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Dicomplemented Lattices. A Contextual Generalization of Boolean Algebras.

DISSERTATION

Gedruckt mit Unterstützung des Deutschen Akademischen Austauschdienstes

Dicomplemented Lattices.

A Contextual Generalization of Boolean Algebras.

Dissertation

zur Erlangung des akademischen Grades

Doktor rerum naturalium

(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften

Technischen Universität Dresden

von

Dipl.-Math. Léonard Kwuida

geboren am 03. Dezember 1970 in Kamerun

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Eingereicht am: **18 Februar 2004**

Tag der Disputation: **29 Juni 2004**

To my parents

Acknowledgments

First of all I would like to thank my supervisor Prof. Bernhard Ganter. His continuous encouragement and support strengthen me during this investigation. I benefit a lot not only from his intuition and hints during our discussions, but also from his readiness to discuss at any time on this subject.

I am grateful to Rudolf Wille, initiator of this research project, for discussions and suggestions. He raised my attention to the connection between Mathematics and Philosophy. My thanks also go to Christian Herrmann, Peter P. Pálfy, Dragan Mašulović, Andreja Tepavčević, Tibor Katriňák, Janis Cirulis, Heiko Reppe and Christian Pech for discussions, suggestions and/or proof-reading of preliminary drafts.

I am very grateful to the German Academic Exchange Service (DAAD) for supporting my studies in Germany. Many thanks to the PhD-program GK334 "Specification of Discrete Processes and Systems of Processes by Operational Models and Logics", who has supported travels to some conferences, where parts of this work have been discussed. I also thank Gesellschaft von Freunden und Förderern der TU Dresden e.V for their financial contribution in the last phase of this project.

I would like to use this occasion to congratulate all the members of Institut für Algebra, TU Dresden for the family and friendly atmosphere. I also congratulate all my friends and relatives for their moral support.

Dresden, February 2004

Léonard Kwuida

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[1]This section follows from a wish of a referee to have an overview of the problems I have considered.

0

Introduction and Preliminaries

0.1 Introduction

The aim of this investigation is to develop a mathematical theory of **concept algebras**. We mainly consider the **representation problem** for this recently introduced class of structures. Motivated by the search of a "negation" on formal concepts, *"concept algebras"* are of considerable interest not only in Philosophy or Knowledge Representation, but also in other fields as Logic or Linguistics. The problem of negation is surely one of the oldest problems of the scientific and philosophic community, and still attracts the attention of many researchers (see [Hl89], [Wa96]). Various types of Logic (defined according to the behaviour of the corresponding negation) can attest this affirmation. In this thesis we focus on *"Contextual Logic"*, a *Formal Concept Analysis* approach, based on **concepts** as units of thought.

As a part of his project to extend Formal Concept Analysis to a broader field (called *Contextual Logic*), Rudolf Wille suggested and started a systematic study of potential "conceptual negations". One of the starting points is that of Boolean algebras, which are most important and useful in *Propositional Calculus*. Is there a natural generalization of Boolean algebras to concept lattices?

Boolean Concept Logic aims to develop a mathematical theory for Logic, based on concept as unit of thought, as a generalization of that developed by George Boole in [Bo54], based on *signs* and *classes*. The main operations of human mind encoded by Boole are conjunction, disjunction, "the universe", "nothing" and the so-called "negation". His abstraction led to algebraic structures which today are known as Boolean algebras. A model is the powerset algebra.

The set of all formal concepts of a given formal context forms a complete lattice. Therefore, apart from the negation, the operations encoded by Boole are without problem encoded by lattice operations. To encode a negation Wille followed Boole's idea, with the requirement that the operation obtained should be internal. He introduced a *weak negation* by taking the concept generated by the complement of the extent and a *weak opposition* by taking the concept generated by the complement of the intent. The concept lattice together with these operations is called **concept algebra** (see [Wi00]). This plays for Boolean Concept Logic the role played by powerset algebras for *Propositional Calculus*.

Boolean algebras abstract powerset algebras. We are looking for such an abstraction for concept algebras. That is an abstract structure defined by a set of identities or quasi-identities such that the equational or quasi-equational theory it generates is that of concept algebras. Such a structure, if it exists, will be called a **dicomplemented lattice**. Characterizing dicomplemented lattices remains an open problem, but substantial results are obtained, especially in the case of finite lattices.

We divide this contribution into five parts. In Chapter 1, we introduce weakly dicomplemented lattices (potential candidates for a representation of concept algebras), and state their basic properties.

In Chapter 2 the main (and still unsolved) problem, the representation problem, is considered in its general form. **Is there any equational theory for concept algebras?** The main tool here is Birkhoff's theorem for equational classes. We present many steps towards a solution. The class of concept algebras is closed under formation of products and complete substructures. Using congruences, we can also prove that homomorphic images of finite concept algebras are again concept algebras. The natural way towards the desired representation theorem is to use the proof strategy that has been successful for Boolean algebras (*Stone representation*). We therefore study primary filters and ideals, the natural analogue of prime filters and ideals in Boolean algebra. Restricted to finite lattices, this give in Chapter 3 the first characterization of finite concept algebras. Chapter 3 continues with a contextual description of the lattice of all concrete weak complementations on a fixed lattice. Furthermore we study the relationship between weakly dicomplemented lattices and other generalizations of Boolean algebras.

In Chapter 4 we restrict our consideration to distributive lattices. Our main result is: ***finite distributive weakly dicomplemented lattices are (isomorphic to) concept algebras***. Their congruence lattices are described. We end with Chapter 5 where we demonstrate how, although we cannot expect a weak negation or a weak opposition to fulfill all *laws of negation*, the restriction to appropriate subsets can reconcile Mathematics and Philosophy.

The tools are from Lattice Theory and Formal Concept Analysis. We recall in the next section some notions used in this work. For further information, the reader is referred to [BS81] for *Universal Algebra*, to [DP02] for *Lattice Theory* and to [GW99] for *Formal Concept Analysis*. The contents of Section 0.2 can be found in these books.

0.2 Preliminaries

0.2.1 Some basic notions from Lattice Theory

There are two ways to define a lattice. A lattice can be seen as an ordered set (usually **poset**[1] for short) for which each pair of elements has an infimum and a supremum. From the *Universal Algebra* point of view, a lattice is an algebra $\underline{L} := (L, \wedge, \vee)$ of type $(2,2)$ such that the operation $\wedge$ and $\vee$, respectively called *meet* and *join*, are commutative, associative, idempotent and satisfy the **absorption laws**.

$$a \vee (a \wedge b) = a = a \wedge (a \vee b) \qquad \text{(absorption laws)}.$$

The relation between the two approaches is given by the equivalences

$$a = a \wedge b \iff a \leq b \iff a \vee b = b.$$

In this work we use lattices as algebras. The main difference for the two approaches lies in the structure theory. For example, all lattice homomorphisms are posets homomorphisms (of the underlying poset), but the converse does not hold. Fortunately there is no difference between their isomorphisms:

Lemma 0.2.1. *Let $\underline{L}_1$ and $\underline{L}_2$ be lattices. A mapping $h\colon L_1 \to L_2$ is a lattice isomorphism if and only if h is an order isomorphism.*

A poset $(L, \leq)$ is said to be **complete** if each subset has an infimum and a supremum. The following lemma is a well-known characterization of complete posets.

Lemma 0.2.2. *A poset $(L, \leq)$ is complete iff each subset of L has an infimum.*

There is a one-to-one correspondence between complete lattices and closure systems. We adopt the following terminology for a unary operation f on a lattice $\underline{L}$.

Definition 0.2.1. We say that f is **monotone** if $x \leq y \implies fx \leq fy$ and **antitone** if $x \leq y \implies fx \geq fy$. We call f **extensive** if $x \leq fx$

[1] partially ordered set

and **intensive** if $x \geq fx$. The unary operation f is **idempotent**[2] if $f^2x = fx$, and an **involution** if $f^2(x) = x$. If $x \leq f^2x$ we say that f is **square-extensive**. The dual notion is **square-intensive**.

With the above terminology, a **closure operator** on $\underline{L}$ is a monotone, extensive and idempotent unary operation f on L. An **element** $x \in L$ is **closed** if $fx = x$. The dual notions are that of **interior operator** (monotone, intensive and idempotent) and of **interior element**.

Definition 0.2.2. A nonempty subset F of a lattice $\underline{L}$ is called **order filter**[3] if

$$x \in F,\ x \leq y \implies y \in F \quad \forall x, y \in L.$$

A **lattice filter** (or **filter** for short) is an order filter such that

$$x, y \in F \implies x \wedge y \in F \quad \forall x, y \in L.$$

A **proper filter** is a filter different from L. Dually the notion of **ideal** is introduced.

Note that the filters form a closure system. For an arbitrary subset X of L, **Filter**(X) will denote the smallest filter and **Ideal**(X) the smallest ideal containing X. The filter generated by a singleton set is the order filter it generates:

$$\text{Filter}(\{x\}) = \{y \mid y \geq x\}.$$

Birkhoff gave a pleasant description of completely distributive complete lattices. An element $a \in L$ is **$\bigvee$-irreducible** (resp. **$\bigwedge$-irreducible**) if

$$a \neq a_* := \bigvee\{x \in L \mid x < a\} \qquad (\text{resp.} \quad a \neq a^* := \bigwedge\{x \in L \mid x > a\}).$$

$J(L)$ denotes the sets of all $\bigvee$-irreducible elements and $M(L)$ the set of all $\bigwedge$-irreducible elements. A subset X of L is called **supremum-dense** (resp. **infimum-dense**) in L if for all $a \in L$,

$$a = \bigvee\{x \in X \mid x \leq a\} \qquad (\text{resp.} \quad a = \bigwedge\{x \in X \mid x \geq a\}).$$

We use the notations

$$\downarrow x := \{y \in L \mid y \leq x\} \quad \text{and} \quad \uparrow x := \{y \in L \mid y \geq x\}.$$

Theorem 0.2.3 (Theorem of Birkhoff). *If $\underline{L}$ is a completely distributive complete lattice in which the set $J(L)$ of $\bigvee$-irreducible elements is supremum dense, then $x \mapsto \downarrow x \cap J(L)$ describes an isomorphism of L onto the closure system of all order ideals of $(J(L), \leq)$. Conversely for every ordered set $(P, \leq)$ the closure system of all order ideals is a completely distributive lattice L in which $J(L) = \{\downarrow x \mid x \in P\}$ is supremum dense.*

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[2]For a binary operation g we use the term idempotent to indicate that $g(x, x) = x$.
[3]This can be defined on posets as well.

Before we switch to *Formal Concept Analysis* we give the definition of Boolean algebras.

Definition 0.2.3. A **Boolean algebra** is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and for all $x \in L$, $x \wedge x' = 0$ and $x \vee x' = 1$. The unary operation $'$ is called **complementation** and x' the **complement** of x. It abstracts the Boolean negation.

One possibility to generalize Boolean algebras is to consider **p-algebras**. These are algebras $(L, \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and for all $x, y \in L$, $x \wedge y = 0 \iff y \leq x^*$. The element x^* is called the **pseudocomplement** of x. The dual notion is that of **dual p-algebras**.

0.2.2 Some basic notions from Formal Concept Analysis

Formal Concept Analysis arose around 1980 from the formalization of the notion of *concept*. Traditional philosophers considered a **concept** to be determined by its extent and its intent. The extent consists of all objects belonging to the concept while the intent is the set of all attributes shared by all objects of the concept. In general, it may be difficult to list all objects or attributes of a concept. Therefore a specific *context* should be set down in order to enable a formalization.

A **formal context** is a triple (G, M, I) of sets such that $I \subseteq G \times M$. The members of G are called **objects** and those of M **attributes**. If $(g, m) \in I$, then the object g is said to have m as an attribute. For subsets $A \subseteq G$ and $B \subseteq M$, A' and B' are defined by

$$A' := \{m \in M \mid \forall g \in A \quad g \, \mathrm{I} \, m\}$$

$$B' := \{g \in G \mid \forall m \in B \quad g \, \mathrm{I} \, m\}.$$

A **formal concept** of the formal context (G, M, I) is a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A' = B$ and $B' = A$. The set A is called the **extent** and B the **intent** of the concept (A, B).

$\mathfrak{B}(G, M, I)$ denotes the set of all formal concepts of the formal context (G, M, I). For concepts (A, B) and (C, D), (A, B) is called a **subconcept** of (C, D) provided that $A \subseteq C$ (which is equivalent to $D \subseteq B$). In this case, (C, D) is a **superconcept** of (A, B) and we write $(A, B) \leq (C, D)$. The relation **subconcept-superconcept** is called the **hierarchy of concepts**. $(\mathfrak{B}(G, M, I); \leq)$ is a complete lattice, called the **concept lattice** of the context (G, M, I), and is denoted by $\underline{\mathfrak{B}}(G, M, I)$.

Theorem 0.2.4 (The Basic Theorem on Concept Lattices). *The concept lattice* $\underline{\mathfrak{B}}(G, M, I)$ *is a complete lattice in which infimum and* *supremum are given by:*

$$\bigwedge_{t\in T}(A_t,B_t)=\left(\bigcap_{t\in T}A_t,\left(\bigcup_{t\in T}B_t\right)''\right)$$

$$\bigvee_{t\in T}(A_t,B_t)=\left(\left(\bigcup_{t\in T}A_t\right)'',\bigcap_{t\in T}B_t\right).$$

A complete lattice $\underline{L}$ is isomorphic to $\underline{\mathfrak{B}}(G,M,I)$ iff there are mappings $\tilde{\gamma}:G\to L$ and $\tilde{\mu}:M\to L$ such that $\tilde{\gamma}(G)$ is supremum-dense, $\tilde{\mu}(M)$ is infimum-dense and $g\,\mathrm{I}\,m \iff \tilde{\gamma}g\le\tilde{\mu}m$ for all $(g,m)\in G\times M$.

Thus all complete lattices are (copies of) concept lattices. Complete sublattices of concept lattices are described by closed subrelations. A relation $J\subseteq I$ is called a **closed subrelation** of the context (G,M,I) if every concept of the context (G,M,J) is also a concept of (G,M,I). In this case we write $J\le I$.

Proposition 0.2.5. *If J is a closed subrelation of (G,M,I), then $\underline{\mathfrak{B}}(G,M,J)$ is a complete sublattice of $\underline{\mathfrak{B}}(G,M,I)$ with*

$$J=\bigcup\{A\times B\mid (A,B)\in\mathfrak{B}(G,M,J)\}.$$

Conversely, for every complete sublattice S of $\underline{\mathfrak{B}}(G,M,I)$ the relation

$$J:=\bigcup\{A\times B\mid (A,B)\in S\}$$

is closed and $\underline{\mathfrak{B}}(G,M,J)=S$.

The below notations is adopted.

$$\gamma g:=(\{g\}'',\{g\}')\quad\text{and}\quad \mu m:=(\{m\}',\{m\}'')$$

The concept γg is called **object concept** and μm **attribute concept**. The set γG is supremum-dense and μM infimum-dense in $\underline{\mathfrak{B}}(G,M,I)$. If γg is supremum-irreducible we say that the **object** g is **irreducible**. In a clarified context an **attribute** m is said **irreducible** if the attribute concept μm is infimum-irreducible. A **formal context** is called **reduced** if all its objects and attributes are irreducible.

For every finite lattice $\underline{L}$ there is, up to isomorphism, a unique reduced context $\mathbb{K}(L):=(J(L),M(L),\le)$ such that $L\cong\underline{\mathfrak{B}}(\mathbb{K}(L))$. This is called the **standard context** of $\underline{L}$. The *finiteness* condition is usually generalized to *doubly foundedness*[4], a condition that can be defined using the **arrow relations**.

$$g\swarrow m:\iff m\notin g' \text{ and } g' \text{ is maximal with respect to } m\notin g',$$

[4] Almost all results obtained for finite lattices can be generalized to doubly founded lattices.

$$g \nearrow m :\iff g \notin m' \text{ and } m' \text{ is maximal with respect to } g \notin m'.$$

Definition 0.2.4. A **context** (G, M, I) is called **doubly founded**, if for every object $g \in G$ and every attribute $m \in M$ with $g \not\mathrel{I} m$, there is an object $h \in G$ and an attribute $n \in M$ with

$$g \nearrow n \text{ and } m' \subseteq n' \quad \text{as well as} \quad h \swarrow m \text{ and } g' \subseteq h'.$$

A **complete lattice** $\underline{L}$ is **doubly founded**, if for any two elements $x < y$ of L, there are elements $s, t \in L$ with:

$$s \text{ is minimal with respect to } s \leq y,\ s \nleq x, \text{ as well as}$$

$$t \text{ is maximal with respect to } t \geq x,\ t \ngeq y.$$

In a doubly founded lattice $\underline{L}$, the set $J(L)$ of join-irreducible elements is supremum dense and the set $M(L)$ of meet-irreducible elements is infimum-dense. This is the main condition we need in almost all our proofs when we assume the lattice to be doubly founded.

We close these preliminaries with the definition of two special context families we will at times refer to.

Contranominal scale. For every set S the context $(S, S, \neq)$ is reduced. The concepts of this context are precisely the pairs $(A, S \setminus A)$ for $A \subseteq S$. Its concept lattice is isomorphic to the power set lattice of S. $(S, S, \neq)$ is called the contranominal scale.

Contraordinal scale. For an arbitrary ordered set $(P, \leq)$ the context $(P, P, \ngeq)$ is reduced. The concepts of this context are precisely the pairs (X, Y) with $X \cup Y = P$, $X \cap Y = \emptyset$, X is an order ideal in $(P, \leq)$ and Y is an order filter in $(P, \leq)$. The concept lattice of $(P, P, \ngeq)$ is isomorphic to the lattice of order ideals of $(P, \leq)$.

1

Dicomplementation

We introduce (weakly) dicomplemented lattices , and state their basic properties. The main (and still unsolved) problem is the representation problem. **Is every (*weakly*) dicomplemented lattice a (*subalgebra of a*) concept algebra?** As a case study we discuss all possible (weak) dicomplementations for the lattices $\underline{2} \times \underline{n}$.

1.1 Weak Dicomplementation.

1.1.1 Definition, motivation and examples.

Definition 1.1.1. A **weakly dicomplemented lattice** is a bounded lattice L equipped with two unary operations $^{\triangle}$ and $^{\triangledown}$ called **weak complementation** and **dual weak complementation**, and satisfying for all $x, y \in L$ the following equations:

(1) $x^{\triangle\triangle} \leq x$,

(2) $x \leq y \implies x^{\triangle} \geq y^{\triangle}$,

(3) $(x \wedge y) \vee (x \wedge y^{\triangle}) = x$,

(1') $x^{\triangledown\triangledown} \geq x$,

(2') $x \leq y \implies x^{\triangledown} \geq y^{\triangledown}$,

(3') $(x \vee y) \wedge (x \vee y^{\triangledown}) = x$.

We call $x^{\triangle}$ the **weak complement** of x and $x^{\triangledown}$ the **dual weak complement** of x. The pair $(x^{\triangle}, x^{\triangledown})$ is called the **weak dicomplement**of x and the pair $(^{\triangle}, ^{\triangledown})$ a **weak dicomplementation** on L. The structure $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ is called a **weakly complemented lattice** and $(L, \wedge, \vee, ^{\triangledown}, 0, 1)$ a **dual weakly complemented lattice**

The motivation comes from concept algebras.

Definition 1.1.2. Let (G, M, I) be a formal context. For a formal concept (A, B) we define

its **weak negation** by $\quad (A,B)^{\triangle} := \big((G \setminus A)'', (G \setminus A)'\big)$

and its **weak opposition** by $\quad (A,B)^{\triangledown} := \big((M \setminus B)', (M \setminus B)''\big)$.

$\underline{\mathfrak{A}}(\mathbb{K}) := \big(\mathfrak{B}(\mathbb{K}); \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1\big)$ is called the **concept algebra** of the formal context $\mathbb{K}$, where $\wedge$ and $\vee$ denote the meet and the join operations of the concept lattice.

These operations satisfy the equations in Definition 1.1.1 (cf. [Wi00]). Thus concept algebras are examples of weakly dicomplemented lattices. If a weakly dicomplemented lattice is isomorphic to a concept algebra of some context, it is said to be **representable** (by this context).

We want to discover the equational or quasi-equational theory of concept algebras, if there is one. An abstract structure, defined by a set of equations or implications that satisfies all equations or quasi-equations valid in all concept algebras, will be called a **dicomplemented lattice**. Since we are not sure that the equations in Definition 1.1.1 are enough to do the job, we prefer to refer to these structures as "*weakly dicomplemented lattices*". It turns out, that, at least for finite distributive lattices, there is no need to distinguish between both notions [cf. Theorem 4.1.7].

As mentioned above, concept algebras are weakly dicomplemented lattices. They are trivially dicomplemented lattices. Examples are also:

Example 1.1.1. Boolean algebras can be made into weakly dicomplemented lattices by defining $x^{\triangle} := x' =: x^{\triangledown}$ (the complement of x).

Example 1.1.2. Each bounded lattice can be endowed with a **trivial weak dicomplementation** by defining $(1,1)$, $(0,0)$ and $(1,0)$ as the dicomplement of 0, 1 and of each $x \notin \{0,1\}$, respectively. This defines a trivial dicomplementation if L is complete. The corresponding formal context is $(L, L, \leq)$.

Example 1.1.3. A **double p-algebra** is an algebra $(L, \wedge, \vee, ^{+}, ^{*}, 0, 1)$ of type $(2,2,1,1,0,0)$ such that $(L, \wedge, \vee, ^{*}, 0, 1)$ is a p-algebra and $(L, \wedge, \vee, ^{+}, 0, 1)$ a dual p-algebra. It is not difficult to see that each distributive double p-algebra is a weakly dicomplemented lattice. Not all double p-algebras are weakly dicomplemented lattices. The example in Figure 1.1,

$$(N_5; \wedge, \vee, ^{+}, ^{*}, 0, 1) \text{ with } a^{**} = a^{++} = a^{*+} = a^{+*} = a,$$

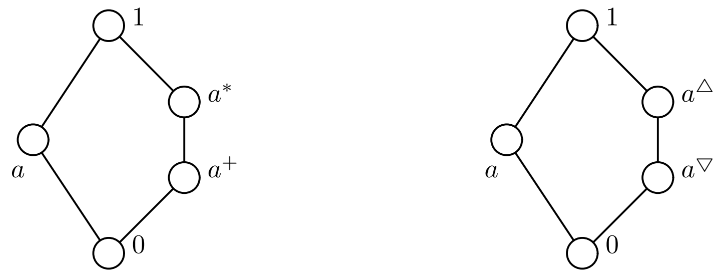


Figure 1.1. Double pseudocomplementation and a dicomplementation on N_5.

is a double p-algebra and cannot be a weakly dicomplemented lattice. However

$$(N_5; \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1) \text{ with } a^{\triangledown\triangledown} = a^{\triangle\triangle} = a,\ a^{\triangledown\triangle} = 1 \text{ and } a^{\triangle\triangledown} = 0$$

on the right of Figure 1.1 is a dicomplemented lattice. The corresponding formal context is its standard context[1].

Remark 1.1.4. A doubly founded lattice can be endowed with different weak dicomplementations while it carries at most one structure of double p-algebra.

1.1.2 Basic properties

Axioms (2) and (2′) can be reformulated as equations. Thus weakly dicomplemented lattices form an equational class. Some basic properties are gathered in the following propositions:

Proposition 1.1.1 ([Wi00]). *Each weakly dicomplemented lattice satisfies the following equations:*

(i) $x^{\triangledown\triangledown\triangledown} = x^{\triangledown} \leq x^{\triangle} = x^{\triangle\triangle\triangle}$,

(ii) $x^{\triangle\triangledown} \leq x^{\triangle\triangle} \leq x \leq x^{\triangledown\triangledown} \leq x^{\triangledown\triangle}$.

Remark 1.1.5. Taking $x := 1$ in axiom (3) of Definition 1.1.1 leads to $y \vee y^{\triangle} = 1$. If we fix y to be 1 we obtain $0^{\triangle} = 1$. Dually $y \wedge y^{\triangledown} = 0$ and $1^{\triangledown} = 0$ always hold. We can also replace x in the same equation by $y^{\triangledown}$. In this case we obtain $y^{\triangledown} \wedge y^{\triangle} = y^{\triangledown}$. Thus $y^{\triangledown} \leq y^{\triangle}$. Note that $0^{\triangledown} = 1^{\triangledown\triangledown} \geq 1$. Thus $0^{\triangledown} = 1$ and $1^{\triangle} = 0$. Moreover, x is comparable with $x^{\triangle}$ or $x^{\triangledown}$ if and only if $\{x^{\triangle}, x^{\triangledown}\}$ and $\{0, 1\}$ have a nonempty intersection.

Proposition 1.1.2. *For any weak dicomplementation* $(^{\triangle}, ^{\triangledown})$ *we have*

[1] The attributes are the meet-irreducible elements, the objects the join irreducible elements and the incidence relation induced by the order relation of the lattice.

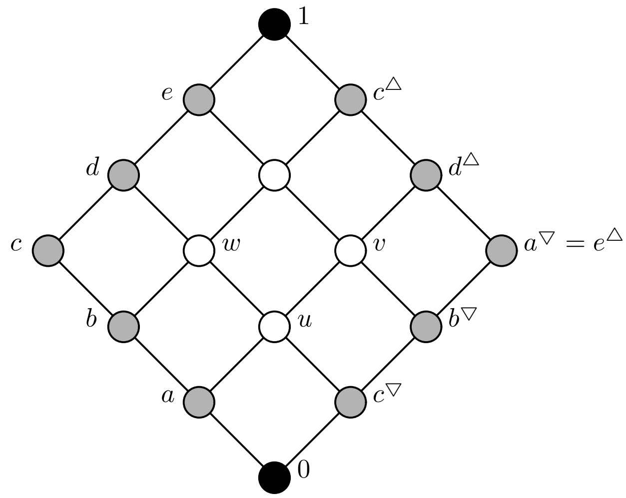


Figure 1.2. A dicomplementation on the product of two 4 element chains. All elements above u are sent to 0 by $^{\triangledown}$. The elements c, b and a are each image (of their image). The operation $^{\triangle}$ is the dual of $^{\triangledown}$.

(4) $(x \wedge x^{\triangle})^{\triangle} = 1$,

(5) $x^{\triangle} \leq y \iff y^{\triangle} \leq x$,

(6) $(x \wedge y)^{\triangle} = x^{\triangle} \vee y^{\triangle}$,

(7) $(x \wedge y)^{\triangle\triangle} \leq x^{\triangle\triangle} \wedge y^{\triangle\triangle}$,

(4') $(x \vee x^{\triangledown})^{\triangledown} = 0$,

(5') $x^{\triangledown} \geq y \iff y^{\triangledown} \geq x$,

(6') $(x \vee y)^{\triangledown} = x^{\triangledown} \wedge y^{\triangledown}$,

(7') $(x \vee y)^{\triangledown\triangledown} \geq x^{\triangledown\triangledown} \vee y^{\triangledown\triangledown}$.

Proof. (4) follows from (6). The equivalence (5) is proved in [Wi00]. Let us prove (6). Obviously $(x \wedge y)^{\triangle} \geq x^{\triangle} \vee y^{\triangle}$. If $a \geq x^{\triangle}$ and $a \geq y^{\triangle}$ then $a^{\triangle} \leq x^{\triangle\triangle} \wedge y^{\triangle\triangle} \leq x \wedge y$. Thus $(x \wedge y)^{\triangle} \leq a^{\triangle\triangle} \leq a$ and (6) is proved. Equation (7) follows from equation (2). The remaining ones are proved dually. □

Remark 1.1.6. The inequalities (7) and (7)′ can be strict. In Figure 1.2,

$$(d \wedge d^{\triangle})^{\triangle\triangle} = 0 < u = d \wedge d^{\triangle} = d^{\triangle\triangle} \wedge d^{\triangle\triangle\triangle}.$$

The weakly dicomplemented lattice on Figure 1.2 is the concept algebra of the formal context $(J(L) \cup \{w, v\}, M(L) \cup \{w, v\}, \leq)$ where $J(L)$ and $M(L)$ are respectively the $\wedge$-irreducible and $\vee$-irreducible elements of the underlying lattice $\underline{L}$.

Proposition 1.1.3. *For any weakly dicomplemented lattice L and $x, y \in L$ we have*

(8) $x \wedge y^{\triangle} \leq y \iff x \leq y,$

(9) $x^{\triangle} \leq y \implies y \vee x = 1,$

(10) $y \wedge x = 0 \implies y^{\triangle} \geq x,$

(11) $x \wedge x^{\triangle} = 0 \implies x = x^{\triangle\triangle}$

(12) $x \vee (x \vee y)^{\triangle} \leq x \vee y^{\triangle}$

(8') $x \vee y^{\triangledown} \geq y \iff x \geq y.$

(9') $x^{\triangledown} \geq y \implies y \wedge x = 0.$

(10') $y \vee x = 1 \implies y^{\triangledown} \leq x.$

(11') $x \vee x^{\triangledown} = 1 \implies x = x^{\triangledown\triangledown}$

(12') $x \wedge (x \wedge y)^{\triangledown} \geq x \wedge y^{\triangledown}$

Remark 1.1.7. For a double p-algebra (9) and (9′) with $^{\triangle}$ and $^{\triangledown}$ replaced by $^{+}$ and * become the equivalences $(\tilde{9})$ and $(\tilde{9}')$[2] defining the dual pseudocomplementation and the pseudocomplementation, while $(\tilde{12})$ and $(\tilde{12}')$ are equalities if the lattice is distributive. Conditions (8), (10) and (11) do not hold in general in double p-algebras. The variety of distributive double p-algebras is defined by the set of identities that defines bounded distributive lattices and the equations $(\tilde{12})$ and $(\tilde{12}')$.

1.1.3 Dicomplementations on the lattice $\underline{2} \times \underline{n}$

As we pointed out in Remark 1.1.4 there may be more than one weak dicomplementation on the same lattice. Lattices with unique weak dicomplementation are described in Subsection 4.3.1. $\mathrm{Wd}(L)$ denotes the set of weak dicomplementations on a lattice $\underline{L}$. As a case study, we shall now determine all weak dicomplementations on the lattice $\underline{L} := \underline{2} \times \underline{n}$, the lattice product of a two element chain and an n element chain. $\underline{L}$ is distributive. The axioms for a weak complementation f can be rewritten as follows (see Definition 1.1.1):

$$f^2x \leq x, \quad x \leq y \implies fx \geq fy \quad \text{and} \quad x \vee fx = 1.$$

Using the notations on Figure 1.3 we obtain that $x \leq t$ implies $fx = 1$ and $r \leq x \leq s$ implies $fx \geq u$, as well as $fu \geq r$. We set

$$A := \{x \in [r, s] \mid fx = 1\}.$$

If A is empty then $fx = u$ for all $x \in [r, s]$. Since $f^2r \leq r$ holds we obtain the equality $fu = r$. We assume that A is not empty and denote by x_1 the greatest element of A. For all x in $[r, s]$, we have

$$x \leq x_1 \implies fx = 1 \text{ and } x > x_1 \implies fx = u.$$

We denote by x_2 the successor of x_1. Since $f^2x_2 \leq x_2$ we obtain $fu \leq x_2$; if $fu < x_2$ we would have $f^2u = 1 > u$, a contradiction. Then $fu = x_2$. Thus

[2] The equation $(\tilde{n})$ in a double p-algebra is obtained from the equation (n) of a weakly dicomplemented lattice by replacing $^{\triangle}$ with $^{+}$ and $^{\triangledown}$ with *.

$(\tilde{9})$ $x^{+} \leq y \iff y \vee x = 1$

$(\tilde{9}')$ $x^{*} \geq y \iff y \wedge x = 0$

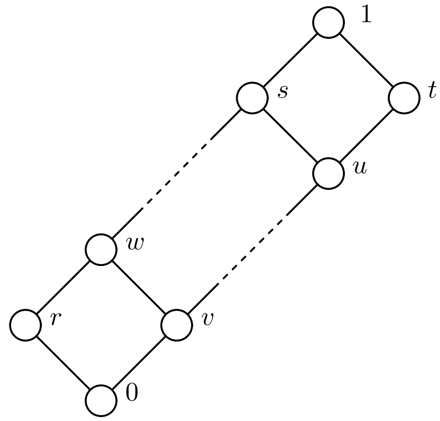


Figure 1.3. $L := \underline{2} \times \underline{n}$

f is completely determined by the image of u in $[r, 1]$. There are exactly n such operations (which are all weak complementations). Similarly there are also n dual weak complementations g on L, each determined by the image of r in $[0, u]$. Thus $\mathrm{Wd}(L)$ is the product of two n element chains (and contains exactly n^2 weak dicomplementations). They all are dicomplementations. In fact for a weak complementation f and a dual weak complementation g on L the concept algebra of the context $(J(L) \cup \{fu\}, M(L) \cup \{gr\}, \leq)$ is (isomorphic to) the weakly dicomplemented lattice $(L, \wedge, \vee, f, g, 0, 1)$ since

$$\begin{aligned} u^{\triangle} &\equiv \big(((J(L) \cup \{fu\}) \setminus \downarrow u)'' , ((J(L) \cup \{fu\}) \setminus \downarrow u)'\big) \\ &= (\downarrow fu \cap (J(L) \cup \{fu\}), \uparrow fu \cap (M(L) \cup \{gr\})) \\ &\equiv fu. \end{aligned}$$

Dually $r^{\triangledown} \equiv gr$.

Theorem 1.1.4. *There are exactly n^2 weak dicomplementations on the lattice $\underline{2} \times \underline{n}$. They all are dicomplementations.*

1.2 Weak Dicomplementations on a Fixed Lattice

1.2.1 Lattice of weak dicomplementations

Let $\underline{L}$ be a doubly founded complete lattice. Up to isomorphism there is a unique reduced context $\mathbb{K}(L)$ (its standard context) the concept lattice of which is isomorphic to $\underline{L}$. The concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ yields a dicomplementation on $\underline{L}$.

Definition 1.2.1. The **standard dicomplementation** of a doubly founded complete lattice $\underline{L}$ is the dicomplementation induced by the concept algebra of its standard context.

Theorem 1.2.1. *For doubly founded complete distributive lattices the double pseudocomplementation*[3] *is the standard dicomplementation.*

Proof. Let $\underline{L}$ be a doubly founded complete distributive lattice. There is a poset $(P, \leq)$ such that $(P, P, \not\geq)$ is the standard context of L. For a set of objects A (resp. attributes B), $A'' = \downarrow A$ (resp. $B'' = \uparrow B$) is the order ideal (resp. order filter) generated by A (resp. B). Moreover we have

$$A' = \overline{\downarrow A} \text{ and } B' = \overline{\uparrow B}.$$

Let (A, B) be a concept; its weak opposition is

$$(A,B)^{\triangledown} = (\bar{B}', \bar{B}'') = (\overline{\uparrow \bar{B}}, \uparrow \bar{B}) = (\overline{\uparrow A}, \uparrow A).$$

The pseudocomplement of (A, B), denoted by $(A, B)^*$, satisfies

$$\begin{aligned}
(A,B)^* &= \bigvee\{(X,Y) \mid (A,B) \wedge (X,Y) = 0\} \\
&= \bigvee\{(X,Y) \mid A \cap X = \emptyset\} \\
&= \left(\left(\bigcup\{X \mid A \cap X = \emptyset\}\right)'', \left(\bigcup\{X \mid A \cap X = \emptyset\}\right)'\right) \\
&= \left(\left(\bigcup\{\downarrow X \mid A \cap \downarrow X = \emptyset\}\right)'', \left(\bigcup\{\downarrow X \mid A \cap \downarrow X = \emptyset\}\right)'\right) \\
&= \left(\bigcup\{\downarrow X \mid A \cap \downarrow X = \emptyset\}, \overline{\bigcup\{\downarrow X \mid A \cap \downarrow X = \emptyset\})}\right) \\
&= \left(\{x \mid A \cap \downarrow x = \emptyset\}, \overline{\{x \mid A \cap \downarrow x = \emptyset\}}\right)
\end{aligned}$$

The set $\{x \mid A \cap \downarrow x = \emptyset\}$ is equal to $\overline{\uparrow A}$. This proves that $(A,B)^* \leq (A,B)^{\triangledown}$. The reverse inequality is immediate since $^{\triangledown}$ is a semicomplementation[4].

Dually $(A,B)^{\vartriangle} = (A,B)^+$, the dual pseudocomplement of (A, B). □

Definition 1.2.2. Let $(^{\vartriangle_1}, ^{\triangledown_1})$ and $(^{\vartriangle_2}, ^{\triangledown_2})$ be two weak dicomplementations on $\underline{L}$. We say that $(^{\vartriangle_1}, ^{\triangledown_1})$ is **finer than** $(^{\vartriangle_2}, ^{\triangledown_2})$ (and write $(^{\vartriangle_1}, ^{\triangledown_1}) \preceq (^{\vartriangle_2}, ^{\triangledown_2})$) if $x^{\vartriangle_1} \leq x^{\vartriangle_2}$ and $x^{\triangledown_1} \geq x^{\triangledown_2}$ for all x in L.

We sometimes say that the weakly dicomplemented lattice

$$(L, \wedge, \vee, ^{\vartriangle_1}, ^{\triangledown_1}, 0, 1) \quad \text{is finer than} \quad (L, \wedge, \vee, ^{\vartriangle_2}, ^{\triangledown_2}, 0, 1).$$

The "finer than" relation is an order relation on the class $Wd(L)$ of all weak dicomplementations on a bounded lattice $\underline{L}$. The poset $(Wd(L), \preceq)$ admits a top element, namely the trivial dicomplementation. In the case of doubly founded complete distributive lattices the "finer than" relation also admits a bottom element, namely its double p-algebra structure [cf. Theorem 1.2.1]. We prove in Lemma 1.2.3 that the standard dicomplementation

[3] See Section 3.4

[4] See a definition in Section 3.4

is the smallest weak dicomplementation. However it is not always a double pseudocomplementation.

Lemma 1.2.2. *For each* $\vee$*-irreducible element* $g \in L$*, each* $\wedge$*-irreducible element* $b \in L$ *and for all* x *in* L *we have*

(i) $g \not\leq x \implies g \leq x^{\triangle}$,

(ii) $b \not\geq x \implies b \geq x^{\triangledown}$

for any weak dicomplementation $(^{\triangle}, ^{\triangledown})$ *on* L.

Proof. Let g be a $\vee$-irreducible element and $x \in L$. Let $(^{\triangle}, ^{\triangledown})$ a weak dicomplementation on L.

(1) Since g is $\vee$-irreducible, the equality $g = (g \wedge x) \vee (g \wedge x^{\triangle})$ implies $g = g \wedge x$ or $g = g \wedge x^{\triangle}$, which means that $g \leq x$ or $g \leq x^{\triangle}$. Then $g \not\leq x \implies g \leq x^{\triangle}$.

(2) is proved similarly.

□

Lemma 1.2.3. *For every doubly founded complete lattice* $\underline{L}$*, the standard dicomplementation is the finest dicomplementation on* $\underline{L}$*.*

Proof. The context $\mathbb{K}(L) := (J(L), M(L), \leq)$ is (a copy of) the standard context of L. Let $(^{\triangle}, ^{\triangledown})$ be a weak dicomplementation and $x \in L$. We denote $\left(x^{\triangle_{\mathbb{K}(L)}}, x^{\triangledown_{\mathbb{K}(L)}}\right)$ the weak dicomplement of x in $\underline{\mathfrak{A}}(\mathbb{K}(L))$. Then

$$x^{\triangle_{\mathbb{K}(L)}} = \bigvee \{g \in J(L) \mid g \not\leq x\} \leq \bigvee \{g \in J(L) \mid g \leq x^{\triangle}\} \leq x^{\triangle}.$$

Similarly, $x^{\triangledown_{\mathbb{K}(L)}} \geq x^{\triangledown}$. Since x is arbitrarily chosen in L, we get the inequalities

$$^{\triangle_{\mathbb{K}(L)}} \leq ^{\triangle} \quad \text{and} \quad ^{\triangledown_{\mathbb{K}(L)}} \geq ^{\triangledown} .$$

Thus the standard weak dicomplementation of $\underline{L}$ is the finest weak dicomplementation on $\underline{L}$. □

Notation. Unless otherwise stated, $\mathbb{K}(L)$ denotes the standard context of of a doubly founded complete lattice $\underline{L}$.

By Lemma 1.2.3, for any doubly founded complete lattice $\underline{L}$ the poset $(\mathrm{Wd}(L), \preceq)$ of weak dicomplementations on $\underline{L}$ is bounded. Moreover the top and bottom elements are dicomplementations. The next theorem gives a better insight.

Theorem 1.2.4. *The class of weak dicomplementations on a fixed doubly founded complete lattice ordered by the "finer than" relation builds a complete lattice.*

Proof. Let $\underline{L}$ be a doubly founded complete lattice and $\{(^{\triangle_i}, ^{\triangledown_i}) \mid i \in I\}$ be a nonempty family of weak dicomplementations on L. We define two new operations $^{\triangle_I}$ and $^{\triangledown_I}$ by

$$x^{\triangle_I} := \bigvee \{x^{\triangle_i} \mid i \in I\} \quad \text{and} \quad x^{\triangledown_I} := \bigwedge \{x^{\triangledown_i} \mid i \in I\}.$$

For any x in L we have

$$x^{\triangle_I\triangle_I} = \left(\bigvee \{x^{\triangle_i} \mid i \in I\}\right)^{\triangle_I} = \bigvee \left\{\left(\bigvee \{x^{\triangle_i} \mid i \in I\}\right)^{\triangle_j} \mid j \in I\right\}.$$

Therefore

$$x^{\triangle_I\triangle_I} \leq \bigvee \left\{\left(x^{\triangle_j}\right)^{\triangle_j} \mid j \in I\right\} = \bigvee \{x^{\triangle_j\triangle_j} \mid j \in I\} \leq x.$$

For $x \leq y$ in L, we have $x^{\triangle_i} \geq y^{\triangle_i}$ for all $i \in I$, and then $x^{\triangle_I} \geq y^{\triangle_I}$. Now x and y are arbitrarily chosen in L; trivially $\left(x \wedge y^{\triangle_I}\right) \vee (x \wedge y) \leq x$ always holds. On the other hand

$$\left(x \wedge y^{\triangle_I}\right) \vee (x \wedge y) = \left(x \wedge \left(\bigvee \{y^{\triangle_i} \mid i \in I\}\right)\right) \vee (x \wedge y)$$

and is greater than $\left(x \wedge y^{\triangle_i}\right) \vee (x \wedge y)$, which equals x, for all $i \in I$. Thus $\left(x \wedge y^{\triangle_I}\right) \vee (x \wedge y) = x$.

Similarly we obtain the following equations for the operation $^{\triangledown_I}$

(i') $x^{\triangledown_I\triangledown_I} \geq x$,

(ii') $x \leq y \implies x^{\triangledown_I} \geq y^{\triangledown_I}$ and

(iii') $\left(x \vee y^{\triangledown_I}\right) \wedge (x \vee y) = x$.

Therefore $(^{\triangle_I}, ^{\triangledown_I})$ is a weak dicomplementation on L. It is the supremum of the family $\{(^{\triangle_i}, ^{\triangledown_i}) \mid i \in I\}$. By a well known characterization of complete posets[5], we conclude that the poset of weak dicomplementations on L is a complete lattice. □

1.2.2 Dependence between weak operations

Although the inequality $x^{\triangledown} \leq x^{\triangle}$ always holds, the weak operations seem to be independent from each other. This is the case for all representable weak dicomplementations. As already mentioned in the motivation, a weakly dicomplemented lattice is said to be representable if it is (isomorphic to) a concept algebra of some context. In this subsection we determine the relation between representable weak dicomplementations on a fixed lattice L.

As clarifying a context does not alter the concept algebra structure, a representable weak dicomplementation on $\underline{L}$ can always be **represented**

[5] Lemma 0.2.2

by a pair of subsets of L; that is a pair (G, M) with $G, M \subseteq L$ such that the concept algebra of the context $(G, M, \leq)$ is isomorphic to the given representable weak dicomplementation. To avoid confusion we sometimes index representable weak operations by their context name[6]. This is usually the case if we deal with more than one context.

Proposition 1.2.5. *Let* $\underline{L}$ *be a complete lattice and* $\mathbb{K} := (G, M, \leq)$ *be a subcontext of* $\mathbb{L} := (L, L, \leq)$ *such that the concept lattice of* $\mathbb{K}$ *is isomorphic to* $\underline{L}$. *For* g *and* m *in* L *we set*

$$\mathbb{K}_g := (G \cup \{g\}, M, \leq) \text{ and } \mathbb{K}^m := (G, M \cup \{m\}, \leq).$$

(i) If a subcontext $\mathbb{H}$ *of* $\mathbb{L}$ *is a supercontext of* $\mathbb{K}$*, then the concept algebra of* $\mathbb{K}$ *is finer*[7] *than that of* $\mathbb{H}$.

(ii) ${}^{\triangle_{\mathbb{K}^m}} = {}^{\triangle_{\mathbb{K}}}$ *and* ${}^{\triangledown_{\mathbb{K}^m}} \leq {}^{\triangledown_{\mathbb{K}}}$

(ii') ${}^{\triangledown_{\mathbb{K}_g}} = {}^{\triangledown_{\mathbb{K}}}$ *and* ${}^{\triangle_{\mathbb{K}_g}} \geq {}^{\triangle_{\mathbb{K}}}$.

Proof. (i) is an immediate consequence of (ii) and (ii'), and (ii') is the dual of (ii). We will prove (ii) and (ii'). For X and Y subsets of L we denote by $\mathrm{UB}(X)(Y)$ the upper bounds of Y in X and $\mathrm{LB}(X)(Y)$ the lower bounds of Y in X. Let x in L; there are elements a, b and c of L such that $x^{\triangle_{\mathbb{K}}} = a$, $x^{\triangle_{\mathbb{K}_g}} = b$ and $x^{\triangle_{\mathbb{K}^m}} = c$. If $g \leq x$ then

$$\uparrow b \cap M = \mathrm{UB}(M)((G \cup \{g\}) \setminus \downarrow x) = \mathrm{UB}(M)(G \setminus \downarrow x) = \uparrow a \cap M.$$

Else

$$\uparrow b \cap M = \mathrm{UB}(M)((G \cup \{g\}) \setminus \downarrow x) = \mathrm{UB}(M)((G \setminus \downarrow x) \cup \{g\}) = \mathrm{UB}(M)(a \vee g)$$

and is a subset of $\uparrow a \cap M$. Thus $a \leq b$ and $x^{\triangle_{\mathbb{K}}} \leq x^{\triangle_{\mathbb{K}_g}}$. Dually $x^{\triangledown_{\mathbb{K}}} \geq x^{\triangledown_{\mathbb{K}^m}}$ and the second parts of (ii) and (ii') are proved. Let us prove the first parts. If m is not an upper bound of $G \setminus \downarrow x$ then

$$\uparrow c \cap (M \cup \{m\}) = \mathrm{UB}(M \cup \{m\})(G \setminus \downarrow x) = \uparrow a \cap M$$

and

$$\mathrm{LB}(G)(\uparrow c \cap (M \cup \{m\})) = \mathrm{LB}(G)(\uparrow a \cap M) = \downarrow a \cap G.$$

Otherwise

$$\uparrow c \cap (M \cup \{m\}) = \mathrm{UB}(M \cup \{m\})(G \setminus \downarrow x) = \mathrm{UB}(M)(G \setminus \downarrow x) \cup \{m\}.$$

This is exactly $(\uparrow a \cap M) \cup \{m\}$, and then

$$\mathrm{LB}(G)(\uparrow c \cap (M \cup \{m\})) = \mathrm{LB}(G)((\uparrow a \cap M) \cup \{m\}) = \downarrow a \cap \downarrow m \cap G = \downarrow a \cap G.$$

Thus $x^{\triangle_{\mathbb{K}^m}} = x^{\triangle_{\mathbb{K}}}$. Dually $x^{\triangledown_{\mathbb{K}_g}} = x^{\triangledown_{\mathbb{K}}}$ □

[6] as we did in the proof of Lemma 1.2.3

[7] Recall that we use “finer” in the sense of “finer or equal”.

Proposition 1.2.5 also tells us that the representation problem can be considered for the two weak operations independently. Thus

Definition 1.2.3. A weak complementation $^\triangle$ on L is **representable by a subset** G of L if G is supremum dense and for all x in L,

$$x^\triangle = \bigvee\{g \in G \mid g \not\leq x\}.$$

Dually, a dual weak complementation $^\triangledown$ on L is representable by $M \subseteq L$ if M is infimum dense and for all x in L,

$$x^\triangledown = \bigwedge\{m \in M \mid m \not\geq x\}.$$

An important problem in this topic is to find a characterization of representable weak dicomplementations. Proposition 1.2.6 gives the best candidate set for a representation.

Definition 1.2.4. For a weak complementation $^\triangle$, an element $u \in L$ is said to be $^\triangle$**-compatible** if $u \leq x$ or $u \leq x^\triangle$ for all $x \in L$. Dually an element $u \in L$ is said to be $^\triangledown$**-compatible** if $u \geq x$ or $u \geq x^\triangledown$ for all $x \in L$.

By Lemma 1.2.2 all $\vee$-irreducible elements are $^\triangle$-compatible.

Proposition 1.2.6. *If G represents $^\triangle$ on $\underline{L}$ then $G \cup H$ also represents $^\triangle$ if and only if all elements of H are $^\triangle$-compatible.*

Proof. $x^\triangle = \bigvee\{g \in G \mid g \not\leq x\} \leq \bigvee\{u \in G \cup H \mid u \not\leq x\}$. The inequality is proper if and only if there is some $u \in H$ with $u \not\leq x$ and $u \not\leq x^\triangle$. Thus u must be $^\triangle$-incompatible. Conversely if u is $^\triangle$-incompatible, then $u \not\leq x$ and $u \not\leq x^\triangle$ for some x, and for this x the inequality is proper. □

For a general approach, $^\triangle$-compatible elements will be replaced by primary filters and $^\triangledown$-compatible element by *primary ideals* (see Definition 2.2.1 and Section 3.1).

1.2.3 Determination of a weak complementation

A weak complementation on $\underline{L}$ can be determined by its values on some subsets of L; this is made clear in the next proposition.

Proposition 1.2.7. *Let M be an $\wedge$-dense subset of L. Let $u \in L$.*

$$\bigvee\left\{m^\triangle \mid m \geq u, m \in M\right\} = u^\triangle = \bigwedge\left\{n \mid n \geq m^\triangle \text{ for all } m \in M, m \geq u\right\}.$$

Proof. $m \geq u$ implies $m^\triangle \leq u^\triangle$, thus $\bigvee\left\{m^\triangle \mid m \geq u, m \in M\right\} \leq u^\triangle$. Suppose $n \geq m^\triangle$ for all $m \in M$, $m \geq u$. Then $m \geq n^\triangle$ for all $m \geq u$ and thus $u = \bigwedge\{m \mid m \in M, m \geq u\}$ is greater than $n^\triangle$, and $n \geq u^\triangle$. This proves that $u^\triangle \leq \bigwedge\left\{n \mid n \geq m^\triangle \text{ for all } m \in M, m \geq u\right\}$. However

$n \geq m^{\triangle}$ for all $m \geq u$ is equivalent to $n \geq \bigvee \{m^{\triangle} \mid m \geq u, m \in M\}$, thus

$$\bigwedge \{n \mid n \geq m^{\triangle} \text{ for all } m \in M, m \geq u\}$$

is equal to

$$\bigwedge \left\{n \mid n \geq \bigvee \{m^{\triangle} \mid m \in M, m \geq u\}\right\}$$

which is exactly

$$\bigvee \{m^{\triangle} \mid m \in M, m \geq u\}$$

since M is $\wedge$-dense in L, and the third expression equals the first. □

Corollary 1.2.8.

(1) Weak complementations are determined by their values on any $\wedge$-dense subset.

(2) Weak complementations are determined by their Υ**-relation** *(on any $\wedge$-dense subset) defined by $m \Upsilon n :\iff n \geq m^{\triangle}$.*

For a context (G, M, I) a **relation** $\perp$ is defined on M by

$$m \perp n :\iff m' \cup n' = G.$$

Lemma 1.2.9. *The relation Υ is an order filter of the relation $\perp$.*

Proof. Let $^{\triangle}$ be a weak complementation on $\underline{L}$ and G the set of $^{\triangle}$-compatible elements of L. G is a $\vee$-dense subset of L. Let M be an $\wedge$-dense subset of L; then $(G, M, \leq)$ is a context the concept lattice of which is isomorphic to $\underline{L}$. For all $x \in L$,

$$x^{\triangle_{(G,M,\leq)}} = \bigvee\{g \in G \mid g \not\leq x\} \leq x^{\triangle}.$$

Let $m, n \in L$. From $m \Upsilon n$ we get

$$m \geq n^{\triangle} \geq n^{\triangle_{(G,M,\leq)}} = \bigvee\{g \in G \mid g \not\leq n\}.$$

For an element g in G, if $g \notin n' = \{h \in G \mid h \leq n\}$ then $g \not\leq n$. Thus $g \leq n^{\triangle} \leq m$ and $g \in m'$. Thus $m \Upsilon n$ implies $m' \cup n' = G$ and $m \perp n$. From $m \Upsilon n$ and $(x, y) \geq (m, n)$ we have $m^{\triangle} \geq x^{\triangle}$ and $y \geq n \geq m^{\triangle} \geq x^{\triangle}$. This means that $y \geq x^{\triangle}$ and $x \Upsilon y$. □

Notation. The relations Υ and $\perp$ are symmetric. In the rest of this contribution we adopt the following notations:

$$\Gamma := \{\{m, n\} \subseteq M \mid m \Upsilon n\} \text{ and } T := \{\{m, n\} \subseteq M \mid m \perp n\}.$$

These two sets play a crucial role in the representation of weak dicomplementation. [See Chapter 3 and Chapter 4].

1.3 Skeletons of Weak Dicomplementations

A model for weakly dicomplemented lattice is the class of distributive double p-algebras. Glivenko[8] proved that their skeletons (defined below) are Boolean algebras. In this section we examine skeletons of weakly dicomplemented lattices.

Proposition 1.3.1. *[Wi00] Let* $\underline{L} := (L, \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1)$ *be a weakly dicomplemented lattice. The mapping* $\phi\colon x \longmapsto x^{\triangle\triangle}$ *is an interior operator while* $\psi\colon x \longmapsto x^{\triangledown\triangledown}$ *is a closure operator on* L.

Definition 1.3.1. The set $S(L)$ of closed elements is called the **skeleton** and the set $\bar{S}(L)$ of interior elements is called the **dual skeleton** of $\underline{L}$.

Thus $S(L) = \{x \in L \mid x^{\triangledown\triangledown} = x\}$ and $\bar{S}(L) = \{x \in L \mid x^{\triangle\triangle} = x\}$. We define the operations $\sqcap$ and $\sqcup$ on L by:

$$x \sqcap y := (x^{\triangledown} \vee y^{\triangledown})^{\triangledown} \quad \text{and} \quad x \sqcup y := (x^{\triangledown} \wedge y^{\triangledown})^{\triangledown}.$$

These operations are from $L \times L$ onto $S(L)$. Dually the operations

$$x \bar{\sqcap} y := (x^{\triangle} \vee y^{\triangle})^{\triangle} \quad \text{and} \quad x \underline{\sqcup} y := (x^{\triangle} \wedge y^{\triangle})^{\triangle}$$

are from $L \times L$ onto $\bar{S}(L)$.

Proposition 1.3.2. *For a weakly dicomplemented lattice* $\underline{L}$ *the following hold.*

(i) $x \bar{\sqcap} y = (x \wedge y)^{\triangle\triangle}$ *and* $x \sqcup y = (x \vee y)^{\triangledown\triangledown}$*;*

(ii) $x \underline{\sqcup} y = x^{\triangle\triangle} \vee y^{\triangle\triangle} \leq (x \vee y)^{\triangle\triangle} \leq x \vee y$*;*

(ii)' $x \sqcap y = x^{\triangledown\triangledown} \wedge y^{\triangledown\triangledown} \geq (x \wedge y)^{\triangledown\triangledown} \geq x \wedge y$.

Proof. $(ii)'$ is the dual of (ii).

(i) $x \bar{\sqcap} y := (x^{\triangle} \vee y^{\triangle})^{\triangle} = (x \wedge y)^{\triangle\triangle}$.

(ii) $x \underline{\sqcup} y := (x^{\triangle} \wedge y^{\triangle})^{\triangle} = x^{\triangle\triangle} \vee y^{\triangle\triangle}$.

□

Remark 1.3.1. On Figure 1.2 taking $x := c$ and $y := c^{\triangledown}$ gives

$$x \underline{\sqcup} y = x^{\triangle\triangle} \vee y^{\triangle\triangle} = c < d = (x \vee y)^{\triangle\triangle}$$

and shows that the inequality can be strict. The interior and closure operators do neither preserve the supremum nor the infimum.

[8] See for example [BD74]

Proposition 1.3.3.

(v) $(x \wedge y)^{\vartriangle} = x^{\vartriangle} \underline{\sqcup} y^{\vartriangle}$ $\qquad$ $(v)'$ $(x \vee y)^{\triangledown} = x^{\triangledown} \sqcap y^{\triangledown}$

(vi) $(x \vee y)^{\vartriangle} \leq x^{\vartriangle} \bar{\sqcap} y^{\vartriangle}$; $\qquad$ $(vi)'$ $(x \wedge y)^{\triangledown} \geq x^{\triangledown} \sqcup y^{\triangledown}$.

Proof. $x^{\vartriangle} \underline{\sqcup} y^{\vartriangle} = (x^{\vartriangle\vartriangle} \wedge y^{\vartriangle\vartriangle})^{\vartriangle} = x^{\vartriangle} \vee y^{\vartriangle} = (x \wedge y)^{\vartriangle}$ and (v) is proved. □

Remark 1.3.2. On Figure 1.2, taking $x := c$ and $y := b^{\triangledown}$ gives

$$(x \vee y)^{\vartriangle} = e^{\vartriangle} < c^{\vartriangle} = x^{\vartriangle} \bar{\sqcap} y^{\vartriangle}.$$

and the inequality (vi) can be strict.

In order to get a better description of the structure of the skeleton we recall the following definition:

Definition 1.3.2. An **orthocomplementation** on a bounded lattice L is an involutorial antitone complementation. An **orthocomplemented lattice** or **ortholattice**, for short, is a bounded lattice equipped with an orthocomplementation. An **orthomodular lattice** is an ortholattice satisfying for all elements a and b

$$a \leq b \implies a \vee (a^{\perp} \wedge b) = b \qquad \text{(the orthomodular law)}$$

where $x \mapsto x^{\perp}$ denoted the orthocomplementation.

Proposition 1.3.4.

(i) The skeleton of L is the set $\{x^{\triangledown} \mid x \in L\}$ and is equal to $\{x \in L \mid x \vee x^{\triangledown} \geq x^{\triangledown\triangledown}\}$. The dual holds for dual skeleton.

(ii) Skeletons and dual skeletons are ortholattices.

Proof. (i) follows obviously from axioms (3). Note that $(\bar{S}(L), \bar{\sqcap}, \underline{\sqcup}, 0, 1)$ is a bounded lattice. Let x and y in L. We have

$$x \underline{\sqcup} y = (x^{\vartriangle} \wedge y^{\vartriangle})^{\vartriangle} = x^{\vartriangle\vartriangle} \vee y^{\vartriangle\vartriangle} = x \vee y,$$

$$(x \bar{\sqcap} y)^{\vartriangle} = (x^{\vartriangle} \vee y^{\vartriangle})^{\vartriangle\vartriangle} = (x \wedge y)^{\vartriangle} = x^{\vartriangle} \vee y^{\vartriangle},$$

$$\text{and } x^{\vartriangle} \bar{\sqcap} y^{\vartriangle} = (x^{\vartriangle\vartriangle} \vee y^{\vartriangle\vartriangle})^{\vartriangle} = (x \vee y)^{\vartriangle} = (x \underline{\sqcup} y)^{\vartriangle}.$$

In addition $x \bar{\sqcap} x^{\vartriangle} = (x^{\vartriangle} \vee x^{\vartriangle\vartriangle})^{\vartriangle} = 0$ and $x \underline{\sqcup} x^{\vartriangle} = (x^{\vartriangle} \wedge x^{\vartriangle\vartriangle})^{\vartriangle} = 1$. The proof for $(S(L), \wedge, \sqcup, ^{\triangledown}, 0, 1)$ is obtained similarly. □

Remark 1.3.3. The skeleton or dual skeleton can be both not distributive and not uniquely complemented even if the lattice L were distributive. As illustration the skeletons of the weakly dicomplemented lattice on Figure 1.2 are given on Figure 1.4. The orthomodular law is not fulfilled. For finite distributive lattices, skeleton and dual skeleton are Boolean algebras

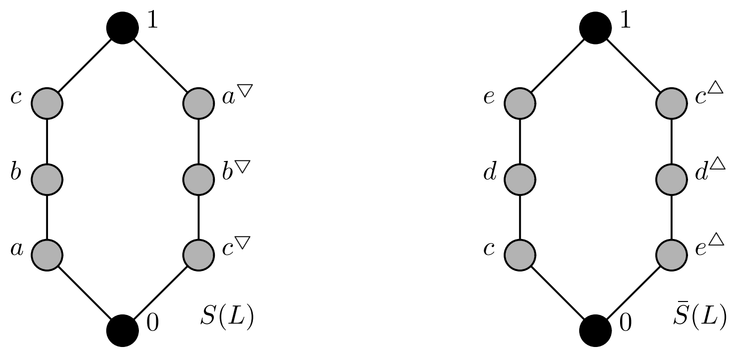


Figure 1.4. Skeleton and dual skeleton of the (weakly) dicomplemented lattice in Figure 1.2. If we reduce the context representing L we obtain a double p-algebra, with skeleton and dual skeleton isomorphic to a 4 element Boolean algebra.

if the dicomplementation is minimal with respect to the "finer than relation". Unfortunately this cannot be extended to finite lattices in general. The context on Figure 5.1 is reduced. Its skeleton and dual skeleton are isomorphic to M_4.

Remark 1.3.4. Although the operation $^{\triangle}$ on the skeleton is a complementation, it is no longer a weak complementation. The axiom

$$(x \wedge y) \vee (x \wedge y^{\triangle}) = x$$

is violated. However the operation $^{\triangle}$ on the skeleton is always a polarity that satisfies the de Morgan laws and is a complementation. Such an operation is sometimes called "**orthonegation**".

Skeletons are not always sublattices of L. Now let us have a look at the interior operator. In the case of distributive double p-algebra it is a homomorphism.

Proposition 1.3.5. *The interior operator* $\phi\colon x \mapsto x^{\triangle\triangle}$ *is a homomorphism from* $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ *onto* $(\bar{S}(L), \bar{\sqcap}, \vee, ^{\triangle}, 0, 1)$ *if and only if* $(x \vee y)^{\triangle\triangle} = x^{\triangle\triangle} \vee y^{\triangle\triangle}$ *for all* $x, y \in L$.

Proof. $\phi(x)\bar{\sqcap}\phi(y) = (x^{\triangle} \vee y^{\triangle})^{\triangle} = (x \wedge y)^{\triangle\triangle} = \phi(x \wedge y)$. Trivially ϕ is onto, $\phi(x^{\triangle}) = \phi(x)^{\triangle}$, $\phi(0) = 0$ and $\phi(1) = 1$. From

$$\phi(x) \vee \phi(y) = x^{\triangle\triangle} \vee y^{\triangle\triangle} \leq (x \vee y)^{\triangle\triangle} = \phi(x \vee y),$$

we get the result. □

Remark 1.3.5. In the example on Figure 1.2, ϕ is not a homomorphism since

$$\phi(c \vee v) = (c \vee v)^{\triangle\triangle} = e > c = c^{\triangle\triangle} \vee v^{\triangle\triangle} = \phi(x) \vee \phi(y).$$

Stone algebras are determined in the class of distributive p-algebras by one of the following equivalent conditions: $x^* \vee x^{**} = 1$, $(x \wedge y)^* = x^* \vee y^*$

or $S(L)$ is a sublattice of L. These conditions are in general not equivalent for any weakly dicomplemented lattice.

Proposition 1.3.6. *The implications* $(i) \implies (ii) \iff (iii) \implies (iv)$ *hold in every weakly complemented lattice* L.

(i) $(x \vee y)^{\triangle} = x^{\triangle} \wedge y^{\triangle}$,

(ii) $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$,

(iii) $\bar{S}(L)$ *is a sublattice of* L,

(iv) $x^{\triangle} \wedge \phi(x) = 0$.

Proof. We start with $(i) \implies (ii)$. We assume that $(x \vee y)^{\triangle} = x^{\triangle} \wedge y^{\triangle}$, for all $x, y \in L$. We have

$$\phi(x \wedge y) = (x \wedge y)^{\triangle\triangle} = (x^{\triangle} \vee y^{\triangle})^{\triangle} = x^{\triangle\triangle} \wedge y^{\triangle\triangle} = \phi(x) \wedge \phi(y).$$

We assume (ii). Let x and y be elements in $\bar{S}(L)$.

$$x \bar{\sqcap} y = (x^{\triangle} \vee y^{\triangle})^{\triangle} = (x \wedge y)^{\triangle\triangle} = \phi(x \wedge y) = \phi(x) \wedge \phi(y) = x \wedge y.$$

Thus $\bar{S}(L)$ is a sublattice of L. For the converse we assume that $\bar{S}(L)$ is a sublattice and set $z^{\triangle} := x^{\triangle\triangle} \wedge y^{\triangle\triangle}$ for some $x, y \in L$. It follows that $z^{\triangle\triangle} \geq x^{\triangle} \vee y^{\triangle} = (x \wedge y)^{\triangle}$ and $z^{\triangle} \leq (x \wedge y)^{\triangle\triangle}$. Thus

$$\phi(x) \wedge \phi(y) \leq \phi(x \wedge y).$$

The reverse inequality is trivial. Thus (ii) is equivalent to (iii).

We assume (iii). Let $x \in L$. We have

$$x^{\triangle} \wedge \phi(x) = \phi(x^{\triangle}) \wedge \phi(x) = \phi(x^{\triangle} \wedge x) = (x^{\triangle} \wedge x)^{\triangle\triangle} = 0$$

and (iv) is proved. □

The lattice M_3 satisfies (ii) but not (i). In Figure 5.1 the condition (iv) holds, but $\bar{S}(L)$ is not a sublattice of L.

Now that we are familiar with weakly dicomplemented lattices and concept algebras, we will start in the next chapter to investigate their structure theory.

1.4 Main Problem Revisited[9]

In this dissertation the main class of algebras under investigation is the class $\mathfrak{C}$ of all concept algebras. This is not an equational class. The (*equational*) class $\mathfrak{D}_w$ of all weakly dicomplemented lattices has been introduced. It contains the class $\mathfrak{C}$. The aim is to find out if the axioms of weakly

[9] This section follows from a wish of a referee to have an overview of the problems I have considered.

dicomplemented lattices generate the equational theory of concept algebras. Therefore we introduce other classes, which are between these two categories.

- The class $\mathfrak{D}_e$ consisting of all models of the equational theory on $\mathfrak{C}$. That is the equational class generated by $\mathfrak{C}$.
- The class $\mathfrak{D}$ of all dicomplemented lattices. This is the quasi-equational class generated by $\mathfrak{C}$.
- The class $\mathfrak{D}_s$ of all algebras that can be embedded into concept algebras.
- The class $\mathfrak{D}_i$ of all algebras that are isomorphic to concept algebras.

The following inclusions hold.

$$\mathfrak{C} \subseteq \mathfrak{D}_i \subseteq \mathfrak{D}_s \subseteq \mathfrak{D} \subseteq \mathfrak{D}_e \subseteq \mathfrak{D}_w.$$

The main problem, ***"finding the equational or quasi-equational theory of concept algebras"***, can split in several parts.

The axiomatization problem. We are looking for a set of formulae that axiomatizes the class $\mathfrak{D}_i$. The problem of describing those weakly dicomplemented lattices that are isomorphic to concept algebras will be referred to as the **strong representation problem**[11] for weakly dicomplemented lattices. Section 3.1 presents a solution for the subclass of finite weakly dicomplemented lattices.

The general equational-axiomatization problem. Is the class of dicomplemented lattices an equational class? In other words, is there any difference between the equational theory and the quasi-equational theory on concept algebras?

The specific equational-axiomatization problem. Do the axioms of weakly dicomplemented lattices axiomatize $\mathfrak{D}$? That is, is every weakly dicomplemented lattice a dicomplemented lattice? Observe that a positive answer would yield a positive answer to the general equational-axiomatization problem.

The concrete embedding problem. Can every weakly dicomplemented lattice be embedded into a concept algebra? That is, do we have $\mathfrak{D}_w \subseteq \mathfrak{D}_s$? Note that we cannot have $\mathfrak{D}_w \subseteq \mathfrak{D}_i$ since weakly dicomplemented lattices need not be complete. A positive answer would yield a positive answer to the specific equational-axiomatization problem. We refer to the *problem of embedding weakly dicomplemented lattices into concept algebras* as the **representation problem**.

[11] I do thank B. A. Davey for the name.

Section 4.1 presents a solution to both the representation and strong representation problems for the subclass of finite distributive weakly dicomplemented lattices. The general case is still open. However Section 2.2 presents a hope towards a solution for the general case.

2
Structure Theory

Do dicomplemented lattices form an equational class? We cannot answer this question completely, but we present many steps towards an answer. A problem remains for the formation of homomorphic images. The natural way towards the desired representation theorem is to use the proof strategy that has been successful for Boolean algebras (*Stone representation*). We therefore study primary filters and ideals, the natural analogue of prime filters and ideals in Boolean algebras. We construct the canonical context, best candidate for a contextual representation of a weakly dicomplemented lattice. Congruences are characterized.

2.1 Structure Theory of Concept Algebras

2.1.1 The class of concept algebras

Is there any description of the class $\mathfrak{D}_i$ of algebras isomorphic to concept algebras? As already stated in Chapter 1 the aim is to get a structure defined by a set of quasi-equations[1]such that all quasi-equations valid in

[1]For an algebra $\underline{A}$ of type τ, an **identity** or **equation** over a set of variables X is an expression of the form $p \simeq q$ where p and q are terms of type τ over X. A **quasi-identity**, also called **quasi-equation** or **implication**, is an expression of the form $(p_1 \simeq q_1 \& \ldots \& p_n \simeq q_n) \rightarrow p \simeq q$. Equational classes are those defined by a set of

concept algebras are exactly those valid in that structure. Such a structure, if it exists, will be called a *dicomplemented lattice*. A celebrated theorem of Birkhoff describes equational classes as varieties [BS81, p. 83]. Recall that **varieties** are classes of structures stable under the operators H, S and P (defined below). Quasi-equational classes are **quasivarieties**: that are classes of algebras closed under I, S and P_R (defined below), and containing the one-element algebras.

Definition 2.1.1. Let $\mathcal{K}$ denote a class of structures of the same type τ. The class $\mathcal{K}$ is said to be stable under an operator O if the image of member(s) of $\mathcal{K}$ is again in $\mathcal{K}$. i.e. $O(\mathcal{K}) \subseteq \mathcal{K}$. The operators I, H, S, P and P_R are defined by:

- $\underline{A} \in I(\mathcal{K})$ iff $\underline{A}$ is isomorphic to an element of $\mathcal{K}$,
- $\underline{A} \in H(\mathcal{K})$ iff there is an epimorphism from an element of $\mathcal{K}$ onto $\underline{A}$,
- $\underline{A} \in S(\mathcal{K})$ iff $\underline{A}$ is a substructure of a an element of $\mathcal{K}$,
- $\underline{A} \in P(\mathcal{K})$ iff $\underline{A}$ is a product of a family of structures of $\mathcal{K}$,
- $\underline{A} \in P_R(\mathcal{K})$ iff $\underline{A}$ is a reduced product[2] of a family of structures of $\mathcal{K}$,

where $\underline{A}$ denotes an arbitrary algebra of type τ.

Tarski proved that varieties can be defined by means of a unique operator, namely $V := HSP$ [BS81, p. 67]. In the following we shall consider under which operators the class of concept algebras is closed. Since concept algebras are complete lattices we can only expect this class to be closed under some restrictions of the operators V and $Q := ISP_R$. With such a restriction we could only expect to have the class of concept algebras as "trace" of some varieties or quasivarieties. However, from a category theory point of vue, its seems important to explore substructures, products, morphisms, ... of concepts algebras.

2.1.2 Product of concept algebras

We shall prove that the class of concept algebras is stable for the operator P. Let $\underline{\mathfrak{A}}(\mathbb{K}_1)$ and $\underline{\mathfrak{A}}(\mathbb{K}_2)$ be concept algebras. Is there any context $\mathbb{K}$ such that $\underline{\mathfrak{A}}(\mathbb{K}_1) \times \underline{\mathfrak{A}}(\mathbb{K}_2)$ is isomorphic to $\underline{\mathfrak{A}}(\mathbb{K})$? We know that the concept lattice of the context

$$\mathbb{K}_1 \oplus \mathbb{K}_2 := (G_1 \oplus G_2, M_1 \oplus M_2, I_1 \cup I_2 \cup G_1 \times M_2 \cup G_2 \times M_1)$$

---

identities while quasi-equational classes are those defined by quasi-identities. The reader is referred to [BS81, Chapter II & Chapter V] for more details.

[2] Given a nonempty family $(\underline{A}_i)_{i\in I}$ of type τ and F a proper filter over I, the relation θ_F, defined on $\Pi_{i\in I} A_i$ by $(a,b) \in \theta_F :\iff \{i \in I \mid a_i = b_i\} \in F$, is a congruence on $\underline{A} := \Pi_{i\in I}\underline{A}_i$. The algebra $\underline{A}/\theta_F$ is called a **reduced product** of $(\underline{A}_i)_{i\in I}$.

is isomorphic to the lattice product $\underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2)$. In fact

Proposition 2.1.1. *[GW99, p. 46]*

(i) $(A, B) \in \mathfrak{B}(\mathbb{K}_1 \oplus \mathbb{K}_2)$ *if and only if* $(A \cap G_i, B \cap M_i) \in \mathfrak{B}(\mathbb{K}_i)$.

(ii) The mapping $(A, B) \mapsto ((A \cap G_1, B \cap M_1), (A \cap G_2, B \cap M_2))$ *is a lattice isomorphism from* $\underline{\mathfrak{B}}(\mathbb{K}_1 \oplus \mathbb{K}_2)$ *onto* $\underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2)$.

We shall prove that $\underline{\mathfrak{A}}(\mathbb{K}_1 \oplus \mathbb{K}_2) \cong \underline{\mathfrak{A}}(\mathbb{K}_1) \times \underline{\mathfrak{A}}(\mathbb{K}_2)$ and get the following result:

Theorem 2.1.2. *A product of concept algebras is isomorphic to a concept algebra.*

Proof. To get this result we have to prove that the unary operations are preserved. We start with the weak complementation. Operations are indexed by their context name.

$$(A, B)^{\triangle_{\mathbb{K}}} = ((G \setminus A)'', (G \setminus A)')$$

and

$$(A \cap G_i, B \cap M_i)^{\triangle_{\mathbb{K}_i}} = ((G_i \setminus A_i)^{\mathrm{I}_i \mathrm{I}_i}, (G_i \setminus A_i)^{\mathrm{I}_i}).$$

We set $A_i := A \cap G_i$ and $B_i := B \cap M_i$. It is enough to prove that

$$(G \setminus A)' \cap M_i = (G_i \setminus A)^{\mathrm{I}_i}.$$

That is what we do for $i = 1$. We first show that $(G_2 \setminus A_2)' \cap M_1 = M_1$. If $G_2 \setminus A_2$ is empty the result follows trivially. We assume that $G_2 \setminus A_2$ is nonempty. Note that

$$g \in G_2 \implies g \,\mathrm{I}\, m \quad \forall m \in M_1.$$

Therefore $M_1 \subseteq G_2'$. For any subset U of G_2 we have $U' \supseteq G_2' \supseteq M_1$. Thus $(G_2 \setminus A_2)' \cap M_1 = M_1$. Now

$$\begin{aligned}(G \setminus A)' \cap M_1 &= ((G_1 \setminus A_1) \cup (G_2 \setminus A_2))' \cap M_1 \\ &= (G_1 \setminus A_1)' \cap M_1 \cap (G_2 \setminus A_2)' \cap M_1 \\ &= (G_1 \setminus A_1)' \cap M_1 \\ &= (G_1 \setminus A_1)^{\mathrm{I}_1} \cap M_1\end{aligned}$$

since for all $m \in M_1$,

$$m \in (G_1 \setminus A_1)' \iff g \,\mathrm{I}\, m, \forall g \in G_1 \setminus A_1 \iff g \,\mathrm{I}_1\, m, \forall g \in G_1 \setminus A_1.$$

Thus $(G \setminus A)' \cap M_i = (G_i \setminus A_i)^{\mathrm{I}_i}$. Using these equalities and the notations $\bar{A} := G \setminus A$ and $\bar{A}_i := G_i \setminus A_i$ we get:

$$\begin{aligned}(A,B)^{\triangle_{\mathbb{K}}} &= ((G \setminus A)'', (G \setminus A)') \\ &\equiv \big((\bar{A}'' \cap G_1, \bar{A}' \cap M_1), (\bar{A}'' \cap G_2, \bar{A}' \cap M_2)\big) \\ &= \Big((\bar{A}_1^{\mathrm{I}_1\mathrm{I}_1} \cap G_1, \bar{A}_1^{\mathrm{I}_1} \cap M_1), (\bar{A}_2^{\mathrm{I}_2\mathrm{I}_2} \cap G_2, \bar{A}_2^{\mathrm{I}_2} \cap M_2)\Big) \\ &= \big((A_1,B_1)^{\triangle_{\mathbb{K}_1}}, (A_2,B_2)^{\triangle_{\mathbb{K}_2}}\big)\end{aligned}$$

Thus $\underline{\mathfrak{A}}(\mathbb{K}) \cong \underline{\mathfrak{A}}(\mathbb{K}_1) \times \underline{\mathfrak{A}}(\mathbb{K}_2)$

This result extends to arbitrary products. That is

$$\underline{\mathfrak{A}}(\oplus_{i\in I}\mathbb{K}_i) \cong \Pi_{i\in I}\underline{\mathfrak{A}}(\mathbb{K}_i)$$

for any family $(\mathbb{K}_i)_{i\in I}$ of contexts. □

2.1.3 Subalgebras of concept algebras

The aim of this subsection is to prove that subalgebras of concept algebras are concept algebras. Since concept algebras are complete lattices, only complete subalgebras of concept algebras can be again concept algebras. We consider a formal context $\mathbb{K} := (G, M, \mathrm{I})$ and $\underline{L}$ a subalgebra of the concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$. Is there any context $\mathbb{K}_{\underline{L}}$ such that $\underline{\mathfrak{A}}(\mathbb{K}_{\underline{L}})$ is isomorphic to $\underline{L}$? Without loss of generality we assume that L is a subset of $\mathfrak{B}(\mathbb{K})$. The elements 0 and 1 belong to L because $\underline{L}$ is a subalgebra. This means that the concepts (G, G') and (M', M) are members of L. A relation J is defined on $G \times M$ by:

$$\mathrm{J} := \bigcup\{A \times B \mid (A,B) \in L\}.$$

$$\text{i.e.}\quad g\,\mathrm{J}\,m \iff \exists (A,B) \in L \text{ such that } g \in A \text{ and } m \in B.$$

The relation J is a closed subrelation of I. Thus the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ is isomorphic to the lattice $(L, \wedge, \vee, 0, 1)$ ([GW99, p. 112]). It remains to prove that the weak negation $^{\triangle_{\mathrm{J}}}$ of the context (G, M, J) is the restriction of the weak negation $^{\triangle_{\mathrm{I}}}$ of the context (G, M, I). Let $(A,B) \in L$. We want to prove that $(A,B)^{\triangle_{\mathrm{J}}} = (A,B)^{\triangle_{\mathrm{I}}}$. We denote by $()^{\mathrm{J}}$ the derivation in (G, M, J). It is enough to prove that $(G \setminus A)'$ is equal to $(G \setminus A)^{\mathrm{J}}$. Let $m \in M$ such that $m \in (G \setminus A)^{\mathrm{J}}$. For all $g \in G \setminus A$, we have $g\,\mathrm{J}\,m$. Therefore $g\,\mathrm{I}\,m$ for all $g \notin A$. i.e. $m \in (G \setminus A)'$. Thus $(G \setminus A)^{\mathrm{J}} \subseteq (G \setminus A)'$ and

$$(A,B)^{\triangle_{\mathrm{J}}} \leq (A,B)^{\triangle_{\mathrm{I}}}.$$

If this inequality were strict, there would exist an m in M with $m \in (G \setminus A)'$ and $m \notin (G \setminus A)^{\mathrm{J}}$. This means that for all $g \notin A$, $g\,\mathrm{I}\,m$ and there is an element $h \notin A$ such that $(h,m) \notin J$. Moreover $h \in G \setminus A$ together with $m \in (G \setminus A)'$ imply $(h,m) \in (G \setminus A)'' \times (G \setminus A)'$. This concept is exactly the weak negation of the concept (A,B). It belongs to L since (A,B) was taken in the subalgebra $\underline{L}$. This would yield $h \, \mathrm{J} \, m$, which is a contradiction. Thus

$$(A, B)^{\triangle_{\mathrm{J}}} = (A, B)^{\triangle_{\mathrm{I}}}.$$

Dually

$$(A, B)^{\triangledown_{\mathrm{J}}} = (A, B)^{\triangledown_{\mathrm{I}}}.$$

Thus

Theorem 2.1.3. *Complete subalgebras of concept algebras are concept algebras.*

Considering the class of concept algebras as a category which objets are concept algebras and morphisms being complete lattice homomorphisms[3] preserving weak negations and weak oppositions, the theorem above means that sub-objects are complete subalgebras. We have seen that the class of concept algebras is stable under the formation of products and complete substructures. Now, is a homomorphic image of a concept algebra again a concept algebra? It is reasonnable to consider only complete homomorphisms.

2.1.4 Homomorphic images of concept algebras

The idea is to prove that a complete homomorphic image of a concept algebra is isomorphic to a concept algebra. This seems to be provable, but up to now no one has succeeded to get the desired result. If this happens to be true, then the class of concept algebras will be closed under formation of products, complete substructures and complete homomorphic images.

To formulate this problem, we consider an algebra $(L, \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ to be the image of a concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$, with $\mathbb{K} := (G, M, \mathrm{I})$, under a complete homomorphism ϕ. Is there any context $\mathbb{K}_\phi$ such that L is isomorphic to $\underline{\mathfrak{A}}(\mathbb{K}_\phi)$?

Without loss of generality, the object set G can be assumed to be the set of $^{\triangle_{\mathbb{K}}}$-compatible elements, M the set of $^{\triangledown_{\mathbb{K}}}$-compatible elements and I the restriction of $\leq$ on $G \times M$. The image of the context $\mathbb{K}$ by ϕ is the context $\mathbb{K}_\phi := (\phi(G), \phi(M), \mathrm{I}_\phi)$ with

$$g \, \mathrm{I} \, m \implies \phi(g) \, \mathrm{I}_\phi \, \phi(m).$$

For any element b in L there is a concept $a := (A, B)$ in $\mathfrak{B}(\mathbb{K})$ such that $b = \phi(a)$. In addition $a = \bigvee A = \bigwedge B$. Thus $\phi(a) = \bigvee \phi(A) = \bigwedge \phi(B)$ meaning that G_ϕ is $\vee$-dense and M_ϕ is $\wedge$-dense in L. Thus $\underline{\mathfrak{B}}(\mathbb{K}_\phi)$ is isomorphic to the lattice $(L, \wedge, \vee, 0, 1)$. To get an isomorphism between $\underline{\mathfrak{A}}(\mathbb{K}_\phi)$ and $(L, \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1)$ it remains to prove that the weak operations are preserved.

[3] That are homomorphisms preserving arbitrary meet and join.

It is not difficult to see that all elements of $\phi(G)$ (resp. of $\phi(M)$) are ${}^{\triangle}$-compatible (resp. ${}^{\triangledown}$-compatible). In fact for $b \in \phi(G)$ and $y \in L$, there is $a \in G$ and $x \in \mathfrak{B}(\mathbb{K})$ such that $\phi(a) = b$ and $\phi(x) = y$. From $a \leq x$ or $a \leq x^{\triangle}$ we get $b \leq y$ or $b \leq y^{\triangle}$ since ϕ preserves ${}^{\triangle}$. Therefore

$$y^{\triangle_{\phi(\mathbb{K})}} = \bigvee\{b \in \phi(G) \mid b \not\leq y\} \leq y^{\triangle}.$$

Thus $({}^{\triangle_{\phi(\mathbb{K})}}, {}^{\triangledown_{\phi(\mathbb{K})}})$ is finer than $({}^{\triangle}, {}^{\triangledown})$. *The reverse inequality is needed to get the desired result. This is up to now an open problem.*

2.2 Prime Ideals

Recall that weakly dicomplemented lattices are defined by some identities valid in all concept algebras. If each weakly dicomplemented lattice embeds into a concept algebra, then the equational theory of concept algebras will be generated by the axioms of weakly dicomplemented lattices. Let $\underline{L}$ be a weakly dicomplemented lattice. How can we construct a context $\mathbb{K}^{\triangle}_{\triangledown}(L)$ such that $\underline{L}$ embeds into $\underline{\mathfrak{A}}(\mathbb{K}^{\triangle}_{\triangledown}(L))$? Rudolf Wille introduced concept algebras in [Wi00] and considered *the problem of finding its quasi-equational theory*. He set down a set of implications to define a *dicomplemented lattice* and tried to embed it into a concept algebra. Unfortunately this attempt did not give the desired result. This problem is still unsolved. The approach we use here is similar to that of Wille. We obtain a characterization of finite dicomplemented lattices. Since we hope that this result can be generalized, we shall, in this section, first present the general construction, and later consider the finite case in Chapter 3. This has been done using the so called "prime ideal theorem". As observed in [BD74], "the prime ideal theorem is one of the most important results in the theory of distributive lattices". It is the cornerstone of well-known representation theorems such as topological representation of Boolean algebras by M. H. Stone [St36], of bounded distributive lattices by H. A. Priestley [Pr70], and of lattices by G. Hartung [Ha92]. A prime ideal theorem is easy to prove for weakly dicomplemented lattices. It may however be insufficient to get an embedding.

2.2.1 A prime ideal theorem for weakly dicomplemented lattices

Definition 2.2.1. A proper filter F is called **primary**[4] if it contains w or $w^{\triangle}$ for all $w \in L$. Dually, a **primary ideal** is a proper ideal which contains w or $w^{\triangledown}$ for all $w \in L$.

$\mathcal{F}_{pr}(L)$ denotes the set of all primary filters and $\mathcal{I}_{pr}(L)$ the set of primary ideals of L.

Lemma 2.2.1 ("Prime ideal theorem").
For every filter F and every ideal I such that $F \cap I = \emptyset$ there is a primary filter G containing F and disjoint from I.

For every ideal I and every filter F such that $I \cap F = \emptyset$ there is a primary ideal J containing I and disjoint from F.

Proof. Set $\mathcal{F}_I := \{G \mid G \text{ filter } \ F \subseteq G,\ G \cap I = \emptyset\}$. The family $\mathcal{F}_I$ contains F and satisfies the conditions of Zorn's lemma. Let $\tilde{G}$ be a maximal element in $(\mathcal{F}_I, \subseteq)$. We claim that $\tilde{G}$ is primary. Otherwise, there exists an element x in L such that neither x nor $x^{\triangle}$ belong to $\tilde{G}$. Then

$$I \cap \mathrm{Filter}(\tilde{G} \cup \{x\}) \neq \emptyset \quad \text{and} \quad I \cap \mathrm{Filter}(\tilde{G} \cup \{x^{\triangle}\}) \neq \emptyset.$$

There must be elements $u, v \in \tilde{G}$ with $u \wedge x \in I$ and $v \wedge x^{\triangle} \in I$. Since $u, v \in \tilde{G}$ we have $u \wedge v \in \tilde{G}$ and consequently

$$u \wedge v \notin I,$$

since $I \cap \tilde{G} = \emptyset$. But $u \wedge x \in I$ implies $u \wedge v \wedge x \in I$ and $v \wedge x^{\triangle} \in I$ implies $u \wedge v \wedge x^{\triangle} \in I$. Since I is an ideal, we get

$$(u \wedge v \wedge x) \vee (u \wedge v \wedge x^{\triangle}) \in I.$$

With axiom (3), this is equal to $u \wedge v$, a contradiction.

The remaining claim follows dually. □

Corollary 2.2.2. *If $x \not\leq y$ in L, then there exists a primary filter F containing x and not y.*

Proof. If $x \not\leq y$ then $\mathrm{Filter}(\{x\}) \cap \mathrm{Ideal}(\{y\}) = \emptyset$. By Lemma 2.2.1 there is a primary filter containing $\mathrm{Filter}(\{x\})$ and disjoint from $\mathrm{Ideal}(\{y\})$. □

For every pair of elements x and y in L, there exists a primary filter containing exactly one of them. The dual statement holds for primary ideals.

[4] In the case of Boolean algebras the notion of a primary filter is equivalent to that of an ultrafilter and of a prime filter. We recall that an **ultrafilter** is a proper filter which is maximal under inclusion; a proper filter F is called **prime filter**

$$\forall x, y \in L,\ x \vee y \in F \implies x \in F \text{ or } y \in F.$$

2.2.2 Canonical context

We start here to prepare a representation of L by a formal context.

Definition 2.2.2. For $x \in L$, define

$$\mathcal{F}_x := \{F \in \mathcal{F}_{pr}(L) \mid x \in F\}$$

and

$$\mathcal{I}_x := \{I \in \mathcal{I}_{pr}(L) \mid x \in I\}.$$

Definition 2.2.3. The **canonical context** of a weakly dicomplemented lattice L is the formal context

$$\mathbb{K}_{\triangledown}^{\vartriangle}(L) := (\mathcal{F}_{pr}(L), \mathcal{I}_{pr}(L), \Box)$$

with $F \,\Box\, I :\iff F \cap I \neq \emptyset$.

Lemma 2.2.3 ("Derivation Lemma").
The derivation in $\mathbb{K}_{\triangledown}^{\vartriangle}(L)$ yields, for every $x \in L$,

$$\mathcal{F}_x' = \mathcal{I}_x \quad \textit{and} \quad \mathcal{I}_{x^{\vartriangle}} \subseteq (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)',$$

$$\mathcal{I}_x' = \mathcal{F}_x \quad \textit{and} \quad \mathcal{F}_{x^{\triangledown}} \subseteq (\mathcal{I}_{pr}(L) \setminus \mathcal{I}_x)'.$$

Proof. If $I \in \mathcal{I}_x$ then $I \in \mathcal{I}_{pr}(L)$ and $x \in I$. But then $x \in I \cap F$ for all $F \in \mathcal{F}_x$, and thus $I \in \mathcal{F}_x'$. This proves $\mathcal{I}_x \subseteq \mathcal{F}_x'$. Conversely, assume that I is an ideal not containing x. Then

$$I \cap \mathrm{Filter}(\{x\}) = \emptyset,$$

and by Lemma 2.2.1 there is some primary filter F disjoint from I and containing x. Thus $I \notin \mathcal{F}_x'$, and $\mathcal{I}_x = \mathcal{F}_x'$ is proved.

Now let $I \in \mathcal{I}_{x^{\vartriangle}}$. Since any primary filter F with $x \notin F$ contains $x^{\vartriangle}$, we have

$$x^{\vartriangle} \in F \quad \text{for all} \quad F \in \mathcal{F}_{pr}(L) \setminus \mathcal{F}_x.$$

Then $F \,\Box\, I$ for all $F \in (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)$ and therefore I belongs to $(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$. i.e.,

$$\mathcal{I}_{x^{\vartriangle}} \subseteq (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'.$$

The remaining claims follow dually. □

Theorem 2.2.4 (Dreamlike embedding).
Let $\underline{L}$ be a weakly dicomplemented lattice. The map

$$\begin{array}{rcl} i\colon L & \to & \underline{\mathfrak{B}}\left(\mathbb{K}_{\triangledown}^{\vartriangle}(L)\right) \\ x & \mapsto & (\mathcal{F}_x, \mathcal{I}_x) \end{array}$$

is a bounded lattice embedding with:

$$i(x^{\triangledown}) \leq i(x)^{\triangledown} \leq i(x)^{\vartriangle} \leq i(x^{\vartriangle}).$$

Proof. By Lemma 2.2.3 the pair $(\mathcal{F}_x, \mathcal{I}_x)$ is a formal concept of $\mathbb{K}_\triangledown^\vartriangle(L)$ for all $x \in L$ and i maps L into $\mathfrak{B}\left(\mathbb{K}_\triangledown^\vartriangle(L)\right)$. From Corollary 2.2.2 we get $x \not\leq y$ implies $\mathcal{F}_x \not\subseteq \mathcal{F}_y$. Thus i is injective.

Let F be a primary filter and I a primary ideal.

$$\begin{aligned} F \in \mathcal{F}_{x \wedge y} &\iff x \wedge y \in F \\ &\iff x,\ y \in F \\ &\iff F \in \mathcal{F}_x \text{ and } F \in \mathcal{F}_y \\ &\iff F \in \mathcal{F}_x \cap \mathcal{F}_y. \end{aligned}$$

i.e., $i(x \wedge y) = i(x) \wedge i(y)$.

$$\begin{aligned} I \in \mathcal{I}_{x \vee y} &\iff x \vee y \in I \\ &\iff x,\ y \in I \\ &\iff I \in \mathcal{I}_x \text{ and } I \in \mathcal{I}_y \\ &\iff I \in \mathcal{I}_x \cap \mathcal{I}_y. \end{aligned}$$

i.e., $i(x \vee y) = i(x) \vee i(y)$.

$$i(0) = 0; \quad i(1) = 1.$$

$$i(x^\triangledown) = (\mathcal{F}_{x^\triangledown}, \mathcal{I}_{x^\triangledown}) \leq \left(\left(\mathcal{I}_{pr}(L) \setminus \mathcal{I}_x\right)', \left(\mathcal{I}_{pr}(L) \setminus \mathcal{I}_x\right)''\right) = i(x)^\triangledown.$$

$$i(x)^\vartriangle = \left(\left(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x\right)'', \left(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x\right)'\right) \leq (\mathcal{F}_{x^\vartriangle}, \mathcal{I}_{x^\vartriangle}) = i(x^\vartriangle).$$

□

We want the map i in Theorem 2.2.4 to be a weakly dicomplemented lattice embedding. In this case $\underline{L}$ would be a copy of a subalgebra of the concept algebra $\underline{\mathfrak{A}}\left(\mathbb{K}_\triangledown^\vartriangle(L)\right)$ and would satisfy all equations valid in all concept algebras. To achieve this, the inclusions in the Derivation Lemma should be equalities. It seems quite natural to check under which conditions we have the equalities. When does the inclusion

$$\mathcal{I}_{x^\vartriangle} \supseteq (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$$

hold? We do a case analysis.

If $x = 1$ then $\mathcal{I}_{x^\vartriangle} = \mathcal{I}_0 = \mathcal{I}_{pr}(L)$ and

$$(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)' = (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_1)' = (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_{pr}(L))' = \emptyset' = \mathcal{I}_{pr}(L).$$

If $x = 0$ then $\mathcal{I}_{x^\vartriangle} = \mathcal{I}_1 = \emptyset$ and

$$(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)' = (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_0)' = (\mathcal{F}_{pr}(L) \setminus \emptyset)' = \mathcal{F}_{pr}(L)' = \emptyset.$$

If $x^\vartriangle = 0$ then $x = 1$ and we get the equality. It remains to prove the inclusion for $x \notin \{0, 1\}$. Let $I \in \mathcal{I}_{pr}(L)$, $I \notin \mathcal{I}_{x^\vartriangle}$. We want to prove that

$I \notin (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$. Note that

$$I \in (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)' \iff I \cap F \neq \emptyset \text{ for all } F \in \mathcal{F}_{pr}(L) \setminus \mathcal{F}_x.$$

Equivalent to this is the statement

$$I \notin (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)' \iff I \cap F = \emptyset \text{ for some } F \in \mathcal{F}_{pr}(L) \setminus \mathcal{F}_x.$$

We distinguish 3 cases:

(a) $x \in I$.
We have $I \cap \text{Filter}(x^{\triangle}) = \emptyset$. By Lemma 2.2.1 there exists a primary filter F disjoint from I and thus not containing x. It follows that $F \in (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)$ and $F \cap I = \emptyset$. Therefore I does not belong to $(\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$.

(b) $x \notin I$ and $x^{\triangle} \notin \text{Ideal}(I \cup \{x \wedge x^{\triangle}\})$.
We have $\text{Filter}(x^{\triangle}) \cap \text{Ideal}(I \cup \{x \wedge x^{\triangle}\}) = \emptyset$. By Lemma 2.2.1 there exists a primary filter F disjoint from $\text{Ideal}(I \cup \{x \wedge x^{\triangle}\})$ and thus not containing x, i.e., $F \in (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)$ and $F \cap I = \emptyset$. Therefore $I \notin (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$.

(c) $x \notin I$ and $x^{\triangle} \in \text{Ideal}(I \cup \{x \wedge x^{\triangle}\})$.
From $x \notin I$ we know that $x \neq 0$. We assume that $x \neq 1$. The statement $x^{\triangle} \in \text{Ideal}(I \cup \{x \wedge x^{\triangle}\})$ implies there exists $u \in I$ such that $x^{\triangle} \leq u \vee (x \wedge x^{\triangle})$. But $x^{\triangle} \notin I$ implies $x \wedge x^{\triangle} \nleq u$. In addition $u \nleq x \wedge x^{\triangle}$; otherwise we would have $x^{\triangle} \leq u \vee (x \wedge x^{\triangle}) \leq x \wedge x^{\triangle}$ forcing x to be 1, which is contrary to the assumption. Thus u and $x \wedge x^{\triangle}$ are incomparable. The hope is to prove that, even in this case there is a primary filter F not containing x with $F \cap I = \emptyset$. This is an open question. The special case of finite lattices is solved in the next chapter.

To close this section we prove the following:

Theorem 2.2.5. *The concept algebra of the canonical context of $\underline{\mathfrak{A}}(\mathbb{K})$ is isomorphic to the concept algebra of $\mathbb{K}$. i.e.*

$$\underline{\mathfrak{A}}\left(\mathbb{K}^{\triangle}_{\triangledown}(\underline{\mathfrak{A}}(\mathbb{K}))\right) \cong \underline{\mathfrak{A}}(\mathbb{K}).$$

Proof. Observe that $\underline{\mathfrak{A}}(\mathbb{K})$ is a (weakly) dicomplemented lattice represented by the context $\mathbb{K} := (G, M, I)$. By Proposition 1.2.6 the set $G \cup H$ represents the weak complementation ${}^{\triangle_{\mathbb{K}}}$ where H is the set of ${}^{\triangle}$-compatible elements of $\underline{\mathfrak{A}}(\mathbb{K})$. Dually $M \cup N$ represents ${}^{\triangledown_{\mathbb{K}}}$ where N is the set of ${}^{\triangledown}$-compatible elements of $\underline{\mathfrak{A}}(\mathbb{K})$. This means that the canonical context of $\underline{\mathfrak{A}}(\mathbb{K})$ represents $({}^{\triangle_{\mathbb{K}}}, {}^{\triangledown_{\mathbb{K}}})$ and the isomorphism is obtained. □

Corollary 2.2.6. *If $\underline{\mathfrak{A}}(\mathbb{K}_1)$ and $\underline{\mathfrak{A}}(\mathbb{K}_2)$ are isomorphic concept algebras then there are isomorphic contexts $\mathbb{K}^1$ and $\mathbb{K}^2$ such that*

$$\mathbb{K}_1 \leq \mathbb{K}^1, \quad \mathbb{K}_2 \leq \mathbb{K}^2 \quad \textit{and} \quad \underline{\mathfrak{A}}(\mathbb{K}_1) \cong \underline{\mathfrak{A}}(\mathbb{K}^1) \cong \underline{\mathfrak{A}}(\mathbb{K}^2) \cong \underline{\mathfrak{A}}(\mathbb{K}_2).$$

Proof. We assume that $\mathbb{K}_1$ and $\mathbb{K}_2$ are clarified contexts.

$$\underline{\mathfrak{A}}(\mathbb{K}_1) \cong \underline{\mathfrak{A}}(\mathbb{K}_2) \implies \mathbb{K}_{\triangledown}^{\triangle}(\underline{\mathfrak{A}}(\mathbb{K}_1)) \cong \mathbb{K}_{\triangledown}^{\triangle}(\underline{\mathfrak{A}}(\mathbb{K}_2))$$

Take $\mathbb{K}^1 := \mathbb{K}_{\triangledown}^{\triangle}(\underline{\mathfrak{A}}(\mathbb{K}_1))$ and $\mathbb{K}^2 := \mathbb{K}_{\triangledown}^{\triangle}(\underline{\mathfrak{A}}(\mathbb{K}_2))$. By Theorem 2.2.5 we get the result. □

2.3 Congruence Theory

2.3.1 Congruences of concept algebras

Concept algebras are of course concept lattices. Each concept algebra congruence is by then a concept lattice congruence with some additional properties.

Definition 2.3.1. A **complete congruence** relation on a complete lattice L is an equivalence relation θ on L such that $x_t \,\theta\, y_t$ for all $t \in T$ implies

$$\bigwedge_{t\in T} x_t \,\theta \bigwedge_{t\in T} y_t \text{ and } \bigvee_{t\in T} x_t \,\theta \bigvee_{t\in T} y_t.$$

Note that $x \,\theta\, y$ if and only if $x \wedge y \,\theta\, x \vee y$. The congruence classes are intervals of L. For an element $x \in L$, we denoted by $[x]\theta$ its congruence class. We denote by x_θ the least element of $[x]\theta$ and by x^θ its greatest element. Thus $[x]\theta$ is the interval $[x_\theta, x^\theta]$. The **factor lattice** $L/_\theta$ is a complete lattice with respect to the order relation defined by:

$$[x]\theta \leq [y]\theta :\iff x\,\theta(x \wedge y).$$

The following proposition gives a characterization of complete congruence relations.

Proposition 2.3.1. *[GW99, pp. 106-107] An equivalence relation θ on a complete lattice L is a complete congruence relation if and only if every equivalence class of θ is an interval of L, the lower bounds of these intervals being closed under suprema and the upper bounds being closed under infima.*

Concept lattice congruences are described by **compatible subcontexts**. For a formal context (G, M, I), a subcontext $(H, N, I \cap H \times N)$, usually denoted by (H, N), is said to be compatible if for all (A, B) in $\mathfrak{B}(G, M, I)$, the pair $(A \cap H, B \cap N)$ is a formal concept of (H, N). The compatible subcontexts are characterized by their induced **projections**.

Proposition 2.3.2. *[GW99, p. 100] The subcontext (H, N) of (G, M, I) is compatible if and only if the mapping*

$$\begin{array}{llll} \Pi_{H,N}: & \mathfrak{B}(G,M,I) & \to & \mathfrak{B}(H,N) \\ & (A,B) & \mapsto & (A \cap H, B \cap N) \end{array}$$

is a surjective complete homomorphism.

The kernel of $\Pi_{H,N}$ is a complete congruence of $\underline{\mathfrak{B}}(G,M,I)$. We denote it by $\theta_{H,N}$. We get

$$\underline{\mathfrak{B}}(H,N) \cong \underline{\mathfrak{B}}(G,M,I)/_{\theta_{H,N}}$$

with

$$(A_1,B_1)\,\theta_{\mathrm{H,N}}(A_2,B_2) \iff A_1\cap H = A_2\cap H \iff B_1\cap N = B_2\cap N.$$

The bounds of congruence classes can be easily identified. In fact for a concept (A,B), the smallest element of $[(A,B)]\theta_{H,N}$ is the concept $((A\cap H)'',(A\cap H)')$ and the greatest element is $((B\cap N)',(B\cap N)'')$. A **complete congruence** θ is said to be **induced by a subcontext** if there is a compatible subcontext (H,N) such that $\theta=\theta_{H,N}$. In the case of doubly founded concept lattice every complete congruence is induced by a subcontext. If in addition the context is reduced then this subcontext is uniquely determined by the congruence.

Definition 2.3.2. A concept lattice congruence θ of $\underline{\mathfrak{B}}(G,M,I)$ is said to be $^{\triangle}$**-compatible** (resp. $^{\triangledown}$**-compatible**) if for all concepts x and y in $\mathfrak{B}(G,M,I)$, $x\,\theta\,y \implies x^{\triangle}\,\theta\,y^{\triangle}$ (resp. $x\,\theta\,y \implies x^{\triangledown}\,\theta\,y^{\triangledown}$). A **concept algebra congruence** is a $^{\triangle}$-compatible and $^{\triangledown}$-compatible concept lattice congruence.

If θ is a congruence of the concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ then θ is a congruence of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ and therefore corresponds to a compatible subcontext of $\mathbb{K}$. Which of these subcontexts enable the preservation of the unary operations? We are going to examine under which conditions a congruence induced by a compatible subcontext preserves the operation $^{\triangle}$ and dualize to get the result for the operation $^{\triangledown}$. We write $m\,\underline{\perp}\,n$ to mean that $m_0\perp n$ for all $m_0\in m''\cap N$, where $m_0\perp n$ stands for $m_0'\cup n'=G$.

Theorem 2.3.3. *The lattice congruence induced by a compatible subcontext* (H,N) *is* $^{\triangle}$*-compatible if and only if*

$$\forall_{m\in M}\,\forall_{n\in N}\quad m\,\underline{\perp}\,n \implies m\perp n. \qquad (*)$$

Proof. ($\Leftarrow$) We assume that the condition $(*)$ holds. We prove that if x and y are concepts such that $x\,\theta_{\mathrm{H,N}}\,y$ then automatically $x^{\triangle}\,\theta_{\mathrm{H,N}}\,y^{\triangle}$. Since $x\,\theta_{\mathrm{H,N}}\,y$ is equivalent to $(x\wedge y)\,\theta_{\mathrm{H,N}}(x\vee y)$, it is enough to prove the assertion only for $x\leq y$. We can even restrict to pairs (x,y) such that x is minimal and y is maximal in their congruence class. Recall that $x\,\theta_{\mathrm{H,N}}\,y$ means

$$ext(x)\cap H = ext(y)\cap H =: A \text{ and } int(x)\cap N = int(y)\cap N =: B,$$

where $ext(x)$ denotes the extent of the concept x and $int(x)$ its intent. As we assume x to be minimal and y maximal, we have $x=(A'',A')$ and $y=(B',B'')$. Reformulating the problem, we have to prove that

$$\Big((G\setminus A'')'',(G\setminus A'')'\Big)\,\theta_{\mathrm{H,N}}\,\Big((G\setminus B')'',(G\setminus B')'\Big).$$

This is equivalent to the equality

$$(G \setminus A'')' \cap N = (G \setminus B')' \cap N.$$

The inclusion

$$(G \setminus A'')' \cap N \subseteq (G \setminus B')' \cap N$$

is immediate since A'' is a subset of B'. Note that for all $n \in N$,

$$n \in (G \setminus B')' \iff n'' \subseteq (G \setminus B')' \iff G \setminus B' \subseteq n' \iff G \setminus n' \subseteq B'.$$

Therefore it suffices to show that

$$\forall_{n \in N} \quad [G \setminus n' \subseteq B' \implies G \setminus n' \subseteq A''] .$$

We know that $B = A' \cap N = \{n \in N \mid A \subseteq n'\}$. To get the above assertion we need to demonstrate that

$$\forall_{n \in N} \quad \left[G \setminus n' \subseteq \bigcap_{m_0 \in N,\, A \subseteq m_0'} m_0' \implies G \setminus n' \subseteq \bigcap_{m \in M,\, A \subseteq m'} m' \right] .$$

i.e.

$$\forall_{n \in N} \; [m_0 \perp n \quad \forall_{m_0 \in N} \text{ with } A \subseteq m_0' \implies m \perp n \quad \forall_{m \in M} \text{ with } A \subseteq m'] .$$

Equivalently we do prove that for all $n \in N$ the assertion

$$\exists_{m \in M} \text{ such that } A \subseteq m' \text{ and } m \not\perp n$$

implies

$$\exists_{m_0 \in N} \text{ such that } A \subseteq m_0' \text{ and } m_0 \not\perp n.$$

If this implication were not true for a certain n in N there would exist an attribute m with $A \subseteq m'$ and $m \not\perp n$ such that for any attribute $m_0 \in N$ with $A \subseteq m_0'$, we have $m_0 \perp n$. All attributes from $m'' \cap N$ belong particularly[5] to these attributes. Therefore $m_0 \perp n$ for all $m_0 \in m'' \cap N$. This is exactly $m \underline{\perp} n$. From $(*)$ we would get $m \perp n$, which would be a contradiction. Thus $x^{\triangle} \, \theta_{\mathrm{H,N}} \, y^{\triangle}$. Since x and y were arbitrary chosen we obtain that $\theta_{\mathrm{H,N}}$ is ${}^{\triangle}$-compatible.

($\Rightarrow$) For the converse we assume that $\theta_{H,N}$ is ${}^{\triangle}$-compatible and want to prove the condition $(*)$. We consider $m \in M$ and $n \in N$ with $m \underline{\perp} n$. When do we have $m \perp n$? The congruence class $[(m', m'')]_{\theta_{\mathrm{H,N}}}$ of the attribute concept (m', m'') is the interval

$$[((m' \cap H)'', (m' \cap H')) , ((m'' \cap N)', (m'' \cap N)'')] .$$

From the ${}^{\triangle}$-compatibility of $\theta_{H,N}$ it follows that

$$((m' \cap H)'', (m' \cap H)')^{\triangle} \, \theta_{\mathrm{H,N}} ((m'' \cap N)', (m'' \cap N)'')^{\triangle}.$$

[5] $m_0 \in m'' \cap N \implies m_0' \supseteq m' \supseteq A$.

i.e.

$$((G\setminus(m'\cap H)'')'',(G\setminus(m'\cap H)'')')\,\theta_{\mathrm{H,N}}((G\setminus(m''\cap N)')'',(G\setminus(m''\cap N)')').$$

Thus

$$(G\setminus(m'\cap H)'')'\cap N=(G\setminus(m''\cap N)')'\cap N.$$

This is equivalent to

$$\forall_{n\in N}\quad G\setminus n'\subseteq(m'\cap H)''\iff G\setminus n'\subseteq(m''\cap N)'$$

which is the same as

$$\forall_{n\in N}\quad G\setminus n'\subseteq(m''\cap N)'\implies G\setminus n'\subseteq(m'\cap H)''$$

since $m'\cap H\subseteq(m''\cap N)'$. From $m\,\underline{\perp}\,n$ we get

$$\forall_{m_0\in N}\quad m_0'\supseteq m'\implies m_0\perp n$$

and furthermore

$$(m''\cap N)'=\bigcap_{m_0\in m''\cap N}m_0'\supseteq G\setminus n'.$$

If $m\not\perp n$ then $G\setminus n'\not\subseteq m'$. But $m'\supseteq(m'\cap H)''$; it follows that $G\setminus n'\not\subseteq(m'\cap H)''$. This is a contradiction since $G\setminus n'\subseteq(m''\cap N)'$ implies $G\setminus n'\subseteq(m'\cap H)''$. This achieves the proof. □

Corollary 2.3.4. *A compatible subcontext* (H,N) *of* (G,M,I) *induces a concept algebra congruence if and only if the conditions* (*i*) *and* (*ii*) *below hold.*

(i) $\forall_{m\in M}\forall_{n\in N}\quad m\,\underline{\perp}\,n\implies m\perp n.$

(ii) $\forall_{g\in G}\forall_{h\in H}\quad g\,\underline{\perp}\,h\implies g\perp h.$[6]

From Proposition 2.3.2 compatible subcontexts correspond to projections that are surjective homomorphisms. Another way to look for concept algebra congruences is to examine compatible subcontexts (H,N) for which the projection $\Pi_{H,N}$ preserves the unary operations. We denote by ^{j} the derivation in the subcontext (H,N). Let $x=(A'',A')$ be a concept of (G,M,I).

$$\Pi_{H,N}(x^{\triangle})=((G\setminus A'')''\cap H,(G\setminus A'')'\cap N)$$

and

$$\Pi_{H,N}(x)^{\triangle}=((H\setminus A'')^{jj},(H\setminus A'')^{j}).$$

Thus $\Pi_{H,N}(x^{\triangle})=\Pi_{H,N}(x)^{\triangle}$ if and only if

[6] $g\perp h:\iff g'\cup h'=M$ and $g\,\underline{\perp}\,h:\iff g_0\perp h\quad\forall g_0\in g''\cap H$

$$(G \setminus A'')' \cap N = (H \setminus A'')^j.$$

This means that for all $n \in N$

$$\left[G \setminus A'' \subseteq n' \iff H \setminus A'' \subseteq n^j\right].$$

Thus $\Pi_{H,N}(x^{\triangle}) = \Pi_{H,N}(x)^{\triangle}$ if and only if for all $n \in N$

$$\left[G \setminus n' \subseteq A'' \iff H \setminus n^j \subseteq A''\right].$$

The above equivalence can be rewritten as

$$\left[G \setminus n' \subseteq \bigcap_{A \subseteq m'} m' \iff H \setminus n^j \subseteq \bigcap_{A \subseteq m'} m'\right].$$

$$i.e \qquad \forall_{m \in M} \quad A \subseteq m',\, n' \cup m' = G \iff n^j \cup m' \supseteq H.$$

Since x was taken arbitrarily in $\mathfrak{B}(G, M, I)$, we obtain the equality $\Pi_{H,N}(x^{\triangle}) = \Pi_{H,N}(x)^{\triangle}$ if and only if for all subset A of G, for all $n \in N$ and for all $m \in M$ with $A \subseteq m'$, the equivalence

$$n \perp_G m \iff n \perp_H m$$

holds. This is equivalent to

$$\forall_{n \in N} \forall_{m \in M} \quad (n \perp_G m \iff n \perp_H m).$$

Thus

Theorem 2.3.5. *A compatible subcontext (H, N) of (G, M, I) induces a congruence of the concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ if and only if the following assertions are valid:*

(i) $\forall_{n \in N} \forall_{m \in M} \quad n \perp_G m \iff n \perp_H m,$

(ii) $\forall_{h \in H} \forall_{g \in G} \quad h \perp_M g \iff h \perp_N g.$

We denote by M_{irr} the set of irreducible attributes of a context (G, M, I). The test of compatibility of subcontexts can just be done on the irreducible elements, as we can see in the next proposition.

Proposition 2.3.6. *The following assertions are equivalent:*

(i) $\forall m \in M, \forall n \in N, n \perp_G m \iff n \perp_H m.$

(ii) $\forall m \in M_{irr}, \forall n \in N \cap M_{irr}, n \perp_G m \iff n \perp_H m.$

Proof. The implication $(i) \implies (ii)$ is obviously true. We are going to prove $(ii) \implies (i)$. We assume that (ii) holds. We need only to prove that for $m \in M$ and $n \in N$, $m \perp_H n \implies m \perp_G n$ since the reverse implication is trivial. Let $m \in M$ and $n \in N$ such that $m \perp_H n$. We want to prove that $m \perp_G n$. If m and n are irreducible then we are done. Else we get

$$m' = \bigcap_{i=0}^{k} m_i' \quad \text{and} \quad n' = \bigcap_{s=0}^{l} n_s' \qquad \text{for } 0 \leq i \leq k \text{ and } 0 \leq s \leq i$$

where m_i and n_s are irreducible. Therefore

$$\begin{aligned}
m \perp_H n \quad &\Longrightarrow \quad (m' \cap H) \cup (n' \cap H) = H \\
&\Longrightarrow \quad (m_i' \cup n_s') \cap H = H \quad \forall (i,j) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Longrightarrow \quad m_i \perp_H n_s \quad \forall (i,j) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Longrightarrow \quad m_i \perp_G n_s \quad \forall (i,j) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Longrightarrow \quad m_i' \cup n_s' = G \quad \forall (i,j) \in \{0,\dots,k\} \times \{0,\dots,l\} \\
&\Longrightarrow \quad \bigcap_{i=0}^{k} m_i' \cup \bigcap_{s=0}^{l} n_s' = G \\
&\Longrightarrow \quad m' \cup n' = G \\
&\Longrightarrow \quad m \perp_G n.
\end{aligned}$$

And (i) is proved. □

This result is a little bit surprising, since the concept algebra structure does not live only on the irreducible elements, but on the $^{\triangle}$-compatible and $^{\triangledown}$-compatible elements. Some of them are from the lattice point of view reducible, but not from the concept algebra point of view.

With our knowledge on congruences let us come back to the problem of homomorphic images of concept algebras. We consider a concept algebra $\underline{\mathfrak{A}}(G, M, I)$, a weakly dicomplemented lattice L and a surjective homomorphism $f : \underline{\mathfrak{A}}(G, M, I) \to L$. The kernel of f is a congruence of $\underline{\mathfrak{A}}(G, M, I)$. We assume that (G, M, I) is doubly founded. Then there exists a compatible subcontext (H, N) of (G, M, I) such that $\theta_{H,N}$ is $\ker f$. Thus (H, N) is a compatible subcontext of $G, M, I)$ and (H, N) induces a concept algebra congruence. By the first isomorphism theorem we obtain that

$$L \cong \underline{\mathfrak{A}}(G, M, I)/_{\ker f} \cong \underline{\mathfrak{A}}(G, M, I)/_{\theta_{H,N}} \cong \underline{\mathfrak{A}}(H, N) \quad \text{.i.e.}$$

Theorem 2.3.7. *Homomorphic images of doubly founded concept algebras are (isomorphic to) concept algebras.*

All finite lattices are complete and doubly founded. Thus the class of (copies of) finite concept algebras is stable under finite products, substructures and homomorphic images. They form a **pseudovariety**. Banaschewski described such classes as "directed unions of equational classes of finite algebras and accordingly the classes of finite algebras defined by filters of sets of equations"[Ba83].

2.3.2 Congruence lattices of concept algebras.

The set $Con\underline{\mathfrak{A}}(\mathbb{K})$ of all complete congruences of a concept algebra $\underline{\mathfrak{A}}(\mathbb{K})$ is a complete sublattice of the lattice of all equivalence relations on $\mathfrak{B}(\mathbb{K})$[7]. It is also a complete sublattice of the congruence lattice $Con\underline{\mathfrak{B}}(\mathbb{K})$ of the concept lattice of $\mathbb{K}$. Thus $Con\underline{\mathfrak{A}}(\mathbb{K})$ is a completely distributive complete lattice. By Birkhoff's theorem there is an ordered set $(P, \leq)$ such that $Con\underline{\mathfrak{A}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{B}}(P, P, \ngeq)$. Finding a good description of the poset $(P, \leq)$ is a problem worthy to be considered, but yet unsolved.

There is another approach which momentarily is more successful, using the characterization of compatible subcontexts by means of arrow relations.

Definition 2.3.3. A subcontext (H, N) of a clarified context (G, M, I) is **arrow-closed** if the following holds: $h \nearrow m$ and $h \in H$ together imply $m \in N$, and dually, $g \swarrow n$ and $n \in N$ together imply $g \in H$.

Proposition 2.3.8. *[GW99, p. 101] Every compatible subcontext is arrow-closed. Every arrow-closed subcontext of a doubly founded context is compatible.*

If the context (G, M, I) is reduced, then arrow-closed subcontexts can be elegantly described in terms of the concepts of a context. For this purpose the transitive closure of the arrow relations is needed.

Definition 2.3.4. For $g \in G$ and $m \in M$ we write $g \mathrel{\swarrow\!\!\!\nearrow} m$ if there are objects $g = g_1, g_2, \ldots, g_k \in G$ and attributes $m_1, m_2, \ldots, m_k = m \in M$ with $g_i \swarrow m_i$ for $i \in \{1, \ldots, k\}$ and $g_j \nearrow m_{j-1}$ for $j \in \{2, \ldots, k\}$. The relation $\mathrel{\swarrow\!\!\!\nearrow}$ is called the transitive closure of the arrow relation and is also denoted by $trans(\swarrow, \nearrow)$. The complement of this relation is denoted by $\not\mathrel{\swarrow\!\!\!\nearrow}$.

Proposition 2.3.9. *[GW99, p. 102] Let (G, M, I) be a reduced doubly founded context. Then (H, N) is an arrow-closed subcontext if and only if $(G \setminus H, N)$ is a concept of the context $(G, M, \not\mathrel{\swarrow\!\!\!\nearrow})$.*

Thus the congruence lattice of $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the concept lattice of the context $(G, M, \not\mathrel{\swarrow\!\!\!\nearrow})$ if (G, M, I) is reduced and doubly founded. This isomorphism exists even if the context is not assumed to be reduced.

We aim to find a similar description for the congruence lattice of concept algebras, restricting our considerations to a reduced finite context (G, M, I). Complete sublattices of concept lattices are described by closed subrelations. The congruence lattice of $\underline{\mathfrak{B}}(\mathbb{K})$ is isomorphic to $\underline{\mathfrak{B}}(G, M, \not\mathrel{\swarrow\!\!\!\nearrow})$. The congruence lattice of $\underline{\mathfrak{A}}(\mathbb{K})$ is a complete sublattice of the congruence lattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Thus there is a closed subrelation $\bowtie$ of $\not\mathrel{\swarrow\!\!\!\nearrow}$ such that $Con\underline{\mathfrak{A}}(\mathbb{K}) \cong \underline{\mathfrak{B}}(G, M, \bowtie)$.

[7] See for example [Ih93, pp. 22-31] for a proof.

Is there a usable characterization of the relation $\bowtie$? When does (g,m) belong to $\bowtie$?

It is enough to find out when (g,m) does not belong to $\bowtie$ (i.e. to consider the complementary relation $\not\bowtie$). Observe that

$$\not\bowtie \ = (\not\bowtie \cap \not\swarrow\!\!\!\nearrow) \cup (\not\bowtie \cap \swarrow\!\!\!\nearrow) = (\not\bowtie \cap \not\swarrow\!\!\!\nearrow) \cup \swarrow\!\!\!\nearrow,$$

since $\bowtie \ \leq \ \not\swarrow\!\!\!\nearrow$ implies that $\not\bowtie \ \supseteq \swarrow\!\!\!\nearrow$. The problem thus reduces to finding $\not\bowtie \cap \not\swarrow\!\!\!\nearrow$. Note that

$$\bowtie = \bigcup_{(A,A')\in\mathfrak{B}(G,M,\bowtie)} A \times A'.$$

i.e. (g,m) is in $\bowtie$ if and only if there exists (A,A') in $\mathfrak{B}(G,M,\bowtie)$ such that $g \in A$ and $m \in A'$. Therefore $(g,m) \notin \bowtie$ if and only if for any concept (A,A') in $\mathfrak{B}(G,M,\bowtie)$, it holds that $g \notin A$ or $m \notin A'$. This is equivalent to $g \in A \implies m \notin A'$ for any concept $(A,A') \in \mathfrak{B}(G,M,\bowtie)$. Thus $(g,m) \notin \bowtie$ if and only if for any compatible subcontext (H,N) which is both $^{\triangle}$- and $^{\triangledown}$-compatible, $g \notin G\backslash H$ or $m \notin N$. In other words, $(g,m) \notin \bowtie$ if and only if for any compatible subcontext (H,N) which is both $^{\triangle}$- and $^{\triangledown}$-compatible, $m \in N \implies g \in H$.

In the doubly founded case, it suffices to consider the irreducible elements:

Remark 2.3.1. Let (G,M,I) be a reduced doubly founded context. Any complete congruence θ of $\underline{\mathfrak{B}}(G,M,I)$ is induced by (H,N) with

$$H = \{g \in G_{irr} \mid \gamma g \not\theta \gamma g_*\}$$

and

$$N = \{m \in M_{irr} \mid \mu m \not\theta \mu m^*\}.$$

Proposition 2.3.10. *Let (G,M,I) be a reduced context, and let $g \in G$ and $m \in M$. Then the following holds:*

$$\gamma g^{\triangle} \leq \mu m \text{ and } \gamma g_*^{\triangle} \not\leq \mu m \implies (g,m) \notin \bowtie \text{ and } (g,m) \notin \mathrm{I}\,.$$

Proof. If $\gamma g^{\triangle} \leq \mu m$ and $g \,\mathrm{I}\, m$ then $\gamma g \leq \mu m$ and $1 = \gamma g^{\triangle} \vee \gamma g \leq \mu m$, which is in contradiction with $\gamma g_*^{\triangle} \not\leq \mu m$. Let (H,N) be a compatible and $\{^{\triangle},^{\triangledown}\}$-compatible subcontext and $\theta_{\mathrm{H,N}}$ be the corresponding congruence. We consider $m \in N$ and want to show that g must be in H.

$$\gamma g \,\theta\, \gamma g_* \implies \gamma g^{\triangle} \,\theta\, \gamma g_*^{\triangle} \implies \gamma g^{\triangle} \vee \mu m \,\theta\, \gamma g_*^{\triangle} \vee \mu m \implies \mu m \,\theta\, \mu m^*.$$

$m \in N$ implies $\mu m \not\theta \mu m^*$ and $\gamma g \not\theta \gamma g_*$. Thus $g \in H$. □

The above proposition gives a sufficient condition for membership in the relation $\not\bowtie$. But it is not necessary. It is possible to have $(g,m) \in \not\bowtie$ and $(g,m) \in \mathrm{I}$. From the proof we also have

$$\gamma g^{\triangledown} \leq \mu m \text{ and } \gamma g_*^{\triangledown} \not\leq \mu m \implies (g,m) \notin \bowtie\ .$$

This leaves us with the open problem to find a necessary and sufficient condition. An answer can be given in the distributive case. Bernhard Ganter gave in [Ga04] a nice description of the congruence lattice of distributive concept algebras (see [Section 4.2]).

2.4 Normal Filters

Tibor Katriňák used the notion of normal filters to switch from congruences of distributive p-algebras to congruences of distributive double p-algebras. The fact that skeletons are Boolean algebras is very helpful. The properties of normal filters needed in his proof are still valid in the case of weakly dicomplemented lattices. These are in Proposition 2.4.3 and Proposition 2.4.4.

2.4.1 Congruences of distributive double p-algebras

Definition 2.4.1. An ideal I of a (distributive) double p-algebra L is called **normal** if $x \in I$ implies $x^{*+} \in I$. Dually a filter F of L is called normal if $x \in F$ implies $x^{+*} \in F$.

Let θ be a congruence of L. $1/\theta$ is a normal filter and $0/\theta$ is a normal ideal. To each normal filter F of L is assigned a pair of congruences

$$(\theta_F, \bar{\theta}_F) \in Con(S(L)) \times Con(\bar{S}(L))$$

as follows

$$x \equiv y(\theta_F) \iff x \wedge v = y \wedge v \text{ for some } v \in F \cap S(L) \text{ in } S(L),$$

$$x \equiv y(\bar{\theta}_F) \iff x \wedge v = y \wedge v \text{ for some } v \in F \cap \bar{S}(L) \text{ in } \bar{S}(L).$$

If F is the congruence class of 1 then θ_F and $\bar{\theta}_F$ are restrictions of θ to $S(L)$ and $\bar{S}(L)$ respectively.

Definition 2.4.2. A **congruence pair** of a p-algebra L is a pair

$$< \theta_1, \theta_2 > \in Con(S(L)) \times Con(D(L))$$

satisfying the condition:

$$x \in S(L),\ y \in D(L),\ y \geq x \text{ and } x \equiv 1(\theta_1) \text{ imply } y \equiv 1(\theta_2).$$

$Con_p(L)$ denotes the set of congruence pairs of L.

For a subset X of L, the restriction of θ on X is denoted by θ_X. For each p-algebra congruence θ, the pair $< \theta_{S(L)}, \theta_{D(L)} >$ is a congruence pair. Lakser gave in [La71] a characterization of p-algebra congruences using congruences of $S(L)$ and $D(L)$. That is

Proposition 2.4.1 (Lakser). *Let*

$$\Phi(L)\colon Con(L) \to Con(S(L)) \times Con(D(L))$$

be defined by

$$\Phi(L)(\theta) := < \theta_{S(L)}, \theta_{D(L)} >$$

where θ_X *denotes the restriction of* θ *on* $X \subseteq L$*. Then* $\Phi(L)$ *determines a lattice isomorphism between* $Con(L)$ *and* $Con_p(L)$*; if* $< \theta_1, \theta_2 > \in Con_p(L)$ *the corresponding* $\theta \in Con(L)$ *is determined by* $x \equiv y(\theta)$ *if and only if*

(i) $x^* \equiv y^*(\theta_1)$ *and*

(ii) $x \vee u \equiv y \vee u(\theta_2)$ *for all* $u \in D(L)$*.*

To extend this result to double p-algebras, Tibor Katriňák called a pair (F, ψ) of $F(L) \times Con(D(L))$ a **filter-congruence pair** if F is a normal filter and the following conditions hold:

$$x^{++} \equiv y^{++}(\bar{\theta}_F) \implies x^{++} \vee d \equiv y^{++} \vee d(\psi) \text{ for all } d \in D(L)$$

and

$$x \equiv y(\psi) \implies x^{++} \equiv y^{++}(\bar{\theta}_F).$$

He then gave a description of congruences of double p-algebras.

Proposition 2.4.2 (T. Katriňák). *For a distributive double p-algebra* L*, every congruence relation* θ *determines a filter-congruence pair* $([1]_\theta, \theta_D)$*. Conversely, every filter-congruence pair* (F, ψ) *uniquely determines a congruence relation* θ *on* L *with* $[1]_\theta = F$ *and* $\theta_D = \psi$ *by the following rule*

$$x \equiv y(\theta) \iff x^* \equiv y^*(\theta_F) \text{ and } x \vee d \equiv y \vee d(\psi) \text{ for all } d \in D(L).$$

The first part of the proof is obvious. For the second part, he started with a filter-congruence pair (F, ψ) of a double p-algebra and got a congruence-pair (θ_F, ψ) of the p-algebra $(L, \wedge, \vee, ^*, 0, 1)$. By means of the result of Lakser he obtained a congruence θ of the p-algebra. He then proved using normality that θ preserves the dual pseudocomplementation operation $^+$. Thus θ is a double p-algebra congruence.

2.4.2 Definition and properties

Definition 2.4.3. Let L be a weakly dicomplemented lattice. An ideal I of L is called **normal** if $x \in I$ implies $x^{\triangledown\vartriangle} \in I$. Dually a filter F of L is called normal if $x \in F$ implies $x^{\vartriangle\triangledown} \in F$.

Of course $1/\theta$ is a normal filter for each congruence of L. Let F be a filter of L; we define $J(F)$ by $J(F) := \{z \in L \mid z \leq x^\triangledown,\ x \in F\}$. Dually we define $F(J)$ for each ideal J of L by $F(J) := \{z \in L \mid z \geq x^\vartriangle,\ x \in J\}$.

Proposition 2.4.3. *Let L be a weakly dicomplemented lattice.*

(i) *For each normal filter F, $J(F)$ is a normal ideal, and $x \in F$ implies $x^{\triangle} \in J(F)$.*

(ii) *For each normal ideal J, $F(J)$ is a normal filter, and $x \in J$ implies $x^{\triangledown} \in F(J)$.*

Proof. The second part of the lemma is obtained as a dual of the first part. Let us prove the first part. Trivially $0 \in J(F)$. Let z and t in $J(F)$; there are x and y in F with $z \leq x^{\triangledown}$ and $t \leq y^{\triangledown}$. Then $z \vee t \leq (x \wedge y)^{\triangledown}$. As $x \wedge y$ is in F, $z \vee t$ would also be in $J(F)$. In addition, $z \leq x^{\triangledown}$ implies $z^{\triangledown\triangle} \leq x^{\triangledown\triangledown\triangle} \leq x^{\triangledown\triangledown\triangle\triangledown\triangledown}$. On the other hand $x \in F$ together with the inequality $x \leq x^{\triangledown\triangledown}$ imply $x^{\triangledown\triangledown} \in F$. Therefore $x^{\triangledown\triangledown\triangle\triangledown}$ belongs to F since F is normal. This proves that $z^{\triangledown\triangle}$ belongs to $J(F)$ whenever z belongs to $J(F)$. Thus $J(F)$ is a normal ideal. Moreover, $x \in F$ implies $x^{\triangle\triangledown} \in F$. Thus $x^{\triangle} \leq x^{\triangle\triangledown\triangledown}$ implies $x^{\triangle} \in J(F)$. □

Proposition 2.4.4. *Let F be a normal filter and J a normal ideal. Then $F = F(J(F))$ and $J = J(F(J))$.*

Proof. For any filter F and any $u \in F(J(F))$, there is $z \in J(F)$ such that $u \geq z^{\triangle}$. Then there exists $x \in F$ such that $z \leq x^{\triangledown}$ and $u \geq z^{\triangle}$; thus $u \geq z^{\triangle} \geq x^{\triangledown\triangle} \geq x$ and u belongs to F.

Let x in a normal filter F; we have $x^{\triangle} \leq x^{\triangle\triangledown\triangledown}$ with $x^{\triangle\triangledown} \in F$. Then $x^{\triangle}$ is an element of $J(F)$. This together with $x \geq x^{\triangle\triangle}$ imply $x \in F(J(F))$. Thus $F(J(F)) = F$. Dually $J = J(F(J))$. □

Remark 2.4.1.

(i) For each filter F and each ideal J, $J(F)$ is an ideal and $F(J)$ is a filter. Are they normal? It seems not to be the case.

(ii) The intersection of normal filters is again a normal filter. Trivially L is a normal filter. Thus each subset, in particular each filter, generates a normal filter, which is the intersection of all normal filters it is contained in. Which relationship exists between the normal filter generated by F and the filter $F(J(F))$ if the later is normal?

(iii) For each ideal J and each filter F of a weakly dicomplemented lattice L we have

$$F \cap J(F) = \emptyset = J \cap F(J) \text{ or } F = L = J$$

Is the pair (F, J) a Galois connection?

Problem 2.4.1. Do normal filters allow to switch from congruences of weakly complemented lattices to congruences of weakly dicomplemented lattices? This problem should be considered together with the construction and characterization problems [Subsection 3.5.2]

3

Representation Results

A challenge in this topic is the representation problem. That is to ***find a set of formulae describing concept algebras***. An approach will be to embed each weakly dicomplemented lattice into a concept algebra. This was the attempt in Theorem 2.2.4. In this case concept algebras would have an equational theory. An alternative would be to find a characterization of representable weak dicomplementations. Our main results are the representation of finite dicomplemented lattices, the description of the lattice of possible weak negations and the characterization of weakly dicomplemented lattices with negation.

3.1 Finite Weakly Dicomplemented Lattices

Recall that the canonical context is the best candidate, up to isomorphism, for a contextual representation of a weak dicomplementation[1]. We restrict the investigation to finite lattices.

Let us now examine the *representation problem* in the case of finite lattices. We start by translating some results of Section 2.2 to finite lattices. To keep the terminology of Definition 2.2.1 we call an element $a \neq 0$ of a weakly dicomplemented lattice L $\vee$-**primary** if $a \not\leq x$ implies $a \leq x^{\triangle}$, and

[1] cf. Proposition 1.2.6

an element $b \neq 1$ $\wedge$-**primary** if $b \not\geq x$ implies $b \geq x^{\triangledown}$. The $\vee$-primary elements are exactly nonzero $^{\triangle}$-compatible elements and $\wedge$-primary elements $^{\triangledown}$-compatible elements not equal to 1. $\mathcal{J}_{pr}(L)$ (resp. $\mathcal{M}_{pr}(L)$) denotes the set of $\vee$-primary (resp. $\wedge$-primary) elements of L.

Proposition 3.1.1. *In a concept algebra, object concepts are $\vee$-primary and attribute concepts are $\wedge$-primary.*

Proof. If $\gamma g \not\leq (A, B)$ in a concept algebra, then $g \notin A$ and therefore $g \in G \setminus A$. This implies

$$\gamma g \leq ((G \setminus A)'', (G \setminus A)') = (A, B)^{\triangle}.$$

The dual statement is obtained similarly. □

Proposition 3.1.2. *A principal filter is $\vee$-primary if and only if it is generated by a $\vee$-primary element. The principal primary ideals are exactly those generated by a $\wedge$-primary element.*

Proof. Let F be a principal primary filter of L. For all $w \in L$, we have $w \in F$ or $w^{\triangle} \in F$. Let a in L with $F = \text{Filter}(\{a\})$. Then

$$\begin{array}{rcll} w \not\geq a & \implies & w \notin F & \\ & \implies & w^{\triangle} \in F & \text{since } F \text{ is a primary filter} \\ & \implies & w^{\triangle} \geq a, & \end{array}$$

and a is therefore $\vee$-primary. Conversely assume a to be a $\vee$-primary element. $F = \text{Filter}(\{a\})$ is a primary filter since

$$\begin{array}{rcll} w \notin F & \implies & w \not\geq a & \\ & \implies & w^{\triangle} \geq a & \text{since } a \text{ is } \vee-\text{primary} \\ & \implies & w^{\triangle} \in F. & \end{array}$$

The proof for primary ideals follows dually. □

In a finite lattice, every primary ideal and every primary filter are principal. We define a property PIP by:

Definition 3.1.1 (PIP). A weakly dicomplemented lattice satisfies the **property PIP**[2] if its primary filters and primary ideals are all principal.

Corollary 3.1.3. *For a weakly dicomplemented lattice L satisfying the property PIP we have:*

$$a \not\leq b \implies \exists t \in \mathcal{J}_{pr}(L), t \leq a \textit{ and } t \not\leq b.$$

$$a \not\geq b \implies \exists t \in \mathcal{M}_{pr}(L), t \geq a \textit{ and } t \not\geq b.$$

Proof. This is immediate from Corollary 2.2.2, assuming PIP. □

[2]PIP:= Primary Implies Principal

Lemma 3.1.4. *For a weakly dicomplemented lattice L satisfying the property PIP, its canonical context $\mathbb{K}_{\triangledown}^{\triangle}(L) := \big(\mathcal{F}_{pr}(L), \mathcal{I}_{pr}(L), \square\big)$ is isomorphic to the context $\mathcal{H}(L) := (\mathcal{J}_{pr}(L), \mathcal{M}_{pr}(L), \leq)$.*

Proof. This follows from the assertions :

$$\mathcal{F}_{pr}(L) = \big\{\text{Filter}(\{a\}) \mid a \in \mathcal{J}_{pr}(L)\big\},$$

$$\mathcal{I}_{pr}(L) = \big\{\text{Ideal}(\{b\}) \mid b \in \mathcal{M}_{pr}(L)\big\}$$

and

$$\begin{aligned} \text{Filter}(\{a\}) \,\square\, \text{Ideal}(\{b\}) &\iff \text{Filter}(\{a\}) \cap \text{Ideal}(\{b\}) \neq \emptyset \\ &\iff a \leq b. \end{aligned}$$

□

Lemma 3.1.5. *For every element x of a weakly dicomplemented lattice L, the following conditions are equivalent:*

(i) $x^{\triangle} = \bigvee\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\}$,

(ii) $\forall y \in L,\ y < x^{\triangle} \implies \exists a \in \mathcal{J}_{pr}(L)$ *such that* $a \nleq x$ *and* $a \nleq y$.

Proof. $x^{\triangle}$ is an upper bound of $\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\}$, since $a \in \mathcal{J}_{pr}(L)$ and $a \nleq x$ together imply $a \leq x^{\triangle}$. Then $x^{\triangle}$ must be the least upper bound of this set because any other upper bound $z \ngeq x^{\triangle}$ would yield an upper bound $z \wedge x^{\triangle} < x^{\triangle}$, which is impossible because *(ii)* implies that no element $y < x^{\triangle}$ can be above all elements of $\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\}$.

For the converse we assume that (i) holds. For any $y < x^{\triangle}$ we have

$$y < \bigvee\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\}$$

and therefore *(ii)*. □

Lemma 3.1.6. *Let I be a principal ideal of a weakly dicomplemented lattice L satisfying PIP and let $x \in L$ be an element satisfying the condition (ii) of Lemma 3.1.5. If neither x nor $x^{\triangle}$ belong to I but $x^{\triangle} \in$ Ideal$(I \cup \{x \wedge x^{\triangle}\})$, then there is a primary filter F with $x \notin F$ and $F \cap I = \emptyset$.*

Proof. The ideal I is principal, so let $I = \text{Ideal}(u)$. If no filter as claimed exists then every primary filter not containing x would have a nonempty intersection with I; i.e., for all $a \in \mathcal{J}_{pr}(L)$, $a \nleq x$ implies $a \leq u$. This would imply

$$x^{\triangle} = \bigvee\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\} \leq u$$

which is impossible since $x^{\triangle}$ does not belong to I. Thus F belongs to $\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x$ and $F \cap I = \emptyset$. □

Under the condition of Lemma 3.1.6, $I \notin (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'$. Thus

$$\mathcal{I}_{x^\triangle} = (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)'.$$

The immediate consequence is a PIP version of the Derivation Lemma.

Lemma 3.1.7. *Let L be a weakly dicomplemented lattice satisfying PIP such that for all $x \in L$ and $y \in L$ the following conditions (ii) and $(ii)'$ hold:*

(ii) *if $y < x^\triangle$ then $a \nleq x,\ a \nleq y$ for some $\vee$-primary element a,*

(ii') *if $y > x^\triangledown$ then $b \ngeq x,\ b \ngeq y$ for some $\wedge$-primary element b.*

The derivation in $\mathbb{K}^{\triangle}_{\triangledown}(L)$ yields, for every $x \in L$,

$$\mathcal{F}'_x = \mathcal{I}_x \quad \text{and} \quad \mathcal{I}_{x^\triangle} = (\mathcal{F}_{pr}(L) \setminus \mathcal{F}_x)',$$

$$\mathcal{I}'_x = \mathcal{F}_x \quad \text{and} \quad \mathcal{F}_{x^\triangledown} = (\mathcal{I}_{pr}(L) \setminus \mathcal{I}_x)'.$$

Theorem 3.1.8. *Let L be a weakly dicomplemented lattice as in Lemma 3.1.7. The map*

$$x \longmapsto i(x) := \left(\left\{a \in \mathcal{J}_{pr}(L) \mid a \leq x\right\}, \left\{c \in \mathcal{M}_{pr}(L) \mid c \geq x\right\}\right)$$

is an embedding from L into the concept algebra $\mathcal{A}\left(\mathcal{J}_{pr}(L), \mathcal{M}_{pr}(L), \leq\right)$.

Proof. Let $u \in \mathcal{M}_{pr}(L)$. Then

$$\begin{aligned} u \in \{a \in \mathcal{J}_{pr}(L) \mid a \leq x\}' \quad &\iff \quad a \leq u, \forall a \in \mathcal{J}_{pr}(L) \text{ with } a \leq x \\ &\iff \quad x \leq u. \end{aligned}$$

(Otherwise there exists $t \in \mathcal{J}_{pr}(L), t \nleq u$ and $t \leq x$.) Therefore

$$\left\{a \in \mathcal{J}_{pr}(L) \mid a \leq x\right\}' = \left\{u \in \mathcal{M}_{pr}(L) \mid u \geq x\right\}$$

Dually $\left\{a \in \mathcal{J}_{pr}(L) \mid a \leq x\right\} = \left\{u \in \mathcal{M}_{pr}(L) \mid u \geq x\right\}'$. So is i a map from L into $\underline{\mathfrak{A}}(\mathcal{H}(L))$. By Theorem 2.2.4, Lemma 3.1.7 and Lemma 3.1.4, i is an embedding. □

Proposition 3.1.9. *Every concept algebra satisfies the conditions (ii) and (ii') of Lemma 3.1.7.*

Proof. Let x be an arbitrary element of a concept algebra $\underline{\mathfrak{A}}(G, M, I)$. By the definition of $x^\triangle$ and by the fact that object concepts are primary, we have

$$x^\triangle = \bigvee\{\gamma g \mid \gamma g \nleq x\} \leq \bigvee\{a \in \mathcal{J}_{pr}(L) \mid a \nleq x\} \leq x^\triangle,$$

which proves condition *(i)* of Lemma 3.1.5 and thereby *(ii)*. Condition *(ii')* is dual. □

Theorem 3.1.10. *A complete weakly dicomplemented lattice L satisfying the property PIP is isomorphic to a concept algebra if and only if it satisfies conditions (ii) and (ii') of Lemma 3.1.7.*

Proof. From Proposition 3.1.9 we know that these conditions are necessary. To see that they are also sufficient, we use Theorem 3.1.8 because PIP is satisfied. It remains to prove that the embedding provided by that theorem,

$$i : L \to \mathcal{A}(\mathcal{J}_{pr}(L), \mathcal{M}_{pr}(L), \leq),$$

is surjective. This follows from the completeness of L.

Let (A, B) be a concept of $(\mathcal{J}_{pr}(L), \mathcal{M}_{pr}(L), \leq)$ and let $x \in L$ be defined by $x := \bigvee A$. Then $A = \{a \in \mathcal{J}_{pr}(L) \mid a \leq x\}$. For any element b of $\mathcal{M}_{pr}(L)$ we have

$$\begin{aligned} b \in B &\iff b \in A' \\ &\iff a \leq b,\ \forall a \in A \\ &\iff \bigvee A \leq b \\ &\iff x \leq b. \end{aligned}$$

Then $B = \{b \in \mathcal{M}_{pr}(L) \mid x \leq b\}$. Therefore $i(x) = (A, B)$ and i is onto. □

As all finite lattices are complete and satisfy PIP an immediate consequence of Theorem 3.1.10 is

Corollary 3.1.11. *A finite weakly dicomplemented lattice L is isomorphic to a concept algebra if and only if it satisfies conditions (ii) and (ii') of Lemma 3.1.7.*

Reformulating this result gives, at least in the finite case, the first characterization of *concept algebras*. To make it self-contained, we replace the notion of "primary" by its definition. Then **finite concept algebras** are (isomorphic to) finite weakly dicomplemented lattices satisfying the following conditions for all x and y:

(ii) $y < x^{\triangle} \implies \exists a \in L$ such that $a \not\leq x$, $a \not\leq y$ and $\forall z \in L$ $a \not\leq z$ implies $a \leq z^{\triangle}$.

(ii)' $y > x^{\triangledown} \implies \exists b \in L$ such that $b \not\geq x$, $b \not\geq y$ and $\forall z \in L$ $b \not\geq z$ implies $b \geq z^{\triangledown}$.

The general infinite case is still open.

3.2 Lattice of Concrete Weak Complementations

In general, we call a **weak dicomplementation representable** if it can be embedded into a concept algebra. We will use the term **concrete** if this embedding is an isomorphism. Representable weak dicomplementations would play for concept algebras the role played by Boolean algebras for powerset algebras. We call a **weak complementation concrete** if it is a weak negation of some context.

Theorem 3.2.1. *The concrete weak complementations on a doubly founded lattice L form a complete sublattice of the complete lattice of all weak complementations on the lattice L.*

Proof. We know from Theorem 1.2.4 that weak dicomplementations on a fixed doubly founded lattice form a complete lattice. It follows that weak complementations on a fixed doubly founded lattice form a complete lattice. The join of two weak complementations, ${}^{\triangle_1}$ and ${}^{\triangle_2}$, computed in the proof of Theorem 1.2.4, is given by

$$x^{\triangle_1 \vee \triangle_2} = x^{\triangle_1} \vee x^{\triangle_2}$$

Now let G_1 and G_2 be two subsets of L representing two weak complementations ${}^{\triangle_1}$ and ${}^{\triangle_2}$ respectively. Without loss of generality we can assume that $G_i\,, i = 1,2$ are the sets of ${}^{\triangle_i}$-compatible elements. We claim that the set $G := G_1 \cup G_2$ represents ${}^{\triangle} := {}^{\triangle_1 \vee \triangle_2}$. Of course G is $\vee$-dense in L. It is easy to verify that all elements of G are ${}^{\triangle}$-compatible. From Proposition 1.2.5 we obtain that ${}^{\triangle_{G_1}} \vee {}^{\triangle_{G_2}}$ is finer than ${}^{\triangle_G}$. It remains to prove the converse. We consider $x \in L$. We shall prove that

$$x^{\triangle} \leq x^{\triangle_1 \vee \triangle_2} = x^{\triangle_1} \vee x^{\triangle_2}. \quad \text{i.e.} \quad (G_1 \setminus \downarrow x)^{I_1} \cap (G_2 \setminus \downarrow x)^{I_2} \subseteq (G \setminus \downarrow x)',$$

where ^{I_i} denotes the derivation in $(G_i, L, \leq)$ $i = 1,2$ and $'$ the derivation in $(G, L, \leq)$.[3] Let $m \in L$ with $m \in (G_1 \setminus \downarrow x)^{I_1} \cap (G_2 \setminus \downarrow x)^{I_2}$. Let $a \in L$ with $a \nleq x$. It holds

$$(a \in G_1 \text{ and } a \nleq x) \quad \text{or} \quad (a \in G_2 \text{ and } a \nleq x).$$

Each of this conjunction implies $a \leq m$. Therefore ${}^{\triangle} \preceq {}^{\triangle_1 \vee \triangle_2}$. This generalizes to arbitrary families. Namely

$$\bigvee_{j \in J} {}^{\triangle_{G_j}} = {}^{\triangle_{\bigcup_{j \in J} G_j}}$$

for an arbitrary set J. Thus concrete weak complementations form a complete sublattice of the complete lattice of all weak complementations on L. □

In the upcoming part we describe this sublattice of concrete weak complementations. We denote by $Ext(\mathbb{K})$ the extents of the context $\mathbb{K} := (G, M, I)$. The context $\mathbb{R} := (Ext(\mathbb{K}), T, \mathcal{R})$ is defined by:

$$U \mathcal{R} \{m, n\} :\iff U \subseteq m' \text{ or } U \subseteq n'.$$

[3] A weak negation does not depend on the set of attributes [Proposition 1.2.5].

We denote by U^c the closure of U in $\mathbb{R}$. Note that

$$U^c = \{W \in Ext(\mathbb{K}) \mid U\mathcal{R}\{m,n\} \implies W\mathcal{R}\{m,n\} \text{ for all } \{m,n\} \in T\}$$

The relation I of $\mathbb{K}$ is extended by I_e on $(G \cup \{U\}, M, I_e)$ as follows:

$$I_e \cap G \times M = I, \quad UI_em :\iff U \subseteq m'.$$

To avoid confusion we will denote by $(-)^{\mathcal{R}}$ the derivation operation in $\mathbb{R}$. Recall that an extent U is said $^{\triangle}$-compatible iff $U \subseteq A$ or $U \subseteq \bar{A}''$ for all extent A, and that the weak negation of an extended context coincides with the old weak negation if and only if all new objects are $^{\triangle}$-compatible in the old context. Here is a characterization of the compatibility by means of the $\perp$-*relation*.

Lemma 3.2.2. *An extent U of (G,M,I) is $^{\triangle}$-incompatible if and only if there are attributes m and n with $m \perp n$, $U \nsubseteq m'$ and $U \nsubseteq n'$.*

Proof. Suppose $m \perp n$, $U \nsubseteq m'$, $U \nsubseteq n'$ and let $A := m'$. Then $U \nsubseteq A$ and, since $\bar{A}'' \subseteq n'$, also $U \nsubseteq \bar{A}''$. Conversely suppose $U \nsubseteq A$ and $U \nsubseteq \bar{A}''$. There are some $m \in A'$ with $U \nsubseteq m'$ and some $n \in \bar{A}'$ with $U \nsubseteq n'$, and $m' \cup n' \supseteq A \cup \bar{A} = G$. □

Lemma 3.2.3. $^{\triangle(G,M,I)} = ^{\triangle(G\cup\{U\},M,I_e)}$ *if and only if $U \in T^{\mathcal{R}}$.*

Proof. The weak negation of the new context $(G \cup \{U\}, M, I_e)$ equals the weak negation of the old context (G,M,I) if and only if U is $^{\triangle}$-compatible in the last context. This means that $U \subseteq m'$ or $U \subseteq n'$ for all $\{m,n\} \in T$, which is equivalent $U\mathcal{R}\{m,n\}$ for all $\{m,n\} \in T$. i.e. $U \in T^{\mathcal{R}}$. □

Such an U can always be reduced in $\mathbb{R}$.

Lemma 3.2.4. *For any family of extents $\mathcal{H} \subseteq Ext(\mathbb{K})$, the following equality holds:*

$$^{\triangle(G\cup\mathcal{H},M,I_e)} = ^{\triangle(G\cup\mathcal{H}^c,M,I_e)}$$

Proof. We denote by $\perp_{\mathcal{H}}$ the corresponding $\perp$-*relation* in $(G \cup \mathcal{H}, M, I_e)$ and by $T_{\mathcal{H}}$ its quotient according to symmetry. For $m,n \in M$ note that

$$\begin{aligned}
\{m,n\} \in \perp_{\mathcal{H}} \quad &\iff \quad m^{\mathcal{R}} \cup n^{\mathcal{R}} = G \cup \mathcal{H} \\
&\iff \quad m' \cup n' = G \text{ and } \mathcal{H} \subseteq m^{\mathcal{R}} \cup n^{\mathcal{R}} \\
&\iff \quad m \perp n \text{ and for all } W \in \mathcal{H},\ W \in m^{\mathcal{R}} \text{ or } W \in n^{\mathcal{R}} \\
&\iff \quad \{m,n\} \in T \text{ and for all } W \in \mathcal{H} \\
&\qquad\quad W \subseteq m^{\mathcal{R}} \text{ or } W \subseteq n^{\mathcal{R}} \\
&\iff \quad \{m,n\} \in T \text{ and for all } W \in \mathcal{H},\ W \subseteq m' \text{ or } W \subseteq n'
\end{aligned}$$

It is enough to prove that each $W \in \mathcal{H}^c$ is ${}^{\triangle}$-compatible in $(G \cup \mathcal{H}, M, I_e)$. We take an extent W and $\{m,n\} \in T_{\mathcal{H}}$.

$$\begin{aligned}
W \in \mathcal{H}^c \quad &\iff \quad W\mathcal{R}\{m,n\} \text{ for all } \{m,n\} \in \mathcal{H}^{\mathcal{R}} \\
&\iff \quad \text{For all } \{m,n\} \in T, \\
&\qquad\quad (U\mathcal{R}\{m,n\} \text{ for all } U \in \mathcal{H} \implies W\mathcal{R}\{m,n\}) \\
&\iff \quad m' \cup n' = G \text{ and } (U \subseteq m' \text{ or } U \subseteq n' \; \forall U \in \mathcal{H} \\
&\qquad\quad \text{implies } W \subseteq m' \text{ or } W \subseteq n') \\
&\iff \quad (m^{I_e} \cup n^{I_e} = G \cup \mathcal{H} \implies W \subseteq m' \text{ or } W \subseteq n') \\
&\iff \quad (\{m,n\} \in T_{\mathcal{H}} \implies W \subseteq m^{I_e} \text{ or } W \subseteq n^{I_e}) \\
&\iff \quad \text{For all } \{m,n\} \in T_{\mathcal{H}} \; W\mathcal{R}_{I_e}\{m,n\} \\
&\iff \quad W \in T_{\mathcal{H}}^{\mathcal{R}_{I_e}}
\end{aligned}$$

Thus $W \in \mathcal{H}^c$ if and only if W is compatible in $(G \cup \mathcal{H}, M, I_e)$. □

In particular ${}^{\triangle(G\cup\{U\},M,I_e)} = {}^{\triangle(G\cup\{U\}^c,M,I_e)}$ for all extent U. While proving Lemma 3.2.4 we have also shown that

Corollary 3.2.5. *For any family $\mathcal{H}$ of extents, the closure $\mathcal{H}^c$ of $\mathcal{H}$ in the context $(Ext(\mathbb{K}), T, \mathcal{R})$ is exactly the set of ${}^{\triangle(G\cup\mathcal{H},M,I_e)}$-compatible extents.*

Lemma 3.2.6. *Let L be a lattice with two weak negations ${}^{\triangle_1}$ and ${}^{\triangle_2}$. If ${}^{\triangle_2} \leq {}^{\triangle_1}$ then every ${}^{\triangle_1}$-compatible element is also ${}^{\triangle_2}$-compatible.*

Proof. Let u be a ${}^{\triangle_1}$-compatible element. For all $x \in L$, $u \leq x$ or $u \leq x^{\triangle_1}$. This implies $u \leq x$ or $u \leq x^{\triangle_2}$ since $x^{\triangle_1} \leq x^{\triangle_2}$. Thus u is ${}^{\triangle_2}$-compatible. □

From now on, we can prove the following result

Theorem 3.2.7. *Let $\mathbb{K}$ be a reduced context. The concept lattice of the context $(Ext(\mathbb{K}), T, \mathcal{R})$ is isomorphic to the lattice of concrete weak complementations on the concept lattice of $\mathbb{K}$.*

Proof. We denote by $Wn(L)$ the set of representable weak complementations on a lattice L. By Lemma 3.2.4 the map

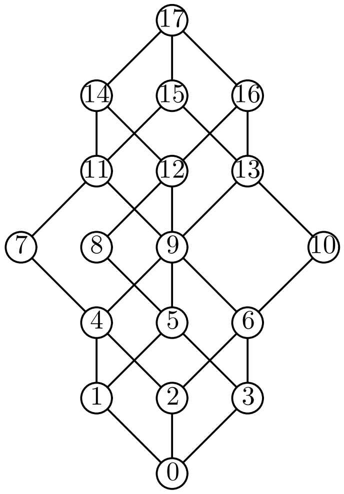
$$\psi : (\mathcal{H}, \mathcal{H}^{\mathcal{R}}) \mapsto {}^{\triangle(G\cup\mathcal{H},M,I_e)}$$

is an increasing map from $\mathfrak{B}(Ext(\mathbb{K}), T, \mathcal{R})$ to $Wn(\mathfrak{B}(\mathbb{K}))$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two families of extents of $\mathbb{K}$. We assume that

$${}^{\triangle(G\cup\mathcal{H}_1,M,I_e)} = {}^{\triangle(G\cup\mathcal{H}_2,M,I_e)}; \quad \text{then} \quad {}^{\triangle_1} := {}^{\triangle(G\cup\mathcal{H}_1^c,M,I_e)} = {}^{\triangle(G\cup\mathcal{H}_2^c,M,I_e)} =: {}^{\triangle_2}.$$

For each extent W of $\mathbb{K}$, by Corollary 3.2.5, W belongs to $\mathcal{H}_1^c$ means that W is ${}^{\triangle_1}$-compatible, which is also equivalent to say that W is ${}^{\triangle_2}$-compatible since ${}^{\triangle_2}$ equals ${}^{\triangle_1}$; Using again Corollary 3.2.5 it means that W belongs to $\mathcal{H}_2^c$. Thus the map

$$\phi\colon {}^{\triangle(G\cup\mathcal{H},M,I_e)} \mapsto (\mathcal{H}^c, \mathcal{H}^{c\mathcal{R}})$$



	14	15	16	7	8	10
1	×	×	×	×	×	
2	×	×	×	×		×
3	×	×	×		×	×
7	×	×		×		
8	×		×		×	
10		×	×			×

Figure 3.1. Free distributive lattice generated by 3 elements and the corresponding reduced context

is well defined from $Wn(\mathfrak{B}(\mathbb{K}))$ to $\mathfrak{B}(Ext(\mathbb{K}), T, \mathcal{R})$. $\psi \circ \phi$ and $\phi \circ \psi$ are identity maps. Thus ψ and ϕ are bijections and inverse each other. To achieve the proof it remains to show that ϕ is also increasing. For, assume that

$$^{\triangle_1} :=^{\triangle_{(G\cup\mathcal{H}_1,M,I_e)}} \leq \ ^{\triangle_{(G\cup\mathcal{H}_2,M,I_e)}} =:^{\triangle_2} .$$

By Lemma 3.2.6 the set $\mathcal{H}_2^c$ of $^{\triangle_2}$-compatible extents contains the set $\mathcal{H}_1^c$ of $^{\triangle_1}$-compatible extents. Therefore $^{\triangle_{(G\cup\mathcal{H}_1,M,I_e)}} \leq \ ^{\triangle_{(G\cup\mathcal{H}_2,M,I_e)}}$ implies $\mathcal{H}_1^c \subseteq \mathcal{H}_2^c$ and ϕ is increasing. Thus ϕ and ψ are order preserving bijections, inverse each other and are, by Lemma 0.2.1, lattice isomorphisms □

Let us look at an example.

Example 3.2.1. We consider the free distributive lattice generated by 3 elements. Its reduced context has the attribute set $M := \{7, 8, 10, 14, 15, 16\}$ and the object set $J := \{1, 2, 3, 7, 8, 10\}$. (See Figure 3.1).

The set of orthogonal pairs of attributes is given by

$$T = \{\{14, 15\}, \{14, 16\}, \{14, 10\}, \{15, 16\}, \{15, 8\}, \{16, 7\}\}.$$

The context $\mathbb{R}$ is given on Figure 3.2. All objects from 0 to 10, and the object 17 are reducible. The resulting context is a copy of the context on Figure 3.1. Thus the lattice of concrete weak complementations on the free lattice generated by 3 element is isomorphic to this lattice.

In the next chapter we show that some properties of the initial lattice can be carried over. Now we are interested by the weakly dicomplemented lattices for which the unary operations coincide. Recall that doubly founded

	$\{14,15\}$	$\{14,16\}$	$\{14,10\}$	$\{15,16\}$	$\{15,8\}$	$\{16,7\}$
0	×	×	×	×	×	×
1	×	×	×	×	×	×
2	×	×	×	×	×	×
3	×	×	×	×	×	×
4	×	×	×	×	×	×
5	×	×	×	×	×	×
6	×	×	×	×	×	×
7	×	×	×	×	×	×
8	×	×	×	×	×	×
9	×	×	×	×	×	×
10	×	×	×	×	×	×
11	×	×	×	×	×	
12	×	×	×	×		×
13	×	×		×	×	×
14	×	×	×			
15	×			×	×	
16		×		×		×
17						

Figure 3.2. Context of all weak negations on the free distributive lattice generated by 3 elements

concept algebras for which the weak negation and the weak opposition coincide are "Boolean algebras".

3.3 Weakly Dicomplemented Lattices with Negation

Example 1.1.1 states that duplicating the complementation of a Boolean algebra leads to a weakly dicomplemented lattice. Does the converse hold? The finite case is easily obtained [cf. Corollary 3.3.2]. The general case is far from obvious.

Definition 3.3.1. A weakly dicomplemented lattice is said to be **with negation** if the unary operations coincide, i.e., if $x^{\bigtriangledown} = x^{\bigtriangleup}$ for all x. In this case we set $x^{\bigtriangleup} =: x' := x^{\bigtriangledown}$.

Proposition 3.3.1. *A weakly dicomplemented lattice with negation is uniquely complemented.*

Proof. $x^{\bigtriangleup\bigtriangleup} \leq x \leq x^{\bigtriangledown\bigtriangledown}$ implies that $x = x''$. Moreover, $x \wedge x' = 0$ and x' is a complement of x. If y is another complement of x then

$$x = (x \wedge y) \vee (x \wedge y') = x \wedge y' \implies x \leq y'$$

$$x = (x \vee y) \wedge (x \vee y') = x \vee y' \implies x \geq y'$$

Then $y' = x$ and $x' = y$. L is therefore a uniquely complemented lattice. □

It can be easily seen that each uniquely complemented atomic lattice is a copy of the power set of the set of its atoms, and therefore distributive. Thus

Corollary 3.3.2. *The finite weakly dicomplemented lattices with negation are exactly the finite Boolean algebras.*

Of course, the natural question will be if the converse of Proposition 3.3.1 holds. That is, can any uniquely complemented lattice be endowed with a structure of weakly dicomplemented lattice with negation? The answer is yes for distributive lattices. If the assertion of Corollary 3.3.2 can be extended to lattices in general (and we do), the answer will unfortunately be no. In fact R. P. Dilworth proved that each lattice can be embedded into a uniquely complemented lattice. The immediate consequence is the existence of non distributive uniquely complemented lattices. They are however infinite If a uniquely complemented lattice could be endowed with a structure of weakly dicomplemented lattice, it would be distributive. This can not be true for non distributive uniquely complemented lattices.

We are going to extend Corollary 3.3.2 on infinite lattices. The first proof is more conceptual and uses subdirect decomposition. The idea is to prove that only the one and two element weakly dicomplemented lattices are indecomposable. The variety they generate will be the variety of distributive lattices with complementation. i.e. the variety of Boolean algebras.[4]

$\underline{L}$ denotes a weakly dicomplemented lattice with negation.

Lemma 3.3.3. *If $c \in L \setminus \{0,1\}$ then the intervals $[c,1]$ and $[0,c']$ are isomorphic sublattices.*

Proof. The mapping $f_c\colon x \mapsto x \wedge c'$ preserves trivially the join operation. Thus f_c is an increasing mapping from $[c,1]$ into $[0,c']$. If $f(x) = f(y)$ then $x \wedge c' = y \wedge c'$. Thus

$$x = (x \wedge c) \vee (x \wedge c') = c \vee (y \wedge c') = (y \wedge c) \vee (y \wedge c') = y$$

and f_c is injective. Let $y \in [0,c']$. $y \vee c \in [c,1]$ and

$$f_c(y \vee c) = (y \vee c) \wedge c' = (y \vee c) \wedge (y \vee c') = y.$$

Thus f_c is an increasing bijection from $[c,1]$ onto $[0,c']$. Similarly the mapping $g_c\colon x \mapsto x \vee c$ is an increasing bijection from $[0,c']$ onto $[c,1]$.

$$\forall x \in [0,c'] \quad f_c(g_c(x)) = (x \vee c) \wedge c' = (x \vee c) \wedge (x \vee c') = x$$

[4] There is another proof using term manipulations. [cf. Proposition 3.3.7]

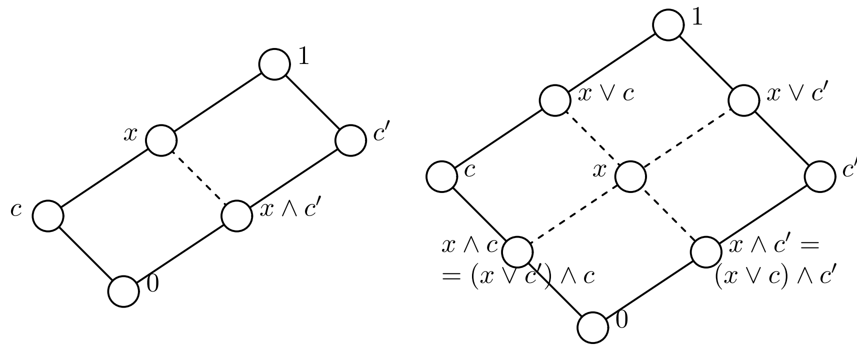


Figure 3.3. Decomposition of weakly dicomplemented lattices with negation

and

$$\forall x \in [c,1] \quad g_c(f_c(x)) = (x \wedge c') \vee c = (x \wedge c') \vee (x \wedge c) = x.$$

Thus f_c and g_c are inverse each other. i.e. f_c is and order isomorphism. By Lemma 0.2.1 f_c is a lattice isomorphism. □

Theorem 3.3.4. *The two element weakly dicomplemented lattice is the unique non-trivial subdirectly irreducible algebra in the variety of weakly dicomplemented lattices with negation.*

Proof. We assume that L contains more than two elements. We are going to prove that L is directly decomposable. For any $d \in L$, we set $f_d(x) := x \wedge d$. We consider an element $c \in L \setminus \{0,1\}$. By Lemma 3.3.3 the intervals $[c,1]$ and $[0,c']$ are isomorphic. Thus the mapping

$$\omega\colon (x,y) \mapsto (f_{c'}(x), f_c(y))$$

is a lattice isomorphism between $[c',1] \times [c,1]$ and $[0,c] \times [0,c']$. On the other hand the mapping

$$\psi\colon x \mapsto (x \vee c, x \vee c')$$

is obviously $\vee$-preserving from L to $[c',1] \times [c,1]$, and similarly the mapping

$$\phi\colon x \mapsto (x \wedge c, x \wedge c')$$

is $\wedge$-preserving from L to $[0,c] \times [0,c']$. In addition, the mapping ϕ is injective; in fact if $\phi(x) = \phi(y)$ then $x \wedge c = y \wedge c$ and $x \wedge c' = y \wedge c'$, and thus $x = (x \wedge c) \vee (x \wedge c') = (y \wedge c) \vee (y \wedge c') = y$. If we can prove that $\phi = \omega \circ \psi$, it will be an embedding since the equalities

$$\phi(x \vee y) = \omega(\psi(x \vee y)) = \omega(\psi(x)) \vee \omega(\psi(y)) = \phi(x) \vee \phi(y)$$

will hold. Let us prove that $\phi = \omega \circ \psi$. First observe that

$$x \wedge c = (x \vee c') \wedge (x \vee c) \wedge c = (x \vee c') \wedge c.$$

Thus

$$\omega \circ \psi(x) = \omega(x \vee c', x \vee c) = ((x \vee c') \wedge c, (x \vee c) \wedge c' = (x \wedge c, x \wedge c') = \phi(x).$$

The mapping ϕ is in fact an isomorphism. To see this what we still have to prove the surjectivity. Let us consider $(u, v) \in [0, c] \times [0, c']$. We set $x := u \vee v$. We have

$$x \wedge c = (u \vee v \vee c') \wedge c = (u \vee c') \wedge c = u \wedge c = u.$$

Similarly we have $x \wedge c' = v$. Thus $\phi(x) = (u, v)$ and ϕ is surjective. Therefore L is isomorphic to $[0, c] \times [0, c']$. □

Corollary 3.3.5. *The weakly dicomplemented lattices with negation are exactly the Boolean algebras.*

Proof. We already observed that each $(L, \wedge, \vee, ', 0, 1)$ is a uniquely complemented lattice. It remains to prove that $(L, \wedge, \vee)$ is distributive. That is what we did in Theorem 3.3.4, since the only nontrivial indecomposable underlying lattice is the two element lattice and, the variety it generates is exactly the variety of distributive lattices. Therefore every weakly dicomplemented lattice with negation is a complemented distributive lattice (a Boolean algebra). □

Weakly dicomplemented lattices can then be considered as **semantical extensions** of Boolean algebras in comparison to other extensions which are more or less obtained by retaining some properties valid in Boolean algebras: **syntactic extensions**. Boole developed in [Bo54] a mathematical theory for logic, based on *signs* and *classes*. He encoded the conjunction, the disjunction, the negation, the universe and "nothing". This gave rise to the so-called Boolean algebras. On concepts the conjunction and disjunction are respectively encoded by the meet and join operations of the concept lattice. The universe is encoded by the greatest element and "nothing" by the least element. To encode a negation the idea of Boole has been followed, with the requirement that the operation obtained should be internal. The weak negation is obtained by taking the concept generated by the complement of the extent and the weak opposition by taking the concept generated by the complement of the intent. The two operations coincide iff we have a Boolean algebra. In this case we get a negation. Thus weakly dicomplemented lattices can be seen as a **contextual generalization** or (**natural extension**) of Boolean algebras. We later compare this generalization to other extensions in Section 3.4

There is another proof of Corollary 3.3.5, using term manipulations. First we can prove that

Lemma 3.3.6. *Each weakly dicomplemented lattice with negation* $\underline{L}$ *satisfies the de Morgan laws.*

Proof. We want to prove that $(x \wedge y)' = x' \vee y'$.

$$(x' \vee y') \vee (x \wedge y) \geq x' \vee (x \wedge y') \vee (x \wedge y) = x' \vee x = 1$$

and

$$(x' \vee y') \wedge (x \wedge y) \leq (x' \vee y') \wedge x \wedge (x' \vee y) = x' \wedge x = 0.$$

So $x' \vee y'$ is a complement of $x \wedge y$, hence by uniqueness it is equal to $(x \wedge y)'$. Dually we have $(x \vee y)' = x' \wedge y'$. □

Now for the distributivity we can show that

Proposition 3.3.7. *$(x \wedge (y \vee z))'$ is a complement of $(x \wedge y) \vee (x \wedge z)$.*

Proof. Since in every lattice the equation

$$x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$$

holds, we have that $(x \wedge (y \vee z))' \leq ((x \wedge y) \vee (x \wedge z))'$; so we have to show only that

$$(x \wedge (y \vee z))' \vee (x \wedge y) \vee (x \wedge z) = 1.$$

Using the de Morgan laws and axiom (3) several times we obtain:

$$\begin{aligned}
(x \wedge (y \vee z))' \vee (x \wedge y) \vee (x \wedge z) &= x' \vee (y' \wedge z') \vee (x \wedge y) \vee (x \wedge z) \\
&= x' \vee (y' \wedge z' \wedge x) \vee (y' \wedge z' \wedge x') \\
&\quad \vee (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge z \wedge y') \\
&= x' \vee (y' \wedge z' \wedge x') \vee (x \wedge y \wedge z) \\
&\quad \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y' \wedge z') \\
&= x' \vee (y' \wedge z' \wedge x') \vee (x \wedge y) \vee (x \wedge y') \\
&= x' \vee (y' \wedge z' \wedge x') \vee x \\
&= 1.
\end{aligned}$$

Thus $(x \wedge (y \vee z))'$ is a complement of $(x \wedge y) \vee (x \wedge z)$. □

Since the complement is unique we get the equality

$$x \wedge (y \vee z) = (x \wedge (y \vee z))'' = (x \wedge y) \vee (x \wedge z).$$

As the equality $x^{\triangle} = x^{\triangledown}$ not always holds, we can look for a maximal subset with this property.

Definition 3.3.2. For any weakly dicomplemented lattice $\underline{L}$, the set $B(L) := \{x \in L \mid x^{\triangle} = x^{\triangledown}\}$ is called the **subset of elements with negation**[5].

As in Definition 3.3.1 we denote by x' the common value of $x^{\triangle}$ and $x^{\triangledown}$.

[5] See Subsection 5.3.3

Corollary 3.3.8. $(B(L), \wedge, \vee, ', 0, 1)$ *is a Boolean algebra that is a subalgebra of the skeleton and the dual skeleton.*

Proof. From $x^{\triangle} = x^{\triangledown}$ we get $x^{\triangle\triangle} = x^{\triangledown\triangle}$ and $x^{\triangle\triangledown} = x^{\triangledown\triangledown}$. Thus

$$x^{\triangle\triangledown} = x^{\triangle\triangle} = x = x^{\triangledown\triangledown} = x^{\triangledown\triangle}$$

and $B(L)$ is closed under the operations $^{\triangle}$ and $^{\triangledown}$. We will prove that $\underline{B}(L)$ is a subalgebra of $\underline{L}$. We consider x and y in $B(L)$. We have

$$(x \wedge y)^{\triangle} = x^{\triangle} \vee y^{\triangle} = x^{\triangledown} \vee y^{\triangledown} \leq (x \wedge y)^{\triangledown} \leq (x \wedge y)^{\triangle} \text{ and}$$

$$(x \vee y)^{\triangledown} = x^{\triangledown} \wedge y^{\triangledown} = x^{\triangle} \wedge y^{\triangle} \geq (x \vee y)^{\triangle} \geq (x \vee y)^{\triangledown}.$$

Thus $x \wedge y$ and $x \vee y$ belong to $B(L)$. $\underline{B}(L)$ is a weakly dicomplemented lattice with negation, and is by Corollary 3.3.5, a Boolean algebra. □

In the proof of Corollary 3.3.8 we showed that $\underline{B}(L)$ a subalgebra of $\underline{L}$. It is the largest Boolean algebra which is subalgebra of both the skeleton and the dual skeleton. We call it the **Boolean part** of the weakly dicomplemented lattice $\underline{L}$. When is the Boolean part equal to the intersection of the skeleton and dual skeleton? In order to define the *Boolean part of a weakly complemented lattice* we give more characterizations of weakly dicomplemented lattices with negation. We need the following lemma.

Lemma 3.3.9. *Each antitone involution satisfies the de Morgan laws.*

Proof. Let $^{\perp}$ be an antitone involution on a lattice $\underline{L}$. Obviously $(x \vee y)^{\perp} \leq x^{\perp} \wedge y^{\perp}$. Let a be a lower bound of $x^{\perp}$ and $y^{\perp}$. We have $a^{\perp} \geq x^{\perp\perp} = x$ and $a^{\perp} \geq y^{\perp\perp} = y$. Thus $a^{\perp} \geq x \vee y$. It follows that $a = a^{\perp\perp} \leq (x \vee y)^{\perp}$. This means that $(x \vee y)^{\perp}$ is the meet of $x^{\perp}$ and $y^{\perp}$, and the join de Morgan law is proved. That is $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$. Similarly we get the meet de Morgan law, $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$. □

Corollary 3.3.10. *Each weakly complemented lattice, the weak complementation of which is an involution is a Boolean algebra.*

Proof. We shall prove that such a weak complementation is also a dual weak complementation. We will then get, by Corollary 3.3.5, a Boolean algebra. Let $^{\triangle}$ be a weak complementation which is an involution. By Lemma 3.3, $^{\triangle}$ satisfies the de Morgan laws. From axiom (3) we get $(x^{\triangle} \vee y^{\triangle}) \wedge (x^{\triangle} \vee y^{\triangle\triangle}) = x^{\triangle}$. Since $^{\triangle}$ is assumed to be an involution, each element of L is of the form $x^{\triangle}$. Thus $^{\triangle}$ is also a dual weak complementation on $\underline{L}$. □

Corollary 3.3.11. *Each weakly complemented lattice, the weak complementation of which is a complementation, is a Boolean algebra.*

Proof. Let $^{\triangle}$ be such a weak complementation. From axiom (3) we obtain

$$x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle} = x^{\triangle\triangle}.$$

Thus $^{\triangle}$ is an involution. By Corollary 3.3.10 we get the result. □

For a weakly complemented lattice $\underline{L}$, the largest Boolean algebra that is a subalgebra of the dual skeleton and of $\underline{L}$ is called its **Boolean part**. Dually is defined the **Boolean part of a dual weakly complemented lattice**. The dual skeleton is, in general, not a sublattice of L. Of course, if it is, then it is necessarily a Boolean algebra. The converse does not hold.

Definition 3.3.3. A weak complementation $^{\triangle}$ on $\underline{L}$ is said to be **0-separating** if only 0 has 1 as weak complement. A dual weak complementation $^{\triangledown}$ is said to be **1-separating** if only 1 has 0 as dual weak complement. A $(0,1)$**-separating weak dicomplementation** is formed by a 0-separating weak complementation and a 1-separating dual weak complementation.

Lemma 3.3.12. *Let $^{\triangle}$ be a weak complementation on a lattice L. The following assertions are equivalent:*

(i) $^{\triangle}$ *is a* 0*-separating weak complementation,*

(ii) $^{\triangle}$ *is a complementation on* L*,*

(iii) $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ *is a p-algebra,*

(iv) The dual skeleton of L *is* L*.*

(v) The mapping $x \mapsto x^{\triangle}$ *is bijective.*

Proof.

- The implication (v)⇒(i) is obvious.
- To prove (i)⇒(ii), we assume that $^{\triangle}$ is 0-separating. From the property (4) of Proposition 1.1.2 we get $x \wedge x^{\triangle} = 0$ and $^{\triangle}$ is a complementation on $\underline{L}$ since $x \vee x^{\triangle} = 1$ always holds. [See Remark 1.1.5]
- For (ii)⇒(iii), note that, if $^{\triangle}$ is a complementation. Therefore the property (10) of Proposition 1.1.3 implies that $y^{\triangle}$ is the pseudocomplement of y.
- Now we shall prove that (iii) implies (iv). We assume that $^{\triangle}$ is a pseudocomplementation. For all $x \in L$, we have

$$x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle} = x^{\triangle\triangle}.$$

  and $\bar{S}(L) = L$.
- For the last implication, note that the mapping $x \mapsto x^{\triangle}$ is surjective since each $x \in L$ is equal to $y^{\triangle}$ for $y := x^{\triangle}$. From

$$x^{\triangle} = y^{\triangle} \implies x = x^{\triangle\triangle} = y^{\triangle\triangle} = y$$

  we get an injection, and by then a bijection.

□

Corollary 3.3.13. *(i) A weak complementation* $^{\triangle}$ *is* 0*-separating if and only if* $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ *is a Boolean algebra.*

(ii) A dicomplementation $(^{\triangle}, ^{\triangledown})$ *is* $(0,1)$*-separating if and only if* $^{\triangle} = ^{\triangledown}$.

Proof. (i) is immediate from Corollary 3.3.11. Let us prove (ii). We assume that $^{\triangle} = ^{\triangledown}$. If $x^{\triangle} = 1$ then $0 = x^{\triangledown} \wedge x = x^{\triangle} \wedge x = x$. Dually $x^{\triangledown} = 0$ implies $x = 1$. Thus $(^{\triangle}, ^{\triangledown})$ is $(0,1)$-separating. For the converse, note that $^{\triangle}$ is 0-separating, and by then, implies that $(L, \wedge, \vee, ^{\triangle}, 0, 1)$ is a Boolean algebra. Duallizing Lemma 3.3.12 and Corollary 3.3.11 proves that $(L, \wedge, \vee, ^{\triangledown}, 0, 1)$ is also a Boolean algebra. The uniqueness of complementation implies $^{\triangle} = ^{\triangledown}$. □

In the forthcoming section we investigate the relationship between this new extension of Boolean algebras and other extensions.

3.4 Boolean Algebras Extension

They are many attempts to generalize Boolean algebras. The main idea up to now has been to retain some properties and drop others: **syntactic extension**. One of the most widely investigated extension is the class of distributive lattices [Grätzer, Birkhoff, Balbes, Davey & Priestley,. . .]. Other extensions are more concerned with some properties of the complementation. Although the more or less fruitful investigations have been made again assuming distributivity, there are, however, some attempts with nondistributivity like orthocomplementation notions. In this section we do not assume distributivity. It will be put in brackets if the initial definition used it. We are going to investigate the interdependence between weak dicomplementation and similar complementation extensions.

3.4.1 Extension by a single unary operation

In a Boolean algebra the complementation satisfies the **de Morgan laws**. If, in a bounded (distributive) lattice, we can define a unary operation that satisfies the de Morgan laws and interchanges 0 and 1 then we obtain what is called an **Ockham algebra**[6]. That is a bounded (distributive) lattice with a unary operation f such that

$$f(x \wedge y) = f(x) \vee f(y), \quad f(x \vee y) = f(x) \wedge f(y), \quad f(0) = 1 \quad \text{and } f(1) = 0.$$

The operation f is sometimes called a (de Morgan) **"negation"**, although this does not satisfy the *law of double negation*. The complementation in a Boolean algebra is also a **polarity** (i.e. an antitone involution). If we

[6]The name Ockham lattices has been introduced in [Ur79] by A. Urquhart with the justification: "the term *Ockham lattice* was chosen because the so-called *de Morgan laws* are due (at least in the case of propositional logic) to William of Ockham" (1290-1349).

require that the unary operation of an Ockham algebra should be an involution, we get the so called the **de Morgan algebras**. This is an algebra $(L, \wedge, \vee, f, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a bounded (distributive) lattice and f a unary operation that satisfies

$$f^2(x) = x, \quad f(x \wedge y) = f(x) \vee f(y) \quad \text{and } f(x \vee y) = f(x) \wedge f(y).$$

De Morgan algebras "*arose in the researches on the algebraic treatment of constructive logic with strong negation" [BV94].* If in addition f satisfies the inequality $x \wedge f(x) \leq y \vee f(y)$ then $(L, \wedge, \vee, f, 0, 1)$ is called a **Kleene algebra**.

Contrary to the above mentioned extensions, where less attention is paid to the property of *being a complementation*, the second approach is more interested with this property. One way of generalizing the notion of complementation is to retain the identity $x \wedge x' = 0$ and drop the other. The operation obtained is a **semicomplementation**. Of considerable interest are those lattices in which, for any element x, the subset of semicomplements of x has a greatest element (the **pseudocomplement** of x): **pseudocomplemented lattices**. A lattice with a **pseudocomplementation** is called **p-algebra**. The dual notions are **dual semicomplementation**, **dual pseudocomplementation** and **dual p-algebra**. Of course if we require that $x \vee x^* = 1$ for every x in a p-algebra L, then x^* becomes a complement of x and, when $\underline{L}$ is distributive, $\underline{L}$ is then a **Boolean lattice**[7]. M. H. Stone suggested a restriction of the equation $x \vee x^* = 1$ to those elements x that are pseudocomplement, i.e. that it will be fruitful to consider the equation $x^* \vee x^{**} = 1$ (Stone identity). (Distributive) pseudocomplemented lattices that satisfy this identity are therefore called **Stone lattices**[8].

Retaining the properties common to de Morgan algebras and Stone algebras defines the so-called **de Morgan-Stone algebra**, or for short **MS-algebra**. That is an algebra $(L, \wedge, \vee, f, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded (distributive) lattice and the unary operation satisfies the equations

$$f1 = 0, \quad f(x \wedge y) = fx \vee fy \quad \text{and} \quad x \leq f^2x$$

for all x and y in L. Similarly a **dual de Morgan-Stone** algebra is defined.

The approach without distributivity appeared in ortholattices, orthomodular lattices[9] and weakly orthocomplemented lattices. A **weak orthocomplementation** is a square-extensive antitone semicomplementation. A **weakly orthocomplemented lattice** is a bounded lattice

[7] The term **Boolean algebra** is used when $x \mapsto x^*$ is considered as a fundamental operation

[8] When $x \mapsto x^*$ is considered as a fundamental operation the term **Stone algebra** is used.

[9] See Definition 1.3.2

with a weak orthocomplementation. Dually is defined a **dual weak orthocomplementation**. All dual weak complementations are weak orthocomplementation. In this contribution two other notions have been introduced: weakly complemented lattices and dual weakly complemented lattices. It is not difficult to see that each weak complementation satisfies the orthomodular law. But, weakly complemented lattices are orthomodular if and only if the weak complementation is also a complementation. In this case we automatically get a Boolean algebra.

There are other extensions by a single operation, worthy to be mentioned, although this operation is not unary. However a unary operation can be deduced. The first class is that of relatively complemented lattices.

Relative complement. Let L be a lattice and $a, b \in L$ with $a < b$. Let c be in $[a, b]$. An element $d \in [a, b]$ is called a **relative complement** of c in the interval $[a, b]$ if $c \vee d = b$ and $c \wedge b = a$. A lattice is called **relatively complemented** if for every $a, b \in L$ with $a < b$, each element $c \in [a, b]$ has at least one relative complement in $[a, b]$. Of course if L has 0 and 1 and is relatively complemented then L is complemented. If L has neither 0 nor 1, a relative complement (if there is one) is neither a semicomplement nor a dual semicomplement.

Heyting algebras. The second class is that of Heyting algebras. In relation to intuitionistic logic they play an analogous role to that played by Boolean algebras to classical logic. A **Heyting algebra** is a bounded lattice $\underline{L}$ such that for x and y in L the set $\{z \in L \mid z \wedge x \leq y\}$ has a largest element (denoted by $x \rightarrow y$). This defines a binary operation called **implication**. A unary operation $\neg$, called **Heyting negation** or **intuitionistic negation**, is obtained by setting $\neg x := x \rightarrow 0$. This is a pseudocomplementation. Dually is defined a **dual Heyting algebra**. Moreover Heyting algebras and dual Heyting algebras are completely distributive.

Before we give the interrelation between all these extensions we stop to some extensions where the negation is captured by two operations.

3.4.2 Extension by two unary operations

We have mentioned p-algebras and dual p-algebras as Boolean algebra extensions. Putting together the two unary operations gives **double p-algebras**. Similarly are defined **double Stone algebras**, weakly dicomplemented lattices (for **double weakly complemented lattices**) and **double Heyting algebras**. For double de Morgan-Stone algebras the two operations should somehow be connected. A **double de Morgan-Stone algebra** or **DMS-algebra** for short is an algebra $(L, \wedge, \vee, ^\perp, ^\circ, 0, 1)$ such that $(L, \wedge, \vee, ^\circ, 0, 1)$ is an MS-algebra, $(L, \wedge, \vee, ^\perp, 0, 1)$ a dual MS-algebra and the equations $x^{\perp\circ} = x^{\perp\perp}$ and $x^{\circ\perp} = x^{\circ\circ}$ hold for all x in L. Note that in a distributive double Stone algebra $(L, \wedge, \vee, ^\circ, ^\perp, 0, 1)$ we have

$$x^{\perp\circ} = x^{\perp\perp} \leq x \leq x^{\circ\circ} = x^{\circ\perp}$$

We have seen that Boolean algebras are weakly dicomplemented lattices such that the unary operations are identic. The same holds for double p-algebras.

Proposition 3.4.1. *Boolean algebras are double p-algebras such that the unary operations coincide.*

Proof. Let $\underline{L}$ be a double p-algebra. If the pseudocomplementation and the dual pseudocomplementation coincide then the skeleton and dual skeleton are equal to L. But O. Frink proved that skeletons of p-algebras are Boolean algebras[10]. Thus $\underline{L}$ is a Boolean algebra. □

Contrary to the case of weakly dicomplemented lattices and double p-algebras, requiring the two unary operations to coincide is not longer enough to get Boolean algebras from DMS-algebras. The class obtained is that of de Morgan algebras.

Remark 3.4.1. A **bi-uniquely complemented lattice** is a bounded lattice in which every element $x \notin \{0, 1\}$ has exactly two complements. The lattice $L12$ in Figure 3.6 is bi-uniquely complemented. The two element Boolean algebra is the unique distributive bi-uniquely complemented lattice. We cannot consider bi-uniquely complemented lattices as an extension of Boolean algebras (with two unary operations).

3.4.3 Attribute exploration

We first consider extensions by a single unary operation. These are the attribute of the context we are looking for. The programm **ConImp**[11] is an adequate tool for a systematic generation of this context. The results for extensions by two unary operations can be deduced, since they are more or less obtained by putting together the unary operations. The definitions we use are summarized in the table on Figure 3.5. The unary operation, denoted by f, is defined on a bounded lattice L and interchanges 0 and 1. The properties of unary operations used are in Figure 3.4.

The following results are some implications valid between these attributes. The proofs are not difficult once you wrote them down. These are just routine exercises for morning mathematic gymnastics.

Proposition 3.4.2. *Any unary operation satisfying one of the de Morgan laws is antitone.*

[10] A proof can be found in [CG00].

[11] ConImp is a DOS program for contexts, concepts, concept lattices and implications written by Peter Burmeister.

Name	Definition	Denomination
ane	$x \leq y \Rightarrow fx \geq fy$	antitone
mMl	$f(x \wedge y) = fx \vee fy$	meet de Morgan law
jMl	$f(x \vee y) = fx \wedge fy$	join de Morgan law
sqex	$x \leq f^2x$	square extensive
sqin	$x \geq f^2x$	square intensive
inv	$x = f^2x$	involution
wcol	$(x \wedge y) \vee (x \wedge fy) = x$	weak complementation's 3^{rd} law
dwcl	$(x \vee y) \wedge (x \vee fy) = x$	dual weak complementation's 3^{rd} law
omol	$x \leq y \Rightarrow x \vee (fx \wedge y) = y$	orthomodular law

Figure 3.4. Some properties of unary operations abstracting a negation.

Name	Definition	Denomination
sco	$x \wedge fx = 0$	semicomplementation
dsc	$x \vee fx = 1$	dual semicomplementation
com	sco & dsc	complementation
uco	$x \wedge y = 0$ & $x \vee y = 1$ $\iff y = fx$	unique complementation
wco	sqin & ane & wcol	weak complementation
dwc	sqex & ane & dwcl	dual weak complementation
Ock	mMl & jMl	Ockham algebra
Mor	Ock & inv	de Morgan algebra
Kle	Mor & $x \wedge fx \leq y \vee fy$	Kleene algebra
pa	$x \wedge y = 0 \iff y \leq fx$	p-algebra
dpa	$x \vee y = 1 \iff y \geq fx$	dual p-algebra
Sa	pa & $fx \vee f^2x = 1$	Stone algebra
dSa	dpa & $fx \wedge f^2x = 0$	dual Stone algebra
MSa	mMl & sqex	de Morgan-Stone algebra
dMSa	jMl & sqin	dual de Morgan-Stone algebra
ola	ane & com & inv	ortholattice
oml	ola & omol	orthomodular lattice
wol	sqex & ane & sco	weak ortholattice
dwol	sqin & ane & dsc	dual weak ortholattice
Ba	all properties above	Boolean algebra

Figure 3.5. Extensions of a Boolean algebra complementation by means of a single unary operation.

Proof. Let f be a unary operation satisfying the join de Morgan law. Let $x \leq y$. We have $fy = f(x \vee y) = fx \wedge fy$. Thus $fx \geq fy$. The proof for meet de Morgan law is obtained similarly. □

Proposition 3.4.3. *Any antitone square-extensive operation satisfies the join de Morgan law. Dually any antitone square intensive operation satisfies the meet de Morgan law.*

Proof. Let x and y be two elements. Obviously $f(x \vee y) \leq fx \wedge fy$ holds. Assume that $a \leq fx$ and $a \leq fy$. We get $fa \geq f^2x \vee f^2y \geq x \vee y$. Thus $a \leq f^2a \leq f(x \vee y)$, and $f(x \vee y)$ is the meet of fx and fy. □

Therefore the de Morgan algebras are exactly bounded lattices with polarities. It also follows that the classes of MS-algebras and of dual MS-algebras is contained in the class of Ockham algebras

Proposition 3.4.4. *A pseudocomplementation is antitone and square-extensive. Dually a dual pseudocomplementation is antitone and square-intensive.*

Proof. Let $x \leq y$ and f a pseudocomplementation. $y \wedge fy = 0$ implies $x \wedge fy = 0$. Thus $fy \leq fx$, and f is antitone. Now $x \wedge fx = 0$ implies $x \leq f^2x$, and f is square-extensive. □

Proposition 3.4.5. *Each square-intensive semicomplementation satisfying one of the de Morgan laws is a complementation. Dually each square-extensive dual semicomplementation satisfying one of the de Morgan laws is a complementation.*

Proof. Let f be a semicomplementation satisfying the meet de Morgan law. We have

$$1 = f0 = f(x \wedge fx) = fx \vee f^2x \leq fx \vee x \implies fx \vee x = 1.$$

Thus f is a complementation. Now we assume that f is a semicomplementation satisfying the join de Morgan law. We get

$$f(fx \vee x) = f^2x \wedge fx \leq x \wedge fx = 0 \implies 1 = f^2(fx \vee x) \leq fx \vee x.$$

Thus $fx \vee x = 1$ and f is a complementation. The remaining claim follows dually. □

Thus each dually semicomplemented MS-algebra is complemented and each semicomplemented dual MS-algebra complemented.

Proposition 3.4.6. *Each dual weak complementation satisfying the meet de Morgan law is a pseudocomplementation. Dually each weak complementation satisfying the join de Morgan law is a dual pseudocomplementation.*

Proof. We shall prove that $x \vee y = 1 \iff x \geq fy$. Obviously $x \vee fx = 1$. We assume that $x \vee y = 1$. We get $fx \wedge fy = 0$ and

$$fy = (fy \wedge x) \vee (fy \wedge fx) = fy \wedge x.$$

Thus $x \geq fy$. The rest is proved similarly. □

Proposition 3.4.7. *[O. Frink, 1969] A unary operation f is a pseudocomplementation if it is square-extensive with $f^2 0 = 0$ and*

$$f(x \wedge y) \wedge f(x \wedge fy) = fx.$$

Proposition 3.4.8. *Each pseudocomplemented de Morgan algebra is a Boolean algebra. Dually each dual pseudocomplemented de Morgan algebra is a Boolean algebra.*

Proof. We first prove that the unary operation is a dual weak complementation. It is square-extensive and antitone since it is a pseudocomplementation. Moreover

$$fx = f(x \wedge y) \wedge f(x \wedge fy) = (fx \vee fy) \wedge (fx \vee f^2 y) = (fx \vee fy) \wedge (fx \vee y)$$

We need to prove that fx can be replaced by x. This is always possible since f is an involution. Thus

$$x = (x \vee fy) \wedge (x \vee y)$$

and f is a dual weak complementaion. By the dual of Corollary 3.3.10 we get the result. □

The notions of complementation, unique complementation, Ockham algebra, de Morgan algebra, Kleene algebra and ortholattice are self dual.

Proposition 3.4.9. *The notion of orthomodular lattice is self dual.*

Proof. It is enough to prove that each orthomodular lattice satisfies

$$x \geq y \implies x \wedge (fx \vee y) = y \qquad \text{(dual orthomodular law)}.$$

We consider $x \geq y$. Since f is antitone we get $fx \leq fy$. From the orthomodular law we have

$$fy = fx \vee (f^2 x \wedge fy) = fx \vee (x \wedge fy).$$

Thus

$$y = f^2 y = f(fx \vee (x \wedge fy)) = f^2 x \wedge (fx \vee f^2 y) = x \wedge (fx \vee y)$$

$$\text{i.e.} \quad x \geq y \implies y = x \wedge (fx \vee y) \quad \text{(dual orthomodular law)}.$$

□

Proposition 3.4.10. *Each pseudocomplemented dual MS-algebra is dual weakly complemented. Dually each dual pseudocomplemented MS-algebra is weakly complemented.*

Proof. From the definition of dual MS-algebra the unary operation is square-intensive and satisfies the join de Morgan law. Thus f is antitone. From Proposition 3.4.4 it is also square-extensive and is by then an involution. Since f is a pseudocomplementation, we have

$$f(x \wedge y) \wedge f(x \wedge fy) = fx.$$

Thus

$$x = f^2 x = f^2 \left[(x \wedge y) \vee (x \wedge fy) \right] = (x \wedge y) \vee (x \wedge fy).$$

The second part follows dually. □

Remark 3.4.2. Each antitone unique complementation defines a Boolean algebra [Sa88, p.48]. By a result of R. P. Dilworth there is a uniquely complemented lattice that is not a Boolean algebra. This lattice is infinite. This complementation is automatically an involution. It cannot be antitone. A representative of this class is example $L15$ in the context in Figure 3.7.

Remark 3.4.3. An implication is still open.

Is any weakly complemented lattice that is a dual Stone algebra also an Ockham algebra?

Note that the meet de Morgan law follows from the weakly complemented lattice. Thus to get this implication we should prove the join de Morgan. In the case of distributive lattices, the dual Stone identity is equivalent to the join de Morgan law. Up to now no counter example is found. We believe that the implication might be true. However, if the conjecture happens to be false, a representative of this class would be $L16$. The dual is $L17$.

We can now write down (Figure 3.7) the context we have got from the attribute exploration. A list of algebras used as counter examples in this exploration is given in Figure 3.6.

The corresponding concept lattice is shown in Figure 3.8. This diagram has been drawn using ANACONDA, a preparator software for TOSCANA. TOSCANA is a Management System for Conceptual Information Systems.

Remark 3.4.4. In [Da00] the author explored the elementary properties of unary mappings used to defined most of the important notions of complementation. Axiom (3) was not considered. He resctricted to finite lattices. In this case you cannot distinguish between Boolean algebra and uniquely complemented lattice.

3.5 Triple Characterization

A model for weakly dicomplemented lattices is the class of distributive double p-algebras. Some p-algebras can be characterized by the structure of their skeleton and the dense elements. We present here some analogies motivating the search of such a characterization for weakly dicomplemented lattices.

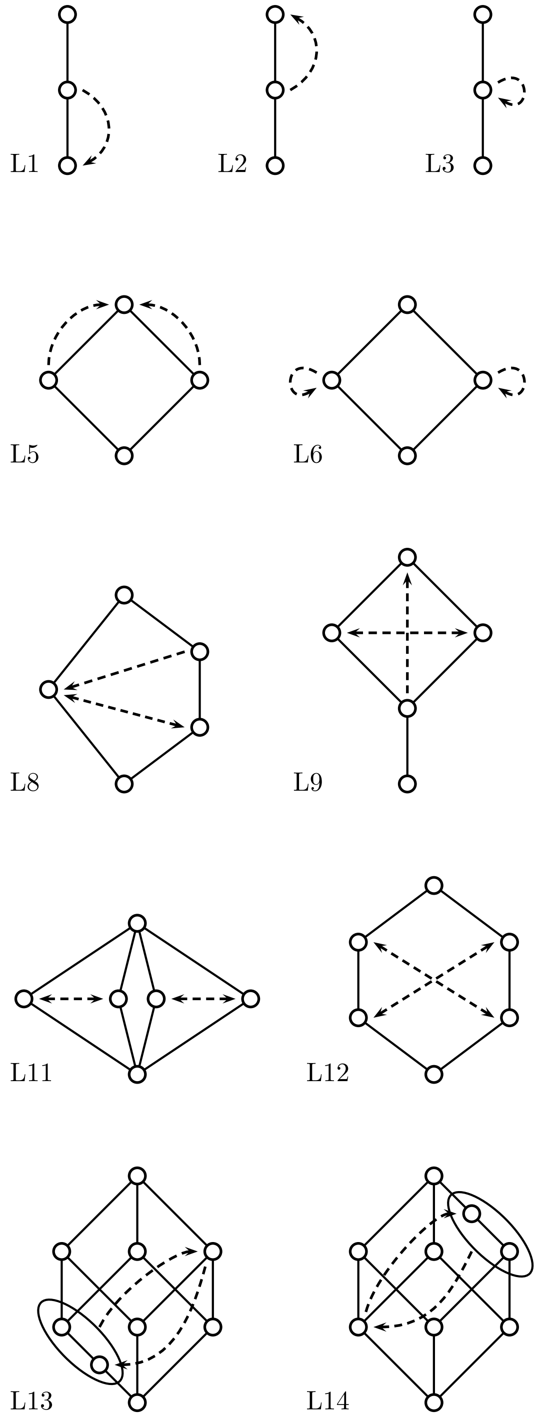


Figure 3.6. Lattice diagrams of algebras used as counter example in the attribute exploration. The dashed lines indicate the image by the unary operation. For the two last case $L13$ and $L14$ the images of other elements are their (unique) complements.

	sco	dsc	com	uco	wco	dwc	Ock	Mor	Kle	pa	dpa	Sa	dSa	MSa	dMSa	ola	oml	wol	dwol	Ba
L1	×					×	×			×		×		×				×		
L2		×			×		×				×		×		×				×	
L3							×	×	×					×	×					
L4	×					×												×		
L5		×			×														×	
L6							×	×						×	×					
L7	×	×	×				×			×		×		×				×		
L8	×	×	×				×				×		×		×				×	
L9		×			×						×								×	
L10	×					×				×								×		
L11	×	×	×				×	×	×					×	×	×	×	×	×	
L12	×	×	×				×	×	×					×	×	×		×	×	
L13	×	×	×								×		×						×	
L14	×	×	×							×		×						×		
L15	×	×	×	×																
L16		×			×						×		×						×	
L17	×					×				×		×						×		

Figure 3.7. Context of the attribute exploration of extensions of the Boolean algebra complementation by a unary operation.

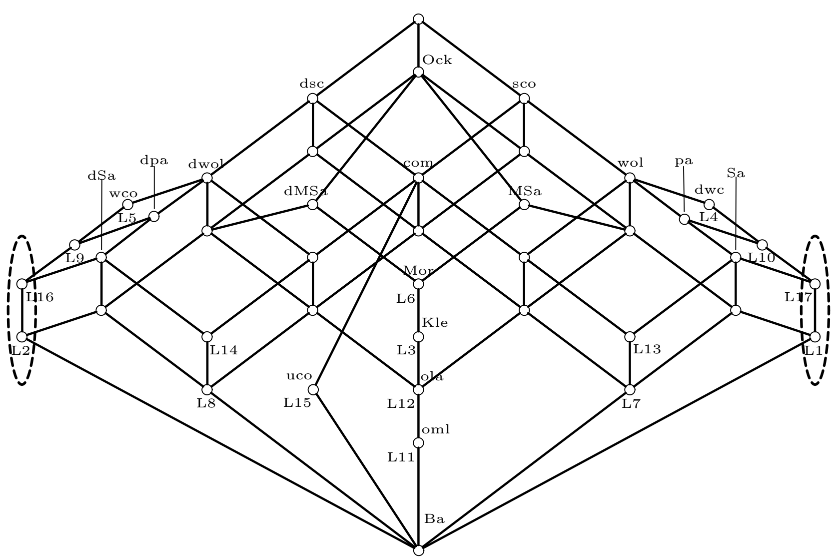


Figure 3.8. Lattice of extensions of the Boolean complementation by a unary operation.

3.5.1 Triple characterization for p-algebras

To every p-algebra L can be assigned a Boolean algebra $S(L)$ (its skeleton) and a lattice filter $D(L)$ (dense elements). Nemitz, Chen and Grätzer observed that a third bit of information is needed in order to characterize L. That is the structure map $\phi(L)$ between $S(L)$ and $D(L)$.

For a distributive Stone algebra, the identity

$$x = x^{**} \wedge (x \vee x^*) \text{ with } (x^{**}, x \vee x^*) \in S(L) \times D(L).$$

holds. The structures of $S(L)$ and $D(L)$ together with the relationship between their elements describe $\underline{L}$. The relationship here is expressed by the homomorphism $\phi(L)\colon S(L) \to \mathcal{D}(D(L))$ defined by

$$\phi(L)(x) := \{y \in D(L) \mid y \geq x^*\},$$

where $\mathcal{D}(D(L))$ is the set of lattice filters of $D(L)$. Chen and Grätzer proved that the triple $< S(L), D(L), \phi(L) >$ characterizes L up to isomorphism. In this case

(i) the map $x \mapsto x \vee a^*$ is a lattice isomorphism between

$$F_a := \{x \in L \mid x^{**} = a\}$$

and

$$D_a := \{x \in D(L) \mid a^* \leq x \leq a \vee a^*\} \subseteq \phi(L)(a).$$

Therefore every element $x \in L$ can be represented by a pair (a, d) in $S(L) \times D(L)$ with $d \in D_a$.

(ii) For $(a, d) \in S(L) \times D(L)$ we have $[d) \cap \phi(L)(a) = [d \vee a^*)$. Thus there are maps $\eta_a^L\colon D(L) \to D(L)$ defined by $\big[\eta_a^L(d)\big) = [d) \wedge \phi(a)$.

(iii) For representations (a, d) and (c, e) of x and y, respectively, we have

$$x \leq y \text{ in } L \iff a \leq b \text{ and } d \leq \eta_a^L(e).$$

To characterize a much larger class, Katriňák called a p-algebra **decomposable** if for every $x \in L$ there is $d \in D(L)$ such that $x = x^{**} \wedge d$. For a decomposable p-algebra a binary relation $\phi_L(a)$ on $D(L)$ can be assigned to each element a of $S(L)$ as follows:

$$d \equiv e(\phi_L(a)) \iff a^* \wedge d = a^* \wedge e.$$

ϕ_L is a $(0,1)$-isotone map from $S(L)$ into the lattice of $\wedge$-compatible equivalence relations on $D(L)$, denoted by $Eq_\wedge(D)$. Note that $\phi(L)(a)$ is the equivalence class $[1]_{\phi_L(a)}$. He further called (S, D, ϕ) an **abstract triple** if S is a Boolean algebra, D is a lattice with 1, and $\phi : S \to Eq_\wedge(D)$ is a $(0,1)$-isotone map such that for any $d, e \in D$ and $a, b \in S$ there exists $t \in D$ such that

$$\big[[d]_{\phi(a)}\big) \cap \big[[e]_{\phi(b)}\big) = \big[[t]_{\phi(a \wedge b)}\big)$$

He defined an **isomorphism between the triples** (S_1, D_1, ϕ_1) and (S_2, D_2, ϕ_2) to be a pair (f, g), where $f : B_1 \cong B_2$, $g : D_1 \cong D_2$ (inducing an isomorphism $\bar{g} : Eq_\wedge(D_1) \cong Eq_\wedge(D_2)$), and $\bar{g} \circ \phi_1 = \phi_2 \circ f$. He then proved these two results, respectively called **triple characterization** and **triple construction** of p-algebras.

(i) Two decomposable p-algebras are isomorphic if and only if their associated triples are isomorphic.

(ii) For a triple (S, D, ϕ) there is a decomposable p-algebra L such that $(S(L), D(L), \phi_L) \cong (S, D, \phi)$.

3.5.2 Characterization for weakly dicomplemented lattices

Now L denotes a weakly dicomplemented lattice. Recall that its skeleton $S(L)$ and dual skeleton $\bar{S}(L)$ are ortholattices, and that its Boolean part $B(L)$ is a Boolean algebra and is a sublattice of $\underline{L}$. How are dense elements? An element x of L is said to be **dense** if $x^\triangledown = 0$. We denote by $D(L)$ the set of dense elements. Dually is defined the set $\bar{D}(L)$ of **dual dense elements**. $D(L)$ is an order filter of $\underline{L}$. For the weak complementation on the lattice $L4$ in Figure 3.7, $D(L4)$ is an order filter that is not a lattice filter. However $D_\triangle(L) := \{x \in L \mid x^\triangle = 0\}$ is a lattice filter contained in $D(L)$. Dually $\bar{D}(L)$ is an order ideal containing $x \wedge x^\triangle$ for all $x \in L$. It is in general not a lattice ideal. $\bar{D}_\triangledown(L) := \{x \in L \mid x^\triangledown = 1\}$ is a lattice ideal contained in $\bar{D}(L)$.

For every $x \in L$, the element $x \vee x^\triangledown$ is in $D(L)$ and $x^{\triangledown\triangledown}$ is in $S(L)$. Moreover the equality $x = x^{\triangledown\triangledown} \wedge (x \vee x^\triangledown)$ holds. This identity can be interpreted, as in the case of Stone algebras, to mean that every $x \in L$ can be represented by a pair $(y, z) \in S(L) \times D(L)$. Such an interpretation suggests that the structures of $S(L)$ and $D(L)$ together with the relationship between their elements may characterize L.

Define a map $\phi(L)$ from $S(L)$ to the power set of $D(L)$ by

$$\phi(L)(a) = \{x \in D(L) \mid x \geq a^\triangledown\}.$$

The equalities $\phi(L)(0) = \{1\}$ and $\phi(L)(1) = D(L)$ hold. $\phi(L)(a)$ is an order filter of $D(L)$. All $\phi(a)$s are lattice filters iff $D(L)$ is a lattice filter. Let $a \leq b$ in $S(L)$. From $a^\triangledown \geq b^\triangledown$ we get $\phi(L)(a) \subseteq \phi(b)$. Thus $\phi(L)$ is a $(0,1)$-isotone map from $S(L)$ into the lattice of order filters of $D(L)$.

We set $F_a := \{x \in L \mid x^{\triangledown\triangledown} = a\}$. F_a is an a $\vee$-semilattice[12] with a greatest element a. The family $(F_a)_{a \in S(L)}$ forms a partition of L. We get $F_0 = \{0\}$ and $F_1 = D(L)$. The mapping

$$\psi_a : x \mapsto x \vee a^\triangledown$$

[12] Is F_a a $\wedge$-semilattice?

maps F_a into $D(L)$. In fact $x \in F_a$ implies

$$(x \vee a^{\triangledown})^{\triangledown} = x^{\triangledown} \wedge a = a^{\triangledown} \wedge a = 0$$ [13].

If $\psi_a(x) = \psi_a(y)$ for x and y in F_a then we have

$$x = (x \vee x^{\triangledown}) \wedge x^{\triangledown\triangledown} = (x \vee a^{\triangledown}) \wedge a = (y \vee a^{\triangledown}) \wedge a = (y \vee y^{\triangledown}) \wedge y^{\triangledown\triangledown} = y.$$

Thus ψ_a is a bijection from F_a onto $\psi_a(F_a)$. In addition the equality[14]

$$\psi_a(x \vee y) = \psi_a(x) \vee \psi_a(y)$$

holds. Each element $x \in F_a$ is completely determined by $a \in S(L)$ and $x \vee a^{\triangledown} \in \psi_a(L)$; that is, by a pair (a, z) with $a \in S(L)$ and $z \in \psi_a(F_a)$[15]. Every such pair determines exactly one element of L, namely $a \wedge z \in F_a$. We want to show that the partial ordering of L can be dertermined by such pairs. Let x and y in L. If they belong to F_a then

$$x \leq y \iff (a, x \vee a^{\triangledown}) \leq (a, y \vee a^{\triangledown})$$

since $x^{\triangledown\triangledown} = a$ and $a \wedge (x \vee a^{\triangledown}) = x$. Without loss of generality we assume $x \in F_a$ and $y \in F_b$. From $x \leq y$ we get $a = x^{\triangledown\triangledown} \leq y^{\triangledown\triangledown} = b$. In addition

$$x \leq y \implies x \vee a \leq y \vee a \text{ and } x \vee a^{\triangledown} \leq y \vee a^{\triangledown}.$$

The converse holds if L is distributive. In this case we have $x \leq y$ is equivalent to $x \vee a^{\triangledown} \leq y \vee a^{\triangledown}$ since $a \vee x = a \leq a \vee y$ is obvious.

Construction problem. To each weakly complemented lattice $\underline{L}$ can be associated an order filter $D(L)$, a Boolean algebra $B(L)$ and an ortholattice $S(L)$. Of course if D is an order filter of a lattice $\underline{L}$ (that is a $\vee$-semilattice with 1) then the lattice $L_D := \{0\} \oplus D$ endowed with the trivial weak complementation has D as set of dense element. It is also evident that each Boolean algebra $\underline{L}$ can be considered as a weakly complemented lattice with $B(L) = L$. The first problem to consider is to find if each ortholattice is the skeleton of some weakly dicomplemented lattice. The construction problem can be formulated as follow:

Given a Boolean algebra B, an ortholattice S and D a $\vee$-semilattice with 1. Is there any weakly complemented lattice $\underline{L}$ such that

$$B(L) = B, \quad S(L) = S, \quad \textbf{and } D(L) = D?$$

Of course these structures should be somehow connected. Finding these conditions is an interresting problem to be considered in future works.

[13] $x^{\triangledown\triangledown} = a \iff x^{\triangledown} = a^{\triangledown}$ for all $a \in S(L)$

[14] If L is distributive then $\psi_a(x \wedge y) = \psi_a(x) \wedge \psi_a(y)$.

[15] On Figure 1.2 (c, v) belongs to $S(L) \times \psi_a(L)$ but $a \wedge v$ does not belong to F_a.

Characterization problem. If $\underline{L}_1$ and $\underline{L}_2$ are isomorphic weakly dicomplemented lattices then their skeletons are isomorphic ortholattices, their Boolean parts are isomorphic Boolean algebras, their dense elements form two isomorphic $\vee$-semilattices and the dual dense elements two isomorphic $\wedge$-semilattices. The correspoding isomorphisms are the restriction of the isomorphism between L_1 and L_2 to the corresponding subsets. **Now if $\underline{L}_1$ and $\underline{L}_2$ are two weakly dicomplemented lattices such that $S(L_1) \cong S(L_2)$, $D(L_1) \cong D(L_2)$, $B(L_1) \cong B(L_2)$ and the dual, under which conditions is L_1 isomorphic to L_2?**

We know that for each $x \in L$ the equalities

$$x^{\triangledown\triangledown} \wedge (x \vee x^{\triangledown}) = x = x^{\vartriangle\vartriangle} \vee (x \wedge x^{\vartriangle})$$

hold. If the isomorphisms $f : S(L_1) \cong S(L_2)$, $g : D(L_1) \cong D(L_2)$, $u : B(L_1) \cong B(L_2)$ and their dual (these are also isomorphisms) $\bar{f} : \bar{S}(L_1) \cong \bar{S}(L_2)$, $\bar{g} : \bar{D}(L_1) \cong \bar{D}(L_2)$ have a common extension $h : L \cong L$, then necessarily

$$f(x^{\triangledown\triangledown}) \wedge g(x \vee x^{\triangledown}) = h(x) = \bar{f}(x^{\vartriangle\vartriangle}) \vee \bar{g}(x \wedge x^{\vartriangle})$$

This can be set as a definition of a candidate map h. As the Boolean part is a subalgebra of the skeleton, the map u should be the restriction of f and $\bar{f}$ on $B(L_1)$. The problem now is to find conditions under which h is an isomorphism. Like the construction problem, it still open and would be considered in future works.

4

Distributive Weakly Dicomplemented Lattices

We restrict to distributive lattices. Here we can prove the representation theorem for finite distributive weakly dicomplemented lattices. Their lattice congruences are described. We give a description of lattices with unique weak dicomplementation. Typical examples are chains.

4.1 Representation of Finite Distributive Weakly Dicomplemented Lattices

The concrete weak complementations on a lattice L form a lattice Wn(L). Some properties of the lattice L can be carried over Wn(L). By means of theses properties we obtain the representation of weak complementations of finite distributive lattices. We start with Boolean algebras.

Theorem 4.1.1. *The lattice of representable weak complementations on a finite Boolean algebra is again a Boolean algebra. It is of cardinality* $2^{\frac{n(n-1)}{2}}$ *where* n *is the number of the atoms of the initial Boolean algebra.*

Proof. Let L be a Boolean algebra. We note G the set of its atoms. L is isomorphic to the power set of G. For $a \in G$ we denote by $\overline{a}$ the complement $G \setminus \{a\}$ of $\{a\}$. Then $M := \{\overline{a} \mid a \in G\}$ is the set of coatoms of L. The standard context of L is $\mathbb{K} := (G, M, \leq)$. The incidence relation is exactly $G \times M \setminus \{(a, \overline{a}) \mid a \in G\}$ and can be expressed by

$$x \leq \overline{y} \iff x \neq y, \; x, y \in G$$

For all $a \in G$, $a' = M \setminus \{a\}$ and $\overline{a}' = G \setminus \{a\} = \overline{a}$. Thus $a \neq b$ implies $\overline{a}' \cup \overline{b}' = G$ and $T = \{\{a,b\} \mid a \neq b,\ a,b \in G\} = \mathcal{P}_2(G)$, where $\mathcal{P}_2(G)$ denotes the set of two element subsets of G. Moreover $Ext(\mathbb{K})$ is the power set of G. For an U in $Ext(\mathbb{K})$, $U\mathcal{R}\{\overline{a},\overline{b}\}$ if and only if $U \subseteq \overline{a}'$ or $U \subseteq \overline{b}'$; i.e iff $U \subseteq \overline{a}$ or $U \subseteq \overline{b}$. Then $U\mathcal{R}\{\overline{a},\overline{b}\}$ if and only if $\{a,b\} \nsubseteq U$. Therefore each U of cardinality less than two is incident to each attribute and is by then reducible. We consider $U := \{x,y\}$ of cadinality two; trivially $U\mathcal{R}\{\overline{a},\overline{b}\}$ if and only if $\{a,b\} \neq \{x,y\}$. For an U of cardinality greater than two,

$$U^{\mathcal{R}} = \bigcap \{\{x,y\}^{\mathcal{R}} \mid x,y \in U, x \neq y\}$$

and is by then reducible. Therefore all objects of cardinality different from two can be removed. By identifying each $\overline{a}$ of M with $a \in G$, the reduced context of $(Ext(\mathbb{K}), T, \mathcal{R})$ is isomorphic to $(\mathcal{P}_2(G), \mathcal{P}_2(G), \neq)$. Its concept algebra is a Boolean algebra whose atoms set is $\mathcal{P}_2(G)$. It has the cardinality $2^{|\mathcal{P}_2(G)|} = 2^{\frac{n(n-1)}{2}}$, where n is the cardinality of G. □

The property "distributivity together with complementation" is carried over to the lattice of representable weak complementations. We next consider distributivity alone. We take L a finite distributive lattice; there is an ordered set $(P, \leq)$ such that $(P, P, \ngeq)$ is the standard context of L. The relation $\perp$ is characterized by

$$\{m,n\} \in T \iff \overline{\uparrow m} \cup \overline{\uparrow n} = P \iff \uparrow m \cap \uparrow n = \emptyset.$$

We define a relation $\leq$ on T by

$$\{x,y\} \leq \{s,t\} :\iff \{x,y\} \subseteq \downarrow\{s,t\}.$$

Lemma 4.1.2. *T is ordered by $\leq$.*

Proof. Reflexivity and transitivity are obvious. To prove antisymmetry, we assume $\{x,y\} \leq \{s,t\}$ and $\{s,t\} \leq \{x,y\}$. Note that x and y cannot together be less than s or t; otherwise s or t would belong to $\uparrow x \cap \uparrow y$ which is empty. Even the assertion "$x \leq s$, $y \leq t$ and $s \leq y$, $t \leq x$" cannot hold; otherwise we would have $x \leq s \leq y \leq t \leq x$ which is a contradiction. Without loss of generality our assumption implies $x \leq s$, $y \leq t$ and $s \leq x$ as well as $t \leq y$; therefore $\{x,y\} = \{s,t\}$ and $\leq$ is antisymmetric. Thus $\leq$ is an order relation on T. □

$\mathbb{K}(L)$ denotes the standard context $(P, P, \ngeq)$ of L.

Lemma 4.1.3. *The intents of the context $(Ext(\mathbb{K}), T, \mathcal{R})$ are order filters of the poset $(T, \leq)$.*

Proof. Let $U \in Ext(\mathbb{K})$ with $\{m,n\} \in U^{\mathcal{R}}$ and $\{x,y\} \geq \{m,n\}$. On one hand $\{m,n\} \in U^{\mathcal{R}}$ if and only if $U \subseteq m'$ or $U \subseteq n'$; up to permutation of

m and n we have

$$\begin{aligned}\{x,y\}\geq\{m,n\} &\iff x\geq m,\ y\geq n\\ &\iff \overline{\uparrow x}\supseteq\overline{\uparrow m},\ \overline{\uparrow y}\supseteq\overline{\uparrow n}\\ &\iff x'\supseteq m',\ y'\supseteq n'\\ &\implies x'\supseteq m'\supseteq U \text{ or } y'\supseteq n'\supseteq U\\ &\implies \{x,y\}\in U^{\mathcal{R}}\end{aligned}$$

$U^{\mathcal{R}}$ is an order filter. In general, for a family $\mathcal{U}\subseteq Ext(\mathbb{K})$, the intent $\mathcal{U}^{\mathcal{R}}$ is the intersection of $U^{\mathcal{R}}$, $U\in\mathcal{U}$ which is again an order filter. This completes the proof since each intent is the intent of a subset of $Ext(\mathbb{K})$. □

If the converse holds, i.e. if any order filter is an intent, then by Birkhoff's theorem[1] the concept lattice of $(Ext(\mathbb{K}),T,\mathcal{R})$ will be distributive. It turns out to be easier to reduce the context $(Ext(\mathbb{K}),T,\mathcal{R})$. The relation $\leq$ on T^2 is extended to $\mathcal{P}_2\times T$. For $U\in Ext(\mathbb{K})$ and $\{m,n\}\in T$ we have

$$U\mathcal{R}\{m,n\}\iff U\subseteq\overline{\uparrow m}\text{ or }U\subseteq\overline{\uparrow m}\iff m\notin U\text{ or }n\notin U.$$

Lemma 4.1.4.

(1) If $\{a,b\}\in Ext(\mathbb{K})$ then $\{a,b\}\mathcal{R}\{m,n\}\iff\{a,b\}\not\geq\{m,n\}$.

(2) For an extent $U\in Ext(\mathbb{K})$ containing less than two elements $U^{\mathcal{R}}$ is equal to T.

(3) If $\{a,b\}\notin T$ then $\{a,b\}^{\not\geq}=T$.

(4) For $a,b\in P$, $\downarrow\{a,b\}^{\mathcal{R}}=\{a,b\}^{\not\geq}$.

(5) For an $U\in Ext(\mathbb{K})$ containing more than two elements,

$$\begin{aligned}U^{\mathcal{R}} &= \bigcap\left\{\downarrow\{a,b\}^{\mathcal{R}}\mid\{a,b\}\subseteq U,\ a\perp b\right\}\\ &= \bigcap\left\{\{a,b\}^{\not\geq}\mid\{a,b\}\subseteq U,\ a\perp b\right\}.\end{aligned}$$

Proof.

(2) Each U of cardinality less than two cannot contain a two element set $\{m,n\}$. Thus $U^{\mathcal{R}}=T$.

(3) If $\{a,b\}\notin T$ then $\uparrow a\cap\uparrow b$ contains an c and $\{a,b\}\geq\{m,n\}$ would imply that $m,n\leq c$ and $c\in\uparrow m\cap\uparrow n=\emptyset$. Thus $\{a,b\}\not\geq\{m,n\}$ for all $\{m,n\}\in T$, and $\{a,b\}^{\not\geq}=T$.

[1] Theorem 0.2.3

(1) Let $\{a,b\} \in Ext(\mathbb{K})$ and $\{m,n\} \in T$.

$$\begin{aligned} \{a,b\}\mathcal{R}\{m,n\} &\iff m \notin \{a,b\} \text{ or } n \notin \{a,b\} \\ &\iff \{m,n\} \nsubseteq \{a,b\} = {\downarrow}\{a,b\} \\ &\iff \{a,b\} \ngeq \{m,n\}. \end{aligned}$$

Thus $\{a,b\}^{\mathcal{R}} = \{m,n\}^{\ngeq}$.

(4) We consider ${\downarrow}\{a,b\} \in Ext(\mathbb{K})$ and $\{m,n\} \in T$. We obtain:

$$\begin{aligned} {\downarrow}\{a,b\} \notin \{m,n\}^{\mathcal{R}} &\iff {\downarrow}\{a,b\} \nsubseteq \overline{{\uparrow}m} \text{ and } {\downarrow}\{a,b\} \nsubseteq \overline{{\uparrow}n} \\ &\iff {\uparrow}m \nsubseteq \overline{{\downarrow}\{a,b\}} \text{ and } {\uparrow}n \nsubseteq \overline{{\downarrow}\{a,b\}} \\ &\iff a \geq m \text{ or } b \geq m \text{ and } a \geq n \text{ or } b \geq n \\ &\iff a \geq m \text{ and } b \geq n \text{ or } a \geq n \text{ and } b \geq m \\ &\iff \{m,n\} \subseteq {\downarrow}\{a,b\} \\ &\iff \{m,n\} \leq \{a,b\} \end{aligned}$$

Thus ${\downarrow}\{a,b\} \in \{m,n\}^{\mathcal{R}} \iff \{a,b\} \ngeq \{m,n\}$ for all $\{a,b\} \subseteq P$, and then $\{a,b\}^{\mathcal{R}} = \{a,b\}^{\ngeq}$.

(5) We now consider an $U \in Ext(\mathbb{K})$ of cardinality greater than three. For $\{m,n\} \in T$ we have

$$\begin{aligned} U\mathcal{R}\{m,n\} &\iff {\uparrow}m \subseteq \overline{U} \text{ or } {\uparrow}n \subseteq \overline{U} \\ &\iff (a \geq m \implies a \notin U) \text{ or } (b \geq n \implies b \notin U) \\ &\iff (a \in U \implies a \ngeq m) \text{ or } (b \in U \implies b \ngeq n) \\ &\iff (\{a,b\} \subseteq U \implies a \ngeq m,\ a \ngeq n,\ b \ngeq m,\ b \ngeq n) \\ &\iff \text{For all } \{a,b\} \subseteq U,\ \{a,b\} \ngeq \{m,n\}. \end{aligned}$$

Thus

$$\begin{aligned} U^{\mathcal{R}} &= \bigcap \left\{ \{a,b\}^{\ngeq} \mid \{a,b\} \subseteq U \right\} \\ &= \bigcap \left\{ \{a,b\}^{\ngeq} \mid \{a,b\} \subseteq U,\ a \perp b \right\}. \end{aligned}$$

□

Theorem 4.1.5. *The lattice of representable weak complementations on a finite distributive lattice is again distributive.*

Proof. Let L be a finite distributive lattice. There is a poset $(P, \leq)$ such that $\mathbb{K} := (P, P \ngeq)$ is its standard context. From Theorem 3.2.7 $\underline{\mathfrak{B}}(Ext(\mathbb{K}), T, \mathcal{R})$ is isomorphic to the lattice of representable weak complementations on L. Lemma 4.1.4 allows us to delete all objects of $(Ext(\mathbb{K}), T, \mathcal{R})$ except those which are of the form ${\downarrow}\{a,b\}$ with $a \perp b$. The mapping

$$(f,h)\colon ({\downarrow}\{a,b\}, \{m,n\}) \mapsto (\{a,b\}, \{m,n\})$$

is a context isomorphism from the new context to $(T, T, \not\geq)$. Therefore the concept lattice $\underline{\mathfrak{B}}(Ext(\mathbb{K}), T, \mathcal{R})$ is isomorphic to $\underline{\mathfrak{B}}(T, T, \not\geq)$, and thus distributive. □

A direct consequence is the reciproque of Lemma 4.1.3.

Corollary 4.1.6. *The order filters of the poset $(T, \leq)$ are exactly the intents of the context $(Ext(\mathbb{K}), T, \mathcal{R})$.*

Proof. In the proof of the Theorem 4.1.5 we showed that a reduce context of the context $(Ext(\mathbb{K}), T, \mathcal{R})$ is isomorphic to $(T, T, \not\geq)$. By Birkoff's theorem we get the result. □

This gives us the representation theorem for finite distributive lattices.

Theorem 4.1.7. *On finite distributive lattices, all weak complementations are representable.*

Proof. Let $^\triangle$ be a weak complementation on a finite distributive lattice L. By Corollary 1.2.8 the weak complementation $^\triangle$ is completely determined by its Υ-relation (on an $\wedge$-dense subset M of L). This relation Υ is, by Lemma 1.2.9, an order filter of the relation $\perp$. The relations Υ and $\perp$ are symmetric and are exactly determined by their respective factorization (with respect to the symmetry)

$$\Gamma = \{\{m, n\} \mid \; m, n \in M, \; m \Upsilon n\}$$

and

$$T = \{\{m, n\} \mid \; m, n \in M, \; m \perp n\}.$$

The factorisation of the order $\leq$ on $\perp$ [cf. Lemma 1.2.9] corresponds to the order $\leq$ on T [cf. Lemma 4.1.2] and turns Γ into an order filter of the poset $(T, \leq)$. By Corollary 4.1.6, Γ is an intent of the concept lattice $\underline{\mathfrak{B}}(Ext(\mathbb{K}), T, \mathcal{R})$, which is by Theorem 3.2.7 the lattice of representable weak complementations on L. □

Dually

Corollary 4.1.8. *All dual weak complementations on finite distributive lattices are representable.*

Theorem 4.1.7 and Corollary 4.1.8 give a better characterization of *finite distributive concept algebras*. Contrary to that given in Corollary 3.1.11 the identities of Definition 1.1.1 are enough. Thus *finite distributive concept algebras* are (isomorphic to) finite distributive lattices $\underline{L}$ equipped with two unary operations $^\triangle$ and $^\triangledown$ that satisfy, for all $x, y \in L$, the following equations:

(1) $x^{\triangle\triangle} \leq x$, (1') $x^{\triangledown\triangledown} \geq x$,

(2) $x \leq y \implies x^\triangle \geq y^\triangle$, (2') $x \leq y \implies x^\triangledown \geq y^\triangledown$,

(3) $(x \wedge y) \vee (x \wedge y^\triangle) = x$, (3') $(x \vee y) \wedge (x \vee y^\triangledown) = x$,

Under these conditions axiom (3) is equivalent to $x \vee x^{\triangle} = 1$ while axiom (3') is equivalent to $x \wedge x^{\bigtriangledown} = 0$.

Remark 4.1.1. We have seen that the lattices of weak complementations of distributive lattices are distributive. Does this holds for all lattices? Up to now, for all examples considered, the lattice of all weak complementations is a distributive.

4.2 Congruence Lattices of Distributive Concept Algebras

Recall that the congruence lattice of a concept algebra $\underline{\mathfrak{A}}(G, M, \mathrm{I})$ is isomorphic to $\underline{\mathfrak{B}}(G, M, \rhd\!\!|\!\!\lhd)$, where $\rhd\!\!|\!\!\lhd$ is a certain closed subrelation of $\not\swarrow\!\!\!\!\nearrow$. In the distributive case this subrelation can effectively be characterized. This is demonstrated in the following part, which is based on a recent publication of Bernhard Ganter [Ga04]. Before we proceed, we study closed relations of such contexts in greater generality.

4.2.1 Quasi-ordered contexts

Let (G, M, I) be a formal context and let $\equiv$ be a subrelation of I such that for each $g \in G$ there is some $m \in M$ with $g \equiv m$, and for each $m \in M$ there is some $g \in G$ with $g \equiv m$.

Slightly misusing notation, we will sometimes write $m \equiv g$ instead of $g \equiv m$, and moreover will combine several relational expressions into single terms. For example, "$g\ I\ m \equiv h$" is short for "$g\ I\ m$ and $h \equiv m$".

Definition 4.2.1. We call (G, M, I) **quasi-ordered** (over $\equiv$) if

$$g_1\ I\ m_1 \equiv g_2\ I\ m_2 \text{ implies } g_1\ I\ m_2.$$

For example, the **ordinal scale** $(P, P, \leq)$ is quasi-ordered over the equality relation $=$.

The reason for calling such a context *quasi-ordered* is the following: We can define relations $\leq_M$ on M and $\leq_G$ on G by

$$\begin{aligned} g_1 \leq_G g_2 \quad &:\Longleftrightarrow \quad g_2\ I\ m_1 \equiv g_1 \quad \text{for some } m_1 \in M \\ m_1 \leq_M m_2 \quad &:\Longleftrightarrow \quad m_2 \equiv g_2\ I\ m_1 \quad \text{for some } g_2 \in G. \end{aligned}$$

Proposition 4.2.1. *If (G, M, I) is quasi-ordered, then both $\leq_G$ and $\leq_M$ are quasi-orders.*

Proof. It suffices to give a proof for $\leq_G$. We have to show that $\leq_G$ is reflexive and transitive.

Reflexivity: For each $g \in G$ there is some $m \in M$ with $g \equiv m$, which implies $g\ I\ m$. Thus we get $g\ I\ m \equiv g$, and consequently $g \leq_G g$.

Transitivity: Suppose $g_3 \leq_G g_2 \leq_G g_1$. Then

$g_2\ I\ m_3 \equiv g_3$ for some $m_3 \in M$ and $g_1\ I\ m_2 \equiv g_2$ for some $m_2 \in M$,

which combines to

$$g_1\ I\ m_2 \equiv g_2\ I\ m_3.$$

Since (G, M, I) is quasi-ordered, this implies $g_1\ I\ m_3 \equiv g_3$, which is

$$g_3 \leq_G g_1.$$

□

From each of these two quasi-orders we get an equivalence relation by defining

$$g_1 \sim_G g_2 \quad :\Longleftrightarrow \quad g_1 \leq_G g_2 \text{ and } g_2 \leq_G g_1,$$
$$m_1 \sim_M m_2 \quad :\Longleftrightarrow \quad m_1 \leq_M m_2 \text{ and } m_2 \leq_M m_1.$$

Proposition 4.2.2.

- $g_1 \equiv m \equiv g_2$ *implies* $g_1 \sim_G g_2$, *and*
- $m_1 \equiv g \equiv m_2$ *implies* $m_1 \sim_M m_2$.

Proof. Since $\equiv$ is a subrelation of I, $g_1 \equiv m \equiv g_2$ implies $g_1\ I\ m \equiv g_2$, which is $g_2 \leq_G g_1$. The rest follows analogously. □

Proposition 4.2.3. $g_1 \equiv m_1 \leq_M m_2 \equiv g_2$ *implies* $g_1 \leq_G g_2$, *and dually.*

Proof. If $m_1 \leq_M m_2$, then there is some $g \in G$ with $m_2 \equiv g\ I\ m_1$. From $g\ I\ m_1 \equiv g_1$ we get $g_1 \leq_G g$ and from $g \equiv m_2 \equiv g_2$ we get $g \leq_G g_2$. □

Proposition 4.2.4. *If* $g_1 \leq_G g_2$ *and* $g_1\ I\ m$, *then* $g_2\ I\ m$. *Dually, if* $m_1 \leq_M m_2$ *and* $g\ I\ m_2$, *then* $g\ I\ m_1$.

Proof. $g_1 \leq_G g_2$ implies that for some $m_1 \in M$ we have $g_2\ I\ m_1 \equiv g_1$ and therefore

$$g_2\ I\ m_1 \equiv g_1\ I\ m,$$

which implies $g_2\ I\ m$. □

It is now apparent what the structure of a quasi-ordered context is: after clarification, it is isomorphic to a context $(P, P, \leq)$ for some ordered set $(P, \leq)$. It is therefore not surprising that we can characterize the formal concepts of the complementary context:

Proposition 4.2.5. *Let* (G, M, I) *be a quasi-ordered context (over* $\equiv$*). Then* (A, B) *is a concept of the complementary context* $(G, M, (G \times M) \setminus I)$ *if and only if the following conditions are fulfilled:*

- A *is an order ideal of* $\leq_G$,
- B *is an order filter of* $\leq_M$,

- *$A = B^{\not\equiv}$, and $B = A^{\not\equiv}$.*

Proof. First suppose that (A, B) satisfies the conditions of the proposition. If $m \in M$ is not in A' (where A' if computed with respect to the formal context $(G, M, (G \times M) \setminus I)$), then $a\ I\ m$ for some $a \in A$. We find some $g \in G$ with $g \equiv m$, which gives $a\ I\ m \equiv g$, thus $g \leq_G a$. Since A is an order ideal, this yields $g \in A$ and therefore $m \notin A^{\not\equiv} = B$. This proves $B \subseteq A'$. For the other inclusion, let $m \notin B$. Then, since $B = A^{\not\equiv}$, we find some $a \in A$ with $a \equiv m$ and thus $a\ I\ m$, which shows that $m \notin A'$.

Conversely, let (A, B) be a formal concept of $(G, M, (G \times M) \setminus I)$. Then

$$B = \{m \in M \mid \neg(a\ I\ m) \text{ for all } a \in A\}.$$

According to Proposition 4.2.4, $m_1 \leq_M m_2$ and $\neg(a\ I\ m_1)$ together imply $\neg(a\ I\ m_2)$. For this reason, B is an order filter. The dual argument shows that A must be an order ideal. Since $\equiv$ is a subrelation of I we clearly have $B = A' \subseteq A^{\not\equiv}$. Suppose $m \in A^{\not\equiv}$, $m \notin B$. Then there is some $a \in A$ such that $a\ I\ m$, and some $g \in G$ with $g \equiv m$. But $a\ I\ m \equiv g$ implies $g \leq_G a$, which implies $g \in A$. But that contradicts $m \in A^{\not\equiv}$. □

Theorem 4.2.6. *If (G, M, I) is a quasi-ordered context over $\equiv$, then*

$$(G \times M) \setminus J$$

is a closed subrelation of $(G, M, (G \times M) \setminus I)$ if and only if $I \subseteq J$ and (G, M, J) is quasi-ordered over $\equiv$.

Proof. If J is a quasi-ordered super-relation of I, then Proposition 4.2.5 can be used to describe the concepts of $(G, M, (G \times M) \setminus J)$. It follows immediately from the definitions that the quasi-orders $\leq_G$ and $\leq_M$ induced by J contain the corresponding quasi-orders induced by I. Therefore order filters induced by J are also order filters for I, and the same holds for order ideals. As a consequence we get that each concept of $(G, M, (G \times M) \setminus J)$ also is a concept of $(G, M, (G \times M) \setminus I)$. This makes $(G \times M) \setminus J$ a closed relation.

For the converse assume that $\bar{J} := (G \times M) \setminus J$ is a closed subrelation of $(G, M, (G \times M) \setminus I)$. Then $\bar{J}$ is the union of sets $A \times B$, where (A, B) is a concept of $(G, M, (G \times M) \setminus I)$. So if

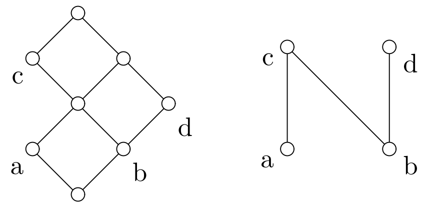
$$g_1 \bar{J} m_2,$$

then there is some formal concept (A, B) of $(G, M, (G \times M) \setminus I)$ such that $g_1 \in A$ and $m_2 \in B$. Now consider arbitrary $g_2 \in G, m_1 \in M$ with $g_2 \equiv m_1$. We have

$$g_2 \in A \quad \text{or} \quad m_1 \in B,$$

because if $g_2 \notin A$ then we find some $b \in B$ with $g_2 \equiv b$ (because $A = B^{\not\equiv}$), and $m_1 \equiv g_2 \equiv b$ enforces that $m_1 \in B$. So we have

$$g_2 \bar{J} m_2 \quad \text{or} \quad g_1 \bar{J} m_1.$$



$\not\geq$	a	b	c	d
a		×	×	×
b	×		×	×
c				×
d	×		×	

Figure 4.1. A small distributive lattice, its ordered set $(P, \leq)$ of irreducibles, and its standard context $(P, P, \not\geq)$.

We have proved

$$g_1 \bar{J} m_2 \Rightarrow \forall_{g_2 \equiv m_1} (g_1 \bar{J} m_1 \text{ or } g_2 \bar{J} m_2).$$

This is logically equivalent to

$$(\exists_{g_2 \equiv m_1}\ g_1\ J\ m_1 \text{ and } g_2\ J\ m_2) \Rightarrow g_1\ J\ m_2,$$

or, in other notation,

$$g_1\ J\ m_1 \equiv g_2\ J\ m_2 \Rightarrow g_1\ J\ m_2,$$

which is precisely the condition of being quasi-ordered for J. □

The theorem can be used to characterize the closed subrelations in the case of doubly founded completely distibutive lattices. Such lattices are concept lattices of **contra-ordinal scales**i.e., formal contexts of the form $(P, P, \not\geq)$, where $(P, \leq)$ is some (quasi-)ordered set.

Corollary 4.2.7. *The closed subrelations of the contra-ordinal scale $(P, P, \not\geq)$, where $(P, \leq)$ is some (quasi-)ordered set, are precisely of the form $\not\sqsupseteq$ for quasi-orders $\sqsubseteq$ containing $\leq$.*

Without proof we mention

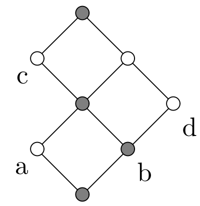
Corollary 4.2.8. *A formal context, the extents of which are precisely the complete sublattices of $\underline{\mathfrak{B}}(P, P, \not\geq)$, is*

$$(\mathfrak{B}(P, P, \not\geq), P \times P, \circ),$$

where

$$(A, B) \circ (p, q) \quad :\Longleftrightarrow \quad \neg(p \in B \text{ and } q \in A).$$

The intents of this context are the quasi-orders containing $\leq$. An example is given below. Consider the lattice in Figure 4.1. Its 35 complete sublattices are the extents of the concept lattice in Figure 4.3.



$\sqsubseteq$	a	b	c	d
a	×		×	×
b	×	×	×	×
c			×	×
d			×	×

$\not\sqsupseteq$	a	b	c	d
a			×	×
b	×		×	×
c				
d				

Figure 4.2. With reference to the shaded element in Figure 4.3: its extent, its intent, and the corresponding closed subrelation of $(P, P, \not\geq)$.

4.2.2 Closed subrelations of $\not\swarrow\!\!\!\swarrow$

With the next corollary we come back to our original theme. Recall that the context $(G, M, \swarrow\!\!\!\swarrow)$ is quasi-ordered over $\nearrow$.

Corollary 4.2.9. *Let (G, M, I) be a doubly founded reduced formal context. The closed subrelations of $(G, M, \not\swarrow\!\!\!\swarrow)$ are precisely of the form $(G \times M) \setminus J$, where $J \subseteq G \times M$ is some relation containing $\swarrow\!\!\!\swarrow$, for which (G, M, J) is quasi-ordered over $\nearrow$.*

The condition of being quasi-ordered is a closure condition. For given relations $R \subseteq G \times M$ and $\equiv$ (as in Subsection 4.2.1) there is always a smallest relation S containing R for which (G, M, S) is quasi-ordered over $\equiv$. In the case of the arrow relations, we write

$$\swarrow\!\!\!\swarrow = \mathrm{trans}(\swarrow, \nearrow),$$

and, more generally, if D and U are relations containing $\swarrow$ and $\nearrow$, respectively, then

$$(g, m) \in \mathrm{trans}(D, U) :\Longleftrightarrow \begin{cases} \text{there are } g_1 = g, g_2, \ldots, g_k \in G \text{ and} \\ m_1, \ldots, m_k, m_k = m \in M \text{ such that} \\ g_i \; D \; m_i \text{ holds for all } i \in \{1, \ldots, k\} \\ \text{and} \\ g_i \; U \; m_{i-1} \text{ holds for all } i \in \{2, \ldots, k\}. \end{cases}$$

Recall that each formal concept (A, B) of $(G, M, \not\swarrow\!\!\!\swarrow)$ defines a compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I) by $H := G \setminus A$, $N := B$. If (A, B) is also a concept of the closed subrelation $(G \times M) \setminus \mathrm{trans}(D, U)$, we will say that $(H, N, I \cap H \times N)$ is compatible with (D, U).

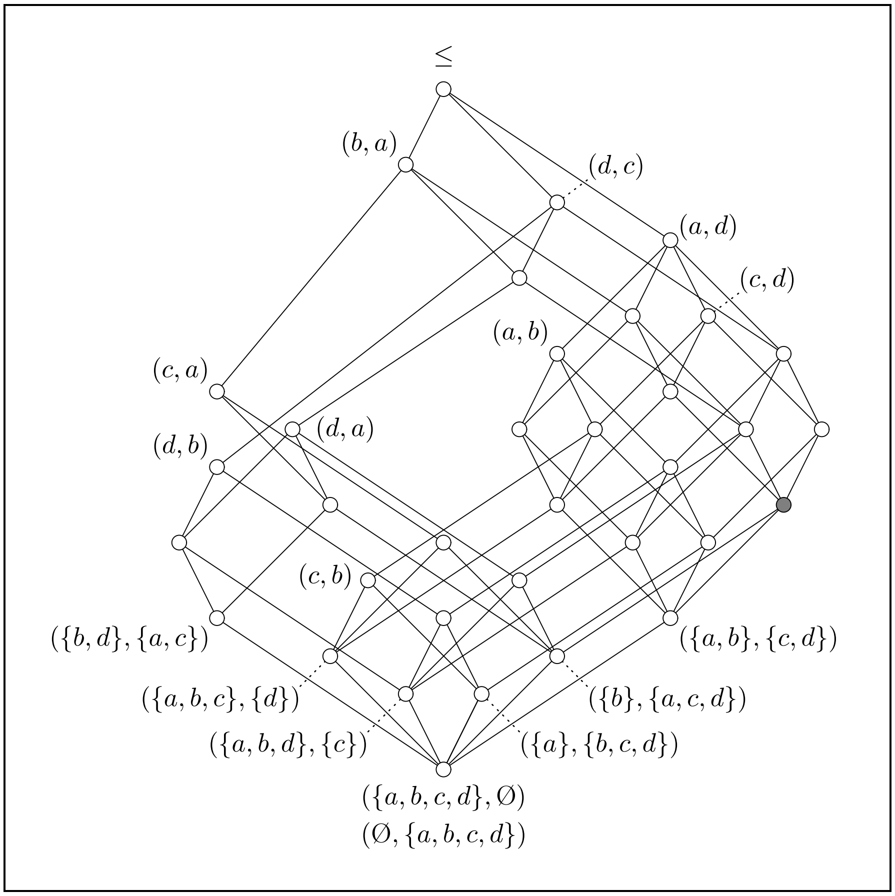


Figure 4.3. The lattice of complete sublattices of the lattice in Figure 4.1. The shaded element serves as an example of how to interprete this lattice. Its meaning is explained in Figure 4.2.

Proposition 4.2.10. *A subcontext* $(H, N, I \cap H \times N)$ *is compatible with* (D, U) *if and only if*

(i) $h \in H$ *and* $h\ U\ m$ *together imply* $m \in N$*, and*

(ii) $n \in N$ *and* $g\ D\ n$ *together imply* $g \in H$*.*

Proof. Let us abbreviate

$$\overline{T} := (G \times M) \setminus \text{trans}(D, U).$$

"⇒": If $(H, N, I \cap H \times N)$ is compatible with (D, U), then $(G \setminus H, N)$ is a formal concept of $(G, M, \overline{T})$. Now if $n \in N$ and $g\ D\ n$, then (g, n) is in trans(D, U) and therefore $(g, n) \notin \overline{T}$. This implies $g \notin N^{\overline{T}} = G \setminus H$. Thus g must be in H.

If $h \in H$ and $h\ U\ m$, then choose some n and g such that $h \swarrow\!\!\!\!\!\nearrow n$ and $g \swarrow\!\!\!\!\!\nearrow m$. Such elements exist since (G, M, I) is reduced. Each subcontext compatible with (D, U) must in particular be compatible and therefore arrow-closed, thus $n \in N$ can be inferred from $h \swarrow\!\!\!\!\!\nearrow n$. Moreover

$$g \swarrow\!\!\!\!\!\nearrow m\ U\ h \swarrow\!\!\!\!\!\nearrow n$$

implies $(g, n) \in \text{trans}(D, U)$ and thus $(g, n) \notin \overline{T}$. Then $g \notin N^{\overline{T}} = G \setminus H$, and consequently $g \in H$ and thus $m \in N$.

"$\Leftarrow$": Conversely let $(H, N, I \cap H \times N)$ be a subcontext that satisfies the conditions of the proposition. We will show that $(G \setminus H, N)$ is a formal concept of $(G, M, \overline{T})$. Let $n \in N$ and $(g, n) \notin \overline{T}$, thus $(g, n) \in \text{trans}(D, U)$. By the definition of $\text{trans}(D, U)$ there must be a sequence

$$g = g_1\ D\ m_1\ U\ g_2\ D\ m_2 \ldots U\ g_k\ D\ m_k = n.$$

Using the conditions along this sequence from right to left, and assuming $n \in N$, we get $g \in H$. Thus $(g, n) \in \overline{T}$ for all $g \notin H$, which proves that $G \setminus H \subseteq N^{\overline{T}}$. If $h \in H$ then there is some $m \in M$ such that $h \swarrow\!\!\!\!\!\nearrow m$, which implies $m \in N$ and $(h, m) \notin \overline{T}$. Thus $G \setminus H = N^{\overline{T}}$.

It remains to show that $(G \setminus H)^{\overline{T}} \subseteq N$. So let $m \notin N$, and consider some $g \swarrow\!\!\!\!\!\nearrow m$. $g \in H$ would contradict the first condition. Therefore $g \in G \setminus H$, $(g, m) \notin \overline{T}$ and thus $m \notin (G \setminus H)^{\overline{T}}$. $\square$

To find the closed relation characterizing concept algebra congruences, we may look for "additional arrow relations" $\curlywedge$ and $\curlyvee$, such that

$$\bar{J} = \text{trans}(\swarrow \cup \curlywedge, \nearrow \cup \curlyvee).$$

We can actually split our considerations and treat the two unary operations separately [Proposition 1.2.5]. We will consider only one, say $^{\triangle}$.

4.2.3 $^{\triangle}$–compatible subcontexts

Note that a compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is called $^{\triangle}$-compatible, if

$$(\varphi_{H,N}(A, B))^{\triangle} := \Pi_{H,N}((A, B)^{\triangle})$$

defines a unary operation on $\underline{\mathfrak{B}}(H, N, I \cap H \times N)$ (see Section 2.3). From this condition it follows automatically that the so defined operation satisfies the equations which are satisfied by $^{\triangle}$.

Lemma 4.2.11. *A compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is $^{\triangle}$-compatible iff*

$$\forall_{n \in N} \quad G \setminus n' \subseteq ((G \setminus n')'' \cap H)''.$$

Proof. We must show that for arbitrary concepts (A_1, B_1) and (A_2, B_2) we have:

$$\text{if } \varphi(A_1, B_1) = \varphi(A_2, B_2) \text{ then } \varphi((A_1, B_1)^{\triangle}) = \varphi((A_2, B_2)^{\triangle}).$$

This can be simplyfied to the case that $A_2 = (A_1 \cap H)''$. The simplified condition then is that

$$\varphi((A, A')^{\triangle}) = \varphi(((A \cap H)'', (A \cap H)')^{\triangle})$$

holds for all concept extents A of (G, M, I), which is equivalent to

$$(G \setminus A)' \cap N = (G \setminus (A \cap H)'')' \cap N \quad \text{for all extents } A.$$

Yet another equivalent reformulation is that for all extents A we have

$$\forall_{n \in N} \ \ (G \setminus A \subseteq n' \iff G \setminus (A \cap H)'' \subseteq n').$$

Since $(A \cap H)'' \subseteq A$, the direction $\Leftarrow$ is always true and it suffices to prove

$$\forall_{n \in N} \ \ (G \setminus A \subseteq n' \Rightarrow G \setminus (A \cap H)'' \subseteq n'),$$

which is equivalent to

$$\forall_{n \in N} \ \ (G \setminus n' \subseteq A \Rightarrow G \setminus n' \subseteq (A \cap H)'').$$

For the special case that $A = (G \setminus n')''$ this is exactly the condition of the lemma. It remains to show that that condition is also sufficient.

Suppose therefore that A is some extent for which $G \setminus n' \subseteq A$. Since A is an extent, we infer $(G \setminus n')'' \subseteq A$, and consequently

$$(G \setminus n')'' \cap H \subseteq A \cap H.$$

If the condition of the lemma holds, then

$$G \setminus n' \subseteq ((G \setminus n')'' \cap H)'' \subseteq (A \cap H)'',$$

as was to be proved. □

In order to better understand the condition of the lemma, let us abbreviate $E := G \setminus n'$. Then the condition is

$$E \subseteq (E'' \cap H)'',$$

which is clearly equivalent to

$$E'' = (E'' \cap H)''.$$

The latter condition states that the extent E'' has a generating system in H. A reformulation of the lemma therefore is:

Lemma 4.2.12. *A compatible subcontext $(H, N, I \cap H \times N)$ of (G, M, I) is ${}^{\triangle}$-compatible iff for each attribute $n \in N$ the set H contains a generating set for the closure of $G \setminus n'$.*

It now becomes clearer how to define the "extra arrows"-relation $\curlywedge$: For each attribute $n \in N$ we must define $\curlywedge$ in such a way that the transitive closure

$$\text{trans}(\swarrow \cup \curlywedge, \nearrow)$$

points from n to some generating set of $(G \setminus n')''$.

But there may be many generating sets, and it is not clear which one to choose in general. However, there is a class of lattices where the generating systems are essentially unique.

An **extremal point** of a concept extent A in a clarified context is an irreducible object $e \in A$ such that

$$e \notin ((A \cap G_{irr}) \setminus \{e\})''.$$

An extremal point of a subset is an extremal point of the extent it generates. Certainly, an extremal point of A must be contained in *every* generating set of A that consists of irreducibles. For certain lattices, the extremal points always form a generating set:

Theorem 4.2.13. *[GW99, Thm. 44] In a finite meet-distributive lattice each concept extent is generated by its extremal points.*

Proposition 4.2.14. *If $\underline{\mathfrak{B}}(G, M, I)$ is finite and meet-distributive, then a compatible subcontext $(H, N, I \cap H \times N)$ with $H \subseteq G_{irr}$, $N \subseteq M_{irr}$ is also ${}^{\triangle}$-compatible iff H contains for each $n \in N$ all extremal points of $G \setminus n'$.*

Proof. This is immediate from Theorem 4.2.13 together with Lemma 4.2.12. □

In the meet-distributive case it is now clear how to define the relation $\curlywedge$:

$$g \curlywedge m :\iff g \text{ is an extremal point of } G \setminus m'.$$

With this notation we get from Propositions 4.2.10 and 4.2.14

Theorem 4.2.15. *If $\underline{\mathfrak{B}}(G, M, I)$ is finite and meet-distributive, then the congruence lattice of the concept algebra $\underline{\mathfrak{A}}(G, M, I)$ is isomorphic to the concept lattice*

$$\underline{\mathfrak{B}}(G_{irr}, M_{irr}, G_{irr} \times M_{irr} \setminus \mathit{trans}(\swarrow \cup \curlywedge, \nearrow)).$$

Combining this with the dual relation

$$g \curlyvee m :\iff m \text{ is extremal in } M \setminus g'$$

we get

Theorem 4.2.16. *The complete congruences of a finite distributive concept algebra $\underline{\mathfrak{A}}(G, M, I)$ are given by those compatible subcontexts of*

$$(G_{irr}, M_{irr}, I \cap G_{irr} \times M_{irr}),$$

which are both $\curlywedgedownarrow$*-closed and* $\curlyveeuparrow$*-closed. The congruence lattice is isomorphic to the concept lattice of*

$$(G_{irr}, M_{irr}, G_{irr} \times M_{irr} \setminus trans(\swarrow \cup \curlywedgedownarrow, \nearrow \cup \curlyveeuparrow)).$$

	a	b	c	d	e
1		×	×	×	×
2	×		×	×	×
3	×	×		×	
4	×	×	×		
5	×	×			

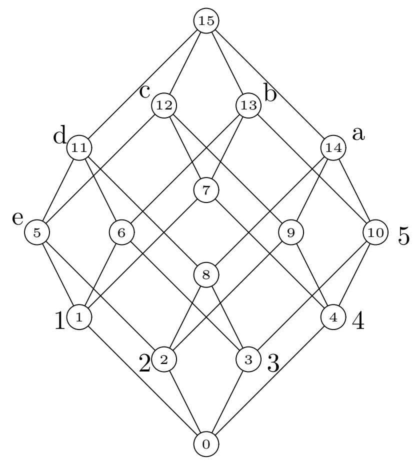


Figure 4.4. A formal context and its concept lattice.

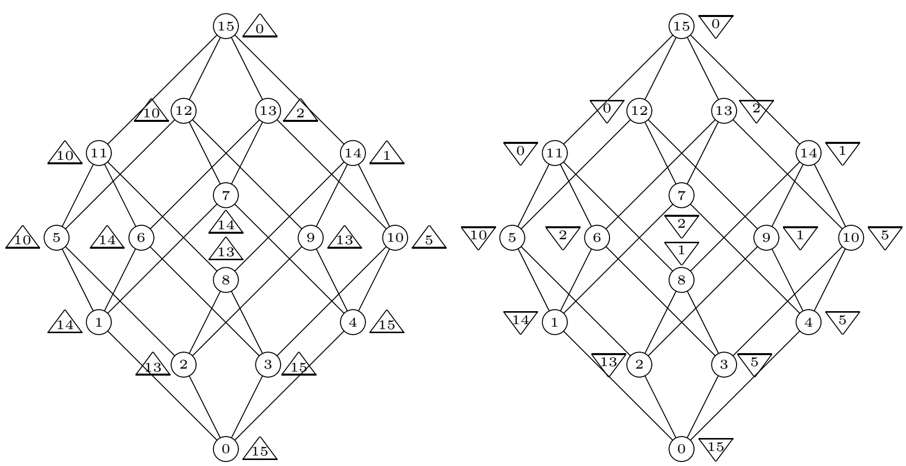


Figure 4.5. The dicomplementation of the concept algebra $\underline{\mathfrak{A}}(G, M, I)$, where (G, M, I) is the formal context from Figure 4.4. The number in the triangle next to a concept gives its weak negation and its weak opposition, respectively.

	a	b	c	d
1	$\swarrow\!\!\!\nearrow$	×	×	×
2	×	$\swarrow\!\!\!\nearrow$	×	×
3	×	×	$\swarrow\!\!\!\nearrow$	×
4	×	×	×	$\swarrow\!\!\!\nearrow$

	a	b	c	d
1				
2				
3			$\curlywedge\!\!\!\curlyvee$	$\curlywedge\!\!\!\curlyvee$
4			$\curlywedge\!\!\!\curlyvee$	$\curlywedge\!\!\!\curlyvee$

	a	b	c	d
1	×			
2		×		
3			×	×
4			×	×

	a	b	c	d
1		×	×	×
2	×		×	×
3	×	×		
4	×	×		

Figure 4.6. Determining the congruence lattice of the concept algebra in Figure 4.4.

We close our considerations by determining the congruence lattice of the concept algebra in Figure 4.4. Figure 4.6 shows four contexts. The first is the reduced context for Figure 4.4, with the arrow relations. The second displays the relations $\curlywedge$ and $\curlyvee$. The third context ist $(G, M, \operatorname{trans}(\swarrow \cup \curlywedge, \nearrow \cup \curlyvee))$. The fourth finally is the the complementary context of the third, which by Theorem 4.2.16 describes the concept algebra congruences.

The congruence lattice obviously is an eight element Boolean lattice. Consider, as an example, the formal concept $(\{1\}, \{b, c, d\})$ of the fourth context. It corresponds to the compatible subcontext $(\{2, 3, 4\}, \{b, c, d\})$. The induced congruence has the classes

$$\{0, 1\}, \{2, 5\}, \{3, 6\}, \{4, 7\}, \{8, 11\}, \{9, 12\}, \{10, 13\}, \{14, 15\}.$$

It can easily be read off from Figure 4.5 that this congruence indeed is compatible with the dicomplementation.

4.3 Varieties Generated by Chains

The variety of all lattices has a unique minimal subvariety, the variety of all distributive lattices, generated by a two-element chain. The two element Boolean algebra generates the variety of all Boolean algebras, and this is the unique minimal subvariety of the variety of all weakly dicomplemented lattices. There are, however, distributive dicomplemented lattices that are not Boolean algebras. How does a minimal variety covering the variety generated by a two element dicomplemented lattice look like?

4.3.1 Lattices with unique weak dicomplementation.

The equations $x^{\triangle} \vee x = 1$ and $x^{\triangledown} \wedge x = 0$ hold in all weakly dicomplemented lattices. Therefore bounded chains can bear only one weak dicomplementation, namely, the trivial weak dicomplementation. We denote by C_n the n-element chain. Trivial weak dicomplementations are directly indecomposable. In fact each product contains $(0,1)$ and, we have

$$(0,0) \neq (1,0) = (0,1)^{\triangledown} = (0,1)^{\triangle} = (1,0) \neq (1,1)$$

which is impossible.

Remark 4.3.1. For trivial weak dicomplementations the congruence classes $0/\theta$ and $1/\theta$ are singletons for each congruence $\theta \neq \triangledown$ (the all relation). In fact if $a \neq 0$ and $a\theta 0$, then $0 = a^{\triangledown}\theta 0^{\triangledown} = 1$ implies $\theta = \triangledown$. Conversely if θ is a congruence on a lattice L such that $|0/\theta| = |1/\theta| = 1$ then θ is a congruence on L equipped with the trivial weak dicomplementation.

Proposition 4.3.1. *Let L be a lattice. The lattice $A := \{0\} \oplus L \oplus \{1\}$ can bear only the trivial weak dicomplementation.*

In this case $x^{\triangledown}$ is the pseudocomplement while $x^{\triangle}$ is the dual pseudocomplement of x. Hence, A is a double p-algebra. Not all lattices bearing only the trivial weak dicomplementation have the presentation in Proposition 4.3.1. For example the lattice M_3 can bear only the trivial weak dicomplementation. Therefore we can ask ourself, if there is a description of such lattices accepting only the trivial weak dicomplementation. We restrict our investigation on finite lattices. For a lattice L we denote by $J(L)$ the set of join-irreducible elements and by $M(L)$ the set of meet-irreducible. $A_t(L)$ denotes the set of atoms of L and $C_t(L)$ the set of coatoms.

Lemma 4.3.2. *A weak complementation $^{\triangle}$ is trivial if and only if $b^{\triangle} = 1$ for each $b \in C_t(L)$ and $a^{\triangledown} = 0$ for each $a \in A_t(L)$.*

Proof. Note that for any element $x \in L$ there are $a \in A_t(L)$ and $b \in C_t(L)$ such that $a \leq x \leq b$. Thus $x^{\triangle} \geq b^{\triangle} = 1$ and $x^{\triangledown} \leq a^{\triangledown} = 0$. □

The set of all weak complementations on a finite lattice L forms a lattice. This lattice has a bottom element, which is the weak negation of the formal context $(J(L), M(L), \leq)$ and a top element, which is the weak negation of the context $(L, L, \leq)$. Thus L can bear only the unique weak complementation if the weak negations of these two contexts coincide. Furthermore the lattice of weak complementations on L is isomorphic[2] to the concept lattice of the context $(Ext(\mathbb{K}(L), T, \mathcal{R})$ where $Ext(\mathbb{K}(L))$ is the set of extents of the reduced context $\mathbb{K}(L) = (J(L), M(L), \leq)$

$$T = \{\{m,n\} \mid m,n \in M(L) \text{ and } m' \cup n' = J(L)\}$$

[2] [see Theorem 3.2.7]

and for any extent U of $\mathbb{K}(L)$ and any pair $\{m,n\} \in T$,

$$U\mathcal{R}\{m,n\} \iff U \subseteq m' \text{ or } U \subseteq n'.$$

Remark 4.3.2. The smallest context with nonempty object and attribute sets with less crosses is the context $(\{g\},\{m\},\emptyset)$. Its concept lattice is isomorphic to the two element chain. Moreover if (G,M,I) is a formal context with $G \neq \emptyset \neq M$ then $\mathfrak{B}(G,M,I)$ contains at least two elements whenever there exists $(g,m) \notin I$. In this case the concepts (G,G') and (M',M) are definitely distinct.

Lemma 4.3.3. *A concept lattice* $\underline{\mathfrak{B}}(G,M,I)$ *has exactly one element if and only if* $G \times M = I$.

Corollary 4.3.4. *A lattice* L *can bear only the trivial weak complementation if and only if for any pair* $\{m,n\} \subseteq M(L)$ *there is* $g \in J(L)$ *such that* $g \nleq m$ *or* $g \nleq n$.

Proof. L can bear only the trivial weak complementation if and only if $\mathfrak{B}(Ext(\mathbb{K}(L),T,\mathcal{R}))$ has exactly one element. Thus $Ext(\mathbb{K}(L)) \times T = \mathcal{R}$. If $T \neq \emptyset$ then for any pair $\{m,n\}$ in T we would have $J(L) \subseteq m'$ or $J(L) \subseteq n'$, and by then $1 \in M(L)$, which is absurd. Thus $T = \emptyset$, and for any pair $\{m,n\}$ of $M(L)$, $m' \cup n' \neq J(L)$. Equivalently, for any pair $\{m,n\} \subseteq M(L)$ there is an element g in $J(L)$ such that $g \nleq m$ or $g \nleq n$. □

Proposition 4.3.5. *Let* L *be a lattice. Then* $A := \{0\} \oplus L \oplus \{1\}$ *is a subdirectly irreducible weakly dicomplemented lattice if and only if* L *is subdirectly irreducible.*

Proof. From Remark 4.3.1 it is easy to see that any congruence relation $\tau \in Con(A)$ can be written in the form

$$\tau = \theta \cup \triangle,$$

where $\theta \in Con(L)$. □

Corollary 4.3.6. *Let* L *be a distributive lattice. Then the weakly dicomplemented lattice* A *from Proposition 4.3.5 is subdirectly irreducible if and only if* $|L| \leq 2$. *In this case* A *is a double Stone algebra.*

Thus C_n is subdirectly irreducible if and only if $n \leq 4$. All C_n are double Stone algebras. In [Ka74, Theorem 1] it was shown that C_n for $1 \leq n \leq 4$ are the only subdirectly double Stone algebras. In addition the lattice of all subvarieties of double Stone algebras is a 4-element chain. There are many other subdirectly irreducible weak dicomplementations whose lattice does not have the presentation in Proposition 4.3.1. For example, there are 4 weak dicomplementations on the lattice $C_2 \times C_2$ among which only the Boolean algebra structure is decomposable; the remaining ones are simple. A subdirect decomposition of C_5 can be visualized on Figure 4.7.

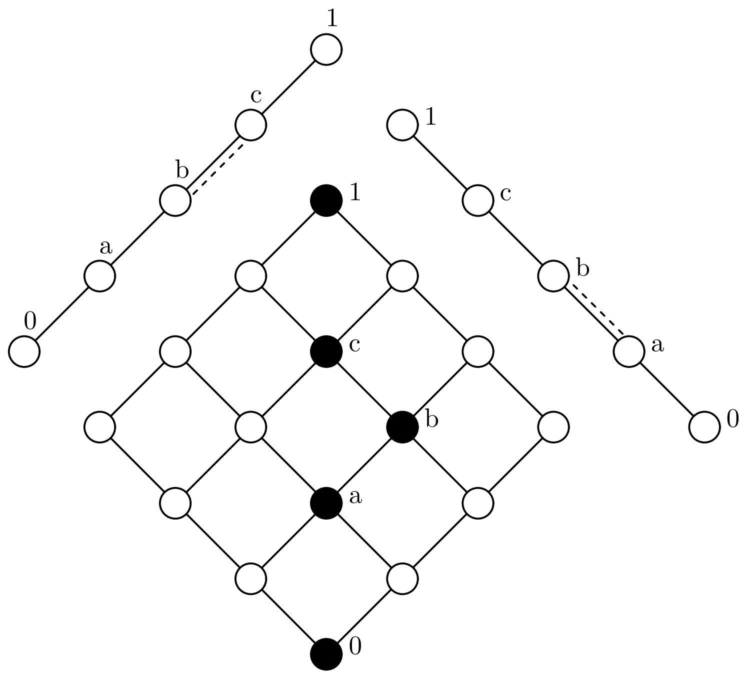


Figure 4.7. Sudirect decomposition of C_5.

4.3.2 Variety generated by C_3

From now on we set $A := C_3$. In term of cardinality, A is the smallest weakly dicomplemented lattice which is not a Boolean algebra. It is simple and is a homomorphic image of all longer chains.

Theorem 4.3.7. *$V(A)$ is a discriminator variety.*

Proof. We are going to prove that the **discriminator function** d defined by

$$d(a,b,c) = \begin{cases} a & : \quad a \neq b \\ c & : \quad a = b \end{cases}$$

is representable by a term. The variety of lattices has a majority term

$$M(x,y,z) := (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

which is also a term on $V(A)$. It is by then enough to show that d preserves subalgebras of A^2 ([BS81, Lemma 10.4 (Baker-Pixley)]). Set $A := \{0, a, 1\}$. The lattice A^2 can be visualized on Figure 4.8, where xy stands for (x, y). Trivially d preserves the subalgebras A^2 and $\{00, 11\}$. It is easy to see that $\{00, aa, 11\}$ and $\{00, 01, 10, 11\}$ are also preserved. It remains to prove that this is again the case for the subalgebras $B_5 := \{00, 11, a0, 01, 10, a1\}$ and $B_6 := \{00, 11, 0a, 01, 10, 1a\}$. If d does not preserves B_5 there would exist

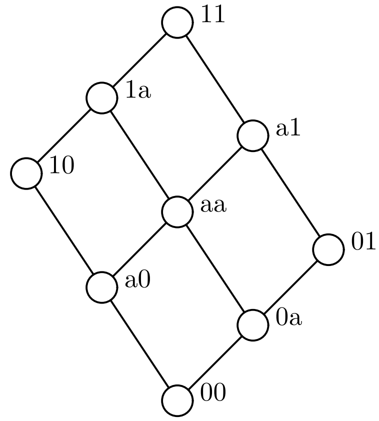


Figure 4.8. $C_3 \times C_3$

$X, Y, Z \in B_5$ such that $d(X, Y, Z)$ is in $\{0a, aa, 1a\}$. Then we would have $d(\pi_2(X), \pi_2(Y), \pi_2(Z)) = a$. This cannot happen since $d(x, y, z) \in \{x, y, z\}$ and for all X in B_5, $\pi_2(X) \neq a$. Then d preserves B_5. Analogeously d preserves B_6. Therefore there is a term representing d on A. □

A is quasiprimal and generates an arithmetical variety. Thus there is a term $m(x, y, z)$ in the variety generated by A such that

$$m(x, y, x) = m(x, y, y) = m(y, y, x) = x.$$

Remark 4.3.3. The variety $V(A)$ is the class of all regular double Stone algebras. In [Ka73] it was shown that every regular double p-algebra is automatically a double Heyting algebra. The relative pseudocomplementation operation *to* is given by

$$a \to b = (a^* \vee b^{**})^{**} \wedge \left[(a \vee a^*)^+ \vee a^* \vee b \vee b^*\right]. \qquad (H)$$

Since Heyting algebras are arithmetical with the term

$$p(x, y, z) = [(x \to y) \to z] \wedge [(z \to y) \to x] \wedge (x \vee y),$$

the term $m(x, y, z)$ in question is obtained by substituting any $c \to d$ in $p(x, y, z)$ by the term from (H).

Corollary 4.3.8. *The algebra A is functionally complete.*

Proof. The algebra A_A is quasiprimal since A is quasiprimal. It has only one automorphism and only one subalgebra (itself). Then A_A is primal ([BS81, Corollary 10.8 (Foster-Pixley)]). i.e A is functionally complete. □

Remark 4.3.4. Theorem 4.3.7 can be deduced from Remark 4.3.3 since $V(A)$ is arithmetical and every subalgebra of A is simple.[3] Corollary 4.3.8

[3]See [Pi72, Theorem 1.2]

also follows because the discriminator of A is a term, and hence, a polynomial.

Then every finitary function on A is representable by a polynomial (a term on A_A).

Corollary 4.3.9. *All unary functions on A preserving the two element subalgebra are term representable.*

Proof. From Corollary 4.3.8 we know that all functions on A are representable by a polynomial since the algebra A is functionally complete. The aim is to prove that the polynomial c_a can be removed in all polynomial functions preserving $\{0,1\}$. All polynomials of the form $p^{\triangle}$ and $p^{\triangledown}$, where p and q are polynomials, preserve the two element subalgebra and the following equivalences hold:

1. $((p \wedge c_a) \vee q)^{\triangle} \equiv q^{\triangle}$,
2. $((p \wedge c_a) \vee q)^{\triangledown} \equiv (p \vee q)^{\triangledown}$,
3. $((p \vee c_a) \wedge q)^{\triangledown} \equiv q^{\triangledown}$,
4. $((p \vee c_a) \wedge q)^{\triangle} \equiv (p \wedge q)^{\triangle}$.

Thus c_a can be removed in such polynomials. We assume that $(p \wedge c_a) \vee q$ is a unary polynomial (function) preserving $\{0,1\}$ with p and q unary term functions. For $x \in \{0,1\}$, if $q(x) = 0$ we will also have $p(x) = 0$. Therefore,

(5) $p(a) \neq a \implies (p \wedge c_a) \vee q \equiv q$,

(6) $p(a) = a$ and $q(a) \neq 0 \implies (p \wedge c_a) \vee q \equiv q$,

(7) $p(a) = a$ and $q(a) = 0 \implies (p \wedge c_a) \vee q \equiv p \vee q$

and c_a can be removed from any unary polynomial function $(p \wedge c_a) \vee q$ preserving $\{0,1\}$. Similarly c_a can be removed in any unary polynomial $(p \vee c_a) \wedge q$ if it preserves $\{0,1\}$. Inductively all c_a can be removed in all unary polynomial functions preserving the two element subalgebra. □

This corollary tells us that the base set of the clone of unary terms on A, which is also the free algebra on one element in the variety generated by A, is the set of all unary functions preserving the subalgebra $\{0,1\}$. The equivalences (1) to (4) in the proof still hold for n-ary polynomials ($n \geq 2$). This observation suggests a generalization to determine the free weakly dicomplemented lattice in the variety generated by A.

Proposition 4.3.10. *The set $\{\wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1\}$ generates all $\{0,1\}$-preserving operations on A.*

Proof. $\rho := \{0,1\}$ a unary relation on $A := \{0,a,1\}$. We denote by $< F >$ the clone generated by $F := \{\wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1\}$. We will prove that $Pol_A(\rho)$

(functions preserving the relation ρ) is $< F >$. From Rosenberg's characterization of maximal clones [Ro70], we know that $Pol_A(\rho)$ [4] is one of the 18 maximal clones over A. Obviously $< F >$ is contained in $Pol_A(\rho)$ since any operation in F preserves ρ. Dietlinde Lau gave in [Lau82] a description of all submaximal clones over A. There are 15 (hereunder denoted by F_1 to F_{15}) and none of them contains F as we can see in the following case studies.

1. $F_1 := Pol_A(\rho) \cap Pol_A\{0\}$ does not contain the nullary operation 1.
2. $F_2 := Pol_A(\rho) \cap Pol_A\{0\}$ does not contain the nullary operation 0.
3. $F_3 := Pol_A(\rho) \cap Pol_A \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ does not contain the unary operation $^{\triangledown}$ since $\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\triangledown} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.
4. $F_4 := Pol_A(\rho) \cap Pol_A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ does not contain the nullary operation 0.
5. $F_5 := Pol_A(\rho) \cap Pol_A \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ does not contain the binary operation $\wedge$ since $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.
6. $F_6 := Pol_A(\rho) \cap Pol_A\{a\}$ does not contain 0.
7. $F_7 := Pol_A(\rho) \cap Pol_A \begin{pmatrix} 0 & a & 1 & 0 & 1 \\ 0 & a & 1 & 1 & 0 \end{pmatrix}$ does not contain the binary operation $\vee$ since $\begin{pmatrix} a \\ a \end{pmatrix} \vee \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$.
8. $F_8 := Pol_A(\rho) \cap Pol_A \begin{pmatrix} 0 & a & 1 & 0 & a & 1 & a \\ 0 & a & 1 & a & 0 & a & 1 \end{pmatrix}$ does not contain $^{\triangledown}$ since $\begin{pmatrix} 0 \\ a \end{pmatrix}^{\triangledown} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
9. $F_9 := Pol_A \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & a \end{pmatrix}$ as F_8 does not contain $^{\triangledown}$.

[4] ρ is a central relation.

10. $F_{10} := Pol_A \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & a \end{pmatrix}$ does not contain the unary operation ${}^{\triangle}$ since $\begin{pmatrix} 1 \\ a \end{pmatrix}^{\triangle} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

11. $F_{11} := Pol_A \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & a \end{pmatrix}$ does not contain the operation $\vee$ since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}$.

12. $F_{12} := Pol_A \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & a \end{pmatrix}$ does not contain the operation $\wedge$ since $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$.

13. $F_{13} := Pol_A \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & a & 1 & a \\ 0 & 1 & 0 & 1 & a & 0 & a & 1 \end{pmatrix}$ does not contain the operation $\vee$ since $\begin{pmatrix} 0 \\ a \end{pmatrix} \vee \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$.

14. $F_{14} := Pol_A \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & a & a & a & a \end{pmatrix}$ does not contain the unary operation ${}^{\triangledown}$ since $\begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix}^{\triangledown} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

15. $F_{15} := Pol_A \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & a & a \end{pmatrix}$ does not contain the operation $\vee$ since $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}$.

□

Corollary 4.3.11. *The set of all n-ary functions on A preserving the subalgebra $\{0,1\}$, endowed with the pointwise defined operations $\wedge, \vee, {}^{\triangle}, {}^{\triangledown}, 0$ and 1, is isomorphic to the free weakly dicomplemented lattice on n generators in the variety generated by A and is of cardinality $2^{2^n} \times 3^{3^n - 2^n}$.*

Proof. Recall that the clone of n-ary term operations on A is a free algebra in $V(A)$ over n generators. This is by Proposition 4.3.10 the clone of all functions preserving the subalgebra $\{0,1\}$. They are exactly $2^{2^n} \times 3^{3^n - 2^n}$ such functions. □

There is a characterization of regular double Stone algebras by T. Hecht and T. Katriňák in [HK74, Theorem 4]. The focus is more on the structure theory. From a computational point of view, Corollary 4.3.11 is worhty to be mentioned.

We cannot expected like in Boolean algebras to have a nice formula for the free algebra generated by n elements. M. Skorsky proved that the weak negation and the weak opposition generate a monoid. An element of this monoid is the composition of the two operations [Sk89]. Thus the free algebras are infinite.

5
Negation and Contextual Logic.

This part discusses the formalization of the notion "*negation of a concept*". The notion of "*concept*" has been formalized in the early eighties and led to the theory of *Formal Concept Analysis*. The formalization of the negation of a concept is needed in order to develop a mathematical theory of human thought based on "*concept as a basic unit of thought*", similar to that developed by Boole, based on signs and classes. Weak negation and weak opposition have been introduced for this purpose. We cannot expect all *laws of negation* to be fulfilled by a weak negation or a weak opposition. In this part we demonstrate how the choice of appropriate subsets can reconcile mathematics and philosophy.

5.1 From Logic to Formal Concept Analysis

5.1.1 From Logic to Lattice Theory

In the first half of the nineteenth century, George Boole's attempt to formalize *Logic*[1] in [Bo54] led to the concept of Boolean algebras. Boole gave himself the task "*to investigate the fundamental laws of those operations*

[1] Kant [1723-1804] considered Logic as "... a science *a priori* of the necessary laws of thinking, not, however, in respect of particular objects but all objects in general: it is a science, therefore of the right use of the understanding and of reason as such, not subjectively, i.e. not according to empirical (psychological) principles of how the

of the mind by which reasoning is performed; to give expression to them in the symbolical language of calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities; and finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind ". The main operations he encoded are conjunction, disjunction, negation, universe and nothing, for which he derived some laws. He elaborated this logic as a theory of symbolic operations applied to classes of objects. Charles Sanders Peirce [1839-1914] and Ernst Schröder [1841-1902] introduced the notion of lattice at the end of nineteenth century as they were investigating the axiomatics of Boolean algebras. Independently Richard Dedekind [1831-1916] got the same concept while working on ideals of algebraic numbers.

The general development of lattice theory really started in the mid-thirties with the work of Garrett Birkhoff [1911-1996]. Other mathematicians like Valére Glivenko, Karl Menger, John von Neumann, Oystein Ore, ... contributed to establish lattice as mathematical theory.

5.1.2 Restructuring Lattice Theory: Formal Concept Analysis

Lattice theory became a successful subject in mathematics and attracted many researchers. But ***why develop lattice theory***? In [Wi82] the author made this observation: "*lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker: the result is that the theory and even many of its parts become more isolated* ". This isolation did not affect only lattice theory, but many other sciences. Wille was influenced by a book of Harmut von Hentig [He72], in which he discussed the status of the humanities and sciences. It was then urgent to "restructure" theoretical developments in order to integrate and rationalize origins, connections, interpretations and applications. Wille understood "restructuring lattice theory" as "*an attempt to unfold lattice theoretical concepts, results, and methods in a continuous relationship with their surroundings*", with the aim "*to promote a better communication between lattice theorists and potential users of lattice theory*".

Even the pioneer did not neglect this aspect. For example Birkhoff wrote: "*lattice theory has helped to simplify, unify and generalize mathematics, and it has suggested many interesting problems.*" In a survey paper on "lattices

understanding thinks, but objectively, i.e. according to *a priori* principles of how it ought to think".

and their applications"[Bi38], he set up a more general viewpoint for lattice theory: "*lattice theory provides a proper vocabulary for discussing order, and especially systems which are in any sense hierarchies*".

The approach to lattice theory outlined in [Wi82] is based on an attempt to reinvigorate the general view of order. He went back to the origin of the lattice concept in the nineteenth-century attempts to formalize logic, where the study of hierarchies of concepts played a central rôle. A *concept* is determined by its *extent* and its *intent*; the extent consists of all objects belonging to the concept while the intent is the multitude of all properties valid for all those objects[2] . The *hierarchy* of concepts is given by the relation of "subconcept" to "superconcept", i.e. the extent of the subconcept is contained in the extent of the superconcept while, reciprocally, the intent of the superconcept is contained in the intent of the subconcept.

5.2 Contextual Logic

5.2.1 Contextual Attribute Logic

Contextual Logic has been introduced with the aim to support knowledge representation and knowledge processing. An attempt to elaborate a contextual logic started with "Contextual Attribute Logic" in [GW99a]. The authors considered "signs" as attributes and outlined how this correspondence may lead to a development of a Contextual Attribute Logic in the spirit of Boole. Contextual Attribute Logic focuses, in a formal context (G, M, I), on the formal attributes and their extents, understood as the formalizations of the extensions of the attributes. It deals with logical combination and relation between attributes. This is a "local logic".

The logical relationships between formal attributes are expressed via their extents. For example they said that "an attribute m implies an attribute n if $m' \subseteq n'$", and that "m and n are incompatible if $m' \cap n' = \emptyset$". In order to have more expressivity in Contextual Attribute Logic the authors introduced compound attributes of a formal context (G, M, I) by using the operational symbols $\neg$, $\bigwedge$ and $\bigvee$.

- For each attribute m they defined its negation, $\neg m$, to be a compound attribute, which has the extent $G \setminus m'$. Thus g is in the extent of $\neg m$ if and only if g is not in the extent of m.
- For each set $A \subseteq M$ of attributes, they defined the conjunction, $\bigwedge A$, and the disjunction, $\bigvee A$, to be the compound attributes that have the extents $\bigcap\{m' \mid m \in A\}$ and $\bigcup\{m' \mid m \in A\}$ respectively.

[2]See [Ei29]

- Iteration of the above compositions leads to further compound attributes, the extents of which are determined in the obvious manner.

Observe that the complement of the extent m' is imposed to be an extent. The new extents are generated by the family $\{m' \mid m \in M\}$ and are closed under complementation, arbitrary union and intersection. Moreover to each concept generated as indicated above, an attribute is assigned. This corresponds to Boole's logic where signs are attributes and classes are extents generated by the family $\{m' \mid m \in M\}$ with respect to complementation, union and intersection. The negation of an attribute is not necessary an attribute of the initial context. The new context is a dichotomic context in which all concepts are attribute concepts.

5.2.2 Contextual Concept Logic

At the second stage, a contextual concept logic should be developed, by mathematizing the doctrines of concepts, judgments and conclusions on which the human thinking is based. Wille divided the development of a contextual logic in three parts: a "Contextual Concept Logic", a "Contextual Judgment Logic", and a "Contextual Conclusion Logic". In [KV03] the authors compared various approaches to Contextual Judgment Logic. We are more concerned with the first step. Wille introduced concept algebras in [Wi00] with the aim "*to show how a Boolean concept Logic may be elaborated as a mathematical theory based on Formal Concept Analysis*".

To extend the Boolean Attribute Logic to a Boolean Concept Logic, the main problem is the negation, since the conjunction and disjunction can be encoded by the meet and join operation of the concept lattice. How can you define a negation of a concept? To define a negation of a sign, Boole first set up a universe of discourse, then took the complement of the class representing the given sign and assigned to the class he obtained a sign that he called the negation of the given sign. Here the universe is encoded by 1 and nothing by 0. Doing an analogy with formal concepts, the first problem is that the class of extents and the class of intents need not be closed under complementation. To have a negation as an operation on concepts[3], you can take as negation of a concept (A, B) of a formal context (G, M, I), the concept generated by the complement of its extent, namely $((G \setminus A)'', (G \setminus A)')$. Although the *principle of excluded middle*[4] is satisfied the *principle of contradiction* does not always hold. So is obtained the weak negation of the concept (A, B) in (G, M, I). On the intent side

[3] There is another approach where the negation of a concept is not necessary a concept. See for example [HLSW], [Vo02] or [Wi00].

[4] See Section 5.3

the operation obtained is the weak opposition. It satisfies the *principle of contradiction* but not in general the *principle of excluded middle.*

5.3 Negation

5.3.1 Philosophical backgrounds

The problem of negation is almost as old as philosophy. It has been handled by many philosophers, with more or less contradicting points of view or confusing statements in one side, and with some attempts to formalize it on other side. Aristoteles [384-322 BC] considered "**negation**" as the **opposite** of "**affirmation**". But what does *affirmation* mean? Even if we consider *affirmation*, as all what we know or can represent, we still need the meaning of *opposite*. According to Georg Friedrich Meier [1718-1777] the negation is the *representation of the absence of something.* Therefore a negation can only be represented in mind. This point of view is shared by Wilhelm Rosenkrantz [1821-1874] who stressed that *a pure negation exists only in thinking*, and only as opposite from an affirmation. Up to now we still need a clear definition of the terms *opposite* and *affirmation.* John Locke [1632-1704] had doubts on the existence of negative representation; according to him the **not** *only means lack of representation.*

Meister Eckhart [1260-1328] stated that each creature has a negation in himself. Although he did not mention how the negation is obtained from a given creature, this implied that each creature should possess a negation. This idea was not welcomed by all philosophers. For example Wilhelm Jerusalem [1854-1923] thought that only a judgment can be rejected, and not a representation, like Brentano [1838-1917] wished.

On the way to formalize negation, we can note the idea of Georg Hagemann [1832-1903] for whom each negation is '...originally an affirmation of being different'. Could we consider each object "A" different of "B" as a negation of B? Not really. But at least each creature should be different from its negation. An even more "formal" definition is given by Adolph Stöhr [1855-1921]. He said that *not*-A is a derived name from A according to the type of opposing derivation meaning what remains after removing A. Coming back to George Boole [1815-1864], he understood a negated sign as the representation of the complement of the class represented by the original sign in the given universe of discourse. The opposing derivation in this case is simply "taking the complement". We will refer to this as **Boolean negation**.

For a concept the negative is generally considered as the opposite or contrary of the positive and means the lacking of such properties. The trend is to assign to a negative concept an intent with negative attributes.

5.3.2 Some properties of a negation

Is there really a formal definition of "*negation*"? The formalization of the negation by Boole, with other operations of human thought led to Boolean algebras. The negation is encoded by a unary operation, that satisfies many nice properties. For example it is an involution, is a complementation, is antitone, satisfies the de Morgan laws,. . . etc. Which properties characterize a negation? In Philosophy some conditions have been highlighted.

Principium exclusi tertii. This principle states that A or *not-*A is true. It is called **principle of excluded middle**. Thus an operation abstracting a negation should be a **dual semicomplementation**. This principle is not accepted by all logicians. It is, for example, rejected by intuitionist logicians.

Principium contradictionis. This principle says that A and *not-*A is false. It is called **principle of contradiction**. Thus an operation abstracting a negation should be a **semicomplementation**.

Duplex negatio affirmat. This is one of the most wished properties for a negation. This law wants *not-not-*A to have the same logical value as A. This means that *a double negation is an affirmation.* It is called **law of double negation**. Thus an operation abstracting a negation should be an **involution**.

De Morgan laws. To the above stated principles, it is always useful to know how to get the negation of complex concepts from the basic ones building this concept. For any pairs of concepts, their disjunction and their conjunction can be represented. The **meet de Morgan law** states that the negation of the conjunction of two concepts is the disjunction of their negations, while for the **join de Morgan law** the negation of the disjunction of two concepts is the meet of the conjunction of their negations. These two laws are called the **de Morgan[5] laws**.

A brief history on the discussion on Logic can be found in the paper "*19th Century Logic between Philosophy and mathematics*" by Volker Peckhaus[6].

5.3.3 Laws of negation and concept algebras

L denotes a concept lattice $\underline{\mathfrak{B}}(G, M, I)$.

Concepts with negation. If there is a negation on a context $\mathbb{K}$ it should not depend on the intent side or extent side definition. Therefore the

[5] De Morgan [1806-1871]

[6] http://www.phil.uni-erlangen.de/~p1phil/personen/peckhaus/texte/logic_phil_math.html

two unary operations should be equal.

$$(A,B)^{\triangle} = (A,B)^{\triangledown} \iff (A,B)^{\triangle} \leq (A,B)^{\triangledown} \iff \bar{A}'' \subseteq \bar{B}'$$

Thus any object g not in A has all attributes not in B. Each concept algebra in which the two unary operations coincide is said **with negation**. Concept algebras with negation are Boolean algebras (Corollary 3.3.5). In addition the subset of elements with negation,

$$B(L) := \{x \in \mathfrak{B}(G,M,I) \mid x^{\triangle} = x^{\triangledown}\},$$

is a Boolean algebra, and is a subalgebra of L [7]. If the equality $^{\triangle} =^{\triangledown}$ fails, we can consider the negation as a partial operation defined on $B(L)$. Its domain $B(L)$ is quite often small.

Law of double negation. The set of elements that satisfy the law of double negation with respect to the weak opposition is called the skeleton[8] and denoted by $S(L)$. The dual skeleton $\bar{S}(L)$ is the set of elements that satisfy the law of double negation with respect to the weak negation. An operation $\sqcup$ has been introduced on $S(L)$ and turned $(S(L),\wedge,\sqcup,^{\triangledown},0,1)$ into an ortholattice. That is a bounded lattice with an antitone complementation which is an involution. With a slight modification of the join operation we obtain a set of concepts on which all the above mentioned laws of negation are satisfied. A similar modification can be made with $S(\bar{L})$. Skeleton and dual skeleton both contain all elements with negation. The skeleton or dual skeleton can be neither distributive nor uniquely complemented even if the lattice L were distributive. Although the operation $^{\triangle}$ on the skeleton is a complementation, it is no longer a weak complementation[9]. Skeletons are generally not sublattices of L. They are sublattices of L if and only if they are Boolean algebras. In particular $S(L) = L$ if and only if $(L,\wedge,\vee,^{\triangledown},0,1)$ is a Boolean algebra. This means that the meet and the join operations are not modified only in the case of a Boolean negation.

Principle of contradiction/principle of excluded middle. The weak negation satisfies the principle of excluded middle. But the principle of contradiction fails. The weak opposition satisfies the principle of contradiction, but the principle of excluded middle fails. One of the weak operations does the job if we only need one of these principles. If the both principle should hold we would automatically get a Boolean negation, since the weak operations would be complementations[10].

[7]This a Boolean part of L. See Corollary 3.3.8
[8]See Section 1.3
[9]See for example the concept algebra in Figure 1.2 with their skeletons in Figure 1.4
[10]See Corollary 3.3.11

Instead of assuming these two principles on the whole concept algebra, we can look for concepts on which the weak negation (resp. weak opposition) respects them. These concepts are again complemented. Their extent complements (resp. their intent complements) are again extents (resp. intents) for a weak negation (resp. a weak opposition). They are consider in Proposition 5.3.1.

De Morgan laws. The weak negation satisfies the meet de Morgan law while the weak opposition satisfies the join de Morgan law. But the join de Morgan law fails for a weak negation and the meet de Morgan law fails for a weak opposition. Assuming the de Morgan laws turns the skeletons into complemented sublattices. Thus the Boolean part and the skeletons are identical. (See Proposition 5.3.2 below).

Is there any characterization of concepts which extent complements are again extents. We denote by E_c the set of those concepts of a context $\mathbb{K}$.

$$E_c = \{(A,B) \in \mathfrak{B}(\mathbb{K}) \mid (\bar{A},\bar{A}') \in \mathfrak{B}(\mathbb{K})\}$$

Proposition 5.3.1. *The following assertions are valid:*

(i) $E_c = \{x \in \mathfrak{B}(\mathbb{K}) \mid x \wedge x^{\triangle} = 0\}$.

(ii) $E_c \subseteq \bar{S}(\mathfrak{B}(\mathbb{K}))$.

(iii) $x \in E_c$ *implies* $x^{\triangle} \in E_c$ *and* $x^{\triangle}$ *is the pseudocomplement of* x *in* L.

(iv) Moreover if $\mathfrak{B}(\mathbb{K})$ *is distributive then*

- *(a)* E_c *is a sublattice of* $\mathfrak{B}(\mathbb{K})$,
- *(b)* E_c *is a sublattice of* $\bar{S}(\mathfrak{B}(\mathbb{K}))$ *and*
- *(c)* E_c *is a Boolean algebra.*

Proof. (i) Let $x \in E_c$. We have $x^{\triangle} = (\bar{A}'', \bar{A}') = (\bar{A}, \bar{A}')$. Thus $x \wedge x^{\triangle} = 0$. Conversely if $x \wedge x^{\triangle} = 0$ for an $x \in \mathfrak{B}(\mathbb{K})$ then $A \cap \bar{A}'' = \emptyset$ and $\bar{A}'' \subseteq \bar{A}$. This means that $\bar{A}'' = \bar{A}$ and $x \in E_c$.

(ii) $x \in E_c \implies x = (x \wedge x^{\triangle}) \vee (x \wedge x^{\triangle\triangle}) = x \wedge x^{\triangle\triangle}$. Thus $x \in E_c \implies x = x^{\triangle\triangle}$. For x and y in E_c does $x \wedge y, x \vee y \in E_c$?

(iii) $x \in E_c$ implies $x^{\triangle} \wedge x^{\triangle\triangle} \leq x^{\triangle} \wedge x = 0$. The rest follows from Lemma 3.3.12.

(iv) We assume that $\mathfrak{B}(\mathbb{K})$ is distributive.

(a) Let $x, y \in E_c$. The assertion

$$(x \vee y) \wedge (x \vee y)^{\triangle} \leq (x \vee y) \wedge x^{\triangle} \wedge y^{\triangle} = 0$$

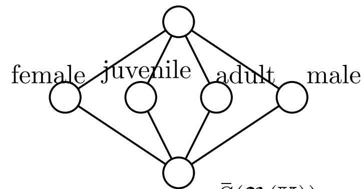
is valid and implies that $x \vee y \in E_c$. It also holds:

$$(x \wedge y) \wedge (x \wedge y)^{\triangle} = (x \wedge y) \wedge (x^{\triangle} \vee y^{\triangle}) = 0.$$

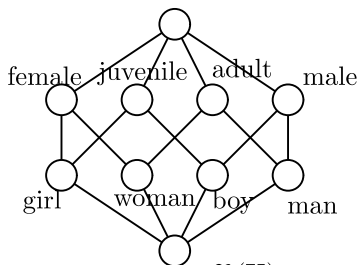
Thus E_c is a sublattice of L.

	female	juvenile	adult	male
girl	×	×		
woman	×		×	
boy		×		×
man			×	×

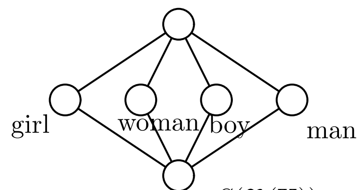
$\mathbb{K}$



$\bar{S}(\mathfrak{B}(\mathbb{K}))$



$\mathfrak{B}(\mathbb{K})$



$S(\mathfrak{B}(\mathbb{K}))$

Figure 5.1. Complemented extents.

(b) To prove that E_c is a sublattice of $\bar{S}(L)$ it remains to show that $\bar{\sqcap}$ is the restriction of $\wedge$. This is immediate since for any x and y in E_c we have

$$x \bar{\sqcap} y = (x^{\triangle} \vee y^{\triangle})^{\triangle} = (x \wedge y)^{\triangle\triangle} = x \wedge y$$

(c) E_c is a complemented sublattice of a distributive lattice and thus a Boolean algebra.

□

If we assume that one of the first three above mentioned principles holds in a concept algebra, we automatically get the other since the unary operation is forced to be a Boolean algebra complementation. If we consider only the elements satisfying the law of double negation, the skeleton, we get an ortholattice. Again here all the above mentioned laws hold. Thus if we want to work on the same context, we can consider a negation as a partial operation defined only for concepts of the skeleton.

Proposition 5.3.2. *If we assume the join de Morgan law for the weak negation then the dual skeleton is a complemented sublattice of L, thus the Boolean part.*

Proof. We assume the join de Morgan law for the weak negation. That is

$$(x \vee y)^{\triangle} = x^{\triangle} \wedge y^{\triangle}.$$

From $x \vee x^{\triangle} = 1$ we get $x^{\triangle} \wedge x^{\triangle\triangle} = 0$. Since all elements of the dual skeletons are of the form $x^{\triangle}$ and $x^{\triangle} \vee x^{\triangle\triangle} = 1$ it follows that the dual skeleton is complemented. We are going to prove that $\bar{S}(L)$ is a sublattice of L. Let x and y be elements of $\bar{S}(L)$. By the join de Morgan law we get

$$x \bar{\sqcap} y = (x^{\triangle} \vee y^{\triangle})^{\triangle} = x^{\triangle\triangle} \wedge y^{\triangle\triangle} = x \wedge y.$$

Thus $x \wedge y$ belongs to $\bar{S}(L)$. We have already seen that $x \vee y$ belongs to $\bar{S}(L)$. Thus $\bar{S}(L)$ is a sublattice of L. □

Dually assuming the meet de Morgan law for the weak opposition turns the skeleton into a complemented sublattice of L.

5.4 Embeddings into Boolean Algebras

In this section we discuss how a context can be enlarged to get "negation" as full operation on concepts. This works quite well in the distributive case, by embedding them into Boolean algebras. But in general we cannot embed a non distributive lattice into a Boolean algebra. We propose in this case an order embedding.

5.4.1 Distributive concept algebras

For every set S the context $(S, S, \neq)$ is reduced. The concepts of this context are precisely the pairs $(A, S \setminus A)$ for $A \subseteq S$. Its concept lattice is isomorphic to the power set lattice of S. How does its concept algebra look like? For each concept $(A, S \setminus A) \in \mathfrak{B}(S, S, \neq)$ we have

$$(A, S \setminus A)^{\triangle} = ((S \setminus A)'', (S \setminus A)') = (S \setminus A, A)$$

and

$$(A, S \setminus A)^{\triangledown} = (A', A'') = (S \setminus A, A) = (A, S \setminus A)^{\triangle}.$$

The operations $^{\triangle}$ and $^{\triangledown}$ are equal. Thus $\big(\mathfrak{B}(S, S, \neq), \wedge, \vee, ^{\triangle}, \emptyset, S\big)$ is a Boolean algebra isomorphic to the powerset algebra of S. To get the converse we make use of the following fact:

Lemma 5.4.1.

(i) *If a and b are incomparable elements of a weakly dicomplemented lattice L such that none of them has 1 as weak complement then $a \vee b$ cannot be $^{\triangle}$-compatible.*

(ii) *If $a \leq c$ and a is not $^{\triangle}$-compatible then c is not $^{\triangle}$-compatible.*

(i)' *Dually if a and b are incomparable elements of a weakly dicomplemented lattice L such that none of them has 0 as dual weak complement then $a \wedge b$ cannot be $^{\triangledown}$-compatible.*

(ii)' If $a \geq c$ and is not $^{\triangledown}$-compatible then c is not $^{\triangledown}$-compatible.

Proof. We set $c := a \vee b$. Obviously $c \not\leq a$. If $c \leq a^{\triangle}$ this would imply $a \leq c \leq a^{\triangle}$ and $a^{\triangle} = 1$ which is a contradiction. □

We obtain

Theorem 5.4.2. *If $\mathbb{K}$ is a clarified context with no full line or empty column and such that $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangle}, 0, 1\big)$ and $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangledown}, 0, 1\big)$ are Boolean algebras then there is a set S such that $\mathbb{K}$ is isomorphic to $(S, S, \neq)$. In this situation the Boolean algebras $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangle}, 0, 1\big)$ and $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangledown}, 0, 1\big)$ are equal.*

Proof. The standard context of the lattice $\mathfrak{B}(\mathbb{K})$ is $(J, M, \leq)$ with J the set of atoms and M the set of coatoms. Note that $|M| = |J|$. Moreover the mapping $i : a \mapsto a'$ is a bijection from J onto M such that

$$a \leq b \iff b \neq i(a) \quad \forall a \in J \text{ and } b \in M$$

Therefore the context $(J, M, \leq)$ is isomorphic to $(J, J, \neq)$ by identifying each element $i(a) \in M$ with $a \in J$. We denote by G the object set of $\mathbb{K}$ and by S the set of its irreducible objects. If $G \neq S$ then there is an object $g \in G$ such that $\gamma g \geq \gamma g_1 \vee \gamma g_2$ with g_1 and g_2 in S. Note that

$$x \in S \implies (\gamma x)^{\triangle} \neq 1.$$

g is, By Lemma 5.4.1, not $^{\triangle}$-compatible. Thus $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangle}, 0, 1\big)$ cannot be isomorphic to $\big(\mathfrak{B}(S, S, \neq), \wedge, \vee, ^{\triangle}, 0, 1\big)$. This contradicts the assumption that $\big(\mathfrak{B}(\mathbb{K}), \wedge, \vee, ^{\triangle}, 0, 1\big)$ is a Boolean algebra. The similar argument using the dual of Lemma 5.4.1 proves that $\mathbb{K}$ is also attribute reduced. □

Corollary 5.4.3. *Clarified contexts without full line and empty column with negation are exactly those isomorphic to $(S, S, \neq)$ for some set S.*

In general contexts are not reduced. To define a negation on a context $\mathbb{K}$, we can first reduce $\mathbb{K}$. If its reduced context is a copy of $(S, S, \neq)$ for a certain set S, then we are done. We define a negation on $\mathbb{K}$ by taking the concept algebra of $(S, S, \neq)$. In this case the set of concepts with negation (Boolean part) is the whole concept lattice.

If $\underline{\mathfrak{B}}(\mathbb{K})$ is not a Boolean lattice, we might assume that our knowledge are not enough to get a negation. One option might be to extent the context to a larger one in which all concepts will have a negation. We should however make sure that doing so does not destroy the structure of elements with negation. This means that concepts having already a negation in the old context should have the same negated concept in the extended context.

Let us examine the distributive case. To each distributive lattice can be assigned a context $(P, P, \not\geq)$ (contra nominal scale) where $(P, \leq)$ is a poset. In the context $(P, P, \neq)$ all concepts have a negation. This context is obtained by extending the relation $\not\geq$ to $\neq$ on P. Of course $\not\geq$ is a closed subrelation of $\neq$. Therefore $\mathfrak{B}(P, P, \ngeq)$ is a sublattice of $\mathfrak{B}(P, P, \neq)$. It remains to verify that the negation is preserved on concepts with negation. This is straightforward since each concept with negation has a complement. In the extended context the complementation is unique and is at the same time the negation. Thus we are again able to define a negation on each context whose concept lattice is distributive. Unfortunately we cannot expect to have a lattice embedding from a nondistributive lattice into a Boolean algebra. An alternative is to find an embedding preserving the hierarchy of concept and the negation on concepts with negation. This can be done using a set representation of lattices.

5.4.2 General case: order embedding

Unless otherwise stated, the lattices considered here are assumed to be doubly founded. L usually denotes a weakly dicomplemented lattice. Since a concept is determined by its intent and its extent, the negation of a context, if there is one, should not depend on the intent- or extent side definition. i.e. the two weak operations should coincide. Recall that a concept is said with negation if its weak negation and its weak opposition coincide. $B(L)$ denotes the set of elements with negation (Boolean part). $\underline{B}$ is a Boolean algebra. The canonical context of $\underline{B}(L)$ is (isomorphic to) a subcontext representing $B(L)$.

Theorem 5.4.4. *Each weakly dicomplemented lattice L can be order embedded in a Boolean algebra B in such a way that the structure of elements with negation are preserved.*

Proof. The set $J(L)$ of join irreducible elements of L is a supremum dense subset of L. Its powerset $\mathcal{P}(J(L))$ is a Boolean algebra. We shall embed L into $\mathcal{P}(J(L))$. We define i by $i(x) = \downarrow x \cap J(L)$. Trivially

$$i(x \wedge y) = i(x) \cap i(y).$$

Therefore i is order preserving. Moreover i is an injective mapping. In fact if $x \nleq y$ then there is an $a \in J(L)$ such that $a \leq x$ and $a \nleq y$. Thus $a \in i(x)$ and $a \notin i(y)$. Therefore

$$i(x) = i(y) \implies x = y.$$

To prove that i is an order embedding we have to show that

$$x \leq y \iff i(x) \leq i(y).$$

We assume that $i(x) \leq i(y)$. We get

$$i(x) = i(x) \cap i(y) = i(x \wedge y).$$

Since i is injective we get $x = x \wedge y$ and $x \leq y$. Thus i is an order embedding.

It remains to prove that the weakly dicomplemented lattice operations on $B(L)$ are preserved.

$$i(0) = \emptyset; \quad i(1) = J(L); \quad i(x \wedge y) = i(x) \cap i(y).$$

Let $x \in B(L)$. We have

$$\emptyset = i(0) = i(x \wedge x^{\bigtriangledown}) = i(x \wedge x^{\triangle}) = i(x) \cap i(x^{\triangle})$$

This equality implies that $i(x^{\triangle}) \subseteq i(x)'$. To prove the converse inclusion we consider an element a in $i(x)'$.

$$a \in J(L) \setminus i(x) = J(L) \setminus \downarrow x.$$

$a \in J(L)$ and is by then ${}^{\triangle}$-compatible. Thus $a \not\leq x$ implies $a \leq x^{\triangle}$. i.e

$$a \in J(L) \cap \downarrow x^{\triangle} = i(x^{\triangle}).$$

Thus

$$i(x^{\bigtriangledown}) = i(x^{\triangle}) = i(x)'$$

and the weak operations restricted on $B(L)$ are preserved. For the join we have

$$i(x \vee y) = i((x \vee y)')' = i(x' \wedge y')' = (i(x') \cap i(y'))' = (i(x)' \cap i(y)')' = i(x) \cup i(y).$$

Therefore $i(B(L))$ is a Boolean algebra isomorphic to $B(L)$. In other words the structure of elements with negation is preserved by the order embedding. □

Remark 5.4.1. The similar construction holds with the set $M(L)$ of meet irreducible elements and the mapping $j\colon x \mapsto j(x) := \uparrow x \cap M(L)$ from L into $\mathcal{P}(M(L))$. If L is a distributive lattice then $M(L)$ is isomorphic to $J(L)$ and $\mathcal{P}(M(L))$ is isomorphic to $\mathcal{P}(J(L))$. In this case there is an isomorphism $\psi : \mathcal{P}(M(L)) \to \mathcal{P}(J(L))$ such that $\psi \circ j = i$.

If we do not assume the distributivity, it might happen that $M(L)$ and $J(L)$ are of different cardinality. Without loss of generality we can assume that $|M(L)| \leq |J(L)|$ holds. Therefore there exists an embedding $\phi : \mathcal{P}(M(L)) \to \mathcal{P}(J(L))$ such that $\psi \circ j = i$.

Remark 5.4.2. If $({}^{\triangle_1}, {}^{\bigtriangledown_1})$ and $({}^{\triangle_2}, {}^{\bigtriangledown_2})$ are weak dicomplementations on a bounded lattice L such that $({}^{\triangle_1}, {}^{\bigtriangledown_1})$ is finer than $({}^{\triangle_2}, {}^{\bigtriangledown_2})$ then for all $x \in L$ we have

$$x^{\bigtriangledown_2} \leq x^{\bigtriangledown_1} \leq x^{\triangle_1} \leq x^{\triangle_2}.$$

Thus $B_2(L) \subseteq B_1(L)$ and $D_2(L) \subseteq D_1(L)$. The finer a weak dicomplementation is, the larger its set of elements with negation is. In the case of doubly founded lattice the finest dicomplementation is induced by the context $(J(L), M(L), \leq)$. Unfortunately $S_1(L)$ and $S_2(L)$ can be incomparable. On Figure 1.4 the context is not reduced but gives the largest skeletons. If we reduce that context the skeleton will be the four element

Boolean algebra. If the context is $(L, L, \leq)$ (largest clarified context), then the skeletons will be (a copy of) the two element Boolean algebra.

In Figure 1.2 the weak dicomplementation has the largest skeleton. But the corresponding context is not reduced. Is there any description of the context giving the largest skeleton?

Concluding Remarks

On our way towards the representation theory of concept algebras we have introduced and investigated the class of weakly dicomplemented lattices. This equational class belongs naturally to the best candidates, that can generate the equational theory of concept algebras. The results we obtained suggest that this variety can be considered on its own. Its mathematical theory should be developed. A next step would be for example to work out this theory similar to that of (distributive) double p-algebras. A sort of "triple" characterization of a weakly dicomplemented lattice (by some of its subsets: skeletons, Boolean part, dense subsets and structure maps) would be a tool of great importance. Congruences could by then be better described. Although the free algebras in this class are infinite, their structure can be examined. Another step might be to develop a duality theory for this class. The Prime Ideal Theorem is a promising starting point. Some problems on weakly dicomplemented lattices could then be translated in other fields and vice versa. There is another notion of negation introduced by K. Deiters and M. Erné [DE98]. This negation is not defined as an operation on concepts, but as an operation on complete lattices. The connections between these two points of view should be worked out. Of course the contribution to Contextual Logic should guide this future works.

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Versicherung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

I affirm that I have written this dissertation without any inadmissible help from any third person and without recourse to any other aids; all sources are clearly referenced. The dissertation has never been submitted in this or similar form before, neither in Germany nor in any foreign country.

Die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der Technischen Universität Dresden vom 20. März 2000 erkenne ich an.

I accept the regulations for the conferral of a doctorate of the faculty for mathematics and natural sciences of the Technical University of Dresden (published on March, 20th 2000).

Die vorliegende Dissertation wurde an der Technischen Universität Dresden unter der wissenschaftlichen Betreuung von Prof. Dr. Bernhard Ganter angefertigt.

I have written this dissertation at the Technical University of Dresden under the scientific supervision of Prof. Dr. Bernhard Ganter.

Dresden, 12. Februar 2004 Léonard Kwuida