.5. *Define the* $k \times k$ *matrix* $S^{(k)}$ *as*

$$S^{(k)} := \sum_{n_1,\dots,n_l=1}^{k} p^{(k)}_{n_1} \dots p^{(k)}_{n_l} S^{(k)}_{n_1,\dots,n_l}, \tag{6.38}$$

and denote by $s^{(k)} = (s^{(k)}_1, \dots, s^{(k)}_k)$ *an arbitrary fixed left eigenvector of* $S^{(k)}$, *i.e.* $s^{(k)}S^{(k)} = s^{(k)}$. *Then the measure*

$$\nu^{(k)} := \sum_{n=1}^{k} s^{(k)}_n \delta_{x_n} \tag{6.39}$$

is a fixed point of the operator $\mathcal{S}_{(k)}$.

Proof. By (6.32), we obtain

$$\mathcal{S}_{(k)}\delta_{x_n} = \sum_{j=1}^{k}\left(\sum_{n_1,\ldots,n_l=1}^{k} p_{n_1}^{(k)}\ldots p_{n_l}^{(k)} S_{n_1,\ldots,n_l,nj}^{(k)}\right)\delta_{x_j},$$

which implies that

$$\begin{aligned}
\mathcal{S}_{(k)}\nu^{(k)} &= \sum_{n=1}^{k} s_n^{(k)}\cdot\mathcal{S}_{(k)}\delta_{x_n}\\
&= \sum_{n=1}^{k} s_n^{(k)}\sum_{j=1}^{k}\left(\sum_{n_1,\ldots,n_l=1}^{k} p_{n_1}^{(k)}\ldots p_{n_l}^{(k)} S_{n_1,\ldots,n_l,nj}^{(k)}\right)\delta_{x_j}\\
&= \sum_{j=1}^{k}\left(\sum_{n=1}^{k} s_n^{(k)}\sum_{n_1,\ldots,n_l=1}^{k} p_{n_1}^{(k)}\ldots p_{n_l}^{(k)} S_{n_1,\ldots,n_l,nj}^{(k)}\right)\delta_{x_j}\\
&= \sum_{j=1}^{k} s_j^{(k)}\delta_{x_j},
\end{aligned}$$

where we use the assumption that $(s_1^{(k)},\ldots,s_k^{(k)})$ is a fixed left eigenvector of $S^{(k)}$ to get the last equality. This completes the proof. □

The following theorem is our main result on the approximation of the time average λ of the IIFS through a computable quantity, together with an explicit error estimate.

Theorem 6.3.6 (Computational Ergodic Theorem for IIFS)**.** *Let* $(\mathbf{f},\mathbf{p})$ *be an* l*-contractive on average IIFS on a compact subset* $X\subset\mathbb{R}^d$ *and* $G: X\to\mathbb{R}$ *a Lipschitz function. Then* $\sum_{n=1}^{k} s_n^{(k)}G(x_n)$ *converges to the time average* (6.25) *of* $(\mathbf{f},\mathbf{p})$ *w.r.t.* G *for* $k\to\infty$*, more precisely, the following estimate holds*

$$\begin{aligned}
\left|\sum_{n=1}^{k} s_n^{(k)}G(x_n)-\lambda\right| \;\le\; & \frac{\mathrm{Lip}(G)}{1-\mathrm{Lip}_l(\mathbf{f})}\Big[2\,\mathrm{diam}(X)(1-(p_1+\cdots+p_k)^l)+\\
& +\big(1+\mathrm{Lip}_l(\mathbf{f}^{(k)})\big)\varepsilon_k\Big]. \qquad (6.40)
\end{aligned}$$

Proof. Since μ and $\nu^{(k)}$ are the fixed points of $\mathcal{P}$ and $\mathcal{S}_{(k)}$, respectively, it follows with Lemma 6.1.13 that

$$\begin{aligned}
d_H(\mu,\nu^{(k)}) &= d_H(\mathcal{P}^l\mu,\mathcal{S}_{(k)}\nu^{(k)})\\
&\le d_H(\mathcal{P}^l\mu,\mathcal{P}^l\nu^{(k)})+d_H(\mathcal{P}^l\nu^{(k)},\mathcal{S}_{(k)}\nu^{(k)})\\
&\le \mathrm{Lip}_l(\mathbf{f})d_H(\mu,\nu^{(k)})+d_H(\mathcal{P}^l\nu^{(k)},\mathcal{S}_{(k)}\nu^{(k)}).
\end{aligned}$$

This implies with Proposition 6.3.4 that

$$d_H(\mu, \nu^{(k)}) \leq \frac{1}{1 - \mathrm{Lip}_l(\mathbf{f})} \Big[2 \operatorname{diam}(X)(1 - (p_1 + \cdots + p_k)^l) + \big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big)\varepsilon_k \Big],$$

which gives

$$\left| \int_X G \, d\mu - \int_X G \, d\nu^{(k)} \right| \leq \frac{\mathrm{Lip}(G)}{1 - \mathrm{Lip}_l(\mathbf{f})} \Big[2 \operatorname{diam}(X)(1 - (p_1 + \cdots + p_k)^l) + \\ + \big(1 + \mathrm{Lip}_l(\mathbf{f}^{(k)})\big)\varepsilon_k \Big].$$

On the other hand, Lemma 6.3.5 implies $\int_X G(z) \; d\nu^{(k)}(z) = \sum_{n=1}^k s_n^{(k)} G(x_n)$ and by Theorem 6.1.16 we have $\int_X G(z) \; d\mu(z) = \lambda$. Hence,

$$\left| \int_X G(z) \; d\nu^{(k)}(z) - \int_X G(z) \; d\mu(z) \right| = \left| \sum_{n=1}^k s_n^{(k)} G(x_n) - \lambda \right|,$$

which completes the proof. □

6.4 Products of Random Matrices

Now we have a rigorous method to compute the invariant measure for place-dependent IFS and IIFS. In this section, we will apply this method to compute Lyapunov exponents of products of random matrices satisfying some additional assumptions. For clarity we deal with the top Lyapunov exponent, but using the Furtensberg-Kersten Theorem (see Theorem 1.3.3) and exterior power (see Section 1.3.2), one can also compute other Lyapunov exponents. For simplicity, throughout this section we always endow $\mathbb{R}^d$ with the standard Euclidean norm.

6.4.1 Products of Random Matrices

Assume that we have a collection of nonsingular matrices $\mathbf{A} = \{A_n\}_{n=1}^\infty$ associated with a collection of probabilities $\mathbf{p} = \{p_n\}_{n=1}^\infty$, i.e., $\sum_{n=1}^\infty p_n = 1$ and $p_n > 0$ for $n = 1, 2, \ldots$.

Definition 6.4.1. A pair $(\mathbf{A}, \mathbf{p})$ defines an RDS on the ergodic MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ defined as in Subsection 6.1.3 in the following way. Define a random map $A : \Omega \to Gl(d)$ by $A(\omega) = A_{\omega_0}$, then the random map A generates the following linear cocycle over the dynamical system θ via

$$\Phi_A(n, \omega) := \begin{cases} A(\theta^{n-1}\omega) \circ \cdots \circ A(\omega), & n > 0, \\ I_d, & n = 0. \end{cases}$$

We are interested in the limits

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_A(n,\omega)v\|, \qquad \text{where } 0 \neq v \in \mathbb{R}^d. \tag{6.41}$$

The following result of Oseledets for one-sided RDS (see, e.g. [3, Theorem 3.4.1]) guarantees that these limits exist and are independent of the random sequence $\omega \in \Omega$.

Theorem 6.4.1 ([3]). *Suppose that the pair* $(\mathbf{A}, \mathbf{p})$ *satisfies the integrability condition, i.e.,*

$$\sum_{n=1}^{\infty} p_n \log^+ \|A_n\|, \quad \sum_{n=1}^{\infty} p_n \log^+ \|A_n^{-1}\| < \infty. \tag{6.42}$$

Then for $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *the following statements hold:*

(i) Limit (6.41) *exists and takes on finitely many (non-random) values*

$$-\infty < \lambda_p < \cdots < \lambda_1 < \infty,$$

as v *is varied over* $\mathbb{R}^d \setminus \{0\}$.

(ii) There exists a pointwise filtration of random subspaces

$$\{0\} \subset V_p(\omega) \subset \cdots \subset V_i(\omega) \subset \cdots \subset V_1(\omega) = \mathbb{R}^d,$$

such that the limit in (6.41) *equals* λ_i *if* $v \in V_i(\omega) \setminus V_{i+1}(\omega)$.

(iii) The subspaces $V_i(\omega)$ *satisfy*

$$A(\omega)V_i(\omega) = V_i(\theta\,\omega).$$

(iv) Denote by d_i*, the non-random value* $\dim V_i(\omega) - \dim V_{i+1}(\omega)$*, the multiplicity of the exponent* λ_i*. If* $\Phi_A^{\wedge k}(n,\omega)$ *denotes the* k^{th} *exterior power of* $\Phi_A(n,\omega)$ *for* $1 \leq k \leq d$ *then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_A^{\wedge k}(n,\omega)\|$$

exists and equals the sum of the k *largest values* λ_i*, counted with multiplicity.*

We establish direct estimates for the top Lyapunov exponent λ_1, and recover the remaining exponents by working with exterior powers. By using the auxiliary measures in the projective space and the study of product of induced matrices on the projective space, Furstenberg and Kifer [62] obtained the following result.

Lemma 6.4.2 ([62]). *Suppose that the only subspace of* $\mathbb{R}^d$ *which is invariant under all of the* A_n *is the trivial subspace, i.e., if* V *is a subspace of* $\mathbb{R}^d$ *satisfying that* $A_nV = V$ *for all* $n = 1, 2, \ldots,$ *then* $V = 0$*. Then for every* $v \in \mathbb{R}^d \setminus \{0\}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi_A(n,\omega)v\| = \lambda_1 \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Remark 6.4.3. In Froyland [56] it is shown that it is very rare for all the A_n to share a common nontrivial invariant subspace. Hence the assumptions in Lemma 6.4.2 hold for almost all pairs $(\mathbf{A}, \mathbf{p})$.

Instead of working with the original dynamical system $(\mathbf{A}, \mathbf{p})$ we will work with an induced dynamical system on S^{d-1}, the unit sphere of $(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$. The reason is that S^{d-1} together with the natural Riemannian metric from $\mathbb{R}^d$ is a compact metric space and we can apply the method which was developed in the previous sections to estimate the invariant measure. Now we explain how to compute the top Lyapunov exponent by using the induced RDS on S^{d-1}.

Denote by $\widetilde{A}_n$, for $n = 1, 2, \dots$, the nonlinear operator on S^{d-1} which is induced by A_n, i.e.,

$$\widetilde{A}_n(v) := \frac{A_n v}{\|A_n v\|} \qquad \text{for all unit vectors } v \in S^{d-1}. \tag{6.43}$$

Let $\mathrm{Lip}(\widetilde{A}_n)$ denote the Lipschitz constant of the function $\mathrm{Lip}(\widetilde{A}_n)$ for each $n = 1, 2, \dots$. We require that the Lipschitz constant on average satisfies

$$\mathrm{Lip}(\mathbf{A}) := \sum_{n=1}^{\infty} p_n \, \mathrm{Lip}(\widetilde{A}_n) < \infty. \tag{6.44}$$

If the pair $(\mathbf{A}, \mathbf{p})$ satisfies (6.44) then we say that this system is *Lipschitz on average.* We define the skew product $\widetilde{\tau} : \Omega \times S^{d-1} \to \Omega \times S^{d-1}$ by

$$\widetilde{\tau}(\omega, v) = (\theta\omega, \widetilde{A}_{\omega_0} v) \qquad \text{for all } (\omega, v) \in \Omega \times S^{d-1}, \tag{6.45}$$

and the scalar measurable function $G : \Omega \times S^{d-1} \to \mathbb{R}$ by

$$G(\omega, v) = \log \|A_{\omega_0} v\| \qquad \text{for all } (\omega, v) \in \Omega \times S^{d-1}. \tag{6.46}$$

An elementary computation yields that for all $(\omega, v) \in \Omega \times S^{d-1}$

$$\begin{aligned} \frac{1}{n} \log \|\Phi_A(n, \omega) v\| &= \frac{1}{n} \log \|A_{\omega_{n-1}} \dots A_{\omega_0} v\| \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \log \frac{\|A_{\omega_k} \dots A_{\omega_0} v\|}{\|A_{\omega_{k-1}} \dots A_{\omega_0} v\|} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} G(\widetilde{\tau}^k(\omega, v)). \end{aligned}$$

This implies with Lemma 6.4.2 that for every $v \in S^{d-1}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} G(\widetilde{\tau}^k(\omega, v)) = \lambda_1 \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{6.47}$$

Definition 6.4.2. Let $\mathcal{M}(S^{d-1})$ be the space of all probability measures on S^{d-1}. A probability measure μ on $\mathcal{M}(S^{d-1})$ is said to be an invariant measure of the pair $(\mathbf{A},\mathbf{p})$ if μ is a fixed point of $\mathcal{P}$, where $\mathcal{P}:\mathcal{M}(S^{d-1})\to\mathcal{M}(S^{d-1})$ is defined by

$$\mathcal{P}\nu=\sum_{n=1}^{\infty}p_n\nu\circ\widetilde{A}_n^{-1}\qquad\text{for all }\nu\in\mathcal{M}(S^{d-1}).\tag{6.48}$$

Some properties of the operator $\mathcal{P}$ as well as the invariant measures of the pair $(\mathbf{A},\mathbf{p})$ are collected in the following remark. Its proof is analog to the proof in Lemma 6.1.13.

Remark 6.4.4. (i) A probability measure μ is invariant if and only if the product measure $\mathbb{P}\times\mu$ on $\Omega\times S^{d-1}$ is invariant with respect to the skew product $\widetilde{\tau}$.
(ii) Let d_H be the Hutchinson metric on $\mathcal{M}(S^{d-1})$. Then we have

$$d_H(\mathcal{P}\nu_1,\mathcal{P}\nu_2)\le\operatorname{Lip}(\mathbf{A})d_H(\nu_1,\nu_2)\qquad\text{for all }\nu_1,\nu_2\in\mathcal{M}(S^{d-1}).$$

Now we prove that the top Lyapunov exponent can be represented by an arbitrary invariant probability measure on $\mathcal{M}(S^{d-1})$.

Lemma 6.4.5. *Let μ be an invariant probability measure in $\mathcal{M}(S^{d-1})$. Suppose that the pair $(\mathbf{A},\mathbf{p})$ satisfies the assumptions of Theorem 6.4.1 and Lemma 6.4.2. Then we have*

$$\lambda_1=\lambda(\mu):=\sum_{n=1}^{\infty}p_n\int_{S^{d-1}}\log\|A_nv\|\,d\mu(v).$$

Proof. Let $G:\Omega\times S^{d-1}\to\mathbb{R}$ be the function which is defined as in (6.46). The integrability condition of Theorem 6.4.1 ensures that

$$\begin{aligned}\int_{\Omega\times S^{d-1}}G^+(\omega,v)\,d(\mathbb{P}\times\mu)(\omega,v)&=\sum_{n=1}^{\infty}p_n\int_{S^{d-1}}\log^+\|A_nv\|\,d\mu(v)\\&\le\sum_{n=1}^{\infty}p_n\log^+\|A_n\|\\&<\infty\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega\times S^{d-1}}G^-(\omega,v)\,d(\mathbb{P}\times\mu)(\omega,v)&=\sum_{n=1}^{\infty}p_n\int_{S^{d-1}}\log^-\|A_nv\|\,d\mu(v)\\&\le\sum_{n=1}^{\infty}p_n\log^+\|A_n^{-1}\|\\&<\infty.\end{aligned}$$

Consequently, the function $G\in\mathcal{L}^1(\Omega\times S^{d-1},\mathcal{F}\otimes\mathcal{B}(S^{d-1}),\mathbb{P}\times\mu)$ and from (6.46) we get

$$\int_{\Omega\times S^{d-1}}G(\omega,v)\,d(\mathbb{P}\times\mu)(\omega,v)=\sum_{n=1}^{\infty}p_n\int_{S^{d-1}}\log\|A_nv\|\,d\mu(v).\tag{6.49}$$

By Remark 6.4.4 and since μ is an invariant measure, the probability measure $\mathbb{P} \times \mu$ is invariant under the skew product $\widetilde{\tau}$ which is defined as in (6.45). Using the Birkhoff ergodic theorem (see Appendix A), there exists an $\mathcal{L}^1$ function $G^* : \Omega \times S^{d-1} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} G(\widetilde{\tau}^k(\omega, v)) = G^*(\omega, v) \qquad \mathbb{P} \times \mu\text{-a.e. } (\omega, v) \in \Omega \times S^{d-1}$$

and

$$\int_{\Omega\times S^{d-1}} G(\omega, v)\, d(\mathbb{P} \times \mu)(\omega, v) = \int_{\Omega\times S^{d-1}} G^*(\omega, v)\, d(\mathbb{P} \times \mu)(\omega, v). \tag{6.50}$$

Using (6.47) and the Fubini theorem, we get

$$\int_{\Omega\times S^1} G^*(\omega, v)\ d(\mathbb{P} \times \mu)(\omega, v) = \int_{S^{d-1}} \left(\int_{\Omega} G^*(\omega, v)\ d\mathbb{P}(\omega) \right)\ d\mu(v) = \lambda_1,$$

which together with (6.49) and (6.50) implies that

$$\lambda_1 = \sum_{n=1}^{\infty} p_n \int_{S^{d-1}} \log \|A_n v\|\, d\mu(v),$$

which completes the proof.

□

6.4.2 An Approximation and Convergence Result

In this subsection, we develop a method to approximate the top Lyapunov exponent of the pair $(\mathbf{A}, \mathbf{p})$.

Lemma 6.4.6. *For each $k = 1, 2, \ldots,$ we define the operator $\mathcal{P}_{(k)} : \mathcal{M}(S^{d-1}) \to \mathcal{M}(S^{d-1})$ by*

$$\mathcal{P}_{(k)}\nu = \sum_{n=1}^{k} p_n^{(k)} \nu \circ \widetilde{A}_n^{-1} \qquad \textit{for all } \nu \in \mathcal{P}(S^{d-1}),$$

where $p_n^{(k)} = \frac{p_n}{p_1+p_2+\cdots+p_k}$. Then we have

$$\sup_{\nu\in\mathcal{M}(S^{d-1})} d_H(\mathcal{P}_{(k)}\nu, \mathcal{P}\nu) \le 4\Big[1 - (p_1 + \cdots + p_k)\Big].$$

Proof. As in the proof of Lemma 6.3.1. □

Proposition 6.4.7. *Partition S^{d-1} into k connected measurable subsets $\{B_1, B_2, \ldots, B_k\}$. Choose and fix a single point $b_i \in B_i, i = 1, 2, \ldots, k$, and construct the $k \times k$ transition matrices $S_{n,ij}^{(k)}$ by*

$$S_{n,ij}^{(k)} = \begin{cases} 1, & \textit{if } \widetilde{A}_n(b_i) \in B_j, \\ 0, & \textit{otherwise.} \end{cases}$$

Define an operator $\mathcal{S}_{(k)} : \mathcal{M}(S^{d-1}) \to \mathcal{M}(S^{d-1})$ *by*

$$\mathcal{S}_{(k)}\nu = \sum_{j=1}^{k} \left(\sum_{n=1}^{k} p_n^{(k)} \sum_{i=1}^{k} \nu(B_i) S_{n,ij}^{(k)} \right) \delta_{b_j} \qquad \text{for all } \nu \in \mathcal{M}(S^{(d-1)}).$$

Then the following estimate holds

$$\sup_{\nu \in \mathcal{M}(S^{d-1})} d_H(\mathcal{S}_{(k)}\nu, \mathcal{P}\nu) \le 4\Big[1 - (p_1 + \cdots + p_k)\Big] + \varepsilon_k\big(1 + \mathrm{Lip}(\mathbf{A}^{(k)})\big),$$

where

$$\varepsilon_k := \max_{1 \le i \le k} \mathrm{diam}(B_i), \quad \mathrm{Lip}(\mathbf{A}^{(k)}) := \sum_{n=1}^{k} p_n^{(k)} \mathrm{Lip}(\widetilde{A}_n).$$

Proof. Analog to the proof of Proposition 6.3.4. □

We now provide a natural way to construct an invariant measure of the operator $\mathcal{S}_{(k)}$.

Lemma 6.4.8. *Define the* $k \times k$ *matrix* $S^{(k)}$ *as*

$$S^{(k)} := \sum_{n=1}^{k} p_n^{(k)} S_n^{(k)}, \tag{6.51}$$

where $S_1^{(k)}, \ldots, S_k^{(k)}$ *are the matrices defined as in Proposition 6.4.7. Denote by* $s^{(k)} = (s_1^{(k)}, \ldots, s_k^{(k)})$ *a fixed left eigenvector of* $S^{(k)}$*. Then the measure*

$$\nu^{(k)} := \sum_{n=1}^{k} s_n^{(k)} \delta_{b_n} \tag{6.52}$$

is a fixed point of the Markov operator $\mathcal{S}_{(k)}$*.*

Proof. Similar to the proof of Lemma 6.3.5. □

Based on the above results, the following theorem provides an algorithm to compute the Lyapunov exponents of products of random matrices.

Theorem 6.4.9. *Suppose that the system* $(\mathbf{A}, \mathbf{p})$ *satisfies the assumptions of Theorem 6.4.1, Lemma 6.4.5 and is Lipschitz on average. Then*

$$\sum_{n=1}^{k} p_n^{(k)} \sum_{i=1}^{k} s_i^{(k)} \log \|A_n(b_i)\| := \lambda^{(k)} \to \lambda_1 \quad \text{as } k \to \infty, \tag{6.53}$$

provided that $\lim_{k\to\infty} \varepsilon_k = 0$*.*

Proof. We first remark that the integrability condition, see (6.42), of Theorem 6.4.1 implies that

$$\sum_{n=1}^{\infty} p_n \left| \int_{S^{d-1}} \log \|A_n v\| \, d\zeta \right| \leq \sum_{n=1}^{\infty} p_n (\log^+ \|A_n\| + \log^+ \|A_n^{-1}\|) < \infty \tag{6.54}$$

for all $\zeta \in \mathcal{M}(S^{d-1})$. As a consequence, we can define the function $\lambda : \mathcal{M}(S^{d-1}) \to \mathbb{R}$ by

$$\lambda(\zeta) = \sum_{n=1}^{\infty} p_n \int_{S^{d-1}} \log \|A_n v\| \, d\zeta \qquad \text{for all } \zeta \in \mathcal{M}(S^{d-1}). \tag{6.55}$$

We show that λ is a continuous function. For an arbitrary but fixed $\varepsilon > 0$ by using (6.54), there exists $N > 0$ such that

$$\sum_{n=N+1}^{\infty} p_n \big(\log^+ \|A_n\| + \log^+ \|A_n^{-1}\| \big) \leq \frac{\varepsilon}{4}. \tag{6.56}$$

Now for each $\zeta \in \mathcal{M}(S^{d-1})$, in view of Remark 6.0.3 (iv) there exists $\delta > 0$ such that

$$\sum_{n=1}^{N} p_n \left| \int_{S^{d-1}} \log \|A_n v\| \, d\zeta - \int_{S^{d-1}} \log \|A_n v\| \, d\nu \right| \leq \frac{\varepsilon}{2} \qquad \text{for all } \nu \in B_\delta(\zeta),$$

which together with (6.56) implies that the statements

$$\begin{aligned} \sum_{n=1}^{\infty} p_n \left| \int_{S^{d-1}} \log \|A_n v\| \, d\zeta - \int_{S^{d-1}} \log \|A_n v\| \, d\nu \right| &\leq \frac{\varepsilon}{2} + 2 \sum_{n=N+1}^{\infty} p_n \log^+ \|A_n\| + \\ &\quad +2 \sum_{n=N+1}^{\infty} p_n \log^+ \|A_n^{-1}\| \\ &\leq \varepsilon \end{aligned}$$

hold for all $\nu \in B_\delta(\zeta)$. As a consequence, λ is a continuous function. By (6.55) and (6.52), we have

$$\begin{aligned} \lambda(\nu^{(k)}) &= \sum_{n=1}^{\infty} p_n \int_{S^{d-1}} \log \|A_n v\| \, d\nu^{(k)} \\ &= \sum_{n=1}^{\infty} p_n \sum_{i=1}^{k} s_i^{(k)} \log \|A_n b_i\|, \end{aligned}$$

which implies that

$$\lim_{k \to \infty} \left| \lambda(\nu^{(k)}) - \lambda^{(k)} \right| = 0.$$

Hence, it remains to show that $\lim_{k\to\infty} \lambda\big(\nu^{(k)}\big) = \lambda_1$. We prove this statement by contradiction, i.e. there exist $\varepsilon > 0$ and a subsequence $\nu^{(n_k)}$ such that

$$\left| \lambda(\nu^{(n_k)}) - \lambda_1 \right| \geq \varepsilon \qquad \text{for all } k \in \mathbb{N}. \tag{6.57}$$

Taking a subsequence of $\{\nu^{(n_k)}\}_{k\in\mathbb{N}}$ if necessary, one can find a probability measure $\xi \in \mathcal{M}(S^{d-1})$ such that

$$\lim_{k\to\infty} d_H\big(\nu^{(n_k)}, \xi\big) = 0,$$

which together with the continuity of λ gives that

$$\lim_{k\to\infty} \lambda\big(\nu^{(n_k)}\big) = \lambda(\xi). \tag{6.58}$$

On the other hand, using Proposition 6.4.7, we obtain

$$\lim_{k\to\infty} \sup_{\nu\in\mathcal{M}(S^{d-1})} d_H(\mathcal{S}_{(k)}\nu, \mathcal{P}\nu) = 0,$$

which together with Remark 6.4.4 (ii) and Lemma 6.4.8 implies that

$$\begin{aligned} d_H(\mathcal{P}\xi, \xi) &\le d_H\big(\mathcal{P}\xi, \mathcal{P}\nu^{(n_k)}\big) + d_H\big(\mathcal{P}\nu^{(n_k)}, \mathcal{S}_{(n_k)}\nu^{(n_k)}\big) + d_H\big(\nu^{(n_k)}, \xi\big) \\ &\le (1+\mathrm{Lip}(\mathbf{A}))d_H(\xi, \nu^{(n_k)}) + d_H\big(\mathcal{P}\nu^{(n_k)}, \mathcal{S}_{(n_k)}\nu^{(n_k)}\big). \end{aligned}$$

Letting $k\to\infty$, we have $\mathcal{P}\xi = \xi$. Hence, according to Lemma 6.4.5 and (6.58), we get

$$\lim_{k\to\infty} \lambda\big(\nu^{(n_k)}\big) = \lambda(\xi) = \lambda_1,$$

which contradicts to inequality (6.57) and the proof is completed. □

6.5 Examples

Example 6.5.1 (Contractive on Average Place-dependent IFS). We consider a place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ on $X = [0,1]$ defined as follows:

$$f_1(x) = x, f_2(x) = \frac{1}{2}x + \frac{1}{2}, \quad p_1(x) = \frac{3-2x}{4}, p_2(x) = \frac{1+2x}{4}.$$

The following inequality

$$p_1(x)|f_1(x) - f_1(y)| + p_2(x)|f_2(x) - f_2(y)| \le \frac{7}{8}|x-y| \qquad \text{for all } x, y \in X$$

implies that the system $(\mathbf{f}, \mathbf{p}(\cdot))$ is contractive on average. Since $f_1(1) = f_2(1) = 1$ and by virtue of Theorem 6.1.9, δ_1 is the unique invariant measure of $(\mathbf{f}, \mathbf{p}(\cdot))$. Let G be a continuous function from X to $\mathbb{R}$. Denote by λ the time average of the place-dependent IFS $(\mathbf{f}, \mathbf{p}(\cdot))$ with respect to the function G. Then

$$\lambda = \int_X G(z)\, d\delta_1(z) = G(1). \tag{6.59}$$

To apply the algorithm described in Section 6.2 we fix $K \in \mathbb{N}$ and partition X into $X_1, X_2, ..., X_K$, where $X_n = [\frac{n-1}{K}, \frac{n}{K})$ for all $1 \le n \le K-1$ and $X_K = [\frac{K-1}{K}, 1]$. For each interval X_n we define the middle point $x_n = \frac{2n-1}{2K} \in X_n$.

For an explicit computation we set $G \equiv \mathrm{id}$. From (6.59) we get $\lambda = 1$. For each $K \in \mathbb{N}$ we numerically compute the matrix $S^{(K)}$ as in (6.22) and a fixed left eigenvector $s^{(K)}$, see [56, Remark 3.10]. Our main result Theorem 6.2.3 implies that

$$\lambda_{\mathrm{approx}}(K) := \sum_{n=1}^{K} s_n^{(K)} G(x_n)$$

converges to $\lambda = 1$ as $K \to \infty$. Figure 6.1 shows how $\lambda_{\mathrm{approx}}(K)$ converges to $\lambda = 1$.

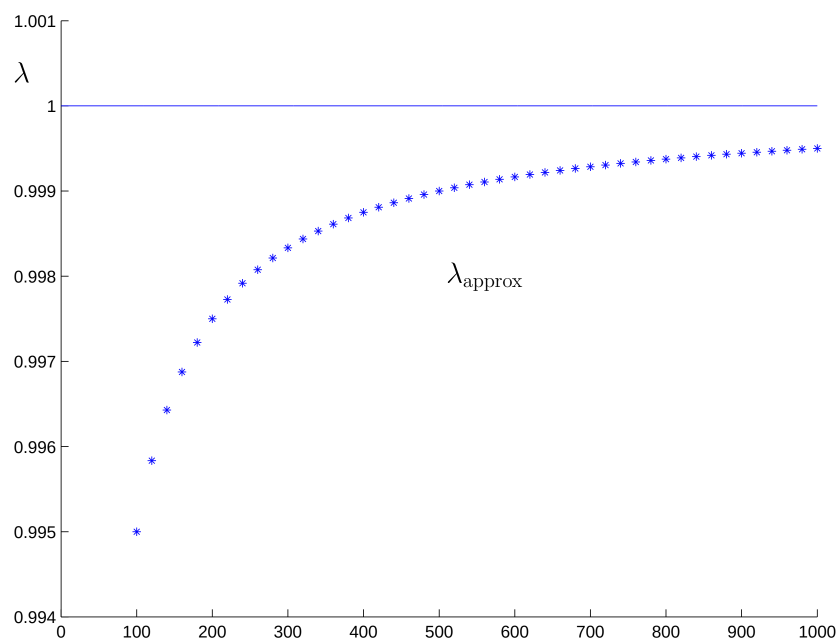

Figure 6.1: Numerical approximation $\lambda_{\mathrm{approx}}$ tends to λ for $G \equiv \mathrm{id}$.

Example 6.5.2 (Contractive on average IIFS). We consider a parameter-dependent family of IIFS

$$(\mathbf{f}_\alpha, \mathbf{p}) \qquad \text{on } X = [0,1] \quad \text{with } \alpha \in \mathbb{R}, \alpha \geq 1$$

where the family of Lipschitz maps $\mathbf{f}_\alpha = \{f_{\alpha,n}\}_{n=1}^{\infty}$ with associated probabilities $\mathbf{p} = \{p_n\}_{n=1}^{\infty}$ is defined as follows:

$$f_{\alpha,n}(x) = \frac{1}{n}x + \frac{n-1}{\alpha n} \qquad \text{and} \qquad p_n = \frac{1}{n} - \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}.$$

It is easy to see that $\mathrm{Lip}(f_{\alpha,n}) = \frac{1}{n}$ and consequently the average expansion rate of the pair $(\mathbf{f}_\alpha, \mathbf{p})$ is independent of α and satisfies

$$\sum_{n=1}^{\infty} p_n \mathrm{Lip}(f_{\alpha,n}) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \frac{1}{n} = \frac{\pi^2}{6} - 1 < 1.$$

Hence the IIFS $(\mathbf{f}_\alpha, \mathbf{p})$ is contractive on average. Since $\frac{1}{\alpha}$ is the fixed point of $f_{\alpha,n}$ for all $n \in \mathbb{N}$, it follows that $\delta_{\frac{1}{\alpha}}$ is the invariant measure of the IIFS $(\mathbf{f}_\alpha, \mathbf{p})$. Let G be a continuous function from X to $\mathbb{R}$. Denote by λ_α the time average of the IIFS $(\mathbf{f}_\alpha, \mathbf{p})$ with respect to the function G. Then

$$\lambda_\alpha = \int_X G(z)\, d\delta_{\frac{1}{\alpha}}(z) = G\Big(\frac{1}{\alpha}\Big). \tag{6.60}$$

To apply the algorithm described in Section 6.3 we fix $k \in \mathbb{N}$ and partition X into $X_1, X_2, ..., X_k$, where $X_n = [\frac{n-1}{k}, \frac{n}{k})$ for all $1 \le n \le k-1$ and $X_k = [\frac{k-1}{k}, 1]$. For each interval X_n we define the middle point $x_n = \frac{2n-1}{2k} \in X_n$.
For an explicit computation we set $G \equiv \mathrm{id}$ and $\alpha = 3$. From (6.60) we get $\lambda_3 = \frac{1}{3}$. For each $k \in \mathbb{N}$ we numerically compute the matrix $S^{(k)}$ as in (6.38) for $l = 1$ and a fixed left eigenvector $s^{(k)}$, see [56, Remark 3.10]. Our main result Theorem 6.3.6 implies that

$$\lambda_{\text{approx}}(k) := \sum_{n=1}^{k} s_n^{(k)} G(x_n)$$

converges to λ_3 as $k \to \infty$. Table 6.1 shows the error $\lambda_{\text{approx}} - \lambda_3$ of the numerical approximation and the upper bound $\mathrm{err}(k)$ of $\lambda_{\text{approx}} - \lambda_3$ provided by the right hand side of (6.40), see also Figure 6.2.

k	Numerical approximation error: $\lambda_{\text{approx}} - \lambda_3$	Upper bound as in (6.40): $\mathrm{err}(k)$
50	3.3×10^{-3}	2×10^{-1}
100	1.7×10^{-3}	1×10^{-1}
500	3.3×10^{-4}	2×10^{-2}
1000	1.7×10^{-4}	1×10^{-2}

Table 6.1: Error of numerical approximation and upper bound as in (6.40) for $G \equiv \mathrm{id}$ and $\alpha = 3$.

Example 6.5.3 (l-Contractive on average IIFS). We consider an IIFS $(\mathbf{f}, \mathbf{p})$ on $X = [-1, 1]$ defined as follows $f_n(x) = a_n x^2 + b_n$, where

$$a_n := \begin{cases} 1 & \text{if } n = 1; \\ \frac{1}{2(n-1)} & \text{if } n \ge 2, \end{cases} \qquad \text{and} \qquad b_n := \frac{1}{2} - a_n,$$

and $p_n := \frac{1}{n} - \frac{1}{n+1}$ for all $n \in \mathbb{N}$. A direct computation yields that

$$\mathrm{Lip}(f_n) = 2a_n \qquad \text{for all } n \in \mathbb{N}.$$

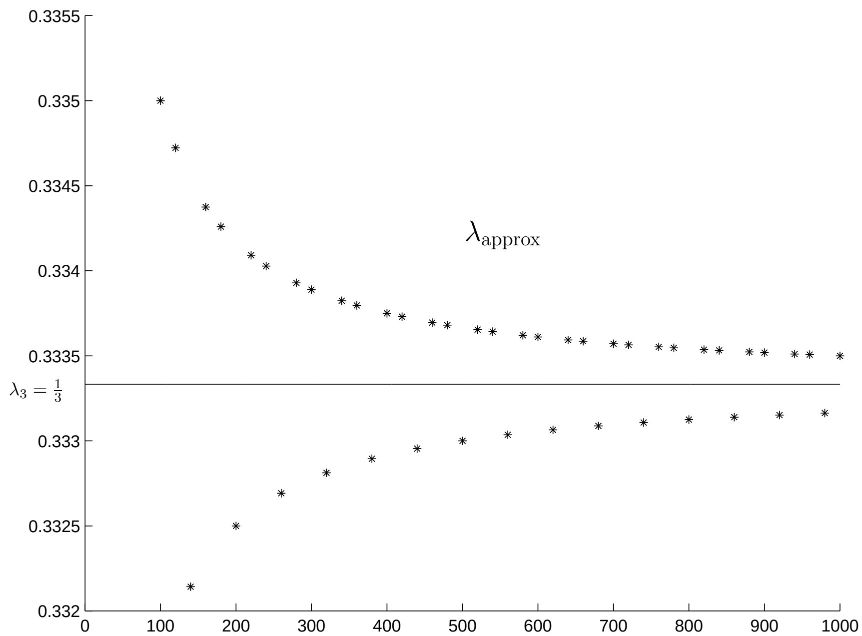

Figure 6.2: Numerical approximation λ_{approx} tends to $\lambda_3 = \frac{1}{3}$ for $G \equiv \text{id}$ and $\alpha = 3$.

Therefore,

$$\begin{aligned}\text{Lip}(\mathbf{f}) &= \sum_{n=1}^{\infty} 2p_n a_n \\ &= 1 + \sum_{n=2}^{\infty}(\frac{1}{n} - \frac{1}{n+1})\frac{1}{n-1} \\ &= \frac{5}{4} > 1,\end{aligned}$$

which implies that the system $(\mathbf{f}, \mathbf{p})$ is not contractive on average. On the other hand, we have

$$\begin{aligned}f_n \circ f_m(x) &= a_n(a_m x^2 + b_m)^2 + b_n \\ &= a_n a_m^2 x^4 + 2a_n a_m b_m x^2 + a_n b_m^2 + b_n,\end{aligned}$$

which gives that

$$\begin{aligned}\text{Lip}(f_n \circ f_m) &= \max_{x \in [-1,1]} |4a_n a_m^2 x^3 + 4a_n a_m b_m x| \\ &= 2a_n a_m \qquad \text{for all } n, m \in \mathbb{N}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{Lip}_2(\mathbf{f}) &= \sum_{n,m=1}^{\infty} p_n p_m \mathrm{Lip}(f_n \circ f_m) \\ &= \frac{1}{2}\sum_{n,m=1}^{\infty} 4 p_n p_m a_n a_m \\ &= \frac{25}{32} < 1,\end{aligned}$$

which ensures that the system $(\mathbf{f}, \mathbf{p})$ is 2-contractive on average. To apply the algorithm described in Section 6.3 we fix $k \in \mathbb{N}$ and partition X into $X_1, X_2, ..., X_k$, where for all $1 \le n \le k-1$, $X_n = \left[-1 + \frac{2(n-1)}{k}, -1 + \frac{2n}{k}\right)$ and $X_k = \left[1 - \frac{2}{k}, 1\right]$. For each interval X_n we define the middle point $x_n = -1 + \frac{2n-1}{k} \in X_n$. For an explicit computation we set $G \equiv \mathrm{id}$. For each $k \in \mathbb{N}$ we numerically compute the matrix $S^{(k)}$ as in (6.38) for $l = 2$ and a fixed left eigenvector $s^{(k)}$, see [56, Remark 3.10]. Our main result Theorem 6.3.6 implies that

$$\lambda_{\mathrm{approx}}(k) := \sum_{n=1}^{k} s_n^{(k)} G(x_n)$$

converges to λ as $k \to \infty$. Table 6.2 shows the approximation $\lambda_{\mathrm{approx}}(k)$ and the upper bound $\mathrm{err}(k)$ of $\lambda_{\mathrm{approx}}(k) - \lambda$ provided by the right hand side of (6.40).

k	Approximation $\lambda_{\mathrm{approx}}(k)$	Upper bound as in (6.40): $\mathrm{err}(k)$
50	-0.0562	1
100	-0.0511	5×10^{-1}
500	-0.0479	1×10^{-1}
1000	-0.0466	5×10^{-2}

Table 6.2: Approximated number and upper bound as in (6.40) for $G \equiv \mathrm{id}$.

Example 6.5.4 (Products of random matrices). Define the pair $(\mathbf{A}, \mathbf{P})$ by

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, p_1 = \frac{1}{2};\ A_n = \begin{pmatrix} 2n-2 & 0 \\ 0 & n-1 \end{pmatrix}, p_n = \frac{1}{2n-2} - \frac{1}{2n}, n = 2, 3, \ldots. \tag{6.61}$$

We first point out that the pair $(\mathbf{A}, \mathbf{P})$ fulfills all assumptions of Theorem 6.4.9. It is easy to see that the only subspace of $\mathbb{R}^2$ which is invariant under all of A_k is the trivial subspace. Since $\sum_{n=1}^{\infty} \frac{\log n}{n^2} < \infty$ it follows that

$$\sum_{n=1}^{\infty} p_n \left(\log^+ \|A_n\| + \log^+ \|A_n^{-1}\|\right) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{\log n + \log 2}{n(n+1)} < \infty,$$

which implies the integrability of $(\mathbf{A}, \mathbf{P})$. Concerning the Lipschitz constant on average of $(\mathbf{A}, \mathbf{P})$, we use the observation that $\mathrm{Lip}(\widetilde{A}_n) = \mathrm{Lip}(\widetilde{A}_2)$ for all $n = 2, 3, \ldots$, where $\widetilde{A}_n$

is the induced map of A_n on S^{d-1}, i.e.

$$\widetilde{A}_n(v) = \frac{A_n(v)}{\|A_n(v)\|} \qquad \text{for all } v \in S^{d-1}.$$

Therefore, by (6.44) we have

$$\mathrm{Lip}(\mathbf{A}) = \sum_{n=1}^{\infty} p_n \,\mathrm{Lip}(\widetilde{A}_n) = \frac{1}{2}\,\mathrm{Lip}(\widetilde{A}_1) + \frac{1}{2}\,\mathrm{Lip}(\widetilde{A}_2) < \infty.$$

Now we compute explicitly an invariant measure of $(\mathbf{A}, \mathbf{p})$. By (6.61), we have

$$\widetilde{A}_1 \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ \cos\varphi \end{pmatrix} \qquad \text{for all } \varphi \in [0, 2\pi]$$

and

$$\widetilde{A}_n \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \frac{1}{\sqrt{4\cos^2\varphi + \sin^2\varphi}} \begin{pmatrix} 2\cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \text{for all } n = 2, 3, \dots \text{ and } \varphi \in [0, 2\pi].$$

Hence, a direct computation yields that $\mu := \frac{1}{2}(\delta_{(1,0)^T} + \delta_{(0,1)^T}) \in \mathcal{M}(S^1)$ is an invariant probability measure under the operator $\mathcal{P} : \mathcal{M}(S^1) \to \mathcal{M}(S^1)$ which is defined as in (6.48), i.e.

$$\mathcal{P}\nu = \sum_{n=1}^{\infty} p_n \nu \circ \widetilde{A}_n^{-1} \qquad \text{for all } \nu \in \mathcal{M}(S^1).$$

Therefore, in view of Lemma 6.4.5 the top Lyapunov exponent λ_1 of $(\mathbf{A}, \mathbf{p})$ can be computed by

$$\begin{aligned}
\lambda_1 &= \sum_{n=1}^{\infty} p_n \int_{S^1} \log\|A_n(v)\|\, d\mu(v) \\
&= \frac{1}{2}\left(\sum_{n=1}^{\infty} p_n \int_{S^1} \log\|A_n(v)\|\, d\delta_{(0,1)^T}(v) + \sum_{n=1}^{\infty} p_n \int_{S^1} \log\|A_n(v)\|\, d\delta_{(1,0)^T}(v)\right) \\
&= \frac{1}{2}\left(\sum_{n=2}^{\infty} p_n \log(n-1)\right) + \frac{1}{2}\left(\sum_{n=2}^{\infty} p_n \log(2n-2)\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{\log n}{n(n+1)} + \frac{\log 2}{4}.
\end{aligned} \tag{6.62}$$

To apply the algorithm described in Subsection 6.4.2 we fix $k \in \mathbb{N}$ and partition S^1 into k disjoint sets $B_1, B_2, \dots, B_k$ by

$$B_n := \left\{ \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} : \varphi \in \left[\frac{2\pi(n-1)}{k}, \frac{2\pi n}{k}\right) \right\}, \qquad n = 1, 2, \dots, k.$$

For each set B_n we define the middle point $b_n = \begin{pmatrix} \cos\frac{\pi(2n-1)}{k} \\ \sin\frac{\pi(2n-1)}{k} \end{pmatrix}$. In the first step of the algorithm we compute the matrix $S^{(k)}$ as in (6.51) and a fixed left eigenvector $s^{(k)}$. By virtue of Theorem 6.4.9 we obtain that

$$\lambda_{\text{approx}}(k) := \sum_{n=1}^{k} p_n^{(k)} \sum_{i=1}^{k} s_i^{(k)} \log \|A_n(b_i)\|$$

converges to λ_1. Figure 6.3 shows how $\lambda_{\text{approx}}(k)$ converges to λ_1 which is determined explicitly by (6.62).

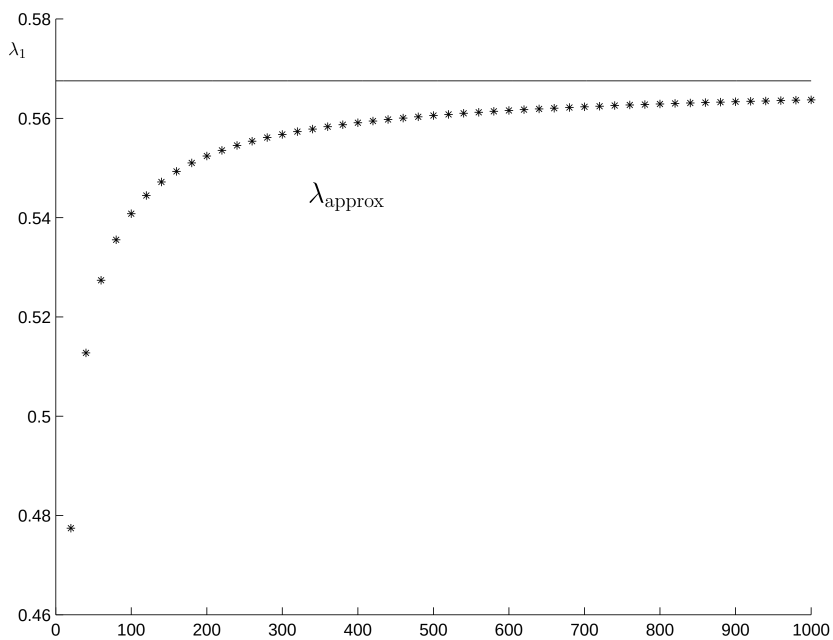

Figure 6.3: Numerical approximation λ_{approx} tends to λ_1.

Chapter 7

Outlook

7.1 One-Sided RDS on Banach Space

As in finite dimensional case, if the RDS is not invertible one can only hope for a flag decomposition of the state space. We use here a similar approach as in Thieullen [136], it means that we enlarge the state space to make the RDS invertible. Then the flag spaces of the original system are obtained as images of the projections of Oseledet spaces of the enlarged system. A technical problem arising here is that the separability of the state space is an important property for which the MET in Banach space in Lian and Lu [89] (see also Theorem 1.4.2) can be applied. The enlarged state space provided in Thieullen [136] is in general not separable. In this section, using positive weight factors, we first enlarge the state space to become a separable Banach space and the corresponding RDS on this space is injective. After that applying the MET in Banach space in Lian and Lu [89] gives the MET for one-sided RDS on Banach space.
Let X be a separable Banach space and $\Phi : \Omega \to \mathcal{L}(X)$ a strongly measurable map satisfying that

$$\log^+ \|\Phi(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{7.1}$$

Recall that the Kuratowski measure of the RDS Φ is determined by

$$l_\alpha(\Phi) := \lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)\|_\alpha \qquad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

For a fixed positive number $\gamma > 0$, the state space X is enlarged by

$$\mathbf{X}_\gamma = \left\{ \mathbf{x} := (x_n)_{n\in\mathbb{N}_0} : \quad \lim_{n\to\infty} e^{-\gamma n} x_n \text{ exists} \right\}.$$

We endow the space $\mathbf{X}_\gamma$ with the following norm

$$\|\mathbf{x}\|_\gamma := \sup_{n\in\mathbb{N}_0} e^{-\gamma n} \|x_n\| \qquad \text{for all } \mathbf{x} \in \mathbf{X}_\gamma.$$

Lemma 7.1.1. *The Banach space $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ is separable.*

Proof. The proof is analog to Lemma 4.1.2. □

We choose and fix a sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ satisfying the following conditions:

1. $\alpha_n > \alpha_{n+1} > 0$ for all $n \in \mathbb{N}_0$.
2. Set

$$\gamma_0 := 0, \quad \gamma_n := \sum_{k=0}^{n-1} \log \alpha_k \qquad \text{for all } n \geq 1 \tag{7.2}$$

and for each $\mu < 0$

$$P_\mu(n) := \sup\{p \in \mathbb{N}_0 : \gamma_p \geq (n+p)\mu\}. \tag{7.3}$$

Then we have

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = -\infty, \quad \text{and } \lim_{n\to\infty} \frac{P_\mu(n)}{n} = 0, \qquad \text{for all } \mu < 0. \tag{7.4}$$

An explicit example of sequence satisfying the above conditions is

$$\alpha_n = e^{-(2n+1)} \qquad \text{for all } n \in \mathbb{N}_0.$$

Associated to the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ we define a mapping $\widetilde{\Phi} : \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ by

$$\widetilde{\Phi}(\omega)(x_0, x_1, \dots) = (\Phi(\omega)x_0, \alpha_0 x_0, \alpha_1 x_1, \dots). \tag{7.5}$$

An explicit form of $\widetilde{\Phi}(n,\omega)$ is given by

$$\widetilde{\Phi}(n,\omega)\mathbf{x} = (\Phi(n,\omega)x_0, \alpha_0\Phi(n-1,\omega)x_0, \dots, \alpha_{n-1}\dots\alpha_0 x_0, \alpha_n \dots \alpha_1 x_1, \dots), \tag{7.6}$$

where $\mathbf{x} = (x_0, x_1, \dots) \in \mathbf{X}_\gamma$.

Lemma 7.1.2. *The mapping $\widetilde{\Phi}$ is strongly measurable and satisfies*

$$\log^+ \|\widetilde{\Phi}(\cdot)\|_\gamma \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Moreover, for each $\omega \in \Omega$ the linear operator $\widetilde{\Phi}(\omega) : \mathbf{X}_\gamma \to \mathbf{X}_\gamma$ is injective.

Proof. Since the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ is positive and decreasing it follows that the limit $\lim_{n\to\infty} \alpha_n$ exists. Hence, for each $\mathbf{x} = (x_n)_{n\in\mathbb{N}_0}$ the limit $\lim_{n\to\infty} e^{-\gamma(n+1)}\alpha_n x_n$ exists and thus $\widetilde{\Phi}(\omega)\mathbf{x} \in \mathbf{X}_\gamma$. Furthermore, a direct computation yields that

$$\|\widetilde{\Phi}(\omega)\mathbf{x}\|_\gamma \leq \max\left\{\|\Phi(\omega)\|, \alpha_0 e^{-\gamma}\right\} \qquad \text{for all } \|\mathbf{x}\|_\gamma = 1.$$

As a consequence, the function $\widetilde{\Phi}$ is well-defined and satisfies the integrability condition, i.e. it satisfies

$$\log^+ \|\widetilde{\Phi}(\cdot)\|_\gamma \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

To prove the strong measurability of $\widetilde{\Phi}$, we choose and fix $\mathbf{x} \in \mathbf{X}_\gamma$. For any $\varepsilon > 0$ and $\mathbf{y} = (y_0, y_1, \dots) \in \mathbf{X}_\gamma$ we consider the following cases.

Case 1: If $|\alpha_n x_n - y_{n+1}| \leq \varepsilon e^{\gamma(n+1)}$ for all $n \in \mathbb{N}_0$ then

$$\left\{\omega \in \Omega : \widetilde{\Phi}(\omega)\mathbf{x} \in B_\varepsilon(\mathbf{y})\right\} = \left\{\omega \in \Omega : \Phi(\omega)x_0 \in B_\varepsilon(y_0)\right\}.$$

Case 2: If there exists $n \in \mathbb{N}_0$ such that $|\alpha_n x_n - y_{n+1}| > \varepsilon e^{\gamma(n+1)}$ then

$$\left\{\omega \in \Omega : \widetilde{\Phi}(\omega)\mathbf{x} \in B_\varepsilon(\mathbf{y})\right\} = \emptyset.$$

Hence, the function $\widetilde{\Phi}$ is strongly measurable. The injectivity of the linear operator $\widetilde{\Phi}(\omega)$ can be easily seen from (7.5) and the proof is complete. □

So far we have enlarged the state space and introduced the corresponding RDS $\widetilde{\Phi}$ on the enlarged state space. We now investigate the relation between the linear cocycles Φ and $\widetilde{\Phi}$. Firstly, we state and prove the following technical lemma.

Lemma 7.1.3. *Let $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(a_n)_{n\in\mathbb{N}_0}$ be sequences such that*

$$\lim_{n\to\infty} \frac{\gamma_n}{n} = -\infty, \qquad \lim_{n\to\infty} \frac{a_n}{n} = a.$$

Then

$$\lim_{n\to\infty} \frac{b_n}{n} = a, \qquad \text{where } b_n := \max_{0\leq j\leq n-1} (\gamma_j + a_{n-j}).$$

Proof. By the definition of the sequence $(b_n)_{n\in\mathbb{N}_0}$ we have

$$\liminf_{n\to\infty} \frac{b_n}{n} \geq \lim_{n\to\infty} \frac{a_n}{n}.$$

Hence, it remains to show that

$$\limsup_{n\to\infty} \frac{b_n}{n} \leq \lim_{n\to\infty} \frac{a_n}{n} + \varepsilon \qquad \text{for all } \varepsilon > 0.$$

Choose $N \in \mathbb{N}$ such that

$$a_n \leq n(a+\varepsilon), \quad \gamma_n \leq na \qquad \text{for all } n \geq N,$$

which implies that for all $n \geq 2N$

$$\begin{aligned}
\frac{b_n}{n} &= \max\left\{\sup_{0\leq j\leq N-1} \frac{\gamma_j + a_{n-j}}{n}, \sup_{N\leq j\leq n-N-1} \frac{\gamma_j + a_{n-j}}{n}, \sup_{n-N\leq j\leq n-1} \frac{\gamma_j + a_{n-j}}{n}\right\}\\
&\leq \max\left\{\sup_{0\leq j\leq N-1} \frac{\gamma_j + (n-j)(a+\varepsilon)}{n}, a+\varepsilon, \sup_{n-N\leq j\leq n-1} \frac{ja + a_{n-j}}{n}\right\}.
\end{aligned}$$

Letting $n \to \infty$ gives that

$$\limsup_{n\to\infty} \frac{b_n}{n} \leq \max\{a+\varepsilon, a+\varepsilon, a\},$$

which completes the proof. □

Lemma 7.1.4. *Suppose that* $(\alpha_n)_{n\in\mathbb{N}_0}$ *be a strictly decreasing and positive sequence satisfying condition* (7.4). *Then* $l_\alpha(\Phi) = l_\alpha(\widetilde{\Phi})$.

Proof. Let $\mathbf{B}_1(0)$ and $B_1(0)$ denote the unit ball in $(\mathbf{X}_\gamma, \|\cdot\|_\gamma)$ and X, respectively. Set

$$\beta_n := \|\Phi(n,\omega)\|_\alpha = \alpha\left(\Phi(n,\omega)B_1(0)\right).$$

From the definition of Kuratowski measure (see (1.14)), there exist finitely many sets $A_1^n, \dots, A_{k_n}^n$ such that

$$\Phi(n,\omega)B_1(0) \subset \bigcup_{i=1}^{k_n} A_i^n, \quad \operatorname{diam}(A_i^n) \leq \beta_n \qquad \text{for all } i = 1, \dots, k_n.$$

For each $n \in \mathbb{N}$ and $i_j = 1, \dots, k_{n-j}$ for $j = 0, \dots, n-1$, we define

$$\begin{aligned} B^n_{i_0,\dots,i_{n-1}} \quad := \quad & \Big\{(x_0, x_1, \dots) \in \mathbf{X}_\gamma : x_j \in \alpha_{j-1}\dots\alpha_{j-n}e^{(j-n)\gamma}B_1(0) \quad \text{for } j \geq n, \\ & x_0 \in A^n_{i_0}, \quad x_j \in \alpha_{j-1}\dots\alpha_0 A^{n-j}_{i_j} \quad \text{for } 1 \leq j \leq n-1\Big\}. \end{aligned}$$

By (7.6), a direct computation yields that

$$\widetilde{\Phi}(n,\omega)\mathbf{B}_1(0) \subset \bigcup_{i_0=1}^{k_n}\bigcup_{i_1=1}^{k_{n-1}}\cdots\bigcup_{i_{n-1}=1}^{k_1} B^n_{i_0,\dots,i_{n-1}} \qquad \text{for all } n \in \mathbb{N},$$

and

$$\operatorname{diam}(B^n_{i_0,\dots,i_{n-1}}) \leq \sup\left\{\beta_n, \alpha_0\beta_{n-1}, \dots, \alpha_{n-2}\dots\alpha_0\beta_1, e^{-\gamma n}\sup_{j\geq n}\alpha_{j-1}\dots\alpha_{j-n}\right\}.$$

As a consequence, we get

$$\|\widetilde{\Phi}(n,\omega)\|_\alpha \leq \max\left\{\beta_n, \alpha_0\beta_{n-1}, \dots, \alpha_{n-2}\dots\alpha_0\beta_1, e^{-\gamma n}\alpha_{n-1}\dots\alpha_0\right\}.$$

Therefore,

$$\log\|\widetilde{\Phi}(n,\omega)\|_\alpha \leq \max\left\{\max_{0\leq j\leq n-1}(\gamma_j + \log\beta_{n-j}), \gamma_n - \gamma n\right\}, \tag{7.7}$$

where $(\gamma_n)_{n\in\mathbb{N}_0}$ is defined as in (7.2). In light of Lemma 7.1.3 we have

$$\lim_{n\to\infty}\frac{\max_{0\leq j\leq n-1}(\gamma_j + \log\beta_{n-j})}{n} = \lim_{n\to\infty}\frac{\log\beta_n}{n} = l_\alpha(\Phi),$$

which together with (7.7) implies that

$$l_\alpha(\widetilde{\Phi}) = \lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\|_\alpha \leq l_\alpha(\Phi).$$

On the other hand, it is easy to verify that

$$\|\Phi(n,\omega)\|_\alpha \leq \|\widetilde{\Phi}(n,\omega)\|_\alpha \qquad \text{for all } \omega \in \Omega, n \in \mathbb{N},$$

which gives that $l_\alpha(\Phi) \leq l_\alpha(\widetilde{\Phi})$ and the proof is complete. □

Lemma 7.1.5. *Let $\widetilde{\Phi}$ be the linear cocycle defined as in* (7.5)*. Then*

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\| = \lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\|_\gamma.$$

Proof. An elementary computation from (7.6) yields that

$$\|\widetilde{\Phi}(n,\omega)\|_\gamma \geq \|\Phi(n,\omega)\| \qquad \text{for all } n\in\mathbb{N}, \omega\in\Omega.$$

On the other hand, from (7.6) we derive

$$\log\|\widetilde{\Phi}(n,\omega)\|_\gamma \leq \max\left\{\sup_{0\leq j\leq n-1}\big(\log\gamma_j + \log\|\Phi(n-j,\omega)\|\big), -\gamma n + \gamma_n\right\},$$

where $(\gamma_n)_{n\in\mathbb{N}_0}$ is defined as in (7.2). According to Lemma 7.1.3, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\|_\gamma \leq \lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)\|,$$

which completes the proof. □

Lemma 7.1.6. *Let $\mathbf{x} = (x_0, x_1, \dots) \in \mathbf{X}_\gamma$ such that $\lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\mathbf{x}\|_\gamma$ exists. Then*

$$\lim_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)x_0\| = \lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\mathbf{x}\|_\gamma.$$

Proof. By (7.6), we get

$$\lambda := \lim_{n\to\infty}\frac{1}{n}\log\|\widetilde{\Phi}(n,\omega)\mathbf{x}\|_\gamma \geq \limsup_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)x_0\|.$$

Hence, it remains to show that

$$a := \liminf_{n\to\infty}\frac{1}{n}\log\|\Phi(n,\omega)x_0\| \geq \lambda. \tag{7.8}$$

For any $\varepsilon > 0$ there exists a sequence $(n_k)_{k\in\mathbb{N}_0}$ such that $n_k < n_{k+1}$ and

$$\frac{1}{n_k}\log\|\Phi(n_k,\omega)x_0\| \leq a+\varepsilon \qquad \text{for all } k\in\mathbb{N}_0. \tag{7.9}$$

Choose and fix $\mu < 0$ such that $\mu \leq a$. Define

$$m_k := n_k + P_\mu(n_k) \qquad \text{for all } k\in\mathbb{N}_0,$$

where $P_\mu(n_k)$ is defined as in (7.3). By considering a new cocycle $\widehat{\Phi}(\omega) := \frac{\Phi(\omega)}{\max\{1,\|\Phi(\omega)\|\}}$ if necessary, we assume w.l.o.g. that $\|\Phi(\omega)\| \leq 1$ for all $\omega\in\Omega$. Therefore, we have

$$\|\Phi(n,\omega)x_0\| \leq \|\Phi(m,\omega)x_0\| \qquad \text{for all } n \geq m,$$

which implies together with (7.9) that for $0 \leq j \leq P_\mu(n_k)$ we get

$$\begin{aligned} \gamma_j + \log \|\Phi(m_k - j, \omega)x_0\| &\leq \sup_{j\geq 0} \gamma_j + \|\Phi(n_k, \omega)x_0\| \\ &\leq \sup_{j\geq 0} \gamma_j + (a+\varepsilon)n_k. \end{aligned} \tag{7.10}$$

On the other hand, for any $m_k - 1 \geq j > P_\mu(n_k)$ we get

$$\gamma_j + \log \|\Phi(m_k - j, \omega)x_0\| \leq (n_k + P_\mu(n_k))\mu + \|x_0\|,$$

which together with (7.10) and (7.6) implies that

$$\begin{aligned} \log \|\widetilde{\Phi}(m_k, \omega)\mathbf{x}\|_\gamma &\leq \max\left\{ \max_{0\leq j\leq m_k-1} \left(\gamma_j + \log \|\Phi(m_k - j, \omega)x_0\|\right), \right. \\ &\qquad \left. , \gamma_{m_k} + \log \|\mathbf{x}\|_\gamma - \gamma m_k \right\} \\ &\leq \max\left\{ (a+\varepsilon)n_k + \sup_{j\geq 0}\gamma_j, m_k\mu + \|x_0\|, \gamma_{m_k} + \log\|\mathbf{x}\|_\gamma - \gamma m_k \right\}, \end{aligned}$$

where $(\gamma_n)_{n\in\mathbb{N}_0}$ is defined as in (7.2). As a consequence, we obtain

$$\liminf_{k\to\infty} \frac{1}{m_k} \log \|\widetilde{\Phi}(m_k, \omega)\mathbf{x}\|_\gamma \leq a + \varepsilon,$$

where we use the assumption $\lim_{k\to\infty} \frac{n_k}{m_k} = \lim_{k\to\infty} \frac{n_k}{n_k + P_\mu(n_k)} = 1$ to obtain the above estimate. Hence, statement (7.8) is proved and the proof is complete. □

Now we are at a position to state and prove the MET for one-sided random dynamical systems on Banach spaces.

Theorem 7.1.7 (MET for One-sided RDS on Banach Space)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an ergodic MDS and X a separable Banach space. Assume that $\Phi : \Omega \to \mathcal{L}(X)$ is a strongly measurable mapping satisfying that*

$$\log^+ \|\Phi(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Suppose additionally that $\kappa(\Phi) > l_\alpha(\Phi)$. Then there exists a θ-invariant subset $\tilde{\Omega} \subset \Omega$ of full measure such that there exist Lyapunov exponents $\lambda_1 > \cdots > \lambda_p > l_\alpha(\Phi)$ *and a filtration*

$$X = V_1(\omega) \supset V_2(\omega) \supset \cdots \supset V_{p+1}(\omega)$$

with the following properties hold:

(i) Invariance: V_j is of finite codimension and $\Phi(\omega)V_j(\omega) \subset V_j(\theta\omega)$.

(ii) Lyapunov exponents: for all $j = 1, \dots, p$

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)v\| = \lambda_j \quad \text{for all } v \in V_j(\omega) \setminus V_{j+1}(\omega)$$

and if p *is infinite then* $\lim_{p\to\infty} \lambda_p = l_\alpha(\Phi)$.

Proof. Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be a strictly decreasing and positive sequence satisfying condition (7.4). We define the enlarged RDS $\widetilde{\Phi} : \Omega \to \mathcal{L}(\mathbf{X}_\gamma)$ as in (7.5). By virtue of Lemma 7.1.4 and Lemma 7.1.5, we get $\kappa(\widetilde{\Phi}) > l_\alpha(\widetilde{\Phi})$. Furthermore, in view of Lemma 7.1.2, the linear cocycle $\widetilde{\Phi}$ fulfills all assumptions of Theorem 1.4.2. We divide the proof into the following cases:

Case 1: The linear cocycle $\widetilde{\Phi}$ has finitely many Lyapunov exponents denoted by $\lambda_1 > \lambda_2 > \cdots > \lambda_p > l_\alpha(\widetilde{\Phi})$. The corresponding Oseledet's splitting is given by

$$\mathbf{X}_\gamma = \mathbf{E}_1(\omega) \oplus \mathbf{E}_2(\omega) \oplus \cdots \oplus \mathbf{E}_p(\omega) \oplus \mathbf{F}(\omega),$$

where

$$\lim_{n\to\infty} \frac{1}{n} \log \|\widetilde{\Phi}(n,\omega)\mathbf{x}\|_\gamma = \lambda_j \qquad \text{for all } 0 \neq \mathbf{x} \in \mathbf{E}_j(\omega) \tag{7.11}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \|\widetilde{\Phi}(n,\omega)|_{\mathbf{F}(\omega)}\|_\gamma \leq l_\alpha(\widetilde{\Phi}). \tag{7.12}$$

Let $\pi : \mathbf{X}_\gamma \to X$ denote the projection onto the first component, i.e.

$$\pi\mathbf{x} = x_0 \qquad \text{for all } \mathbf{x} = (x_0, x_1, \dots) \in \mathbf{X}_\gamma.$$

Define

$$V_j(\omega) := \pi\left(\mathbf{F}(\omega) \oplus \bigoplus_{j\leq i\leq p} \mathbf{E}_i(\omega)\right) \qquad \text{for all } 1 \leq j \leq p+1.$$

Using the fact that

$$\Phi(\omega)\pi\mathbf{x} = \pi\widetilde{\Phi}(\omega)\mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbf{X}_\gamma,$$

we obtain that the subspace $V_j(\omega)$ is invariant under Φ. Using (7.11) and (7.12) and in view of Lemma 7.1.6, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Phi(n,\omega)x\| = \lambda_j \qquad \text{for all } x \in V_j(\omega) \setminus V_{j+1}(\omega),$$

which completes the proof in this case.

Case 2: The linear cocycle $\widetilde{\Phi}$ has infinitely many Lyapunov exponents $\lambda_1 > \lambda_2 > \dots$, where $\lim_{p\to\infty} \lambda_p = l_\alpha(\widetilde{\Phi})$. The proof in this case is analog to Case 1 and we obtain the desired conclusion. □

7.2 Lyapunov norm for RDS on Banach Space

Invariant manifold theory for RDS based on the MET is an important part of *smooth ergodic theory*. It was started in 1976 with the pioneering work of Pesin [115, 116]. He constructed the classical stable and unstable manifolds of a deterministic diffeomorphism on a compact Riemannian manifold preserving a measure which is absolutely continuous with respect to the Riemannian volume. His technique is to cope with the non-uniformity of the MET (random norms, ε-slowly varying functions). This technique is also used in Wanner [139] and Arnold [3] to construct invariant manifolds for RDS on finite dimensional space. In this chapter, based on the non-uniformity of the MET for RDS on Banach space we construct the Lyapunov norms corresponding to a linear difference equation with random delay which is the investigated object in Chapter 4. The work in this subsetion can be considered as the first technical step toward the nonlinear theory of difference equations with random delay.

We start this chapter by introducing the notion of *tempered* and *ε-slowly varying random variable.*

Definition 7.2.1 (Tempered, Slowly Varying Random Variables)**.** Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an MDS.

(i) A random variable $R : \Omega \to (0, \infty)$ is called *tempered* with respect to θ if

$$\lim_{n\to\pm\infty} \frac{1}{n} R(\theta^n\omega) = 0 \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

(ii) For a given $\varepsilon \geq 0$, a random variable $R : \Omega \to (0, \infty)$ is called *ε-slowly varying* with respect to θ if $\mathbb{P}$-a.s.

$$e^{-\varepsilon|n|}R(\omega) \leq R(\theta^n\omega) \leq e^{\varepsilon|n|}R(\omega) \qquad \text{for all } n \in \mathbb{Z}.$$

A relation between tempered and slowly varying random variables is the content of the following lemma.

Lemma 7.2.1 (Tempered Versus Slowly Varying)**.** *(i) If R_ε is ε-slowly varying for some $\varepsilon \geq 0$ then it is tempered.*

(ii) Conversely, if $f : \Omega \to (0, \infty)$ is tempered then for any $\varepsilon > 0$ there is an ε-slowly varying random variable R_ε for which

$$\frac{1}{R_\varepsilon(\omega)} \leq f(\omega) \leq R_\varepsilon(\omega).$$

Proof. A proof can be found in Arnold [3, Proposition 4.3.3]. □

Now we discuss about the Lyapunov norm associated with a linear difference equation with random delay. We consider a linear difference equation of the form

$$x_{n+1} = A(\theta^n\omega)x_n + B(\theta^n\omega)x_{n-r(\theta^n\omega)}, \tag{7.13}$$

where $A, B : \Omega \to \mathbb{R}^{d\times d}$ are measurable functions and $r : \Omega \to \mathbb{N}$ is measurable, on the state space $\mathbf{X}_\gamma$. Let Φ denote the linear cocycle generated (7.13). Throughout this chapter, we assume that Φ satisfies the integrability condition of the MET, i.e.

$$\log^+ \|\Phi(\cdot)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

By virtue of Theorem 4.2.3, let $-\gamma < \lambda_p < \cdots < \lambda_1$ be the Lyapunov exponents of Φ and

$$\mathbf{X}_\gamma = E_1(\omega) \oplus E_2(\omega) \oplus \cdots \oplus E_p(\omega) \oplus F(\omega)$$

denote the corresponding Oseledet splitting. Choose any j with $1 \le j < p$, let

$$\Lambda_j^+ := \{\lambda_1 > \cdots > \lambda_j\}, \quad E_j^+(\omega) := \bigoplus_{1\le i\le j} E_i(\omega), \quad F_j(\omega) := \bigoplus_{j+1\le i\le p} E_i(\omega) \oplus F(\omega).$$

Let $\pi_j(\omega)$ denote the projection onto $F_j(\omega)$ along $E_j^+(\omega)$. By virtue of MET, we obtain that the random variable $\|\pi_j(\cdot)\|$ is tempered. The following theorem is a direct consequence of the non-uniformity of the MET for RDS on Banach space which is proved in Lian and Lu [89].

Theorem 7.2.2 (Non-Uniformity of MET is Slowly Varying, [89])**.** *Suppose that system* (7.13) *fulfills all assumptions in Theorem 4.2.3. Then there exists an ε-slowly varying random variable $R_\varepsilon : \Omega \to [1, \infty)$ such that on the invariant set $\widetilde{\Omega}$ of the MET the cocycle Φ has the following properties:*

$$\frac{1}{R_\varepsilon(\omega)} e^{\lambda_j n - \varepsilon|n|}\|\mathbf{x}\| \le \|\Phi(n,\omega)\mathbf{x}\| \le R_\varepsilon(\omega) e^{\lambda_j n + \varepsilon|n|}\|\mathbf{x}\|, \qquad \mathbf{x} \in E_j^+(\omega),$$

and

$$\|\Phi(n,\omega)\mathbf{x}\| \le R_\varepsilon(\omega) e^{\lambda_{j+1} n + \varepsilon|n|}\|\mathbf{x}\|, \qquad \mathbf{x} \in F_j(\omega).$$

We choose and fix a positive constant κ such that $\lambda_j - \kappa > \lambda_{j+1} + \kappa$ and construct the following norms

$$\|\mathbf{x}^u\|_\omega := \sum_{n=0}^{\infty} e^{(\lambda_j - \kappa)n} \|\Phi(-n,\omega)\mathbf{x}^u\| \qquad \text{for all } \mathbf{x}^u \in E_j^+(\omega),$$

$$\|\mathbf{x}^s\|_\omega := \sum_{n=0}^{\infty} e^{-(\lambda_{j+1} + \kappa)n} \|\Phi(n,\omega)\mathbf{x}^s\| \qquad \text{for all } \mathbf{x}^s \in F_{j+1}(\omega).$$

For any $\mathbf{x} \in \mathbf{X}_\gamma$ and $\omega \in \Omega$ with $\mathbf{x} = \mathbf{x}^s + \mathbf{x}^u$, where $\mathbf{x}^s \in E_j^+(\omega)$ and $\mathbf{x}^u \in F_{j+1}(\omega)$, we set

$$\|\mathbf{x}\|_\omega = \max\{\|\mathbf{x}^u\|_\omega, \|\mathbf{x}^s\|_\omega\}.$$

The norm $\|\cdot\|_\omega$ is usually called the *Lyapunov norm.* In the following theorem, we provide some fundamental properties of the above norm.

Theorem 7.2.3. *The norm $\|\cdot\|_\omega$ is strongly measurable, i.e. for each fixed $\mathbf{x}\in\mathbf{X}_\gamma$ the scalar valued function $\Omega\to\mathbb{R}^+$, $\omega\mapsto\|\mathbf{x}\|_\omega$ is measurable, and satisfies that for each $\varepsilon>0$ there exists an ε-slowly varying random variable $D_\varepsilon:\Omega\to[1,\infty)$ such that*

$$\frac{1}{D_\varepsilon(\omega)}\|\cdot\|\le\|\cdot\|_\omega\le D_\varepsilon(\omega)\|\cdot\|. \tag{7.14}$$

Furthermore, we obtain that for all $n\ge 0$

$$\begin{aligned}\|\Phi(n,\omega)\mathbf{x}^s\|_{\theta^n\omega} &\le e^{(\lambda_{j+1}+\kappa)n}\|\mathbf{x}^s\|_\omega \qquad \textit{for all } \mathbf{x}^s\in F_{j+1}(\omega),\\ \|\Phi(-n,\omega)\mathbf{x}^u\|_{\theta^{-n}\omega} &\le e^{-(\lambda_j-\kappa)n}\|\mathbf{x}^u\|_\omega \qquad \textit{for all } \mathbf{x}^u\in E_i^+(\omega).\end{aligned}$$

Proof. The fact that $\|\cdot\|_\omega$ is strongly measurable is a direct consequence of the strong measurability of the projection $\pi_j(\cdot)$. W.l.o.g. we assume that $\varepsilon<\frac{\kappa}{2}$. We first verify the estimate of $\|\cdot\|_\omega$ from above. In view of Theorem 7.2.2, there exists an $\frac{\varepsilon}{2}$-slowly varying random variable $R_{\frac{\varepsilon}{2}}:\Omega\to[1,\infty)$ such that for all $n\ge 0$

$$\begin{aligned}\|\Phi(-n,\omega)\mathbf{x}^u\| &\le R_{\frac{\varepsilon}{2}}(\omega)e^{-(\lambda_j-\varepsilon)n}\|\mathbf{x}^u\| \qquad \text{for all } \mathbf{x}^u\in E_j^+(\omega),\\ \|\Phi(n,\omega)\mathbf{x}^s\| &\le R_{\frac{\varepsilon}{2}}(\omega)e^{(\lambda_{j+1}+\varepsilon)n}\|\mathbf{x}^s\| \qquad \text{for all } \mathbf{x}^s\in F_j(\omega).\end{aligned}$$

As a consequence, a direct computation yields that

$$\|\mathbf{x}^u\|_\omega\le\frac{e^\varepsilon}{e^\varepsilon-1}R_{\frac{\varepsilon}{2}}(\omega)\|\mathbf{x}^u\|,\quad \|\mathbf{x}^s\|_\omega\le\frac{e^\varepsilon}{e^\varepsilon-1}R_{\frac{\varepsilon}{2}}(\omega)\|\mathbf{x}^s\|,$$

which implies that

$$\|\mathbf{x}\|_\omega\le\frac{e^\varepsilon}{e^\varepsilon-1}R_{\frac{\varepsilon}{2}}(\omega)(1+2\|\pi_j(\omega)\|)\|\mathbf{x}\|.$$

On the other hand, from the definition of $\|\cdot\|_\omega$ we derive

$$\|\mathbf{x}\|_\omega\ge\frac{1}{2}(\|\mathbf{x}^u\|_\omega+\|\mathbf{x}^s\|_\omega)\ge\frac{1}{2}\|\mathbf{x}\|.$$

Define $D_\varepsilon(\omega):=\frac{e^\varepsilon}{e^\varepsilon-1}R_{\frac{\varepsilon}{2}}(\omega)(1+2\|\pi_j(\omega)\|)$. Note that $\|\pi_j(\cdot)\|$ is tempered and by virtue of Lemma 7.2.1, the random variable D_ε is thus ε-slowly varying. Obviously, D_ε satisfies inequality (7.14). For the remainder of the proof, we need to estimate

$$\|\Phi(n,\omega)\mathbf{x}^s\|_{\theta^n\omega},\quad \|\Phi(-n,\omega)\mathbf{x}^u\|_{\theta^{-n}\omega}\qquad \text{for all } n\ge 0.$$

Using the fact that $\Phi(n,\omega)F_j(\omega)\subset F_j(\theta^n\omega)$, we obtain

$$\begin{aligned}\|\Phi(n,\omega)\mathbf{x}^s\|_{\theta^n\omega} &= \sum_{k=0}^{\infty}e^{-(\lambda_{j+1}+\kappa)k}\|\Phi(k,\theta^n\omega)\Phi(n,\omega)\mathbf{x}^s\|\\ &= e^{(\lambda_{j+1}+\kappa)n}\sum_{k=0}^{\infty}e^{-(\lambda_{j+1}+\kappa)(k+n)}\|\Phi(k+n,\omega)\mathbf{x}^s\|\\ &\le e^{(\lambda_{j+1}+\kappa)n}\|\mathbf{x}^s\|_\omega.\end{aligned}$$

Similarly, we also have

$$\|\Phi(-n,\omega)\mathbf{x}^u\|_{\theta^{-n}\omega}\le e^{-(\lambda_j-\kappa)n}\|\mathbf{x}^u\|_\omega,$$

which completes the proof. □

Appendix A

Birkhoff Ergodic Theorem

A central aspect of ergodic theory is the long-term behavior of a dynamical system and the relation to the spacial behavior. This is expressed through the Birkhoff ergodic theorem which asserts that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. To make the thesis self-contained we state and prove the Birkhoff ergodic theorem in this Appendix. Here we follow the presentation to prove the Birkhoff ergodic theorem in Walter [138].

Theorem A.0.4 (Birkhoff Ergodic Theorem)**.** *Let θ be a measure preserving transformation of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a function $f^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(\theta^k \omega) = f^*(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Moreover, f^* is invariant under θ, i.e. $f^* \circ \theta = f^*$, and $\int_\Omega f^* \; d\mathbb{P} = \int_\Omega f \; d\mathbb{P}$. In particular, if θ is an ergodic transformation then

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(\theta^k \omega) = \int_\Omega f(\omega) \; d\mathbb{P}(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Before we prove the Birkhoff ergodic theorem we show a preparatory theorem which is well known under the name maximal ergodic theorem.

Theorem A.0.5 (Maximal Ergodic Theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all real-valued integrable functions on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $U : \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ is a positive linear operator, i.e. $Uf \geq 0$ for all $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $f \geq 0$, and $\|U\| \leq 1$. Let $N \geq 0$ be an integer and $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Define the sequence of functions $(f_n)_{n\in\mathbb{N}_0}$ by*

$$f_0 := 0, \quad f_n := f + Uf + \cdots + U^{n-1} f \qquad \text{for all } n \geq 1,$$

and define $F_N = \max_{0 \le n \le N} f_n$. *Then the following statement holds*

$$\int_{\{\omega : F_N(\omega) > 0\}} f \, d\mathbb{P} \geq 0. \tag{A.1}$$

Proof. Clearly $F_N \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. For any $0 \le n \le N$, we have $F_N \ge f_n$ so $UF_N \ge Uf_n$ by the positivity of U. Hence, we obtain

$$UF_N + f \geq Uf_n + f = f_{n+1} \qquad \text{for all } 0 \le n \le N.$$

Therefore,

$$UF_N(\omega) + f(\omega) \geq \max_{1 \le n \le N} f_n(\omega).$$

In particular, if $F_N(\omega) > 0$ then

$$UF_N(\omega) + f(\omega) = \max_{0 \le n \le N} f_n(\omega) = F_N(\omega),$$

which implies that $f(\omega) \ge F_N(\omega) - UF_N(\omega)$ on the set $\{\omega : F_N(\omega) > 0\}$. As a consequence, we get

$$\begin{aligned}
\int_{\{\omega : F_N(\omega) > 0\}} f \, d\mathbb{P} &\geq \int_{\{\omega : F_N(\omega) > 0\}} F_N \, d\mathbb{P} - \int_{\{\omega : F_N(\omega) > 0\}} UF_N \, d\mathbb{P} \\
&\geq \int_{\Omega} F_N \, d\mathbb{P} - \int_{\{\omega : F_N(\omega) > 0\}} UF_N \, d\mathbb{P},
\end{aligned}$$

On the other hand, since $F_N \ge 0$ it follows that $UF_N \ge 0$ by positivity of U and we therefore obtain the following estimate

$$\int_{\{\omega : F_N(\omega) > 0\}} f \, d\mathbb{P} \geq \int_{\Omega} F_N \, d\mathbb{P} - \int_{\Omega} UF_N \, d\mathbb{P}.$$

This implies with $\|U\| \le 1$ inequality (A.1) and completes the proof. □

Remark A.0.6. For any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a transformation $\theta : \Omega \to \Omega$ preserving the probability $\mathbb{P}$, we can define an operator $U : L^1(\mathbb{P}) \to \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ by

$$Uf(\omega) = f(\theta\omega) \qquad \text{for all } f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}), \omega \in \Omega.$$

Then it is easy to see that U is a positive linear operator and $\|U\| = 1$.

Corollary A.0.7. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\theta : \Omega \to \Omega$ *a transformation preserving the probability* $\mathbb{P}$. *Let* $g \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ *and define*

$$B_\alpha := \{\omega \in \Omega : \sup_{n \ge 1} \frac{1}{n} \sum_{i=0}^{n-1} g(\theta^i \omega) > \alpha\}$$

Then

$$\int_{B_\alpha \cap A} g(\omega) \, d\mathbb{P}(\omega) \geq \alpha \, \mathbb{P}(B_\alpha \cap A)$$

if A *is a measurable set which is invariant under* θ, *i.e.* $\theta^{-1}A = A$.

Proof. We first prove this result under the assumption $A = X$. Define $f := g - \alpha$, the sequence of functions $(f_n)_{n\geq 0}$ from Ω to $\mathbb{R}$ by

$$f_0 := 0, \quad f_n := f + f \circ \theta + \cdots + f \circ \theta^{n-1} \qquad \text{for } n \geq 1,$$

and $F_N = \max_{0\leq n\leq N} f_n$. Hence,

$$B_\alpha = \bigcup_{N=0}^{\infty} \{\omega : F_N(\omega) > 0\}.$$

By virtue of Theorem A.0.5 and Remark A.0.6, we get

$$\int_{B_\alpha} f(\omega)\, d\mathbb{P}(\omega) \geq 0.$$

Consequently,

$$\int_{B_\alpha} g(\omega)\, d\mathbb{P}(\omega) \geq \alpha\mathbb{P}(B_\alpha).$$

In the general case, we apply the above result to $\theta|_A$ to get

$$\int_{A\cap B_\alpha} g(\omega)\, d\mathbb{P}(\omega) \geq \alpha\mathbb{P}(A \cap B_\alpha),$$

and this completes the proof. □

Proof of Theorem A.0.4. Define

$$f^*(\omega) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(\theta^i\omega) \qquad \text{for } \omega \in \Omega,$$

and

$$f_*(\omega) = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(\theta^i\omega) \qquad \text{for } \omega \in \Omega.$$

It is easy to see that $f^*\theta = f^*$ and $f_*\theta = f_*$. Now, we show that $f^* = f_*$ $\mathbb{P}$-a.e. and that they belong to $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. For real numbers α, β with $\alpha < \beta$, define

$$E_{\alpha,\beta} := \{\omega \in \Omega : f_*(\omega) < \beta \text{ and } \alpha < f^*(\omega)\}.$$

Note that

$$\{\omega : f_*(\omega) < f^*(\omega)\} = \bigcup_{\alpha,\beta\in\mathbb{Q},\beta<\alpha} E_{\alpha,\beta}.$$

Hence, to prove $f^* = f_*$ $\mathbb{P}$-a.e. it is sufficient to show that $\mathbb{P}(E_{\alpha,\beta}) = 0$. Clearly, the set $E_{\alpha,\beta}$ is invariant under θ, i.e. $\theta^{-1}E_{\alpha,\beta} = E_{\alpha,\beta}$, and if we put

$$B_\alpha := \{\omega \in \Omega : \sup_{n\geq 1} \frac{1}{n}\sum_{i=0}^{n-1} f(\theta^i\omega) > \alpha\}$$

then $E_{\alpha,\beta} \cap B_\alpha = E_{\alpha,\beta}$. By Corollary (A.0.7) we obtain

$$\int_{E_{\alpha,\beta}} f(\omega)\, d\mathbb{P}(\omega) = \int_{E_{\alpha,\beta}\cap B_\alpha} f(\omega)\, d\mathbb{P}(\omega) \geq \alpha\mathbb{P}(E_{\alpha,\beta} \cap B_\alpha) = \alpha\mathbb{P}(E_{\alpha,\beta}).$$

Consequently,

$$\int_{E_{\alpha,\beta}} f(\omega)\, d\mathbb{P}(\omega) \geq \alpha\mathbb{P}(E_{\alpha,\beta}). \tag{A.2}$$

On the other hand, if we replace f, α, β by $-f, -\beta, -\alpha$, respectively, then since $(-f)^* = -f_*$ and $(-f)_* = -f^*$, we get

$$\int_{E_{\alpha,\beta}} f(\omega)\, d\mathbb{P}(\omega) \leq \beta\mathbb{P}(E_{\alpha,\beta}). \tag{A.3}$$

Combining (A.2) and (A.3), we get $\alpha\mathbb{P}(E_{\alpha,\beta}) \leq \beta\mathbb{P}(E_{\alpha,\beta})$. This implies that $\mathbb{P}(E_{\alpha,\beta}) = 0$. Hence, $f^* = f_*$ $\mathbb{P}$-a.e., and then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\theta^i\omega) = f^*(\omega) = f_*(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

To show $f^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ we use the Fatou lemma (see Lang [85, pp. 141]) that asserts $\lim_{n\to\infty} g_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ if $(g_n)_{n\in\mathbb{N}}$ is a pointwise convergent sequence of nonnegative integrable functions with $\liminf \int_\Omega g_n\, d\mathbb{P}(\omega) < \infty$. Define

$$g_n(\omega) = \frac{1}{n} \sum_{i=0}^{n-1} f(\theta^i\omega) \qquad \text{for } \omega \in \Omega.$$

It is easily seen that

$$\int_\Omega g_n(\omega)\, d\mathbb{P}(\omega) \leq \int_\Omega |f(\omega)|\, d\mathbb{P}(\omega) \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, we can apply the Fatou lemma together with the fact that $\lim_{n\to\infty} g_n(\omega) = |f^*(\omega)|$ to conclude that $f^* \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. It remains to show that $\int_\Omega f(\omega)\, d\mathbb{P}(\omega) = \int_\Omega f^*(\omega)\, d\mathbb{P}(\omega)$. For this purpose, we define

$$D_k^n := \left\{ \omega \in \Omega : \frac{k}{n} \leq f^*(\omega) < \frac{k+1}{n} \right\},$$

where $k \in \mathbb{Z}, n = 1, 2, \ldots$. For each small $\varepsilon > 0$, we have $D_k^n \cap B_{\frac{k}{n}-\varepsilon} = D_k^n$ and by Corollary A.0.7 we get

$$\int_{D_k^n} f(\omega)\, d\mathbb{P}(\omega) \geq \left(\frac{k}{n} - \varepsilon\right) \mathbb{P}(D_k^n).$$

This implies that

$$\int_{D_k^n} f^*(\omega)\, d\mathbb{P}(\omega) \leq \frac{k+1}{n}\mathbb{P}(D_k^n) \leq \frac{1}{n}\mathbb{P}(D_k^n) + \int_{D_k^n} f(\omega)\, d\mathbb{P}(\omega).$$

Summing this inequality over k gives $\int_\Omega f^*(\omega)\, d\mathbb{P}(\omega) \leq \frac{1}{n} + \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$. Since this holds for all $n \geq 1$, we have

$$\int_\Omega f^*(\omega)\, d\mathbb{P}(\omega) \leq \int_\Omega f(\omega)\, d\mathbb{P}(\omega). \tag{A.4}$$

Applying (A.4) to $-f$ instead of f gives

$$\int_\Omega (-f)^*(\omega)\, d\mathbb{P}(\omega) \leq -\int_\Omega f(\omega)\, d\mathbb{P}(\omega).$$

Hence,

$$\int_\Omega f_*(\omega)\, d\mathbb{P}(\omega) \geq \int_\Omega f(\omega)\, d\mathbb{P}(\omega). \tag{A.5}$$

Combining (A.4) and (A.5) and the fact that $f^* = f_*$ $\mathbb{P}$-a.e., we get $\int_\Omega f^*(\omega)\, d\mathbb{P}(\omega) = \int_\Omega f(\omega)\, d\mathbb{P}(\omega)$. For the remaining part of the proof, we consider the case that θ is an ergodic transformation. Since $f^*\theta = \theta$ it follows that the function f^* is almost surely constant. Consequently, the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f(\theta^k\omega) = f^*(\omega) = \int_\Omega f^*(\omega)\, d\mathbb{P}(\omega) = \int_\Omega f(\omega)$$

holds for $\mathbb{P}$-a.e. $\omega \in \Omega$ and this completes the proof. □

Appendix B

Kingman Subadditive Ergodic Theorem

A substantial generalization of Birkhoff ergodic theorem was obtained by Kingman, who proved an ergodic theorem for subadditive stationary processes in [78]. This result has also been reproved with elegant proofs in Burkholder [25], Steele [129]. Here we follow the materials in Steele [129] to state and prove the Kingman subadditive ergodic theorem.

Theorem B.0.8 (Kingman subadditive ergodic theorem)**.** *Let θ be a measure preserving transformation of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{g_n\}_{n=1}^{\infty}$ a subadditive sequence of random variables over the MDS $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$, i.e.,*

$$g_{n+m}(\omega) \leq g_n(\omega) + g_m(\theta^n \omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{B.1}$$

Suppose that $g_1^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, where $a^+ := \max(0, a)$. Then, with probability one, we have

$$\lim_{n\to\infty} \frac{g_n(\omega)}{n} = g(\omega) \geq -\infty,$$

and $g(\omega)$ is an invariant measurable function, i.e. $g(\theta\omega) = g(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. In particular, if θ is an ergodic transformation then

$$\lim_{n\to\infty} \frac{1}{n} g_n(\omega) = \lim_{n\to\infty} \frac{1}{n} \int_{\Omega} g_n(\omega)\, d\mathbb{P}(\omega).$$

Proof. We first deal with the case that $\{g_n\}_{n=1}^{\infty}$ is a subadditive sequence of integrable random variables. For $n \in \mathbb{N}$ we define a new process $g_n^{'}(\cdot) : \Omega \to \mathbb{R}$ by

$$g_n^{'}(\omega) = g_n(\omega) - \sum_{k=1}^{n-1} g_1(\theta^k \omega) \qquad \text{for all } \omega \in \Omega.$$

Due to (B.1) one has that $g_n^{'}(\omega) \leq 0$ for all $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover, the sequence $\{g_n^{'}\}_{n=1}^{\infty}$ again satisfies inequality (B.1). Since the Birkhoff ergodic theorem (see Theorem A.0.4) can be applied to the second term of $g_n^{'}$ it follows that the almost sure convergence of $g_n^{'}/n$ implies the almost sure convergence of g_n/n. Thus, we can assume w.l.o.g. that $g_n(\omega) \leq 0$. Now we define a function $g : \Omega \to \mathbb{R}$ by

$$g(\omega) := \liminf_{n\to\infty} \frac{g_n(\omega)}{n} \qquad \text{for all } \omega \in \Omega.$$

The function g is clearly measurable and we show that g is furthermore invariant under θ, i.e. $g(\theta\omega) = g(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. By (B.1) we have

$$\frac{g_{n+1}(\omega)}{n} \leq \frac{g_1(\omega)}{n} + \frac{g_n(\theta\omega)}{n} \qquad \text{for all } \omega \in \Omega, n \in \mathbb{N}.$$

By taking the limit inferior we get $g(\omega) \leq g(\theta\omega)$. Assume that there exists $\alpha \in \mathbb{R}$ such that $\mathbb{P}(E_\alpha) > 0$, where $E_\alpha = \{\omega \in \Omega : g(\theta\omega) - g(\omega) > \alpha\}$. Since

$$E_\alpha \subset \bigcup_{p/q\in\mathbb{Q}} \left\{\omega \in \Omega : g(\omega) < \frac{p}{q} \quad \text{and} \quad g(\theta\omega) > \frac{p}{q} + \frac{\alpha}{2}\right\},$$

it follows that there exists $p/q \in \mathbb{Q}$ with

$$\mathbb{P}\left(\left\{\omega \in \Omega : g(\omega) < \frac{p}{q} \quad \text{and} \quad g(\theta\omega) > \frac{p}{q} + \frac{\alpha}{2}\right\}\right) > 0. \tag{B.2}$$

Since $g(\theta\omega) \geq g(\omega)$ it follows that

$$\left\{\omega \in \Omega : g(\omega) > \frac{p}{q}\right\} \subset \left\{\omega \in \Omega : g(\theta\omega) \geq \frac{p}{q}\right\} = \theta^{-1}\left\{\omega \in \Omega : g(\omega) \geq \frac{p}{q}\right\}.$$

On the other hand, due to θ is a preserving measure we have

$$\mathbb{P}\left(\left\{\omega \in \Omega : g(\omega) \geq \frac{p}{q}\right\}\right) = \mathbb{P}\left(\left\{\omega \in \Omega : g(\theta\omega) \geq \frac{p}{q}\right\}\right)$$

Hence,

$$\begin{aligned} 0 &= \mathbb{P}\left(\left\{\omega \in \Omega : g(\theta\omega) \geq \frac{p}{q}\right\} \setminus \left\{\omega \in \Omega : g(\omega) \geq \frac{p}{q}\right\}\right) \\ &= \mathbb{P}\left(\left\{\omega \in \Omega : g(\theta\omega) \geq \frac{p}{q} \quad \text{and} \quad g(\omega) < \frac{p}{q}\right\}\right), \end{aligned}$$

which contradicts to (B.2). Therefore, we can assume w.l.o.g. that $g(\theta^k\omega) = g(\omega)$ for all $k \in \mathbb{N}$ and for all $\omega \in \Omega$. Now we show that

$$\limsup_{n\to\infty} \frac{g_n(\omega)}{n} \leq \liminf_{n\to\infty} \frac{g_n(\omega)}{n} = g(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{B.3}$$

For any $\varepsilon > 0, 1 < N < \infty$ and $M > 0$ we define

$$G_M(\omega) = \max\{-M, g(\omega)\} \qquad \text{for all } \omega \in \Omega,$$

and consider the set

$$B(N,M)=\left\{\omega\in\Omega:\frac{g_k(\omega)}{k}>G_M(\omega)+\varepsilon\quad\text{for all }1\le k\le N\right\},\tag{B.4}$$

and its complement $A(N,M)=B(N,M)^c$. For any $\omega\in\Omega$ and $n\ge N$ we decompose the integer set $\{1,2,\dots,n-1\}$ into the union of three disjoint sets U,V,W by the following algorithm:
Begin with $l=1$. If l is the smallest integer in $\{1,2,\dots,n-1\}$ which is not in a set already constructed, then consider θ^l. There are two cases:

Case 1: If $\theta^l\omega\in A(N,M)$: Let k be the smallest integer number in $\{1,\dots,N\}$ such that $g_k(\theta^l\omega)/k\le G_M(\theta^l\omega)+\varepsilon=G_M(\omega)+\varepsilon$. We have two subcases here:

- *Case 1a* If $l+k\le n$: Then we put the points $l,l+1,\dots,l+k-1$ in the set U.
- *Case 1b* If $l+k>n$: Then we just put the point l in the set W.

Case 2: If $\theta^l\omega\in B(N,M)$: Then we put the point l in the set V.

Thus, for any ω we have a decomposition of the set $\{1,2,\dots,n-1\}$ into the set U containing $\{l_i,l_i+1,\dots,l_i+k_i-1\}$, where $g_{k_i}(\theta^{l_i}\omega)/k_i\le G_M(\omega)+\varepsilon$ with $1\le k_i\le N$, the set V containing singletons l_i for which $\chi_{B(N,M)}(\theta^{l_i}\omega)=1$ and the second set of singletons W contained in the set $\{n-N+1,n-N+2,\dots,n-1\}$. By subadditive inequality (B.1), our decomposition of thet set $\{1,2,\dots,n-1\}$, one has the following estimate:

$$g_n(\omega)\le g_1(\omega)+\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}g_{k_i}(\theta^{l_i}\omega)+\sum_{l_i\in V}g_1(\theta^{l_i}\omega)+\sum_{l_i\in W}g_1(\theta^{l_i}\omega).$$

This implies together with $g_1(\omega)\le 0$ for all $\omega\in\Omega$ that

$$g_n(\omega)\le\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}g_{k_i}(\theta^{l_i}\omega).$$

Using the definition of the set U, we have

$$g_n(\omega)\le(G_M(\omega)+\varepsilon)\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}k_i\le n\varepsilon+G_M(\omega)\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}k_i.\tag{B.5}$$

Also, by the construction of the sets U,V,W we have

$$\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}k_i\le n-\sum_{l=1}^{n}\chi_{B(N,M)}(\theta^l\omega).$$

Hence, by Birkhoff ergodic theorem (see Theorem A.0.4) we get

$$\limsup_{n\to\infty}\frac{\sum_{\{l_i,\dots,l_i+k_i-1\}\subset U}k_i}{n}\le 1-h_{N,M}(\omega)\qquad\text{for }\mathbb{P}\text{-a.e. }\omega\in\Omega,$$

where the nonnegative integrable function $h_{N,M} : \Omega \to \mathbb{R}$ is defined by

$$h_{N,M}(\omega) := \lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \chi_{B(N,M)}(\theta^l \omega).$$

By (B.5) we then conclude that

$$\limsup_{n\to\infty} \frac{g_n(\omega)}{n} \le G_M(\omega)(1 - h_{N,M}(\omega)) + \varepsilon \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Letting $N \to \infty$, with the observation that the definition of $B(N,M)$ as in (B.4) guarantees that $\chi_{B(N,M)} \to 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, we obtain

$$\limsup_{n\to\infty} \frac{g_n(\omega)}{n} \le G_M(\omega) + \varepsilon \qquad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{B.6}$$

Since estimate (B.6) holds for arbitrary $M > 0$ and $\varepsilon > 0$ estimate (B.3) follows and thus we have $\lim_{n\to\infty} g(\theta^n\omega)/n = g(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Hence, we get the desired conclusions in the case that $g_n \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $n \in \mathbb{N}$. We refer to Ruelle [120, Appendix A] for how the theorem can be reduced to the case $g_1^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

□

Appendix C

Baire Category and Baire Class of Functions

The following material is taken from Oxtoby [110] and Munkres [106].
Baire Category:
Let X be a topological space. A set $A \subset X$ is called *nowhere dense* if the interior of its closure is empty. A set $B \subset X$ is said to be of *first category* if B can be represented as a countable union of the nowhere dense sets. A set $C \subset X$ is of *second category* if it is not of first category. A subset A of X is called *residual* if it is a complement of a set of first category in X, i.e. A contains a countable intersection of open dense subsets of X. The topological space X is said to be a *Baire space* if every non-empty open set $U \subset X$ is of second category. A Baire space has the following characteristic properties (see Munkres [106]):

(i) Every intersection of countably many dense open sets is dense.

(ii) The countable union of any collection of closed sets with empty interior has empty interior.

As a consequence, in a Baire space a set is residual if and only if it is of second category and dense in X. A property of a function f defined on X is called *generic* if it holds on a residual subset of X. In the following theorem, we state the Baire category theorem and refer to Munkres [106, Theorem 48.2] for a proof of the theorem.

Theorem C.0.9 (Baire category theorem)**.** *Let X be either a complete semimetric space or a locally compact Hausdorff space. Then X is a Baire space.*

Baire Class of Functions:
An effective tool for investigating analytic properties of real-valued functions on topological spaces is the notion of *Baire classes* of functions. Let X be a topological space and $f : X \to \mathbb{R}$ a real-valued function. The function f is said to be of the *first Baire class* if f can be represented as a pointwise limit of continuous functions. The function f is said to be of the *second Baire class* if f can be represented as a pointwise limit of functions

of the first Baire class. Inductively, one defines all other Baire classes of functions. We refer to Goffman [67] for more details on Baire functions. Note that the Baire classes of functions are closed with respect to the operations of taking sums, differences, products and quotients (if the denominator is nowhere vanishing) of their elements.

A function of the first Baire class does not need to be continuous, as simple examples show. For instance, the functions $f_n(x) = \max(0, 1 - n|x|)$ are continuous and the sequence converges pointwise to the discontinuous function $f(x) = 1$ or 0 according to whether $x = 0$ or $x \neq 0$. However, the following theorem shows that a function of the first Baire class cannot be everywhere discontinuous and we refer to Oxtoby [110, Theorem 7.3] for a proof of the theorem.

Theorem C.0.10 (Baire theorem on functions of the first Baire class). *Let $f : X \to \mathbb{R}$ be a real-valued function of the first Baire class. Then f is continuous except on a set of first category.*

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor any other country.

The present thesis was started at the Institute for Mathematic of Johann Wolfgang Goethe University at Frankfurt am Main and finished at the Institute for Analysis of Technical University of Dresden under the supervision of Prof. Dr. Stefan Siegmund.

I accept the rules for obtaining a PhD (Promotionsordnung) of the Faculty of Science at Dresden University of Technology, issued March 20, 2000.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Dissertation wurde am Mathematischen Institut der Johann Wolfgang Goethe Universität in Frankfurt am Main unter der Betreuung von Herrn Prof. Dr. Stefan Siegmund begonnen und am Institut für Analysis der Technischen Universität Dresden, ebenfalls unter der Betreuung von Herrn Prof. Dr. Stefan Siegmund, fertig gestellt.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20. März 2000 an.

Dresden, den

Doan Thai Son