# Belief Change in Reasoning Agents 

\author{

- Axiomatizations, Semantics and Computations -
}

Dissertation<br>zur Erlangung des akademischen Grades<br>Doktor rerum naturalium (Dr. rer. nat.)<br>vorgelegt an der<br>Technischen Universität Dresden<br>Fakultät Informatik<br>eingereicht von<br>Dipl. Inf. Yi Jin<br>geboren am 08 Februar, 1978 in Ningbo/China<br>Gutachter: Prof. Michael Thielscher<br>Technische Universität Dresden<br>Prof. Franz Baader<br>Technische Universität Dresden<br>Dr. Dongmo Zhang<br>University of Western Sydney

Belief Change in Reasoning Agents Copyright (c) 2007 by Yi Jin

## Abstract

The capability of changing beliefs upon new information in a rational and efficient way is crucial for an intelligent agent. Belief change therefore is one of the central research fields in Artificial Intelligence (AI) for over two decades. In the AI literature, two different kinds of belief change operations have been intensively investigated: belief update, which deal with situations where the new information describes changes of the world; and belief revision, which assumes the world is static. As another important research area in AI, reasoning about actions mainly studies the problem of representing and reasoning about effects of actions. These two research fields are closely related and apply a common underlying principle, that is, an agent should change its beliefs (knowledge) as little as possible whenever an adjustment is necessary. This lays down the possibility of reusing the ideas and results of one field in the other, and vice verse. This thesis aims to develop a general framework and devise computational models that are applicable in reasoning about actions. Firstly, I shall propose a new framework for iterated belief revision by introducing a new postulate to the existing AGM/DP postulates, which provides general criteria for the design of iterated revision operators. Secondly, based on the new framework, a concrete iterated revision operator is devised. The semantic model of the operator gives nice intuitions and helps to show its satisfiability of desirable postulates. I also show that the computational model of the operator is almost optimal in time and space-complexity. In order to deal with the belief change problem in multi-agent systems, I introduce a concept of mutual belief revision which is concerned with information exchange among agents. A concrete mutual revision operator is devised by generalizing the iterated revision operator. Likewise, a semantic model is used to show the intuition and many nice properties of the mutual revision operator, and the complexity of its computational model is formally analyzed. Finally, I present a belief update operator, which takes into account two important problems of reasoning about action, i.e., disjunctive updates and domain constraints. Again, the updated operator is presented with both a semantic model and a computational model.

## Acknowledgments

I wish to thank my supervisor, Michael Thielscher, for providing direction, ideas and proofreading all along the way this thesis comes into its current form; and the rest of the Computational Logic Group, particularly Yves Martin, Conrad Drescher, Matthias Fichtner, Stephan Schiffel, for their constructive suggestions and criticisms. I am also grateful to Dongmo Zhang, who gets me started in the field of belief change through an excellent introductory course.

Take this opportunity, I would like to express my deep appreciation to all my family members, who are always willing to help in any way they could. My love to them is simply beyond words.

This work was financially supported by Deutsche Forschungsgemeinschaft under grant no. Gr 334/3. My appreciations also go to Gesellschaft von Freunden und Förderern der TU Dresden and University of Western Sydney for offering travel allowance of my academic visit to Sydney.

## Table of Contents

1 Introduction ..... 1
1.1 Belief Change ..... 1
1.2 Reasoning about Actions ..... 3
1.3 Motivations and Problems ..... 4
1.4 Structure of the Thesis ..... 7
2 The Classical Belief Change ..... 9
2.1 The AGM Theory ..... 10
2.2 System of Spheres ..... 24
2.3 The KM Theory ..... 28
2.4 Related Research Fields ..... 31
3 Iterated Belief Revision: General Frameworks ..... 36
3.1 Why is it Difficult to Iterate ..... 37
3.2 The DP Theory ..... 40
3.3 The Problem of Implicit Dependence ..... 46
3.4 A General Framework for Iterated Revision ..... 52
3.5 Discussions and Related Work ..... 59
4 Iterated Belief Revision: Computational Approaches ..... 68
4.1 Cut Base Revision ..... 69
4.2 Reinforcement Base Revision ..... 71
4.3 Discussions and Related Work ..... 80
5 Mutual Belief Revision: Semantics and Computation ..... 93
5.1 OCF Model of Mutual Belief Revision ..... 94
5.2 Computational Model ..... 99
5.3 Discussions and Related Work ..... 104
6 Belief Update, Revisited ..... 106
6.1 The PMA ..... 107
6.2 The WSS, and Extensions ..... 111
6.3 Updating Possibilistic Beliefs ..... 121
6.4 Discussions and Related Work ..... 126
7 Conclusions and Future Work ..... 128
Appendices
A Proofs of Results of Chapter 3 ..... 131
B Proofs of Results of Chapter 4 ..... 144
C Proofs of Results of Chapter 5 ..... 150
D Proofs of Results of Chapter 6 ..... 155
E Algorithms ..... 162

## Chapter 1

## Introduction

This thesis is mainly concerned with general frameworks and computational approaches of belief change which can be applied in reasoning about actions. This work can be considered as an attempt towards combining two fields of AI, reasoning about actions and belief change, to obtain a framework, which can deal with more realistic scenarios than the state-of-the-art action theories. Such a framework could be potentially employed to build the high-level reasoning and control components of intelligent agents which are allowed to have both fallible beliefs and fallible sensors.

In this chapter, I will first briefly sketch the basic ideas of belief change and reasoning about actions. Then motivations and problems of combining the two fields are discussed. Thereafter, the overall structure of the thesis is outlined.

### 1.1 Belief Change

As a very young field, belief change has not been recognized as a subject of its own until the middle of the 1980's [Hansson, 1997]. Since it so new, it does not even have a well established name. Belief change is just one name of the field among others such as: database updating, theory change, belief dynamics and belief revision. In general, belief change is about changing the beliefs of minds and the data of databases to accommodate new information. As already have been pointed out by [Keller and Winslett, 1985] that there are usually two types of reasons why an agent should change its beliefs. One is because the world has been changed, and the other is
that the agent has made a new observation of the static world. ${ }^{1}$ The first type, change-recording incorporation of new information, is often called belief update. The term belief revision is reserved for the second type, knowledge-adding incorporation of new information. ${ }^{2}$

The research subject which we now call belief change has mainly two origins [Hansson, 1997]. In philosophy, belief change has been studied to investigate the revision of scientific theories and logical theory. The first milestone of philosophical researches on belief change is the series studies of Levi [1977; 1980] in the 1970's, which have underpinned the major concerns of the field and provided the basic formal framework. The next milestone is the AGM theory (named after its originators Alchourrón, Gärdenfors and Makinson) which has provided a more general and versatile formal framework for studying belief change [Alchourrón and Makinson, 1982; Alchourrón et al., 1985; Gärdenfors, 1988]. In a nutshell, the AGM theory assumes the beliefs of an agent are represented by a deductively closed set of sentences (or, a belief set) of some logical language, and mainly studies how to incorporate (remove) a new sentence into (from) a belief set in a rational way. The AGM trio have studied belief change mainly in two ways. They first have introduced the so-called rationality postulates, which they claimed should be respected by any rational belief change operator. The guiding criterion of the AGM postulates is the so-called minimal change principle, that is to change the belief set as little as possible. Also, they have proposed models of constructing concrete rational belief change operators. The advent of the AGM theory finally helped the field to grow up as an important subject of its own. Since then belief change becomes a flourishing and interdisciplinary field of researches. Many researchers from different fields find the value of belief change in their own fields and thus get involved in the development of belief change. The second origin of belief change is computer science. Specifically, database theorist are interested in models of database update which are more sophisticated than those of the usual relational database [Winslett, 1990]. One important development in this direction is Doyle's 'Truth Maintenance Systems" [Doyle, 1979]. Also, the problem of updating a belief set (base) is an important topic in AI [Herzig and Rifi, 1999; Dalal, 1988]. Parallel to the AGM theory, Katsuno and Mendelzon (KM) have proposed a general framework for belief update [Katsuno and Mendelzon, 1991a].

Later on, various extensions of the classical belief change have been proposed and ex-

[^0]tensively investigated. In particular, many belief revisionists are interested in iterated belief revision [Darwiche and Pearl, 1997; Nayak, 1994b], that is how should an agent revise its beliefs in response to a sequence of new information. Recently, non-prioritized belief revision, in which the new information is not always accepted in the revise belief set, has also drawn considerable attentions from the community of belief change [Hansson, 1999; Booth, 2001]. Non-prioritized revision can handle more realistic domains where there is no strict correlation between the chronology of the information and the credibility of their contents. The classical belief change is concerned with the beliefs of a single agent. There are also extended models of belief change designed for multi-agent scenarios, e.g., both belief merging [Konieczny and Pérez, 1998; Gauwin et al., 2005] and belief arbitration [Revesz, 1997; Liberatore and Schaerf, 1998] are about how to extract the coherent common beliefs out of a set of (possibly mutually inconsistent) belief sets.

### 1.2 Reasoning about Actions

Reasoning about actions is one of the most important research areas, whose history can date back to the very early stage of AI [McCarthy and Hayes, 1969]. The field mainly studies the problem of representing and reasoning about effects of actions that can be performed by intelligence agents. It is now commonly believed that declarative, logic-based approaches will be the key to the advent of agents with high-level reasoning capabilities. A variety of logic-based theories of actions exist, among which are the situation calculus [McCarthy, 1963; Reiter, 2001a] the fluent calculus [Thielscher, 1999] and causal theories [McCain, 1997]. These approaches have recently provided the basis for the high-level, logic-based agent programming languages and systems GOLOG [Levesque et al., 1997; Reiter, 2001b], FLUX [Thielscher, 2004a] and CCalc [Giunchiglia et al., 2004], respectively. An important extension of these basic action theories allows agents to reason about their knowledge and knowledge-producing actions (or, sensing), e.g., [Thielscher, 2000; Scherl and Levesque, 2003]. In declarative, logic-based approaches, an agent's knowledge of the world is represented by sentences of some logical language. The agent is equipped with a reasoning mechanism usually built upon a logical inference engine. With its reasoning mechanism, the agent can perform several tasks autonomously: verifying the executability of actions, executing complex strategies, planning ahead, etc.

As proposed by McCarthy and Hayes [1969], the work on reasoning about actions has focused on a number of fundamental problems among which the frame problem is the most im-
portant. In reasoning about actions, the frame problem is mainly concerned with representing the effects of actions without explicitly mentioning the invariance (frame) of the world; since otherwise the domain specifications will be inefficient or even unmanageable [Reiter, 1991]. The key idea of almost all solutions to the frame problem employed in various action theories is the so-called law of inertia or law of persistence, that is, the world does not change unless it is forced to. As another important topic in reasoning about actions, the so-called ramification problem [Ginsberg and Smith, 1987; Lifschitz, 1990] names the challenge to accommodate actions whose execution causes indirect effect. Indirect effects typically are consequences of domain constraints, namely, additional, general knowledge of domain specific dependencies between world description components. It is now commonly believed that merely taking into account formalizations of domain constraints as pure logical sentences is not adequate, since these sentences do not include causal information [Thielscher, 1997; McCain and Turner, 1995]. In fact, most state-of-the-art action theories take into account both domain constraints and causal information.

### 1.3 Motivations and Problems

The aforementioned two research fields are closely related, since essentially both the principle of minimal change and the law of inertia require an agent to change its beliefs (knowledge) as little as possible whenever an adjustment is necessary. This suggests the possibility of reusing the ideas and results of one field in the other, and vice verse. Recently, several researchers have attempted to combine belief change with reasoning about actions [Shapiro et al., 2000; Shapiro and Pagnucco, 2004; Jin and Thielscher, 2004; Hunter and Delgrande, 2005]. The main motivation of doing so is that belief change can help frameworks of reasoning about actions to deal with more realistic scenarios, in which agents are allowed to have both fallible beliefs and fallible sensors.

As one major disadvantage of the most state-of-the-art action theories (e.g., [Thielscher, 2000; Scherl and Levesque, 2003]), the beliefs (knowledge) of the agent are assumed infallible, that is, what the agent believes (knows) must be true. ${ }^{3}$ Usually, these action theories can only deal with trivial situations where the sensing information is consistent with the agent's current beliefs. In realistic scenarios, however discrepancies between the agent's be-

[^1]liefs and the real world can occur for many reasons: fallible sensors, unexpected changes of the world, failures of action executions, etc. Also, an agent may very well be making observations which contradict its current beliefs.

To deal with more realistic situations, frameworks of reasoning about actions must be enhanced with methods from belief change. As a simple solution, I propose to use belief revision to deal with sensing actions, and belief update to deal with ontic actions (i.e., actions with physical effects). More precisely, the agent maintains a (fallible) belief state; whenever a sensing (ontic) action has been performed, the belief state is accordingly revised (updated, respectively). 4 E.g., assume the agent's initial beliefs are represented by a belief state $K_{0}$. Then, after having executed sensing action $a_{1}$, the agent's belief state evolves to $K_{1}$ through revising $K_{0}$ by the sensing results of $a_{1}$ (cf. Figure 1.1). A successive execution of ontic action $a_{2}$ causes a further evolution of the agent's beliefs to $K_{2}$, this time $K_{1}$ is updated by the physical effects of $a_{2}$.


Figure 1.1: Combining belief change and reasoning about actions

Note that the application of belief change in reasoning about actions offers a nice solution to the frame problem, as we only need to specify the effects of actions; moreover, it also adds more formality to reasoning about actions, as so far solutions to the frame problem in most action theories are usually illustrated only by examples.

Unfortunately, belief change of its current status cannot be directly integrated into reasoning about actions, since there are several important problems that need to be addressed. Firstly, for the incremental adaptation of beliefs, the AGM postulates proved to be overly weak too [Darwiche and Pearl, 1994; Darwiche and Pearl, 1997]. This has led to the development

[^2]of additional postulates for iterated belief revision by Darwiche and Pearl (DP), among others [Freund and Lehmann, 1994; Lehmann, 1995; Boutilier, 1993]. Still, however, the AGM and DP postulates together are too permissive in that they support belief revision operators which assume arbitrary dependencies among the pieces of information which an agent acquires along its way. These operators have a drastic effect when the agent makes an observation which contradicts its currently held beliefs: The agent is forced to cancel everything it has learned up to this point [Nayak et al., 1996a; Nayak et al., 2003]. Secondly, the mainstream study of belief revision has been focused on how a single agent revises its beliefs to incorporate new information. This research normally assumes that new information is fully accepted. Obviously such an assumption is not applicable to multi-agent systems. There have been a variety of approaches which have been proposed in the literature to deal with the problem of belief revision in multi-agent systems [Revesz, 1997; Liberatore and Schaerf, 1998; Konieczny and Pérez, 1998]. Multiple belief revision is another one of the most important variations and extensions of the AGM theory, which is closely related to iterated belief revision and mutual belief revision [Fuhrmann and Hansson, 1994; Zhang, 1996; Zhang and Foo, 2001]. Unlike in the AGM theory, the new information is represented by a set of (possibly infinite) sentences in multiple belief revision. Thirdly, the agents considered in the classical belief change are infinite beings, without any limitation in memory, time, or deductive ability. When implementation is concerned, one has to consider additionally that any realistic agent is a finite being and that calculations take time. Therefore, the beliefs of an agent should be represented by a finite belief state, and a satisfactory revision operator should not only behave rationally but also consume less time and space. Adapting belief revision to less idealized agents is far from trivial, as we need an approach which takes these characteristics of finiteness, memory and time limitations into account [Wassermann, 1999]. Moreover, the classical studies have focused on formal and general aspects of belief change. It is commonly believed that there is no general purpose means of belief change that will do the right thing under all circumstance, and we must be explicit about the "ontology" or "scenario" underlying the belief change process [Winslett, 1990; Friedman and Halpern, 1996]. To be applied in reasoning about actions, the KM theory has unfortunately been shown problematic with both disjunctive updates and domain constraints [Herzig and Rifi, 1999]. Note that the above-mentioned problems are by no means the only ones, but in this thesis only these most important problems will be tackled.

The major contributions of the thesis are as follows. I first give a formal analysis of this
problem of implicit dependence, and then I present, as a solution, an Independence postulate for iterated belief revision. I give a representation theorem for the new postulate and prove its consistency by defining a concrete belief revision operator. The independence postulate together with the AGM/DP postulates give us a new elegant framework for iterated belief revision. I also contrast the new framework to the most prominent existing approaches to iterated revision and argue that it is so far the most satisfactory. I propose a computational iterated revision operator which satisfies the AGM/DP and independence postulates. A formal assessment shows that this revision operator is almost optimal in computational complexity and space-consumption. To clearly display the intuition, I also present a possible world-based semantics for the revision operator. It is worth mentioning that the revision operator can deal with beliefs with different reliability degrees, as it exploits richer representation of beliefs. In order to deal with multiagent scenarios, I introduce a concept of mutual belief revision which studies the problem of information exchange among agents. By generalizing the revision operator, a concrete mutual revision operator has been devised. I show formally the nice logical properties of the mutual revision operator, as well as its computational complexity. To apply belief update in reasoning about actions, I first analyze the KM theory's problems with disjunctive update and domain constraints, and then I propose an update operator which does not suffer from these problems.

### 1.4 Structure of the Thesis

As this work is interdisciplinary in nature, I assume the reader has basic knowledge on formal logic, and computer science (specially, complexity theory). Ideas of belief change will however be explained in detail throughout this thesis. The rest of the thesis is organized as follows. ${ }^{5}$

- In Chapter 2, I give a survey on the classic frameworks for belief change. Firstly, the AGM theory (for belief revision) is presented in detail. Then, the basic ideas of the KM theory (for belief update) are sketched. Finally, the connections between belief change and some other research fields are briefly discussed.
- Chapter 3 is mainly concerned with general frameworks for iterated belief revision. I will start by showing why iterated belief revision is a difficult problem. Thereafter, I first give a formal analysis of this problem of implicit dependence, and then I present, as a solution,

[^3]an Independence postulate for iterated belief revision. In the end, the new framework is compared with the most prominent existing approaches to iterated belief revision.

- In Chapter 4, computational approaches to iterated belief revision are studied. Firstly, based on Nebel's cut base revision, the problems of pure qualitative approaches to iterated revision are discussed. Then, I propose a computational revision operator, which is based on a very compact representation of belief states. This revision operator is formally assessed in terms of its logical properties and computational complexity. Thereafter, I will give a detailed comparison between my proposal with other well-known approaches.
- Chapter 5 introduces a concept of mutual belief revision, which is concerned with how agents can exchange their beliefs in a multi-agent system. To model mutual revision processes, I first introduce two difference models: a possible world-based model, which clearly shows the intuition and semantics, and a computational model. Then, with the help of the semantic model, I show many nice properties of mutual belief revision. The complexity of the computational model is also formally analyzed.
- Chapter 6 is about belief update operators, which take into account two important problems of reasoning about actions, i.e., disjunctive updates and domain constraints. Firstly, I recall the well-known PMA and discuss its problems with disjunctive updates and domain constraints. Then, I present another renown update operator, i.e., the WSS, which does not suffer from these problems. Finally, I propose an update operator for possibilistic belief states, based on ideas of the WSS.
- Finally, I present in Chapter 7 the conclusions and some possible directions of the future work.


## Chapter 2

## The Classical Belief Change

The capability of gathering information about the world and changing its beliefs based on the new information is crucial for an intelligent agent. Belief change therefore is one of the central research fields in AI for over two decades. Technically, belief change studies the process an agent adapts its beliefs to accommodate new, or more reliable information that is possibly inconsistent with the existing beliefs.

Before taking about belief change, one fundamental notion needs to be clarified, that is, what are beliefs? In other words, what are the objects to be changed? This is not an easy question. If cognitive states of human minds are considered directly, it would be obviously very difficult, if not impossible, to study formally the way how they are being changed. Therefore, we should abstract from irrelevant accompaniments of real cognitive states, in order to study and reveal the essences of belief change. Informally, a belief state (or, an epistemic state) is a rational idealization of a cognitive state of some individual at a given point of time [Gärdenfors, 1988]. As objects to be changed, belief states are the central entities of belief change. In the classical belief change, very abstract models of belief states are studied. Usually, beliefs of an agent are represented by a set of sentences of some logical language. From a black-box point of view, an idealized interpretation of belief states is that they are supposed to be in reflective equilibrium. Changes of a belief state should be caused by some external forces (also referred to as epistemic inputs). In the classical belief change, such epistemic inputs are usually encoded by logical sentences. In this chapter, the classical studies on belief change will be surveyed.

The rest of the chapter is organized as follows. In Section 2.1, I will recall the AGM theory for belief revision, including the AGM postulates and two standard constructive models. Then,
another interesting construction for belief revision, i.e., Grove's system of spheres, is presented in Section 2.2. Thereafter, I recapitulate briefly in Section 2.3 the KM theory for belief update. Finally, two research fields, which are intimately related to belief change, are discussed in Section 2.4.

### 2.1 The AGM Theory

Most formal studies on belief change take as starting point the work of Alchourrón, Gärdenfors, and Makinson (AGM) during the first half of 1980s [Alchourrón et al., 1985; Alchourrón and Makinson, 1985], which is now commonly referred to as the AGM theory. In the literature, the AGM theory are also called the AGM framework, the AGM paradigm and the AGM model.

The AGM theory formally studies idealized models of belief change. Given an underlying logical language $\mathcal{L}$, it is assumed that beliefs of an agent can be represented by sentences of $\mathcal{L}$. For the sake of generality, the concrete structure of the underlying language $\mathcal{L}$ is deliberatively unspecified, except it is assumed closed under logical connectives (i.e., $\neg, \rightarrow, \wedge, \vee, \leftrightarrow$ ). То be precise, if $\alpha, \beta$ are members of $\mathcal{L}$, so are $\neg \alpha, \alpha \rightarrow \beta, \alpha \wedge \beta, \alpha \vee \beta$ and $\alpha \leftrightarrow \beta$. Usually, sentences and sets of sentences of $\mathcal{L}$ are denoted by lower case Greek letters (e.g., $\alpha, \beta, \cdots$ ) and capital Roman letters (e.g., $A, B, K, \cdots$ ), respectively.

The logic of the underlying language $\mathcal{L}$ is identified with a so-called logical consequence operator. Formally, a logical consequence operator $C n$ is a function on subsets of $\mathcal{L}$ that satisfies the following properties:

- $A \subseteq C n(A)$
(inclusion)
- If $A \subseteq B$, then $C n(A) \subseteq C n(B)$
- $C n(A)=C n(C n(A))$
(iteration)
- If $A$ derives $\alpha$ in the classical truth-functional way, then $\alpha \in C n(A)$ (super-classicality)
- If $\alpha \in C n(A)$, then there is a finite subset $A^{\prime} \subseteq A$ s.t. $\alpha \in C n\left(A^{\prime}\right)$
(compactness)
- $\beta \in C n(A \cup\{\alpha\})$ iff $\alpha \rightarrow \beta \in C n(A)$
(deduction theorem)

Note that the AGM theory is a very general framework, as it applies to any logical language equipped with an logical consequence operator (e.g., propositional logics and first order predicate logics).

Now, we are ready to define some basic notions. A sentence $\alpha$ is called a logical consequence of a set $A$ of sentences (written as $A \vdash \alpha$ ) iff $\alpha \in C n(A)$. In the limiting case, a sentence $\alpha$ is called a tautology (written as $\vdash \alpha$ ) iff $\alpha \in C n(\})$. Two sentences $\alpha, \beta$ are said to be logically equivalent (denoted by $\alpha \equiv \beta$ ) iff $\vdash \alpha \leftrightarrow \beta$. A sentence $\alpha$ is inconsistent iff $\vdash \neg \alpha$; and a set $A$ of sentences is inconsistent iff there exists a sentence $\beta$ such that $A \vdash \beta$ and $A \vdash \neg \beta$. A set $A$ of sentences is called complete (or maximal), if for any sentence $\alpha$ either $\alpha \in A$ or $\neg \alpha \in A$. A set $A$ of sentences is called logically closed (or deductively closed) iff $A=C n(A)$. It is not difficult to see that (due to properties of $C n$ ) a logically closed set of sentence $A$ is inconsistent iff $A=\mathcal{L}$. Moreover, all tautologous (inconsistent) sentences are logically equivalent. Therefore, we will use $\top(\perp)$ to denote an arbitrary tautologous (inconsistent respectively) sentence. For the sake of succinctness, a singleton set is identified with it unique element, e.g., $C n(\alpha)$ and $\alpha \vdash \beta$ represent $C n(\{\alpha\})$ and $\beta \in C n(\{\alpha\})$ respectively.

Ideally, an agent is supposed to be aware of and responsible for all logical consequences of it beliefs. In the AGM theory, beliefs of an ideal agent are therefore represented by a set sentences which is logically closed (also called a belief set). Technically, a belief set is just a logical theory as it is called by logicians, and it is usually denoted by $K$ (possibly indexed).

The AGM theory formally studies three types of belief change operators on belief sets: 1

- Expansion: A new sentence $\alpha$ is simply added to a belief set $K$, regardless of whether it is inconsistent with $K$. The belief set that results from expanding $K$ by $\alpha$ is denoted $K+\alpha$.
- Contraction: A sentence $\alpha$ is retracted from a belief set $K$ without acquiring any new sentences. The result of contracting $\alpha$ from $K$ is denoted $K-\alpha$.
- Revision: A new sentence is incorporated to a belief set $K$ without rendering the new belief set inconsistent. The result of revising $K$ by $\alpha$ is denoted $K * \alpha$.

In the AGM theory, expansion is the simplest type of belief change, and it is formally defined

[^4]as follows:
$$
K+\alpha=C n(K \cup\{\alpha\})
$$

The problem of expansion is that it is not consistency preserving. Note that, if $\neg \alpha \in K$ then $K+\alpha$ is also inconsistent, even if $K$ is initially consistent. Therefore, expansion cannot be considered as a rational operator on its own. The reason of introducing expansion is merely because it is used in studies of other belief change operators.

In contrast to expansion, as inconsistency cannot be caused by removing a sentence, contraction at first glance seems more innocent. Naively, one might define contraction as to remove a sentence directly as follows:

$$
\begin{equation*}
K-\alpha=C n(K \backslash\{\alpha\}) \tag{2.1}
\end{equation*}
$$

Unfortunately, in general the operation defined by (2.1) does not guarantee a successful removal of the specified sentence. To see this, let us consider the following counterexample:

Example 1. Let $K=C n(\{\alpha, \beta, \alpha \wedge \beta \rightarrow \gamma\})$ and $\gamma$ the sentence to be contracted. By inclusion, we have $\{\alpha, \beta, \alpha \wedge \beta \rightarrow \gamma\} \subseteq K$. It follows that $\{\alpha, \beta, \alpha \wedge \beta \rightarrow \gamma\} \subseteq K \backslash\{\gamma\}$. By super-classicality, we obtain $\gamma \in C n(K \backslash\{\gamma\})$. Therefore $\gamma$ is regained in $K-\gamma$.

A fact revealed by Example 1 is that we might have to remove some other sentences in order to prevent regaining the sentence to be contracted. What makes the situation more complicated is that there are often several choices on which sentences should be removed. For instance, in Example 1, it is sufficient to remove any of $\alpha, \beta, \alpha \wedge \beta \rightarrow \gamma$. Obviously, as far as pure logical strength is considered, it is not at all clear how to make a right selection. The AGM trio argue that an contraction operator should make use of some extra-logical preference information in order to choose the right sentences to be removed. As we will see soon, several quite different forms of extra-logical preference information are introduced in the AGM theory, in order to construct concrete contraction operators.

Unlike expansion, revision incorporates a new sentence $\alpha$ in a belief set $K$ in a consistency preserving way, i.e., the revised belief set $K * \alpha$ should be consistent. Obviously, if $\neg \alpha \in K$, some sentences of $K$ need to be given up in order to give way to $\alpha$.

Example 2. Let $K$ as defined in Example 1. If $K$ is revised by $\neg \gamma$, then again at least one of $\alpha, \beta, \alpha \wedge \beta \rightarrow \gamma$ should be removed, otherwise $K * \neg \gamma$ will become inconsistent.

Analogous to contraction, in revision there are also several choices on which sentences
should be removed. Therefore. constructions of concrete revision operators also should exploit some kind of extra-logical preference information.

In the AGM theory, belief change are mainly studied in two aspects. On the one hand, the AGM trio have introduced the so-called rationality postulates for contraction and revision. On the other hand, also several concrete constructions of belief change operators have been proposed which satisfy these postulates.

### 2.1.1 The AGM Postulates

It is unrealistic that an unique contraction (or revision) operator is applicable to all application domains. The AGM trio therefore have introduced the so-called rationality postulates for both contraction and revision which they claim should be satisfied by all rational belief change operators. The guiding principle of the AGM postulates is that of economy of information or minimal change, which means neither to give up currently held beliefs nor to acquire new beliefs, unless forced to, in a process of belief change.

## Postulates for Contraction

As contraction is a function on belief sets, $K-\alpha$ should be logically closed:

$$
(\mathrm{K}-1) \quad K-\alpha=C n(K-\alpha)
$$

(closure)

Contraction should not introduce any new sentences:

$$
(\mathrm{K}-2) \quad K-\alpha \subseteq K
$$

If $\alpha \notin K$, then $K-\alpha$ should be same as $K$, as no change is certainly the minimal change:

$$
\text { (K-3) } \quad \text { If } \alpha \notin K \text {, then } K-\alpha=K
$$

> (vacuity)

Unless it is tautologous, $\sqrt[2]{ } \alpha$ should be successfully expelled from $K$.
(K-4) If $\nvdash \alpha$, then $\alpha \notin K-\alpha$
(success)

When $K-\alpha$ is expanded by $\alpha$, all previously discarded sentences should be recovered:

[^5]$$
(\mathrm{K}-5) \quad K \subseteq(K-\alpha)+\alpha
$$

Contraction is syntax irrelevant:

$$
\text { (K-6) If } \alpha \equiv \beta \text {, then } K-\alpha=K-\beta
$$

Usually, (K-1)-(K-6) are called the basic postulates for contraction. Among them, Postulate (K-5) is the only controversial one, which has been criticized by several researchers [Hansson, 1991; Niederée, 1991; Fermé, 1998].

As suggested by [Makinson, 1987], a simple response to these criticisms is to completely drop (K-5). Contraction operators satisfies all the basic postulates, but (K-5), are called withdrawal operators [Makinson, 1987].

However, there are also strong justifications for (K-5). First of all, Postulate (K-5) corresponds closely to the minimal change principle, as it requires the change from $K$ to $K-\alpha$ be small enough so that a subsequent $\alpha$-expansion suffices to recover all discarded sentences. Also, [Hansson, 1991] has argued that recovery is a prominent feature by showing that, in the presence of other five basic postulates, Postulate (K-5) can be derived from following Postulate of core-retainment:

$$
\begin{aligned}
& \text { If } \beta \in K \text { and } \beta \in K-\alpha \text {, then there is a set } A \text { such that } A \subset K \text { and } \\
& \alpha \in C n(A) \text { and } \alpha \in C n(A \cup\{\beta\}) \quad \\
& \text { (core-retainment) }
\end{aligned}
$$

Informally, core-retainment says "beliefs that do not in any way contributes to the fact that $K$ implies $\alpha$ are retained in $K-\alpha$ ".

Another support for (K-5) is that, together with (K-2), it implies the following property:

$$
\begin{equation*}
\text { If } \vdash \alpha \text {, then } K-\alpha=K \tag{failure}
\end{equation*}
$$

Obviously, failure is a promising property, since it is the minimal change to leave the original belief set unchanged when required to retract a sentence that cannot be removed.

In addition to be basic postulates, the AGM trio have also proposed two complementary postulates, (K-7) and (K-8), for the cases where the sentence to be contracted is a conjunction.

Intuitively, the expulsion of $\alpha$ or $\beta$ requires more changes than the expulsion of $\alpha \wedge \beta$, since $\alpha \wedge \beta$ must be given up in order to give up $\alpha$ or $\beta$. It is therefore reasonable to require that beliefs contained in both $K-\alpha$ and $K-\beta$ be included in $K-\alpha \wedge \beta$ :
(K-7) $\quad K-\alpha \cap K-\beta \subseteq K-(\alpha \wedge \beta)$
(conjunctive overlap)

As argued Nayak [1994a], if $\delta$ needs to be removed in order to remove $\alpha$, and $\alpha$ needs to be removed in order to remove $\alpha \wedge \beta$, then $\delta$ needs to be removed in order to remove $\alpha \wedge \beta$. This gives support to the following postulates:
(K-8) If $\alpha \notin K-(\alpha \wedge \beta)$ then $K-(\alpha \wedge \beta) \subseteq K-\alpha \quad$ (conjunctive inclusion)

## Postulates for Revision

Symmetrically, for revision there are also six basic postulates, $\left(\mathrm{K}^{*} 1\right)-\left(\mathrm{K}^{*} 6\right)$, and two complementary postulates, $\left(\mathrm{K}^{*} 7\right)$ and $\left(\mathrm{K}^{*} 8\right)$.

As revision is a function on belief sets, $K * \alpha$ should be logically closed:

$$
\left(\mathrm{K}^{*} 1\right) \quad K * \alpha=C n(K * \alpha)
$$

(closure)

The new information should be successfully accepted $3^{3}$

$$
\left(\mathbf{K}^{*} 2\right) \quad \alpha \in K * \alpha
$$

(success)

The revised belief set $K * \alpha$ should not contain more sentences than the result of $K$ expanded by $\alpha$ :

$$
(\mathrm{K} * 3) \quad K * \alpha \subseteq K+\alpha
$$

If $\alpha$ is consistent with $K$, we should not remove any sentence from $K$ :
$(\mathrm{K} * 4) \quad$ If $\neg \alpha \notin K$ then $K+\alpha \subseteq K * \alpha$
(preservation)

The revised belief set $K * \alpha$ is consistent, unless $\alpha$ is inconsistent:
( $\mathrm{K} * 5$ ) $\quad K * \alpha$ is inconsistent, only if $\vdash \neg \alpha$
(consistency)

Revision should also be syntax irrelevant:

[^6]$$
\left(\mathrm{K}^{*} 6\right) \quad \text { If } \alpha \equiv \beta \text { then } K * \alpha=K * \beta
$$

Like those for contraction, the complementary postulates for revision also deal with conjunctions:

$$
\begin{array}{ll}
(\mathrm{K} * 7) & K *(\alpha \wedge \beta) \subseteq(K * \alpha)+\beta \\
(\mathrm{K} * 8) & \text { If } \neg \beta \notin K * \alpha \text { then }(K * \alpha)+\beta \subseteq K *(\alpha \wedge \beta) \quad \text { (conjunctive inclusion) }
\end{array}
$$

In the presence of the basic postulates, Postulates $\left(\mathrm{K}^{*} 7\right)$ and $(\mathrm{K} * 8)$ are equivalent to the following postulate:

$$
\begin{equation*}
\text { If } \neg \beta \notin K * \alpha \text { then }(K * \alpha)+\beta=K *(\alpha \wedge \beta) \tag{iteration}
\end{equation*}
$$

Note that the postulate of iteration gives a basic account for iterated belief revision, which will be extensively investigated in Chapter 3.

As already mentioned, the AGM postulates are motivated by the minimal change principle. Indeed, Postulates $\left(\mathrm{K}^{*} 3\right)$ and $\left(\mathrm{K}^{*} 4\right)$ implies that if $\alpha$ is consistent with $K$ then no elements of $K$ are removed:

$$
\text { If } \neg \alpha \notin K \text {, then } K * \alpha=K+\alpha
$$

Such trivial revision is also called mild revision [Freund and Lehmann, 1994]. More interesting and complicated are situations where the new sentence $\alpha$ is inconsistent with the belief set $K$, in which case some elements of $K$ have to be removed in order to accommodate the new belief. This kind of revision is referred to as severe revision or belief contravening revision.

## Inter-definability between Contraction and Revision

Note that, according to Postulate (K-2), contraction should not introduce any new beliefs. This is obviously not very realistic, as we usually give up a belief only if there is evidence against it.

Example 3. [Hansson, 1997] When I came home yesterday, I believed that my copy of Rousseau's The Social Contract was on the kitchen table. When I saw the empty kitchen table, I gave up the belief.

In this example, my belief in "The Social Contract was on the kitchen table" was given up due to its negation was observed. Therefore, it is essential not contracted. In fact, Hansson [1997] argues that it is very difficult, if not impossible, to find realistic examples of such pure contraction, which satisfies (K-2). Then why we study contraction at all? The main reason is that contraction can used to construct more realistic belief change operators as stated in Levi's decomposition principle [Levi, 1977]:4

Every legitimate belief change is decomposable into a sequence of incorporation (expansions) and contractions.

In fact, one major result of the AGM theory is that rational revision and contraction operators can be defined in terms of each other. Formally, given a contraction operator - , we can construct a revision operator as follows:

Levi identity: $K * a=(K-\neg \alpha)+\alpha$.

Note that the Levi identity is a direct support of Levi's decomposition principle.
If the given contraction operator - is rational, then the revision operator $*$ constructed via the Levi identity is also rational.

Theorem 2.1. Alchourrón et al., 1985; Gärdenfors, 1988 If the contraction - satisfies ( $K-1$ )-$(K-4)$ and ( $K-6$ ), then the revision operator $*$ obtained via the Levi identity satisfies ( $K * 1$ )$(K * 6) .{ }^{5}$

Theorem 2.2. [Alchourrón et al., 1985; Gärdenfors, 1988] If the contraction - satisfies ( $K-1$ ) and ( $K-8$ ), then the revision operator obtained via the Levi identity satisfies $\left(K^{*} 1\right)-\left(K^{*} 8\right)$.

Conversely, given a revision operator, we can construct a contraction operator as follows:
Harper identity: $K-\alpha=K \cap(K * \neg \alpha)$.
Similarly, if the given revision operator $*$ is rational, so is the contraction operator - constructed via the Harper identity.

[^7]Theorem 2.3. [Alchourrón et al., 1985; Gärdenfors, 1988] If the revision operator satisfies $\left(K^{*} 1\right)-\left(K^{*} 6\right)$, then the contraction operator obtained via the Harper identity satisfies ( $\left.K-1\right)$ ( $K$ - 6 ).

Theorem 2.4. [Alchourrón et al., 1985; Gärdenfors, 1988] If the revision operator satisfies $\left(K^{*} 1\right)-\left(K^{*} 8\right)$, then the contraction operator obtained via the Harper identity satisfies $(K-1)$ -(K-8).

### 2.1.2 Maximal Consistent Subsets

As the rationality postulates does not uniquely determine a contraction operator, the AGM theory also studies approaches to constructing concrete rational contraction operators. Due to the inter-definability between revision and contraction, these approaches also can be used to construct rational revision operators.

The constructive model of the so-called partial meet contraction is directly motivated by the minimal change principle. Intuitively, the contracted belief set $K-\alpha$ should be a maximal subset of $K$ which does not include $\alpha$. Formally,

Definition 2.5. [Alchourrón and Makinson, 1981] Let $A$ be a set of sentences and $\alpha$ a sentence. Then $A$ 's remainder set modulo $\alpha$ (denoted by $A \downarrow \alpha$ ) consists of subsets of $A$ s.t., $X \in A \downarrow \alpha$ iff:

1. $X \subseteq A$
2. $\alpha \notin C n(X)$
3. There is no set $X^{\prime}$ s.t., $X \subset X^{\prime} \subseteq A$ and $\alpha \notin C n\left(X^{\prime}\right)$.

It is worth to mention that an element of $K \downarrow \alpha$ is also a belief set (meaning that it is logically closed) for any belief set $K$ and any sentence $\alpha$.

Example 4. Assume $\mathcal{L}$ is the propositional logic generated from atoms $p$ and $q$. Let $K=$ $C n(\{p, q\})$. In Figure 2.1, each node denotes an element of $K$ (modulo logical equivalence); and the upwards edges denote the deduction relation, that is, there is an edge from $e_{1}$ to $e_{2}$ iff $e_{1} \vdash e_{2}$. Suppose $q$ is to be contracted. It is not difficult to see that there are two maximal subsets of $K$ which do not include $q: K \downarrow q=\{\operatorname{Cn}(p), C n(p \leftrightarrow q)\}$. Note that the remainder set $K \downarrow \alpha$ contains multiple elements. In order to obtain a contracted belief set $K-q$, a
naive approach is taking the intersection of all elements of $K \downarrow q$. Note that, by doing so, an $q$-contraction will remove surprisingly also $p$, since $\bigcap\{C n(p), C n(p \leftrightarrow q)\}=C n(p \vee \neg q)$.


Figure 2.1: Internal structure of belief set $K=C n(\{p, q\})$

Formally, Alchourrón and Makinson, 1982] has proposed the so-called full meet contraction (denoted by $\sim$ ), which is defined as follows:

$$
K \sim \alpha= \begin{cases}K & \text { if } \vdash \alpha \\ \bigcap K \downarrow \alpha & \text { otherwise }\end{cases}
$$

Note that $\sim$ is indeed a function on belief sets, as it is well-known that the intersection of logic theories is also a logical theory.

Full meet contraction has been shown too cautious, in the sense, that a sentence of $K$ will remain in $K-\alpha$ iff it is a logic consequence of $\neg \alpha$ :

Observation 2.6. [Alchourrón and Makinson, 1982] Let $K$ a belief set. Then for any sentence $\alpha \in K$ :

$$
K \sim \alpha=K \cap C n(\{\neg \alpha\})
$$

Via the Levi identity, a revision operator, called full meet revision, can be constructed basedon full meet contraction. Also, full meet revision has been shown overly cautious.

Observation 2.7. [Alchourrón and Makinson, 1982] Let $K$ be a belief set and $*$ the revision operator generated from full meet contraction via the Levi identity. Then for any sentence $\alpha$ :

$$
K * \alpha= \begin{cases}K+\alpha & \text { if } \neg \alpha \notin K \\ C n(\{\alpha\}) & \text { otherwise }\end{cases}
$$

Obviously, both full meet contraction and revision are almost of no importance in practice. In order to construct more reasonable operators, the AGM trio have introduced the notion of selection functions. Intuitively, when $K$ is contracted by $\alpha$, a selection function helps to select most plausible elements of $K \downarrow \alpha$. Formally,

Definition 2.8. Alchourrón et al., 1985] Let $K$ be belief set. A selection function on $K$ is a function $f$ s.t., for any sentence $\alpha$ :

1. If $K \downarrow \alpha$ is non-empty, then $f(K \downarrow \alpha)$ is a non-empty subset of $K \downarrow \alpha$.
2. If $K \downarrow \alpha$ is empty, then $f(K \downarrow \alpha)=\{K\}$.

Given a selection function $f$ on belief set $K$, the partial meet contraction, written as $\sim_{f}$, on $K$ is defined as follows [Alchourrón et al., 1985]:

$$
K \sim_{f} \alpha=\bigcap f(K \downarrow \alpha)
$$

As shown by [Alchourrón et al., 1985], partial meet contraction satisfies Postulates (K-1)-(K-6). Conversely, any contraction operator satisfying (K-1)-(K-6) can be reconstructed as a partial meet contraction. In other words, partial meet contraction operators are exactly characterized by the basic postulates for contraction.

Theorem 2.9. Alchourrón et al., 1985] Let $K$ be a belief set, an operator - on $K$ satisfies Postulates $(K-1)-(K-6)$ iff there is a selection function $f$ on $K$ s.t., for any sentence $\alpha$ :

$$
K-\alpha=K \sim_{f} \alpha
$$

By the Levi identity, we also can construct the so-called partial meet revision. According to Theorem 2.3, partial meet revision should satisfy Postulates ( $\left.\mathrm{K}^{*} 1\right)-\left(\mathrm{K}^{*} 6\right)$.

Note that full meet contraction is just a special case of partial meet contraction, where the selection function always selects all elements of the remainder set. Therefore, it also satisfies all basic postulates for contraction.

A maxichoice contraction on belief set $K$ is another special case of partial meet contraction, which is based a selection function on $K$ selecting exactly one element of $K \downarrow \alpha$ for any sentence $\alpha$ [Alchourrón and Makinson, 1982].

In contrast to full meet contraction, a maxichoice contraction is extremely incautious. This is particular obvious, if we consider the behavior of the revision operator, called a maxichoice revision, generated from a maxichoice contraction operators via the Levi identity.

Observation 2.10. [Alchourrón and Makinson, 1982] Let $K$ be a belief set and $*$ an operator generated from a maxichoice contraction on $K$ via the Levi identity. Then it holds for all sentences $\alpha$ and $\beta$ that

$$
\text { If } \neg \alpha \in K \text { then either } \beta \in K * \alpha \text { or } \neg \beta \in K * \alpha \text {. }
$$

Put in words, a maxichoice revision will render the revised belief set $K * \alpha$ to be complete, even if $K$ is initially not complete.

As already mentioned, a selection function $f$ on $K$ is assumed to determine the most plausible elements of $K \downarrow \alpha$. So as if there is ordering of plausibility on $K \downarrow \alpha$. Obviously, this plausibility ordering should be independent of which sentence is to be contracted, in other words, it is should be an ordering on all maximal subsets of $K$ (i.e., $\bigcup\{K \downarrow \alpha \mid \alpha \in \mathcal{L}\}$ ).

Formally, a selection function $f$ on $K$ is called relational if it is induced from a relation $\prec$ on all maximal subsets of $K$ as follows:

$$
f(K \downarrow \alpha)=\left\{K^{\prime} \in K \downarrow \alpha \mid K^{\prime \prime} \prec K^{\prime} \text { for all } K^{\prime \prime} \in K \downarrow \alpha\right\}
$$

It is also natural to require an ordering to be transitive. Formally, a selection function on $K$ is called transitively relational if it is induced from a transitive relation $\prec$ on all maximal subsets of $K$.

The following result shows that the transitively relational partial meet contraction operators are exactly characterized by the AGM postulates. ${ }^{6}$

Theorem 2.11. [Alchourrón et al., 1985] Let $K$ be a belief set. An operator - on $K$ is a transitively relational partial meet contraction on $K$ iff it satisfies $(K-1)-(K-8)$.

[^8]Note that while the construction of (transitively relational) partial meet contraction and revision are mathematically very elegant, they are not suitable for direct computer realizations. In next section, I will present a constructive model which is more amenable to implementation.

### 2.1.3 Epistemic Entrenchment

As already mentioned, a concrete contraction (revision) operator should make use of some extralogical preference information. In partial meet contraction (revision) such extra-logical preference information is encoded by a selection function.

To make the realization easier, [Gärdenfors and Makinson, 1988] have proposed to use a total pre-order on the underlying language $\mathcal{L}$ as the extra-logical preference information. ${ }^{7}$

Definition 2.12. [Gärdenfors and Makinson, 1988] Let $K$ be a belief set. An epistemic entrenchment $(\mathrm{EE}$, for short $) \leq_{K}$ on $K$ is a binary relation over $\mathcal{L}$ obeying following conditions:

```
(EE1) \(\quad\) If \(\alpha \leq_{K} \beta\) and \(\beta \leq_{K} \gamma\) then \(\alpha \leq_{K} \gamma\)
(EE2) If \(\alpha \vdash \beta\) then \(\alpha \leq_{K} \beta\)
(EE3) \(\quad\) For any \(\alpha\) and \(\beta, \alpha \leq_{K} \alpha \wedge \beta\) or \(\beta \leq_{K} \alpha \wedge \beta\)
(EE4) When \(K\) is consistent, \(\alpha \notin K\) iff \(\alpha \leq_{K} \beta\) for all \(\beta\)
(EE5) If \(\beta \leq_{K} \alpha\) for all \(\beta\) then \(\vdash \alpha\)
```

It is easy to see that an EE is indeed a total pre-order. The transitivity is required by the condition (EE1). Since $\alpha \vdash \alpha$, it follows from (EE2) that $\leq_{K}$ is also reflexive. Assume $\alpha, \beta$ are arbitrary sentences. It follows from (EE3) either $\alpha \leq_{K} \alpha \wedge \beta$ or $\beta \leq_{K} \alpha \wedge \beta$. Assume, without loss of generality, $\alpha \leq_{K} \alpha \wedge \beta$. Since $\alpha \wedge \beta \vdash \beta$, it follows from (EE2) that $\alpha \wedge \beta \leq_{K} \beta$. By transitivity, we can obtain $\alpha \leq_{K} \beta$. Hence, $\leq_{K}$ is total.

Note that an $\mathrm{EE} \leq_{K}$ is correlated with a belief set $K$. According to (EE4), if $K$ is consistent then the minimal elements of $\mathcal{L}$ wrt. $\leq_{K}$ are precisely those not contained in $K$. Therefore, in general $\leq_{K}$ cannot be an EE on another belief set $K^{\prime}$. Furthermore, according to (EE2) and (EE5), a sentence is a maximal element of $\mathcal{L}$ wrt. $\leq_{K}$ iff it is a tautology.

[^9]Intuitively, an $\mathrm{EE} \leq_{K}$ is a preference relation on $\mathcal{L}$. If $\alpha \leq_{K} \beta$ then $\beta$ is at least as entrenched as $\alpha$, which means when one of them has to be given up it is preferred to retract $\alpha: 8$

$$
(\mathrm{C} \leq) \quad \alpha \leq_{K} \beta \text { iff } \alpha \notin K-(\alpha \wedge \beta) \text { or } \vdash \alpha \wedge \beta
$$

Theoretically, Condition $(\mathrm{C} \leq)$ can be understood as a way of constructing an $\mathrm{EE} \leq_{K}$ from a contraction operator - on $K$ : for any $\alpha, \beta \in \mathcal{L}$, we can determine $\alpha \leq_{K} \beta$ by simply checking whether $\alpha \notin K-(\alpha \wedge \beta)$ or $\vdash \alpha \wedge \beta$ ?

From a pragmatic point of view, it is more interesting to construct a contraction operator from an EE. Based on Condition ( $\mathrm{C} \leq$ ), Gärdenfors has proposed the so-called entrenchmentbased contraction: 9
(C-) $\quad K-\alpha= \begin{cases}K & \text { if } \vdash \alpha \\ \left\{\beta \in K \mid \alpha<_{K} \alpha \vee \beta\right\} & \text { otherwise }\end{cases}$
Like transitively relational partial meet contraction, G ardenfors' entrenchment-based contraction is also exactly characterized by the AGM postulates.

Theorem 2.13. [Gärdenfors, 1988]

1. Let - be a contraction operator on a belief set $K$ satisfying ( $K-1)-(K-8)$. Then relation $\leq_{K}$ derived from $(C \leq)$ satisfies (EE1)-(EE5).
2. Let $\leq_{K}$ be an EE on a belief set $K$. Then entrenchment-base contraction operator defined by ( $C-)$ satisfies ( $K-1$ )-( $K-8$ ).

Rott has argued that (C-) is counterintuitive. Therefore, he has suggested instead the following more natural definition [Rott, 1991]:
(C-R) $\quad K-\alpha= \begin{cases}K & \text { if } \vdash \alpha \\ \left\{\beta \in K \mid \alpha<_{K} \beta\right\} & \text { otherwise }\end{cases}$
Rott's entrenchment-based contraction has been shown satisfying all the AGM postulates, but the controversial (K-5).

It turns out surprisingly that, although the two entrenchment-based contraction operators are distinct, they obtain the same revision operator by the Levi identity [Rott, 1991].

[^10]To avoid reference to a contraction operator, Gärdenfors has also proposed the so-called entrenchment-base revision:
(C*) $\quad K * \alpha= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \left\{\beta \in \mathcal{L} \mid \neg \alpha<_{K} \neg \alpha \vee \beta\right\} & \text { otherwise }\end{cases}$
It is been shown that entrenchment-base revision is exactly characterized by the AGM postulates:

Theorem 2.14. [Gärdenfors, 1988] Let $K$ be a belief set. Then a revision operator $*$ satisfies Postulates ( $K^{*}$ ) $)$-( $K^{*} 8$ ) iff there exists an $E E \leq_{K}$ s.t., Condition ( $C^{*}$ ) holds for any sentence $\alpha$.

In comparison to partial meet revision, entrenchment-based revision is more like a procedure in computer science. However, it is not possible to implement entrenchment-base revision directly, since if the underlying language $\mathcal{L}$ is infinite so is an EE. Moreover, even for a finite $\mathcal{L}$ the size of an $\mathrm{EE} \leq_{K}$ could be very large.

### 2.2 System of Spheres

In this section, I present another interesting possible world-based constructive model constructive model of belief change proposed by [Grove, 1988]. This model provides us nice and clear pictures of the behavior of belief change. Strictly speaking, this model is not an original part of the AGM theory, as it is not a contribution of the AGM trio.

Intuitively, a possible world represents a snapshot of the world at some point in time. In our setting, a possible world, denoted by $W$ (possible indexed), can be represented by a maximal consistent set of sentences of $\mathcal{L}$. Recall that $K \downarrow \alpha$ are the maximal subsets of $K$ which does include $\alpha$, therefore the set of possible worlds, denoted by $\Theta_{\mathcal{L}}$, of $\mathcal{L}$ can be formally defined as follows:

$$
\Theta_{\mathcal{L}}:=\mathcal{L} \downarrow \perp
$$

Note that, a possible world $W \in \Theta_{\mathcal{L}}$ is not only consistent but also complete, that is, for any sentence $\alpha$, either $\alpha \in W$ or $\neg \alpha \in W$.

Example 5. Assume $\mathcal{L}$ is the propositional language generated from two atoms $p$ and $q$. Then it can be shown that:

$$
\Theta_{\mathcal{L}}=\{C n(\{p, q\}), C n(\{p, \neg q\}), C n(\{\neg p, q\}), C n(\{\neg p, \neg q\})\}
$$

A possible world $W$ is called a model of a sentence $\alpha$ (written as $W \models \alpha$ ) iff $\alpha \in W$, and the set of all models of $\alpha$ is denoted by $[\alpha]$, that is, $[\alpha]=\left\{W \in \Theta_{\mathcal{L}} \mid W \models \alpha\right\}$. Similarly, a possible world $W$ is a model of a set $A$ of sentences (written as $W \models A$ ) iff $A \subseteq W$, and $[A]=\left\{W \in \Theta_{\mathcal{L}} \mid W \models A\right\}$. Conversely, given a set of possible worlds $\mathcal{W} \subseteq \Theta_{\mathcal{L}}$, we denote by $\operatorname{Th}(\mathcal{W})$ the set of sentences which are true in all elements of $\mathcal{W}$ :

$$
\begin{equation*}
\operatorname{Th}(\mathcal{W}):=\{\alpha \in \mathcal{L} \mid W \models \alpha \text { for all } W \in \mathcal{W}\} \tag{2.2}
\end{equation*}
$$

It is not difficult to see that for any set $\mathcal{W}$ of possible worlds $\operatorname{Th}(\mathcal{W})$ is a belief set since it is logically closed.

As frequently repeated, a contraction (or revision) operator needs to employ some extralogical preference information to uniquely determine its contraction (revision) policy. For this purpose, we might think, in additional to a belief set $K$, there is a system of concentric spheres surrounding $[K]$.

Definition 2.15. [Grove, 1988] Let $\mathcal{W}$ be a set of possible worlds. A system of spheres $\mathfrak{S}$ (SOS, for short) centered on $\mathcal{W}$ is a family of subsets of $\Theta_{\mathcal{L}}$ such that:

1. If $S_{1}, S_{2} \in \mathfrak{S}$, then either $S_{1} \subseteq S_{2}$ or $S_{2} \subseteq S_{1}$
2. $\mathcal{W} \in \mathfrak{S}$, and $\mathcal{W} \subseteq S$ for all $S \in \mathfrak{S}$
3. $\Theta_{\mathcal{L}} \in \mathfrak{S}$
4. For any consistent sentence $\alpha$, there exists a smallest sphere $S$ such that $[\alpha] \cap S \neq \emptyset$

Note that the last condition is not redundant. It essential requires the set $\{S \in \mathbb{S} \mid S \cap[\alpha] \neq$ $\emptyset\}$ to be well-ordered wrt. $\subseteq$, for any consistent sentence $\alpha$.

Given a belief set $K$ and a SOS $\mathfrak{S}$ centered on $[K]$, those possible worlds which are closer to $[K]$ are considered more plausible. 10

The plausibility relation encoded by a SOS has been show by Grove [1988] sufficient to determine uniquely a contraction operator, as well as a revision operator.

Suppose belief set $K$ is contracted by $\alpha$. As the contracted belief set $K-\alpha$ should be a subset of $K$, we expect that $[K-\alpha]$ to be a superset of $[K]$. Since the only motivation of

[^11]

Figure 2.2: A SOS centered on $[K]$.
accepting more possible worlds is to exclude $\alpha$ (in other words, to make $\neg \alpha$ possible), it is reasonable that $[K]$ is only enlarged by elements of $[\neg \alpha]$. As accepting $[\neg \alpha]$ wholly might be too radical, we would like to enlarge $[K]$ by the most plausible elements of $[\alpha]$, i.e., those closest to $[K]$. Such a contraction process is intuitively depicted in Figure 2.3, where $[K-\alpha]$ is denoted by the filled part.


Figure 2.3: SOS-base contraction

For any consistent sentence $\alpha$, let $\min _{\mathfrak{S}}(\alpha)$ denote the smallest sphere of $\mathfrak{S}$ that intersects $[\alpha]$. Formally, a contraction operator - on a belief set $K$ is called a sphere-based iff there is a SOS $\mathfrak{S}$ centered on $[K]$ s.t., for any sentence $\alpha$ :

$$
K-\alpha= \begin{cases}K & \text { if } \vdash \alpha \\ \operatorname{Th}\left([K] \cup\left(\min _{\mathfrak{S}}(\neg \alpha) \cap[\neg \alpha]\right)\right) & \text { otherwise }\end{cases}
$$

The following theorem shows that sphere-based contraction is characterized by the AGM postulates.

Theorem 2.16. [Grove, 1988] Let $K$ be a belief set. Then a contraction operator on $K$ satisfies $(K-1)-(K-8)$ iff it is a sphere-based contraction operator on $K$.

Suppose belief set $K$ is revised by $\alpha$. Since $\alpha$ has to be incorporated in $K * \alpha,[K * \alpha]$ should be a subset of $[\alpha]$. As we do not want $K * \alpha$ to differ more from the $K$ than what is motivated by $\alpha$, it is reasonable to assume that $[K * \alpha]$ consists of the most plausible elements of $[\alpha]$. Such a revision process is intuitively depicted in Figure 2.4, where $[K * \alpha]$ is denoted by the filled part.


Figure 2.4: SOS-based revision

Formally, a revision operator $*$ on belief set $K$ is called sphere-based revision iff there is a SOS $\mathfrak{S}$ centered on for $[K]$, s.t., for any sentence $\alpha$ :

$$
K * \alpha= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \left.\operatorname{Th}\left(\min _{\mathfrak{S}}(\alpha) \cap[\alpha]\right)\right) & \text { othwise }\end{cases}
$$

It has been shown that sphere-based revision is also characterized by the AGM postulates.

Theorem 2.17. [Grove, 1988] Let $K$ be a belief set. Then a revision operator on $K$ satisfies $\left(K^{*}\right)-\left(K^{* 8}\right)$ iff it is a sphere-based revision operator on $K$.

Note that both sphere-base contraction and revision seem quite natural and promising, therefore they can be considered as supports for the rationality of the AGM postulates.

### 2.3 The KM Theory

The distinction between update and revision was first informally made by Keller and Winslett [1985]. As argued by several researchers, revision is not suitable for situations where the new information describes a change of the world [Ginsberg and Smith, 1987; Winslett, 1988b]. In particular, Winslett has proposed an example of belief change, with which revision will obtain undesirable results.

Example 6. [Winslett, 1988b] Initially, all we believed about a certain table is that there is either a book on the table or a magazine on the table, but not both. How should we change our beliefs after a robot is instructed to put the book on the table?

Assume "the book (the magazine) is on the table" is represented by $b$ and ( $m$ respectively). Let $K=C n(\{\neg b \leftrightarrow m\})$. The new information can be represented by $b$. If $K$ is to be revised by $b$, then the revised belief set $K * b=C n(\{\neg b \leftrightarrow m, b\})$, provided $*$ satisfies the postulates $(K * 3)$ and (K4). So we believe now that the magazine is not on the table $(\neg m)$. This is obviously unreasonable, as by common-sense the magazine should remain where it was.

Belief update is a type of belief change operation designed for situations where the new information reflects changes in the world. Formally, an update operation is a function $\diamond$ which maps a belief set $K$ and a sentence $\alpha$ to the updated belief set $K \diamond \alpha$. It is now well-known that there is a fundamental difference between update and revision [Katsuno and Mendelzon, 1991a], which can be explained intuitively as follows. Given a belief set $K$, the real world is conceived to be among the possible worlds $[K]$. When $K$ is revised by $\alpha$, the world has assumedly not been changed. Therefore, it is reasonable that the revised belief set $K * \alpha$ is determined by elements of [ $\alpha$ ] which are closest to [ $K$ ] (cf. Figure 2.4), in other words, $[K]$ is viewed as a whole. When $K$ is updated by $\alpha$. It is assumed that $\alpha$ is caused by changes in the world. Since we do not know which possible world in $[K]$ is the real world, it is better to give each of them equal consideration. Therefore, the updated belief set $K \diamond \alpha$ should be obtained by changing each possible world in $[K]$ "as little as possible" so that it becomes a model $\alpha$.

Informally, let $W \diamond \alpha$ denote the set of possible worlds obtained from changing $W$ as little as possible to accommodate $\alpha$. The above argument leads to the well-known Winslett identity [Winslett, 1990]:

Winslett identity: $\quad K \diamond \alpha= \begin{cases}\mathcal{L} & \text { if }[K]=\emptyset \text { or }[\alpha]=\emptyset \\ \operatorname{Th}\left(\bigcup_{W \in[K]} W \diamond \alpha\right) & \text { otherwise }\end{cases}$

### 2.3.1 The KM Postulates

In [Katsuno and Mendelzon, 1991a], the difference between update and revision has been formally studied. In particular, Katsuno and Mendelzon also have introduced a set of rationality postulates, numbered (U1)-(U9), to constrain the behavior of belief update operators. In the original KM theory, the beliefs of an agent are represented as a sentence of some finitary propositional logic and update is modeled as a binary function over sentences.

By generalizing the KM theory, Peppas et al,. have proposed a framework for belief update, which is not restricted to finitary propositional logics. [Peppas and Williams, 1995; Peppas et al., 1996]:11

$$
\begin{array}{ll}
(K \diamond 1) & K \diamond \alpha=C n(K \diamond \alpha) \\
(K \diamond 2) & \alpha \in K \diamond \alpha \\
(K \diamond 3) & \text { If } \alpha \in K \text { then } K \diamond \alpha=K \\
(K \diamond 4) & K \text { or } \alpha \text { is inconsistent iff } K \diamond \alpha \text { is inconsistent } \\
(K \diamond 5) & \text { If } \alpha \equiv \beta \text { then } K \diamond \alpha=K \diamond \beta \\
(K \diamond 6) & K \diamond(\alpha \wedge \beta) \subseteq(K \diamond \alpha)+\beta \\
(K \diamond 7) & \text { If } K \text { is complete and } \neg \beta \notin K \diamond \alpha \text { then }(K \diamond \alpha)+\beta \subseteq K \diamond(\alpha \wedge \beta) \\
(K \diamond 8) & \text { If }[K] \neq \emptyset, \text { then } K \diamond \alpha=T h\left(\bigcup_{W \in[K]} W \diamond \alpha\right)
\end{array}
$$

Readers should be familiar with $K \diamond 1, K \diamond 2, K \diamond 5$ and $K \diamond 6$, since they also occur in the AGM postulates for revision.

[^12]Note that there are at least three major differences between the KM and AGM postulates. First, $\left(K^{*} 4\right)$ is replaced by a weaker version of $(K \diamond 3)$. So even $\alpha$ is consistent with $K$, it is not necessary that $K \diamond \alpha=K+\alpha$. Recall Example 6, it is in fact ( $\mathrm{K} * 4$ ) which forces to obtain undesirable results. Hence the weakening of ( $\mathrm{K} * 4$ ) seems necessary and justified. The second difference is that when the inconsistent belief set $K$ (or, equivalently $\mathcal{L}$ ) updated by a consistent sentence $\alpha$, according to ( $K \diamond 4$ ), results in the inconsistent belief set; whereas if $K$ is revised by $\alpha$, according to ( $K * 5$ ), the revised belief set $K * \alpha$ should be consistent. This can be explained as follows. The inconsistent belief set can be remedied with revision by adding new information that supersedes the inconsistency. We can never repair the inconsistent belief set using update, because specifies a change in the world. If there is no possible world compatible with the belief set, we have no way of recording the change in the world. Another important difference is that $(K \diamond 8)$ only occurs in the KM postulates. The postulate $(K \diamond 8)$ directly corresponds to the Winslett identity. ${ }^{12}$ An immediate consequence of $K \diamond 8$ is the so-called monotonicity [Katsuno and Mendelzon, 1991a]:
$(\mathrm{K} \diamond \mathrm{M}) \quad$ For any belief sets $K, K^{\prime}$ and any sentence $\alpha$, if $K \subseteq K^{\prime}$ then $K \diamond \alpha \subseteq K^{\prime} \diamond \alpha$.

As we will see in Section 2.4.1, monotonicity is incompatible with the AGM postulates.

### 2.3.2 Similarity Structure

Based on the work of [Katsuno and Mendelzon, 1991a], Peppas et al., have proposed a constructive model for belief update. The idea is quite similar to Grove's SOS. Note that, to revise a belief set $K$, we only need a SOS centered on $[K]$. This is not sufficient for updating $K$, since in this case we would like to change each possible world in $[K]$ point-wise. Instead, we need a SOS centered on each possible world. Formally, a similarity structure is a function that assigns to every possible world $W \in \Theta_{\mathcal{L}}$ a SOS $\mathfrak{S}_{W}$ Recall that, for any consistent sentence $\alpha, \min _{\mathfrak{S}_{W}}(\alpha)$ denotes the minimal sphere which intersects $[\alpha]$. Given a similarity structure, we can easily construct an update operator by letting $W \diamond \alpha=\min _{\mathfrak{S}_{W}}(\alpha) \cap[\alpha]$.

It has been shown that update operators induced from similarity structures are precisely those satisfying the KM postulates.

[^13]Theorem 2.18. [Peppas et al., 1996] An update operator $\diamond$ satisfies $(K \diamond 1)-(K \diamond 8)$ iff there exists a similarity structure s.t., for any belief set $K$ and any sentence $\alpha$ :

$$
K \diamond \alpha= \begin{cases}\mathcal{L} & \text { if }[\alpha]=\emptyset \text { or }[K]=\emptyset \\ \operatorname{Th}\left(\bigcup_{W \in[K]} \min _{\mathfrak{S}_{W}}(\alpha) \cap[\alpha]\right) & \text { otherwise }\end{cases}
$$

An update operator based on a similarity structure can be nicely depicted by Figure 2.5. Note that there are two major differences between Figure 2.5 and Figure 2.4. Firstly, in Figure 2.4 there is only one SOS centered on $[K]$, whereas in Figure 2.5 there is a family of SOSs (one for each possible world in $[K]$ ). Secondly, in Figure 2.5 the centers of all SOSs are singleton sets.


Figure 2.5: Similarity structure-base update

It is worth to mention that, unlike extra-logical preference information employed in belief revision (e.g., an EE), a similarity structure is not correlated to a fixed belief set. This means, given a similarity structure, we can define an update operator which applies to any belief set $K$ and any sentence $\alpha$. Therefore, the update operator induced from a similarity structure is truly a binary function. Such binary operators are called by Hansson [1998] global, whereas an operator for a fixed belief set (e.g., an EE-based revision) is said to be local. In fact, many researchers believe that it is more appropriate to view a local revision operator as a unary function (with the belief set in background) which maps the new information to the revised belief set.

### 2.4 Related Research Fields

In this section, I will present two research fields, i.e., conditionals and non-monotonic reasoning, which are closed related to belief change.

### 2.4.1 Conditionals

One major motivation of the philosophic study on belief change is to develop an epistemic semantic model for conditional sentences. A conditional sentences is of the form $\alpha \gg \beta$, which can be read as "If $\alpha$, then $\beta$ " [Gärdenfors, 1988]. If $\alpha$ contradicts what is already accepted in a given belief set $K$, the conditional is called a counterfactuals (relative to $K$ ); otherwise it is called an open conditional (relative to $K$ ).

The most widely accepted proposal of the epistemic semantics for conditionals is based on well-known Ramsey test:
(RT) Accept a sentence of the form "If $\alpha$, then $\beta$ " in a belief set $K$ iff the minimal change of $K$ needed to accept $\alpha$ also requires accepting $\beta$.

As suggested by [Gärdenfors, 1988], if we consider $\gg$ as a binary connective of $\mathcal{L}$, then the Ramsey test can be reformulated in a more condensed way:

$$
\left(\mathrm{RT}^{*}\right) \quad \alpha \gg \beta \in K \text { iff } \beta \in K * \alpha
$$

Gärdernfors has shown that if a global revision operator $*$ satisfies (RT*), then it must be monotonic, that is, 13
$(\mathrm{K} * \mathrm{M}) \quad$ For any belief sets $K, K^{\prime}$ and any sentence $\alpha$,

$$
\text { if } K \subseteq K^{\prime} \text { then } K * \alpha \subseteq K^{\prime} * \alpha
$$

Unfortunately, Gärdenfors' well-known triviality theorem shows that $\left(\mathrm{K}^{*} \mathrm{M}\right)$ is not compatible with the AGM postulates. Formally, a belief set $K$ is called non-trivial iff there are at least three pairwise disjoint sentences $\alpha, \beta, \gamma$ that are consistent with $K$, that is, $\neg \alpha \notin K, \neg \beta \notin K$, and $\neg \gamma \notin K .{ }^{14}$ The triviality theorem essentially says there does not exist non-trivial global revision operator which satisfies the AGM postulates and ( $\mathrm{K}^{*} \mathbf{M}$ ) simultaneously:

Theorem 2.19. [Gärdenfors, 1988] Let $*$ be a global revision operator. Then the following two conditions are incompatible:

## 1. There exists a non-trivial belief set $K$.

[^14]2. $*$ satisfies $\left(K^{*} 2\right),\left(K^{*} 4\right),\left(K^{*} 5\right)$ and $\left(K^{*} M\right)$.

Since Postulates $(\mathrm{K} * 2),(\mathrm{K} * 4),(\mathrm{K} * 5)$ are commonly accepted and very plausible, we consider $\left(\mathrm{K}^{*} \mathrm{M}\right)$ (hence $\left.\left(\mathrm{RT}^{*}\right)\right)$ as the culprit of the incompatibility.

An important lesson we learn from the triviality theorem is that it is improper to include conditionals into the underlying language $\mathcal{L}$, as far as we accept the AGM postulates. In principle, a set of conditionals can be considered as one particular form of extra-logical preference information. Therefore, I suggest to distinguish a belief set from a belief state. The latter contains, in addition to a belief set, the extra-logical preference information (e.g., an EE) which can used to determine the belief change strategy. I will argue in Chapter 3 that it is more appropriate to consider belief change operators as functions on belief states (instead of belief sets).

Note that an update operator $\diamond$ can be used in the Ramsey test to avoid the trivialization. Since in the KM postulates, one source of the incompatibility (i.e., $K * 4$ ) is weakened, monotonicity is not any more a problem. In fact, monotonicity is a consequence of the KM postdates (cf. $(K \diamond M)$ of Section 2.3.1) and considered a desirable property for update. The immunity of the update to triviality theorem has attracted some researchers to study the connections between belief update and conditionals [Grahne, 1991; Katsuno and Satoh, 1991].

### 2.4.2 Non-monotonic Reasoning

The triviality theorem reveals that the AGM revision is intrinsically non-monotonic. In fact, belief revision and non-monotonic reasoning are often considered as "two sides of the same coins". The direct relation between revision and non-monotonic reasoning is formally studied by [Makinson and Gärdenfors, 1989].

Let $K$ be a belief set and $*$ a revision operator on $K$, a non-monotonic inference relation $\sim_{K, *}$ can be induced as follows:

$$
\text { (NM) } \quad \alpha \sim_{K, *} \beta \text { iff } \beta \in K * \alpha
$$

Intuitively, $\alpha \sim_{K, *} \beta$ means $\beta$ can be non-monotonically inferred from $\alpha$, given the background belief set $K$.

Freund and Lehmann have shown that properties of a revision operator $*$ corresponds directly to well-known properties of the inference relation $\mid \sim_{K, *}$ induced by
(NM) [Freund and Lehmann, 1994] (cf. Table 2.115)

| AGM postulates | Non-monotonic inference |
| :---: | :---: |
| ( $\mathrm{K}^{*}$ ) | If $\vdash \alpha \rightarrow \beta$ and $\gamma \vdash \alpha$, then $\gamma \vdash \beta$ If $\alpha \nsim \beta$ and $\alpha \sim \gamma$, then $\alpha \nsim \beta \wedge \gamma$ |
| (K*2) | $\alpha \sim \alpha$ |
| (K*5) | If $\alpha \nvdash \perp$, then $\alpha \nVdash \perp$ |
| (K*6) | If $\vdash \alpha \leftrightarrow \beta$ and $\alpha \sim \gamma$, then $\beta \vdash \gamma$ |
| (K*7) | If $\alpha \nsim \gamma$ and $\beta \downarrow \gamma$, then $\alpha \vee \beta \vdash \gamma$ |
| (K*8) | If $\alpha \wedge \beta \vdash \gamma$ then $\alpha \sim \beta \rightarrow \gamma$ |

Table 2.1: The AGM postulates and properties of non-monotonic inference.

Readers are referred to [Makinson and Gärdenfors, 1989; Freund and Lehmann, 1994] for the explanations of the listed (non-monotonic) properties.

A non-monotonic inference relation is called rational and consistency preserving iff it satisfies all properties in Table 2.1. Freund and Lehmann have shown that there is an one-to-one correspondence between rational and consistency preserving inference relations and revision operators satisfying the AGM theory.

Theorem 2.20. [Freund and Lehmann, 1994]

1. Let $K$ be a belief set and $*$ a revision operator on $K$. If $*$ satisfies Postulates $(K * 1)-(K * 8)$, then inference relation $\sim_{K, *}$ defined by (NM) is rational and consistency preserving.
2. Let $\mid \sim$ be a rational and consistency preserving inference relation. Then there exists a belief set $K$ and a revision operator $*$ on $K$, s.t.,

$$
\alpha \nsim \beta \text { iff } \beta \in K * \alpha
$$

The direct translation between non-monotonic reasoning and belief revision suggests we can use techniques of the one field to solve the problems of the other field. In particular, algorithms for belief revision can be used to implement non-monotonic reasoning and vice verse.

[^15]
## Summary

In this chapter, I have surveyed two classic frameworks for belief change, i.e., the AGM theory for belief revision and the KM theory for belief update. Revision and update are different approaches to changing the beliefs of an agent in response to different types of new information. Update is suitable for recording changes in the world; whereas revision is to add knowledge about the static world. In both frameworks, belief change are studied from two perspectives. On the one hand, a set of rationality postulates are introduced to constrain the general behavior of belief change operators. On the other hand, there are also proposals of various constructive models.

## Chapter 3

## Iterated Belief Revision: General Frameworks

In many situations, an agent needs to adapt its beliefs incrementally in response to a sequence of observations. Therefore, iterated belief revision is a very important topic in belief change, that has been studied by many researchers [Freund and Lehmann, 1994; Darwiche and Pearl, 1997; Nayak et al., 1996a; Boutilier, 1993; Konieczny and Pérez, 2000; Jin and Thielscher, 2005b].

The classical AGM theory seems suitable and sufficient for one-step belief revision. However, for the incremental adaptation of beliefs, the AGM postulates proved to be overly weak [Darwiche and Pearl, 1994; Darwiche and Pearl, 1997]. This has led to the development of additional postulates for iterated belief revision by Darwiche and Pearl (DP), among others [Lehmann, 1995; Boutilier, 1993; Nayak et al., 2003].

Still, however, the AGM and DP postulates together are too permissive in that they support belief revision operators which assume arbitrary dependencies among the pieces of information which an agent acquires along its way. These operators have a drastic effect when the agent makes an observation which contradicts its currently held beliefs: The agent is forced to cancel everything it has learned up to this point. In this chapter, I first give a formal analysis of this problem of implicit dependence, and then I present, as a solution, an independence postulate for iterated belief revision. I give a representation theorem for the new postulate and prove its consistency by defining a concrete belief revision operator. I also contrast the postulate of independence to the so-called Recalcitrance postulate of [Nayak et al., 1996a; Nayak et al., 2003] and argue that the latter is too strict in that it rejects reasonable belief revi-
sion operators. The main contribution of this chapter is a general framework for iterated belief revision, which is more satisfactory than any other proposals in the literature.

The rest of the chapter is organized as follows. In the next section, I give discussions on why iterated belief revision is a difficult problem, followed by the approach of [Darwiche and Pearl, 1994] for iterated belief revision. In Section 3.3, I formally analyze the problem of the DP postulates to be overly permissive and propose an additional postulate to overcome this deficiency. I give a representation theorem for the postulate along with a concrete revision operator. In Section 3.5, a detailed comparison to related work will be given.

### 3.1 Why is it Difficult to Iterate

The AGM theory seems suitable and sufficient for one-step belief revision. But by a closer look at the constructive models, we can observe immediately its problem for iterated belief revision at least from a technical point of view. To make the argument more grounded, let us consider the constructive model based on $\mathrm{EE},{ }^{1}$ in which, to revise a belief set $K$, an $\mathrm{EE} \leq_{K}$ on $K$ is exploited. Recall, if $K$ is revised by $\alpha$, Condition ( $\mathrm{C}^{*}$ ) (cf. Section 2.1.3) can uniquely determine the revised belief set $K * \alpha$. Now suppose we want to subsequently revise $K * \alpha$ by $\beta$. Clearly, for revising $K * \alpha$, an EE on $K * \alpha$ is needed. Unfortunately, provided $K \neq K * \alpha$, it is impossible to reuse $\leq_{K}$ due to Condition (EE4) (of Definition 2.12). Nor does Condition (C*) tell us how to construct an EE for the revised belief set $K * \alpha$. Therefore, the revision process can not be iterated. The problem of the EE-based revision is that it takes a belief set and some extra-logical preference information (i.e., an EE), but only produces a revised belief set. This is referred to as the problem of categorial mis-matching by Hansson [2003].

Technically, iterated belief revision is possible only if we solve the problem of categorial mis-matching. A naive solution of the problem of categorial mis-matching is to assume that there exists an external source which assigns extra-logical preference information (e.g., an EE or a selection function) to each belief set (cf. [Rott, 1992; Hansson, 1989]). Revision operators constructed by using such an external source are called external revisions. The problem of external revisions is that they are based on some information source which is external to the agent's beliefs. Therefore, the agent is supposed to adhere to the same revision policy regardless of its actual beliefs. Moreover, it is not at all clear where the external extra-logical preference

[^16]information comes from and how it is to be interpreted. Thus, this kind of revision operators has later on been criticized by Rott as embodying a bad philosophy [Rott, 2003].

A more promising approach is to design belief revision operators which not only revise the belief set but also the extra-logical preference information, so that the revised extra-logical preference information can be used for the subsequent revision of the revised belief set. Recall, due to Gärdenfors' triviality theorem (see Section 2.4.1), we should distinguish a belief set from a belief state. For the sake of generality, we consider a belief state as an abstract object from which we can induce a belief set (or, the propositional beliefs) and the extra-logical preference information (required by a revision operator). Canonically, the extra-logical preference information can be represented by a set of conditionals, although it might appear in different forms. Therefore, the extra-logical preference information of a belief state is also called its conditional beliefs. Usually, a belief state is denoted by $\mathcal{K}$ (possibly indexed). The belief set induced from $\mathcal{K}$ is denoted by $\operatorname{Bel}(\mathcal{K})$. A belief state is said to be consistent iff its belief set is consistent. Two belief states $\mathcal{K}_{1}, \mathcal{K}_{2}$ are called statically equivalent (written as $\mathcal{K}_{1} \equiv \mathcal{K}_{2}$ ), if they have the same belief set, i.e., $\operatorname{Bel}\left(\mathcal{K}_{1}\right)=\operatorname{Bel}\left(\mathcal{K}_{2}\right)$. For the sake of succinctness, we often write $\alpha \in \mathcal{K}$ instead of $\alpha \in \operatorname{Bel}(\mathcal{K})$. The above analysis suggests it is more appropriate to regard an iterated belief revision operator as a function on belief states (rather than on belief sets), i.e., it maps a belief state $\mathcal{K}$ and the new information $\alpha$ to the revised belief state $\mathcal{K} * \alpha$. Note that the notion of belief state is rather abstract at moment, we will see a more concrete form of belief states in Chapter 4, when a computational iterated revision operator is concerned.

Following the AGM trio, we might argue that it is unrealistic to have a unique iterated belief revision operator which makes sense in all domains. A more theoretically sufficient solution is to provide a general framework for iterated belief revision. For this purpose, we need to consider the problem of how to revise a belief state in a rational way. Furthermore, we would like to postulate the behavior of rational iterated revision operations in a general way, which is independent of the concrete form of belief states. Obviously, this is much more difficult than to devise a concrete iterated revision operator. An iterated revision operator should revise a belief state in a rational way. It seems reasonable to require that a rational iterated revision operator will change the propositional beliefs as little as possible. To make this idea precise, we expect any rational iterated revision operator to satisfy the following modified AGM postulates:

$$
\begin{array}{ll}
(\mathcal{K} * 1) & \operatorname{Bel}(\mathcal{K} * \alpha)=\operatorname{Cn}(\operatorname{Bel}(\mathcal{K} * \alpha)) \\
(\mathcal{K} * 2) & \alpha \in \operatorname{Bel}(\mathcal{K} * \alpha)
\end{array}
$$

$$
\begin{array}{ll}
(\mathcal{K} * 3) & \operatorname{Bel}(\mathcal{K} * \alpha) \subseteq \operatorname{Bel}(\mathcal{K})+\alpha \\
(\mathcal{K} * 4) & \text { If } \neg \alpha \notin \operatorname{Bel}(\mathcal{K}) \text { then } \operatorname{Bel}(\mathcal{K})+\alpha \subseteq \operatorname{Bel}(\mathcal{K} * \alpha) \\
(\mathcal{K} * 5) & \mathcal{K} * \alpha \text { is inconsistent, only if } \nvdash \neg \alpha \\
(\mathcal{K} * 6) & \text { If } \alpha \equiv \beta \text { then } \mathcal{K} * \alpha \equiv \mathcal{K} * \beta \\
(\mathcal{K} * 7) & \operatorname{Bel}(\mathcal{K} *(\alpha \wedge \beta)) \subseteq \operatorname{Bel}(\mathcal{K} * \alpha)+\beta \\
& \\
(\mathcal{K} * 8) & \text { If } \neg \beta \notin \operatorname{Bel}(\mathcal{K} * \alpha) \text { then } \operatorname{Bel}(\mathcal{K} * \alpha)+\beta \subseteq \operatorname{Bel}(\mathcal{K} *(\alpha \wedge \beta))
\end{array}
$$

In the rest of this thesis, an iterated revision operator satisfying the modified AGM postulates is called an AGM revision operators.

Note that an iterated revision operator is a global operator, in the sense, it does not only apply (locally) to one belief state. Suppose $*$ is an iterated revision operator. Given an belief state $\mathcal{K}$, we can induced a local revision operator $*_{\mathcal{K}}$ on belief set $\operatorname{Bel}(\mathcal{K})$, such that for any sentence $\alpha$ :

$$
\operatorname{Bel}(\mathcal{K}) *_{\mathcal{K}} \alpha=\operatorname{Bel}(\mathcal{K} * \alpha)
$$

Obviously, an iterated revision operator satisfies Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ iff the induced (local) revision operator for any belief state satisfies Postulates $\left(K^{*} 1\right)-\left(K^{*} 8\right)$.

While minimizing the change of the belief set, unfortunately, the (modified) AGM postulates impose almost no constraint on the change of the conditional beliefs. To see this, assume, without loss of generality, a belief state is of the form $\left\langle K, \leq_{K}\right\rangle$, where $K$ is a belief set and $\leq_{K}$ is an EE on $K$. Suppose $*$ is an iterated revision operator that satisfies the modified AGM postulates. When $\left\langle K, \leq_{K}\right\rangle$ is revised by $\alpha$, the revised belief set $K * \alpha$ is uniquely determined, however, $*$ can arbitrarily change $\leq_{K}$, as long as the revised $\mathrm{EE} \leq_{K * \alpha}$ satisfies Conditions (EE1)-(EE5) wrt. $K * \alpha$. It is clear that such excessive freedom on the change of the conditional beliefs is not desirable, and therefore the modified AGM postulates are too weak. But it is far from trivial how the (modified) AGM postulates should be extended to impose reasonable constraints on the change of conditional beliefs. In fact, this is the main problem I am going to tackle in the rest of this chapter.

### 3.2 The DP Theory

The problem of iterated belief revision has been studied by many researchers. In this section, I recall a general framework for iterated belief revision proposed by Darwiche and Pearl (DP, for short) [1997], which is the most commonly accepted in the literature.

In fact, the idea of regarding iterated revision operators as functions on belief states is originally put forward by Darwiche and Pearl. However they represent the propositional beliefs of a belief state by a single sentence (instead of a belief set), as the underlying language $\mathcal{L}$ is assumed to be a propositional logic generated from a finite set of propositions (atoms). For such belief states, Darwiche and Pearl also proposed a reformulation of the AGM postulates, which is essentially equivalent to the modified AGM postulates (of Section 3.1), provided $\mathcal{L}$ is finite. Base on that of [Katsuno and Mendelzon, 1991b], Darwiche and Pearl provided a representation theorem for their reformulated AGM postulates.

For the sake of generality, I present a generalization of the Darwiche and Pearl's result, which requires $\mathcal{L}$ to be neither finite nor propositional.

Definition 3.1. Given a belief state $\mathcal{K}$, a faithful ranking on $\mathcal{K}$ is a total pre-order $\preceq_{\mathcal{K}}$ on the possible worlds $\Theta_{\mathcal{L}}$, s.t., for any possible worlds $W_{1}, W_{2}$ :

1. If $W_{1}, W_{2} \models \mathcal{K}$ then $W_{1}=\mathcal{K} W_{2}$
2. If $W_{1} \models \mathcal{K}$ and $W_{2} \not \vDash \mathcal{K}$, then $W_{1} \prec_{\mathcal{K}} W_{2}$
3. For any consistent sentence $\alpha$, there exists a set (denote by $\min \left([\alpha], \preceq_{\mathcal{K}}\right)$ ) of the minimal elements of $[\alpha]$ wrt. $\preceq_{\mathcal{K}}$.
where $W \models \mathcal{K}$ abbreviates $W \models \operatorname{Bel}(\mathcal{K})$. A faithful assignment $h$ is a function that maps a belief state $\mathcal{K}$ to a faithful ranking $\preceq_{\mathcal{K}}^{h}$ on $\mathcal{K}$.

The intuitive meaning of $W_{1} \preceq_{\mathcal{K}} W_{2}$ is that $W_{1}$ is at least as plausible as $W_{2}$. It is also worth to mention that, for an infinite $\mathcal{L}$, the last condition of Definition 3.1 is not redundant.

Note that in [Darwiche and Pearl, 1997] a possible world is represented by a propositional interpretation. Since there is a one-to-one correspondence between interpretations and maximal consistent sets in a propositional logic, my definition indeed generalizes that of [Darwiche and Pearl, 1997]. See [Peppas and Williams, 1995] for another generalization of

Darwiche and Pearl's, where a so-called nice order is defined as a total pre-order on (modeltheoretic) first order interpretations. I can argue that my proposal is more general, in the sense it does not refer to the model-theoretic definition of first order interpretations.

A faithful ranking $\preceq_{\mathcal{K}}$ essentially represents a SOS centered on $[\mathcal{K}]$. It does not come as a surprise that the following representation theorem can be directly derived from the result of [Grove, 1988].

Theorem 3.2. Suppose $*$ is an iterated revision operator. Then $*$ satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ iff there exists a faithful assignment h, s.t., for any belief state $\mathcal{K}$ and any sentence $\alpha$ :

$$
\operatorname{Bel}(\mathcal{K} * \alpha)= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \operatorname{Th}\left(\min \left([\alpha], \preceq_{\mathcal{K}}^{h}\right)\right) & \text { otherwise }\end{cases}
$$

The faithful assignment $h$ (described in Theorem 3.2) is called a faithful assignment corresponding to $*$. Moreover, $\preceq_{\mathcal{K}}^{h}$ is called a faithful ranking on $\mathcal{K}$ corresponding to $*$. In situations where $h$ is not relevant, $\preceq_{\mathcal{K}}^{h}$ is simply written as $\preceq_{\mathcal{K}}$.

In general there could be more than one faithful assignment corresponding to an iterated revision operator $*$; however, if the underlying language $\mathcal{L}$ is finite (modulo logical equivalence), then it must be unique and is called the faithful assignment corresponding to $*$.

### 3.2.1 The DP Postulates

Darwiche and Pearl first show by examples that the (modified) AGM postulates permit improper response to the sequence of new information due to the excessive freedom they permit on the change of the conditional beliefs. To save the space, I only present one of such examples. 2

Example 7. [Darwiche and Pearl, 1997] We are introduced to a lady $X$ who sounds smart and looks rich, so we believe that $X$ is smart and $X$ is rich. Since we profess of no prejudice, we also maintain that $X$ is smart even if found to be poor and conversely, $X$ is rich even if found to be not smart. Now, we obtain some evidence that $X$ is in fact not smart, we remain of course convinced that $X$ is rich. Still, it would be strange for use to say, "if the evidence turns out false, and $X$ turns out smart after all, we would no longer believe that $X$ is rich. If we currently believe $X$ is smart and rich, then evidence first refuting then supporting that $X$ is smart should not in any way change our opinion about $X$ being rich.

[^17]The following scenario shows that the modified AGM postulates do permit to change our beliefs in a strange manner. Suppose $s$ and $r$ represent respectively $X$ is smart and $X$ is rich, and $\mathcal{L}$ is the propositional logic generated from $s$ and $r$. Suppose our initial belief state $\mathcal{K}$, and the revised belief state $\mathcal{K} * \neg s$ after we learned that $X$ is not smart are as described in Table 3.1 (which is allowed by the modified AGM postulates). It is easy to see that, according to our initially belief state, we indeed maintain that $X$ is smart even if found to be poor and $X$ is rich even if found to be not smart, since $\min ([\neg r], \preceq \mathcal{K})=\left\{W_{2}\right\}$ and $\min \left([\neg s], \preceq_{\mathcal{K}}=\left\{W_{3}\right\}\right)$. But if $\mathcal{K} * \neg s$ is subsequently revised by $s$, we will surprisingly lose our belief in $X$ is rich, since $\left\{W_{2}\right\}=\min \left([s], \preceq_{\mathcal{K} * \neg s}\right)$.

| possible worlds | $\preceq_{\mathcal{K}}$ | $\preceq_{\mathcal{K} * \neg s}$ |
| :--- | :---: | :---: |
| $W_{1}=\operatorname{Cn}(\{s, r\})$ | 0 | 2 |
| $W_{2}=C n(\{s, \neg r\})$ | 1 | 1 |
| $W_{3}=C n(\{\neg s, r\})$ | 1 | 0 |
| $W_{4}=C n(\{\neg s, \neg r\})$ | 2 | 1 |

Table 3.1: An example of undesirable revision

Motivated by their insufficiency, as supplements to the (modified) AGM postulates, Darwiche and Pearl have proposed four postulates for iterated belief revision. The underlying principle of the DP populates is quite similar to the principle of minimal change: An iterated revision operator should retain not only the propositional beliefs but also the conditional beliefs as much as possible. As before, the original DP postulates are also tailored for finite languages. Here, I present a generalization of the DP postulates [Darwiche and Pearl, 1997].

$$
\text { (DP1) If } \beta \vdash \alpha \text {, then }(\mathcal{K} * \alpha) * \beta \equiv \mathcal{K} * \beta \text {. }
$$

When two pieces of information arrive in tandem such that the second is more specific, then the first is redundant; that is the second information alone would yield the same belief set.

$$
\text { (DP2) } \quad \text { If } \beta \vdash \neg \alpha \text {, then }(\mathcal{K} * \alpha) * \beta \equiv \mathcal{K} * \beta \text {. }
$$

When two pieces of contradictory information arrive, the last one prevails; that is the second information alone would yield the same propositional beliefs.
(DP3) If $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$, then $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$.

An information $\alpha$ should be retained after accommodating a more recent information $\beta$ that implies $\alpha$ given the current beliefs.
(DP4) If $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \beta)$, then $\neg \alpha \notin \operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$.

An information $\alpha$ can not contribute to its own demise. If $\alpha$ does not contradict $\beta$ given the current beliefs, then the revision of the current beliefs by $\alpha$ should not make $\alpha$ contradictory with $\beta$.

To see the power of the DP postulates, let us recall Example 7: Suppose now the iterated revision operator $*$ satisfies (DP2). Then $(\mathcal{K} * \neg s) * s \equiv \mathcal{K} * s$, due to $s \vdash \neg \neg s$. Since $\operatorname{Bel}(\mathcal{K})=$ $C n(\{s, r\})$, it follows from $(\mathcal{K} * 4)$ that $r \in \mathcal{K} * s$. Therefore it turns out $r \in(\mathcal{K} * \neg s) * s$, in other words, we keep believing $X$ is rich after first learn that she is not smart, then the opposite. Hence Postulate (DP2) guarantees that we will change our beliefs as expected.

To provide formal justifications, Darwiche and Pearl have given a representation theorem for Postulates (DP1)-(DP4). Their result however again depends on the assumption that $\mathcal{L}$ is a finitary propositional logic.

Theorem 3.3. [Darwiche and Pearl, 1997] Suppose $\mathcal{L}$ is a finitary propositional logic. Let * be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulates (DP1)-(DP4) iff its corresponding faithful assignment satisfies the following conditions:
(DPR1) If $W_{1}, W_{2} \models \alpha$, then $W_{1} \preceq \mathcal{K} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.
(DPR2) If $W_{1}, W_{2} \not \vDash \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.
(DPR3) If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \prec_{\mathcal{K}} W_{2}$ implies $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$.
(DPR4) If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ implies $W_{1} \preceq \mathfrak{K}_{* \alpha} W_{2}$.

This theorem gives an elegant characterization of the seemingly natural constraints that the DP postulates impose on the change of the conditional beliefs: When $\mathcal{K}$ is revised by $\alpha$, Conditions (DPR1) and (DPR2) require not to change the relative plausible ordering of any two $\alpha$-worlds ( $\neg \alpha$-worlds, respectively); Conditions (DPR3) and (DPR4) require that if an $\alpha$-world $W_{1}$ is (strictly) more plausible than a $\neg \alpha$-world $W_{2}$, then $W_{2}$ continues to be (strictly) more plausible than $W_{2}$.

### 3.2.2 Two Radical Revision Operators

The modified AGM postulates and the DP postulates together do not uniquely determine an iterated revision operator. In the rest of this thesis, I will call an iterated revision operator a DP revision operator, if it satisfies the modified AGM postulates as well as the DP postulates. In this section, I present two interesting instances of DP revision operators, which are somehow dual to each other.
[Boutilier, 1993] has proposed a specific revision operator (known as natural revision) which satisfies the modified AGM postulates and the following condition:

$$
\text { (CB) If } \neg \beta \in \operatorname{Bel}(\mathcal{K} * \alpha) \text {, then }(\mathcal{K} * \alpha) * \beta \equiv \mathcal{K} * \beta \text {. }
$$

Along the line of Theorem 3.3, [Darwiche and Pearl, 1997] have given a semantical characterization of Postulate (CB):

$$
\text { (CBR) If } W_{1}, W_{2} \not \models \mathcal{K} * \alpha \text { then } W_{1} \preceq_{\mathcal{K}} W_{2} \text { iff } W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}
$$

Note that now $W_{1}, W_{2} \not \models \mathcal{K} * \alpha$ is the only case, where the relative ordering of $W_{1}, W_{2}$ in $\mathcal{K} * \alpha$ is not determined, since $\preceq_{\mathcal{K} * \alpha}$ must satisfy conditions of Definition 3.1. Therefore, Condition (CBR) imposes absolute minimization on the change of a faithful ranking permitted by the modified AGM postulates.

It is easy to see that the DP postulates are a weakening of Postulate (CB), in the sense Postulate (CB) implies all of the DP postulates but not vice versa [Darwiche and Pearl, 1994; Zhang, 2004].

At first glance, it seems that Condition (CBR) complies with the principle of minimal change. However, the following example shows that Postulate (CB) enforces very radical behavior.

Example 8. Suppose $\mathcal{K}$ is an initial belief state. Assume $\left\langle\alpha_{1} ; \cdots ; \alpha_{n}\right\rangle$ is a sequence of sentences, s.t., $\alpha_{1} \wedge \cdots \wedge \alpha_{n}$ is consistent, which represents the sequence of the observations the agent have made along its way. The current belief state is represented by $\left(\left(\mathcal{K} * \alpha_{1}\right) \cdots\right) * \alpha_{n}$. Suppose now the agent observes $\neg \alpha_{1}$. It is not difficult to see that by applying Postulate (CB) repeatedly, we have $\left(\left(\left(\mathcal{K} * \alpha_{1}\right) \cdots\right) * \alpha_{n}\right) * \neg \alpha_{1} \equiv \mathcal{K} * \neg \alpha_{1}$, provided $*$ satisfies Postulates $(\mathcal{K} * 1)$ - $(\mathcal{K} * 8)$. In other words, the agent cancels all evidences $\alpha_{1}, \cdots, \alpha_{n}$, simply because $\neg \alpha_{1}$ is observed.

The above example shows that Postulate (CB) is too strict a criterion for iterated belief revision operators and the most conservative way of changing the conditional beliefs is not desirable in general.

The problem of natural revision can be informally explained as follows. On the one hand natural revision respects the principle of primacy of the new information, so that the information is accepted; on the other hand it assigns the lowest plausibility to the new information, so that it will be easily canceled by a subsequent revision.

While natural revision is the most conservative of all possible DP (more precisely, AGM) revision operators, another revision operator, called lexicographic revision (with "naked evidence") [Nayak, 1994b], sits exactly on the opposite side of the spectrum. Lexicographic revision satisfies, in addition to Postulate (DP1) and (DP2), another so-called postulate of Recalcitrance:

$$
\text { (Rec) } \quad \text { If } \beta \nvdash \neg \alpha \text {, then } \alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \beta) \text {. }
$$

Semantically, Postulate (Rec) corresponds to the following condition [Nayak et al., 2003]:

$$
\text { (RecR) } \quad \text { If } W_{1} \models \alpha \text { and } W_{2} \models \neg \alpha \text {, then } W_{1} \prec \mathcal{K} * \alpha ~ W_{2} .
$$

According to (RecR), all possible worlds satisfying the new information become more reliable than those violating the new information, hence ( Rec ) is also said to impose the principle of strong primacy of update [Konieczny and Pérez, 2000], which is arguably only suitable when the agent has full confidence in the new information. Based on Conditions (DPR1), (DPR2) and (RecR), it it easy to see that lexicographic revision is the least conservative of all possible DP revision operators, effecting most changes in the faithful ranking permitted by the AGM and DP postulates [Booth et al., 2005].

It is easy to easy that, in the presence of Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$, we can derive $(\mathrm{DP} 3)$ and (DP4) from (Rec) [Nayak et al., 2003].

Unfortunately, Postulate (Rec) also enforces very radical behavior. According to Postulate (Rec), as long as $\beta$ does not logically deduce $\neg \alpha$, the sentence $\beta \rightarrow \neg \alpha$ should be canceled after a successive revision by $\alpha$ followed by $\beta$, no matter how strong the initial belief in $\beta \rightarrow \neg \alpha$. A simple example shows that this behavior may not be reasonable:

Example 9. All her childhood, Alice was taught by her parents that a person who has told a lie is not a good person. So Alice believed, initially, that if Bob has told a lie then he is not a good
person. After her first date with Bob, she began to believe that he is a good guy. Then a reliable friend of Alice warns her that Bob is in fact a liar, and Alice chooses to believe her. Now, should Alice still believe that Bob is a good guy?

Suppose $l$ and $g$ represent, respectively, Bob is a liar and Bob is a good person. Since $l \nvdash \neg g$, it follows from Postulate (Rec) that $g \in \operatorname{Bel}((\mathcal{K} * g) * l)$, in other words, Alice should not challenge Bob's morality and still believe he is good, and hence to disbelieve what her parents taught her. But by common-sense it is at least as reasonable to give up the belief that Bob is good. This shows that Postulate (Rec) is too strict a criterion for iterated belief revision operators. As Postulate (Rec) has been shown corresponding to the least conservatism in the change of conditional beliefs, the above argument is also a criticism of the least conservatism. In [Zhang, 2004], it has also been argued that Postulate (Rec) is too radical because only those revision operators are admissible which assign the highest plausibility to the new information.

Based on above arguments, I depict in Figure 3.1 a map of iterated revision operators. that natural revision is also the most conservative AGM revision operator. It is however of no practical interest to investigate the least conservative AGM operators because of the total freedom on the change of conditional beliefs allowed by the modified AGM postulates.


Figure 3.1: A map of iterated revision operators.

### 3.3 The Problem of Implicit Dependence

Although the DP theory seems a quite acceptable extension to the AGM theory for iterated belief revision, it is not without problems. Specifically, the DP postulates are consistent with (CB),
hence they do not block counter-examples against natural revision, like the following one, proposed by Darwiche and Pearl themselves:

Example 10. [Darwiche and Pearl, 1997] We encounter a strange new animal and it appears to be a bird, so we believe the animal is a bird. As it comes closer to our hiding place, we see clearly that the animal is red, so we believe that it is a red bird. To remove further doubts about the animal birdhood, we call in a bird expert who takes it for examination and concludes that it is not really a bird but some sort of mammal. The question now is whether we should still believe that the animal is red.

As argued in [Darwiche and Pearl, 1997], we have every reason to keep our belief that the animal is red, since birdhood and color are not correlated. However, natural revision enforces us to give up the belief of the animal's color. Suppose $\operatorname{Bel}(\mathcal{K})=C n(\{b i r d\})$. According to Postulate (CB), since $\mathcal{K} *$ red $\vdash \neg(\neg$ bird $)$ it follows that $(\mathcal{K} *$ red $) * \neg$ bird $\equiv \mathcal{K} * \neg$ bird.

In being compatible with (CB), the DP postulates are not strong enough to guarantee that the belief of the animal's color is retained. This can be intuitively explained as follows: After observing the animal's color, we are actually acquiring a new conditional belief, namely, that the animal is red even if it were not a bird i.e., $\neg$ bird $\gg$ red. However, the DP postulates do not enforce the acquisition of conditional beliefs.

In the sequel, I first give a formal analysis of this weakness of the DP postulates, and then I present an additional postulate by which this problem is overcome (at least partially).

As pointed out in Gärdenfors' preservation criterion, the dependences between the beliefs play an important role in the process of belief revision [Gärdenfors, 1990]:

If a belief state is revised by a sentence $\alpha$, then all sentences that are independent of the validity of $\alpha$ should be retained in the revised state of belief.

The principle of minimal change that have been used in the AGM theory is based on almost exclusively logical considerations. However, the dependence relations are extra-logical preference factors of the belief states. It is therefore necessary to formally study the problem of dependence.

To begin with, I define the notion of dependence between sentences wrt. a belief state. Formally, ${ }^{3}$

[^18]Definition 3.4. Suppose $\mathcal{K}$ is a belief state. Let $\alpha, \beta$ two sentences. Then $\beta$ is called to depend on $\alpha$ in $\mathcal{K}$ iff $\beta \in \operatorname{Bel}(\mathcal{K})$ and $\beta \notin \operatorname{Bel}(\mathcal{K} * \neg \alpha)$. Two sentences $\alpha, \beta$ are said to be dependent in $\mathcal{K}$ if either $\alpha$ depends on $\beta$ or $\beta$ depends on $\alpha$ in $\mathcal{K}$.

I will first show that there is a so-called problem of implicit dependence in the AGM theory. Then I argue that problem of implicit dependence is overlooked in the DP theory. Without loss of generality, assume that a belief state $\mathcal{K}$ is of the form $\left\langle K, \leq_{K}\right\rangle$, such that $\leq_{K}$ is an EE on $K$. For easy reference, I present again Condition (C*) used in the constructive model based on EE.
(C*) $\beta \in K * \alpha$ iff either $\vdash \neg \alpha$ or $\neg \alpha<_{K}(\neg \alpha \vee \beta)$
Consider, now, a (non-tautological) new evidence $\alpha$. Whenever $\beta \in K$, Condition ( $\mathrm{C}^{*}$ ) implies that if $\alpha \not{ }_{K} \alpha \vee \beta$, then $\beta$ is (implicitly) dependent on $\alpha$ in $\mathcal{K}$. Informally, we can consider that sentences with lower plausibility have the tendency to depend on sentences with higher plausibility. This kind of dependency could be problematic. In particular, it is possible that two initially independent sentences become, undesirably, dependent after a revision step. In Example 10, for instance, red becomes dependent on bird after revising by red, since red is assigned by natural revision the lowest plausibility.

The general problem of natural revision is that it assigns the lowest plausibility to the new information without asserting conditional beliefs for independence. Thus the new information depends on all other beliefs which survive the revision process. This explains why severe revision always cancels all previous evidences. Of course, this is not merely a problem of natural revision: In the revised belief state $\left\langle K * \alpha, \leq_{K * \alpha}\right\rangle$, regardless of the plausibility of the new information $\alpha$, a belief $\beta$ (logically unrelated to $\alpha$ ) with a lower plausibility will depend on $\alpha$, unless the revision operator explicitly makes the condition $\alpha<_{K * \alpha} \alpha \vee \beta$ true. If initially $\alpha \not_{K} \alpha \vee \beta$, the validation of $\alpha<_{K * \alpha} \alpha \vee \beta$ means the revision operator should assert explicitly the conditional belief $\neg \alpha \gg \beta$. Symmetrically, a rational revision operator also should take care of the implicit dependence of the new information on other beliefs with higher plausibility.

The analysis in the previous section shows that in order to overcome the problem of implicit dependence, the revision operator must explicitly assert some conditional beliefs. It is easy to see that the DP postulates only require the preservation of conditional beliefs when a belief state $\mathcal{K}$ is revised with $\alpha$ : Postulates (DP1) and (DP2) neither require to add nor to remove certain conditional beliefs (namely, those conditioned on $\beta$ ) in case $\beta \vdash \alpha$ or $\beta \vdash \neg \alpha$; Postulate (DP3) requires to retain the conditional belief $\beta \gg \alpha$; finally, Postulate (DP4) requires not to obtain the
new conditional belief $\beta \gg \neg \alpha$. Since none of the DP postulates requires to make independence assumptions, a new postulates is necessary to avoid undesirable dependencies.

### 3.3.1 Postulate of Independence

As already mentioned, the revision process may introduce undesirable dependencies in both directions. That is, it could be that the new information becomes dependent on existing beliefs, or that it is the other way around. Prior to stating the new postulate, I show that the DP postulates impose some constraints on the retention of the independence information in one direction. In the presence of the AGM postulates, Postulate (DP2) implies the following:

$$
\text { (WDP2) } \quad \text { If } \beta \in \operatorname{Bel}(\mathcal{K} * \neg \alpha) \text {, then } \beta \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \alpha)
$$

Suppose $\beta \in \operatorname{Bel}(\mathcal{K})$, Postulate (WDP) guarantees that if $\beta$ is not dependent on the new information $\alpha$ in $\mathcal{K}$, then it also does not depend on $\alpha$ in $\mathcal{K} * \alpha$.

Observation 3.5. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If $*$ satisfies (DP2), then it also satisfies (WDP2).

In order to ensure the explicit assertion of independence information in the other direction, I propose the following postulate of Independence ( weak version) dual to (WDP2):
(WInd) If $\alpha \in \operatorname{Bel}(\mathcal{K} * \neg \beta)$ then $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \beta)$
Suppose $\alpha \in \operatorname{Bel}(\mathcal{K})$, Postulate (WInd) guarantees that if the new information $\alpha$ does not depend on $\beta$ in $\mathcal{K}$, then it also does not depend on $\beta$ in $\mathcal{K} * \alpha$.

As it is too much to require that the new information $\alpha$ is already believed (i.e, $\alpha \in \operatorname{Bel}(\mathcal{K})$ ), I propose the following postulate of Independence (strong version):
(Ind) If $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \neg \beta)$ then $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \beta)$

Postulate (Ind) says that if the conditional belief $\neg \beta \gg \neg \alpha$ is not held in $\mathcal{K}$, then $\alpha$ does not depend on $\beta$ in $\mathcal{K} * \alpha$.

It is not difficult to see that (Ind) is a strengthening of (WInd), in the presence of the modified AGM postulates:

Observation 3.6. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If * satisfies (Ind), then it also satisfies (WInd).

Postulate (Ind) is sufficient to overcome the problem of implicit dependence, as can be shown by reconsidering Example 10 (in which $\operatorname{Bel}(\mathcal{K})=C n(\{b i r d\})$ ). According to (Ind), $(\mathcal{K} *$ red $) * \neg$ bird $\vdash$ red, given that $\mathcal{K} * \neg$ bird $\nvdash \neg$ red. This shows that the new postulate blocks unreasonable behavior which are admitted by the DP postulates.

### 3.3.2 Representation Theorems

In order to formally justify the new postulate, I will provide a representation theorem along the line of Theorem 3.3.

Theorem 3.7. Suppose $\mathcal{L}$ is a finitary propositional logic. Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulate (Ind) iff its corresponding faithful assignment satisfies the following condition:

$$
\text { (IndR) } \quad \text { If } W_{1} \models \alpha \text { and } W_{2} \models \neg \alpha \text {, then } W_{1} \preceq_{\mathcal{K}} W_{2} \text { implies } W_{1} \prec_{\mathcal{K} * \alpha} W_{2} \text {. }
$$

Theorem 3.7 shows that Postulate (Ind) is quite natural and not overly constrained: Suppose $\mathcal{K}$ is revised by $\alpha$, Condition (IndR) requires a world $W_{1}$ conforming the new information $\alpha$ to become more plausible than a world $W_{2}$ violating $\alpha$, if $W_{1}$ was at least as plausible as $W_{2}$.

It is easy to see that Postulate (Ind) implies both (DP3) and (DP4), in the presence of the modified AGM postulates:

Observation 3.8. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If $*$ satisfies (Ind), then it also satisfies (DP3) and (DP4).

As we will see in Section 3.4.2 that Postulate (Ind) is also consistent with the modified AGM, (DP1) and (DP2).

Base on above arguments, I suggest to use the modified AGM postulates along with Postulates (DP1), (DP2), and (Ind) to govern iterated belief revision. It is worth to mention that Postulate (Ind) has later, and independently, been proposed in [Booth et al., 2005], where an iterated revision operator is called admissible iff it satisfies all above-mentioned postulates. Booth et al., also proposed an interesting instance of admissible revision operators, namely, the so-called restrained revision, which additionally satisfies the following postulate:
(D) If $\neg \alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$ and $\neg \beta \in \operatorname{Bel}(\mathcal{K} * \alpha)$, then $\neg \alpha \in(\mathcal{K} * \alpha) * \beta$.

In [Booth et al., 2005], two sentences $\alpha, \beta$ are also said to be counteracting if they satisfy the premise of Postulate (D). It has also been shown that Postulate (D) is semantically characterized by the following condition:
(DR) If $W_{1} \not \models \alpha, W_{2} \models \alpha$ and $W_{2} \not \models \mathcal{K} * \alpha$, then $W_{1} \prec \mathcal{K} W_{2}$ implies $W_{1} \prec \mathcal{K}_{*_{\alpha}} W_{2}$

It is easy to see that (D) is a weakening of (CB). However, Postulate (D) is criticized in [Darwiche and Pearl, 1997] by the following example.

Example 11. [Darwiche and Pearl, 1997] We believe exactly one of John and Mary committed a murder. Now we get persuasive evidence indicating that John is the murderer. This is followed by persuasive information indicating that Mary is the Murderer. According to (D), we are forced to conclude that Mary, but not John, was involved in the murder.

The following theorem shows that restrained revision is the most conservative admissible revision operator, effecting least changes in the faithful ranking permitted by reformulated AGM postulates, (DP1), (DP2) and (Ind).

Theorem 3.9. [Booth et al., 2005] Suppose $\mathcal{L}$ is a finitary propositional logic. Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulates (DP1), (DP2), (Ind) and (D) iff its corresponding faithful assignment satisfy the following condition:
(R) If $W_{1}, W_{2} \not \vDash \mathcal{K} * \alpha$, then $W_{1} \preceq_{K * \alpha} W_{2}$ precisely when

$$
\left\{\begin{array}{l}
W_{1} \prec \mathcal{K} W_{2} \text { or }, \\
W_{1} \preceq \mathcal{K} W_{2} \text { and } W_{1} \models \alpha, W_{2} \not \models \alpha
\end{array}\right.
$$

Suppose $\mathcal{K}$ is revised by $\alpha$. Condition ( R ) says the relative ordering of the possible worlds that are not models of $\mathcal{K} * \alpha$ does not change, except for $\alpha$-worlds and $\neg \alpha$-worlds on the same plausibility level: They are split into two levels with $\alpha$-worlds being (strictly) more plausible than $\neg \alpha$-worlds.

A refined map of iterated revision operators can be found in Figure 3.2. Based above arguments, all rational iterated revision operators should be located in the innermost ellipse.


Figure 3.2: A refined map of iterated revision operators.

### 3.4 A General Framework for Iterated Revision

In the representation theorem of the DP postulates (i.e., Theorem 3.3), as well as that of Postulate (Ind) (i.e., Theorem 3.7), the underlying language $\mathcal{L}$ is assumed to be a finitary propositional logic. An interesting question is can we extend those results to a general logic language, which needs not to be finitary nor propositional?

### 3.4.1 Representation Theorems for Infinite Languages

After a careful analysis of the proof in [Darwiche and Pearl, 1997], I found that the restriction of $\mathcal{L}$ being propositional can be lifted without any problem. Moreover, the finiteness of $\mathcal{L}$ is exploited only in one direction of the proof. Therefore, the following sufficient condition for the DP postulates still holds even if $\mathcal{L}$ is neither finitary nor propositional.

Theorem 3.10. Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulates (DP1)-(DP4) if it is induced from a faithful assignment that satisfies the following conditions:
(DPR1)
If $W_{1}, W_{2} \models \alpha$, then $W_{1} \preceq \mathcal{K} W_{2}$ iff $W_{1} \preceq \mathcal{K} * \alpha W_{2}$.
(DPR2)
If $W_{1}, W_{2} \not \vDash \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.
(DPR3) If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \prec_{\mathcal{K}} W_{2}$ implies $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$.
(DPR4) If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ implies $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.

Similarly, I am able to obtain the following sufficient condition for Postulates (Ind) for general languages.

Theorem 3.11. Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulate (Ind) if it is induced from a faithful assignment that satisfies the following condition:

$$
\text { (IndR) } \quad \text { If } W_{1} \models \alpha \text { and } W_{2} \models \neg \alpha \text {, then } W_{1} \preceq \mathcal{K} W_{2} \text { implies } W_{1} \prec_{\mathcal{K} * \alpha} W_{2} .
$$

Unfortunately, the other direction of Theorem 3.11 (as well as Theorem 3.11) does not hold in general. The main reason is that, for infinite languages, there is no one-to-one correspondence between an iterated revision operator and a faithful assignment: Let $\mathcal{K}_{1}$ be a belief state and $\preceq \mathcal{K}_{1}$ a faithful ranking on $\mathcal{K}_{1}$. Suppose $W_{1} \prec \mathcal{K}_{1} W_{2}$ and there does not exist consistent sentence $\alpha$ such that $W_{1}, W_{2} \models \alpha$ and $W_{1} \in \min \left([\alpha], \preceq \mathcal{K}_{1}\right)$. It is not difficult to see that by swapping the relative order of $W_{1}$ and $W_{2}$ (the rest of $\preceq \mathcal{K}_{1}$ remains unchanged), we obtain another faithful ranking $\preceq_{\mathcal{K}_{1}}^{\prime}$, such that for any consistent sentence $\alpha, \min \left([\alpha], \preceq \mathcal{K}_{1}\right)=\min \left([\alpha], \preceq^{\prime} \mathcal{K}_{1}\right)$. In other words, the relative ordering of $W_{1}$ and $W_{2}$ is not relevant wrt. $\preceq \mathcal{K}_{1}$. Suppose now $h$ is a faithful assignment with $\preceq_{\mathcal{K}_{1}}^{h}=\preceq_{\mathcal{K}_{1}}$, and $*$ is the iterated revision induced from $h$. Obviously, we can construct another faithful assignment $h^{\prime}$ which also corresponds to $*$, such that, if $\mathcal{K}=\mathcal{K}_{1}$ then $h^{\prime}(\mathcal{K})=\preceq^{\prime} \mathcal{K}_{1}$; otherwise $h^{\prime}(\mathcal{K})=h(\mathcal{K})$.

The following example shows that there indeed exists a faithful ranking in which some possible worlds are irrelevantly ordered.

Example 12. Suppose $\mathcal{L}$ contains countably infinite sentences $\left(\alpha_{i}\right)_{i \geq 0}$, such that $\alpha_{i} \vdash \neg \alpha_{j}$ iff $i>j$. Let $K=C n\left(\left\{\alpha_{0}\right\}\right)$. Then by defining $S_{i}=\left[\alpha_{i}\right]$ for any consistent sentence $\alpha_{i}$, we can construct a SOS $\mathfrak{S}$ centered on $[K]$ s.t., the minimal sphere intersects $\left[\alpha_{i}\right]$ is exactly $S_{i}$. Since $\mathcal{L}$ is infinite, each sphere $S_{i}$ infinitely many elements, hence contains at least two elements (this is not the case if $\mathcal{L}$ is finite). Take any two possible $W_{i}, W_{j}$ such that $W_{i} \in S_{i}$, $W_{j} \in S_{j}$ and $0<i<j$. Removing $W_{i}, W_{j}$ from $S_{i}, S_{j}$ and adding two spheres $\Theta_{\mathcal{L}} \backslash\left\{W_{i}\right\}$ and $\Theta_{\mathcal{L}} \backslash\left\{W_{i}, W_{j}\right\}$, we can construct a modified $\operatorname{SOS} \mathfrak{S}^{\prime}$. It is not difficult to see that $W_{i}, W_{j}$ is irrelevant ordered wrt. the faithful ranking corresponding to $\mathfrak{S}^{\prime}$.

As we have seen changing the relative plausibility order of two irrelevant ordered possible worlds will not affect the way an iterated revision operator revises the propositional beliefs in the subsequent revision. Therefore, we should not expect the DP postulates and Postulate (Ind)
restrict the changes of irrelevant ordering. An interesting problem is what constraints these postulates impose on the change of relevant ordering? In order to do a formal analysis of the problem, we first need to formally define the notion of relevant ordering:

Definition 3.12. Let $\preceq_{\mathcal{K}}$ be a faithful ranking and $W_{1}, W_{2}$ two possible worlds. Then $W_{1}$ is said to be relevantly at least as plausible as $W_{2}$ (denoted by $W_{1} \preceq_{\mathcal{K}} W_{2}$ ) wrt. $\preceq_{\mathcal{K}}$ iff there is a sentence $\gamma$ such that $W_{1}, W_{2} \models \gamma$ and $W_{1} \in \min \left([\gamma], \preceq_{\mathcal{K}}\right)$. $W_{1}$ is called relevantly more plausible than $W_{2}\left(\right.$ denoted by $\left.W_{1} \prec_{\mathcal{K}}^{\triangleleft} W_{2}\right)$ iff $W_{1} \preceq_{\mathcal{K}}^{\natural} W_{2}$ and $W_{1} \prec_{\mathcal{K}} W_{2}$.

Two possible worlds $W_{1}, W_{2}$ are called relevantly ordered in $\preceq_{\mathcal{K}}$ iff either $W_{1} \preceq_{\mathcal{K}} W_{2}$ or $W_{2} \preceq_{\mathcal{K}}^{\triangleleft} W_{1}$. As we have seen, in general, not all pairs of possible worlds are relevantly ordered wrt. a given faithful ranking.

The following results given necessary and sufficient conditions of (DP1), (DP2) and (Ind) in terms of the constraints they impose on the changes of relevant plausibility ordering.

Theorem 3.13. Suppose that an iterated revision operator satisfies Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. The operator satisfies Postulates (DP1),(DP2) and (Ind) iff the operator and its corresponding faithful assignment satisfy:

$$
\begin{aligned}
& \text { ( } \text { DPRI }^{\triangleleft} \text { ) If } W_{1}, W_{2} \models \alpha \text {, then } W_{1} \preceq_{\mathcal{K}}^{\wedge} W_{2} \text { iff } W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2} \text {. } \\
& \text { (DPR2 }{ }^{\triangleleft} \text { ) If } W_{1}, W_{2} \not \vDash \alpha \text {, then } W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2} \text { iff } W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2} \text {. } \\
& \left(\text { Ind } R^{\triangleleft}\right) \quad \text { If } W_{1} \models \alpha \text { and } W_{2} \not \models \alpha \text {, then } W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2} \text { implies either } W_{1}, W_{2} \\
& \text { are not relevantly ordered wrt. } \preceq_{\mathcal{K} * \alpha} \text { or } W_{1} \prec_{\mathcal{K} * \alpha}^{\triangleleft} W_{2} \text {. }
\end{aligned}
$$

Theorem 3.13 shows Postulates (DP1), (DP2) and (Ind) also make a lot of sense in the setting of infinite languages. Suppose $\mathcal{K}$ is revised by $\alpha$. Conditions (DPR1 ${ }^{\triangleleft}$ ) and (DPR2 ${ }^{\triangleleft}$ ) say if two $\alpha$-worlds ( $\neg \alpha$-worlds) are relevantly ordered after revision iff they were relevantly ordered, and their relative plausible ordering should not change; Condition (IndR ${ }^{\triangleleft}$ ) says if a $\alpha$-world $W_{1}$ was relevantly at least plausible as a $\neg \alpha$-world $W_{2}$ then $W_{1}$ is relevantly (strictly) more plausible than $W_{1}$ after revision, unless $W_{1}, W_{2}$ become not relevantly ordered.

It is in principle possible that a belief state contains some irrelevant extra-logical preference information. But in practice, we can hardly image that we will make use of such kind of belief states. An interesting question is how we could guarantee that a belief state will not contain irrelevant extra-logical preference information? The following result shows that any two possible worlds are relevantly ordered in a faithful ranking for finite languages:

Observation 3.14. Suppose $\mathcal{L}$ is a finite language. Let $\preceq_{\mathcal{K}}$ be a faithful ranking on some belief state $\mathcal{K}$. Then for any $W_{1}, W_{2} \in \Theta_{\mathcal{L}}, W_{1}$ and $W_{2}$ are relevantly ordered in $\preceq_{\mathcal{K}}$.

Given the above result, it is not difficult to see that Theorem 3.3 (the part regarding (DP1) and (DP2)) and Theorem 3.7 can be directly derived from Theorem 3.13.

It is also worth to mention that a finite underlying language is not necessary condition to have belief states without irrelevant extra-logical preference information. We will see in Chapter 4 how to define belief states without irrelevant extra-logic information for infinite languages.

### 3.4.2 An Iterated OCF Revision Operator

As limiting cases of admissible revision, i.e., restrained revision and lexicographic revision, are shown to be too radical. In this section, I present an admissible revision operator which lays between the two radical operators. The operator is based on Spohn's proposal of ordinal conditional functions (OCF, for short) [Spohn, 1988].

Originally, an ordinal conditional function (OCF) $k$ is a mapping from the set of possible worlds $\Theta_{\mathcal{L}}$ to the class of ordinals. Like in [Spohn, 1991], for mathematical simplicity, we consider the range of an OCF $k$ is $\mathbb{N}^{4} 4^{4}$ For any possible world $W, k(W)$ is called the rank of $W$. Intuitively, the rank of a world represents its degree of implausibility. The lower its rank, more plausible is the possible world.

An OCF $k$ is a general belief state from which we can induce a belief set and a faithful ranking $\preceq_{k}$ (according to ranks of possible worlds), where $\operatorname{Bel}(k)$ is the set of sentences which hold in all worlds of rank 0 :

$$
\begin{equation*}
\operatorname{Bel}(k)=\operatorname{Th}(\{W \mid k(W)=0\}) \tag{3.1}
\end{equation*}
$$

Note that, unlike in [Spohn, 1988], it is not required that the set of possible worlds with rank 0 be non-empty. Therefore, the approach presented here can also deal with inconsistent belief sets.

[^19]An OCF is extended to a ranking of sentences as follows:

$$
k(\beta)= \begin{cases}\infty & \text { if } \vdash \beta  \tag{3.2}\\ \min \{k(W) \mid W \models \neg \beta)\} & \text { otherwise }\end{cases}
$$

Put in words, the rank of a sentence is the lowest rank of a world in which the sentence does not hold $\sqrt{5}$ Hence, the higher the rank of a sentence, the firmer the belief in it, and the belief set consists of all sentences with rank greater than 0 . In fact, it is not hard to see that an OCF $k$ determines an EE as follows:

$$
\begin{equation*}
\alpha \leq_{k} \beta \text { iff } k(\alpha) \leq k(\beta) \tag{3.3}
\end{equation*}
$$

Observation 3.15. Given an $O C F k$, the binary relation $\leq_{k}$ defined by (3.3) satisfies (EE1)(EE5).

Now, I present a generalized revision operator (named reinforcement OCF revision) which also allows to assign different evidence degrees to new information. An OCF $k$ is revised according to new information $\alpha$ with evidence degree $m \in \mathbb{N}^{+}$as follows:

$$
\left(k_{\alpha, m}^{r, *}\right)(W)= \begin{cases}k(W)-k(\neg \alpha) & \text { if } W \models \alpha  \tag{3.4}\\ k(W)+m & \text { otherwise }\end{cases}
$$

Intuitively, reinforcement OCF revision can be depicted in Figure 3.3, where circles on the left (right) side of the vertical dotted line represent $\alpha$-worlds ( $\neg \alpha$-worlds respectively); the vertical coordinate of a possible world denotes its rank. Moreover, changes of ranks are denoted by the arrows: ranks of $\alpha$-worlds decreases $k(\neg \alpha)$, whereas ranks of $\neg \alpha$-worlds increases $m$.

To see an concrete example, let us recall Example 7. Assume now our initial belief state is represented by an OCF $k$ as shown in the second column of Table $3.2^{6}$ Since $k(\neg \neg s)=1$, according to (3.4) and we have learn that $X$ is not smart with evidence degree 2 . It is easy to see that $k(\neg \neg s)=k(s)=1$. Therefore, our revised belief state $k_{\neg s, 2}^{*}$ is the one shown in the last column. As $W_{1}$ is most plausible $s$-world in $k_{\neg s, 2}^{*}$, when $k_{\neg s, 2}^{*}$ is revised by $s$ we will still believe that $X$ is rich.

[^20]

Figure 3.3: Reinforcement OCF revision

| possible worlds | $k$ | $k_{\neg s, 2}^{*}$ |
| :--- | :---: | :---: |
| $W_{1}=C n(\{s, r\})$ | 0 | 2 |
| $W_{2}=C n(\{s, \neg r\})$ | 1 | 3 |
| $W_{3}=C n(\{\neg s, r\})$ | 1 | 0 |
| $W_{4}=C n(\{\neg s, \neg r\})$ | 2 | 1 |

Table 3.2: An example of reinforcement OCF revision.

## Formal Properties

It is easy to see that the value of the evidence degree will not affect the logical contents of the revised OCF.

Observation 3.16. Let $k$ be an $O C F$ and $\alpha$ a sentence, then for any $m_{1}, m_{2} \in \mathbb{N}^{+}$,

$$
\operatorname{Bel}\left(k_{\alpha, m_{1}}^{r, *}\right)=\operatorname{Bel}\left(k_{\alpha, m_{2}}^{r, *}\right)
$$

Assuming an arbitrary but fixed evidence degree for any new information, a standard iterated revision operator is obtained and the satisfiability of the modified AGM postulates along with Postulates (DP1), (DP2), and (Ind) is a direct consequence of Theorem 3.2, 3.10 and 3.11 .

Theorem 3.17. Assume an arbitrary but fixed evidence degree for any new information. Then reinforcement OCF revision satisfies all AGM postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ as well as (DP1), (DP2) and (Ind).

Incidentally, Theorem 3.17 shows that Postulate (Ind) is consistent with the modified AGM and DP postulates. It is also obvious that reinforcement OCF revision does not satisfy (CB), (D) or (Rec).

I can even prove a stronger result with varying evidence degrees. To show the satisfiability of the postulates in case of varying evidence degrees, we need the following result, which fully characterizes the change of belief degrees of non-tautological sentences.

Observation 3.18. Let $k$ be an $O C F$ and $\langle\alpha, m\rangle$ be any new information. Then for any nontautological sentence $\beta$,

$$
k_{\alpha, m}^{r, *}(\beta)= \begin{cases}k(\beta)+m & \text { if } \vdash \alpha \rightarrow \beta \\ k(\alpha \rightarrow \beta)-k(\neg \alpha) & \text { else if } k(\alpha \rightarrow \beta)=k(\beta) \\ \min (k(\alpha \rightarrow \beta)-k(\neg \alpha), k(\beta)+m) & \text { else }\end{cases}
$$

Since the belief degree of a tautological sentence is always $\infty$, the above result actually gives a full characterization of the change of beliefs degrees for all sentences.

As a direct consequence of Observation 3.18, it can be seen that the reinforcement OCF revision indeed has a reinforcement effect, in the sense that, the evidence degrees of the new information are accumulated.

Observation 3.19. Let $k$ be an arbitrary $O C F$ and $\alpha$ a new non-tautological sentence with evidence degree $m \in \mathbb{N}^{+}$. Then

$$
k_{\alpha, m}^{r, *}(\alpha)=k(\alpha)+m
$$

From a pragmatic point of view, this is a desirable property in particular for domains where several independent information sources provide new information. In this case, it is appropriate to sum up the evidence degrees of the same information from different sources.

Finally, with the help of Observation 3.18, I am able to prove that reinforcement base revision satisfies Postulates (DP1), (DP2) and (Ind), regardless of evidence degrees.

Theorem 3.20. For arbitrary $m_{1}, m_{2} \in \mathbb{N}^{+}$, reinforcement $O C F$ revision satisfies the following conditions:

$$
\begin{array}{ll}
\text { (EDP1) } & \text { If } \beta \vdash \alpha, \text { then }\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *} \equiv k_{\beta, m_{2}}^{r, *} . \\
\text { (EDP2) } & \text { If } \beta \vdash \neg \alpha \text {, then }\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *} \equiv k_{\beta, m_{2}}^{r, *} . \\
\text { (EInd) } & \text { If there exists } m \text { such that } k_{\neg \beta, m}^{r, *} \nvdash \neg \alpha, \text { then }\left(k_{\alpha, m_{1}}^{r, *}\right)_{\neg \beta, m_{2}}^{r, *} \vdash \alpha .
\end{array}
$$

It is worth mentioning that revisions based on OCFs are particularly suitable for implementations of belief revision. For instance, in Chapter 4 I will present a algorithm for the revision of belief bases which is equivalent to the belief revision defined by (3.4).

### 3.5 Discussions and Related Work

In the previous sections, I argue that a rational iterated revision operator should be a function on belief states, which satisfies the modified AGM postulates, as well as, Postulates (DP1), (DP2), and (Ind).

The notion of (iterated) revision operator itself is not uncontroversial among the belief revisionists. As in Darwiche and Pearl's original work [1994], revision operators are most commonly viewed as binary functions which map a belief set and the new information to the revised belief set. This is problematic in two aspects. First of all, the revision operators studied in the AGM theory are local in the sense that a fixed belief set is assumed. Such revision operators are more appropriately considered as unary functions, which map the new information $\alpha$ to a revised belief set $K * \alpha$, with the understanding that $K$ is taken to be the background knowledge [Rott, 1999]. Secondly, the extra-logical preference information should play a role in the revision process. Based on the characterization of revision operators as unary functions, [Nayak et al., 2003] have proposed to view belief revision as dynamic, in the sense that the operator (i.e., the revision policy) itself evolves after each revision by taking the revised belief set as the new background knowledge. 7 While theoretically sound, the idea of dynamic revision is technically quite confusing in the sense that realizing a dynamic revision seems like devising an algorithm which evolves after each run. Most belief revisionist now consider that (iterated) revision operators to be functions on belief state [Lehmann, 1995; Rott, 1999; Darwiche and Pearl, 1994; Williams, 1994b; Konieczny and Pérez, 2000], although there is no consensus on what is a belief state.

Furthermore, while Postulate (DP1) is almost universally accepted, Postulate (DP2) seems to be more problematic. In fact, it is mainly different attitudes to Postulate (DP2) which provoke the disputation on the framework for iterated belief revision. In defense of the new framework, I argue that, according to the semantical characterization (Conditions (DP1R) and (DPR2)), Postulate (DP2) seems as reasonable as Postulate (DP1). If being informed about $\alpha$ does not

[^21]change the relative ordering of $\alpha$-worlds, why should the relative ordering of $\neg \alpha$-worlds be changed? This idea is also supported by Spohn, who argues that it is only reasonable to change the relative ordering between $\alpha$-worlds and $\neg \alpha$-worlds [Spohn, 1988].

To provide further support of Postulate (DP2), I show that it plays an important role in the principle of minimal change. Recently, Rott [1999; 2000] has pointed out that " it is a pure myth that minimal change principles are the foundation of existing theories of belief revision, as least as far as the AGM tradition is concerned". His argument is mainly based on the fact that full meet revision (see Observation 2.7) perfectly satisfies all AGM postulates [Alchourrón and Makinson, 1982; Rott, 2000]. Recall, full meet revision is defined as follows:

$$
K *_{a} \alpha= \begin{cases}K+\alpha & \text { if } K \nvdash \neg \alpha  \tag{3.5}\\ C n(\alpha) & \text { otherwise }\end{cases}
$$

Full meet revision is also called amnesic revision by Rott, as it completely "forgets" the prior beliefs in a severe revision. It is worth to mention that amnesic revision does suffer the problem of categorial mis-matching, as a belief set $K$ and the new information $\alpha$ uniquely determine the revised belief set $K *_{a} \alpha$. Therefore, it is already an iterated revision which satisfies the modified AGM postulates. Despite its radical behavior, amnesic revision surprisingly satisfies all DP postulates, but (DP2).

Observation 3.21. Amnesic revision $*_{a}$ satisfies (DP1), (DP3), and (DP4), but violates (DP2).

The above result shows Postulate (DP2) helps to minimize on the change of propositional beliefs.

At first glance, the existence of amnesic revision challenges the idea that a belief state should contain some extra-logical preference information. I argue that it is more appropriate to consider that (in amnesic revision) the extra-logical preference information contained in a belief state $\mathcal{K}$ is uniquely determined by its belief set $\operatorname{Bel}(\mathcal{K})$. It is not difficult to see that amnesic revision is induced from the following faithful assignment $h$, such that, for any belief state $\mathcal{K}$ :

$$
W_{1} \preceq_{\mathcal{K}}^{h} W_{2} \text { iff } W_{1} \models \mathcal{K} \text { or } W_{2} \not \vDash \mathcal{K}
$$

Put in worlds, $\preceq_{\mathcal{K}}^{h}$ splits $\Theta_{\mathcal{L}}$ into only two levels, such that all possible worlds (not) in $[\operatorname{Bel}(\mathcal{K})]$ are in the same partition.

In the sequel, I will give a detailed comparison of the present framework with the most prominent existing approaches to iterated revision.

### 3.5.1 Freund and Lehmann's Proposals

Freund and Lehmann [1994] were the first to point out that the AGM postulates are inconsistent with the following original version of Postulate (DP2) (called (C2) in [Darwiche and Pearl, 1994]), when revision operators are considered as binary functions on belief sets.
(C2) If $\beta \vdash \neg \alpha$ then $(K * \alpha) * \beta=K * \beta$.

To avoid the inconsistency, they have suggested to replace the original DP postulates by the so-called minimal influence postulate:
$(\mathrm{K} * 9) \quad$ If $K_{1} \vdash \neg \alpha$ and $K_{2} \vdash \neg \alpha$, then $K_{1} * \alpha=K_{2} * \alpha$.

According to $\left(\mathrm{K}^{*} 9\right)$, the revised belief set $K * \alpha$ does not depend on $K$ at all in the case of a severe revision. This is of course a very strong restriction, which violates the intuition that the prior beliefs should play a major role. Furthermore, in the presence of the AGM postulates, $\left(\mathrm{K}^{*} 9\right)$ implies (C1), (C3), (C4) (original version of (DP1), (DP3) and (DP4)) and the following weakening of Postulates (C2):

$$
\left(\mathrm{C} 2^{\prime}\right) \quad \text { If } K \vdash \neg \beta \text { and } \beta \vdash \neg \alpha \text {, then }(K * \alpha) * \beta=K * \beta
$$

Strong as it is, Postulate $\left(\mathrm{K}^{*} 9\right)$ on the other hand is, on the other hand, too weak to even rule out amnesic revision.

However, we have seen that the modified AGM postulates are consistent with (DP2), which means the culprit of the inconsistency between the AGM postulates and (C2) is the assumption that revision operators are binary functions on belief sets. As already discussed, this assumption is not accepted, if not denied, by many researchers. Therefore, the proposal of $\left(\mathrm{K}^{*} 9\right)$ is in some sense not well-grounded.

A conclusion Freund and Lehmann have drawn is that the AGM framework is not the right one in which to study iterated revision. In a later work, Lehmann therefore has proposed another framework for iterated revision, in which a belief state $\mathcal{K}$ is a finite sequence of consistent (propositional) sentences $\left\langle\beta_{1}: \ldots: \beta_{n}\right\rangle$ (the revision history of the agent) [Lehmann, 1995]. In Lehmann's framework, the iterated revision operator is trivial: $\mathcal{K} * \alpha$ is simply defined as the concatenation $\langle\mathcal{K}: \alpha\rangle$ of $\mathcal{K}$ and $\alpha$. Similarly, we might denote $\left\langle\mathcal{K}_{1}: \mathcal{K}_{2}\right\rangle$ by $\mathcal{K}_{1} * \mathcal{K}_{2}$. What
seems more difficult to define, however, is a mapping "Bel" from a belief state to its belief set. For this purpose, Lehmann has proposed the following set of postulates:
(I1) $\operatorname{Bel}(\mathcal{K})$ is a consistent logical theory
(I2) $\quad \alpha \in \operatorname{Bel}(\mathcal{K} * \alpha)$
(I3) If $\beta \in \operatorname{Bel}(\mathcal{K} * \alpha)$, then $\alpha \rightarrow \beta \in \operatorname{Bel}(\mathcal{K})$
(I4) If $\alpha \in \operatorname{Bel}(\mathcal{K})$, then $\mathcal{K} * \mathcal{K}_{1} \equiv(\mathcal{K} * \alpha) * \mathcal{K}_{1}$
(I5) If $\beta \vdash \alpha$, then $((\mathcal{K} * \alpha) * \beta) * \mathcal{K}_{1} \equiv(\mathcal{K} * \beta) * \mathcal{K}_{1}$
(I6) If $\neg \beta \notin \operatorname{Bel}(\mathcal{K} * \alpha)$, then $((\mathcal{K} * \alpha) * \beta) * \mathcal{K}_{1} \equiv((\mathcal{K} * \alpha) * \alpha \wedge \beta) * \mathcal{K}_{1}$
(I7) $\quad \operatorname{Bel}((\mathcal{K} * \neg \beta) * \beta) \subseteq \operatorname{Bel}(\mathcal{K})+\beta$

Readers are referred to [Lehmann, 1995] for the relation between Lehmann's postulates and the AGM postulates. It is worth to mention that Postulate (I5) is in fact just an adaption of Postulate (DP1).

To provide a constructive model, Lehmann has shown that his postulates characterize the so-called widening ranked revision. A widening ranked model is a function $g$ which maps an ordinal to a non-empty subset of $\Theta_{\mathcal{L}}$ s.t.,

1. for any $n$, $m$, if $n \leq m$, then $g(n) \subseteq g(m)$, and
2. for any $W \in \Theta_{\mathcal{L}}$, there exists $n$ with $W \in g(n)$.

Given a widening ranked model $g$, we can inductively define a rank $r(\mathcal{K})$ and a set of worlds $p(\mathcal{K})$ for any belief state $\mathcal{K}$ :

- $r(\rangle)=0$ and $p(\rangle)=g(0)$,
- $r(\langle\mathcal{K}: \alpha\rangle)=\min _{O}(\mathcal{K}, \alpha)$ and $p(\langle\mathcal{K}: \alpha\rangle)=g(r(\langle\mathcal{K}: \alpha\rangle)) \cap[\alpha]$,
where $\min _{O}(\mathcal{K}, \alpha)$ is the minimal ordinal $n$ s.t., $n \geq r(\mathcal{K})$ and $g(n) \cap[\alpha] \neq \emptyset$.
The widening ranked revision (essentially, the mapping $B e l$ ) is then defined as follows:

$$
\operatorname{Bel}(\mathcal{K} * \alpha)=\operatorname{Th}(p(\langle\mathcal{K}: \alpha\rangle))
$$

Lehmann has shown that the widening ranked revision generated from a widening ranked model satisfies Postulates (I1)-(I7). Conversely, any revision operator that satisfies Postulates (I1)-(I7) can be constructed as widening ranked revision. A major problem with widening ranked revision is that it is based on a fixed widening ranked model which is external to the agent's beliefs. Therefore, it is a kind of external revision which has been criticized by Rott as embodying a bad philosophy (cf. Section 3.1).

### 3.5.2 Revision Operators with Memory

Konieczny and Pérez have proposed yet another framework for iterated revision, which considers as the agent's belief state the sequence of consistent sentences the agent has learned [Konieczny and Pérez, 2000]. Like in Lehmann's approach, the revised belief state $\mathcal{K} * \alpha$ is just the concatenation of $\mathcal{K}$ and $\alpha$. However, Konieczny and Pérez have suggested a different set of postulates for iterated belief revision, which are essentially a reformulation of the AGM postulates along with the following one $:^{8}$

$$
\begin{equation*}
\mathcal{K} * \mathcal{K}_{1} \equiv \mathcal{K} *\left(\wedge \operatorname{Bel}\left(\mathcal{K}_{1}\right)\right) \tag{H7}
\end{equation*}
$$

Postulate (H7) is a sort of associativity law, which expresses the strong confidence in the new information. It is not difficult to see that (H7) implies (Rec).

The postulates proposed by Konieczny and Pérez characterize the so-called revision operators with memory. Formally, a faithful assignment over $\mathcal{L}$ is a function which maps a sentence $\alpha$ to a total pre-order $\preceq_{\alpha}$, such that,

- If $W_{1}, W_{2} \models \alpha$, then $W_{1}={ }_{\alpha} W_{2}$.
- If $W_{1} \models \alpha$ and $W_{2} \models \neg \alpha$, then $W_{1} \prec_{\alpha} W_{2}$.

Note that $\preceq_{\alpha}$ is nothing but a faithful ranking on $C n(\alpha)$.
Given a faithful assignment over $\mathcal{L}$, we can inductively define a ranking $\preceq_{\mathcal{K}}$ of the possible worlds for any belief state $\mathcal{K}$ :

$$
\text { - } \preceq_{\langle \rangle}=\Theta_{\mathcal{L}} \times \Theta_{\mathcal{L}},
$$

[^22]- for any $W_{1}, W_{2}: W_{1} \preceq_{\langle\mathcal{K}: \alpha\rangle} W_{2}$ iff $W_{1} \prec_{\alpha} W_{2}$ or $W_{1}={ }_{\alpha} W_{2}$ and $W_{1} \preceq_{\mathcal{K}} W_{2}$,

The revision operator with memory is then defined as follows:

$$
\begin{equation*}
\operatorname{Bel}(\mathcal{K} * \alpha)=\operatorname{Th}(\min ([\alpha], \preceq \mathcal{K})) \tag{3.6}
\end{equation*}
$$

A revision operator based on memory satisfies all DP postulates, except (DP2). Just like Lehmann's revision, a revision operator with memory assumes a fixed (external) faithful assignment, which means that the agent never changes its revision policy. Hence, Rott's criticism regarding widening ranked revisions also applies to revision operators with memory.

As a special case, the so-called basic memory operator is generated from the basic faithful assignment over $\mathcal{L}$ which additionally satisfies the following condition:

- If $W_{1}, W_{2} \models \neg \alpha$, then $W_{1}={ }_{\alpha} W_{2}$.

Put in words, $\preceq_{\alpha}$ partitions $\Theta_{\mathcal{L}}$ into two levels: the lower level contains all $\alpha$-worlds while the other level contains all $\neg \alpha$-worlds.

In fact, basic memory operator is equivalent to Nayak's lexicographic revision (with "naked evidence") (cf. Section 3.2.2). Not surprisingly, therefore, Konieczny and Pérez were able to show that basic memory operator satisfies all DP Postulates.

In their later work, Konieczny and Pérez [2002] have suggested to lift the unrealistic restriction by allowing the faithful ranking of the new evidence to be dynamic, meaning that logically equivalent evidences may come with distinct faithful rankings. The new revision operator has therefore been named dynamic revision operators with memory.

Konieczny and Pérez have shown that dynamic revision operator with memory satisfies (DP1), (DP3) and (DP4), but violates (DP2). Based on this, they have criticized (DP2) as too strong [Konieczny and Pérez, 2000]. In particular, they have suggested the following counterexample:

Example 13. Consider an electric circuit containing an adder and a multiplier. The atomic propositions adder_ok and multiplier_ok denote respectively that the adder and the multiplier are working. Initially we have no information about this circuit $(\operatorname{Bel}(\mathcal{K})=C n(\{ \})$ ), and we then learn that the adder and the multiplier are working ( $\alpha=$ adder_ok $\wedge$ multiplier_ok). Thereafter, someone tells us that the adder is actually not working ( $\beta=\neg$ adder_ok). There
is no reason to "forget" that the multiplier is working, whereas imposed by (DP2) we have $(\mathcal{K} * \alpha) * \beta \equiv \mathcal{K} * \beta \equiv \neg a d d e r \_o k$, since $\beta \vdash \neg \alpha$.

In favor of (DP2), I give a counterargument of Konieczny and Pérez's criticism. First I observe that a (dynamic) revision operator with memory is not a single revision operator, unlike the AGM framework attempts to model. Since the new information is coupled with a faithful ranking, a revision operator with memory (except basic memory revision) essentially is a multiple revision operator which revises a belief state with another belief state. After observing that, it is no surprise that (DP2) is violated since this postulate is only intended for single revision operators. This argument is supported by the fact that basic memory revision does satisfy (DP2). From the perspective of single revision, the behavior imposed by (DP2) in Example 13 is perfectly reasonable, since the evidence $\alpha$ is supposed to be an atomic piece of information. Note that, in case we learned adder_ok and mutiplier_ok subsequently, thanks to Postulate (Ind), we will retain multiplier_ok after the $\neg$ adder_ok-revision. In fact it is not difficult to see that if we want the revision operator with memory to exhibit the behavior expected by Konieczny and Pérez, then the faithful ranking that comes with $\alpha$ should encode the independence of multiplier_ok and adder_ok. This somehow highlights the subtle distinction between revising by a conjunction of sentences and revising by a set of sentences (with different plausibility degrees) (cf. the discussion in [Nayak et al., 1996b]), which will be further cultivated in Chapter 5. Based on the above argument, I consider (DP2) a well justified postulate for single revision operators, although it could be too strong for multiple revision operators.

### 3.5.3 Dynamic Revision Operators

Independently, [Nayak et al., 1996a] have also noticed the inconsistency between (C2) and the original AGM postulates. Their solution to avoid inconsistency has been to view belief revision as dynamic, which has been mentioned already. By doing so, it becomes possible to safely accept the DP postulates. The framework of dynamic revision operators is not too different from the DP framework, except that the former makes explicit the idea of evolutionary revision policy.

The problem of the DP postulates to be overly permissive has been also studied by Nayak et al., [1996a; 2003]. They have suggested to strengthen the DP postulates by the following so-called postulate of Conjunction: ${ }^{9}$

[^23](Conj) If $\alpha \nvdash \neg \beta$, then $(\mathcal{K} * \alpha) * \beta \equiv \mathcal{K} *(\beta \wedge \alpha)$.

In the presence of the modified AGM postulates, (Conj) is strong enough to imply (DP1), (DP3), (DP4) and (Rec).

I argue that (Conj) is too strong, as (Rec) already is criticized in Section 3.2.2 of being too strong. With regard to the postulate I have proposed, it is easy to see that (Ind) is a weakening of Postulate (Rec). This raises the question whether Postulate (Ind) weakens too much. Let us consider an example, taken from [Nayak et al., 1996a], which, at first glance, seems to show that this is indeed the case.

Example 14. Our agent believes that Tweety is a singing bird. However, since there is no strong correlation between singing and birdhood, the agent is prepared to retain the belief that Tweety sings even after accepting the information that Tweety is not a bird, and conversely, if the agent were to be informed that Tweety does not sing, she would still retain the belief that Tweety is a bird. Imagine that the agent first receives the information that Tweety is in fact not a bird, and later learns that Tweety does not sing.

Nayak et al. claimed that it is only reasonable to assume that the agent should, in the end, always believe that Tweety is a non-singing non-bird. Indeed, with $\operatorname{Bel}(\mathcal{K})=C n(\operatorname{singing} \wedge$ bird) it follows from Postulate (Rec) that $(\mathcal{K} * \neg$ bird $) * \neg$ singing $\vdash \neg$ bird, since $\nvdash \neg$ singing $\rightarrow$ bird. Postulate (Ind), on the other hand, does not apply in this case. But the behavior which is claimed to be the only reasonable one is not generally justified. Suppose, for example, the agent initially believes firmly that $\neg$ singing $\rightarrow$ bird. It is then possible, after revising by $\neg$ bird, that the belief in $\neg$ singing $\rightarrow$ bird is stronger than the belief in $\neg$ bird. In this case, after further revising by $\neg$ singing, the agent believes that Tweety is a bird after all.

## Summary

In this chapter, I have formally analyzed the problem of implicit dependence which is intrinsic to belief revision but largely overlooked in the community over the past decade. As (at least a partial) solution to the problem, I have proposed to strengthen the DP theory by a new postulate of independence. The resulting framework for iterated belief revision now consists of the (modified) AGM postulates, (DP1), (DP2) and (Ind). I have informally argued for the new postulate

[^24](Ind) by means of examples, and I have given a formal justification by an elegant semantic characterization. Also, a detailed comparison to related work has shown that the new framework is the most satisfactory one thus far in the literature. As a conclusion, I argue that the new framework provides better criteria for the design of rational iterated belief revision operators.

## Chapter 4

## Iterated Belief Revision: Computational Approaches

In the classical belief revision, the agents considered are infinite beings, without any limitation of memory, time, or deductive ability. When realization is concerned, one has to consider additionally that any realistic agent is a finite being and that calculations take time. Therefore, the beliefs of an agent should be represented by a finite belief state, and a satisfactory revision operator should not only behave rationally but also consume less time and space [Nebel, 1994; Williams, 1995]. Adapting belief revision to less idealized agents is far from trivial, as we need an approach which takes these characteristics of finiteness, memory and time limitations into account [Wassermann, 1999]. In order to construct a realizable revision operator, we first need to find a feasible representation of belief states. Note that a belief state induces a belief set (the logical contents) and the extra-logical preference information. Apparently, it is not feasible to represent directly a belief set on a computer, since it is infinite in general; even if the underlying language $\mathcal{L}$ is finite, the size of a belief set could also be very huge as it is logically closed. Many researcher [Wobck, 1992; Nebel, 1994] have therefore suggested to represent the logical contents of a belief state by a belief base, that is, a finite set of sentences (not necessary logically closed). Moreover, [Nebel, 1998] has argued that the size of the extra-logical preference information of a representationally feasible belief state should be polynomially bounded in the size of its belief base. 1

[^25]In this chapter, computational approaches to iterated belief revision will be studied. I will propose a concrete computational iterated revision operator, which satisfies the AGM, DP and Independence postulates. To clearly displays the intuition of the operator, I present an OCFbased semantics. Also, a formal assessment shows that the operator is optimal in computational complexity and space-consumption.

The rest of the chapter is organized as follows. In the next section, Nebel's cut base revision is recapitulated along with its problems. Then, I present in Section 4.2 the so-called reinforcement base revision. I will give a formal assessment of reinforcement base revision in terms of logical properties, computational complexity, and etc. Finally, Section 4.3 contains a detailed comparison to related work.

### 4.1 Cut Base Revision

Based on ideas of [Rott, 1991], Nebel has proposed so-called cut base revision which uses a very compact representation of belief states. Formally, a prioritized base, denoted by $\left\langle B, \leq_{B}\right\rangle$, consists of a belief base $B$ and a total pre-order $\leq_{B}$ on $B \cdot{ }^{2}$ Note that, given a belief state $\mathcal{K}$ of the form $\left\langle B, \leq_{B}\right\rangle$, the induced belief set $\operatorname{Bel}(\mathcal{K})$ consists of all logical consequences of $B$, that is, $\operatorname{Bel}(\mathcal{K})=C n(B)$.

Given a prioritized base $\left\langle B, \leq_{B}\right\rangle$ and a sentence $\alpha$, the cut-set of $\alpha$, denoted by $c u t_{<_{B}}(\alpha)$, is defined as follows:

$$
\begin{equation*}
\text { cut }_{<_{B}}(\alpha)=\left\{\beta \in B \mid\left\{\gamma \in B \mid \beta \leq_{B} \gamma\right\} \nvdash \alpha\right\} \tag{4.1}
\end{equation*}
$$

Put in words, the cut-set of $\alpha$ consists of all sentences in classes of $\left(B_{i}, \cdots, B_{n}\right)$, such that, adding next lower class $B_{i-1}$ leads to the implication of $\alpha$.

Cut base revision is then formally defined as follows:

$$
\begin{equation*}
\left\langle B, \leq_{B}\right\rangle \circledast_{c u t} \alpha=c u t_{<_{B}}(\neg \alpha) \cup\{\alpha\} \tag{4.2}
\end{equation*}
$$

Note that cut $_{<_{B}}(\neg \alpha)$ is subset of $B$, therefore the size of $\left\langle B, \leq_{B}\right\rangle \circledast_{\text {cut }} \alpha$ is linearly bounded in the size of $B$ and $\alpha$.

[^26]Nebel has shown that the total $\leq_{B}$ on a belief base $B$ can be generalized to an $\mathrm{EE} \leq_{C_{n}(B)}$ on the belief set $C n(B)$ :

Observation 4.1. [Nebel, 1994] Suppose $\left\langle B, \leq_{B}\right\rangle$ is a prioritized base. Let $\leq_{C n(B)}$ be a binary relation on $\mathcal{L}$, such that, for any sentences $\alpha, \beta$ :

$$
\begin{equation*}
\alpha \leq_{C_{n}(B)} \beta \text { iff cut }_{<_{B}}(\beta) \subseteq \text { cut }_{<_{B}}(\alpha) \tag{4.3}
\end{equation*}
$$

Then $\leq_{C n(B)}$ satisfies (EE1)-(EE5), assuming $K=C n(B)$.
The following result shows that cut base revision applied to a prioritized base $\left\langle B, \leq_{B}\right\rangle$ will obtain essentially the same result as EE-based revision applied to the belief set $C n(B)$.

Observation 4.2. [Nebel, 1994; Williams, 1994a] Suppose $\left\langle B, \leq_{B}\right\rangle$ is a prioritized base. Let $\leq_{C n(B)}$ be the EE on $C n(B)$ as defined by (4.3). Then

$$
C n\left(\left\langle B, \leq_{B}\right\rangle \circledast_{\text {cut }} \alpha\right)= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \left\{\beta \in \mathcal{L} \mid \neg \alpha<_{C n(B)} \neg \alpha \vee \beta\right\} & \text { otherwise }\end{cases}
$$

Nebel's proposal seems a nice step from theory to computation. Unfortunately, cut base revision is not an iterated revision operator, since it maps a prioritized base and the new information to a revised belief base, instead of a revised prioritized base. As the revised belief base is not ordered, it is therefore impossible to do a subsequent revision. This problem is also referred to by [Hansson, 2003] as the problem of categorial mis-matching. It is not difficult to see that both constructive models of Section 2.1 also suffer from the problem of categorial mis-matching.

It is an interesting question whether we can construct a satisfactory iterated revision operator by a slight modification of cut base revision? Unfortunately, the following discussion will show that al least native approaches do not work. To make the presentation easier, I assume a prioritized base is represented by a totally ordered family of classes of sentences $\left(B_{1}, \cdots, B_{n}\right)$.

Suppose a prioritized base $\left(B_{1}, \cdots, B_{n}\right)$ is revised by $\alpha$ and cut $_{<_{B}}(\neg \alpha)=$ $\bigcup\left\{B_{i}, \cdots, B_{n}\right\}$. According to (4.2), the revised belief base consists of $c^{c u t}{<_{B}}(\neg \alpha)$ and $\alpha$. Arguably, we have every reason to assume that sentences in $\mathrm{cut}_{<_{B}}(\neg \alpha)$ are ordered as before. The only problem is where to put the new information $\alpha$ ? Naively, we have two options: let $\alpha$ be less plausible than all sentences of $\mathrm{cut}_{<_{B}}(\neg \alpha)$ (cf. Figure 4.1(a)), or let $\alpha$ be more plausible than all sentences of $\mathrm{cut}_{<_{B}}(\neg \alpha)$ (cf. Figure 4.1(b)).

Base on above discussion, we can define so-called skeptical cut base revision as follows:

$$
\left\langle B, \leq_{B}\right\rangle \circledast_{c u t}^{s} \alpha=\left\langle B_{1}=\text { cut }_{<_{B}}(\neg \alpha) \cup\{\alpha\}, \leq_{B_{1}}\right\rangle
$$

where $\beta \leq_{B_{1}} \gamma$ iff $\beta=\alpha$ or $\beta, \gamma \in$ cut $_{<_{B}}(\neg \alpha)$ and $\beta \leq_{B} \gamma$.
As direct consequences of Theorem 2.14 and Observation 4.2, skeptical cut base revision satisfies the AGM postulates. It is easy to see that skeptical cut base revision satisfies Postulate (CB)

Similarly, so-called credulous cut base revision is defined as follows:

$$
\left\langle B, \leq_{B}\right\rangle \circledast_{c u t}^{c} \alpha=\left\langle B_{1}=\text { cut }_{<_{B}}(\neg \alpha) \cup\{\alpha\}, \leq_{B_{1}}\right\rangle
$$

where $\beta \leq_{B_{1}} \gamma$ iff $\gamma=\alpha$ or $\beta, \gamma \in$ cut $_{<_{B}}(\neg \alpha)$ and $\beta \leq_{B} \gamma$.

| $B_{n}$ |
| :---: |
| $\vdots$ |
| $B_{i}$ |
| $\alpha$ |

(a)

(b)

Figure 4.1: Two radical extensions of cut base revision

Analogously, credulous cut base satisfies the AGM postulates. It is also easy to see that credulous cut base revision satisfies (DP1) and (Rec), but violates (DP2).

Since both ( CB ) and ( Rec ) are too radical, I argue that the two iterated revision operators obtained by modifying cut base revision naively are not satisfactory.

### 4.2 Reinforcement Base Revision

The two radical cases in last section suggests that we might need to exploit additional (quantitative) information in order to obtain more reasonable iterated revision operators based on cut base revision.

In the sequel, I present such a revision operator in which a belief state is represented by a finite set of integer-weighted sentences. Formally,

Definition 4.3. [Jin and Thielscher, 2005a] An epistemic entrenchment base (EE base, for short), denoted by $\Xi=\langle B, f\rangle$, consists of a belief base $B$ and a mapping $f$ from $B$ to $\mathbb{N}^{+} \cdot \sqrt[3]{\text { For }}$ any sentence $\beta \in B, f(\beta)$ is called its evidence degree.

Given an EE base $\Xi=\langle B, f\rangle$, we denote by $\left.\Xi\right|_{m}$ the set of sentences in $B$ whose evidence degrees are exactly $m$ :

$$
\left.\Xi\right|_{m}=\{\beta \in B \mid f(\beta)=m\}
$$

Moreover, $\Xi^{m}$ denotes the set of sentences in $B$ whose evidence degrees are at lease $m$ :

$$
\Xi^{m}=\bigcup\left\{\left.\Xi\right|_{i} \mid i \geq m\right\}
$$

The belief degree (also called $r a n k$ ) of a sentence $\beta$ in an EE base $\Xi=\langle B, f\rangle$ is defined as follows:

$$
\operatorname{Rank}_{\Xi}(\beta)= \begin{cases}0 & \text { if } B \nvdash \beta  \tag{4.4}\\ \infty & \text { else if } \vdash \beta \\ \max \left(\left\{m \mid \Xi^{m} \vdash \beta\right\}\right) & \text { else }\end{cases}
$$

Note that, for a sentence $\beta \in B$, it is possible $\operatorname{Rank}_{\Xi}(\beta)>f(\beta)$; in this case $\beta$ is called redundant in $\Xi$. Therefore, the evidence degree $f(\beta)$ of a sentence $\beta \in B$ is only the lower bound of its belief degree. It is not difficult to see that redundant sentences can be removed from an EE base without affecting belief degrees of all sentences.

The belief set $\operatorname{Bel}(\Xi)$ of an EE base $\Xi=\langle B, f\rangle$ is identified with $C n(B)$, which is also exactly the set of all sentences with ranks greater than 0 .

EE bases are generalized prioritized bases, in the sense, a prioritized base $\left\langle B, \leq_{B}\right\rangle$ can be induced from an EE base $\Xi=\langle B, f\rangle$ by letting:

$$
\alpha \leq_{B} \beta \text { iff } f(\alpha) \leq f(\beta)
$$

Given an EE base $\Xi=\langle B, f\rangle$, the cut-set of a sentence $\alpha$, denoted by $\operatorname{cut}_{\Xi}(\alpha)$, is obtained

[^27]as follows:
\[

$$
\begin{equation*}
\operatorname{cut}_{\Xi}(\alpha)=\left\{\beta \in B \mid \operatorname{Rank}_{\Xi}(\alpha)<f(\beta)\right\} \tag{4.5}
\end{equation*}
$$

\]

Note that the notion of cut-set defined by (4.5) generalizes that defined by (4.1). More precisely, let $\Xi=\langle B, f\rangle$ be an EE base and $\left\langle B, \leq_{B}\right\rangle$ be the prioritized base induced from $\Xi$, then $\operatorname{cut}_{\Xi}(\beta)=\operatorname{cut}_{<_{B}}(\beta)$ for any sentence $\beta$.

Not surprisingly, given an EE base $\Xi=\langle B, f\rangle$, we can also induce an $\mathrm{EE} \leq_{\operatorname{Bel}(\Xi)}$ on $\operatorname{Bel}(\Xi)$ by stipulating:

$$
\begin{equation*}
\alpha \leq_{B e l(\Xi)} \beta \text { iff } \operatorname{Rank}_{\Xi}(\alpha) \leq \operatorname{Rank}_{\Xi}(\beta) \tag{4.6}
\end{equation*}
$$

Observation 4.4. [Wobck, 1992] Given an EE base $\Xi=\langle B, f\rangle$, the induced binary relation $\leq_{B e l(\Xi)}$ defined by (4.6) is an $E E$ on $\operatorname{Bel}(B)$.

It is worth mentioning that the quantitative nature of EE bases allows to represent more finegrained beliefs, e.g., we can encode information like " $\alpha$ is much plausible than $\beta$ ". Also, the quantitative nature of EE bases endow us to have a fine-grained control on belief revision.

In the current setting, an iterated revision operator now should be a function which maps an EE base and the new information to the revised EE base. The discussion in Section 4.1 suggests that the major problem is to find an appropriate evidence degree for the new information in the revised EE base. Obviously, if the new information is purely a sentence $\alpha$, then the revision operator has to assign $\alpha$ an evidence degree via a fixed scheme. It is unlikely that there exists such a fixed scheme suitable for all different kinds of applications. Therefore, based on the same considerations of [Spohn, 1988], I will consider a more general revision schema where the new information consists of a sentence $\alpha$ and an evidence degree $m \in \mathbb{N}^{+}$; standard AGM/DP revision is easily obtained as a special case by using the same evidence degree for all new information.

By a slight modification of cut base revision, I now define an iterated revision operator, named reinforcement base revision, as follows:

$$
\begin{align*}
\langle B, f\rangle \circledast_{r}\langle\alpha, m\rangle= & \left\{\left\langle\beta, f(\beta)-\operatorname{Rank}_{\Xi}(\neg \alpha)\right\rangle \mid \beta \in \operatorname{cut}_{\Xi}(\neg \alpha)\right\} \\
& \cup\{\langle\alpha \vee \beta, f(\beta)+m\rangle \mid \beta \in B\}  \tag{4.7}\\
& \cup\{\langle\alpha, m\rangle\}
\end{align*}
$$

Note that, the evidence degrees of all sentences in $\operatorname{cut}_{\Xi}(\neg \alpha)$ are degraded $\operatorname{Rank}_{\Xi}(\neg \alpha)$ and the new sentence $\alpha$ is assigned evidence degree $m$. Unlike cut base revision, reinforcement base
revision additionally adds a disjunction $\beta \vee \alpha$ with evidence degree $f(\beta)+m$ for every sentence $\beta \in B$. Obviously, these disjunctions will not affect the logical contents of the revised EE base. For those who are familiar with implicit dependence, note that these new added disjunctions are necessary to avoid (undesirable) implicit dependence; since if $\beta \vee \alpha$ is more plausible than $\alpha$ and $\beta$, it will disqualify $\alpha$ and $\beta$ to be implicitly dependent.

It is easy to see that the size of $\langle B, f\rangle \circledast_{r}\langle\alpha, m\rangle$ is linearly bounded in the size of $\langle B, f\rangle$ and $\langle\alpha, m\rangle$. Arguably, this is all one can expect from any satisfactory revision operator in terms of space-consumption.

To see a concrete example, let us recall again Example 7. Assume our initial belief state is now encoded by $\Xi=\{\langle r, 1\rangle,\langle s, 1\rangle,\langle r \vee s, 2\rangle\}$ and we first have learnt that $X$ is not smart with evidence degree 2. Since $\operatorname{Rank}_{\Xi}(\neg \neg s)=1$, according to (4.7), the revised EE base $\Xi_{1}=\Xi \circledast_{r}$ $\langle\neg s, 2\rangle=\{\langle r \vee s, 1\rangle,\langle\neg s, 2\rangle,\langle r \vee \neg s, 3\rangle\}{ }^{4}$ Suppose now we learn that $X$ is smart with evidence 2. Since $\operatorname{Rank}_{\Xi_{1}}(\neg s)=2$, we obtain $\Xi_{2}=\Xi_{1} \circledast_{r}\langle s, 2\rangle=\{\langle r \vee \neg s, 1\rangle,\langle s, 2\rangle,\langle r \vee s \vee s, 3\rangle\}$. It is not difficult to see that $\operatorname{Rank} \Xi_{1}(r)=2$, which means we continue to believe that $X$ is rich (as expected).

### 4.2.1 Formal Assessment

I give in the sequel a formal assessment of reinforcement base revision. First of all, I will show that reinforcement base revision is essentially equivalent to reinforcement OCF revision; hence it shares all nice properties of the latter. Then, the degree of syntax relevance of and the computational complexity of reinforcement base revision is analyzed.

## Equivalence Result

To show the equivalence of reinforcement base revision and OCF revision, we need to define a mapping from EE bases to OCFs. Formally, given EE base $\Xi=\langle B, f\rangle$, we can induce an OCF

[^28]$k_{\Xi}$ by letting $\sqrt{5}$
\[

k_{\Xi}(W)= $$
\begin{cases}0 & \text { if } W \models B  \tag{4.8}\\ \max (\{f(\beta) \mid \beta \in B \text { and } W \not \models \beta\}) & \text { otherwise }\end{cases}
$$
\]

Put in words, the rank of a possible world is the maximal evidence degree of all sentences it does not satisfy.

The following result shows that the OCF induced from any EE base does not contain irrelevant extra-logic preference information.

Observation 4.5. Suppose $\Xi=\langle B, f\rangle$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $W_{1}, W_{2}$ be arbitrary two possible worlds. Then $W_{1}, W_{2}$ are relevantly ordered.

Also, it is not difficult to see that an EE base and its induced OCF encode essentially the same belief state.

Observation 4.6. Suppose $\Xi=\langle B, f\rangle$ is an $E E$ base, and $k_{\Xi}$ is the induced $O C F$ as defined by (4.8). Then for any sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi}(\beta)=k_{\Xi}(\beta)
$$

Let us recall Example 7. It is not difficult to see that the EE base $\Xi=\{\langle r, 1\rangle,\langle s, 1\rangle,\langle r \vee$ $s, 2\rangle\}$ induces the OCF $k$ shown in the second column of Table 3.2 and the revised OCF $k_{\neg s, 2}^{r, *}$ corresponds exactly to the revised EE base $\Xi_{1}=\{\langle r \vee s, 1\rangle,\langle\neg s, 2\rangle,\langle r \vee \neg s, 3\rangle\}$.

To show the above example is not a coincidence, I need to exhibit that for any EE base $\Xi$ and new information $\langle\alpha, m\rangle$, the OCF induced from the revised EE base $\Xi \circledast_{r}\langle\alpha, m\rangle$ is exactly the revised OCF $k_{\Xi}^{\Xi_{\alpha, m}^{r, *}}$ (cf. the commute diagram in Figure 4.2).

Observation 4.7. Suppose $\Xi$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $\langle\alpha, m\rangle$ be any new information. Then for any possible world $W$ :

$$
k_{\Xi_{1}}(W)=k_{\Xi}^{\Xi_{\alpha, m}^{r, *}}(W)
$$

where $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$.

[^29]For the current purpose, it is sufficient to have a mapping from EE bases to OCFs.


Figure 4.2: Equivalence of reinforcement base and OCF revision

As a direct consequence of Observation 4.6 and Observation 4.7, we have the following equivalence result.

Theorem 4.8. Suppose $\Xi$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $\langle\alpha, m\rangle$ be any new information. The for any sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi_{1}}(\beta)=k_{\Xi}^{\Xi_{\alpha, m}^{r, *}}(\beta)
$$

where $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$.

Naturally, since the two are essentially equivalent, all formal properties of reinforcement OCF revision are inherited by reinforcement base revision, e.g., it follows immediately from Corollary 4.8 and Theorem 3.17 that reinforcement base revision satisfies all desirable postulates.

Theorem 4.9. Assume an arbitrary but fixed evidence degree for any new information. Then reinforcement base revision satisfies all AGM postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ as well as (DP1), (DP2) and (Ind), but violates (Rec).

Analogously, we can obtain the following formal properties of reinforcement base revision.
Observation 4.10. Suppose $\Xi$ is an EE base and $\langle\alpha, m\rangle$ is any new information. Let $\Xi_{1}=$ $\Xi \circledast_{r}\langle\alpha, m\rangle$. Then for any non-tautologous sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi_{1}}(\beta)= \begin{cases}t+m & \text { if } \vdash \alpha \rightarrow \beta \\ t^{\prime}-\bar{r} & \text { else if } t^{\prime}=t \\ \min \left(t^{\prime}-\bar{r}, t+m\right) & \text { else }\end{cases}
$$

where $\bar{r}=\operatorname{Rank}_{\Xi}(\neg \alpha), t=\operatorname{Rank}_{\Xi}(\beta)$ and $t^{\prime}=\operatorname{Rank}_{\Xi}(\alpha \rightarrow \beta)$.

Observation 4.11. Suppose $\Xi$ is an EE base and $\alpha$ is a new non-tautological sentence with evidence degree $m \in \mathbb{N}^{+}$. Let $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$. Then

$$
\operatorname{Rank}_{\Xi_{1}}(\alpha)=\operatorname{Rank}_{\Xi}(B, \alpha)+m
$$

Theorem 4.12. For arbitrary $m_{1}, m_{2} \in \mathbb{N}^{+}$, reinforcement base revision satisfies the following conditions:

$$
\begin{array}{ll}
\left(E D P 1^{\prime}\right) & \text { If } \beta \vdash \alpha, \text { then }\left(\Xi \circledast_{r}\left\langle\alpha, m_{1}\right\rangle\right) \circledast_{r}\left\langle\beta, m_{2}\right\rangle \equiv \Xi \circledast_{r}\left\langle\beta, m_{2}\right\rangle . \\
\left(E D P 2^{\prime}\right) & \text { If } \beta \vdash \neg \alpha, \text { then }\left(\Xi \circledast_{r}\left\langle\alpha, m_{1}\right\rangle\right) \circledast_{r}\left\langle\beta, m_{2}\right\rangle \equiv \Xi \circledast_{r}\left\langle\beta, m_{2}\right\rangle . \\
\left(E I n d^{\prime}\right) & \text { If there exists } m \text { such that } \Xi \circledast_{r}\langle\beta, m\rangle \nvdash \neg \alpha \text {, then } \\
& \left(\Xi \circledast_{r}\left\langle\alpha, m_{1}\right\rangle\right) \circledast_{r}\left\langle\beta, m_{2}\right\rangle \vdash \alpha .
\end{array}
$$

## Degree of Syntax Irrelevance

Strictly speaking, reinforcement base revision violates Dalal's principle of Irrelevance of Syntax [Dalal, 1988], in the sense, the revised belief state is not purely determined by the logical contents of the original belief state. However, it will be shown that reinforcement base revision does not really depend on the syntax of EE bases.

Two EE bases $\Xi_{1}$ and $\Xi_{2}$ are called epistemically equivalent iff their induced EEs (as defined by (4.6)) are equivalent, that is, for any sentences $\alpha, \beta$ :

$$
\alpha \leq_{\operatorname{Bel}\left(\Xi_{1}\right)} \beta \text { iff } \alpha \leq_{\operatorname{Bel}\left(\Xi_{2}\right)} \beta
$$

The following result shows two epistemically equivalent EE bases will yield logically equivalent revised EE bases, when revised by same new sentence.

Theorem 4.13. Let $\Xi_{1}, \Xi_{2}$ be two epistemically equivalent EE bases, then for any sentence $\alpha$ and evidence degrees $m_{1}, m_{2} \in \mathbb{N}^{+}$:

$$
\operatorname{Bel}\left(\Xi_{1}^{\prime}\right)=\operatorname{Bel}\left(\Xi_{2}^{\prime}\right)
$$

where $\Xi_{1}^{\prime}=\Xi_{1} \circledast_{r}\left\langle\alpha, m_{1}\right\rangle$ and $\Xi_{2}^{\prime}=\Xi_{2} \circledast_{r}\left\langle\alpha, m_{2}\right\rangle$.

Two EE bases $\Xi_{1}$ and $\Xi_{2}$ are called equivalent, denoted by $\Xi_{1} \cong \Xi_{2}$, iff for any sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi_{1}}(\beta)=\operatorname{Rank}_{\Xi_{2}}(\beta)
$$

It is not difficult to see that two equivalent EE bases will lead to equivalent revised EE bases, when revised by same new information.

Theorem 4.14. Let $\Xi_{1}, \Xi_{2}$ be two equivalent EE bases and $\langle\alpha, m\rangle$ any new information. Then

$$
\Xi_{1} \circledast_{r}\langle\alpha, m\rangle \cong \Xi_{2} \circledast_{r}\langle\alpha, m\rangle
$$

### 4.2.2 Computational Complexity

In computational settings, complexity is a very important criterion of evaluating revision operators. ${ }^{6}$ Like in [Eiter and Gottlob, 1992], I will consider the so-called problem of counterfactual (CF, for short), which decides " $\mathcal{K} * \alpha \vdash \beta$ ?" for arbitrary belief state $\mathcal{K}$ and sentences $\alpha, \beta$. In complexity theory, problems like CF, which only can have answers "yes" or "no", are called decision problems. It is obvious that the problem of CF is harder than the implication problem (IMPL, for short). ${ }^{7}$ Therefore, if underlying language $\mathcal{L}$ (e.g., a first order predicate logic) is undecidable, so is the problem of CF. In the sequel, I therefore assume a propositional underlying language $\mathcal{L}$.
[Nebel, 1992] has shown that both SAT and VALID can be polynomially (many-to-one) reduced to the problem of CF , given a revision operator satisfies Postulates $(\mathcal{K} * 4)$ and $(\mathcal{K} * 5)$. This means the problem of CF is at least both NP-hard and coNP-hard.

Observation 4.15. [Nebel, 1992] For any revision operation satisfying Postulates $(\mathcal{K} * 4)$ and $(\mathcal{K} * 5)$, the problem of CF is NP -hard and coNP-hard.

It is follows immediately that in general the complexity of the problem of CF is beyond NP and coNP, provided $N P \neq$ coNP.

For most well-known belief change operators in the literature, the problem of CF has been shown located at the lower end of the so-called polynomial hierarchy [Eiter and Gottlob, 1992;

[^30]Nebel, 1994; Liberatore, 1997]. Therefore, I will first briefly sketch the notion of polynomial hierarchy. Let $X$ be a class of decision problem. Then $\mathrm{P}^{X}$ denotes the class of decision problems that can be decided in polynomial time by a deterministic Turing machine $T$ that is allowed to use a procedure (also referred to as an oracle) for deciding a problem $Q \in X$, whereby executing the procedure only costs constant time. Similarly, $\mathrm{NP}^{X}$ denotes the class of decision problems that can be decided in polynomial time by a non-deterministic Turing machine $T$ that is allowed to use an oracle for deciding a problem $Q \in X$. Based on these notions, the complexity classes $\Delta_{k}^{p}, \Sigma_{k}^{p}$ and $\Pi_{k}^{p}$ are formally defined as follows:

$$
\begin{aligned}
& \Delta_{0}^{p}=\Sigma_{0}^{p}=\Pi_{0}^{p}=\mathrm{P} \\
& \Delta_{k+1}^{p}=\mathrm{P}^{\Sigma_{k}^{p}} \\
& \Sigma_{k+1}^{p}=\mathrm{NP}^{\Sigma_{k}^{p}} \\
& \Pi_{k}^{p}=\operatorname{co} \Sigma_{k+1}^{p}
\end{aligned}
$$

Note that $\Sigma_{1}^{p}=\mathrm{NP}$ and $\Pi_{1}^{p}=$ coNP. The polynomial hierarchy is then defined as $\mathrm{PH}=$ $\bigcup_{k \geq 0} \Delta_{k}^{p}=\bigcup_{k \geq 0} \Sigma_{k}^{p}=\bigcup_{k \geq 0} \Pi_{k}^{p} \subseteq$ PSPACE. It is unknown whether the inclusion between PH and PSPACE is proper, although strongly believed.

For problems in $\Delta_{2}^{p}$, it is often difficult to determine their exact complexity. But by restricting the number of oracle calls, we obtain an important special class $\Delta_{2}^{p}[\mathrm{O}(\log n)]$ of problems that can be decided in polynomial time by using only logarithmically many times of oracle calls. Furthermore, inside $\Delta_{2}^{p}[\mathrm{O}(\log n)]$ are the classes of so-called boolean hierarchy. The classes $\mathrm{NP}(k)$ and $\operatorname{coNP}(k)$ are defined as follows:

$$
\begin{aligned}
& \mathrm{NP}(0)=\mathrm{P} \\
& \mathrm{NP}(2 k+1)=\{S \cup T \mid S \in \mathrm{NP}(2 k), T \in \mathrm{NP}\} \\
& \mathrm{NP}(2 k+2)=\{S \cap T \mid S \in \mathrm{NP}(2 k+1), T \in \mathrm{NP}\} \\
& \operatorname{coNP}(k)=\{S \mid \bar{S} \in \mathrm{NP}(k)\}
\end{aligned}
$$

The boolean hierarchy $\mathrm{BH}=\bigcup_{k \geq 0} \mathrm{NP}(k)$ is equivalent to the class of problems that can be solved in deterministic polynomial time using a constant number of oracle calls.

Nebel has identified the exact complexity for cut base revision.

Theorem 4.16. [Nebel, 1994] For cut base revision, the problem of CF is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$ complete.

Since reinforcement base revision generalizes cut base revision, not surprisingly, I am able to show that the complexity of the former is also $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Theorem 4.17. For reinforcement base revision, the problem of $C F$ is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.
This theorem shows that reinforcement base revision is computationally optimal, since $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$ is the lowest class beyond NP and coNP. Moreover, if the underlying language is constrained to Horn sentences, then the problem of CF for reinforcement base revision becomes tractable (i.e., can be solved in polynomial time).

It is also interesting to know how hard is it to compute the revised belief state, given an arbitrary belief state and new information? Unlike the problem of CF, now we deal with a so-called function problem, which can have more answers than "yes" or "no". The complexity classes for decision problems have natural counterparts for function problems: e.g., $\mathrm{FP}^{\mathrm{NP}}$ (also referred to as NP-easy) represents the set of all function problems which can be solved in polynomial time by a deterministic Turing machine that is allowed to invoke a NP-oracle. It turns out that the problem of computing a revised belief state (EE base) for reinforcement base revision falls in an interesting complexity class, called NP-equivalent. Formally, a function problem is called NP-equivalent iff it is both NP-easy and NP-hard. Note that NP-equivalent is the analogue of NP-complete for function problems, in the sense that as far as a NP-equivalent problem can be solved in polynomial time so are all other NP-equivalent problems.

Theorem 4.18. For reinforcement base revision, the problem of computing a revised belief state is NP-equivalent.

### 4.3 Discussions and Related Work

I have proposed an iterated revision operator which not only satisfies all desirable rational postulates but also is optimal in terms of computational complexity and space-consumption. In this section, I will give a detailed comparison of my proposal with the most well-known existing computational approaches to iterated revision.

### 4.3.1 Syntax Irrelevant Operators

In the literature, there are revision operators which do not exploit any explicit extra-logical preference information. From a representation point of view, this seems an advantage; but
these revision operators are also criticized as inflexible [Nebel, 1998], in the sense, they have less control over what sentences are discarded and what sentences are retained. Obviously, such revision operators obey Dalal's principle of Irrelevance of Syntax [Dalal, 1988]. The principle of irrelevance of syntax is motivate by Newell's influential proposal which states that the behavior of an intelligent system should be specifiable on the knowledge level [Newell, 1982], i.e., the behavior of the system is determined entirely by the contents of the its knowledge (beliefs) despite their symbolic representation. In the sequel, I present two well-known syntax irrelevant revision operators.

## Full Meet Base Revision

The so-called full meet base revision [Nebel, 1994] is obtained directly from full meet revision:

$$
B \circledast_{a} \alpha=\left\{\begin{array}{lc}
B \cup\{\alpha\} & \text { if } B \nvdash \neg \alpha \\
\{\alpha\} & \text { otherwise }
\end{array}\right.
$$

Radical as it is, full meet base revision satisfies all AGM postulates and Postulate (DP1), but violates Postulate (DP2) and (Ind).

The complexity of full meet base revision has also been shown by Nebel.

Theorem 4.19. [Nebel, 1994] For full meet base revision, the problem of $C F$ is $\operatorname{coNP}(3)$ complete.

Although, the complexity of full meet base revision is lower than that of reinforcement base revision, the former is obviously of no practice use due to its radical behavior; and I conjecture that any useful revision operator will have a complexity beyond BH .

## Dalal's Operator

Based on a notion of distances between possible worlds, [Dalal, 1988] has proposed another syntax irrelevant revision operator. According to Dalal, when a belief base $B$ is revised by $\alpha$, the revised belief base should be determined by models of $\alpha$ that are "closest" to models of $B$.

Formally, the distance between two possible worlds $W_{1}, W_{2}$, denote by $\|\Delta\|\left(W_{1}, W_{2}\right)$, is
the cardinality of their symmetric difference:

$$
\|\Delta\|\left(W_{1}, W_{2}\right)=\left\|W_{1} \backslash W_{2} \cup W_{2} \backslash W_{1}\right\|
$$

The distance between a possible world $W$ and a (consistent) belief base $B$, denoted by $\|\Delta\|^{\min }(B, W)$, is the minimal distance between $W$ and models of $B$ :

$$
\|\Delta\|^{\min }(B, W)=\min \left(\left\{\|\Delta\|\left(W_{1}, W\right) \mid W_{1} \in[B]\right\}\right)
$$

Given a (consistent) belief base $B$, we can induce a faithful ranking $\preceq_{C n(B)}$ on $C n(B)$ by stipulating:

$$
\begin{equation*}
W_{1} \preceq_{C n(B)} W_{2} \text { iff }\|\Delta\|^{\min }\left(B, W_{1}\right) \leq\|\Delta\|^{\min }\left(B, W_{2}\right) \tag{4.9}
\end{equation*}
$$

Dalal's operator is then defined as follows:

$$
C n\left(B \circledast_{d} \alpha\right)= \begin{cases}\operatorname{Th}\left(\min \left([\alpha], \preceq_{C n(B)}\right)\right. & \text { if } B \text { is consistent }  \tag{4.10}\\ \operatorname{Cn}(\{\alpha\}) & \text { otherwise }\end{cases}
$$

where $\preceq_{C n(B)}$ is as defined by (4.9).
Note that Dalal's operator does not tell us explicitly how to construct a revised belief base. Unfortunately, [Cadoli et al., 1995] have shown a negative result which says that the size of the revised belief base $B \circledast_{d} \alpha$ could be much larger than that of $B$ and $\alpha$.

Theorem 4.20. [Cadoli et al., 1995] Suppose there exists a polynomial p such that for any belief base $B$ and sentence $\alpha$, there is a revised belief base $B \circledast_{d} \alpha$ as defined by (4.10) and $\left|B \circledast_{d} \alpha\right| \leq$ $p(|B|+|\alpha|)$. Then $\Sigma_{2}^{p}=\Pi_{2}^{p}=\mathrm{PH}$.

As $\Sigma_{2}^{p}=\Pi_{2}^{p}=\mathrm{PH}$ is commonly believed very unlikely, the above theorem essentially shows that Dalal's operator will cause super-polynomial space explosion.

It has been shown by [Katsuno and Satoh, 1991] that Dalal's operator satisfies all AGM postulates.. Moreover, Eiter and Gottlob have shown that the complexity of Dalal's operator is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete

Theorem 4.21. [Eiter and Gottlob, 1992] For Dalal's operator, the problem of CF is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

This result shows that in general Dalal's operator has the same complexity as reinforcement base revision. However, unlike the latter, even if the underlying language only consists of Horn sentences, the problem of CF for Dalal's operator remains $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log n)]$ complete [Eiter and Gottlob, 1992].

### 4.3.2 Theory Base Transmutation

Theory base transmutation is a class of iterated revision operators proposed by [Williams, 1994b], among which, conditionalization and adjustment are most prominent. Like reinforcement base revision, both conditionalization and adjustment have intuitive OCF-based semantics. Note that there are two major differences between reinforcement base revision and theory base transmutation: first of all, both conditionalization and adjustment violate Postulate (Ind), and the latter also violates Postulate (DP2); secondly, theory base transmutation allows the input evidence degree to be 0 , and in this case they behave like contraction operators since in theory base transmutation the input evidence degree will be the rank of the input sentence. To facilitate the comparison between reinforcement revision and theory base transmutation, I assume the input evidence degree is always greater than 0 .

## Conditionalization

OCF conditionalization was originally introduced by [Spohn, 1988], which can be viewed as a qualitative version of Jeffrey's Rule of probabilistic conditioning [Goldszmidt, 1992]:

$$
\left(k_{\alpha, m}^{c, *}\right)(W)= \begin{cases}k(W)-k(\neg \alpha) & \text { if } W \models \alpha  \tag{4.11}\\ k(W)-k(\alpha)+m & \text { otherwise }\end{cases}
$$

It is easy to see that OCF conditionalization resembles very much reinforcement OCF revision. In fact, if $k(\alpha)=0$ then the two coincide.

The follow result shows that in fact OCF conditionalization can be decomposed as a sequence of reinforcement OCF revision.

Observation 4.22. Let $k$ be an $O C F$ and $\langle\alpha, m\rangle$ be any new information. Then for any possible worlds $W$,

$$
k_{\alpha, m}^{c, *}(W)= \begin{cases}k_{\alpha, m-k(\alpha)}^{r, *}(W) & \text { if } k(\alpha)<m \\ \left(\left(k_{\neg \alpha, m^{\prime}}^{r, *}\right)_{\alpha, m}^{r, *}\right)(W) & \text { otherwise }\end{cases}
$$

## where $m^{\prime} \in \mathbb{N}^{+}$is an arbitrary positive integer.

I give an intuitive picture of OCF conditionalization in Figure 4.3, in which two cases are distinguished depending whether $k(\alpha)>m$ or $k(\alpha) \leq m$.


Figure 4.3: OCF conditionalization

OCF conditionalization changes the ranks of $\alpha$-worlds in exactly the same way as reinforcement OCF revision. Depending whether $k(\alpha)>m$ or $k(a) \leq m$, all $\neg \alpha$-worlds are uniformly moved downwards (or upwards respectively) to let the most plausible $\neg \alpha$-worlds have the rank $m$, which is necessary to make $k_{\alpha, m}^{c, *}(\alpha)=m$.

It is not difficult to see that OCF conditionalization satisfies (DP1) and (DP2), but violates (Ind) (in case $k(\alpha)>m$ ).

Formally, conditionalization is an EE base revision operator defined as follows [Williams, 1992; Benferhat et al., 2002]:

$$
\langle B, f\rangle \circledast_{c}\langle\alpha, m\rangle= \begin{cases}\langle B, f\rangle \circledast_{r}\langle\alpha, m\rangle & \text { if } B \nvdash \alpha \\
\left\{\begin{array}{l}
\left\{\left\langle\beta, f(\beta)-\operatorname{Rank}_{\Xi}(\alpha)+m\right\rangle \mid f(\beta)>\operatorname{Rank}_{\Xi}(\alpha)\right\} \\
\cup\{\langle\neg \alpha \vee \beta, f(\beta)\rangle \mid \beta \in B\} \\
\cup\{\langle\alpha, m\rangle\}
\end{array}\right. & \text { othewise }\end{cases}
$$

Note that, when $B \nvdash \alpha$ conditionalization coincides with reinforcement base revision.
It is not difficult to see that $\Xi \circledast_{r} \alpha \equiv \Xi \circledast_{c} \alpha$ for any EE base $\Xi$ and new information $\langle\alpha, m\rangle$. Therefore, the complexity of conditionalization is same as that of reinforcement base revision.

Theorem 4.23. For conditionalization, the problem of $C F$ is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Since it violates Postulates (Ind), it is not difficult to find a counterexample of conditionalization. Recall Example 7. Suppose now $\Xi=\{\langle r, 6\rangle,\langle s, 5\rangle,\langle r \vee s, 7\rangle\}$ and we first have learnt that $X$ is smart with evidence degree 1 . Since $\operatorname{Rank}_{\Xi}(s)=5$, this yields $\Xi_{1}=\Xi \circledast_{c}\langle s, 1\rangle=$ $\{\langle r \vee s, 3\rangle,\langle r \vee \neg s, 6\rangle,\langle s, 1\rangle\}$. Then we learn that $X$ is not rich with evidence degree 2 . Since $\operatorname{Rank}_{\Xi}(\neg \neg r)=3$, we obtain $\Xi_{2}=\Xi_{1} \circledast_{c}\langle\neg r, 2\rangle=\{\langle r \vee \neg s, 3\rangle\},\langle\neg r, 2\rangle,\langle s \vee \neg r, 3\rangle$. So strange enough, we now believe $X$ is not smart, since $\operatorname{Rank}_{\Xi_{2}}(\neg s)=2$.

## Adjustment

OCF adjustment is an operator based on an absolute measure of minimal change [Williams, 1994b]:

$$
\left(k_{\alpha, m}^{j *}\right)(W)= \begin{cases}\left(k_{\alpha}^{-}\right)_{\alpha, m}^{\times}(W) & \text { if } m<k(\alpha)  \tag{4.12}\\ k_{\alpha, m}^{\times}(W) & \text { otherwise }\end{cases}
$$

where

$$
\begin{aligned}
& \left(k_{\alpha}^{-}\right)(W)= \begin{cases}0 & \text { if } W \models \neg \alpha \text { and } K(W)=k(\alpha) \\
k(W) & \text { otherwise }\end{cases} \\
& \left(k_{\alpha, m}^{\times}\right)(W)= \begin{cases}0 & \text { if } W \models \alpha \text { and } K(W)=k(\neg \alpha) \\
i & \text { else if } W \models \neg \alpha \text { and } K(W)<i \\
k(W) & \text { else }\end{cases}
\end{aligned}
$$

Although its definition seems quite complicated, pictures in Figure 4.4 give us a nice intuition of OCF adjustment. Unlike OCF conditionalization and reinforcement OCF revision, OCF adjustment only changes the ranks of the most plausible $\alpha$-worlds. Moreover, when $k(a) \leq m$, the relative plausibility ordering of $\neg \alpha$-worlds are not always preserved (cf. Figure 4.4(b)).

It is easy to see that OCF adjustment only satisfies Postulate (DP1), but violates (DP2) and (Ind).

Formally, adjustment is an EE base revision operator defined as follows [Williams, 1992;


Figure 4.4: OCF adjustment

Benferhat et al., 2002]:

$$
\langle B, f\rangle \circledast_{j}\langle\alpha, m\rangle= \begin{cases}\left\{\begin{array}{ll}
\left\{\langle\beta, f(\beta)\rangle \mid f(\beta)>\operatorname{Rank}_{\Xi}(\neg \alpha)\right\} & \\
\cup\left\{\langle\alpha \vee \beta, f(\beta)\rangle \mid m<f(\beta) \leq \operatorname{Rank}_{\Xi}(\neg \alpha)\right\} & \text { if } B \nvdash \alpha \\
\cup\{\langle\alpha, m\rangle\} & \\
\left\{\langle\langle\beta, f(\beta)\rangle| f(\beta)>\operatorname{Rank}_{\Xi}(\alpha)\right\} & \\
\cup\left\{\langle\neg \alpha \vee \beta, f(\beta)\rangle \mid f(\beta) \leq \operatorname{Rank}_{\Xi}(\alpha)\right\} & \text { othewise } \\
\cup\{\langle\alpha, m\rangle\} &
\end{array}\right. \text {. }\end{cases}
$$

Like conditionalization, adjustment also has the same complexity as reinforcement base revision.

Theorem 4.24. For adjustment, the problem of $C F$ is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

To see a counterexample of adjustment, let us recall again Example 7. As in Section 4.2, assume $\Xi=\{\langle r, 1\rangle,\langle s, 1\rangle,\langle r \vee s, 2\rangle\}$ and we first learn that $X$ is not smart with evidence degree 2 , then the opposite with the same evidence degree. Now, since $\operatorname{Rank}_{\Xi}(\neg \neg s)=1$, we have $\Xi_{1}=\Xi \circledast_{j}\langle\neg s, 2\rangle=\{\langle r \vee s, 2\rangle,\langle\neg s, 2\rangle\}$. Moreover, since $\operatorname{Ran}_{\Xi^{\prime}}(\neg s)=2$, we yield $\Xi_{2}=\Xi_{1} \circledast J\langle s, 2\rangle=\{\langle s, 2\rangle\}$. Strangely, we do not believe any more that $X$ is rich.

### 4.3.3 Revising Possibilistic Beliefs

A problem of reinforcement revision (as well as of theory base transmutation) is that sentences are weighted with integers. In particular, it is unclear how the evidence degree of the input
information is obtained. Possibility theory is a more practical framework, in which the input sentence comes with a possibility degree within interval [0, 1] [Benferhat et al., 2002].

In possibility theory, a belief state is represented by a possibility distribution $\pi$ which is a mapping from $\Theta_{\mathcal{L}}$ to interval $[0,1]$. Contrary to an OCF, a possible distribution $\pi$ assigns more plausible worlds higher values, so that $\pi(W)=0$ means $W$ is impossible and $\pi(W)=1$ means that nothing prevents $W$ from being the real world. A possibility distribution $\pi$ is called finite iff its range is finite. In this thesis, we will only consider finite possibility distributions.

The belief set $\operatorname{Bel}(\pi)$ encoded by a possibility distribution $\pi$ are sentences which are true in all most plausibly possible worlds:

$$
\begin{equation*}
\operatorname{Bel}(\pi)=\operatorname{Th}(\{W \mid \pi(W)=1\}) \tag{4.13}
\end{equation*}
$$

Moreover, given a possibility distribution $\pi$, we can define two different measures on sentences of the language:

- the possibility degree $\Pi_{\pi}(\beta)$ evaluates the extent to which $\beta$ is consistent with $\pi$ :

$$
\Pi_{\pi}(\beta)= \begin{cases}0 & \text { if } \vdash \neg \beta  \tag{4.14}\\ \max (\{\pi(W) \mid W \models \beta\}) & \text { otherwise }\end{cases}
$$

- the necessity degree $N_{\pi}(\beta)$ evaluates the extent to which $\beta$ is entailed by $\pi$ :

$$
N_{\pi}(\beta)= \begin{cases}\infty & \text { if } \vdash \beta  \tag{4.15}\\ 1-\Pi_{\pi}(\neg \beta) & \text { otherwise }\end{cases}
$$

In the context, where $\pi$ is obvious, we might denote $N_{\pi}(\beta)$ and $\Pi_{\pi}(\beta)$, respectively, by $N(\beta)$ and $\Pi(\beta)$.

## Possibilistic Conditionings

The revision of a possibility distribution $\pi$ by a total reliable information $\alpha$ can be modeled by Bayesian-style conditioning, i.e., for any sentence $\alpha$ :

$$
\pi_{\alpha}^{*}(W)=\pi(W \mid \alpha)
$$

where $\pi(. \mid \alpha)$ is the posterior possibility distribution conditioned on $\alpha$.
Mainly, two methodologies of conditioning a possibility distribution are proposed. The socalled minimum-based conditioning defined as follows is more of qualitative nature:

$$
\pi\left(\left.W\right|_{m} \alpha\right)= \begin{cases}1 & \text { if } \pi(W)=\Pi(W) \text { and } W \models \alpha \\ \pi(W) & \text { else if } \pi(W)<\Pi(W) \text { and } W \models \alpha \\ 0 & \text { else }\end{cases}
$$

Much closer to the genuine Bayesian is the so-called product-based conditioning which re-scales all models of $\alpha$ upwards:

$$
\pi\left(\left.W\right|_{\times} \alpha\right)= \begin{cases}\frac{\pi(W)}{\Pi(\alpha)} & \text { if } W \models \alpha \text { and } \Pi(\alpha) \neq 0 \\ 1 & \text { else if } W \models \alpha \text { and } \Pi(\alpha)=0 \\ 0 & \text { else }\end{cases}
$$

Both $\pi\left(.\left.\right|_{m} \alpha\right)$ and $\pi\left(.\left.\right|_{\times} \alpha\right)$ are justified to be called conditioning, since $\pi(W)=\pi(W \mid \alpha) \otimes$ $\Pi(\alpha)$, where $\otimes$ is respective the minimum operator min for minimum-based conditioning and the product operator $\times$ for produce-based conditioning.

Based on Jeffrey's rule in probability theory, revision of a possibility distribution $\pi$ by an uncertain input $\langle\alpha, w\rangle$ (with $w \in[0,1]$ ) is achieved using the following definition:

$$
\pi_{\langle\alpha, w\rangle}^{*}(W)=\pi(W \mid\langle\alpha, w\rangle)
$$

where

$$
\pi(W \mid\langle\alpha, w\rangle)= \begin{cases}\pi(W \mid \alpha) & \text { if } W \models \alpha \\ (1-w) \otimes \pi(W \mid \neg \alpha) & \text { otherwise }\end{cases}
$$

Berferhat et al., have pointed out that there is an intimate relation between revision operators based on OCFs and conditionings of possibility distributions. In particular, they proposed a translation function form an OCF $k$ to a possibility distribution $\pi_{k}$ by letting:

$$
\begin{equation*}
\pi_{k}(W)=e^{-k(W)} \tag{4.16}
\end{equation*}
$$

The following theorem shows that adjustment and conditionalization in OCF corresponds, respectively, to possibilistic minimum-based and product-based conditioning with uncertain in-
put.
Theorem 4.25. [Benferhat et al., 2002] Let $k$ be an OCF, and $\langle\alpha, m\rangle\left(\right.$ with $m \in \mathbb{N}^{+}$) be the input information. Then for all possible worlds $W$,

- $\left.\pi_{k_{\langle\alpha, m\rangle}^{c, *}}(W)=\pi_{k}\left(\left.W\right|_{\times}\left\langle\alpha, 1-e^{-m}\right\rangle\right)\right)(W)$
- $\left.\pi_{k_{\langle\alpha, m\rangle}^{j, *}}(W)=\pi_{k}\left(\left.W\right|_{m}\left\langle\alpha, 1-e^{-m}\right\rangle\right)\right)(W)$

Note that the input information $\langle\alpha, m\rangle$ for OCF revision is mapped to as $\left\langle\alpha, 1-e^{-m}\right\rangle$ for possibilistic conditionings.

## Possibilistic Reinforcement Revision

In the section, I present the possibilistic versions for both reinforcement OCF and base revision.

## Reinforcement Possibility Distribution Revision

By slightly modifying product-based conditioning, I obtain the following reinforcement possibility distribution revision:

$$
\pi\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)= \begin{cases}\pi\left(\left.W\right|_{\times} \alpha\right) & \text { if } W \models \alpha  \tag{4.17}\\ (1-w) \times \pi(W) & \text { otherwise }\end{cases}
$$

The following theorem shows that reinforcement possibility distribution revision corresponds exactly to reinforcement OCF revision (see Figure 4.5).


Figure 4.5: Correspondence between reinforcement OCF and possibilistic revision

Theorem 4.26. Suppose $k$ is an $O C F$, and $\langle\alpha, m\rangle$ (with $m \in \mathbb{N}^{+}$) is the input information. Let $k_{\langle\alpha, m\rangle}^{*}$ be the revised OCF using reinforcement OCF revision. Then for any possible world $W$,

$$
\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)
$$

## Reinforcement Possibility Base Revision

In the possibilistic version of reinforcement base revision, a belief state is represented by a possibilistic base $\Sigma$ which consists of a finite set of weighted sentences. Formally, a possibilistic base $\Sigma$ consists of a belief base $B$ and mapping $g$ from $B$ to $[0,1]$. Like an EE base, we might represent a possibilistic base $\Sigma=\langle B, g\rangle$ by a finite set of pairs: $\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$, such that $\beta_{i} \in B$ and $g\left(\beta_{1}\right)=w_{i}$. Now, $g\left(\beta_{i}\right)$ is called the possibilistic degree of $\beta_{i}$. Like in an EE base, the higher its possibilistic degree, the more certain the sentence.

Given a possibilistic base $\Sigma=\langle B, g\rangle$, the set of sentences which has at least possibility degree $w$ is denoted by $\Sigma^{w}$, i.e.,

$$
\Sigma^{w}=\left\{\beta_{i} \in B \mid g\left(\beta_{i}\right) \geq w\right\}
$$

Moreover, we denote by $\Sigma^{>w}$ the set of sentences with possibility degree greater than $w$, i.e.,

$$
\Sigma^{>w}=\left\{\beta_{i} \in B \mid g\left(\beta_{i}\right)>w\right\}
$$

Given a possibilistic base $\Sigma=\langle B, g\rangle$, the necessary degree of $\beta$ (denoted by $N_{\Sigma}(\beta)$ ) is defined as follows:

$$
N_{\Sigma}(\beta)= \begin{cases}0 & \text { if } B \nvdash \beta  \tag{4.18}\\ \infty & \text { else if } \vdash \beta \\ \max \left(\left\{w \mid \Sigma^{w} \vdash \beta\right\}\right) & \text { else }\end{cases}
$$

Given a possibilistic base $\Sigma=\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$ and input information $\langle\alpha, w\rangle$ (with $w \in[0,1]$ ), the revised possibilistic base $\Sigma_{1}=\Sigma$ 囷 $_{r}\langle\alpha, w\rangle$ is obtained as follows:

$$
\begin{align*}
\Sigma_{1}= & \left\{\left.\left\langle\beta_{i}, \frac{w_{i}-\bar{w}}{1-\bar{w}}\right\rangle \right\rvert\, w_{i}>\bar{w}\right\} \\
& \cup\{\langle\alpha, w\rangle\}  \tag{4.19}\\
& \cup\left\{\left\langle\alpha \vee \beta_{i}, w_{i}+w-w \times w_{i}\right\}\right.
\end{align*}
$$

where $\bar{w}=N_{\Sigma}(\neg \alpha)$.
To show that possibilistic reinforcement base revision is equivalent to reinforcement possibility distribution revision, I define the mapping from a possibilistic base $\Sigma=$ $\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$, to a possibility distribution $\pi_{\Sigma}$ as follows:

$$
\pi_{\Sigma}(W)= \begin{cases}1 & \text { if } W \models\left\{\beta_{i}, \cdots, \beta_{n}\right\}  \tag{4.20}\\ 1-\max \left(\left\{w_{i} \mid W \not \models \beta_{i}\right\}\right) & \text { otherwise }\end{cases}
$$

A possibilistic base $\Sigma$ and its induced possibility distribution $\pi_{\Sigma}$ encode essentially the same belief state.

Observation 4.27. Suppose $\Sigma$ is a possibilistic base, and $\pi_{\Sigma}$ is the induced possibility distribution as defined by (4.20). Then for any sentence $\beta$ :

$$
N_{\Sigma}(\beta)=N_{\pi_{\Sigma}}(\beta)
$$

Moreover, I can show that reinforcement possibilistic base revision is equivalent to reinforcement possibility distribution revision (see Figure 4.6).


Figure 4.6: Equivalence of reinforcement possibilistic base and distribution revision

Theorem 4.28. Let $\Sigma$ be a possibilistic base and $\langle\alpha, w\rangle$ be the input information. Then for all possible worlds $W$,

$$
\pi_{\Sigma}\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)=\pi_{\Sigma_{1}}(W)
$$

where $\Sigma_{1}=\Sigma$ 粵 $_{r}\langle\alpha, w\rangle$.

As reinforcement possibility distribution revision has been shown corresponding to reinforcement OCF revision, the above equivalence result implies that reinforcement possibilistic
satisfies Postulates (DP1), (DP2) and (Ind). Since the necessary degree of a sentence wrt. a possibilistic base is defined symmetrically to the rank of a sentence in an EE base, Algorithm E. 1 also can be used to compute necessary degree of a sentence. The algorithm of possibilistic reinforcement base revision can be found in Appendix E.

## Summary

In this chapter, several computational approaches to iterated belief revision are studied. Firstly, I have discussed the limitations of pure qualitative approaches. Then, I have proposed a so-called reinforcement base revision based on a compact representation (named, EE base) of quantitative belief states. With the help of its semantic model (i.e., reinforcement OCF revision), I have shown that reinforcement base revision satisfies the postulate of independence (Ind) along with the AGM/DP postulates. Note that, reinforcement base revision is applicable to any decidable underlying language. Particularly, for propositional logics, I have shown that the computational complexity of reinforcement base revision is optimal. As a more practical solution to iterated belief revision, I have presented a so-called possibilistic revision, in which possibility degrees of beliefs are real numbers in the unit interval $[0,1]$.

## Chapter 5

## Mutual Belief Revision: Semantics and Computation

It is an interesting problem how people understand each other through information exchange. Bypassing the communication and psychology issues, the problem can be described as a pure AI issue as "how epistemic agents in a multi-agent system revise their beliefs as a result of belief exchange". The mainstream study of belief revision has been focused on how a single agent revises its beliefs to incorporate new information. This research normally assumes that new information is fully accepted, no matter whether it is represented as a single formula or as a set of sentences. Obviously such an assumption is not applicable to multi-agent systems. There have been a variety of approaches which have been proposed in the literature to deal with the problem of belief revision in multi-agent systems. The approach of non-prioritized belief revision allows an agent to revise its beliefs by partially accepting new information [Fermé and Hansson, 1999; Hansson, 1999]. This research sheds light on belief change with defeasible information resources but still focuses on the formalization of belief change from a single agent point of view. The acceptance or rejection of information resources is purely determined by the epistemic subject rather than decided by all participated agents. The study of belief merging or knowledge arbitration directly accounts for multi-agent belief change by pursuing a "fair" process that are able to incorporate beliefs of agents into coherent group beliefs [Revesz, 1997; Liberatore and Schaerf, 1998; Konieczny and Pérez, 1998]. These approaches sometimes force agents to accept some democracy rules, such as majority, social welfare maximization and so on, but disregard agents' "personal" view of group consent.

In this chapter, I propose an approach which takes another point of view to deal with the problem of multi-agent belief revision. Belief changes in a multi-agent system proceed with two stages. In the first stage, all agents sit together to work out a mutually acceptable points of view (a common understanding of the world) through a sequence of interchanges of respective views. This process is similar to belief merging. Once such a common understanding has been reached, in the second stage of belief change, each agent will adjust its original belief state in order to form a new view of the world as a result of belief interchange.

To make the exploration simple, I will focus on the belief revision problem in the setting of two-agent systems, so the mutual belief revision. I will present a semantic model based on OCF to specify the above mentioned two stages of mutual belief revision. While the OCF model is conceptually clear and constructively simple, it is not computation-friendly because it requires to take an exponential input of possible worlds from each agent. In order to have an estimation of computational complexity, I will present another model of mutual belief revision for EE bases. I will prove that this model is essentially equivalent to the OCF model.

The plan of the chapter is as follows. In next section, I will present an OCF-based semantic model for mutual belief revision operation, followed by a discussion of its formal properties. In Section 5.2, I present a computational model for mutual belief revision and some preliminary results on its computational complexity. Finally I conclude the chapter with a brief discussion of related work.

### 5.1 OCF Model of Mutual Belief Revision

In this section, I will explain the concept of mutual belief revision and present a semantic model to specify the concept by using Spohn's OCF, which has been widely employed in the literature [Spohn, 1988] and as well in Section 3.4.2.

Recall that with OCF a belief state of an agent is represented as a function $k$ which maps a possible world to a natural number, i.e., $k: \Theta_{\mathcal{L}} \rightarrow \mathbb{N}^{+}$. The set of all OCFs is denoted by $\kappa$. OCF representation of belief provides a richer structure than the set representation of beliefs in the sense that it encodes both a belief set and the plausibility of beliefs. This is not only necessary for modeling iterated belief revision (cf. Section 3.4.2) but also, as we will see, for modeling mutual belief revision.

### 5.1.1 Reaching a Common Understanding

I consider that a mutual belief revision procedure takes two stages. In the first stage, two agents try to reach a common understanding (assuming to be logically consistent) through a sequence of belief interchange. Once such a common understanding is reached, each agent performs a belief revision process to adapt its belief state to the information it learnt from the other agent. I will model these two stages separately.

The first stage of mutual belief revision is very similar to belief merging. To reach a consistent common understanding, several rounds of "belief interchange" might be needed. In each round, each agent receives more information from the other and get more understanding each other. To represent such a sequence of belief interchange, I will introduce the following concepts.

Given an OCF $k$ and a natural number $i$, the set of worlds with ranks smaller than or equal to $i$ is called a sphere of $i$ with radius $i$, denoted by $k^{-}(i)$. Formally,

$$
\begin{equation*}
k^{-}(i)=\{W \mid k(W) \leq i\} \tag{5.1}
\end{equation*}
$$

In particular, $k^{-}(0)$ is called the core of $k$. The belief set of a belief state $k$, denoted by $\operatorname{Bel}(k)$, is the set of sentences which hold in the core of $k$ (cf. Section 3.4.2). Therefore, two OCFs $k_{1}, k_{2}$ are called epistemically equivalent, denoted by $k_{1} \equiv k_{2}$, iff $k_{1}^{-}(0)=k_{2}^{-}(0)$. Two OCFs $k_{1}, k_{2}$ are consistent iff $k_{1}^{-}(0) \cap k_{2}^{-}(0) \neq \emptyset$.

The inconsistency degree, denoted by $d_{\text {inc }}\left(k_{1}, k_{2}\right)$, of two OCFs $k_{1}, k_{2}$ measures their extent of inconsistency:

$$
\begin{equation*}
d_{\text {inc }}\left(k_{1}, k_{2}\right)=\min \left\{n \mid k_{1}^{-}(n) \cap k_{2}^{-}(n) \neq \emptyset\right\} \tag{5.2}
\end{equation*}
$$

Given two pairs of subsets of possible worlds $\langle s, t\rangle$ and $\left\langle s^{\prime}, t^{\prime}\right\rangle$, we say that $\left\langle s^{\prime}, t^{\prime}\right\rangle$ is closer than $\langle s, t\rangle$, denoted by $\langle s, t\rangle<\left\langle s^{\prime}, t^{\prime}\right\rangle$, iff $s \subseteq s^{\prime}, t \subseteq t^{\prime}$ and $s \cup t \subset s^{\prime} \cup t^{\prime}$.

Consider two belief states $k_{1}$ and $k_{2}$ from two agents. We start with the cores of $k_{1}$ and $k_{2}$; if they do not intersect, the next round of belief interchange will continue on the least radius $r$ such that $\left\langle k_{1}^{-}(0), k_{2}^{-}(0)\right\rangle<\left\langle k_{1}^{-}(r), k_{2}^{-}(r)\right\rangle$ and check the intersection of the corresponding two spheres $k_{1}^{-}(r)$ and $k_{2}^{-}(r)$. The process continues until it reaches a radius big enough such that the corresponding spheres intersect. The procedure can be represented by the sequence of belief interchange: $\left\langle k_{1}^{-}(0), k_{2}^{-}(0)\right\rangle, \ldots,\left\langle k_{1}^{-}\left(r_{n}\right), k_{2}^{-}\left(r_{n}\right)\right\rangle$, where

1. $r_{i+1}=\min \left\{r \mid\left\langle k_{1}^{-}\left(r_{i}\right), k_{2}^{-}\left(r_{i}\right)\right\rangle<\left\langle k_{1}^{-}(r), k_{2}^{-}(r)\right\rangle\right\}$;
2. $r_{n}=\min \left\{r \mid k_{1}^{-}(r) \cap k_{2}^{-}(r) \neq \emptyset\right\}$.

This sequence clearly shows the procedure of "mutual understanding": each agents gradually broadens their views (possible worlds) in order to reach a common understanding. Afterwards, the belief state of each agent with the common understanding can be represented, respectively, by belief states $k_{1}-r_{n}$ and $k_{2}-r_{n}$, where

$$
(k-r)(W)= \begin{cases}0, & \text { if } k(W) \leq r  \tag{5.3}\\ k(W), & \text { otherwise }\end{cases}
$$

Therefore, the first stage of mutual belief revision defines a function $\gamma$ which takes a pair of belief states (possibly inconsistent each other) and returns a pair of consistent belief states:

$$
\begin{equation*}
\gamma\left(k_{1}, k_{2}\right)=\left\langle k_{1}-r_{n}, k_{2}-r_{n}\right\rangle \tag{5.4}
\end{equation*}
$$

where $r_{n}=d_{\text {inc }}\left(k_{1}, k_{2}\right)$.

### 5.1.2 Performing a Mutual Revision

As described in the previous section, the first stage of mutual belief revision results a pair of weakened belief states, which represents the (mutually consistent) remaining beliefs of the two agents. In the second stage of mutual belief revision, the weakened belief state of one agent will be sent to the other agent as the new information, and vice verse. Both agent will review the new information they just received and their original beliefs to form a new view of the world. Such a process is similar to the normal single agent belief revision operation. The only difference is that while an agent revises its beliefs, it not only tries to incorporate the information received form other agent but also takes the other agent's view of the information into account. In other words, the agent views the new information as a belief state rather than a single sentence or a set of beliefs. In order to model such a process, I will define a belief revision operator which allows to revise an OCF by another OCF.

Recall that I have proposed in Section 3.4.2 so-called reinforcement revision OCF which can revise an OCF by a new evidence with an evidence degree. Reinforcement OCF revision has been shown satisfying all AGM/DP postulates, as well as the independence postulate. By
generalizing it, now I define an operator which allows to revise an OCF by another OCF.
Definition 5.1. Given two OCFs $k$ and $\lambda$, the revision of $k$ by $\lambda$, denoted by $k \otimes \lambda$, is defined as follows:

$$
(k \otimes \lambda)(W)= \begin{cases}k(W)+\lambda(W), & \text { if } \lambda(W)>0  \tag{5.5}\\ k(W)-m_{k, \lambda}, & \text { otherwise }\end{cases}
$$

where $m_{k, \lambda}$ is the smallest radius of a sphere of $k$ which intersects the core of $\lambda$ :

$$
\begin{equation*}
m_{k, \lambda}=\min \left\{i \mid k^{-}(i) \cap \lambda^{-}(0) \neq \emptyset\right\} \tag{5.6}
\end{equation*}
$$

The idea behind the operator is the following. For any world the other agent disbelieves, the agent will degrade the world according to the other agent's degree of disbeliefs. Contrarily, for those worlds the other agent believes, the agent will upgrade these worlds according to its original degree of belief.

We say that an OCF $k$ encodes a sentence $\alpha$ with evidence degree $m>0$ iff for any possible world $W$,

$$
k(W)= \begin{cases}0, & \text { if } W \models \alpha  \tag{5.7}\\ m, & \text { otherwise }\end{cases}
$$

It is easy to see that an OCF $k$ encodes $\alpha$ with plausibility $m$ iff $\operatorname{Bel}(k)=C n(\{\alpha\})$ and $\operatorname{Rank}_{k}(\alpha)=m$. It is not difficult to see that the revision operator defined by (5.5) is indeed a generalization of that defined by (3.4).

Observation 5.2. Let $k$ be an arbitrary $O C F$ and $\lambda$ be an $O C F$ encodes $\langle\alpha, m\rangle$. Then for any possible $W$,

$$
\left(k_{\alpha, m}^{*}\right)(W)=(k \otimes \lambda)(W)
$$

In the second stage of mutual belief revision, each agent revises its belief state by mutually accepting the point of views reached in the first stage. The construction is similar to [Zhang et al., 2004]'s approach. Formally,

Definition 5.3. A function $M: \kappa \times \kappa \rightarrow \kappa \times \kappa$ is an OCF mutual belief revision if for any $k_{1}, k_{2}$,

$$
\begin{equation*}
M\left(k_{1}, k_{2}\right)=\left\langle k_{1} \otimes \gamma_{2}\left(k_{1}, k_{2}\right), k_{2} \otimes \gamma_{1}\left(k_{1}, k_{2}\right)\right\rangle \tag{5.8}
\end{equation*}
$$

where $\gamma$ is defined by (5.4). ${ }^{1}$

[^31]The diagrams in Figure 5.1 give an intuitive depiction of how mutual belief revision works. The left diagram shows a special case where $k_{1}$ and $k_{2}$ are consistent. It is obvious that the first stage ends in just one round and $\gamma\left(k_{1}, k_{2}\right)=\left\langle k_{1}, k_{2}\right\rangle$. Therefore, $M\left(k_{1}, k_{2}\right)=\left\langle k_{1} \otimes k_{2}, k_{2} \otimes\right.$ $\left.k_{1}\right\rangle$, and the core of $M_{1}\left(k_{1}, k_{2}\right)$ (as well as $M_{2}\left(k_{1}, k_{2}\right)$ ) is identified with the intersection of $k_{1}^{-}(0)$ and $k_{2}^{-}(0)$. The right diagram illustrates general situations where $k_{1}$ and $k_{2}$ are inconsistent. In the example displayed, it is assumed that the belief interchange takes place only three rounds and $\gamma\left(k_{1}, k_{2}\right)=\left\langle k_{1}-r_{2}, k_{2}-r_{2}\right\rangle$. Then $M\left(k_{1}, k_{2}\right)=\left\langle k_{1} \otimes\left(k_{2}-r_{2}\right), k_{2} \otimes\left(k_{1}-r_{2}\right)\right\rangle$. The core of $M_{1}\left(k_{1}, k_{2}\right)$ is exactly the intersection of $k_{2}^{-}\left(r_{2}\right)$ and the innermost sphere of $k_{1}, 2$ Analogously, the core $M_{2}\left(k_{1}, k_{2}\right)$ is the intersection of $k_{1}^{-}\left(r_{2}\right)$ and the innermost sphere of $k_{2}$. It is easy to observe that $k_{1}^{-}\left(r_{2}\right) \cap k_{2}^{-}\left(r_{n}\right)$ contains both cores of $M_{1}\left(k_{1}, k_{2}\right)$ and $M_{2}\left(k_{1}, k_{2}\right)$, in other words, the common understanding is accepted by both agents after mutual belief revision.


Figure 5.1: OCF-based mutual belief revision

### 5.1.3 Formal Properties

To justify the OCF model, I discuss in this section some of its formal properties. The following result shows that belief states of agents become more consistent after mutual belief revision unless no agent gives up any of their original beliefs.

Observation 5.4. The OCF mutual belief revision defined by Definition 5.3 satisfies the following properties :

1. $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right) \leq d_{\text {inc }}\left(k_{1}, k_{2}\right)$;
2. $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)=d_{\text {inc }}\left(k_{1}, k_{2}\right)$ iff $\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$ and $\operatorname{Bel}\left(k_{2}\right) \subseteq$ $\operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$.
[^32]The simple construction of the OCF model allows us to prove more interesting properties:
Observation 5.5. The OCF mutual belief revision defined by Definition 5.3 satisfies the following properties (for each $i \in\{1,2\}$ ):

```
(M1) \(\quad \operatorname{Bel}\left(\gamma_{1}\left(k_{1}, k_{2}\right)\right)+\operatorname{Bel}\left(\gamma_{2}\left(k_{1}, k_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)\);
(M2) if \(k_{1}\) and \(k_{2}\) are consistent, then \(\operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)=\operatorname{Bel}\left(k_{1}\right)+\operatorname{Bel}\left(k_{2}\right)\);
(M3) \(\quad k_{i}\) and \(M_{i}\left(k_{1}, k_{2}\right)\) are consistent iff \(\operatorname{Bel}\left(k_{i}\right) \subseteq \operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)\),
(M4) \(\quad M_{1}\left(k_{1}, k_{2}\right)=M_{1}\left(k_{1}, \gamma_{2}\left(k_{1}, k_{2}\right)\right), M_{2}\left(k_{1}, k_{2}\right)=M_{2}\left(\gamma_{1}\left(k_{1}, k_{2}\right), k_{2}\right)\);
\((M 5) \quad\) if \(\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)\) and \(\operatorname{Bel}\left(k_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)\),
    then \(M_{i}\left(k_{1}, k_{2}\right) \equiv M_{i}\left(M\left(k_{1}, k_{2}\right)\right)\).
```

(M1) ensures that the common understanding to be accepted by both agents. (M2) captures the cooperative attitude of agents: if two agents have no disagreement, then each of them will accept the beliefs of another agent. Whereas, (M3) captures the self-interest features of agents, which says if an agent is not going to accept any counter beliefs that contradicts its own, it does not need to give up any of its beliefs.
(M4) shows that the information an agent gains from mutual belief revision is no more than what it agrees with. In fact, (M1) and (M4) are two principal properties of mutual belief revision: both agents benefit from mutual belief revision without loss of diversity of views. In general, it is not necessary that two agents' belief states will merge together, as described by (M5), they may stuck in a stand-off (fixed-point) if both of them are not willing to make concessions (see Example 15 of Section 5.2 .1 for a concrete example).

### 5.2 Computational Model

An OCF is a function over possible worlds, the OCF model for mutual belief revision is not computationally friendly. In this section, I present another construction of mutual revision operator to investigate its computational properties. I will show that these two models are essentially equivalent.

As in Section 4.2, a belief state is represented by an EE base in the computational model of mutual belief revision.

Given two EE bases $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle, \Xi_{2}=\left\langle B_{2}, f_{2}\right\rangle$, we define the inconsistency degree, written as $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$, of $\Xi_{1}$ and $\Xi_{2}$ as follows

$$
d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)= \begin{cases}0, & \text { if } B_{1} \cup B_{2} \nvdash \perp ;  \tag{5.9}\\ \max \left\{i \mid \Xi_{1}^{i} \cup \Xi_{2}^{i} \vdash \perp\right\} . & \text { otherwise } .\end{cases}
$$

Recall that we can relate an EE base $\Xi$ to an OCF $k_{\Xi}$ by the mapping defined by (4.8). It is not difficult to see that inconsistency degree of EE bases closely resembles that of OCFs.

Observation 5.6. Suppose $\Xi_{1}, \Xi_{2}$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
d_{i n c}\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)=d_{i n c}\left(\Xi_{1}, \Xi_{2}\right)
$$

Given two (possibly mutual inconsistent) EE bases $\Xi_{1}, \Xi_{2}$, in order to reach a common understanding, it is reasonable that both agents gradually give up least plausible sentences. It is obvious that once all sentences with evidence degrees not greater than $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$ are discarded from $\Xi_{1}$ and $\Xi_{2}$, the two agents will reach a common understand.

Formally, $i$-cut of an EE base $\Xi=\langle B, f\rangle$, denoted by $\Xi-i$, is a new EE base obtained from $\Xi$ by removing sentences with evidence degrees not greater than $i$ :

$$
\Xi-i=\{\langle\beta, f(\beta)\rangle \mid \beta \in B \text { and } f(\beta)>i\}
$$

The above discussion leads to the following definition of function $\gamma$ which maps a pair of (possibly inconsistent each other) EE bases to a pair of mutually consistent ones:

$$
\begin{equation*}
\gamma\left(\Xi_{1}, \Xi_{2}\right)=\left\langle\Xi_{1}-r, \Xi_{2}-r\right\rangle \tag{5.10}
\end{equation*}
$$

where $r=d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$.
The follows result shows that the function $\gamma$ defined by (5.10) is essentially equivalent to that defined by (5.4).

Observation 5.7. Suppose $\Xi_{1}, \Xi_{2}$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
\gamma\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)=\left\langle k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}\right\rangle
$$

Where $k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}$ are OCFs induced respectively from $\gamma_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $\gamma_{2}\left(\Xi_{1}, \Xi_{2}\right)$.

### 5.2.1 EE base Mutual Revision

Similar to the OCF model, in the second stage of mutual belief revision we need an operator which revises an belief base (new, however an EE base) by another belief base.

Here, I present a natural generalization of reinforcement base revision proposed in Section 4.2. Suppose $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle$ and $\Xi_{2}=\left\langle B_{2}, f_{2}\right\rangle$ are two EE bases. The result of $\Xi$ revised by $\Xi_{2}$, denoted by $\Xi_{1} \otimes \Xi_{2}$, is defined as:

$$
\begin{align*}
\Xi_{1} \otimes \Xi_{2}= & \left\{\left\langle\beta, f_{1}(\beta)-r\right\rangle \mid \beta \in B_{1}\right\} \\
& \cup \Xi_{2}  \tag{5.11}\\
& \cup\left\{\left\langle\beta \vee \alpha, f_{1}(\beta)+f_{2}(\alpha)\right\rangle \mid \beta \in B_{1} \text { and } \alpha \in B_{2}\right\}
\end{align*}
$$

where $r=\operatorname{Rank}_{\Xi_{1}}\left(\neg \bigwedge B_{2}\right)$, assuming $\bigwedge \emptyset=\top$.
It is easy to see that if $\Xi_{2}$ only contains a single sentence, then the above EE base revision operator coincides with that defined by (4.7). Moreover, the following results show that it is essentially equivalent to the OCF-based revision operator defined by (5.5).

Theorem 5.8. Suppose $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle, \Xi_{2}=\left\langle B_{2}, f_{2}\right\rangle$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then for any possible world w:

$$
\left(k_{\Xi_{1}} \otimes k_{\Xi_{2}}\right)(W)=k_{\Xi}(W)
$$

Where $\Xi=\Xi_{1} \otimes \Xi_{2}$ and $k_{\Xi}$ is the OCF induced from $\Xi$.

Now we are ready to define a mutual belief revision operator on EE bases.

Definition 5.9. The EE base mutual belief revision operator $M$ is defined as follows:

$$
\begin{equation*}
M\left(\Xi_{1}, \Xi_{2}\right)=\left\langle\Xi_{1} \otimes \gamma_{2}\left(\Xi_{1}, \Xi_{2}\right), \Xi_{2} \otimes \gamma_{1}\left(\Xi_{1}, \Xi_{2}\right)\right\rangle \tag{5.12}
\end{equation*}
$$

where $\gamma$ is defined by (5.10).

Let's see some examples to check how this operator works.

Example 15. Suppose $\Xi_{1}=\{\langle q, 3\rangle,\langle p, 2\rangle\}$ and $\Xi_{2}=\{\langle q, 3\rangle\langle\neg p, 2\rangle\}$. It is easy to see that $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)=2$ and $\Xi_{1}-2=\Xi_{2}-2=\{\langle q, 3\rangle\}$. According to (5.11), we have $M_{1}\left(\Xi_{1}, \Xi_{2}\right)=$ $\Xi_{1} \otimes\left(\Xi_{2}-2\right)=\{\langle q, 6\rangle,\langle p, 2\rangle\}$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)=\Xi_{2} \otimes\left(\Xi_{1}-2\right)=\{\langle q, 6\rangle,\langle\neg p, 2\rangle\}$, after removing redundant sentences.

We may notice that the belief set of each agent remains the same but the evidence degree of the common belief is increased as a result of the mutual belief revision. Different from belief merging, the conflicting opinions of $\Xi_{1}$ and $\Xi_{2}$ on $p$ is not solved. It is not difficult to see that the inconsistency remains even operator $M$ is applied repeatedly.

Example 16. Suppose $\Xi_{1}=\{\langle q, 3\rangle,\langle p, 1\rangle\}$ and $\Xi_{2}=\{\langle\neg p, 3\rangle,\langle q, 1\rangle\}$. In such a case, we have $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)=1, \Xi_{1}-1=\{\langle q, 3\rangle\}$ and $\Xi_{2}-1=\{\langle\neg p, 3\rangle\}$. According to (5.11), we have $M_{1}\left(\Xi_{1}, \Xi_{2}\right)=\{\langle q \vee \neg p, 6\rangle,\langle\neg p, 3\rangle,\langle q, 2\rangle\}$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)=\{\langle q \vee \neg p, 6\rangle,\langle q, 4\rangle,\langle\neg p, 3\rangle\}$.

In this example, both agents confirm the common beliefs $q \vee \neg p$ with high degree. Agent 1 is convinced by agent 2 of $\neg p$ since agent 2 believes it in high degree and also has a common understanding with it on $q$. Agent 2 does not gain any new belief but gets $q$ more confirmed. As a result of mutual belief revision, two agents reach a consensus.

The following equivalence theorem (as illustrated in Fig. 5.2) is a direct consequence of Observation 5.7 and Theorem 5.8.

Theorem 5.10. Suppose $\Xi_{1}, \Xi_{2}$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
M\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)=\left\langle k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}\right\rangle
$$

where $\left\langle\Xi_{1}^{\prime}, \Xi_{2}^{\prime}\right\rangle=M\left(\Xi_{1}, \Xi_{2}\right)$ and $k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}$ are the OCFs induced from $\Xi_{1}^{\prime}$ and $\Xi_{2}^{\prime}$, respectively.


Figure 5.2: Equivalence of EE base and OCF Mutual Belief Revision

It follows directly from Theorem 4.6 and Theorem 5.10 that the EE base operator shares all nice logical properties of the OCF-based operator.

Observation 5.11. The EE base mutual belief revision defined by Definition 5.9 satisfies the following properties :

1. $d_{\text {inc }}\left(M\left(\Xi_{1}, \Xi_{2}\right)\right) \leq d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$;
2. $d_{\text {inc }}\left(M\left(\Xi_{1}, \Xi_{2}\right)\right)=d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$ iff $\operatorname{Bel}\left(\Xi_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(\Xi_{1}, \Xi_{2}\right)\right)$ and $\operatorname{Bel}\left(\Xi_{2}\right) \subseteq$ $\operatorname{Bel}\left(M_{2}\left(\Xi_{1}, \Xi_{2}\right)\right)$.

Observation 5.12. The EE base mutual belief revision defined by Definition 5.9 satisfies the following properties (for each $i \in\{1,2\}$ ):

$$
\begin{array}{ll}
\left(M 1^{\prime}\right) & \operatorname{Bel}\left(\gamma_{1}\left(\Xi_{1}, \Xi_{2}\right)\right)+\operatorname{Bel}\left(\gamma_{2}\left(\Xi_{1}, \Xi_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{i}\left(\Xi_{1}, \Xi_{2}\right)\right) ; \\
\left(M 2^{\prime}\right) & \text { if } \Xi_{1} \text { and } \Xi_{2} \text { are consistent, then } \operatorname{Bel}\left(M_{i}\left(\Xi_{1}, \Xi_{2}\right)\right)=\operatorname{Bel}\left(\Xi_{1}\right)+\operatorname{Bel}\left(\Xi_{2}\right) ; \\
\left(M 3^{\prime}\right) & \Xi_{i} \text { and } M_{i}\left(\Xi_{1}, \Xi_{2}\right) \text { are consistent iff } \operatorname{Bel}\left(\Xi_{i}\right) \subseteq \operatorname{Bel}\left(M_{i}\left(\Xi_{1}, \Xi_{2}\right)\right) ; \\
\left(M 4^{\prime}\right) & M_{1}\left(\Xi_{1}, \Xi_{2}\right)=M_{1}\left(\Xi_{1}, \gamma_{2}\left(\Xi_{1}, \Xi_{2}\right)\right), M_{2}\left(\Xi_{1}, \Xi_{2}\right)=M_{2}\left(\gamma_{1}\left(\Xi_{1}, \Xi_{2}\right), \Xi_{2}\right) ; \\
\left(M 5^{\prime}\right) & \text { if } \operatorname{Bel}\left(\Xi_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(\Xi_{1}, \Xi_{2}\right)\right) \text { and } \operatorname{Bel}\left(\Xi_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(\Xi_{1}, \Xi_{2}\right)\right), \\
& \text { then } M_{i}\left(\Xi_{1}, \Xi_{2}\right) \equiv M_{i}\left(M\left(\Xi_{1}, \Xi_{2}\right)\right) .
\end{array}
$$

### 5.2.2 Computational Complexity

In this section, I present the complexity results of two related problems. First of all, we are interested in how hard is it to compute the result of $M\left(\Xi_{1}, \Xi_{2}\right)$, given two arbitrary EE bases $\Xi_{1}, \Xi_{2}$. It turns out that the problem is NP-equivalent.

Theorem 5.13. The problem of computing $M\left(\Xi_{1}, \Xi_{2}\right)$, for arbitrary EE bases $\Xi_{1}$ and $\Xi_{2}$, is NP-equivalent.

The second problem is to decide whether an arbitrary sentence $\beta$ is entailed by both $M_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)$. It turns out this decision problem inhabits the very low level of the polynomial hierarchy.

Theorem 5.14. The problem of deciding whether both $M_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)$ entail $\beta$, for arbitrary EE bases $\Xi_{1}, \Xi_{2}$ and sentence $\beta$, is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

### 5.3 Discussions and Related Work

I have introduced the concept of mutual belief revision. We consider that a process of mutual belief revision takes two stages: in the first stage, two agents get together trying to reach a common understanding. This stage is quite similar to belief merging [Revesz, 1997; Liberatore and Schaerf, 1998; Konieczny and Pérez, 1998]. Once such a common understanding has formed, two agents revise their belief states in order to incorporate the common agreed views into their own belief states. This idea is in the spirit of [Zhang et al., 2004] and also quite similar to the so-called credulous conciliation operation [Gauwin et al., 2005]. Note that there is a significant difference between belief merging and our work: the result of belief merging is simply a coherent set of group beliefs; whereas mutual belief revision is concerned with the evolution of individual beliefs of agents. Moreover, unlike belief merging, our approach does not force agents to loss of diversity of views. For instance, the two agents in Example 15 still have conflicting views regarding $p$ after mutual belief revision. With belief merging, however at least one agent has to give up its belief regarding $p$. In our opinion, diversity of views is very important, otherwise it does not make too much sense to let agents have individual beliefs. In such a sense, mutual belief revision is more subtle an information exchange process than belief merging. To model such a process, I introduce two difference models: an OCF-based model, which clearly shows the intuition and semantics of the operation, and an EE-based model for calculation. These models are generalizations of reinforcement OCF revision and base revision, respectively. Note that it is also not difficult to obtain possibilistic models of mutual belief revision by generalizing possibilistic models of belief revision of Section 4.3.3 in the same spirit.

In the AI literature, the concept of mutual belief revision has be used from different perspectives. [van der Meyden, 94] names the process of reaching common awareness of knowledge as mutual belief revision based on a Kripkean semantics. Different from this approach, I do not concern how knowledge or beliefs is formed. Instead, the current approach is concerned with how to maintain consistency of existing beliefs. This also differentiates the belief revision approaches from modal logic approaches in knowledge representation and reasoning. [Zhang et al., 2004] presents a formalism in which negotiation is viewed as a process of mutual belief revision. In this work, a set of AGM-like postulates are proposed to specify mutual belief revision operation. However, no semantic model or computational model is given. In addition, I do not think we can view negotiation and mutual belief revision as the same concept though they might share some similar ways in operation. In particular, Zhang et. al., have proposed
a so-called the postulate of no recantation for negotiation which can be reformulated in our setting as follows:

$$
\begin{align*}
& \operatorname{Bel}\left(k_{2}\right) \cap \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)  \tag{M6}\\
& \operatorname{Bel}\left(k_{1}\right) \cap \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)
\end{align*}
$$

It states that every agent should commit itself to keeping its original demands/offers once they have been are accepted by the other side. It is not difficult to see that, in the presence of (M3), Postulate (M6) implies the following postulate:

$$
\begin{aligned}
& \text { (M7) If } \operatorname{Bel}\left(k_{1}\right) \nsubseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right) \text { and } \operatorname{Bel}\left(k_{2}\right) \nsubseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right) \text {, } \\
& \text { then } M_{1}\left(k_{1}, k_{2}\right) \equiv M_{2}\left(k_{1}, k_{2}\right)
\end{aligned}
$$

Postulate (M7) says that the beliefs of two agents will converge if both of them are willing to make concessions. This might be a desirable property for negotiation, since the focus of negotiation is how to reach an mutually beneficial agreement, once such an agreement is formed, the divergence of beliefs among the agents will become unimportant. However, the divergence of beliefs is always the focus of mutual belief revision no matter before or after the process, and indeed the OCF mutual belief revision does not satisfies Postulate (M7). In such a sense, negotiation is more similar to belief merging than mutual belief revision.

## Summary

In this chapter, I have introduced the concept of mutual belief revision. A process of mutual belief revision takes two stages; in the first stage, two agents get together trying to reach a common understanding; in the second stage, two agents revise their belief states in order to incorporate the common agreed views into their own belief states. In particular, I have presented both a semantic model and a computational model for a concrete mutual revision operator by generalizing reinforcement OCF revision and reinforcement base revision, respectively. I have shown several desirable properties of the mutual revision operator, as well the complexity of the computational model.

## Chapter 6

## Belief Update, Revisited

In the previous chapters, we have mainly focused on belief revision. As pointed out by [Katsuno and Mendelzon, 1991a] that belief update is more appropriate when the new information reflects changes of the world; it is therefore very useful for modeling actions with physical effects. In Section 2.3, I have already briefly sketched the basic ideas of belief update, especially, the KM postulates. Now it is time to pay a revisit to belief update from a perspective of reasoning about actions. Recall that, unlike belief revision, the main idea of belief update is to change individually possible worlds as little as possible to accommodate the new information. In particular, the constructive model of the KM postulates is based on a similarity structure which maps a possible world $W$ to a SOS $\mathfrak{S}_{W}$ centered on $W$; and the result of updating $W$ by a sentence $\alpha$ is the intersection of $[\alpha]$ with the minimal sphere of $\mathfrak{S}_{W}$ that intersects $[\alpha]$ (cf. Figure 2.5). Unlike extra-logical preference information exploited in belief revision (e.g., an EE or a faithful ranking), a similarity structure is not specific to a particular belief set. At first glance, this seems an advantage as a single similarity structure can be used for any belief set; therefore the constructive model does not suffer from the problem of iteration (cf. the discussion in Section 3.1). However, since a similarity structure is external to the agent's beliefs and the new information, the construction essentially defines a kind of external operator. Such external operators have been criticized by [Rott, 2003] as embodying a bad philosophy. Also, several researchers [Herzig, 1996; Doherty et al., 1998; Zhang and Foo, 1996] have lately pointed out that the KM postulates are problematic with disjunctive updates and domain constraints, which are however considered useful in reasoning about actions. In this chapter, I first study systematically the above-mentioned problems of the

KM postulates. Then, I will present an update operator for possibilistic belief states, which does not suffer from these problems.

In this chapter, I assume that the underlying language $\mathcal{L}$ is a propositional logic built on a (possibly infinite) set AT of atoms. ${ }^{1}$ A literal is either an atom or its negation. The set of all literals is denoted by LI, that is, $\mathrm{LI}=\mathrm{AT} \cup\{\neg p \mid p \in \mathrm{AT}\}$. As usual, a (propositional) interpretation is an assignment from AT to truth values $\{$ true, false $\}$. For the sake of succinctness, I will represent an interpretation by the set of atoms to which it assigns true. In the obvious way, an interpretation can be truth functionally extended to be a mapping from $\mathcal{L}$ to $\{$ true, false $\}$. So far, we have considered maximal consistent sets of sentences as possible worlds. Obviously, there is an one-to-one correspondence between maximal consistent sets of sentences and propositional interpretations. Therefore, I will simply use interpretations to represent possible worlds. To avoid confusions, I will use $I$ (possibly indexed) to denote a possible world (an interpretation). In the sequel, the notions regarding possible worlds (as defined in Section 2.2) will be reused, except now a possible world $I$ is said to be a model of a sentence $\alpha$ iff it maps $\alpha$ to true.

The rest of the chapter is organized as follows. I first recall in the next section the wellknown possible models approach (PMA), along with an analysis of its problems with disjunctive updates and domain constraints. Then, in Section 6.2 I present Winslett's standard semantics (WSS) and its some extensions. I will show that the WSS can appropriately handle both disjunctive updates and domain constraints. Thereafter, in Section 6.3 I will introduce an update operator for possibilistic belief states by generalizing the WSS. Finally, I conclude with discussions and related work.

### 6.1 The PMA

The possible models approach (PMA, for short) was introduced by [Winslett, 1990] in the context of reasoning about actions. Like Dalal's operator (cf. Section 4.3.1), the main idea of the PMA is based on a notion of distance between possible worlds.

Formally, the distance, denoted by $\Delta\left(I_{1}, I_{2}\right)$, between two possible worlds $I_{1}, I_{2}$ is simply

[^33]their symmetrical difference, that is, ${ }^{2}$
$$
\Delta\left(I_{1}, I_{2}\right)=I_{1} \backslash I_{2} \cup I_{2} \backslash I_{1}
$$

Given a possible world $I$ and a consistent sentence $\alpha$, we denote by $\Delta(I, \alpha)$ the set of all possible distances between $I$ and $\alpha$-worlds:

$$
\Delta(I, \alpha)=\left\{\Delta\left(I, I^{\prime}\right) \mid I^{\prime} \in[\alpha]\right\}
$$

The distance can be used to measure the degree of the changes from one possible world to another. When a possible world $I$ is updated by a sentence $\alpha$, the set of all successor worlds, denoted by $I \diamond_{P M A} \alpha$, consists of all $\alpha$-worlds whose distance to $I$ is minimal:

$$
I \diamond_{P M A} \alpha=\left\{I^{\prime} \in[\alpha] \mid \Delta\left(I, I^{\prime}\right) \in \min (\Delta(I, \alpha), \subseteq)\right\}
$$

By the Winslett identity, the PMA is then formally defined as follows:

$$
K \diamond_{P M A} \alpha= \begin{cases}\mathcal{L} & \text { if }[K]=\emptyset \text { or }[\alpha]=\emptyset \\ T h\left(\bigcup_{I \in[K]} I \diamond_{P M A} \alpha\right) & \text { otherwise }\end{cases}
$$

It is well-known that the PMA satisfies all of the original KM postulates.
Observation 6.1. [Katsuno and Mendelzon, 1991] The PMA satisfies $(K \diamond 1)-(K \diamond 6),(K \diamond 8)$, $\left(U 6^{\prime}\right)$ and $\left(U 7^{\prime}\right) \cdot{ }^{3}$

Note that, in general $\Delta(I, \alpha)$ is not totally ordered wrt. $\subseteq$. Therefore, we can not construct a SOS centered on $I$ as shown in Figure 2.5. This explains why Postulate ( $K \diamond 7$ ) is not satisfied by the PMA.

To see how the PMA works, let us consider Example 6 again. Recall that the belief set $K=C n(\{\neg b \leftrightarrow m\})$ and the new information is $b$. Assuming $b, m$ are the only atoms, we need to consider only four possible worlds, viz,. $I_{1}=\{b, m\}, I_{2}=\{b\}, I_{3}=\{m\}$ and $I_{4}=\{ \}$. Obviously, $[K]=\left\{I_{2}, I_{3}\right\}$ and $[b]=\left\{I_{1}, I_{2}\right\}$. Since $\Delta\left(I_{2}, I_{1}\right)=\{m\}$ and $\Delta\left(I_{2}, I_{2}\right)=\{ \}$, we have $I_{2} \diamond_{P M A} b=\left\{I_{2}\right\}$. Similarly, since $\Delta\left(I_{3}, I_{1}\right)=\{b\}$ and $\Delta\left(I_{3}, I_{2}\right)=\{b, m\}$, we have

[^34]$I_{3} \diamond_{P M A} b=\left\{I_{1}\right\}$. It follows that $K \diamond_{P M A} b=\operatorname{Th}\left(\left\{I_{1}, I_{2}\right\}\right)$. Therefore, unlike AGM revision operators, the PMA does not conclude unintuitively that the magazine is not the table.

### 6.1.1 Two Criticisms on the PMA

Despite its satisfiability of the original KM postulates, the PMA has been criticized by several researchers as being inadequate mainly for two reasons [Herzig, 1996; Doherty et al., 1998; Zhang and Foo, 1996].

## Problems with Disjunctive Updates

The following example (originally contributed by Reiter) is often used to show that the PMA fails to handle disjunctive updates.

Example 17. Suppose we throw a coin onto a chess board. Before this action, the coin isn't touching any squares, but when it comes to rest on the chess board, it could be touching just a white square, it could be touching just a black square, or it could be touching both.

Let $w$ and $b$ represent, respectively, that the coin touches a white square and the coin touches a black square. Then our initial beliefs can be represented by $K=C n(\{\neg w, \neg b\})$, and the new information is represented by $w \vee b$. It is not difficult to see that $K \diamond_{P M A}(w \vee b)=$ $\operatorname{Th}(\{\{w\},\{b\}\})$. So strangely enough, the PMA excludes the possibility that the coin touches squares of both colors.

The above example shows that the PMA undesirably interprets "inclusive or" as "exclusive or'. Unfortunately, Herzig and Rifi have shown that this is not a specific problem of the PMA but of the KM postulates in general.

Observation 6.2. [Herzig and Rifi, 1998] Suppose $\diamond$ is an update operator that satisfies ( $K \diamond 2$ ), $(K \diamond 5)$ and $(K \diamond 6)$. Then $\diamond$ also satisfies the following condition. 4

$$
\text { (XOR) } \quad \text { If } \neg \beta \in K \diamond \alpha \text { and } \neg \alpha \in K \diamond \beta \text { then } \alpha \oplus \beta \in K \diamond(\alpha \vee \beta)
$$

We can argue that, leave along Example 17, Condition (XOR) is also not desirable in terms of expressiveness, as "exclusive or" can be expressed with "inclusive or", but not vice versa.

[^35]Therefore, we suggest to drop $(K \diamond 6)$ as a general property of belief update, as $(K \diamond 2)$ and $(K \diamond 5)$ are quite promising and almost uncontroversial.

## Problems with Domain Constraints

Besides the aforementioned problem, the PMA is also often criticized as unable to handle domain constraints. Recall that domain constraints are general dependence relations between components of the world. In reasoning about actions, handling domain constraints is one of the center topics; in particular, the ramification problem is one fundamental problem of reasoning about actions, which is concerned with handling effects that are indirectly derived from domain constraints.

As usual in reasoning about actions, I assume domain constraints of a particular domain are indefeasible, and can be represented by a finite set $D C$ of sentences. In principle, an update operator, denoted by $\diamond^{\mathrm{DC}}$, is said to take into account domain constraints DC iff it satisfies the following condition:

$$
\text { (DC) } \quad \mathrm{DC} \subseteq K \diamond^{\mathrm{DC}} \alpha
$$

To obtain such an update operator $\diamond^{\mathrm{DC}}$, [Katsuno and Mendelzon, 1991a] have proposed to simply extend a (standard) update operator $\diamond$ as follows:

$$
\begin{equation*}
K \diamond^{\mathrm{DC}} \alpha=K \diamond(\alpha \wedge \bigwedge \mathrm{DC}) \tag{6.1}
\end{equation*}
$$

Unfortunately, the above naive extension does not work with the PMA. To show this, let us consider the following benchmark example of the ramification problem:

Example 18. [Thielscher, 1997] Consider an electric circuit (depicted in Figure 6.1) consists of a battery, two switches, and a light bulb. Now suppose we toggle the first switch in the particular state displayed, where both the first switch and the light are off. Then, besides the direct effect of the first switch becoming on, we also expect that the light turns on.

Let $s w_{1}$ and $s w_{2}$ mean respectively that "switch 1 is on" and "switch 2 is on", and $l t$ means that "the light is on". Then the obvious connection between the components of the circuit can be formally described by domain constraints $\mathrm{DC}=\left\{l i \leftrightarrow s w_{1} \wedge s w_{2}\right\}$, and the initial state of the circuit can be described by belief set $K=C n\left(\left\{\neg s w_{1}, s w_{2}, \neg l t\right\}\right)$. However, it is not difficult


Figure 6.1: A simple circuit
to see that $K \diamond_{P M A}\left(s w_{1} \wedge\left(l i \leftrightarrow s w_{1} \wedge s w_{2}\right)\right)=\operatorname{Th}\left(\left\{\left\{s w_{1}, s w_{2}, l t\right\},\left\{s w_{1}\right\}\right\}\right)$. Therefore, the PMA also admits $\left\{s w_{1}\right\}$ as a possible successor world, in which $s w_{2}$ magically becomes false. This result of course contradicts our intuition.

Note that the PMA imposes a kind of absolute minimal change policy. The above discussions show that absolute minimal change is not suitable for dealing with disjunctive updates and domain constraints. Independently, the studies in reasoning about actions also suggest that a solution to the ramification problem requires a suitable weakened version of law of persistence [Lifschitz, 1990; Thielscher, 1997]. In next section, I present an update operator, i.e., the WSS, which is based a weaken version of minimal change policy. I will show that the WSS can deal with both disjunctive updates and domain constraints in a suitable way. Afterwards, I will present a new update operator for possibilistic belief states by generalizing the WSS.

### 6.2 The WSS, and Extensions

The so-called Winslett's standard semantics (WSS, for short) is another update operator proposed by [Winslett, 1988a] which has drawn a lot of attention from researchers of reasoning about actions [Herzig and Rifi, 1999; Liberatore, 1997]. In this section, I first show the semantics-based definition of the WSS, then a computational model of the WSS.

### 6.2.1 Semantic Model

Instead of minimizing changes, the WSS constrains changes of possible worlds to be in some set of exceptions computed from the new information. More precisely, when a possible world $I$
is updated by a sentence $\alpha$, the truth values of all atoms are not allowed to change, except those occurred in $\alpha$ :

$$
I \diamond_{W S S} \alpha=\left\{I^{\prime} \in[\alpha] \mid \Delta\left(I, I^{\prime}\right) \subseteq \operatorname{Atm}(\alpha)\right\}
$$

where $\operatorname{Atm}(\alpha)$ denotes the set of all atoms occurring in $\alpha$.
One problem of the WSS is that it is dependent on the syntax of the new information, e.g., it will obtain different successor worlds when a possible world $I=\{h\}$ is updated, respectively, by two logical equivalent sentences $h \vee \neg h$ and $\top$, since $\operatorname{Atm}(h \vee \neg h)=\{h\}$ and $\operatorname{Atm}(\mathrm{T})=\emptyset .5^{5}$ Obviously, this problem can be easily solved by preprocessing the new information to remove all redundant atoms. Formally, an atom $p$ is called redundant in a sentence $\alpha$ iff $\alpha[p / \top] \equiv \alpha[p / \perp]$, where $\alpha[p / \beta]$ is obtained by substituting all occurrences of $p$ in $\alpha$ with $\beta$. Given a sentence $\alpha$, we denote by $A t m_{\downarrow}(\alpha)$ the set of non-redundant atoms of $\alpha$. E.g., $A t m_{\downarrow}(h \vee \neg h)=\emptyset$, since $h$ is redundant.

Based on above notion, [Herzig, 1996] has suggested a modification of the WSS: ${ }^{6}$

$$
K \diamond_{W S S_{\downarrow}} \alpha= \begin{cases}\mathcal{L} & \text { if }[K]=\emptyset \text { or }[\alpha]=\emptyset \\ \operatorname{Th}\left(\bigcup_{I \in[K]} I \diamond_{W S S_{\downarrow}} \alpha\right) & \text { otherwise }\end{cases}
$$

where

$$
I \diamond_{W S S_{\downarrow}} \alpha=\left\{I^{\prime} \in[\alpha] \mid \Delta\left(I, I^{\prime}\right) \subseteq \operatorname{Atm}_{\downarrow}(\alpha)\right\}
$$

To see that the $\mathrm{WSS}_{\downarrow}$ handles disjunctive updates properly, let us consider Example 17 again. It is not difficult to see that now $K \diamond_{W S S_{\downarrow}}(w \vee b)=\operatorname{Th}(\{\{w, b\},\{w\},\{b\}\})$, since $K=C n(\neg w \wedge \neg b)$ and $\operatorname{Atm}_{\downarrow}(w \vee b)=\{w, b\}$. Therefore, the WSS $_{\downarrow}$ admits the possibility that the coin touches squares with both colors.

Formally, [Herzig and Rifi, 1999] have shown that the WSS ${ }_{\downarrow}$ does not satisfy all of the KM postulates. ${ }^{7}$

Observation 6.3. [Herzig and Rifi, 1999] The $W S S_{\downarrow}$ satisfies $(K \diamond 1),(K \diamond 2),(K \diamond 4),(K \diamond 5)$

[^36]and $(K \diamond 8)$, but violates $(K \diamond 3),(K \diamond 6),(K \diamond 7),\left(U 6^{\prime}\right)$ and $\left(U 7^{\prime}\right)$.

On the other hand, the $\mathrm{WSS}_{\downarrow}$ does capture all intuitively expected successor possible worlds with minimal distance to the original possible world:

Observation 6.4. [Herzig and Rifi, 1999] Let I be a possible world and $\alpha$ a consistent sentence . Then,

$$
I \diamond_{P M A} \alpha \subseteq I \diamond_{W S S_{\downarrow}} \alpha
$$

As illustrated by Example 17, in general the $\mathrm{WSS}_{\downarrow}$ is distinct from the PMA. But I am able to show that the two coincide when the new information is not disjunctive:

Observation 6.5. Let I be a possible world, and $\alpha$ a consistent conjunction of literals. Then,

$$
I \diamond_{P M A} \alpha=I \diamond_{W S S_{\downarrow}} \alpha
$$

## An Alternative Model

As we have seen that the $\mathrm{WSS}_{\downarrow}$ allows more changes than the PMA in general. It is an interesting question whether the former is too liberal? At first glance, the $\mathrm{WSS}_{\downarrow}$ indeed appears to be very liberal, as it allows to change the truth values of all non-redundant atoms of the new information regardless of their positions (meaning that how they occur in the new information). To show that the $\mathrm{WSS}_{\downarrow}$ is in fact not as liberal as it appears, I present in the sequel an alternative model of the $\mathrm{WSS}_{\downarrow}$. I will show that the seemingly more promising alternative model is equivalent to the original definition of the $\mathrm{WSS}_{\downarrow}$. The following notions are needed for the presentation of the alternative model.

Given a sentence $\alpha$, we denote by $\operatorname{Lit}(\alpha)$ the set of all literals made of non-redundant atoms of $\alpha$, that is,

$$
\operatorname{Lit}(\alpha)=\operatorname{Atm}_{\downarrow}(\alpha) \cup\left\{\neg p \mid p \in \operatorname{Atm}_{\downarrow}(\alpha)\right\}
$$

A partial model of a sentence $\alpha$ is a maximal consistent subset of $\operatorname{Lit}(\alpha)$ that entails $\alpha$. Formally, we denote by $\operatorname{PM}(\alpha)$ the set of all partial models of $\alpha:{ }^{8}$

$$
P M(\alpha)=\{L \mid L \in(\operatorname{Lit}(\alpha) \downarrow \perp) \text { and } L \vdash \alpha\}
$$

[^37]Note that, each element of $P M(\alpha)$ corresponds to a model of $\alpha$ restricted to its non-redundant atoms, e.g., $P M(w \vee b)=\{\{w, b\},\{w, \neg b\},\{\neg w, b\}\}$.

The so-called partial model assertion is an update operator gives all partial models of the new information equal treatment. More precisely, when a possible world $I$ is update by a sentence $\alpha$, each partial model of $\alpha$ will be inserted into $I$ :

$$
I \diamond_{P M} \alpha=\bigcup_{L \in P M(\alpha)} \operatorname{Insert}(L, I)
$$

where $\operatorname{Insert}(L, I)=I \backslash\{p \in \mathrm{AT} \mid \neg p \in L\} \cup\{p \in \mathrm{AT} \mid p \in L\} .9$
Obviously, $\operatorname{Insert}(L, I)$ is the result of changing $I$ as little as possible to accommodate $L$. Moreover, it is also quite reasonable to treat equally all partial models of the new information. Therefore, we argue that partial model assertion is a natural and justified update operator.

As a concrete example, let us recall again Example 17. Since $[K]=[C n(\neg w \wedge \neg b)]=$ $\left\{\}\}\right.$, we have $K \diamond_{P M} w \vee b=\operatorname{Th}(\{\operatorname{Insert}(L,\{ \}) \mid L \in P M(w \vee b)\})$. As we have already seen that $P M(w \vee b)=\{\{w, b\},\{w, \neg b\},\{\neg w, b\}\}$. From $\operatorname{Insert}(\{w, b\},\{ \}\}=$ $\{w, b\}, \operatorname{Insert}(\{w, \neg b\},\{ \})=\{w\}$ and $\operatorname{Insert}\{\neg w, b\},\{ \}=\{b\}$, it follows that $K \diamond_{P M} \alpha=$ $\operatorname{Th}(\{\{w, b\},\{w\},\{b\}\})$. Note that, this result is same as that obtained with the WSS $_{\downarrow}$.

The following result shows that partial model assertion is equivalent to the WSS $_{\downarrow}$.
Theorem 6.6. Let I be a possible world. Then for any sentence $\alpha$,

$$
I \diamond_{W S S_{\downarrow}} \alpha=I \diamond_{P M} \alpha
$$

This theorem gives a strong support to the $\mathrm{WSS}_{\downarrow}$. In particular, it indicates that the $\mathrm{WSS}_{\downarrow}$ is not as liberal as it appears.

## Dealing with Domain Constraints

Unfortunately, the $\mathrm{WSS}_{\downarrow}$ still cannot deal with domain constraints properly at least in the way defined by (6.1). Let us recall Example 18. It is not difficult to see that $A t m_{\downarrow}\left(s w_{1} \wedge\right.$ $\left.\left(s w_{1} \wedge s w_{2} \leftrightarrow l t\right)\right)=\left\{s w_{1}, s w_{2}, l t\right\}$. Therefore, $K \diamond_{W s s_{\downarrow}}\left(s w_{1} \wedge\left(s w_{1} \wedge s w_{2} \leftrightarrow l t\right)\right)=$

[^38]$\left\{\left\{s w_{1}, s w_{2}, l t\right\},\left\{s w_{2}\right\}\right\}$, where $K=C n\left(\left\{\neg s w_{1}, s w_{2}, \neg l t\right\}\right)$. This obviously does not give the expected results.

The problem which is well understood by now in the community of reasoning about actions is that domain constraints alone are not sufficient to solve the ramification problem as they do contain causality information. 10 Based on this idea, Herzig and Rifi have therefore suggested to extend the $\mathrm{WSS}_{\downarrow}$ by a so-called dependence function.

Formally, a dependence function, denoted by $D e p$, is a function which maps an atom to a set of atoms whose truth value depends on it [Thielscher, 1996]. As an atom's truth value depends on itself, a minimal requirement for a dependence function is that $p \in \operatorname{Dep}(p)$. Note that a dependence function is part of the domain specification supplements the domain constraints. Regarding Example 18, in addition to $\left.\mathrm{DC}=\left\{l t \leftrightarrow s w_{1} \wedge s w_{2}\right)\right\}$, we have $D e p\left(s w_{1}\right)=$ $\left\{s w_{1}, l t\right\}, \operatorname{Dep}\left(s w_{2}\right)=\left\{s w_{2}, l t\right\}, D e p(l t)=\{l t\}$, since $l t$ depends both on $s w_{1}$ and $s w_{2}$, but $s w_{1}, s w_{2}$ are independents of each other.

A dependence function $D e p$ can be extended to general sentences by stipulating:

$$
\operatorname{Dep}(\alpha)=\bigcup_{p \in A t m_{\downarrow}(\alpha)} D e p(p)
$$

[Herzig and Rifi, 1999] have then suggested to extend the $\mathrm{WSS}_{\downarrow}$ as follows:

$$
\begin{equation*}
I \diamond_{W S S \downarrow}^{D e p} \alpha=\left\{I^{\prime} \in[\alpha] \mid \Delta\left(I, I^{\prime}\right) \subseteq D e p(\alpha)\right\} \tag{6.2}
\end{equation*}
$$

Note that the extended $\mathrm{WSS}_{\downarrow}$ is a generalization of $\mathrm{WSS}_{\downarrow}$, in the sense, that the two coincide when $\operatorname{Dep}(p)=\{p\}$ for all $p \in \mathrm{AT}$.

The extended $\mathrm{WSS}_{\downarrow}$ can deal with domain constraints in a very straightforward way:

$$
K \diamond_{W S S_{\downarrow}}^{\mathrm{DC}} \alpha=\operatorname{Th}\left(\bigcup_{I \in[K]} I \diamond_{W S S_{\downarrow}}^{D e p} \alpha\right)+\bigwedge \mathrm{DC}
$$

or equivalently,

$$
K \diamond_{W S S_{\downarrow}}^{\mathrm{DC}} \alpha=\operatorname{Th}\left(\left\{J \in[\mathrm{DC}] \mid J \in \bigcup_{I \in[K]} I \diamond_{W S S}^{D e p} \alpha\right\}\right)
$$

[^39]To see how the extended $\mathrm{WSS}_{\downarrow}$ works, let us recall once again Example 18. Since $\operatorname{Dep}\left(s w_{1}\right)=\left\{s w_{1}, l t\right\}$, we now have $K \diamond_{W S S_{\downarrow}}^{\mathrm{DC}} s w_{1}=\operatorname{Th}\left(\left\{\left\{s w_{1}, s w_{2}, l t\right\}\right\}\right)$. This shows that the extended $\mathrm{WSS}_{\downarrow}$ produces the desirable results.

In fact, I am able to show that the extended $\mathrm{WSS}_{\downarrow}$ also works with the another renown benchmark example of the ramification problem.

Example 19. [Thielscher, 1997] We argument Example 18 by introducing a third switch, plus a relay (cf. Figure 6.2.1). If the relay is activated (represented by re), it will force the second switch to jump open. The relay is controlled by the first and third switch together.


Figure 6.2: An extended circuit

Let us investigate the particular state depicted. The expected result of toggling the first switch, $s w_{1}$, is that the relay becomes activated, which in turn causes the second switch, $s w_{2}$, jumping its position; hence the light bulb stays off. 11

Formally, the dependencies among all components of the extended circuit are described by the following domain constraints and dependence function:

$$
\begin{aligned}
\mathrm{DC} & =\left\{s w_{1} \wedge s w_{2} \leftrightarrow l t, s w_{1} \wedge s w_{3} \leftrightarrow r e, r e \rightarrow \neg s w_{2}\right\} \\
\operatorname{Dep}\left(s w_{1}\right) & =\left\{s w_{1}, s w_{2}, r e, l t\right\} \\
\operatorname{Dep}\left(s w_{2}\right) & =\left\{s w_{2}, l t\right\} \\
\operatorname{Dep}\left(s w_{3}\right) & =\left\{s w_{3}, s w_{2}, r e, l t\right\} \\
\operatorname{Dep}(r e) & =\left\{r e, s w_{2}, l t\right\}
\end{aligned}
$$

[^40]The state displayed can be described by belief set $K=C n\left\{\neg s w_{1}, s w_{2}, s w_{3}, \neg r e, \neg l t\right\}$. It is not difficult to see that $K \diamond_{W S S_{\downarrow}}^{\mathrm{DC}} s w_{1}=\operatorname{Th}\left\{\left\{s w_{1}, s w_{3}, r e\right\}\right\}$. This result is obviously what we expected.

## Dealing with Conditional Effects

So far, the new information is assumed to be represented by a sentence. In reasoning about action, we however often need to deal with actions with conditional effects. Therefore, the $\mathrm{WSS}_{\downarrow}$ needs to be further extended to deal with conditional update.

Formally, a conditional update is of the form $\varphi \Rightarrow \alpha$, where $\varphi, \alpha$ are sentences and $\varphi$ can be considered as the precondition of $\alpha$. When a possible world $I$ is updated by $\varphi \Rightarrow \alpha: I$ will remain unchanged when $I \not \vDash \varphi$; otherwise $I$ is updated by $\alpha$ :

$$
I \diamond_{W S S \downarrow}^{D e p}(\varphi \Rightarrow \alpha)= \begin{cases}I \diamond_{W S S \downarrow}^{D e p} \alpha & \text { if } I \models \varphi  \tag{6.3}\\ \{I\} & \text { otherwise }\end{cases}
$$

Note that, if $\varphi=\top$ then $I \diamond_{W S S_{\downarrow}}^{D e p} \varphi \Rightarrow \alpha$ will be same as $I \diamond_{W S S \downarrow}^{D e p} \alpha$.
Accordingly, the $\mathrm{WSS}_{\downarrow}$ can be extended as follows:

$$
\begin{equation*}
K \diamond_{W S S_{\downarrow}}^{\mathrm{DC}}(\varphi \Rightarrow \alpha)=\operatorname{Th}\left(\bigcup_{I \in[K]} I \diamond_{W S S_{\downarrow}}^{D e p}(\varphi \Rightarrow \alpha)\right)+\bigwedge \mathrm{DC} \tag{6.4}
\end{equation*}
$$

### 6.2.2 Computational Model

It is clear that the (extended) $\mathrm{WSS}_{\downarrow}$ is not suitable for computation, as it is based on possible worlds. In a computational setting, the beliefs of an agent should be represented by a belief base, and an update operator should directly operate on belief bases. In the sequel, a computational update operator, which maps a belief base and the new information to an updated belief base, is denote by © (possibly indexed).

Ideally, a computational update operator is semantically characterized by the $\mathrm{WSS}_{\downarrow}$. An update operator © is called a computational model or syntactical characterization of the WSS $\downarrow$ iff it satisfies the following condition:

$$
\begin{equation*}
C n(B \odot \alpha)=C n(B) \diamond_{W S S_{\downarrow}} \alpha \tag{6.5}
\end{equation*}
$$

Put in words, for any belief base $B$ and sentence $\alpha$, the updated belief base $B \odot \alpha$ is logically equivalent to the result of $C n(B)$ updated by $\alpha$ using the $\mathrm{WSS}_{\downarrow}$.

## Proposal of Doherty et. al.,

[Doherty et al., 1998] have proposed a succinct syntactical characterization of the $\mathrm{WSS}_{\downarrow}$, based on the notion of so-called eliminant. Formally, given a sentence $\alpha$ and an atom $p$, we write $\exists p . \alpha$ (called an eliminant of $p$ in $\alpha$ ) to denote the sentence $\alpha[p / T] \vee \alpha[p / \perp]$. If $P=\left\{p_{1}, \cdots, p_{n}\right\}$ is a set of atoms then $\exists P . \alpha$ stands for $\exists p_{1} \cdots \exists p_{n} . \alpha$, which is called an eliminant of $P$ in $\alpha$. Intuitively, an eliminant of $P$ in $\alpha$ can be viewed as a sentence representing the same knowledge of $\alpha$ about all atoms from AT $\backslash P$ and provide no information about atoms in $P$.

The syntactical characterization of the $\mathrm{WSS}_{\downarrow}$ is then defined as follows

$$
\begin{equation*}
B \odot_{D H} \alpha=\{\exists P . \beta \mid \beta \in B\} \cup\{\alpha\} \tag{6.6}
\end{equation*}
$$

where $P=A t m_{\downarrow}(\alpha)$.
Note that, there is a natural correspondence between the $\mathrm{WSS}_{\downarrow}$ and the update operator defined (6.6), in the sense that both of them try to liberate changes of the truth values of nonredundant atoms of the new information.

Not surprisingly, [Doherty et al., 1998] therefore were able to show that the latter is indeed a syntactical characterization of the $\mathrm{WSS}_{\downarrow}$.

Theorem 6.7. [Doherty et al., 1998] Let $\odot_{D H}$ be as defined by (6.6). Then for any belief base $B$ and new information $\alpha$ :

$$
C n\left(B \odot_{D H} \alpha\right)=C n(B) \diamond_{W S S_{\downarrow}} \alpha
$$

## Problems with Space Explosion

One major disadvantage of the approach proposed by Doherty et al., is that the size of the updated belief base $B \diamond_{D H} \alpha$ could be exponentially larger than the size of $B$ and $\alpha$; more precisely, in the worst case we would have $\left|B \diamond_{D H} \alpha\right|=|B|^{|\alpha|+1}+|\alpha|$. An interesting question is that could we find a more space-economic approach? A bad news is that it is unlikely we can avoid
super-polynomial space explosion. To show this, we need to introduce the notion of so-called non-uniform complexity classes, which is based on advice-taking machines [Johnson, 1990].

Formally, an advice-taking Turing machine is a Turing machine with an advice oracle, which can be considered as a function $a$ from positive integers to strings. On input $x$, the machine loads string $a(|x|)$ and then continues as usual based on two inputs $x$ and $a(|x|)$. Note that, the oracle string $a(|x|)$ only depends on the size of the input $x$. We call an advice oracle $a$ polynomial iff $|a(n)|<p(n)$ for some fixed polynomial $p$ and all positive integers $n$. If X is a usual complexity class defined in terms of resource-bounded machines (e.g., P or NP) then $\mathrm{X} /$ poly is the class of the problem that can be decided on machines with the same resource bound augmented by polynomial advice oracles. Any class X/poly is also known as the nun-uniform $X$. In particular, $\mathrm{P} /$ poly appears to be much more powerful than P . However, it has been shown very unlikely that $\mathrm{NP} \subseteq \mathrm{P} /$ poly, otherwise the polynomial hierarchy would collapse at $\Sigma_{2}^{\mathrm{p}}$ [Karp and Lipton, 1980].

The result regarding the connection between NP and P /poly be used to show that it is unlikely that there exists a computational model of the WSS $_{\downarrow}$ with polynomially space bound.

Theorem 6.8. Suppose there exist a polynomial pand a computational model $\odot$ of the $W S S_{\downarrow}$ such that $|B \odot \alpha| \leq p(|B|+|\alpha|)$ for any belief base $B$ and sentence $\alpha$. Then $\mathrm{NP} \subseteq \mathrm{P} /$ poly.

The above theorem shows that in general super-polynomial space growth is inevitable for any computational model of the WSS $\downarrow$.

In reasoning about actions, it is reasonable to assume that an action will only change a small part of the world, in other words, the size of new information $\alpha$ is usually very small compares to the size of the belief base $B$. In particular, we may assume that the size of the new information is bounded by a constant $k$. By doing so, the size of an updated belief base $B \diamond_{D H} \alpha$ is theoretically polynomially bounded by the size of $B$. But in practice this is still problematic: suppose the new information of any action is encoded by a sentence of length 4 (that is $k=4$ ); then after performing just two actions, we might end up with a new belief base which is approximately 25 times larger. Therefore, we need to find a more space-economic update operator for practice.

## Proposal of Winslett

To avoid above-mentioned space explosion, [Winslett, 1990] has proposed a computational update operator $\odot_{W}$ which approximates the WSS $_{\downarrow}$, in the sense, that an updated belief base
$B \odot_{W} \alpha$ is query equivalent (instead of logical equivalent) to the result of $C n(B)$ updated by $\alpha$ using the WSS $\downarrow .{ }^{12}$

Formally, two sets of sentences $A$ and $B$ are called query equivalent wrt. another set of sentences $C$ iff for any sentence $\beta \in C$ :

$$
A \vdash \beta \text { iff } B \vdash \beta
$$

Note that, $A$ and $B$ are logically equivalent precisely when they are query equivalent wrt. the underlying language $\mathcal{L}$.

The main idea of Winslett's computational approach is to introduce for each atom $p$ a socalled history atom $p^{\prime}$. Given a set of atoms $P$ and a sentence $\alpha$, we denoted by $\alpha\left[P / P^{\prime}\right]$ the sentence obtained by substituting every atom $p \in P$ in $\alpha$ with its history atom $p^{\prime}$.

Then, Winslett's computational update operator is formally defined as follows:

$$
\begin{equation*}
B \odot_{W} \alpha=\left\{\beta\left[P / P^{\prime}\right] \mid \beta \in B\right\} \cup\{\alpha\} \tag{6.7}
\end{equation*}
$$

where $P=A t m_{\downarrow}(\alpha)$.
Example 20. Recall Example 17. Suppose now $B=\{\neg w \wedge \neg b\}$ and $\alpha=w \vee b$. Then $B \odot_{W} \alpha=\left\{\neg w^{\prime} \wedge \neg b^{\prime}, w \vee b\right\}$, since $\operatorname{Atm}(\alpha)=\{b, w\}$.

It is easy to see that the size of the update belief base $B \odot_{W} \alpha$ is linear to the size of original belief base $B$ and the new information $\alpha$.

Obviously, in general $B \odot_{W} \alpha$ is not logical equivalent to $C n(B) \diamond_{W S S_{\downarrow}} \alpha$, as the former might contain history atoms. However, it has been shown that the two are query equivalent wrt. sentences without history atoms.

Theorem 6.9. [Winslett, 1990; Liberatore, 1997] Let B be a belief base and $\alpha$ be a sentence. Then for any sentence $\beta$ which does not contain history atoms:

$$
B \odot_{W} \alpha \vdash \beta \text { iff }\left(C n(B) \diamond_{W S S_{\downarrow}} \alpha\right) \vdash \beta
$$

For the extended WSS $_{\downarrow}$ defined by (6.4), Winslett has proposed an extended computational

[^41]update operator:
\[

$$
\begin{align*}
B \odot_{W}^{\mathrm{DC}}(\varphi \Rightarrow \alpha)= & \left\{\beta\left[P / P^{\prime}\right] \mid \beta \in B\right\} \\
& \cup\left\{\varphi\left[P / P^{\prime}\right] \rightarrow \alpha\right\} \cup \mathrm{DC}  \tag{6.8}\\
& \cup\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right) \mid p \in P\right\}
\end{align*}
$$
\]

where $P=\operatorname{Dep}(\alpha)$.
Also, the extended computational update operator defined by (6.8) has been shown approximating the extended WSS $_{\downarrow}$.

Theorem 6.10. Winslett, 1990] Let DC be the domain constraints and Dep be a dependence function. Assume $B$ is a belief base and $\varphi \Rightarrow \alpha$ is a conditional update. Then for any sentence $\beta$ without history atoms:

$$
B \odot_{W}^{\mathrm{DC}}(\varphi \Rightarrow \alpha) \vdash \beta \text { iff }\left(C n(B) \diamond_{W S S_{\downarrow}}^{\mathrm{DC}}(\varphi \Rightarrow \alpha)\right) \vdash \beta
$$

### 6.3 Updating Possibilistic Beliefs

InSection 4.3.3, I have presented operators for revising possibilistic belief states, i.e., possibility distributions and possibilistic bases. Note that, possibilistic belief states are arguably more practical than any other representations of beliefs (e.g., belief sets and OCFs) in the literature. In this section, I will present update operators for possibilistic belief states. Like in Section 4.3.3, I give both a semantic model for updating possibility distributions, ${ }^{13}$ and a computational model for updating possibilistic bases. As we will see that the semantic model is based on the (extended) $\mathrm{WSS}_{\downarrow}$, whereas the computational model generalizes the proposal of Winslett.

### 6.3.1 Updating Possibility Distributions

Recall that a possibility distribution $\pi$ is a mapping from the set of all possible worlds $\Theta_{\mathcal{L}}$ to the unit interval $[0,1]$, such that more plausible worlds are assigned higher weights. In this section, we are interested in update operators which can deal with domain constraints. Therefore, I will first show how domain constraints can be seamlessly encoded in a possibility distribution. Note

[^42]that domain constraints are assumed indefeasible, possible worlds that violate them are therefore considered as implausible. In a possibility distribution, such implausible worlds are assigned the minimal weight 0 .

Formally, a possibility distribution $\pi$ respects (encodes) domain constraints DC iff for any possible world $W \in \Theta_{\mathcal{L}}, \pi(W)=0$ precisely when $W \not \vDash \mathrm{DC}$.

Given a possibility distribution $\pi$ that respects DC , according to (4.15), for any sentence $\beta$ the following condition holds:

$$
N_{\pi}(\beta)=1 \text { iff } \mathrm{DC} \vdash \beta
$$

Put in words, all sentences must have necessary degrees less than 1 , except logical consequences of the domain constraints.

Formally, an update operator is a function which maps a possibility distribution $\pi$ and a conditional update $\varphi \Rightarrow \alpha$ to an updated possibility distribution $\pi \diamond(\varphi \Rightarrow \alpha)$. To define such an update operator, we essentially need to determine the weight of every possible world in the updated possibility distribution.

Given a possible world $W$, we denote by $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)$ the set of plausible worlds whose successor worlds contain $W$, when they are updated by $\varphi \Rightarrow \alpha: 1^{14}$

$$
\operatorname{Sup}(\varphi \Rightarrow \alpha, W)=\left\{W^{\prime} \mid W^{\prime} \models \mathrm{DC} \text { and } W \in W^{\prime} \diamond_{W S S_{\downarrow}}^{D e p} \varphi \Rightarrow \alpha\right\}
$$

Intuitively, $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)$ is the support of $W$ being the actual world after update.
It is clear that the weight of a plausible world $W$ in $\pi \diamond(\varphi \Rightarrow \alpha)$ should be obtained from the weights of its supporting plausible worlds in $\pi$. For this purpose, we need to consider two situations:

1. If $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)$ contains more than one supporting plausible worlds, then the strongest support prevails, that is, $(\pi \diamond(\varphi \Rightarrow \alpha))(W)=\max \left\{\pi\left(W^{\prime}\right) \mid W^{\prime} \in \operatorname{Sup}(\varphi \Rightarrow \alpha, W)\right\}$.
2. If $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)=\emptyset$, then $W$ is considered highly implausible. Therefore, we define $(\pi \diamond(\varphi \Rightarrow \alpha))(W)=w_{\mathbf{\Delta}}$, where $w_{\mathbf{\Delta}}$ is a relatively small weight. ${ }^{15}$

Based on above discussion, the update operator for possibility distributions is then defined

[^43]as follows:
\[

(\pi \diamond(\varphi \Rightarrow \alpha))(W)= $$
\begin{cases}0 & \text { if } W \not \models \mathrm{DC}  \tag{6.9}\\ w_{\mathbf{\Delta}} & \text { else if } \operatorname{Sup}(W)=\emptyset \\ \max \left(\left\{\pi\left(W^{\prime}\right) \mid W^{\prime} \in \operatorname{Sup}(W)\right\}\right) & \text { else }\end{cases}
$$
\]

where $\operatorname{Sup}(W)$ is a shorthand of $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)$.
Due to its definition, it is clear that an updated possibility distribution $\pi \diamond(\varphi \Rightarrow \alpha)$ also respects domain constraints DC.

Observation 6.11. Suppose $\pi$ is a possibility distribution that respects DC and $\varphi \Rightarrow$ is a conditional update. Then the updated possibility distribution $\pi \diamond(\varphi \Rightarrow \alpha)$ also respects DC.

To see an concrete example, let us recall Example 18. Suppose now our initial beliefs are encoded by a possibility distribution $\pi$ (as shown in Table 6.1). According to (4.13), we have $\operatorname{Bel}(\pi)=\operatorname{Th}\left(\left\{W_{2}\right\}\right)$. Note that possible worlds $W_{2}, W_{4}, W_{6}$ and $W_{7}$ are weighted 0 in $\pi$, since they violate the domain constraint: $\mathrm{DC}=\left\{s w_{1} \wedge s w_{2} \leftrightarrow l t\right\}$. According to (6.9), they should also be assigned 0 in $\pi \diamond s w_{1}$. Recall that $\operatorname{Dep}\left(s w_{1}\right)=\left\{s w_{1}, l t\right\}$. The results of updating every plausible world by $s w_{1}$ are represented in the $3 r d$ column of Table 6.1 (with implausible worlds crossed out). It is not difficult to see that $W_{1}$ and $W_{3}$ are not supported by any plausible world. They are therefore assigned in $\pi \diamond s w_{1}$ the relatively small weight $w_{\mathbf{\Delta}}$. Moreover, according to (6.9), $\left(\pi \diamond s w_{1}\right)\left(W_{5}\right)=\max \left\{\pi\left(W_{1}\right), \pi\left(W_{5}\right)\right\}=0.5$ since $\operatorname{Sup}\left(s w_{1}, W_{5}\right)=\left\{W_{1}, W_{5}\right\}$. Similarly, we have $\left(\pi \diamond s w_{1}\right)\left(W_{8}\right)=\max \left\{\pi\left(W_{8}\right), \pi\left(W_{3}\right)\right\}=1$. Finally, we obtain an updated possibility distribution $\pi \diamond s w_{1}$ as represented in last column. It is easy to see that this is indeed a desirable result, since $\operatorname{Bel}\left(\pi \diamond s w_{1}\right)=\operatorname{Th}\left(\left\{W_{8}\right\}\right)$ according to (4.13).

| Possible worlds | $\pi$ | $W \diamond_{W S S_{1}}^{D e p} s w_{1}$ | $\pi \diamond s w_{1}$ |
| :--- | :--- | :--- | :--- |
| $W_{1}=\{ \}$ | 0.5 | $W_{5}, W_{6}$ | $w_{\mathbf{1}}$ |
| $W_{2}=\{l t\}$ | 0 |  | 0 |
| $W_{3}=\left\{s w_{2}\right\}$ | 1 | $W_{7}, W_{8}$ | $w_{\mathbf{1}}$ |
| $W_{4}=\left\{s w_{2}, l t\right\}$ | 0 |  | 0 |
| $W_{5}=\left\{s w_{1}\right\}$ | 0.5 | $W_{5}, W_{6}$ | 0.5 |
| $W_{6}=\left\{s w_{1}, l t\right\}$ | 0 |  | 0 |
| $W_{7}=\left\{s w_{1}, s w_{2}\right\}$ | 0 |  | 0 |
| $W_{8}=\left\{s w_{1}, s w_{2}, l t\right\}$ | 0.6 | $W_{7}, W_{8}$ | 1 |

Table 6.1: An example of updating possibility distribution.

## 6．3．2 Updating Possibilistic Bases

In this section，I present a computational approach to updating possibilistic bases．Recall that a possibilistic base is a finite set of real number－weighted sentences．Like a possibility distri－ bution，a possibilistic base can also encode domain constraints DC in a straightforward way． Formally，we say that a possibilistic base $\Sigma=\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$ respects domain con－ straints DC iff $\mathrm{DC} \subseteq\left\{\beta_{1}, \cdots, \beta_{n}\right\}$ and $w_{i}=1$ precisely when $\beta_{i} \in \mathrm{DC}$ ．

Given a possibilistic base $\Sigma$ that respects DC ，it is not difficult to see that its induced possi－ bility distribution $\pi_{\Sigma}$ also respects DC．

Observation 6．12．Suppose $\Sigma$ is a possibilistic base that respects DC．Then the possibility distribution $\pi_{\Sigma}$ induced from $\Sigma$（as defined by（4．20）also respects DC．

Suppose now we are to update a possibilistic base $\Sigma$ by $\varphi \Rightarrow \alpha$ ，and the updated possibilistic base is denoted by $\Sigma_{1}=\Sigma$ 回 $(\varphi \Rightarrow \alpha)$ ．Let $\pi_{\Sigma}$ be the possibility distribution induced from $\Sigma$ and $\pi_{1}=\pi_{\Sigma} \diamond(\varphi \Rightarrow \alpha)$ ．Ideally，it should hold that for any sentence $\beta$ ：

$$
N_{\Sigma_{1}}(\beta)=N_{\pi_{1}}(\beta)
$$

Unfortunately，the discussion in Section 6.2 suggests that such an update operator $⿴ 囗 口$ will in－ evitable cause super－polynomial space explosion．

Therefore，in the same spirit of Winslett＇s computational update operator，I present here an approximation of the update operator for possibility distributions．Let $\sigma=$ $\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$ be a possibilistic base that respects DC and $\varphi \Rightarrow \alpha$ a conditional update．The updated possibilistic base，denoted by $\Sigma_{1}=\Sigma$ 回 $\left.\varphi \Rightarrow \alpha\right)$ ，is formally defined as follows：

$$
\begin{align*}
\Sigma_{1}= & \{\langle\beta, 1\rangle \mid \beta \in \mathrm{DC}\} \cup\left\{\left\langle\beta\left[P / P^{\prime}\right], 1\right\rangle \mid \beta \in \mathrm{DC}\right\} \\
& \cup\left\{\left\langle\beta_{i}\left[P / P^{\prime}\right], w_{i}\right\rangle \mid w_{i}<1\right\}  \tag{6.10}\\
& \cup\left\{\left\langle\varphi\left[P / P^{\prime}\right] \rightarrow \alpha, w \stackrel{\rightharpoonup}{ }\right)\right. \\
& \cup\left\{\left\langle\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right), w \stackrel{\rightharpoonup}{ }\right\rangle \mid p \in P\right\}
\end{align*}
$$

where $P=\operatorname{Dep}(\alpha) .{ }^{16}$

[^44]Recall Example 18 again. Now suppose our initial beliefs are encoded by a possibilistic base $\Sigma=\left\{\left\langle l t \leftrightarrow s w_{1} \wedge s w_{2}, 1\right\rangle,\left\langle\neg s w_{1}, 0.5\right\rangle,\left\langle s w_{2}, 0.5\right\rangle,\langle\neg l t, 0.6\rangle\right\}$. Since $\operatorname{Dep}\left(s w_{1}\right)=\left\{s w_{1}, l t\right\}$, we have $\Sigma_{1}=\Sigma$ 回 $s w_{1}=\left\{\left\langle l t \leftrightarrow s w_{1} \wedge s w_{2}, 1\right\rangle,\left\langle l t^{\prime} \leftrightarrow\right.\right.$ $\left.\left.s w_{1}^{\prime} \wedge s w_{2}, 1\right\rangle,\left\langle\neg s w_{1}^{\prime}, 0.5\right\rangle,\left\langle s w_{2}, 0.5\right\rangle,\left\langle\neg l t^{\prime}, 0.3\right\rangle,\left\langle s w_{1}, w_{\checkmark}\right\rangle\right\}$. It is not difficult to see that $\Sigma$ maps exactly to the possibility distribution $\pi$ represented in the $2 n d$ column of Table 6.1. However, the updated possibilistic base $\Sigma_{1}$ is not equivalent to the updated possibility distribution $\pi \diamond s w_{1}$, since the former contains history atoms.

The following result shows that the update operator for possibilistic bases indeed approximates the update operator for possibility distribution.

Theorem 6.13. Suppose $\Sigma$ is a possibilistic base that respects DC and $\pi_{\Sigma}$ is the possibility distribution induced from $\Sigma$ as defined by (4.20). Let $\pi_{1}=\pi \diamond(\varphi \Rightarrow \alpha)$ and $\Sigma_{1}=\Sigma$ 回 $\left.\varphi \Rightarrow \alpha\right)$. Then for any sentence $\beta$ that contains no history atom:

$$
N_{\Sigma_{1}}(\beta)=N_{\pi_{1}}(\beta)
$$

One advantage of my update operator is that an updated possibilistic base can be computed in polynomial time (cf. Algorithm E. 6 in Appendix E), provided so is the dependence function Dep.

Observation 6.14. Assume the dependence function Dep can be computed in polynomial time. Then for the update operator defined by (6.10), the problem of computing an updated possibilistic base is in FP .

It is also not difficult to see that the CF problem wrt. the possibilistic base update operator is coNP-complete.

Observation 6.15. Assume the dependence function Dep can be computed in polynomial time. Then for the update operator defined by (6.10), the CF problem is coNP-complete.

It is worthy mentioning that complexities of most well-known update operators have been shown beyond coNP-compete [Liberatore, 1997; Eiter and Gottlob, 1992].

[^45]
### 6.4 Discussions and Related Work

I have first shown in detail the problems of the PMA with disjunctive updates and domain constraints. In reasoning about actions, both disjunctive updates and domain constraints are however considered very important; in particular, the capability of handling domain constraints is commonly believed the key to solve the ramification problem [Shanahan, 1999; Thielscher, 1997; Ginsberg and Smith, 1987]. Then, I have presented another well-known update operator (i.e., the WSS) which is based on a weakened version of minimal change policy. By generalizing the WSS, an operator for updating possibility distributions has been proposed, which can properly handle disjunctive updates and domain constraints.

It is well understood in reasoning about actions that causality information is necessary for dealing with indirect effects. The causality information exploited by the current update operator is encoded by a dependence function. There are also proposals of other forms of causality information in the literature, e.g., causal relationships [Thielscher, 1997] or causal rules [McCain and Turner, 1995]; it is an interesting future work to do a comparison of all these proposals.

As we have seen that the update operator for possibility distributions handles properly domain constraints. Now, I show that possibilistic reinforcement revision (cf. Section 4.3.3) also can deal with domain constraints. Suppose a possibility distribution is to be revised by new information $\langle\alpha, w\rangle$. As domain constraints DC are indefeasible, it is reasonable to assume that $\alpha$ is consistent with DC and $w<1$. Then the revised possibility distribution also respects DC:

Observation 6.16. Suppose $\pi$ is a possibility distribution that respects DC. Let $\langle\alpha, w\rangle$ be the input information such that $\alpha$ is consistent with DC and $w<1$. Then the revised possibility distribution $\pi\left(.\left.\right|_{r}\langle\alpha, w\rangle\right)$ also respects DC.

Also, it is worthy mentioning that we can easily obtain an operator for updating OCFs with a slight modification of the current proposal. To my knowledge, there is no work on approaches to updating OCFs in the literature, except [Kudo et al., 1999]. Note that, a major problem of the update operator proposed by Kudo et. al., is that it suffers from the same problems of the PMA, since it satisfies all of the KM postulates.

## Summary

In this chapter, I have studied approaches of belief update that are suitable for reasoning about actions. Firstly, I have shown that the classical PMA is problematic with disjunctive updates and domain constraints. Then, based on the WSS, I have presented a (semantic) possibilistic distribution update operator, which can handle properly disjunctive updates and domain constraints. I have shown that super-polynomial space explosion is inevitable for computational models of possibilistic distribution update operator. To avoid this, I have proposed instead an approximation of the possibilistic distribution update operator.

## Chapter 7

## Conclusions and Future Work

In this thesis, I have mainly studied general frameworks and computational approaches of belief change that can be potentially applied in reasoning about actions. Firstly, the motivations of integrating belief change into reasoning about actions are described, along with the problems that need to be solved. Then, the classical work in belief change is recapitulated, including the distinction between belief revision and belief update, rationality postulates that should be satisfied by belief change operators and various constructive models. Also, I have discussed the relation between belief change and relevant research fields. The results obtained in this thesis might be also useful for these related fields, in particular, for non-monotonic reasoning. A detailed investigation on this issue is left as future work.

One of the main contributions of the thesis is a new postulate of independence for iterated belief revision. I have discussed why iterated belief revision is a difficult problem in principle. Then, I have formally analyzed the problem of implicit dependence which is intrinsic to belief revision but largely overlooked in the belief change community. Discussions show that the AGM/DP postulates are too weak, hence recommend the inception of the independence postulate. To provide formal justification, I have also presented an elegant semantic characterization of the independence postulate. As a result, a new general framework for iterated belief revision is obtained. A detailed comparison to related work suggests that the new framework is so far the most satisfactory one in the literature.

I have also studied computational approaches to iterated belief revision. First of all, the limitations of pure qualitative approaches (in particular, cut base revision) are discussed. Then, I have proposed a so-called reinforcement base revision based on a compact representation
of beliefs. To show its intuition and satisfiability of desirable postulates, I have presented a semantic model of reinforcement base revision, i.e., reinforcement OCF revision. I have also given an algorithm of reinforcement base revision which can be implemented straightforwardly, provided the underlying language is decidable. For propositional logics, I have shown that the computational complexity of reinforcement base revision is optimal. It is an interesting topic of future research to investigate the complexity of belief revision for some suitable decidable fragments of first order logic (e.g., the guarded predicate logic of [Andréka et al., 1998]). Note that reinforcement base revision has a reinforcement effect (meaning that it accumulates the evidence degree of the new information) which is arguably a desirable feature for domains with multi independent information sources. As a more practical solution to iterated belief revision, I have presented a so-called possibilistic revision, in which possibility degrees of beliefs are real numbers in the unit interval $[0,1]$.

To study formally the problem of "how agents exchange their beliefs in a rational manner?", I have introduced the concept of mutual belief revision. A process of mutual belief revision takes two stages; in the first stage, two agents get together trying to reach a common understanding; in the second stage, two agents revise their belief states in order to incorporate the common agreed views into their own belief states. In particular, I have presented both a semantic model and a computational model for a concrete mutual revision operator by generalizing reinforcement OCF revision and reinforcement base revision, respectively. I have shown several desirable properties of the mutual revision operator, as well the complexity of the computational model. Note that the work presented here is just the starting point of our research in formal study of information exchange. The properties listed in this thesis can be considered as the minimal requirements for any rational mutual revision operators. It is of our great interest to further study other desirable properties, and another interesting avenue of further research is to obtain a set of AGM-style rationality postulates for mutual belief revision.

For reasoning about actions, belief update is very important to model actions with physical effects. I have shown that the classical KM theory is problematic with disjunctive updates and domain constraints. Based on the WSS, I have presented a (semantic) possibilistic distribution update operator, which can handle properly disjunctive updates and domain constraints. I have shown that super-polynomial space explosion is inevitable for computational models of possibilistic distribution update operator. To avoid this, I have proposed instead a computational approximation of the possibilistic distribution update operator. Note that, the current approaches of belief update are specific to propositional languages, and extending these approaches to more
expressive languages is an interesting topic of future research.
Note that both types of belief change studied in this thesis have their own expectations on the reasons of the new information: belief update assumes that the new information reflects changes of the world, whereas the world is assumed static in belief revision. However, in many situations, the new information received by the agent might be not labeled with such a reason. There are also examples which require non-elementary interactions between revision and update [Boutilier, 1995; Hunter and Delgrande, 2005]. It is of course interesting future work to investigate the possibility of extending the current approaches to handle such situations.

## Appendix A

## Proofs of Results of Chapter 3

To prove Theorem 3.2, I first show by Observation A. 1 and A. 2 that, given a belief state $\mathcal{K}$, there is a one-to-one correspondence between a SOS centered $[\mathcal{K}]$ and a faithful ranking on $\mathcal{K}$.

Observation A.1. Let $\mathcal{K}$ be a belief state. Given a SOS $\mathfrak{S}$ centered on $[\mathcal{K}]$, the binary relation $\preceq_{\mathcal{K}}$ on $\Theta_{\mathcal{L}}$ generated from $\mathfrak{S}$ via the following condition:
(SSToFR) $\quad W_{1} \preceq_{\mathcal{K}} W_{2}$ iff for all $S \in \mathfrak{S}$ if $W_{2} \in S$ then $W_{1} \in S$
is a faithful ranking on $\mathcal{K}$, s.t., for any consistent sentence $\alpha$ :

$$
\min _{\mathfrak{S}}(\alpha) \cap[\alpha]=\min \left([\alpha], \preceq_{\mathcal{K}}\right)
$$

 $\preceq_{\mathcal{K}}$ is transitive, assume $W_{1} \preceq_{\mathcal{K}} W_{2}$ and $W_{2} \preceq_{\mathcal{K}} W_{3}$. We have to show $W_{1} \in S$ for all $S \in \mathfrak{S}$ such that $W_{3} \in S$. Suppose $W_{3} \in S$. It follows from $W_{2} \preceq_{\mathcal{K}} W_{3}$ that $W_{2} \in S$. Subsequently from $W_{1} \preceq_{\mathcal{K}} W_{2}$, it follows that $W_{1} \in S$. The connectiveness of $\preceq_{\mathcal{K}}$ is shown by contradiction. Assume there exist $W_{1}, W_{2}$ such that neither $W_{1} \preceq_{\mathcal{K}} W_{2}$ nor $W_{2} \preceq_{\mathcal{K}} W_{1}$. It follows that there are $S_{1}, S_{2} \in \mathfrak{S}$ such that $W_{2} \in S_{1}, W_{1} \notin S_{1}$ and $W_{1} \in S_{2}, W_{2} \notin S_{s}$. Obviously, then neither $S_{1} \subseteq S_{2}$ nor $S_{2} \subseteq S_{1}$, which contradicts the fact that $\mathfrak{S}$ is a SOS (see Definition 2.15).

We show next that $\preceq_{\mathcal{K}}$ satisfies all conditions of Definition 3.1. Assume $W_{1}, W_{2} \models \mathcal{K}$. Since for all $S \in \mathfrak{S},[\mathcal{K}] \subseteq S$, it follows that $W_{1} \in S$ for all $S \in \mathfrak{S}$. It follows immediately that $W_{1} \preceq_{\mathcal{K}} W$ for all $W \in \Theta_{\mathcal{L}}$. Similarly, $W_{2} \preceq_{\mathcal{K}} W$ for all $W \in \Theta_{\mathcal{L}}$. Hence, we have $W_{1}=\mathcal{K} W_{2}$. Suppose $W_{1} \models \mathcal{K}, W_{2} \not \models \mathcal{K}$. Since $[\mathcal{K}] \in \mathfrak{S}$, it follows from (SSToFR) that
$W_{2} \preceq_{\mathcal{K}} W_{1}$. Hence $W_{1} \prec_{\mathcal{K}} W_{2}$. What remains to show is that there exist a set of minimal elements $\min \left([\alpha], \preceq_{\mathcal{K}}\right)$, for any consistent sentence $\alpha$. Suppose $\alpha$ is a consistent sentence. Since there exists a minimal sphere $\min _{\mathfrak{S}}(\alpha)$ intersecting $[\alpha]$, it suffice to show that $\min _{\mathfrak{S}}(\alpha) \cap[\alpha]=$ $\min \left([\alpha], \preceq_{\mathcal{K}}\right)$ :

1. To show by contradiction that $\min _{\mathfrak{S}}(\alpha) \cap[\alpha] \subseteq \min ([\alpha], \preceq \mathcal{K})$, assume there exists a possible world $W$ such that $W \in \min _{\mathfrak{S}}(\alpha) \cap[\alpha]$ and $W \notin \min \left([\alpha], \preceq_{\mathcal{K}}\right)$. It follows from $W \notin \min \left([\alpha], \preceq_{\mathcal{K}}\right)$ that there is a $W_{1} \in[\alpha]$ and $W \preceq_{\mathcal{K}} W_{1}$. According to (SSToFR), there exists $S \in \mathfrak{S}$ such that $W_{1} \in S$ and $W \notin S$. Since $S \cap[\alpha] \neq \emptyset$, we have $\min _{\mathfrak{S}}(\alpha) \subseteq S$, this contradicts $W \in \min _{\mathfrak{S}}(\alpha)$.
2. Similarly, to show $\min \left([\alpha], \preceq_{\mathcal{K}}\right) \subseteq \min _{\mathcal{S}}(\alpha) \cap[\alpha]$, we assume there exists a possible world $W$ such that $W \in \min \left([\alpha], \preceq_{\mathcal{K}}\right)$ and $W \notin \min _{\mathfrak{S}}(\alpha) \cap[\alpha]$. It follows immediately that $W \notin \min _{\mathfrak{S}}(\alpha)$. According to (SSToFR) we have $W \npreceq \mathcal{K} W_{1}$, for all $W_{1} \in \min _{\mathfrak{S}}(\alpha)$. Since $\min _{\mathfrak{S}}(\alpha) \cap[\alpha] \neq \emptyset$, it follows there exists a possible $W_{1}$ such that $W \npreceq_{\mathcal{K}} W_{1}$ and $W_{1} \in[\alpha]$, this contradicts $W \in \min ([\alpha], \preceq \mathcal{K})$.

Observation A.2. Let $\mathcal{K}$ be a belief state. Given faithful ranking $\preceq_{\mathcal{K}}$ on $\mathcal{K}$ the family of sets of possible worlds $\mathfrak{S}$ constructed from $\preceq_{\mathcal{K}}$ via the following condition
(FRToSS) $\left\{\begin{array}{l}\text { For all } W \in \Theta_{\mathcal{L}},\left\{W^{\prime} \in \Theta_{\mathcal{L}} \mid W^{\prime} \preceq \mathcal{K} W\right\} \in \mathfrak{S} \\ \Theta_{\mathcal{L}} \in \mathfrak{S} \\ \text { If }[\mathcal{K}]=\emptyset \text { then } \emptyset \in \mathfrak{S} .\end{array}\right.$
is a SOS centered on $[\mathcal{K}]$, s.t., for any consistent sentence $\alpha$ :

$$
\min _{\mathfrak{S}}(\alpha) \cap[\alpha]=\min \left([\alpha], \preceq_{\mathcal{K}}\right)
$$

Proof. We need to show that $\mathfrak{S}$ satisfies all conditions of Definition 2.15, Let $S_{1}$ and $S_{2}$ be two spheres in $\mathfrak{S}$. Without loss of generality, we assume $S_{1}=\left\{W \in \Theta_{\mathcal{L}} \mid W \preceq_{\mathcal{K}} W_{1}\right\}$ and $S_{2}=\left\{W \in \Theta_{\mathcal{L}} \mid W \preceq_{\mathcal{K}} W_{2}\right\}$. Since $\preceq_{\mathcal{K}}$ is total, we have either $W_{1} \preceq_{\mathcal{K}} W_{2}$ or $W_{2} \preceq_{\mathcal{K}} W_{1}$. It follows that either $S_{1} \subseteq S_{2}$ or $S_{2} \subseteq S_{1}$. If $\mathcal{K}=\emptyset$, it is obvious that $[\mathcal{K}] \in \mathfrak{S}$ and $\emptyset \subseteq S$ for all $S \in \mathfrak{S}$. Suppose $\mathcal{K}$ is consistent and $W \in[\mathcal{K}]$. It follows from the properties of $\preceq_{\mathcal{K}}$ that $W_{1} \preceq$
$W$ iff $W_{1} \in[\mathcal{K}]$. Therefore $[\mathcal{K}]=\left\{W_{1} \mid W_{1} \preceq \mathcal{K} W\right\} \in \mathfrak{S}$ and $[\mathcal{K}] \subseteq S$ for all $S \in \mathfrak{S}$. Hence $\mathfrak{S}$ satisfies the second condition. The third condition holds trivially. To show that $\mathfrak{S}$ satisfies the fourth condition, for any consistent sentence $\alpha$, we consider $\min _{\mathfrak{S}}(\alpha)=\left\{W_{1} \mid W_{1} \preceq_{\mathcal{K}} W\right\}$, where $W \in \min \left([\alpha], \preceq_{\mathcal{K}}\right)$. It follows immediately that $\min _{\mathfrak{S}}(\alpha) \cap[\alpha]=\min \left([\alpha], \preceq_{\mathcal{K}}\right)$. What remains to show is that $\min _{\mathfrak{S}}(\alpha)$ is indeed the minimal sphere intersects $[\alpha]$. Suppose there is a sphere $S$ such that $S \subset \min _{\mathfrak{S}}(\alpha)$ and $S \cap[\alpha] \neq \emptyset$. Without loss of generality, we assume $S=\left\{W_{1} \mid W_{1} \preceq_{\mathcal{K}} W_{2}\right\}$. It follow from $S \subset \min _{\mathfrak{S}}(\alpha)$ that $W_{2} \prec_{\mathcal{K}} W$. Since $S \cap[\alpha] \neq \emptyset$, there must be a possible world $W_{1}$ such that $W_{1} \in[\alpha]$ and $W_{1} \preceq_{\mathcal{K}} W_{2}$, which contradicts $W \in \min \left([\alpha], \preceq_{\mathcal{K}}\right)$.

Theorem 3.2. Suppose $*$ is an iterated revision operator. Then $*$ satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ iff there exists a faithful assignment h, s.t., for any belief state $\mathcal{K}$ and any sentence $\alpha$ :

$$
\operatorname{Bel}(\mathcal{K} * \alpha)= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \operatorname{Th}\left(\min \left([\alpha], \preceq_{\mathcal{K}}^{h}\right)\right. & \text { otherwise }\end{cases}
$$

Proof.
$\Leftarrow$ Assume $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Obviously, for any belief state $\mathcal{K}$, we can induce a (local) revision operator for $\operatorname{Bel}(\mathcal{K})$, which satisfies $\left(\mathrm{K}^{*} 1\right)$ $(K * 8)$. It follows from Theorem 2.17 that there exists a SOS $\mathfrak{S}$ centered on $[\mathcal{K}]$ such that for any sentence $\alpha$,

$$
\operatorname{Bel}(\mathcal{K} * \alpha)= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \left.\operatorname{Th}\left(\min _{\mathfrak{S}}(\alpha) \cap[\alpha]\right)\right) & \text { othwise }\end{cases}
$$

By Observation A.1, we can generate from $\mathfrak{S}$ a faithful ranking $\preceq_{\mathcal{K}}$ on $\mathcal{K}$ such that for any consistent sentence $\alpha, \min _{\mathfrak{S}}(\alpha) \cap[\alpha]=\min ([\alpha], \preceq \mathcal{K})$. Thus for any belief state $\mathcal{K}$, we can induce (assign) a faithful ranking $\preceq_{\mathcal{K}}$ on $\mathcal{K}$ such that, for any sentence $\alpha$,

$$
\operatorname{Bel}(\mathcal{K} * \alpha)= \begin{cases}\mathcal{L} & \text { if } \vdash \neg \alpha \\ \operatorname{Th}(\min ([\alpha], \preceq \mathcal{K})) & \text { othwise }\end{cases}
$$

$\Rightarrow$ Suppose $h$ is a faithful assignment as required. For any belief state $\mathcal{K}$, by Observation A.2, we can construct a SOS $\mathfrak{S}$ centered on $[\mathcal{K}]$ from $\preceq_{\mathcal{K}}^{h}$, such that for any consis-
tent sentence $\alpha, \min _{\mathcal{S}}(\alpha) \cap[\alpha]=\min \left([\alpha], \preceq_{\mathcal{K}}^{h}\right)$. It follows from Theorem 2.17, that the local revision operator (induced from $*$ ) on $\operatorname{Bel}(\mathcal{K})$ satisfies $\left(\mathrm{K}^{*} 1\right)-\left(\mathrm{K}^{*} 8\right)$. Thus $*$ satisfies $(\mathcal{K} * 1)$ $(\mathcal{K} * 8)$.

Observation 3.5. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If $*$ satisfies (DP2), then it also satisfies (WDP2).

Proof. Suppose $*$ satisfies (DP2) and $\beta \in \operatorname{Bel}(\mathcal{K} * \neg \alpha)$. Since $\alpha \vdash \neg \neg \alpha$, it follows from (DP2) that $\mathcal{K} * \neg \alpha \equiv(\mathcal{K} * \alpha) * \neg \alpha$. Hence $\beta \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \alpha)$.

Observation 3.6. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If $*$ satisfies (Ind), then it also satisfies (WInd).

Proof. Suppose $*$ satisfies (Ind) and $\alpha \in \operatorname{Bel}(\mathcal{K} * \neg \beta)$. Assume $\vdash \beta$. Then $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) *$ $\neg \beta)=\mathcal{L}$, due to $(\mathcal{K} * 1)$ and $(\mathcal{K} * 8)$. Assume $\nvdash \beta$. If follows from $(\mathcal{K} * 5)$ that $\operatorname{Bel}(\mathcal{K} * \neg \beta)$ is consistent. Thus $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \neg \beta)$. According to (Ind), we have $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \beta)$.

To prove Theorem 3.7, we need the following Observation A.3':

Observation A.3. Suppose $*$ is an iterated revision operator that satisfies Postulates $(\mathcal{K} * 1)$ $(\mathcal{K} * 8)$. If $\beta$ is consistent, then $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$ precisely when there exists a world $W_{1}$ such that $W_{1} \models \alpha \wedge \beta$ and $W_{1} \prec_{\mathcal{K}} W_{2}$ for all $W_{2} \models \neg \alpha \wedge \beta$, where $\preceq_{\mathcal{K}}$ is a faithful ranking on $\mathcal{K}$ corresponding to $*$.

## Proof.

$\Rightarrow$ Suppose $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$. Since $\beta$ is consistent, $\min \left([\beta], \preceq_{\mathcal{K}}\right) \neq \emptyset$. Assume $W_{1}$ is possible worlds such that $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K}}\right)$. It is obvious $W_{1} \models \alpha$, since $\alpha \in \bigcap \min \left([\beta], \preceq_{\mathcal{K}}\right)$. Assume there is $W_{2}$ such that $W_{2} \models \neg \alpha \wedge \beta$ and $W_{1} \prec_{\mathcal{K}} W_{2}$. Since $\preceq_{\mathcal{K}}$ is total, we have $W_{2} \preceq_{\mathcal{K}} W_{1}$. Hence $W_{2} \in \min \left([\beta], \preceq_{\mathcal{K}}\right)$, which contradicts $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$.
$\Leftarrow$ Assume there exists a $W_{1}$ such that $W_{1} \models \alpha \wedge \beta$ and $W_{1} \prec_{\mathcal{K}} W_{2}$ for any $W_{2} \models \neg \alpha \wedge \beta$. It follows that for all $W \in \min ([\beta], \preceq \mathcal{K}), W \not \models \neg \alpha$, i.e., $W \models \alpha$. Thus $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$.

Theorem 3.7. Suppose $\mathcal{L}$ is a finitary propositional logic. Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulate (Ind) iff its corresponding faithful assignment satisfies the following condition:

$$
\text { (IndR) If } W_{1} \models \alpha \text { and } W_{2} \models \neg \alpha \text {, then } W_{1} \preceq_{\mathcal{K}} W_{2} \text { implies } W_{1} \prec_{\mathcal{K} * \alpha} W_{2} \text {. }
$$

## Proof.

$\Leftarrow$ Assume $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \neg \beta)$. From Observation A.3, it follows that for any world $W \models$ $\neg \beta \wedge \neg \alpha$, there exists another world $W^{\prime} \models \neg \beta \wedge \alpha$ such that $W^{\prime} \preceq_{\mathcal{K}} W$. Hence, since $\preceq_{\mathcal{K}}$ is total, there must be a world $W_{1}$ such that $W_{1} \models \alpha \wedge \neg \beta$ and $W_{1} \preceq_{\mathcal{K}} W_{2}$ for all $W_{2} \models$ $\neg \alpha \wedge \neg \beta$. Condition (IndR) then implies that $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$ for all $W_{2} \models \neg \alpha \wedge \neg \beta$. Due to Observation A.3, we have $(\mathcal{K} * \alpha) * \neg \beta \vdash \alpha$.
$\Rightarrow$ Assume $W_{1} \models \alpha, W_{2} \models \neg \alpha$, and $W_{1} \preceq \mathcal{K} W_{2}$. Since $\mathcal{L}$ is finite, there exists $\neg \beta$ such that $[\neg \beta]=\left\{W_{1}, W_{2}\right\}$ (i.e., $\neg \beta=\wedge W_{1} \vee \wedge W_{2}$ ). From Theorem 3.2, it follows that $W_{1} \in[\mathcal{K} * \neg \beta]$. Hence $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \neg \beta)$. Postulate (Ind) implies $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \neg \beta)$. Due to Postulates $(\mathcal{K} * 2)$ and $(\mathcal{K} * 5),[(\mathcal{K} * \alpha) * \neg \beta]=\left\{W_{1}\right\}$. From Theorem 3.2, it follows that $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$.

Observation 3.8. Suppose $*$ is an iterated revision operator that satisfies $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. If $*$ satisfies (Ind), then it also satisfies (DP3) and (DP4).

Proof. Assume $\alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$. If $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \beta)$, then (Ind) implies $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$. If $\neg \alpha \in \operatorname{Bel}(\mathcal{K} * \beta)$, then $\operatorname{Bel}(\mathcal{K} * \beta)$ is inconsistent. According to $(\mathcal{K} * 5), \beta$ is inconsistent. Hence $\alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$, due to $\left(\mathcal{K}^{*} 1\right)$ and $(\mathcal{K} * 2)$. Thus $*$ satisfies (DP3).

Assume $\neg \alpha \notin \operatorname{Bel}(\mathcal{K} * \beta)$. It follows from ( $\mathcal{K} * 2$ ) that $\beta$ is consistent. (Ind) implies $\alpha \in$ $\operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$. Assume $\neg \alpha \in \operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$. Then $\operatorname{Bel}((\mathcal{K} * \alpha) * \beta)$ is inconsistent, which contradicts ( $\mathcal{K} * 5$ ). Thus $*$ satisfies (DP4).

Theorem 3.10, Let $*$ be an iterated revision operator satisfying Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Then $*$ satisfies Postulates (DP1)-(DP4) if it is induced from a faithful assignment that satisfies the following conditions:
(DPR1) If $W_{1}, W_{2} \models \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.
(DPR2) If $W_{1}, W_{2} \not \models \alpha$, then $W_{1} \preceq_{\mathcal{K}} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.
(DPR3) If $W_{1} \models \alpha$ and $W_{2} \not \vDash \alpha$, then $W_{1} \prec_{\mathcal{K}} W_{2}$ implies $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$.
(DPR4) If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \preceq \mathcal{K} W_{2}$ implies $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$.

Proof.
$(\mathrm{DPR} 1) \Rightarrow(\mathrm{DP} 1)$ :
Suppose that (DPR1) holds. Assume $\beta \vdash \alpha$. It follows immediately that $[\beta] \subseteq[\alpha]$. It follows from (DPR1), $W_{1} \preceq_{\mathcal{K}} W_{2}$ iff $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$ for all $W_{1}, W_{2} \in[\beta]$. Hence, $\bigcap \min \left([\beta], \preceq_{\mathcal{K}}\right)=$ $\bigcap \min \left([\beta], \preceq_{\mathcal{K} * \alpha}\right)$. According to Theorem 3.2, $\mathcal{K} * \beta \equiv(\mathcal{K} * \alpha) * \beta$.
$(\mathrm{DPR} 2) \Rightarrow(\mathrm{DP} 2)$ :
Symmetric to the one above.
$(\mathrm{DPR} 3) \Rightarrow(\mathrm{DP} 3)$ :
Suppose that (DPR3) holds. Assume $\mathcal{K} * \beta \vdash \alpha$. If $\beta$ is inconsistent, then by $(\mathcal{K} * 1)$ and $(\mathcal{K} * 2)$ we have $(\mathcal{K} * \alpha) * \beta \vdash \alpha$. Assume $\beta$ is consistent. By Observation A.3, there exists $W_{1} \models \beta \wedge \alpha$ and $W_{1} \prec_{\mathcal{K}} W_{2}$ for all $W_{2} \models \beta \wedge \neg \alpha$. Therefore, by (DPR3) there exist $W_{1} \vDash \beta \wedge \alpha$ and $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$ for all $W_{2} \models \beta \wedge \neg \alpha$. Apply Observation A.3 in another direction, we obtain $(\mathcal{K} * \alpha) * \beta \vdash \alpha$.
(DPR4) $\Rightarrow$ (DP4):
Suppose that (DPR4) holds. In the case $\beta$ is inconsistent, (DP4) holds vacuously, since $\mathcal{K} *$ $\beta \nvdash \neg \alpha$ contradicts with $\left(\mathcal{K}^{*} 1\right)$ and $(\mathcal{K} * 2)$. Assume $\mathcal{K} * \beta \nvdash \neg \alpha$ and $\beta$ is consistent. By Observation A.3, for all $W_{2} \models \beta \wedge \neg \alpha$, there exists $W_{1} \models \beta \wedge \alpha$ such that $W_{1} \preceq_{\mathcal{K}} W_{2}$. It follows from (DPR4), for all $W_{2} \models \beta \wedge \neg \alpha$, there exists $W_{1} \models \beta \wedge \alpha$ such that $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$. Apply Observation A. 3 in another direction, we obtain $(\mathcal{K} * \alpha) * \beta \nvdash \neg \alpha$.

Theorem 3.13. Suppose that an iterated revision operator satisfies Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. The operator satisfies Postulates (DP1),(DP2) and (Ind) iff the operator and its corresponding faithful assignment satisfy:

$$
\begin{array}{ll}
\left(D P R l^{\triangleleft}\right) & \text { If } W_{1}, W_{2} \models \alpha, \text { then } W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2} \text { iff } W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2} . \\
\left(D P R 2^{\triangleleft}\right) & \text { If } W_{1}, W_{2} \not \vDash \alpha, \text { then } W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2} \text { iff } W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2} .
\end{array}
$$

(IndR $\left.{ }^{\triangleleft}\right) \quad$ If $W_{1} \models \alpha$ and $W_{2} \not \vDash \alpha$, then $W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2}$ implies either $W_{1}, W_{2}$ are not relevantly ordered $w r t$. $\preceq_{\mathcal{K} * \alpha}$ or $W_{1} \prec_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$.

Proof.

1. (DP1) is equivalent to $\left(\mathrm{DPR}^{\triangleleft}\right)$.
$(\mathrm{DP} 1) \Rightarrow\left(\mathrm{DPR}^{\triangleleft}\right)$ :
Suppose that (DP1) holds. Assume $W_{1}, W_{2} \models \alpha$ and $W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2}$. It follows that there exists $\gamma$ such that $W_{1}, W_{2} \models \gamma$ and $W_{1} \in \min \left([\gamma], \preceq_{\mathcal{K}}\right)$. It follows from $\alpha \wedge \gamma \vdash \alpha$ and (DP1) that $\min \left([\alpha \wedge \gamma], \preceq_{\mathcal{K}}\right)=\min \left([\alpha \wedge \gamma], \preceq_{\mathcal{K} * \alpha}\right)$. Since $W_{1} \models \gamma$ and $W_{1} \in$ $\min \left([\alpha], \preceq_{\mathcal{K}}\right)$, we have $W_{1} \in \min \left([\alpha \wedge \gamma], \preceq_{\mathcal{K}}\right)$. Hence $W_{1} \in \min ([\alpha \wedge \gamma], \preceq \mathcal{K} * \alpha)$. If follows from $W_{2} \in[\alpha \wedge \gamma]$, that $W_{1} \preceq_{\mathcal{K} * \alpha} W_{2}$. Together with $W_{1}, W_{2} \models \alpha \wedge \gamma$, and $W_{1} \in \min \left([\alpha \wedge \gamma], \preceq_{\mathcal{K} * \alpha}\right)$, we obtain $W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$. Analogously, we also have that $W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$ implies $W_{1} \preceq^{\triangleleft}{ }_{\mathcal{K}} W_{2}$.
$\left(\mathrm{DPR}^{\triangleleft}\right) \Rightarrow(\mathrm{DP} 1):$
Suppose that $\left(\mathrm{DPR} 1^{\triangleleft}\right)$ holds. Assume $\beta \vdash \alpha$. We want to prove that $(\mathcal{K} * \alpha) * \beta \equiv$ $\mathcal{K} * \beta$. We do this by showing $\min ([\beta], \preceq \mathcal{K})=\min \left([\beta], \preceq_{\mathcal{K} * \alpha}\right)$. Given any $W_{1}$ such that $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K}}\right)$, we want to show $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K} * \alpha}\right)$. Assume $W_{2} \in[\beta]$. It is easy to see that $W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2}$. Since $\beta \vdash \alpha$, it follows from (DPR1 ${ }^{\triangleleft}$ ) that $W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$. Thus $W_{1} \in \min ([\beta], \preceq \mathcal{K} * \alpha)$. Analogously, we also have $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K} * \alpha}\right)$ implies $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K}}\right)$.
2. (DP2) is equivalent to $\left(\mathrm{DPR}^{\triangleleft}\right)$.

Symmetrical to the one above.
3. (Ind) is equivalent to ( $\operatorname{IndR}^{\triangleleft}$ ).
$($ Ind $) \Rightarrow\left(\operatorname{IndR} 3^{\triangleleft}\right)$ :
Suppose that (Ind) holds. Consider any two possible worlds such that $W_{1} \models \alpha$ and $W_{2} \not \vDash \alpha$. Assume $W_{1} \preceq_{\mathcal{K}}^{\triangleleft} W_{2}$ and $W_{1}, W_{2}$ are relevantly ordered in $\preceq_{\mathcal{K} * \alpha}$, we have to show that $W_{1}<_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$, i.e., $W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$ and $W_{1}<_{\mathcal{K} * \alpha} W_{2}$. Suppose $W_{1} \AA_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$. Since $W_{1}, W_{2}$ are relevantly ordered in $\preceq_{\mathcal{K} * \alpha}$, we have $W_{2} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{1}$. It follows that there is a sentence $\gamma^{\prime}$ such that $W_{1}, W_{2} \models \gamma^{\prime}$ and $W_{2} \in \min \left(\left[\gamma^{\prime}\right], \preceq \mathcal{K} * \alpha\right)$. Since $W_{1} \preceq^{\triangleleft} W_{2}$, there is also a sentence $\gamma$ such that $W_{1}, W_{2} \models \gamma$ and $W_{1} \in \min ([\gamma], \preceq \mathcal{K})$. It is easy to see that $W_{1} \in \min \left(\left[\gamma \wedge \gamma^{\prime}\right], \preceq_{\mathcal{K}}\right)$ and $W_{2} \in \min \left(\left[\gamma \wedge \gamma^{\prime}\right], \preceq_{\mathcal{K} * \alpha}\right)$. Since $W_{1} \models \alpha$, it follows
that $\mathcal{K} *\left(\gamma \wedge \gamma^{\prime}\right) \nvdash \neg \alpha$. By (Ind), we have $(\mathcal{K} * \alpha) *\left(\gamma \wedge \gamma^{\prime}\right) \vdash \alpha$. This contradicts $W_{2} \in \min \left(\left[\gamma \wedge \gamma^{\prime}\right], \preceq_{\mathcal{K} * \alpha}\right)$ and $W_{2} \models \neg \alpha$. Hence, we can conclude $W_{1} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$. What remain to show is that $W_{1} \prec_{\mathcal{K} * \alpha} W_{2}$. Assume $W_{1} \not_{\mathcal{K} * \alpha} W_{2}$. Since $\preceq_{\mathcal{K} * \alpha}$ is total, we have $W_{2} \preceq_{\mathcal{K} * \alpha} W_{1}$. From $W_{1} \preceq_{\mathcal{K} * \alpha}^{\wedge} W_{2}$, it follows that there is a sentence $\gamma^{\prime \prime}$ such that $W_{1}, W_{2} \models \gamma^{\prime \prime}$ and $W_{1} \in \min \left(\left[\gamma^{\prime \prime}\right], \preceq \mathcal{K} * \alpha\right)$. By a similar argument to the above one, we have $\mathcal{K} * \gamma \wedge \gamma^{\prime \prime} \nvdash \neg \alpha$. By (Ind), it follows that $(\mathcal{K} * \alpha) * \gamma \wedge \gamma^{\prime \prime} \vdash \alpha$. On the other hand, we have $W_{1} \in \min \left(\left[\gamma \wedge \gamma^{\prime \prime}\right], \preceq_{\mathcal{K} * \alpha}\right)$ and $W_{2} \preceq_{\mathcal{K} * \alpha} W_{1}$. It follows that $W_{2} \in \min \left(\left[\gamma \wedge \gamma^{\prime \prime}\right], \preceq \mathcal{K}_{* \alpha}\right)$. Since $W_{2} \models \neg \alpha$, it contradicts $(\mathcal{K} * \alpha) * \gamma \wedge \gamma^{\prime \prime} \vdash \alpha$.
$\left(\right.$ IndR $\left.^{\triangleleft}\right) \Rightarrow($ Ind $):$
Suppose that ( $\operatorname{IndR}^{\triangleleft}$ ) holds. Assume $\mathcal{K} * \beta \nvdash \neg \alpha$, and $(\mathcal{K} * \alpha) * \beta \nvdash \alpha$. It follows from $\mathcal{K} * \beta \nvdash \neg \alpha$ that there exists $W_{1} \in \min ([\beta], \preceq \mathcal{K}) ~ s u c h ~ t h a t ~_{W_{1}}^{㇒}{ }^{\models}$. It follows from $(\mathcal{K} * \alpha) * \beta \nvdash \alpha$ that there exists $W_{2} \in \min \left([\beta], \preceq_{\mathcal{K} * \alpha}\right)$ such that $W_{2} \models \neg \alpha$. Since $W_{2} \models \beta$ and $W_{1} \in \min \left([\beta], \preceq_{\mathcal{K}}\right)$, we have $W_{1} \preceq_{\mathcal{K}} W_{2}$. Similarly, we have $W_{2} \preceq_{\mathcal{K} * \alpha}^{\triangleleft} W_{1}$. Since $W_{1} \models \alpha$ and $W_{2} \models \neg \alpha$, this contradicts ( $\operatorname{IndR}^{\triangleleft}$ ), which says that $W_{1}, W_{2}$ are well ordered in $\preceq_{\mathcal{K} * \alpha}$ then it must the case $W_{1} \prec_{\mathcal{K} * \alpha}^{\triangleleft} W_{2}$.

Observation 3.14. Suppose $\mathcal{L}$ is a finite language. Let $\preceq_{\mathcal{K}}$ be a faithful ranking on some belief state $\mathcal{K}$. Then for any $W_{1}, W_{2} \in \Theta_{\mathcal{L}}, W_{1}$ and $W_{2}$ are relevantly ordered in $\preceq_{\mathcal{K}}$.

Proof. Since $\preceq_{\mathcal{K}}$ is total, we have either $W_{1} \preceq_{\mathcal{K}} W_{2}$ or $W_{2} \preceq_{\mathcal{K}} W_{1}$. Without loss of generality, we assume $W_{1} \preceq_{\mathcal{K}} W_{2}$. Since $\mathcal{L}$ is finite, we can construct a sentence $\gamma=\wedge W_{1} \vee \wedge W_{2}$, such that $[\gamma]=\left\{W_{1}, W_{2}\right\}$. It is easy to see that $\left.W_{1} \in \min ([\gamma]), \preceq_{\mathcal{K}}\right)$. Hence $W_{1} \preceq_{\mathcal{K}}^{\mathcal{\mathcal { K }}} W_{2}$.

Observation 3.15. Given an $O C F k$, the binary relation $\leq_{k}$ defined by (3.3) satisfies (EE1)(EE5).

Proof. Due to the transitivity of $\leq$ on $\mathbb{N}^{+}, \leq_{k}$ satisfies (EE1).
Assume $\alpha \vdash \beta$. By contra-position, we have $\neg \beta \vdash \neg \alpha$. Hence, for any possible world $W$ if $W \models \neg \beta$ then $W \models \neg \alpha$. According to (3.2), we have $k(\alpha) \leq k(\beta)$, i.e., $\alpha \leq_{k} \beta$. Thus $\leq_{k}$ satisfies (EE2).

Assume $k(\alpha)>k(\alpha \wedge \beta)$ and $k(\beta)>k(\alpha \wedge \beta)$. From (3.2), it follows that there exists $W$, s.t., $k(W)=k(\alpha \wedge \beta)$ and $W \models \neg \alpha \vee \neg \beta$. Since $k(\alpha)>k(\alpha \wedge \beta)$, according to (3.2), we have $W \not \models \neg \alpha$, i.e., $W \models \alpha$. From $W \models \neg \alpha \vee \neg \beta$, it follows that $W \models \neg \beta$. It follows from (3.2), that $k(\beta) \leq k(\alpha \wedge \beta)$, which contradicts $k(\beta)>k(\alpha \wedge \beta)$. Thus, $\leq_{k}$ satisfies (EE3).

Assume $\operatorname{Bel}(k)$ is consistent. Suppose $\alpha \notin \operatorname{Bel}(k)$. According to (3.1) and (3.2), $\alpha \notin$ $\operatorname{Bel}(k)$ iff there exists $W$, s.t., $k(W)=0$ and $W \models \neg \alpha$, i.e., $k(\alpha)=0$. Therefore, we have $\alpha \notin \operatorname{Bel}(k)$ iff $k(\alpha) \leq k(\beta)$, for any $\beta$. Thus, $\leq_{k}$ satisfies (EE4).

Assume $\nvdash \alpha$. According to (3.2), $k(T)>k(\alpha)$. Hence, by contra-position, $\leq_{k}$ satisfies (EE5).

Observation 3.16. Let $k$ be an OCF and $\alpha$ a sentence, then for any $m_{1}, m_{2} \in \mathbb{N}^{+}$,

$$
\operatorname{Bel}\left(k_{\alpha, m_{1}}^{r, *}\right)=\operatorname{Bel}\left(k_{\alpha, m_{2}}^{r, *}\right)
$$

Proof. According to (3.4), $k_{\alpha, m}^{r, *}(W)=0$ iff $W \models \alpha$ and $k(W)=k(\neg \alpha)$, which means the value of $m$ does not affect the set of worlds with rank 0 in the revised OCF. From (3.1), it follows immediately that $\operatorname{Bel}\left(k_{\alpha, m_{1}}^{r, *}\right)=\operatorname{Bel}\left(k_{\alpha, m_{2}}^{r, *}\right)$.

Theorem 3.17. Assume an arbitrary but fixed evidence degree for any new information. Then reinforcement OCF revision satisfies all AGM postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$ as well as (DP1), (DP2) and (Ind).

Proof. Obviously, each OCF $k$ can induce a faithful ranking $\preceq_{k}$ on $k$ by letting :

$$
W_{1} \preceq_{k} W_{1} \text { iff } k\left(W_{1}\right) \leq k\left(W_{2}\right)
$$

According to (3.4), $k_{\alpha, m}^{*}(W)=0$ iff $W \models \alpha$ and $k(W)=k(\neg \alpha)$. From (3.2), it is easy to see that $k_{\alpha, m}^{*}(W)=0$ iff $W \in \min \left([\alpha], \preceq_{k}\right)$. As a direct consequence of Theorem 3.2, reinforcement OCF revision satisfies Postulates $(\mathcal{K} * 1)-(\mathcal{K} * 8)$. Assume $m$ is an arbitrary positive integer. From (3.4), it it not difficult to see that the following conditions are satisfied:

- If $W_{1}, W_{2} \models \alpha$, then $W_{1} \preceq_{k} W_{2}$ iff $W_{1} \preceq_{K_{\alpha, m}^{r, *}} W_{2}$
- If $W_{1}, W_{2} \not \models \alpha$, then $W_{1} \preceq_{k} W_{2}$ iff $W_{1} \preceq_{K_{\alpha, m}^{r, *}} W_{2}$
- If $W_{1} \models \alpha$ and $W_{2} \not \models \alpha$, then $W_{1} \preceq_{k} W_{2}$ implies $W_{1} \prec_{K_{\alpha, m}^{r, *}} W_{2}$

As a direct consequence of Theorem 3.10 and 3.11, reinforcement OCF revision satisfies (DP1), (DP2) and (Ind).

Observation 3.18. Let $k$ be an $O C F$ and $\langle\alpha, m\rangle$ be any new information. Then for any nontautological sentence $\beta$,

$$
k_{\alpha, m}^{r, *}(\beta)= \begin{cases}k(\beta)+m & \text { if } \vdash \alpha \rightarrow \beta \\ k(\alpha \rightarrow \beta)-k(\neg \alpha) & \text { else if } k(\alpha \rightarrow \beta)=k(\beta) \\ \min (k(\alpha \rightarrow \beta)-k(\neg \alpha), k(\beta)+m) & \text { else }\end{cases}
$$

Proof. Assume $\vdash \alpha \rightarrow \beta$. From $\nvdash \beta$ and (3.2), it follows that there exists $W_{1} \models \neg \beta$, s.t., $k\left(W_{1}\right)=k(\beta)$ and $k(W) \geq k\left(W_{1}\right)$ for any $W \models \neg \beta$. Since $\vdash \alpha \rightarrow \beta$, we have $W_{1} \models \neg \alpha$. According to (3.4), $k_{\alpha, m}^{r, *}\left(W_{1}\right)=k\left(W_{1}\right)+m$. Similarly, for any $W \models \neg \beta$ we have $k_{\alpha, m}^{r, *}(W)=$ $k(W)+m$. Again according to (3.2) we have $k_{\alpha, m}^{r, *}(\beta)=k(\beta)+m$.

Assume $\nvdash \alpha \rightarrow \beta$ and $k(\alpha \rightarrow \beta)=k(\beta)$. From $\nvdash \alpha \rightarrow \beta$ and (3.2), it follows that there exists $W_{1} \models \alpha \wedge \neg \beta$, s.t., $k\left(W_{1}\right)=k(\alpha \rightarrow \beta)$. According to (3.4), $k_{\alpha, m}^{r, *}\left(W_{1}\right)=k\left(W_{1}\right)-$ $k(\neg \alpha)$. Since $k(\alpha \rightarrow \beta)=k(\beta)$, according to (3.2) we have $k(W) \geq k\left(W_{1}\right)$ for any $W \models \neg \beta$. It follows from (3.4) that for any $W \models \neg \beta, k_{\alpha, m}^{r, *}(W)$ is either $k(W)-k(\neg \alpha)$ or $k(W)+m$. Therefore, according to (3.2) we have $k_{\alpha, m}^{r, *}(\beta)=k(\beta)-k(\neg \alpha)=k(\alpha \rightarrow \beta)-k(\neg \alpha)$.

Assume $\nvdash \alpha \rightarrow \beta$ and $k(\alpha \rightarrow \beta) \neq k(\beta)$. It is not difficult to see, according to (3.2), that this is possible only if $k(\alpha \rightarrow \beta)>k(\beta)$. From $\nvdash \alpha \rightarrow \beta$ and (3.2), it follows that there exists $W_{1} \models \alpha \wedge \neg \beta$, s.t., $k\left(W_{1}\right)=k(\alpha \rightarrow \beta)$ and $k(W) \geq k\left(W_{1}\right)$ for any $W \models \alpha \wedge \neg \beta$. Analogously, there exists $W_{2} \models \neg \beta$, s.t., $k\left(W_{2}\right)=k(\beta)$ and $k(W) \geq k\left(W_{2}\right)$ for any $W \models \neg \beta$. We consider two cases:

1. Assume $k(\alpha \rightarrow \beta)-k(\neg \alpha) \leq k(\beta)+m$. According to (3.4), $k_{\alpha, m}^{r, *}\left(W_{1}\right)=k\left(W_{1}\right)-$ $k(\neg \alpha)$. For any $W \models \neg \beta$, according to (3.4), if $W \models \alpha$, then $k_{\alpha, m}^{r, *}(W)=k(W)-$
$k(\neg \alpha) \geq k\left(W_{1}\right)-k(\neg \alpha)$; otherwise $k_{\alpha, m}^{r, *}(W)=k(W)+m \geq k\left(W_{2}\right)+m \geq k\left(W_{1}\right)-$ $k(\neg \alpha)$. From (3.2), it follows that $k_{\alpha, m}^{r, *}(\beta)=k(\alpha \rightarrow \beta)-k(\neg \alpha)$.
2. Assume $k(\alpha \rightarrow \beta)-k(\neg \alpha)>k(\beta)+m$. Since $k(\alpha \rightarrow \beta)>k(\beta)$, according to (3.2), we have $W_{2} \models \neg \alpha$. From (3.4), it follows that $k_{\alpha, m}^{r, *}\left(W_{2}\right)=k\left(W_{2}\right)+m$. For any $W \models \neg \beta$, according to (3.4), if $W \models \alpha$, then $k_{\alpha, m}^{r, *}(W)=k(W)-k(\neg \alpha) \geq k\left(W_{1}\right)-k(\neg \alpha)>$ $k\left(W_{2}\right)+m$; otherwise $k_{\alpha, m}^{r, *}(W)=k(W)+m \geq k\left(W_{2}\right)+m$. From (3.2), it follows that $k_{\alpha, m}^{r, *}(\beta)=k(\beta)+m$. Therefore, $k_{\alpha, m}^{r, *}(\beta)=\min (k(\alpha \rightarrow \beta)-k(\neg \alpha), k(\beta)+m)$.

Observation 3.19. Let $k$ be an arbitrary $O C F$ and $\alpha$ a new non-tautological sentence with evidence degree $m \in \mathbb{N}^{+}$. Then

$$
k_{\alpha, m}^{r, *}(\alpha)=k(\alpha)+m
$$

Proof. A direct consequence of Observation 3.18.

Theorem 3.20. For arbitrary $m_{1}, m_{2} \in \mathbb{N}^{+}$, reinforcement $O C F$ revision satisfies the following conditions.

```
(EDP1) If \(\beta \vdash \alpha\), then \(\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *} \equiv k_{\beta, m_{2}}^{r, *}\).
(EDP2) If \(\beta \vdash \neg \alpha\), then \(\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *} \equiv k_{\beta, m_{2}}^{r, *}\).
(EInd) If there exists \(m\) such that \(k_{\neg \beta, m}^{r, *} \nvdash \neg \alpha\), then \(\left(k_{\alpha, m_{1}}^{r, *}\right)_{\neg \beta, m_{2}}^{r, *} \vdash \alpha\).
```

Proof. If $\vdash \neg \beta$, Condition (EDP1) holds trivially. Assume that $\beta \vdash \alpha$ and $\nvdash \neg \beta$. By (3.4),

$$
\begin{equation*}
k_{\beta, m_{2}}^{r, *}(W)=0 \text { iff } W \models \beta \text { and } k(W)=k(\neg \beta) \tag{A.1}
\end{equation*}
$$

Likewise,

$$
\begin{equation*}
\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *}(W)=0 \text { iff } W \models \beta \text { and } k_{\alpha, m_{1}}^{r, *}(W)=k_{\alpha, m_{1}}^{r, *}(\neg \beta) \tag{A.2}
\end{equation*}
$$

Since $\beta \vdash \alpha$, for any $W \models \beta$ we have $k_{\alpha, m_{1}}^{r, *}(W)=k(W)-k(\neg \alpha)$ by (3.4). Since $\alpha \rightarrow \neg \beta \equiv$ $\neg \beta$ and $\nvdash \neg \beta$, it follows from Observation 3.18 that $k_{\alpha, m_{1}}^{r, *}(\neg \beta)=k(\neg \beta)-k(\neg \alpha)$. Hence, (A.2) is equivalent to

$$
\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *}(W)=0 \text { iff } W \models \beta \text { and } k(W)=k(\neg \beta)
$$

This and (A.1) implies $\left(k_{\alpha, m_{1}}^{r, *}\right)_{\beta, m_{2}}^{r, *} \equiv k_{\beta, m_{2}}^{r, *}$. Condition (EDP2) can be proved analogously.
We prove Condition (EInd) by contradiction. To begin with, from the assumption that $k_{\neg \beta, m}^{r, *} \nvdash \neg \alpha$ it follows that $\nvdash \beta$ and $\nvdash \alpha \rightarrow \beta$. Furthermore, there exists $W$ such that $k_{\neg \beta, m}^{r, *}(W)=0, W \models \neg \beta \wedge \alpha$, and $k(W)=k(\beta)$. With the help of (3.2), this implies $k(\beta)=k(\alpha \rightarrow \beta)$.

Now assume that $\left(k_{\alpha, m_{1}}^{r, *}\right)_{\neg \beta, m_{2}}^{r, *} \nvdash \alpha$. It follows that there exists $W^{\prime}$ such that $\left(k_{\alpha, m_{1}}^{r, *}\right)_{\neg \beta, m_{2}}^{r, *}\left(W^{\prime}\right)=0, W^{\prime} \models \neg \beta \wedge \neg \alpha$, and $k_{\alpha, m_{1}}^{r, *}\left(W^{\prime}\right)=k_{\alpha, m_{1}}^{r, *}(\beta)$. Since $k(W)=k(\beta)$ and $W^{\prime} \models \neg \beta$, we have $k\left(W^{\prime}\right) \geq k(W)$. Hence by (3.4), $k_{\alpha, m_{1}}^{r, *}\left(W^{\prime}\right)=k\left(W^{\prime}\right)+m_{1}>k(\beta)$. But from Observation 3.18 it follows that $k_{\alpha, m_{1}}^{r, *}(\beta) \leq k(\beta)$, since $\nvdash \beta$ and $\nvdash \alpha \rightarrow \beta$. This contradicts $k_{\alpha, m_{1}}^{r, *}\left(W^{\prime}\right)=k_{\alpha, m_{1}}^{r, *}(\beta)$.

Observation 3.21. The amnesic revision $*_{a}$ satisfies (DP1), (DP3) and (DP4), but violates (DP2).

Proof. Note, in the case of the amnesic revision $*_{a}$ a belief state is identified with its propositional beliefs.

Assume $\vdash \neg \beta$. According to (3.5), we have $\left(K *_{a} \alpha\right) *_{a} \beta=\beta$ and $K *_{a} \beta=\beta$. Hence, $*_{a}$ satisfies (DP1), (DP2) and (DP3). Moreover, (DP4) is vacuously satisfied. In the rest of the proof, we consider the case $\nvdash \neg \beta$.

Assume $\beta \vdash \alpha$. We consider two cases: 1) Assume $K \nvdash \neg \alpha$. It follows from $\beta \vdash \alpha$, that we have $K \wedge \alpha \nvdash \neg \beta$. According to (3.5), if $K \nvdash \neg \beta$ then $\left(K *_{a} \alpha\right) *_{a} \beta=(K \wedge \alpha) *_{a} \beta=K \wedge \alpha \wedge \beta$ and $K *_{a} \beta=K \wedge \beta$; otherwise $\left(K *_{a} \alpha\right) *_{a} \beta=(K \wedge \alpha) *_{a} \beta=\beta$ and $K *_{a} \beta=\beta$. 2) Assume $K \vdash \neg \alpha$. From $\beta \vdash \alpha$, if follows that $K \vdash \neg \beta$. Since $\nvdash \neg \beta$ and $\beta \vdash \alpha$, we have $\alpha \nvdash \neg \beta$. According to (3.5), $\left(K *_{a} \alpha\right) *_{a} \beta=\alpha *_{a} \beta=\alpha \wedge \beta$ and $K *_{a} \beta=\beta$. Therefore $*_{a}$ satisfies (DP1).

Assume $K *_{a} \beta \vdash \alpha$. We consider two cases: 1) Assume $K \nvdash \neg \beta$. According to (3.5),
$K *_{a} \beta=K \wedge \beta$. It follows from $K \wedge \beta \vdash \alpha$ and $K \nvdash \neg \beta$, that we have $K \wedge \alpha \nvdash \neg \beta$. According to (3.5), $\left(K *_{a} \alpha\right) *_{a} \beta$ is either $K \wedge \alpha \wedge \beta$ or $\alpha \wedge \beta$. 2) Assume $K \vdash \neg \beta$. According to (3.5), $K *_{a} \beta=\beta$. From $K *_{a} \beta \vdash \alpha$, it follows that $\beta \vdash \alpha$. Since $*_{a}$ satisfies (*1), we have $\left(K *_{a} \alpha\right) *_{a} \beta \vdash \alpha$, Therefore $*_{a}$ satisfies (DP3).

Assume $K *_{a} \beta \nvdash \neg \alpha$. Obviously, we have $\nvdash \neg \alpha$. Consider two cases: 1) Assume $K \nvdash \neg \beta$. According to (3.5), $K *_{a} \beta=K \wedge \beta$. From $K \wedge \beta \nvdash \neg \alpha$ and $K \nvdash \neg \beta$, it follows $K \wedge \alpha \nvdash \beta$. According to (3.5), $\left(K *_{a} \alpha\right) *_{a} \beta=K \wedge \alpha \wedge \beta$. From $\nvdash \neg \alpha$, it follows $K \wedge \alpha \wedge \beta \nvdash \neg \alpha$. 2) Assume $K \vdash \neg \beta$. According to (3.5), $K *_{a} \beta=\beta$. Since $*_{a}$ satisfies ( $* 1$ ), we have $\left(K *_{a} \alpha\right) *_{a} \beta \vdash \beta$. From $\nvdash \neg \beta$ and $\beta \nvdash \neg \alpha$, it follows $\left(K *_{a} \alpha\right) *_{a} \beta \nvdash \neg \alpha$. Therefore $*_{a}$ satisfies (DP4).

The following counterexample shows that $*_{a}$ violates (DP2). Let $\alpha, \beta$ and $K$ be, respectively, $p, \neg p$ and $C n(p \vee q)(p, q$ are propositional atoms). Obviously, $\beta \vdash \neg \alpha$ holds. According to (3.5), $\left(K *_{a} \alpha\right) *_{a} \beta=C n(\neg q)$ and $K *_{a} \beta=C n((p \vee q) \wedge \neg q)$. Therefore $*_{a}$ violates (DP2).

## Appendix B

## Proofs of Results of Chapter 4

Observation 4.5. Suppose $\Xi=\langle B, f\rangle$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $W_{1}, W_{2}$ be arbitrary two possible worlds. Then $W_{1}, W_{2}$ are relevantly ordered.

Proof. Let $k_{\Xi}\left(W_{1}\right)=i$ and $k_{\Xi}\left(W_{2}\right)=j$. According to (4.8), there exists two sentences $\beta_{i}, \beta_{j}$ such that $W_{1} \models \neg \beta_{i}, W_{j} \models \neg \beta_{j}, f\left(\beta_{i}\right)=i$ and $f\left(\beta_{j}\right)=j$. Without loss of generality, we assume $i \leq j$. We want to show that there there exists a sentence $\gamma$ such that $W_{1}, W_{2} \models \gamma$ and $W_{1} \in \min \left([\gamma], \preceq_{\Xi}\right)$. If $i=0$, then let $\gamma=$ T. It is obvious $W_{1}, W_{2} \models \gamma$ and $W_{1} \in$ $\min \left([\gamma], \preceq_{\xi}\right)$. If $i>0$, then let $\gamma=\neg \beta_{i} \vee \neg \beta_{j}$. It is obvious that $W_{1}, W_{2} \models \gamma$. Suppose there exists a possible world $W$ such that $k_{\Xi}(W)<k_{\Xi}\left(W_{1}\right)$ and $W \models \gamma$. According to (4.8), we have $W \models \beta_{i} \wedge \beta_{j}$, which contradicts $W \models \gamma$.

Observation 4.6. Suppose $\Xi=\langle B, f\rangle$ is an EE base, and $k_{\Xi}$ is the induced OCF as defined by (4.8). Then for any sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi}(\beta)=k_{\Xi}(\beta)
$$

Proof. Assume $\vdash \beta$. Then $\operatorname{Rank}_{\Xi}(\beta)=k_{\Xi}(\beta)=\infty$.
Assume $\nvdash \beta$ and $\operatorname{Rank}_{\Xi}(\beta)=i$. It follows from (4.4) that $\Xi^{i+1} \nvdash \beta$ and $\Xi^{i} \vdash \beta$. Let $W$ be a possible world such that $W \models \Xi^{i+1} \cup\{\neg \beta\}$. From $\Xi^{i} \vdash \beta$, it follows that $W \nvdash \Xi^{i}$. Therefore, there is a sentence $\left.\beta_{i} \in \Xi\right|_{i}$ such that $W \not \vDash \beta_{i}$. It follows from (4.8) that $k_{\Xi}(W)=i$. Let $W_{1}$
be any possible world such that $k_{\Xi}\left(W_{1}\right)<k_{\Xi}(W)$. It follows from (4.8) that $W_{1} \vdash \Xi^{i}$; hence $W_{1} \vdash \beta$. According to (3.2), we have $k_{\Xi}(\beta)=i$.

Observation 4.7. Suppose $\Xi$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $\langle\alpha, m\rangle$ be any new information. Then for any possible world $W$ :

$$
k_{\Xi_{1}}(W)=k_{\Xi_{\alpha, m}}^{r, *}(W)
$$

where $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$.

Proof. Assume $k_{\Xi}(W)=i$. By (4.8), we have $W \models \Xi^{i+1}$ and there exists a sentence $\left.\beta_{i} \in \Xi\right|_{i}$ such that $W \models \neg \beta_{i}$.

If $W \models \neg \alpha$, then (3.4) implies that $k_{\Xi}{ }_{\alpha, m}^{r, *}(W)=i+m$. It follows from (4.7) that $\Xi_{1}^{i+m+1}=$ $\Xi^{i+m+1} \cup\left\{\beta \vee \alpha \mid \beta \in \Xi^{i+1}\right\}$ and $\left.\alpha \vee \beta_{i} \in \Xi_{1}\right|_{i+m}$. Obviously, we have $W \models \Xi_{1}^{i+m+1}$ and $W \not \vDash \alpha \vee \beta_{i}$. Hence, according to (4.8), $k_{\Xi_{1}}(W)=i+m$.
 follows from (4.7) that $\Xi_{1}^{i-r+1} \subseteq \Xi^{i+1} \cup\{\beta \vee \alpha \mid \beta \in B\} \cup\{\alpha\}$ and $\left.\beta_{i} \in \Xi_{1}\right|_{i-r}$. It is obvious that $W \models \Xi_{1}^{i-r+1}$. It follows from (4.8) that $k_{\Xi_{1}}(W)=i-r=i-\operatorname{Ran} k_{\Xi}(\neg \alpha)$. According to Observation 4.6, we obtain that $k_{\Xi_{1}}(W)=i-k_{\Xi}(\neg \alpha)$.

Theorem 4.8. Suppose $\Xi$ is an EE base and $k_{\Xi}$ is the OCF induced from $\Xi$. Let $\langle\alpha, m\rangle$ be any new information. The for any sentence $\beta$ :

$$
\operatorname{Rank}_{\Xi_{1}}(\beta)=k_{\Xi}^{\Xi_{\alpha, m}^{r, *}}(\beta)
$$

where $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$.

Proof. A direct consequence of Observation 4.6 and Observation 4.7.

Theorem 4.13. Let $\Xi_{1}, \Xi_{2}$ be two epistemically equivalent EE bases, then for any sentence $\alpha$
and evidence degrees $m_{1}, m_{2} \in \mathbb{N}^{+}$:

$$
\operatorname{Bel}\left(\Xi_{1}^{\prime}\right)=\operatorname{Bel}\left(\Xi_{2}^{\prime}\right)
$$

where $\Xi_{1}^{\prime}=\Xi_{1} \circledast_{r}\left\langle\alpha, m_{1}\right\rangle$ and $\Xi_{2}^{\prime}=\Xi_{2} \circledast_{r}\left\langle\alpha, m_{2}\right\rangle$.

Proof. Suppose $\beta \in \operatorname{Bel}\left(\Xi_{1}^{\prime}\right)$. It follows that $\operatorname{Rank}_{\Xi_{1}^{\prime}}(\beta)>0$. According to Observation 4.10, then it must be the case that either $\vdash \alpha \rightarrow \beta$ or $\operatorname{Rank}_{\Xi_{1}}(\alpha \rightarrow \beta)>\operatorname{Rank}_{\Xi_{1}}(\neg \alpha)$.

If $\vdash \alpha \rightarrow \beta$, then according to Observation 4.10, we have $\operatorname{Rank}_{\Xi_{2}^{\prime}}(\beta)=\operatorname{Rank}_{\Xi_{2}}(\beta)+m_{2}>$ 0.

If $\operatorname{Rank}_{\Xi_{1}}(\alpha \rightarrow \beta)>\operatorname{Rank}_{\Xi_{1}}(\neg \alpha)$ then also $\operatorname{Rank}_{\Xi_{2}}(\alpha \rightarrow \beta)>\operatorname{Ran} \xi_{\Xi_{2}}(\neg \alpha)$, since $\Xi_{2}$ is epistemically equivalent to $\Xi_{1}$. Again, it follows from Observation 4.10 that $\operatorname{Rank}_{\Xi_{2}^{\prime}}(\beta)>$ $\operatorname{Rank}_{\Xi_{1}}(\alpha \rightarrow \beta)-\operatorname{Rank}_{\Xi_{1}}(\neg \alpha)>0$.

The above discussion shows that $\operatorname{Bel}\left(\Xi_{1}^{\prime}\right) \subseteq \operatorname{Bel}\left(\Xi_{2}^{\prime}\right)$. Symmetrically, we also can show that $\operatorname{Bel}\left(\Xi_{2}^{\prime}\right) \subseteq \operatorname{Bel}\left(\Xi_{1}^{\prime}\right)$.

Theorem 4.14. Let $\Xi_{1}, \Xi_{2}$ be two equivalent EE bases and $\langle\alpha, m\rangle$ any new information. Then

$$
\Xi_{1} \circledast \circledast_{r}\langle\alpha, m\rangle \cong \Xi_{2} \circledast_{r}\langle\alpha, m\rangle
$$

Proof. A direct consequence of Observation 4.10.

Theorem 4.17. For reinforcement base revision, the problem of CF is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Proof. We first show that the problem is in $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$. It is easy to see that to compute a revised EE base $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle$, we mainly need to calculate Rank ${ }_{\Xi}(\neg \alpha)$ (cf. Algorithm E. 2 in Appendix). Apparently, Algorithm E. 1 shows that $\operatorname{Ran}_{\Xi}(\neg \alpha)$ can be computed with at most logarithmic many times of NP-oracle calls. Once computed the revised EE base $\Xi_{1}$, we just need one additional NP-oracle to decide whether $\Xi_{1}$ entails $\beta$. Therefore, the problem is in $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$.

To show that the problem is $\Delta_{2}^{\mathrm{P}}[\mathrm{O}(\log \mathrm{n})]$-hard, we give a polynomial (many-to-one) reduction from the problem of CF for cut base revision to it. Given any prioritized base $\left\langle B, \leq_{B}\right\rangle$, we can construct a EE base $\Xi=\langle B, f\rangle$ by assigning evidence degree 1 to all sentences in the lowest class and evidence degree 2 to all sentences in the next higher class, and so on. It is easy to see that for any sentences $\alpha$ and $\beta:\left\langle B, \leq_{B}\right\rangle \circledast_{\text {cut }} \alpha \vdash \beta$ iff $\Xi_{1}$ entails $\beta$ where $\Xi_{1}=\Xi \circledast_{r}\langle\alpha, 1\rangle$. From Theorem 4.16, it follows that the problem of CF for reinforcement base revision is $\Delta_{2}^{\mathrm{P}}[\mathrm{O}(\log \mathrm{n})]$-hard

Theorem 4.18. For reinforcement base revision, the problem of computing a revised belief state is NP-equivalent.

Proof. The proof of Theorem 4.17 already shows that the problem is NP-easy. It is not difficult to see that for any sentence $\alpha$ : $\alpha$ is satisfiable iff $\alpha \in\{\langle\alpha, 1\rangle\} \circledast_{r}\langle\top, 1\rangle$. Therefore, the problem is also NP-hard.

Theorem 4.22. Let $k$ be an OCF and $\langle\alpha, m\rangle$ be any new information. Then for any possible worlds $W$,

$$
k_{\alpha, m}^{c, *}(W)= \begin{cases}k_{\alpha, m-k(\alpha)}^{r, *}(W) & \text { if } k(\alpha)<m \\ \left(\left(k_{\neg \alpha, m^{\prime}}^{r, *}\right)_{\alpha, m}^{r, *}\right)(W) & \text { otherwise }\end{cases}
$$

where $m^{\prime} \in \mathbb{N}^{+}$is an arbitrary positive integer.

Proof.
Assume $k(\alpha)<m$, it follows directly from (4.11) and (3.4) that $k_{\alpha, m}^{c, *}(W)=$ $k_{\alpha, m-k(\alpha)}^{r, *}(W)$.

Assume $k(\alpha) \geq m$. Suppose $W \models \alpha$. According to (4.11), $k_{\alpha, m}^{c, *}(W)=k(W)-k(\neg \alpha)$,
 plies that $k_{\left\langle\neg \alpha, m^{\prime}\right\rangle}^{r, *}(W)=k(W)+m^{\prime}$. From Observation 3.19, it follows that $k_{\neg \alpha, m^{\prime}}^{r, *}(\neg \alpha)=$ $k(\neg \alpha)+m^{\prime}$. Therefore, $\left(\left(k_{\neg \alpha, m^{\prime}}^{r, *}\right)_{\alpha, m}^{r, *}\right)(W)=k(W)-k(\neg \alpha)$. Thus, $\left(\left(k_{\neg \alpha, m^{\prime}}^{r, *}{ }_{\alpha, m}^{r, *}\right)(W)=\right.$ $k_{\langle\alpha, m\rangle}^{c, *}(W)$. Suppose $W \not \vDash \alpha$. It follows from (4.11) that $k_{\alpha, m}^{c, *}(W)=k(W)-k(\alpha)+m$, whereas (3.4) implies that $\left(\left(k_{\neg \alpha, m^{\prime}}^{r, *}\right)_{\alpha, m}^{r, *}\right)(W)=k_{\rightarrow \alpha, m^{\prime}}^{r, *}(W)+m$. According to (3.4),



Theorem 4.23. For conditionalization, the problem of CF is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Proof. It suffice to show that $\Xi \circledast_{r} \alpha \equiv \Xi \circledast_{c} \alpha$ for any EE base $\Xi$ and new information $\langle\alpha, m\rangle$. If $B \nvdash \alpha$, this holds trivially. Assume $B \vdash \alpha$. Then $\Xi \circledast_{r} \alpha \equiv \Xi \equiv \Xi \circledast_{c} \alpha$, since both conditionalization and reinforcement base revision satisfy the AGM postulates.

Theorem 4.24. For adjustment, the problem of CF is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Proof. It suffice to show that $\Xi \circledast_{r} \alpha \equiv \Xi \circledast_{j} \alpha$ for any EE base $\Xi$ and new information $\langle\alpha, m\rangle$. Assume $B \vdash \alpha$. Then $\Xi \circledast_{r} \alpha \equiv \Xi \equiv \Xi \circledast_{j} \alpha$, since both adjustment and reinforcement base revision satisfy the AGM postulates. Assume $B \nvdash \alpha$. It is easy to see that the only difference between $\Xi \circledast_{r} \alpha$ and $\Xi \circledast_{j} \alpha$ is that the former contains some more sentences of the form $\alpha \vee \beta_{i}$. Since both of them contain $\alpha$, therefore $\Xi \circledast_{r} \alpha \equiv \Xi \equiv \Xi \circledast_{j} \alpha$.

To prove Theorem 4.26, we need the following observation.
Observation B.1. Let $k$ be an $O C F$, and $\mathrm{P}(k)$ the possibility distribution as defined by (4.16). Then for any sentence $\alpha$,

$$
\Pi_{\pi_{k}}(\alpha)=e^{-k(\neg \alpha)}
$$

Proof. Recall that $\Pi_{\pi}(\alpha)$ is defined as $\max (\{\pi(W) \mid W \models \alpha\})$. Let $W_{1}$ be a possible world such that $\pi_{k}\left(W_{1}\right)=\Pi_{\pi_{k}}(\alpha)=\max \left(\left\{\pi_{k}(W) \mid W \models \alpha\right\}\right)$. Since $f(x)=e^{-x}$ is monotonically decreasing, we have $k\left(W_{1}\right)=\min (\{k(W) \mid W \models \alpha\})$. It follows from (3.2) that $k\left(W_{1}\right)=$ $k(\neg \alpha)$. Thus $\Pi_{\pi_{k}}(\alpha)=e^{-k(\neg \alpha)}$.

Theorem 4.26. Suppose $k$ is an $O C F$, and $\langle\alpha, m\rangle$ (with $m \in \mathbb{N}^{+}$) is input information. Let $k_{\langle\alpha, m\rangle}^{*}$ be the revised OCF using reinforcement OCF revision. Then for any possible world $W$,

$$
\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)
$$

Proof. Assume $W \models \alpha$. According to (4.16), $\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=e^{-k_{\alpha, m}^{*}(W)}$. It follows from (3.4) that $e^{-k_{\alpha, m}^{*}(W)}=e^{-(k(w)-k(\neg \alpha))}$. On the other hand, (4.17) implies $\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)=$
$\frac{\pi_{k}(W)}{\Pi_{\pi_{k}}(\alpha)}$. It follows from (4.16) and Observation B.1 that $\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)=\frac{e^{-k(w)}}{e^{-k(\neg \alpha)}}$. Hence $\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)$.

Assume $W \not \models \alpha$. Similarly, according to (4.16) and (3.4), $\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=e^{-(k(W)+m)}$. It follows from (4.17) that $\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)=\pi_{k}(W) \times e^{-m}$. (4.16) implies $\pi_{k}\left(\left.W\right|_{r}\langle\alpha, 1-\right.$ $\left.\left.e^{-m}\right\rangle\right)=e^{-k(W)} \times e^{-m}$. Hence $\pi_{k_{\langle\alpha, m\rangle}^{*}}(W)=\pi_{k}\left(\left.W\right|_{r}\left\langle\alpha, 1-e^{-m}\right\rangle\right)$.

Observation 4.27. Suppose $\Sigma$ is a possibilistic base, and $\pi_{\Sigma}$ is the induced possibility distribution as defined by (4.20). Then for any sentence $\beta$ :

$$
N_{\Sigma}(\beta)=N_{\pi_{\Sigma}}(\beta)
$$

Proof. Analogous to the proof of Observation 4.6.

Theorem 4.28. Let $\Sigma$ be a possibilistic base and $\langle\alpha, w\rangle$ be the input information. Then for all possible worlds $W$,

$$
\pi_{\Sigma}\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)=\pi_{\Sigma_{1}}(W)
$$

where $\Sigma_{1}=\Sigma$ 㘢 $_{r}\langle\alpha, w\rangle$.

Proof. Analogous to the proof of Theorem 4.7.

## Appendix C

## Proofs of Results of Chapter 5

Observation 5.2. Let $k$ be an arbitrary $O C F$ and $\lambda$ be an OCF encodes $\langle\alpha, m\rangle$. Then for any possible $W$,

$$
\left(k_{\alpha, m}^{*}\right)(W)=(k \otimes \lambda)(W)
$$

Proof. Assume $W \models \neg \alpha$. It follows from (5.7) that $\lambda(W)=m>0$. According to (3.4) and $(5.5),\left(k_{1}^{*}{ }_{\alpha, m}^{*}\right)(W)=\left(k_{1} \otimes \lambda\right)(W)=k_{1}(W)+m$.

Assume $W \models \alpha$. It follows from (5.7) that $\lambda(W)=0$ and $\lambda^{-}(0)=\left\{W^{\prime} \mid W^{\prime} \models \alpha\right\}$. According to (5.6) and (3.2), $m_{k_{1}, \lambda}=\min \left\{n \mid k_{1}^{-}(n) \cap\left\{W^{\prime} \mid W^{\prime} \models \alpha\right\} \neq \emptyset\right\}=k_{1}(\neg \alpha)$. Therefore, we have $\left(k_{1}{ }_{\alpha, m}^{*}\right)(W)=k_{1}(W)-k_{1}(\neg \alpha)=\left(k_{1} \otimes \lambda\right)(W)=k_{1}(W)-m_{k_{1}, \lambda}$.

Observation 5.4. The OCF mutual belief revision defined by Definition 5.3 satisfies the following properties :

1. $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right) \leq d_{\text {inc }}\left(k_{1}, k_{2}\right)$;
2. $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)=d_{\text {inc }}\left(k_{1}, k_{2}\right)$ iff $\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$ and $\operatorname{Bel}\left(k_{2}\right) \subseteq$ $\operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$.

Proof. Assume $d_{\text {inc }}\left(k_{1}, k_{2}\right)=r$. It follows from (5.2) that $k_{1}^{-}(r-1) \cap k_{2}^{-}(r-1)=\emptyset$ and $k_{1}^{-}(r) \cap k_{2}^{-}(r) \neq \emptyset$.

From (5.5) and (5.8), it follows that $k_{1}^{-}(r) \cap k_{2}^{-}(r) \subseteq\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(r)$ and $k_{1}^{-}(r) \cap$ $k_{2}^{-}(r) \subseteq\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}(r)$. According to (5.2), we then have $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right) \leq r$.

Suppose $\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$ and $\operatorname{Bel}\left(k_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$. It follows from (5.5) and (5.8) that $m_{k_{1}, k_{2}}=m_{k_{2}, k_{1}}=0$. Therefore, we have $\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(r-1) \subseteq k_{1}^{-}(r-1)$ and $\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}(r-1) \subseteq k_{2}^{-}(r-1)$. From $\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(r-1) \cap\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(r-$ 1) $\subseteq k_{1}^{-}(r-1) \cap k_{2}^{-}(r-1)=\emptyset$, it follows that $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)>r-1$. As we have already shown $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right) \leq r$, it follows that $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)=r$. To show the other direction, we assume without loss of generality that $\operatorname{Bel}\left(k_{1}\right) \nsubseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$. It follows from (5.5) and (5.8) that $m_{k_{1}, k_{2}}=r_{1}>0$ and $k_{1}^{-}(r) \cap k_{2}^{-}(r) \subseteq\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}\left(r-r_{1}\right)$. We consider two cases: 1). Assume $\operatorname{Bel}\left(k_{2}\right) \nsubseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$. It follows that $m_{k_{2}, k_{1}}=$ $r_{2}>0$ and $k_{1}^{-}(r) \cap k_{2}^{-}(r) \subseteq\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}\left(r-r_{2}\right)$. Therefore, according to (5.2), we have $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)<r$. 2). Assume $\operatorname{Bel}\left(k_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$. It follows from (5.5) and (5.8) that $M_{2}\left(\left(k_{1}, k_{2}\right)\right)^{-}(0)=k_{1}^{-}(r) \cap k_{2}^{-}(0) \neq \emptyset$. From $k_{1}^{-}(r) \cap k_{2}^{-}(0) \subseteq k_{1}^{-}(r) \cap k_{2}^{-}(r) \subseteq$ $\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}\left(r-r_{1}\right)$, it follows $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right) \leq r-r_{1}<r$.

Observation 5.5. The OCF mutual belief revision defined by Definition 5.3 satisfies the following properties (for each $i \in\{1,2\}$ ):
$(M 1) \quad \operatorname{Bel}\left(\gamma_{1}\left(k_{1}, k_{2}\right)\right)+\operatorname{Bel}\left(\gamma_{2}\left(k_{1}, k_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)$;
(M2) if $k_{1}$ and $k_{2}$ are consistent, then $\operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)=\operatorname{Bel}\left(k_{1}\right)+\operatorname{Bel}\left(k_{2}\right)$;
(M3) $\quad k_{i}$ and $M_{i}\left(k_{1}, k_{2}\right)$ are consistent iff $\operatorname{Bel}\left(k_{i}\right) \subseteq \operatorname{Bel}\left(M_{i}\left(k_{1}, k_{2}\right)\right)$;
(M4) $\quad M_{1}\left(k_{1}, k_{2}\right)=M_{1}\left(k_{1}, \gamma_{2}\left(k_{1}, k_{2}\right)\right), M_{2}\left(k_{1}, k_{2}\right)=M_{2}\left(\gamma_{1}\left(k_{1}, k_{2}\right), k_{2}\right)$;
(M5) if $\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$ and $\operatorname{Bel}\left(k_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$, then $M_{i}\left(k_{1}, k_{2}\right) \equiv M_{i}\left(M\left(k_{1}, k_{2}\right)\right)$.

Proof. It is easy to see that (M1)-(M4) hold from the construction of the OCF model (cf. Figure 5.1).

To show $($ M5 $)$, we assume $\operatorname{Bel}\left(k_{1}\right) \subseteq \operatorname{Bel}\left(M_{1}\left(k_{1}, k_{2}\right)\right)$ and $\operatorname{Bel}\left(k_{2}\right) \subseteq \operatorname{Bel}\left(M_{2}\left(k_{1}, k_{2}\right)\right)$. Assume $d_{\text {inc }}\left(k_{1}, k_{2}\right)=r$. From Observation 5.4, it follows that $d_{\text {inc }}\left(M\left(k_{1}, k_{2}\right)\right)=r$. According to (5.5) and (5.8), we have for all $r^{\prime} \leq r,\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}\left(r^{\prime}\right)=k_{2}^{-}(r) \cap k_{1}^{-}\left(r^{\prime}\right)$ and $\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}\left(r^{\prime}\right)=k_{1}^{-}(r) \cap k_{2}^{-}\left(r^{\prime}\right)$. It follows that $\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}(r)=k_{1}^{-}(r) \cap k_{2}^{-}(r)$ and
$\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(0)=k_{2}^{-}(r) \cap k_{1}^{-}(0)$. Therefore, $\left(M_{1}\left(M\left(k_{1}, k_{2}\right)\right)^{-}(0)=\left(M_{2}\left(k_{1}, k_{2}\right)\right)^{-}(r) \cap\right.$ $\left(M_{1}\left(k_{1}, k_{2}\right)\right)^{-}(0)=\left(M_{1}\left(k_{1}, K_{2}\right)\right)^{-}(0)$. It follows that $M_{1}\left(k_{1}, k_{2}\right) \equiv M_{1}\left(M\left(k_{1}, k_{2}\right)\right)$. Similarly, we have $M_{2}\left(k_{1}, k_{2}\right) \equiv M_{2}\left(M\left(k_{1}, k_{2}\right)\right)$.

Observation 5.6. Suppose $\Xi_{1}, \Xi_{2}$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
d_{i n c}\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)=d_{i n c}\left(\Xi_{1}, \Xi_{2}\right)
$$

Proof. Assume $i=d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)$. According to (5.1) and (4.8), we have $k_{\Xi_{1}}^{-}(n)=\{W \mid W \models$ $\left.\Xi_{1}^{n+1}\right\}$ and $k_{\Xi_{2}}^{-}(n)=\left\{W \mid W \models \Xi_{2}^{n+1}\right\}$ for any $n$. It follows from (5.9) that $\Xi_{1}^{i+1} \cup \Xi_{2}^{i+1} \nvdash \perp$ and $\Xi_{1}^{i} \cup \Xi_{2}^{i} \vdash \perp$. Since $\Xi_{1}^{i+1} \cup \Xi_{2}^{i+1} \nvdash \perp$, there exists a possible world $W \models \Xi_{1}^{i+1} \cup \Xi_{2}^{i+1}$. It follows that $k_{\Xi_{1}}^{-}(i) \cap k_{\Xi_{2}}^{-}(i) \neq \emptyset$. Since $\Xi_{1}^{i} \cup \Xi_{2}^{i} \vdash \perp$, it follows that $k_{\Xi_{1}}^{-}(n) \cap k_{\Xi_{2}}^{-}(n)=\emptyset$ for any $n<i$. Therefore, $i=\min \left\{n \mid k_{\Xi_{1}}^{-}(n) \cap k_{\Xi_{2}}^{-}(n) \neq \emptyset\right\}$, which is equal to $d_{\text {inc }}\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)$ according to (5.2).

To prove Observation 5.7, we need the following result.

Observation C.1. Suppose $\Xi$ is an EE base and $k_{\Xi}$ its induced OCF. Let $r$ be any natural number. Then for any possible world $W$ :

$$
\left(k_{\Xi}-r\right)(W)=k_{\Xi^{\prime}}(W)
$$

where $\Xi^{\prime}=\Xi-r$ and $k_{\Xi^{\prime}}$ is the OCF induced from $\Xi^{\prime}$.

Proof. Assume $k_{\Xi}(W) \leq r$. According to (5.3), we have $\left(k_{\Xi}-r\right)(W)=0$. It follows from (4.8) that $W \models \Xi^{r+1}$. According to (5.2), $\Xi^{r+1}$ contains all sentences of $\Xi_{1}$. Therefore, we have $k_{\Xi_{1}}(W)=0$, due to (4.8). Assume $k_{\Xi}(W)=i>r$. According to (5.3), $\left(k_{\Xi}-r\right)(W)=i$. It follows from (4.8) that $W \models \Xi^{i+1}$ and there is a sentence $\left.\alpha \in \Xi\right|_{i}$ such that $W \models \neg \alpha$. According to (5.2), we have $\Xi_{1}^{i+1}=\Xi^{i+1}$ and $\Xi_{1}|i=\Xi| i$. Therefore, (4.8) implies $k_{\Xi_{1}}(W)=$ $i$.

Observation 5.7. Suppose $\Xi_{1}, \Xi_{2}$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
\gamma\left(k_{\Xi_{1}}, k_{\Xi_{2}}\right)=\left\langle k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}\right\rangle
$$

Where $k_{\Xi_{1}^{\prime}}, k_{\Xi_{2}^{\prime}}$ are OCFs induced respectively from $\gamma_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $\gamma_{2}\left(\Xi_{1}, \Xi_{2}\right)$.

Proof. It follows directly from Observation 5.6 and C.1.

We need the following observation for the proof of Theorem 5.8.

Observation C.2. Suppose $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle, \Xi_{2}=\left\langle B_{2} . f_{2}\right\rangle$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then

$$
m_{k_{\Xi_{1}}, k_{\Xi_{2}}}=\operatorname{Rank}_{\Xi_{1}}(\neg \bigwedge B
$$

Proof. Assume $m_{k_{\Xi_{1}}, k_{\Xi_{2}}}=i$. According to (5.1) and (4.8), $W \in k_{\Xi_{2}}^{-}(0)$ iff $W \models B_{2}$, for any possible world $W$. It follows from (5.6) that $i=\min \left\{n \mid k_{\Xi_{1}}^{-}(n) \cap\left\{W \mid W \models B_{2}\right\} \neq\right.$ $\emptyset\}$. Therefore, (5.1) implies $i=\min \left\{k_{\Xi_{1}}(W) \mid W \models B_{2}\right\}$, which is equal to $k_{\Xi_{1}}\left(\neg \wedge B_{2}\right)$ according to (3.2). It follows from Theorem 4.6 that $i=\operatorname{Rank}_{\Xi_{1}}(\neg \wedge B 2)$.

Theorem 5.8. Suppose $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle, \Xi_{2}=\left\langle B_{2}, f_{2}\right\rangle$ are two EE bases and $k_{\Xi_{1}}, k_{\Xi_{2}}$ are the OCFs induced from $\Xi_{1}$ and $\Xi_{2}$, respectively. Then for any possible world $w$ :

$$
\left(k_{\Xi_{1}} \otimes k_{\Xi_{2}}\right)(W)=k_{\Xi}(W)
$$

Where $\Xi=\Xi_{1} \otimes \Xi_{2}$ and $k_{\Xi}$ is the OCF induced from $\Xi$.

Proof. Assume $k_{\Xi_{1}}(W)=i$. It follows from (4.8) that $W \models \Xi_{1}^{i+1}$ and there exists a sentence $\left.\psi \in \Xi_{1}\right|_{i}$ such that $W \models \neg \psi$. Let $r=\operatorname{Rank}_{\Xi_{1}}\left(\neg \bigwedge B_{2}\right)$. Due to Observation C.2, it follows that see that $r=\min \left\{k_{\Xi_{1}}(W) \mid W \models B_{2}\right\}=m_{k_{\Xi_{1}}, k_{\Xi_{2}}}$. We distinguish two cases:
1). Suppose $k_{\Xi_{2}}(W)=0$. According to (5.5), $\left(k_{\Xi_{1}} \otimes k_{\Xi_{2}}\right)(W)=i-r$. Moreover, according to (5.11), $\Xi^{i+1-r}=\Xi_{1}^{i+1} \cup \Xi_{2}^{i+1-r} \cup\left\{\beta \vee \alpha \mid \beta \in B_{1}, \alpha \in B_{2}\right.$ and $\left.f_{1}(\beta)+f_{2}(\alpha) \geq i+1-r\right\}$. Since $k_{\Xi_{2}}(W)=0$, it follows from (4.8) that $W \models \Xi_{2}^{1}=B_{2}$. Therefore, it is easy to see that
$W \models \Xi^{i+1-r}$, since $W \models \Xi_{1}^{i+1}$. According to (5.11), we have $\left.\psi \in \Xi\right|_{i-r}$. It follows from (4.8), we have $k_{\Xi}(W)=i-r$.
2). Suppose $k_{\Xi_{2}}(W)=j>0$. According to (5.5), $\left(k_{\Xi_{1}} \otimes k_{\Xi_{2}}\right)(W)=i+j$. It follows from (4.8) that $W \models \Xi_{2}^{j+1}$ and there exists a sentence $\left.\varphi \in \Xi_{2}\right|_{j}$ such that $W \models \neg \varphi$. According to (5.11), $\Xi^{i+j+1}=\Xi_{1}^{i+j+1+r} \cup \Xi_{2}^{i+j+1} \cup\left\{\beta \vee \alpha \mid \beta \in B_{1}, \alpha \in B_{2}\right.$ and $f_{1}(\beta)+f_{2}(\alpha) \geq$ $i+j+1\}$. Since $W \models \Xi_{1}^{i}$ and $W \models \Xi_{2}^{j}$, it is obvious that $W \models \Xi^{i+j+1}$. According to (5.11), we have $\left.\psi \vee \varphi \in \Xi\right|_{i+j}$. Since $W \models \neg(\psi \vee \varphi)$, it follows from (4.8) that $k_{\Xi}(W)=i+j$.

Theorem 5.13. The problem of computing $M\left(\Xi_{1}, \Xi_{2}\right)$, for arbitrary EE bases $\Xi_{1}$ and $\Xi_{2}$, is NP-equivalent.

Proof. The main computation lays in calculating $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right), \operatorname{Rank}_{\Xi_{1}}\left(\neg \wedge B_{2}^{\prime}\right)$ and $\operatorname{Rank}_{\Xi_{2}}\left(\neg \wedge B_{1}^{\prime}\right)$, where $B_{1}^{\prime}$ and $B_{2}^{\prime}$ are belief bases of $\gamma_{1}\left(\Xi_{1}, X i_{2}\right)$ and $\gamma_{2}\left(\Xi_{1}, X i_{2}\right)$ respectively. It is not difficult to see that $d_{\text {inc }}\left(\Xi_{1}, \Xi_{2}\right)=\operatorname{Rank}_{\Xi}(\perp)$, where $\Xi=\Xi_{1} \cup \Xi_{2}$. Therefore, according to Algorithm E.1, they can be computed in polynomial time, calling at most logarithmic times the decision procedure. Hence, the problem is in $\mathrm{F} \Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})] \subseteq \mathrm{F} \Delta_{2}^{\mathrm{p}}$.

To show NP-hardness, it suffices to observe that for an arbitrary sentence $\alpha, \alpha$ is satisfiable iff $\alpha$ occurs in $M_{1}(\{\langle\alpha, 1\rangle\},\{\langle\top, 2\rangle\})$.

Theorem 5.14. The problem of deciding whether both $M_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)$ entail $\beta$, for arbitrary EE bases $\Xi_{1}, \Xi_{2}$ and sentence $\beta$, is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-complete.

Proof. It is easy to see that the size of $M\left(\Xi_{1}, \Xi_{2}\right)$ is polynomial to that of $\left\langle\Xi_{1}, \Xi_{2}\right\rangle$. Once we have computed $M\left(\Xi_{1}, \Xi_{2}\right)$, two additional calls of the decision procedure suffice to solve the problem. Therefore, it is in $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$ (see the proof of Theorem 5.13).

The $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-hardness follows from the fact that the current problem generalizes the CF problem wrt. reinforcement base revision which is $\Delta_{2}^{\mathrm{p}}[\mathrm{O}(\log \mathrm{n})]$-hard (cf. Theorem 4.17). Suppose $\Xi_{1}=\left\langle B_{1}, f_{1}\right\rangle$ is an EE base and $\alpha, \beta$ any sentences. In order to decide $\Xi_{1} \circledast_{r} \alpha \vdash \beta$, we construct $\Xi_{2}=\{\langle\alpha, m\rangle\}$ such that $m>\max \left\{f_{1}\left(\beta_{i}\right) \mid \beta_{i} \in B_{1}\right\}$. It is not difficult to see that $\operatorname{Bel}\left(M_{2}\left(\Xi_{1}, \Xi_{2}\right)\right) \subseteq \operatorname{Bel}\left(M_{1}\left(\Xi_{1}, \Xi_{2}\right)\right)$ and $\Xi_{1} \alpha \vdash \beta$ iff $M_{1}\left(\Xi_{1}, \Xi_{2}\right)$ entails $\beta$. Therefore $\Xi_{1} \circledast_{r} \alpha \vdash \beta$ iff $\beta$ is implied by both $M_{1}\left(\Xi_{1}, \Xi_{2}\right)$ and $M_{2}\left(\Xi_{1}, \Xi_{2}\right)$.

## Appendix D

## Proofs of Results of Chapter 6

Observation 6.5. Let I be a possible world, and $\alpha$ a consistent conjunction of literals. Then,

$$
I \diamond_{P M A} \alpha=I \diamond_{W S S \downarrow} \alpha
$$

Proof. Thanks to Observation 6.4, we only need to show that $I \diamond_{W S S_{\downarrow}} \alpha \subseteq I \diamond_{P M A} \alpha$. Assume there is possible world $I^{\prime}$ such that $I^{\prime} \in I \diamond_{W S S_{\downarrow}} \alpha$ and $I^{\prime} \notin I \diamond_{P M A} \alpha$. It follows that there is a possible world $I^{\prime \prime}: I^{\prime \prime} \in[\alpha]$ and $\Delta\left(I, I^{\prime \prime}\right) \subset \Delta\left(I, I^{\prime}\right)$. Therefore, there exists at least one atom $p$ such that $p \notin \Delta\left(I, I^{\prime \prime}\right)$ and $p \in \Delta\left(I, I^{\prime}\right)$. Assume, without loss of generality, that $p \in I$. Then we have $p \in I^{\prime \prime}$ and $p \notin I^{\prime}$. From $I^{\prime}, I^{\prime \prime} \in[\alpha]$ and $\alpha$ is a consistent conjunction of literals, it follows that $p \notin \operatorname{Atm}_{\downarrow}(\alpha)$, which contradicts $I^{\prime} \in I \diamond_{W S S \downarrow} \alpha$.

Theorem 6.6. Let I be a possible world. Then for any sentence $\alpha$,

$$
I \diamond_{W S S_{\downarrow}} \alpha=I \diamond_{P M} \alpha
$$

Proof. We first show $I \diamond_{P M} \alpha \subseteq I \diamond_{W S S_{\downarrow}} \alpha$. Assume $I^{\prime} \in I \diamond_{P M} \alpha$. Then there exists $L \in P M(\alpha)$ and $I^{\prime}=\operatorname{Insert}(L, I)$. From $L \vdash \alpha$ and $\operatorname{Insert}(L, I) \models L$, it follows that $I^{\prime} \in[\alpha]$. Moreover,
$L \in P M(\alpha)$ implies $\operatorname{Atm}(U) \subseteq \operatorname{Atm}_{\downarrow}(\alpha)$. It is follows immediately that $\Delta(I, \operatorname{Insert}(L, I)) \subseteq$ $\operatorname{Atm}_{\downarrow}(\alpha)$. Therefore, $I^{\prime} \in I \diamond_{W S S_{\downarrow}} \alpha$.

What remains to show is that $I \diamond_{W S S_{\downarrow}} \alpha \subseteq I \diamond_{P M} \alpha$. Assume $I^{\prime} \in I \diamond_{W S S_{\downarrow}} \alpha$. It follows that $I^{\prime} \in[\alpha]$ and $\Delta\left(I, I^{\prime}\right) \subseteq \operatorname{Atm}_{\downarrow}(\alpha)$. Let $T=\left\{p \mid p \in I^{\prime}\right.$ and $\left.p \in \operatorname{Atm}_{\downarrow}(\alpha)\right\}$ and $L=$ $T \cup\left\{\neg p \mid p \notin T\right.$ and $\left.p \in \operatorname{Atm}_{\downarrow}(\alpha)\right\}$. Obviously, $L$ is a maximal consistent subset of $\operatorname{Lit}(\alpha)$. From $I^{\prime} \models \alpha$, it follows that $L \vdash \alpha$. Therefore, $L \in P M(\alpha)$. We will show by contradiction that $I^{\prime}=\operatorname{Insert}(L, I)$. Assume, without loss of generality, there is an atom $p$ such that $p \in I^{\prime}$ and $p \notin \operatorname{Insert}(L, I)$. We distinguish two cases:

1. Assume $p \in \operatorname{Atm}_{\downarrow}(\alpha)$. From $p \in I^{\prime}$, it follows that $p \in L$. Therefore, $p \in \operatorname{Insert}(L, I)$, which contradicts $p \notin \operatorname{Insert}(L, I)$.
2. Assume $p \notin \operatorname{Atm}_{\downarrow}(\alpha)$. From $\Delta\left(I, I^{\prime}\right) \subseteq \operatorname{Atm}_{\downarrow}(\alpha)$, it follows that $p \in I$. It is easy to see that $\Delta(I, \operatorname{Insert}(L, I)) \subseteq A \operatorname{tm}_{\downarrow}(\alpha)$. Therefore, it must be the case that $p \in \operatorname{Insert}(L, I)$, which contradicts our assumption.

Hence, we obtain $I^{\prime}=\operatorname{Insert}(L, I)$. It follows that $I^{\prime} \in I \diamond_{P M} \alpha$.

Theorem 6.8. Suppose there exist a polynomial pand a computational model © of the $W S S_{\downarrow}$ such that $|B \odot \alpha| \leq p(|B|+|\alpha|)$ for any belief base $B$ and sentence $\alpha$. Then $\mathrm{NP} \subseteq \mathrm{P} /$ poly.

Proof. This proof is based on ideas of the proof of Theorem 8 in [Cadoli et al., 1995]. We will show that if there exists a polynomially space bounded computational model of the $\mathrm{WSS}_{\downarrow}$, then $3 \mathrm{SAT}^{1}{ }^{1}$ is in $\mathrm{P} /$ poly. The proof consists of two steps.

Step 1: For any integer $n$, we first construct a belief set $B_{n}$ and a sentence $\alpha_{n}$, whose sizes are polynomial wrt. $n$. Let $X=\left\{x_{1}, \cdots, x_{n}\right\}$ and $Y=\left\{y_{1}, \cdots, y_{n}\right\}$ be two disjoint set of atoms and let $C$ be a set of new atoms for each 3-literal clause over $X$, i.e., $C=\left\{c_{1} \mid \gamma_{i}\right.$ is a 3-literal clause of $\left.X\right\}$. We define $B_{n}$ and $\alpha_{n}$ as follows:

$$
\begin{aligned}
& B_{n}=\left\{x_{i} \not \equiv y_{i}, \gamma_{i} \vee \neg c_{i} \mid 1 \leq i \leq n\right\} \\
& \alpha_{n}=\bigwedge_{i=1}^{n}\left(\neg x_{1} \wedge \neg y_{i}\right)
\end{aligned}
$$

It is easy to see that $\left|B_{n}\right| \in \mathrm{O}\left(n^{3}\right)$ and $\left|\alpha_{n}\right| \in \mathrm{O}(n)$.

[^46]Then we show that for any 3 CNF $\beta$ of size $n$, there exists an interpretation $I_{\beta}$ (on atoms $X \cup Y \cup C$ ) such that $I_{\beta} \models B \diamond_{W S S_{\downarrow}} \alpha$ iff $\beta$ is satisfiable. We assume, without loss of generality, that $\operatorname{Atm}(\beta) \subseteq X,{ }^{2}$ Then $I_{\beta}$ can be constructed as follows:

$$
I_{\beta}=\left\{c_{i} \in C \mid \gamma_{i} \text { is a clause of } \beta\right\}
$$

We now show that $\beta$ is satisfiable iff $I_{\beta} \models B_{n} \diamond_{W S S_{\downarrow}} \alpha_{n}$. It is easy to see that $\operatorname{Atm}_{\downarrow}\left(\alpha_{n}\right)=$ $X \cup Y$ and $I_{\beta} \models \alpha_{n}$.
$\Rightarrow$ Assume $\beta$ is satisfiable and $I$ is a model of $\beta$. We construct another interpretation $I^{\prime}=$ $(I \cap X) \cup I_{\beta} \cup\left\{y_{i} \mid x_{i} \notin I\right\}$. It is easy to see that $I^{\prime} \models B_{n}$ and $\Delta\left(I^{\prime}, I_{\beta}\right) \subseteq A t m_{\downarrow}\left(\alpha_{n}\right)$. From the definition of the $\mathrm{WSS}_{\downarrow}$, it follows that $I_{\beta} \models B \diamond_{W S S_{\downarrow}} \alpha$.
$\Leftarrow$ Assume $I_{\beta} \models B_{n} \diamond_{W S S_{\downarrow}} \alpha_{n}$. Then there exists an interpretation $I$ such that $I \models B_{n}$ and $\Delta\left(I, I_{\beta}\right) \subseteq \operatorname{Atm}_{\downarrow}\left(\alpha_{n}\right)$. We claim that $I \models \beta$. Assume $I \not \models \beta$. Then there exists a 3-literal clause $\gamma_{i}$ of $\beta$ such that $I \not \vDash \gamma_{i}$. From $\Delta\left(I, I_{\beta}\right) \subseteq A t m_{\downarrow}\left(\alpha_{n}\right)$ and $c_{i} \in I_{\beta}$, if follows that $c_{i} \in I$. This implies $I \not \vDash \gamma_{i} \vee \neg c_{i}$, which contradicts $I \models B_{n}$. Thus, $\beta$ is satisfiable.

Step 2: Suppose © is a polynomial space bounded WSS base update operator. Then 3SAT can be solved by an advice taking Turing machine in this way: given a arbitrary 3 CNF $\beta$ of size $n$, the machine computes $I_{\beta}$ after loading the advice $B_{n} \odot \alpha_{n}$, then it verifies $I_{\beta} \models B_{n} \odot \alpha_{n}$ ? From $\left|B_{n}\right| \in \mathrm{O}\left(n^{3}\right),\left|\alpha_{n}\right| \in \mathrm{O}(n)$ and $\odot$ is polynomially space bounded, it follows that $B_{n} \odot \alpha_{n}$ has polynomial size wrt. $n$. Therefore, $I_{\beta} \models B_{n} \odot \alpha_{n}$ can be verified in polynomial time, which shows that $3 \mathrm{SAT} \in \mathrm{P} /$ poly. Since 3 SAT is NP-complete, this implies $\mathrm{NP} \subseteq \mathrm{P} /$ poly.

Observation 6.11. Suppose $\pi$ is a possibility distribution that respects DC and $\varphi \Rightarrow$ is a conditional update. Then the updated possibility distribution $\pi \diamond(\varphi \Rightarrow \alpha)$ also respects DC .

Proof. We need to show that $\pi \diamond \varphi \Rightarrow \alpha(W)=0$ iff $W \not \vDash$ DC. Suppose $W$ is an implausible world such that $W \not \models \mathrm{DC}$. It follows directly from (6.9) that $\pi \diamond \varphi \Rightarrow \alpha(W)=0$. Suppose $W$ is a possible world such that $W \models$ DC. Assume $W \nLeftarrow \alpha$. According to (6.3), we have $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)=\emptyset$. It follows from (6.9) that $\pi \diamond \varphi \Rightarrow \alpha(W)=w_{\mathbf{\Delta}}>0$. Assume $W \models \alpha$.

[^47]Then, according to (6.3), $W \in \operatorname{Sup}(\varphi \Rightarrow \alpha, W)$. Since $\pi(W)>0$, it follows from (6.9) that $\pi \diamond \varphi \Rightarrow \alpha(W)=\max \left(\left\{\pi\left(W^{\prime}\right) \mid W^{\prime} \in \operatorname{Sup}(\varphi \Rightarrow \alpha, W)\right\}\right)>0$.

Observation 6.12. Suppose $\Sigma$ is a possibilistic base that respects DC. Then the possibility distribution $\pi_{\Sigma}$ induced from $\Sigma$ (as defined by (4.20) also respects DC.

Proof. Let $\Sigma=\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$. Suppose $W$ is a possible world such that $W \models$ DC. Since $\Sigma$ respects DC, it follows that $W \not \vDash \beta_{i}$ implies $w_{i}<1$ for any $\beta_{i}$. Then according to (4.20), we have $\pi_{\Sigma}(W)>0$. Suppose $W$ is an implausible worlds such that $W \not \vDash \mathrm{DC}$. Then there exists a sentence $\beta_{i} \in \mathrm{DC}$ with $W \not \vDash \beta_{i}$. Since $\Sigma$ respects DC, it follows that $w_{i}=1$. According to (4.20), we have $\pi_{\Sigma}(W)=1-1=0$.

Theorem 6.13. Suppose $\Sigma$ is a possibilistic base that respects DC and $\pi_{\Sigma}$ is the possibility distribution induced from $\Sigma$ as defined by (4.20). Let $\pi_{1}=\pi \diamond(\varphi \Rightarrow \alpha)$ and $\Sigma_{1}=\Sigma \square(\varphi \Rightarrow \alpha)$. Then for any sentence $\beta$ that contains no history atom:

$$
N_{\Sigma_{1}}(\beta)=N_{\pi_{1}}(\beta)
$$

Proof. Assume $\Sigma=\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$ and $P=\operatorname{Dep}(\alpha)$.
If $\vdash \beta$, then $N_{\Sigma_{1}}(\beta)=N_{\pi_{1}}(\beta)=\infty$, according to (4.15) and (4.18). In the rest of the proof, we assume $\nvdash \beta$.

Assume $\mathrm{DC} \vdash \beta$. It follows from (4.18) and (6.10) that $N_{\Sigma_{1}}(\beta)=1$. Moreover, Observation 6.12 and Observation 6.11 imply that $\pi_{1}$ also respects DC, i.e., for any possible world $W: \pi_{1}(W)=0$ iff $W \not \vDash \mathrm{DC}$. From $\mathrm{DC} \vdash \beta$, it follows that for all possible world $W$ : if $W \models \neg \beta$ then $\pi_{1}(W)=0$. Therefore, $N_{\pi_{1}}(\beta)=1-\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right)=1$.

Assume $\mathrm{DC} \nvdash \beta$ and $N_{\Sigma_{1}}(\beta)=w_{\mathbf{\vee}}$. It follows from (4.18) that $\Sigma_{1}^{w \boldsymbol{\rightharpoonup}} \vdash \beta$. We need to show that $N_{\pi_{1}}(\beta)=1-\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right)=w_{\mathbf{V}}$. Since $\mathrm{DC} \nvdash \beta$, there exists a possible world $W_{1}$ such that $W_{1} \models \mathrm{DC}$ and $W_{1} \models \neg \beta$. Since $\pi_{1}$ respects DC, we have $\pi_{1}\left(W_{1}\right)>0$. Hence, $W \in \operatorname{argmax}_{W \models \neg \beta} \pi_{1}(W)$ implies $W \models \mathrm{DC}$, We show next that $W \in$ $\operatorname{argmax}_{W \models \neg \beta} \pi_{1}(W)$ implies $W \not \models \varphi \rightarrow \alpha$. Suppose there is a possible world $W_{2}$ such that $W_{2} \in \operatorname{argmax}_{W \models \neg \beta} \pi_{1}(W)$ and $W_{2} \models \varphi \rightarrow \alpha$. It follows immediately that $W_{2} \models \mathrm{DC}$. Then
we construct another possible world $W_{2}^{\prime}$ from $W_{2}$ by changing the truth value of the history atom $p^{\prime}$ to be same as $p$ for all $p \in P$, that is, $W_{2}^{\prime}=\operatorname{Insert}\left(U, W_{2}\right)$ where $U=\left\{p^{\prime} \mid p \in P\right.$ and $p \in$ $\left.W_{2}\right\} \cup\left\{\neg p^{\prime} \mid p \in P\right.$ and $\left.p \notin W_{2}\right\}$. It follows immediately that $W_{2}^{\prime} \models\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p^{\prime} \leftrightarrow\right.\right.$ p) $\mid p \in A\}$. From $W_{2} \models\{\varphi \rightarrow \alpha, \neg \beta\} \cup \mathrm{DC}$, it follows that $W_{2}^{\prime} \models\left\{\varphi\left[P / P^{\prime}\right] \rightarrow \alpha, \neg \beta\right\} \cup$ $\mathrm{DC} \cup \mathrm{DC}\left[P / P^{\prime}\right]$. Therefore, we obtain $W_{2}^{\prime} \models \Sigma_{1}^{w \boldsymbol{v}}$ and $W_{2}^{\prime} \models \neg \beta$, which contradicts $\Sigma_{1}^{w \mathbf{v}} \vdash \beta$. It is easy to see that for any possible world $W$ : if $W \not \vDash \varphi \rightarrow \alpha$ then $\operatorname{Sup}(\varphi \Rightarrow \alpha, W)=\emptyset$. Therefore, according to (6.9), we have $N_{\pi_{1}}(\beta)=1-\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right)=1-w_{\mathbf{\Delta}}$. Therefore, we have $N_{\Sigma_{1}}(\beta)=N_{\pi_{1}}(\beta)$, since it is assumed that $1-w_{\mathbf{\Delta}}=w_{\mathbf{\nabla}}$.

Assume $N_{\Sigma_{1}}(\beta)=w<w_{\mathbf{v}}$. It follows from (4.18) that $\Sigma_{1}^{>w} \nvdash \beta$ and $\Sigma_{1}^{w} \vdash \beta$. In order to show $N_{\pi_{1}}(\beta)=1-\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right)$, we will prove in two steps that $\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right)=1-w$.

1. We claim that $\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right) \geq 1-w$. Since $\Sigma_{1}^{>w} \nvdash \beta$, there is a possible world $W_{1}$ such that $W_{1} \models \Sigma_{1}^{>w}$ and $W_{1} \models \neg \beta$. From $\Sigma_{1}^{w} \vdash \beta$, it follows that there is a sentence $\gamma$ such that $\langle\gamma, w\rangle \in \Sigma,\left\langle\gamma\left[P / P^{\prime}\right], w\right\rangle \in \Sigma_{1}$ and $W_{1} \models \neg \gamma\left[P / P^{\prime}\right]$. From $W_{1} \models$ $\Sigma_{1}^{>w}$, it follows that $W_{1} \models\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right) \mid p \in P\right\}$ and $W_{1} \models \varphi\left[P / P^{\prime}\right] \rightarrow \alpha$. We distinguish two cases:

- Assume $W_{1} \models \neg \varphi\left[P / P^{\prime}\right]$. Since $W_{1} \models\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right) \mid p \in P\right\}$, we have $W_{1} \models\left\{p \leftrightarrow p^{\prime} \mid p \in P\right\}$. It follows immediately that $W_{1} \models \neg \varphi$. From $W_{1} \models \Sigma_{1}^{>w}$ and $W_{1} \models \neg \gamma\left[P / P^{\prime}\right]$, it follows that $W_{1} \models \Sigma^{>w}$ and $W_{1} \models \neg \gamma$. According to (4.20), we have $\pi_{\Sigma}\left(W_{1}\right)=1-w$. Since $W_{1} \models \neg \varphi$ and $W_{1} \models \mathrm{DC}$, according to (6.3), we have $W_{1} \in \operatorname{Sup}\left(\varphi \Rightarrow \alpha, W_{1}\right)$. By (6.9), we have $\pi_{1}\left(W_{1}\right) \geq$ $\pi_{\Sigma}\left(W_{1}\right)=1-w$.
- Assume $W_{1} \models \varphi\left[P / P^{\prime}\right]$. From $W_{1} \models \varphi\left[P / P^{\prime}\right] \rightarrow \alpha$, it follows that $W_{1} \models \alpha$. We construct another possible world $W_{1}^{\prime}$ from $W_{1}$ by changing the truth value of atom $p$ to be same as $p^{\prime}$ for all $p \in P$, that is, $W_{1}^{\prime}=\operatorname{Insert}\left(U, W_{1}\right)$ where $U=$ $\left\{p \mid p \in P\right.$ and $\left.p^{\prime} \in W_{1}\right\} \cup\left\{\neg p \mid p \in P\right.$ and $\left.p^{\prime} \notin W_{1}\right\}$. From $W_{1} \models \Sigma_{1}^{>w}$ and $W_{1} \models \neg \gamma\left[P / P^{\prime}\right]$, it follows that $W_{1}^{\prime} \models \Sigma^{>w}$ and $W_{1}^{\prime} \models \neg \gamma$. In particular, we have $W_{1}^{\prime} \models \mathrm{DC}$. Then, according to (4.20), we have $\pi_{\Sigma}\left(W_{1}^{\prime}\right)=1-w$. If follows from $\Delta\left(W_{1}^{\prime}, W_{1}\right) \subseteq A$ and $W_{1} \models \alpha$, that $W_{1}^{\prime} \in \operatorname{Sup}\left(\varphi \Rightarrow \alpha, W_{1}\right)$. According to (6.9), we have $\pi_{1}\left(W_{1}\right) \geq 1-w$. From $W_{1} \models \neg \beta$, it follows that $\max \left(\left\{\pi_{1}(W) \mid W \models\right.\right.$ $\neg \beta\}) \geq 1-w$.

2. Then we show that $\max \left(\left\{\pi_{1}(W) \mid W \models \neg \beta\right\}\right) \leq 1-w$. Suppose there exists a possible
world $W_{2}$ such that $W_{2} \models \neg \beta$ and $\pi_{1}\left(W_{2}\right)>1-w$. According to (6.9), there must be another possible world $W_{2}^{\prime}$ such that $\pi_{\Sigma}\left(W_{2}^{\prime}\right)>1-w$ and $W_{2}^{\prime} \in \operatorname{Sup}\left(\varphi \Rightarrow \alpha, W_{2}\right)$, that is, $W_{2}^{\prime} \models \mathrm{DC}$ and $W_{2} \in W_{2}^{\prime} \diamond_{W S S_{\downarrow}}^{D e p} \varphi \Rightarrow \alpha$. Since $\pi_{\Sigma}\left(W_{2}^{\prime}\right)>1-w$, according to (4.20) we have $W_{2}^{\prime} \models \Sigma^{w}$. Again, we distinguish two cases:

- Assume $W_{2}^{\prime} \models \neg \varphi$. It follows that $W_{2}^{\prime} \triangleleft_{W S S_{\downarrow}}^{D e p} \varphi \Rightarrow \alpha=\left\{W_{2}^{\prime}\right\}$. Therefore, we have $W_{2}^{\prime}=W_{2}$, since $W_{2} \in W_{2}^{\prime} \diamond_{W S S_{\downarrow}}^{D e p} \varphi \Rightarrow \alpha$. We construct a possible world $W_{2}^{\prime \prime}$ from $W_{2}^{\prime}$ by changing the truth value of the history atoms $p^{\prime}$ to be same as $p$ for all $p \in P$, that is, $W_{2}^{\prime \prime}=\operatorname{Insert}\left(U, W_{2}^{\prime}\right)$ where $U=\left\{p^{\prime} \mid p \in P\right.$ and $\left.p \in W_{2}^{\prime}\right\} \cup\left\{\neg p^{\prime} \mid p \in\right.$ $P$ and $\left.p \notin W_{2}^{\prime}\right\}$. It follows immediately that $W_{2}^{\prime \prime} \models\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right) \mid p \in\right.$ $P\}$. From $W_{2}^{\prime} \models \Sigma^{w}$ and $W_{2}^{\prime} \models \neg \varphi$, it follows that $W_{2}^{\prime \prime} \models\left\{\beta_{i}\left[P / P^{\prime}\right] \mid w \leq w_{i}<\right.$ $\left.w_{\vee}\right\}$ and $W_{2}^{\prime \prime} \models \varphi\left[P / P^{\prime}\right] \rightarrow \alpha$. Since $W_{2}^{\prime} \models \mathrm{DC}$, we have $W_{2}^{\prime \prime} \models \mathrm{DC} \cup \mathrm{DC}\left[P / P^{\prime}\right]$. Hence, $W_{2}^{\prime \prime} \models \Sigma_{1}^{w}$. From $W_{2}=W_{2}^{\prime} \models \neg \beta$, it follows that $W_{2}^{\prime \prime} \models \neg \beta$. This contradicts $\Sigma_{1}^{w} \vdash \beta$.
- Assume $W_{2}^{\prime} \models \varphi$. It follows that $W_{2} \models \alpha$ and $\Delta\left(W_{2}, W_{2}^{\prime}\right) \subseteq P$. We construct a possible world $W_{2}^{\prime \prime}$ from $W_{2}$ by change the truth value of the history atoms $p^{\prime}$ to be same as the truth value of $p$ in $W_{2}^{\prime}$ for all $p \in P$, that is, $W_{2}^{\prime \prime}=\operatorname{Insert}\left(U, W_{2}\right)$ where $U=\left\{p^{\prime} \mid p \in P\right.$ and $\left.p \in W_{2}^{\prime}\right\} \cup\left\{\neg p^{\prime} \mid p \in P\right.$ and $\left.p \notin W_{2}^{\prime}\right\}$. It is easy to see that $\Delta\left(W_{2}^{\prime}, W_{2}^{\prime \prime}\right) \subseteq P$, since $\Delta\left(W_{2}, W_{2}^{\prime}\right) \subseteq P$. From $W_{2} \models \mathrm{DC} \cup\{\alpha, \neg \beta\}$ and $W_{2}^{\prime} \models$ $\mathrm{DC} \cup\{\varphi\}$, it follows that $W_{2}^{\prime \prime} \models \mathrm{DC} \cup \mathrm{DC}\left[P / P^{\prime}\right] \cup\left\{\alpha, \neg \beta, \varphi\left[P / P^{\prime}\right]\right\}$. Similarly, we have $W_{2}^{\prime \prime} \models\left\{\beta_{i}\left[P / P^{\prime}\right] \mid w \leq w_{i}<w_{\mathbf{V}}\right\}$, since $W_{2}^{\prime} \models\left\{\beta_{i} \mid w \leq w_{i}<w_{\mathbf{\vee}}\right\}$. From $W_{2}^{\prime \prime} \models \varphi\left[P / P^{\prime}\right]$, it follows that $W_{2}^{\prime \prime} \models\left\{\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right) \mid p \in A\right\}$. Therefore, we have $W_{2}^{\prime \prime} \models \Sigma_{1}^{w} \cup\{\neg \beta\}$, which contradicts $\Sigma_{1}^{w} \vdash \beta$.

Observation 6.15. Assume the dependence function Dep can be computed in polynomial time. Then for the update operator defined by (6.10), the CF problem is coNP-complete.

Proof. Since an updated possibilistic base can be computed in polynomial time, it is obvious that the CF problem is in coNP.

To show the coNP-hardness of the problem, it suffices to observe that for any sentences $\alpha, \beta$ we have $\alpha \vdash \beta$ iff $\{\langle\alpha, 0.5\rangle\}$ 回 T entails $\beta$.

Observation 6.16. Suppose $\pi$ is a possibility distribution that respects DC. Let $\langle\alpha, w\rangle$ be the input information such that $\alpha$ is consistent with DC and $w<1$. Then the revised possibility distribution $\pi\left(.\left.\right|_{r}\langle\alpha, w\rangle\right)$ also respects DC.

Proof. As $\alpha$ is consistent with DC, there exits a possible world $W_{1}$ such that $W_{1} \models \mathrm{DC}$ and $W_{1} \models \alpha$. Since $\pi$ respects DC, we have $\pi(W)=0$ iff $W \not \models \mathrm{DC}$. It follows immediately that $\pi\left(W_{1}\right)>0$ and $\Pi_{\pi}(\alpha)>0$. We need to show that $\pi\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)=0$ iff $W \not \vDash$ DC. Suppose $W$ is an implausible world such that $W \not \models \mathrm{DC}$. Since $\Pi_{\pi}(\alpha)>0$, it follows from (4.17) that $\pi\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)=\frac{\pi(w)}{\Pi_{\pi}(\alpha)}=0$. Suppose $W$ is a possible world such that $W \models \mathrm{DC}$. As $\pi$ respects DC, we have $\pi(W)>0$. Since $w<1$, it follows from (4.17) that $\pi\left(\left.W\right|_{r}\langle\alpha, w\rangle\right)=$ $(1-w) \times \pi(W)>0$.

## Appendix E

## Algorithms

```
Input : \(\Xi=\left\{\left\langle\beta_{1}, e_{i}\right\rangle, \ldots,\left\langle\beta_{n}, e_{n}\right\rangle\right\}\) such that \(e_{i} \leq e_{i+1}, \beta\)
Output: \(r=\operatorname{Rank}_{\Xi}(\beta)\)
begin
    if \(\vdash \beta\) then
        \(r=\infty\);
    else if \(\left\{\beta_{1}, \cdots, \beta_{n}\right\} \nvdash \beta\) then
        \(r=0 ;\)
    else
        \(i=1 ;\)
        \(j=n\)
        while \(i \leq j\) do
            \(k=i+\left\lceil\frac{j-i}{2}\right\rceil\);
            if \(\Xi^{e_{k}} \vdash \beta\) then
                    if \(\Xi^{e_{k+1}} \nvdash \beta\) then
                                    \(r=e_{k} ;\)
                                    return \(r\);
                    else
                        \(i=k+1 ;\)
            endif
                else
                    \(j=k ;\)
            endif
        endw
    endif
    return \(r\);
end
E.1: Algorithm of computing rank of a sentence
```

```
Input : \(\Xi=\left\{\left\langle\beta_{1}, e_{i}\right\rangle, \ldots,\left\langle\beta_{n}, e_{n}\right\rangle\right\}, \alpha, m\)
Output: \(\Xi_{1}\) such that \(\Xi_{1}=\Xi \circledast_{r}\langle\alpha, m\rangle\)
begin
    \(\Xi_{1}=\{ \} ;\)
    \(\bar{r}=\operatorname{Rank}_{\Xi}(\neg \alpha)\);
    for \(i=1 \ldots n\) do
        if \(e_{i}>\bar{r}\) then
            \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i}, e_{i}-\bar{r}\right\rangle\right\} ;\)
        endif
        \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \alpha, e_{i}+m\right\rangle\right\} ;\)
    endfor
    \(\Xi_{1}=\Xi_{1} \cup\{\langle\alpha, m\rangle\} ;\)
    return \(\Xi_{1}\);
end
```

E.2: Reinforcement base revision

```
Input : \(\Xi=\left\{\left\langle\beta_{1}, e_{i}\right\rangle, \ldots,\left\langle\beta_{n}, e_{n}\right\rangle\right\}, \alpha, m\)
Output: \(\Xi_{1}\) such that \(\Xi_{1}=\Xi \circledast_{c}\langle\alpha, m\rangle\)
begin
    \(\Xi_{1}=\{ \} ;\)
    if \(\left\{\beta_{1}, \cdots, \beta_{n}\right\} \vdash \alpha\) then
        \(r=\operatorname{Rank}_{\Xi}(\alpha)\);
        for \(i=1 \ldots n\) do
            if \(e_{i}>r\) then
                \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \alpha, e_{i}-r+m\right\rangle\right\} ;\)
            endif
            \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \neg \alpha, e_{i}\right\rangle\right\} ;\)
        endfor
    else
            \(\bar{r}=\operatorname{Rank}_{\Xi}(\neg \alpha) ;\)
            for \(i=1 \ldots n\) do
            if \(e_{i}>\bar{r}\) then
                        \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i}, e_{i}-\bar{r}\right\rangle\right\} ;\)
            endif
            \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \alpha, e_{i}+m\right\rangle\right\} ;\)
            endfor
        endif
        \(\Xi_{1}=\Xi_{1} \cup\{\langle\alpha, m\rangle\} ;\)
        return \(\Xi_{1}\);
end
E.3: Base conditionalization
```

```
Input : \(\Xi=\left\{\left\langle\beta_{1}, e_{i}\right\rangle, \ldots,\left\langle\beta_{n}, e_{n}\right\rangle\right\}, \alpha, m\)
Output: \(\Xi_{1}\) such that \(\Xi_{1}=\Xi \circledast_{j}\langle\alpha, m\rangle\)
begin
    \(\Xi_{1}=\{ \} ;\)
    if \(\left\{\beta_{1}, \cdots, \beta_{n}\right\} \vdash \alpha\) then
        \(r=\operatorname{Rank}_{\Xi}(\alpha)\);
        for \(i=1 \ldots n\) do
            if \(e_{i}>r\) then
                    \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i}, e_{i}\right\rangle\right\} ;\)
            else
                \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \neg \alpha, e_{i}\right\rangle\right\} ;\)
            endif
        endfor
    else
        \(\bar{r}=\operatorname{Rank}_{\Xi}(\neg \alpha) ;\)
        for \(i=1 \ldots n\) do
            if \(e_{i}>\bar{r}\) then
                    \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i}, e_{i}\right\rangle\right\} ;\)
            else if \(e_{i}>m\) then
                    \(\Xi_{1}=\Xi_{1} \cup\left\{\left\langle\beta_{i} \vee \alpha, e_{i}\right\rangle\right\} ;\)
            endif
        endfor
    endif
    \(\Xi_{1}=\Xi_{1} \cup\{\langle\alpha, m\rangle\} ;\)
    return \(\Xi_{1}\);
end
```

E.4: Base adjustment

Input : $\Sigma=\left\{\left\langle\beta_{1}, w_{i}\right\rangle, \ldots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}, \alpha, w$
Output: $\Sigma_{1}$ such that $\Sigma_{1}=\Sigma$ * $_{r}\langle\alpha, w\rangle$
begin
$\Sigma_{1}=\{ \} ;$
$\bar{r}=N_{\Sigma}(\neg \alpha)$;
for $i=1 \ldots n$ do
if $w_{i}>\bar{r}$ then
$\Sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\beta_{i}, \frac{w_{i}-\bar{r}}{1-\bar{r}}\right\rangle\right\} ;$
endif
$\Sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\beta_{i} \vee \alpha, w_{i}+w-w \times w_{i}\right\rangle\right\} ;$
endfor
$\Sigma_{1}=\Sigma_{1} \cup\{\langle\alpha, w\rangle\} ;$
return $\Sigma_{1}$;
end
E.5: Possibilistic reinforcement base revision

```
Input : \(\Sigma=\left\{\left\langle\beta_{1}, w_{i}\right\rangle, \ldots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}, \varphi \Rightarrow \alpha\), Dep
Output: \(\Sigma_{1}\) such that \(\Sigma_{1}=\Sigma\) 回 \((\varphi \Rightarrow \alpha)\)
begin
    \(\Sigma_{1}=\{ \} ;\)
    \(P=\operatorname{Dep}(\alpha)\);
    for \(i=1 \ldots n\) do
        if \(w_{i}=1\) then
            \(\Sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\beta_{i}, w_{i}\right\rangle\right\} ;\)
        else
            \(\Sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\beta_{i}\left[P / P^{\prime}\right], w_{i}\right\rangle\right\} ;\)
        endif
    endfor
    foreach \(p \in P\) do
        \(\sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\neg \varphi\left[P / P^{\prime}\right] \rightarrow\left(p \leftrightarrow p^{\prime}\right), w_{\mathbf{\vee}}\right\rangle\right\} ;\)
    endfch
    \(\Sigma_{1}=\Sigma_{1} \cup\left\{\left\langle\varphi\left[P / P^{\prime}\right] \rightarrow \alpha, w_{\boldsymbol{\rightharpoonup}}\right\rangle\right\} ;\)
    return \(\Sigma_{1}\);
end
E.6: Possibilistic base update
```


## Bibliography

[Alchourrón and Makinson, 1981] C. E. Alchourrón and D. Makinson. Hierarchies of regulations and their logic. New Studies in Deontic Logic, pages 125-148, 1981.
[Alchourrón and Makinson, 1982] C. E. Alchourrón and D. Makinson. On the logic of theory change: contraction functions and their associated revision functions. Theoria, 48:14-37, 1982.
[Alchourrón and Makinson, 1985] C. E. Alchourrón and D. Makinson. On the logic of theory change: Safe contraction. Studia Logica, 44, pages 405-422, 1985.
[Alchourrón et al., 1985] C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. The Journal of Symbolic Logic 50(2), pages 510-530, 1985.
[Andréka et al., 1998] H. Andréka, J. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. Journal of Philosophical Logic, 27:217-274, 1998.
[Benferhat et al., 2002] S. Benferhat, D. Dubois, H. Prade, and M.-A. Williams. A practical approach to revising prioritized knowledge bases. Studia Logica, 70(1):105-130, 2002.
[Booth et al., 2005] R. Booth, S. Chopra, and T. Meyer. Restrained revision. In Proceedings of NRAC05, Sixth Workshop on Nonmonotonic Reasoning, Action and Change, 2005.
[Booth, 2001] R. Booth. A negotiation-style framework for non-prioritised revision. In Proceedings of TARK: Theoretical Aspects of Reasoning about Knowledge, volume 8, pages 137-150. Morgan Kaufmann Publishers Inc., 2001.
[Boutilier, 1993] C. Boutilier. Revision sequences and nested conditionals. In Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), pages 519-525, 1993.
[Boutilier, 1995] C. Boutilier. Generalized update: Belief change in dynamic settings. In Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence (IJCAI95), pages 1550-1556, Montreal, 1995.
[Cadoli et al., 1995] M. Cadoli, F. M. Donini, P. Liberatore, and M. Schaerf. The size of a revised knowledge base. In PODS '95: Proceedings of the fourteenth ACM SIGACT-SIGMODSIGART symposium on Principles of database systems, pages 151-162, New York, NY, USA, 1995. ACM Press.
[Dalal, 1988] M. Dalal. Updates in propositional databases. Technical Report DCS-TR 222, Rutgers University, Dept. of Computer Science, Feb 1988.
[Darwiche and Pearl, 1994] A. Darwiche and J. Pearl. On the logic of iterated belief revision. In R. Fagin, editor, TARK: Theoretical Aspects of Reasoning about Knowledge, pages 5-23. Kaufmann, San Francisco, CA, 1994.
[Darwiche and Pearl, 1997] A. Darwiche and J. Pearl. On the logic of iterated belief revision. Artificial Intelligence 89, 1-29, 1997.
[Doherty et al., 1998] P. Doherty, W. Łukaszewicz, and E. Madalińska-Bugaj. The PMA and relativizing change for action update. In Anthony G. Cohn, Lenhart Schubert, and Stuart C. Shapiro, editors, KR'98: Principles of Knowledge Representation and Reasoning, pages 258-269. Morgan Kaufmann, San Francisco, California, 1998.
[Doyle, 1979] J. Doyle. A truth maintenance system. Artifical Intelligence, 12:231-272, 1979.
[Eiter and Gottlob, 1992] T. Eiter and G. Gottlob. On the complexity of propositional knowledge base revision, updates, and counterfactuals. artificial intelligence, 57:227-270, 1992.
[Fariñas del Cerro and Herzig, 1996] L. Fariñas del Cerro and A. Herzig. Belief change and dependence. In Yoav Shoham, editor, TARK: Theoretical Aspects of Reasoning about Knowledge, pages 147-162. Morgan Kaufmann, 1996.
[Fermé and Hansson, 1999] E. L. Fermé and S. O. Hansson. Selective revision. Studia Logica, 63(3):331-342, 1999.
[Fermé, 1998] E. Fermé. On the logic of theory change: Contraction without recovery. Journal of Logic, Language and Information, 7(2):127-137, 1998.
[Freund and Lehmann, 1994] M. Freund and D. Lehmann. Belief revision and rational inference. Technical Report TR-94-16, Institute of Computer Science, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel, 1994.
[Friedman and Halpern, 1996] N. Friedman and J. Y. Halpern. Belief revision: A critique. In Luigia Carlucci Aiello, Jon Doyle, and Stuart Shapiro, editors, KR'96: Principles of Knowledge Representation and Reasoning, pages 421-431. Morgan Kaufmann, San Francisco, California, 1996.
[Fuhrmann and Hansson, 1994] A. Fuhrmann and S. O. Hansson. A survey of multiple contractions. Journal of Logic, Language, and Information, 3:39-76, 1994.
[Gärdenfors and Makinson, 1988] P. Gärdenfors and D. Makinson. Revisions of knowledge systems using epistemic entrenchment. In Proceedings of TARK: Theoretical Aspects of Reasoning about Knowledge, pages 83-95, Asilomar, CA, 1988.
[Gärdenfors, 1988] P. Gärdenfors. Knowledge in Flux: Modeling the Dynamics of Epistemic States. MIT Press, 1988.
[Gärdenfors, 1990] P. Gärdenfors. Belief revision and irrelevance. PSA, 2:349-356, 1990.
[Gauwin et al., 2005] O. Gauwin, S. Konieczny, and P. Marquis. Conciliation and consensus in iterated belief merging. In Proceedings of 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'05), pages 514-526, Barcelona, 2005.
[Ginsberg and Smith, 1987] M.L. Ginsberg and D.E. Smith. Reasoning about action i: a possible world approach. In M.L. Ginsberg, editor, Readings in Nonmonotonic Reasoning, pages 434-463. Morgan Kaufmann, 1987.
[Giunchiglia et al., 2004] E. Giunchiglia, J. Lee, V. Lifschitz, N. McCain, and H. Turner. Nonmonotonic causal theories. Artificial Intelligence, 153:49-104, 2004.
[Goldszmidt, 1992] M. Goldszmidt. Qualitative probabilities: a normative framework for commonsense reasoning. PhD thesis, University of California at Los Angeles, 1992.
[Grahne, 1991] G. Grahne. Updates and counterfactuals. In James F. Allen, Richard Fikes, and Erik Sandewall, editors, KR'91: Principles of Knowledge Representation and Reasoning, pages 269-276. Morgan Kaufmann, San Mateo, California, 1991.
[Grove, 1988] A. Grove. Two modellings for theory change. Journal of Philosophical Logic, 17, pages 157-180, 1988.
[Hansson, 1989] S. O. Hansson. New operators for thoery change. Theoria, 55:115-132, 1989.
[Hansson, 1991] S. O. Hansson. Belief contraction without recovery. Studia Logica, 50(2):251260, 1991.
[Hansson, 1997] S. O. Hansson. A Textbook of Belief Dynamics. Kluwer Academic Publishers, 1997.
[Hansson, 1998] S. O. Hansson. Revision of belief sets and belief bases. Handbook of Defeasible Reasoning and Uncertainty Management Systems, 3:17-75, 1998.
[Hansson, 1999] S. O. Hansson. A survey of non-prioritized belief revision. Erkenntnis, 50:413-427, 1999.
[Hansson, 2003] S. O. Hansson. Ten philosophical problems in belief revision. Journal of Logic Computation, 13(1):37-49, 2003.
[Hegner, 1987] S. J. Hegner. Specification and implementation of programs for updating incomplete information databases. In Proceedings of the Sixth ACM SIGACT-SIGMODSIGART Symposium on Principles of Database Systems, March 23-25, 1987, San Diego, California, pages 146-158. ACM, 1987.
[Herzig and Rifi, 1998] A. Herzig and O. Rifi. Update operations: a review. In H. Prade, editor, Proc. Eur. Conf. on Artificial Intelligence (ECAI'98), August 1998.
[Herzig and Rifi, 1999] A. Herzig and O. Rifi. Propositional belief base update and minimal change. Artificial Intelligence, 115(1):107-138, 1999.
[Herzig, 1996] A. Herzig. The pma revisited. In L. C. Aiello and S. Shapiro, editors, Proc. Int. Conf. on Knowledge Representation and Reasoning (KR'96), pages 40-50, 1996.
[Hunter and Delgrande, 2005] A. Hunter and J. P. Delgrande. Using ranking functions to determine plausible action histories. In IJCAI-05 Workshop on Nonmonotonic Reasoning, Action, and Change (NRAC'05), pages 59-64, Edinburgh, Scotland, August 2005.
[Jin and Thielscher, 2004] Y. Jin and M. Thielscher. Representing beliefs in the fluent calculus. In R. L. de Mántaras and L. Saitta, editors, Proceedings of the European Conference on Artificial Intelligence (ECAI), pages 823-827, Valencia, Spain, August 2004. IOS Press.
[Jin and Thielscher, 2005a] Y. Jin and M. Thielscher. Actions and belief revision: A computational approach. In J. Delgrande, J. Lang, H. Rott, and J. Tallon, editors, Belief Change in Rational Agents: Perspectives from Artificial Intelligence, Philosophy, and Economics, number 05321 in Dagstuhl Seminar Proceedings, 2005.
[Jin and Thielscher, 2005b] Y. Jin and M. Thielscher. Iterated belief revision, revised. In Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), pages 478483, Edinburgh, Scotland, August 2005.
[Johnson, 1990] D. S. Johnson. A catalog of complexity classes. In J. van Leeuwen, editor, Handbook of Theoretical Computer Science: Volume A: Algorithms and Complexity, pages 67-161. Elsevier, Amsterdam, 1990.
[Karp and Lipton, 1980] R. M. Karp and R. J. Lipton. Some connections between non-uniform and uniform complexity classes. In Proc. of the 12th ACM sym. on Theory of Computing (STOC-80), pages 302-309, 1980.
[Katsuno and Mendelzon, 1991a] H. Katsuno and A. Mendelzon. On the difference between updating a knowledge base and revising it. In J. F. Allen, R. Fikes, and E. Sandewall, editors, KR'91: Principles of Knowledge Representation and Reasoning, pages 387-394. Morgan Kaufmann, San Mateo, California, 1991.
[Katsuno and Mendelzon, 1991b] H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. Artificial Intelligence, 52(3):263-294, 1991.
[Katsuno and Satoh, 1991] H. Katsuno and K. Satoh. A unified view of consequence relation, belief revision and conditional logic. In Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), pages 406-412, 1991.
[Keller and Winslett, 1985] A. M. Keller and M. Winslett. On the use of an extended relational model to handle changing incomplete information. In IEEE Transactions on Software Engineering SE-11:7, pages 620-633, July 1985.
[Konieczny and Pérez, 1998] S. Konieczny and R. P. Pérez. On the logic of merging. In Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98), pages 488-498, 1998.
[Konieczny and Pérez, 2000] S. Konieczny and R. P. Pérez. A framework for iterated revision. Journal of Applied Non-Classical Logics, 10(3-4), 2000.
[Konieczny and Pérez, 2002] S. Konieczny and R. P. Pérez. Dynamical revision operators with memory. In S. Benferhat and E. Giunchiglia, editors, Ninth International Workshop on NonMonotonic Reasoning (NMR'02), pages 171-179, 2002.
[Kudo et al., 1999] Y. Kudo, T. Murai, and T. Da-te. Iterated belief update based on ordinal conditional functions. In Proceedings of the Third International Conference on KnowledgeBased Intelligent Information Engineering Systems, pages 526-529, 1999.
[Lehmann, 1995] D. J. Lehmann. Belief revision, revised. In C. S. Mellish, editor, Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), pages 1534-1540, 1995.
[Levesque et al., 1997] H. J. Levesque, R. Reiter, Y. Lespérance, F. Lin, and R. B. Scherl. GOLOG: A logic programming language for dynamic domains. Journal of Logic Programming, 31(1-3):59-83, 1997.
[Levi, 1977] I. Levi. Subjunctives, dispositions and chances. Synthese, 34(Issue 4):423-455, April 1977.
[Levi, 1980] I. Levi. The Enterprise of Knowledge. The MIT Press, 1980.
[Liberatore and Schaerf, 1998] P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). IEEE Transactions on Knowledge and Data Engineering, 10(1):76-90, 1998.
[Liberatore, 1997] P. Liberatore. The complexity of belief update. In Proceedings of the Fifteenth International Joint Conference on Artificial Intelligence (IJCAI'97), pages 68-73, 1997.
[Lifschitz, 1990] V. Lifschitz. Frames in the space of situations (research note). Artif. Intell., 46(3):365-376, 1990.
[Makinson and Gärdenfors, 1989] D. Makinson and P. Gärdenfors. Relations between the logic of theory change and nonmonotonic logic. In André Fuhrmann and Michael Morreau, editors, The Logic of Theory Change, pages 185-205, 1989.
[Makinson, 1987] D. Makinson. On the status of the postulate of recovery in the logic of theory change. Journal of Philosophical Logic, 1987.
[McCain and Turner, 1995] N. McCain and H. Turner. A causal theory of ramifications and qualifications. In Chris Mellish, editor, Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence, pages 1978-1984, San Francisco, 1995. Morgan Kaufmann.
[McCain, 1997] N. C. McCain. Causality in commonsense reasoning about actions. Technical report, Austin, TX, USA, 1997.
[McCarthy and Hayes, 1969] J. McCarthy and P. J. Hayes. Some philosophical problems from the standpoint of artificial intelligence. Machine Intelligence, 4:463-502, 1969.
[McCarthy, 1963] J. McCarthy. Situations and Actions and Causal Laws. Stanford Artificial Intelligence Project, Memo 2, Stanford University, CA, 1963.
[Nayak et al., 1996a] A. Nayak, N. Foo, M. Pagnucco, and A. Sattar. Changing conditional belief unconditionally. In Proceedings of the Sixth Conference on Theoretical Aspects of Rationality and Knowledge, pages 119-135. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1996.
[Nayak et al., 1996b] A. Nayak, P. Nelson, and H. Polansky. Belief change as change in epistemic entrenchment. Synthese, 109:143-174, 1996.
[Nayak et al., 2003] A. Nayak, M. Pagnucco, and P. Peppas. Dynamic belief revision operators. Artificial Intelligence, 146(2):193-228, 2003.
[Nayak, 1994a] A. Nayak. Foundational belief change. Journal of Philosophical Logic, 23:495-533, 1994.
[Nayak, 1994b] A. Nayak. Iterated belief change based on epistemic entrenchment. Erkenntnis, 4:353-390, 1994.
[Nebel, 1992] B Nebel. Syntax-based approaches to belief revision. In P. Gärdenfors, editor, Belief Revision, volume 29, pages 52-88. Cambridge University Press, Cambridge, UK, 1992.
[Nebel, 1994] B. Nebel. Base revision operations and schemes: Semantics, representation and complexity. In ECAI, pages 341-345, 1994.
[Nebel, 1998] B. Nebel. How hard is it to revise a belief base? In Didier Dubois and Henri Prade, editors, Handbook of Defeasible Reasoning and Uncertainty Management Systems, Volume 3: Belief Change, pages 77-145. Kluwer Academic Publishers, Dordrecht, 1998.
[Newell, 1982] A. Newell. The knowledge level. Artificial Intelligence, 18:87-127, 1982.
[Niederée, 1991] R. Niederée. Mutiple contraction: A further case against gärdenfors' principle of recovery. In A. Fuhrmann and M. Morreau, editors, The Logic of Theory Change. Springer, Berlin, 1991.
[Papadimitriou, 1994] C. Papadimitriou. Computational Complexity. Addison-Wesley, 1994.
[Peppas and Williams, 1995] P. Peppas and M-A. Williams. Constructive modellings for theory change. Notre Dame Journal of Formal Logic, 36(1), 1995.
[Peppas et al., 1996] P. Peppas, A. Nayak, M. Pagnucco, N. Foo, R. Kwok, and M. Prokopenko. Revision vs. update: Taking a closer look. In Proceedings of the 12th European Conference on Artificial Intelligence, pages 95-99, Budapest, 1996. Wiley \& Sons.
[Reiter, 1991] R. Reiter. The frame problem in the situation calculus: a simple solution (sometimes) and a completeness result for goal regression. In V. Lifschitz, editor, Artificial Intelligence and Mathematical Theory of Computation: Papers in Honor of John McCarthy, pages 359-380. Academic Press, 1991.
[Reiter, 2001a] R. Reiter. Logic in Action. MIT Press, 2001.
[Reiter, 2001b] R. Reiter. On knowledge-based programming with sensing in the situation calculus. ACM Transactions on Computational Logic, 2(4):433-457, 2001.
[Revesz, 1997] P. Z. Revesz. On the Semantics of Arbitration. Journal of Algebra and Computation, 7 (2):133-160, 1997.
[Rott, 1991] H. Rott. Two methods of constructing contractions and revisions of knowledge systems. Journal of Philosophical Logic, 20:149-173, 1991.
[Rott, 1992] H. Rott. Preferential belief change using generalized epistemic entrenchment. Journal of Logic, Language and Information, 1(1):45-78, 1992.
[Rott, 1999] H. Rott. Coherence and conservatism in the dynamics of belief. part i: Finding the right framework. Erkenntnis, 50:387-412, 1999.
[Rott, 2000] H. Rott. Two dogmas of belief revision. Journal of Philosophy, 97(9):503-522, 2000.
[Rott, 2003] H. Rott. Coherence and conservatism in the dynamics of belief ii: Iterated belief change without dispositional coherence. Journal of Logic Computation, 13(1):111-145, 2003.
[Scherl and Levesque, 2003] R. Scherl and H. Levesque. Knowledge, action, and the frame problem. Artificial Intelligence, 144(1):1-39, 2003.
[Shanahan, 1999] M. Shanahan. The event calculus explained. Lecture Notes in Computer Science, 1600:409-430, 1999.
[Shapiro and Pagnucco, 2004] S. Shapiro and M. Pagnucco. Iterated belief change and exogenous actions in the situation calculus. In R. López de Mántaras and L. Saitta, editors, Proceedings of the 16th European Conference on Artificial Intelligence (ECAI-04), pages 878-882, Amsterdam, 2004. IOS Press.
[Shapiro et al., 2000] S. Shapiro, M. Pagnucco, and H. J. Levesque. Iterated belief change in the situation calculus. In Anthony G. Cohn, Fausto Giunchiglia, and Bart Selman, editors, KR2000: Principles of Knowledge Representation and Reasoning, pages 527-538, San Francisco, 2000. Morgan Kaufmann.
[Spohn, 1988] W. Spohn. Ordinal conditional functions: A dynamic theory of epistemic state. In W. L. Harper and B. Skyrms, editors, Causation in Decision: Belief, Change and Statistics: Proceedings of the Irvine Conference on Probability and Causation, volume II, pages 105134, Dordrecht, 1988. Kluwer Academic Publisher.
[Spohn, 1991] W. Spohn. A reason for explanation: Explanations provide stable reasons. In W. Spohn et al., editor, Existence and Explanation, pages 165-196. Kluwer Academic Publisher, 1991.
[Thielscher, 1996] M. Thielscher. Causality and the qualification problem. In L. C. Aiello, J. Doyle, and S. C. Shapiro, editors, Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning (KR), pages 51-62, Cambridge, MA, November 1996. Morgan Kaufmann.
[Thielscher, 1997] M. Thielscher. Ramification and causality. Artificial Intelligence Journal, 89(1-2):317-364, 1997.
[Thielscher, 1999] M. Thielscher. From situation calculus to fluent calculus: State update axioms as a solution to the inferential frame problem. Artificial Intelligence, 111(1-2):277-299, 1999.
[Thielscher, 2000] M. Thielscher. Representing the knowledge of a robot. In A. Cohn, F. Giunchiglia, and B. Selman, editors, Proc. of the International Conference on Principles of Knowledge Representation and Reasoning ( $K R$ ), pages 109-120, Breckenridge, CO, April 2000. Morgan Kaufmann.
[Thielscher, 2004a] M. Thielscher. FLUX: A logic programming method for reasoning agents. Theory and Practice of Logic Programming, 2004. Available for download at www.fluxagent.org.
[Thielscher, 2004b] M. Thielscher. Logic-based agents and the frame problem: A case for progression. In V. Hendricks, editor, First-Order Logic Revisited: Proceedings of the Conference 75 Years of First Order Logic (FOL75), pages 323-336, Berlin, Germany, 2004. Logos.
[van der Meyden, 94] R. van der Meyden. Mutual belief revision (preliminary report). In Proceedings of $K R$, pages 595-606, 94.
[Wassermann, 1999] R. Wassermann. Resource-Bounded Belief Revision. PhD thesis, ILLC, University of Amsterdam, 1999.
[Williams, 1992] M. Williams. Two operators for theory base change. In Proceedings of the Fifth Australian Joint Conference on Artificial Intelligence, pages 259-265, 1992.
[Williams, 1994a] M.-A. Williams. On the logic of theory base change. In C. MacNish, D. Pearce, and L.M. Pereira, editors, Logics in Artificial Intelligence, volume LNCS No 835, pages 86-105. Springer Verlag, 1994.
[Williams, 1994b] M.-A. Williams. Transmutations of knowledge systems. In J. Doyle, E. Sandewall, and P. Torasso, editors, Principles of Knowledge Representation and Reasoning (KR), pages 619-629, Bonn, Germany, 1994. Morgan Kaufmann.
[Williams, 1995] M.-A. Williams. Iterated theory base change: A computational model. In C. S. Mellish, editor, Proceedings of the International Joint Conference on Artificial Intelligence (IJCAI), pages 1541-1547, 1995.
[Winslett, 1988a] M. Winslett. A model-based approach to updating databases with incomplete information. ACM Trans. Database Syst., 13(2):167-196, 1988.
[Winslett, 1988b] M. Winslett. Reasoning about action using a possible models approach. In The Seventh National Conference on Artificial Intelligence, pages 89-93, 1988.
[Winslett, 1990] M. Winslett. Updating logical databases. Cambridge University Press, 1990.
[Wobck, 1992] W. R. Wobck. On the use of epistemic entrenchment in nonmonotonic reasoning. In Proceedings of the Tenth European Conference on Artificial Intelligence, pages 324-328, 1992.
[Zhang and Foo, 1996] Y. Zhang and N. Foo. Updating knowledge bases with disjunctive information. In Proceedings of the AAAI National Conference on Artificial Intelligence, pages 563-568, 1996.
[Zhang and Foo, 2001] D. Zhang and N. Foo. Infinitary belief revision. Journal of Philosophical Logic, 30(6):525-570, 2001.
[Zhang et al., 2004] D. Zhang, N. Foo, T. Meyer, and R. Kwok. Negotiation as mutual belief revision. In Proceedings of the AAAI National Conference on Artificial Intelligence, pages 317-322, 2004.
[Zhang, 1996] D. Zhang. Belief revision by sets of sentences. Computer Science and Technology 11(2):119, 1996.
[Zhang, 2004] D. Zhang. Properties of iterated multiple belief revision. In V. Lifschitz and I. Niemelä, editors, Proceedings of the International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR), pages 314-325. Springer, 2004.


[^0]:    ${ }^{1}$ see Section 2.3 for a detailed discussion on why such a distinction is necessary.
    ${ }^{2}$ In the literature, the word "belief revision" in commonly used in two distinct senses. Often, it is used as a synonym of belief change to refer to the research field in general. Here, as well in the rest of this thesis, belief revision means a particular type of belief change operators.

[^1]:    ${ }^{3}$ In this thesis, we do not distinguish between the beliefs and knowledge of an agent, and they both refer to agent's mental model of the world.

[^2]:    ${ }^{4}$ In reasoning about actions, there are also proposals in which the agent does not maintain a belief (knowledge) state [Levesque et al., 1997, Giunchiglia et al., 2004]. E.g., in the situation calculus, only the agent's initial knowledge of the world is explicitly specified; in order to to verify whether a sentence (denoting certain facts of the world) holds after some actions, we need first to regress the sentence to another (possibly very large) sentence according to the effects of performed actions; then this regressed sentence is verified against the agent's initial knowledge. Not surprisingly, such regression-based approaches will have very bad computational behavior [Thielscher, 2004b; Thielscher, 2004a].

[^3]:    ${ }^{5}$ For the sake of readability, proofs are not given in the main body of the thesis, and they can be found in the appendices.

[^4]:    ${ }^{1}$ For the time-being, let us simply consider a belief change operator as a function which maps a belief set and a sentence to a new belief set. In fact, it is quite controversial in the belief change community, what is the right notion of belief change operators? More discussions on this issue can be found in Section 3.1.

[^5]:    ${ }^{2}$ Note that a tautology is contained in any belief set.

[^6]:    ${ }^{3}$ Postulate $\left(\mathrm{K}^{*} 2\right)$ reflects the principle of primacy of the new information, that is, the new information is more reliable (hence prioritized) than the original belief set. It is worth to mention that non-prioritized belief revision in which the new information is not always accepted is a relatively recent research topic [Fermé and Hansson, 1999; Booth, 2001; Hansson, 1999].

[^7]:    ${ }^{4}$ The decomposition principle does not mean in practice all belief change operators should be constructed as sequences of contraction and expansion, although it is possible in principle.
    ${ }^{5}$ Note that the satisfiability of controversial Postulate (K-5) is not required here.

[^8]:    ${ }^{6}$ A selection function is called connectively relational iff it is induced from a total relation $\prec$. Analogously, a partial meet contraction operator - on $K$ is transitively and connectively relational iff it is generated from a transitively and connectively relational selection function. It is interesting to mention that the class of partial meet contraction operators induced from transitively relational selection functions is same to those induced from transitively and connectively relational selection functions, that is, the connectivity condition on $\prec$ essentially does not impose any additional constraint[Gärdenfors, 1988].

[^9]:    ${ }^{7}$ A total pre-order $\leq$ is a reflexive, transitive binary relation, s.t., for all $u, v$ either $u \leq v$ or $v \leq u$. If $u \leq v$ and $v \not \leq u$, we also write $u<v$, while $u \leq v$ and $v \leq u$ is abbreviated as $u=v$. We denote by $\min (\bar{A}, \leq)$ the minimal elements of $A$ wrt $\leq$, that is, $\min (A, \leq)=\{u \in A \mid u \leq v$ for all $v \in A\}$.

[^10]:    ${ }^{8}$ Note that, if $\vdash \alpha \wedge \beta$ then $\alpha \leq_{K} \beta$ due to (EE2) and (EE5).
    ${ }^{9}$ See [Gärdenfors and Makinson, 1988] for details on how (C-) is obtained from (C $\leq$ ).

[^11]:    ${ }^{10} \mathrm{~A}$ SOS centered on $[K]$ is depicted in Figure 2.2, where the rectangle surface represents $\Theta_{\mathcal{L}}$ (i.e., each point on the surface denotes a possible world).

[^12]:    ${ }^{11}$ To be precise, the original KM postulates consists of (U1)-(U8), and (U9) is a complementary postulate which implies (U6) and (U7). Here, $(K \diamond 2)-(K \diamond 6)$ and $(K \diamond 8)$ correspond respectively to $(U 1)-(U 5)$ and $(U 8)$; whereas $(U 9)$ is reformulated as $(K \diamond 7)$. Similarly, $(U 6)$ and $(U 7)$ can be reformulated as follows:
    (U6') If $\beta \in K \diamond \alpha$ and $\alpha \in K \diamond \beta$, then $K \diamond \alpha=K \diamond \beta$
    (U7') If $K$ is complete then $K \diamond(\alpha \vee \beta) \subseteq K \diamond \alpha \cap K \diamond \beta$

[^13]:    ${ }^{12}$ Observe, in $(K \diamond 8)$ the same update operator $\diamond$ is applied to the belief set $K$ and each possible world $W \in[K]$. This is well-defined, since a possible world is essentially also a belief set.

[^14]:    ${ }^{13}$ The proof is not difficult. Suppose $K \subseteq K^{\prime}$. For any sentence $\beta$, if $\beta \in K * \alpha$ then according to the "if" part of (RT) $\alpha \gg \beta \in K \subseteq K^{\prime}$. It follows from the "only if" part of (RT), $\beta \in K^{\prime} * \alpha$.
    ${ }^{14}$ Two sentences $\alpha, \beta$ are disjoint iff $\vdash \neg(\alpha \wedge \beta)$

[^15]:    ${ }^{15}$ Note that, $\left(\mathrm{K}^{*} 3\right)$ and $\left(\mathrm{K}^{*} 4\right)$ do not appear in the first column of Table 2.1, since they follow respectively from $\left(\mathrm{K}^{*} 7\right.$ ) and ( $\mathrm{K} * 8$ ) (by assuming $\beta=\mathrm{T}$ ). For the sake of conciseness, $\sim_{K, *}$ is abbreviated as $\sim_{\text {. }}$.

[^16]:    ${ }^{1}$ It is not difficult to see that the following arguments apply to other constructive models as well.

[^17]:    ${ }^{2}$ Readers are referred to [Darwiche and Pearl, 1997] for other examples.

[^18]:    ${ }^{3}$ see [Fariñas del Cerro and Herzig, 1996] for a similar definition.

[^19]:    ${ }^{4}$ Here, $\mathbb{N}$ represents the set of all natural numbers and the set of all positive integers (i.e., natural number greater than 0 ) is denoted by $\mathbb{N}^{+}$.

[^20]:    ${ }^{5}$ In Spohn's original proposal, the rank of a sentence is the lowest rank of a world in which it is true. So the rank of $\beta$ there is equal to $k(\neg \beta)$ here.
    ${ }^{6}$ For the sake of simplicity, $s$ and $r$ are assumed the only atoms.

[^21]:    ${ }^{7}$ Essentially, they consider the extra-logical preference information as part of the revision operator.

[^22]:    ${ }^{8}$ As $\mathcal{L}$ is assumed finite in [Konieczny and Pérez, 2000], the conjunction $\wedge \operatorname{Bel}\left(\mathcal{K}_{1}\right)$ is a well-defined sentence.

[^23]:    ${ }^{9}$ In [Nayak et al., 2003], (Conj) is written as "If $\alpha \nvdash \neg \beta$, then $(\mathcal{K} * \alpha) *{ }^{\mu} \beta \equiv \mathcal{K} *(\beta \wedge \alpha)$. ", where $*^{\alpha}$ denotes

[^24]:    the evolved operator after a $\alpha$-revision. Accordingly, they have reformulated the DP postulates in the same spirit.

[^25]:    ${ }^{1}$ Formally, the size of $S$, denoted by $|S|$, is the number of symbols occurring in $S$; whereas the cardinality of a set $S$, denoted by $\|S\|$, is the number of elements of $S$.

[^26]:    ${ }^{2}$ Note that a prioritized base $\left\langle B, \leq_{B}\right\rangle$ can also be represented by a totally ordered family of classes of sentences $\left(B_{1}, \cdots, B_{n}\right)$, with $\alpha \leq_{B} \beta$ iff there exist $i, j$ such that $\alpha \in B_{i}, \beta \in B_{j}$ and $i \leq j$.

[^27]:    ${ }^{3}$ Note that an EE base $\Xi=\langle B, f\rangle$ can also be represented as a finite set of pairs: $\left\{\left\langle\beta_{1}, e_{1}\right\rangle, \cdots,\left\langle\beta_{n}, e_{n}\right\rangle\right\}$ with $\beta_{i} \in B$ and $f\left(\beta_{i}\right)=e_{i}$.

[^28]:    ${ }^{4}$ For simplicity, redundant sentences are removed and I will always do so in the rest of the paper

[^29]:    ${ }^{5}$ Being interested in computation, I assume that an EE base is finite. Without this assumption, an EE base $\Xi=\langle B, f\rangle$ can be induced from an OCF $k$ as follows:

    - $f(\alpha)=\min \{k(W) \mid k \notin W\}$ for any non-tautologous sentence $\alpha$
    - $B=\{\alpha \mid \alpha$ is non-tautologous and $f(\alpha)>0\}$

[^30]:    ${ }^{6}$ I assume the reader has basic knowledge on complexity theory, or otherwise can be found in [Papadimitriou, 1994]
    ${ }^{7}$ IMPL decides " $B \vdash \alpha$ ?" for a finite set $B$ of sentences and a sentence $\alpha$.

[^31]:    ${ }^{1}$ Given a function $f$, its $i$-th projection is denoted by $f_{i}$.

[^32]:    ${ }^{2}$ Observe the analogy to the belief revision model based on SOS [Grove, 1988].

[^33]:    ${ }^{1}$ Although it is also assumed in Section 4.2.2 that the underlying language is propositional, the approaches proposed in Chapter 4 actually apply to any decidable languages. However, the approaches that will be presented in this chapter are specific to propositional languages, as their definitions exploit specific features of propositional languages.

[^34]:    ${ }^{2}$ Notice that Dalal's notion of distance is cardinality-based.
    ${ }^{3}$ These postulates correspond exactly to (U1)-(U8) of [Katsuno and Mendelzon, 1991a].

[^35]:    ${ }^{4}$ Here, $\oplus$ denotes the exclusive or, that is, $\alpha \oplus \beta=\alpha \wedge \neg \beta \vee \neg \alpha \wedge \beta$

[^36]:    ${ }^{5}$ Some researchers believe that the syntax of the new information is relevant. E.g., they think the unpredictable effect of "toss a coin" should be represented by $h \vee \neg h$, where $h$ stands for "the coin lands with head side up". However, in my opinion, it is more suitable to deal with such (unpredictable) actions using a so-called (symmetric) erasure [Katsuno and Mendelzon, 1991a] instead of update, so the agent should erase its beliefs regarding $h$ after tossing a coin. Note that erasure is to contraction as update is to revision, and I leave a comprehensive study on erasure as future work.
    ${ }^{6}$ It is worth mentioning that the same operator has also been proposed in [Doherty et al., 1998], where it is named the modified PMA.
    ${ }^{7}$ Note that, since the $W S S_{\downarrow}$ does not satisfy ( $K \diamond 6$ ), Condition of (XOR) of Observation 6.2 does not apply.

[^37]:    ${ }^{8}$ Note that, according to Definition 2.5, $\operatorname{Lit}(\alpha) \downarrow \perp$ denotes the set of all maximal consistent subsets of $\operatorname{Lit}(\alpha)$.

[^38]:    ${ }^{9}$ This definition of Insert is borrowed from [Hegner, 1987].

[^39]:    ${ }^{10}$ Readers are referred to [Thielscher, 1997] for an excellent discussion on why causality information is necessary for dealing with indirect effects.

[^40]:    ${ }^{11}$ It might be however happen that the light turns on for a very short period of time, depending on the time it takes to activate the relay and to affect the second switch. Nevertheless, the light is definitely off in the final stable state.

[^41]:    ${ }^{12}$ However, Winslett has not formally shown that super-polynomial space explosion is inevitable for computational models of the $\mathrm{WSS}_{\downarrow}$

[^42]:    ${ }^{13}$ Due to the correspondence between possibilistic distributions and OCFs, the semantic model can also be applied to OCFs with a slight modification.

[^43]:    ${ }^{14}$ Note that, we need only to update plausible worlds.
    ${ }^{15}$ Technically, it is required that $0<w_{\mathbf{\Delta}}<\min \left\{\pi\left(W^{\prime}\right) \mid W^{\prime} \models \mathrm{DC}\right\}$, in order to distinguish $W$ from those supported possible worlds and implausible worlds.

[^44]:    ${ }^{16}$ To express that the new information is relatively plausible，I assume $w_{\boldsymbol{v}}$ is real number such that max $\left\{w_{i} \mid w_{i}<\right.$ $1\}<w_{\boldsymbol{v}}<1$ ．Note that values of $w_{\mathbf{v}}$ and $w_{\boldsymbol{\Delta}}$（which is used in updating possibility distributions）are not explicitly specified．Let $\Sigma=\left\{\left\langle\beta_{1}, w_{1}\right\rangle, \cdots,\left\langle\beta_{n}, w_{n}\right\rangle\right\}$ that respects DC and $\pi_{\Sigma}$ the possibility distribu－ tion induced from $\Sigma$ ．Then for any real number $w_{\mathbf{v}}$ such that $\max \left\{w_{i} \mid w_{i}\right\}<w_{\mathbf{v}}<1$ ，it is obvious that

[^45]:    $0<1-w_{\mathbf{\nabla}}<\min \left\{\pi_{\Sigma}\left(W^{\prime}\right) \mid W^{\prime} \models \mathrm{DC}\right\}$. In the sequel, I assume $w_{\mathbf{\Delta}}=1-w_{\mathbf{\nabla}}$ in situations where possibility distribution update and possibilistic base update are related.

[^46]:    ${ }^{1}$ A 3-literal clause is clause consists of precisely 3 literals and a 3CNF is a conjunction of 3-literal clauses. 3SAT is the satisfiability problem for 3CNFs, which has been shown NP-complete.

[^47]:    ${ }^{2}$ Note that, we can always substitute atoms of $\beta$ respectively by elements of $X$ to obtain a new sentence $\beta_{X}$ such that $\beta$ is satisfiable iff $\beta_{X}$ is satisfiable.

