Positive-off-diagonal Operators on Ordered Normed Spaces and Maximum Principles for M-Operators

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Introduction

M-matrices and their generalizations are extensively employed in numerical procedures, and occur as well in the theory of linear operators on vector spaces which are equipped with a partial ordering. A rich theory on M-matrices is developed e. g. in [BNS89] and [BP94]. Certain generalizations of M-matrices are known as "generalized M-matrices" in the finite-dimensional setting and "M-operators" in a general case. The infinite-dimensional theory, however, is not yet as developed as the finite-dimensional one. We extend results, which are well-known for M-matrices or generalized M-matrices, to a more general setting. We study two different notions of an M-operator on an ordered normed space and establish some of their properties.

In the thesis a real ordered normed space $(X, K, \|\cdot\|)$ is considered. The partial ordering $\leq$ induced by the cone K (i. e. $x \leq y$ provided $y - x \in K$ for $x, y \in X$) is in accord with the vector space operations. The norm will be related to the ordering by means of additional topological assumptions on the cone K. We deal with the set $\mathcal{L}(X)$ of all linear continuous operators on X and, in particular, with the set of positive operators in $\mathcal{L}(X)$. There are several options to define an "M-operator" on an ordered normed space. We single out two of them, where the first, classical, one is related to operators that are dominated by a multiple of the identity, and the second, more general, one employs positive-off-diagonal operators. An operator $A \in \mathcal{L}(X)$ is called

- a positive-off-diagonal operator, if for each $x \in K$ and each positive linear continuous functional f with $f(x) = 0$ the inequality $f(Ax) \geq 0$ is valid.

- an M_1-operator, if there is a positive operator $B \in \mathcal{L}(X)$ and a number $s > r(B)$ such that $A = sI - B$. Here $r(B)$ denotes the spectral radius of B.

- an M_2-operator, if A has a positive inverse $A^{-1} \in \mathcal{L}(X)$ and the operator $-A$ is positive-off-diagonal.

Positive-off-diagonal operators are studied in the theory of operator semigroups, e. g. in [CHA$^+$87]. In a finite-dimensional setting, for a positive-off-diagonal operator the terms "cross-positive matrix" or "exponentially non-negative matrix" are used. Some investigations can be found in [SV70], [Tam76], [BNS89] and [GKT95].

Each M_1-operator is an M_2-operator. In our investigation of M-operators (i. e. M_1- and M_2-operators) we focus on two questions:

1. For which types of ordered normed spaces do the both notions coincide ?
2. Which conditions on an M-operator ensure that its (positive) inverse satisfies certain maximum principles ?

Question 1 is motivated by results (which establish that the both notions are equal) known in the case that the underlying space is a Banach lattice, as well as in a finite-dimensional setting, where the cone is finitely generated. The thesis aims to show that the both notions of an M-operator coincide for certain more general classes of ordered normed spaces.

Question 2 is of interest since there are certain results on M-matrices concerning maximum principles, in particular, the maximum principle for inverse column entries, cf. [Sto82], [Sto86], [Win89]. For maximum principles of this type a general approach for ordered normed spaces $(X, K, \|\cdot\|)$ is developed under the assumption that the cone K possesses a non-empty interior. Moreover, the corresponding maximum principles are geometrically characterized and sufficient conditions for M-operators are provided such that their inverses satisfy these maximum principles.

The exposition consists of three chapters.

In Chapter 1 the basic theory on ordered normed spaces is presented, where we mainly refer to [Jam70], [Vul77], [Vul78]. We focus on the link between the ordering and the norm and, in particular, on cones with a non-empty interior. Certain results on vector lattices are listed as well, based on [AB85] and [AB99].

Chapter 2 is dedicated to the study of question 1. We discuss positive-off-diagonal operators in ordered normed spaces and the related class of operators that dominate a multiple of the identity. If these both classes of operators are equal, then the notions M_1-operator and M_2-operator coincide.
It is known that the both classes of operators are equal provided the underlying space is a Banach lattice [Are86]. We provide some techniques to relax the assumption on the underlying space to be a Banach lattice. We use the embedding theory to answer question 1 confirmatively, provided

- X is a dense linear subspace of a Banach lattice, where the induced cone in X has to satisfy additional conditions (Theorem 2.2.7 and Corollary 2.2.8).

Inspired by the technique in the finite-dimensional case with the standard ordering, we show that the both classes of operators are equal, provided

- X is an ordered Banach space with a shrinking positive Schauder basis (Proposition 2.2.10).

Using certain properties of the dual space, in particular, a total set of positive functionals with directed kernel, we state the equivalence of the both operator classes if

- X is an ordered normed space that satisfies the Riesz decomposition property, with additional conditions on the link between the ordering and the norm (Theorem 2.3.11 and Corollary 2.3.12).

We apply our results on positive-off-diagonal operators to list ordered normed spaces where the both notions M_1-operator and M_2-operator coincide (Corollary 2.4.5).

Maximum principles are defined and studied in Chapter 3. The following maximum principle for an $n \times n$-matrix $B = (b_{ij})_{i,j}$, which is developed in [Sto82] in connection with the study of discrete approximations for differential equations, serves as a starting point:

- For any $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n_+ \setminus \{0\}$ with $Bx = y$ there follows $x \geq 0$ and, moreover,
$$\max_{i \in N} x_i = \max_{i \in N_+(y)} x_i \,, \tag{1}$$
where $N = \{1, \ldots, n\}$ and $N_+(y) = \{i \in N \colon\; y_i > 0\}$.

This so-called *maximum principle for inverse column entries* is based on the standard ordering in $\mathbb{R}^n$. It is generalized to the case of arbitrary cones in $\mathbb{R}^n$ in [PSW98]. For a given operator equation, the maximum principle can roughly be described as follows.

> *A positive input causes a positive output, and the maximum response takes place in that part of the system where the influence is non-zero.*

In the thesis we consider maximum principles for positive operators on an ordered normed space $(X, K, \|\cdot\|)$. The method to consider "maximizing positive functionals" on a given element works whenever the cone of all positive functionals in the dual space X' has an appropriate base. Assume that the cone K possesses a non-empty interior. Then for a fixed interior point u of K the set

$$F = \{f \in X' \colon f(K) \subseteq [0, \infty) \;\text{ and }\; f(u) = 1\}$$

is a $\sigma(X', X)$-compact base of the cone in X'. The two maximum principles that we consider are defined as follows: A positive operator $A \in \mathcal{L}(X)$ is said to satisfy the maximum principle

- MP with respect to u, if for each non-zero element x in K there is a functional $f \in F$ which is maximizing on Ax (i. e. $f(Ax)$ is the maximum of all $g(Ax)$, whenever $g \in F$) and strictly positive on x (i. e. $f(x) > 0$);

- SMP with respect to u, if for each non-zero element x in K every functional in F is strictly positive on x, whenever f is maximizing on Ax.

The maximum principle SMP ("strong maximum principle") implies MP. Concerning (1), a regular matrix B satisfies the maximum principle for inverse column entries if and only if B^{-1} is positive and satisfies MP with respect to $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$. In a similar sense, MP covers the maximum principles studied in [TW95] and [PSW98].

For the maximum principle MP in a finite-dimensional setting some geometrical characterizations are known in the case that K is a circular or a finitely generated cone. In the thesis geometrical characterizations of the maximum principles MP and SMP are given in the general case (Theorem 3.3.10). These geometrical characterizations describe how the operator maps the faces of the cone. This provides a convenient tool to show MP or SMP for a given operator.

Concerning question 2, we consider the inverses of M-operators and list sufficient conditions such that they satisfy the introduced maximum principles. The following result on M-matrices is given in [TW95]:

- Let $\mathbb{R}^n$ be equipped with the standard cone $\mathbb{R}^n_+$, let u be an interior point of $\mathbb{R}^n_+$ and B an M-matrix such that $Bu \in \mathbb{R}^n_+$. Then B^{-1} satisfies MP with respect to u.

As a generalization of this result we show:

- If $\mathbb{R}^n$ is equipped with a circular or a finitely generated cone K, u is an interior point of K and B is an M_2-operator with $Bu \in K$, then B^{-1} satisfies MP with respect to u (Theorem 3.5.17, Corollaries 3.5.18 and 3.5.19).

The latter result is not true for arbitrary cones in $\mathbb{R}^n$. We present a counterexample (Example 3.5.11), where we use a cone in $\mathbb{R}^3$ that possesses an extreme ray which is not exposed. If we strengthen the assumption on the operator B to map the interior point u into the interior of the cone K, we get a stronger result, even for the infinite-dimensional case:

- Let $(X, K, \|\cdot\|)$ be an ordered normed space, where K is closed and has a non-empty interior, let u be an interior point of K and let $B \in \mathcal{L}(X)$ be an M_2-operator such that Bu is an interior point of K. Then B^{-1} satisfies SMP with respect to u (Theorem 3.5.5).

For an M_1-operator a different approach is investigated to get a sufficient condition such that the inverse of the operator satisfies a maximum principle. One of the obtained criteria reads as follows (Propositions 3.2.7 and 3.2.9, Theorem 3.5.1):

- If $(X, K, \|\cdot\|)$ is an ordered normed space such that K has a non-empty interior and the norm $\|\cdot\|$ is semi-monotone, then for a fixed interior point u of K the following statements are true:
  (i) There are constants m and M such that for each $x \in K$ the inequalities
  $$m\|x\| \leq \max\{f(x)\colon f \in F\} \leq M\|x\|$$
  are valid.

(ii) Let $B \in \mathcal{L}(X)$ be a positive operator with

$$\|B\| < \frac{m}{M}\,.$$

Then the inverse of the M_1-operator $I - B$ satisfies SMP with respect to u.

A commentary with some conclusions and open problems is included after Chapter 3. Some results of the thesis have been published in [KW00a], [Kal03a] as well as [Kal03b].

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Chapter 1

The Framework

The objective of this chapter is to introduce the basic concepts and results that will be used throughout the text. The definitions and statements are well-known and can be found in almost any textbook on partially ordered vector spaces or vector lattices. They are included for reference and to leave no room for ambiguity. It is not intended to give a survey of the literature or to point out the origin of the results. Detailed references or proofs are included for the readers' convenience, only.
The survey treats partially ordered vector spaces, vector lattices, norms on partially ordered vector spaces and its properties, partially ordered normed spaces where the positive cone has a non-empty interior, the set of positive functionals on a partially ordered vector space, and finally, basic properties of positive operators. For certain mathematical standard notations see the Appendix A.1.

1.1 Partially Ordered Vector Spaces

This section lists the standard terminology on partially ordered vector spaces and vector lattices. We refer mainly to [Vul77], [Vul78] and [AB85]. All vector spaces that appear in this text are assumed to be (non-trivial) vector spaces over the real numbers.

Definition 1.1.1 Let X be a vector space.

(i) A non-empty subset $K \neq \{0\}$ of X is called a *wedge* in X if $x, y \in K$, $\lambda, \mu \in [0, \infty)$ imply $\lambda x + \mu y \in K$. If K is a wedge, then the set $K \cap (-K)$ is called its *blade*. For a subset $M \subseteq X$ the (by inclusion) smallest wedge which contains M (i. e. the positive-linear hull of M) is denoted by $\operatorname{pos} M$, i. e.

$$\operatorname{pos} M = \{x \in X \colon\ x = \sum_{i=1}^{n} \lambda_i x_i \ \text{ with } \ \lambda_i \in [0, \infty),\ x_i \in M \ \text{ for all } \ i = 1, \dots, n\}.$$

(ii) If K is a wedge in X with the additional property $K \cap (-K) = \{0\}$, then K is called a *cone* in X.

(iii) A partial ordering (i. e. a reflexive, transitive and antisymmetric relation) $\leq$ on X is called a *vector space ordering* if

(a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,

(b) $x \in X$, $0 \leq x$, $\lambda \in [0, \infty)$ implies $0 \leq \lambda x$.

$x \leq y$ will also be written as $y \geq x$. Instead of $x \leq y$, $x \neq y$ we write $x < y$ (or $y > x$). An element $x \in X$ is called *positive* (*negative*) if $0 \leq x$ ($x \leq 0$, respectively). For given $a, b \in X$, $a \leq b$, we denote $[a, b] = \{x \in X \colon a \leq x \leq b\}$ and call $[a, b]$ an *order intervall.*

In a vector space the notions of a cone and a vector space ordering are closely related.

Proposition 1.1.2 *Let X be a vector space.*

(i) *Let K be a cone in X and $\leq$ the binary relation on X defined by means of $x \leq y$ if $y - x \in K$. Then $\leq$ is a vector space ordering.*

(ii) *Let $\leq$ be a vector space ordering on X. Then the set $K = \{x \in X \colon 0 \leq x\}$ of all positive elements in X is a cone in X.*

If in a vector space X a cone[1] K is given, then we equip X with the vector space ordering $\leq$ introduced in Proposition 1.1.2 (i) and call (X, K) a *partially ordered vector space.* Occasionally we write loosely X instead of (X, K), provided that K is fixed in advance.

Definition 1.1.3 Let (X, K) be a partially ordered vector space.

(i) A subset $M \subseteq X$ is called *directed* if for any $x, y \in M$ there is $z \in M$ such that $x \leq z$, $y \leq z$.

(ii) A subset $M \subseteq X$ is called *order bounded* if there are $a, b \in X$ such that $M \subseteq [a, b]$.

(iii) The cone[2] K is called *generating* if each $x \in X$ can be represented as $x = y - z$, $y, z \in K$.

(iv) An element $u > 0$ is called a *unit* if for every $x \in X$ there is a $\lambda \in (0, \infty)$ such that $x \in [-\lambda u, \lambda u]$.

(v) A subset D of K is called a *base* of K if D is a (non-empty) convex set such that every element $x \in K$ with $x \neq 0$ has a unique representation $x = \lambda y$ with $y \in D$ and $\lambda \in (0, \infty)$.

[1] If K is a wedge in the vector space X and $\leq$ the binary relation introduced in Proposition 1.1.2 (i), then the relation $\leq$ is reflexive and transitive and satisfies the properties (a) and (b) of Definition 1.1.1 (iii). On the other hand, if X is a vector space and $\leq$ a reflexive and transitive relation on X that satisfies the properties (a) and (b) of Definition 1.1.1 (iii), then the set of all elements that are greater or equal zero is a wedge in X.

[2] Analogously, a wedge H in X is called *generating* if $H - H = X$.

(vi) A *face* of K is a proper non-empty subcone $H \neq \{0\}$ of K such that $x \in H$ and $0 \leq y \leq x$ imply $y \in H$.

(vii) An element $0 < x \in X$ is called an *atom*[3] if the ray $\mathrm{r}(x) = \{\lambda x \colon \lambda \in [0, \infty)\}$ is a face of K.

(viii) X has the *Riesz decomposition property* if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_1 \leq x_1$, $y_2 \leq x_2$.

(ix) X is called a *vector lattice* if $\leq$ is a lattice ordering on X, i. e. for every x, y in X the set $\{x, y\}$ has a least upper bound (*supremum*) and a greatest lower bound (*infimum*), denoted by $x \vee y$ and $x \wedge y$, respectively.

(x) A net $\{x_\alpha\} \subseteq X$ is called *decreasing* (in symbols, $x_\alpha \downarrow$, or, if the index is stressed, $x_\alpha \downarrow_\alpha$), whenever $\alpha \geq \beta$ implies $x_\alpha \leq x_\beta$, and *increasing* if the net $\{-x_\alpha\}$ is decreasing. For $x \in X$ the notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ both hold, i. e. x is the greatest lower bound of the set $\{x_\alpha\}$ in X. The meanings of $x_\alpha \uparrow$ and $x_\alpha \uparrow x$ are analogous.

(xi) X is called *Archimedean* if for every $x, y \in X$ such that $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$.

(xii) X is called *Dedekind complete*, if every directed set that is bounded from above has a supremum, and *σ-Dedekind complete* if every increasing sequence that is bounded from above has a supremum.

In (X, K) the cone K is generating if and only if X is directed. The existence of a unit u in X implies that K is generating, since for every $x \in X$ there is $\lambda \in (0, \infty)$ such that $\lambda u - x \in K$, moreover $\lambda u \in K$ and $x = \lambda u - (\lambda u - x)$.

A set $H \subset K$, $H \neq \{0\}$, is a face of K if and only if it is an extreme subset of K. For properties of extreme subsets of a convex set see the Appendix A.2. If x is an atom, then each element λx with $\lambda \in (0, \infty)$ is an atom. Let K have a base D with extreme points. It is straightforward that every element of the set $\operatorname{ext} D$ of extreme points of D is an atom of K. On the other hand, let x be an atom and let $y \in D$ be the unique element such that $x = \lambda y$ for some $\lambda \in (0, \infty)$, then $y \in \operatorname{ext} D$.

X is Archimedean if and only if for every $x \in K$ one has that $\inf\{\frac{1}{n}x \colon n \in \mathbb{N}\}$ exists and equals zero[4] [Vul77, Theorem I.3.1]. Clearly, if X is Dedekind complete, then X is σ-Dedekind complete. Moreover, if X is σ-Dedekind complete, then X is Archimedean [Vul77, Theorem I.3.2].

X possesses the Riesz decomposition property if and only if for any $v_1, v_2 \in K$ one has

$$[0, v_1 + v_2] = [0, v_1] + [0, v_2]\,. \tag{1.1}$$

[3] In the literature also the notions *discrete element* [AB85, p. 105] or *extremal element* [KLS89, p. 25] are used.

[4] Cf. [AB99, Definition 7.2] or [AB85, p. 7].

Another characterization of the Riesz decomposition property is the so-called *Riesz separation property*[5], which is stated next.

Proposition 1.1.4 [Vul77, Lemma 1, V.1] *A partially ordered vector space* (X, K) *has the Riesz decomposition property if and only if for any four elements* $a_1, a_2, b_1, b_2 \in X$ *that satisfy the inequalities* $a_i \leq b_j$ $(i, j = 1, 2)$ *there is an element* $c \in X$ *such that*

$$a_i \leq c \leq b_j \quad (i, j = 1, 2)\,.$$

Every vector lattice has the Riesz decomposition property. On the other hand, if X is a Dedekind complete partially ordered vector space with a generating cone K and has the Riesz decomposition property, then (X, K) is a Dedekind complete vector lattice [Vul77, Theorem V.2.1].

Definition 1.1.5 Let (X, K) be a vector lattice.

(i) For an element $x \in X$, the element $x^+ = x \vee 0$ is called the *positive part* of x, $x^- = (-x) \vee 0$ the *negative part* and $|x| = x \vee (-x)$ the *modulus* of x.

(ii) The elements x and y of X are called *disjoint*, in symbols $x \perp y$, if $|x| \wedge |y| = 0$. The *disjoint complement* of a non-empty set $M \subseteq X$ is the set $M^d = \{x \in X\colon\ x \perp y \text{ for all } y \in M\}$.

(iii) A set $M \subseteq X$ is called *solid* if $y \in X$, $x \in M$, $|y| \leq |x|$ imply $y \in M$. A solid vector subspace of a vector lattice is called an *ideal*. The *ideal generated by a subset* M of X is the (with respect to inclusion) smallest ideal that contains M. For an element $x \in X$ the ideal generated by $\{x\}$ is called a *principal ideal* and will be denoted by I_x.

(iv) A net $\{x_\alpha\}_{\alpha \in A}$ in X is *order convergent* to some x, written $x_\alpha \xrightarrow{o} x$, if there exists a net $\{y_\alpha\}_{\alpha \in A}$ in X satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for each $\alpha \in A$. A set $M \subseteq X$ is called *order closed* if for every net $\{x_\alpha\}_{\alpha \in A}$ in M and $x \in X$ from $x_\alpha \xrightarrow{o} x$ follows $x \in M$.

(v) An order closed ideal is called a *band*. The *band generated by a subset* M of X is (with respect to inclusion) the smallest band that contains M. For an element $x \in X$ the band generated by $\{x\}$ is called a *principal band* and will be denoted by B_x.

(vi) A band B in X is called a *projection band* if $X = B \oplus B^d$, i. e. every $x \in X$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. By

$$P_B\colon X \to B\,, \quad x \mapsto x_1\,,$$

a projection operator is defined, which is called a *band projection*. X is said to have the *principal projection property* if for each $x \in X$ the according principal band B_x is a projection band.

[5] Cf. [Wic95].

For an element x of a vector lattice X one has $|x| = x^+ + x^-$, $x^+ \wedge x^- = 0$ and

$$x = x^+ - x^-. \tag{1.2}$$

If $x = y - z$ with $y, z \in K$, then $x^+ \leq y$, $x^- \leq z$. Moreover, the decomposition in (1.2) is unique in the sense that if $x = y - z$ with $y, z \in K$ and $y \wedge z = 0$, then $y = x^+$ and $z = x^-$. The principal ideal of x is given by

$$I_x = \{\, y \in X\colon \text{ there is } s > 0 \text{ such that } |y| \leq s|x| \,\}, \tag{1.3}$$

whereas the according principal band equals the set

$$B_x = \{\, y \in X\colon \ |y| \wedge n|x| \uparrow_n |y| \,\} \tag{1.4}$$

[AB85, Theorem 3.4]. The following is straightforward:

$$\text{For } \ x, y \geq 0 \ \text{ with } \ x \perp y \ \text{ one has } \ [0, y] \cap I_x = \{0\}\,. \tag{1.5}$$

Observe for elements $x, y, z \in X$ the subsequent inequalities [AB99, Corollary 7.8]:

$$|x \wedge y - z \wedge y| \leq |x - z|\,, \tag{1.6}$$

$$\big||x| - |y|\big| \leq |x - y|\,. \tag{1.7}$$

A vector lattice X is Archimedean if and only if for every $x, y \in X$ with $0 \leq nx \leq y$ for all $n \in \mathbb{N}$ one has $x = 0$, cf. [AB99, Definition 7.2]. Indeed, let X be Archimedean and let $x, y \in X$ be such that $0 \leq nx \leq y$ for all $n \in \mathbb{N}$. Then $0 \leq x \leq \frac{1}{n}y$ for all $n \in \mathbb{N}$, and, since $y \in K$, one has $x \leq \inf\{\frac{1}{n}y\colon\ n \in \mathbb{N}\} = 0$, so $x = 0$. Vice versa, let $x, y \in X$ be such that $nx \leq y$ for all $n \in \mathbb{N}$. Then

$$0 \leq nx^+ = (nx) \vee 0 \leq y \vee 0 = y^+$$

for all $n \in \mathbb{N}$, so $x^+ = 0$, and, consequently, $x \leq 0$.

A vector lattice X is Dedekind complete if and only if every nonempty subset that is order bounded from above has a supremum [Vul77, Theorem I.5.1].

In an Archimedean vector lattice every band B satisfies $B = B^{dd}$, and the band generated by a set $M \subseteq X$ is the set M^{dd} [AB99, Theorem 7.18].

Proposition 1.1.6 [AB85, Consequence of Theorem 3.13] *Any σ-Dedekind complete vector lattice has the principal projection property.*

Proposition 1.1.7 *Let (X, K) be an Archimedean vector lattice and $0 < x \in X$. The element x is an atom if and only if $B_x = \{\lambda x\colon\ \lambda \in \mathbb{R}\}$.*

Proof Let $0 < x \in X$ be an atom and let $y \in B_x$. Then by (1.4) one has

$$|y| = \sup\{\,|y| \wedge n|x|\colon\ n \in \mathbb{N}\,\}\,.$$

Since x is an atom, the relations $0 \le |y| \wedge n|x| \le n|x| = nx$ imply the existence of a number $\alpha_n \in \mathbb{R}$ for any $n \in \mathbb{N}$ such that $|y| \wedge n|x| = \alpha_n x$. We show that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is bounded. If the contrary is assumed, then for each $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $mx \le \alpha_n x = |y| \wedge n|x| \le |y|$. Since X is Archimedean, we conclude $x \le 0$, which is a contradiction. If C denotes an upper bound of $(\alpha_n)_{n\in\mathbb{N}}$, then

$$|y| = \sup\{\,|y| \wedge n|x|\colon\ n \in \mathbb{N}\,\} = \sup\{\,\alpha_n x\colon\ n \in \mathbb{N}\,\} \le Cx\,.$$

Since x is an atom, we get $|y| = \alpha x$ for some $\alpha \ge 0$. Finally, y^+ and y^- are multiples of x as well because of $0 \le y^+, y^- \le |y| = \alpha x$. Hence $y = y^+ - y^- = \lambda x$ for some $\lambda \in \mathbb{R}$. Vice versa, let $x \in K$ and $B_x = \{\lambda x\colon\ \lambda \in \mathbb{R}\}$. Obviously, since B_x is solid, $0 \le y \le x$ implies $y \in B_x$, hence $y = \lambda x$. $\square$

We list some properties of linear operators between partially ordered vector spaces, which we need later on.

Definition 1.1.8 Let (X_1, K_1) and (X_2, K_2) be partially ordered vector spaces and $S, T\colon X_1 \to X_2$ linear operators.

(i) S is called *positive* if $S(K_1) \subseteq K_2$. We write $S \le T$ if $T - S$ is positive.

Let, in addition, (X_1, K_1) and (X_2, K_2) be vector lattices.

(ii) S is called a *lattice homomorphism* if for all $x, y \in X_1$ one has

$$S(x \vee y) = (Sx) \vee (Sy)\,.$$

The set of all positive linear operators is a wedge in the vector space of all linear operators. Moreover, if K_1 is a generating cone in X_1, then the set of all positive linear operators is a cone in the set of all linear operators, so the space of all linear operators becomes a partially ordered vector space.

If $X_1 = X_2 = X$, then, as usual, $I\colon X \to X$ denotes the identity operator. If B is a projection band in a vector lattice X, then for the projection P_B one has the relations

$$0 \le P_B \le I\,.$$

The set B^d is a projection band as well, where

$$P_{B^d} = I - P_B\,.$$

We consider now the special case $(X_2, K_2) = (\mathbb{R}, \mathbb{R}_+)$, where the partially ordered vector space (X_1, K_1) is written as (X, K). Denote by X^* the algebraic dual of X, i. e. the set of all linear functionals on X, and let

$$K^* = \{f \in X^*\colon f(K) \subseteq [0, \infty)\} \tag{1.8}$$

be the set of all positive linear functionals on X. K^* is a wedge in the vector space X^*.

Definition 1.1.9 [KLS89, p. 102] A non-empty subset $M \subseteq K^*$ is called *total* if $x \in X$ and $f(x) \geq 0$ for every $f \in M$ imply $x \in K$.

The following statement is the HAYES Theorem [Jam70, Theorem 1.8.1] (cf. also [KLS89, Theorem 3.3]) applied to vector lattices.

Proposition 1.1.10 *Let (X, K) be a vector lattice and $0 < f \in X^*$. The functional f is an atom of K^* if and only if f is a lattice homomorphism.*

Finally, we state an extension result for a map which is additive on the cone.

Proposition 1.1.11 [AB85, Theorem 1.7] *Let (X, K) be a vector lattice and let*

$$f_0 \colon K \to [0, \infty)$$

be an additive map (i. e. $f_0(x + y) = f_0(x) + f_0(y)$ holds for all $x, y \in K$). Then f_0 extends uniquely to a positive linear functional f on X, where $f(x) = f_0(x^+) - f_0(x^-)$ for every $x \in X$.

1.2 Ordered Normed Spaces

In this section we consider a partially ordered vector space (X, K) that is equipped with a norm $\|\cdot\|$, where we simply write $(X, K, \|\cdot\|)$. Instead of using the terminology *partially ordered normed space* we write shortly *ordered normed space*. The closure of a subset $M \subseteq X$ in the norm topology is denoted by $\overline{M}$. An interesting theory of ordered normed spaces requires certain relations between the ordering and the norm.

Definition 1.2.1 Let $(X, K, \|\cdot\|)$ be an ordered normed space.

(i) K is called *weakly generating* if $X = \overline{K - K}$.

(ii) The norm $\|\cdot\|$ is called *semi-monotone* if there is a constant N (the *constant of semi-monotony*) such that for every $x, y \in X$ with $0 \leq x \leq y$ one has that $\|x\| \leq N\|y\|$. If $\|\cdot\|$ is semi-monotone, then the cone K is called *normal*.

(iii) K is called *non-flat*[6] if there is a constant (the *constant of non-flatness*) $\kappa > 0$ such that each $x \in X$ possesses a representation $x = y - z$ with $y, z \in K$ and $\|y\|, \|z\| \leq \kappa\|x\|$.

An important property is the closedness of the cone K.

Proposition 1.2.2 [Vul77, Theorems II.3.2, II.1.3 and III.2.1] *Let $(X, K, \|\cdot\|)$ be an ordered normed space.*

[6]The definition is stated according to [KLS89, Section 1.8]. In the related English literature such a cone is said to *give an open decomposition*, see e. g. [Jam70, Section 3.3].

(i) *If K is closed, then X is Archimedean.*

(ii) *If X is Archimedean and K has an interior point, then K is closed.*

(iii) [KREIN-ŠMULJAN] *If X is, in addition, a Banach space and K is closed and generating, then K is non-flat.*

Proposition 1.2.3 *Let (X, K) be a vector lattice and $\|\cdot\|$ a norm on X. K is normal and non-flat if and only if there is a constant C such that for all $x, y \in X$ with $|x| \leq |y|$ one has $\|x\| \leq C\|y\|$.*

Proof Let K be normal and non-flat, where the according constants of semi-monotony and non-flatness are N and κ, respectively. Note that for $x \in X$ one has $x^+, x^- \leq |x|$, so $\|x^+\|, \|x^-\| \leq N\| \, |x| \, \|$. Moreover,

$$\|x\| = \|x^+ - x^-\| \leq \|x^+\| + \|x^-\| \leq 2N\| \, |x| \, \| \, .$$

On the other hand, there are $y, z \in K$ such that $x = y - z$ and $\|y\|, \|z\| \leq \kappa\|x\|$. Due to $x^+ = x \vee 0 \leq y$ one has

$$\|x^+\| \leq N\|y\| \leq N\kappa\|x\| \quad \text{and, analogously,} \quad \|x^-\| \leq N\kappa\|x\| \, . \tag{1.9}$$

So,

$$\| \, |x| \, \| = \|x^+ + x^-\| \leq \|x^+\| + \|x^-\| \leq 2N\kappa\|x\| \, .$$

Now let $x, y \in X$ with $|x| \leq |y|$, i. e. $\| \, |x| \, \| \leq N\| \, |y| \, \|$. Then

$$\|x\| \leq 2N\| \, |x| \, \| \leq 2N^2\| \, |y| \, \| \leq 4N^3\kappa\|y\| \, ,$$

hence $C = 4N^3\kappa$.
Vice versa, assume that there is a constant C such that for all $x, y \in X$ with $|x| \leq |y|$ one has $\|x\| \leq C\|y\|$. Clearly, K is normal, where as a constant of semi-monotony $N = C$ can be taken. For $x \in X$ one has $x^+, x^- \leq |x|$, so $\|x^+\|, \|x^-\| \leq C\|x\|$, hence K is non-flat, where the constant of non-flatness is $\kappa = C$. □

By means of this proposition we show the following result.

Lemma 1.2.4 *Let (X, K) be a vector lattice and $\|\cdot\|$ a norm on X such that K is a normal and non-flat cone. Let $u > 0$. An element $x > 0$ is contained in the norm closure of the principal ideal I_u if and only if*

$$\lim_{n\to\infty} \|x - nu \wedge x\| = 0 \, .$$

Moreover, $\overline{I_u} \subseteq B_u$.

Proof Let $x > 0$ be contained in $\overline{I_u}$, i. e. there is a sequence $(x_k)_{k\in\mathbb{N}} \subset I_u$ such that $\lim_{k\to\infty} \|x - x_k\| = 0$. Due to (1.3) for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $|x_k| \leq nu$ for every $n \geq n_k$, i. e. $|x_k| = |x_k| \wedge nu$. The inequalities (1.6) and (1.7) imply

$$\begin{aligned} \big||x_k| - nu \wedge x\big| &= \big||x_k| \wedge nu - x \wedge nu\big| \\ &\leq \big||x_k| - x\big| \\ &\leq |x_k - x| \,. \end{aligned}$$

Due to Proposition 1.2.3 there is a constant C such that $\|\,|x_k| - nu \wedge x\| \leq C\|x_k - x\|$ and $\|\,|x_k| - x\| \leq C\|x_k - x\|$. Hence,

$$\begin{aligned} \|x - nu \wedge x\| &= \|x - |x_k| + |x_k| - nu \wedge x\| \\ &\leq \|x - |x_k|\,\| + \|\,|x_k| - nu \wedge x\| \\ &\leq 2C\|x_k - x\| \,. \end{aligned}$$

Summing up, for each $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that $\|x_{k_0} - x\| \leq \frac{\varepsilon}{2C}$ and there is $n_{k_0} \in \mathbb{N}$ such that for all $n \geq n_{k_0}$ one has $\|x - nu \wedge x\| \leq 2C\|x_{k_0} - x\| \leq \varepsilon$, which implies $\lim_{n\to\infty} \|x - nu \wedge x\| = 0$.
Vice versa, let $x > 0$ be such that $\lim_{n\to\infty} \|x - nu \wedge x\| = 0$. Due to $0 \leq nu \wedge x \leq nu$ one has $nu \wedge x \in I_u$ for every $n \in \mathbb{N}$, so $x \in \overline{I_u}$.
Finally, we justify $\overline{I_u} \subseteq B_u$. Let $x \in \overline{I_u}$ and y be an upper bound of the set $\{nu \wedge x \colon n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$ one has $nu \wedge x \leq x \wedge y$, so $\|x \wedge y - nu \wedge x\| \leq C\|x - nu \wedge x\|$, which implies $\lim_{n\to\infty} \|x \wedge y - nu \wedge x\| = 0$. Since the limit of the sequence $(nu \wedge x)_{n\in\mathbb{N}}$ is unique, one gets $x = x \wedge y$, so $x \leq y$. This means $x = \sup\{nu \wedge x \colon n \in \mathbb{N}\}$, so $x \in B_u$ due to (1.4). $\square$

Definition 1.2.5 Let (X, K) be a vector lattice. A norm $\|\cdot\|$ on X is called a *lattice norm*, if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for arbitrary $x, y \in X$. If $\|\cdot\|$ is a lattice norm on X, the space $(X, K, \|\cdot\|)$ is called a *normed vector lattice*. If, in addition, X is norm complete, then X is called a *Banach lattice*.

Let (X, K) be a vector lattice and $\|\cdot\|$ a norm on X. If there is a constant C such that for all $x, y \in X$ with $|x| \leq |y|$ one has $\|x\| \leq C\|y\|$, it is straightforward that

$$\|x\|_l = \sup\{\|y\| \colon\; 0 \leq y \leq |x|\} \tag{1.10}$$

defines a lattice norm on X which is equivalent to the norm $\|\cdot\|$.

We collect some properties of a lattice norm. In the approximation theory the notion of a proximinal set in a normed space X is widely used. For a non-empty subset $M \subset X$ and an element $x \in X$ denote

$$\operatorname{dist}(x, M) = \inf\{\, \|x - v\| \colon\; v \in M\} \,.$$

Definition 1.2.6 Let $(X, \|\cdot\|)$ be a normed space. A non-empty subset $M \subset X$ is called *proximinal* if for each $x \in X$ there is an element $y \in M$ such that $\|x - y\| = \operatorname{dist}(x, M)$.

If $M \subset X$ is proximinal, then M is closed[7]. Indeed, x is in the norm closure of M if and only if $\operatorname{dist}(x, M) = 0$. Since there is an element $y \in M$ with $\|x - y\| = \operatorname{dist}(x, M) = 0$, one gets $x = y \in M$.

Proposition 1.2.7 *If $(X, K, \|\cdot\|)$ is a normed vector lattice, then*

(i) *K is normal and non-flat, where the constants of semi-monotony and of non-flatness are both equal to 1;*

(ii) *K is proximinal and, in particular, closed.*

Proof (i) This is an immediate consequence of Proposition 1.2.3 (second part of the proof).
(ii) Let $x \in X$ be a fixed element, then for every $y \in K$ we have $x \geq x - y$, hence $x^- \leq (x - y)^- \leq |x - y|$ and $\|x - x^+\| = \|x^-\| \leq \|x - y\|$. Consequently,

$$\operatorname{dist}(x, K) = \|x - x^+\| .$$

Hence K is proximinal and therefore closed. □

Remark 1.2.8 If (X, K) is a vector lattice equipped with a norm $\|\cdot\|$ such that K is normal and non-flat, then due to Proposition 1.2.3 we get a lattice norm $\|\cdot\|_l$ on X according to (1.10). Due to Proposition 1.2.7 (ii) K is closed with respect to the norm $\|\cdot\|_l$, and since the norms $\|\cdot\|$ and $\|\cdot\|_l$ are equivalent, K is also closed with respect to the norm $\|\cdot\|$.

If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, then the set of all continuous linear operators $A\colon X_1 \to X_2$ is denoted by $\mathcal{L}(X_1, X_2)$. In the vector space $\mathcal{L}(X_1, X_2)$ for an element $A \in \mathcal{L}(X_1, X_2)$ the usual *operator norm* is defined by

$$\|A\| = \sup_{\|x\|_1 = 1} \|Ax\|_2 .$$

If $(X_1, K_1, \|\cdot\|_1)$ and $(X_2, K_2, \|\cdot\|_2)$ are ordered normed spaces, then the set of all positive operators in $\mathcal{L}(X_1, X_2)$ is denoted by $\mathcal{L}_+(X_1, X_2)$, which is a wedge in $\mathcal{L}(X_1, X_2)$. We list some properties of the set $\mathcal{L}_+(X_1, X_2)$.

Proposition 1.2.9 *Let $(X_1, K_1, \|\cdot\|_1)$ and $(X_2, K_2, \|\cdot\|_2)$ be ordered normed spaces.*

(i) *If K_1 is weakly generating, then $\mathcal{L}_+(X_1, X_2)$ is a cone in $\mathcal{L}(X_1, X_2)$.*

(ii) *If K_2 is closed, then $\mathcal{L}_+(X_1, X_2)$ is closed in $\mathcal{L}(X_1, X_2)$.*

[7] If $M \subset X$ is convex, closed and weakly locally compact, then M is proximinal [Köt66, 26.2.(1)]. In particular, if $(X, \|\cdot\|)$ is a reflexive Banach space and $M \subset X$ is convex and closed, then M is proximinal [Köt66, 26.2.(2)].

Proof (i) Let K_1 be weakly generating and $T, -T \in \mathcal{L}_+(X_1, X_2)$. For $x \in K_1$ one has $Tx, -Tx \in K_2$, so $Tx = 0$. Therefore $T(K_1 - K_1) = \{0\}$, and, since T is continuous, $T(X) = T(\overline{K_1 - K_1}) = \{0\}$, so $T = 0$.
(ii) Let K_2 be closed and consider a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}_+(X_1, X_2)$ such that

$$\lim_{n\to\infty} \|A_n - A\| = 0$$

for some $A \in \mathcal{L}(X_1, X_2)$. For every $x \in X_1$ one has

$$\lim_{n\to\infty} \|A_n x - Ax\| \le \|x\| \lim_{n\to\infty} \|A_n - A\| = 0\,.$$

Moreover, if $x \in K_1$, then $A_n x \in K_2$ and $Ax \in K_2$ since K_2 is closed. So,

$$A \in \mathcal{L}_+(X_1, X_2)\,. \ \square$$

For a normed space $(X, \|\cdot\|)$ abbreviate $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$. For $A \in \mathcal{L}(X)$ there exists the limit

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = r(A)\,,$$

the *spectral radius* of A, where $r(A) \le \|A\|$.
As usual, X' denotes the *dual space*, i. e. the vector space $\mathcal{L}(X, \mathbb{R})$ of all continuous linear functionals on X. X' equipped with the operator norm[8] becomes a Banach space. Moreover, on X and X' the weak topology $\sigma(X, X')$ and the weak* topology $\sigma(X', X)$ are considered, respectively. (X, X') is a bilinear system with

$$\langle x, f\rangle = f(x)$$

for $x \in X$, $f \in X'$. For an operator $A \in \mathcal{L}(X)$ via

$$\langle Ax, f\rangle = \langle x, A'f\rangle \tag{1.11}$$

the *dual operator* $A'\colon X' \to X'$ is defined. Then $A' \in \mathcal{L}(X')$ and $\|A'\| = \|A\|$.

If $(X, K, \|\cdot\|)$ is an ordered normed space, then $\mathcal{L}_+(X, X)$ is abbreviated by $\mathcal{L}_+(X)$. According to (1.8), denote

$$K' = K^* \cap X'\,.$$

K' is called the *dual wedge*. The statements (i) and (ii) in the following proposition are immediate consequences of Proposition 1.2.9.

Proposition 1.2.10 *Let $(X, K, \|\cdot\|)$ be an ordered normed space.*

(i) *If K is weakly generating, then K' is a cone.*

(ii) *K' is closed in X'.*

[8] We use the notation $\|\cdot\|$ for both the norm in X and in X', which norm is ment follows from the respective context.

(iii) [Vul77, Theorem IV.5.1, Krein] *K' is generating if and only if K is normal.*

If K' is a cone in X', then K' is called the *dual cone*, and (X', K') becomes a partially ordered vector space. Combining the statements (i) and (ii) in Proposition 1.2.10 with Proposition 1.2.2 (i), for an ordered normed space with a weakly generating cone K the dual space $(X', K', \|\cdot\|)$ is an Archimedean ordered Banach space. If K is, in addition, normal, then K' is non-flat due to Proposition 1.2.2 (iii).

Proposition 1.2.11 *Let $(X, K, \|\cdot\|)$ be an ordered normed space. If K is non-flat with the constant κ, then the dual cone K' is normal, where a constant of semi-monotony of the dual norm is 2κ.*

Proof Let $f, g \in K'$ such that $g \leq f$. Let $x \in X$ with $\|x\| = 1$. There are $y, z \in K$ such that $x = y - z$ and $\|y\|, \|z\| \leq \kappa \|x\|$. So,

$$\begin{aligned} |g(x)| &= |g(y) - g(z)| \\ &\leq g(y) + g(z) \\ &\leq f(y) + f(z) \\ &\leq \|f\|(\|y\| + \|z\|) \\ &\leq \|f\| 2\kappa \|x\| \,. \end{aligned}$$

Hence, $\|g\| \leq 2\kappa \|f\|$. $\square$

We sum up the previous statements.

Corollary 1.2.12 *Let $(X, K, \|\cdot\|)$ be an ordered normed space with a normal and non-flat cone K. Then K' is a normal and non-flat cone in X'.*

Next we address the issue under which assumption the dual cone becomes a total set, i. e. $x \in X$ and $f(x) \geq 0$ for each $f \in K'$ imply $x \in K$.

Proposition 1.2.13 [Vul77, Section II.4] *If K is closed, then for each $x \in X \setminus K$ there is a functional $f \in K'$ such that $f(x) < 0$.*

If $x > 0$, then $-x \in X \setminus K$, so the following is obvious.

Corollary 1.2.14 *If K is closed, then for any $x > 0$ there is a functional $f \in K'$ such that $f(x) > 0$.*

Corollary 1.2.15 *If K is closed, then K' is a total set.*

Moreover, if K is closed and K' has a base F, then F is a total set.

In an ordered normed space $(X, K, \|\cdot\|)$ certain faces of K are specified.

Definition 1.2.16 A non-empty subset $H \neq \{0\}$ of K is called *exposed* if there is a functional $f \in K'$ such that $H = f^{-1}(0) \cap K$. An exposed subset of K which is a ray is called an *exposed ray*.

Every exposed subset of K is a face of K, but not vice versa (cf. Example 3.5.11 below).

Now conditions on an ordered normed space are stated which ensure that the dual space is a vector lattice.

Theorem 1.2.17 [Vul77, Theorem V.3.1, RIESZ-KANTOROVIČ] *If an ordered normed space $(X, K, \|\cdot\|)$ with a non-flat and normal cone K satisfies the Riesz decomposition property, then (X', K') is a Dedekind complete vector lattice, where the lattice operations for $x \in K$ and $f, g \in X'$ are given by*

$$\begin{aligned} (f \wedge g)(x) &= \inf\{f(y) + g(x-y)\colon\ y \in [0,x]\} \\ \text{and}\quad (f \vee g)(x) &= \sup\{f(y) + g(x-y)\colon\ y \in [0,x]\}\,. \end{aligned}$$

Clearly, under the assumptions of this theorem for each $f \in X'$ one has $|f| \geq f$, $|f| \geq -f$, and so for $x \in K$

$$|f|(x) \geq f(x) \vee (-f(x)) = |f(x)|\,. \tag{1.12}$$

Some inverse result to Theorem 1.2.17 is stated next.

Theorem 1.2.18 [Vul77, Theorem V.4.1] *Let $(X, K, \|\cdot\|)$ be an ordered Banach space, where K is closed and generating. If (X', K') is a vector lattice, then (X, K) satisfies the Riesz decomposition property and K is normal*[9].

Corresponding results can also be found in [Wic74].

The next statement contains some explicit estimate for the constant of non-flatness of K', provided the necessary constants of K are given.

Proposition 1.2.19 *Let $(X, K, \|\cdot\|)$ be an ordered normed space which satisfies the assumptions of Theorem 1.2.17, where N and κ are the constants of semi-monotony and of non-flatness of K, respectively. Then as a constant of non-flatness of K' one can take $2N\kappa$.*

Proof Let $f \in X'$ and $x \in X$ with $\|x\| = 1$. Since K is non-flat, there are elements $y, z \in K$ such that $x = y - z$, where $\|y\|, \|z\| \leq \kappa$. So,

$$\begin{aligned} |f^+(x)| &= |f^+(y) - f^+(z)| \\ &\leq f^+(y) + f^+(z) \\ &= \sup\{f(v)\colon v \in [0,y]\} + \sup\{f(w)\colon w \in [0,z]\} \\ &\leq \sup\{\|f\|\,\|v\|\colon v \in [0,y]\} + \sup\{\|f\|\|w\|\colon w \in [0,z]\} \\ &\leq \|f\|N\|y\| + \|f\|N\|z\| \\ &\leq \|f\|N2\kappa\,. \end{aligned}$$

Hence, $\|f^+\| \leq 2N\kappa\|f\|$. Analogously, $\|f^-\| \leq 2N\kappa\|f\|$. □

A similar technique can be applied for band projections in normed vector lattices.

[9] The non-flatness of K follows by the KREIN-ŠMULJAN theorem, see Proposition 1.2.2 (iii).

Proposition 1.2.20 *Let (X, K) be a vector lattice equipped with a norm $\|\cdot\|$ such that K is normal and non-flat, where the constant of semi-monotony is N and the constant of non-flatness is κ. If B is a projection band in X, then for the projection $P_B\colon X \to B$ one has $\|P_B\| \leq 2N\kappa$.*

Proof Let $x \in X$ with $\|x\| = 1$, then there are elements $y, z \in K$ with $x = y - z$ and $\|y\|, \|z\| \leq \kappa$. So,

$$\|P_B(x)\| = \|P_B(y) - P_B(z)\| \leq \|P_B(y)\| + \|P_B(z)\|.$$

Due to $0 \leq P_B \leq I$ one has $0 \leq P_B(y) \leq y$, hence

$$\|P_B(y)\| \leq N\|y\| \leq N\kappa$$

and, analogously, $\|P_B(z)\| \leq N\kappa$. So, $\|P_B(x)\| \leq 2N\kappa$, which implies $\|P_B\| \leq 2N\kappa$. $\square$

We recall some properties of the dual operator.

Proposition 1.2.21 *Let $(X, \|\cdot\|, K)$ be an ordered normed space and $A \in \mathcal{L}(X)$. If A is positive, then A' is positive. The converse is true provided K is closed.*

Proof If $A \geq 0$, then for each $f \in K'$ and $x \in K$ one has $(A'f)(x) = f(Ax) \geq 0$, so $A'f \in K'$. Hence, $A' \geq 0$.
On the other hand, let $A' \geq 0$ and $x \in K$. For any $g \in K'$ one has $g(Ax) = (A'g)(x) \geq 0$. Due to Corollary 1.2.15 the set K' is total, so $Ax \in K$. Hence, $A \geq 0$. $\square$

At some places below we deal with operator semigroups, where, as usual, for an operator $A \in \mathcal{L}(X)$ the operator e^{tA} defined as

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \quad \text{for} \quad t \geq 0$$

belongs to $\mathcal{L}(X)$. Here the series converges with respect to the operator norm in $\mathcal{L}(X)$, and $A^0 = I$.

Proposition 1.2.22 [CHA$^+$87, Theorem 3.16,(iv)-(vi)] *Let $(X, \|\cdot\|)$ be a Banach space and $A \in \mathcal{L}(X)$. Then $(e^{tA})' = e^{tA'}$.*

If $(Y, \|\cdot\|)$ is a normed space, X a (norm) dense linear subspace of Y and $(Z, \|\cdot\|_Z)$ a Banach space, then the spaces $\mathcal{L}(Y, Z)$ and $\mathcal{L}(X, Z)$ are isomorphic. If $\hat{A} \in \mathcal{L}(Y, Z)$, then $A = \hat{A}|_X \in \mathcal{L}(X, Z)$. Vice versa, if $A \in \mathcal{L}(X, Z)$ and $y \in Y$, then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ which converges to y, so $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since A is continuous, $(A(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Z, i. e. the element $\lim_{n\to\infty} A(x_n)$ is defined in Z. For each sequence in X that converges to y one gets the same limit in Z. Therefore, by

$$\hat{A}(y) = \lim_{n\to\infty} A(x_n) \tag{1.13}$$

an operator $\hat{A}$ is well-defined on Y, and $\hat{A}|_X = A$. It is straightforward that $\hat{A}$ is continuous and $\|\hat{A}\| = \|A\|$.
If K_Y is a cone in Y, K_X a cone in X such that $K_X \subseteq K_Y$, and K_Z a cone in Z, then

$$\hat{A} \in \mathcal{L}_+(Y,Z) \text{ implies } A \in \mathcal{L}_+(X,Z)\,. \tag{1.14}$$

If, in addition, $K_Y \subseteq \overline{K_X}$ and K_Z is closed, then the converse implication in (1.14) is true as well. Indeed, for $A \in \mathcal{L}_+(X,Z)$ and $y \in K_Y$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ in K_X which converges to y. One has $Ax_n \in K_Z$ for all n, so $\hat{A}(y) \in K_Z$, since K_Z is closed. Summing up, we have the following:

Proposition 1.2.23 *Let $(Y, K_Y, \|\cdot\|)$ be an ordered normed space, X a dense linear subspace of Y and K_X a cone in X such that $K_X \subseteq K_Y \subseteq \overline{K_X}$. Let $(Z, K_Z, \|\cdot\|_Z)$ be an ordered Banach space with a closed cone K_Z. Then*

$$\hat{A} \in \mathcal{L}_+(Y,Z) \text{ if and only if } A \in \mathcal{L}_+(X,Z)\,.$$

If $Z = \mathbb{R}$, then one obtains the statement (i) in the subsequent corollary. For the statement (ii) put $(Z, K_Z, \|\cdot\|_Z) = (Y, K_Y, \|\cdot\|)$.

Corollary 1.2.24 (i) *Let $(Y, K_Y, \|\cdot\|)$ be an ordered normed space, X a dense linear subspace of Y and K_X a cone in X such that $K_X \subseteq K_Y \subseteq \overline{K_X}$. Then Y' and X' are isomorphic, and one can identify $(K_Y)'$ and $(K_X)'$.*

(ii) *Let $(Y, K_Y, \|\cdot\|)$ be an ordered Banach space with a closed cone K_Y, let X be a dense linear subspace of Y and K_X a cone in X such that $\overline{K_X} = K_Y$. Then $\mathcal{L}(Y)$ and $\mathcal{L}(X,Y)$ are isomorphic, and one can identify $\mathcal{L}_+(Y)$ and $\mathcal{L}_+(X,Y)$.*

1.3 Cones with Non-empty Interior

In this section let $(X, K, \|\cdot\|)$ be an ordered normed space such that the cone K has a non-empty interior. Clearly, if $u \in \operatorname{int} K$ and $x > 0$, then $x + u \in \operatorname{int} K$, and $su \in \operatorname{int} K$ for any $s \in (0,\infty)$. An element $u > 0$ is an order unit in X if and only if $u \in \operatorname{int} K$.
Fix $u \in \operatorname{int} K$, then there is $t_0 > 0$ such that the closed ball $\mathrm{B}(u, t_0)$ belongs to K. Then for $x \in X \setminus \{0\}$ one has $u + \frac{t_0}{\|x\|}x \geq 0$, so $x + \frac{\|x\|}{t_0}u \geq 0$ and

$$x = \left(x + \tfrac{\|x\|}{t_0}u\right) - \tfrac{\|x\|}{t_0}u \in K - K\,.$$

Moreover,

$$\|x + \tfrac{\|x\|}{t_0}u\| \leq (1 + \tfrac{\|u\|}{t_0})\|x\|\,, \tag{1.15}$$

which yields the following statement.

Proposition 1.3.1 *If K has a non-empty interior, then K is non-flat.*

In particular, K is generating. Applying Proposition 1.2.10 (i), the following is obvious.

Corollary 1.3.2 *If K has a non-empty interior, then K' is a cone.*

Proposition 1.3.3 [Vul77, Section II.2] *Let* $\operatorname{int} K \neq \varnothing$*. Then an element $u \in K$ is an interior point of K if and only if $f(u) > 0$ for each non-zero functional $f \in K'$.*

Fix an element $u \in \operatorname{int} K$ and consider the set

$$F = F_u = \{f \in K' : f(u) = 1\}\,. \tag{1.16}$$

Theorem 1.3.4 [Vul78, Theorem II.3.2] *The set F is a $\sigma(X', X)$-compact base of K'.*

The set F is a non-empty convex compact set in the locally convex Hausdorff space $(X', \sigma(X', X))$. The Krein-Milman Theorem (A.11) implies $\operatorname{ext} F \neq \varnothing$ and

$$F \;= \overline{\operatorname{co}}^{\sigma(X',X)}(\operatorname{ext} F)\,. \tag{1.17}$$

For the fixed element $u \in \operatorname{int} K$ denote

$$\gamma = \sup\{\, t \in \mathbb{R}_+ : \ \ \mathrm{B}(u, t) \subset K\,\}\,. \tag{1.18}$$

Then $\gamma > 0$, and γ is finite due to $K \cap (-K) = \{0\}$.

In the following let K be closed. For $x \in X \setminus \{0\}$ one has

$$u \pm \tfrac{\gamma}{\|x\|} x \geq 0\,, \tag{1.19}$$

and for every $f \in F$ one gets

$$0 \leq f(u \pm \tfrac{\gamma}{\|x\|} x) = 1 \pm \gamma \tfrac{f(x)}{\|x\|}\,,$$

which implies

$$|f(x)| \leq \tfrac{1}{\gamma} \|x\| \tag{1.20}$$

(which also holds for $x = 0$).

Due to Corollary 1.2.15 the closedness of K implies that F is a total set. Moreover, (1.17) yields the following assertion.

Proposition 1.3.5 *If K is closed, then* $\operatorname{ext} F$ *is a total set.*

In the space $\mathbb{R}^n$, each generating cone has a non-empty interior.

Proposition 1.3.6 [Vul77, Theorems II.3.3 and II.1.2] *Let $(X, K, \|\cdot\|)$ be a finite-dimensional Banach space.*

(i) *X is Archimedean if and only if K is closed.*

(ii) *K has an interior point if and only if K is generating.*

We list certain cones in $\mathbb{R}^n$ which we need frequently later on. The *standard cone* in the space $\mathbb{R}^n$ is the set

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n)^T \in \mathbb{R}^n\colon\ x_i \geq 0 \ \text{ for all } i = 1, \dots, n\}\,.$$

The unit vectors in $\mathbb{R}^n$ are denoted by $e^{(j)}$ for $j = 1, \dots, n$, i. e. $e^{(j)} = (x_1, \dots, x_n)^T$ with $x_j = 1$ and $x_i = 0$ for all $i \neq j$.

Definition 1.3.7 Let the space $\mathbb{R}^n$ be equipped with the Euclidean norm $\|\cdot\|$ and let K be a closed[10] cone in $\mathbb{R}^n$.

(i) K is called *finitely generated* if there is a non-empty finite set $S \subseteq \mathbb{R}^n$ such that $K = \operatorname{pos} S$.

(ii) The cone

$$K_n = \{tx\colon\ x = (x_1, \dots, x_n)^T \in \mathbb{R}^n\,,\ \sum_{i=1}^{n-1} x_i^2 \leq 1\,,\ t \geq 0\}$$

is called the *n-dimensional ice-cream cone*. If T is a linear homeomorphism in $\mathbb{R}^n$, then $K = T(K_n)$ is called *ellipsoidal*.

(iii) K is called a *circular* cone if there is an element $z \in \mathbb{R}^n$ with $\|z\| = 1$ and a number $r > 0$ such that

$$K = \{tx\colon\ x \in \mathbb{R}^n,\ \langle x, z\rangle = 1,\ \|x - z\| \leq r,\ t \geq 0\}\ .$$

(iv) K is called *strictly convex* if each of its faces is one-dimensional.

(v) K is called *smooth* if for each $x \in \partial K$ there is a (up to scalar multiplication) unique $f \in K'$ such that $f(x) = 0$.

The n-dimensional ice-cream cone is a circular cone in $\mathbb{R}^n$, where $z = e^{(n)}$ and $r = 1$.

For a circular cone given in (iii), the cone K^* is obtained by

$$K^* = \{tx\colon\ x \in \mathbb{R}^n,\ \langle x, z\rangle = 1,\ \|x - z\| \leq \tfrac{1}{r},\ t \geq 0\}\ . \tag{1.21}$$

For more details see e. g. [PSW98, Lemma 8].

Next we specify the extreme points of a base of the dual cone in two examples.

Example 1.3.8 Let the space $X = \mathbb{R}^n$ be equipped with the standard cone $K = \mathbb{R}^n_+$ and the Euclidean norm. It is straightforward that $K' = \mathbb{R}^n_+$. We fix $e = (1, 1, \dots, 1)^T \in \mathbb{R}^n$, which is an interior point of K. The corresponding base of K' according to (1.16) is

$$F_e = \{(x_1, \dots, x_n)^T \in \mathbb{R}^n_+\colon\ \langle e, x\rangle = \sum_{i=1}^{n} x_i = 1\}\,. \tag{1.22}$$

[10] In (i), (ii) and (iii) the definition automatically implies that the cone is closed.

The set $\operatorname{ext} F_e$ consists of the unit vectors, i. e.

$$\operatorname{ext} F_e = \{e^{(i)} : i = 1, \dots, n\} .$$

The value of the functional $e^{(i)}$ at a point $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ is the coordinate x_i of the vector x.
Now fix an arbitrary vector $u = (u_1, \dots, u_n)^T \in \operatorname{int} \mathbb{R}^n_+$, i. e. $u_i > 0$ for all $i = 1, \dots, n$. Then

$$F_u = \{(x_1, \dots, x_n)^T \in \mathbb{R}^n_+ : \sum_{i=1}^{n} u_i x_i = 1\} \text{ and } \tag{1.23}$$

$$\operatorname{ext} F_u = \{\tfrac{1}{u_i} e^{(i)} : i = 1, \dots, n\} .$$

Example 1.3.9 Given a compact Hausdorff space T, we deal with the vector space $X = C(T)$ of all continuous real-valued functions on T. We equip X with the maximum norm $\|x\|_\infty = \max\{|x(t)| : t \in T\}$ and consider the usual order on X with respect to the cone

$$K = C_+(T) = \{x \in X : x(t) \geq 0 \text{ for all } t \in T\}$$

of all non-negative functions in X. In this situation we will say that $C(T)$ is equipped with its natural norm and ordering. $(X, K, \|\cdot\|)$ becomes a Banach lattice, where the lattice operations are defined pointwise. All elements $u \in K$ with $u(t) > 0$ for each $t \in T$ (and thus $\min\{u(t) : t \in T\} > 0$) are interior points of K. Due to the Riesz representation theorem one can identify X' with the vector space $M(T)$ of all finite signed regular Borel measures μ on T normed by $|\mu|(T)$, i. e. each functional $f \in X'$ corresponds to a measure $\mu \in M(T)$ such that $\|f\| = |\mu|(T)$ and $f(x) = \int_T x d\mu$ for each $x \in X$. The constant function $\mathbb{1}$, i. e. $\mathbb{1}(t) = 1$ for all $t \in T$, is an interior point of K. Then

$$F_{\mathbb{1}} = \{\mu \in M(T) : \mu \geq 0, \mu(T) = 1\} .$$

The set $\operatorname{ext} F_{\mathbb{1}}$ corresponds to the collection of the evaluation maps ε_t determined by the points $t \in T$, where $\varepsilon_t : X \to \mathbb{R}$, $x \mapsto x(t)$ [Köt66, Section 25.2, p. 337]. Due to the remarks after Definition 1.1.3, the set of atoms of K' is

$$\{\lambda \varepsilon_t : \lambda \in (0, \infty), t \in T\} . \tag{1.24}$$

A natural question in the theory of partially ordered vector spaces is to provide conditions under which such a space is isomorphic (preservation of algebraic, topological and order structure) to a subspace of $C(T)$ for some compact Hausdorff space T.

Proposition 1.3.10 [Vul78, Theorem I.7.1] *Let $(X, K, \|\cdot\|)$ be an ordered normed space such that K is closed, normal and non-flat, and let $T = K' \cap \mathrm{B}'$ be the intersection of the dual cone with the dual unit ball, endowed with the topology $\sigma(X', X)$. Then T is $\sigma(X', X)$-compact, and X is isomorphic to a subspace of $C(T)$, where $x \in X$ is represented as the function $\hat{x} \in C(T)$ with $\hat{x}(f) = f(x)$ for all $f \in T$.*

If K, in addition, has an interior point u, then K is non-flat due to Proposition 1.3.1, and K' possesses a $\sigma(X', X)$-compact base F defined according to (1.16). A similar statement to Proposition 1.3.10 holds, where F is used as the set T. This serves as a motivation for a technique used in Section 3.2.

Proposition 1.3.11 *Let $(X, K, \|\cdot\|)$ be an ordered normed space such that $\operatorname{int} K \neq \varnothing$. Furthermore let K be closed and normal. Then X is isomorphic to a subspace of $C(F)$, where $x \in X$ is represented as the function $\hat{x} \in C(F)$ with $\hat{x}(f) = f(x)$ for all $f \in F$.*

Proof We use an analogous argumentation as in [Vul78, proof of Theorem I.7.1], and only sketch the line of reasoning here. Let $i\colon X \to C(F)$, $i(x) = \hat{x}$, be the representation mapping. Obviously, i is linear. Let $\hat{x} = 0$, i. e. $\hat{x}(f) = f(x) = 0$ for every $f \in F$, consequently $f(x) = 0$ for every $f \in K'$. Since K is normal, due to Proposition 1.2.10 (iii) the dual cone K' is generating, hence $f(x) = 0$ for every $f \in X'$ and therefore $x = 0$, i. e. i is injective. If x is positive, then $i(x)$ is positive. The converse is true since K is closed and hence F is total. The inequality (1.20) ensures the continuity of i. On the other hand, Corollary 1.2.12 yields that K' is non-flat. Therefore the continuity[11] of i^{-1} is justified by Proposition 3.2.12 that will be shown later on. □

1.4 The Riesz Decomposition Property: Examples

In Section 2.3 we will deal with ordered vector spaces that possess the Riesz decomposition property. In the present section we consider examples of such spaces. First we describe the situation in a finite-dimensional partially ordered vector space.

Proposition 1.4.1 [Vul77, Sections I.5 and V.3] *Let $\mathbb{R}^n$ be ordered by a closed generating cone K. Then the following conditions are equivalent:*

(i) $(\mathbb{R}^n, K)$ *is a vector lattice.*

(ii) $(\mathbb{R}^n, K)$ *has the Riesz decomposition property.*

(iii) K *is a finitely generated cone with n extreme rays.*

Observe that the space $\mathbb{R}^n$ ordered by the n-dimensional ice-cream cone does not satisfy the Riesz decomposition property.

We continue with an example of a subspace of $C[0, 1]$ that is not a vector lattice, but has the Riesz decomposition property.

Let $\mathcal{A}$ be an arbitrary set. For a family of real numbers $\{r_\alpha \in (0, \infty)\colon \alpha \in \mathcal{A}\}$ define, as usual,

$$\sum_{\alpha \in \mathcal{A}} r_\alpha = \sup \Big\{ \sum_{\alpha \in \mathcal{A}_0} r_\alpha\colon\ \mathcal{A}_0 \subseteq \mathcal{A},\, \mathcal{A}_0 \text{ finite} \Big\}.$$

We need the following well-known statement.

[11] Using the notations in Section 3.2, one can write $\|\hat{x}\| = \max\{|f(x)|\colon\ f \in F\} = \max\{|\alpha(x)|, |\beta(x)|\}$.

Lemma 1.4.2 *Let $\mathcal{A}$ be a set and $\{r_\alpha \in (0,\infty)\colon \alpha \in \mathcal{A}\}$ a family of real numbers. If the family is summable, i. e.*

$$\sum_{\alpha\in\mathcal{A}} r_\alpha < \infty\,,$$

then $\mathcal{A}$ is a finite or countable set.

Proof Put $\mathcal{A}_n = \{\alpha \in \mathcal{A}\colon\ r_\alpha \geq \frac{1}{n}\}$ for every $n \in \mathbb{N}$. Since

$$\sum_{\alpha\in\mathcal{A}} r_\alpha < \infty\,,$$

the set $\mathcal{A}_n$ is finite for every $n \in \mathbb{N}$. Moreover,

$$\bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \mathcal{A}\,,$$

i. e. the set $\mathcal{A}$ is a countable union of finite sets. Therefore, $\mathcal{A}$ is finite or countable. □

Example 1.4.3 We consider the space $X = C^1[0,1]$ of all continuously differentiable functions on $[0,1]$, which is a linear subspace of $C[0,1]$. The derivatives at the points 0 and 1 are the corresponding one-sided derivatives. Let X be equipped with the natural norm and ordering of $C[0,1]$. Then X is not a vector lattice. By means of Proposition 1.1.4 we show that X has the Riesz decomposition property. Let $a_1, a_2, b_1, b_2 \in X$ be elements such that $a_i \leq b_j$ $(i,j = 1,2)$ and denote $a = a_1 \vee a_2$ and $b = b_1 \wedge b_2$, which are elements of the vector lattice $C[0,1]$ and satisfy $a \leq b$. We are going to construct an element $c \in X$ such that $a \leq c \leq b$. Put

$$S = \{s \in (0,1)\colon\ a \text{ is not differentiable at the point } s\}\,.$$

We list some properties of the element a and the set S.

(i) For each $s \in S$ there is $\varepsilon_s > 0$ such that a is continuously differentiable on

$$(s-\varepsilon_s, s) \cup (s, s+\varepsilon_s)\,.$$

Indeed, let $s \in S$. Then, in particular, $a_1(s) = a_2(s)$ but $a_1'(s) \neq a_2'(s)$. Without loss of generality assume $a_1'(s) < a_2'(s)$. Since a_1' and a_2' are continuous functions, there is $\varepsilon_s > 0$ such that $a_1'(t) < a_2'(t)$ for all $t \in (s-\varepsilon_s, s+\varepsilon_s)$. So, the function $a_2 - a_1$, which vanishes at the point s, is strictly monotone increasing on $(s-\varepsilon_s, s+\varepsilon_s)$. This yields

$$a_2(t) - a_1(t) < 0 \ \text{ for all } \ t \in (s-\varepsilon_s, s) \ \text{ and } \ a_2(t) - a_1(t) > 0 \ \text{ for all } \ t \in (s, s+\varepsilon_s)\,.$$

Hence, by definition of a as $a = a_1 \vee a_2$, one obtains both $a(t) = a_1(t)$ for all $t \in (s-\varepsilon_s, s)$ and $a(t) = a_2(t)$ for all $t \in (s, s+\varepsilon_s)$, i. e. the function a is continuously differentiable on the set $(s-\varepsilon_s, s) \cup (s, s+\varepsilon_s)$.

(ii) S is a countable set.

Indeed, consider the following collection of (non-empty) open intervals $(s,t) \subset [0,1]$, which turn out to be maximal intervals on which the function a is continuously differentiable:

$$\mathcal{A} = \{(s,t)\colon\ s,t \in S \cup \{0,1\} \text{ and } a \text{ is continuously differentiable on } (s,t)\}\ .$$

For $I = (s,t) \in \mathcal{A}$ put $r_I = t - s$, where (i) implies $r_I > 0$. Since $\mathcal{A}$ consists of pairwise disjoint subintervals of $[0,1]$, one has

$$\sum_{I \in \mathcal{A}} r_I \le 1\,,$$

such that Lemma 1.4.2 implies the at most countability of the set $\mathcal{A}$. Due to (i), for each $s \in S$ there is $I_s \in \mathcal{A}$ such that $(s, s+\varepsilon_s) \subseteq I_s$. Moreover, for $s,u \in S$ with $s \neq u$ one has $I_s \neq I_u$, so S is a countable set.

(iii) If for some $s \in (0,1)$ one has $a(s) = b(s)$, then a is differentiable at s.

Indeed, let $s \in (0,1)$ be such that $a(s) = b(s)$. In the case $a_1(s) \neq a_2(s)$, the function a coincides with the greater one of a_1 and a_2 and, therefore, is differentiable in s. It remains to consider the case $a_1(s) = a_2(s) = a(s) = b(s)$. Now there is a $j_0 \in \{1,2\}$ such that $a_1(s) = a_2(s) = b_{j_0}(s)$. For each $i \in \{1,2\}$ the function $d_i = b_{j_0} - a_i$ satisfies the relations $d_i(t) \ge 0$ for all $t \in [0,1]$ and $d_i(s) = 0$. So, both functions d_i attain their minimum at the point s, which implies $d_i'(s) = 0$, i. e.

$$a_i'(s) = b_{j_0}'(s) \ \text{ for } \ i = 1,2\,.$$

Consequently, $a_1'(s) = a_2'(s)$, i. e. a is differentiable at the point s.

Now we specify the construction of the required element c. If $S = \varnothing$, then put $c = a$, and we are done. Otherwise, since S is countable, let

$$S = \{s_k\colon\ k \in \mathbb{N}\}\,.$$

If S is finite, i. e. $S = \{s_k\colon\ k = 1,\dots,n\}$ for a fixed $n \in \mathbb{N}$, adapt the subsequent construction.

For every $k \in \mathbb{N}$ put

$$T_k = [0,1] \setminus \{s_j\colon\ j \ge k\}\,,$$

so $T_1 = [0,1] \setminus S$. Let $c_1 = a$. For every $k \in \mathbb{N}$ we establish a function

$$c_k\colon [0,1] \to \mathbb{R}\,,$$

which is continuously differentiable on every open interval of the set T_k. If c_k is available, then the function c_{k+1} is constructed such that it becomes continuously differentiable in a neighbourhood of s_k.

Due to (i) and (ii), for the point s_k there are intervals $I_1, I_2 \in \mathcal{A}$ such that

$$(s_k - \varepsilon_{s_k}, s_k) \subseteq I_1 \ \text{ and } \ (s_k, s_k + \varepsilon_{s_k}) \subseteq I_2\,.$$

Moreover, from (iii) follows $a(s_k) < b(s_k)$. Due to the continuity of a and b there is $\delta_0 > 0$ such that $a(t) < b(t)$ for all $t \in [s_k - \delta_0, s_k + \delta_0]$. Put

$$\delta = \min\{\tfrac{1}{3}r_{I_1}\,,\ \tfrac{1}{3}r_{I_2}\,, \delta_0\}$$

and let $t_1 = s_k - \delta,\ t_2 = s_k + \delta$ and $\varepsilon = \min\{\, b(t) - a(t) \colon\ t \in [t_1, t_2]\,\}$. Define

$$c_{k+1}(t) = c_k(t) \quad \text{for all} \quad t \in [0,1] \setminus [t_1, t_2]\,.$$

The set

$$X_1 = \{\, x \in C^1[t_1, t_2] \colon\ x'(t_1) = x'(t_2) = 0\,\}$$

is a subalgebra of $C[t_1, t_2]$ which contains the constant functions and separates the points of $[t_1, t_2]$. Due to the Stone-Weierstrass theorem, the space X_1 is dense in $C[t_1, t_2]$. We consider the function $a + \frac{\varepsilon}{2}\mathbb{1}$ on the interval $[t_1, t_2]$. In the $\frac{\varepsilon}{4}$-neighbourhood of this function a function $d \in X_1$ can be found, i. e.

$$a(t) < (a + \tfrac{\varepsilon}{4}\mathbb{1})(t) \le d(t) \le (a + \tfrac{3\varepsilon}{4}\mathbb{1})(t) < b(t)$$

holds for $t \in [t_1, t_2]$. Let $\hat{t}_1 = s_k - \frac{\delta}{2},\ \hat{t}_2 = s_k + \frac{\delta}{2}$ and put

$$c_{k+1}(t) = d(t) \quad \text{for all} \quad t \in [\hat{t}_1, \hat{t}_2]\,.$$

On the interval $[t_1, \hat{t}_1]$ one proceeds as follows: A polynomial p of third degree is uniquely defined on $[t_1, \hat{t}_1]$ by the conditions $p(t_1) = 0,\ p'(t_1) = 0,\ p(\hat{t}_1) = 1,\ p'(\hat{t}_1) = 0$. We combine the continuously differentiable functions c_k and d on $[t_1, \hat{t}_1]$ by defining

$$c_{k+1}(t) = c_k(t)(1 - p(t)) + d(t)p(t) \quad \text{for all} \quad t \in [t_1, \hat{t}_1]\,.$$

Then $c_{k+1}(t_1) = c_k(t_1),\ c'_{k+1}(t_1) = c'_k(t_1),\ c_{k+1}(\hat{t}_1) = d(\hat{t}_1),\ c'_{k+1}(\hat{t}_1) = d'(\hat{t}_1)$. Proceed analogously on the interval $[\hat{t}_2, t_2]$. One gets $c_{k+1} \in C^1[t_1, t_2]$ such that

$$a \le c_k \le c_{k+1} \le d < b \quad \text{on} \quad [t_1, t_2]\,.$$

The function c_{k+1} is continuously differentiable on all open intervals of T_{k+1}, moreover $a \le c_k \le c_{k+1} < b$ on $[0, 1]$. Clearly, $\bigcup_{k\in\mathbb{N}} T_k = [0, 1]$. Define finally

$$c(t) = \sup\{c_k(t) \colon\ k \in \mathbb{N}\} \quad \text{for each} \quad t \in [0,1]\,,$$

then $c \in C^1[0, 1]$ and $a \le c \le b$.

Chapter 2

Positive-off-diagonal Operators and M-Operators

In this chapter we first study operators which appear to be a generalization of square matrices with non-negative off-diagonal entries. Such matrices map the faces of the standard cone of $\mathbb{R}^n$ in a specific manner. The latter property is used to define "positive-off-diagonal" operators if $\mathbb{R}^n$ is equipped with an arbitrary cone, as well as for arbitrary ordered normed spaces.

In the first section we collect the relevant definitions and certain basic properties of positive-off-diagonal operators. The second section is dedicated to the class of all positive-off-diagonal operators on a given ordered normed space and to related classes of operators, in particular, to the class of operators which dominate a multiple of the identity. We survey the corresponding literature and establish certain sufficient conditions on the underlying space such that the two mentioned classes of operators coincide. In the third section we consider ordered normed spaces with the Riesz decomposition property. Again certain sufficient conditions are provided such that the two classes of operators are equal.

Matrices with non-positive off-diagonal entries appear in the theory of M-matrices. As a generalization of M-matrices, in the fourth section M_1- and M_2-operators on ordered normed spaces are defined. We apply the results on positive-off-diagonal operators in order to list some spaces where the notions M_1- and M_2-operator coincide.

2.1 Positive-off-diagonal Operators

We start with the well-known notions from the finite-dimensional theory.

Definition 2.1.1 [BF04] A matrix $A = (a_{ij})_{n,n}$ is called a METZLER *matrix* if $a_{ij} \geq 0$ for $i \neq j$.

In the subsequent definition let the space $\mathbb{R}^n$ be equipped with the standard cone $\mathbb{R}^n_+$, i. e. for an $n \times n$-matrix $B = (b_{ij})_{n,n}$ one has $B \geq 0$ if and only if $b_{ij} \geq 0$ for all $i, j = 1, \ldots, n$. As usual, $r(B)$ denotes the spectral radius of the matrix B.

Definition 2.1.2 [BP94, Chapter 6, Definition (1.2)] A matrix $A = (a_{ij})_{n,n}$ is called an M-*matrix* if $A = sI - B$, where $B \geq 0$ and $s > r(B)$.

We only consider non-singular M-matrices. Singular M-matrices, i. e. matrices of the form $A = r(B)I - B$, are not relevant in our investigation. If A is an M-matrix, then $-A$ is a METZLER matrix. On the other hand, if for a matrix $A = (a_{ij})_{n,n}$ the matrix $-A$ is a METZLER matrix, and A is strictly row diagonally dominant, i. e. $|a_{ii}| - \sum_{j \neq i} |a_{ij}| > 0$ for $i = 1, \ldots, n$, and has only strictly positive diagonal elements, then A is an M-matrix. Recall the following characterization of M-matrices.

Proposition 2.1.3 [BP94, Chapter 6, Theorem (2.3), N_{38}] *Let $A = (a_{ij})_{n,n}$ be a matrix such that A^{-1} exists. A is an* M-*matrix if and only if $a_{ij} \leq 0$ for all $i \neq j$ and $A^{-1} \geq 0$.*

Generalizations of M-matrices will be studied in Section 2.4.

In the present section we deal with the generalization of METZLER matrices to the case of linear operators on ordered normed spaces, which leads to positive-off-diagonal operators. We refer to [CHA$^+$87, Definition 7.18].

Definition 2.1.4 Let $(X, K, \|\cdot\|)$ be an ordered normed space, let K' be the dual wedge and let D be a linear subspace of X. A linear operator $A \colon D \to X$ is called *positive-off-diagonal,* if for every $x \in K \cap D$ and each $f \in K'$ such that $f(x) = 0$ one has $f(Ax) \geq 0$. The operator A is called *negative-off-diagonal* if $-A$ is positive-off-diagonal.

If there are different cones in consideration, we write *positive-off-diagonal with respect to* K. Positive-off-diagonal operators appear in the literature in a broad variety of different names. In the theory of operator semigroups the notion *positive-off-diagonal* is used in [CHA$^+$87], whereas in [Are86, Chapter CII, Definition 1.6] a positive-off-diagonal operator is said to *satisfy the positive minimum principle*. In the matrix theory a positive-off-diagonal operator with respect to a closed and generating cone K in $\mathbb{R}^n$ is called a *cross-positive* matrix. Such matrices are discussed e. g. in [SV70], [Tam75], [Tam76] and, more recently, in [GKT95]. In [BNS89, Definition 3.2], the notion *subtangential* matrix is used.

In the non-linear theory an operator $A \colon D \to X$ $(D \subseteq X)$ is called *quasimonotone increasing* if for every $x, y \in D$ with $x \leq y$ and $f \in K'$ with $f(x) = f(y)$ follows $f(Ax) \leq f(Ay)$, see e. g. [Vol72]. Clearly, if D is a linear subspace of X and A is a linear operator, then A is quasimonotone increasing if and only if A is positive-off-diagonal.

If $\operatorname{int} K \neq \varnothing$, then only for points x that belong to the boundary of K there are non-trivial functionals $f \in K'$ such that $f(x) = 0$ (Proposition 1.3.3). So, the behaviour of the boundary points of K under a linear operator is responsible for the positive-off-diagonality of this operator. For the special case $K = (\operatorname{int} K) \cup \{0\}$, each linear operator $A \colon D \to X$ is positive-off-diagonal. The condition $\operatorname{int} K \neq \varnothing$ implies that K' is a cone (Corollary 1.3.2), and for a fixed element $u \in \operatorname{int} K$ there is a $\sigma(X', X)$-compact base

$F = F_u$ in K' (Proposition 1.3.4). To show that in this case an operator is positive-off-diagonal, only the extreme points of F have to be attracted. For the next statement we use the fact that for an element $x \in \partial K$ the set

$$F_0(x) = \{f \in F\colon f(x) = 0\}$$

is non-empty and can be represented as

$$F_0(x) = \overline{\mathrm{co}}^{\sigma(X',X)}(F_0(x) \cap \operatorname{ext} F)\,. \tag{2.1}$$

This will be discussed in detail in Section 3.2, cf. (3.6).

Proposition 2.1.5 *Let $(X, K, \|\cdot\|)$ be an ordered normed space such that $\operatorname{int} K \neq \varnothing$. Let D be a linear subspace of X and F a base of K'. Then a linear operator*

$$A\colon D \to X$$

is positive-off-diagonal if and only if for every $x \in \partial K \cap D$ and each $f \in \operatorname{ext} F$ such that $f(x) = 0$ one has $f(Ax) \geq 0$.

Proof We have to show only the sufficiency of the condition. Consider $x \in \partial K \cap D$ and an arbitrary functional $g \in K'$, $g \neq 0$, with $g(x) = 0$. Since F is a base of K', there is a functional $f \in F$ such that $g = \lambda f$ for some $\lambda > 0$. Due to (2.1) there is a net $\{f_\alpha\}_{\alpha\in J} \subset \mathrm{co}(F_0(x) \cap \operatorname{ext} F)$ which converges pointwise to f. For each functional f_α, $\alpha \in J$, there is a natural number $m = m(\alpha)$ such that f_α is represented as follows:

$$f_\alpha = \sum_{j=1}^{m} \mu_j h_j\,, \quad h_j \in F_0(x) \cap \operatorname{ext} F \text{ and } \mu_j \geq 0 \text{ for } j = 1, \ldots, m\,, \quad \sum_{j=1}^{m} \mu_j = 1\,.$$

By assumption, we have $h_j(Ax) \geq 0$ and, therefore, $f_\alpha(Ax) \geq 0$, hence $f(Ax) \geq 0$ and $g(Ax) \geq 0$. $\square$

In general, D may differ from X and A may not be continuous. In the following we consider mostly the case $D = X$ and $A \in \mathfrak{L}(X)$, i. e. A is a linear continuous operator on X. Clearly, each positive operator is positive-off-diagonal. For a given ordered normed space $(X, K, \|\cdot\|)$ we denote

$$\Sigma(X, K) = \{A \in \mathfrak{L}(X)\colon\ A \text{ is positive-off-diagonal }\}\,.$$

Occasionally, we abbreviate $\Sigma(X, K)$ by $\Sigma(K)$.

Next we consider the situation as in Example 1.3.8, i. e. the space $\mathbb{R}^n$ is ordered by the cone $\mathbb{R}^n_+$. The base $F_e = \{x \in \mathbb{R}^n_+\colon \langle e, x\rangle = 1\}$ of the dual cone, according to (1.22), has the collection of all unit vectors $e^{(i)}$, $i = 1, \ldots, n$, as its set of extreme points. A linear operator $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is identified with its matrix representation with respect to the standard basis in $\mathbb{R}^n$ consisting of the unit vectors. Due to the previous proposition, $A = (a_{ij})_{n,n}$ is positive-off-diagonal with respect to $\mathbb{R}^n_+$ if and only if $x \in \mathbb{R}^n_+$ and $i \in \{1, \ldots, n\}$ with $\langle x, e^{(i)}\rangle = x_i = 0$ imply $\langle Ax, e^{(i)}\rangle = (Ax)_i \geq 0$.

Corollary 2.1.6 *The linear operator $A = (a_{ij})_{n,n}$ on $\mathbb{R}^n$ is positive-off-diagonal with respect to $\mathbb{R}^n_+$ if and only if it is a* METZLER *matrix.*

Proof Let $A = (a_{ij})_{n,n}$ be positive-off-diagonal. Since

$$a_{ij} = \langle Ae^{(j)}, e^{(i)} \rangle$$

and $\langle e^{(j)}, e^{(i)} \rangle = 0$ for $i \neq j$, the assumption implies $a_{ij} \geq 0$.
Vice versa, let $a_{ij} \geq 0$ for $i \neq j$ and $x \in \mathbb{R}^n_+$ with $x_i = 0$ for some fixed i. Then

$$(Ax)_i = \sum_j a_{ij} x_j = \sum_{j \neq i} a_{ij} x_j \geq 0\,,$$

so A is positive-off-diagonal. $\square$

If one considers an arbitrary cone K in $\mathbb{R}^n$ and a matrix A that is positive-off-diagonal with respect to K, then it is not surprising that the off-diagonal entries of A need not be non-negative, as the next example illustrates, cf. [BNS89, Section 4 (1.6)].

Example 2.1.7 We consider the space $X = \mathbb{R}^3$, ordered by the 3-dimensional ice-cream cone

$$K = K_3 = \left\{ t \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix} \right) : \; x_1^2 + x_2^2 \leq 1,\, t \geq 0 \right\}.$$

The set

$$D = \left\{ \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix} \right) : \; x_1^2 + x_2^2 \leq 1 \right\}$$

is a base of K. Clearly, $X^* = \mathbb{R}^3$ and $K^* = K$, cf. (1.21). Whenever

$$x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix} \right) \in D \quad \text{and} \quad f = \left(\begin{smallmatrix} w_1 \\ w_2 \\ w_3 \end{smallmatrix} \right) \in K^*$$

are such that $f(x) = \langle x, f \rangle = 0$, where without loss of generality $w_3 = 1$, then

$$\begin{aligned} 0 &\leq (x_1 + w_1)^2 + (x_2 + w_2)^2 \\ &= (x_1^2 + x_2^2) + (w_1^2 + w_2^2) + 2(x_1 w_1 + x_2 w_2) \\ &\leq 2 + 2(\langle x, f \rangle - 1) \\ &= 0\,, \end{aligned}$$

which implies $w_1 = -x_1$ and $w_2 = -x_2$. Moreover, one gets $(x_1^2 + x_2^2) + (w_1^2 + w_2^2) = 2$, and so $x_1^2 + x_2^2 = 1$ and $w_1^2 + w_2^2 = 1$.
Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

One has

$$f(Ax) = \langle Ax, f \rangle = \left\langle \left(\begin{smallmatrix} -x_1 + x_2 \\ -x_1 - x_2 \\ -1 \end{smallmatrix} \right), \left(\begin{smallmatrix} -x_1 \\ -x_2 \\ 1 \end{smallmatrix} \right) \right\rangle = x_1^2 + x_2^2 - 1 = 0\,,$$

i. e. A is positive-off-diagonal with respect to K. Observe that A is negative-off-diagonal with respect to K as well.

If $(X, K, \|\cdot\|)$ is a Banach lattice, then there is a convenient characterization of positive-off-diagonal operators.

Theorem 2.1.8 [Are86, Theorem 1.11] *If $(X, K, \|\cdot\|)$ is a Banach lattice and $A \in \mathcal{L}(X)$, then the following conditions are equivalent:*

(i) $A \in \Sigma(K)$*, and*

(ii) $A + \|A\|I \geq 0$.

In view of the technique in Corollary 2.1.6, we consider positive-off-diagonal operators on an ordered Banach space $(X, K, \|\cdot\|)$ that has a Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$, i. e. for each $x \in X$ there is a unique sequence $(x_j)_{j\in\mathbb{N}}$ of real numbers such that

$$x = \sum_{j=1}^{\infty} x_j b^{(j)} ,$$

where the series converges in norm. In the Appendix A.1 the corresponding notations and statements are listed. In particular, for each $j \in \mathbb{N}$, a (biorthogonal) coordinate functional $f^{(j)}$ on X is defined by $f^{(j)} \colon X \to \mathbb{R}$, $x \mapsto x_j$ (cf. also the formulas (A.2) and (A.3)).

Definition 2.1.9 [AB99, Section 15.3] A Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$ of an ordered Banach space $(X, K, \|\cdot\|)$ is called *positive* if for every $x \in X$ one has $x \in K$ if and only if $f^{(j)}(x) \geq 0$ for all $j \in \mathbb{N}$.

If $(b^{(j)})_{j\in\mathbb{N}}$ is a positive Schauder basis, then $b^{(j)} \in K$ for all $j \in \mathbb{N}$ due to (A.5). Clearly, $f^{(j)} \in K'$ for each $j \in \mathbb{N}$. To an operator $A \in \mathcal{L}(X)$ a matrix

$$(a_{ij})_{i,j\in\mathbb{N}}$$

is associated by means of (A.7). For sake of convenience, we write $A = (a_{ij})_{i,j\in\mathbb{N}}$. If A is positive, then $Ab^{(j)} \in K$ for all $j \in \mathbb{N}$, so $a_{ij} = f^{(i)}(Ab^j) \geq 0$ for all $i, j \in \mathbb{N}$. Vice versa, if $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$, then $x \in K$ implies

$$f^{(i)}(Ax) = \sum_{j=1}^{\infty} a_{ij} f^{(j)}(x) \geq 0$$

for all $i \in \mathbb{N}$, where we use (A.9). So, $Ax \in K$, and we get the following:

Proposition 2.1.10 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a positive Schauder basis. Then $A \in \mathcal{L}(X)$ is positive if and only if $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$.*

The positive-off-diagonality of a linear continuous operator in the Banach lattices $\boldsymbol{c}_0$ or $\boldsymbol{l}_p$ for $p \in [1, \infty)$ can be characterized similarly to the finite-dimensional case, as the next example shows.

Example 2.1.11 We consider sequence spaces X according to Definition A.1.1, i. e. X is equipped with the standard ordering. For each $j \in \mathbb{N}$ denote

$$e^{(j)} = (x_i)_{i\in\mathbb{N}} \text{ with } x_j = 1 \text{ and } x_i = 0 \text{ for all } i \neq j\,. \tag{2.2}$$

If X equals one of the Banach lattices $\boldsymbol{c}_0$ or $\boldsymbol{l}_p$ for $p \in [1, \infty)$, then the sequence $(e^{(j)})_{j\in\mathbb{N}}$ is a positive Schauder basis in X [AB99, Theorems 15.12 and 15.21].

Consider an operator $A \in \mathcal{L}(X)$ and its matrix representation $(a_{ij})_{i,j\in\mathbb{N}}$. A combination of Theorem 2.1.8, Proposition 2.1.10, (A.8) and (A.10) implies that A is positive-off-diagonal if and only if $a_{ij} \geq 0$ for all $i \neq j$.

In the next statement we make use of the Definition A.1.4 of a shrinking Schauder basis for a characterization of the positive-off-diagonality of linear continuous operators.

Proposition 2.1.12 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a shrinking positive Schauder basis. Then $A \in \mathcal{L}(X)$ is positive-off-diagonal if and only if $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$.*

Proof Let $A \in \mathcal{L}(X)$ be positive-off-diagonal. Since $b^{(j)} \in K$, $f^{(i)} \in K'$ and

$$f^{(i)}(b^{(j)}) = 0$$

provided $i \neq j$, one gets $a_{ij} = f^{(i)}(Ab^{(j)}) \geq 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

Vice versa, let $A \in \mathcal{L}(X)$ be such that $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Fix $x \in K$ and $f \in K'$ such that $f(x) = 0$. Since the Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$ is shrinking, we have

$$f = \sum_{i=1}^{\infty} f(b^{(i)}) f^{(i)}$$

according to (A.6). Due to

$$0 = f(x) = \sum_{i=1}^{\infty} f(b^{(i)}) f^{(i)}(x)\,,$$

$f(b^{(i)}) \geq 0$ and $f^{(i)}(x) \geq 0$ for all $i \in \mathbb{N}$, one has

$$f(b^{(i)}) f^{(i)}(x) = 0$$

for all $i \in \mathbb{N}$. Now we conclude

$$\begin{aligned} f(Ax) &= \sum_{i=1}^{\infty} f(b^{(i)}) f^{(i)}(Ax) \\ &= \sum_{i=1}^{\infty} f(b^{(i)}) \sum_{j=1}^{\infty} a_{ij} f^{(j)}(x) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} f(b^{(i)}) f^{(j)}(x) \\ &= \sum_{i=1}^{\infty} \sum_{\substack{j=1 \\ j \neq i}}^{\infty} a_{ij} f(b^{(i)}) f^{(j)}(x) \\ &\geq 0 \,. \quad \square \end{aligned}$$

Next we investigate the relation between a positive-off-diagonal operator and its restriction on a dense subspace. We use the notation in (1.13) and employ Corollary 1.2.24 (i).

Proposition 2.1.13 (i) *Let $(Y, K_Y, \|\cdot\|)$ be an ordered normed space and let K be a cone in Y with $K \subseteq K_Y \subseteq \overline{K}$. Then*

$$\Sigma(Y, K_Y) \subseteq \Sigma(Y, K) \,. \tag{2.3}$$

(ii) *Let $(Y, K_Y, \|\cdot\|)$ be an ordered Banach space, let X be a dense linear subspace of Y and let K_X be a cone in X such that $K_X \subseteq K_Y \subseteq \overline{K_X}$. If $\hat{A} \in \Sigma(Y, K_Y)$, then $A \in \Sigma(X, K_X)$.*

(iii) *If, in addition to the assumptions in* (ii)*, for each $f \in K'_Y$ one has*

$$f^{-1}(0) \cap K_Y \subseteq \overline{f^{-1}(0) \cap K_X} \,, \tag{2.4}$$

then $\hat{A} \in \Sigma(Y, K_Y)$ if and only if $A \in \Sigma(X, K_X)$.

Proof (i) Let $A \in \Sigma(Y, K_Y)$, $x \in K$ and $f \in K'$ be such that $f(x) = 0$. Corollary 1.2.24 (i) implies that $f \in K'_Y$. So, $f(Ax) \geq 0$, i. e. $A \in \Sigma(Y, K)$.

(ii) Let $\hat{A} \in \Sigma(Y, K_Y)$, $x \in K_X$ and $f \in K'_X$ such that $f(x) = 0$. Then $f \in K'_Y$ due to Corollary 1.2.24 (i). Since $\hat{A}$ is positive-off-diagonal with respect to K_Y, one has $f(\hat{A}x) \geq 0$. Clearly, $\hat{A}x = Ax$, so $f(Ax) \geq 0$. Hence, $A \in \Sigma(X, K_X)$.

(iii) Let $A \in \Sigma(X, K_X)$, $y \in K_Y$ and $f \in K'_Y$ such that $f(y) = 0$. Due to the assumption, there is a sequence $(x_n)_{n \in \mathbb{N}} \subset f^{-1}(0) \cap K_X$ that converges to y. Now $f \in K'_X$ and $f(x_n) = 0$ for all $n \in \mathbb{N}$ imply $f(Ax_n) \geq 0$, i. e. $f(\hat{A}x_n) \geq 0$, for all $n \in \mathbb{N}$. Since the sequence $(f(\hat{A}x_n))_{n \in \mathbb{N}}$ converges to $f(\hat{A}y)$, one gets $f(\hat{A}y) \geq 0$, i. e. $\hat{A} \in \Sigma(Y, K_Y)$. $\square$

Remark 2.1.14 If the cone K_Y has an interior point, then K'_Y has a base F, and due to Proposition 2.1.5, in (iii) it suffices to require the assumption (2.4) only for all $f \in \operatorname{ext} F$. An illustration will be given in Example 2.2.9.

We illustrate the statement (i) of Proposition 2.1.13 by the following example.

Example 2.1.15 Let the space $\boldsymbol{l}_1$ be ordered by the cone $\boldsymbol{\varphi}^+$ of all finite sequences with non-negative coordinates. $\boldsymbol{\varphi}^+$ is not closed in $\boldsymbol{l}_1$, the closure of $\boldsymbol{\varphi}^+$ is $\boldsymbol{l}_1^+$. Let $A = (a_{ij})_{i,j\in\mathbb{N}} \in \mathcal{L}(X)$ be such that $a_{ij} \geq 0$ for all $i \neq j$. Example 2.1.11 justifies that A is positive-off-diagonal with respect to $\boldsymbol{l}_1^+$. Due to (2.3) the operator A is also positive-off-diagonal with respect to $\boldsymbol{\varphi}^+$.

Clearly, the converse implication in (2.3) is not true in general. Consider e. g. $Y = \mathbb{R}^2$, $K_Y = \mathbb{R}^2_+$ and $K = (\operatorname{int} \mathbb{R}^2_+) \cup \{0\}$. A matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $b < 0$ is positive-off-diagonal with respect to K, but it is not positive-off-diagonal with respect to K_Y due to Corollary 2.1.6.

2.2 Related Classes of Operators

In this section we study properties of the set $\Sigma(X, K)$ of all positive-off-diagonal operators on a given ordered normed space $(X, K, \|\cdot\|)$. We consider subsets of $\mathcal{L}(X)$ that are related to $\Sigma(X, K)$, where we use the standard notations that are employed in the finite-dimensional theory, cf. e. g. [SV70], [GKT95]. As usual, $\mathcal{L}_+(X)$ denotes the set of all positive operators in $\mathcal{L}(X)$. We consider the set

$$e(X, K) = \{A \in \mathcal{L}(X)\colon\ e^{tA} \geq 0 \ \text{ for all } t \in [0, \infty)\}$$

of operators in $\mathcal{L}(X)$ that are generators of a positive operator semigroup $\{e^{tA}\}_{t\geq 0}$, as well as the set

$$\Pi_1(X, K) = \mathcal{L}_+(X) + \mathbb{R}I\,.$$

Definitions and basic statements concerning operator semigroups are listed in the Appendix A.3. We occasionally abbreviate $e(X, K)$ and $\Pi_1(X, K)$ by $e(K)$ and $\Pi_1(K)$, respectively. If $A \in \Pi_1(K)$, i. e.

$$A = B + sI \tag{2.5}$$

for some operator $B \geq 0$ and $s \in \mathbb{R}$, then $A \geq sI$, i. e. $\Pi_1(K)$ is the class of operators that dominate a multiple of the identity. Instead of (2.5) we will occasionally write $A + tI \geq 0$ for a certain constant $t > 0$. Theorem 2.1.8 states that for a Banach lattice X the sets $\Sigma(K)$ and $\Pi_1(K)$ coincide.

Let $(X, K, \|\cdot\|)$ be an ordered normed space and $A \in \mathcal{L}(X)$ be given according to (2.5). If $x \in K$ and $f \in K'$ are such that $f(x) = 0$, then

$$f(Ax) = f(Bx) + sf(x) = f(Bx) \geq 0\,,$$

i. e. $A \in \Sigma(K)$. Hence we obtain the following.

Proposition 2.2.1 *The sets* $\Pi_1(K)$ *and* $\Sigma(K)$ *are wedges in* $\mathcal{L}(X)$*, where the inclusions*

$$\mathcal{L}_+(X) \subseteq \Pi_1(K) \subseteq \Sigma(K)$$

are satisfied.

Since the identity operator I and the operator $-I$ are both in $\Pi_1(K)$, the set $\Pi_1(K)$ is not a cone in $\mathcal{L}(X)$, so $\Sigma(K)$ is not a cone in $\mathcal{L}(X)$ as well. The blade of the the wedge $\Pi_1(K)$ is the set

$$\{A \in \mathcal{L}(X)\colon \text{ there is } s \in \mathbb{R}_+ \text{ such that } -sI \leq A \leq sI\},$$

which is known as the *centre* of $\mathcal{L}(X)$.

Proposition 2.2.2 *The set* $\Sigma(K)$ *is closed in* $\mathcal{L}(X)$ *with respect to the uniform operator topology.*

Proof The set $\Sigma(K)$ is closed in $\mathcal{L}(X)$ with respect to the weak operator topology. Indeed, let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma(K)$ that converges to $A \in \mathcal{L}(X)$ in the weak operator topology, i. e. for each $x \in X$ and $f \in X'$ one has

$$\lim_{n\to\infty} f(A_n x) = f(Ax)\,.$$

If $x \in K$ and $f \in K'$ are such that $f(x) = 0$, then $f(A_n x) \geq 0$ for each $n \in \mathbb{N}$, so $f(Ax) \geq 0$ and A is positive-off-diagonal.
The closedness of $\Sigma(K)$ with respect to the weak operator topology implies that $\Sigma(K)$ is closed in $\mathcal{L}(X)$ with respect to the uniform operator topology. □

Some properties of the set $e(K)$ can be obtained e. g. by applying results given in [HL98] to our case. In that paper the corresponding statements are shown for elements of a real Banach algebra (with unit) that is ordered by a closed algebra cone. If $\mathcal{A}$ is an algebra with unit $\mathbf{1}$ and $\mathcal{A}_+$ is a cone in $\mathcal{A}$, then $\mathcal{A}_+$ is called an *algebra cone* if $\mathbf{1} \in \mathcal{A}_+$, and $a, b \in \mathcal{A}_+$ implies $a * b \in \mathcal{A}_+$, where $*$ denotes the multiplication in $\mathcal{A}$.
Given an ordered normed space $(X, K, \|\cdot\|)$, it is well-known that the set $\mathcal{L}(X)$, equipped with the operator norm, is a normed algebra, where the composition of operators corresponds to the algebra multiplication. The identity operator $I \in \mathcal{L}(X)$ is a unit in the algebra $\mathcal{L}(X)$. If X is a Banach space, then $\mathcal{L}(X)$ is a Banach algebra. If K is weakly generating, then due to Proposition 1.2.9 the set $\mathcal{L}_+(X)$ is a cone in $\mathcal{L}(X)$. If K is closed, then $\mathcal{L}_+(X)$ is closed. One has $I \in \mathcal{L}_+(X)$, moreover $A, B \in \mathcal{L}_+(X)$ implies $A(B(K)) \subseteq A(K) \subseteq K$, hence the composition of A and B is in $\mathcal{L}_+(X)$, which yields that $\mathcal{L}_+(X)$ is an algebra cone in $\mathcal{L}(X)$. So, we can apply the results in [HL98] to our setting.

Proposition 2.2.3 *If* X *is a Banach space and* K *a weakly generating closed cone in* X*, then the following statements hold:*

(i) *The inclusions* $\Pi_1(K) \subseteq e(K) \subseteq \Sigma(K)$ *are valid.*

(ii) *The set* $e(K)$ *is a wedge in* $\mathcal{L}(X)$.

(iii) *The set* $e(K)$ *is the closure of* $\Pi_1(K)$ *in* $\mathcal{L}(X)$ *with respect to the uniform operator topology.*

The inclusion $\Pi_1(K) \subseteq e(K)$ in (i) is given in [HL98, Remark 1], the inclusion $e(K) \subseteq \Sigma(K)$ is a special case of Proposition A.3.3. The assertions (ii) and (iii) are stated in [HL98, Theorem 1, (1), (6)].
Note that in [HL98, Theorem 1, (2), (3), (8) and Example 2] it is shown that the centre of $\mathcal{L}(X)$, i. e. the blade of the wedge $\Pi_1(K)$, is a closed subalgebra of $\mathcal{L}(X)$, whereas the blade of the wedge $e(K)$ is a closed subspace of $\mathcal{L}(X)$, but need not be a subalgebra in general.

In view of Proposition 2.2.3 (i), we list two examples, where the corresponding inclusions are proper.

Example 2.2.4 Consider the 3-dimensional ice-cream cone $K = K_3$ in the space $X = \mathbb{R}^3$ and the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

as in Example 2.1.7. We show that $A \in e(K)$, but $A \notin \Pi_1(K)$. We diagonalize the matrix A and get $A = SDS^{-1}$, where

$$D = \begin{pmatrix} -1+\mathrm{i} & 0 & 0 \\ 0 & -1-\mathrm{i} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & \mathrm{i} & 0 \\ \mathrm{i} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} & 0 \\ -\mathrm{i} & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For every $k \in \mathbb{N}$ one has $A^k = SD^kS^{-1}$ and so for $t \geq 0$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = S e^{tD} S^{-1}.$$

With

$$e^{tD} = \begin{pmatrix} e^{(-1+\mathrm{i})t} & 0 & 0 \\ 0 & e^{(-1-\mathrm{i})t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} = e^{-t}\begin{pmatrix} \cos t + \mathrm{i}\sin t & 0 & 0 \\ 0 & \cos t - \mathrm{i}\sin t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

it is straightforward that

$$e^{tA} = e^{-t}\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix}\right) \in K$, i. e. $x_1^2 + x_2^2 \leq 1$. For every $t \geq 0$ one has

$$e^{tA}x = e^{-t}\left(\begin{smallmatrix} x_1\cos t + x_2\sin t \\ -x_1\sin t + x_2\cos t \\ 1 \end{smallmatrix}\right) \in K$$

because of $(x_1 \cos t + x_2 \sin t)^2 + (-x_1 \sin t + x_2 \cos t)^2 = x_1^2 + x_2^2 \le 1$. So, e^{tA} is a positive operator for each $t \ge 0$, hence $A \in e(K)$.

On the other hand, there is no number s such that $sI + A$ is positive. Indeed, for

$$v = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \in K \quad \text{one has} \quad y(s) = (sI + A)v = \begin{pmatrix} -1 \\ 1-s \\ s-1 \end{pmatrix} .$$

If $y(s)$ would belong to K for some $s \in \mathbb{R}$, then we would get $(-1)^2 + (1-s)^2 \le (s-1)^2$, a contradiction. Hence, $y(s) \notin K$ for every $s \in \mathbb{R}$, consequently $A \notin \Pi_1(K)$. So, $A \in e(K) \setminus \Pi_1(K)$.

Example 2.2.5 Consider the ordered Banach space as in Example 2.1.15, i. e. the space $X = \boldsymbol{l}_1$ ordered by the cone $K = \boldsymbol{\varphi}^+$ of all finite sequences with non-negative coordinates. For a sequence $b = (b_i)_{i \in \mathbb{N}} \in \boldsymbol{l}_1^+ \setminus K$ consider the matrix $A = (a_{ij})_{i,j \in \mathbb{N}}$ with

$$\begin{aligned} a_{ii} &= 1\,, \quad i \in \mathbb{N}\,, \\ a_{i1} &= b_{i-1}\,, \quad i \in \mathbb{N}\,, i \ge 2\,, \\ a_{ij} &= 0 \quad \text{otherwise}\,, \end{aligned}$$

$$\text{i. e.} \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ b_1 & 1 & 0 & 0 & \\ b_2 & 0 & 1 & 0 & \\ b_3 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} .$$

For $x \in \boldsymbol{l}_1$ and $i \ge 2$ one has

$$\begin{aligned} (Ax)_i &= b_{i-1} x_1 + x_i \ , \quad \text{hence} \\ \|Ax\|_1 &= \sum_{i=1}^{\infty} |(Ax)_i| \\ &\le \sum_{i=2}^{\infty} b_{i-1} |x_1| + \sum_{i=1}^{\infty} |x_i| \\ &\le (\|b\|_1 + 1) \|x\|_1 \ , \end{aligned}$$

so $A \in \mathfrak{L}(X)$. One has $a_{ij} \ge 0$ for all $i, j \in \mathbb{N}$, so due to Example 2.1.15, the operator A is positive-off-diagonal with respect to K. Observe that A is not positive, since $e^{(1)} \in K$, but $A(e^{(1)}) \notin K$.

We justify that $A \notin e(K)$. It is straightforward that for A^n $(n \geq 1)$ one has

$$\begin{aligned} (A^n)_{ii} &= 1\,, \quad i \in \mathbb{N}\,, \\ (A^n)_{i1} &= nb_{i-1}\,, \quad i \in \mathbb{N}\,, i \geq 2\,, \\ (A^n)_{ij} &= 0 \quad \text{otherwise}\,, \end{aligned}$$

$$\text{i. e.} \qquad A^n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ nb_1 & 1 & 0 & 0 & \\ nb_2 & 0 & 1 & 0 & \\ nb_3 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

So, the operator

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

is determined by

$$\begin{aligned} (e^{tA})_{ii} &= e^t\,, \quad i \in \mathbb{N}\,, \\ (e^{tA})_{i1} &= te^t b_{i-1}\,, \quad i \in \mathbb{N}\,, i \geq 2\,, \\ (e^{tA})_{ij} &= 0 \quad \text{otherwise}\,, \end{aligned}$$

$$\text{i. e.} \qquad e^{tA} = \begin{pmatrix} e^t & 0 & 0 & 0 & \dots \\ te^t b_1 & e^t & 0 & 0 & \\ te^t b_2 & 0 & e^t & 0 & \\ te^t b_3 & 0 & 0 & e^t & \\ \vdots & & & & \ddots \end{pmatrix}.$$

For $t = 1$ and the element $e^{(1)} \in K$ this implies

$$e^A(e^{(1)}) = (e, eb_1, eb_2, \dots) \notin K\,,$$

since $(b_i)_{i \in \mathbb{N}} \notin K$. Hence, $A \in \Sigma(K) \setminus e(K)$.

The next proposition indicates some conditions on an ordered Banach space that guarantee the equality $e(K) = \Sigma(K)$.

Proposition 2.2.6 *In an ordered Banach space $(X, K, \|\cdot\|)$ with a closed cone K each of the conditions*

(a) $\operatorname{int} K \neq \varnothing$, *or*

(b) K *is proximinal*

implies $e(K) = \Sigma(K)$.

For (a) see [CHA$^+$87, Theorem 7.27] (cf. also Proposition A.3.4). For (b) the statement is shown in [EHO79, Theorem 1.11].

For a Banach lattice $(X, K, \|\cdot\|)$ the stronger statement $\Pi_1(K) = \Sigma(K)$ is satisfied due to Theorem 2.1.8. In particular, we have $e(K) = \Sigma(K)$, which is a special case of (b), since the cone in a normed vector lattice is proximinal, cf. Proposition 1.2.7 (ii).

In what follows, we consider certain ordered normed spaces $(X, K, \|\cdot\|)$ and investigate whether the equality

$$\Pi_1(X, K) = \Sigma(X, K)$$

is satisfied. In particular, we look for statements similar to Theorem 2.1.8. We start with an investigation of dense subspaces of Banach lattices. In the proof of the subsequent theorem we use the notation (1.13).

Theorem 2.2.7 *Let $(Y, K_Y, \|\cdot\|)$ be an ordered Banach space and let X be a dense linear subspace of Y ordered by the cone $K_X = X \cap K_Y$, where $K_Y \subseteq \overline{K_X}$. Assume that for each $f \in (K_Y)'$ the condition (2.4) is satisfied, i. e.*

$$f^{-1}(0) \cap K_Y \subseteq \overline{f^{-1}(0) \cap K_X}\,.$$

If $\Pi_1(Y, K_Y) = \Sigma(Y, K_Y)$, then $\Pi_1(X, K_X) = \Sigma(X, K_X)$.

Proof We have to show $\Sigma(X, K_X) \subseteq \Pi_1(X, K_X)$. Let $A \in \Sigma(X, K_X)$. Due to Proposition 2.1.13 (iii), one has

$$\hat{A} \in \Sigma(Y, K_Y) = \Pi_1(Y, K_Y)\,,$$

i. e. there is $s \in \mathbb{R}$ and $\hat{B} \in \mathcal{L}_+(Y)$ such that $\hat{A} = s\hat{I} + \hat{B}$, where $\hat{I}$ is the identity operator in $\mathcal{L}(Y)$. Since $A = \hat{A}|_X \in \mathcal{L}(X)$, one gets $B = \hat{B}|_X \in \mathcal{L}(X)$. Moreover, $B(K_X) = \hat{B}(K_X) \subseteq \hat{B}(K_Y) \subseteq K_Y$. Consequently, $B(K_X) \subseteq X \cap K_Y = K_X$, i. e. $B \in \mathcal{L}_+(X)$, which implies $A = sI + B \in \Pi_1(X)$. $\square$

Corollary 2.2.8 *Let $(Y, K_Y, \|\cdot\|)$ be a Banach lattice and let X be a dense linear subspace of Y, ordered by the cone $K_X = X \cap K_Y$, where $K_Y \subseteq \overline{K_X}$. Let for each $f \in (K_Y)'$ the condition (2.4) be satisfied. Then for an operator $A \in \mathcal{L}(X)$ the following conditions are equivalent:*

(i) *$A \in \Sigma(X, K_X)$, and*

(ii) *$A + \|A\|I \geq 0$.*

Thus, $\Pi_1(X, K_X) = \Sigma(X, K_X)$.

Since Y is a Banach lattice, K_Y is closed, so $K_Y \subseteq \overline{K_X}$ implies $K_Y = \overline{K_X}$.

In the case that K_Y has an interior point, according to Remark 2.1.14 it is sufficient to verify the assumption $f^{-1}(0) \cap K_Y \subseteq \overline{f^{-1}(0) \cap K_X}$ in Theorem 2.2.7 (and Corollary 2.2.8, respectively) for all functionals $f \in \operatorname{ext} F$, where F is a base of $(K_Y)'$.

Example 2.2.9 The space $X = C^1[0,1]$ is a dense linear subspace of the Banach lattice $Y = C[0,1]$, equipped with its natural norm and ordering. One can identify the corresponding dual spaces (cf. Corollary 1.2.24 (i)). So, the set of atoms in X' is given in (1.24). We show that for each $t_0 \in [0,1]$ the inclusion

$$\varepsilon_{t_0}^{-1}(0) \cap K_Y \subseteq \overline{\varepsilon_{t_0}^{-1}(0) \cap K_X}$$

holds. Let $x \in \varepsilon_{t_0}^{-1}(0) \cap K_Y$, i. e. $x \geq 0$ and $x(t_0) = 0$. For each $n \in \mathbb{N}$ we construct a function $x_n \in C^1[0,1]$ such that $x_n \geq 0$, $x_n(t_0) = 0$ and $\|x - x_n\| \leq \frac{1}{n+1}$, where we use a similar argument as in Example 1.4.3. Let $r = \frac{1}{n+1}$, then by density there is $v \in C^1[0,1]$ in the $\frac{r}{4}$-neighbourhood of the function $x - \frac{r}{2}\mathbb{1}$. Put $a = 0 \vee v$, then $a \leq x$, and there is a neighbourhood of t_0 such that a equals 0 there, which yields that a is continuously differentiable there. Due to 1.4.3 (ii) there are at most countably many points in $[0,1]$ where a is not differentiable. If a is not differentiable at some point $s \in [0,1]$, then there is $\delta > 0$ such that a is continuously differentiable on $(s-\delta, s) \cup (s, s+\delta)$, and $a(t) < x(t)$ for all $t \in [s-\delta, s+\delta]$, so $t_0 \notin [s-\delta, s+\delta]$. This is sufficient to employ the construction given in 1.4.3, which yields the desired function x_n. The sequence $(x_n)_{n\in\mathbb{N}}$ is contained in $\varepsilon_{t_0}^{-1}(0) \cap K_X$ and converges to x. So, as a consequence of Corollary 2.2.8, for any operator $A \in \Sigma(X, K_X)$ one has $A + \|A\|I \geq 0$.

In the previous section we considered ordered Banach spaces that possess a shrinking positive Schauder basis. In this case the class of positive-off-diagonal operators and the class of operators that dominate a multiple of the identity operator coincide. This result is a consequence of the Propositions 2.1.10 and 2.1.12, where (A.8) and (A.10) are used.

Proposition 2.2.10 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a shrinking positive Schauder basis that has the basis constant s. For an operator $A \in \mathcal{L}(X)$ the following conditions are equivalent:*

(i) *$A \in \Sigma(K)$, and*

(ii) *$A + 2s\|A\|I \geq 0$.*

Thus, $\Pi_1(K) = \Sigma(K)$.

We conclude the section with some remarks concerning positive-off-diagonal operators on the space $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed with the Euclidean norm and ordered by an arbitrary closed generating cone K. One has $\operatorname{int} K \neq \varnothing$ due Proposition 1.3.6 (ii), so Proposition 2.2.6 implies $e(K) = \Sigma(K)$. Now Proposition 2.2.3 (iii) yields that $\Sigma(K)$ is the closure of $\Pi_1(K)$ in $\mathcal{L}(\mathbb{R}^n)$. To establish the equality $\Pi_1(K) = \Sigma(K)$, it has to be shown that $\Pi_1(K)$ is closed. This is true for certain cones in $\mathbb{R}^n$ which need not be lattice cones. $(\mathbb{R}^n, K)$ is a vector lattice if and only if K has exactly n extreme rays. If K is a finitely generated cone in $\mathbb{R}^n$, i. e. K has finitely many extreme rays, it turns out that both $\mathcal{L}_+(X)$ and $\Pi_1(K)$ are the positive-linear hull of finitely many operators, so $\Pi_1(K)$ is closed. This line of reasoning has been used to establish the following statement.

Proposition 2.2.11 [SV70, Theorem 8] *If K is a generating and finitely generated cone in $\mathbb{R}^n$, then*

$$\Pi_1(K) = \Sigma(K)\,.$$

However, the equality $\Pi_1(K) = \Sigma(K)$ for a generating cone K in $\mathbb{R}^n$ does not imply that K is finitely generated. An example of a cone K that is not finitely generated, but yields $\Pi_1(K) = \Sigma(K)$, is described in [GKT95, Example 6.2]. If $n \geq 3$ and K is a strictly convex or smooth cone in $\mathbb{R}^n$, in particular, if K is a circular cone, then $\Pi_1(K)$ is not closed [GKT95, Theorem 4.3], so $\Pi_1(K) \neq \Sigma(K)$.
We finally remark that in the latter case a weaker relation is investigated. For an arbitrary ordered normed space $(X, K, \|\cdot\|)$ one has

$$\Pi_1(K) = \mathcal{L}_+(X) + \mathbb{R}I \subseteq \mathcal{L}_+(X) + \Sigma^b(K) \subseteq \Sigma(K)\,,$$

where $\Sigma^b(K)$ denotes the blade of the wedge $\Sigma(K)$. The question arises for which ordered normed spaces the equality

$$\mathcal{L}_+(X) + \Sigma^b(K) = \Sigma(K) \tag{2.6}$$

holds. This equality is satisfied provided K is an ellipsoidal cone in $\mathbb{R}^n$, see [SW94, Theorem 4.2 and Remark 4.2]. In this case

$$\Sigma^b(K) = \{A \in \mathcal{L}(X)\colon\ e^{At}(\partial K) \subseteq \partial K \text{ for all } t \geq 0\}\,,$$

cf. [SW94, Lemma 4.1], or [SW91]. That the equality (2.6) fails for many cones K even in $\mathbb{R}^n$ is shown in [GKT95].

2.3 Positive-off-diagonal Operators on Spaces with the Riesz Decomposition Property

This section is dedicated to ordered normed spaces that, in addition, have the Riesz decomposition property. For instance, the space $C^1[0,1]$ belongs to this class of spaces, cf. Example 1.4.3. A density argument has been used to establish that the set of all positive-off-diagonal operators on $C^1[0,1]$ and the set of operators that dominate a multiple of the identity operator coincide, cf. Example 2.2.9.
We study ordered normed spaces $(X, K, \|\cdot\|)$ with the Riesz decomposition property, where we investigate which additional conditions on the cone allow to show the equality $\Sigma(K) = \Pi_1(K)$ directly. We will use different techniques to obtain results that are similar to Theorem 2.1.8, where (ii) is replaced by $A + C\|A\|I \geq 0$ for a certain constant $C \in \mathbb{R}$, which depends on constants of the cone in the underlying space.

In the first approach a total set of certain linear functionals is required. We present an extended version of our investigation published in [Kal03b]. Let (X, K) be a partially ordered vector space. Consider for a linear positive functional f the set $f^{-1}(0) \cap K$, which is a cone in the hyperplane $f^{-1}(0)$ (or equals $\{0\}$). We are interested in such positive linear functionals f for which the set $f^{-1}(0) \cap K$ is a generating cone in $f^{-1}(0)$. In this case, the kernel $f^{-1}(0)$ of f, equipped with the ordering induced by X, is a directed partially ordered vector space.

Definition 2.3.1 A functional $f \in K^*$, $f \neq 0$, is called a *functional with directed kernel*[1], if $f^{-1}(0) = (f^{-1}(0) \cap K) - (f^{-1}(0) \cap K)$.

Proposition 2.3.2 *A functional with directed kernel is an atom of K^*.*

Proof Let $f \in K^*$ be a functional with directed kernel and $g \in K^*$ with $0 \leq g \leq f$. Let $x \in f^{-1}(0)$, then there are $x_1, x_2 \in f^{-1}(0) \cap K$ with $x = x_1 - x_2$. One has $0 = f(x_1) \geq g(x_1) \geq g(x)$. Similarly, $f(-x) = 0$ yields $0 \geq g(-x)$. So, $g(x) = 0$, which implies $f^{-1}(0) \subseteq g^{-1}(0)$. From this $g = \lambda f$ follows for some $\lambda \in [0, \infty)$. $\square$

In a vector lattice the converse statement to Proposition 2.3.2 holds as well.

Proposition 2.3.3 *If (X, K) is a vector lattice, then $0 < f \in X^*$ is a functional with directed kernel if and only if f is an atom of K^*.*

Proof Let f be an atom of K^* and $x \in f^{-1}(0)$. Due to Proposition 1.1.10 f is a lattice homomorphism, so

$$f(x^+) = f(x \vee 0) = f(x) \vee f(0) = 0\,,$$

and, analogously, $f(x^-) = 0$. Hence, $x^+, x^- \in f^{-1}(0) \cap K$, and $x = x^+ - x^-$. $\square$

In general partially ordered vector spaces, the atoms of K^* need not have a directed kernel. We carry this out for the following example which we need later on.

Example 2.3.4 Let the space $X = \mathbb{R}^3$ be ordered by the 3-dimensional ice-cream cone $K = K_3$ (cf. Example 2.1.7). We already observed that $K^* = K$. The functional

$$f = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in K^*$$

is an atom of K^*. Indeed, let

$$g = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in X^*$$

be such that $0 \leq g \leq f$. Then $v_3 \geq 0$ and $v_1^2 + v_2^2 \leq v_3^2$ imply

$$|v_2| \leq v_3\,. \tag{2.7}$$

On the other hand, one has

$$f - g = \begin{pmatrix} -v_1 \\ 1-v_2 \\ 1-v_3 \end{pmatrix} \in K^*\,,$$

which (by definition) means $v_3 \leq 1$ and

$$v_1^2 + (1 - v_2)^2 \leq (1 - v_3)^2\,, \quad \text{i. e.} \quad v_1^2 \leq (-2 + v_2 + v_3)(v_3 - v_2)\,.$$

[1]Functionals with directed kernel are applied e. g. in [Sch77].

Now (2.7) and $v_3 \leq 1$ imply that $-2 + v_2 + v_3 \leq 0$, and therefore $v_3 - v_2 \leq 0$. Together with (2.7) this gives $v_2 = v_3$, and, hence, $v_1 = 0$. So,

$$g = \begin{pmatrix} 0 \\ v_2 \\ v_2 \end{pmatrix} = v_2 f \,,$$

i. e. f is an atom. The kernel of f is

$$f^{-1}(0) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \, x_3 = -x_2 \right\} .$$

For the induced cone in $f^{-1}(0)$ one gets

$$f^{-1}(0) \cap K = \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : \, t \in [0, \infty) \right\} .$$

So, $f^{-1}(0)$ is not directed. For instance, the vector

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in f^{-1}(0)$$

can not be represented as the difference of two vectors of $f^{-1}(0) \cap K$.

For spaces enjoying the Riesz decomposition property, a statement similar to Proposition 2.3.3 can not be expected in general. The next two examples show that for a partially ordered vector space the property of atoms in K^* to have a directed kernel and the Riesz decomposition property are independent traits of the space.

Example 2.3.5 Let $X = \mathbb{R}^3$ and let K be that finitely generated cone which is generated by the set $\{x^{(i)} \colon i = 1, 2, 3, 4\}$, where

$$x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} , \; x^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} , \; x^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} , \; x^{(4)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} .$$

It is straightforward that $K^* = \text{pos}\{f^{(i)} \colon i = 1, 2, 3, 4\}$, where

$$f^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \; f^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} , \; f^{(3)} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} , \; f^{(4)} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} .$$

The set of all atoms of K^* is

$$\{\lambda f^{(i)} \colon \; \lambda \in (0, \infty), \; i = 1, 2, 3, 4\} .$$

The kernel of the atom $f^{(1)}$ is the set $\text{span}\{x^{(3)}, x^{(4)}\}$, which is directed. Analogously, all atoms of K^* have a directed kernel. From Proposition 1.4.1 it follows that the space X does not possess the Riesz decomposition property.

Example 2.3.6 Consider the space $X = C^1[0, 1]$ with the norm and ordering induced from $C[0, 1]$. The evaluation maps ε_t for $t \in [0, 1]$ are atoms in X'. If $t_0 \in (0, 1)$, then $\varepsilon_{t_0}^{-1}(0)$ is not directed. Indeed, let $x(t) = t - t_0 \in \varepsilon_{t_0}^{-1}(0)$. Assume that there are $x_1, x_2 \in \varepsilon_{t_0}^{-1}(0)$ with $x_1, x_2 \geq 0$ such that $x = x_1 - x_2$. Since the functions x_1 and x_2 attain their minimum at the point t_0, for the derivatives at t_0 one has $x_1'(t_0) = x_2'(t_0) = 0$. Now $x'(t_0) = x_1'(t_0) - x_2'(t_0) = 0$ yields a contradiction, since $x'(t_0) = 1$. Hence, ε_{t_0} does not have a directed kernel. However, according to Example 1.4.3, the space X has the Riesz decomposition property.

In order to prove the first main result of this section (Theorem 2.3.11) we need some preliminary properties of functionals with directed kernel. We consider an ordered normed space $(X, K, \|\cdot\|)$. Clearly, if $f \in K'$ is an atom of K^*, then f is an atom of K'. So, by Proposition 2.3.2 each functional $f \in K'$ with directed kernel is an atom of K'.

Lemma 2.3.7 *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property, where K is a normal cone. Then for every functional $f \in K'$ with directed kernel there is a constant $C > 0$ such that for every $y \in f^{-1}(1) \cap K$ there is an element $z \in f^{-1}(1)$ with $z \le y$ and $\|z\| \le C$.*

Proof Let N be the constant of semi-monotony of the norm $\|\cdot\|$ and let $f \in K'$ be a functional with directed kernel. If $f^{-1}(1) \cap K \neq \varnothing$, fix an element $y_0 \in f^{-1}(1) \cap K$. We show that for the required constant one can take $C = N\|y_0\|$. Let y be an arbitrary element of $f^{-1}(1) \cap K$. The element $x = y - y_0$ lies in the kernel $f^{-1}(0)$ and can be decomposed into $x = x_1 - x_2$, where $x_1, x_2 \ge 0$ and $x_1, x_2 \in f^{-1}(0)$. Hence we get $0 \le y \le y + x_2 = x_1 + y_0$. Due to the Riesz decomposition property there are elements $w, z \in K$ such that $y = w + z$, where $w \le x_1$ and $z \le y_0$. Now $f(x_1) = 0$ implies $f(w) = 0$, hence $f(z) = 1$. Obviously, $z \le y$ and $\|z\| \le N\|y_0\| = C$. $\square$

Before applying Lemma 2.3.7 for the proof of the next result, we provide two examples which show that in general such a constant C need not exist, and, in another situation, how to calculate it.

Example 2.3.8 Consider the space $X = \mathbb{R}^3$, equipped with the 3-dimensional ice-cream cone $K = K_3$ and the Euclidean norm. The functional

$$f = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in X'$$

is an atom of K', which was shown in Example 2.3.4. One has

$$f^{-1}(0) \cap K = \left\{ \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix} :\ t \ge 0 \right\} \text{ and } f^{-1}(1) \cap K = \left\{ \begin{pmatrix} s \\ 1-r \\ r \end{pmatrix} :\ s^2 \le 2r - 1 \right\}.$$

The set $f^{-1}(0) \cap K$ is a ray. For a given element $y \in f^{-1}(1) \cap K$ we investigate the set $f^{-1}(1) \cap (y - K)$. For $r \ge \frac{1}{2}$ consider the element

$$y = \begin{pmatrix} \sqrt{2r-1} \\ 1-r \\ r \end{pmatrix} \in f^{-1}(1) \cap K\,. \tag{2.8}$$

For an element $z \in f^{-1}(1) \cap (y - K)$, i. e. $z \in f^{-1}(1)$ and $z \le y$, one has

$$y - z \in f^{-1}(0) \cap K$$

and so $y - z = \begin{pmatrix} 0 \\ -t \\ t \end{pmatrix}$ for some number $t \ge 0$. This implies

$$z = \begin{pmatrix} \sqrt{2r-1} \\ 1-r+t \\ r-t \end{pmatrix}.$$

For each element $z \in f^{-1}(1) \cap (y - K)$ one has $\|z\| \ge \sqrt{2r-1}$.
If now $C > 0$ is an arbitrary number and y is given according to (2.8) with $r > C^2 + 1$, then for all elements $z \in f^{-1}(1) \cap (y - K)$ one has $\|z\| > C$.

Example 2.3.9 Consider the Banach lattice $X = C[0,1]$ equipped with its natural norm and ordering, where $K = C_+[0,1]$. Due to Proposition 2.3.3 each atom of K' has a directed kernel. Let f be an atom of K', then according to (1.24) there are $t_0 \in [0,1]$ and $s \in (0,\infty)$ such that $f = s\varepsilon_{t_0}$. For every $y \in K$ with $f(y) = sy(t_0) = 1$ there is a function z such that $f(z) = 1$, i. e. $z(t_0) = \frac{1}{s}$, $0 \leq z(t) \leq y(t)$ for all $t \in [0,1]$ and $\|z\| = \frac{1}{s}$. So, the according constant C for the atom f is $C = \frac{1}{s}$.

Now we turn back to the main line and apply Lemma 2.3.7 in order to show the next result.

Theorem 2.3.10 *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property where K is a normal non-flat cone. Let $f \in K'$ be a functional with directed kernel. If $g \in X'$ is such that $f \perp g$ and $g(x) \geq 0$ for each $x \in f^{-1}(0) \cap K$, then $g \in K'$.*

Proof By means of Theorem 1.2.17, the assumptions imply, in particular, that (X', K') is a vector lattice. Let the functional $f \in K'$ have a directed kernel and let $g \in X'$, $g \neq 0$, be such that $f \perp g$ and $g(x) \geq 0$ for each $x \in f^{-1}(0) \cap K$. Let $x \in K$, $x \neq 0$. Because of

$$0 = (f \wedge |g|)(x) = \inf\{f(y) + |g|(x-y)\colon\ y \in [0,x]\}$$

for every $n \in \mathbb{N}$ there is an element $x_n \in [0,x]$ such that

$$f(x_n) + |g|(x - x_n) \leq \tfrac{1}{n}\,. \tag{2.9}$$

This implies $f(x_n) \leq \frac{1}{n}$ and due to (1.12) also

$$|g(x) - g(x_n)| = |g(x - x_n)| \leq |g|(x - x_n) \leq \tfrac{1}{n}\ . \tag{2.10}$$

In the case $f(x_n) = 0$, the positivity of g on $f^{-1}(0) \cap K$ ensures $g(x_n) \geq 0$. If $f(x_n) > 0$, then we obtain a lower bound for $g(x_n)$ as follows: Since f has a directed kernel, there is a constant $C > 0$ for f according to Lemma 2.3.7 such that for the element

$$\tfrac{1}{f(x_n)} x_n \in f^{-1}(1) \cap K$$

there exists an element $z_n \in f^{-1}(1)$ satisfying the inequalities

$$z_n \leq \tfrac{1}{f(x_n)} x_n \quad \text{and} \quad \|z_n\| \leq C\,.$$

Then the element $w_n = x_n - f(x_n) z_n$ lies in $f^{-1}(0) \cap K$, and one has

$$\|x_n - w_n\| = f(x_n)\|z_n\| \leq f(x_n) C \leq \tfrac{C}{n}\,.$$

Moreover, $g(w_n) \geq 0$. Therefore, from

$$|g(x_n) - g(w_n)| \leq \|g\|\,\|x_n - w_n\| \leq \|g\| \tfrac{C}{n}$$

we conclude

$$g(x_n) \geq -\tfrac{\|g\|C}{n} \,. \tag{2.11}$$

Now we prove the assertion by way of contradiction. Suppose that there is a vector $x > 0$ such that $g(x) < 0$. Take

$$n > \tfrac{\|g\|C+1}{-g(x)}$$

and the according x_n such that (2.9) is satisfied. Then $-g(x) > \tfrac{\|g\|C}{n} + \tfrac{1}{n}$, and the inequality (2.11) implies

$$g(x_n) - g(x) > \tfrac{-\|g\|C}{n} + \tfrac{\|g\|C}{n} + \tfrac{1}{n} = \tfrac{1}{n}\,,$$

a contradiction to (2.10). □

Given an ordered normed space $(X, K, \|\cdot\|)$, we come back to the representation of positive-off-diagonal operators as elements of the class $\Pi_1(K)$. Considering the statement in Theorem 2.1.8, we replace the assumption on the underlying space to be a Banach lattice by the Riesz decomposition property, in combination with additional conditions on the cone and the norm. In the proof we make use of the normality and the non-flatness of the dual cone. These properties are guaranteed according to Corollary 1.2.12 by the same properties required for the cone K.

Theorem 2.3.11 *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property and let K be a closed normal non-flat cone. Let $\hat{N}$ be the constant of semi-monotony of the norm in X' and $\hat{\kappa}$ the constant of non-flatness of the cone K'. Suppose that there is a total set $M \subseteq K'$ such that each $f \in M$ possesses a directed kernel. Then for an operator $A \in \mathfrak{L}(X)$ the following conditions are equivalent:*

(i) $A \in \Sigma(K)\,,$

(ii) $A + \hat{N}\hat{\kappa}\|A\|I \geq 0\,.$

Proof We already mentioned that condition (ii) implies condition (i). Now assume that A is a positive-off-diagonal operator. We have to show $\hat{N}\hat{\kappa}\|A\|x + Ax \in K$ for every $x \in K$. Since there is a total set $M \subseteq K'$, it suffices to show

$$f(\hat{N}\hat{\kappa}\|A\|x + Ax) \geq 0$$

for each $f \in M$. Fix a functional $f \in M$. Due to Theorem 1.2.17 the dual space (X', K') is a Dedekind complete vector lattice. Therefore any principal band in X' is a projection band. So, for the principal band B_f one has $X' = B_f \oplus B_f^d$. Since f has a directed kernel, Proposition 2.3.2 in combination with the remarks after Example 2.3.6 yield that f is an atom of K'. Since (X', K') is Dedekind complete and hence Archimedean, Proposition 1.1.7 yields the representation

$$B_f = \{\lambda f\colon\ \lambda \in \mathbb{R}\}\,.$$

This allows us to decompose the element $A'f$ as $A'f = f_1 + f_2$, where $f_1 = \lambda f$ for some $\lambda \in \mathbb{R}$, $f_2 \perp f$, and A' denotes the dual operator defined by (1.11). If we show the both properties
(a) $f_2 \geq 0$ and
(b) if $\lambda < 0$, then $|\lambda| \leq \hat{N}\hat{\kappa}\|A\|$,
then we can conclude

$$\begin{aligned} f(\hat{N}\hat{\kappa}\|A\|x + Ax) &= \hat{N}\hat{\kappa}\|A\|f(x) + (A'f)(x) \\ &= \hat{N}\hat{\kappa}\|A\|f(x) + \lambda f(x) + f_2(x) \\ &= (\hat{N}\hat{\kappa}\|A\| + \lambda)f(x) + f_2(x) \\ &\geq 0\,. \end{aligned}$$

Property (a): According to Theorem 2.3.10 it suffices to show $f_2(x) \geq 0$ only for any $x \in f^{-1}(0) \cap K$. If $x \in f^{-1}(0) \cap K$, then $f_1(x) = 0$, and $f(Ax) \geq 0$ since A is a positive-off-diagonal operator. By using the decomposition of $A'f$, one has

$$f_2(x) = (A'f)(x) - f_1(x) = f(Ax) \geq 0\,.$$

Consequently, $f_2 \in K'$.
Property (b): If $\lambda < 0$, then the above representation of $A'f$ can be written as

$$A'f = f_2 - (-\lambda f)\,,$$

where $f_2, -\lambda f \in K'$ and $f_2 \perp (-\lambda f)$. Since such a decomposition is unique, we have $f_2 = (A'f)^+$ and $-\lambda f = (A'f)^-$. Now (1.9) yields

$$|\lambda|\,\|f\| = \|-\lambda f\| \leq \hat{N}\hat{\kappa}\|A'f\| \leq \hat{N}\hat{\kappa}\|A'\|\,\|f\| = \hat{N}\hat{\kappa}\|A\|\,\|f\|\,,$$

so $|\lambda| \leq \hat{N}\hat{\kappa}\|A\|$. □

If the constants N of semi-monotony of $\|\cdot\|$ and κ of non-flatness of the cone K are given, then according to the Propositions 1.2.11 and 1.2.19 for the constants $\hat{N}$ and $\hat{\kappa}$ one gets

$$\hat{N} = 2\kappa \quad \text{and} \quad \hat{\kappa} = 2N\kappa\,.$$

Therefore the last result can be given in the following formulation.

Corollary 2.3.12 *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property and let K be a closed normal non-flat cone, where N is the constant of semi-monotony of the norm $\|\cdot\|$ and κ is the constant of non-flatness of K. Suppose that there is a total set $M \subseteq K'$ such that each $f \in M$ possesses a directed kernel. Then for an operator $A \in \mathfrak{L}(X)$ the following conditions are equivalent:*

(i) $A \in \Sigma(K)$,

(ii) $A + 4N\kappa^2\|A\|I \geq 0$.

Example 2.3.13 Consider the space $C[0,2]$, equipped with its natural norm and ordering, and its subspace

$$X = \{x \in C[0,2]\colon\ x(1) = x(0) + x(2)\}\,,$$

where K is the induced cone in X. In [Nam57, Example 8.10] it is shown that (X,K) is not a vector lattice, but satisfies the Riesz decomposition property. Moreover, X is a Banach space and K is closed.
We investigate further properties of this well-known space X concerning the assumptions in Theorem 2.3.11. The cone K is normal, where the constant N of semi-monotony equals 1. K has a non-empty interior. Indeed, the function

$$e(t) = \begin{cases} t+1 & \text{for } t \in [0,1] \\ -t+3 & \text{for } t \in [1,2] \end{cases}$$

is an interior point of K, where $\|e\| = 2$ and the closed ball $\mathrm{B}(e,1)$ belongs to K. Due to Proposition 1.3.1 the cone K is non-flat, where (1.15) yields the constant of non-flatness $\kappa = 3$. We investigate the set

$$M = \{\varepsilon_t \colon t \in [0,1) \cup (1,2]\}$$

of evaluation maps. M is a total subset of K'. We show that each functional in M has a directed kernel.
Fix $s \in [0,1) \cup (1,2]$. In order to show that $\varepsilon_s^{-1}(0) \cap K$ is a generating cone in the hyperplane $\varepsilon_s^{-1}(0) = \{x \in X\colon\ x(s) = 0\}$, we have to decompose each element of $\varepsilon_s^{-1}(0)$ into the difference of two elements of $\varepsilon_s^{-1}(0) \cap K$. Fix an element $x \in \varepsilon_s^{-1}(0)$. The element x belongs to the vector lattice $C[0,2]$, where x can be decomposed as $x = x^+ - x^-$ with the non-negative functions

$$x^+(t) = \max\{0, x(t)\} \quad \text{and} \quad x^-(t) = \max\{0, -x(t)\}\,.$$

Note that $x^+(s) = x^-(s) = 0$. In general, the functions x^+, x^- may not belong to X. Consider the following two cases:

Case (a): If the values $x(0)$ and $x(2)$ are both non-negative, then x^+ and x^- both belong to the space X. Indeed, one has $x^+(0) = x(0)$, $x^+(2) = x(2)$ and $x(1) = x(0) + x(2) \geq 0$, hence $x^+(1) = x(1) = x^+(0) + x^+(2)$. So, $x^+ \in X$. Because of

$$x^-(0) = x^-(2) = x^-(1) = 0$$

one has $x^- \in X$. The case $x(0) \leq 0$, $x(2) \leq 0$ is considered analogously.

Case (b): If $x(0)$ and $x(2)$ have different signs, say $x(0) > 0$ and $x(2) < 0$, then x^+ and x^- may not belong to X. However, one can find another representation $x = x_1 - x_2$ such that $x_1, x_2 \geq 0$, $x_1, x_2 \in X$, $x_1(s) = x_2(s) = 0$. First observe that

$$x(0) - x^+(1) \geq 0\,.$$

Indeed, if $x(1) < 0$, then $x^+(1) = 0 \le x(0)$. If $x(1) \ge 0$, then

$$x^+(1) = x(1) = x(0) + x(2) \le x(0)\,.$$

Take an element $w \in C[0,2]$ such that $w \ge 0$, $w(0) = w(2) = w(s) = 0$ and

$$w(1) = x(0) - x^+(1)\,.$$

The required decomposition of x is defined as

$$x_1 = x^+ + w \quad\text{and}\quad x_2 = x_1 - x\,.$$

Indeed, one has $x_1 \ge 0$ and $x_2 = x^+ + w - (x^+ - x^-) = w + x^- \ge 0$. Moreover, $x_1(s) = x_2(s) = 0$. It remains to show that $x_1, x_2 \in X$. For x_1 this follows from

$$\begin{aligned} x_1(0) + x_1(2) &= x^+(0) + w(0) + x^+(2) + w(2) \\ &= x(0) \\ &= x(0) - x^+(1) + x^+(1) \\ &= w(1) + x^+(1) \\ &= x_1(1)\,, \end{aligned}$$

i. e. $x_1 \in X$. For x_2 we use that $x_1 \in X$ and have

$$\begin{aligned} x_2(0) + x_2(2) &= x_1(0) - x(0) + x_1(2) - x(2) \\ &= x_1(0) + x_1(2) - (x(0) + x(2)) \\ &= x_1(1) - x(1) \\ &= x_2(1)\,, \end{aligned}$$

therefore $x_2 \in X$.

Notice that for the functional ε_1 one has $\varepsilon_1 = \varepsilon_0 + \varepsilon_2$, so it is not an atom. Due to Proposition 2.3.2, the functional ε_1 does not have a directed kernel. Consider e. g. an element $x \in \varepsilon_1^{-1}(0)$ with $x(0) \neq 0$, and assume $x = x_1 - x_2$ for some $x_1, x_2 \in \varepsilon_1^{-1}(0) \cap K$. Then $0 = x_1(1) = x_1(0) + x_1(2) \ge 0$ implies, in particular, $x_1(0) = 0$. Analogously, $x_2(0) = 0$. Finally $x(0) = x_1(0) - x_2(0) = 0$ yields a contradiction.

To sum up, the space X satisfies all assumptions of Theorem 2.3.11. Due to Corollary 2.3.12 for each positive-off-diagonal operator $A \in \mathcal{L}(X)$ one has $A + 36\|A\|I \ge 0$.

We now investigate another approach to show for a space $(X, K, \|\cdot\|)$ and an operator $A \in \mathcal{L}(X)$ the equivalence of the statements $A \in \Sigma(K)$ and $A + C\|A\|I \ge 0$ for a certain constant $C \in \mathbb{R}$, where we employ a modification of a technique which is given in the proofs of [Are86, Lemma 1.10 and Theorem 1.11]. For the proof of the subsequent theorem we need the following result.

Lemma 2.3.14 *Let (X, K) be a vector lattice and $\|\cdot\|$ a norm on X such that K is normal and non-flat. Let D be a linear subspace of X and let $A\colon D \to X$ be a linear operator. The operator A is positive-off-diagonal if and only if for every $x \in D \cap K$ the element $(Ax)^-$ belongs to the closure of the principal ideal I_x.*

Proof Let $A\colon D \to X$ be a positive-off-diagonal operator. Fix $x \in D \cap K$ and consider the principal ideal I_x. By way of contradiction, we assume $(Ax)^- \notin \overline{I_x}$. Then there is a functional $f \in X'$ such that $f(z) = 0$ for each $z \in \overline{I_x}$ and $f((Ax)^-) \neq 0$. Since I_x is a vector sublattice of X, for any element $z \in I_x$ one has $z = z^+ - z^-$ with $z^+, z^- \in I_x \cap K$, so

$$f^+(z^+) = (f \vee 0)(z^+) = \sup\{f(y)\colon\ y \in [0, z^+]\} = 0$$

and, analogously, $f^+(z^-) = 0$, so $f^+(z) = 0$. Similarly, one has $f^-(z) = 0$. The relation $f((Ax)^-) \neq 0$ implies that at least one of the numbers $f^+((Ax)^-)$, $f^-((Ax)^-)$ is positive. Without loss of generality assume $f^+((Ax)^-) > 0$. Otherwise replace f by $-f$. Define the functional g_0 on K by means of

$$g_0(v) = \sup\{f^+(w)\colon\ w \in [0, v] \cap I_{(Ax)^-}\} \text{ for all } v \in K\,.$$

For any $v \in K$ one has $0 \leq g_0(v) \leq f^+(v)$. For x and $(Ax)^-$ we get

$$0 \leq g_0(x) \leq f^+(x) = 0$$

$$\text{and } \ g_0((Ax)^-) = f^+((Ax)^-) > 0\,.$$

Since $(Ax)^+$ and $(Ax)^-$ are disjoint, according to (1.5) there follows

$$g_0((Ax)^+) = \sup\{f^+(w)\colon\ w \in [0, (Ax)^+] \cap I_{(Ax)^-}\} = f^+(0) = 0\,.$$

Since X has the Riesz decomposition property, for any $v_1, v_2 \in K$ the equality (1.1) is satisfied, and it is straightforward that

$$g_0(v_1 + v_2) = g_0(v_1) + g_0(v_2)\,.$$

Applying Proposition 1.1.11, there is a unique positive linear functional g on X which extends g_0. For $v \in X$ we have

$$\begin{aligned} |g(v)| &= |g_0(v^+) - g_0(v^-)| \\ &\leq f^+(v^+) + f^+(v^-) \\ &\leq \|f^+\|\,\|v^+\| + \|f^+\|\,\|v^-\| \\ &\leq \|f^+\|C\|v\| \end{aligned}$$

for a constant $C > 0$ according to (1.9), so $g \in X'$. One has $g(x) = 0$ and

$$g(Ax) = g((Ax)^+) - g((Ax)^-) < 0\,,$$

which contradicts the assumption that A is positive-off-diagonal.
Vice versa, let $x \in D \cap K$ be such that $(Ax)^- \in \overline{I_x}$, and consider $f \in K'$ with $f(x) = 0$. Then $f(y) = 0$ for every $y \in \overline{I_x}$. In particular, $f((Ax)^-) = 0$, so

$$f(Ax) = f((Ax)^+ - (Ax)^-) \geq 0\,. \quad \square$$

Theorem 2.3.15 *Let (X, K) be a vector lattice with the principal projection property and $\|\cdot\|$ a norm on X such that K is normal and non-flat with the according constants N and κ, respectively. Then for an operator $A \in \mathfrak{L}(X)$ the following conditions are equivalent:*

(i) $A \in \Sigma(K)$,

(ii) $A + 2N^2\kappa\|A\|I \geq 0$.

Proof Let $A \in \mathfrak{L}(X)$ be a positive-off-diagonal operator. If for some $\lambda \geq 0$ the operator $A + \lambda I$ is not positive, then for some $x \in K$ one has $y = (A + \lambda I)(x) \notin K$, i. e. $y^- > 0$. We show that in this case $\lambda \leq 2N^2\kappa\|A\|$.
Let B be the principal band generated by y^-, i. e. $B = \{y^-\}^{dd}$. Due to Remark 1.2.8 the cone K is closed, so the space X is Archimedean, which implies $B^{dd} = B$. One has $X = B \oplus B^d$ with the band projections $P_B \colon X \to B$ and $P_{B^d} \colon X \to B^d$, where $P_{B^d} = I - P_B$. So, one has

$$\begin{aligned} 0 &> -y^- \\ &= P_B(y) \\ &= P_B(Ax + \lambda x) \\ &= \lambda P_B(x) + P_B A P_B(x) + [P_B A(x) - P_B A P_B(x)] \end{aligned} \tag{2.12}$$

$$\text{and} \quad \begin{aligned} P_B A(x) - P_B A P_B(x) &= P_B A(I - P_B)(x) \\ &= P_B A P_{B^d}(x) . \end{aligned}$$

Let $w = P_{B^d}(x)$. Clearly, $w \geq 0$. Hence, $w \in B^d$ yields the inclusion $B_w = \{w\}^{dd} \subseteq B^d$, where B_w is the principal band generated by w. Now,

$$\begin{aligned} P_B A P_{B^d}(x) &= P_B A(w) \\ &= P_B((Aw)^+) - P_B((Aw)^-) , \end{aligned}$$

where due to Lemma 2.3.14 and Lemma 1.2.4 one has $(Aw)^- \in \overline{I_w} \subseteq B_w \subseteq B^d$ and hence $P_B((Aw)^-) = 0$. Consequently, $P_B A P_{B^d}(x) \geq 0$, and due to (2.12) one has

$$0 > \lambda P_B(x) + P_B A P_B(x) = (\lambda I + P_B A) P_B(x) .$$

In particular, this yields $P_B(x) \neq 0$. The inequality $0 \leq \lambda P_B(x) < -P_B A P_B(x)$, the semi-monotony of the norm and the estimation $\|P_B\| \leq 2N\kappa$ from Proposition 1.2.20 imply

$$\begin{aligned} \lambda\|P_B(x)\| &\leq N\|P_B A P_B(x)\| \\ &\leq N\|P_B\|\,\|A\|\,\|P_B(x)\| \\ &\leq 2N^2\kappa\|A\|\,\|P_B(x)\| \end{aligned}$$

and hence $\lambda \leq 2N^2\kappa\|A\|$.

Until now we have shown that for every $s > 2N^2\kappa\|A\|$ the operator $A + sI$ is positive. Since K is closed, the cone $\mathfrak{L}_+(X)$ is also closed. The sequence

$$\left(A + \left(2N^2\kappa\|A\| + \tfrac{1}{n}\right) I\right)_{n\in\mathbb{N}}$$

of positive operators converges to the operator $A + 2N^2\kappa\|A\|I$, therefore

$$A + 2N^2\kappa\|A\|I \geq 0\,. \quad \square$$

In order to obtain an analogue to the Theorem 2.3.15 for a non-lattice case, we start with a result ensuring the equality $e(K) = \Pi_1(K)$.

Theorem 2.3.16 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space such that K is closed and $(X', K', \|\cdot\|')$ is a vector lattice with the principal projection property, where $\|\cdot\|'$ is semi-monotone with the constant $\hat{N}$ of semi-monotony, and K' is non-flat with the constant $\hat{\kappa}$. For an operator $A \in \mathfrak{L}(X)$ the following conditions are equivalent:*

(i) $A \in e(K)\,,$

(ii) $A + 2\hat{N}^2\hat{\kappa}\|A\|I \geq 0\,.$

Proof Let e^{At} be the positive semigroup generated by A. Due to Proposition 1.2.21, for each $t \geq 0$ one has $(e^{At})' \geq 0$. Moreover, Proposition 1.2.22 implies $e^{A't} \geq 0$, so due to Proposition A.3.3 the operator A' is positive-off-diagonal. Since $(X', K', \|\cdot\|')$ satisfies all assumptions of Theorem 2.3.15, one gets

$$A' + 2\hat{N}^2\hat{\kappa}\|A'\|I' \geq 0\,.$$

Since K is closed and $(A+2\hat{N}^2\hat{\kappa}\|A\|I)' \geq 0$, Proposition 1.2.21 yields $A+2\hat{N}^2\hat{\kappa}\|A\|I \geq 0$. $\square$

Corollary 2.3.17 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with the Riesz decomposition property such that K is closed, non-flat with the constant κ and normal, where the constant of semi-monotony of the norm is N. For an operator $A \in \mathfrak{L}(X)$ the following conditions are equivalent:*

(i) $A \in e(K)\,,$

(ii) $A + 16N\kappa^3\|A\|I \geq 0\,.$

Proof Due to Theorem 1.2.17 the dual space (X', K') is a Dedekind complete vector lattice, due to Proposition 1.1.6 it has the principal projection property. Proposition 1.2.11 yields the constant 2κ of the semi-monotony of the dual norm, whereas Proposition 1.2.19 provides the constant $2N\kappa$ of non-flatness of K'. For an operator $A \in \mathfrak{L}(X)$ Theorem 2.3.16 implies $A \in e(K)$ if and only if

$$A + 2(2\kappa)^2 2N\kappa\|A\|I = A + 16N\kappa^3\|A\|I \geq 0\,. \ \square$$

The final result is a combination of Corollary 2.3.17 (where the equality $e(K) = \Pi_1(K)$ is established) and Proposition 2.2.6 (where $e(K) = \Sigma(K)$ is ensured).

Theorem 2.3.18 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with the Riesz decomposition property such that K is closed, non-flat with the constant κ and normal, where the constant of semi-monotony of the norm is N. Each of the conditions*

(a) $\operatorname{int} K \neq \varnothing$, *or*

(b) K *is proximinal*

ensures that for an operator $A \in \mathcal{L}(X)$ the following assertions are equivalent:

(i) $A \in \Sigma(K)$,

(ii) $A + 16N\kappa^3\|A\|I \geq 0$.

Remark 2.3.19 The ordered normed space in Example 2.3.13 satisfies the assumptions, in particular condition (a), of Theorem 2.3.18. Nevertheless, for this example Corollary 2.3.12 provides a stronger statement than Theorem 2.3.18.

2.4 M-Operators

We provide two different notions of an M-operator on an ordered vector space. If one considers the space $(\mathbb{R}^n, \mathbb{R}^n_+)$, then both definitions coincide with the notion of a (non-singular) M-matrix. The Definition 2.4.1 below is motivated by the Definition 2.1.2 of an M-matrix, whereas the Definition 2.4.3 is indicated by the statement in Proposition 2.1.3. As usual, $r(B)$ denotes the spectral radius of the operator B.

Definition 2.4.1 Let $(X, K, \|\cdot\|)$be an ordered normed space. An operator $A \in \mathcal{L}(X)$ is called an M_1-*operator* if there is an operator $B \in \mathcal{L}_+(X)$ and a number $s > r(B)$ such that $A = sI - B$.

The notion "M_1-operator" corresponds to the notion "non-singular M-operator" given in [MS90].

Remark 2.4.2 If an operator A is represented as $A = sI - B$ for some operator $B \in \mathcal{L}(X)$, then the condition $s > r(B)$ is satisfied if and only if the series

$$\sum_{k=0}^{\infty} \tfrac{1}{s^{k+1}} B^k \tag{2.13}$$

converges in the operator norm (where $B^0 = I$). In this case there exists the inverse $A^{-1} \in \mathcal{L}(X)$, where A^{-1} is given by (2.13). Moreover, if $B \geq 0$, then $A^{-1} \geq 0$.

Definition 2.4.3 Let $(X, K, \|\cdot\|)$ be an ordered normed space. An operator $A \in \mathcal{L}(X)$ is called an M_2-*operator* if A is negative-off-diagonal, there exists $A^{-1} \in \mathcal{L}(X)$ and $A^{-1} \geq 0$.

Clearly, every M_1-operator is negative-off-diagonal, and so Remark 2.4.2 implies that every M_1-operator is an M_2-operator. The inverse implication is not true in general.

Example 2.4.4 We provide an example of an M_2-operator that is not an M_1-operator. Consider the space $X = \mathbb{R}^3$, equipped with the 3-dimensional ice-cream cone K as in Example 2.1.7, and the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In 2.1.7 it is shown that A is negative-off-diagonal with respect to K. Moreover, A is invertible, where

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \geq 0.$$

Indeed, let without loss of generality

$$y = \left(\begin{smallmatrix} y_1 \\ y_2 \\ 1 \end{smallmatrix}\right) \in K,$$

i. e. $y_1^2 + y_2^2 \leq 1$, and let $x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right)$ be such that $x = A^{-1}y$, then $x_3 = 1$ and

$$x_1^2 + x_2^2 = \tfrac{1}{4}(y_1 + y_2)^2 + \tfrac{1}{4}(-y_1 + y_2)^2 = \tfrac{1}{2}(y_1^2 + y_2^2) \leq \tfrac{1}{2} < x_3,$$

so $x \in K$. Consequently, A is an M_2-operator. On the other hand, in Example 2.2.4 it is shown that there is no real number s such that $B = sI - A$ is positive, hence A is not an M_1-operator.

Due to Proposition 2.1.3 and Corollary 2.1.6, for the space $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$ the notions (non-singular) M-matrix, M_1-operator and M_2-operator coincide. We list some ordered normed spaces with the property that every M_2-operator is an M_1-operator, where we apply the results on positive-off-diagonal operators given in the previous sections.
If in an ordered normed space for every positive-off-diagonal operator A there is a number $s \in \mathbb{R}$ such that $A + sI \geq 0$, then for every negative-off-diagonal operator A there is a positive operator B and a number $s \in \mathbb{R}$ such that $A = sI - B$. Due to Remark 2.4.2, the existence of $A^{-1} \in \mathcal{L}(X)$ implies $s > r(B)$, so in such spaces A is an M_1-operator if and only if A is an M_2-operator. The following is a consequence of Proposition 2.2.11, Corollary 2.2.8, Proposition 2.2.10, Theorem 2.3.11 and Theorem 2.3.18, respectively.

Corollary 2.4.5 *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies one of the following conditions:*

(a) *$X = \mathbb{R}^n$ and K is generating and finitely generated;*

(b) *X is a dense linear subspace of a Banach lattice $(Y, K_Y, \|\cdot\|)$, ordered by the cone $K_X = X \cap K_Y$ such that $K_Y \subseteq \overline{K_X}$, and for each $f \in K_Y'$ one has*

$$f^{-1}(0) \cap K_Y \subseteq \overline{f^{-1}(0) \cap K_X}\,;$$

(c) *X is an ordered Banach space with a shrinking positive Schauder basis;*

(d) *X has the Riesz decomposition property, K is closed, normal and non-flat, and there is a total set $M \subseteq K'$ such that every $f \in M$ has a directed kernel;*

(e) *X is a Banach space that has the Riesz decomposition property, K is closed and normal, and one has $\operatorname{int} K \neq \varnothing$;*

(f) *X is a Banach space that has the Riesz decomposition property, K is proximinal, normal and generating.*

Then the notions M_1-operator and M_2-operator coincide.

The results in Corollary 2.4.5 provide only certain sufficient conditions for the both classes of operators to be equal, and might be a subject for further investigations.

Chapter 3

Maximum Principles

In the matrix theory, certain maximum principles are investigated and conditions are established for some classes of matrices to satisfy these principles, where mostly the natural, i. e. componentwise, order is considered. However, the so-called maximum principle for inverse column entries is generalized to the case of arbitrary cones in $\mathbb{R}^n$ in the literature. Moreover, for certain special cones in $\mathbb{R}^n$ geometrical characterizations are known.
This chapter provides a general approach to define corresponding maximum principles for positive operators that act on an ordered normed space, where the cone has a non-empty interior. We give geometrical conditions which are necessary and sufficient for an operator to satisfy a maximum principle. Several examples are listed.
Presuming the theory on maximum principles for M-matrices, it is a straightforward quest to get similar statements for M-operators on an ordered normed space. We provide certain sufficient conditions, such that the (positive) inverse of an M_1- or M_2-operator satisfies the introduced maximum principles.

3.1 Introduction: Maximizing Functionals

For certain normed spaces one has an intuitive understanding on which "position" a given element has its "maximum". Consider e. g. a linear subspace $X(T)$ of $C(T)$ (where T is a compact Hausdorff space) with the induced norm and ordering. For an element $x \in X(T)$ we define the "maximum of x" by

$$\alpha(x) = \max\{x(t) \colon\ t \in T\}\,. \tag{3.1}$$

The set T corresponds to the "set of all possible positions". We get the "set of all maximizing positions of x" by

$$T_\alpha(x) = \{t \in T \colon\ x(t) = \alpha(x)\}\,. \tag{3.2}$$

Now we consider an arbitrary ordered normed space $(X, K, \|\cdot\|)$. What would be a provitable way of introducing the "maximum" of an arbitrary element $x \in X$? The idea is to use a certain set M of positive functionals as the "set of all possible positions". In [PSW98] it is proposed to take for M a base of the dual cone. Before we follow this line, let us take a more general view first. We consider the locally convex Hausdorff space $(X', \sigma(X', X))$ and fix a subset M in the dual cone K'. Since we want to be able to "maximize" an element $x \in X$ by means of M, the set M needs additional properties. For each $x \in X$ the mapping

$$\hat{x}\colon X' \to \mathbb{R} \ \text{ defined by } \ f \mapsto f(x)$$

is $\sigma(X', X)$-continuous, hence $\sigma(X', X)$-compact subsets M of K' will be appropriate. Let M be a convex $\sigma(X', X)$-compact subset of K'. Standard arguments that are collected in the Appendix A.2 imply now that the mapping $\hat{x}$ attains its maximum (and minimum, respectively) on M, where for $x \in X$ we denote the maximum[1] by $\alpha_M(x)$ and the minimum by $\beta_M(x)$:

$$\begin{aligned} \alpha_M(x) &= \max\{\hat{x}(f)\colon f \in M\} = \max\{f(x)\colon f \in M\} \\ \text{and } \beta_M(x) &= \min\{\hat{x}(f)\colon f \in M\} = \min\{f(x)\colon f \in M\}. \end{aligned} \tag{3.3}$$

For every $x \in X$ denote by

$$\begin{aligned} M_\alpha(x) &= \{f \in M\colon f(x) = \alpha_M(x)\} \\ \text{and } M_\beta(x) &= \{f \in M\colon f(x) = \beta_M(x)\} \end{aligned} \tag{3.4}$$

the sets of all maximizers and all minimizers, respectively. Due to Proposition A.2.3, for every $x \in X$ the sets $M_\alpha(x)$ and $M_\beta(x)$ are non-empty compact extreme subsets of M.

The connection between the definitions of the "maximum of x" in (3.1) and (3.3) is now nearly obvious. Take a glance at the representation of an ordered normed space in Theorem 1.3.10. An ordered normed space $(X, K, \|\cdot\|)$ with a closed, normal and non-flat cone K can be considered as a subspace of $C(K' \cap \mathrm{B}')$, where B' is the unit ball in the norm dual X'. Since an element $x \in X$ is represented as the function $\hat{x} \in C(K' \cap \mathrm{B}')$ with $\hat{x}(f) = f(x)$ for all $f \in K' \cap \mathrm{B}'$, one can take $M = K' \cap \mathrm{B}'$. Then for $x \in X$ the maximum $\alpha_M(x)$ according to (3.3) corresponds to $\alpha(\hat{x})$ for $\hat{x} \in C(K' \cap \mathrm{B}')$ in the sense of (3.1).

If the interior of the cone K is non-empty, then the dual cone K' possesses a base, say F, according to (1.16), and the representation in Theorem 1.3.11 can be used. This indicates to put $M = F$, and one has that $\alpha_F(x)$ according to (3.3) corresponds to $\alpha(\hat{x})$ for $\hat{x} \in C(F)$ according to (3.1). Our subsequent considerations will use this approach, where we extend the results in [PSW98]. A part of the material in the Sections 3.2 and 3.3 has been published in [KW00a].

3.2 An Approach for Cones with Non-empty Interior

In the following let $(X, K, \|\cdot\|)$ be an ordered normed space, where the cone K is closed and has a non-empty interior. We fix an element $u \in \operatorname{int} K$, the subsequent definitions

[1] A consistent notation according to (A.13) would be $\alpha_M(\hat{x})$, but for sake of readability we use $\alpha_M(x)$.

depend on u. The dual cone K' possesses a $\sigma(X',X)$-compact base

$$F = F_u = \{f \in K' \colon f(u) = 1\}$$

defined according to (1.16). For an element $x \in X$ we use the notions introduced in the previous section with respect to the set $M = F$. Since F is fixed, we abbreviate $\alpha_F(x)$ by $\alpha(x)$ and $\beta_F(x)$ by $\beta(x)$, i. e.

$$\begin{aligned} \alpha(x) &= \max\{f(x) \colon f \in F\} \\ \text{and } \beta(x) &= \min\{f(x) \colon f \in F\}. \end{aligned}$$

The corresponding sets of maximizers and minimizers are

$$\begin{aligned} F_\alpha(x) &= \{f \in F \colon f(x) = \alpha(x)\} \\ \text{and } F_\beta(x) &= \{f \in F \colon f(x) = \beta(x)\}, \end{aligned}$$

respectively. Note that

$$\begin{aligned} \alpha(u) &= \beta(u) = 1 \\ \text{and } F_\alpha(u) &= F_\beta(u) = F. \end{aligned}$$

The following is an immediate consequence of Proposition A.2.3.

Proposition 3.2.1 *For every $x \in X$ the sets $F_\alpha(x)$ and $F_\beta(x)$ are non-empty $\sigma(X',X)$-compact extreme subsets of F. They contain extreme points of F and can be represented as*

$$F_\alpha(x) = \overline{\mathrm{co}}^{\sigma(X',X)}(F_\alpha(x) \cap \operatorname{ext} F) \quad \text{and} \quad F_\beta(x) = \overline{\mathrm{co}}^{\sigma(X',X)}(F_\beta(x) \cap \operatorname{ext} F).$$

Since K is closed, for $x \notin K$ there is a functional $f \in F$ with $f(x) < 0$ (see Proposition 1.2.13), i. e. $\beta(x) < 0$. Analogously, $-x \notin K$ implies the existence of a functional $g \in F$ with $g(-x) < 0$, i. e. $g(x) > 0$ and $\alpha(x) > 0$. So, the following is obvious.

Lemma 3.2.2 *For $x \in X$ one has*

(i) *$\alpha(x) \leq 0$ if and only if $x \in (-K)$; and*

(ii) *$\beta(x) \geq 0$ if and only if $x \in K$.*

In particular, for $x > 0$ one has $\alpha(x) > 0$, and for $x < 0$ one has $\beta(x) < 0$.

For $x \in K$ denote

$$F_0(x) = \{f \in F \colon f(x) = 0\}. \tag{3.5}$$

Due to Proposition 1.3.3, for $x \in \operatorname{int} K$ the set $F_0(x)$ is empty, whereas for $x \in \partial K$ the set $F_0(x)$ is non-empty. In the latter case we have

$$F_0(x) = F_\beta(x).$$

Consequently, for $x \in \partial K$ the set $F_0(x)$ is a non-empty $\sigma(X', X)$-compact extreme subset of F that contains extreme points of F and can be represented as

$$F_0(x) = \overline{\mathrm{co}}^{\sigma(X',X)}(F_0(x) \cap \operatorname{ext} F)\,. \tag{3.6}$$

For $x \in K$ denote $F_+(x) = F \setminus F_0(x)$. Then $x > 0$ implies

$$F_\alpha(x) \subseteq F_+(x)\,. \tag{3.7}$$

We provide certain decompositions of an element $x \in X$ that will be used later on. Figure 3.1 illustrates the subsequent statements.

Proposition 3.2.3 *For every $x \in X$ there are $z_1, z_2 \in K$ such that*

(R_1) $\quad x = \alpha(x)u - z_1$, *where* $F_\alpha(x) = F_0(z_1)$; *and*

(R_2) $\quad x = \beta(x)u + z_2$, *where* $F_\beta(x) = F_0(z_2)$.

Proof (R_1): Fix $x \in X$ and let $z_1 = \alpha(x)u - x$. Assume $z_1 \notin K$. Since K is closed, the sets K and $\{z_1\}$ can be separated by a hyperplane, i. e. there is a functional $f \in F$ with $f(z_1) < 0$. Then due to

$$f(x) = \alpha(x)f(u) - f(z_1) = \alpha(x) - f(z_1) > \alpha(x)$$

a contradiction is obtained. Hence $z_1 \in K$. The equality $F_\alpha(x) = F_0(z_1)$ is obvious. (R_2) is proved analogously. □

Clearly, the vectors z_1 and z_2 are contained in ∂K.

We continue with a statement on negative-off-diagonal operators which we apply later on (cf. Remark 3.5.6).

Theorem 3.2.4 *Let $(X, K, \|\cdot\|)$ be an ordered normed space such that K is closed and $\operatorname{int} K \neq \varnothing$. For an operator $A \in \mathcal{L}(X)$ that possesses an inverse $A^{-1} \in \mathcal{L}(X)$ consider the conditions*

(i) $A^{-1} \geq 0$;

(ii) *there is an element $u \in \operatorname{int} K$ such that $Au \in \operatorname{int} K$.*

Then (i) *implies* (ii), *and if A is negative-off-diagonal, then* (i) *and* (ii) *are equivalent.*

Proof Let the operator A be positively invertible, i. e. $A^{-1}K \subseteq K$, so $K \subseteq AK$. Since $\operatorname{int} K \neq \varnothing$, there is an element $y \in \operatorname{int} K \subseteq AK$, so $y = Ax$ for some $x > 0$. Let $r > 0$ be such that $\mathrm{B}(y, r) \subset \operatorname{int} K$, and put

$$u = x + \tfrac{r}{\|A\|\,\|y\|}\, y\,.$$

Clearly, $u \in \operatorname{int} K$ and

$$\|Au - y\| = \|A(u - x)\| \leq \|A\| \tfrac{r}{\|A\|\,\|y\|} \|y\| = r\,,$$

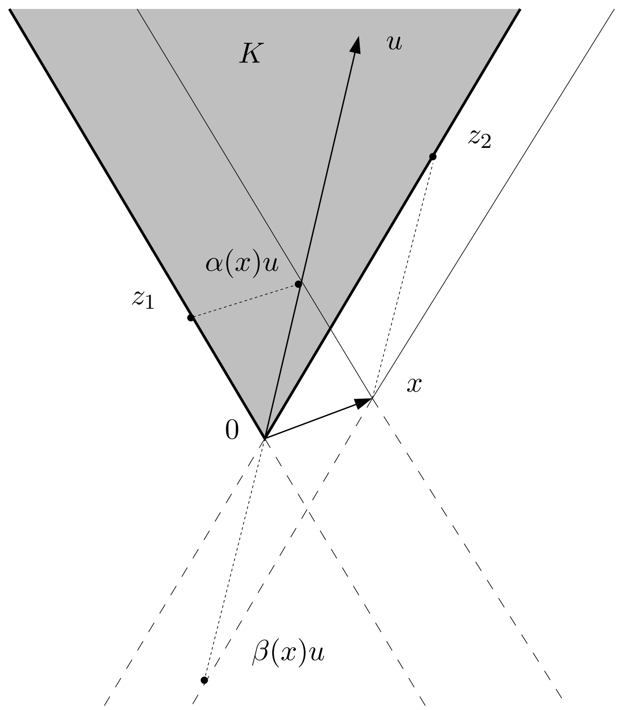

Figure 3.1: Illustration of Proposition 3.2.3.

i. e. $Au \in \mathrm{B}(y, r)$, so $Au \in \operatorname{int} K$.

Vice versa, let A, in addition, be negative-off-diagonal, let $u \in \operatorname{int} K$ be such that $Au \in \operatorname{int} K$ and let $F = F_u$. Suppose $A^{-1} \not\geq 0$, i. e. there is $x_0 \in K$ with $A^{-1}x_0 \notin K$. The element $x = A^{-1}x_0$ satisfies $x \in X \setminus K$ and $Ax \in K$. Due to Lemma 3.2.2 one has $\beta(x) < 0$. The decomposition (R_2) of x in Proposition 3.2.3 provides the element

$$z_2 = x - \beta(x)u \in K\,.$$

For a functional $f \in F_{\beta}(x) \cap \operatorname{ext} F$ one has $f(z_2) = 0$ and, since A is negative-off-diagonal, $f(Az_2) \leq 0$. Due to $Au \in \operatorname{int} K$ it follows $f(Au) > 0$, so

$$f(Ax) = f(Az_2) + \beta(x) f(Au) < 0\,,$$

which contradicts $Ax \in K$ since K is closed. □

Remark 3.2.5 Hence, if an operator $A \in \mathfrak{L}(X)$ is negative-off-diagonal, has an inverse $A^{-1} \in \mathfrak{L}(X)$ and satisfies (ii), then A is an M_2-operator. For the special case $(\mathbb{R}^n, \mathbb{R}^n_+)$ and an $n \times n$-matrix A, the statement of Theorem 3.2.4 is shown in [BP94, Theorem 6(2.3), conditions (N_{38}) and (I_{27})].

We introduce some notations for subsets of X that will be convenient later on. For any non-empty set $E \subset \operatorname{ext} F$ denote

$$H_E = \{x \in K\colon\ f(x) = 0 \text{ for every } f \in E\}\ . \tag{3.8}$$

For $f \in \operatorname{ext} F$ we abbreviate $H_{\{f\}}$ by H_f. For every non-empty set $E \subset \operatorname{ext} F$ the set H_E is a closed extreme subset of the cone K, i. e. $H_E \subset \partial K$. Moreover,

$$H_E = \bigcap_{f \in E} H_f\,.$$

We already mentioned that for every $x \in \partial K$ there is a functional $f \in \operatorname{ext} F$ such that $f(x) = 0$. Hence,

$$\partial K = \bigcup_{f \in \operatorname{ext} F} H_f\,.$$

For a fixed element $x \in X$ we have introduced the set $F_\alpha(x)$ of all maximizers. Now we change the point of view: We fix a functional $f \in \operatorname{ext} F$ and describe the set of all elements $x \in X$ for which f is a maximizer. Denote

$$X_{\alpha,f} = \bigcup_{\lambda \in \mathbb{R}} \{\lambda u - H_f\} \quad \text{and} \quad X_{\beta,f} = \bigcup_{\lambda \in \mathbb{R}} \{\lambda u + H_f\} \tag{3.9}$$

for every $f \in \operatorname{ext} F$.

Proposition 3.2.6 *Let $x \in X$ and $f \in \operatorname{ext} F$. Then*

(i) $f \in F_\alpha(x)$ *if and only if* $x \in X_{\alpha,f}$.

(ii) $f \in F_\beta(x)$ *if and only if* $x \in X_{\beta,f}$.

(iii) $X = \bigcup_{f \in \operatorname{ext} F} X_{\alpha,f} = \bigcup_{f \in \operatorname{ext} F} X_{\beta,f}$.

Proof (i) If $x \in X$ and $f \in F_\alpha(x)$, i. e. $f(x) = \alpha(x)$, then (i) of Proposition 3.2.3 implies $x = \alpha(x)u - z_1$, where $z_1 \in K$ and $f(z_1) = 0$, i. e. $z_1 \in H_f$. So, $x \in X_{\alpha,f}$.
Vice versa, let $x \in X_{\alpha,f}$, i. e. $x = \lambda u - z$ for some $\lambda \in \mathbb{R}$ and $z \in K$ with $f(z) = 0$. Assume that $f \notin F_\alpha(x)$, then $\alpha(x) > f(x) = \lambda f(u) - f(z) = \lambda$. For a functional $g \in F_\alpha(x) \cap \operatorname{ext} F$ (which exists by Proposition 3.2.1) one obtains

$$\alpha(x) = g(x) = g(\lambda u - z) = \lambda - g(z)$$

and so $g(z) = \lambda - \alpha(x) < 0$, a contradiction to $z \geq 0$ and $g \geq 0$. So, $f \in F_\alpha(x)$.
(ii) is shown analogously. (iii) is an immediate consequence of (i), (ii) and Proposition 3.2.1. $\square$

In particular, one gets a decomposition of the cone K by means of the sets

$$K_f = X_{\alpha,f} \cap K \quad \text{for all} \quad f \in \operatorname{ext} F\,.$$

Obviously, K_f is a subcone of K for each $f \in \operatorname{ext} F$, and

$$K = \bigcup_{f \in \operatorname{ext} F} K_f\,. \tag{3.10}$$

For $x \in K$ one has $\alpha(x) \geq 0$, therefore

$$K_f = \bigcup_{\lambda \geq 0} \{\lambda u - H_f\} \cap K\,. \tag{3.11}$$

For $x \in K$ and $f \in \operatorname{ext} F$ one has $f \in F_\alpha(x)$ if and only if $x \in K_f$, i. e. K_f is the collection of all elements x in K that have f as a maximizer.

Next we collect some estimates for the maximum and minimum values that we need later on. For an element $x \in X$ the numbers $\alpha(x)$ and $\beta(x)$ are estimated with respect to the norm of x and with respect to certain constants depending on u and on the geometry of the cone K. For the fixed interior point u of K we use the constant

$$\gamma = \sup\{t \in \mathbb{R}_+ :\; \mathrm{B}(u,t) \subset K\}\,.$$

An immediate consequence of the inequality (1.20) is the following.

Proposition 3.2.7 *For every $x \in X$ one has*

$$\max\{|\alpha(x)|, |\beta(x)|\} \leq \tfrac{1}{\gamma}\|x\|\,,$$

in particular $\alpha(x) \leq \frac{1}{\gamma}\|x\|$.

We will also need the converse inequality

$$m\|x\| \leq \alpha(x) \quad \text{for all} \quad x \in K$$

for a certain constant $m > 0$. We calculate such a constant m in a simple case.

Example 3.2.8 Let $X = C[0,1]$, fix $u \in \operatorname{int} K$ (i. e. $\min\{u(t) :\; t \in [0,1]\} > 0$) and define $F = F_u$ corresponding to (1.16). The set of atoms of K' is given in (1.24), and the intersection of this set with F yields $\operatorname{ext} F$. A straightforward calculation shows that $\operatorname{ext} F$ is the set of maps f_t determined by the points $t \in [0,1]$, where

$$f_t(x) = \tfrac{x(t)}{u(t)} \quad \text{for each} \quad x \in C[0,1]\,.$$

So,

$$\alpha(x) = \max\{f_t(x) :\; t \in [0,1]\}\,.$$

For every $x \in K$ one has

$$\tfrac{1}{\|u\|}\|x\| = \max\left\{\tfrac{x(t)}{\|u\|} :\; t \in [0,1]\right\} \leq \max\left\{\tfrac{x(t)}{u(t)} :\; t \in [0,1]\right\} = \alpha(x)\,,$$

hence we get $m = \frac{1}{\|u\|}$. One has $\gamma = \min\{u(t) :\; t \in [0,1]\}$ and, obviously,

$$\alpha(x) \leq \max\left\{\tfrac{x(t)}{\gamma} :\; t \in [0,1]\right\} = \tfrac{1}{\gamma}\|x\|\,.$$

Clearly, in the special case $u \equiv \mathbb{1}$ we obtain $m = 1$, $\gamma = 1$ and $\alpha(x) = \|x\|$ for $x \in K$.

In general, the constant m can be ascertained if the norm $\|\cdot\|$ is assumed to be semi-monotone[2] or, equivalently, if K' is assumed to be non-flat.

Proposition 3.2.9 *If the norm $\|\cdot\|$ is semi-monotone with the constant N, then for each $x \in K$ one has*

$$\tfrac{1}{N\|u\|}\|x\| \leq \alpha(x)\,.$$

Proof Fix $x \in K \setminus \{0\}$ and consider $z = \alpha(x)u - x$. Due to (R_1) in Proposition 3.2.3 one has $z \in K$ and therefore $x + z \geq x$. The semi-monotony of the norm yields

$$\|u\| = \|\tfrac{1}{\alpha(x)}(x+z)\| \geq \tfrac{1}{\alpha(x)}\,\tfrac{1}{N}\|x\|\ ,$$

hence $\frac{1}{N\|u\|}\|x\| \leq \alpha(x)$. □

Remark 3.2.10 If (X, K) is a normed vector lattice, then $N = 1$ and $m = \frac{1}{\|u\|}$.

We summarize Proposition 3.2.7 and Proposition 3.2.9: Let $(X, K, \|\cdot\|)$ be an ordered normed space with a semi-monotone norm $\|\cdot\|$ and a closed cone K that possesses a non-empty interior. Put (with the previous notions)

$$M = \tfrac{1}{\gamma} \quad\text{and}\quad m = \tfrac{1}{N\|u\|}\,. \tag{3.12}$$

Then for each $x \in K$ we get

$$m\|x\| \leq \alpha(x) \leq M\|x\|\,. \tag{3.13}$$

We will make use of this statement in subsequent sections. For sake of completeness we continue with a statement which is similar to Proposition 3.2.9, where the constant of non-flatness of the dual cone is involved.

Proposition 3.2.11 *If the dual cone K' is non-flat with the constant $\hat{\kappa}$, then for each $x \in K$ one has*

$$\tfrac{1}{\hat{\kappa}\|u\|}\|x\| \leq \alpha(x)\,.$$

Proof First we remark that due to $1 = f(u) \leq \|f\|\,\|u\|$ for each $f \in F$ one has

$$\|f\| \;\geq\; \tfrac{1}{\|u\|}\,. \tag{3.14}$$

Let $g \in K'$ with $\|g\| = \frac{1}{\|u\|}$ and let f be the corresponding element of the base F of K' such that $g = \lambda f$ for some $\lambda > 0$. This yields

$$\lambda\|f\| = \|\lambda f\| = \|g\| = \tfrac{1}{\|u\|} \leq \|f\|$$

[2] If the order interval $[0, u]$ is norm bounded, then the norm is semi-monotone, see e. g. [Vul77, Theorem IV.2.3].

and hence $\lambda \le 1$. For any $x \in K$ we get $g(x) \le f(x)$, and therefore

$$\sup\{\, g(x):\; g \in K',\; \|g\| = \tfrac{1}{\|u\|}\,\} \le \sup\{\, f(x):\; f \in F\,\} = \alpha(x)\,. \tag{3.15}$$

Since K' is non-flat, for each $h \in X'$ there are $h_1,\, h_2 \in K'$ such that $h = h_1 - h_2$ and $\|h_1\|,\, \|h_2\| \le \hat\kappa\|h\|$. Notice that for $x \in X$ one has

$$\begin{aligned} \|x\| &= \sup\{\, |h(x)|:\; h \in X',\; \|h\| \le 1\,\} \\ &\le \sup\{\, |h_1(x) - h_2(x)|:\; h_1,\, h_2 \in K',\; \|h_1\|, \|h_2\| \le \hat\kappa\,\}\,. \end{aligned} \tag{3.16}$$

Now fix $x \in K$, then $h_1(x)$ and $h_2(x)$ are non-negative numbers, and one gets

$$\begin{aligned} \|x\| &\le \sup\{\, |f(x)|:\; f \in K',\; \|f\| \le \hat\kappa\,\} \\ &= \sup\{\, f(x):\; f \in K',\; \|f\| \le \hat\kappa\,\}\,. \end{aligned} \tag{3.17}$$

In (3.17) it suffices to consider elements f of K' with $\|f\| = \hat\kappa$, hence

$$\begin{aligned} \|x\| &\le \sup\{\, f(x):\; f \in K',\; \|f\| = \hat\kappa\,\} \\ &= \sup\{\, f(x):\; f \in K',\; \|\tfrac{1}{\hat\kappa\|u\|} f\| = \tfrac{1}{\|u\|}\,\}\,. \end{aligned}$$

Let $g = \frac{1}{\hat\kappa\|u\|} f$, consequently

$$\begin{aligned} \|x\| &\le \sup\{\, \hat\kappa\,\|u\|\, g(x):\; g \in K',\; \|g\| = \tfrac{1}{\|u\|}\,\} \\ &= \hat\kappa\,\|u\| \sup\{\, g(x):\; g \in K',\; \|g\| = \tfrac{1}{\|u\|}\,\}\,. \end{aligned}$$

We use (3.15) and get $\|x\| \le \hat\kappa\,\|u\|\,\alpha(x)$ and hence $\frac{1}{\hat\kappa\|u\|}\|x\| \le \alpha(x)$. □

Proposition 3.2.12 *If the dual cone K' is non-flat with the constant $\hat\kappa$, then for each $x \in X$ one has*

$$\|x\| \le 2\,\hat\kappa\,\|u\| \max\{\, |\alpha(x)|,\, |\beta(x)|\,\}\,.$$

Proof The argument is analogous to the proof of Proposition 3.2.11. We use (3.14) and observe for $g \in K'$ with $\|g\| = \frac{1}{\|u\|} = \lambda\|f\|$ (where f is the corresponding element in F) and for $x \in X$, that $|g(x)| \le |f(x)|$ holds, and hence

$$\begin{aligned} \sup\{\, |g(x)|:\; g \in K',\; \|g\| = \tfrac{1}{\|u\|}\,\} &\le \sup\{\, |f(x)|:\; f \in F\,\} \\ &= \max\{\, |\alpha(x)|,\, |\beta(x)|\,\}\,. \end{aligned}$$

Furthermore, (3.16) implies for every $x \in X$

$$\|x\| \le \sup\{\, 2\,|f(x)|:\; f \in K',\; \|f\| \le \hat\kappa\,\}\,.$$

A staightforward calculation yields

$$\|x\| \le 2\,\hat\kappa\,\|u\| \sup\{\, |g(x)|:\; g \in K',\; \|g\| = \tfrac{1}{\|u\|}\,\}\,,$$

which completes the proof. □

Remark 3.2.13 Proposition 3.2.12 completes the proof of the representation result in Proposition 1.3.11.

3.3 The Maximum Principles and their Geometrical Characterizations

We start the section by listing examples of maximum principles for linear systems on ordered vector spaces, that are established in the literature. For a given operator equation these principles can roughly be described as follows:

> *A positive input causes a positive output, and the maximum response takes place in that part of the system where the influence is non-zero.*

Example 3.3.1 We consider the space $\mathbb{R}^n$ equipped with the standard cone $\mathbb{R}^n_+$. In connection with the study of discrete approximations for differential equations in [Sto82] the following maximum principle for an $n \times n$-matrix $B = (b_{ij})_{i,j}$ is introduced: For any $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n_+ \setminus \{0\}$ with $Bx = y$ follows $x \geq 0$ and, moreover,

$$\max_{i \in N} x_i = \max_{i \in N_+(y)} x_i \,, \tag{3.18}$$

where $N = \{1, \ldots, n\}$ and $N_+(y) = \{i \in N\colon \; y_i > 0\}$. In [Win89, Definition 3.31], this maximum principle is called the *maximum principle for inverse column entries* . If the matrix B is invertible, denote $B^{-1} = (a_{ij})_{i,j}$. For the i-th unit vector $y = e^{(i)}$ the element $x = B^{-1}y$ is the i-th column of B^{-1}, and if B is positively invertible and satisfies (3.18), then B^{-1} is weakly diagonally dominant with respect to its column entries, i. e. $a_{ii} \geq a_{ji}$ for all $i, j \in N$.
In [Sto86] the following statement is shown: If B is an M-matrix, then B satisfies (3.18) if and only if $Be \in \mathbb{R}^n_+$, where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$.

An extension of the maximum principle described in Example 3.3.1 is given in [TW95]. Consider the space $(\mathbb{R}^n, \mathbb{R}^n_+)$ and let $\gamma = (\gamma_1, \ldots, \gamma_n)^T \in \operatorname{int} \mathbb{R}^n_+$. An $n \times n$-matrix B is said to satisfy the γ*-maximum principle* if for any $y \in \mathbb{R}^n_+ \setminus \{0\}$ with $Bx = y$ one has $x \geq 0$ and

$$\max_{i \in N} \gamma_i x_i = \max_{i \in N^+(y)} \gamma_i x_i \,. \tag{3.19}$$

In [TW95, Theorem 5] the following is shown.

Proposition 3.3.2 *An* M*-matrix B satisfies the γ-maximum principle if and only if $Bu \in \mathbb{R}^n_+$ for*

$$u = \left(\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n}\right)^T \,.$$

The γ-maximum principle is generalized for arbitrary generating cones in $\mathbb{R}^n$ in [PSW98], using a base of the dual cone.

Example 3.3.3 Let $X = C(T)$ according to Example 1.3.9 and let $B\colon C(T) \to C(T)$ be a linear operator that possesses a positive continuous inverse. A maximum principle can be formulated as follows: For every positive nonzero function y there is a point $t \in T$ at which the function $x = B^{-1}y$ attains its maximum and $y(t) > 0$ holds as well.

Inspired by [PSW98], we define maximum principles for a positive operator on an ordered normed space $(X, K, \|\cdot\|)$, using a base of the dual cone. Our general setting corresponds to the one in Section 3.2, i. e. K is closed and $\operatorname{int} K \neq \varnothing$. For some fixed element $u \in \operatorname{int} K$ we consider the base $F = F_u$ of K' according to (1.16). For $x \in K$ the sets

$$F_\alpha(x) = \{f \in F\colon\ f(x) = \max\{g(x)\colon\ g \in F\}\} \quad \text{and} \quad F_+(x) = \{f \in F\colon\ f(x) > 0\}$$

have been introduced.

The following maximum principle MP has been defined in [KW00a]. In addition, we establish a strong version of this principle.

Definition 3.3.4 An operator $A \in \mathcal{L}_+(X)$ is said to satisfy

- the maximum principle MP with respect to u, if for every $x > 0$ one has

$$F_\alpha(Ax) \cap F_+(x) \neq \varnothing\,;$$

- the (strong) maximum principle SMP with respect to u, if for every $x > 0$ one has

$$F_\alpha(Ax) \subseteq F_+(x)\,.$$

An operator A that satisfies MP (SMP, respectively) with respect to u is called an MP-operator (SMP-operator, respectively) with respect to u.

We remark that the investigation of MP with respect to different interior points of K leads to a so-called *weighted* maximum principle which is studied in [PSW98] (in the finite-dimensional case) and in [KW00b].

For the maximum principles it is sufficient to consider the extreme points of F, as the subsequent statement reveals.

Lemma 3.3.5 *Let $A \in \mathcal{L}_+(X)$ and $x > 0$. Then*

(i) *$F_\alpha(Ax) \cap F_+(x) \neq \varnothing$ if and only if $F_\alpha(Ax) \cap F_+(x) \cap \operatorname{ext} F \neq \varnothing$;*

(ii) *$F_\alpha(Ax) \subseteq F_+(x)$ if and only if $F_\alpha(Ax) \cap \operatorname{ext} F \subseteq F_+(x)$.*

Proof (i) Let $x > 0$ be such that $F_\alpha(Ax) \cap F_+(x) \neq \varnothing$. Assume that for every $f \in F_\alpha(Ax) \cap \operatorname{ext} F$ one has $f(x) = 0$. The representation

$$F_\alpha(Ax) = \overline{\operatorname{co}}^{\sigma(X',X)}(F_\alpha(Ax) \cap \operatorname{ext} F)$$

provided in Proposition 3.2.1 implies $f(x) = 0$ for each $f \in F_\alpha(Ax)$. So, one gets $F_\alpha(Ax) \cap F_+(x) = \varnothing$, a contradiction.

(ii) Let $x > 0$ be such that $F_\alpha(Ax) \cap \operatorname{ext} F \subseteq F_+(x)$, i. e.

$$F_\alpha(Ax) \cap F_0(x) \cap \operatorname{ext} F = \varnothing\,. \tag{3.20}$$

Assume $F_\alpha(Ax) \nsubseteq F_+(x)$, i. e. the set $G = F_\alpha(Ax) \cap F_0(x)$ is non-empty. The sets $F_\alpha(Ax)$ and $F_0(x)$ are $\sigma(X', X)$-compact convex subsets of X' (cf. Proposition 3.2.1 and (3.6)), hence G is a $\sigma(X', X)$-compact convex subset of X'. Therefore,

$$G = \overline{\mathrm{co}}^{\sigma(X',X)}(\mathrm{ext}\, G) \tag{3.21}$$

due to (A.11). Furthermore, since $F_\alpha(Ax)$ and $F_0(x)$ are extreme subsets of F, the set G is an extreme subset of $F_\alpha(Ax)$ (cf. Lemma A.2.1). Lemma A.2.2 implies then

$$\begin{aligned} \mathrm{ext}\, G &= G \cap \mathrm{ext}\, F_\alpha(Ax) \\ &= F_0(x) \cap \mathrm{ext}\, F_\alpha(Ax) \\ &= F_0(x) \cap F_\alpha(Ax) \cap \mathrm{ext}\, F\,. \end{aligned}$$

Now (3.20) yields $\mathrm{ext}\, G = \varnothing$, and due to (3.21) we get $G = \varnothing$, a contradiction. □

We verify that the maximum principle MP covers the maximum principles in the examples above. We start with Example 3.3.1, i. e. the space $\mathbb{R}^n$ is ordered by its natural cone $\mathbb{R}^n_+$, and we consider the base $F = F_e$ according to (1.22) of the dual cone. The extreme points of F are the unit vectors $e^{(i)}$, where $i \in N$. Proposition 3.2.1 implies that a vector $x = (x_1, \ldots, x_n)^T$ attains its maximum $\alpha(x)$ at an extreme point of F, i. e. there is an index $i \in N$ such that $\alpha(x) = \langle x, e^{(i)} \rangle = x_i$. The value of $\alpha(x)$ corresponds to the maximum of the coordinates of x, and the set of maximizers of x is

$$F_\alpha(x) = \mathrm{co}\left\{ e^{(j)} \colon\ j \in N \ \text{ and } \ x_j = \max\{x_i \colon\ i \in N\} \right\}. \tag{3.22}$$

Let B be a regular $n \times n$-matrix such that B^{-1} is positive. The matrix B satisfies the maximum principle (3.18) if and only if for every $y \in \mathbb{R}^n_+ \setminus \{0\}$ and $x = B^{-1}y$ there is an index $i \in N$ such that x_i is a maximal coordinate of x and $y_i > 0$, i. e. there is an index $i \in N$ such that $\langle x, e^{(i)} \rangle = \alpha(x)$ and $\langle y, e^{(i)} \rangle > 0$, which means

$$e^{(i)} \in F_\alpha(x) \cap F_+(y)\,. \tag{3.23}$$

Hence, if B satisfies (3.18), then B^{-1} satisfies MP with respect to e. On the other hand, if B^{-1} satisfies MP with respect to e, then Lemma 3.3.5 (i) implies (3.23) for some $i \in N$. So, the following statement is valid.

Proposition 3.3.6 *Let the space $\mathbb{R}^n$ be ordered by the cone $\mathbb{R}^n_+$ and let B be an invertible $n \times n$-matrix with $B^{-1} \geq 0$. Then B satisfies the maximum principle (3.18) if and only if B^{-1} satisfies* MP *with respect to e.*

We continue with the γ-maximum principle. Fix

$$\gamma = (\gamma_1, \ldots, \gamma_n)^T \in \mathrm{int}\, \mathbb{R}^n_+\,,$$

then the vector $u = (u_1, \ldots, u_n)^T$ with

$$u_i = \tfrac{1}{\gamma_i} \ \text{ for all } \ i \in N$$

also belongs to $\operatorname{int}\mathbb{R}^n_+$. The base $F = F_u$ is given according to (1.23). For a vector $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ one has

$$\alpha(x) = \max\{\langle x, \tfrac{1}{u_i} e^{(i)}\rangle \colon i \in N\} = \max\{\gamma_i x_i \colon i \in N\}.$$

An analogue argument as above for the maximum principle (3.18) yields the following generalized version of Proposition 3.3.6.

Proposition 3.3.7 *Let the space $\mathbb{R}^n$ be ordered by the cone $\mathbb{R}^n_+$ and let B be an invertible $n \times n$-matrix with $B^{-1} \geq 0$. Then B satisfies the γ-maximum principle if and only if B^{-1} satisfies* MP *with respect to*

$$u = (\tfrac{1}{\gamma_1}, \ldots, \tfrac{1}{\gamma_n})^T.$$

Finally, we examine the maximum principle described in Example 3.3.3. Consider the space $C(T)$ and the function $u = \mathbb{1} \in \operatorname{int} C_+(T)$. The set of extreme points of $F_{\mathbb{1}}$ is given in Example 1.3.9. For $x \in C(T)$ one has

$$\alpha(x) = \max\{\varepsilon_t(x) = x(t) \colon t \in T\}.$$

Applying Lemma 3.3.5(i), we get the following result.

Proposition 3.3.8 *Let $B \colon C(T) \to C(T)$ be such that $B^{-1} \in \mathcal{L}_+(C(T))$. B satisfies the maximum principle formulated in 3.3.3 if and only if B^{-1} satisfies* MP *with respect to $u = \mathbb{1}$.*

In the remaining part of this section let $(X, K, \|\cdot\|)$ be an ordered normed space, where K is closed and $u \in \operatorname{int} K$ is fixed. Let A be a positive linear continuous operator on X. For an interior point x of K one has

$$F_\alpha(Ax) \subseteq F = F_+(x),$$

so in this case the conditions in Definition 3.3.4 do not require anything. Hence MP and SMP are conditions how to transform boundary points of K. We provide some geometrical conditions G and SG that are equivalent to the maximum principles MP and SMP, respectively. These geometrical conditions describe how the operator maps faces of the cone K with respect to certain subcones of K. We use the notations introduced in Section 3.2, in particular (3.8) and (3.11).

Definition 3.3.9 *An operator $A \in \mathcal{L}_+(X)$ is said to satisfy*

– *the condition* G *with respect to u, if for any non-empty subset $E \subset \operatorname{ext} F$ one has*

$$A(H_E) \subseteq \bigcup_{g \in (\operatorname{ext} F)\setminus E} K_g\,;$$

– *the condition* SG *with respect to* u, *if for any non-empty subset* $E \subset \operatorname{ext} F$ *one has*

$$A(H_E \setminus \{0\}) \subseteq K \setminus \bigcup_{g \in E} K_g \,.$$

Note that the inclusions are trivial for the subsets E of $\operatorname{ext} F$ with $H_E = \{0\}$. Furthermore, for each non-empty subset $E \subset \operatorname{ext} F$ one has

$$K \setminus \bigcup_{g \in E} K_g \subseteq \bigcup_{g \in (\operatorname{ext} F) \setminus E} K_g \,,$$

hence it is obvious that for a fixed $u \in \operatorname{int} K$ the condition SG for some operator A implies the condition G for A.

The condition G generalizes certain geometrical conditions that are known in the literature. For the space $(\mathbb{R}^n, \mathbb{R}^n_+)$ a geometrical interpretation of the maximum principle (3.18) is provided in [Sto86]. For the γ-maximum principle a similar geometrical condition is introduced in [TW95]. In [PSW98] a geometrical characterization of (a finite-dimensional version of) MP is given in the cases of a finitely generated cone or a circular cone K in $\mathbb{R}^n$, where a certain decomposition of a base of K is employed. The geometrical condition G has been introduced in [KW00a].

Next we show the equivalence of the maximum principles and the according geometrical conditions. Recall that for some element $x \in K$ and some functional $f \in \operatorname{ext} F$ one has $f \in F_\alpha(x)$ if and only if $x \in K_f$.

Theorem 3.3.10 *For an operator* $A \in \mathcal{L}_+(X)$ *the following is satisfied:*

(i) A *satisfies* MP *with respect to* u *if and only if* A *satisfies* G *with respect to* u.

(ii) A *satisfies* SMP *with respect to* u *if and only if* A *satisfies* SG *with respect to* u.

Proof (i) Suppose that A satisfies MP with respect to u. Due to Lemma 3.3.5 (i) for every $x > 0$ one has $F_\alpha(Ax) \cap F_+(x) \cap \operatorname{ext} F \neq \varnothing$, i. e. there is a functional $f \in \operatorname{ext} F$ such that $Ax \in K_f$ and $x \notin H_f$. Now assume that A does not satisfy G with respect to u, i. e. there is a non-empty subset $E \subset \operatorname{ext} F$ such that

$$A(H_E) \nsubseteq \bigcup_{g \in (\operatorname{ext} F) \setminus E} K_g \,.$$

This implies $H_E \neq \{0\}$. Furthermore, there is an element $x > 0$, $x \in H_E$, such that

$$Ax \notin \bigcup_{g \in (\operatorname{ext} F) \setminus E} K_g \,,$$

i. e. $Ax \notin K_g$ for every $g \in (\operatorname{ext} F) \setminus E$. On the other hand, there is a functional $f \in \operatorname{ext} F$ such that $x \notin H_f$, i. e. $f \notin E$, and $Ax \in K_f$, which yields a contradiction.

Vice versa, suppose that A satisfies G with respect to u and consider an arbitrary element $x > 0$. If $x \in \operatorname{int} K$, then $F_+(x) = F$. Proposition 3.2.1 yields $F_\alpha(Ax) \neq \varnothing$, and one immediately gets $F_\alpha(Ax) \cap F_+(x) \neq \varnothing$. In the case $x \in \partial K$ let

$$E = \{f \in \operatorname{ext} F\colon\ f(x) = 0\}\,.$$

Due to (3.6) one has $E \neq \varnothing$. Clearly, $x \in H_E$. The condition G implies

$$Ax \in \bigcup_{g \in (\operatorname{ext} F) \setminus E} K_g\,,$$

i. e. there is a functional $g \in (\operatorname{ext} F) \setminus E$ such that $Ax \in K_g$, which means $g \in F_\alpha(Ax)$. Obviously, $g \in F_+(x)$, hence $g \in F_\alpha(Ax) \cap F_+(x)$.

(ii) Let the operator A satisfy SMP with respect to u. Assume that A does not satisfy SG with respect to u, i. e. there is a non-empty subset $E \subset \operatorname{ext} F$ such that

$$A(H_E \setminus \{0\}) \nsubseteq K \setminus \bigcup_{g \in E} K_g\,.$$

Hence $H_E \neq \{0\}$ and, furthermore, there is an element $x > 0$, $x \in H_E$, such that

$$Ax \notin K \setminus \bigcup_{g \in E} K_g\,.$$

One has $Ax \neq 0$, and, since $Ax \in K$, there is a functional $g \in E$ such that $Ax \in K_g$. This implies $g \in F_\alpha(Ax)$ and $g \in F_0(x)$, which contradicts SMP.

Vice versa, assume that A satisfies SG with respect to u and consider some element $x > 0$. If $x \in \operatorname{int} K$, then $F_\alpha(Ax) \subseteq F = F_+(x)$. In the case $x \in \partial K$ let

$$E = \{f \in \operatorname{ext} F\colon\ f(x) = 0\}\,,$$

hence $E \neq \varnothing$ and $x \in H_E$. The condition SG

$$A(H_E \setminus \{0\}) \subseteq K \setminus \bigcup_{g \in E} K_g$$

implies $Ax \notin K_g$ for each $g \in E$. One gets $g \notin F_\alpha(Ax)$ for each $g \in F_0(x) \cap \operatorname{ext} F$, consequently $F_\alpha(Ax) \cap \operatorname{ext} F \subseteq F_+(x)$ and, due to Lemma 3.3.5 (ii), $F_\alpha(Ax) \subseteq F_+(x)$. □

The geometrical characterizations of the maximum principles provide a convenient tool in order to show the maximum principles for a given operator, as we will see in some examples in the next section.

3.4 MP- and SMP-Operators

In this section we collect some properties of MP- and SMP-operators (see Definition 3.3.4) as well as investigate some further examples. Consider an ordered normed space $(X, K, \|\cdot\|)$ such that K is closed and $\operatorname{int} K \neq \varnothing$, as in Section 3.2. We start with two simple observations, where the statement (i) of the next proposition is an immediate consequence of (3.7).

Proposition 3.4.1 *Let $(X, K, \|\cdot\|)$ be an ordered normed space such that K is closed and $\operatorname{int} K \neq \varnothing$.*

(i) *The identity operator I on X always satisfies* SMP *(and hence* MP*) with respect to any arbitrary element $u \in \operatorname{int} K$.*

(ii) *Let $A \in \mathfrak{L}_+(X)$ be an* MP*-operator (*SMP*-operator, respectively) with respect to $u \in \operatorname{int} K$. Then for any $\lambda > 0$ the operator λA is an* MP*-operator (*SMP*-operator, respectively) with respect to u.*

Fix an element $u \in \operatorname{int} K$ and let the norm $\|\cdot\|$ be semi-monotone with the constant N. We come back to the inequalities (3.13), i. e. the constants $m = \frac{1}{N\|u\|}$ and $M = \frac{1}{\gamma}$ (where $\gamma = \sup\{t \in \mathbb{R}_+ : \ \mathrm{B}(u,t) \subset K\}$) ensure

$$m\|x\| \leq \alpha(x) \leq M\|x\|$$

for each $x \in K$. This yields a sufficient condition to satisfy SMP for certain positive operators.

Theorem 3.4.2 *Let $(X, K, \|\cdot\|)$ be an ordered normed space, where $u \in \operatorname{int} K$ and the norm $\|\cdot\|$ is semi-monotone. If for an operator $C \in \mathfrak{L}_+(X)$ one has $\|C\| < \frac{m}{M}$, then $I + C$ satisfies* SMP *with respect to u.*

Proof For fixed $x > 0$ we have to show $f(x) > 0$ for each $f \in F_\alpha(x + Cx)$. First note that for any $g \in F_\alpha(x)$ by virtue of the left inequality above one has

$$0 < \|x\| \leq \tfrac{\alpha(x)}{m} = \tfrac{g(x)}{m} \leq \tfrac{g(x)}{m} + \tfrac{g(Cx)}{m} = \tfrac{g(x+Cx)}{m} \,. \tag{3.24}$$

If for some fixed $f \in F_\alpha(x + Cx)$ we suppose $f(x) = 0$, then

$$g(x + Cx) \leq f(x + Cx) = f(Cx)\,.$$

From (3.24) we conclude

$$0 < \|x\| \leq \tfrac{f(Cx)}{m} \leq \tfrac{\alpha(Cx)}{m} \leq \tfrac{M\,\|Cx\|}{m} \leq \tfrac{M}{m}\,\|C\|\,\|x\| < \|x\| \,,$$

which yields a contradiction. Hence, $I + C$ satisfies SMP with respect to u. □

Note the following immediate consequence of Theorem 3.4.2 and Proposition 3.4.1 (ii).

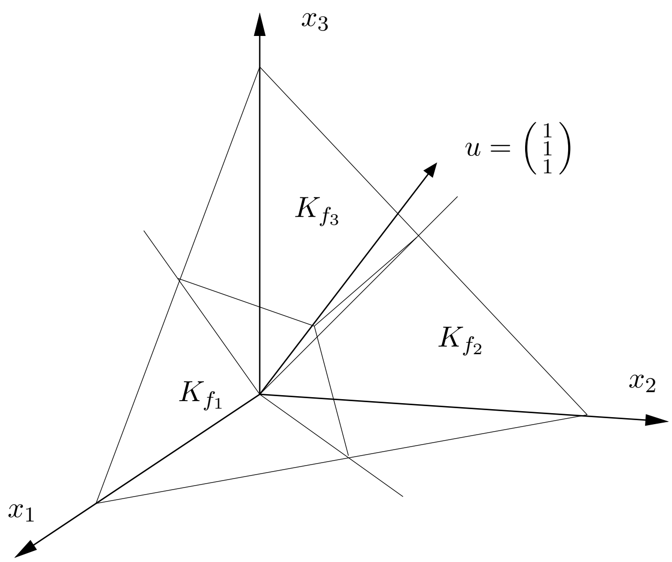

Figure 3.2: The subcones K_{f_i} of K according to (3.25).

Corollary 3.4.3 *If $C \in \mathfrak{L}_+(X)$ and $s > \frac{M}{m}\|C\|$, then the operator $C + sI$ is a* SMP-*operator with respect to u.*

We list several examples of MP- and SMP-operators on the space $(\mathbb{R}^3, \mathbb{R}^3_+, \|\cdot\|)$ (cf. Example 1.3.8). To show that an operator satisfies MP or SMP, we use the geometrical characterizations of the maximum principles that are given in the previous section in Theorem 3.3.10. We consider MP and SMP with respect to

$$u = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) \in \operatorname{int}\mathbb{R}^3_+ .$$

The dual cone is again $\mathbb{R}^3_+$, and the set of extreme points of the base F_u is

$$\operatorname{ext} F_u = \left\{ f_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right),\ f_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right),\ f_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) \right\}.$$

Hence we have to consider the following faces of $\mathbb{R}^3_+$:

$$H_{f_1} = \left\{ \left(\begin{smallmatrix}0\\b\\c\end{smallmatrix}\right):\ b, c \geq 0 \right\},\ H_{f_2} = \left\{ \left(\begin{smallmatrix}a\\0\\c\end{smallmatrix}\right):\ a, c \geq 0 \right\},\ H_{f_3} = \left\{ \left(\begin{smallmatrix}a\\b\\0\end{smallmatrix}\right):\ a, b \geq 0 \right\},$$

$$H_{\{f_1,f_2\}} = \left\{ \left(\begin{smallmatrix}0\\0\\c\end{smallmatrix}\right):\ c \geq 0 \right\},\ H_{\{f_1,f_3\}} = \left\{ \left(\begin{smallmatrix}0\\b\\0\end{smallmatrix}\right):\ b \geq 0 \right\},\ H_{\{f_2,f_3\}} = \left\{ \left(\begin{smallmatrix}a\\0\\0\end{smallmatrix}\right):\ a \geq 0 \right\}.$$

For the calculation it is sufficient to consider the relative interior of these sets, i. e. the sets where the corresponding numbers a, b or c are strictly positive. Clearly,

$$K_{f_i} = \left\{ \left(\begin{smallmatrix}x_1\\x_2\\x_3\end{smallmatrix}\right):\ x_i \geq x_k \geq 0 \ \text{ for } \ k = 1, 2, 3 \right\}. \tag{3.25}$$

We carry out the calculation for one operator in detail, for the further examples it is then obvious.

Example 3.4.4 The matrix

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is an MP-operator with respect to u, but not an SMP-operator with respect to u. Indeed,

$$\begin{aligned} A_1\begin{pmatrix} 0 \\ b \\ c \end{pmatrix} &= \begin{pmatrix} c \\ 2b+c \\ c \end{pmatrix} \in K_{f_2} \subseteq \bigcup_{g\in\{f_2,f_3\}} K_g\,, \\ A_1\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} &= \begin{pmatrix} a+c \\ a+c \\ c \end{pmatrix} \in K_{f_1} \subseteq \bigcup_{g\in\{f_1,f_3\}} K_g\,, \\ A_1\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} &= \begin{pmatrix} a \\ a+2b \\ 0 \end{pmatrix} \in K_{f_2} \subseteq \bigcup_{g\in\{f_1,f_2\}} K_g\,, \end{aligned}$$

$$A_1\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \end{pmatrix} \in K_{f_3}\,, \quad A_1\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2b \\ 0 \end{pmatrix} \in K_{f_2}\,, \quad A_1\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix} \in K_{f_1}\,,$$

which shows that A_1 satisfies the condition G, hence A_1 is an MP-operator with respect to u. On the other hand, A_1 does not satisfy SG, since e. g.

$$A_1\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \end{pmatrix} \notin K \setminus (K_{f_1} \cup K_{f_2})\,,$$

consequently A_1 is not an SMP-operator.

Example 3.4.5 In order to show that the matrix

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

is an SMP-operator with respect to u, we check the condition SG:

$$B\begin{pmatrix} 0 \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ 2b \\ 2c \end{pmatrix} \in K \setminus K_{f_1}\,, \quad B\begin{pmatrix} a \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 2a+c \\ a \\ 2c \end{pmatrix} \in K \setminus K_{f_2}\,, \quad B\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ a+2b \\ 0 \end{pmatrix} \in K \setminus K_{f_3}\,,$$

$$B\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 2c \end{pmatrix} \in K \setminus (K_{f_1} \cup K_{f_2})\,, \quad B\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2b \\ 0 \end{pmatrix} \in K \setminus (K_{f_1} \cup K_{f_3})\,,$$

$$B\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ a \\ 0 \end{pmatrix} \in K \setminus (K_{f_2} \cup K_{f_3})\,.$$

Remark 3.4.6 The operator B can be considered as $B = I + B_1$, where

$$B_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us have a glance at Theorem 3.4.2. If we equip $\mathbb{R}^n$ with the maximum norm, we have $\|u\| = 1$, $\gamma = 1$, $N = 1$, consequently $m = M = 1$. The according matrix norm is the maximum absolute row sum norm, hence $\|B_1\| = 2 \not< \frac{m}{M}$, but $I + B_1$ is an SMP-operator. This shows that the sufficient condition to satisfy SMP in Theorem 3.4.2 is not a necessary one.

The subsequent examples show that the collections of all MP- and SMP-operators, respectively, do not own certain simple structures.

(i) The composition of two MP-operators need not be an MP-operator.

Example 3.4.7 Consider the MP-operator A_1 in Example 3.4.4 and the operator

$$(A_1)^2 = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix} .$$

Since for the element $x = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ one has

$$(A_1)^2 x = \left(\begin{smallmatrix} 3 \\ 7 \\ 1 \end{smallmatrix}\right) \notin K_{f_1} \cup K_{f_3} ,$$

the operator $(A_1)^2$ does not satisfy G, and so it is not an MP-operator. A straightforward calculation shows that for any $k \in \mathbb{N}$, $k > 2$, one has $(A_1)^k x \notin K_{f_1} \cup K_{f_3}$, hence no power of A_1 satisfies MP.

(ii) The composition of two SMP-operators need not be an SMP-operator.

Example 3.4.8 Consider the SMP-operator B in Example 3.4.5 and the operator

$$B^2 = \begin{pmatrix} 4 & 0 & 4 \\ 4 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} .$$

For the element $x = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ one has

$$B^2 x = \left(\begin{smallmatrix} 4 \\ 4 \\ 0 \end{smallmatrix}\right) \notin K \setminus (K_{f_2} \cup K_{f_3}) .$$

The operator B^2 does not satisfy SG, hence it is not an SMP-operator.

(iii) The sum of two MP-operators need not be an MP-operator.

Example 3.4.9 Consider the MP-operator A_1 in Example 3.4.4 and the operator

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} ,$$

which is an MP-operator as well. For the element $x = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ one has

$$(A_1 + A_2)x = \left(\begin{smallmatrix} 3 \\ 4 \\ 3 \end{smallmatrix}\right) \notin K_{f_1} \cup K_{f_3} ,$$

hence $A_1 + A_2$ is not an MP-operator. This also points out that a convex combination of MP-operators, e. g. $\frac{1}{2}A_1 + \frac{1}{2}A_2$, is, in general, not an MP-operator.

(iv) An operator that is bounded from both sides by MP-operators need not be an MP-operator.

Example 3.4.10 The operator

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies $I \leq C \leq A_1$, i. e. C is bounded from both sides by MP-operators. However, C is not an MP-operator, since for $x = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ one has

$$Cx = \left(\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}\right) \notin K_{f_1} \cup K_{f_3} .$$

In the previous examples we considered a fixed interior point of the cone and asked if the sets of MP- and SMP-operators with respect to that interior point have a certain structure. In the next example we change the point of view. We consider a fixed positive operator and ask for which interior points u of the cone the operator is an MP-operator (or SMP-operator, respectively) with respect to u. We use the following notation: For an operator $A \in \mathfrak{L}_+(X)$ let $\mathrm{MP}(A)$ ($\mathrm{SMP}(A)$, respectively) be the set of all points $u \in \operatorname{int} K$ such that A is an MP-operator (SMP-operator, respectively) with respect to u. In the subsequent example we calculate the sets $\mathrm{MP}(A)$ and $\mathrm{SMP}(A)$ for a 2×2-matrix A. We will refer to this example later on (see Example 3.5.3 and Figure 3.3).

Example 3.4.11 Consider $X = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .$$

For an element $u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) \in \operatorname{int} K$ the set F_u has the extreme points $f_1 = \left(\begin{smallmatrix} 1/u_1 \\ 0 \end{smallmatrix}\right)$ and $f_2 = \left(\begin{smallmatrix} 0 \\ 1/u_2 \end{smallmatrix}\right)$. Moreover,

$$H_{f_1} = \left\{ \left(\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right) :\ b \geq 0 \right\}, \quad H_{f_2} = \left\{ \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) :\ a \geq 0 \right\} \quad \text{and}$$

$$K_{f_1} = \left\{ \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) :\ 0 \leq x_1,\ x_2 \leq \tfrac{u_2}{u_1} x_1 \right\}, \quad K_{f_2} = \left\{ \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) :\ 0 \leq x_1,\ \tfrac{u_2}{u_1} x_1 \leq x_2 \right\} .$$

Consider the condition G: One has

$$A\left(\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) b \in K_{f_2}$$

if and only if $\frac{u_2}{u_1} b \leq 2b$, i. e. $u_2 \leq 2u_1$. Analogously,

$$A\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) a \in K_{f_1}$$

if and only if $\frac{1}{2} u_1 \leq u_2$. Consequently,

$$\mathrm{MP}(A) = \left\{ u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) :\ u_1 > 0,\ \tfrac{1}{2} u_1 \leq u_2 \leq 2u_1 \right\} .$$

Concerning the condition SG, we get

$$\mathrm{SMP}(A) = \left\{ u = \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right) :\ u_1 > 0,\ \tfrac{1}{2} u_1 < u_2 < 2u_1 \right\} .$$

We conclude the section with a simple example of an operator in the space $C(T)$.

Example 3.4.12 Consider the space $X = C(T)$ in Example 1.3.9 with its natural norm and ordering. We fix an arbitrary element $u \in \operatorname{int} K$, i. e. $u(t) > 0$ for all $t \in T$, and get the corresponding $F = F_u$. Furthermore, let $v \in \operatorname{int} K$ and $A \in \mathcal{L}(X)$ such that $Ax = vx$ for each $x \in X$. The operator A is positive, $x > 0$ implies $Ax > 0$, and hence $\alpha(Ax) > 0$. If for some $f \in F$ we have $f(x) = 0$, then $f(Ax) = 0$. We get $F_\alpha(Ax) \subseteq F_+(x)$, consequently A is an SMP-operator with respect to u.

3.5 Maximum Principles for M-Operators

In Section 2.4 we introduced M_1-operators and M_2-operators in an ordered normed space. Any operator belonging to one of these classes possesses a positive inverse. Inspired by the results on M-matrices that are stated in Example 3.3.1 and Proposition 3.3.2, the following question arises:

> *Under which conditions does the (positive) inverse of an* M_1*-operator or an* M_2*-operator satisfy the maximum principle* MP *or* SMP *?*

The following is an extended version of our investigation in [Kal03a]. We start with a sufficient condition for M_1-operators. Let $(X, K, \|\cdot\|)$ be an ordered normed space, where $\operatorname{int} K \neq \varnothing$, and let an element $u \in \operatorname{int} K$ be fixed. We use the notations in the Sections 3.2 and 3.3. Let the norm $\|\cdot\|$ be semi-monotone. Then for $x \in K$ the inequalities

$$m\|x\| \le \alpha(x) \le M\|x\|$$

hold, where the constants m, M are defined corresponding to (3.12). Let

$$A = I - B \in \mathcal{L}(X)$$

be an M_1-operator, i. e. $B \ge 0$ and $r(B) < 1$. Then the inverse A^{-1} exists, is positive, and, moreover,

$$A^{-1} = (I - B)^{-1} = I + \sum_{k=1}^{\infty} B^k .$$

The operator

$$C = \sum_{k=1}^{\infty} B^k$$

is positive, and one has

$$\|C\| = \|\sum_{k=1}^{\infty} B^k\| \le \sum_{k=1}^{\infty} \|B\|^k = \tfrac{\|B\|}{1-\|B\|} ,$$

provided $\|B\| < 1$. Applying Theorem 3.4.2, we get a sufficient condition for the operator

$$A^{-1} = (I - B)^{-1} = I + C$$

to satisfy SMP as follows: The condition

$$\tfrac{\|B\|}{1-\|B\|} < \tfrac{m}{M}$$

implies $\|C\| < \frac{m}{M}$, hence in this case $I + C$ satisfies SMP with respect to u. In other words, if $B \geq 0$ is such that

$$\|B\| < \tfrac{m}{m+M}\,,$$

then $(I - B)^{-1}$ is an SMP-operator with respect to u. This sufficient condition can be weakened.

Theorem 3.5.1 *Let* $(X, K, \|\cdot\|)$ *be an ordered normed space with* $\operatorname{int} K \neq \varnothing$ *and a semi-monotone norm* $\|\cdot\|$. *If for* $u \in \operatorname{int} K$ *and a positive operator* $B \in \mathcal{L}(X)$ *one has*

$$\|B\| < \tfrac{m}{M}\,,$$

then $(I - B)^{-1}$ *satisfies* SMP *with respect to* u.

Proof Assume that $T = (I - B)^{-1}$ is not an SMP-operator with respect to u, i. e. there exist $x > 0$ and a functional $f \in F_\alpha(Tx)$ such that $f(x) = 0$. Put $y = Tx$, then $f(Tx) = \alpha(Tx) = \alpha(y)$, and we get

$$0 = f(x) = f(y - By) = \alpha(y) - f(By)\,.$$

The inequalities (3.13) imply

$$m\|y\| \leq \alpha(y) = f(By) \leq \alpha(By) \leq M\|By\| \leq M\|B\|\,\|y\|\,,$$

which yields

$$\tfrac{m}{M} \leq \|B\|\,,$$

a contradiction to the assumption. □

Example 3.5.2 Consider the operator $B\colon C[0,1] \to C[0,1]$, defined by

$$(Bx)(s) = k(s)\int_0^1 x(t)dt\,,$$

where $k \in C_+[0,1]$. The operator $I - B$ is invertible provided $\int_0^1 k(t)dt \neq 1$. In [KW00a, Section 5, Example 3] it is shown by a direct calculation that the condition $\|B\| < 1$ is sufficient for the operator $A = (I - B)^{-1}$ to satisfy MP with respect to the function $u \equiv \mathbb{1}$. The constants according to u are $m = 1$ and $M = 1$ (cf. Example 3.2.8). Applying Theorem 3.5.1, the condition $\|B\| < \frac{m}{M} = 1$ ensures that A satisfies even SMP with respect to $u \equiv \mathbb{1}$.

The sufficient condition $\|B\| < \frac{m}{M}$ in Theorem 3.5.1 is not a necessary one for the statement there.

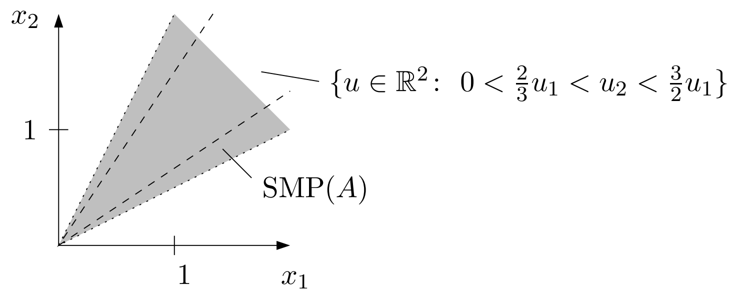

Figure 3.3: Illustration of Example 3.5.3.

Example 3.5.3 Let $\mathbb{R}^2$ be equipped with the maximum norm and the standard cone. Clearly, the constant of semi-monotony is $N = 1$, and $\|u\| = \max\{u_1, u_2\}$, so

$$m = \tfrac{1}{\|u\|} = \tfrac{1}{\max\{u_1,u_2\}}\,.$$

Moreover, $\gamma = \sup\{t \in \mathbb{R}_+\colon\ \mathrm{B}(u,t) \subset K\} = \min\{u_1, u_2\}$, so

$$M = \tfrac{1}{\min\{u_1,u_2\}}\,,$$

see (3.12). In Example 3.4.11 the set SMP(A) is calculated for the operator

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since the matrix A is the inverse of the M_1-operator $I - B$, where

$$B = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $\|B\| = r(B) = \frac{2}{3} < 1$, Theorem 3.5.1 implies that $u = \binom{u_1}{u_2} \in \mathrm{SMP}(A)$ provided

$$\|B\| = \tfrac{2}{3} < \tfrac{m}{M} = \tfrac{\min\{u_1,u_2\}}{\max\{u_1,u_2\}}\,,$$

which is assured if the components of u satisfy the inequalities $u_1 > 0$, $\frac{2}{3}u_1 < u_2 < \frac{3}{2}u_1$. This set of interior points of K is strictly smaller than the set SMP(A), see Figure 3.3.

Next we study a second approach to get a sufficient condition for the inverse of an M-operator to satisfy some maximum principle. This approach is motivated by the finite-dimensional case (with the standard ordering). In Proposition 3.3.2 a sufficient condition for an M-matrix B to satisfy the γ-maximum principle is given. Proposition 3.3.7 provides the connection between the γ-maximum principle for B and the maximum principle MP for B^{-1}. Combining both statements, we get the following.

Corollary 3.5.4 *Let the space $\mathbb{R}^n$ be equipped with the cone $\mathbb{R}^n_+$, let $u \in \operatorname{int}\mathbb{R}^n_+$ and let B be an* M*-matrix. Then B^{-1} satisfies* MP *with respect to u if and only if $Bu \in \mathbb{R}^n_+$.*

This gives reason to look for a corresponding condition in the general case. We formulate the question:

> *Let $(X, K, \|\cdot\|)$ be an ordered normed space, $u \in \operatorname{int} K$ a fixed element and $B \in \mathcal{L}(X)$ an* M_2*-operator. Does B^{-1} satisfy* MP *(or even* SMP*, respectively) with respect to u provided $Bu \in K$?*

We will not get a confirmative answer in general. We collect certain positive results as well as give counterexamples in other cases. First we examine the following situation: We strengthen the condition $Bu \in K$ to $Bu \in \operatorname{int} K$ and expect SMP for B^{-1}.

Theorem 3.5.5 *Let $(X, K, \|\cdot\|)$ be an ordered normed space with a closed cone K that has a non-empty interior, let $u \in \operatorname{int} K$ be fixed and let $B \in \mathcal{L}(X)$ be an* M_2*-operator. If $Bu \in \operatorname{int} K$, then B^{-1} satisfies* SMP *with respect to u.*

Proof For any element $y > 0$ one has $x = B^{-1}y > 0$ and hence $\alpha(x) > 0$. For any $f \in F_\alpha(x)$ one has to show $f \in F_+(y)$. The representation (R_1) in Proposition 3.2.3 yields $x = \alpha(x)u - z$, where $z \in K$ and $f(z) = 0$. Since B is a negative-off-diagonal operator, one has $f(Bz) \leq 0$. The assumption $Bu \in \operatorname{int} K$ ensures $f(Bu) > 0$, hence $f(y) = f(Bx) = \alpha(x)f(Bu) - f(Bz) > 0$. So, B^{-1} satisfies SMP with respect to u. □

If $Bu \in \partial K$, then B^{-1} does not satisfy SMP with respect to u. Indeed, $Bu \in \partial K$ implies the existence of a functional $f \in F$ such that $f(Bu) = 0$. This contradicts $F = F_\alpha(u) \subseteq F_+(Bu)$.

Remark 3.5.6 In view of Theorem 3.2.4, one can formulate the result in Theorem 3.5.5 as follows: If $B \in \mathcal{L}(X)$ is a negative-off-diagonal operator that possesses an inverse $B^{-1} \in \mathcal{L}(X)$ and satisfies $Bu \in \operatorname{int} K$, then B^{-1} is positive and satisfies SMP with respect to u.

Remark 3.5.7 If B is an M_1-operator represented as $B = I - C$, $C \geq 0$, $r(C) < 1$, the condition in Theorem 3.5.5 means $Bu = u - Cu \in \operatorname{int} K$. In particular, if $\|Cu\| < \sup\{t \in \mathbb{R}_+ \colon \mathrm{B}(u,t) \subset K\}$, then B^{-1} satisfies SMP with respect to u.

A slight modification of the proof of Theorem 3.5.5 yields the following.

Proposition 3.5.8 *Let X_0 be a subspace of X such that $X_0 \cap \operatorname{int} K \neq \varnothing$. Let M be an arbitrary non-empty subset of $\operatorname{ext} F$ and let $B \colon X_0 \to X$ be a linear operator such that for every $f \in M$ and $x \in X_0 \cap K$ with $f(x) = 0$ one has $f(Bx) \leq 0$. Let $S \colon X \to S(X) \subseteq X_0$ be an invertible operator such that $S^{-1} = B|_{S(X)}$. If $u \in X_0 \cap \operatorname{int} K$ is such that $f(Bu) > 0$ for all $f \in M$, then for each $y \in X$, $y > 0$, one has $x = Sy > 0$ and $F_\alpha(x) \cap M \subseteq F_+(y)$.*

Indeed, if $x \in S(X) \subseteq X_0$ and $u \in X_0$, then $z = \alpha(x)u - x \in X_0 \cap K$. For a functional $f \in F_\alpha(x) \cap M = F_0(z) \cap M$ we get $f(Bz) \leq 0$ due to the assumption. Since $f(Bu) > 0$, the argument can be completed as in the proof of Theorem 3.5.5.

Example 3.5.9 For this example we refer to the classical theory on the Dirichlet problem established e. g. in [GT01, Chapter 6]. We consider the following homogeneous boundary value problem for an elliptic operator

$$\begin{cases} Bx &= y \quad \text{in} \quad \Omega \\ x &= 0 \quad \text{on} \quad \partial\Omega \end{cases}, \tag{3.26}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a sufficiently smooth boundary $\partial\Omega$, and B denotes the operator

$$(Bx)(t) = -\sum_{i,j=1}^{n} a_{ij}(t) D_{ij} x + \sum_{i=1}^{n} b_i(t) D_i x + c(t) x$$

with the coefficient-functions $a_{ij} = a_{ij}(t)$ satisfying $a_{ij} = a_{ji}$. We assume that a_{ij}, b_i, c, y are Hölder-continuous functions with the exponent $\mu \in (0,1)$, i. e. belong to the space

$$C^{\mu}(\overline{\Omega}) = \{v \in C(\overline{\Omega}) \colon \exists L > 0 \text{ such that } |v(s) - v(t)| \leq L\|s-t\|^{\mu} \text{ for any } s,t \in \overline{\Omega}\}$$

and satisfy the conditions $c(t) \geq 0$, $y(t) \geq 0$ for $t \in \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω with respect to the norm $\|\cdot\|$ in $\mathbb{R}^n$. Denote by $\operatorname{supp}(y)$ the closure of the set $\{t \in \Omega \colon y(t) > 0\}$ and, in order to avoid the trivial case, assume $\operatorname{supp}(y) \neq \overline{\Omega}$.
The operator B is said to be *strongly (uniformly) elliptic*, if there is a constant $\lambda > 0$ such that

$$0 < \lambda\|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j$$

for any $t \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n$, $\xi \neq 0$.
For $k \in \mathbb{N}$ we denote by $C^k(\overline{\Omega})$ the vector space of all uniformly continuous functions $v \colon \Omega \to \mathbb{R}$, for which all partial derivatives $D^{\alpha}v$ of order $|\alpha| \leq k$ exist and are also uniformly continuous on Ω. Therefore they possess unique continuous extensions onto $\overline{\Omega}$. These extensions are again denoted by $D^{\alpha}v$.
Under the made assumptions for any function $y \in C^{\mu}(\overline{\Omega})$ the boundary value problem (3.26) has a unique solution u, which belongs to the Hölder space

$$C^{2,\mu}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \ D^{\alpha}v \in C^{\mu}(\overline{\Omega}) \quad \text{for} \quad 0 \leq |\alpha| \leq 2, \ \mu \in (0,1)\}$$

(cf. also [Ama72], [Ama76]). So the solution operator

$$S \colon C^{\mu}(\overline{\Omega}) \to C^{2,\mu}(\overline{\Omega})$$

is defined, where Sy denotes the unique solution of the boundary value problem (3.26). The operator S is strongly positive (cf. also [PW67]), i. e. any non-zero element of the cone of the non-negative functions is mapped by S into the interior of this cone. From this we get $(Sy)(t) = x(t) > 0$, $t \in \Omega$, and a standard argument shows that x satisfies the condition

$$\max_{t \in \overline{\Omega}} x(t) = \max_{t \in \operatorname{supp}(y)} x(t) \,. \tag{3.27}$$

By straightforward calculation[3] one gets even the following: If $c(t) > 0$ for all $t \in \Omega$ and if the solution x of the boundary value problem (3.26) has a local maximum at the point $t^* \in \Omega$, then $y(t^*) > 0$ and $x(t^*) \le \frac{y(t^*)}{c(t^*)}$.
Put $X = C^\mu(\overline{\Omega})$, $X_0 = C^{2,\mu}(\overline{\Omega})$ and let K be the set of all pointwise non-negative functions in X. Let $\mathbb{1}(t) = 1$ for all $t \in \overline{\Omega}$ and note that $\mathbb{1} \in X_0$. Our assumptions ensure that X is dense in $C(\overline{\Omega})$, hence there is a bijective map $i\colon X' \to (C(\overline{\Omega}))'$ such that for each $f \in X'$ we have $i(f)|_X = f$. Recall Example 1.3.9 and let M be the set of all evaluation maps ε_t determined by the points $t \in \Omega$. Then the above result means that the condition $c(t) > 0$ for all $t \in \Omega$ is sufficient for each $y \in X$, $y > 0$ to imply $F_\alpha(Sy) \cap M \subseteq F_+(y)$. The same can be derived by the following considerations:
An arbitrary function $v \in X_0 \cap K$ with $v(t^*) = 0$ for some $t^* \in \Omega$ has a local minimum in t^*. From the calculus it is known that at the point t^* one has $(D_i v)(t^*) = 0$ for $i = 1, \dots, n$ and

$$\sum_{i,j=1}^{n} (D_{ij} v)(t^*) \xi_i \xi_j \ge 0$$

for any vector $\xi \in \mathbb{R}^n$. Due to the (strong) ellipticity of the operator B the inequality

$$\sum_{i,j=1}^{n} a_{ij}(t^*) \xi_i \xi_j \ge 0$$

holds, which yields

$$\sum_{i,j=1}^{n} a_{ij}(t^*) (D_{ij} v)(t^*) \ge 0$$

(cf. [Hel60, III.1.1]). Consequently, $(Bv)(t^*) \le 0$. For the function $\mathbb{1}$ we get

$$(B\mathbb{1})(t) = c(t)\mathbb{1}(t) = c(t) > 0$$

for each $t \in \Omega$. Hence all conditions in Proposition 3.5.8 are satisfied. So, for each $y \in X$, $y > 0$ we have $F_\alpha(Sy) \cap M \subseteq F_+(y)$.

Now we apply the sufficient condition of Theorem 3.5.5 to an example that will be a crucial counterexample later on.

Example 3.5.10 Consider in $\mathbb{R}^3$ the the 3-dimensional ice-cream cone

$$K_1 = \operatorname{pos}\left\{ \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} : \ x_1^2 + x_2^2 \le 1 \right\}$$

and the finitely generated cone

$$K_2 = \operatorname{pos}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

[3] Cf. also [KW00b, Lemma 1].

then $K = K_1 \cup K_2$ is again a cone in $\mathbb{R}^3$. For $v = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) \in \operatorname{int} K$ consider the according base

$$D_v = \{x \in K\colon\ \langle x, v\rangle = 1\}$$

of K. A base of the dual cone K' is

$$F_v = \{x \in K'\colon\ \langle x, v\rangle = 1\}\,.$$

The sets D_v and F_v are the intersections of K and K' with the plane $x_3 = 1$, respectively, and are illustrated in Figure 3.4.

Consider the operator

$$B = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

B is a M_2-operator, since

(i) B is negative-off-diagonal with respect to K.

Indeed, consider the following cases of an extreme point $\tilde f$ of F_v and an appropriate element

$$x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix}\right) \in D_v$$

such that $\tilde f(x) = \langle x, \tilde f\rangle = 0$:

(1) For $\tilde f_1 = \left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right)$ we get $\langle x, \tilde f_1\rangle = 0$ if and only if $x_1 = 1$ and $0 \le x_2 \le 1$. Hence

$$\tilde f_1(Bx) = \left\langle B\left(\begin{smallmatrix} 1 \\ x_2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right)\right\rangle = \left\langle \left(\begin{smallmatrix} 2x_2-1 \\ x_2-2 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right)\right\rangle = -2x_2 \le 0\,.$$

(2) Let $\tilde f_2 = \left(\begin{smallmatrix} 0 \\ -1 \\ 1 \end{smallmatrix}\right)$. Then $\langle x, \tilde f_2\rangle = 0$ if and only if $x_2 = 1$ and $0 \le x_1 \le 1$. We conclude

$$\tilde f_2(Bx) = \left\langle B\left(\begin{smallmatrix} x_1 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \\ 1 \end{smallmatrix}\right)\right\rangle = \left\langle \left(\begin{smallmatrix} x_1 \\ -2x_1+1 \\ -2x_1+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \\ 1 \end{smallmatrix}\right)\right\rangle = 0\,.$$

(3) Finally, let $\tilde f_3 = \left(\begin{smallmatrix} w_1 \\ w_2 \\ 1 \end{smallmatrix}\right)$ with $w_1^2 + w_2^2 = 1$ and at least one $w_i > 0$. Moreover, let

$$\langle x, \tilde f_3\rangle = x_1 w_1 + x_2 w_2 + 1 = 0\,.$$

If one assumes $x_1 > 0$, $x_2 > 0$ and, without loss of generality, $w_1 > 0$, then

$$-x_2 w_2 = x_1 w_1 + 1 > 1$$

yields a contradiction since $|x_2| \le 1$, $|w_2| \le 1$. Hence, at least one $x_i \le 0$ and, since x belongs to D_v, $x_1^2 + x_2^2 \le 1$. So,

$$\begin{aligned} 0 &\le (x_1 + w_1)^2 + (x_2 + w_2)^2 \\ &= 2(x_1 w_1 + x_2 w_2) + (w_1^2 + w_2^2) + (x_1^2 + x_2^2) \\ &\le 2(x_1 w_1 + x_2 w_2 + 1) = 2\langle x, \tilde f_3\rangle = 0\,. \end{aligned}$$

Hence, $x_1 = -w_1$ and $x_2 = -w_2$. This yields

$$\tilde{f}_3(Bx) = \left\langle B\left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -x_1 \\ -x_2 \\ 1 \end{smallmatrix}\right)\right\rangle = \left\langle \left(\begin{smallmatrix} x_1+2x_2-2 \\ -2x_1+x_2 \\ -2x_1+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -x_1 \\ -x_2 \\ 1 \end{smallmatrix}\right)\right\rangle = -x_1^2 - x_2^2 \le 0\,.$$

(ii) The operator

$$B^{-1} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 4 \\ 2 & -4 & 5 \end{pmatrix}$$

is positive with respect to K.

We show $B^{-1}(K_1) \subseteq K_1$ and $B^{-1}(K_2) \subseteq K_2$. To determine $B^{-1}(K_1)$, we look for points

$$x = \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix}\right) \;\text{ which satisfy }\; Bx = \left(\begin{smallmatrix} x_1+2x_2-2 \\ -2x_1+x_2 \\ -2x_1+1 \end{smallmatrix}\right) \in K_1\,.$$

This means $-2x_1 + 1 \ge 0$, i. e. $x_1 \le \frac{1}{2}$, and

$$(x_1 + 2x_2 - 2)^2 + (-2x_1 + x_2)^2 \le (-2x_1 + 1)^2\,.$$

So, $x_1^2 + 5x_2^2 - 8x_2 + 3 \le 0$, which is equivalent to

$$\frac{x_1^2}{\left(\frac{1}{\sqrt{5}}\right)^2} + \frac{\left(x_2 - \frac{4}{5}\right)^2}{\left(\frac{1}{5}\right)^2} \le 1\,.$$

Hence, $B^{-1}(K_1)$ is the cone spanned by the ellipse in the plane $x_3 = 1$ with the centre $(0, \frac{4}{5}, 1)$ and the half-axises $\frac{1}{\sqrt{5}}$ and $\frac{1}{5}$:

$$B^{-1}(K_1) = \operatorname{pos}\left\{ \left(\begin{smallmatrix} x_1 \\ x_2 \\ 1 \end{smallmatrix}\right) \colon \;\frac{x_1^2}{\left(\frac{1}{\sqrt{5}}\right)^2} + \frac{\left(x_2 - \frac{4}{5}\right)^2}{\left(\frac{1}{5}\right)^2} \le 1 \right\}.$$

It is straightforward that $B^{-1}(K_1) \subseteq K_1$.

To ascertain $B^{-1}(K_2)$, we map the extremals $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right)$ of K_2 and get

$$\begin{aligned} B^{-1}(K_2) &= \operatorname{pos}\left\{ \left(\begin{smallmatrix} 2 \\ 4 \\ 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 \\ 6 \\ 7 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 3 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right) \right\} \\ &= \operatorname{pos}\left\{ \left(\begin{smallmatrix} 2/5 \\ 4/5 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3/7 \\ 6/7 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1/3 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right) \right\} \subseteq K_2\,. \end{aligned}$$

Finally, $B^{-1}(K) \subseteq K$. In Figure 3.4 the base

$$G_v = \{x \in B^{-1}(K) \colon \langle x, v\rangle = 1\}$$

of the cone $B^{-1}(K)$ is illustrated.

Now let $u \in \operatorname{int} B^{-1}(K) = B^{-1}(\operatorname{int} K)$ (see (A.1)), then $Bu \in \operatorname{int} K$, and by Theorem 3.5.5 the operator B^{-1} satisfies SMP with respect to u.

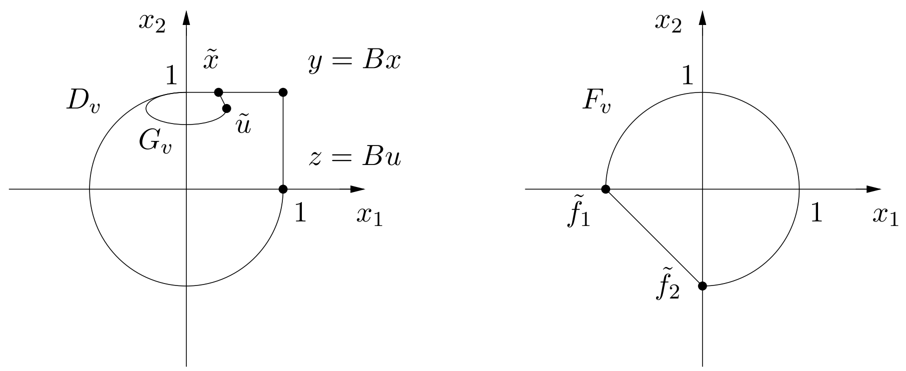

Figure 3.4: Illustration of the Examples 3.5.10 and 3.5.11.

The last theorem gives reason to specify our above question.

> *Let $(X, K, \|\cdot\|)$ be an ordered normed space, $u \in \operatorname{int} K$ a fixed element and $B \in \mathcal{L}(X)$ an* M_2*-operator. Does B^{-1} satisfy* MP *with respect to u provided $Bu \in \partial K$?*

In general, this question has to be answered negatively. First we provide a counterexample, afterwards we show that for certain spaces the question can be answered confirmatively.

We start with a consideration how a counterexample in general should be constructed. Let $(X, K, \|\cdot\|)$ be an ordered normed space, let $B \in \mathcal{L}(X)$ be an M_2-operator and $u \in K$ a fixed element such that $Bu \in \partial K$. The operator B^{-1} does not satisfy MP if there is an element $y > 0$ such that for $x = B^{-1}y > 0$ one has $F_\alpha(x) \cap F_+(y) = \varnothing$, which means $F_\alpha(x) \subseteq F_0(Bx)$, i. e. for each $f \in F_\alpha(x)$ one has $f(Bx) = 0$. By Proposition 3.2.3 one gets $x = \alpha(x)u - z$ with some vector $z \in \partial K$, where $F_\alpha(x) = F_0(z)$. So, $Bx = \alpha(x)Bu - Bz$. For each $f \in F_\alpha(x)$ one has

$$0 = f(Bx) = \alpha(x)f(Bu) - f(Bz)\,.$$

Moreover, $f(Bu) \geq 0$ since $Bu \in K$, and $f(Bz) \leq 0$, since $f(z) = 0$ and B is negative-off-diagonal. So,

$$0 \geq f(Bz) = \alpha(x)f(Bu) \geq 0\,.$$

Due to $\alpha(x) > 0$ we infer $f(Bz) = f(Bu) = 0$.

Summarizing these estimates, for constructing a counterexample we need a vector $z \in \partial K$ such that

$$F_0(z) \subseteq F_0(Bz) \ \text{ and } \ F_0(z) \subseteq F_0(Bu)\,. \tag{3.28}$$

The subsequent example is a continuation of Example 3.5.10.

Example 3.5.11 Consider the cone $K = K_1 \cup K_2$ in $\mathbb{R}^3$ as in Example 3.5.10, we already showed that the matrix

$$B = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

is an M_2-operator with respect to K. Let

$$z = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad u = B^{-1}z = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix}.$$

Then $u \in \operatorname{int} K$, $Bu = z \in \partial K$ and

$$Bz = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}.$$

The elements $\tilde{u} = \frac{1}{7}u \in D_v$ and $z \in D_v$ are illustrated in Figure 3.4. We show that B^{-1} does not satisfy MP with respect to u, although $Bu \in \partial K$. The base

$$F = F_u = \{x \in K' \colon \langle x, u \rangle = 1\}$$

of K' possesses the following set of extreme points:

$$\operatorname{ext} F = \Big\{ \tfrac{1}{3w_1+6w_2+7} \begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix} \colon \; w_1^2 + w_2^2 = 1\,, \;\; w_1 \geq 0 \;\text{ or }\; w_2 \geq 0 \Big\}.$$

In particular, $f_1 = \frac{1}{4}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ are extreme points of F. One has

$$F_0(z) = F_0(Bu) = \{f_1\}$$

and $f_1(Bz) = 0$, therefore

$$F_0(z) \subseteq F_0(Bz)\,.$$

So, the conditions in (3.28) are satisfied. Observe that the ray $\mathrm{r}(z)$ is a face of K, but it is not an exposed ray of K. If one takes now

$$x = u - z = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix} \in K\,, \quad \text{then} \quad y = Bx = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in K\,.$$

The elements $\tilde{x} = \frac{1}{6}x \in D_v$ and $y \in D_v$ are illustrated in Figure 3.4. Due to Proposition 3.2.3 one has

$$F_\alpha(x) = F_0(z) = \{f_1\}\,,$$

and moreover, $f_1(y) = 0$, consequently $F_\alpha(x) \subseteq F_0(Bx)$. So $F_\alpha(B^{-1}y) \cap F_+(y) = \varnothing$, and B^{-1} does not satisfy MP with respect to u.

To get an affirmative answer for the above question we need some more assumptions on the structure of ∂K. The subsequent results are shown in a finite-dimensional setting. We consider the space $X = \mathbb{R}^n$, equipped with a closed generating cone K (i. e. K has a non-empty interior). As above, fix some element $u \in \operatorname{int} K$ and the base F of K' corresponding to (1.16). For a subset $H \subseteq K$ we define

$$F_0(H) = \{f \in F \colon \; f(x) = 0 \text{ for all } x \in H\}\,,$$

and for $x \in K$ we abbreviate $F_0(\{x\})$ by $F_0(x)$, as defined in (3.5). For a set $H \subseteq K$ put

$$E_H = \bigcap \{\, f^{-1}(0) \colon \;\; f \in F_0(H) \,\}\,.$$

The set E_H is a closed linear subspace of X. For an element $z \in K$ we abbreviate $E_{\{z\}}$ by E_z. If $z \in H$, then

$$F_0(H) \subseteq F_0(z) \quad \text{and} \quad E_z \subseteq E_H \, .$$

If H is a face of K, then the inclusions

$$H \subseteq K \cap E_H \quad \text{and} \quad H - H \subseteq E_H$$

are obvious. An example for $H - H \neq E_H$ we get from 3.5.11 by setting $H = \mathrm{r}(z)$. In this case $H - H$ coincides with the line through 0 and z, and E_H is the plane through 0, z and $(1,1,1)^T$.

The next two lemmata describe some properties of faces of a closed generating cone in $\mathbb{R}^n$.

Lemma 3.5.12 *Let K be a closed generating cone in $\mathbb{R}^n$ and let H be a face of K with $H - H = E_H$. Then H is an exposed subset of K.*

Proof First notice that $H = (H - H) \cap K$. Indeed, if $x = x_1 - x_2 \geq 0$ with $x_1, x_2 \in H$, then $x_1 \geq x \geq 0$ and, since H is a face, $x \in H$.
For $\varepsilon > 0$ put $S_\varepsilon = \{x \in \mathbb{R}^n \colon \ \mathrm{dist}(E_H, x) < \varepsilon\}$. For the closed unit ball B in $\mathbb{R}^n$ one has

$$\mathrm{B} = (\mathrm{B} \cap E_H) \cup \big[\, \mathrm{B} \setminus \bigcap_{f \in F_0(H)} f^{-1}(0) \big] \subseteq (\mathrm{B} \cap S_\varepsilon) \cup \bigcup_{f \in F_0(H)} (\mathrm{B} \setminus f^{-1}(0))$$

which provides an open covering for B in the induced topology. Since B is compact we get a finite covering, i. e. there exist functionals $f_i \in F_0(H)$, $i = 1, \ldots, k$, such that

$$\mathrm{B} \subseteq (\mathrm{B} \cap S_\varepsilon) \cup \bigcup_{i=1}^{k} (\mathrm{B} \setminus f_i^{-1}(0)) = (\mathrm{B} \cap S_\varepsilon) \cup \big(\mathrm{B} \setminus \bigcap_{i=1}^{k} f_i^{-1}(0) \big) \, .$$

This implies

$$\mathrm{B} \cap \bigcap_{i=1}^{k} f_i^{-1}(0) \subseteq \mathrm{B} \cap S_\varepsilon$$

for every $\varepsilon > 0$. Hence $\bigcap_{i=1}^{k} f_i^{-1}(0) \subseteq \overline{E_H}$ and, since E_H is closed,

$$E_H = \bigcap_{i=1}^{k} f_i^{-1}(0) \ .$$

Put $g = \frac{1}{k} \sum_{i=1}^{k} f_i$ and observe that $g^{-1}(0) \cap K = E_H \cap K$. Now, by assumption, $H = (H - H) \cap K = E_H \cap K = g^{-1}(0) \cap K$, hence H is exposed. $\square$

For a face H of K an element $z \in H$ is a relative interior point of H, denoted $z \in \mathrm{ri}(H)$, if there is a number $\varepsilon > 0$ such that $\mathrm{B}(z, \varepsilon) \cap (H - H) \subseteq H$. The smallest face G of K that contains z is the set

$$G = \{x \in X \colon \ \text{there is a number } \lambda \geq 0 \text{ such that } 0 \leq x \leq \lambda z\} \ .$$

This implies $F_0(z) = F_0(G)$ and hence

$$E_z = E_G \,. \tag{3.29}$$

Lemma 3.5.13 *Let K be a closed generating cone in $\mathbb{R}^n$, H a face of K and $z \in H$. Then the properties*

(i) $z \in \mathrm{ri}(H)$,

(ii) *H is the smallest face of K that contains z*

are equivalent. If, in addition, every face of K is an exposed subset of K, then (ii) *is equivalent to the property*

(iii) $H = K \cap E_z$.

Proof For the equivalence of (i) and (ii) see [Brø83, Theorem 5.6].

We show that (ii) implies (iii). Due to (3.29) we immediately get $H \subseteq K \cap E_H = K \cap E_z$. Vice versa, since H is exposed, there exists a functional $g \in K'$ such that $H = g^{-1}(0) \cap K$. From $g(z) = 0$ we get $g \in F_0(z)$ and $E_z \subseteq g^{-1}(0)$. Hence $K \cap E_z \subseteq K \cap g^{-1}(0) = H$.

Now we assume (iii) and show (ii). Let $H = K \cap E_z$ and let G be the smallest face of K that contains z. The inclusion $G \subseteq H$ is obvious. By assumption, G is exposed, i. e. there exists a functional $g \in K'$ such that $G = g^{-1}(0) \cap K$. Since $z \in G$ we get $g \in F_0(z)$ and $E_z \subseteq g^{-1}(0)$, hence $H \subseteq K \cap g^{-1}(0) = G$. □

Note that for a face H of K with $\dim(H) = n - 1$ the set $F_0(H)$ contains only one element, say f, and $f^{-1}(0) = H - H$. Furthermore, $f^{-1}(0) \cap K = H$, hence H is exposed. For example, if K is a closed generating cone in $\mathbb{R}^3$ and every extreme ray of K is exposed, then K has the property required in Lemma 3.5.13 that every face of K is an exposed subset of K.

Next we deal with finitely generated cones in $\mathbb{R}^n$.

Lemma 3.5.14 *Let K be a finitely generated cone in $\mathbb{R}^n$ that is generating. Then every face H of K is exposed and $H - H = E_H$.*

Proof The cone K has a base D with $\operatorname{ext} D = \{x_1, x_2, \ldots, x_r\}$, and K' possesses a base F with $\operatorname{ext} F = \{f_1, f_2, \ldots, f_s\}$ (for details see e. g. [PSW98]). If G is a face of K, one has $G = \mathrm{pos}\{x_{i_1}, \ldots x_{i_k}\}$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$, $F_0(G) = \mathrm{co}\{f_{j_1}, \ldots f_{j_l}\}$ for some $\{j_1, \ldots, j_l\} \subseteq \{1, \ldots, s\}$ and $G = f^{-1}(0) \cap K$ for

$$f = \tfrac{1}{l} \sum_{m=1}^{l} f_{j_m} \in F \,.$$

Hence every face of K is an exposed subset of K.

Now let H be a face of K, $z \in \mathrm{ri}(H)$ and $v \in E_H \setminus \{0\}$, we show $v \in H - H$. Observe that $F_0(z) = F_0(H) \subseteq F_0(v)$. Since $F^+(z) \cap \operatorname{ext} F$ is a finite set one has

$$b(z) = \min \{ \, g(z) \colon \; g \in F^+(z) \cap \operatorname{ext} F \, \} > 0 \,.$$

Put $a(v) = \max\{\,|f(v)|\colon\ f \in F\,\}$, then $a(v) > 0$ since K is generating and closed. We get $1 \geq \frac{g(v)}{a(v)} \geq -1$ for each $g \in F$. Put

$$\varepsilon = \tfrac{b(z)}{2a(v)} \quad\text{and}\quad w = z + \varepsilon v$$

and note that $\varepsilon > 0$. For every $g \in F^+(z) \cap \operatorname{ext} F$ one has

$$g(w) = g(z) + \tfrac{b(z)}{2}\tfrac{g(v)}{a(v)} \geq b(z) - \tfrac{b(z)}{2} > 0\,.$$

If $g \in F_0(z)$ then $g \in F_0(v)$, and we get $g(w) = 0$. Consequently, $g(w) \geq 0$ for every $g \in F$ and, since K is closed, $w \in K$. We conclude $w \in K \cap E_H$ and apply Lemma 3.5.13 and (3.29). Since every face of K is exposed we get $w \in K \cap E_z = H$. This yields

$$v = \tfrac{w}{\varepsilon} - \tfrac{z}{\varepsilon} \in H - H\,,$$

and hence $E_H = H - H$. $\square$

The next statement, which concerns circular cones in $\mathbb{R}^n$, is straightforward.

Lemma 3.5.15 *For a circular cone K in $\mathbb{R}^n$ every face H of K is an exposed ray.*

Before we turn to the main theorem we provide a technical statement.

Proposition 3.5.16 *Let E be a linear subspace of $\mathbb{R}^n$ and $B\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear invertible operator such that $B(E) \subseteq E$. Then $B|_E\colon E \to E$ is surjective.*

Proof Since B is invertible, it is surjective. Assume that $B|_E\colon E \to E$ is not surjective. Then $\dim E = k < n$, and there is an element $x_1 \in \mathbb{R}^n \setminus E$ such that $Bx_1 \in E$. Put

$$E_1 = \operatorname{span}\{E, x_1\}\,.$$

So, $\dim E_1 = k + 1$ and $B|_{E_1}\colon E_1 \to E_1$ is not surjective. Either $E_1 = \mathbb{R}^n$, or one can continue analogously. Consequently, B is not surjective, which yields a contradiction. $\square$

Theorem 3.5.17 *Let K be a closed generating cone in $\mathbb{R}^n$. Let every extreme ray of K be exposed, and for every face H of K with $\dim H > 1$ let $H - H = E_H$. Furthermore, let $u \in \operatorname{int} K$ be fixed and let $B\colon \mathbb{R}^n \to \mathbb{R}^n$ be an M_2-operator in $(\mathbb{R}^n, K, \|\cdot\|)$. If $Bu \in K$, then B^{-1} satisfies MP with respect to u.*

Proof Let B be an M_2-operator with $Bu \in K$ and suppose that B^{-1} does not satisfy MP with respect to u. Then there exists an element $y > 0$ such that for every functional $f \in F_\alpha(B^{-1}y)$ follows $f(y) = 0$. Now the idea is to find a linear subspace E of $\mathbb{R}^n$ that is invariant with respect to B, i. e. $B(E) \subseteq E$, such that $u \notin E$ and $Bu \in E$. Then Proposition 3.5.16 implies that there exists an element $e \in E$ such that $Be = Bu$. Since B is injective we get a contradiction.
Put $x = B^{-1}y$. Hence $\alpha(x) > 0$. Due to Proposition 3.2.3 there exists an element $z \in K$ such that $x = \alpha(x)u - z$ and $F_0(z) = F_\alpha(x)$. Note that $z \neq 0$, otherwise

$F = F_0(z) = F_\alpha(x) \subseteq F_0(y)$ which contradicts $y \neq 0$. Let $f \in F_0(z)$. Since B is a negative-off-diagonal operator, we get $f(Bz) \leq 0$. The condition $Bu \in K$ implies $f(Bu) \geq 0$. Since $0 = f(y) = \alpha(x)f(Bu) - f(Bz)$, $f(Bu) \geq 0$ and $-f(Bz) \geq 0$ one has $f(Bu) = f(Bz) = 0$ for every $f \in F_0(z)$. The set $H = K \cap E_z$ is a face of K with $\mathrm{r}(z) \subseteq H$ and $Bu \in H$. Consider the following cases:

(i) $H = \mathrm{r}(z)$. Since $y \in H$ one has $Bz = \alpha(x)Bu - y \in H - H$. If v is an arbitrary element of $H - H$, then $v = \lambda z$ for some $\lambda \in \mathbb{R}$ and $Bv = \lambda Bz$, i. e. Bv belongs to $H - H$. Hence the linear subspace $E = H - H$ of $\mathbb{R}^n$ is invariant with respect to B, furthermore $Bu \in E$ and, obviously, $u \notin E$.

(ii) $H \neq \mathrm{r}(z)$. Lemma 3.5.12 and the assumption ensure that every face of K is exposed. Hence, due to Lemma 3.5.13, the element z is a relative interior point of H, i. e. there exists a number $\varepsilon > 0$ such that $\mathrm{B}(z, \varepsilon) \cap (H - H) \subseteq H$. Furthermore $E_z = E_H$, and, by assumption, $E_z = H - H$. Consider an element $v \in E_z$ with $\|v\| \leq \varepsilon$. We get $z \pm v \in \mathrm{B}(z, \varepsilon) \cap E_z = \mathrm{B}(z, \varepsilon) \cap (H - H) \subseteq H$ and, in particular, $f(z \pm v) = 0$ for every $f \in F_0(z)$. Since B is a negative-off-diagonal operator one has $0 \geq f(B(z \pm v)) = f(Bz) \pm f(Bv) = \pm f(Bv)$ and hence $f(Bv) = 0$ for each $f \in F_0(z)$. This yields $Bv \in E_z$. Hence $E = E_z$ is an invariant linear subspace of $\mathbb{R}^n$ with respect to the operator B with $Bu \in E$ and $u \notin E$. □

Lemma 3.5.14 implies the following consequence of Theorem 3.5.17.

Corollary 3.5.18 *Let K be a finitely generated cone in $\mathbb{R}^n$ that is generating, let $u \in \operatorname{int} K$ and $B \colon \mathbb{R}^n \to \mathbb{R}^n$ an M_2-operator. If $Bu \in K$ then B^{-1} satisfies* MP *with respect to u.*

Finally, we apply Lemma 3.5.15 and get the following statement.

Corollary 3.5.19 *Let K be a circular cone in $\mathbb{R}^n$, let $u \in \operatorname{int} K$ and $B \colon \mathbb{R}^n \to \mathbb{R}^n$ an M_2-operator. If $Bu \in K$ then B^{-1} satisfies* MP *with respect to u.*

The results of the thesis are summarized in the subsequent commentary.

Commentary

Inspired by results on M-matrices, this text has investigated two classes of M-operators, focussing on properties like satisfying certain maximum principles. This section aims to give a short summary with some comments on the main results.

In the thesis certain types of operators on an ordered normed space have been considered. We have investigated the class $\Pi_1(K)$ of operators that dominate a multiple of the identity, and the class $\Sigma(K)$ of positive-off-diagonal operators. The notions M_1- and M_2-operator have been defined and investigated on an ordered normed space as two different generalizations of the notion M-matrix.
An M_1-operator is dominated by a multiple of the identity, whereas an M_2-operator is defined using the negative-off-diagonal property. Hence, the class of M_1-operators is related to $\Pi_1(K)$, whereas the class of M_2-operators is related to $\Sigma(K)$. If in some ordered normed space the classes $\Pi_1(K)$ and $\Sigma(K)$ are equal, then the notions M_1- and M_2-operator coincide. Certain sufficient conditions have been listed which ensure that the classes of operators $\Pi_1(K)$ and $\Sigma(K)$ are equal.
Even in the finite-dimensional case it is still an open problem to give a condition on the underlying ordered space which is necessary and sufficient for $\Pi_1(K)$ and $\Sigma(K)$ to coincide. As a first step, other classes of ordered normed spaces than the ones listed in Corollary 2.4.5 should be investigated to find sufficient conditions for $\Pi_1(K)$ and $\Sigma(K)$ to be equal. Moreover, the question arises for which ordered normed spaces the equality (2.6) is valid.

As a second main topic we have dealt with certain maximum principles for operators on ordered normed spaces. Inspired by the maximum principle for inverse column entries and its generalizations in a finite-dimensional setting, we have considered the maximum principles MP and SMP for positive operators on an ordered normed space, where the cone has a non-empty interior. Here the question arises if it is possible to give profitable definitions for maximum principles under a weaker assumption on the cone. The ideas sketched in Section 3.1 could serve as a starting point.
For the introduced maximum principles geometrical characterizations have been stated. These characterizations are necessary and sufficient for an operator to satisfy a maximum principle, and provide a tool to show a maximum principle for a given operator.
For the sets of MP-operators and SMP-operators many desired structures fail. It could be of interest if the sets MP(A) and SMP(A) for an operator $A \in \mathcal{L}_+(X)$, as defined in Section 3.4, own a certain structure.

Finally, the (positive) inverses of M_1- or M_2-operators have been investigated. Certain sufficient conditions have been shown for these positive operators to satisfy MP or SMP. For M_1-operators a technique has been developed which uses for an element of the underlying space the link between its norm and its maximum value. For M_2-operators conditions are established which use that the operator maps an interior point of the cone into the cone (or its interior, respectively).
So, the inverses of M-operators own interesting properties in connection with certain maximum principles, which leaves room for further investigations.

Appendix A

A.1 Some Standard Notations

Let X be a real vector space. The linear (affine, convex) hull of a subset M of X is denoted by $\operatorname{span} M$ ($\operatorname{aff} M$, $\operatorname{co} M$, respectively). Denote by $\dim M$ the dimension of the linear subspace $\operatorname{span} M$ generated by M. For $x \in X \setminus \{0\}$ let $\mathrm{r}(x) = \{\lambda x \colon \lambda \geq 0\}$ be the ray generated by x.

If (X, τ) is a topological space and M an arbitrary subset of X, then by $\operatorname{int} M$ and ∂M we denote the sets of all interior and boundary points of C, respectively. $\overline{M}$ (or, more precisely, $\overline{M}^{\tau}$) denotes the closure of M.

Now let $\|\cdot\|$ be a norm on X. For $x \in X$ and $r > 0$ let

$$\mathrm{B}(x, r) = \{v \in X \colon \ \|x - v\| \leq r\}$$

be the closed ball centered at x with radius r. We abbreviate $\mathrm{B}(0, 1)$ by B and denote the closed unit ball in the dual space X' by B'. Let $M \subseteq X$ be a convex set; an element $x \in M$ is a *relative interior point* of M, written $x \in \operatorname{ri} M$, if x is an interior point of M in $\operatorname{aff} M$, i. e. if there is a number $r > 0$ such that

$$\mathrm{B}(x, r) \cap \operatorname{aff} M \subseteq M\,.$$

$\mathfrak{L}(X)$ denotes the vector space of all linear continuous operators $A \colon X \to X$, where $I \colon X \to X$, $x \mapsto x$, denotes the identity operator. If $M \subset X$ and $A \in \mathfrak{L}(X)$ is such that $A^{-1} \in \mathfrak{L}(X)$, then

$$A(\operatorname{int} M) = \operatorname{int} A(M)\,. \tag{A.1}$$

Indeed, one has $A(\operatorname{int} M) \subseteq A(M)$, and $A(\operatorname{int} M)$ is open since A^{-1} is continuous, so $A(\operatorname{int} M) \subseteq \operatorname{int} A(M)$. On the other hand, $A^{-1}(\operatorname{int} A(M)) \subseteq A^{-1}(A(M)) = M$, and $A^{-1}(\operatorname{int} A(M))$ is open since A is continuous, so $A^{-1}(\operatorname{int} A(M)) \subseteq \operatorname{int} M$ and hence $\operatorname{int} A(M) \subseteq A(\operatorname{int} M)$.

A bilinear system is denoted by (X, Y), i. e. X and Y are real vector spaces and

$$\langle \cdot, \cdot \rangle \colon X \times Y \to \mathbb{R}$$

is a bilinear form.

If $X = Y = \mathbb{R}^n$, then we use $\langle\cdot,\cdot\rangle$ for the standard dot product, i. e. for vectors $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$ we have

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \, .$$

As usual, $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of all natural numbers, and $\mathbb{R}^{\mathbb{N}}$ is the vector space of all real sequences on $\mathbb{N}$. The space $\mathbb{R}^{\mathbb{N}}$ is ordered by the cone

$$\mathbb{R}^{\mathbb{N}}_+ = \{(x_i)_{i\in\mathbb{N}} \colon\ x_i \geq 0 \ \text{ for all } \ i \in \mathbb{N}\} \, .$$

Definition A.1.1 A linear subspace X of $\mathbb{R}^{\mathbb{N}}$ that contains the space of finite sequences

$$\boldsymbol{\varphi} = \{(x_i)_{i\in\mathbb{N}} \colon\ x_i \neq 0 \ \text{ for at most finitely many } \ i\}$$

is called a *sequence space*, where X is ordered by the cone $X^+ = X \cap \mathbb{R}^{\mathbb{N}}_+$.

Example A.1.2 The real sequence space

$$\boldsymbol{l}_\infty = \{(x_i)_{i\in\mathbb{N}} \colon\ (x_i)_{i\in\mathbb{N}} \ \text{ is bounded}\}$$

is equipped with the supremum norm

$$\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i| \ \text{ for } \ x = (x_i)_{i\in\mathbb{N}} \in l_\infty \, .$$

The sequence spaces

$$\boldsymbol{c} = \{(x_i)_{i\in\mathbb{N}} \colon\ (x_i)_{i\in\mathbb{N}} \ \text{ converges}\} \ \text{ and } \ \boldsymbol{c}_0 = \{(x_i)_{i\in\mathbb{N}} \colon\ \lim_{i\to\infty} x_i = 0\}$$

are linear subspaces of $\boldsymbol{l}_\infty$. The sequence space

$$\boldsymbol{l}_p = \{(x_i)_{i\in\mathbb{N}} \colon\ \sum_{n=1}^{\infty} |x_i|^p < \infty\} \, ,$$

where $1 \leq p < \infty$, is equipped with the norm

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_i|^p)^{1/p} \ \text{ for } \ x = (x_i)_{i\in\mathbb{N}} \in \boldsymbol{l}^p \, .$$

The spaces $\boldsymbol{l}_\infty$, $\boldsymbol{c}$, $\boldsymbol{c}_0$, $\boldsymbol{l}_p$ are Banach spaces. The space $\boldsymbol{\varphi}$ can be considered as a subspace of $\boldsymbol{l}_p$ for a certain $p \in [1, \infty]$, where $\boldsymbol{\varphi}$ is not a Banach space.

Let $(X, \|\cdot\|)$ be a Banach space that has a Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$, i. e. for each $x \in X$ there is a unique sequence $(x_j)_{j\in\mathbb{N}}$ of real numbers such that

$$x = \sum_{j=1}^{\infty} x_j b^{(j)} \, , \tag{A.2}$$

where the series converges in norm. In what follows, we list some results given in [LT96, Sections 1.a, 1.b]. For each $n \in \mathbb{N}$ the natural projection

$$P^{(n)}\colon X \to X\,,\ x \mapsto \sum_{j=1}^{n} x_j b^{(j)}\,,$$

is a linear continuous operator, where $\sup\{\|P^{(n)}\|\colon n \in \mathbb{N}\} < \infty$.

Definition A.1.3 The constant

$$s = \sup\{\|P^{(n)}\|\colon n \in \mathbb{N}\}$$

is called the *basis constant* associated to the Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$.

For each $j \in \mathbb{N}$ the mapping

$$f^{(j)}\colon X \to \mathbb{R}\,,\ x \mapsto x_j\,, \tag{A.3}$$

is well-defined, linear and continuous, where

$$\|f^{(j)}\| \le \tfrac{2s}{\|b^{(j)}\|}\,. \tag{A.4}$$

For all $i, j \in \mathbb{N}$ one has

$$f^{(i)}(b^{(j)}) = (b^{(j)})_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}\,, \tag{A.5}$$

i. e. $f^{(i)}$ is *biorthogonal*.

Definition A.1.4 The Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$ of the Banach space X is called *shrinking*, if for every $f \in X'$ one has

$$\lim_{n\to\infty} \|f|_{\operatorname{span}\{b^{(j)}\colon j\in\mathbb{N},\, j\ge n\}}\| = 0\,.$$

Proposition A.1.5 *The sequence $(f^{(j)})_{j\in\mathbb{N}} \subset X'$ defined according to (A.3) is a Schauder basis of X' if and only if the Schauder basis $(b^{(j)})_{j\in\mathbb{N}}$ of X is shrinking.*

In this case, for $f \in X'$ one has the representation

$$f = \sum_{i=1}^{\infty} f(b^{(i)}) f^{(i)}\,, \tag{A.6}$$

where the sequence converges in the dual norm.
Let $A \in \mathcal{L}(X)$ and define an infinite matrix $(a_{ij})_{i,j\in\mathbb{N}}$ by means of

$$a_{ij} = f^{(i)}(Ab^{(j)}) = (Ab^{(j)})_i \quad \text{for all} \quad i, j \in \mathbb{N}\,. \tag{A.7}$$

Observe that for every $i \in \mathbb{N}$ one has

$$|a_{ii}| = |f^{(i)}(Ab^{(i)})| \leq \|f^{(i)}\| \, \|A\| \, \|b^{(i)}\| \leq 2s\|A\| \tag{A.8}$$

due to (A.4). For $x \in X$ one gets

$$\begin{aligned} |f^{(i)}(Ax) - \sum_{j=1}^{n} a_{ij} f^{(j)}(x)| &= |f^{(i)}(Ax) - \sum_{j=1}^{n} f^{(i)}(Ab^{(j)}) f^{(j)}(x)| \\ &= |f^{(i)}(Ax) - f^{(i)}(A \sum_{j=1}^{n} f^{(j)}(x) b^{(j)})| \\ &\leq \|f^{(i)}\| \, \|A\| \, \|x - \sum_{j=1}^{n} x_j b^{(j)}\| \,, \end{aligned}$$

which converges to 0 as $n \to \infty$. So,

$$(Ax)_i = f^{(i)}(Ax) = \sum_{j=1}^{\infty} a_{ij} x_j \,. \tag{A.9}$$

Clearly, if $A = I$, then (A.5) and (A.7) imply

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \,. \tag{A.10}$$

A.2 Extreme Subsets

Let X be a real vector space and let M be a non-empty convex subset of X. A non-empty subset $E \subset M$ is called an *extreme subset* of M if E is convex and for every $y, z \in M$ with $\lambda y + (1-\lambda) z \in E$ for some $\lambda \in (0,1)$ one has $y, z \in E$. The following is straightforward.

Lemma A.2.1 *If E_1 and E_2 are extreme subsets of M such that $E_1 \cap E_2 \neq \varnothing$, then $E_1 \cap E_2$ is an extreme subset of E_1.*

An element $x \in M$ is an *extreme point* of M if $\{x\}$ is an extreme subset of M. The set of all extreme points of M is denoted by $\operatorname{ext} M$. Obviously, $x \in M$ belongs to $\operatorname{ext} M$ if and only if $x = \lambda y + (1-\lambda) z$ implies $x = y = z$ whenever $y, z \in M$ and $\lambda \in (0,1)$. For the next statement see [Wer02, Lemma VIII.4.2].

Lemma A.2.2 *Let M be a non-empty convex subset of X. If E_1 is an extreme subset of M and E_2 is an extreme subset of E_1, then E_2 is an extreme subset of M. In particular, $\operatorname{ext} E_1 = E_1 \cap \operatorname{ext} M$.*

Now let (X,τ) be a real locally convex Hausdorff space and let M be a non-empty convex compact subset of X. The Krein-Milman Theorem guarantees both the condition $\operatorname{ext} M \neq \varnothing$ and the representation

$$M = \overline{\text{co}}^{\tau}(\operatorname{ext} M)\,. \tag{A.11}$$

Here $\overline{\text{co}}^{\tau}(N)$ denotes the τ-closure of the convex hull of a subset $N \subset X$. Let E be a compact extreme subset of M. Due to (A.11) and Lemma A.2.2 we have

$$E = \overline{\text{co}}^{\tau}(\operatorname{ext} E) = \overline{\text{co}}^{\tau}(E \cap \operatorname{ext} M)\ . \tag{A.12}$$

Now consider a real continuous functional f defined at least on M. Due to the Weierstrass Theorem we know that f achieves its maximum and minimum values on M. Let

$$\alpha_M(f) = \max\{f(x)\colon\ x \in M\} \ \text{ and } \ \beta_M(f) = \min\{f(x)\colon\ x \in M\}\,. \tag{A.13}$$

Denote the set $\{x \in M\colon\ f(x) = \alpha_M(f)\}$ of all maximizers by $M_\alpha(f)$ and, analogously, the set $\{x \in M\colon\ f(x) = \beta_M(f)\}$ of all minimizers by $M_\beta(f)$. $M_\alpha(f)$ and $M_\beta(f)$ are non-empty compact sets [AB99, Theorem 2.40].
Now, in addition, let f be linear. Then $M_\alpha(f)$ and $M_\beta(f)$ are extreme subsets of M [AB99, Lemma 5.113]. Every compact extreme subset of M contains an extreme point of M [AB99, Lemma 5.114]. This yields the statement of the Bauer maximum principle that f attains its maximum on M at an extreme point of M, hence

$$\alpha_M(f) = \max\{f(x)\colon\ x \in \operatorname{ext} M\} \ \text{ and } \ \beta_M(f) = \min\{f(x)\colon\ x \in \operatorname{ext} M\}\,. \tag{A.14}$$

Combining these statements with (A.12), we get the following.

Proposition A.2.3 *Let (X,τ) be a real locally convex Hausdorff space, M a non-empty convex compact subset of X and f a real linear continuous functional on X. Then the sets $M_\alpha(f)$ and $M_\beta(f)$ are non-empty compact extreme subsets of M, they contain extreme points of M and can be represented as*

$$M_\alpha(f) = \overline{\text{co}}^{\tau}(M_\alpha(f) \cap \operatorname{ext} M) \ \text{ and } \ M_\beta(f) = \overline{\text{co}}^{\tau}(M_\beta(f) \cap \operatorname{ext} M)\,.$$

A.3 Operator Semigroups

Even if in the text only operator semigroups are considered that have a continuous generator, we survey some facts that hold in a more general setting.
Let $(X, \|\cdot\|)$ be a real Banach space and $\{T(t)\}_{t\geq 0}$ a semigroup of continuous linear operators on X, i. e.

$$\begin{aligned} &T(0) = I\,,\\ &T(s+t) = T(s)T(t) \ \text{ for all } s,t \geq 0\,. \end{aligned}$$

$\{T(t)\}_{t\geq 0}$ is called *strongly continuous* or a C_0*-semigroup* if $t \to T(t)$ is continuous with respect to the strong operator topology of $\mathcal{L}(X)$, i. e.

$$\lim_{t\downarrow 0} \|T(t)x - x\| = 0 \quad \text{for all } x \in X\,.$$

For a given C_0-semigroup $\{T(t)\}_{t\geq 0}$ the operator $A\colon X \supseteq \mathcal{D}(A) \to X$ with

$$\mathcal{D}(A) = \{x \in X\colon \ \lim_{t\downarrow 0} \tfrac{T(t)x-x}{t} \ \text{ exists } \} \ \text{ and}$$

$$Ax = \lim_{t\downarrow 0} \tfrac{T(t)x-x}{t} \quad \text{for all } \ x \in \mathcal{D}(A)$$

is called its *generator*. The existence of a non-trivial generator for every C_0-semigroup is a basic fact from the theory of C_0-semigroups. The domain $\mathcal{D}(A)$ is then a dense linear subspace of X and A is a linear closed operator. For every $t \geq 0$ the operator $T(t)$ leaves $\mathcal{D}(A)$ invariant, and $AT(t)x = T(t)Ax$ for all $x \in \mathcal{D}(A)$. For every $x \in \mathcal{D}(A)$ the mapping $u : \mathbb{R}_+ \to X$, given by

$$u(t) = T(t)x\,,\ t \geq 0\,,$$

is differentiable on $[0, \infty)$. Moreover, u is the unique solution to the abstract Cauchy problem

$$\tfrac{du(t)}{dt} = Au(t)\,,\ t \geq 0\,,\ u(0) = x\,.$$

For an operator $A \in \mathcal{L}(X)$ let

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\,,$$

where $\{e^{tA}\}_{t\geq 0}$ is a C_0-semigroup with generator A. This semigroup is even *uniformly continuous*, i. e.

$$\lim_{t\downarrow 0} \|e^{tA} - I\| = 0\,.$$

Let $(X, K, \|\cdot\|)$ be an ordered Banach space. A semigroup $\{T(t)\}_{t\geq 0}$ of continuous linear operators is called *positive* if $T(t) \geq 0$ for all $t \geq 0$.

Definition A.3.1 A linear operator $A\colon X \supseteq \mathcal{D}(A) \to X$ is called *resolvent positive* if there is $v_0 \in \mathbb{R}$ such that $(v_0, \infty) \subset \rho(A)$, and the resolvent operator

$$R(v, A) = (vI - A)^{-1}$$

is positive for all $v > v_0$.

If A is the generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$, then the *type* (or *growth bound*) $\omega(A)$ is defined by means of

$$\omega(A) = \inf\{v \in \mathbb{R}\colon \ \text{there is } M \geq 1 \text{ such that } \|T(t)\| \leq Me^{vt} \text{ for all } t \geq 0\}\,.$$

Let A be resolvent positive and denote

$$s(A) = \inf\{v \in \mathbb{R}\colon \ (v, \infty) \subset \rho(A) \text{ and } R(v, A) \geq 0\}\,.$$

One has $s(B) \leq \omega(B) < \infty$, but $s(B) \neq \omega(B)$ is possible, see e. g. [GVW81], [Wol80].

Proposition A.3.2 [CHA+87, Proposition 7.1] *Let $(X, K, \|\cdot\|)$ be an ordered Banach space where K is closed, let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup and A its generator. Then $\{T(t)\}_{t\geq 0}$ is positive if and only if A is resolvent positive.*

In the next statement we need the Definition 2.1.4 of a positive-off-diagonal operator, where the according linear subspace of X is $D = \mathcal{D}(A)$.

Proposition A.3.3 *Let $(X, K, \|\cdot\|)$ be an ordered Banach space, $\{T(t)\}_{t\geq 0}$ a C_0-semigroup and A its generator. If $\{T(t)\}_{t\geq 0}$ is positive, then A is positive-off-diagonal.*

Indeed, let $x \in \mathcal{D}(A) \cap K$ and $f \in K'$ with $f(x) = 0$, then one has

$$f(Ax) = \lim_{t\downarrow 0} \tfrac{f(T(t)x)-f(x)}{t} = \lim_{t\downarrow 0} \tfrac{f(T(t)x)}{t} \geq 0\,.$$

An example of an (unbounded) operator A which is positive-off-diagonal, but generates a semigroup of continuous operators which is not positive, is given e. g. in [Are86, Section 3].

Proposition A.3.4 [CHA+87, Theorem 7.27] *Let $(X, K, \|\cdot\|)$ be an ordered Banach space with a closed cone K and $\operatorname{int} K \neq \varnothing$, and let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup with generator A. Then $\{T(t)\}_{t\geq 0}$ is positive if and only if A is positive-off-diagonal.*

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor any other country.

The present thesis was written at the Institute for Analysis at Dresden University of Technology under the supervision of Prof. Dr. rer. nat. habil. Martin R. Weber.

I accept the rules for obtaining a PhD (Promotionsordnung) of the Faculty of Science at Dresden University of Technology, issued March 20, 2000.

Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Dissertation wurde am Institut für Analysis der Technischen Universität Dresden unter der Betreuung von Herrn Prof. Dr. rer. nat. habil. Martin R. Weber angefertigt.

Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU Dresden vom 20. März 2000 an.

Dresden, den 4.1.2006

Anke Kalauch