Layer-adapted meshes for convection-diffusion problems

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Preface

This is a book on numerical methods for singular perturbation problems—in particular stationary convection-dominated convection-diffusion problems. More precisely it is devoted to the construction and analysis of layer-adapted meshes underlying these numerical methods.

An early important contribution towards the optimization of numerical methods by means of special meshes was made by N. S. Bakhvalov [13] in 1969. His paper spawned a lively discussion in the literature with a number of further meshes being proposed and applied to various singular perturbation problems. However, in the mid 1980s this development stalled, but was enlivend again by G. I. Shishkin's proposal of piecewise-equidistant meshes in the early 1990s [93, 74]. Because of their very simple structure they are often much easier to analyse than other meshes, although they give numerical approximations that are inferior to solutions on competing meshes. Shishkin meshes for numerous problems and numerical methods have been studied since and they are still very much in vogue.

With this contribution we try to counter this development and lay the emphasis on more general meshes that—apart from performing better than piecewise-uniform meshes—provide a much deeper insight in the course of their analysis.

In this monograph a classification and a survey are given of layer-adapted meshes for convection-diffusion problems. It tries to give a comprehensive review of state-of-the art techniques used in the convergence analysis for various numerical methods: finite differences, finite elements and finite volumes.

While for finite difference schemes applied to one-dimensional problems a rather complete convergence theory for arbitrary meshes is developed, the theory is more fragmentary for other methods and problems and still requires the restriction to certain classes of meshes.

The roots of this monograph are a survey lecture presented at the Oberwolfach seminar *Numerical Methods for Singular Perturbation Problems*, 8-14 April 2001 organized by Pieter W. Hemker, Hans-Görg Roos and Martin Stynes and a review article [59] invited by Thomas J. R. Hughes. I am indebted to their invitations and their continued encouragement.

Contents

Chapter 1

Introduction

Recently much attention has been paid to the construction and analysis of layer-adapted meshes for singularly perturbed boundary-value problems such as convection-dominated stationary convection-diffusion problems like

$$-\varepsilon u'' - bu' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1$$

and its two-dimensional analogue

$$-\varepsilon \Delta u - \boldsymbol{b} \cdot \nabla u + cu = f \quad \text{in} \quad \Omega \subset I\!R^2, \quad u|_{\partial\Omega} = g$$

with a small positive parameter ε. These problems may be regarded as linearised versions of the Navier-Stokes equations. They provide an excellent paradigm for numerical techniques in computational fluid dynamics. Classical convergence results for numerical methods for these problems have the structure

$$\|u - U\| \le Kh^k,$$

where the constant K depends on a certain derivative of u and typically tends to infinity as the perturbation parameter ε approaches zero. This means that the maximal step size h has to be chosen proportional to some positive power of ε which is impractible. Therefore we are looking for so-called *uniform* or *robust* methods where the numerical costs are independent of the perturbation parameter ε. More precisely, we are looking for robust methods in the sense of the following definition.

Definition 1.1 (Uniform/robust convergence)**.** *Let u_ε be the solution of a singularly perturbed problem and let U_ε^N be a numerical approximation of u_ε obtained by a numerical method with N degrees of freedom. The numerical method is said to be* uniformly convergent *or* robust *with respect to the perturbation parameter ε in the norm $\|\cdot\|$ if*

$$\left\|u_\varepsilon - U_\varepsilon^N\right\| \le \vartheta(N) \quad \textit{for} \quad N \ge N_0$$

with a function ϑ and a threshold value $N_0 > 0$ that are both independent of ε and

$$\lim_{N\to\infty} \vartheta(N) = 0.$$

Well-developed techniques are available for the computation of solutions outside layers [75, 89], but the problem of resolving layers—which is of great practical importance—is still under investigation. This field has witnessed a stormy development. Layer-adapted meshes have first been proposed by Bakhvalov [13] in the context of reaction-diffusion problems. In the late 1970s and early 1980s special meshes for convection-diffusion problems were investigated by Gartland [29], Liseikin [67, 70, 71], Vulanović [102, 103, 104, 105] and others in order to achieve uniform convergence. The discussion has been livened up by the introduction of special piecewise-uniform meshes by Shishkin [93]. They will be described in more detail in Section 1.3. Because of their simple structure they have attracted much attention and are now widely referred to as Shishkin meshes. A small survey of these meshes can be found in the monograph [89], while [64, 74] and [83] are devoted exclusively to them.

The performance of Shishkin meshes is however inferior to that of Bakhvalov meshes, which has prompted efforts to improve them while retaining some of their simplicity, in particular the mesh uniformity outside the layers and the choice of mesh transition point where the mesh changes from fine to coarse: Vulanović [108] uses a piecewise-uniform mesh with more than one transition point. Linß [48, 49] combines the ideas of Bakhvalov and Shishkin while Beckett and Mackenzie [15] combine an equidistribution idea [21] with a Shishkin-type transition point. With all these various mesh-construction ideas a natural question is: can a general theory be derived that allows one to deduce immediately the robust convergence of standard methods and a guaranteed rate of convergence? A first attempt towards this can be found in [85], where a first-order upwind scheme and a Galerkin FEM are studied on a class of so-called Shishkin-type meshes. A more general criterion was derived in [51, 52] for an upwind-difference scheme in one dimension.

The main purpose of this paper is to give a survey of developments since the monographs [74] and in particular [83] and [64] were published. We not only present the results obtained so far, but also give brief descriptions of the techniques used for the convergence analysis of uniformly convergent numerical methods. In Chapter 2 one-dimensional convection-diffusion problems with regular layers are studied, for which the theory is most advanced. For some of the methods we are able to present fairly general convergence criteria, while for others we have to restrict ourselves to the class of Shishkin-type meshes introduced in [85]. The focus of Chapter 4 is on one-dimensional problems with turning point layers. Finally, two-dimensional problems are considered in Chapter 5. Here we shall refrain from giving detailed analyses since the differences from one-dimensional problems are only of a technical nature and the flavour of the techiques used is given in Chapter 2, though the number of merely technical details increases significantly. Practical issues in the construction of layer-adapted meshes for fairly general situations are extensively discussed in the monograph [68].

Notation

Throughout this paper we use C to denote a generic positive constant that is independent of both the perturbation parameter ε and of the number of degrees of freedom. Given a function $g \in C^0[0,1]$ and a set of mesh points $\{x_i\} \in [0,1]$, let $g_i := g(x_i)$. Similarly we use the notation $g_i = g(\boldsymbol{x}_i)$ and $g_{ij} = g(x_i, y_j)$ for functions in two dimensions. Numerical approximation are indicated by capital letters, for example G as an approximation to g with $G_i \approx g_i$. Various norms are introduced in the course of the paper with a subscript ω indicating discrete norms.

1.1 Mesh-construction ideas

Let us consider the linear convection-diffusion problem

$$-\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{1.1}$$

where ε is a small positive parameter, $b(x) \geq \beta > 0$. The boundary value problem (1.1) has a unique solution that typically has an exponential boundary layer at $x = 0$ which behaves like $\exp(-\beta x/\varepsilon)$. Using (1.1) as a model problem we now review some standard mesh-construction ideas.

Before presenting a few of the most important mesh-construction ideas from the literature we have to recall a basic concept for describing layer-adapted meshes.

Definition 1.2 (Mesh generating function). *A strictly monotone function $\varphi : [0,1] \to [0,1]$ that maps a uniform mesh in ξ onto a layer-adapted mesh in x by $x = \varphi(\xi)$ is called a* mesh generating function.

A related approach is that of *stretching functions* or *layer-damping transformations* [31, 69, 70] which are used to transform a problem with layers into a problem whose derivatives are bounded.

For a given mesh generating function $\varphi \in W^{1,1}(0,1)$ the local mesh step sizes can be computed using the formula

$$h_k = x_k - x_{k-1} = \varphi(\xi_k) - \varphi(\xi_{k-1}) = \int_{(k-1)/N}^{k/N} \varphi'(\xi) d\xi.$$

Another important concept is that of *mesh equidistribution.*

Definition 1.3 (Equidistribution principle)**.** *Let $M : [0,1] \to I\!R$ be a strictly positive function. A mesh $\omega : 0 = x_0 < \cdots < x_N = 1$ is said to equidistribute the* monitor function *M if*

$$\int_{x_{k-1}}^{x_k} M(s) ds = \frac{1}{N} \int_0^1 M(s) ds \quad \textit{for} \quad k = 1, \ldots, N.$$

Given a monitor function M the associated mesh generating function is implicitly defined by

$$\int_0^{\varphi(\xi)} M(s) ds = \xi \int_0^1 M(s) ds \quad \text{for} \quad \xi \in [0,1]$$

and its derivative by

$$\varphi'(\xi) = \frac{1}{M(\varphi(\xi))} \int_0^1 M(s) ds \quad \text{for} \quad \xi \in [0,1].$$

A quantity that will appear frequently in our later convergence estimates is

$$\vartheta_\kappa(\omega) := \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} \left(1 + \varepsilon^{-1} e^{-\beta s/\kappa\varepsilon}\right) ds. \tag{1.2}$$

For example, in Section 2.2 we shall establish for the nodal error of a simple upwind difference scheme on an arbitrary mesh ω

$$\|u - U\|_{\infty,\omega} \le C\vartheta_1(\omega) \quad \text{with} \quad \|v\|_{\infty,\omega} := \max_{i=0,\ldots,N} |v_i|.$$

Noting that

$$\int_0^1 \left(1 + \varepsilon^{-1} e^{-\beta s/\kappa\varepsilon}\right) ds \le C,$$

we see that an opitmal mesh—optimal with respect to the order of convergence—equidistributes

$$M(s) = 1 + \varepsilon^{-1} e^{-\beta s/\kappa\varepsilon}.$$

1.2 Bakhvalov-type meshes

Bakhvalov's idea [13] is to use an equidistant ξ-grid near $x = 0$, then to map this grid back onto the x-axis by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by

$$q\left(1 - \exp\left(-\frac{\beta x_i}{\sigma\varepsilon}\right)\right) = \xi_i = \frac{i}{N} \quad \text{for} \quad i = 0, 1, \ldots,$$

where the scaling parameters $q \in (0,1)$ and $\sigma > 0$ are user chosen: q is the ratio of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. Away from the layer a uniform mesh in x is used with the transition point τ such that the resulting mesh generating function is $C^1[0,1]$, i. e.,

$$\varphi(\xi) = \begin{cases} \chi(\xi) := -\frac{\sigma\varepsilon}{\beta} \ln\left(1 - \frac{\xi}{q}\right) & \text{for} \quad \xi \in [0,\tau], \\ \pi(\xi) := \chi(\tau) + \chi'(\tau)(\xi - \tau) & \text{for} \quad \xi \in [\tau, 1], \end{cases}$$

where the point τ satisfies

$$\chi'(\tau) = \frac{1 - \chi(\tau)}{1 - \tau}. \tag{1.3}$$

Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point $(1,1)$; see Figure 1.1. The nonlinear equation (1.3) cannot be solved explicitely. However, the iteration

$$\tau_0 = 0, \qquad \chi'(\tau_{i+1}) = \frac{1 - \chi(\tau_i)}{1 - \tau_i}, \quad i = 0, 1, 2 \dots$$

is fastly converging. Moreover, the mesh obtained when the exact τ is replaced by the first iterate has very similar properties; see, e.g., [4, 17]. In that case

$$\tau_1 = q - \frac{\sigma\varepsilon}{\beta} \qquad \text{and} \qquad \chi(\tau_1) = \frac{\sigma\varepsilon}{\beta} \ln \frac{\beta q}{\sigma\varepsilon},$$

are the mesh transition points in the ξ and x coordinates.

Alternatively [52] the Bakhvalov mesh can be generated by equidistributing the monitor function

$$M_{Ba}(x) := \max\left\{1, \frac{K\beta}{\varepsilon} \exp\left(-\frac{\beta}{\sigma\varepsilon}\right)\right\} \quad \text{for} \quad s \in [0,1].$$

Clearly, for $\kappa \le \sigma$ and arbitrary K there exists a constant $C = C(\sigma, K)$ with

$$1 + \varepsilon^{-1} e^{-\beta s/\kappa\varepsilon} \le C \max\left\{1, \frac{K\beta}{\varepsilon} \exp\left(-\frac{\beta s}{\sigma\varepsilon}\right)\right\} = C M_{Ba}(s).$$

Thus

$$\vartheta_\kappa(\omega) = \max_{k=0,\dots,N-1} \int_{x_k}^{x_{k+1}} \left(1 + \varepsilon^{-1} e^{-\beta s/\kappa\varepsilon}\right) ds \le \frac{C}{N} \int_0^1 M_{Ba}(s) ds \le \frac{C}{N} \tag{1.4}$$

for a Bakhvalov mesh with $\sigma \ge \kappa$, since $\int_0^1 M_{Ba}(s)ds \le C$.

Because (1.3) cannot be solved explicitely Vulanović [102] proposed to replace the exponential in the above construction by its $(0,1)$-Padé approximation. Thus in (1.3) we would take

$$\chi(\xi) = \frac{\sigma\varepsilon}{\beta} \frac{\xi}{\xi - q}.$$

Meshes that arise from an approximation of Bakhvalov's mesh generating function are called *meshes of Bakhvalov type* (B-type meshes). To this class belong the meshes proposed by Liseikin and Yanenko [71] (quadratic function outside layer) and meshes generated by equidistribution of monitor functions which have been extensively studied by the group of Sloan and Mackenzie [15, 72, 81, 82], the graded mesh of Gartland [29] and its modification by Roos and Skalický [88]. For these meshes (1.4) holds too.

From these considerations a typical convergence result from (1.5) for simple upwinding on B-type meshes is

$$\|u - U\|_{\infty,\omega} \le CN^{-1} \quad \text{with} \quad \|v\|_{\infty,\omega} := \max_{i=0,\dots,N} |v_i|, \tag{1.5}$$

i. e., uniform first-order convergence in the discrete maximum; see Section 2.2.

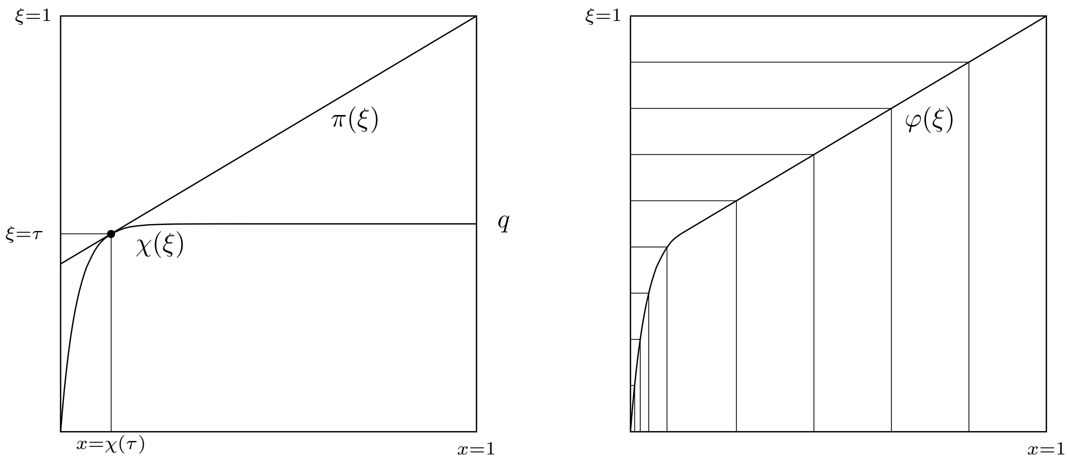


Figure 1.1: Bakhvalov mesh: Construction of the mesh generating function (left) and the mesh generated (right).

1.3 Shishkin-type meshes

Another frequently-studied mesh is the so-called Shishkin mesh [74, 93]. This is because of its simplicity—it is piecewise uniform. We describe this mesh for problem (1.1). Let $q \in (0,1)$ and $\sigma > 0$ be two mesh parameters. We define a mesh transition point λ by

$$\lambda = \min\left\{q, \frac{\sigma\varepsilon}{\beta}\ln N\right\}.$$

Then the intervals $[0,\lambda]$ and $[\lambda,1]$ are divided into qN and $(1-q)N$ equidistant subintervals (assuming that qN is an integer). This mesh may be regarded as generated by the mesh generating function

$$\varphi(\xi) = \begin{cases} \dfrac{\sigma\varepsilon}{\beta}\tilde{\varphi}(\xi) \quad \text{with} \quad \tilde{\varphi}(\xi) = \ln N\,\dfrac{\xi}{q} & \text{for} \quad \xi \in [0,q], \\ 1 - \left(1 - \dfrac{\sigma\varepsilon}{\beta}\ln N\right)\dfrac{1-\xi}{1-q} & \text{for} \quad \xi \in [q,1] \end{cases} \tag{1.6}$$

if $q \geq \lambda$; see Figure 1.2. Again the parameter q is the amount of mesh points used to resolve the layer. The mesh transition point λ has been chosen such that the layer term $\exp(-\beta x/\varepsilon)$ is smaller than $N^{-\sigma}$ on $[\lambda,1]$. Typically σ will be chosen equal to the formal order of the method or sufficiently large to accommodate the error analysis.

Note that unlike the Bakhvalov mesh (and Vulanović's modification of it) the underlying mesh generating function is only piecewise $C^1[0,1]$ and depends on N, the number of mesh points. For simplicity we shall assume throughout that $q \geq \lambda$ as otherwise N is exponentially large compared to $1/\varepsilon$ and a uniform mesh is sufficient to cope with the problem.

Although Shishkin meshes have a simple structure and numerical methods using them are easier to analyse than methods using say B-type meshes, they give numerical results that are inferior to those obtained by B-type meshes:

$$\|u - U\|_{\infty,\omega} \leq CN^{-1}\ln N, \tag{1.7}$$

for the afore-mentioned simple upwind scheme. The convergence deteriorates by a logarithmic factor.

This drawback prompted some work on improving Shishkin meshes. Vulanović [108] proposed the introduction of additional mesh transition points

$$\lambda_0 = 1 \geq \lambda_1 = \frac{\sigma\varepsilon}{\beta}\ln N \geq \lambda_2 = \frac{\sigma\varepsilon}{\beta}\ln\ln N \geq \cdots \geq \lambda_\ell = \frac{\sigma\varepsilon}{\beta}\underbrace{\ln\ln\cdots\ln}_{\ell\ \text{times}} N \geq \lambda_{\ell+1} = 0$$

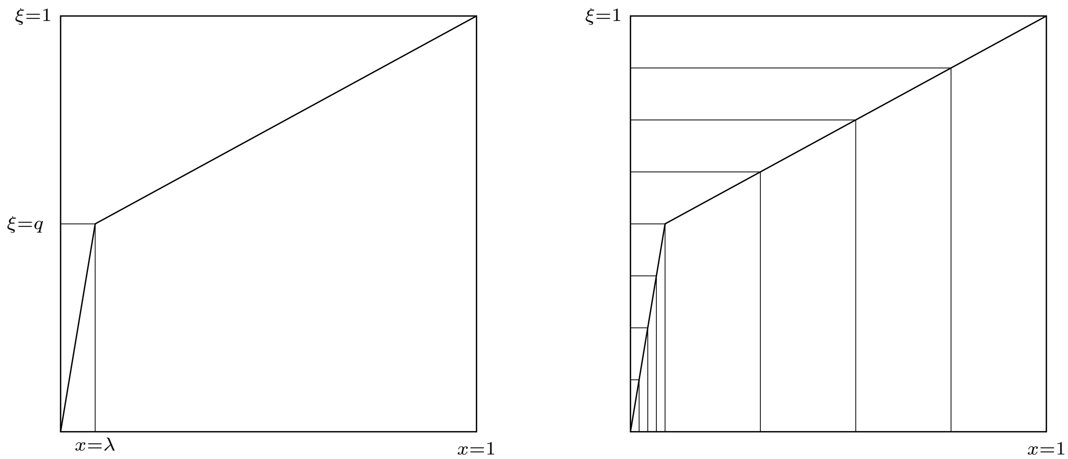


Figure 1.2: Shishkin mesh: mesh generating function (left) and the mesh generated (right).

and to dissect each of the intervals $[\lambda_{i+1}, \lambda_i]$, $i = 0, \ldots, \ell$ uniformly. As a result the convergence is improved to

$$\|u - U\|_{\infty,\omega} \leq CN^{-1} \underbrace{\ln \ln \cdots \ln}_{\ell \text{ times}} N. \tag{1.8}$$

Linß [48, 49] uses Bakhvalov's idea of inverting $\exp(-\beta x/\sigma\varepsilon)$ on $[0, \lambda]$, while using a uniform mesh on $[\lambda, 1]$. The corresponding mesh generating function is given by (1.6) with

$$\tilde{\varphi}(\xi) = -\ln\left(1 - \left(1 - \frac{1}{N}\right)\frac{\xi}{q}\right).$$

Again the Vulanovi'c's idea of replacing the exponential by its $(0, 1)$-Padé approximation can be used. In this case we get

$$\tilde{\varphi}(\xi) = \frac{\frac{\xi}{q}\ln N}{1 + \left(1 - \frac{\xi}{q}\right)\ln N}$$

Meshes that have a transition point $\lambda = \frac{\sigma\varepsilon}{\beta}\ln N$ and that are (quasi-)uniform on $[\lambda, 1]$ are called *meshes of Shishkin type* (S-type meshes). Roos and Linß [85] derive a classification for this class of meshes. Let the mesh be generated by (1.6) with a monotone $\tilde{\varphi}$ satisfying

$$\tilde{\varphi}(0) = 0 \quad \text{and} \quad \tilde{\varphi}(q) = \ln N.$$

We introduce the *mesh characterizing* function $\psi(\xi) = \exp(-\tilde{\varphi}(\xi))$ for $\xi \in [0, q]$. This function is monotonically decreasing with $\psi(0) = 1$ and $\psi(q) = N^{-1}$. In Section 2.2.6, under certain assumptions on ψ we shall prove for simple upwinding that

$$\|u - U\|_{\infty,\omega} \leq C\left(h + \max_{\xi\in[0,q]} |\psi'(\xi)|\, N^{-1}\right), \tag{1.9}$$

where h is the maximal step size. Examples for the mesh characterizing function are

Shishkin mesh [74, 93]

$$\psi(\xi) = \exp\left(-\frac{\xi \ln N}{q}\right) \quad \text{with} \quad \max|\psi'| = \frac{\ln N}{q}, \quad h \leq CN^{-1}$$

Bakhvalov-Shishkin mesh [48, 49]

$$\psi(\xi) = 1 - \left(1 - \frac{1}{N}\right)\frac{\xi}{q} \quad \text{with} \quad \max|\psi'| = \frac{1}{q}\left(1 - \frac{1}{N}\right) \le \frac{1}{q}, \quad h \le C\left(\varepsilon + N^{-1}\right).$$

Thus for the simple upwind scheme on this mesh we get from (1.9)

$$\|u - U\|_{\infty,\omega} \le C\left(\varepsilon + N^{-1}\right). \tag{1.10}$$

Vulanović-Shishkin mesh in the above sense

$$\psi(\xi) = \exp\left(-\frac{\frac{\xi}{q}\ln N}{1 + \left(1 - \frac{\xi}{q}\right)\ln N}\right) \quad \text{with} \quad \max|\psi'| \le \frac{4}{q}, \quad h \le C\left(1 + \varepsilon \ln^2 N\right)N^{-1}.$$

General S-type meshes. Two properties of the mesh generating function that will be assumed later when analysing various numerical schemes are

$$\max_{\xi\in[0,q]} \tilde\varphi'(\xi) \le CN \tag{1.11}$$

and

$$\int_0^q \tilde\varphi'(\xi)^2 d\xi \le CN \tag{1.12}$$

These two conditions are not only satisfied by the above mentioned meshes, but all of the S-type meshes we shall meet later.

Eq. (1.11) implies that

$$h_i \le C\varepsilon \quad \text{and} \quad e^{\beta h_i/\varepsilon} \le C \quad \text{for} \quad i = 1, \dots, qN, \tag{1.13}$$

while (1.12), the representation

$$\frac{\beta h_k}{\varepsilon} = \sigma \int_{(k-1)/N}^{k/N} \tilde\varphi'(\xi) d\xi$$

and the Cauchy-Schwarz inequality yield

$$\sum_{k=1}^{qN}\left(\frac{\beta h_k}{\varepsilon}\right)^2 \le \sigma^2 N^{-1}\int_0^q \tilde\varphi'(\xi)^2 d\xi \le C. \tag{1.14}$$

Furthermore, the above representation gives

$$\frac{\beta h_i}{\sigma\varepsilon}\, e^{-\beta x_i/(\sigma\varepsilon)} \le N^{-1}\max|\psi'| \quad \text{for } i = 1, \dots, qN, \tag{1.15}$$

because

$$\tilde\varphi' = -\frac{\psi'}{\psi} \quad \text{and} \quad \min_{[t_{i-1},t_i]} \psi(t) = \psi(t_i) = e^{-\beta x_i/(\sigma\varepsilon)}.$$

Finally, we like to give bounds for the characteristic quantity ϑ_κ. Let $\kappa \le \sigma$. For $k = qN, \dots, N-1$ we have

$$\int_{x_k}^{x_{k+1}}\left(1 + \varepsilon^{-1}e^{-\beta s/\kappa\varepsilon}\right) ds = h_{k+1} - \frac{\kappa}{\beta}e^{-s\beta/\kappa\varepsilon}\Big|_{x_k}^{x_{k+1}} \le h + CN^{-\sigma/\kappa} \le h + CN^{-1},$$

by the choice of the transition point λ. Using (1.13) and (1.15), we get for $k = 0, \dots, qN-1$

$$\int_{x_k}^{x_{k+1}}\left(1 + \varepsilon^{-1}e^{-\beta s/\kappa\varepsilon}\right) ds \le h + CN^{-1}\max_{\xi\in[0,q]}|\psi'(\xi)| \quad \text{if} \quad \sigma \ge \kappa.$$

Therefore on a S-type meshes we have

$$\vartheta_\kappa(\omega) \le h + CN^{-1}\max_{\xi\in[0,q]}|\psi'(\xi)| \;\; \text{if} \;\; \kappa \le \sigma. \tag{1.16}$$

Chapter 2

Finite difference schemes for problems with regular boundary layers

Throughout this chapter we consider the stationary linear convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{2.1}$$

where ε is a small positive parameter and $b \geq \beta > 0$ on $[0,1]$. For the mere sake of simplicity we will also assume that $c \geq 0$ and $b' + c \geq 0$ on $[0,1]$. The results hold without these restrictions too, see [6], but the arguments become more complicated. Note that if $\beta > 0$ then the conditions $c \geq 0$ and $b' + c \geq 0$ can always be ensured for ε smaller than a certain threshold value ε_0 by a simple transformation $u = \hat{u}e^{\delta x}$ with δ chosen appropriately.

Using (2.1) as a model problem we derive a general convergence theory for first- and second-order upwinded difference schemes on layer-adapted meshes. Thereby highlighting the close relationship between the continuous operator $\mathcal{L}$ and certain upwind discretizations.

2.1 The continuous problem

2.1.1 Stability of the continuous operator

An important tool for studying the stability of differential operators are maximum and comparison principles. Consider the general second-order differential operator

$$\tilde{\mathcal{L}}u := -u'' + bu' + cu \quad \text{in} \quad (0,1).$$

Lemma 2.1 (Maximum principle [80])**.** *Assume there exists a function $v \in C^2(0,1) \cap C[0,1]$ with*

$$v(x) > 0 \quad \textit{for} \quad x \in [0,1] \quad \textit{and} \quad \tilde{\mathcal{L}}v > 0 \quad \textit{for} \quad x \in (0,1).$$

Then the operator $\tilde{\mathcal{L}}$ with Dirichlet boundary conditions satisfies a maximum principle. That is, $u(0) \leq 0$, $u(1) \leq 0$ and $\tilde{\mathcal{L}}u(x) \leq 0$ for $x \in (0,1)$ imply $u(x) \leq 0$ for $x \in [0,1]$.

An immediate consequence is the following result.

Corollary 2.2 (Comparison principle)**.** *Let the assumptions of Lemma 2.1 hold. Then if two functions $\check{u}$ and $\hat{u}$ satisfy $\check{u}(0) \leq \hat{u}(0)$, $\check{u}(1) \leq \hat{u}(1)$ and $\tilde{\mathcal{L}}\check{u}(x) \leq \tilde{\mathcal{L}}\hat{u}(x)$ in $(0,1)$ then $\check{u}(x) \leq \hat{u}(x)$ on $[0,1]$.*

Using the test function $v(x) = 1-x$, we see that the operator $\mathcal{L}$ of (2.1) satisfies the assumptions of Lemma 2.1 because $\mathcal{L}v \geq \beta > 0$. Consequently, Corollary 2.2 yields for the solution of (2.1)

$$|u(x)| \leq \max\{|\gamma_0|, |\gamma_1|\} + \frac{1-x}{\beta} \max_{x\in[0,1]} |f(x)| \quad \text{for} \quad x \in [0,1].$$

Letting $(v,w) := \int_0^1 (vw)(s)ds$ denote the L_2-inner product, we have for any given arbitrary function v with $v(0) = v(1) = 0$

$$v(x) = \big(\mathcal{G}(x,\cdot), \mathcal{L}v\big) \quad \text{for} \quad x \in [0,1], \tag{2.2}$$

where $\mathcal{G}(x,\xi)$, the Green's function associated with $\mathcal{L}$ and Dirichlet boundary conditions, solves for fixed $\xi \in (0,1)$

$$\left(\mathcal{L}\mathcal{G}(\cdot,\xi)\right)(x) = \delta(x-\xi) \quad \text{for} \quad x \in (0,1), \quad G(0,\xi) = G(1,\xi) = 0, \tag{2.3}$$

where δ denotes the Dirac-δ function. Therefore (2.3) has to be read in the context of distributions. Equivalently we may seek a solution $\mathcal{G}(\cdot,\xi) \in C^2((0,1)\setminus\{\xi\}) \cap C[0,1]$ with

$$\left(\mathcal{L}\mathcal{G}(\cdot,\xi)\right)(x) = 0 \quad \text{for} \quad x \in (0,1)\setminus\{\xi\}, \quad \mathcal{G}(0,\xi) = \mathcal{G}(1,\xi) = 0, \quad -\varepsilon[\mathcal{G}(\cdot,\xi)'](\xi) = 1, \tag{2.3$'$}$$

where $[v](d) := v(d+0) - v(d-0)$ denotes the jump of v at d.

The Green's function can also be defined using the adjoint operator to $\mathcal{L}$ with respect to the inner product $(\cdot,\cdot)$:

$$\mathcal{L}^* v = -\varepsilon v'' + (bv)' + cv.$$

For fixed $x \in (0,1)$ the Green's function solves

$$\left(\mathcal{L}^*\mathcal{G}(x,\cdot)\right)(\xi) = \delta(\xi - x) \quad \text{for} \quad \xi \in (0,1), \quad \mathcal{G}(x,0) = \mathcal{G}(x,1) = 0. \tag{2.4}$$

Similar to Corollary 2.2 we have a comparison principle for the operator defined by (2.3′):

$$\left.\begin{array}{r} \check{u}(0) \leq \hat{u}(0) \\ \check{u}(1) \leq \hat{u}(1) \\ \mathcal{L}\check{u}(x) \leq \mathcal{L}\hat{u}(x) \ \text{ in } \ (0,1)\setminus\{\xi\} \\ -\varepsilon[\check{u}'](\xi) \leq -\varepsilon[\hat{u}'](\xi) \end{array}\right\} \implies \check{u}(x) \leq \hat{u}(x) \text{ on } [0,1].$$

Figure 2.1 dipicts the typical behaviour of the Green's function $\mathcal{G}$: It is monotonically increasing for $\xi < x$ and decreasing for $\xi > x$. Also note the layers just left of x and 1. This comparison

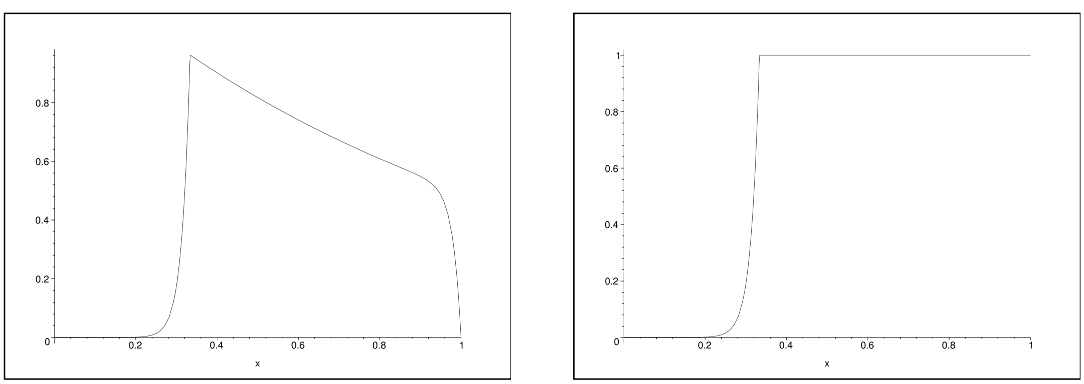


Figure 2.1: Green's function $\mathcal{G}(x,\cdot)$ associated with x (left) and its bound $\hat{\mathcal{G}}(x,\cdot)$ (right).

principle with the test functions, for a plot of $\hat{\mathcal{G}}$ see Figure 2.1,

$$\check{\mathcal{G}} \equiv 0 \quad \text{and} \quad \hat{\mathcal{G}} = \frac{1}{\beta}\begin{cases} 1 & \text{for} \quad 0 \leq x \leq \xi, \\ e^{-\beta(x-\xi)/\varepsilon} & \text{for} \quad \xi \leq x \leq 1 \end{cases}$$

yields

$$0 \le \mathcal{G}(x,\xi) \le \beta^{-1} \quad \text{for} \quad x,\xi \in [0,1].$$

Apart from these bounds of $\mathcal{G}$ we shall also need monotonicity properties of $\mathcal{G}$ later. Since $\mathcal{G}(x,\xi) \ge 0$ and $\mathcal{G}(x,0) = 0$ for $x,\xi \in [0,1]$ we have $\mathcal{G}_\xi(x,0) \ge 0$ for $x \in [0,1]$. Integrating (2.4) over $[0,\xi]$, we get

$$-\varepsilon\mathcal{G}_\xi(x,\xi) + \varepsilon\mathcal{G}_\xi(x,0) + b(\xi)\mathcal{G}(x,\xi) = -\int_0^\xi c(s)\mathcal{G}(x,s)ds \le 0 \quad \text{for} \quad \xi < x. \tag{2.5}$$

Thus

$$\varepsilon\mathcal{G}_\xi(x,\xi) \ge \varepsilon\mathcal{G}_\xi(x,0) + b(\xi)\mathcal{G}(x,\xi) \ge 0 \quad \text{for} \quad \xi < x$$

because $\mathcal{G}(x,\xi) \ge 0$ and $\mathcal{G}_\xi(x,0) \ge 0$. Thus $\mathcal{G}(x,\cdot)$ increases monotonically on $(0,x)$.

On the other hand, since $\mathcal{G}(x,\xi) \ge 0$ and $\mathcal{G}(x,1) = 0$ for $x,\xi \in [0,1]$we have $\mathcal{G}_\xi(x,1) \le 0$ for $x \in [0,1]$. Then inspecting the differential equation (2.4), we see that $v = \mathcal{G}_\xi(x,\cdot)$ satisfies

$$-\varepsilon v' + bv = -(b'+c)G \le 0 \quad \text{for} \quad x \in (\xi,1) \quad \text{and} \quad v(1) \le 0,$$

because $b' + c$ is assumed to be positive. Application of a maximum principle for first-order operators yields $v \le 0$ on $[x,1]$. Thus $\mathcal{G}(x,\cdot)$ decreases monotonically on $(x,1)$.

Similarly one can prove that $G_x(x,\xi) \ge 0$ for $0 \le x < \xi \le 1$ and $G_x(x,\xi) \le 0$ for $0 \le \xi < x \le 1$. Thus

$$G_{x\xi}(x,0) \le 0 \quad \text{and} \quad G_{x\xi}(x,1) \le 0 \quad \text{for} \quad x \in [0,1].$$

because $G_x(x,0) = G_x(x,1) = 0$ for $x \in [0,1]$, Differentiating (2.5) with respect to x, we get

$$-\varepsilon\mathcal{G}_{x\xi}(x,\xi) + \varepsilon\mathcal{G}_{x\xi}(x,0) + b(\xi)\mathcal{G}_x(x,\xi) - b(0)\mathcal{G}_x(x,0) = -\int_0^\xi c(s)\mathcal{G}_x(x,s)ds \quad \text{for} \quad \xi < x.$$

Therefore $G_{x\xi}(x,\xi) \le 0$ for $0 \le \xi < x \le 1$ because $G_x(x,\xi) \le 0$, $G_{x\xi}(x,0) \le 0$ and $G_x(x,0) = 0$. For $x < \xi$, differentiate (2.4) to see that $v = \mathcal{G}_{x\xi}(x,\cdot)$ satisfies

$$-\varepsilon v' + bv = -(b'+c)G_x \le 0 \quad \text{for} \quad x \in (0,\xi) \quad \text{and} \quad v(1) \le 0,$$

because $b' + c \ge 0$ and $G_x(x,\xi) \ge 0$ for $x \le \xi$. Application of a maximum principle for first-order operators yields $\mathcal{G}_{x\xi}(x,\cdot) \le 0$ on $[x,1]$.

We summarize our results.

Theorem 2.3. *The Green's function $\mathcal{G}$ associated with the operator $\mathcal{L}$ and Dirichlet boundary conditions satisfies*

$$\begin{aligned} 0 \le \mathcal{G}(x,\xi) &\le \beta^{-1} \quad \textit{for} \quad x,\xi \in [0,1], \\ \mathcal{G}_\xi(x,\xi) &\ge 0 \quad \textit{for} \quad 0 \le \xi < x \le 1, \\ \mathcal{G}_\xi(x,\xi) &\le 0 \quad \textit{for} \quad 0 \le x < \xi \le 1 \end{aligned}$$

and

$$\mathcal{G}_{x\xi}(x,\xi) \le 0 \quad \textit{for} \quad x,\xi \in [0,1], \ x \ne \xi.$$

For our further investigations, let us introduce the supremum norm

$$\|v\|_\infty := \operatorname*{ess\,sup}_{x \in [0,1]} |v(x)|$$

the L_1 norm

$$\|v\|_1 := \int_0^1 |v(x)|dx$$

and the negative norm

$$\|v\|_* := \min_{V:V'=v} \|V\|_\infty.$$

Note that since

$$\|v\|_* = \min_{c\in I\!R} \left\| \int_\cdot^1 v(s)ds + c \right\|_\infty ,$$

this norm is well-defined. Furthermore,

$$\|v\|_* = \sup_{u\in W_0^{1,1}} \frac{\langle u, v\rangle}{|u|_{1,1}}.$$

Thus $\|\cdot\|_*$ is a norm in $W^{-1,\infty} = \left(W_0^{1,1}\right)'$.

For fixed $x \in [0,1]$ we compute the following norms of the Green's function and its derivatives. Theorem 2.3 yields

$$\|\mathcal{G}(x,\cdot)\|_1 \le \|\mathcal{G}(x,\cdot)\|_\infty \le \beta^{-1}, \tag{2.6a}$$

$$\|\mathcal{G}_\xi(x,\cdot)\|_1 = \int_0^x \mathcal{G}_\xi(x,\xi)d\xi - \int_x^1 \mathcal{G}_\xi(x,\xi)d\xi = 2\mathcal{G}(x,x) \le 2\beta^{-1} \tag{2.6b}$$

and

$$\|\mathcal{G}_{x\xi}(x,\cdot)\|_1 = 2\varepsilon^{-1}, \tag{2.6c}$$

because $G_x\xi \le 0$ for $x \ne \xi$ and $[G_x(x,\xi)](x) = \varepsilon^{-1}$. These norms are used to establish stabily properties for the operator $\mathcal{L}$.

Theorem 2.4. *The operator $\mathcal{L}$ satisfies*

$$\beta\|v\|_\infty \le \|\mathcal{L}v\|_\infty \quad \text{for all} \quad v \in W_0^{1,\infty}(0,1)\cap W^{2,\infty}(0,1), \tag{2.7a}$$

$$\beta\|v\|_\infty \le \|\mathcal{L}v\|_1 \quad \text{for all} \quad v \in W_0^{1,1}(0,1)\cap W^{2,1}(0,1) \tag{2.7b}$$

and

$$|||v|||_{\varepsilon,\infty} := \frac{\beta}{2}\|v\|_\infty + \frac{\varepsilon}{2}\|v'\|_\infty \le \|\mathcal{L}v\|_* \quad \text{for all} \quad v \in W_0^{0,\infty}(0,1)\cap W^{1,\infty}(0,1). \tag{2.7c}$$

Proof. First, the representation (2.2), the Hölder inequality and (2.6a) give (2.7a,b).

Next, let $V \in W^{0,\infty}(0,1)$ be an arbitrary function with $V' = \mathcal{L}v$. Integrating (2.2) by parts, we obtain

$$v(x) = -\int_0^1 \mathcal{G}_\xi(x,\xi)V(\xi)d\xi \quad \text{for} \quad x \in (0,1)$$

and

$$v'(x) = -\int_0^1 \mathcal{G}_{x\xi}(x,\xi)V(\xi)d\xi \quad \text{for} \quad x \in (0,1).$$

The Hölder inequality, (2.6b,c) and the definition of the negative norm yield (2.7c). □

Remark 2.5. *(i) Note that since*

$$\|v\|_* \le \|v\|_1 \le \|v\|_\infty \quad \text{for all} \quad v \in L_\infty(0,1)$$

the negative-norm stability (2.7c) is the strongest of the three stability inequalities of Theorem 2.4.
(ii) The same stability results hold true for the differential operator in conservative form, i. e.,

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1. \tag{2.8}$$

2.1.2 Derivative bounds and solution decomposition

The boundary value problem (2.1) has a unique solution that typically has an exponential boundary layer at $x = 0$: u and its derivatives up to an arbitrary prescribed order q can be bounded by

$$\left|u^{(k)}(x)\right| \leq C\left\{1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}\right\} \quad \text{for } k = 0, 1, \ldots, q \text{ and } x \in [0, 1], \tag{2.9}$$

where the maximal order q depends on the smoothness of the data, see [36].

On a number of occations, e. g. for the error analysis of a finite difference scheme in Section 2.2.6 or of a finite element method in Section 3.2 we need more detailed information on u and its derivatives. In particular we need a splitting of u into a regular component and a boundary layer component. This will be derived now.

Following [56], we construct the decomposition as follows. Let v and w be the solution of the boundary-value problems

$$\mathcal{L}v = f \text{ in } (0, 1), \quad (-bv' + cv)(0) = f(0), \ v(1) = \gamma_1 \tag{2.10a}$$

and

$$\mathcal{L}w = 0 \text{ in } (0, 1), \quad w(0) = \gamma_0 - v(0), \ w(1) = 0. \tag{2.10b}$$

First we study the regular solution component v. The operator $\mathcal{L}$ satisfies maximum and comparison principles [80]. For example, if two functions $\check{v}$ and $\hat{v}$ satisfy $\mathcal{L}\check{v}(x) \leq \mathcal{L}\hat{v}(x)$ in $(0, 1)$ and $\big(-a\check{v}' + b\check{v}\big)(0) \leq \big(-a\hat{v}' + b\hat{v}\big)(0)$ and $\check{v}(1) \leq \hat{v}(1)$, then $\check{v}(x) \leq \hat{v}(x)$ on $[0, 1]$. Using this comparison principle with

$$v^{\pm}(x) := \pm\left(\beta^{-1}\|f\|_\infty(1 - x) + |\gamma_1|\right),$$

we get

$$|v(x)| \leq C \text{ for } x \in [0, 1].$$

To derive bounds on the derivatives of v, we set $h := f - cu$ and write v as

$$v(x) = \int_x^1 H_v(s)ds + \frac{h(0)}{b(0)} \int_x^1 e^{-B(s)}ds + \gamma_1,$$

where

$$B(x) := \frac{1}{\varepsilon}\int_0^x b(s)ds \quad \text{and} \quad H_v(x) := \frac{1}{\varepsilon}\int_0^x h(s)e^{B(s)-B(x)}ds.$$

Differentiating once, we get

$$v'(x) = -H_v(x) - \frac{h(0)}{b(0)}e^{-B(x)}$$

which gives

$$|v'(x)| \leq C \text{ for } x \in [0, 1]$$

because

$$H_v(x) \leq \frac{C}{\varepsilon}\int_0^x e^{\beta(s-x)/\varepsilon}ds = \frac{C}{\beta}\left(1 - e^{-\beta x/\varepsilon}\right) \leq C. \tag{2.11}$$

Invoking the differential equation we get

$$|v''(x)| \leq C\varepsilon^{-1} \text{ for } x \in [0, 1].$$

However, if $b, f \in C^1(0,1)$ then integration by parts and the boundary condition imposed on v at $x = 0$ yield

$$v''(x) = -\frac{b(x)}{\varepsilon} \int_0^x \left(\frac{h}{b}\right)'(s) e^{B(s)-B(x)} ds,$$

from which the sharper estimate

$$|v''(x)| \le C \quad \text{for} \quad x \in [0,1]$$

can be derived using (2.11). A bound for the third-order derivative is readily obtained from the differential equation and the bounds on v' and v'':

$$|v'''(x)| \le C\varepsilon^{-1} \quad \text{for} \quad x \in [0,1].$$

This completes our analysis of the regular part of u.

Now let us consider the boundary-layer term w. The operator $\mathcal{L}$ satisfies another comparison principle: if two functions $\breve{w}$ and $\hat{w}$ satisfy $\mathcal{L}\breve{w}(x) \le \mathcal{L}\hat{w}(x)$ in $(0,1)$ and $\breve{w}(x) \le \hat{w}(x)$ for $x = 0$ and $x = 1$, then $\breve{w}(x) \le \hat{w}(x)$ on $[0,1]$; see [80]. Using this comparison principle with

$$w^{\pm}(x) := \pm \left|\gamma_0 - v(0)\right| e^{-\beta x/\varepsilon},$$

we see that

$$|w(x)| \le Ce^{-\beta x/\varepsilon} \quad \text{for} \quad x \in [0,1]. \tag{2.12}$$

To bound the derivatives of w we use the fact that

$$w(x) = \int_x^1 H_w(s) ds + \kappa \int_x^1 e^{-B(s)} ds$$

with

$$H_w(x) = -\frac{1}{\varepsilon} \int_0^x (bw)(s) e^{B(s)-B(x)}.$$

Estimates for H_w are obtained using (2.12)

$$|H_w(x)| \le \frac{C}{\varepsilon} \int_0^x e^{-\beta s/\varepsilon} e^{B(s)-B(x)} \le \frac{C}{\varepsilon} \exp(-\beta x/\varepsilon).$$

The coefficient κ is determined by the boundary condition for $w(0)$:

$$\kappa = \frac{1}{\alpha} \left(\gamma_0 - v(0) - \int_0^1 \vartheta_w(s) ds \right),$$

where

$$\alpha = \int_0^1 e^{-B(s)} ds \ge \int_0^1 e^{-\|b\|_\infty s/\varepsilon} ds \ge \frac{\varepsilon}{\|b\|_\infty}.$$

Thus

$$|\kappa| \le C\varepsilon^{-1}.$$

For w' we have

$$w'(x) = -H_w(x) - \kappa e^{-B(x)}$$

and therefore

$$|w'(x)| \le C\varepsilon^{-1}e^{-\beta x/\varepsilon} \quad \text{for} \quad x \in [0,1],$$

by the above bounds for κ and H_w.

Using the differential equation and our estimates for w and w', we get

$$|w''(x)| \le C\varepsilon^{-2}e^{-\beta x/\varepsilon} \quad \text{for} \quad x \in [0,1].$$

If $a, b \in C^1(0,1)$ then we differentiate (2.10b) and apply our bounds for w, w' and w''. Thus

$$|w'''(x)| \le C\varepsilon^{-3}e^{-\beta x/\varepsilon} \quad \text{for} \quad x \in [0,1].$$

We summarize the results.

Theorem 2.6. *Let $b, c, f \in C^k[0,1]$ with $k \in \{0,1\}$. Then $u \in C^{k+2}[0,1]$ can be decomposed as $u = v + w$, where the regular solution component v satisfies*

$$(\mathcal{L}v)(x) = f(x) \quad \textit{and} \quad \left|v^{(i)}(x)\right| \le C\bigl(1 + \varepsilon^{k+1-i}\bigr) \quad \textit{for} \quad i = 0,1,\dots,k+2, \; x \in (0,1), \tag{2.13a}$$

$v(1) = \gamma_1$, while the boundary layer component w satisfies

$$(\mathcal{L}w)(x) = 0 \quad \textit{and} \quad \left|w^{(i)}(x)\right| \le C\varepsilon^{-i}e^{-\beta x/\varepsilon} \quad \textit{for} \quad i = 0,1,\dots,k+2, \; x \in (0,1) \tag{2.13b}$$

and $w(1) = 0$.

Remark 2.7. *A similar decomposition is given in [20, 74], however the construction there requires more smoothness of the data of the problem because the regular solution component v is defined via solutions of first-order problems.*

Remark 2.8. *Some applications, e. g., the analysis of higher-order schemes in Section 2.3.3 or [97] or of extrapolation schemes [76], require decompositions with bounds for derivatives of order greater than three. To derive them note that our boundary condition $(-av' + bv)(0) = f(0)$ imposed on v corresponds to $v''(0) = 0$. To prove Theorem 2.6 for $k = 2$ we would impose the boundary condition*

$$\Bigl(-\left(b - \varepsilon(b' - c)\right)v' + \left(c - \varepsilon c'\right)v\Bigr)(0) = \left(f - \varepsilon f'\right)(0)$$

instead. This corresponds to setting $v'''(0) = 0$. The operator $\mathcal{L}$ with this boundary condition satisfies a comparison principle too, provided that ε is smaller than a certain threshold value ε_0. We use this principle to prove the boundedness of v first. Then we proceed as above to get bounds for the derivatives.

2.2 A simple upwind difference scheme

In this section we study a first-order difference scheme for the discretization of (2.1) on an arbitrary mesh $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ with local mesh sizes $h_i := x_i - x_{i-1}$ and maximal mesh size $h := \max_i h_i$: Find $U \in \mathbb{R}^{N+1}$ such that

$$[LU]_i := -\varepsilon U_{\bar{x}x;i} - b_i U_{x;i} + c_i U_i = f_i \quad \text{for} \quad i = 1,\dots,N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1 \tag{2.14}$$

with

$$v_{x;i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad \text{and} \quad v_{\bar{x};i} := \frac{v_i - v_{i-1}}{h_i}.$$

At a first glance the discretization of the second-order derivative is a bit non-standard because on arbitrary meshes it is not consistent in the mesh nodes, but it has advantages that become clearer in the course of our analysis. More frequently used is the central difference approximation

$$u_i'' \approx u_{\bar{x}\hat{x};i} \quad \text{with} \quad v_{\hat{x};i} := \frac{v_{i+1} - v_i}{\hbar_i} \quad \text{and} \quad \hbar_i := \frac{h_i + h_{i+1}}{2}.$$

An upwind scheme based on this discretization of the second-order derivative will be studied in Section 2.2.6 because the technique used there becomes more important in 2D, see Section 5.2.

2.2.1 Stability of the discrete operator

For the mere sake of simplicity we assume in this section that $c \geq 0$ and $b' \geq 0$ on $[0,1]$. The results hold without these restrictions too, see [6], but the arguments become more complicated.

Lemma 2.9 (***M***-matrix criterion [79])**.** *Let $A \in I\!R^{n\times n}$ be a matrix with positive diagonal and nonpositive offdiagonal entries. Assume there exists a vector $v \in I\!R^N$ with*

$$v_i > 0 \quad \textit{for} \quad i = 1,\dots,n \quad \textit{and} \quad (Av)_i > 0 \quad \textit{for} \quad i = 1,\dots,n.$$

Then the matrix A is inverse monotone. That is, $(Au)_i \leq 0$ for $i = 1,\dots,n$ implies $u \leq 0$ for $i = 1,\dots,n$.

An immediate consequence is the following result.

Corollary 2.10 (Comparison principle)**.** *Let the assumptions of Lemma 2.9 hold. Then if two vectors $\check{u}$ and $\hat{u}$ satisfy $(A\check{u})_i \leq (A\hat{u})_i$ for $i = 1,\dots,n$ then $\check{u}_i \leq \hat{u}_i$ for $i = 1,\dots,n$.*

Note that these results are discrete analogues of Lemma 2.1 and Corollary 2.2.

Using the test vector v with $v_i = 1 - x_i$, we see that after eliminating the boundary conditions the operator L of (2.14) satisfies the assumptions of Lemma 2.9 because $[Lv]_i \geq \beta > 0$ for $i = 1,\dots,N-1$. Consequently, Corollary 2.10 yields for the solution of (2.14)

$$|U_i| \leq \max\bigl\{|\gamma_0|, |\gamma_1|\bigr\} + \frac{1 - x_i}{\beta} \max_{i=1,\dots,N-1} |f_i| \quad \text{for} \quad i = 0,\dots,N. \tag{2.15}$$

For mesh functions $v, w \in I\!R_0^{N+1}$ define the inner product

$$(v,w)_\omega := \sum_{j=1}^{N-1} h_{j+1} v_i w_i.$$

Then given an arbitrary mesh function $v \in I\!R_0^{N+1}$, we have the representation

$$v_i = \bigl(G_{i,\cdot} . Lv\bigr)_\omega \quad \text{for} \quad i = 0,\dots,N, \tag{2.16}$$

where $G_{i,j} = G(x_i, \xi_j)$, the discrete Green's function associated with L and Dirichlet boundary conditions, solves for fixed $j = 1,\dots,N-1$

$$[LG_{\cdot,j}]_i = \delta_{i,j} \quad \text{for} \quad i = 1,\dots,N-1, \quad G_{0,j} = G_{N,j} = 0, \tag{2.17}$$

where

$$\delta_{i,j} := \begin{cases} h_{i+1}^{-1} & \text{if} \quad i = j, \\ 0 & \text{otherwise} \end{cases}$$

is the discrete Dirac-δ function. Let L^* be the adjoint operator to L with respect to the discrete inner product $(\cdot,\cdot)_\omega$:

$$[L^* v]_j := -\varepsilon v_{\bar{\xi}\xi;j} + (bv)_{\breve{\xi};j} + c_j v_j \quad \text{with} \quad v_{\breve{x};j} := \frac{v_i - v_{j-1}}{h_{j+1}}.$$

Then the discrete Green's function G solves, for fixed $i = 1,\dots,N-1$,

$$[L^* G_{i,\cdot}]_j = \delta_{i,j} \quad \text{for} \quad j = 1,\dots,N-1, \quad G_{i,0} = G_{i,N} = 0. \tag{2.18}$$

The comparison principle of Corollary 2.10 with the test functions

$$\check{G} \equiv 0 \quad \text{and} \quad \hat{G}_{i,j} = \frac{1}{\beta} \begin{cases} 1 & \text{for} \quad 0 \leq i \leq j \leq N, \\ \displaystyle\prod_{k=j+1}^{i} \left(1 + \frac{\beta h_{k+1}}{\varepsilon}\right)^{-1} & \text{for} \quad 0 \leq j < i \leq N \end{cases}$$

yields

$$0 \le G_{i,j} \le \beta^{-1} \quad \text{for} \quad i,j = 0,\dots,N.$$

Similar to our analysis of the continuous Green's funtion we need monotonicity properties of the discrete Green's function. Since $G_{i,j} \ge 0$ and $G_{i,0} = 0$ for $i,j = 0,\dots,N$ we have $G_{\xi;i,0} \ge 0$ for $i = 0,\dots,N$. Multiplying (2.18) by h_{j+1} and summing over j, we get

$$-\varepsilon G_{\xi;i,j} + \varepsilon G_{\xi;i,0} + b_j G_{i,j} - b_0 G_{i,0} = -\sum_{k=1}^{j} h_{k+1} c_k G_{i,k} \quad \text{for} \quad j = 1,\dots,i-1. \tag{2.19}$$

Hence

$$\varepsilon G_{\xi;i,j} \ge \varepsilon G_{\xi;i,0} + b_j G_{i,j} \ge 0 \quad \text{for} \quad j = 1,\dots,i-1$$

because $G_{i,j} \ge 0$ and $G_{\xi;i,0} \ge 0$. This means $G_{i,j}$ is monontonically increasing for $j = 0,\dots,i$.

On the other hand, since $G_{i,j} \ge 0$ and $G_{i,N} = 0$ for $i,j = 0,\dots,N$ we have $G_{\xi;i,N-1} \le 0$ for $i = 0,\dots,N$. Then inspecting the difference equation (2.18), we see that $v_j := G_{\xi;i,j}$ satisfies

$$\begin{aligned} -\frac{\varepsilon}{h_{j+1}}\left(v_j - v_{j-1}\right) + \frac{h_j}{h_{j+1}} b_j v_{j-1} = -\frac{b_j - b_{j-1}}{h_{j+1}} G_{i,j-1} - c_j G_{i,j} \le 0 \\ \text{for} \quad j = i+1,\dots,N-1. \end{aligned} \tag{2.20}$$

because b' and c are assumed to be positive. Since $v_N \le 0$, induction for $j = N-1,\dots,i$ yields $v_j \le 0$ for $j = i,\dots,N$. Thus $G_{i,j}$ decreases monotonically for $j = i,\dots,N$.

Similarly one can prove that $G_{x;i,j} \ge 0$ for $i = 0,\dots,j-1$ and $G_{x;i,j} \le 0$ for $i = j,\dots,N-1$. Thus

$$G_{x\xi;i,0} \le 0 \quad \text{and} \quad G_{x\xi;i,N-1} \le 0 \quad \text{for} \quad i = 0,\dots,N-1.$$

because $G_{x;i,0} = G_{x;i,N} = 0$ for $i = 0,\dots,N$. Taking differences of (2.18) with respect to i and summing over j, we get

$$-\varepsilon G_{x\xi;i,j} + \varepsilon G_{x\xi;i,0} + b_j G_{x;i,j} - b_0 G_{x;i,0} + \sum_{k=1}^{j} h_{k+1} c_k G_{x;i,k} = -\delta_{i,j} \quad \text{for} \quad j = 1,\dots,i.$$

Therefore

$$G_{x\xi;i,j} \le 0 \quad \text{for} \quad 0 \le j < i < N \quad \text{and} \quad G_{x\xi;i,i} \le \frac{1}{\varepsilon h_{i+1}} \quad \text{for} \quad 0 \le i < N$$

because $G_{x;i,j} \ge 0$, $G_{x\xi;i,0} \le 0$ and $G_{x;i,0} = 0$.

For $i < j$, take differences of (2.20) to see that $v_j = G_{x\xi;i,j}$ satisfies

$$\begin{aligned} -\frac{\varepsilon}{h_{j+1}}\left(v_j - v_{j-1}\right) + \frac{h_j}{h_{j+1}} b_j v_{j-1} = -\frac{b_j - b_{j-1}}{h_{j+1}} G_{x;i,j-1} - c_j G_{x;i,j} \le 0 \\ \text{for} \quad j = i+2,\dots,N-1. \end{aligned}$$

because $b', c \ge 0$ and $G_{x;i,j} \ge 0$ for $i < j$. We get $G_{x\xi;i,j} \le 0$ for $0 \le i < j < N$. Finally, for $i = j$, we use that

$$\sum_{j=0}^{N-1} h_{j+1} G_{x\xi;i,j} = G_{x;i,N} - G_{x;i,0} = 0$$

in order to obtain

$$h_{i+1} G_{x\xi;i,i} = -\sum_{\substack{j=0 \\ i \ne j}}^{N-1} h_{j+1} G_{x\xi;i,j} \ge 0.$$

We summarize our results.

Theorem 2.11. *The Green's function G associated with the discrete operator L and Dirichlet boundary conditions satisfies*

$$\begin{aligned} 0 \le G_{i,j} &\le \beta^{-1} && \textit{for } \ i,j = 0,\dots,N, \\ G_{\xi;i,j} &\ge 0 && \textit{for } \ j = 0,\dots,i-1, \\ G_{\xi;i,j} &\le 0 && \textit{for } \ j = i,\dots,N-1, \\ G_{x\xi;ij} &\le 0 && \textit{for } \ i,j = 0,\dots,N-1, \ i \ne j \end{aligned}$$

and

$$0 \le G_{x\xi;ii} \le \frac{1}{\varepsilon h_{i+1}} \quad \textit{for } \ i = 0,\dots,N-1.$$

Analogously to the continuous case, we introduce the discrete maximum norm

$$\|v\|_{\infty,\omega} := \max_{i=0,\dots,N-1} |v_i|$$

the ℓ_1 norm

$$\|v\|_{1,\omega} := \sum_{j=0}^{N-1} h_{j+1}|v_j|$$

and the discrete negative norm

$$\|v\|_{*,\omega} := \min_{V:V_x=v} \|V\|_\infty = \min_{C\in\mathbb{R}} \left\| \sum_{j=\cdot}^{N-1} h_{j+1}v_j + C \right\|_{\infty,\omega}.$$

For fixed $i = 1,\dots,N-1$ we compute the following norms of the discrete Green's function G. Theorem 2.11 yields

$$\|G_{i,\cdot}\|_{1,\omega} \le \|G_{i,\cdot}\|_{\infty,\omega} \le \beta^{-1}, \tag{2.21a}$$

$$\|G_{\xi;i,\cdot}\|_{1,\omega} = \sum_{j=0}^{i-1} h_{j+1}G_{\xi;i,j} - \sum_{j=1}^{N-1} h_{j+1}G_{\xi;i,j} = 2G_{i,i} \le 2\beta^{-1} \tag{2.21b}$$

and

$$\|G_{x\xi;i,\cdot}\|_{1,\omega} = -\sum_{\substack{j=0\\ i\ne j}}^{N-1} h_{j+1}G_{x\xi;i,j} + h_{i+1}G_{x\xi;i,i} = 2h_{i+1}G_{x\xi;i,i} \le \frac{2}{\varepsilon}. \tag{2.21c}$$

These norms are used to establish stabily properties of the difference operator L.

Theorem 2.12. *The operator L satisfies*

$$\beta\|v\|_{\infty,\omega} \le \|Lv\|_{\infty,\omega} \quad \textit{for all } \ v \in \mathbb{R}_0^{N+1}, \tag{2.22a}$$

$$\beta\|v\|_{\infty,\omega} \le \|Lv\|_{1,\omega} \quad \textit{for all } \ v \in \mathbb{R}_0^{N+1}, \tag{2.22b}$$

and

$$|||v|||_{\varepsilon,\infty,\omega} := \frac{\beta}{2}\|v\|_{\infty,\omega} + \frac{\varepsilon}{2}\|v_x\|_{\infty,\omega} \le \|Lv\|_{*,\omega} \quad \textit{for all } \ v \in \mathbb{R}_0^{N+1}. \tag{2.22c}$$

Proof. First, the representation (2.16), a discrete Hölder inequality and (2.21a) give (2.22a,b).

Next, let $V \in \mathbb{R}^{N+1}$ be an arbitrary mesh function with $V_x = Lv$. Summing (2.16) by parts, we obtain

$$v_i = -\sum_{j=0}^{N-1} h_{j+1} G_{\xi;i,j} V_j \quad \text{for} \quad i = 0, \ldots, N,$$

and

$$v_{x;i} = -\sum_{j=0}^{N-1} h_{j+1} G_{x\xi;i,j} V_j \quad \text{for} \quad i = 0, \ldots, N-1.$$

The discrete Hölder inequality, (2.21b,c) and the definition of the discrete negative norm yield (2.22c). □

Remark 2.13. (i) *Since*

$$\|v\|_{*,\omega} \le \|v\|_{1,\omega} \le \|v\|_{\infty,\omega} \quad \textit{for all} \quad v \in \mathbb{R}_0^{N+1}$$

the negative-norm stability (2.22c) is the strongest of the three stability inequalities of Theorem 2.12.

(ii) *The same stability results hold true if the convection-diffusion problem in conservative form (2.8) is discretized by*

$$\left[L^c U\right]_i := -\varepsilon U_{\bar{x}x;i} - (bU)_{x;i} + c_i U_i = f_i \quad \textit{for} \quad i = 1, \ldots, N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1. \tag{2.23}$$

(iii) *The (ℓ_∞, ℓ_1) stability (2.22b) was first given by Andreev and Savin [5] for a modification of Samarskii's scheme [91]. It has been used in a number of publications to establish uniform convergence on S-type and B-type meshes; see, e. g., [3, 5, 61, 98]. Details of a convergence analysis can be found in Section 2.2.5. This stability result can be generalized to study two-dimensional problems; see Section 5.4.2.*

(iv) *The $(\ell_\infty, w_{-1,\infty})$ stability (2.22c) was derived by Andreev and Kopteva [4] though the proof there is different. A systematic approach can be found in [6], where stability of both the continuous operator $\mathcal{L}$ and of its discrete counterpart L is investigated. So far the $(\ell_\infty, w_{-1,\infty})$-stability inquality gives the sharpest error bounds for one-dimensional problem. But unlike the (ℓ_∞, ℓ_1) stability, it is unclear whether it can be generalized to higher dimensions*

2.2.2 A priori error bounds

Let us consider the approximation error of the simple upwind scheme (2.14) applied to the boundary value problem (2.1). We give a convergence analysis based on the negative norm stability of Theorem 2.12.

Introduce the continuous and discrete operators and functions

$$\mathcal{A}v := \varepsilon v' + bv + \int_\cdot^1 \big((b' + c)v\big)(s)ds, \quad \mathcal{F} := \int_\cdot^1 f(s)ds$$

and

$$Av := \varepsilon v_{\bar{x}} + bv + \sum_{k=\cdot}^{N-1} h_{k+1} \left(b_{x;k} v_{k+1} + c_k v_k\right), \quad F := \sum_{k=\cdot}^{N-1} h_{k+1} f_k.$$

Note that $\mathcal{L}v = -(\mathcal{A}v)'$ and $f = -\mathcal{F}'$ on $(0,1)$, and $Lv = -(Av)_x$ and $f = -F_x$ on ω. Thus

$$\mathcal{A}u - \mathcal{F} \equiv \alpha \quad \text{on} \quad (0,1) \quad \text{and} \quad AU - F \equiv a \quad \text{on} \quad \omega \tag{2.24}$$

with constants α and a.

In view of the stability inequality (2.22c) we have

$$|||u-U|||_{\varepsilon,\infty,\omega} \le \|L(u-U)\|_{*,\omega} = \min_{c\in\mathbb{R}} \|A(u-U)+c\|_{\infty,\omega}.$$

Taking $c = a-\alpha$, where a and α are the constants from (2.24), we get

$$|||u-U|||_{\varepsilon,\infty,\omega} \le \|Au - \mathcal{A}u - F + \mathcal{F}\|_{\infty,\omega}. \tag{2.25}$$

Furthermore

$$\begin{aligned}(Au-\mathcal{A}u-F+\mathcal{F})_i = \varepsilon\left(u_{\bar{x}}-u'\right)_i &+ \sum_{k=i}^{N-1} h_{k+1}\left(c_k u_k - f_k\right) - \int_{x_i}^{x_N} \left(cu-f\right)(x)dx \\ &+ \sum_{k=i}^{N-1} h_{k+1} b_{x;k} u_{k+1} - \int_{x_i}^{x_N} \left(b'u\right)(x)dx.\end{aligned} \tag{2.26}$$

Taylor expansions with the integral form of the remainder give

$$\begin{aligned}h_{k+1}\left(c_k u_k - f_k\right) - \int_{x_k}^{x_{k+1}} \left(cu-f\right)(x)dx &= \int_{x_k}^{x_{k+1}} \int_x^{x_k} \left(cu-f\right)'(s)ds\,dx \\ h_{k+1} b_{x;k} u_{k+1} - \int_{x_k}^{x_{k+1}} \left(b'u\right)(x)dx &= \int_{x_k}^{x_{k+1}} b'(x) \int_x^{x_{k+1}} u'(s)ds\,dx\end{aligned}$$

and

$$\varepsilon\left(u_{\bar{x}}-u'\right)_k = \frac{\varepsilon}{h_k}\int_{x_k}^{x_{k-1}} \int_{x_k}^{x} u''(s)ds\,dx = \frac{1}{h_k}\int_{x_k}^{x_{k-1}} \int_x^{x_k} \big(bu'-cu+f\big)(s)ds\,dx$$

by (2.1). Combining these representations with (2.25) and (2.26) we get the following general convergence result.

Theorem 2.14. *Let u be the solution of (2.1) and U that of (2.14). Then*

$$|||u-U|||_{\varepsilon,\infty,\omega} \le \max_{k=0,\dots,N-1} \int_{x_k}^{x_{k+1}} \left(C_1\left|u'(x)\right| + C_2|u(x)| + C_3\right)dx$$

with the constants

$$C_1 := \|c\|_\infty + \|b'\|_\infty + \|b\|_\infty, \quad C_2 := \|c\|_\infty + \|c'\|_\infty \quad \textit{and} \quad C_3 := \|f\|_\infty + \|f'\|_\infty. \tag{2.27}$$

Remark 2.15. *A similar result is given in [52] for the discretization of the conservative form (2.8). When using the conservative form the last two terms in (2.26) which involve b_x and b' disappear.*

Corollary 2.16. *Theorem 2.14 and the* a priori *bounds (2.9) yield*

$$|||u-U|||_{\varepsilon,\infty,\omega} \le C\vartheta_1(\omega),$$

where the characteristic quantity $\vartheta_\kappa(\omega)$ has been defined on p. 9:

$$\vartheta_\kappa(\omega) := \max_{k=0,\dots,N-1} \int_{x_k}^{x_{k+1}} \left(1+\varepsilon^{-1}e^{-\beta s/\kappa\varepsilon}\right)ds.$$

The mesh function U can be extended to a function U^I defined on $[0,1]$ using linear interpolation. Then the L_∞-error bound

$$\left|\left|\left|u-U^I\right|\right|\right|_{\varepsilon,\infty} \le C\vartheta_1(\omega)$$

can be derived using standard techniques.

From this we immediately get estimates for both S-type and B-type. For example,

$$|||u - U^I|||_{\varepsilon,\infty} \le C \begin{cases} N^{-1} & \text{for Bakhvalov meshes with } \sigma \ge 1, \\ h + N^{-1} \max\limits_{\xi \in [0,q]} |\psi'(\xi)| & \text{for S-type meshes with } \sigma \ge 1, \end{cases}$$

by (1.4) and (1.16). Hence the scheme is uniformly convergent of (almost) first order if $\sigma \ge 1$ is chosen in the construction of the mesh.

A numerical example. Table 2.1 displays numerical results for the upwind scheme (2.14) on a Bakhvalov mesh applied to the test problem

$$-\varepsilon u'' - u' + 2u = e^{x-1}, \quad u(0) = u(1) = 0. \tag{2.28}$$

In our computations we have fixed the parameter q and varied σ to illustrate the sharpness of our theoretical results. We see that choosing $\sigma < 1$ adversely affects the order of convergence. Similar observations can be made for the Shishkin mesh and other meshes.

	$\sigma = 0.2$		$\sigma = 0.4$		$\sigma = 0.6$		$\sigma = 0.8$		$\sigma = 1.0$	
N	error	rate	error	rate	error	rate	error	rate	error	rate
2^7	2.246e-2	0.23	1.173e-2	0.47	6.856e-3	0.69	4.658e-3	0.87	3.995e-3	0.97
2^8	1.913e-2	0.22	8.482e-3	0.45	4.258e-3	0.68	2.547e-3	0.88	2.036e-3	0.98
2^9	1.641e-2	0.21	6.201e-3	0.44	2.662e-3	0.67	1.388e-3	0.87	1.030e-3	0.99
2^{10}	1.416e-2	0.21	4.576e-3	0.43	1.675e-3	0.66	7.586e-4	0.87	5.193e-4	0.99
2^{11}	1.224e-2	0.20	3.403e-3	0.42	1.062e-3	0.65	4.155e-4	0.87	2.611e-4	0.99
2^{12}	1.063e-2	0.20	2.545e-3	0.41	6.784e-4	0.64	2.281e-4	0.86	1.310e-4	1.00
2^{13}	9.226e-3	0.20	1.911e-3	0.41	4.361e-4	0.63	1.256e-4	0.86	6.568e-5	1.00
2^{14}	8.030e-3	0.20	1.439e-3	0.41	2.819e-4	0.62	6.937e-5	0.85	3.290e-5	1.00
2^{15}	6.969e-3	0.21	1.086e-3	0.40	1.830e-4	0.62	3.846e-5	0.85	1.647e-5	1.00
2^{16}	6.026e-3	—	8.207e-4	—	1.193e-4	—	2.139e-5	—	8.245e-6	—

Table 2.1: Simple upwinding on a Bakhvalov mesh ($q = 1/2$).

2.2.3 Error expansion

In the previous section we have seen that the error of the simple upwind scheme (2.14) satisfies

$$|||u - U|||_{\varepsilon,\infty,\omega} \le C\vartheta_1(\omega).$$

Now an expansion of the error of this scheme is constructed. We shall show there exists a function ψ, the leading term of the error, such that

$$u - U = \psi + \text{second order terms}.$$

This result can be applied to analyse, e.g., derivative approximations, defect correction and Richardson extrapolation, see Sections 2.2.9 and 2.2.10.

For the sake of simplicity, we study the conservative form

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1 \tag{2.8}$$

and its discretization by

$$[L^c U]_i := -\varepsilon U_{\bar{x}x;i} - (bU)_{x;i} + c_i U_i = f_i \quad \text{for} \quad i = 1, \dots, N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1. \tag{2.23}$$

Analogously to Section 2.2.2 we introduce

$$\mathcal{A}^c v := \varepsilon v' + bv + \int_\cdot^1 (cv)(s)ds \quad \text{and} \quad A^c v := \varepsilon v_{\bar{x}} + bv + \sum_{k=\cdot}^{N-1} h_{k+1} c_k v_k, \tag{2.29}$$

Note that $\mathcal{L}^c v = -(\mathcal{A}^c v)'$ on $(0,1)$ and that $L^c v = -(A^c v)_x$ on ω.

2.2.3.1 Construction of the error expansion

We define the leading term of the error expansion as the solution of

$$(\mathcal{L}^c \psi)(x) = \Psi', \quad \psi(0) = \psi(1) = 0, \quad \Psi(x) = \varepsilon \frac{h(x)}{2} u''(x) - \int_x^1 h(s) g'(s) ds, \tag{2.30}$$

where $h(x) := x - x_{k-1}$ for $x \in (x_{k-1}, x_k)$ and $g := f - cu$. As Ψ is discontinuous at the mesh nodes $\mathcal{L}^c\psi$ may have such singularities as the Dirac-delta function. Therefore (2.30) has to be interpreted in the context of distributions. Or we may seek a solution $\psi \in C^2((0,1) \setminus \omega) \cap C[0,1]$ such that

$$\mathcal{L}^c \psi = \Psi' \quad x \in (0,1) \setminus \omega, \quad \psi(0) = \psi(1) = 0, \quad -\varepsilon[\psi'](x_i) = [\Psi](x_i) \quad \text{for} \quad x_i \in \omega.$$

Since $\mathcal{A}^c \psi = -\Psi$ on $(0,1) \setminus \omega$, we have

$$\left[A^c \psi\right]_i = \varepsilon \left(\psi_{\bar{x};i} - \psi'_{i-0}\right) + \sum_{k=i}^{N-1} h_{k+1} c_k \psi_k - \int_{x_i}^1 (c\psi)(s) ds + \Psi_{i-0}.$$

Thus

$$\begin{aligned} \left[A^c(u - \psi - U)\right]_i = \varepsilon \left(u_{\bar{x};i} - u'_i + \frac{h_i}{2} u''_i\right) + \int_{x_i}^1 \left(g(x) - h(x) g'(x)\right) dx - \sum_{k=i}^{N-1} h_{k+1} g_k \\ - \varepsilon \left(\psi_{\bar{x};i} - \psi'_{i-0}\right) - \sum_{k=i}^{N-1} h_{k+1} c_k \psi_k + \int_{x_i}^1 (c\psi)(s) ds. \end{aligned} \tag{2.31}$$

The function ψ has been designed such that the terms on the right-hand side that involve u are of second order. Those involving ψ are formally only first-order terms, but second order is gained since ψ itself is first order.

In order to bound the terms on the right-hand side bounds for the derivatives of u up to order three are needed. These are provided by (2.9). The following theorem gives bounds for the leading term ψ of the error expansion and its derivatives up to order two which are also required. Because of the number of technical details its proof is defered to the end of this section.

Lemma 2.17. *Let ψ be the solution of the boundary-value problem (2.30). Assume that $b, c-b', f \in C^1[0,1]$ and $c' \in L_\infty(0,1)$. Then ψ and its first-order derivative satisfy*

$$|\psi^{(k)}(x)| \le C\vartheta_2(\omega) \left(1 + \varepsilon^{-k} e^{-\beta x/2\varepsilon}\right) \quad \text{for} \quad x \in (0,1) \setminus \omega \quad \text{and} \quad k = 0, 1, \tag{2.32a}$$

while for the second-order derivative we have

$$\varepsilon|\psi''(x)| \le C\vartheta_2(\omega) \left(1 + \varepsilon^{-1} e^{-\beta x/2\varepsilon}\right) \quad \text{for} \quad x \in (0,1) \setminus \omega. \tag{2.32b}$$

Later we shall shown that (2.9), (2.31) and Lemma 2.17 yield

$$\|A^c(u - \psi - U)\|_{\infty,\omega} \le C\vartheta_2(\omega)^2. \tag{2.33}$$

Then Theorem 2.12 yields our main result of this section.

Theorem 2.18. *Let u, U and ψ be the solutions of (2.8), (2.23) and (2.30), respectively. Assume that $b, c - b', f \in C^1[0,1]$ and that $c, f \in W^{2,\infty}(0,1)$. Then*

$$|||u - \psi - U|||_{\varepsilon,\infty,\omega} \le C\vartheta_2(\omega)^2.$$

2.2.3.2 Detailed proofs

Proof of Lemma 2.17. Now we derive bounds for the derivatives of the leading term ψ in the error expansion. The following auxiliary result will be used several times in the subsequent analysis.

Proposition 2.19. *Let $x \in (x_{k-1}, x_k)$ and $\sigma > 0$ be arbitrary. Then*

$$h(x)\left(1+\varepsilon^{-1}e^{-\beta x/\sigma\varepsilon}\right) \leq \int_{x_{k-1}}^{x} \left(1+\varepsilon^{-1}e^{-\beta s/\sigma\varepsilon}\right) ds$$

Proof. Let

$$F(x) := h(x)\left(1+\varepsilon^{-1}e^{-\beta x/\sigma\varepsilon}\right) \quad \text{and} \quad G(x) := \int_{x_{k-1}}^{x} \left(1+\varepsilon^{-1}e^{-\beta s/\sigma\varepsilon}\right) ds.$$

Clearly $F(x_{k-1}) = G(x_{k-1}) = 0$ and

$$F'(x) = 1+\varepsilon^{-1}e^{-\beta x/\varepsilon} - \frac{h(x)\beta}{\sigma\varepsilon^2}e^{-\beta x/\sigma\varepsilon} \leq 1+\varepsilon^{-1}e^{-\beta x/\sigma\varepsilon} = G'(x)$$

for $x \in (x_{k-1}, x_k)$. The result follows. □

First (2.9) implies

$$|\Psi(x)| \leq C\varepsilon h(x)\left(1+\varepsilon^{-2}e^{-\beta x/\varepsilon}\right) + C\int_x^1 h(s)\left(1+\varepsilon^{-1}e^{-\beta s/\varepsilon}\right) ds.$$

This inequality, (2.22c) and Proposition 2.19 yield (2.32a) for $k = 0$.

Next, we derive bounds on ψ'. Set

$$B(x) := \frac{1}{\varepsilon}\int_0^x b(s)ds, \quad a(x) := \Psi'(x) + (c-b')(x)\psi(x)$$

and

$$\chi(x) := \frac{1}{\varepsilon}\int_0^x a(s)e^{B(s)-B(x)}ds.$$

Then ψ can be written as

$$\psi(x) = \int_x^1 \chi(s)ds + \kappa\int_x^1 e^{-B(s)}ds \quad \text{with} \quad \kappa = -\frac{\int_0^1 \chi(s)ds}{\int_0^1 e^{-B(s)}ds}.$$

For ψ' we get

$$\psi'(x) = -\chi(x) - \kappa\exp(-B(x)). \tag{2.34}$$

Apparently the critical point is to derive bounds on χ. Integration by parts and the definition of Ψ yield

$$2\chi(x) = \left(hu''\right)(x) - \zeta(x) \tag{2.35}$$

with

$$\zeta(x) := \frac{1}{\varepsilon}\int_0^x \big(hbu'' - 2h(f-cu)' - 2(c-b')\psi\big)(s)e^{B(s)-B(x)}ds.$$

For the first term on the right-hand side of (2.35) we have by (2.9) and Proposition 2.19

$$\left|(hu'')(x)\right| \leq Ch(x)\left(1+\varepsilon^{-1}e^{-\beta x/2\varepsilon}\right)^2 \leq C\vartheta_2(\omega)\left(1+\varepsilon^{-1}e^{-\beta x/2\varepsilon}\right). \tag{2.36}$$

To bound $\zeta(x)$, the second term in (2.35), we use (2.9), (2.32a) for $k=0$ and (2.36):

$$\begin{aligned}|\zeta(x)| &\leq \frac{C}{\varepsilon}\int_0^x \left[h(s)\left(1+\varepsilon^{-2}e^{-\beta s/\varepsilon}\right)+\vartheta_2(\omega)\right]e^{\beta(s-x)/\varepsilon}ds \\ &\leq C\vartheta_2(\omega)\int_0^x\left(1+\varepsilon^{-1}e^{\beta s/2\varepsilon}\right)e^{\beta(s-x)/\varepsilon}ds \leq C\vartheta_2(\omega)\left(1+\varepsilon^{-1}e^{-\beta x/2\varepsilon}\right).\end{aligned}$$

This, equation (2.35) and inequality (2.36) give

$$|\chi(x)| \leq C\vartheta_2(\omega)\left(1+\varepsilon^{-1}e^{-\beta x/2\varepsilon}\right). \tag{2.37}$$

Integrating (2.37) we obtain

$$|\kappa| \leq C\varepsilon^{-1}\vartheta_2(\omega), \tag{2.38}$$

because $\int_0^1 e^{-B(s)}ds \geq \varepsilon/\|b\|_\infty$. Combining (2.34)-(2.38), we get (2.32a) for $k=1$.

Finally the bound (2.32b) for the second-order derivative of ψ follows from (2.30), (2.9), (2.32a) and Proposition 2.19.

Proof of (2.33). We now bound the terms on the right-hand side of (2.31). For the first two terms a Taylor expansion with the integral form of the remainder yields

$$\varepsilon\left|u_{\bar{x};i}-u_i'+\frac{h_i}{2}u_i''\right| \leq C\int_{x_{i-1}}^{x_i}(x-x_{i-1})\left(1+\varepsilon^{-2}e^{-\beta x/\varepsilon}\right)dx$$

by (2.9). To estimate the right-hand side we use the inequality [18]

$$\int_a^b g(\xi)\,(\xi-a)\;d\xi \leq \frac{1}{2}\left\{\int_a^b g(\xi)^{1/2}\,d\xi\right\}^2, \tag{2.39}$$

which holds true for any positive monotonically decreasing function g on $[a,b]$. This can be verified by considering the two integrals as functions of the upper integration limit. We get

$$\varepsilon\left|u_{\bar{x};i}-u_i'+\frac{h_i}{2}u_i''\right| \leq C\left\{\int_{x_{i-1}}^{x_i}\left(1+\varepsilon^{-1}e^{-\beta x/2\varepsilon}\right)dx\right\}^2 \leq C\vartheta_2(\omega)^2. \tag{2.40}$$

Next we bound the third term in (2.31). Assuming $c,f\in W^{2,\infty}(0,1)$, we have

$$\int_{x_k}^{x_{k+1}}\left(g(x)-(x-x_k)g'\right)dx-h_{k+1}g_k = \int_{x_k}^{x_{k+1}}\int_{x_k}^{x}(s-x_k)g''(s)ds.$$

Thus

$$\begin{aligned}&\left|\int_{x_k}^{x_{k+1}}\left(g(x)-(x-x_k)g'\right)dx-h_{k+1}g_k\right| \\ &\qquad\leq Ch_{k+1}\int_{x_k}^{x_{k+1}}(s-x_k)\left(1+\varepsilon^{-2}e^{-\beta s/\varepsilon}\right)ds \leq Ch_{k+1}\vartheta_2(\omega)^2,\end{aligned}$$

by (2.9) and (2.39). Hence

$$\left|\int_{x_i}^1\left(g(x)-h(x)g'(x)\right)dx-\sum_{k=i}^{N-1}h_{k+1}g_k\right| \leq C\vartheta_2(\omega)^2. \tag{2.41}$$

To bound the remaining terms we use the bounds on ψ and its derivatives from Lemma 2.17. A Taylor expansion and (2.32b) yield

$$\varepsilon \left|\psi_{\bar{x};k} - \psi'_{k-0}\right| \le \varepsilon \int_{x_{k-1}}^{x_k} |\psi''(x)|\, dx \le C\vartheta_2(\omega)^2. \tag{2.42}$$

Finally,

$$\left|\int_{x_k}^{x_{k+1}} (c\psi)(s)ds - h_{k+1}(c\psi)_k\right| \le h_{k+1} \int_{x_k}^{x_{k+1}} |(c\psi)'(s)|\, d\xi ds \le Ch_{k+1}\vartheta_2(\omega)^2,$$

by (2.32a). Thus

$$\left|\sum_{k=i}^{N-1} h_{k+1}c_k\psi_k - \int_{x_i}^{1} (c\psi)(s)ds\right| \le C\vartheta_2(\omega)^2. \tag{2.43}$$

Applying (2.40)–(2.43) to (2.31) and taking the maximum over $i = 0, \dots, N-1$ we get (2.33).

2.2.4 A posteriori error estimation and adaptivity

In Section 2.2.2 the stability of the *discrete operator* L was used to bound the error in the *discrete* maximum norm in terms of the derivative of the *exact solution.* Now, in the first part of this section, roles are interchanged and the stability of the *continuous operator* $\mathcal{L}$ is used to bound the error in the *continuous* maximum norm in terms of finite differences of the *numerical solution.* We follow [41].

2.2.4.1 A posteriori error bounds

Let U^I be the piecewise-linear interpolant to the solution U of (2.14). Then (2.7c) yields

$$|||u - U^I|||_{\varepsilon,\infty} \le \left\|\mathcal{L}\left(u - U^I\right)\right\|_* = \min_{c\in I\!R} \left\|\mathcal{A}\left(u - U^I\right) + c\right\|_\infty .$$

Clearly

$$\min_{c\in I\!R} \left\|\mathcal{A}\left(u - U^I\right) + c\right\|_\infty \le \left\|\mathcal{A}\left(u - U^I\right) + a - \alpha\right\|_\infty , \tag{2.44}$$

where a and α are the constants from (2.24). Furthermore, for any $x \in (x_{i-1}, x_i) \subset (0,1) \setminus \omega$,

$$\mathcal{A}\left(u - U^I\right) + a - \alpha = [AU]_i - \left(\mathcal{A}U^I\right)(x) - F_i + \mathcal{F}(x).$$

We bound the two terms on the right-hand side.

Since $\left(U^I\right)' = U_{\bar{x},i}$ for all $x \in (x_{i-1}, x_i)$, we have

$$\begin{aligned}[AU]_i - \left(\mathcal{A}U^I\right)(x) = &\sum_{k=i}^{N-1} h_{k+1}b_{x;k}U_{k+1} - \int_{x_k}^{1} \left(b'U^I\right)(s)ds + \int_{x}^{x_i} \left(b\left(U^I\right)'\right)(s)ds \\ &- \int_{x}^{x_i} \left(cU^I\right)(s)ds - \int_{x_i}^{1} \left(cU^I\right)(s)ds + \sum_{k=i}^{N-1} h_{k+1}c_kU_k,\end{aligned}$$

by the definitions of A and $\mathcal{A}$ and by integration by parts. For the terms on the right-hand side we have the bounds

$$\left|h_{k+1}b_{x;k}U_{k+1} - \int_{x_k}^{x_{k+1}} \left(b'U^I\right)(s)ds\right| \le h_{k+1}\|b'\|_\infty \left|U_{k+1} - U_k\right|,$$

$$\left|\int_{x}^{x_k} b(s)\left(U^I\right)'(s)ds\right| \le \|b\|_\infty \left|U_k - U_{k-1}\right|,$$

$$\left|h_{k+1}c_kU_k - \int_{x_k}^{x_{k+1}} \left(cU^I\right)(s)ds\right| \le h_{k+1}^2\|c'\|_\infty \max\left\{|U_{k+1}|, |U_k|\right\} + h_{k+1}\|c\|_\infty \left|U_{k+1} - U_k\right|$$

and

$$\left| \int_x^{x_k} \left(cU^I\right)(s) ds \right| \le h_k \|c\|_\infty \max \{U_k, U_{k-1}\}.$$

Thus

$$\begin{aligned} \left|[AU]_i - \left(\mathcal{A}U^I\right)(x)\right| \le \{\|c\|_\infty + \|b'\|_\infty + \|b\|_\infty\} \max_{k=0,\ldots,N-1} |U_{k+1} - U_k| \\ + h\{\|c\|_\infty + \|c'\|_\infty\} \|U\|_{\infty,\omega}. \end{aligned} \tag{2.45}$$

Next we bound $F - \mathcal{F}$.

$$\left| \int_x^{x_k} f(s) ds \right| \le h_k \|f\|_\infty$$

and

$$\left| h_{k+1} f_k - \int_{x_k}^{x_{k+1}} f(s) ds \right| \le h_{k+1}^2 \|f'\|_\infty$$

yield

$$|F_k - \mathcal{F}(x)| \le \{\|f\|_\infty + \|f'\|_\infty\} h.$$

Combining this bound with (2.45), then taking the supremum over all $x \in (x_{i-1}, x_i) \subset (0,1) \setminus \omega$, we get

$$\left\|\mathcal{L}\left(u - U^I\right)\right\|_* \le C_1 \max_{k=0,\ldots,N-1} |U_{k+1} - U_k| + h\left(C_2 \|U\|_{\infty,\omega} + C_3\right) \tag{2.46}$$

with the constants C_1, C_2 and C_3 from (2.27).

Finally, use (2.44) in order to obtain the main result of this section.

Theorem 2.20. *Let u be the solution of (2.1) and U that of (2.14). Then*

$$|||u - U|||_{\varepsilon,\infty} \le C_1 \max_{k=0,\ldots,N-1} |U_{k+1} - U_k| + h\left(C_2 \|U\|_{\infty,\omega} + C_3\right).$$

Corollary 2.21. *Theorem 2.20 and the* a priori *bound (2.15) for $\|U\|_{\infty,\omega}$ yield*

$$|||u - U|||_{\varepsilon,\infty} \le C \max_{k=0,\ldots,N-1} h_{k+1}\left(1 + |U_{x;k}|\right).$$

Note the analogy to Theorem 2.14 and Corollary 2.16.

2.2.4.2 An adaptivite method

From Theorem 2.14 it is easily concluded that the error of our upwind scheme satisfies

$$\|u - U\|_{\infty,\omega} \le C \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} \sqrt{1 + u'(x)^2} dx.$$

On the other hand we have $\int_0^1 \sqrt{1 + u'(x)^2} dx \le C$ by (2.9). Thus if the mesh is designed so that

$$\int_{x_{k-1}}^{x_k} \sqrt{1 + u'(x)^2} dx = \int_{x_k}^{x_{k+1}} \sqrt{1 + u'(x)^2} dx \quad \text{for} \quad k = 1, \ldots, N-1, \tag{2.47}$$

i. e., if the mesh equidistributes the arc length of the exact solution, then

$$\|u - U\|_{\infty,\omega} \le CN^{-1}. \tag{2.48}$$

However u' is not available. An idea that leads to an adaptive method is to approximate the integrals in (2.47) by the mid-point quadrature rule and $u'(x_{k-1/2})$ by a central difference quotient and finally to replace u by the numerical solution U. We get

$$\int_{x_{k-1}}^{x_k} \sqrt{1+u'(x)^2}dx \approx h_k\sqrt{1+\left(U_{\bar{x};k}\right)^2} =: Q_k.$$

Thus (2.47) is replaced by $Q_k = Q_{k+1}$ for $k = 1, \dots, N-1$ or, what is equivalent,

$$(x_k - x_{k-1})^2 + (U_k - U_{k-1})^2 = (x_{k+1} - x_k)^2 + (U_{k+1} - U_k)^2 \quad \text{for} \quad k = 1, \dots, N-1. \tag{2.49}$$

Now solving the difference equation (2.14) and the discretised equidistribution principle (2.49) simultaniously, we get an adaptive procedure.

A question that arises naturally is: Does the nonlinear system of equations (2.14) and (2.49) posses a (unique) solution? As (2.47) is not solved exactly, does the error bound (2.48) nevertheless hold true? Kopteva and Stynes [43] proved there exists a solution and the error of the adaptive method satisfies $\|u-U\|_{\infty,\omega} \le CN^{-1}$. A crucial ingredient is the *a posteriori* error bound of Theorem 2.20.

Finally, how can the nonlinear system be solved efficiently? Beckett [14] uses a method that decouples the two sets of equations. He starts with an initial (uniform) mesh, solves (2.14) on this mesh for U, extends U linearly to a function defined on $[0, 1]$ and equidistributes the arc length of this function to get a new mesh that is better adapted to the layer structure of the problem. This process is repeated until the nonlinear system is solved to a desired accuracy. Unfortunately this process becomes numerically unstable when the solution of the nonlinear system is approached. The mesh starts to oscillate: Mesh points moved into the layer region in one iteration are moved back out of it in the next iteration. Beckett applies various damping procedures to suppress these oscillations.

To avoid these oscillations, in [53] the author treats the system as a map $(0, 1] \to I\!R^{2(N-1)} : \varepsilon \mapsto (\omega_\varepsilon, U_\varepsilon)$ and applies a continuation method combining an explicit Euler method (predictor) with a Newton method (corrector). The iteration matrices in each Newton step are seven diagonal and in an example the numerical costs are approximately of order $N\,|\ln(N\varepsilon)|$. However convergence of this method is not proved in [53].

Kopteva and Stynes [43] realized that it is not necessary to solve the equidistribution principle (2.49) exactly. They use the decoupling technique with a modified stopping criterion: The iteration is stopped when

$$Q_i \le \frac{\gamma}{N}\sum_{k=1}^{N} Q_k \quad \text{for} \quad i = 1, \dots, N$$

with a user chosen constant $\gamma > 1$. Note that for $\gamma = 1$ this is equivalent to (2.49). Furthermore in [43] it is shown that this stopping criterion is met after $\mathcal{O}\left(\ln(1/\varepsilon)\right)$ iterations and the error of the numerical solution obtained satisfies (2.48) with a constant $C = C(\gamma)$.

2.2.5 An alternative convergence proof

In this section we shall demonstrate how the (ℓ_∞, ℓ_1) stability (2.22b) can be exploited to study convergence of the scheme (2.14) on S-type meshes. The results are less general than those of Section 2.2.2, but can be generalized to two dimensions; cf. Section 5.4.2. In our presentation we follow [61].

By (2.22b), we have

$$\|u-U\|_{\infty,\omega} \le \frac{1}{\beta}\,\|Lu - f\|_{1,\omega}\,. \tag{2.50}$$

Thus the maximal nodal error is bounded by a discrete ℓ_1 norm of the truncation error $\tau := Lu-f$:

$$\|\tau\|_{1,\omega} = \sum_{j=0}^{N-1} h_{j+1}|\tau_j|.$$

Using the solution decomposition $u = v + w$ of Theorem (2.6) and a triangle inequality, we can bound the truncation error pointwise:

$$|\tau_i| \le \left|\big[Lv\big]_i - f_i\right| + \left|\big[Lw\big]_i\right|.$$

Separate Taylor expansions for the two solution components and the derivative bounds of Theorem (2.6) yield

$$h_{i+1}\,|\tau_i| \le C\left(h_{i+1} + h_i + e^{-\beta x_{i-1}/\varepsilon}\right), \tag{2.51a}$$

and

$$\begin{aligned} h_{i+1}\,|\tau_i| \le C\Big\{ &|h_{i+1}-h_i|\left(1+\varepsilon^{-1}e^{-\beta x_{i-1}/\varepsilon}\right) \\ &+\left(h_i^2 + h_{i+1}\right)\left(1+\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon}\right)\Big\}. \end{aligned} \tag{2.51b}$$

For our further analysis let us assume that the mesh generating function $\tilde{\varphi}$ of our S-type mesh satisfies (1.11) and that $\sigma \ge 2$. For the sake of simplicity suppose $\tilde{\varphi}'$ is nondecreasing. This leads to a mesh that does not condense on $[0,\lambda]$ as we move away from the layer, i.e., $h_i \le h_{i+1}$ for $i = 1,\dots,qN-1$. Which is reasonable for the given problem.

Now let us bound the ℓ_1 norm of the truncation error. Apply (2.51a) to bound $h_{i+1}\,|\tau_i|$ for $i = qN, qN+1$ and (2.51b) otherwise.

$$\begin{aligned} \|\tau\|_{1,\omega} \le C\sum_{i=1}^{qN-1} &\left\{(h_{i+1}-h_i)\left(1+\varepsilon^{-1}e^{-\beta x_i/\varepsilon}\right) + \left(h_i^2+h_{i+1}^2\right)\left(1+\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon}\right)\right\} \\ &+ C\left(h + e^{-\beta x_{qN-1}/\varepsilon} + e^{-\beta x_{qN}/\varepsilon}\right) + C\sum_{i=qN+2}^{N-1} N^{-2}\left(1+\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon}\right). \end{aligned} \tag{2.52}$$

We bound the terms on the right-hand side separately in reverse order.

Letting H denote the (constant) mesh size on $[\lambda, 1]$, we have for $i = qN+2,\dots,N-1$

$$\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon} \le \varepsilon^{-2}e^{-\beta H/\varepsilon}e^{-\beta\lambda/\varepsilon} \le C\big(H/\varepsilon\big)^2 e^{-\beta H/\varepsilon} \le C,$$

since $x_{i-1} \ge x_{N/2} + H = \lambda + H$ and $\sigma \ge 2$. Thus

$$\sum_{i=qN+2}^{N-1} N^{-2}\left(1+\varepsilon^{-2}e^{-\beta x_{i-1}/\varepsilon}\right) \le CN^{-1}. \tag{2.53}$$

Furthermore

$$h + e^{-\beta x_{qN-1}/\varepsilon} + e^{-\beta x_{qN}/\varepsilon} \le h + \left(1+e^{\beta h_{qN}/\varepsilon}\right)e^{-\beta x_{qN}/\varepsilon} \le h + CN^{-\sigma}, \tag{2.54}$$

by (1.13).

Next we bound the first sum in (2.52). We have

$$\sum_{i=1}^{qN-1}\left(h_{i+1} - h_i + h_i^2 + h_{i+1}^2\right) \le 3h \tag{2.55}$$

and

$$\sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, e^{-\beta x_i/\varepsilon} = -h_1 e^{-\beta x_1/\varepsilon} + \sum_{i=2}^{qN-1} h_i \left(e^{-\beta x_{i-1}/\varepsilon} - e^{-\beta x_i/\varepsilon} \right) + h_{qN} e^{-\beta x_{qN-1}/\varepsilon}.$$

Since the mean value theorem, (1.13) and (1.15) imply

$$\left| e^{-\beta x_{i-1}/\varepsilon} - e^{-\beta x_i/\varepsilon} \right| \le h_i \frac{\beta}{\varepsilon} e^{-\beta x_{i-1}/\varepsilon} \le C\varepsilon N^{-1} \max|\psi'| e^{-\beta x_{i-1}/(2\varepsilon)},$$

it follows that

$$\varepsilon^{-1} \sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, e^{-\beta x_i/\varepsilon} \le CN^{-1} \max|\psi'| \sum_{i=1}^{qN} \frac{h_i}{\varepsilon} e^{-\beta x_{i-1}/(2\varepsilon)}.$$

Ineq. (1.13) also gives

$$\sum_{i=1}^{qN} \frac{h_i}{\varepsilon} e^{-\beta x_{i-1}/(2\varepsilon)} \le C \int_0^{\lambda} \varepsilon^{-1} e^{-\beta x/(2\varepsilon)} dx \le C.$$

Thus

$$\varepsilon^{-1} \sum_{i=1}^{qN-1} (h_{i+1} - h_i)\, e^{-\beta x_i/\varepsilon} \le CN^{-1} \max|\psi'|. \tag{2.56}$$

Similar calculations yield

$$\varepsilon^{-2} \left| \sum_{i=1}^{qN-1} \left(h_i^2 + h_{i+1}^2 \right) e^{-\beta x_{i-1}/\varepsilon} \right| \le CN^{-1} \max|\psi'|. \tag{2.57}$$

Substituting (2.53)–(2.57) into (2.52) and applying the stability inequality (2.50), we get the uniform error bound

$$\|u - U\|_{\infty,\omega} \le C \left(h + N^{-1} \max|\psi'| \right).$$

In [61] the authors proceed—using more detailed bounds on the discrete Green's function—to prove the sharper bound

$$\|u - U\|_{\infty,\omega\cap[\lambda,1]} \le CN^{-1}$$

for the error outside of the layer region if (1.12) is satisfied by the mesh generating function.

2.2.6 The truncation error and barrier function technique

We now consider the convection-diffusion problem

$$-\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

discretized by

$$[\hat{L}U]_i := -\varepsilon U_{\bar{x}\hat{x};i} - b_i U_{x;i} + c_i U_i = f_i \quad \text{for} \quad i = 1, \ldots, N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1 \tag{2.58}$$

with

$$v_{x;i} := \frac{v_{i+1} - v_i}{h_{i+1}}, \quad \text{and} \quad v_{\bar{x};i} := \frac{v_i - v_{i-1}}{h_i}.$$

In contrast to the scheme (2.14) of Section 2.2 this scheme is first-order consistent in the mesh nodes on arbitrary meshes.

The analysis of this section uses the truncation error and barrier function technique developed by Kellogg and Tsan [36]. This was adapted to the analysis of Shishkin meshes by Stynes and Roos [97] and later used for other meshes also [85]. This technique can be used for problems in two dimensions too; see Section 5.2.1 or [48, 62]. We demonstrate this technique by sketching the convergence analysis for S-type meshes. For more details the reader is referred to [85].

The matrix associated with $\hat{L}$ is an M-matrix. Therefore we have the following comparison principle for two mesh functions $\hat{u}, \check{u} \in I\!R^{N+1}$:

$$\left.\begin{aligned} \left|[\hat{L}\check{u}]_i\right| &\le [\hat{L}\hat{u}]_i \quad \text{for} \quad i = 1, \dots, N-1, \\ |\check{u}_0| &\le \hat{u}_0, \\ |\check{u}_N| &\le \hat{u}_N \end{aligned}\right\} \quad \Longrightarrow \quad |\check{u}_i| \le \hat{u}_i \quad \text{for} \quad i = 0, \dots, N. \tag{2.59}$$

We call $\hat{u}$ a *barrier function* of $\check{u}$.

Theorem 2.22. *Let ω be a S-type mesh with $\sigma \ge 2$; see Section 1.3. Assume that the function $\tilde{\varphi}$ is piecewise differentiable and satisfies (1.11) and (1.12). Then the error of the simple upwind scheme satisfies*

$$|u_i - U_i| \le \begin{cases} C\left(h + N^{-1} \max|\psi'|\right) & \text{for} \quad i = 0, \dots, qN-1, \\ C\left(h + N^{-1}\right) & \text{for} \quad i = qN, \dots, N. \end{cases}$$

Proof. The numerical solution U is split analogously to the splitting of $u = v + w$ of Theorem 2.6: $U = V + W$ with

$$[\hat{L}V]_i = f_i \quad \text{for} \quad i = 1, \dots, N-1, \quad V_0 = v(0),\ V_N = v(1) = \gamma_1$$

and

$$[\hat{L}W]_i = 0 \quad \text{for} \quad i = 1, \dots, N-1, \quad W_0 = w(0),\ W_N = w(1) = 0.$$

Then the error is $u - U = (v - V) + (w - W)$ and we can estimate the error in v and w separately. For the regular solution component v Taylor expansions and (2.13a) give

$$\left|\hat{L}(v - V)_i\right| = \left|[\hat{L}v]_i - (\mathcal{L}v)_i\right| \le Ch \quad \text{for} \quad i = 1, \dots, N-1.$$

Furthermore $(v - V)_0 = (v - V)_N = 0$. Then the comparison principle (2.59) with the barrier function $C(1-x)h$ yields

$$\|v - V\|_{\infty,\omega} \le Ch. \tag{2.60}$$

Using the M-matrix property of $\hat{L}$, one can show that

$$|W_i| \le \bar{W}_i := C \prod_{k=1}^{i} \left(1 + \frac{\beta h_k}{2\varepsilon}\right)^{-1} \quad \text{for} \quad i = 0, \dots, N. \tag{2.61}$$

For $\xi \ge 0$ we have $\ln(1+\xi) \ge \xi - \xi^2/2$ which implies

$$\bar{W}_i \le \bar{W}_{qN} \le N^{-\sigma/2} \exp\left(\frac{1}{2} \sum_{k=1}^{qN} \left(\frac{\beta h_k}{2\varepsilon}\right)^2\right) \le CN^{-1} \quad \text{for} \quad i = qN, \dots, N,$$

by (1.14). Hence

$$|w_i - W_i| \le |w_i| + |W_i| \le CN^{-1} \quad \text{for} \quad i = qN, \dots, N, \tag{2.62}$$

where we have used (2.13b).

For the truncation error with respect to the layer part w, Taylor expansions and (2.13b) give

$$\begin{aligned}\tau_i := \left|[\hat{L}(w-W)]_i\right| = \left|[\hat{L}w]_i\right| &\le C\varepsilon^{-2}\left(h_i+h_{i+1}\right)e^{-\beta x_{i-1}/\varepsilon}dx \le C\varepsilon^{-1}e^{-\beta x_i/(2\varepsilon)}N^{-1}\max|\psi'| \\ &\le C\varepsilon^{-1}\bar{W}_i N^{-1}\max|\psi'|. \text{ for } i=1,\dots,qN-1,\end{aligned}$$

by (1.13) and (1.15). Finally, application of a discrete comparison principle with the barrier function

$$C\left\{N^{-1}+\bar{W}_i N^{-1}\max|\psi'|\right\}$$

and sufficiently large C yields

$$|w_i-W_i| \le CN^{-1}\max|\psi'| \quad \text{for} \quad i=0,\dots,qN-1.$$

Combine (2.60) and (2.62) with the last inequality to complete the proof. □

Corollary 2.23. *For Shishkin's mesh and Vulanović's modification of it we have $h \le 1/(1-q)N$ and application of Theorem 2.22 gives the afore-mentioned results (1.7) and (1.8). In general, assumption (1.11) implies only $h \le C\left(\varepsilon+N^{-1}\right)$. For the Bakhvalov-Shishkin mesh we have $h_{qN}=\mathcal{O}(\varepsilon)$, which gives the error bound (1.10). Numerical experiments show that for this mesh the convergence stalls when $N^{-1} \ll \varepsilon$ as the theory predicts, however in practice one typically has $\varepsilon \ll N^{-1}$.*

Remark 2.24. (i) *We are not aware of any results for B-type meshes that make use of this truncation error and barrier function technique. Also note that this technique needs $\sigma \ge 2$, while in Section 2.2.2 only $\sigma \ge 1$ was assumed.*

(ii) *The technique of Section 2.2.2 also provides error estimates for the approximation of the first-order differences:*

$$\varepsilon\|(u-U)_x\|_{\infty,\omega} \le C\vartheta_1(\omega).$$

In [22] the authors use the barrier function technique to establish that the upwind scheme (2.58) on standard Shishkin meshes satisfies

$$\varepsilon\left|(U-u)_{x;i}\right| \le \begin{cases} CN^{-1}\ln N & \text{for } i=0,\dots,qN-1, \\ CN^{-1} & \text{for } i=qN,\dots,N-1. \end{cases}$$

However the technique in [22] makes strong use of the piecewise uniformity of the mesh.

2.2.7 Discontinuous coefficients and point sources

Consider the convection-diffusion problem in conservative form with a point source:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \quad \text{in } (0,1), \quad u(0)=\gamma_0, \quad u(1)=\gamma_1, \tag{2.63}$$

where δ_d is the shifted Dirac-delta function $\delta_d(x)=\delta(x-d)$ with $d \in (0,1)$. The coefficient b may also have a discontinuity at $x=d$. We assume that $b \ge \beta_1 > 0$ on $(0,d)$ and $b \ge \beta_2 > 0$ on $(d,1)$ and set $\beta = \min\{\beta_1,\beta_2\}$. For the sake of simplicity we shall also assume that $c \ge 0$ and $c-b' \ge 0$ on $(0,1)$. The argument follows [55].

Problem (2.63) has to be read in a distributional context. Or, we may seek a solution $u \in C[0,1] \cap C^2((0,d)\cup(d,1))$ with

$$\mathcal{L}^c u = f, \quad \text{in } (0,d)\cup(d,1), \quad u(0)=\gamma_0, \quad u(1)=\gamma_1 \quad \text{and} \quad -\varepsilon[u'](d) - [b](d)\,u(d) = \alpha.$$

The solution u typically has an exponential boundary layer at the outflow boundary $x=0$ and an internal layer at $x=d$ caused by the concentrated source or the discontinuity of the convective

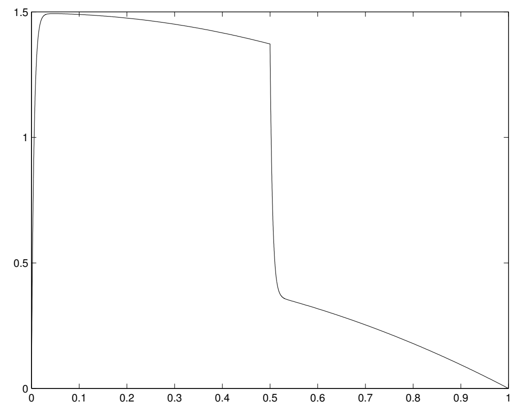


Figure 2.2: Typical solution of (2.1).

field. Figure 2.2 depicts a typical solution of (2.1). Using stability inequality (2.7c), we obtain $\|u\|_\infty \le C$ Therefore, on $(0,d)$ we can interpret u as the solution of

$$\mathcal{L}u = f \quad \text{in} \quad (0,d), \; u(0)=0, \; u(d)=\varrho,$$

while on $(d,1)$ it solves

$$\mathcal{L}u = f \quad \text{in} \quad (d,1), \; u(d)=\varrho, \; u(1)=0$$

with a $|\varrho| \le C$.

Apply separately (2.9) on each of the two subintervals in order to obtain

$$\left|u^{(k)}(x)\right| \le C\left[1+\varepsilon^{-k}\left\{\exp\left(-\frac{\beta_1 x}{\varepsilon}\right)+H_d(x)\exp\left(-\frac{\beta_2(x-d)}{\varepsilon}\right)\right\}\right] \qquad \text{for} \quad x\in(0,d)\cup(d,1) \quad \text{and} \quad k=0,1,\dots,q, \tag{2.64}$$

where H_d denotes the shifted *Heaviside* function, i.e.,

$$H_d(x) = \begin{cases} 0 & \text{for } x<d, \\ 1 & \text{for } x>d. \end{cases}$$

We generalize the difference scheme (2.23) by seeking a solution $U \in I\!R^{N+1}$ with

$$\begin{aligned}[L^c U]_i := -\varepsilon U_{\bar{x}x;i} - (b^- U)_{x;i} + c_i U_i = f_i + \Delta_{d,i} \quad \text{for} \quad i=1,\dots,N-1, \\ U_0=\gamma_0, \quad U_N=\gamma_1,\end{aligned} \tag{2.65}$$

where $v_i^- := v(x_i - 0)$ and

$$\Delta_{d;i} := \begin{cases} h_{i+1}^{-1} & \text{if } d\in[x_i,x_{i+1}), \\ 0 & \text{otherwise} \end{cases}$$

is an approximation of the shifted *Dirac*-delta function.

The discrete operator L^c enjoys the stability property (2.22c). Therefore it is sufficient to derive bounds for the truncation error $\|L^c(u-U)\|_{*,\omega}$. Extending the notation from Section 2.2.2, we set

$$\mathcal{F}(x) := \int_x^1 f(s)ds + \begin{cases} \alpha & \text{if } x_i \le d, \\ 0 & \text{if } x_i > d \end{cases} \qquad \text{and} \quad F_i := \sum_{k=i}^{N-1} h_{k+1} f_k + \begin{cases} \alpha & \text{if } x_i \le d, \\ 0 & \text{if } x_i > d. \end{cases}$$

Inspecting (2.63) and (2.65), we see

$$\mathcal{A}u - \mathcal{F} \equiv \text{const} \quad \text{on} \quad (0,1) \quad \text{and} \quad AU - F \equiv \text{const} \quad \text{on} \quad \omega$$

with A^c defined in (2.29). Then, analogoulsy to (2.26), we obtain

$$(A^c u - \mathcal{A}^c u - F + \mathcal{F})_i = \varepsilon\left(u_{\bar{x}} - u'\right)_i + \sum_{k=i}^{N-1} h_{k+1}\left(c_k u_k - f_k\right) - \int_{x_i}^{x_N} (cu - f)(x)dx$$

since the contributions from the δ functions and its discretization cancel.

Proceeding along the lines of Section 2.2.2, we get.

Theorem 2.25. *Let u be the solution of (2.63). Then the error of the simple upwind scheme (2.65) satisfies*

$$|||u - U|||_{\varepsilon,\infty,\omega} \le C \max_{k=0,\ldots,N-1} \int_{x_k}^{x_{k+1}} \left(1 + |u'(x)|\right) dx.$$

Corollary 2.26. *Theorem 2.25 and the* a priori *bounds (2.64) yield*

$$|||u - U|||_{\varepsilon,\infty,\omega} \le C \max_k \int_{x_k}^{x_{k+1}} \left[1 + \frac{1}{\varepsilon}\left\{\exp\left(-\frac{\beta_1 x}{\varepsilon}\right) + H_d(x)\exp\left(-\frac{\beta_2(x-d)}{\varepsilon}\right)\right\}\right] dx.$$

Comparing this result with Corollary 2.16 and the construction of Shishkin meshes and Bakhvalov meshes for problems with a single boundary layer (see Sections 1.2 and 1.3), we can devise appropriate layer-adapted meshes for the discretization of (2.63).

Shishkin meshes. Let $q_i \in (0,1)$, $i = 1,\ldots,4$ with $\sum q_i = 1$ and $\sigma_1, \sigma_2 > 0$ be mesh parameters. We set

$$\lambda_1 = \min\left\{q_1,\ldots,q_4, \frac{\sigma_1\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \lambda_2 = \min\left\{q_1,\ldots,q_4, \frac{\sigma_2\varepsilon}{\beta_2}\ln N\right\}.$$

Then the subintervals $I_1 = [0,\lambda_1]$, $I_2 = [\lambda_1, d]$, $I_3 = [d, d+\lambda_2]$ and $I_4 = [d+\lambda_2, 1]$ are divided into $q_i N$ equidistant subintervals (assuming that $q_i N$ are integers). The simplest choice is to take $q_i = 1/4$, $i = 1,\ldots,4$, and an $N > 0$ that is divisible by 4.

Bakhvalov meshes for (2.63) can be generated by equidistributing the *monitor function*

$$M_{Ba}(x) = \max\left\{1, \frac{K_1\beta_1}{\varepsilon}\exp\left(-\frac{\beta_1 x}{\sigma_1\varepsilon}\right), \frac{K_2\beta_2}{\varepsilon} H_d(x)\exp\left(-\frac{\beta_2(x-d)}{\sigma_2\varepsilon}\right)\right\}.$$

The quantities $K_i > 0$ determine the number of mesh points used to resolve the two layers, while the $\sigma_i > 0$ determine the grading of the mesh in the layer regions.

Corollary 2.26 yields for $\sigma_1, \sigma_2 \ge 1$ the error estimate

$$\|u - U\|_{\infty,\omega} \le \begin{cases} CN^{-1}\ln N & \text{for the Shishkin mesh and} \\ CN^{-1} & \text{for the Bakhvalov mesh.} \end{cases}$$

Numerical results. Let us briefly verify experimentally the theoretical result of Theorem 2.25. Our test problem is

$$-\varepsilon u'' - u' = x + \delta_{1/2} \quad \text{in} \quad (0,1), \quad u(0) = u(1) = 0.$$

The results presented in Table 2.2 are in fair agreement with Theorem 2.25.

	Bakhvalov mesh		Shishkin mesh	
N	error	rate	error	rate
2^7	2.822e-2	0.95	3.898e-2	0.78
2^8	1.458e-2	0.97	2.277e-2	0.81
2^9	7.447e-3	0.98	1.299e-2	0.84
2^{10}	3.779e-3	0.99	7.280e-3	0.85
2^{11}	1.909e-3	0.99	4.027e-3	0.87
2^{12}	9.610e-4	0.99	2.204e-3	0.88
2^{13}	4.828e-4	1.00	1.197e-3	0.89
2^{14}	2.422e-4	1.00	6.454e-4	0.90
2^{15}	1.214e-4	1.00	3.462e-4	0.91
2^{16}	6.080e-5	—	1.848e-4	—

Table 2.2: The upwind difference scheme for (2.63)

The traditional truncation error and barrier function technique of Section 2.2.6 can also be applied to problems with interior layers. Farrell et al. [25] consider the problem of finding $u \in C^2((0,d) \cap (d,1)) \cup C^1[0,1]$ such that

$$-\varepsilon u'' - bu' = f \quad \text{in} \quad (0,d) \cup (d,1), \quad u(0) = u(1) = 0,$$

where at the point $d \in (0,1)$ the convection coefficient changes sign:

$$b(x) > 0 \quad \text{for } x \in (0,d), \quad b(x) < 0 \quad \text{for } x \in (d,1) \text{ and } \ |b(x)| \geq \beta > 0.$$

The solution u and its derivatives satisfy

$$\left|u^{(k)}(x)\right| \leq C \left\{1 + \varepsilon^{-k} e^{-\beta|x-d|/\varepsilon}\right\} \quad \text{for } k = 0,1,\ldots,q \ \text{ and } x \in [0,1],$$

where the maximal order q depends on the smoothness of the data. Using the barrier function technique of Section 2.2.6, in [25] the authors establish the error bound

$$\|u - U\|_{\infty,\omega} \leq CN^{-1} \ln N$$

for the simple upwind scheme (2.14) on a Shishkin mesh.

2.2.8 Quasilinear problems

We now extend the results of Sections 2.1.1 and 2.2.1 to the class of quasilinear problems described by

$$\mathcal{T}^c u := -\varepsilon u'' - b(x,u)' + c(x,u) = 0 \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1 \tag{2.66}$$

with $0 < \varepsilon \ll 1$, $b_u \geq \beta > 0$ and $c_u \geq 0$ and its simple upwind discretization

$$[T^c U]_i := -\varepsilon U_{\bar{x}x;i} - b(\,\cdot\,,U)_{x;i} + c(\,\cdot\,,U)_i = 0 \quad \text{for} \quad i = 1,\ldots,N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1.$$

First, for the solution u of (2.66) and its derivatives the bounds (2.9) hold true too; see [105]:

$$\left|u^{(k)}(x)\right| \leq C \left\{1 + \varepsilon^{-k} e^{-\beta x/\varepsilon}\right\} \quad \text{for } k = 0,1,\ldots,q \ \text{ and } x \in [0,1],$$

where the maximal order q depends on the smoothness of the data.

Next let us consider stability properties of $\mathcal{T}^c$. For two function $v, w \in W^{1,\infty}(0,1)$ with $v(0) = w(0)$ and $v(1) = w(1)$ define the linear operator

$$\tilde{\mathcal{L}}^c y = \tilde{\mathcal{L}}^c[v,w]y := -\varepsilon y'' - (py)' + qy$$

with

$$p(x) = \int_0^1 b_u\big(x, w(x) + s(v-w)(x)\big)ds \ge \beta$$

and

$$q(x) = \int_0^1 c_u\big(x, w(x) + s(v-w)(x)\big)ds \ge 0.$$

The linearized operator $\tilde{\mathcal{L}}^c$ is constructed such that $\mathcal{L}^c(v-w) = \mathcal{T}^c v - \mathcal{T}^c w$ on $(0,1)$. Furthermore it satisfies the assumptions of Theorem 2.4. Therefore

$$|||v-w|||_{\varepsilon,\infty} \le \|\mathcal{T}^c v - \mathcal{T}^c w\|_* \quad \text{for all} \quad v, w \in W^{1,\infty} \quad \text{with} \quad v - w \in W_0^{0,\infty}.$$

Similarly we linearize T^c. For arbitrary mesh functions $v, w \in I\!R^{N+1}$ with $v_0 = w_0$ and $v_N = w_N$ set

$$[\tilde{L}^c y]_i = \big[\tilde{L}^c[v,w]y\big]_i := -\varepsilon y_{\bar{x}x;i} - (py)_{x;i} + q_i y_i$$

with p and q as defined above. Again the linearized operator L^c is constructed such that $L^c(v-w) = T^c v - T^c w$ on ω. Then Theorem 2.12 yields

$$|||v-w|||_{\varepsilon,\infty,\omega} \le \|Tv - Tw\|_{*,\omega} \quad \text{for all} \quad v, w \in I\!R^{N+1} \quad \text{with} \quad v - w \in I\!R_0^{N+1}.$$

To conduct an error analysis we take $v = u$ and $w = U$ and proceed as in Section 2.2.2 for a priori error bounds or as in Section 2.2.4 to obtain a posteriori error bounds.

Remark 2.27. *There are also analyses based on the truncation error and barrier function technique of Section 2.2.6 [24, 95] and on the (ℓ_∞, ℓ_1) stability (2.22b); see [61].*

2.2.9 Derivative approximation

In a number of applications the user is more interested in the approximation of the gradient or of the flow than in the solution itself. In Section 2.2.2 the following error bound for the weighted derivative was established.

$$\varepsilon\big\|(u-U)_x\big\|_{\infty,\omega} + \varepsilon\big\|\big(u-U^I\big)'\big\|_\infty \le C\vartheta_1(\omega)$$

Note that $u'(0) \approx \varepsilon^{-1}$ by (2.9). Therefore multiplying by ε in this estimate is the correct weighting.

However, looking at the bounds (2.9) for the derivative of u, we see that the derivative is bounded uniformly away from the layer, where we therefore expect that a similar bound holds without the weighting by ε. More insight is gained using the error expansion of Section 2.2.3.

$$(u-U)_{x;i} = \frac{u_{i+1} - \psi_{i+1} - U_{i+1} - (u_i - \psi_i - U_i)}{h_{i+1}} + \frac{\psi_{i+1} - \psi_i}{h_{i+1}}$$

Then

$$\big|(u-U)_{x;i}\big| \le C\frac{\vartheta_2(\omega)^2}{h_{i+1}} \tag{2.67}$$

by Lemma 2.17 and Theorem 2.18. Furthermore

$$\begin{aligned}\big|u'_{i+1} - u_{x;i}\big| &= \frac{1}{h_{i+1}}\left|\int_{x_i}^{x_{i+1}} (s-x_i)u''(s)ds\right| \\ &\le \frac{C}{h_{i+1}}\int_{x_i}^{x_{i+1}} (s-x_i)\left(1 + \varepsilon^{-2}e^{-\beta s/\varepsilon}\right)ds \le C\frac{\vartheta_2(\omega)^2}{h_{i+1}},\end{aligned}$$

by (2.9) and (2.39). Finally a triangle inequality yields

$$\big|u'_{i+1} - U_{x;i}\big| \le C\frac{\vartheta_2(\omega)^2}{h_{i+1}}. \tag{2.68}$$

Let us illustrate (2.68) by applying it to two standard layer-adapted meshes.

Bakhvalov meshes (see Section 1.2) may be regarded as generated by equidistributing

$$M_{Ba}(\xi) = \max\left\{1, \frac{K\beta}{\varepsilon} \exp\left(-\frac{\beta\xi}{\sigma\varepsilon}\right)\right\} \quad \text{for} \quad \xi \in [0,1].$$

Clearly M_{Ba} is continuous and monotonically decreasing. Therefore

$$\frac{1}{N}\int_0^1 M_{Ba}(s)ds = \int_{x_i}^{x_{i+1}} M_{Ba}(s)ds \le h_{i+1}M_{Ba;i}.$$

Thus

$$\frac{1}{h_{i+1}} \le CNM_{Ba;i} = CN\max\left\{1, \frac{K\beta}{\varepsilon} \exp\left(-\frac{\beta x_i}{\sigma\varepsilon}\right)\right\}.$$

Now, (1.4) and (2.68) yield

$$\left|u'_{i+1} - U_{x;i}\right| \le CN^{-1}\max\left\{1, \frac{K\beta}{\varepsilon} \exp\left(-\frac{\beta x_i}{\sigma\varepsilon}\right)\right\} \quad \text{if} \quad \sigma \ge 2.$$

A very similar result was established by Kopteva and Stynes [39] through a different technique.

Shishkin meshes (see Section 1.3). For these meshes the local step sizes satisfy

$$h_i = \frac{\sigma\varepsilon}{q\beta}\frac{\ln N}{N} \quad \text{for} \quad i = 1,\dots,qN \quad \text{and} \quad h_i \ge N^{-1} \quad \text{for} \quad i = qN+1,\dots,N.$$

Hence

$$\left|u'_{i+1} - U_{x;i}\right| \le \begin{cases} C\varepsilon^{-1}N^{-1}\ln N & \text{for} \quad i = 1,\dots,qN, \\ CN^{-1}\ln^2 N & \text{for} \quad i = qN+1,\dots,N. \end{cases}$$

Outside the layer region this result is slightly suboptimal. Both in [27] and in [39] it was shown by means of barrier function techniques that the approximation is a factor of $\ln N$ better, i.e.,

$$\left|u'_{i+1} - U_{x;i}\right| \le CN^{-1}\ln N \quad \text{for} \quad i = qN+1,\dots,N.$$

2.2.10 Convergence acceleration techniques

Because simple-upwind schemes are only first order convergent there is a need to improve their accuracy. Possible approaches to higher-order schemes include

- the combination of two (or more) solutions on nested meshes by means of the Richardson extrapolation technique
- their combination with higher-order unstabilized schemes using defect correction.

Both approaches have the advantage that linear problems involving only stabilized operators have to be solved.

Already in the early 1980's Hemker [33] proposed the use of defect-correction methods when solving singularly perturbed problems. However the first rigorous proof of uniform convergence of a defect-correction scheme was not published before 2001 [27]. Various analyses by Nikolova and Axelsson [11, 78] are at least not rigorous with regard to the robustness, i. e. the ε-independence of the error constants, while the analysis by Fröhner and Roos [28] turned out to be technically unsound [26].

2.2.10.1 Defect correction

Let us consider the defect correction method from [27] for our model convection-diffusion problem in conservative form

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1. \tag{2.8}$$

It is based on the upwind scheme

$$[L^c U]_i := -\varepsilon U_{\bar{x}x;i} - (bU)_{x;i} + c_i U_i = f_i \tag{2.23}$$

combined with the unstabilized second-order central difference scheme

$$\left[\widehat{L}^c U\right]_i := -\varepsilon U_{\bar{x}\hat{x};i} - (bU)_{\mathring{x};i} + c_i U_i = f_i,$$

where

$$v_{\hat{x};i} := \frac{v_{i+1} - v_i}{\hbar_i}, \quad v_{\mathring{x};i} := \frac{v_{i+1} - v_{i-1}}{2\hbar_i} \quad \text{and} \quad \hbar_i := \frac{h_i + h_{i+1}}{2} \quad \text{for} \quad i = 1, \ldots, N-1.$$

We also set $\hbar_N = \frac{h_N}{2}$ and denote by $x_{i+1/2} := (x_i + x_{i+1})/2$ and $x_{i-1/2} := (x_i + x_{i-1})/2$ the midpoints of the two mesh cells adjacent to x_i.

With this notation we can formulate the defect correction method. This two-stage method is the following:

1. Compute an initial first-order approximation U using simple upwinding:

$$[L^c U]_i = f_i \quad \text{for} \quad i = 1, \ldots, N-1, \quad U_0 = \gamma_0, \quad U_N = \gamma_1. \tag{2.69a}$$

2. Estimate the defect τ in the differential equation by means of the central difference scheme:

$$\tau_i = [\widehat{L}^c U]_i - f_i. \tag{2.69b}$$

3. Compute the defect correction δ by solving

$$[L^c \delta]_i = \frac{\hbar_i}{h_{i+1}} \tau_i \quad \text{for} \quad i = 1, \ldots, N-1, \quad \delta_0 = \delta_N = 0. \tag{2.69c}$$

4. Then the final computed solution is

$$U^{DC} = U - \delta. \tag{2.69d}$$

In the analysis of the method we use the following notation

$$\mathcal{A}^c v := \varepsilon v' + bv + \int_\cdot^1 (cv)(s) ds, \quad \mathcal{F} := \int_\cdot^1 f(s) ds, \tag{2.70a}$$

$$A^c v := \varepsilon v_{\bar{x}} + bv + \sum_{k=\cdot}^{N-1} h_{k+1} (cv)_k, \quad F := \sum_{k=\cdot}^{N-1} h_{k+1} f_k \tag{2.70b}$$

and

$$\widehat{A}^c v := \varepsilon v_{\bar{x}} + \frac{bv + (bv)_-}{2} + \sum_{k=\cdot}^{N} \hbar_k (cv)_k, \quad \widehat{F} := \sum_{k=\cdot}^{N} \hbar_k f_k \tag{2.70c}$$

with $v_{-;i} = v_{i-1}$. The differential equation (2.8) yields

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha = \text{const}, \tag{2.71}$$

while (2.69b) and (2.69c) imply

$$A^c\delta - (\widehat{A}^c U - \widehat{F}) \equiv a = \text{const}, \tag{2.72}$$

The negative norm stability (2.22c) of the operator L^c yields for the error of the defect-correction method

$$\left|\!\left|\!\left| u - U^{DC} \right|\!\right|\!\right|_{\varepsilon,\infty,\omega} \le \min_{c\in I\!R} \left\| A^c(u-(U-\delta)) + c \right\|_{\infty,\omega} \le \min_{c\in I\!R} \left\| A^c u - A^c U + \widehat{A}^c U - \widehat{F} + c \right\|_{\infty,\omega},$$

by (2.72). Thus

$$\left|\!\left|\!\left| u - U^{DC} \right|\!\right|\!\right|_{\varepsilon,\infty,\omega} \le \left\| (A^c - \widehat{A}^c)(u-U) \right\|_{\infty,\omega} + \left\| \widehat{A}^c u - \widehat{F} - \alpha \right\|_{\infty,\omega}, \tag{2.73}$$

where α is the constant from (2.71).

The second term in (2.73) is the truncation error of the central difference scheme. It is formally of second order. The first term is the so called *relative consistency error*. While the error $u-U$ of the simple upwind scheme is only of first order, the hope is that A^c and $\widehat{A}^c$ are sufficiently close to gain second order in this term too.

We consider the relative consistency error first. Let $\eta := u - U$ denote the error of the simple upwind scheme. A straight-forward calculation and summation by parts give

$$\left[(A^c - \widehat{A}^c)\eta\right]_i = \frac{(b\eta)_i - (b\eta)_{i-1}}{2} + \sum_{k=i+1}^{N-1} h_{k+1} \frac{(c\eta)_{k-1} - (c\eta)_k}{2} - \frac{h_i}{2}(c\eta)_i,$$

which can be bounded by

$$\left| \left[(A^c - \widehat{A}^c)\eta\right]_i \right| \le \left(\|b\|_\infty + \frac{\|c\|_\infty}{2} \right) \max_{i=1,\dots,N} |\eta_i - \eta_{i-1}| + h \left(\|b'\|_\infty + \frac{\|c'\|_\infty + \|c\|_\infty}{2} \right) \|\eta\|_{\infty,\omega}.$$

Thus

$$\left\| (A^c - \widehat{A}^c)\eta \right\|_{\infty,\omega} \le C \left(\max_{i=1,\dots,N} |\eta_i - \eta_{i-1}| + h\|\eta\|_{\infty,\omega} \right) \le C\vartheta_2(\omega)^2, \tag{2.74}$$

by (2.67) and because $h \le \vartheta_1(\omega) \le \vartheta_2(\omega)$. The first term, the maximum difference of the error of the upwind scheme in two adjacent mesh points, constituted the main difficulty in [27]. With the error expansion of Section 2.2.3 this has become a simple task.

Next, let us bound the truncation error $\widehat{A}^c u - \widehat{F} - \alpha$ of the central difference scheme. By (2.71) we have $(\widehat{A}^c u - \widehat{F})_i - \alpha = (\widehat{A}^c u - \widehat{F})_i - (\mathcal{A}u - \mathcal{F})_{i-1/2}$. Hence

$$\left| (\widehat{A}^c u - \widehat{F})_i - \alpha \right| \le \varepsilon \left| u_{\bar{x};i} - u'_{i-1/2} \right| + \left| \frac{(bu)_i + (bu)_{i-1}}{2} - (bu)_{i-1/2} \right| + \left| \sum_{k=i}^{N} \hbar_k g_k - \int_{x_{i-1/2}}^1 g(s)ds \right| \tag{2.75}$$

with $g = f - cu$. Using Taylor expansions for u, u' and $(bu)'$ about $x = x_i$, we obtain

$$\varepsilon \left| u_{\bar{x};i} - u'_{i-1/2} \right| \le \frac{3\varepsilon}{2} \int_{x_{i-1}}^{x_i} (s - x_{i-1}) \left| u'''(s) \right| ds \le C\vartheta_2(\omega)^2 \tag{2.76}$$

and

$$\left| \frac{(bu)_i + (bu)_{i-1}}{2} - (bu)_{i-1/2} \right| \le \frac{3}{2} \int_{x_{i-1}}^{x_i} (s - x_{i-1}) \left| (bu)''(s) \right| ds \le C\vartheta_2(\omega)^2,$$

by (2.8), (2.9) and (2.39).

For the last term in (2.75) a Taylor expansion gives

$$\begin{aligned}\left|\frac{h_k}{2}g_k - \frac{h_k^2}{8}g'_{k-1/2} - \int_{x_{k-1/2}}^{x_k} g(s)ds\right| \\ \le \frac{h_k^3}{8}\|g''\|_{\infty,(x_{k-1/2},x_k)} \le Ch_k^3\left(1+\varepsilon^{-2}e^{-\beta x_{k-1/2}/\varepsilon}\right) \le Ch_k\vartheta_2(\omega)^2,\end{aligned} \tag{2.77a}$$

where we have used (2.9) and Proposition 2.19 with $x = x_{k-1/2}$ and $\sigma = 2$. Furthermore, we have

$$\begin{aligned}\left|\frac{h_{k+1}}{2}g_k + \frac{h_{k+1}^2}{8}g'_{k+1/2} - \int_{x_k}^{x_{k+1/2}} g(s)ds\right| \\ \le h_{k+1}\int_{x_k}^{x_{k+1/2}}(\sigma - x_k)|g''(\sigma)|d\sigma \le Ch_{k+1}\vartheta_2(\omega)^2,\end{aligned} \tag{2.77b}$$

by (2.9) and (2.39). Combine these two estimates:

$$\left|\sum_{k=i}^N \hbar_k g_k - \int_{x_{i-1/2}}^1 g(s)ds\right| \le C\left\{\vartheta_2(\omega)^2 + h_i^2\left(1+\varepsilon^{-2}e^{-\beta x_{i-1/2}/\varepsilon}\right)\right\} \le C\vartheta_2(\omega)^2,$$

by Proposition 2.19.

Therefore

$$\left\|\widehat{A}^c u - \widehat{F} - \alpha\right\|_{\infty,\omega} \le C\vartheta_2(\omega)^2,$$

by (2.8), (2.9) and (2.39).

Combine (2.73), (2.74) and the last inequality to get the main result of this section.

Theorem 2.28. *Let u be the solution of (2.8) and U^{DC} that of the defect correction method (2.69). Then*

$$\left|\left|\left|u - U^{DC}\right|\right|\right|_{\varepsilon,\infty,\omega} \le C\vartheta_2(\omega)^2.$$

2.2.10.2 Richardson extrapolation

Richardson extrapolation on layer-adapted meshes was first analysed by Natividad and Stynes [76]. They study a simple first-order upwind scheme on a Shishkin mesh and prove that Richardson extrapolation improves the accuracy to almost second order although the underlying scheme is only of first order. The analysis in [76] is based on comparison principles and barrier function techniques.

Here we shall persue an alternative approach similar to the one in [60] that is based on the $(l_\infty, w_{-1,\infty})$ stability and on the error expansion of Section 2.2.3. Consider the conservative form of our model problem:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1. \tag{2.8}$$

Let $\omega : 0 = x_1 < x_2 < \cdots < x_N = 1$ be an arbitrary mesh and let $\tilde\omega : 0 = x_{1/2} < x_1 < x_{1+1/2} < \cdots < x_N = 1$ be the mesh obtained by uniformly bisecting ω. Let U be the solution of the upwind scheme (2.23) on ω and $\tilde U$ with elements $\tilde U_0, \tilde U_{1/2}, \tilde U_1, \ldots$ that of the difference scheme on $\tilde\omega$. Since (2.23) is a first-order scheme we combine U and $\tilde U$ by

$$U_i^R := 2\tilde U_i - U_i \quad \text{for} \quad i = 0, \ldots, N,$$

in order to get a second-order approximation defined on the coarser mesh ω.

In addition to the notation introduced in (2.70) let us set

$$\left[\tilde{A}^c v\right]_i := 2\varepsilon\frac{v_i - v_{i-1}}{h_i} + b_i v_i + \sum_{k=i}^{N-1} h_{k+1}\frac{c_k v_k + c_{k+1/2}v_{k+1/2}}{2}, \tilde{F}_i := \sum_{k=i} h_{k+1}\frac{f_k + f_{k+1/2}}{2}$$

The differential equation (2.8) and (2.23) yield

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha = \text{const}, \quad A^c U - F \equiv a = \text{const} \quad \text{and} \quad \tilde{A}^c\tilde{U} - \tilde{F} \equiv \tilde{a} = \text{const}\,.$$

A direct calculation gives

$$\begin{aligned} A^c(2\tilde{U} - U - u)_i = &-\varepsilon\left(\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}\right) + \left(b_i(\tilde{U}_i - u_i) - b_{i-1/2}(\tilde{U}_{i-1/2} - u_{i-1/2})\right) \\ &- \left\{\frac{h_i}{2}(c\tilde{U} - cu)_{i-1/2} + \sum_{k=i}^{N-1} h_{k+1}\left[(c\tilde{U} - cu)_{k+1/2} - (c\tilde{U} - cu)_k\right]\right\} \\ &+ \left\{\int_{i-1/2}^{1} g(s)ds - \frac{h_i}{2}g_{i-1/2} - \sum_{k=i}^{N-1} h_{k+1}g_{k+1/2}\right\} \end{aligned}$$

with $g = cu - f$. The first term on the right-hand side is bounded by $C\vartheta_2(\omega)^2$, see (2.76) The second and third term can be bounded by $C\vartheta_2(\tilde{\omega})^2$ using the technique that yielded (2.74). The last term is also bounded by $C\vartheta_2(\omega)^2$ since similar to (2.77) we have

$$\left|\frac{h_k}{2}g_{k-1/2} - \frac{h_k^2}{8}g'_{k-1/2} - \int_{x_{k-1/2}}^{x_k} g(s)ds\right| \le Ch_k\vartheta_2(\omega)^2,$$

and

$$\left|\frac{h_{k+1}}{2}g_{k+1/2} + \frac{h_{k+1}^2}{8}g'_{k+1/2} - \int_{x_k}^{x_{k+1/2}} g(s)ds\right| \le Ch_{k+1}\vartheta_2(\omega)^2.$$

Finally, using the stability inequality (2.22c) we obtain the following convergence result.

Theorem 2.29. *Let U_i^R be the approximate solution to (2.8) obtained by the Richardson extrapolation technique applied to the simple upwind scheme (2.23). Then*

$$\left|\left|\left|u - U^R\right|\right|\right|_{\varepsilon,\infty,\omega} \le C\vartheta_2(\omega)^2.$$

2.2.10.3 A numerical example

The following table gives the results of test computations using both the defect correction method and Richardson extrapolation applied to the test problem (2.28) with $\varepsilon = 10^{-8}$. For our tests we have taken $\sigma = 2$, $\beta = 1$ and $q = K = 1/2$ in the definition of the meshes. The numerical results are clear illustrations of the convergence estimates of Theorems 2.28 and 2.29.

2.3 Second-order upwind schemes

As simple upwinding yields only low accuracy it is natural to look for higher-order alternatives. For one-dimensional problems inverse-monotone schemes exist that are second-order accurate. Because of their stability properties they can be analysed with the techniques similar to those of Section 2.2. Consider the convection-diffusion problem in conservative form

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f \quad \text{in} \quad (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1. \tag{2.8}$$

	defect correction				Richards extrapolation			
	Bakhvalov mesh		Shishkin mesh		Bakhvalov mesh		Shishkin mesh	
N	error	rate	error	rate	error	rate	error	rate
2^4	2.691e-3	1.71	2.925e-3	1.38	1.165e-3	1.82	2.654e-3	1.16
2^5	8.255e-4	1.84	1.127e-3	1.41	3.294e-4	1.91	1.186e-3	1.33
2^6	2.305e-4	1.92	4.250e-4	1.48	8.777e-5	1.95	4.702e-4	1.46
2^7	6.101e-5	1.96	1.522e-4	1.58	2.267e-5	1.98	1.710e-4	1.55
2^8	1.570e-5	1.98	5.082e-5	1.65	5.765e-6	1.99	5.821e-5	1.63
2^9	3.984e-6	1.99	1.623e-5	1.70	1.453e-6	1.99	1.883e-5	1.68
2^{10}	1.003e-6	1.99	4.998e-6	1.74	3.648e-7	2.00	5.872e-6	1.72
2^{11}	2.517e-7	2.00	1.500e-6	1.76	9.140e-8	2.00	1.782e-6	1.75
2^{12}	6.305e-8	2.00	4.415e-7	1.78	2.287e-8	2.00	5.298e-7	1.77
2^{13}	1.578e-8	2.00	1.281e-7	1.79	5.721e-9	2.00	1.558e-7	1.75
2^{14}	3.946e-9	—	3.699e-8	—	1.430e-9	—	4.646e-8	—

Table 2.3: Defect correction and Richardson extrapolation on layer-adapted grids

Let ϱ_i, $i = 1, \dots, N$ be arbitrary with $\varrho_i \in [1/2, 1]$. Define the weighted step sizes

$$\chi_i = \varrho_{i+1} h_{i+1} + (1 - \varrho_i) h_i \quad \text{for} \quad i = 1, \dots, N-1, \quad \chi_0 = \chi_N = 0.$$

Then following Andreev and Kopteva [4], our discretisation is: Find $U \in I\!R^{N+1}$ with

$$[L^\varrho U]_i := -\varepsilon U_{\bar{x}\breve{x};i} - (\varrho bU + (1-\varrho)(bU)_-)_{\breve{x};i} + (cU)_{\varrho;i} = f_{\varrho;i} \quad \text{for} \quad i = 1, \dots, N-1 \tag{2.78}$$

with boundary conditions $U_0 = \gamma_0$ and $U_N = \gamma_1$. Here

$$v_{\breve{x};i} = \frac{v_{i+1} - v_i}{\chi_i}, \quad v_{-;i} = v_{i-1} \quad \text{and} \quad v_{\varrho;i} = \frac{\varrho_{i+1} v_{i+1} + (1 - \varrho_{i+1} + \varrho_i) v_i + (1 - \varrho_i) v_{i-1}}{2}.$$

The approximation of the first-order derivative is a weighted combination of upwinded and downwinded operators. At a first glance the approximation of the lowest-order term and of the right-hand side seems to be very non-standard. It is chosen such that

$$\chi_i (cu - f)_{\varrho;i} \quad \text{is a second-order approximation of} \quad \int_{x_{\varrho;i-1/2}}^{x_{\varrho;i+1/2}} (cu - f)(x) dx$$

with $x_{\varrho;i-1/2} = x_{i-1} + \varrho_i h_i$. For $\varrho \equiv 1/2$ we obtain a central difference scheme, while for $\varrho \equiv 1$ the mid-point upwind scheme is recovered.

2.3.1 Stability of the discrete operator

The stability analysis of the operator L^ϱ is complicated by the positive contribution of the discretization $(cU)_{\varrho;i}$ of the lowest order term to the offdiagonal entries of the system matrix. It is difficult to ensure the correct sign pattern for the application of the M-matrix criterion (Lemma 2.9). Instead we follow [52] which adapts the technique from [4].

Set

$$[A^\varrho v]_i := \varepsilon v_{\bar{x};i} + \varrho (bv)_i + (1 - \varrho_i)(bv)_{i-1} - \sum_{j=1}^{i-1} \chi_j (cv)_{\varrho;j} \quad \text{for} \quad i = 1, \dots, N.$$

This operator is related to L^ϱ by $(A^\varrho v)_{\breve{x}} = -L^\varrho v$. Then any function $v \in I\!R_0^{n+1}$ can be represented as

$$v_i = \frac{W_N}{V_N} V_i - W_i$$

where V and W are the solution of the difference equations

$$[A^\varrho V]_i = 1, \quad \text{for} \quad i = 1, 2, \dots, N, \ V_0 = 0$$

and

$$[A^\varrho W]_i = [A^\varrho v]_i + c \quad \text{for} \quad i = 1, 2, \dots, N, \ W_0 = 0$$

for any constant $c \in I\!R$.

Lemma 2.30. *Assume that*

$$1 \geq \varrho_i \geq \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon}{b_{i-1}h_i}\right\} \quad \textit{for} \quad i = 1, \dots, N \tag{2.79a}$$

and

$$\|c\|_\infty h \leq \beta/4. \tag{2.79b}$$

Then the matrix associated with A^ϱ is an M-matrix.

Proof. First (2.79a) ensures that the offdiagonal entries of A^ϱ are nonpositive, while (2.79b) implies that the diagonal entries are positive.

For any monotonically increasing mesh function $z_i \geq 0$ we have

$$[A^\varrho z]_i > \varrho_i b_i z_i - \frac{\|c\|_\infty}{2} \sum_{j=1}^{i-1} \chi_j \left(z_{j+1} + z_j\right) \geq \frac{\beta}{4} z_i - \|c\|_\infty \sum_{j=1}^{i-2} \chi_j z_{j+1},$$

by (2.79).

Now let

$$z_0 = z_1 = z_2 = 1, \quad \text{and} \quad z_i = \prod_{k=3}^{i} \left(1 + \frac{4\|c\|_\infty}{\beta} \chi_{k-2}\right) \quad \text{for} \quad i = 3, \dots, N. \tag{2.80}$$

Clearly $z_i \leq e^{4\|c\|_\infty/\beta}$ and

$$\frac{\beta}{4} z_i - \|c\|_\infty \chi_{i-2} z_{i-1}, \geq \frac{\beta}{4} z_{i-1}, \quad \text{by (2.79b)}.$$

Then induction for i yields

$$[A^\varrho z]_i > \frac{\beta}{4} \quad \text{for} \quad i = 1, \dots, N.$$

Finally application of the M-matrix criterion (Lemma 2.9) with the test function $v_i = z_i$ completes the proof. ∎

The M-matrix property of A^ϱ and the function z from (2.80) can now be used to establish bounds on V and W:

$$0 < V_i \leq \frac{4}{\beta} z_i \leq \frac{4}{\beta} e^{4\|c\|/\beta} \quad \text{and} \quad |W_i| \leq V_i \left\|A^\varrho v + c\right\|_{\infty,\omega} \quad \text{for} \quad i = 1, \dots, N.$$

We get our final stability result.

Theorem 2.31. *Let ϱ and h satisfy (2.79). Then*

$$\|v\|_{\infty,\omega} \leq \frac{8}{\beta} e^{4\|c\|_\infty/\beta} \min_{c \in I\!R} \|A^\varrho v + c\|_{\infty,\omega} \quad \textit{for all} \quad v \in I\!R_0^{N+1}.$$

Remark 2.32. *The (l_∞, l_1) stability*

$$\|v\|_{\infty,\omega} \le C \sum_{k=1}^{N-1} \chi_k \big|[L^\varrho v]_k\big|$$

is an immediate consequence of the negative-norm stability.

Analyses of second-order upwind schemes based on (l_∞, l_1)-stability properties were given by Andreev and Savin [5] for a modification of Samarskii's scheme on a Shishkin mesh [5] and on Bakhvalov meshes [2] and by Linß [54] for quasilinear problems and S-type meshes. However, here we shall follow [4, 52] and base our subsequent analysis on the stronger $(l_\infty, w_{-1,\infty})$ stability.

2.3.2 A priori error analysis

We now study the approximation error of the scheme (2.78) with

$$\varrho_i = \begin{cases} 1/2 & \text{if } h_i \le 2\varepsilon/b_{i-1}, \\ 1 & \text{otherwise.} \end{cases} \tag{2.81}$$

This choice satisfies the assumptions of Theorem 2.31. Therefore,

$$\|u - U\|_{\infty,\omega} \le C \min_{c\in I\!R} \|A^\varrho(u-U) + c\|_{\infty,\omega}\,. \tag{2.82}$$

Set

$$\mathcal{A}^c v := \varepsilon v' + bv - \int_{x_{\varrho;1/2}}^{\cdot} (cv)(s)ds, \quad \mathcal{F} := -\int_{x_{\varrho;1/2}}^{\cdot} f(s)ds \quad \text{and} \quad F_i^\varrho := -\sum_{k=1}^{i-1} \chi_k f_{\varrho;k}$$

Inspecting (2.8) and (2.78), we see that

$$\mathcal{A}^c u - \mathcal{F} \equiv \alpha \quad \text{on} \quad (0,1) \quad \text{and} \quad A^\varrho U - F^\varrho \equiv a \quad \text{on} \quad \omega$$

with constants α and a because $\mathcal{L}^c v = -(\mathcal{A}^c v)'$ and $f = -\mathcal{F}'$ on $(0,1)$, and $L^\varrho v = (A^\varrho v)_{\check{x}}$ and $f = F^\varrho_{\check{x}}$ on ω. Take $c = a - \alpha$ in (2.82) in order to get

$$\|u - U\|_{\infty,\omega} \le C \max_{i=1,\dots,N} \big|[A^\varrho u]_i - (\mathcal{A}^c u)(x_{\varrho;i}) + \mathcal{F}(x_{\varrho;i}) - F_i^\varrho\big|. \tag{2.83}$$

Set

$$[B^\varrho u]_i := \varepsilon u_{\bar{x};i} + \varrho_i b_i u_i + (1-\varrho_i) b_{i-1} u_{i-1}, \quad \mathcal{B}(x) := \varepsilon u'(x) + \big(bu\big)(x) \quad \text{and} \quad g := cu - f.$$

Then

$$\begin{aligned} &[A^\varrho u]_i - (\mathcal{A}^c u)(x_{\varrho;i}) + \mathcal{F}(x_{\varrho;i}) - F_i^\varrho \\ &\quad = [B^\varrho u]_i - (\mathcal{B}^c u)(x_{\varrho;i}) + \int_{x_{\varrho;1/2}}^{x_{\varrho;i-1/2}} g(s)ds - \sum_{j=1}^{i-1} \chi_j g_{\varrho;j}. \end{aligned} \tag{2.84}$$

When bounding the first term on the right-hand side of (2.84) we have to distinguish two cases: $\sigma_i = 1$ and $\sigma_i = 1/2$.

For $\sigma_i = 1$ we have

$$\big[B^\varrho u\big]_i - \big(\mathcal{B}u\big)(x_{\varrho;i}) = \varepsilon \left\{ \frac{u_i - u_{i-1}}{h_i} - u_i' \right\} = \frac{\varepsilon}{h_i} \int_{x_{i-1}}^{x_i} u''(t)(t - x_{i-1})dt.$$

Thus

$$\big|\big[B^h u\big]_i - \big(Bu\big)(x_{\varrho,i})\big| \le C \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-2} e^{-\beta t/\varepsilon}\right)(t - x_{i-1})dt, \tag{2.85}$$

by (2.9) and because $\varepsilon/h_i < \|b\|_\infty/2$ for $\varrho_i = 1$.

Next, let $\sigma_i = 1/2$. Then

$$\left[B^\varrho u\right]_i - \left(\mathcal{B}u\right)(x_{\varrho,i}) = \varepsilon\left\{\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}\right\} + \frac{b_i u_i + b_{i-1}u_{i-1}}{2} - b_{i-1/2}u_{i-1/2},$$

where $x_{i-1/2} = (x_i + x_{i-1})/2$ and $u_{i-1/2} = u(x_{i-1/2})$. Taylor expansions for u and u' about x_i give

$$\varepsilon\left|\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}\right| \le \frac{3\varepsilon}{2}\int_{x_{i-1}}^{x_i}\left|u'''(t)\right|(t - x_{i-1})t$$

and

$$\left|\frac{b_i u_i + b_{i-1}u_{i-1}}{2} - b_{i-1/2}u_{i-1/2}\right| \le \frac{3}{2}\int_{x_{i-1}}^{x_i}\left|(bu)''(t)\right|(t - x_{i-1})dt.$$

From this and (2.9) we see that (2.85) holds for $\sigma_i = 1/2$ too.

Finally we bound the second term of the right-hand side of (2.84):

$$\int_{x_{\varrho,j-1/2}}^{x_{\varrho,j+1/2}} g(s)ds - \chi_{\varrho,j}g_{\varrho,j} = \int_{x_{\varrho,j-1/2}}^{x_{\varrho,j+1/2}}\left(g(s)ds - g_{\varrho,j}\right)ds$$

The representation

$$g(s) = g_{j+1} - g'_{j+1}(x_{j+1} - s) + \int_s^{x_{j+1}} g''(t)(t-s)dt$$

yields

$$\left|g_{\varrho,j} - g(s)ds - (x_{\varrho,j} - s)g'_{j+1}\right| \le 2\int_{x_{j-1}}^{x_{j+1}}\left|g''(t)\right|(t - x_{j-1})dt.$$

Then

$$\begin{aligned}\left|\int_{x_{\varrho,j-1/2}}^{x_{\varrho,j+1/2}} g(s)ds - \chi_{\varrho,j}g_{\varrho,j}\right| &\le 2(h_j + h_{j+1})\int_{x_{j-1}}^{x_{j+1}}\left|g''(t)\right|(t - x_{j-1})dt \\ &\le C(h_j + h_{j+1})\int_{x_{j-1}}^{x_{j+1}}\left(1 + \varepsilon^{-2}e^{-\beta t/\varepsilon}\right)(t - x_{j-1})dt\end{aligned}$$

by (2.9) and because $g = cu - f$.

Combining this estimate with (2.83), (2.82) and (2.85), we get

$$\|u - U\|_{\infty,\omega} \le C\max_{i=1,\ldots,N-1}\int_{x_{i-1}}^{x_{i+1}}\left(1 + \varepsilon^{-2}e^{-\beta t/\varepsilon}\right)(t - x_{i-1})dt.$$

Finally, using (2.39) we obtain the following convergence result.

Theorem 2.33. *Let U be the approximate solution to (2.8) obtained by the difference scheme (2.78) with ϱ chosen according to (2.81). Assume $\|c\|_\infty h \le \beta/4$. Then*

$$\|u - U\|_{\infty,\omega} \le C\vartheta_2(\omega)^2.$$

Quasilinear problems. The conclusion of the Theorem also holds when (2.78) is adapted to discretize the quasilinear problem

$$\mathcal{T}^c u := -\varepsilon u'' - b(x,u)' + c(x,u) = 0 \ \text{ in } \ (0,1), \quad u(0)=\gamma_0, \quad u(1)=\gamma_1 \tag{2.66}$$

with $0<\varepsilon\ll 1$, $b_u \ge \beta > 0$ and $c_u \ge 0$. The scheme reads: Find $U \in I\!R^{n+1}$ such that $U_0 = \gamma_0$, $U_N = \gamma_1$ and

$$\left[T^\varrho U\right]_i := -\varepsilon U_{\bar{x}\breve{x};i} - \big(\varrho b(\cdot,U) + (1-\varrho) b(\cdot,U)_-\big)_{\breve{x};i} + c(x_{\varrho;i}, U_{\varrho;i}) = 0 \ \text{ for } \ i=1,\dots,N-1$$

with the stabilization parameter chosen to satisfy, e. g,

$$\varrho_i = \begin{cases} 1/2 & \text{if } h_i \le 2\varepsilon/\|b\|_\infty, \\ 1 & \text{otherwise.} \end{cases}$$

Discontinuous coefficients and point sources. Consider the convection-diffusion problem (2.63) with a point source:

$$\mathcal{L}^c u := -\varepsilon u'' - (bu)' + cu = f + \alpha\delta_d, \ \text{ in } (0,1), \quad u(0)=\gamma_0, \quad u(1)=\gamma_1,$$

with the shifted Dirac-delta function $\delta_d(x) = \delta(x-d)$. The coefficient b may also have a discontinuity at $x=d$. Assume that $b \ge \beta_1 > 0$ on $(0,d)$ and $b \ge \beta_2 > 0$ on $(d,1)$ and set $\beta = \min\{\beta_1,\beta_2\}$.

Using (2.78) we seek an approximation $U \in I\!R^{n+1}$ with

$$\left[L^\varrho U\right]_i = f_{\varrho;i} + \delta_{d,\varrho;i} \ \text{ for } \ i=1,\dots,N-1,$$

with

$$\Delta_{d,\varrho;i} := \begin{cases} \chi_i^{-1} & \text{if } d \in [x_{\varrho;i-1/2}, x_{\varrho;i+1/2}), \\ 0 & \text{otherwise} \end{cases}$$

Then the above technique and the *a priori* bounds (2.64) for the derivatives of u yield the error estimate

$$\|u-U\|_{\infty,\omega} \le C\left\{\max_k \int_{x_k}^{x_{k+1}} \left[1 + \frac{1}{\varepsilon}\left\{\exp\left(-\frac{\beta_1 x}{2\varepsilon}\right) + H_d(x)\exp\left(-\frac{\beta_2(x-d)}{2\varepsilon}\right)\right\}\right] dx\right\}^2;$$

see [55].

A posteriori error estimates in the maximum norm for (2.8) discretized by (2.78) can be derived using the $(L_\infty, W^{-1,\infty})$-stability (2.7c) of the continuous operator $\mathcal{L}^c$. However compared to Section 2.2.4 the analysis becomes more technical. Therefore we refer the reader to the original article by Kopteva [41].

2.3.3 The barrier function technique

Stynes and Roos [97] study a hybrid difference scheme on a Shishkin mesh (with $q=1/2$ and $\sigma > 4$). Their scheme uses central differencing on the fine part of the mesh and the mid-point upwind scheme on the coarse part.

Let us consider the convection-diffusion problem

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \ \text{ in } \ (0,1), \quad u(0)=\gamma_0, \quad u(1)=\gamma_1, \tag{2.1}$$

with $b \geq \beta > 0$ and $c \geq 0$ on $[0,1]$. This is discretized on a Shishkin mesh—see Section 1.3—using the difference scheme

$$[LU]_i = \tilde{f}_i \quad \text{for} \quad i = 1, \dots, N-1, \ U_0 = \gamma_0, \ U_N = \gamma_1 \tag{2.86}$$

with

$$[Lv]_i := \begin{cases} -\varepsilon v_{\bar{x}\hat{x};i} - b_i v_{\tilde{x};i} + c_i v_i & \text{if } b_i h_i \leq 2\varepsilon, \\ -\varepsilon v_{\bar{x}\hat{x};i} - b_{i+1/2} v_{x;i} + (c_i v_i + c_{i+1} v_{i+1})/2 & \text{otherwise,} \end{cases}$$

$v_{\tilde{x};i} := (v_{i+1} - v_{i-1})/(2\hbar_i)$ and

$$\tilde{f}_i := \begin{cases} f_i & \text{if } b_i h_i \leq 2\varepsilon, \\ f_{i+1/2} & \text{otherwise.} \end{cases}$$

Clearly for N larger than a certain threshold value N_0 the matrix associated with L is an M-matrix and central differencing is used exclusively on the fine part of the mesh.

The next estimates are used later to bound the truncation error. When $2\varepsilon < b_i h_i$ we have the bound

$$\left|[Lg]_i - (\mathcal{L}g)_{i+1/2}\right| \leq C\left\{\varepsilon \int_{x_{i-1}}^{x_{i+1}} |g'''(s)|\, ds + h_{i+1} \int_{x_i}^{x_{i+1}} \left[|g'''(s)| + |g''(s)|\right] ds\right\} \tag{2.87a}$$

otherwise we use

$$\left|[Lg]_i - (\mathcal{L}g)_i\right| \leq C\left\{\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon|g'''(s)| + |g''(s)|\right] ds\right\} \tag{2.87b}$$

and

$$\left|[Lg]_i - (\mathcal{L}g)_i\right| \leq C h_i\left\{\int_{x_{i-1}}^{x_{i+1}} \left[\varepsilon|g^{(4)}(s)| + |g'''(s)|\right] ds\right\} \quad \text{if } h_i = h_{i+1} \tag{2.87c}$$

For the analysis we split the numerical solution U analogously to the splitting $u = v + w$ of Theorem 2.6 and Remark 2.8: $U = V + W$ with

$$[LV]_i = \tilde{f}_i \quad \text{for} \quad i = 1, \dots, N-1, \quad V_0 = v(0), \ V_N = v(1)$$

and

$$[LW]_i = 0 \quad \text{for} \quad i = 1, \dots, N-1, \quad W_0 = w(0), \ W_N = w(1).$$

Then the error is $u - U = (v - V) + (w - W)$ and we can estimate the error in v and w separately.

Regular solution component. Theorem 2.6, Remark 2.8 and (2.87) give

$$\left|[L(v - V)]_i\right| = \left|[Lv]_i - \tilde{f}_i\right| \leq \begin{cases} CN^{-1} & \text{for} \quad i = qN, \\ CN^{-2} & \text{otherwise.} \end{cases}$$

Furthermore $(v - V)_0 = (v - V)_N = 0$. Now set

$$\varphi_i = \begin{cases} 1 & \text{for} \quad i = 0, \dots, qN, \\ \displaystyle\prod_{k=qN+1}^{i} \left(1 + \frac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for} \quad i = qN+1, \dots, N. \end{cases}$$

Clearly $\varphi_0 \geq 0$ and $\varphi_N \geq 0$. Furthermore

$$[L\varphi]_i \geq \begin{cases} 0 & \text{for } \ i \neq qN, \\ \dfrac{\beta}{2h_{qN+1}} \geq \dfrac{\beta(1-q)}{2}N & \text{for } \ i = qN. \end{cases}$$

Then application of comparison principle with the barrier function $CN^{-2}(1-x+\varphi)$ yields

$$\|v - V\|_{\infty,\omega} \leq CN^{-2} \tag{2.88}$$

since the matrix associated with L is inverse monotone as mentioned before.

Layer component. Let

$$\psi_i := \begin{cases} \displaystyle\prod_{k=1}^{i}\left(1+\frac{\beta h_k}{\varepsilon}\right)^{-1} + \prod_{k=1}^{qN}\left(1+\frac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for } \ i = 1,\dots,qN, \\ \displaystyle 2\prod_{k=1}^{i}\left(1+\frac{\beta h_k}{\varepsilon}\right)^{-1} & \text{for } \ i = qN,\dots,N. \end{cases}$$

The inverse monotonicity of the discrete operator L yields

$$|W_i| \leq |v(0) - \gamma_0|\,\psi_i \quad \text{for} \quad i = 0,\dots,N$$

because $L\psi \geq 0$. Furthermore $|w_i| \leq Ce^{-\beta x_i/\varepsilon} \leq C\psi_i$. Thus

$$|w_i - W_i| \leq C\psi_i \ \ \text{for} \ \ i = 0,\dots,N.$$

Now the argument that lead to (2.62) is used to establish

$$|w_i - W_i| \leq CN^{-2} \ \ \text{for} \ \ i = qN,\dots,N. \tag{2.89}$$

if $\sigma \geq 2$ in the construction of the Shishkin mesh (Section 1.3).

For $i = 1,\dots,qN-1$ the truncation error with respect to w satisfies

$$\left|[L(w-W)]_i\right| \leq CN^{-2}\ln^2 N\varepsilon^{-1}e^{-\beta x_{i-1}/\varepsilon} \leq CN^{-2}\ln^2 N\varepsilon^{-1}\tilde{\psi}_i,$$

by (2.87c), Theorem 2.6 and Remark 2.8 with

$$\tilde{\psi}_i := \prod_{k=1}^{i}\left(1+\frac{\beta h_k}{2\varepsilon}\right)^{-1}.$$

Then the inverse monotonicity of L gives

$$|(w-W)_i| \leq CN^{-2}\ln^2 N\tilde{\psi}_i \quad \text{for} \ \ i = 1,\dots,qN-1, \tag{2.90}$$

because $\left[L\tilde{\psi}\right]_i \geq C\varepsilon^{-1}\tilde{\psi}_i$ for $i = 1,\dots,qN-1$ and $|(w-W)_0|, |(w-W)_{qN}| \leq CN^{-2}$.

Finally, combine (2.88)–(2.90) to get our final convergence result.

Theorem 2.34. *Let ω be a Shishkin mesh with $\sigma \geq 2$; see Section 1.3. Then the error of the upwinded scheme (2.86) applied to (2.1) satisfies*

$$|u_i - U_i| \begin{cases} CN^{-2}\ln^2 N & \textit{for } i = 0,\dots,qN-1, \\ CN^{-2} & \textit{for } i = qN,\dots,N \end{cases}$$

if N is larger than a certain threshold value.

A similar scheme generated by a streamline-diffusion stabilisation was analysed by Stynes and Tobiska [98] with special emphasis on the choice of the mesh parameter σ. There it was first established that σ should be chosen equal to the formal order of the scheme. Clavero et al. [19] study second- and third-order compact schemes generated by the HODIE technique on Shishkin meshes.

2.4 Central differencing

In numerical experiments [23, 32, 73] it was observed that central differencing on Shishkin meshes yields almost second-order accuracy. A first analysis was conducted by Andreev and Kopteva [3] who prove that central differencing on a Shishkin mesh is (l_∞, l_1) stable. This result was later generalized by Kopteva [40]. Consider the discretisation

$$[LU]_i := -\varepsilon U_{\bar{x}\hat{x};i} - (bU)_{\mathring{x};i} + c_i U_i = f_i \quad \text{for} \quad i = 1, \dots, N-1, \quad U_0 = U_N = 0 \tag{2.91}$$

of (2.8). The central difference operator L is (l_∞, l_1) stable with

$$\|v\|_{\infty,\omega} \le \frac{81}{4\beta} \sum_{i=1}^{N-1} \hbar_i \left|[Lv]_i\right|. \tag{2.92}$$

if

$$\left| \prod_{i=1}^{N} \left(\frac{\varepsilon}{h_i b_{i-1}} - \frac{1}{2} \right) \Big/ \left(\frac{\varepsilon}{h_i b_i} + \frac{1}{2} \right) \right| \le \frac{1}{4}$$

and $h_i \le \mu h_j$ for $i \le j$ with some constant μ [40]. Kopteva also proves an $(l_\infty, w_{-1,\infty})$-stability result for L applied to special mesh functions: Let m be such that $h_i \le 2\varepsilon/b_{i-1}$ for $i = 1, \dots, m$ and $h_{m+1} > 2\varepsilon/b_m$. Suppose v satisfies $[Lv]_i = 0$ for $i > m$. Then

$$\|v\|_{\infty,\omega} \le \frac{11}{2\beta} \max_{j=1,\dots,N-1} \left| \sum_{k=j}^{N-1} \hbar_k \, [Lv]_k \right|.$$

Based on these two stability inequalities she proves the convergence result

$$\|u - U\|_{\infty,\omega} \le \begin{cases} CN^{-2} & \text{for Bakhvalov meshes with } \sigma > 2, \\ CN^{-2} \ln^2 N & \text{for Shishkin meshes with } \sigma > 2. \end{cases}$$

The (l_∞, l_1) stability (2.92) can be used [86] to prove

$$\|u - U\|_{\infty,\omega} \le C \left(h + \max|\psi'| \, N^{-1} \right)^2 \tag{2.93}$$

on S-type meshes with $\sigma \ge 3$. A similar result was proved by Kopteva and Linss [42] for certain quasilinear problems of type (2.66).

Another approach to study central differencing on Shishkin meshes is that of Lenferink [45, 46]. He eliminates every other unknown to get a scheme whose system matrix is an M-matrix. The same technique is used in [46] to study a fourth-order scheme generated by a Galerkin finite element method using quadratic test and trial functions. For this scheme on a Shishkin mesh pointwise convergence in the mesh nodes of order $N^{-4} \ln^4 N$ is estabilished.

A drawback of central difference approximations and other unstabilized methods is their lack of stability. The use of layer-adapted meshes induce some additional stability, however the discrete systems are difficult to solve efficiently by means of iterative solvers. The system matrices have eigenvalues with large imaginary parts. This becomes a particularly important issue when solving higher-dimensional problems.

2.4.1 Derivatives

For the central-difference scheme (2.91) on S-type meshes with $\sigma \ge 3$ we have the second order bound

$$\varepsilon \left| U_{\bar{x};i} - u'_{i-1/2} \right| \le C\big(h + N^{-1} \max|\psi'|\big)^2.$$

The proof in [86] uses the bound (2.93) for the discretisation error, then interprets the scheme as a finite element method with inexact integration and finally applies a finite element technique [111] to get the bound for the derivative approximation.

Chapter 3

Finite element and finite volume methods

We now consider finite element discretisations of

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{3.1}$$

with $b \geq \beta > 0$ and

$$c + b'/2 \geq \gamma > 0. \tag{3.2}$$

The latter condition ensures that the bilinear form in the variational formulation of (3.1) is coercive. If $b \geq \beta > 0$ (3.2) can be ensured by a transformation $\bar{u}(x) = u(x)e^{\delta x}$ with δ chosen appropriately. We assume this transformation has been carried out.

We start our investigations with interpolation-error estimates and a Galerkin discretizations of (3.1)—including aspects of convergence, superconvergence, and postprocessing of the derivatives. Then stabilized finite element methods are considered. We finish with an upwinded finite volume method.

3.1 The interpolation error

In this section we study the error in linear interpolation. The argument follows [51]. Let ω be an arbitrary mesh. Let u^I denote the piecewise-linear function that interpolates to u at the nodes of ω. Using a Taylor expansion at x_i, we can write the interpolation error for $x \in [x_{i-1}, x_i]$ as

$$\left(u^I - u\right)(x) = \frac{x_i - x}{h_i} \int_{x_i}^{x_{i-1}} u''(\xi)\,(x_{i-1} - \xi)\; d\xi - \int_{x_i}^{x} u''(\xi)\,(x - \xi)\; d\xi.$$

Thus

$$\left|\left(u^I - u\right)(x)\right| \leq 2 \int_{x_{i-1}}^{x_i} |u''(\xi)|\,(\xi - x_{i-1})\; d\xi. \tag{3.3}$$

To bound the right-hand side we apply (2.39) and (2.9):

$$\left|\left(u^I - u\right)(x)\right| \leq C \left\{ \int_{x_{i-1}}^{x_i} \left(1 + \varepsilon^{-1} e^{-\beta\xi/2\varepsilon}\right) d\xi \right\}^2 \quad \text{for } \; x \in [x_{i-1}, x_i]. \tag{3.4}$$

Another sensible norm for measuring the uniform accuracy of numerical methods for (2.1) is the ε-weighted H^1-norm $|||\cdot|||_\varepsilon$ defined by

$$|||v|||_\varepsilon := \left\{ \varepsilon \left\| v' \right\|_0^2 + \left\| v \right\|_0^2 \right\}^{1/2}, \quad \|v\|_{0,D} := \left\{ \int_D v(x)^2 dx \right\}^{1/2}, \quad \|v\|_0 := \|v\|_{0,(0,1)}.$$

Bounds for the L_2-norm of the interpolation error $\left\|u-u^I\right\|_0$ are easily obtained from the L_∞-estimate (3.4). For the error in the H^1 seminorm, integration by parts yields

$$\left|u^I-u\right|_1^2 := \left\|\left(u^I-u\right)'\right\|_0^2 = -\int_0^1 \left(u^I-u\right) u''\,dx \le C\varepsilon^{-1}\left\|u^I-u\right\|_{L_\infty},$$

by (2.9).

Applying these results to S-type meshes with (1.11) we get

$$\left|(u^I-u)(x)\right| \le \begin{cases} C\left(h+N^{-1}\max|\psi'|\right)^2 & \text{for } x\in[0,\lambda), \\ C\left(N^{-2}+N^{-\sigma}\right) & \text{for } x\in[\lambda,1], \end{cases}$$

while for Bakhvalov meshes we have

$$\left\|u^I-u\right\|_{L_\infty} \le C\left(N^{-2}+N^{-\sigma}\right).$$

Thus $\sigma\ge 2$ is the correct choice when selecting the appropriate mesh.

3.2 Linear Galerkin FEM

We start our investigation with the weak formulation of (3.1): Find $u\in H_0^1(0,1)$ such that

$$a(u,v)=f(v) \quad \text{for all} \quad v\in H_0^1(0,1),$$

where

$$a(u,v)=\varepsilon(u',v')-(bu',v)+(cu,v) \quad \text{and} \quad f(v)=(f,v) \quad \text{with} \quad (u,v):=\int_0^1 u(x)v(x)dx.$$

Our assumption (3.2) ensures that the bilinear form $a(\cdot,\cdot)$ is coercive with respect to the energy norm:

$$a(v,v)\ge \min\{1,\gamma\}\,|||v|||_\varepsilon^2 \quad \text{for all} \quad v\in H_0^1(0,1).$$

Therefore the variational formulation posseses a unique solution $u\in H_0^1(0,1)$.

Let ω be an arbitrary mesh and let V^ω denote the space of continuous, piecewise-linear functions on ω that vanish for $x=0$ and $x=1$. Then our discretisation is: Find $U\in V^\omega$ such that

$$a(U,v)=f(v) \quad \text{for all} \quad v\in V^\omega.$$

Again the coercivity of $a(\cdot,\cdot)$ guarantees the existence of a unique solution $U\in V^\omega$.

3.2.1 Convergence

Based on the interpolation error bounds of Section 3.1 we can conduct an error analysis for the Galerkin FEM on S-type meshes. It was first derived by Stynes and O'Riordan [96] for standard Shishkin meshes and later generalized to S-type meshes by Linß and Roos [49, 85]. The technique can be generalized to discretisations of two-dimensional problems using triangular or rectangular elements on tensor-product S-type meshes; see Section 5.3.2.1.

Theorem 3.1. *Let ω be an S-type mesh with $\sigma\ge 2$ whose mesh generating function $\tilde\varphi$ satisfies (1.11) and*

$$\max|\psi'|\ln^{1/2}N \le CN. \tag{3.5}$$

Then

$$|||u-U|||_\varepsilon \le C\left(h+N^{-1}\max|\psi'|\right)$$

for the error of the Galerkin FEM.

Remark 3.2. *The additional assumption (3.5) does not constitute a major restriction. For example both the standard Shishkin mesh and the Bakhvalov-Shishkin mesh satisfy this condition.*

Proof of Theorem 3.1. Let $\eta = u^I - u$ and $\chi = u^I - U$. For η we have from Section 3.1

$$|||\eta|||_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right). \tag{3.6}$$

To bound χ we start from the coercivity of $a(\cdot,\cdot)$ and the orthogonality of the Galerkin method:

$$\begin{aligned}\min\{1,\gamma\}\,|||\chi|||_\varepsilon^2 &\le a(\chi,\chi) = a(\eta,\chi) = \varepsilon(\eta',\chi') + (b\eta,\chi') + ((c+b')\eta,\chi)\\ &\le C\,|||\eta|||_\varepsilon\,|||\chi|||_\varepsilon + C\left(\|\eta\|_{L_\infty(0,\lambda)}\|\chi'\|_{L_1(0,\lambda)} + \|\eta\|_{L_\infty(\lambda,1)}\|\chi'\|_{L_1(\lambda,1)}\right)\end{aligned}$$

On $(0,\lambda)$ we use

$$\|\chi'\|_{L_1(0,\lambda)} \le C\sqrt{\lambda}\,\|\chi'\|_{0,(0,\lambda)} \le C\ln^{1/2}N\,|||\chi|||_\varepsilon\,,$$

while on $(\lambda,1)$ we use an inverse inequality to estimate

$$\|\chi'\|_{L_1(\lambda,1)} \le CN\,\|\chi\|_{L_1(\lambda,1)} \le CN\,|||\chi|||_\varepsilon\,,$$

These two bounds and the interpolation results of Section 3.1 yield

$$\min\{1,\gamma\}\,|||\chi|||_\varepsilon \le C\left\{h + N^{-1}\max|\psi'| + \left(h + N^{-1}\max|\psi'|\right)^2\ln^{1/2}N + N^{-1}\right\}.$$

Thus

$$|||\chi|||_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right),$$

where we have used (3.5). Applying a triangle inequality and the bounds for $|||\eta|||_\varepsilon$ and $|||\chi|||_\varepsilon$, we complete the proof. □

3.2.2 Superconvergence

The phenomenon that the convergence rate in a discrete (semi-) norm say $\|\cdot\|_{*,\omega}$ exceeds the rate of convergence in its continuous counterpart $\|\cdot\|_*$ is called superconvergence. In the preceding section we have seen that the Galerkin FEM is (almost) first-order convergent in the ε-weighted energy norm. Now we prove that $\left|\left|\left|u^I - U\right|\right|\right|_\varepsilon$ converges faster than $|||u-U|||_\varepsilon$—a superconvergence property. Our analysis follows [50, 115] where two-dimensional problems are studied.

Theorem 3.3. *Let ω be an S-type mesh with $\sigma \ge 5/2$ whose mesh generating function $\tilde\varphi$ satisfies (1.11). Then*

$$\left|\left|\left|u^I - U\right|\right|\right|_\varepsilon \le C\left(h^2\ln^{1/2}N + N^{-2}\max|\psi'|^2\right) \tag{3.7}$$

for the solution of the Galerkin FEM.

Proof. For the sake of simplicity we assume that b is constant. Let again $\eta = u^I - u$ and $\chi = u^I - U$. Then

$$a(\eta,\chi) = \varepsilon(\eta',\chi') - (b\eta',\chi) + (c\eta,\chi)$$

For the diffusion term, integration by parts gives

$$\int_{x_{i-1}}^{x_i}\eta'\chi' = \eta\chi'\big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i}\eta\chi'' = 0$$

because $\eta(x_{i-1}) = \eta(x_i) = 0$ and because χ is linear. Thus $(\eta', \chi') = 0$. The reaction term is easily bounded using Hölder's inequality:

$$|(c\eta, \chi)| \le C \left\|\eta\right\|_{L_\infty} \left\|\chi\right\|_{L_1} \le C \left(h + N^{-1} \max|\psi'|\right)^2 \left\|\chi\right\|_0 .$$

We are left with the convection term. Recalling the decomposition (2.13), we split

$$(\eta', \chi) = -\int_0^\lambda (w^I - w)\chi' - \int_0^\lambda (v^I - v)\chi' - \int_\lambda^1 (w^I - w)\chi' + \int_\lambda^1 (v^I - v)'\chi. \tag{3.8}$$

The four terms on the right-hand side are bounded separately.

(i) A standard interpolation error result and our bounds for the derivatives of w give

$$\begin{aligned}\left\|w^I - w\right\|^2_{0,(0,\lambda)} &\le C \sum_{i=1}^{qN} h_i^4 \int_{x_{i-1}}^{x_i} \varepsilon^{-4} e^{-2\beta x/\varepsilon} dx \le C \sum_{i=1}^{qN} \left(\frac{h_i}{\varepsilon}\right)^4 h_i e^{-2\beta x_{i-1}/\varepsilon} \\ &\le C \left(N^{-1} \max|\psi'|\right)^4 \sum_{i=1}^{qN} h_i e^{(4/\sigma - 2)\beta x_i/\varepsilon},\end{aligned}$$

since assumption (1.11) implies $h_i \le C\varepsilon$ for $i = 1, \dots, qN$ and because

$$h_i = \frac{\sigma\varepsilon}{\beta} \int_{(i-1)/N}^{i/N} \tilde{\varphi}'(t) dt \le \frac{\sigma\varepsilon}{\beta} N^{-1} \max|\psi'| e^{\beta x_i/\sigma\varepsilon}. \tag{3.9}$$

Thus

$$\left\|w^I - w\right\|^2_{0,(0,\lambda)} \le C \left(N^{-1} \max|\psi'|\right)^4 \int_0^\lambda e^{(4/\sigma - 2)\beta x/\varepsilon} dx \le C\varepsilon \left(N^{-1} \max|\psi'|\right)^4 ,$$

where we have used $\sigma > 2$. This result and the Cauchy-Schwarz inequality yield

$$\left|\int_0^\lambda (w^I - w)\chi'\right| \le C \left(N^{-1} \max|\psi'|\right)^2 |||\chi|||_\varepsilon . \tag{3.10}$$

(ii) To bound the second term we proceed as follows:

$$\left\|v^I - v\right\|^2_{0,(0,\lambda)} \le C \sum_{i=1}^{qN} h_i^4 \int_{x_{i-1}}^{x_i} v''(x)^2 dx \le C h^4 \int_0^\lambda v''(x)^2 dx \le C h^4 \varepsilon \ln N,$$

since $|v''(x)| \le C$ on $(0, 1)$. Hence

$$\left|\int_0^\lambda (w^I - w)\eta\chi'\right| \le C h^2 \ln^{1/2} N \, |||\chi|||_\varepsilon . \tag{3.11}$$

(iii) Now we consider $\int_\lambda^1 (w - w^I)\chi'$. The argument splits the integral once more, but first let us recall that the mesh on $(\lambda, 1)$ is uniform with mesh diameter $H \in [N^{-1}, N^{-1}/(1-q)]$. We have

$$\left\|w^I - w\right\|^2_{0,(x_{qN}, x_{qN+1})} \le C N^{-1} e^{-2\beta\lambda/\varepsilon} \le C N^{-6}$$

since $\sigma \ge 5/2$. Thus

$$\left|\int_\lambda^{x_{qN+1}} (w - w^I)\chi'\right| \le C N^{-2} \|\chi\|_0,$$

by an inverse inequality. Next we have

$$\left\|w^I - w\right\|^2_{0,(x_{qN+1},1)} \le C \sum_{i=qN+1}^{N-1} He^{-2\beta x_i/\varepsilon} \le C \int_\lambda^{x_{N-1}} e^{-2\beta x/\varepsilon} dx \le C\varepsilon N^{-5}.$$

Thus

$$\left|\int_\lambda^1 (w - w^I)\chi'\right| \le CN^{-2} \left\|\left|\chi\right|\right\|_\varepsilon . \tag{3.12}$$

(iv) To bound the last term in (3.8) we use

$$\int_{x_{i-1}}^{x_i} \left(v - v^I\right)' \chi = \frac{1}{6} \int_{x_{i-1}}^{x_i} v''' \left(E_i^2\right)' \chi' - \frac{1}{3}\left(\frac{h_i}{2}\right)^2 \int_{x_{i-1}}^{x_i} v'''\chi + \frac{1}{3}\left(\frac{h_i}{2}\right)^2 v''\chi\Big|_{x_{i-1}}^{x_i} \tag{3.13}$$

with

$$E_i(x) = \frac{1}{2}\left((x - x_{i-1/2})^2 + \left(\frac{h_i}{2}\right)^2\right)$$

which holds true for arbitrary functions $v \in C^3[x_{i-1}, x_i]$ and linear functions χ; cf. [47]. We get

$$\int_\lambda^1 \left(v - v^I\right)' \chi = \frac{1}{6}\int_\lambda^1 v''' \left(E^2\right)' \chi' - \frac{H^2}{12}\left(v''\chi\right)(\lambda) - \frac{H^2}{12}\int_\lambda^1 v''\chi.$$

Assuming more regularity of the data, the decompostion (2.13) can be sharpened to give $|v'''| \le C$. This yields

$$\left|\int_\lambda^1 \left(v - v^I\right)' \chi\right| \le CH^3 \left\|\chi'\right\|_{L_1(\lambda,1)} + CH^2 \left|\chi(\lambda)\right| + CH^2 \left\|\chi\right\|_{L_1(\lambda,1)} \le CH^2 \left(\left\|\left|\chi\right|\right\|_\varepsilon + \left|\chi(\lambda)\right|\right),$$

by an inverse inequality. Finally, we estimate

$$\left|\chi(\lambda)\right| = \left|\int_0^\lambda \chi'\right| \le \sqrt{\lambda}\left\|\chi'\right\|_{0,(0,\lambda)} \le C \ln^{1/2} N \left\|\left|\chi\right|\right\|_\varepsilon .$$

Thus

$$\left|\int_\lambda^1 \left(v - v^I\right)' \chi\right| \le CH^2 \ln^{1/2} N \left\|\left|\chi\right|\right\|_\varepsilon . \tag{3.14}$$

Combine (3.8)-(3.14) to get for the convection term

$$\left|(\eta', \chi)\right| \le C\left(h^2 \ln^{1/2} N + N^{-2} \max\left|\psi'\right|^2\right) \left\|\left|\chi\right|\right\|_\varepsilon .$$

This inequality, the bounds for the diffusion and reaction terms and the coercivity of $a(\cdot, \cdot)$ yield the proposition of the theorem. □

Remark 3.4. *Surprisingly, the major difficulty in the proof does not arise from the layer term, but from the regular solution component. To cope with this the special integral expansion formula (3.13) by Lin had to be used.*

Another superconvergence result for Shishkin meshes was derived by Zhang [114]. He uses finite elements with piecewise polynomials of degree $p \ge 1$ on a Shishkin mesh with transition point

$$\lambda = \min\left\{\frac{1}{2}, \frac{\varepsilon(p + 3/2)}{\beta} \ln(N + 1)\right\}$$

and estabilishes

$$|||u - U|||_{\varepsilon,\omega} \le C \left(\left(\frac{\ln(N+1)}{N} \right)^{p+1} + N^{-p} \right)$$

with $|||z|||^2_{\varepsilon,\omega} = \varepsilon \sum_{i=1}^{N} Q_i^p((z')^2) + \|z\|_0^2$, where $Q_i^p(z)$ is the p-point Gauss-Lobatto formula for $\int_{x_{i-1}}^{x_i} z(x)dx$. If the regular solution component v lies in the finite element space then the stronger bound

$$|||u - U|||_{\varepsilon,\omega} \le C \left(\frac{\ln(N+1)}{N} \right)^{p+1}$$

holds true. This too illustrates the technical difficulties with the regular solution component mentioned in Remark 3.4, but unlike (3.13) for linear elements no expansion formulae for the convection term are available for quadratic or higher-order elements.

3.2.3 Gradient recovery and *a posteriori* error estimation

As pointed out in [1] for instance, superconvergence properties like Theorem 3.3 are basic ingredients for the superconvergent recovery of gradients. Furthermore, if a superconvergent recovery operator is available, then it is possible to define an *a posteriori* error estimator that is asymptotically exact.

First, we define for a given $v \in V^\omega$ a recovery operator for the derivative. With $K_i := (x_{i-1}, x_i)$ we set

$$(Rv)(x) := \alpha_{i-1} \frac{x_i - x}{h_i} + \alpha_i \frac{x - x_{i-1}}{h_i} \quad \text{for } x \in K_i,\ i = 2, \dots, N-1,$$

where α_i denotes the weighted average of the constant values of v' on the subintervals adjacent to x_i:

$$\alpha_i := \frac{h_{i+1}}{h_i + h_{i+1}} v'\big|_{K_i} + \frac{h_i}{h_i + h_{i+1}} v'\big|_{K_{i+1}}.$$

For the boundary intervals we simply extrapolate the well-defined linear function of the adjacent interval.

Our aim is to prove a superconvergence estimate for $\varepsilon^{1/2}\left\|u' - RU\right\|_0$ that is superior to that of Theorem 3.1 for $\varepsilon^{1/2}\left\|u' - U'\right\|_0$. In our presentation we follow [86]. The key ingredients are Theorem 3.3, and the consistency and stability of the operator R.

Consistency: Let v be a quadratic function on $\tilde{K}_i$, the union of K_i and its adjacent mesh intervals. Then

$$R\left(v^I\right) = v' \quad \text{on} \quad K_i. \tag{3.15a}$$

Stability:

$$\|Rv\|_{0,K_i} \le C \|v'\|_{0,\tilde{K}_i} \quad \text{for all} \quad v \in V^\omega. \tag{3.15b}$$

We start our analysis from a triangle inequality:

$$\|u' - RU\|_0 \le \left\|u' - R\left(u^I\right)\right\|_0 + \left\|R\left(u^I - U\right)\right\|_0.$$

The second term in this inequality can be bounded using Theorem 3.3 and (3.15b). Thus we are left with the problem of estimating $\left\|u' - R\left(u^I\right)\right\|_0$. To take advantage of the consistency property (3.15a) we introduce a quadratic approximation of u on $\tilde{K}_i$: $Q_i u$. Using a triangle inequality, we obtain

$$\left\|u' - R\left(u^I\right)\right\|_{0,K_i} \le \left\|u' - (Q_i u)'\right\|_{0,K_i} + \left\|(Q_i u)' - R\left((Q_i u)^I\right)\right\|_{0,K_i} + \left\|R\left((Q_i u - u)^I\right)\right\|_{0,K_i}.$$

The second term vanishes because of (3.15a). The last term can be bounded using (3.15b) and the stability of the linear interpolation in H^1, i. e., $|v^I|_1 \le C|v|_1$. We get

$$\left\|u' - R\left(u^I\right)\right\|_0^2 \le C \sum_{i=1}^N \left\|u' - (Q_i u)'\right\|_{0,\tilde{K}_i}^2.$$

Note this H^1 stability of the interpolation operator holds true only in the one-dimensional case. In two dimensions the L_∞ stability of the interpolation operator has to be used instead, see Section 5.3.2.5.

Choosing $Q_i u$ to be, e. g., that bilinear function that coincides with u at the midpoint and both endpoints of $\tilde{K}_i$ and estimating the interpolation error carefully, see [86], we obtain

$$\varepsilon \sum_{i=1}^N \left\|u' - (Q_i u)'\right\|_{0,\tilde{K}_i}^2 \le C\left(h + N^{-1} \max|\psi'|\right)^4 \quad \text{if} \quad \sigma \ge 2.$$

Combining these results, we get

Theorem 3.5. *Let ω be an S-type mesh with $\sigma \ge 5/2$ whose mesh generating function $\tilde{\varphi}$ satisfies (1.11). Then the error of the recovered gradient of the Galerkin FEM satisfies*

$$\varepsilon^{1/2}\left\|u' - RU\right\|_0 \le C\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right).$$

Remark 3.6. *Using RU instead of u', we get an asymptotically exact error estimator for the weighted H^1-seminorm of the finite element error $\varepsilon^{1/2}\left\|u' - U'\right\|_0$ on S-type meshes:*

$$\varepsilon^{1/2}\left\|u' - U'\right\|_0 = \varepsilon^{1/2}\left\|RU - U'\right\|_0 + \mathcal{O}\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right).$$

This error estimator is asymptotically exact for $N \to \infty$ because in the generic case

$$\varepsilon^{1/2}\left\|u' - U'\right\|_0 = \mathcal{O}\left(h + N^{-1} \max|\psi'|\right).$$

3.2.4 A numerical example

Let us briefly illustrate our theoretical results for the linear Galerkin FEM on S-type meshes when applied to the test problem

$$-\varepsilon u'' - u' + 2u = e^{x-1} \quad \text{in} \quad (0,1), \quad u(0) = u(1) = 0.$$

For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem.

Apart from the two meshes introduced in section 1.3, we also consider a S-type mesh with the rational mesh-characterizing function

$$\psi_R(t) = \frac{1}{1 + (N-1)(2t)^\ell} \quad \text{with} \quad \ell \ge 2$$

because it particularly emphasises the sharpness of the theoretical results. For this mesh we have $|\psi'| \le N^{1/\ell}$. The results in Tables 3.1–3.3 are clear illustrations of the estimates given in Theorems 3.1–3.5.

	$\|\|\|u-U\|\|\|_\varepsilon$		$\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$		$\varepsilon^{1/2}\left\|u'-RU\right\|_0$	
N	error	rate	error	rate	error	rate
2^8	1.158e-2	0.50	5.889e-4	0.99	3.289e-3	0.98
2^9	8.213e-3	0.50	2.959e-4	1.00	1.673e-3	0.99
2^{10}	5.817e-3	0.50	1.483e-4	1.00	8.435e-4	0.99
2^{11}	4.117e-3	0.50	7.424e-5	1.00	4.236e-4	1.00
2^{12}	2.912e-3	0.50	3.714e-5	1.00	2.123e-4	1.00
2^{13}	2.060e-3	0.50	1.858e-5	1.00	1.062e-4	1.00
2^{14}	1.456e-3	—	9.290e-6	—	5.315e-5	—

Table 3.1: Galerkin FEM on a Shishkin mesh with rational ψ ($\ell = 2$).

	$\|\|\|u-U\|\|\|_\varepsilon$		$\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$		$\varepsilon^{1/2}\left\|u'-RU\right\|_0$	
N	error	rate	error	rate	error	rate
2^8	6.166e-3	0.83	1.624e-4	1.66	8.045e-4	1.64
2^9	3.470e-3	0.85	5.151e-5	1.69	2.585e-4	1.69
2^{10}	1.928e-3	0.86	1.592e-5	1.72	8.025e-5	1.72
2^{11}	1.060e-3	0.87	4.818e-6	1.75	2.432e-5	1.75
2^{12}	5.784e-4	0.88	1.434e-6	1.77	7.242e-6	1.77
2^{13}	3.133e-4	0.89	4.211e-7	1.79	2.126e-6	1.79
2^{14}	1.687e-4	—	1.221e-7	—	6.164e-7	—

Table 3.2: Galerkin FEM on a standard Shishkin mesh.

3.3 Stabilized FEM

We have seen that the Galerkin FEM on S-type meshes has good approximation properties. Unfortunately the linear systems generated are difficult to solve iteratively. Therefore stabilization is essential.

3.3.1 Artificial viscosity stabilization

The simplest way to stabilize discretisation methods for convection-diffusion problems consists of altering the diffusion coefficient a priori, the extra diffusion added being called *artificial viscosity*. Typically artificial viscosity proportional to the stepsize is used. This yields the stabilized finite element formulation: Find $U \in V^\omega$ such that

$$a_\kappa(U,v) = f(v) \quad \text{for all} \quad v \in V^\omega,$$

where

$$a_\kappa(u,v) := \big((\varepsilon + \kappa\hbar)u', v'\big) - (bu' - cu, v) \quad \text{and} \quad \hbar(x) := h_i \quad \text{for} \quad x \in (x_{i-1}, x_i)$$

	$\|\|\|u-U\|\|\|_\varepsilon$		$\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$		$\varepsilon^{1/2}\left\|u'-RU\right\|_0$	
N	error	rate	error	rate	error	rate
2^8	1.357e-3	1.00	5.382e-6	1.99	4.173e-5	2.00
2^9	6.800e-4	1.00	1.353e-6	2.00	1.043e-5	2.00
2^{10}	3.403e-4	1.00	3.393e-7	2.00	2.610e-6	2.00
2^{11}	1.702e-4	1.00	8.497e-8	2.00	6.528e-7	2.00
2^{12}	8.514e-5	1.00	2.126e-8	2.00	1.632e-7	2.00
2^{13}	4.258e-5	1.00	5.317e-9	2.01	4.082e-8	2.00
2^{14}	2.129e-5	—	1.321e-9	—	1.020e-8	—

Table 3.3: Galerkin FEM on a Bakhvalov-Shishkin mesh.

and $\kappa > 0$ is an arbitrary constant. The bilinear form $a_\kappa(\cdot,\cdot)$ is coercive with respect to the norm

$$|||v|||_\kappa := \left\{\left((\varepsilon+\kappa\hbar)v',v'\right)+(v,v)\right\}^{1/2},$$

which is stronger than the ε-weighted energy norm and the reason for the improved stability.

Because of the artificial viscosity the method does not satisfy the orthogonality property which complicates the convergence analysis. Assume an S-type mesh is used and Let again $\eta = u^I - u$ and $\chi = u^I - U$. Then

$$\min\{1,\gamma\}\,|||\chi|||_\kappa^2 \le a_\kappa(\chi,\chi) = a(\eta,\chi) + \left(\kappa\hbar(u^I)',\chi'\right) = a(\eta,\chi) + \kappa\left(\hbar\eta',\chi'\right) + \kappa\left(\hbar u',\chi'\right).$$

Bounds for the first term have been derived in Section 3.2. The second term $(\hbar\eta',\chi')$ vanishes, while the last term, which is the inconsistency of the method, satisfies

$$\kappa\left|(\hbar u',\chi')\right| \le C\kappa\left(h\ln^{1/2}N + N^{-1}\max|\psi'|\right)|||\chi|||_\varepsilon.$$

The proof recycles some ideas from Sections 3.2.1 and 3.2.2 and is therefore omitted. For more details see [92]. We get

$$\left|\left|\left|u^I - U\right|\right|\right|_\kappa \le C\left\{h(h+\kappa)\ln^{1/2}N + \left(\kappa + N^{-1}\max|\psi'|\right)N^{-1}\max|\psi'|\right\}. \tag{3.16}$$

Thus if we choose $\kappa = \mathcal{O}(1)$, i. e., we add artificial viscosity proportional to the local mesh size, we get

$$\left|\left|\left|u^I - U\right|\right|\right|_\kappa + |||u-U|||_\varepsilon \le C\left(h\ln^{1/2}N + N^{-1}\max|\psi'|\right),$$

by the interpolation error estimate (3.6).

Comparing (3.7) and (3.16), we see that the order of accuracy of the Galerkin FEM is not affected if we take $\kappa = \mathcal{O}\left(N^{-1}\right)$. This results in improved stability compared to the Galerkin method and the discrete systems—in particular for higher-dimensional problems—are slightly easier to solve by means of standard iterative methods. On the other hand the method is not as stable as if $\kappa = \mathcal{O}(1)$ were chosen.

3.3.2 Streamline-diffusion stabilization

The most popular and most frequently studied stabilized FEM is the streamline-diffusion finite element method (SDFEM) which is also referred to as the streamline-upwind Petrov-Galerkin method (SUPG). This kind of stabilization was introduced by Hughes and Brooks [34]. Given a mesh ω and a finite element space V^ω, this method can be written as: Find $U \in V^\omega$ such that

$$a(U,v) + \sum_{i=1}^{N}\delta_i\int_{x_{i-1}}^{x_i}(f-\mathcal{L}U)\,bv' = (f,v) \quad \text{for all} \quad v \in V^\omega, \tag{3.17}$$

where the stabilization parameters δ_i are chosen according to the local mesh Peclét number:

$$\delta_i = \begin{cases} \kappa_0 h_i & \text{if } Pe_i > 1, \\ \kappa_1 h_i^2\varepsilon^{-1} & \text{if } Pe_i \le 1, \end{cases} \qquad \text{with} \quad Pe_i = \frac{\|b\|_{\infty,(x_{i-1},x_i)}h_i}{2\varepsilon}$$

with user chosen positive constants κ_0 and κ_1. In contrast to the artificial-viscosity stabilization this method is consistent with (3.1) since u satisfies (3.17) for all $v \in H_0^1(0,1)$. Another advantage—though it becomes relevant only in higher dimensions—is the reduction of crosswind smear because artificial viscosity is added only in the streamline direction.

The second-order upwind schemes of Section 2.3 may be regarded as versions of the SDFEM with linear test and trial functions and inexact intergration. While in the one-dimensional case it is always possible to chose the stabilization parameters δ_i such that the resulting scheme is inverse monotone, this is in general impossible in higher dimensions. Therefore alternative techniques have to be developed to study the SDFEM. Here we shall consider convergence in the streamline-diffusion norm $|||\cdot|||_{SD}$ naturally associated with the bilinear form of the method. This technique can be extended to two-dimensional problems; see [100].

We shall follow [100] and study the SDFEM on S-type meshes. For the sake of simplicity we consider (3.1) with constant b. Let $V^\omega \subset H_0^1(0,1)$ be the space of piecewise-linear functions on ω. We rewrite (3.17) as: Find $U \in V^\omega$ such that

$$a_{SD}(U,v) := a(U,v) + a_{stab}(U,v) = (f,v) - \delta b \int_\lambda^1 fv' \quad \text{for all} \quad v \in V^\omega$$

where $a(\cdot,\cdot)$ is the bilinear form of the Galerkin FEM,

$$a_{stab}(U,v) := -\delta \sum_{i=qN+1}^{N} \int_{x_{i-1}}^{x_i} (-\varepsilon U'' - bU' + cU) bv'$$

and

$$\delta = \begin{cases} \kappa_0 H & \text{if} \quad bH/2\varepsilon > 1, \\ \kappa_1 H^2/\varepsilon & \text{otherwise.} \end{cases}$$

Here H denotes again the mesh size on the coarse part of the mesh.

We define the streamline-diffusion norm naturally associated with $a_{SD}(\cdot,\cdot)$ by

$$|||v|||_{SD}^2 := \varepsilon \|v'\|_0^2 + \gamma \|v\|_0^2 + \|\delta^{1/2} bv'\|_{0,(\lambda,1)}^2.$$

Provided the maximum step size h is smaller than some threshold value the bilinear form $a_{SD}(\cdot,\cdot)$ is coercive with respect to the streamline-diffusion norm:

$$a_{SD}(v,v) \geq \tfrac{1}{2} |||v|||_{SD}^2 .$$

The bilinear form also satisfies the Galerkin-orthogonality property

$$a_{SD}(u-U,v) = 0 \quad \text{for all} \quad v \in V^\omega.$$

This is the starting point of our error analysis. Letting again $\eta = u^I - u$ and $\chi = u^I - U$, we have

$$\tfrac{1}{2} |||\chi|||_{SD}^2 \leq a(\eta,\chi) + a_{stab}(\eta,\chi). \tag{3.18}$$

For the first term we have from the proof of Theorem 3.3

$$|a(\eta,\chi)| \leq C \left(h^2 \ln^{1/2} + N^{-2} \max |\psi'|^2 \right) |||\chi|||_\varepsilon .$$

It remains to bound $a_{stab}(\eta,\chi)$. We have

$$a_{stab}(\eta,\chi) = \delta b \int_\lambda^1 (\varepsilon u'' + b\eta' + c\eta)\, \chi'.$$

Elementwise integration by parts yields $\int_\lambda^1 b\eta'\chi' = 0$. Furthermore we have

$$\left| \delta \int_\lambda^1 c\eta b\chi' \right| \leq C\delta^{1/2} \|\eta\|_{0,(\lambda,1)} \|\delta^{1/2} b\chi'\|_0 \leq C\delta^{1/2} N^{-2} \|\delta^{1/2} b\chi'\|_0 \leq CN^{-2} |||\chi|||_{SD} ,$$

by our earlier bounds for the interpolation error.

To bound the remaining term $\int \eta''\chi'$ we use the decomposition of u into a regular and a layer component. For the regular component v we have

$$\int_\lambda^1 v''\chi' = -\int_\lambda^1 v'''\chi - \int_0^\lambda v''\chi'.$$

Hence

$$\left|\int_\lambda^1 v''\chi'\right| \le C\|\chi\|_0 + C\left(\varepsilon \ln N\right)^{1/2} \|\chi'\|_0,$$

by our bounds for v and its derivatives. Thus

$$\left|\varepsilon\delta \int_\lambda^1 v''\chi'\right| \le CN^{-2} \ln^{1/2} N \, |||\chi|||_{SD} \, ,$$

since the choice of δ implies $\varepsilon\delta \le CH^2 \le CN^{-2}$.

For the layer component w we estimate as follows:

$$\varepsilon\delta\left|\int_\lambda^1 w''b\chi'\right| \le \varepsilon\delta^{1/2}\|w''\|_{L_1(\lambda,1)}\|\delta^{1/2}b\chi'\|_{\infty,(\lambda,1)} \le C\delta^{1/2}N^{-5/2}H^{-1/2}\|\delta^{1/2}b\chi'\|_0,$$

from an inverse inequality and $\sigma \ge 5/2$. We get

$$\varepsilon\delta\left|\int_\lambda^1 w''b\chi'\right| \le CN^{-2} \, |||\chi|||_{SD} \, .$$

Collecting these results, the second term in (3.18) is bounded by

$$|a_{stab}(\eta,\chi)| \le CN^{-2} \ln^{1/2} N \, |||\chi|||_\varepsilon \, .$$

We summarize our results.

Theorem 3.7. *Let ω be an S-type mesh with $\sigma \ge 5/2$ whose mesh generating function $\tilde{\varphi}$ satisfies (1.11). Then the error of the SDFEM satisfies*

$$\left|\left|\left|u^I - U\right|\right|\right|_{SD} \le C\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right).$$

Remark 3.8. *This is a superconvergence result just like Theorem 3.3. Similarly to Section 3.2.3 it is possible to construct a recovery operator to obtain higher-order approximations of the gradient of the exact solution; see [100].*

Finishing this section, let us mention an article by Roos and Zarin [90] who study the streamline diffusion FEM on Shishkin and on Bakhvalov-Shishkin meshes for the discretisation of a problem with a point source.

3.4 An upwind finite volume method

Let us finish this chapter by considering finite volume discretizations of our standard model problem with a regular layer: Find $u \in C^2(0,1) \cup C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{3.19}$$

with $b \ge \beta > 0$.

Although the construction of finite volume methods differs from finite difference and finite element methods they are typically analysed as special finite difference methods or—more often—as nonconforming finite element methods. Here we like to highlight both approaches. In particular this section is intended to prepare our later investigation of the FVM in two dimensions in Section 5.4. Therefore we shall assume

$$c \geq \gamma > 0, \quad c + b' \geq \gamma > 0 \qquad \text{when studying the FVM as a FDM,} \tag{3.20}$$

and

$$c + b'/2 \geq \gamma > 0 \qquad \text{in the FEM context.} \tag{3.21}$$

In the latter case the variational formulation of (3.19) will be used: Find $u \in H_0^1(0,1)$

$$a(u,v) := \varepsilon(u',v') - (bu',v) + (cu,v) = (f,v) =: f(v) \quad \text{for all} \quad v \in H_0^1(0,1).$$

Given an arbitrary mesh $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ our FVM reads: Find $U \in \mathbb{R}_0^{N+1} = \{v \in \mathbb{R}^{N+1} : v_0 = v_N = 0\}$ such that

$$\begin{aligned} [\Lambda_\varrho U]_i := -\varepsilon\left(\frac{U_{i+1}-U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i}\right) - \varrho\left(-\frac{b_{i+1/2}h_{i+1}}{\varepsilon}\right) b_{i+1/2}\big(U_{i+1}-U_i\big) \\ - \varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right) b_{i-1/2}\big(U_i - U_{i-1}\big) + \hbar_i c_i U_i = \hbar_i f_i, \end{aligned} \tag{3.22}$$

with $b_{i+1/2} = b\big((x_i + x_{i+1})/2\big)$. It can also be written in variational form: Find $U \in \mathbb{R}_0^{N+1}$ such that

$$a_\varrho(U,v) := \sum_{i=1}^{N-1} [\Lambda_\varrho U]_i \, v_i = f_\varrho(v) := \sum_{i=1}^{N-1} \hbar_i f_i v_i, \quad \text{for all } v \in \mathbb{R}_0^{N+1},$$

The crucial point is the choice of the controlling function $\varrho : \mathbb{R} \to [0,1]$. It has to provide the correct weighting between the two one-sided difference approximations for the first-order derivative. Possible choices for ϱ include

$$\varrho_I(t) = \begin{cases} \dfrac{1}{t}\left(1 - \dfrac{t}{\exp t - 1}\right) & \text{for} \quad t \neq 0, \\ \dfrac{1}{2} & \text{for} \quad t = 0, \end{cases}$$

$$\varrho_S(t) = \begin{cases} 1/(2+t) & \text{for} \quad t \geq 0, \\ (1-t)/(2-t) & \text{for} \quad t < 0, \end{cases}$$

and

$$\varrho_{U,m}(t) = \begin{cases} 0 & \text{for} \quad t > m, \\ \frac{1}{2} & \text{for} \quad t \in [-m,m], \\ 1 & \text{for} \quad t < -m, \end{cases} \qquad \text{with} \quad m \geq 0.$$

The full upwind stabilization $\varrho_{U,0}$ is due to Baba and Tabata [12], while $\varrho_{U,m}$ with $m > 0$ was introduced by Angermann [8]. For ϱ_I and ϱ_S we get slight modifications of the schemes of Il'in [35] and of Samarski [91]. Further choices of ϱ are mentioned in [8] and [37] where also a detailed derivation of the method in two dimensions can be found.

The constant choice $\varrho \equiv \frac{1}{2}$ generates a central difference scheme, while the choice $\varrho_{U,0}$ gives a scheme with upwinded one-sided difference approximation of the first-order derivative which is very similar to the upwind scheme analysed in Section 2.2. If a different ϱ is used—in particular when ϱ

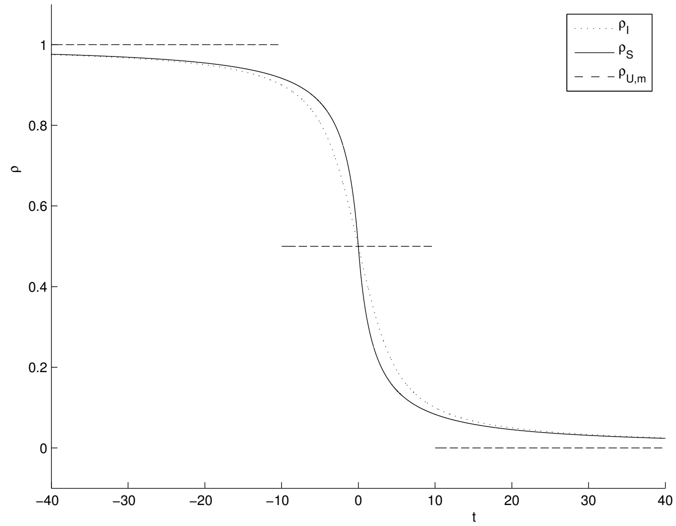


Figure 3.1: The stabilizing functions ϱ_I, ϱ_S and $\varrho_{U,m}$

is Lipschitz continuous in a neighbourhood of 0—then the first-order derivatives are approximated by weighted combinations of upwinded and downwinded operators. This weighting provides an adaptive transition from an upwinded to a central difference approximation when the local mesh size is small enough. In this case higher accuracy is achieved while retaining the good stability of the scheme.

Important properties of ϱ are

$$\begin{array}{lrl} (\varrho_0) & t \mapsto t\varrho(t) & \text{is Lipschitz continuous,} \\ (\varrho_1) & \big[\varrho(t)+\varrho(-t)-1\big]t = 0 & \text{for all } t \in I\!R, \\ (\varrho_2) & \big[1/2-\varrho(t)\big]t \geq 0 & \text{for all } t \in I\!R, \\ (\varrho_3) & 1 - t\varrho(t) \geq 0 & \text{for all } t \in I\!R. \end{array}$$

Condition (ϱ_1) ensures both the consistency of the scheme and the local conservation of the fluxes, while (ϱ_2) guarantees the coercivity of the bilinear form $a_\varrho(\cdot,\cdot)$ and (ϱ_3) the inverse monotonicity of the scheme.

3.4.1 Stability of the FVM

Coercivity of the bilinear form $a_\varrho(\cdot,\cdot)$. The consistency condition (ϱ_1) and summation by parts yield

$$\begin{aligned} a_\varrho(v,v) = \varepsilon \sum_{i=1}^{N} \frac{(v_i - v_{i-1})^2}{h_i} + \sum_{i=1}^{N} \left[\frac{1}{2} - \varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right] b_{i-1/2}(v_i - v_{i-1})^2 \\ + \sum_{i=1}^{N-1} \left[\hbar_i c_i + \frac{1}{2}\left(b_{i+1/2} - b_{i-1/2}\right)\right] v_i^2. \end{aligned} \tag{3.23}$$

Assume b' is Hölder continuous with coefficient $\alpha \in (0,1]$. Then

$$\left|b_{i+1/2} - b_{i-1/2} - \hbar_i b_i'\right| \leq \hbar_i h^\alpha \, \|b\|_{C^{1,\alpha}[0,1]} \,. \tag{3.24}$$

Thus, if (3.21) is satisfied

$$\sum_{i=1}^{N-1} \left[\hbar_i c_i + \frac{1}{2}\left(b_{i+1/2} - b_{i-1/2}\right)\right] v_i^2 \geq \frac{\gamma}{2} \sum_{i=1}^{N-1} \hbar_i v_i^2 \tag{3.25}$$

provided the maximal mesh size h is smaller than some threshold value h^*.

Let

$$|||v|||_\varrho^2 := \varepsilon\, |v|_{1,\omega}^2 + |v|_{\varrho,\omega}^2 + \frac{\gamma}{2}\, \|v\|_{0,\omega}^2$$

with

$$|v|_{1,\omega}^2 := \sum_{i=1}^{N} \frac{(v_i - v_{i-1})^2}{h_i},$$

$$|v|_{\varrho,\omega}^2 := \sum_{i=1}^{N} \left[\frac{1}{2} - \varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right] b_{i-1/2}(v_i - v_{i-1})^2 \quad \text{and} \quad \|v\|_{0,\omega} := \sum_{i=1}^{N-1} \hbar_i v_i^2$$

which is a well-defined norm when (ϱ_2) is satisfied. Now the coercivity follows immediately from (3.23) and (3.25).

Theorem 3.9. *Assume conditions (ϱ_1), (ϱ_2) and (3.21) are satisfied. Let $b \in C^{1,\alpha}[0,1]$ with Hölder exponent $\alpha \in (0,1]$. Then the bilinear form $a_\varrho(\cdot,\cdot)$ is coercive with respect to the FV-norm $|||\cdot|||_\varrho$, i.e.,*

$$a_\varrho(v,v) \ge |||v|||_\varrho^2 \quad \textit{for all} \quad v \in I\!R_0^{N+1}$$

provided the maximum mesh size h is smaller than some threshold value which is independent of the perturbation parameter ε.

Note when $\varrho \equiv \frac{1}{2}$ the stabilization is switched off. Nonetheless Theorem 3.9 states coercivity of the bilinear form with respect to the discrete ε-weighted energy norm $|||v|||_{\varepsilon,\omega}^2 := \varepsilon|v|_{1,\omega}^2 + \frac{\gamma}{2}\|v\|_{0,\omega}^2$. However in the case $\varrho \not\equiv \frac{1}{2}$ the scheme is coercive with respect to a stronger norm which results in enhanced stability of the method.

Inverse monotonicity. Let r^+, r^-, $q \ge \gamma > 0$ and $\chi > 0$ be arbitrary mesh functions with

$$r_i^+ \ge \frac{\beta h_{i+1}}{\alpha\varepsilon} \quad \text{and} \quad 1 \ge r_i^- \ge 0 \quad \text{for } i = 1, \dots, N-1 \tag{3.26}$$

with a constant $\alpha > 0$. Consider the difference operator

$$[L_\chi U]_i := \frac{\varepsilon}{\chi_i}\left[1 + r_i^+\right]\frac{U_i - U_{i+1}}{h_{i+1}} + \frac{\varepsilon}{\chi_i}\left[1 - r_i^-\right]\frac{U_i - U_{i-1}}{h_i} + q_i U_i. \tag{3.27}$$

We study this more general situation because it will also serve as an auxiliary result in Section 5.4.2 when two dimensional problems will be investigated. The FVM (3.22) belongs to this class of schemes provided that (ϱ_3) holds.

Clearly, $1 + r_i^+ \ge 1$ and $1 - r_i^- \ge 0$. Hence the system matrix associated with L_χ possesses positiv diagonal entries and nonnegativ offdiagonal ones and therefore is an L_0-matrix. Application of the M-matrix criterion with the test function $v \equiv 1$ yields the inverse monotonicity of L_χ. This can be used to study the Green's function associated with L_χ and derive stability inequalities similar to those of Section 2.2.1.

$(\boldsymbol{\ell_\infty}, \boldsymbol{\ell_1})$ stability. For $j = 1, \dots, N-1$ the Green's function $G_{\cdot,j}$ associated with the mesh node x_j satisfies

$$[L_\chi G_{\cdot,j}]_i = \delta_{i,j} := \begin{cases} \chi_i^{-1} & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for} \quad i = 1, \dots, N-1, \quad G_{0,j} = G_{N,j} = 0.$$

Given G, we can represent any function $v \in I\!R_0^{N+1}$ as

$$v_i = \sum_{j=1}^{N-1} \chi_j G_{i,j} \left[L_\chi v\right]_j . \tag{3.28}$$

Let

$$\hat{G}_{i,j} = \begin{cases} \dfrac{\alpha}{\beta} & \text{for } i = 0, \dots, j, \\ \dfrac{\alpha}{\beta} \displaystyle\prod_{k=j+1}^{i} \left(1 + \dfrac{\beta h_k}{\alpha \varepsilon}\right)^{-1} & \text{for } i = j+1, \dots, N. \end{cases}$$

Lemma 3.10. *Assume condition (3.26) holds. Then the Green's function G associated with L_χ satisfies*

$$0 \le G_{i,j} \le \hat{G}_{i,j} \le \alpha/\beta \quad \textit{for} \quad i, j = 0, \dots, N.$$

Proof. We have just seen that when (3.26) holds the operator L_χ is inverse monotone and therefore satisfies a discrete comparison principle. The lower bound on G is easily verified using the barrier function $v \equiv 0$.

In order to prove the upper bound it is sufficient to show that for all $j = 1, \dots, N-1$ we have

$$\left[L_\chi G_{\cdot,j}\right]_i = \delta_{i,j} \le \left[L_\chi \hat{G}_{\cdot,j}\right]_i \quad \text{for} \quad i = 1, \dots, N-1, \; G_{0,j} \le \hat{G}_{0,j} \; \text{ and } \; G_{N,j} \le \hat{G}_{N,j}.$$

(i) First we check the boundary conditions. Clearly $\hat{G}_{i,j} > 0$ for $i, j = 0, \dots, N$. Thus

$$0 = G_{0,j} \le \hat{G}_{0,j} \;\text{ and }\; 0 = G_{N,j} \le \hat{G}_{N,j} \;\text{ for }\; j = 1, \dots, N-1.$$

(ii) Next, for $i < j$ we have $\left[L_\chi \hat{G}_{\cdot,j}\right]_i = q_i \hat{G}_{i,j} \ge 0$ since both q and $\hat{G}$ are positive.
(iii) For $i > j$ we have

$$\frac{\hat{G}_{i,j} - \hat{G}_{i+1,j}}{h_{i+1}} = \frac{\beta}{\alpha\varepsilon + \beta h_{i+1}} \hat{G}_{i,j} \quad \text{and} \quad \frac{\hat{G}_{i,j} - \hat{G}_{i-1,j}}{h_i} = -\frac{\beta}{\alpha\varepsilon} \hat{G}_{i,j}.$$

Thus

$$\left[L_\chi \hat{G}_{\cdot,j}\right]_i \ge \frac{\varepsilon}{\chi_i} \left\{ \frac{\left(1 + r_i^+\right)\beta}{\alpha\varepsilon + \beta h_{i+1}} - \frac{\left(1 - r_i^-\right)\beta}{\alpha\varepsilon} \right\} \hat{G}_{i,j} \ge 0,$$

by (3.26).

(iv) For $i = j$ a combination of the arguments from *(ii)* and *(iii)* yields

$$\left[L_\chi \hat{G}_{\cdot,j}\right]_j \ge \frac{\varepsilon}{\chi_j} \frac{\left(1 + r_i^+\right)\beta}{\alpha\varepsilon + \beta h_{j+1}} \hat{G}_{j,j} \ge \frac{1}{\chi_j} = \left[L_\chi G_{\cdot,j}\right]_j,$$

by (3.26). □

This Lemma and (3.28) give the (ℓ_∞, ℓ_1) stability of the method:

Theorem 3.11. *Suppose (3.26) holds. Then the operator L_χ defined in (3.27) satisfies the stability inequality*

$$\|v\|_{\infty,\omega} \le \frac{\alpha}{\beta} \sum_{i=1}^{N-1} \chi_i \left|\left[L_\chi v\right]_i\right| \quad \textit{for all } v \in I\!R_0^{N+1}.$$

Remark 3.12. *An error analysis of the upwind FVM using this (ℓ_∞, ℓ_1) stability can be conducted along the lines of Section 2.2.5; see also [57].*

$(\ell_\infty, w_{-1,\infty})$ stability. Now let us restrict our attention to difference operators of the type

$$[L_\kappa U]_i := \frac{\varepsilon}{\hbar_i}\left[1+\varrho_i^+\right]\frac{U_i - U_{i+1}}{h_{i+1}} + \frac{\varepsilon}{\hbar_i}\left[1-\varrho_i^-\right]\frac{U_i - U_{i-1}}{h_i} + c_i U_i$$

with

$$\varrho_i^+ := \varrho\left(-\frac{b_{i+\kappa}h_{i+1}}{\varepsilon}\right)\frac{b_{i+\kappa}h_{i+1}}{\varepsilon}, \quad \varrho_i^- := \varrho\left(\frac{b_{i-1+\kappa}h_i}{\varepsilon}\right)\frac{b_{i-1+\kappa}h_i}{\varepsilon},$$

and

$$\kappa \in [0,1], \quad \chi_i = \kappa h_{i+1} + (1-\kappa)h_i \quad \text{and} \quad b_{i+\kappa} = b(x_i + \kappa h_{i+1}).$$

The FVM (3.22) is recovered for $\kappa = 1/2$, while the finite difference scheme of Section 2.2 is obtained when $\kappa = 1$ and $\varrho = \varrho_{U,0}$.

Remark 3.13. *Condition (3.26) with $\alpha = \sup_{t<0} 1/\varrho(t)$ follows from (ϱ_3).*

Assuming that (ϱ_1) holds, the Green's function G solves for fixed $i = 1, \dots, N-1$

$$\left[L_\kappa^* G_{i,\cdot}\right]_j = \delta_{i,j} \qquad \text{for } j = 1, \dots, N-1, \quad G_{i,0} = G_{i,N} = 0$$

with the adjoint operator

$$\left[L_\kappa^* v\right]_j = \frac{\varepsilon}{\chi_j}\left\{\left(1-\varrho_{j+1}^-\right)\frac{v_j - v_{j+1}}{h_{j+1}} + \left(1+\varrho_{j-1}^+\right)\frac{v_j - v_{j-1}}{h_j}\right\} + \left(c_j + \frac{b_{j+\kappa} - b_{j-1+\kappa}}{\chi_j}\right)v_j.$$

Assume $c + b' \geq \gamma > 0$ and let b' be Hölder continuous with coefficient $\alpha \in (0,1]$. Then

$$c_j + \frac{b_{j+\kappa} - b_{j-1+\kappa}}{\chi_j} \geq 0$$

if the maximum step size h is sufficiently small, independent of ε; cf. (3.24). Proceeding as in Section 2.2.1, one can show

$$G_{i,j} \geq G_{i,j-1} \quad \text{for } j = 1, \dots, i \quad \text{and} \quad G_{i,j} \leq G_{i,j-1} \ \text{ for } j = i+1, \dots, N,$$

i.e., $G_{i,\cdot}$ is piecewise monotone.

Theorem 3.14. *Suppose conditions (ϱ_1), (ϱ_3) and (3.20) hold. Assume $b \in C^{1,\alpha}[0,1]$ with Hölder exponent $\alpha \in (0,1]$. Then the operator L_κ satisfies the stability inequality*

$$\|v\|_{\infty,\omega} \leq \frac{2\alpha}{\beta}\min_{C\in\mathbb{R}}\left\|\sum_{j=\cdot}^{N-1}\chi_j\left[L_\kappa v\right]_j + C\right\|_{\infty,\omega} \qquad \textit{for all } v \in \mathbb{R}_0^{N+1},$$

with $\alpha = 1/\inf_{t<0}\varrho(t) \leq 2$, if the maximum step size h is smaller than some threshold value that is independent of ε.

3.4.2 Convergence in the energy norm

In this section we study the convergence in the energy norm $|||\cdot|||_\varrho$ of the finite volume method on S-type meshes (see Section 1.3 with $\sigma \geq 2$. The controlling function ϱ is assumed to satisfy conditions (ϱ_0), (ϱ_1) and (ϱ_2).

Our analysis starts from Theorem 3.9 and follows the standard approach of the Strang Lemma [89, III.3.1.2]. Let $\eta = u^I - u$ and $\chi = u^I - U$, where we use U for both the pointwise defined solution of (3.22) and its piecewise-linear extension on the mesh ω. Then

$$\begin{aligned} |||\chi|||_\varrho^2 &\le a_\varrho(\chi,\chi) \le |a(\eta,\chi)| + \left|a(u^I,\chi) - a_\varrho(u^I,\chi)\right| + |f_\varrho(\chi) - f(\chi)| \\ &\le |a(\eta,\chi)| + |f_\varrho(\chi) - f(\chi)| + \left|r(u^I,\chi) - r_\varrho(u^I,\chi)\right| + \left|c(u^I,\chi) - c_\varrho(u^I,\chi)\right|. \end{aligned} \tag{3.29}$$

with

$$r(u^I,\chi) = \int_0^1 cu^I\chi, \qquad r_\varrho(u^I,\chi) = \sum_{i=1}^{N-1} \hbar_i c_i u_i \chi_i, \qquad c(u^I,\chi) = -\int_0^1 b(u^I)'\chi,$$

and

$$c_\varrho(u^I,\chi) = -\sum_{i=1}^{N-1} \left\{ \varrho\left(\frac{b_{i+1/2}h_{i+1}}{\varepsilon}\right) b_{i+1/2}\left(u_{i+1} - u_i\right) + \varrho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) b_{i-1/2}\left(u_i - u_{i-1}\right) \right\} \chi_i.$$

The four terms on the right-hand side of (3.29) will be bounded separately.

(i) The first term has already been analysed in Section 3.2. We have under the assumptions of Theorem 3.1

$$|a(\eta,\chi)| \le C\left(h + N^{-1}\max|\psi'|\right) |||\chi|||_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right) |||\chi|||_{\varepsilon,\omega}, \tag{3.30}$$

because the discrete and continuous energy norms are equivalent for functions from $I\!R_0^{N+1}$.

(ii) Next we bound the error arising from the discretization of the right-hand side f. Denoting by φ_i the usual basis functions for linear finite elements, we have

$$\left|\int_{x_{i-1}}^{x_i} (f\varphi_i)(x)dx - \frac{h_i}{2}f_i\right| = \left|\int_{x_{i-1}}^{x_i} \left\{f_i + \int_{x_i}^{x} f'(s)ds\right\}\varphi_i(x)dx - \frac{h_i}{2}f_i\right| \le \frac{h_i^2}{2}\|f'\|_\infty.$$

Thus

$$\begin{aligned} \left|f(\chi) - f_h(\chi)\right| &= \left|\sum_{i=1}^{N-1} \chi_i\left\{\int_{x_{i-1}}^{x_{i+1}} (f\varphi_i)(x)dx - \hbar_i f_i\right\}\right| \le \|f'\|_\infty h \sum_{i=1}^{N-1} \hbar_i|\chi_i| \\ &\le \|f'\|_\infty h \|\chi\|_{0,\omega}. \end{aligned} \tag{3.31}$$

(iii) The next term in line is $r(u^I,\chi) - r_\varrho(u^I,\chi)$. By the definition of $r_\varrho(\cdot,\cdot)$ and $r(\cdot,\cdot)$, we have

$$\begin{aligned} r_\varrho(u^I,\chi)_i - r(u^I,\chi) = &\sum_{i=1}^{N-1} \left\{\int_{x_{i-1}}^{x_i} \left(cu^I\varphi_i\right)(x)dx - \frac{h_i}{2}c_iu_i\right\}\chi_i \\ &+ \sum_{i=1}^{N-1} \left\{\int_{x_i}^{x_{i+1}} \left(cu^I\varphi_i\right)(x)dx - \frac{h_{i+1}}{2}c_iu_i\right\}\chi_i. \end{aligned} \tag{3.32}$$

Clearly

$$s_i^- := \int_{x_{i-1}}^{x_i} \left(cu^I\varphi_i\right)(x)dx - \frac{h_i}{2}c_iu_i = \int_{x_{i-1}}^{x_i} \left[\left(cu^I\right)(x) - c_iu_i\right]\varphi_i(x)dx$$

and

$$\left|\left(cu^I\right)(x) - c_iu_i\right| \le \int_x^{x_i} \left|\left(cu^I\right)'\right| ds \le C\int_{x_{i-1}}^{x_i} \left\{1 + \varepsilon^{-1}e^{-\beta s/\varepsilon}\right\} ds \le C\vartheta_1(\omega)$$

for $x \in [x_{i-1}, x_i]$. (The quantity $\vartheta_\kappa(\omega)$ has been introduced in Section 2.2.2.) Hence

$$s_i^- \le C\vartheta_1(\omega) h_i.$$

We obtain

$$\left| \sum_{i=1}^{N-1} s_i^- \chi_i \right| \le C\vartheta_1(\omega) \sum_{i=1}^{N-1} \hbar_i |\chi_i| \le C\vartheta_1(\omega) \|\chi\|_{0,\omega}.$$

For the second sum in (3.32) we have an identical bound. Thus

$$\left| r_\varrho(u^I, \chi)_i - r(u^I, \chi) \right| \le C\vartheta_1(\omega) \|\chi\|_{0,\omega}. \tag{3.33}$$

(iv) Finally consider the convection term. We have

$$\begin{aligned} &c_\varrho(u^I, \chi) - c(u^I, \chi) \\ &\quad = \sum_{i=1}^{N} \Bigg\{ \int_{x_{i-1}}^{x_i} \big(b (u^I)' \chi\big)(x)\, dx \\ &\qquad\qquad - \left[\varrho\left(-\frac{b_{i-1/2} h_i}{\varepsilon} \right) \chi_{i-1} + \varrho\left(\frac{b_{i-1/2} h_i}{\varepsilon} \right) \chi_i \right] b_{i-1/2} \big(u_i - u_{i-1}\big) \Bigg\} \end{aligned}$$

and

$$\begin{aligned} &\int_{x_{i-1}}^{x_i} \big(b (u^I)' \chi\big)(x)\, dx \\ &\quad = b_{i-1/2} \big(u_i - u_{i-1}\big) \frac{\chi_i + \chi_{i-1}}{2} + \int_{x_{i-1}}^{x_i} \left\{ \int_{x_{i-1/2}}^{x} b'(s)\, ds \frac{u_i - u_{i-1}}{h_i} \chi(x) \right\} dx. \end{aligned}$$

Combine these two equations and use (ϱ_1).

$$\begin{aligned} c_\varrho(u^I, \chi) - c(u^I, \chi) = &\sum_{i=1}^{N} \left[\frac{1}{2} - \varrho\left(\frac{b_{i-1/2} h_i}{\varepsilon} \right) \right] \big(\chi_i - \chi_{i-1}\big) \big(u_i - u_{i-1}\big) b_{i-1/2} \\ &+ \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left\{ \int_{x_{i-1/2}}^{x} b'(s)\, ds \frac{u_i - u_{i-1}}{h_i} \chi(x) \right\} dx. \end{aligned} \tag{3.34}$$

For the second sum use $|u_i - u_{i-1}| \le C\vartheta_1(\omega)$ in order to obtain

$$\left| \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} \left\{ \int_{x_{i-1/2}}^{x} b'(s) ds \frac{u_i - u_{i-1}}{h_i} \chi(x) \right\} dx \right| \le C\vartheta_1(\omega) \left\| \chi \right\|_{0,\omega}. \tag{3.35}$$

Next we bound the first sum in (3.34). For $i \le qN$ use $|u_i - u_{i-1}| \le C\vartheta_1(\omega)$ again to obtain

$$\begin{aligned} &\left| \sum_{i=1}^{qN} \left[\frac{1}{2} - \varrho\left(\frac{b_{i-1/2} h_i}{\varepsilon} \right) \right] (\chi_i - \chi_{i-1}) (u_i - u_{i-1}) \, b_{i-1/2} \right| \\ &\qquad \le C\vartheta_1(\omega) \sum_{i=1}^{qN} |\chi_i - \chi_{i-1}| \le C\vartheta_1(\omega) \varepsilon^{1/2} \ln^{1/2} N \left| \chi \right|_{1,\omega}, \end{aligned} \tag{3.36}$$

by a discrete Cauchy-Schwarz inequality.

For $i > qN$ we use the splitting $u = v + w$ of the exact solution. Starting with the layer term w, we have $w_i \le CN^{-2}$ for $i \ge qN$. Hence

$$\begin{aligned}\left|\sum_{i=qN+1}^{N}\left[\frac{1}{2}-\varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right](\chi_i-\chi_{i-1})(w_i-w_{i-1})\,b_{i-1/2}\right|\\ \le CN^{-2}\sum_{i=qN+1}^{N}(|\chi_i|+|\chi_{i-1}|)\le CN^{-1}\|\chi\|_{0,\omega},\end{aligned} \tag{3.37}$$

by a discrete Cauchy-Schwarz inequality and because $h_i = \mathcal{O}\left(N^{-1}\right)$ for $i > qN$. Finally, consider the regular solution component v. To simplify the notation let

$$\gamma_{i-1/2} := b_{i-1/2}\left[\frac{1}{2}-\varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right].$$

Summation by parts yields

$$\begin{aligned}\sum_{i=qN+1}^{N}\gamma_{i-1/2}\big(v_i-v_{i-1}\big)\big(\chi_i-\chi_{i-1}\big) &= \gamma_{qN+1/2}\,(v_{qN}-v_{qN+1})\,\chi_{qN}\\ &-\sum_{i=qN+1}^{N-1}\gamma_{i-1/2}\,(v_{i+1}-2v_i+v_{i-1})\,\chi_i+\sum_{i=qN+1}^{N-1}\big(\gamma_{i-1/2}-\gamma_{i+1/2}\big)\,(v_{i+1}-v_i)\,\chi_i.\end{aligned}$$

Taylor expansions for v give $|v_{i+1}-2v_i+v_{i-1}| \le CN^{-2}$ and $|v_i - v_{i-1}| \le CN^{-1}$, while (ϱ_0) implies $\left|\gamma_{i-1/2}-\gamma_{i+1/2}\right| \le CN^{-1}$. Thus

$$\begin{aligned}\left|\sum_{i=qN+1}^{N}\gamma_{i-1/2}\,(v_i-v_{i-1})\,(\chi_i-\chi_{i-1})\right|\\ \le CN^{-1}\left(\|\chi\|_{0,\omega}+|\chi_{qN}|\right)\le CN^{-1}\ln^{1/2}N\,|||\chi|||_{\varrho}\,,\end{aligned} \tag{3.38}$$

because

$$|\chi_{qN}| \le \sum_{i=1}^{qN}|\chi_i-\chi_{i-1}| \le C\ln^{1/2}N\varepsilon^{1/2}\,|\chi|_{1,\omega}\,.$$

Collecting (3.34)–(3.38), we get

$$\left|c_{\varrho}(u^I,\chi)-c(u^I,\chi)\right| \le C\vartheta_1(\omega)\ln^{1/2}N\,|||\chi|||_{\varrho}\,. \tag{3.39}$$

Now all terms on the right-hand side of (3.29) have been bounded; see (3.30), (3.31), (3.35) and (3.39). Dividing by $|||\chi|||_{\varrho}$ and recalling that $\vartheta_1(\omega) \le C\,(h+\max|\psi'|)$ for S-type meshes with $\sigma \ge 1$ and the interpolation error bounds of Section 3.3, we can state the main result of this section.

Theorem 3.15. *Let ω be an S-type mesh with $\sigma \ge 2$ whose mesh generating function $\tilde{\varphi}$ satisfies (1.11) and $\max|\psi'|\ln^{1/2}N \le CN$. Assume (ϱ_0), (ϱ_1) and (ϱ_2) hold. Then*

$$|||u-U|||_{\varepsilon}+\left|\left|\left|u^I-U\right|\right|\right|_{\varrho} \le C\left(h+N^{-1}\max|\psi'|\right)\ln^{1/2}N$$

for the error of the upwind FVM (3.22).

3.4.3 Convergence in the maximum norm

With the results of Section 3.4.1 at hand the simplest maximum-norm analysis is based on the $(\ell_\infty, w_{-1,\infty})$ stability. Setting

$$[A_\varrho v]_i := \varepsilon \left\{ 1 + \frac{b_{i-1/2}h_i}{\varepsilon} \left[\varrho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right]\right\} \frac{v_i - v_{i-1}}{h_i} + b_{i-1/2}\frac{v_i + v_{i-1}}{2}$$
$$+ \sum_{j=i}^{N-1} \left(\hbar_j c_j + b_{j+1/2} - b_{j-1/2}\right) v_j,$$

we have, if condition (ϱ_1) is satisfied,

$$[L_\varrho v]_i = -\frac{[A_\varrho v]_{i+1} - [A_\varrho v]_i}{\hbar_i}$$

and Theorem 3.14 yields

$$\|v\|_{\infty,\omega} \le \frac{4}{\beta} \min_{a\in I\!R} \|A_\varrho v + a\|_{\infty,\omega} \quad \text{for all } v \in I\!R_0^{N+1}.$$

Integration of the differential equation (3.19) yields

$$\varepsilon u'_{i-1/2} + (bu)_{i-1/2} + \int_{x_{i-1/2}}^{x_{N-1/2}} \left((c+b')\, u - f\right)(s) ds \equiv \alpha \quad \text{for all} \quad i = 1,\dots,N.$$

Thus

$$\begin{aligned} \|u - U\|_{\infty,\omega} \le \frac{4}{\beta} \max_{i=1,\dots,N} \Bigg| & \varepsilon\left(\frac{u_i - u_{i-1}}{h_i} - u'_{i-1/2}\right) + b_{i-1/2}\left(\frac{u_i + u_{i-1}}{2} - u_{i-1/2}\right) \\ & + \sum_{j=i}^{N-1} \left(\hbar_j c_j + b_{j+1/2} - b_{j-1/2}\right) u_j - \int_{x_{i-1/2}}^{x_{N-1/2}} \left(c + b'\right)(s) u(s) ds \\ & - \sum_{j=i}^{N-1} \hbar_j f_j + \int_{x_{i-1/2}}^{x_{N-1/2}} f(s) ds \\ & + b_{i-1/2}\left[\varrho\left(-\frac{b_{i-1/2}h_i}{\varepsilon}\right) - \varrho\left(\frac{b_{i-1/2}h_i}{\varepsilon}\right)\right](u_i - u_{i-1})\Bigg|. \end{aligned} \tag{3.40}$$

All terms except for the last one can be bounded by $\vartheta_1(\omega)$ using the technique from Section 2.2.2. For the last term note that $\varrho(t) \in [0,1]$ and

$$|u_i - u_{i-1}| \le \int_{x_{i-1}}^{x_i} |u'(s)|\, ds \le C\vartheta_1(\omega).$$

Theorem 3.16. *Suppose (ϱ_1) and (ϱ_3) hold. Then the error of the upwind FVM (3.22) satisfies*

$$\|u - U\|_{\infty,\omega} \le C\vartheta_1(\omega).$$

It was mentioned earlier that the accuracy of the scheme is improved when the function ϱ is Lipschitz continuous in a neighbourhood of $t = 0$, say on an interval $[-m, m]$. We will briefly illustrate this using a standard Shishkin mesh with mesh parameter $\sigma \ge 2$. For this recall the decomposition $u = v + w$ of the exact solution into a regular solution component v and the layer term w, see Theorem 2.6, and split the numerical solution $U = V + W$, where

$$[L_\varrho V]_i = f_i \quad \text{for } i = 1,\dots,N-1, \quad V_0 = v(0),\ V_N = v(1)$$

and

$$[L_\varrho W]_i = 0 \quad \text{for } i = 1, \dots, N-1, \quad W_0 = w(0),\ W_N = w(1)$$

Imitating the argument that led to (3.40) with u and U replaced by v and V, we obtain

$$\|v - V\|_{\infty,\omega} \le CN^{-1}.$$

When bounding the error in the layer term, use the barrier function technique of Section 2.2.6 to establish—similar to (2.62)—

$$|w_i - W_i| \le |w_i| + |W_i| \le CN^{-1} \quad \text{for} \quad i = qN, \dots, N.$$

Inside the layer region we employ the $(\ell_\infty, w_{-1,\infty})$ stability to establish analogously to (3.40)

$$\begin{aligned} \|w - W\|_{\infty,\omega\cap[0,\lambda]} \le \frac{4}{\beta} \max_{i=1,\dots,qN} \Bigg| & \varepsilon \left(\frac{w_i - w_{i-1}}{\hbar} - w'_{i-1/2} \right) + b_{i-1/2} \left(\frac{w_i + w_{i-1}}{2} - w_{i-1/2} \right) \\ & + \sum_{j=i}^{qN-1} \left(\hbar c_j + b_{j+1/2} - b_{j-1/2} \right) w_j - \int_{x_{i-1/2}}^{x_{qN-1/2}} (c + b')(s) w(s) ds \\ & + b_{i-1/2} \left[\varrho \left(-\frac{b_{i-1/2}\hbar}{\varepsilon} \right) - \varrho \left(\frac{b_{i-1/2}\hbar}{\varepsilon} \right) \right] (w_i - w_{i-1}) \Bigg| + CN^{-1}, \end{aligned}$$

where

$$\hbar := \frac{\sigma\varepsilon}{\beta q N} \ln N$$

The last term in the error bound arises from the error at x_{qN}. This time we use arguments from Section 2.2.10 in order to bound the first four terms by $\vartheta_2(\omega)^2$. The Lipschitz continuity of ϱ on $[-m, m]$ yields

$$\left| \varrho \left(-\frac{b_{i-1/2}\hbar}{\varepsilon} \right) - \varrho \left(\frac{b_{i-1/2}\hbar}{\varepsilon} \right) \right] \le CN^{-1} \ln N \quad \text{for } i = 1, \dots, qN$$

if N is sufficiently large. Furthermore

$$|w_i - w_{i-1}| \le \int_{x_{i-1}}^{x_i} |w'(s)|\, ds \le C\vartheta_1(\omega).$$

Hence

$$\|w - W\|_{\infty,\omega} \le C \left\{ N^{-1} + \vartheta_2(\omega)^2 + N^{-1} \ln N\, \vartheta_1(\omega) \right\} \le CN^{-1}.$$

Finally we obtain

$$\|u - U\|_{\infty,\omega} \le CN^{-1}.$$

Thus on a standard Shishkin mesh the use of a Lipschitz continous function ϱ improves the accuracy from $N^{-1} \ln N$ to N^{-1}.

3.4.4 A numerical example

Table 3.4 displays the results of test computations using the upwind FVM with various stabilizing functions ϱ when applied to the test problem (2.28) and contains the maximum nodal errors. For our tests we have chosen a standard Shishkin mesh with $\sigma = 1$ and $q = 1/2$. The results of the numerical tests are in agreement with our theoretical findings of the previous section. Comparing the numbers for $\varrho_{U,0}$ with those for other ϱ's, we clearly see an improvement in the accuracy when ϱ is Lipschitz continuous in a neighbourhood of $t = 0$. Also notice there is no (visible) difference in using either of those Lipschitz continuous ϱ's.

	$\varrho_{U,0}$		$\varrho_{U,10}$		ϱ_S		ϱ_I	
N	error	rate	error	rate	error	rate	error	rate
2^7	4.236e-3	0.84	3.855e-3	0.99	3.855e-3	0.99	3.855e-3	0.99
2^8	2.364e-3	0.86	1.942e-3	0.99	1.942e-3	0.99	1.942e-3	0.99
2^9	1.303e-3	0.87	9.745e-4	1.00	9.745e-4	1.00	9.745e-4	1.00
2^{10}	7.111e-4	0.89	4.882e-4	1.00	4.882e-4	1.00	4.882e-4	1.00
2^{11}	3.850e-4	0.89	2.443e-4	1.00	2.443e-4	1.00	2.443e-4	1.00
2^{12}	2.071e-4	0.90	1.222e-4	1.00	1.222e-4	1.00	1.222e-4	1.00
2^{13}	1.108e-4	0.91	6.115e-5	1.00	6.115e-5	1.00	6.115e-5	1.00
2^{14}	5.904e-5	0.91	3.059e-5	1.00	3.059e-5	1.00	3.059e-5	1.00
2^{15}	3.133e-5	0.92	1.531e-5	1.00	1.531e-5	1.00	1.531e-5	1.00
2^{16}	1.658e-5	—	7.672e-6	—	7.672e-6	—	7.672e-6	—

Table 3.4: The upwind FVM on a standard Shishkin mesh

Chapter 4

Problems with turning point layers

Turning point layers are associated with zeros of the convection coefficient. Let us consider the convection-diffusion problem

$$-\varepsilon u'' - pbu' + c(\cdot, u) = 0 \quad \text{in} \quad (q, 1), \quad u(q) = \gamma_q, \; u(1) = \gamma_1$$

with $q \in \{-1, 0\}$. We assume that $b(x) \geq \beta > 0$, $c_u \geq 0$ and $\operatorname{sign} p(x) = \operatorname{sign} x$ for $x \in (q, 1)$. The assumption on p implies that the point $x = 0$ is a turning point. If $q = 0$ then the turning point coincides with a boundary; we call this a boundary turning point problem. When $q = -1$ we have an interior layer.

In a couple of papers in the 1980s turning point problems with $c_u(0, \cdot) > 0$ were considered. For interior turning point problems this additional assumption implies that the solution of the reduced problem is continuous and therefore no strong layer is present. This means the problem is not singularly perturbed in the maximum norm. For boundary turning points the situation is different since the solution of the reduced problem will in general not match the boundary condition prescribed at the outflow boundary. However if $c_u(0, \dots) > 0$ then the dominating feature of the problem is the relation between the diffusion and reaction terms and the problem has the character of a reaction-diffusion problem which are not the subject of this book. Consequently we restrict ourselves here to the case $c_u(0, \cdot) = 0$.

In particular we consider the semilinear convection-diffusion problem

$$\mathcal{T} u(x) := -\varepsilon u''(x) - x^\kappa b(x) u'(x) + x^\kappa c(x, u(x)) = 0 \quad \text{for} \quad x \in (0, 1), \tag{4.1a}$$

$$u(0) = \gamma_0, \; u(1) = \gamma_1, \tag{4.1b}$$

where $\kappa > 0$, $b \geq \beta > 0$, $c_u \geq 0$ for $x \in [0, 1]$, $b \in C^1[0, 1]$ and $c \in C^1([0, 1] \times \mathbb{R})$.

We are aware of four publications analysing numerical methods for this problem with $\kappa = 1$. Liseikin [67] constructs a special transformation and solves the transformed problem on a uniform mesh. The method obtained is proven to be first-order uniformly convergent in the discrete maximum norm. Vulanović [106] studies an upwind-difference scheme on a layer-adapted Bakhvalov-type mesh and proves convergence in a discrete ℓ_1 norm. This result is generalized in [107] for quasilinear problems. In [66] the authors establish almost first-order convergence for an upwind-difference scheme on a Shishkin mesh. Here we follow [58] and study (4.1) with arbitrary $\kappa > 0$.

4.1 Derivative bounds and solution decomposition

We follow [58] to derive a decomposition of u into a regular part v and a layer part w for general $\kappa > 0$. This is later used in our analysis of a simple-upwind scheme for (4.1) in Section 4.2.2.

The construction of the decomposition is similar to the one in Section 2.1.2. Let v and w be the solutions of the boundary-value problems

$$\mathcal{T}v = 0 \quad \text{for} \quad x \in (0,1), \quad \mathcal{B}_0 := -b(0)v'(0) + c(0, v(0)) = 0, \; v(1) = \gamma_1 \tag{4.2a}$$

and

$$\tilde{\mathcal{T}}w := -\varepsilon w'' - x^p b w' + x^p \tilde{c}(x,w) = 0 \quad \text{for} \quad x \in (0,1), \quad w(0) = \gamma_0 - v(0), \; w(1) = 0. \tag{4.2b}$$

where $\tilde{c}(x,w) := c(x, v+w) - c(x,v)$.

Preliminaries. Before starting the main argument, let us provide some auxiliary results. Let

$$B(x) := \frac{1}{\varepsilon}\int_0^x s^p b(s) ds$$

and let β^* with $b(x) \geq \beta^* > 0$ be arbitrary. For our analysis we need bounds for a number of integral expressions involving B. First of all we have

$$B(s) - B(x) \leq \frac{\beta^*}{\varepsilon}\frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa+1} \quad \text{for} \quad 0 \leq s \leq x \leq 1. \tag{4.3}$$

From this, for arbitrary $\nu \geq 0$ we get

$$\frac{\beta^*}{\varepsilon}\int_0^x s^{(\kappa+\nu)} \exp(B(s) - B(x)) ds \leq \frac{\beta^*}{\varepsilon}\int_0^x s^\kappa \exp\left(\frac{\beta^*}{\varepsilon}\frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa+1}\right) ds \leq 1. \tag{4.4}$$

With $\mu := \varepsilon^{1/(\kappa+1)}$ we shall also use

$$\begin{aligned}\int_0^1 \exp(-B(s)) ds &\geq \int_0^1 \exp\left(-\frac{\|b\|_\infty s^{\kappa+1}}{(\kappa+1)\varepsilon}\right) ds = \mu\int_0^{1/\mu} \exp\left(-\frac{\|b\|_\infty t^{\kappa+1}}{(\kappa+1)}\right) dt \\ &\geq \mu\int_0^1 \exp\left(-\frac{\|a\|_\infty t^{\kappa+1}}{(\kappa+1)}\right) dt = C\mu.\end{aligned} \tag{4.5}$$

Lemma 4.1. *For arbitrary $\kappa > 0$ there exists a constant $C = C(\kappa)$ such that*

$$\frac{x^\kappa}{\varepsilon}\int_0^x \exp\left(\frac{\beta^*}{\varepsilon}\frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa+1}\right) ds \leq C \quad \textit{for all} \quad x \geq 0, \; \varepsilon > 0.$$

Proof. Using the transformations

$$x = \left(\varepsilon(\kappa+1)t/\beta^*\right)^{1/(\kappa+1)} \quad \text{and} \quad s = \left(\varepsilon(\kappa+1)\sigma/\beta^*\right)^{1/(\kappa+1)},$$

we see that

$$\frac{\beta^* x^\kappa}{\varepsilon}\int_0^x \exp\left(\frac{\beta^*}{\varepsilon}\frac{s^{\kappa+1} - x^{\kappa+1}}{\kappa+1}\right) ds = e^{-t} t^{\kappa/(\kappa+1)} \int_0^t e^\sigma \sigma^{-\kappa/(\kappa+1)} d\sigma := F_\kappa(t).$$

Clearly $F_\kappa \in C^0[0,\infty)$ and $F_\kappa(0) = 0$ for $\kappa > 0$. On the other hand we have $\lim_{t\to\infty} F_\kappa(t) = 1$. Thus there exists a constant $C(\kappa) > 0$ with $\max_{t\in[0,\infty)} F_\kappa(t) \leq C(\kappa)$. □

The regular solution component. The operator $\mathcal{T}$ satisfies certain comparison principles [80] which ensures the existence of a unique solution: If two functions $\check{u}$ and $\hat{u}$ satisfy $\mathcal{T}\check{u}(x) \leq \mathcal{T}\hat{u}(x)$ in $(0,1)$, $\mathcal{B}_0\check{u} \leq \mathcal{B}_0\hat{u}$ and $\check{u}(1) \leq \hat{u}(1)$, then $\check{u}(x) \leq \hat{u}(x)$ on $[0,1]$. Using this comparison principle with

$$v^\pm = \pm\left(\frac{1-x}{\beta}\max_x |c(x,0)| + \gamma_1\right),$$

we get

$$\left|v(x)\right| \le C \quad \text{for} \quad x \in (0,1).$$

Now let us bound the derivatives of v. The function v can be written as

$$v(x) = \int_x^1 H_v(s)ds - \frac{c(0,v(0)}{b(0)} \int_x^1 \exp(-B(s))ds + \gamma_1,$$

where

$$B(x) := \frac{1}{\varepsilon}\int_0^x s^\kappa b(s)ds \quad \text{and} \quad H_v(x) = -\frac{1}{\varepsilon}\int_0^x s^\kappa c(s,v(s)) \exp(B(s)-B(x))ds.$$

From this representation we immediately get

$$v'(x) = \frac{1}{\varepsilon}\int_0^x s^\kappa c(s,v(s)) \exp(B(s)-B(x))ds + \frac{c(0,v(0))}{b(0)} \exp(-B(x)). \tag{4.6}$$

Hence

$$\left|v'(x)\right| \le C \quad \text{for} \quad x \in (0,1),$$

because of (4.4).

Differentiating (4.6) once and using integration by parts, we get

$$v''(x) = \frac{x^\kappa b(x)}{\varepsilon}\int_0^x \left(\frac{c(\cdot,v)}{b}\right)'(s) \exp(B(s)-B(x))ds.$$

Therefore

$$\left|v''(x)\right| \le C\frac{x^\kappa}{\varepsilon}\int_0^x \exp(B(s)-B(x))ds \le C\frac{x^\kappa}{\varepsilon}\int_0^x \exp\left(\frac{\beta}{\varepsilon}\frac{s^{\kappa+1}-x^{\kappa+1}}{\kappa+1}\right)ds$$

and

$$\left|v''(x)\right| \le C \quad \text{for} \quad x \in (0,1)$$

by Lemma 4.1.

A bound for the third-order derivative is obtained from the differential equation and the bounds on v' and v'':

$$-\varepsilon v''' = x^\kappa \big(bv - c(\cdot,v)\big)' + \kappa x^{\kappa-1}\big(bv' - c(\cdot,v)\big).$$

Let $F(x) := bv' - c(\cdot,v)$. Eq. (2.10a) implies $F(0) = 0$. On the other hand we have

$$|F'(x)| = \left|\big(bv' - c(\cdot,v)\big)'(x)\right| \le C,$$

by our earlier bounds for v, v' and v''. Thus $\left|F(x)\right| \le Cx$. We get

$$\varepsilon\left|v'''(x)\right| \le Cx^\kappa \quad \text{for} \quad x \in (0,1).$$

This completes our analysis of the regular part of u.

The boundary layer component. Let β_i be arbitrary but fixed constants with

$$\min_{x\in[0,1]} b(x) = \beta_1 > \beta_2 > \beta_3 > \tilde{\beta} > 0.$$

Recall that the layer component solves

$$\tilde{\mathcal{T}} w(x) = 0 \quad \text{for} \quad x \in (0,1), \quad w(0) = \gamma_0 - v(0), \;\; w(1) = 0.$$

The operator $\tilde{\mathcal{T}}$ with Dirichlet boundary conditions also satisfies a comparison principle [80]: If two functions $\check{u}$ and $\hat{u}$ satisfy $\tilde{\mathcal{T}}\check{u}(x) \le \tilde{\mathcal{T}}\hat{u}(x)$ in $(0,1)$ and $\check{u}(x) \le \hat{u}(x)$ for $x = 0, 1$, then $\check{u}(x) \le \hat{u}(x)$ on $[0,1]$. This comparison principle guarantees the existence of a unique solution. Using the barrier functions

$$w^{\pm} = \pm|\gamma_0 - v(0)| \exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{p+1}}{p+1}\right),$$

we obtain

$$\left|w(x)\right| \le C \exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right) \quad \text{for} \quad x \in (0,1). \tag{4.7}$$

To bound the derivatives of w we use

$$w(x) = \int_x^1 H_w(s)ds - \frac{v(0) - \gamma_0 + \displaystyle\int_0^1 H_w(s)ds}{\displaystyle\int_0^1 \exp(-B(s))ds} \int_x^1 \exp(-B(s))ds,$$

where

$$H_w(x) = -\frac{1}{\varepsilon}\int_0^x s^{\kappa}\tilde{c}(s, w(s)) \exp(B(s) - B(x))ds.$$

Thus

$$w'(x) = -H_w(x) + \frac{v(0) - \gamma_0 + \displaystyle\int_0^1 H_w(s)ds}{\displaystyle\int_0^1 \exp(-B(s))ds} \exp(-B(x)). \tag{4.8}$$

We have

$$\left|\tilde{b}(s, w(s))\right| = \left|c(s, v(s) + w(s)) - c(s, v(s))\right| \le C|w(s)| \le C \exp\left(-\frac{\beta_1}{\varepsilon}\frac{s^{\kappa+1}}{\kappa+1}\right),$$

by (4.7). Using this bound and (4.3) with $\beta^* = \beta_1$, we obtain

$$\left|H_w(x)\right| \le C\frac{x^{\kappa+1}}{\varepsilon} \exp\left(-\frac{\beta_1}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right) \le C \exp\left(-\frac{\beta_2}{\varepsilon}\frac{x^{\kappa+1}}{\kappa+1}\right) \quad \text{for} \quad x \in (0,1). \tag{4.9}$$

From (4.4), (4.5), (4.8) and (4.9) we get

$$\left|w'(x)\right| \le C\mu^{-1} \exp\left(-\frac{\beta_2 x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for} \quad x \in (0,1).$$

Use the differential equation, the estimates for w and w' and Lemma 4.1 to get

$$\left|w''(x)\right| \le C\mu^{-2} \exp\left(-\frac{\beta_3 x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for} \quad x \in (0,1).$$

Differentiate (2.10b), apply the bounds for w, w' and w'' and use Lemma 4.1 again in order to get

$$\left|w'''(x)\right| \le C\mu^{-3} \exp\left(-\frac{\tilde{\beta}x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \text{for} \quad x \in (0,1).$$

The following theorem summarizes the results of our analysis.

Theorem 4.2. *Let $b \in C^1[0,1]$ and $c \in C^1([0,1] \times I\!R)$. Assume $b > \tilde{\beta} > 0$ on $[0,1]$ and $c_u \ge 0$ on $[0,1] \times I\!R$. Then (4.1) has a unique solution $u \in C^3[0,1]$ and this solution can be decomposed as $u = v + w$, where the regular solution component v satisfies*

$$\mathcal{T}v = 0, \quad \left|v'(x)\right| + \left|v''(x)\right| \le C \quad \textit{and} \quad \varepsilon\left|v'''(x)\right| \le Cx^\kappa \quad \textit{for} \quad x \in (0,1),$$

while the boundary layer component w satisfies

$$\tilde{\mathcal{T}}w := -\varepsilon w'' - x^\kappa b w' + x^\kappa \tilde{c}(x,w) = 0, \quad \tilde{c}(x,w) = c(x,v+w) - c(x,v)$$

and

$$\left|w^{(i)}(x)\right| \le C\mu^{-i} \exp\left(-\frac{\tilde{\beta}x^{\kappa+1}}{\varepsilon(\kappa+1)}\right) \quad \textit{for} \quad i = 0,1,2,3, \quad x \in (0,1)$$

with $\mu = \varepsilon^{1/(\kappa+1)}$.

4.2 A first-order upwind scheme

Consider the semilinear problem

$$-\varepsilon u''(x) - p(x)b(x)u'(x) + c(x,u(x)) = 0 \;\text{ in }\; (0,1), \quad u(0) = \gamma_0, \; u(1) = \gamma_1,$$

where $c_u(x,u) \ge 0$,

$$p(x) > 0 \;\text{ is monotonically increasing and }\; b(x) \ge \beta > 0 \;\text{ on }\; (0,1). \tag{4.10}$$

Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh with local mesh size $h_i := x_i - x_{i-1}$ and maximal mesh size $h := \max_i h_i$. The boundary-value problem is discretized using simple upwinding: Find $u \in I\!R^{N+1}$ such that

$$[TU]_i := -\varepsilon U_{\bar{x}x;i} - p_i b_i U_{x;i} + c(x_i,U_i) = 0 \;\text{ for }\; i = 1,\dots,N-1, \quad U_0 = \gamma_0, \; U_N = \gamma_1. \tag{4.11}$$

4.2.1 Stability of the discretization

For the later convergence analysis of the scheme we require a stability estimate of the discrete operator T. This will be derived now. The argument follows [66].

First, let us consider the linear operator L defined by

$$[Lv]_i := -\varepsilon v_{\bar{x}x;i} - p_i b_i v_{x;i} + \bar{c}_i v_i,$$

where p and b satisfy (4.10) and $\bar{c} \ge 0$ on $(0,1)$.

Given an arbitrary mesh function $v \in I\!R_0^{N+1}$ we have

$$v_i = \sum_{j=1}^{N-1} h_{j+1} G_{ij} \, [Lv]_j \quad \text{for} \quad i = 1,\dots,N-1, \tag{4.12}$$

where $G : \omega \times \omega \to I\!R$, $G_{ij} = G(x_i,\xi_j)$, is the Green's function associated with the discrete operator L. For fixed $\xi_j \in \omega$ it solves

$$-\varepsilon G_{\bar{x}x;ij} - (bG)_{x;ij} + c_i G_{ij} = \delta_{ij} \;\text{ for }\; i = 1,\dots,N-1, \quad G_{0j} = G_{Nj} = 0$$

with the discrete δ function

$$\delta_{ij} := \begin{cases} h_{j+1}^{-1} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The operator L satisfies a discrete comparison principle since the matrix associated with it is an M-matrix. This is easily verified using the M-matrix criterion with the test function $v_i = 1 - x_i$.

We construct a barrier function for G now. Let $\beta_i = \beta p_i$,

$$R_{ij} := \begin{cases} 1 & \text{for} \quad i = j+1, \\ \displaystyle\prod_{\mu=j+1}^{i-1} \left(1 + \frac{\beta_\mu h_{\mu+1}}{\varepsilon}\right)^{-1} & \text{for} \quad i = j+2, \dots, N, \end{cases}$$

$$Q_{ij} := \begin{cases} 0 & \text{for} \quad i = 0, \dots, j, \\ \displaystyle\frac{1}{\varepsilon + \beta_j h_{j+1}} \sum_{\nu=j+1}^{i} h_\nu R_{\nu j} & \text{for} \quad i = j+1, \dots, N, \end{cases}$$

and

$$B_{ij} := \begin{cases} Q_{Nj} & \text{for } i = 0, \dots, j, \\ Q_{Nj} - Q_{ij} & \text{for } i = j+1, \dots, N. \end{cases}$$

Clearly, B_{ij} satisfies

$$0 \le B_{ij} \le Q_{Nj} \quad \text{for} \quad i = 0, \dots, N, \tag{4.13}$$

since Q_{ij} monotonically increases with i.

Now we shall show that B is a barrier function for G. We have

$$B_{\bar{x};ij} = \begin{cases} 0 & \text{for} \quad i = 1, \dots, j, \\ -\displaystyle\frac{R_{ij}}{\varepsilon + \beta_j h_{j+1}} & \text{for} \quad i = j+1, \dots, N. \end{cases}$$

Thus

$$\begin{aligned} \left[LB_{\cdot j}\right]_i &= \bar{c}_i B_{ij} \ge 0 \quad \text{for } i = 1, \dots, j-1, \\ \left[LB_{\cdot j}\right]_j &= -\frac{\varepsilon + b_j p_j h_{j+1}}{h_{j+1}} B_{x;jj} + \bar{c}_j B_j^j \ge \frac{1}{h_{j+1}}, \end{aligned}$$

and

$$\begin{aligned} \left[LB_{\cdot j}\right]_i &= -\frac{\varepsilon + b_i p_i h_{i+1}}{h_{i+1}} B_{x;ij} + \frac{\varepsilon}{h_{i+1}} B_{\bar{x};ij} + \bar{c}_i B_{ij} \\ &\ge \frac{\left(\varepsilon + b_i p_i h_{i+1}\right) R_{i+1,j} - \varepsilon R_{ij}}{h_{i+1}\left(\varepsilon + \beta_j h_{j+1}\right)} \ge 0 \quad \text{for } i = j+1, \dots, N-1, \end{aligned}$$

because $\varepsilon R_{ij} = \left(\varepsilon + \beta_i h_{i+1}\right) R_{i+1,j}$. Hence

$$\left[LB_{\cdot j}\right]_i \ge \delta_{ij} h_{i+1}^{-1} \quad \text{for} \quad i = 1, \dots, N-1, \quad B_{0j} \ge 0 \quad \text{and} \quad B_{Nj} \ge 0.$$

Since L satisfies the discrete comparison principle, from (4.13), we get

$$0 \le G_{ij} \le B_{ij} \le Q_{Nj} \quad \text{for} \quad i, j = 1, \dots, N-1. \tag{4.14}$$

Next we show that

$$Q_{Nj} \le \frac{1}{\beta_j} \quad \text{for} \quad j = 1, \dots, N-1. \tag{4.15}$$

From the definition of Q we have

$$Q_{NN} = 0 \quad \text{and} \quad Q_{N,j-1} = \frac{1}{\beta_{j-1}} + \frac{\varepsilon}{\beta_{j-1}} \frac{\beta_{j-1} Q_{N,j} - 1}{\varepsilon + \beta_{j-1} h_j}.$$

Induction for $j = N, N-1, \dots, 2$ yields (4.15) because of the monotonicity of p.

Finally, combine (4.12), (4.14) and (4.15) in order to get

$$\|v\|_{\infty,\omega} \le \frac{1}{\beta} \sum_{j=1}^{N-1} \frac{h_{j+1}}{p_j} \left| [Lv]_j \right| = \frac{1}{\beta} \left\| \frac{Lv}{p} \right\|_{1,\omega} \quad \text{for all} \quad v \in I\!R_0^{N+1}. \tag{4.16}$$

Note, that for $p \equiv 1$ we recover the stability result (2.22b) from Section 2.2.1.

Next, our result for the linear operator L is used to derive a stability inequality for the nonlinear operator T from (4.11).

Theorem 4.3. *Assume that b and p satisfy (4.10) and that $c_u \ge 0$. Then*

$$\|v - w\|_{\infty,\omega} \le \frac{1}{\beta} \left\| \frac{Tv - Tw}{p} \right\|_{1,\omega} \tag{4.17}$$

for all $v, w \in I\!R^{N+1}$ with $v_0 = w_0$ and $v_N = w_N$.

Proof. Let v and w be the two mesh functions for which we want to prove (4.17). Following the usual practice, we define the discrete linear operator

$$[Ly]_i := -\varepsilon y_{\bar{x}x;i} - p_i b_i y_{x;i} + \bar{c}_i y_i, \quad y_0 = y_N = 0,$$

where

$$\bar{c}_i = \int_0^1 c_u\big(x_i, w_i + s(v_i - w_i)\big)\, ds \ge 0.$$

The operators L and T are related by $L\,(v - w) = Tv - Tw$. Since $v - w \in I\!R_0^{N+1}$ and L satisfies the necessary assumptions, we can apply (4.16) to complete the proof. $\square$

Remark 4.4. *An immediate consequence of Theorem 4.3 for the simple upwind scheme is*

$$\|u - U\|_{\infty,\omega} \le \frac{1}{\beta} \left\| \frac{Tu}{p} \right\|_{1,\omega}.$$

Thus the error of the numerical solution in the maximum norm is bounded by an ℓ_1-type norm of the truncation error weighted with the inverse of the coefficient of the convection term. This was used in [66] to establish uniform almost first-order convergence on Shishkin meshes for $\kappa = 1$.

4.2.2 Convergence on Shishkin meshes

Now, let us study the accuracy of the upwind scheme (4.11) applied to (4.1) with arbitrary $\kappa > 0$. Bounds for the derivatives of u are provided by Theorem 4.2. The transition point λ in our mesh is chosen such that the layer term w is of order $N^{-\sigma}$ on $[\lambda, 1]$. Hence we choose

$$\lambda = \min \left\{ q, \left(\sigma \frac{\varepsilon(\kappa + 1)}{\tilde{\beta}} \right)^{1/(\kappa+1)} \right\}. \tag{4.18}$$

Assuming that $J = qN$ is an integer, we subdivide the interval $(0, q)$ into qN equidistant subintervals and $(q, 1)$ into $(1-q)N$ ones. For simplicity we assume that $x_J = \lambda \le q$, since the scheme can be analyzed in a classical manner otherwise. We denote by

$$h = \frac{\lambda}{J} \quad \text{and} \quad H = \frac{1-\lambda}{N-J} \le \frac{1}{(1-q)N}$$

the local mesh sizes on the fine and coarse parts of the mesh.

It follows from Section 4.2.1 that the error of the simple upwind scheme (4.11) applied to (4.1) satisfies

$$\|u - U\|_{\infty,\omega} \le \sum_{j=1}^{N-1} h_{j+1} Q_{Nj} \big|[Tu]_j\big| \le \sum_{j=1}^{N-1} \frac{h_{j+1}}{\beta x_j^\kappa} \big|[Tu]_j\big|. \tag{4.19}$$

However for our analysis we need a sharper bound on Q_j for $j = 1, \dots, J-1$. From the definition of Q we have

$$Q_{N,j} = \frac{\varepsilon}{\varepsilon + \beta_{j-1} h_j}\left(Q_{N,j} + \frac{h_j}{\varepsilon}\right) \le Q_{N,j} + \frac{h_j}{\varepsilon} = Q_{N,j} + \frac{\lambda}{\varepsilon J}, \quad j = 1, \dots, J.$$

Thus

$$Q_{N,j} \le Q_J + \frac{(J-j)\lambda}{\varepsilon J} \le \frac{1}{\beta\lambda^\kappa}\frac{\lambda}{\varepsilon} \le \left(1 + \frac{1}{\sigma(\kappa+1)}\right)\frac{\lambda}{\varepsilon} \le C\frac{\lambda}{\varepsilon} \quad \text{for} \quad j = 1, \dots, J, \tag{4.20}$$

by (4.15).

Theorem 4.5. *Let u be the solution of (4.1) and U that of (4.11) on the Shishkin mesh defined by (4.18). Then*

$$\|u - U\|_{\infty,\omega} \le CN^{-1}\ln^{2/(\kappa+1)} N \quad \textit{if} \;\; \sigma \ge 2.$$

Proof. The solution decomposition $u = v + w$ of Theorem 4.2 gives

$$\big[Tu\big] = \tau^v + \tau^w \quad \text{with} \quad \tau_i^g := \varepsilon(g_i'' - g_{\bar{x}x,i}) + x_i^\kappa b_i(g_i' - g_{x,i}).$$

Thus

$$\|u - U\|_{\infty,\omega} \le \sum_{j=1}^{N-1} h_{j+1} Q_{Nj}\big|[Tu]_j\big| \le \sum_{j=1}^{N-1} h_{j+1} Q_{Nj}\big|\tau_j^v\big| + \sum_{j=1}^{N-1} h_{j+1} Q_{Nj}\big|\tau_j^w\big|, \tag{4.21}$$

by (4.19). The two error contributions from the regular solution component and from the layer are analysized separately.

Regular solution component. When studying $|\tau_j^v|$ we shall distinguish three cases: $j < J$, $j = J$ and $j > J$. Taylor expansions for the truncation error give

$$\big|\tau_j^g\big| \le C(h_j + h_{j+1}\left(\max_{[x_{j-1},x_{j+1}]} |\varepsilon g'''| + x_j^\kappa \max_{[x_j,x_{j+1}]} |g''|\right) \tag{4.22a}$$

and

$$\big|\tau_j^g\big| \le C\left(\max_{[x_{j-1},x_{j+1}]} |\varepsilon g''| + h_{j+1} x_j^\kappa \max_{[x_j,x_{j+1}]} |g''|\right). \tag{4.22b}$$

(i) For $j = 1, \dots, J-1$ (4.22a) and (4.15) give

$$h_{j+1} Q_{N,j}\left|\tau_j^v\right| \le Ch^2\left\{\left(\frac{x_{j+1}}{x_j}\right)^\kappa + 1\right\} \le Ch^2,$$

because $x_{j+1}/x_j = (j+1)/j \le 2$ for $j < J$.
(ii) For $j = J$ we have

$$h_{j+1} Q_{N,j} \left|\tau_j^v\right| \le C \left(\frac{H\varepsilon}{\lambda^\kappa} + H^2 \right) \le C \left(H\mu + H^2 \right) \le CH.$$

by (4.22b) and (4.15).
(iii) For $j = J+1, \dots, N-1$ (4.22a) (4.15) yield Thus

$$\frac{h_{j+1}}{x_j^p} \left|\tau_j^v\right| \le CH^2 \left\{ \left(\frac{\lambda + (j+1-J)H}{\lambda + (j-J)H} \right)^\kappa + 1 \right\} \le CH^2,$$

because $\lambda + (j+1-J)H \le 2\left(\lambda + (j-J)H\right)$.
Combining the last three estimates, we get

$$\sum_{j=1}^{N-1} h_{j+1} Q_{Nj} \left|\tau_j^v\right| \le CN^{-1}. \tag{4.23}$$

Layer component. We shall distinguish two cases: $j < J$ and $j \ge J$.
(i) For $j = 1, \dots, J-1$ we have by (4.22a)

$$\left|\tau_j^w\right| \le Ch \left(\max_{[x_{j-1},x_{j+1}]} |\varepsilon w'''| + x_j^\kappa \max_{[x_j,x_{j+1}]} |w''| \right) \le Ch\mu^{-2} \left(\varepsilon\mu^{-1} + x_j^\kappa \right) \exp\left(-\frac{\tilde\beta x_{j-1}^\kappa}{\varepsilon(\kappa+1)} \right),$$

by Theorem 4.2. This and (4.20) give

$$h_{j+1} Q_j \left|\tau_j^w\right| \le C \frac{\Lambda^3}{J^2} \exp\left(-\frac{\tilde\beta x_{j-1}^\kappa}{\varepsilon(\kappa+1)} \right).$$

where $\Lambda := (\ln N)^{1/(\kappa+1)}$. For any $m > 0$ there exists a constant $\bar C = \bar C(m)$ such that

$$\exp\left(-\frac{\tilde\beta x^\kappa}{\varepsilon(\kappa+1)} \right) \le \bar C \exp\left(-m\frac{x}{\mu} \right).$$

This yields

$$h_{j+1} Q_j \left|\tau_j^w\right| \le C \frac{\Lambda^3}{J^2} \exp\left(-\frac{\Lambda}{J} \right)^{j-1}.$$

Thus

$$\sum_{j=1}^{J-1} h_{j+1} Q_j \left|\tau_j^w\right| \le C \frac{\Lambda^3}{J^2} \frac{1}{1 - \exp\left(-\frac{\Lambda}{J}\right)} \le C \frac{\Lambda^2}{J} \le C \frac{\Lambda^2}{N}, \tag{4.24}$$

since $\lim_{z\to 0} z/\bigl(1-\exp(-z)\bigr) = 1$ and $\lim_{N\to\infty} \Lambda/J = 0$.
(ii) For $j = J, \dots, N-1$ we have

$$\left| \frac{w_j - w_{j-1}}{h_j} \right| \le \max_{[x_{j-1},x_j]} |w'(x)| \le C\mu^{-1} \exp\left(-\frac{\tilde\beta x_{J-1}^{\kappa+1}}{\varepsilon(\kappa+1)} \right)$$

and

$$|w(x)| \le C \exp\left(-\frac{\tilde\beta x_J^{\kappa+1}}{\varepsilon(\kappa+1)} \right),$$

by Theorem 4.2. Thus

$$
\begin{aligned}
h_{j+1}Q_j \left|\tau_j^w\right| \le \frac{h_{j+1}}{x_j^\kappa}\left|\tau_j^w\right| &\le C\left(\frac{\varepsilon}{\lambda^\kappa\mu}+1+H\right)\exp\left(-\frac{\tilde\beta x_{J-1}^{\kappa+1}}{\varepsilon(\kappa+1)}\right)\\
&\le C\exp\left(-\frac{\tilde\beta x_J^{\kappa+1}}{\varepsilon(\kappa+1)}\right)\exp\left(\frac{\tilde\beta\left(x_J^{\kappa+1}-x_{J-1}^{\kappa+1}\right)}{\varepsilon(\kappa+1)}\right)\\
&\le C\exp\left(-\frac{\tilde\beta x_J^{\kappa+1}}{\varepsilon(\kappa+1)}\right)\exp\left(\lambda_0(\kappa+1)\frac{\ln N}{J}\right) \le CN^{-\lambda_0},
\end{aligned}
$$

since $\ln N/J \le C$. We get

$$\sum_{j=1}^{J-1} h_{j+1}Q_j\left|\tau_j^w\right| \le CN^{-1}$$

This and (4.24) yield

$$\sum_{j=1}^{N-1} h_{j+1}Q_{Nj}\left|\tau_j^w\right| \le C\Lambda^2N^{-1}.$$

Finally combine the last estimate this with (4.21) and (4.23) in order to complete the proof. $\square$

4.2.3 A numerical example

We verify experimentally the convergence result of Theorem 4.5. Our test problem is

$$-\varepsilon u'' - x^\kappa(2-x)u' + x^\kappa e^u = 0 \quad \text{for} \quad x\in(0,1), \quad u(0)=u(1)=0.$$

The exact solution of this problem is not available. We therefore estimate the accuracy of the numerical solution by comparing it with the numerical solution on a higher order method: Richardson extrapolation. For our tests we take $\tilde\beta = 1$ and $q = 1/2$.

Indicating by U_ε^N that the numerical approximation depends on both N and ε, we estimate the uniform error by

$$\eta^N := \max_{\varepsilon=1,10^{-1},\ldots,10^{-12}}\left\|U_\varepsilon^N - \tilde U_\varepsilon^N\right\|_\infty,$$

where $\tilde U_\varepsilon^N$ is the approximate solution of the Richardson extrapolation method. The rates of convergence are computed using the standard formula $r^N = \log_2\left(\eta^N/\eta^{2N}\right)$.

4.3 Interior turning points

Let us now briefly discuss the case of interior turning points. For this purpose we consider the boundary-value problem

$$\mathcal{T}u(x) := -\varepsilon u''(x) - \operatorname{sign} x\cdot|x|^\kappa b(x)u'(x) + |x|^\kappa c(x,u(x)) = 0 \quad \text{for} \quad x\in(-1,1), \tag{4.25a}$$

$$u(-1)=\gamma_{-1},\ u(1)=\gamma_1. \tag{4.25b}$$

Again, we assume that $0<\varepsilon\ll 1$ is a small constant, $\kappa>0$, $b(x)\ge\beta>0$, $c_u\ge 0$ for $x\in[-1,1]$, $b\in C^1[0,1]$ and $c\in C^1([-1,1]\times I\!R)$. Because the convection coefficient changes sign at an interior point of the domain u has an interior layer.

The operator $\mathcal{T}$ enjoys a comparison principle which can be used to conclude $|u(x)|\le C$ for $x\in(-1,1)$. Then $u^+ := u|_{[0,1]}$ and $u^- := u|_{[-1,0]}$ solve

$$\mathcal{T}u^+ = 0 \ \text{ in } \ (0,1), \quad u^+(0)=u(0), \quad u^+(1)=\gamma_1$$

	$\kappa = 1/2$		$\kappa = 1$		$\kappa = 2$		$\kappa = 3$	
N	η^N	r^N	η^N	r^N	η^N	r^N	η^N	r^N
2^6	1.114e-2	0.85	9.899e-3	0.89	8.879e-3	0.93	8.465e-3	0.94
2^7	6.171e-3	0.88	5.335e-3	0.92	4.675e-3	0.95	4.411e-3	0.96
2^8	3.358e-3	0.90	2.829e-3	0.93	2.426e-3	0.96	2.270e-3	0.97
2^9	1.803e-3	0.91	1.484e-3	0.94	1.249e-3	0.96	1.160e-3	0.97
2^{10}	9.592e-4	0.92	7.737e-4	0.95	6.401e-4	0.97	5.899e-4	0.98
2^{11}	5.069e-4	0.93	4.014e-4	0.95	3.269e-4	0.97	2.993e-4	0.98
2^{12}	2.666e-4	0.93	2.075e-4	0.96	1.666e-4	0.98	1.516e-4	0.98
2^{13}	1.396e-4	0.94	1.070e-4	0.96	8.473e-5	0.98	7.669e-5	0.98
2^{14}	7.292e-5	0.94	5.506e-5	0.96	4.305e-5	0.98	3.876e-5	0.99
2^{15}	3.798e-5	0.94	2.828e-5	0.96	2.185e-5	0.98	1.958e-5	0.99
2^{16}	1.973e-5	—	1.451e-5	—	1.108e-5	—	9.881e-6	—

Table 4.1: Simple upwinding on Shishkin meshes for turning point problems

and

$$\mathcal{T}u^- = 0 \ \text{ in } \ (-1,0), \quad u^-(-1) = \gamma_{-1}, \quad u^-(0) = u(0).$$

Hence u^+ and u^- can be regarded as solutions of boundary-turning point problems of the type considered in Section 4.1. This gives us immediately bounds for the derivatives of u and a decomposition into regular and layer components.

The simple upwind scheme for (4.25) on the mesh $\omega : -1 = x_0 < x_1 < \cdots < x_N = 1$ is

$$\left[TU\right]_i = 0 \ \text{ for } \ i = 1, \ldots, N-1, \ U_0 = \gamma_{-1}, \ U_N = \gamma_1,$$

where

$$\left[TU\right]_i := \begin{cases} -\varepsilon U_{\bar{x}x,i} - x_i^\kappa b_i U_{x,i} + x_i^\kappa c(x_i, U_i) & \text{if } \ x_i \geq 0, \\ -\varepsilon U_{\bar{x}\bar{x},i} + |x_i|^\kappa b_i U_{\bar{x},i} + |x_i|^\kappa c(x_i, U_i) & \text{if } \ x_i < 0. \end{cases}$$

The technique from Section 4.2.1 can be used to prove that for any mesh functions v and w with $v_0 = w_0$ and $v_N = w_N$, one has

$$\|v - w\|_{\omega,\infty} \leq \sum_{j=1}^{N-1} \chi_j \tilde{Q}_j \left| \left[Tv - Tw\right]_j \right|, \quad \chi_j := \begin{cases} h_{j+1} & \text{if } x_j \geq 0, \\ h_j & \text{otherwise,} \end{cases}$$

with

$$\tilde{Q}_N = 0, \quad \tilde{Q}_{j-1} = \left(1 + \frac{b_j x_{j-1}^\kappa h_j}{\varepsilon}\right)^{-1} \left(\tilde{Q}_j + \frac{h_j}{\varepsilon}\right) \quad \text{for} \quad x_{j-1} \geq 0,$$

and

$$\tilde{Q}_0 = 0, \quad \tilde{Q}_j = \left(1 + \frac{b_j |x_j|^\kappa h_j}{\varepsilon}\right)^{-1} \left(\tilde{Q}_{j-1} + \frac{h_j}{\varepsilon}\right) \quad \text{for} \quad x_j < 0.$$

The convergence analysis then follows along the lines of Section 4.2.2.

Chapter 5

Two dimensional problems

We now consider the two-dimensional convection-diffusion problem

$$-\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \Gamma = \partial\Omega. \tag{5.1}$$

Its solution may typically exhibit three different types of layers: interior layers, parabolic boundary layers and regular boundary layers. Let us assume that Ω is a domain with a regular boundary that has a uniquely defined outward normal n almost everywhere. Then the boundary can be divided into three parts:

$$\begin{aligned}
\Gamma^- &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n < 0\right\} && \textit{inflow boundary,}\\
\Gamma^0 &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n = 0\right\} && \textit{characteristic boundary}\text{ and}\\
\Gamma^+ &:= \left\{\boldsymbol{x}\in\Gamma : \boldsymbol{b}^T n > 0\right\} && \textit{outflow boundary.}
\end{aligned}$$

With this notation the layers can be classified as follows.

Regular Boundary Layers occur at the outflow boundary Γ^+ and have a width of $\mathcal{O}\left(\varepsilon\ln(1/\varepsilon)\right)$. They are often also called *exponential boundary layers*.

Parabolic Boundary Layers occur at characteristic boundaries Γ^0 where the boundary is parallel to the characteristics of the vector field $\boldsymbol{b}$. They are therefore also called *characteristic boundary layers*. In the nondegenerate case, their width is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$.

Interior Layers arise, e. g., from discontinuities in the boundary data at the inflow boundary Γ^- and are propagated across the domain along the characteristics of the vector field $\boldsymbol{b}$. They are similar in nature to parabolic boundary layers and therefore also called *characteristic or parabolic interior layers*. Their thickness is $\mathcal{O}\left(\sqrt{\varepsilon}\ln(1/\varepsilon)\right)$.

We restrict ourselves to problems with regular layers.

5.1 Asymptotic expansion and solution decomposition

In this and the following sections we consider the model problem

$$\mathcal{L}u := -\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in} \quad \Omega = (0,1)^2, \quad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega, \tag{5.2}$$

i. e., (5.1) on the unit square with homogeneous Dirichlet boundary conditions. We assume that $(b_1, b_2) > (\beta_1, \beta_2) > 0$ on $\bar{\Omega}$ with constants β_1 and β_2. These assumptions on $\boldsymbol{b}$ imply that the solution has exponential layers along the sides $x = 0$ and $y = 0$.

The regularity of the solution of (5.2) was studied by Han and Kellogg [30]. Provided that $\boldsymbol{b}$ and c are sufficiently smooth they established that u lies in the Hölder space $C^{1,\alpha}(\bar{\Omega})$ iff $f \in C^{0,\alpha}(\bar{\Omega})$; if $f \in C^{k,\alpha}(\bar{\Omega})$ with $k \in \{0,1\}$ then $u \in C^{k+2,\alpha}(\bar{\Omega})$ iff f satisfies the compatibility conditions

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0. \tag{5.3}$$

Conditions on the data of the problem that ensure higher regularity of the solution are in general not available, see [30, §3].

For the construction of layer-adapted meshes and the analysis of numerical methods precise knowledge of the behaviour of the solution and its derivatives is essential. A standard method to gain insight into the layer structure of the solution is the method of matched asymptotic expansions. In [63] this approach is complemented with a careful analysis of the remainder term of the expansion to establish

Theorem 5.1. *Let $f \in C^{4,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$. Let $n \geq 2$ be an integer. Suppose that f satisfies the compatibility conditions (5.3), that*

$$\left(\frac{f}{b_1}\right)_y (1,1) = \left(\frac{f}{b_2}\right)_x (1,1),$$

$$\left(\left(\frac{f}{b_1}\right)_x - \mathcal{D}_0\left(\frac{f}{b_1}\right)\right)_y (1,1) = \left(\frac{f}{b_2}\right)_{xx} (1,1),$$

$$\left(\left(\frac{f}{b_1}\right)_{xx} - \mathcal{D}_0\left(\left(\frac{f}{b_1}\right)_x - \mathcal{D}_0\left(\frac{f}{b_1}\right)\right) - 2\mathcal{D}_1\left(\frac{f}{b_1}\right)\right)_y (1,1) = \left(\frac{f}{b_2}\right)_{xxx} (1,1)$$

and

$$\left(b_2\left(\frac{f}{b_2}\right)_{xx}\right)(1,1) = \left(b_1\left(\frac{f}{b_1}\right)_{yy}\right)(1,1),$$

where $\mathcal{D}_0 v := -v_y b_2/b_1 + vc/b_1$ and $\mathcal{D}_1 v := v_y (b_2/b_1)_x - v(c/b_1)_x$. If $n \geq 4$ we assume in addition

$$b_{2,x}(0,0) = b_{1,y}(0,0).$$

Then the boundary value problem (5.2) has a solution $u \in C^{3,\alpha}(\bar\Omega)$, and this solution can be decomposed as $u = v + w_1 + w_2 + w_{12}$, where

$$\|v\|_{C^2(\bar\Omega)} + \varepsilon^\alpha |v|_{C^{2,\alpha}(\bar\Omega)} \leq C$$

while for all $x, y \in [0,1]$ we have

$$\left\|\frac{\partial^i w_1}{\partial x^i}(x,\cdot)\right\|_{C^{\nu,\alpha}(\{x\}\times[0,1])} \leq C\varepsilon^{-i} e^{-\beta_1 x/\varepsilon},$$

$$\left\|\frac{\partial^j w_2}{\partial y^j}(\cdot,y)\right\|_{C^{\mu,\alpha}([0,1]\times\{y\})} \leq C\varepsilon^{-j} e^{-\beta_2 y/\varepsilon}$$

and

$$\left|\frac{\partial^{i+j} w_{12}}{\partial x^i \partial y^j}(x,y)\right| \leq C\varepsilon^{-(i+j)} e^{-(\beta_1 x + \beta_2 y)/\varepsilon}$$

for $0 \leq \mu, \nu \leq 2$ and $0 \leq i, j \leq n$. Moreover, for all $(x,y) \in \Omega$ we have

$$|\mathcal{L} w_1(x,y)| \leq C\varepsilon e^{-\beta_1 x/\varepsilon}, \quad |\mathcal{L} w_2(x,y)| \leq C\varepsilon e^{-\beta_2 y/\varepsilon}$$

and

$$|\mathcal{L} w_{12}(x,y)| \leq C\varepsilon e^{-(\beta_1 x + \beta_2 y)/\varepsilon}.$$

The regular solution component is defined via solutions of hyperbolic problems. Unlike elliptic operators these first-order operators do not possess smoothing properties. Because of this we have to assume high regularity of f and a large number of compatibility conditions in Theorem 5.1, but we expect such a decomposition to exist under less restrictive assumptions. Similar ideas have been pursued in [20] and [74], but compatibility issues are either not considered or dealt with incorrectly; see Remarks 5.1, 5.2 and 5.5 in [63].

If for the analysis of a scheme less regularity of the various components of the decompostion is required then some of the compatibility conditions can be discarded, see [63, Remark 5.3].

A different approach is used by Roos [84]. He defines the regular solution component as the solution of an elliptic problem on an extended domain. Therefore the construction requires less regularity and compatibility of the data, but only bounds for the first order derivatives of the components of u are obtained in [84].

5.2 Finite difference methods

We shall consider discretisations of (5.2) on a tensor product mesh $\omega_x \times \omega_y$ with $\omega_x : 0 = x_0 < x_1 < \cdots < x_N = 1$, $\omega_y : 0 = y_0 < y_1 < \cdots < y_N = 1$, with local mesh sizes $h_i = x_i - x_{i-1}$ and $k_j = y_j - y_{j-1}$ and maximal mesh size $h = \max\{h_i, k_j\}$.

The simple upwind scheme for (5.2) is: Find $U \in I\!R_0^{(N+1)^2}$ such that

$$[LU]_{ij} := -\varepsilon \left(U_{\bar{x}\hat{x};ij} + U_{\bar{y}\hat{y};ij}\right) - b_{1;ij}U_{x;ij} - b_{2;ij}U_{y;ij} + c_{ij}U_{ij} = f_{ij} \quad \text{for} \quad i,j = 1,\ldots,N-1 \tag{5.4}$$

with

$$v_{x;ij} = \frac{v_{i+1,j} - v_i}{h_{i+1}}, \quad v_{\bar{x};ij} = \frac{v_{ij} - v_{i-1,j}}{h_i} \quad \text{and} \quad v_{\hat{x};ij} = \frac{v_{i+1,j} - v_{ij}}{\hbar_i}, \tag{5.5}$$

$\hbar_i = (h_i + h_{i+1})/2$ and analogous definitions for $v_{y;ij}$, $v_{\bar{y};ij}$, $v_{\hat{y};ij}$ and $\bar{k}_j$. The matrix associated with L is inverse monotone for arbitrary meshes and therefore satisfies a comparison principle.

5.2.1 Pointwise error bounds

This scheme on layer-adapted meshes was first studied by Shishkin who established the maximum-norm error estimate

$$\|u - U\|_{\infty,\omega} \le CN^{-1} \ln^2 N$$

on Shishkin meshes; see [74]. He also proved [94, §3, Theorem 2.3]

$$\|u - U\|_{\infty,\omega} \le C\big(N^{-1} \ln^2 N\big)^p$$

with $p = 1/4$ and $p = 1/8$ (depending on the precise assumptions on the data) if the solution is less smooth.

Here we shall present the technique from [62] which gives a sharper error estimate. This technique is an extension of the truncation error and barrier function technique from Section 2.2.6 to two dimensions.

Theorem 5.2. *Assume the solution u of (5.2) can be decomposed as in Theorem 5.1 with $\alpha = 1$ and $n = 3$. Let the mesh be a tensor-product S-type mesh with mesh transition parameters*

$$\lambda_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\right\} \quad \textit{and} \quad \lambda_y := \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\right\} \quad \textit{with} \quad \sigma \ge 2, \quad q \in (0,1).$$

Let the mesh generating function $\tilde{\varphi}$ be piecewise differentiable satisfying (1.11) and (1.12). Then the error of the simple upwind scheme satiesfies

$$|u_{ij} - U_{ij}| \le \begin{cases} C\left(h + N^{-1}\right) & \text{for } i, j = qN, \dots, N, \\ C\left(h + N^{-1} \max|\psi'|\right) & \text{otherwise.} \end{cases}$$

Proof. Recalling the decomposition of Theorem 5.1, we split the numerical solution in a similar manner. We set $U = V + W_1 + W_2 + W_{12}$, where we define V, W_1, W_2 and W_{12} by

$$[LV]_{ij} = (\mathcal{L}v)_{ij}\,, \quad [LW_1]_{ij} = (\mathcal{L}w_1)_{ij}\,, \quad [LW_2]_{ij} = (\mathcal{L}w_2)_{ij}\,, \quad [LW_{12}]_{ij} = (\mathcal{L}w_{12})_{ij} \qquad \text{for } i, j = 1, \dots, N-1$$

and

$$V_{ij} = v_{ij}, \quad W_{1;ij} = w_{1;ij}, \quad W_{2;ij} = w_{2;ij}, \quad W_{12;ij} = w_{12;ij} \quad \text{on} \quad \partial\Omega.$$

For the regular solution component a Taylor expansion, the derivative bounds of Theorem 5.1 and the inverse monotonicity of L give

$$\|v - V\|_{\infty,\omega} \le Ch.$$

For the term representing the layer at $x = 0$ we have, similarly to (2.61),

$$0 \le W_{1;ij} \le \bar{W}_{1;i} := C \prod_{k=1}^{i} \left(1 + \frac{\beta_1 h_k}{2\varepsilon}\right)^{-1} \quad \text{for} \quad i, j = 0, \dots, N.$$

Thus

$$|w_{1;ij} - W_{1;ij}| \le CN^{-1} \quad \text{for} \quad i = qN, \dots, N, \ j = 0, \dots, N;$$

see the argument that led to (2.62). Now let $i < qN$. A Taylor expansions give

$$|L(w_1 - W_1)_{ij}| \le C\left(h + \varepsilon^{-1}\bar{W}_{1;i}N^{-1} \max|\psi'|\right).$$

Application of a discrete comparison principle with the barrier function

$$C\left(N^{-1} + h + \bar{W}_{1;i}N^{-1} \max|\psi'|\right)$$

with C sufficiently large yields

$$|w_{1;ij} - W_{1;ij}| \le C\left(h + N^{-1} \max|\psi'|\right) \quad \text{for} \quad i = 0, \dots, qN-1, \ j = 0, \dots, N.$$

For the boundary layer at $y = 0$ the same argument is used in order to obtain

$$|w_{2;ij} - W_{2;ij}| \le CN^{-1} \quad \text{for} \quad i = 0, \dots, N, \ j = qN, \dots, N$$

and

$$|w_{2;ij} - W_{2;ij}| \le C\left(h + N^{-1} \max|\psi'|\right) \quad \text{for} \quad i = 0, \dots, N, \ j = 0, \dots, qN-1.$$

Finally for the corner layer term one first shows

$$|w_{12;ij} - W_{12;ij}| \le \bar{W}_{12;ij} := C \prod_{k=1}^{i} \left(1 + \frac{\beta_1 h_k}{2\varepsilon}\right)^{-1} \prod_{l=1}^{j} \left(1 + \frac{\beta_2 k_l}{2\varepsilon}\right)^{-1} \quad \text{for} \quad i, j = 0, \dots, N,$$

	simple upwinding						hybrid scheme	
	standard Shishkin mesh		Shishkin mesh with 2 transition points		Bakhvalov-Shishkin mesh		standard Shishkin mesh	
N	error	rate	error	rate	error	rate	error	rate
16	9.6379e-2	0.50	9.0430e-2	0.73	9.3261e-2	0.74	1.1072e-1	0.88
32	6.8194e-2	0.59	5.4533e-2	0.76	5.5803e-2	0.90	5.9962e-2	0.94
64	4.5364e-2	0.66	3.2138e-2	0.79	2.9916e-2	0.93	3.1328e-2	0.97
128	2.8636e-2	0.72	1.8606e-2	0.84	1.5665e-2	0.97	1.6031e-2	0.98
256	1.7360e-2	0.77	1.0416e-2	0.87	8.0140e-3	0.98	8.1081e-3	0.99
512	1.0182e-2	0.80	5.6941e-3	0.90	4.0529e-3	0.99	4.0768e-3	1.00
1024	5.8286e-3	0.83	3.0602e-3	0.91	2.0379e-3	1.00	2.0440e-3	1.00
2048	3.2776e-3	—	1.6247e-3	—	1.0219e-3	—	1.0234e-3	—

Table 5.1: Upwind and hybrid difference scheme on S-type meshes

which implies

$$|w_{12;ij} - W_{12;ij}| \le CN^{-1} \quad \text{if} \quad i \ge qN \text{ or } j \ge qN.$$

In a second step the truncation error is estimated using Taylor expansions:

$$|L(w_{12} - W_{12})_{ij}| \le C\varepsilon^{-1}\bar{W}_{12;ij}N^{-1}\max|\psi'|.$$

And the discrete comparison principle yields

$$|w_{12;ij} - W_{12;ij}| \le CN^{-1}\max|\psi'| \quad \text{for} \quad i, j = 0, \dots, qN-1.$$

Collecting the bounds for the various components, we are done. □

Remark 5.3. *In [62] a modified, hybrid scheme on a standard Shishkin mesh is considered. It is based on simple upwinding, but employs central differencing whenever the mesh allows one to do this without losing stability. For this scheme the above technique gives the maximum-norm error bound*

$$\|u - U\|_{\infty,\omega} \le CN^{-1}.$$

The improved bound is because central differencing improves the error terms of order $N^{-1}\ln N$ in the above proof to order $N^{-2}\ln^2 N$.

A numerical example. We briefly illustrate our theoretical findings for the simple upwind difference scheme on S-type meshes and for the hybrid scheme when applied to the test problem

$$-\varepsilon\Delta u - (2+x)u_x - (3+y^2)u_y + u = f(x,y) \quad \text{in} \quad \Omega = (0,1)^2, \tag{5.6a}$$
$$u = 0 \qquad \text{on} \quad \Gamma = \partial\Omega, \tag{5.6b}$$

where the right-hand side is chosen such that

$$u(x,y) = \cos\frac{\pi x}{2}\left(1 - e^{-2x/\varepsilon}\right)(1-y)^3\left(1 - e^{-3y/\varepsilon}\right) \tag{5.6c}$$

is the exact solution. This function exhibits typical boundary layer behaviour. For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem.

5.2.2 Error expansion

Kopteva [44] derives an error expansion for the simple upwind scheme (5.4) on standard Shishkin meshes. Let H and h denote the coarse and fine mesh sizes in the Shishkin mesh. Provided that $\varepsilon \le CN^{-1}$ she proves that the error can be expanded as

$$U_{ij} - u_{ij} = H\Phi_{ij} + \frac{h}{\varepsilon}\Psi_{ij} + R_{ij}$$

with

$$\Phi(x,y) = \varphi - \varphi(0,y)e^{-b_1(0,y)x/\varepsilon} - \varphi(x,0)e^{-b_2(x,0)y/\varepsilon} + \varphi(0,0)e^{-(b_1(0,0)x+b_2(0,0)y)/\varepsilon}$$

$$\Psi(x,y) = \frac{x}{\varepsilon}\frac{b_1^2(0,y)\tilde{w}_1 + b_1^2(0,0)\tilde{w}_{12}}{2} + \frac{y}{\varepsilon}\frac{b_2^2(x,0)\tilde{w}_2 + b_2^2(0,0)\tilde{w}_{12}}{2}$$

where the $\tilde{w}$'s satisfy bounds similar to those of Theorem 5.1 and $\|\varphi\|_{C^{1,1}} \le C$, while for the remainder we have

$$R_{ij} \le \begin{cases} CN^{-2} & \text{for } i,j = qN,\dots,N, \\ CN^{-2}\ln^2 N & \text{otherwise.} \end{cases}$$

This expansion is used in [44] to derive error bounds for Richardson extrapolation and for the approximation of derivatives.

Derivative approximation. In [44] the bounds

$$|(U-u)_{x;ij}| \le C \begin{cases} N^{-1} & \text{for } i,j = qN,\dots,N-1, \\ N^{-1}\ln^2 N & \text{for } i = qN,\dots,N-1,\ j = 0,\dots,qN-1, \\ \varepsilon^{-1}N^{-1}\ln N & \text{otherwise} \end{cases}$$

are given with analogous results for $(U-u)_y$.

Richardson extrapolation. Let $\tilde{U}$ be the upwind difference solution on the mesh obtained by uniformly bisecting the original mesh ω and let $\Pi\tilde{U}$ be the obvious restriction of $\tilde{U}$ to ω. Then

$$\left|\left([2\Pi\tilde{U} - U] - u\right)_{ij}\right| \le C \begin{cases} N^{-2} & \text{for } i,j = qN,\dots,N-1, \\ N^{-2}\ln^2 N & \text{otherwise [44].} \end{cases}$$

These results are neatly illustrated by the numbers in Table 5.2 which display the results of the Richardson extrapolation applied to our test problem (5.6).

	fine mesh region		coarse mesh region	
N	error	rate	error	rate
16	1.3869e-2	1.08	3.7171e-3	1.44
32	6.5448e-3	1.23	1.3733e-3	1.74
64	2.7918e-3	1.38	4.1086e-4	1.87
128	1.0703e-3	1.49	1.1271e-4	1.93
256	3.8049e-4	1.58	2.9616e-5	1.96
512	1.2701e-4	1.64	7.5975e-6	1.98
1024	4.0623e-5	—	1.9234e-6	—

Table 5.2: Richardson extrapolation on a Shishkin mesh

5.3 Finite element methods

This section is concerned with finite element discretisations for

$$\mathcal{L}u := -\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in} \quad \Omega = (0,1)^2, \quad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega.$$

We assume that $(b_1, b_2) > (\beta_1, \beta_2) > 0$ on $\bar{\Omega}$ with constants β_1 and β_2 and

$$c + \tfrac{1}{2}\operatorname{div}\boldsymbol{b} \geq \gamma > 0. \tag{5.7}$$

The last condition guarantees the coercivity of the bilinear form in the weak formulation and therefore the existence of a unique solution.

Finite element methods are based on the weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \quad \text{for all} \quad v \in H_0^1(\Omega),$$

where

$$a(u,v) = \varepsilon(\nabla u, \nabla v) - (\boldsymbol{b}\cdot\nabla u, v) + (cu, v) \quad \text{and} \quad f(v) = (f, v)$$

with

$$(u,v) := \int_\Omega u(x,y)v(x,y)dxdy.$$

Because of (5.7) we have

$$a(v,v) \geq |||v|||_\varepsilon^2 := \varepsilon\left(\|v_x\|_0^2 + \|v_y\|_0^2\right) + \gamma\|v\|_0^2 \quad \text{for all} \quad v \in H_0^1(\Omega),$$

i. e., the bilinear form is coercive and the variational formulation posseses a unique solution $u \in H_0^1(\Omega)$.

Let $V^\omega \subset H_0^1(\Omega)$ be a finite-element space. Then our discretisation is: Find $U \in V^\omega$ such that

$$a(U,v) = f(v) \quad \text{for all} \quad v \in V^\omega.$$

Again the coercivity of $a(\cdot,\cdot)$ guarantees the existence of a unique solution $U \in V$.

For the discretization we shall restrict ourselves to tensor-product meshes $\omega := \omega_x \times \omega_y$ as in Section 5.2.1. We shall consider both bilinear elements on rectangles and linear elements on triangles with the triangulation obtained by drawing either diagonal in each of the mesh rectangles; see Figure 5.1.

5.3.1 The interpolation error

The first important results are bounds for the interpolation error. We denote by u^I the piecewise-linear/bilinear function that interpolates to u at the nodes of the mesh ω. The meshes we consider are characterised by high aspect ratios of the mesh elements. Because of this anisotropy standard interpolation theory cannot be applied. There have been a number of contributions to extend the theory to anisotropic elements, e. g., [10, 109, 110]. The first uniform interpolation error estimates for layer-adapted meshes, namely Shishkin meshes, were derived by Stynes and O'Riordan [96] and Dobrowolski and Roos [20]. Here we shall give the more general results from [51].

Theorem 5.4. *Suppose the assumptions of Theorem 5.1 are satisfied. Then the maximum-norm error of bilinear interpolation on a tensor-product mesh satisfies*

$$\left|(u^I - u)(x,y)\right| \leq C\left\{\int_{x_{i-1}}^{x_i}\left(1 + \varepsilon^{-1}e^{-\beta_1 x/(2\varepsilon)}\right)dx + \int_{y_{j-1}}^{y_j}\left(1 + \varepsilon^{-1}e^{-\beta_2 y/(2\varepsilon)}\right)dy\right\}^2$$
$$\textit{for} \quad (x,y) \in T_{ij} := [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

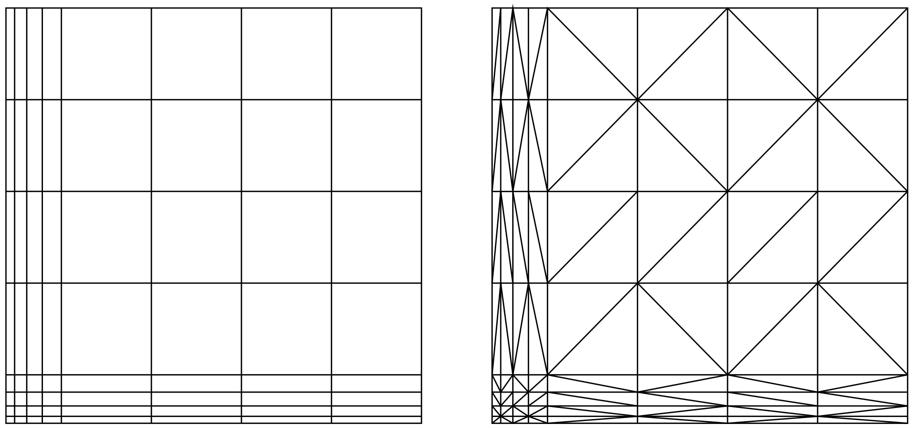


Figure 5.1: Triangulations into rectangles and triangles on tensor-product meshes

and for the ε-weighted energy norm

$$|||u^I - u|||_\varepsilon \le C \max_{i,j=1,\ldots,N} \left\{ \int_{x_{i-1}}^{x_i} \left(1+\varepsilon^{-1}e^{-\beta_1 x/(2\varepsilon)}\right) dx + \int_{y_{j-1}}^{y_j} \left(1+\varepsilon^{-1}e^{-\beta_2 y/(2\varepsilon)}\right) dy \right\}.$$

Proof of Theorem 5.4. First Theorem 5.1 implies

$$\left|\frac{\partial^{i+j}u}{\partial x^i \partial y^j}(x,y)\right| \le C\left(1+\varepsilon^{-i}e^{-\beta_1 x/\varepsilon}\right) \times \left(1+\varepsilon^{-j}e^{-\beta_2 y/\varepsilon}\right) \quad \text{for } i+j \le 2. \tag{5.8}$$

(i) Let τ be a mesh triangle/rectangle that has (x_{i-1}, y_{j-1}) and (x_i, y_j) as two of its vertices. Then for $(x,y) \in \tau$ Taylor expansions yield

$$u(x,y) = u(x_i,y) + (x-x_i)u_x(x_i,y) + \int_{x_i}^{x}\int_{x_i}^{t} u_{xx}(s,y)dsdt,$$
$$u(x_i,y) = u(x_i,y_j) + (y-y_j)u_y(x_i,y_j) + \int_{y_j}^{y}\int_{y_j}^{t} u_{yy}(x_i,s)dsdt$$

and

$$u_x(x_i,y) = u_x(x_i,y_j) + \int_{y_j}^{y} u_{xy}(x_i,t)dt.$$

Combining these three equations, we get

$$u(x,y) = u(x_i,y_j) + (x-x_i)u_x(x_i,y_j) + (y-y_j)u_y(x_i,y_j) + \int_{x_i}^{x} u_{xx}(s,y)(x-s)ds + \int_{y_j}^{y} u_{yy}(x_i,s)(y-s)ds + (x-x_i)\int_{y_j}^{y} u_{xy}(x_i,s)ds.$$

Thus

$$\|u^I - u\|_{\infty,\tau} \le C\Big\{ \max_{y\in[y_{j-1},y_j]} \int_{x_{i-1}}^{x_i} |u_{xx}(s,y)|(s-x_{i-1})ds + \int_{y_{j-1}}^{y_j} |u_{yy}(x_i,s)|(s-y_{j-1})ds + h_i \int_{y_{j-1}}^{y_j} |u_{xy}(x_i,s)|ds \Big\}.$$

Bounds for the first two integrals are easily obtained using the technique from Section 3.1 and (5.8), while for the third term we have

$$h_i \int_{y_{j-1}}^{y_j} \left|u_{xy}(x_i, s)\right| ds \le C h_i \Big(1 + \varepsilon^{-1} e^{-\beta_1 x_i/\varepsilon}\Big) \int_{y_{j-1}}^{y_j} \big(1 + \varepsilon^{-1} e^{-\beta_2 y/\varepsilon}\big) dy$$
$$\le C \int_{x_{i-1}}^{x_i} \big(1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon}\big) dx \int_{y_{j-1}}^{y_j} \big(1 + \varepsilon^{-1} e^{-\beta_2 y/\varepsilon}\big) dy,$$

since $e^{-\beta_1 x_i/\varepsilon} \le e^{-\beta_1 x/\varepsilon}$ for $x \le x_i$. Hence

$$\left\|u^I - u\right\|_{\infty,\tau} \le C \left\{ \int_{x_{i-1}}^{x_i} \big(1 + \varepsilon^{-1} e^{-\beta_1 x/(2\varepsilon)}\big) dx + \int_{y_{j-1}}^{y_j} \big(1 + \varepsilon^{-1} e^{-\beta_2 y/(2\varepsilon)}\big) dy \right\}^2,$$

by (2.39). This is the first bound of the theorem.

(ii) To bound the interpolation error in the H^1 seminorm, integration by parts is used. We get

$$\left\|(u^I - u)_x\right\|_0^2 = \int_\Omega u_{xx}(x,y)\big(u^I - u\big)(x,y) dx\, dy + \sum_{i=1}^{N-1} \int_0^1 \big(u^I - u\big)(x_i, y) J_i(y) dy \tag{5.9}$$

where

$$J_i(y) := u_x^I(x_i - 0, y) - u_x^I(x_i + 0, y)$$

For $y \in [y_{j-1}, y_j]$ we have

$$J_i(y) = \frac{y - y_{j-1}}{k_j} \left(\frac{u_{ij} - u_{i-1,j}}{h_i} - \frac{u_{i+1,j} - u_{ij}}{h_{i+1}} \right)$$
$$+ \frac{y_j - y}{k_j} \left(\frac{u_{i,j-1} - u_{i-1,j-1}}{h_i} - \frac{u_{i+1,j-1} - u_{ij-1}}{h_{i+1}} \right)$$

By the mean-value theorem there exists a $\xi_{i,j}$ with $x_{i-1} \le \xi_{i,j} \le x_i$ such that

$$\frac{u_{i,j} - u_{i-1,j}}{h_i} = u_x(\xi_{i,j}, y_j).$$

Thus

$$\left| \frac{u_{i,j} - u_{i-1,j}}{h_i} - \frac{u_{i+1,j} - u_{ij}}{h_{i+1}} \right| = \left|u_x(\xi_{i,j}, y_j) - u_x(\xi_{i+1,j}, y_j)\right| \le \int_{x_{i-1}}^{x_{i+1}} \left|u_{xx}(\xi, y_j)\right| d\xi$$

We get the bound

$$|J_i(y)| \le \max_{y\in[0,1]} \int_{x_{i-1}}^{x_{i+1}} \left|u_{xx}(\xi, y)\right| d\xi$$

This and a Hölder inequality applied to (5.9) yield

$$\left\|(u^I - u)_x\right\|_0^2 \le \left\|u^I - u\right\|_\infty \left\{ \int_\Omega |u_{xx}(x,y)|\, dx\, dy + 2 \max_{y\in[0,1]} \int_0^1 |u_{xx}(\xi, y)|\, d\xi \right\} \le \frac{C}{\varepsilon} \left\|u^I - u\right\|_\infty,$$

by (5.8). The interpolation error in the L_2 norm is bounded by its L_∞ norm. We get the second bound of the theorem. □

Remark 5.5. *Error bounds for particular layer-adapted meshes can be derived using the results from Sections 1.2 and 1.3.*

The second part of the proof when the H^1 seminorm is considered works for bilinear elements only. However, for S-type meshes and linear elements the conclusions of the theorem hold too; see [49, 85].

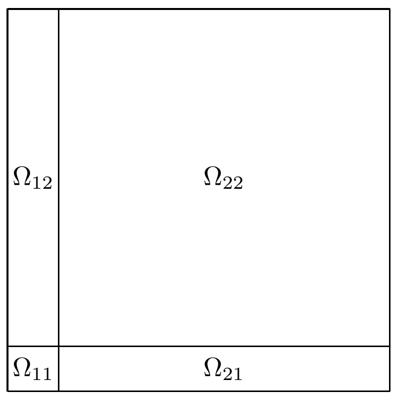
5.3.2 Galerkin FEM

5.3.2.1 Convergence

Convergence of the Galerkin FEM on standard Shishkin meshes was first studied by Stynes and O'Riordan [96]. Their technique was later adapted by Linß and Roos to the analysis of more general S-type meshes [49, 85]: Let

$$\lambda_x := \min\left\{q, \frac{\sigma\varepsilon}{\beta_1}\ln N\right\} \quad \text{and} \quad \lambda_y := \min\left\{q, \frac{\sigma\varepsilon}{\beta_2}\ln N\right\}$$

with $\sigma > 0$ and $q \in (0,1)$ arbitrary, but fixed with $qN \in I\!N$. Divide the domain Ω as in Figure 5.2: $\bar{\Omega} = \Omega_{11} \cup \Omega_{21} \cup \Omega_{12} \cup \Omega_{22}$.



$$\begin{aligned}\Omega_{11} &= [0,\lambda_x] \times [0,\lambda_y],\\ \Omega_{21} &= [\lambda_x,1] \times [0,\lambda_y],\\ \Omega_{12} &= [0,\lambda_x] \times [\lambda_y,1],\\ \Omega_{22} &= [\lambda_x,1] \times [\lambda_y,1].\end{aligned}$$

Figure 5.2: Dissection of Ω for tensor-product S-type meshes

Corollary 5.6. *Let $\omega_x \times \omega_y$ with $\sigma \geq 2$ be a S-type mesh. Then Theorem 5.4 and Remark 5.5 imply*

$$\left\|u - u^I\right\|_{\infty,\Omega\setminus\Omega_{22}} \leq C\left(h + N^{-1}\max|\psi'|\right)^2, \quad \left\|u - u^I\right\|_{\infty,\Omega_{22}} \leq CN^{-2},$$
$$\left|\left\|u - u^I\right\|\right|_{\varepsilon} \leq C\left(h + N^{-1}\max|\psi'|\right)$$

and, by the Cauchy-Schwarz inequality,

$$\left\|u - u^I\right\|_{0,\Omega\setminus\Omega_{22}} \leq C\varepsilon^{1/2}\ln^{1/2}N\left(h + N^{-1}\max|\psi'|\right)^2 \quad \text{and} \quad \left\|u - u^I\right\|_{0,\Omega_{22}} \leq CN^{-2}.$$

Theorem 5.7. *Let $\omega = \omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \geq 2$ whose mesh generating function $\tilde{\varphi}$ satisfies (1.11) and*

$$\left(h + N^{-1}\max|\psi'|\right)\ln^{1/2}N \leq C. \tag{5.10}$$

Then

$$|\|u - U\||_{\varepsilon} \leq C\left(h + N^{-1}\max|\psi'|\right)$$

for both linear elements on triangles and bilinear elements on rectangles.

Remark 5.8. *The additional condition (5.10) does not constitute a major restriction. For example, it is satisfied by both the standard Shishkin mesh and the Bakhvalov-Shishkin mesh.*

Proof of Theorem 5.7. The proof is along the lines of Section 3.2.1 using the tensor-product structure of the mesh and the solution decomposition of Theorem 5.1; see also [49]. Let $\eta = u^I - u$ and $\chi = u^I - U$. A bound for the interpolation error η is provided by Corollary 5.6.

Bounding χ, we start from the coercivity of $a(\cdot,\cdot)$ and the orthogonality of the Galerkin method, i. e.,

$$\begin{aligned}|||\chi|||_\varepsilon^2 &\le a(\chi,\chi) = a(\eta,\chi) = \varepsilon(\nabla\eta,\nabla\chi) + (\eta, \boldsymbol{b}^T\nabla\chi) + \big((\operatorname{div} b + c)\eta, \chi\big)\\ &\le C\,|||\eta|||_\varepsilon\,|||\chi|||_\varepsilon + C\left(\|\eta\|_{0,\Omega_{22}}\|\nabla\chi\|_{0,\Omega_{22}} + \|\eta\|_{L_\infty(\Omega\setminus\Omega_{22})}\|\nabla\chi\|_{L_1(\Omega\setminus\Omega_{22})}\right).\end{aligned}$$

On $\Omega\setminus\Omega_{22}$ the Cauchy-Schwarz inequality yields

$$\|\nabla\chi\|_{L_1(\Omega\setminus\Omega_{22})} \le C\sqrt{\lambda_x+\lambda_y}\,\|\nabla\chi\|_0 \le C\ln^{1/2}N\,|||\chi|||_\varepsilon\,,$$

while on Ω_{22} an inverse inequality yields

$$\|\nabla\chi\|_{0,\Omega_{22}} \le CN\,\|\chi\|_{0,\Omega_{22}} \le CN\,|||\chi|||_\varepsilon\,.$$

because $H \ge N^{-1}$. These two bounds and the interpolation results of Corollay 5.6 give

$$|||\chi|||_\varepsilon \le C\left\{h + N^{-1}\max|\psi'| + \left(h + N^{-1}\max|\psi'|\right)^2\ln^{1/2}N + N^{-1}\right\}.$$

Thus

$$|||\chi|||_\varepsilon \le C\left(h + N^{-1}\max|\psi'|\right),$$

where we have used (5.10). Applying a triangle inequality and the bounds for $|||\eta|||_\varepsilon$ and $|||\chi|||_\varepsilon$, we complete the proof. □

5.3.2.2 Superconvergence

Similar to the one dimensional case the Galerkin FEM using bilinear elements on rectangular S-type meshes enjoys a superconvergence property; see [50, 115]. Note that this superconvergence result generally does not hold for linear elements on triangles as numerical experiments confirm [64].

In contrast to the one-dimensional case where we have $((u^I-u)',\chi') = 0$ for arbitrary $\chi\in V^\omega$, we do *not* have $(\nabla(u^I-U),\nabla\chi) = 0$ here because u^I-u vanishes in the mesh points only, but not at the inter-element boundaries. This complicates the analysis and requires higher regularity of the solution. In particular we shall assume that the solution u can be decomposed as $u = v + w_1 + w_2 + w_{12}$, where

$$\begin{aligned}\left|\frac{\partial^{i+j}v}{\partial x^i\partial y^j}(x,y)\right| \le C,\quad \left|\frac{\partial^{i+j}w_1}{\partial x^i\partial y^j}(x,y)\right| \le C\varepsilon^{-i}e^{-\beta_1 x/\varepsilon},\\ \left|\frac{\partial^{i+j}w_2}{\partial x^i\partial y^j}(x,y)\right| \le C\varepsilon^{-j}e^{-\beta_2 y/\varepsilon}\quad\text{and}\quad\left|\frac{\partial^{i+j}w_{12}}{\partial x^i\partial y^j}(x,y)\right| \le C\varepsilon^{-(i+j)}e^{-(\beta_1 x+\beta_2 y)/\varepsilon}\end{aligned}\tag{5.11}$$

for $i+j\le 3$ and $x,y\in(0,1)$.

Theorem 5.9. *Let $\omega_x\times\omega_y$ be a tensor-product S-type mesh with $\sigma\ge 5/2$ that satisfies (1.11). Then the Galerkin-FEM solution U satisfies*

$$\left|\left|\left|u^I-U\right|\right|\right|_\varepsilon \le C\left(h^2\ln^{1/2}N + N^{-2}\max|\psi'|^2\right).$$

Proof. The coercivity and Galerkin orthogonality of $a(\cdot,\cdot)$ give

$$\begin{aligned}\left|\left|\left|u^I-U\right|\right|\right|_\varepsilon^2 &\le \left|a\big(u-u^I, u^I-U\big)\right|\\ &\le \varepsilon\left|\big(\nabla(u-u^I),\nabla(u^I-U)\big)\right| + \left|\left(\boldsymbol{b}^T\nabla(u-u^I) - c(u-u^I), u^I-U\right)\right|.\end{aligned}$$

In the Section 5.3.2.3 we shall show that

$$\varepsilon\left|\big(\nabla(u-u^I),\nabla\chi\big)\right| \le C\left(h^2 + N^{-2}\max|\psi'|^2\right)|||\chi|||_\varepsilon \tag{5.12}$$

and

$$\left|\left(\boldsymbol{b}^T\nabla(u-u^I)-c(u-u^I),\chi\right)\right| \le C\left(h^2\ln^{1/2}N+N^{-2}\max|\psi'|^2\right)|||\chi|||_\varepsilon \tag{5.13}$$

for all $\chi \in V^\omega$. Thus

$$|||u^I-U|||_\varepsilon^2 \le C\left(h^2\ln^{1/2}N+N^{-2}\max|\psi'|^2\right)|||u^I-U|||_\varepsilon$$

Divide by $|||u^I-U|||_\varepsilon$ to complete the proof. □

Corollary 5.10. *Theorem 5.9 yields*

$$|||u^I-U|||_\varepsilon \le \begin{cases} CN^{-2}\ln^2 N & \text{for the standard Shishkin mesh and} \\ C(\varepsilon^2+N^{-2})\ln^{1/2}N & \text{for the Bakhvalov-Shishkin mesh.}\end{cases}$$

Another superconvergence result was established by Zhang [115] who studied convergence of the Galerkin FEM on Shishkin meshes in a discrete version of the energy norm where $\nabla(u-U)$ is replaced by a piecewise-constant approximation based on the midpoints of the rectangles of the triangulation.

5.3.2.3 Detailed analysis, proofs of (5.12) and (5.13)

In the analysis we require error estimates for interpolation on anisotropic elements which were derived by Apel and Dobrowolski [10]. Furthermore a sharp bound for the L_2-norm error of the interpolation error for the layer terms. We shall also use special error expansion formulae derived by Lin [47].

Preliminaries. Let $T_{ij}=(x_{i-1},x_i)\times(y_{j-1},y_j)$ be an element of the triangulation. Set

$$F_i(x)=\frac{(x-\tilde{x}_i)^2}{2}-\frac{h_i^2}{8} \qquad \text{and} \qquad G_j(y)=\frac{(y-\tilde{y}_j)^2}{2}-\frac{k_j^2}{8},$$

where $(\tilde{x}_i,\tilde{y}_j)$ is the midpoint of the mesh rectangle T_{ij}. Denote the east, north, west and south edges of T_{ij} by $l_{k;ij}$ for $k=1,\dots,4$ respectively.

Lemma 5.11 (Lin Identities [47])**.** *For any function $g\in C^3(\overline{T}_{ij})$ and any $\chi\in V^\omega$ we have the identities*

$$\int_{T_{ij}}(g-g^I)_x\chi_x=\int_{T_{ij}}\left[G_j\chi_x-\frac{1}{3}(G_j^2)'\chi_{xy}\right]g_{xyy}, \tag{5.14a}$$

$$\begin{aligned}\int_{T_{ij}}(g-g^I)_x\chi_y=&\int_{T_{ij}}\left[G_jg_{xyy}\left(\chi_y-F_i'\chi_{xy}\right)+F_jg_{xxy}\chi_x\right]\\&+\left(\int_{l_{4;ij}}-\int_{l_{2;ij}}\right)F_jg_{xx}\chi_x,\end{aligned} \tag{5.14b}$$

and

$$\begin{aligned}\int_{T_{ij}}(g-g^I)_x\chi=&\left(\int_{l_{1;ij}}-\int_{l_{2;ij}}\right)\frac{h_i^2}{12}\chi g_{xx}+\int_{T_{ij}}\left[\frac{1}{6}\left(F_i^2\right)'\chi_x-\frac{h_i^2}{12}\chi\right]g_{xxx}\\&+\int_{T_{ij}}\left[G_j\left(\chi-F_i'\chi_x\right)-\frac{1}{3}\left(G_j^2\right)'\left(\chi_y-F_i'\chi_{xy}\right)\right]g_{xyy}.\end{aligned} \tag{5.14c}$$

An immediate consequence of (5.14a) is

$$\left|\left((g-g^I)_x,\chi_x\right)_{T_{ij}}\right| \le \frac{k_j^2}{8}\int_{T_{ij}} |\chi_x|\ |g_{xyy}| + \frac{k_j^3}{24}\int_{T_{ij}} |\chi_{xy}|\ |g_{xyy}|\,.$$

with the Cauchy-Schwarz and an inverse inequality giving

$$\left|\left((g-g^I)_x,\chi_x\right)_{T_{ij}}\right| \le Ck_j^2\,\|\chi_x\|_{0,T_{ij}}\,\|g_{xyy}\|_{0,T_{ij}}\,. \tag{5.15}$$

Lemma 5.12 ([10, Theorem 3]). *Let $T_{ij}\in\Omega^N$ and $p\in[1,\infty]$. Assume that g lies in $W_p^2(T_{ij})$. Let g^I denote the bilinear function that interpolates to g at the vertices of T_{ij}. Then*

$$\|(g-g^I)_x\|_{L_p(T_{ij})} \le C\left(h_i\|g_{xx}\|_{L_p(T_{ij})} + k_j\|g_{xy}\|_{L_p(T_{ij})}\right).$$

Proposition 5.13. *Let $\omega_x\times\omega_y$ be a tensor-product S-type mesh that satisfies (1.11). Then for $w=w_1+w_2+w_{12}$*

$$\left\|w-w^I\right\|_{0,\Omega_{22}} \le C\left(\varepsilon^{1/2}N^{-\sigma}+N^{-\sigma-1/2}\right) \tag{5.16a}$$

and

$$\left\|w-w^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(h+N^{-1}\max|\psi'|\right)^2 \quad \textit{if } \sigma>2. \tag{5.16b}$$

Proof. *(i)* When prooving (5.16a), we bound $\|w\|_{0,\Omega_{22}}$ and $\left\|w^I\right\|_{0,\Omega_{22}}$ separately and apply a triangle inequality. Clearly

$$\|w\|_{0,\Omega_{22}} \le C\varepsilon^{1/2}N^{-\sigma}, \tag{5.17}$$

by (5.11) and a direct calculation.

In order to bound the L_2 norm of w^I we split Ω_{22} into two subdomains

$$S:=[x_{qN+1},1]\times[y_{qN+1},1] \qquad \text{and} \qquad \Omega_{22}\setminus S.$$

Note that $\Omega_{22}\setminus S$ consists of only one ply of mesh rectangles along the interface between the coarse and the fine mesh regions. Therefore

$$\left\|w^I\right\|_{0,\Omega_{22}\setminus S}^2 \le \big(2(1-q)N-1\big)h_{qN+1}k_{qN+1}\left\|w^I\right\|_{\infty,\Omega_{22}}^2 \tag{5.18}$$

Thus

$$\left\|w^I\right\|_{0,\Omega_{22}\setminus S} \le CN^{-\sigma-1/2}. \tag{5.19}$$

For $T_{ij}\subset S$ we estimate as follows

$$\left\|w^I\right\|_{0,T_{ij}}^2 \le h_ik_j\left\|w^I\right\|_{\infty,T_{ij}}^2 \le C\int_{x_{i-2}}^{x_{i-1}}\int_{y_{j-2}}^{y_{j-1}}\left(e^{-2\beta_1x/\varepsilon}+e^{-2\beta_2y/\varepsilon}+e^{-2(\beta_1x+\beta_2y)/\varepsilon}\right),$$

by (5.11) and since the mesh on Ω_{22} is uniform. We get

$$\left\|w^I\right\|_{0,S}^2 \le C\int_{\lambda_x}^1\int_{\lambda_y}^1\left(e^{-2\beta_1x/\varepsilon}+e^{-2\beta_2y/\varepsilon}+e^{-2(\beta_1x+\beta_2y)/\varepsilon}\right)$$

Hence

$$\left\|w^I\right\|_{0,S} \le C\varepsilon^{1/2}N^{-\sigma}. \tag{5.20}$$

Collecting (5.17)–(5.20), we get (5.16a).

(ii) Before starting the proof of (5.16b) note that by (3.9) and (1.13)

$$\frac{h_i}{\varepsilon} \le \frac{\sigma}{\beta} N^{-1} \max|\psi'|\, e^{\beta_1 x_i/\sigma\varepsilon} \le C\frac{\sigma}{\beta} N^{-1} \max|\psi'|\, e^{\beta_1 x/\sigma\varepsilon} \quad \text{for} \quad x \in [x_{i-1}, x_i].$$

(α) First let us study $w_1 - w_1^I$. For $T_{ij} \subset \Omega_{12} \cup \Omega_{11}$ we have by Lemma 5.12 and (5.11)

$$\begin{aligned}\left\|w_1 - w_1^I\right\|_{0,T_{ij}}^2 &\le C\Bigg\{ h_i^4 k_j \int_{x_{i-1}}^{x_i} \varepsilon^{-4} e^{-2\beta_1 x/\varepsilon} dx \\ &\qquad + h_i^2 k_j^3 \int_{x_{i-1}}^{x_i} \varepsilon^{-2} e^{-2\beta_1 x/\varepsilon} dx + k_j^5 \int_{x_{i-1}}^{x_i} e^{-2\beta_1 x/\varepsilon} dx \Bigg\} \\ &\le C\left(N^{-1}\max|\psi'| + h\right)^4 k_j \int_{x_{i-1}}^{x_i} e^{-(2-4/\sigma)\beta_1 x/\varepsilon} dx.\end{aligned}$$

Summing over all elements in $\Omega_{11} \cup \Omega_{12}$, we get

$$\left\|w_1 - w_1^I\right\|_{0,\Omega_{11}\cup\Omega_{12}}^2 \le C\left(N^{-1}\max|\psi'| + h\right)^4 \int_0^{\lambda_x} e^{-(2-4/\sigma)\beta_1 x/\varepsilon} dx$$

Thus

$$\left\|w_1 - w_1^I\right\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\varepsilon^{1/2}\left(N^{-1}\max|\psi'| + h\right)^2$$

because $\sigma > 2$ is assumed.

On Ω_{21} we estimate as follows

$$\begin{aligned}\left\|w_1 - w_1^I\right\|_{0,\Omega_{21}} &\le \sqrt{\operatorname{meas}\Omega_{21}}\,\left\|w_1 - w_1^I\right\|_{\infty,\Omega_{21}} \le \sqrt{\operatorname{meas}\Omega_{21}}\,\|w_1\|_{\infty,\Omega_{21}} \\ &\le C\varepsilon^{1/2}\ln^{1/2} N e^{-\beta_1\lambda_x/\varepsilon} \le C\varepsilon^{1/2} N^{-\sigma}\ln^{1/2} N \le C\varepsilon^{1/2} N^{-2}\end{aligned}$$

because $\sigma > 2$.

Therefore

$$\left\|w_1 - w_1^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(N^{-1}\max|\psi'| + h\right)^2 \tag{5.21}$$

since $\max|\psi'| \ge 1$.

(β) Clearly the same argument yields

$$\left\|w_2 - w_2^I\right\|_{0,\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/2}\left(N^{-1}\max|\psi'| + h\right)^2 \tag{5.22}$$

because of the structural symmetry.

(γ) Last let us bound $w_{12} - w_{12}^I$. For $T_{ij} \subset \Omega_{11}$ we have by Lemma 5.12 and (5.11)

$$\begin{aligned}\left\|w_{12} - w_{12}^I\right\|_{0,T_{ij}}^2 &\le C\left(h_i^2 + k_j^2\right)^2 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \varepsilon^{-4} e^{-2\beta_1 x/\varepsilon} e^{-2\beta_2 y/\varepsilon} dy dx \\ &\le C\left(N^{-1}\max|\psi'| + h\right)^4 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} e^{-(2-4/\sigma)\beta_1 x/\varepsilon} e^{-(2-4/\sigma)\beta_2 y/\varepsilon} dy dx\end{aligned}$$

Summing over all elements in Ω_{11}, we get

$$\left\|w_{12} - w_{12}^I\right\|_{0,\Omega_{11}}^2 \le C\left(N^{-1}\max|\psi'| + h\right)^4 \int_0^{\lambda_x}\int_0^{\lambda_y} e^{-(2-4/\sigma)\beta_1 x/\varepsilon} e^{-(2-4/\sigma)\beta_2 y/\varepsilon} dy dx$$

Hence

$$\left\| w_{12} - w_{12}^I \right\|_{0,\Omega_{11}} \le C\varepsilon \left(N^{-1} \max |\psi'| + h \right)^2 . \tag{5.23}$$

On $\Omega_{12} \cup \Omega_{21}$ we estimate as follows

$$\begin{aligned} \left\| w_{12} - w_{12}^I \right\|_{0,\Omega_{12}\cup\Omega_{21}} &\le \sqrt{\operatorname{meas} \Omega_{12} \cup \Omega_{21}} \left\| w_{12} - w_{12}^I \right\|_{\infty,\Omega_{12}\cup\Omega_{21}} \\ &\le \varepsilon^{1/2} N^{-\sigma} \ln^{1/2} N \left\| w_{12} \right\|_{\infty,\Omega_{12}\cup\Omega_{21}} \\ &\le C\varepsilon^{1/2} N^{-\sigma} \ln^{1/2} N \le C\varepsilon^{1/2} N^{-2} \end{aligned} \tag{5.24}$$

because $\sigma > 2$.

Collect (5.21)–(5.24) to complete the proof. □

Proof of (5.12). *(i)* Using (5.15), we obtain for $T_{ij} \subset \Omega_{11} \cup \Omega_{21}$

$$\begin{aligned} \left| \left((u - u^I)_x, \chi_x \right)_{T_{ij}} \right| &\le C k_j^2 \left\| \left(1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} \right) \left(1 + \varepsilon^{-2} e^{-\beta_2 y/\varepsilon} \right) \right\|_{0,T_{ij}} \left\| \chi_x \right\|_{0,T_{ij}} \\ &\le C k_j^2 \left(1 + \varepsilon^{-2} e^{-\beta_2 y_{j-1}/\varepsilon} \right) \left\| 1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} \right\|_{0,T_{ij}} \left\| \chi_x \right\|_{0,T_{ij}} \\ &\le C \left(h + N^{-1} \max |\psi'| \right)^2 \left\| 1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} \right\|_{0,T_{ij}} \left\| \chi_x \right\|_{0,T_{ij}} , \end{aligned}$$

by (3.9) and since $e^{\beta_2 k_j/\varepsilon} \le C$ because of (1.11). Application of a discrete Cauchy-Schwarz inequality yields

$$\varepsilon \left| \left((u - u^I)_x, \chi_x \right)_{\Omega_{11}\cup\Omega_{21}} \right| \le C\varepsilon \left(h + N^{-1} \max |\psi'| \right)^2 \left\| 1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} \right\|_{0,\Omega_{11}\cup\Omega_{21}} \left\| \chi_x \right\|_{0,\Omega_{11}\cup\Omega_{21}}$$

Hence

$$\varepsilon \left| \left(u - u^I \right)_x, \chi_x \right)_{\Omega_{11}\cup\Omega_{21}} \right| \le C \left(h + N^{-1} \max |\psi'| \right)^2 |||\chi|||_\varepsilon \tag{5.25}$$

(ii) An argument similar to *(i)* gives

$$\begin{aligned} \varepsilon \left| \left((v + w_1) - (v + w_1)^I)_x, \chi_x \right)_{\Omega_{12}\cup\Omega_{22}} \right| &\le C\varepsilon h^2 \left\| 1 + \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} \right\|_{0,\Omega_{12}\cup\Omega_{22}} \left\| \chi_x \right\|_{0,\Omega_{12}\cup\Omega_{22}} \\ &\le C h^2 |||\chi|||_\varepsilon \end{aligned} \tag{5.26}$$

(iii) Next we consider $w := w_2 + w_{12}$ for $T_{ij} \subset \Omega_{12}$. The stability of the interpolation operator and our bounds on the derivatives of w_2 and w_{12} yield

$$\left\| (w - w^I)_x \right\|_{\infty,T_{ij}} \le \left\| w_x \right\|_{\infty,T_{ij}} + \left\| w_x^I \right\|_{\infty,T_{ij}} \le C \left\| \nabla w \right\|_{\infty,T_{ij}} \le C\varepsilon^{-1} N^{-\sigma}.$$

Thus

$$\varepsilon \left| \left(\left(w - w^I \right)_x , \chi_x \right)_{\Omega_{12}} \right| \le C N^{-\sigma} \left\| \chi_x \right\|_{L_1(\Omega_{12})} \le C N^{-\sigma} \varepsilon^{1/2} \ln^{1/2} N \, |||\chi|||_\varepsilon$$

since $\operatorname{meas} \Omega_{12} = \mathcal{O}(\varepsilon \ln N)$. Therefore

$$\varepsilon \left| \left((w_2 + w_{12}) - (w_2 + w_{12})^I)_x, \chi_x \right)_{\Omega_{12}} \right| \le C N^{-2} |||\chi|||_\varepsilon , \tag{5.27}$$

because $\sigma > 2$.

(iv) Finally, let us bound the terms involving w_2 and w_{12} on Ω_{22}. Using Lemma 5.12 and (5.11) we get

$$\left\| (w_2 - w_2^I)_x \right\|_{0,\Omega_{22}} \le C\varepsilon^{-1/2} N^{-\sigma} \quad \text{and} \quad \left\| (w_{12} - w_{12}^I)_x \right\|_{0,\Omega_{22}} \le C\varepsilon^{-1} N^{-2\sigma}$$

Thus

$$\varepsilon \left|\left(w_2 - w_2^I\right)_x, \chi_x\right)_{\Omega_{22}}\right| \le CN^{-2} \left\|\left|\chi\right|\right\|_\varepsilon \tag{5.28}$$

and

$$\varepsilon \left|\left(w_{12} - w_{12}^I\right)_x, \chi_x\right)_{\Omega_{22}}\right| \le CN^{-2\sigma} \left\|\chi_x\right\|_{0,\Omega_{22}} \le CN^{-2\sigma+1} \left\|\chi\right\|_{0,\Omega_{22}}, \tag{5.29}$$

by an inverse inequality.

Collect (5.25)-(5.29) to obtain

$$\varepsilon \left|\left((u-u^I)_x, \chi_x\right)\right| \le C\left(h + N^{-1}\max|\psi'|\right)^2 \left\|\left|\chi\right|\right\|_\varepsilon \quad \text{for all} \quad \chi \in V^\omega.$$

Clearly we have an identical bound for $\left|\left((u-u^I)_y, \chi_y\right)\right|$ which completes the proof of (5.12).

Proof of (5.13). Recalling the decomposition (5.11), we set $w = w_1 + w_2 + w_{12}$. Then integration by parts yields

$$\begin{aligned} &\left|-\left(\boldsymbol{b}^T\nabla(u-u^I),\chi\right) + \left(c(u-u^I),\chi\right)\right| \\ &\quad \le \left|\left(\boldsymbol{b}^T\nabla(v-v^I),\chi\right)\right| + \left|\left(w-w^I,\boldsymbol{b}^T\nabla\chi\right)\right| + \left|\left(c(v-v^I,\chi\right) + \left((c+\operatorname{div}\boldsymbol{b})(w-w^I,\chi\right)\right| \end{aligned} \tag{5.30}$$

The terms on the right-hand side are analysed separately.

First

$$\left|\left(c(v-v^I,\chi\right) + \left((c+\operatorname{div}\boldsymbol{b})(w-w^I,\chi\right)\right| \le C\left(\|v-v^I\|_0 + \|w-w^I\|_0\right)\|\chi\|.$$

Adapting the technique from Section 5.3.1 it is easily shown that

$$\|v-v^I\|_0 + \|w-w^I\|_0 \le C\left(h + N^{-1}\max|\psi'|\right)^2,$$

since v and w satisfy derivative bounds similar to those of u; cf. Corollary 5.6. Thus

$$\left|\left(c(v-v^I,\chi\right) + \left((c+\operatorname{div}\boldsymbol{b})(w-w^I,\chi\right)\right| \le C\left(h + N^{-1}\max|\psi'|\right)^2 \left\|\left|\chi\right|\right\|_\varepsilon. \tag{5.31}$$

Next let us bound the second and third term in (5.30). The Cauchy-Schwarz inequality and Proposition 5.13 yield

$$\begin{aligned} \left|\left(w-w^I,\boldsymbol{b}^T\nabla\chi\right)\right| &\le C\|w-w^I\|_{0,\Omega_{22}}\|\nabla\chi\|_{0,\Omega_{22}} + \|w-w^I\|_{0,\Omega\setminus\Omega_{22}}\|\nabla\chi\|_{0,\Omega\setminus\Omega_{22}} \\ &\le C\left(\varepsilon^{1/2}N^{-5/2} + N^{-3}\right)\|\nabla\chi\|_{0,\Omega_{22}} + C\varepsilon^{1/2}\left(h + N^{-1}\max|\psi'|\right)^2\|\nabla\chi\|_{0,\Omega\setminus\Omega_{22}} \\ &\le C\left(h + N^{-1}\max|\psi'|\right)^2 \left\|\left|\chi\right|\right\|_\varepsilon, \end{aligned} \tag{5.32}$$

where we have used an inverse inequality and that on Ω_{22} the mesh is uniform with mesh size $\mathcal{O}\left(N^{-1}\right)$.

Finally we study the term $\left(\boldsymbol{b}^T\nabla(v-v^I),\chi\right)$. Let $b_{1;ij} = b_1(x_i,y_j)$ for $i,j = 0,\ldots,N$. Using the second identity of Lemma 5.11, we get

$$
\begin{aligned}
&\big(b_1(v-v^I)_x,\chi\big)\\
&\quad=\sum_{T_{ij}\in\Omega^N}\Big\{\big(b_{1;ij}(v-v^I)_x,\chi\big)_{T_{ij}}+\big((b_1-b_{1;ij})(v-v^I)_x,\chi\big)_{T_{ij}}\Big\}\\
&\quad=\sum_{T_{ij}\in\Omega^N} b_{1;ij}\int_{T_{ij}}\Big\{\Big[\frac{1}{6}\big(F_i^2\big)'\chi_x-\frac{1}{12}h_i^2\chi\Big]v_{xxx}\\
&\qquad\qquad+\Big[G_j\big(\chi-F_i'\chi_x\big)-\frac{1}{3}\big(G_j^2\big)'\big(\chi_y-F_{ij}'\chi_{xy}\big)\Big]v_{xyy}\Big\}\\
&\qquad+\frac{1}{12}\sum_{i=1}^{N-1}\sum_{j=1}^{N}\big(b_{1;i+1,j}h_{i+1}^2-b_{1;ij}h_i^2\big)\int_{y_{j-1}}^{y_j}\big(\chi v_{xx}\big)(x_i,y)dy\\
&\qquad+\sum_{T_{ij}\in\Omega^N}\big((b_1-b_{1;ij})(v-v^I)_x,\chi\big)_{T_{ij}}\\
&\quad=:I_1+I_2+I_3.
\end{aligned}
\tag{5.33}
$$

Use (5.11) to obtain

$$
\begin{aligned}
|I_1|\le C\sum_{T_{ij}\in\Omega^N}\Big\{&\big(h_i^2+k_j^2\big)\big(\|\chi\|_{L_1(T_{ij})}+h_i\|\chi_x\|_{L_1(T_{ij})}\big)\\
&+k_j^2\big(k_j\|\chi_y\|_{L_1(T_{ij})}+h_ik_j\|\chi_{xy}\|_{L_1(T_{ij})}\big)\Big\}.
\end{aligned}
$$

Thus

$$
|I_1|\le C\sum_{T_{ij}\in\Omega^N}\big(h_i^2+k_j^2\big)\|\chi\|_{L_1(T_{ij})}\le Ch^2\|\chi\|_{L_1(\Omega)}\le Ch^2\|\chi\|_0. \tag{5.34}
$$

For I_2 we proceed as follows. First,

$$
\int_{y_{j-1}}^{y_j}\big(\chi v_{xx}\big)(x_i,y)dy=\sum_{k=1}^{i}\int_{y_{j-1}}^{y_j}\int_{x_{k-1}}^{x_k}\big(\chi_xv_{xx}+\chi v_{xxx}\big)(x,y)dxdy
$$

yields

$$
\begin{aligned}
&\sum_{i=1}^{qN}\big(b_{1;i+1,j}h_{i+1}^2-b_{1;ij}h_i^2\big)\int_{y_{j-1}}^{y_j}\big(\chi v_{xx}\big)(x_i,y)dy\\
&\quad=\sum_{i=1}^{qN}\big(b_{1;qN+1,j}h_{qN+1}^2-b_{1;ij}h_i^2\big)\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\big(\chi_xv_{xx}+\chi v_{xxx}\big)(x,y)dxdy.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\Big|\sum_{i=1}^{qN}\sum_{j=1}^{N}\big(b_{1;i+1,j}h_{i+1}^2-b_{1;ij}h_i^2\big)\int_{y_{j-1}}^{y_j}\big(\chi v_{xx}\big)(x_i,y)dy\Big|\le Ch^2\big\|\chi_xv_{xx}+\chi v_{xxx}\big\|_{L_1(\Omega_{11}\cup\Omega_{12})}\\
&\qquad\le Ch^2\varepsilon^{1/2}\ln^{1/2}N\big(\|\chi_x\|_0+\|\chi\|_0\big)\le Ch^2\ln^{1/2}N\,|||\chi|||_\varepsilon\,,
\end{aligned}
\tag{5.35}
$$

by (5.11). Furthermore, for $i=qN+1,\dots,N$, we have

$$
\Big|\big(b_{1;i+1,j}h_{i+1}^2-b_{1;ij}h_i^2\big)\int_{y_{j-1}}^{y_j}\big(\chi v_{xx}\big)(x_i,y)dy\Big|\le Ch_i^2\|\chi\|_{L_1(T_{ij})},
$$

because $\left|b_{1;i+1,j} - b_{1;ij}\right| \le Ch_{i+1}$, $h_i = h_{i+1} \le h$ and

$$\int_{y_{j-1}}^{y_j} \left(\chi v_{xx}\right)(x_i, y)dy \le Ch_i^{-1} \|\chi\|_{L_1(T_{ij})},$$

by an inverse inequality. We get

$$\Big|\sum_{i=qN+1}^{N-1} \sum_{j=1}^{N} \left(b_{1;i+1,j} h_{i+1}^2 - b_{1;ij} h_i^2\right) \int_{y_{j-1}}^{y_j} \left(\chi v_{xx}\right)(x_i, y)dy\Big| \le Ch^2 \|\chi\|_0. \tag{5.36}$$

For I_3 we have the following bound:

$$|I_3| \le \sum_{T_{ij} \in \Omega_N} \left\|b_1 - b_{1;ij}\right\|_{\infty,T_{ij}} \Big(h_i \left\|v_{xx}\right\|_{\infty,T_{ij}} + k_j \left\|v_{xy}\right\|_{\infty,T_{ij}}\Big) \left\|\chi\right\|_{L_1(T_{ij})} \le Ch^2 \|\chi\|_0, \tag{5.37}$$

by Lemma 5.12 and (2.13b).

Collect (5.33)–(5.37) to obtain

$$\left|\left(b_1 (v - v^I)_x, \chi\right)\right| \le Ch^2 \ln^{1/2} N \, |||\chi|||_\varepsilon \tag{5.38}$$

with the analogously bound

$$\left|\left(b_2 (v - v^I)_y, \chi\right)\right| \le Ch^2 \ln^{1/2} N \, |||\chi|||_\varepsilon \,. \tag{5.39}$$

Substituting (5.31), (5.32), (5.38) and (5.39) into (5.30), we are done.

5.3.2.4 Maximum-norm error bounds

In this section we use Theorem 5.9 and the interpolation error bounds from Corollary 5.6 to obtain bounds for the error of the Galerkin method in the maximum norm.

Start with the region Ω_{22}, where the mesh is quasi-uniform with mesh size $\mathcal{O}\left(N^{-1}\right)$:

$$\left\|u^I - U\right\|_{\infty,\Omega_{22}} \le CN \left\|u^I - U\right\|_{0,\Omega_{22}} \le C \left(Nh^2 \ln^{1/2} N + N^{-1} \max |\psi'|^2\right).$$

Thus on a standard Shishkin mesh, where $h = \mathcal{O}\left(N^{-1}\right)$, one gets

$$\|u - U\|_{\infty,\Omega_{22}} \le CN^{-1} \ln^2 N,$$

where Corollary 5.6 was also used. For the Bakhvalov-Shishkin mesh we get

$$\|u - U\|_{\infty,\Omega_{22}} \le CN^{-1} \ln^{1/2} \quad \text{if} \quad \varepsilon \le CN^{-1},$$

because for this mesh $h = \mathcal{O}\left(\max\{N^{-1}, \varepsilon\}\right)$.

Now let (x_i, y_j) be any mesh node in Ω_{21}. Then following [96, pp. 11,12] we obtain

$$\begin{aligned}
\left|(u^I - U)(x_i, y_j)\right| &= \left|\int_0^{x_i} (u^I - U)(x, y_j)dx\right| \le CN \int_{[0,\lambda_x]\times[y_{j-1},y_j]} |(u^I - U)_x| \\
&\le CN \left(\varepsilon N^{-1} \ln N\right)^{1/2} \left\|\nabla(u^I - U)\right\|_{0,[0,\lambda_x]\times[y_{j-1},y_j]} \\
&\le CN^{1/2} \ln^{1/2} N \left|\left|\left|u^I - U\right|\right|\right|_\varepsilon .
\end{aligned}$$

Thus

$$\|u - U\|_{\infty,\Omega_{21}} \le CN^{1/2} \ln^{1/2} N \left(h^2 \ln^{1/2} N + N^{-2} \max |\psi'|^2\right),$$

by Corollary 5.6 and Theorem 5.9. Clearly identical bounds hold on Ω_{12}.

Apply this result to get bounds for our S-type meshes:

$$\|u - U\|_{\infty,\Omega_{12}\cup\Omega_{21}} \le \begin{cases} CN^{-3/2} \ln^{5/2} N & \text{for standard Shishkin meshes} \\ CN^{-3/2} \ln N & \text{for Bakhvalov-Shishkin meshes with } \varepsilon \le CN^{-1}. \end{cases}$$

5.3.2.5 Gradient recovery

Similar to Section 3.2.3 a gradient recovery operator can be defined for the bilinear Galerkin FEM that gives approximations of the gradient which are superior to those of Theorem 5.7. We follow [87].

Let T be a rectangle of Ω^N and let $\widetilde{T}$ be the patch associated with T, consisting of all rectangles that have a common corner with T (see Fig. 5.3). We define for $v \in V^\omega$ the recovered gradient

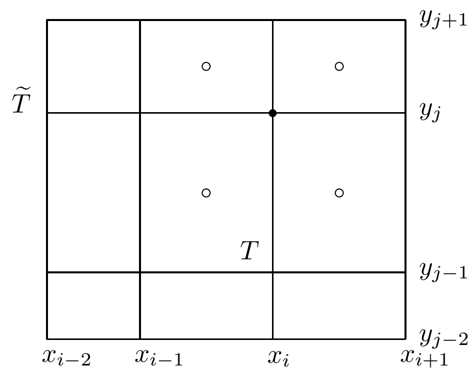


Figure 5.3: Mesh rectangle T and associated patch $\widetilde{T}$.

$\boldsymbol{R}v$ as follows. First we compute the gradient of v at the midpoints of the mesh rectangles $\big(\boldsymbol{\gamma}_{ij} := \nabla v(x_{i-1/2}, y_{j-1/2})\big)$. Then these values are bilinearly interpolated to give the values of Rv at the mesh points of the triangulation, viz.,

$$\big(\boldsymbol{R}v\big)_{ij} = \boldsymbol{\alpha}_{ij} := \frac{\boldsymbol{\gamma}_{ij} h_{i+1} k_{j+1} + \boldsymbol{\gamma}_{i+1,j} h_i k_{j+1} + \boldsymbol{\gamma}_{i,j+1} h_{i+1} k_j + \boldsymbol{\gamma}_{i+1,j+1} h_i k_j}{(h_i + h_{i+1})(k_j + k_{j+1})}. \tag{5.40}$$

Bilinear interpolation is again used to extend the recovered gradient from the mesh nodes to the whole of Ω:

$$\begin{aligned}\big(\boldsymbol{R}w^N\big)(x,y) := {}& \boldsymbol{\alpha}_{i-1,j-1} \frac{x_i - x}{h_i} \frac{y_j - y}{k_j} + \boldsymbol{\alpha}_{i,j-1} \frac{x - x_{i-1}}{h_i} \frac{y_j - y}{k_j} + \boldsymbol{\alpha}_{i-1,j} \frac{x_i - x}{h_i} \frac{y - y_{j-1}}{k_j} \\ & + \boldsymbol{\alpha}_{i,j} \frac{x - x_{i-1}}{h_i} \frac{y - y_{j-1}}{k_j} \qquad \text{for } (x,y) \in T_{ij}, \quad i,j = 2,\dots,N-1.\end{aligned}$$

For the boundary rectangles we simply extrapolate the well-defined bilinear function of the adjacent rectangles.

Lemma 5.14. $\boldsymbol{R} : V^\omega \to V^\omega \times V^\omega$ *is a linear operator with the following properties:*

(locality) $\boldsymbol{R}v$ *on* T *depends only on values of* v *on the patch* $\widetilde{T}$,

(stability)
$$\big\|\boldsymbol{R}v\big\|_{\infty,T} \le C\big\|v\big\|_{1,\infty,\widetilde{T}} \quad \text{for all } v \in V^\omega, \tag{5.41a}$$
$$\big\|\boldsymbol{R}v\big\|_{0,T} \le C\big\|v\big\|_{1,\widetilde{T}} \quad \text{for all } v \in V^\omega, \tag{5.41b}$$

(consistency)
$$\boldsymbol{R}v^I = \nabla v \quad \text{on } T \quad \text{for all } v \text{ that are quadratic on } \widetilde{T}. \tag{5.41c}$$

Proof. The first three properties are immediate consequences of the definition of $\boldsymbol{R}$, while (5.41c) is easily verified by a Taylor expansion of v. □

Now, given any continuous function v on $\widetilde{T}$, we denote by Qv the quadratic function on $\widetilde{T}$ with

$$\big(Qv\big)(P_k) = v(P_k) \qquad (k = 1,\dots,6)$$

(see Fig. 5.4). It is easy to show that this set of degrees of freedom is unisolvent and thus our Lagrange interpolant Qv is well defined.

The decompostion (5.11) and a careful analysis yield the following bounds for quadratic interpolation.

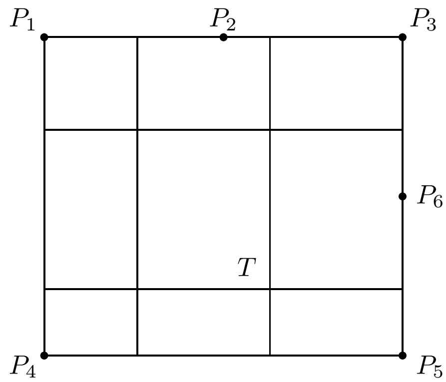


Figure 5.4: Definition of the quadratic interpolant on the patch $\widetilde{T}$.

Lemma 5.15. *Let $\omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \geq 3$ that satisfies (1.11). Assume that the solution u of (5.2) can be decomposed as in (5.11). Then*

$$\varepsilon \big\| u - Qu \big\|_{1,\infty,\widetilde{T}} \leq \begin{cases} C\big(h + N^{-1} \max|\psi'|\big)^2 & \textit{for} \quad T \subset \Omega \setminus \Omega_{22}, \\ CN^{-2} & \textit{for} \quad T \subset \Omega_{22}, \end{cases}$$

$$\big\| u - Qu \big\|_{\infty,\widetilde{T}} \leq \begin{cases} C\big(h + N^{-1} \max|\psi'|\big)^3 & \textit{for} \quad T \subset \Omega \setminus \Omega_{22}, \\ CN^{-3} & \textit{for} \quad T \subset \Omega_{22} \end{cases}$$

and

$$\varepsilon^{1/2} \big\| u - Qu \big\|_{1,\widetilde{T}} \leq \begin{cases} C(\operatorname{meas} \tilde{T})^{1/2}\big(h + N^{-1} \max|\psi'|\big)^2 & \textit{for} \quad T \subset \Omega \setminus \Omega_{22}, \\ CN^{-3} & \textit{for} \quad T \subset \Omega_{22}\,. \end{cases}$$

We would like to estimate the difference between the gradient and the recovered gradient in the ε-weighted H^1 seminorm. We start from

$$\varepsilon^{1/2} \big\| \nabla u - \boldsymbol{R}U \big\|_0 \leq \varepsilon^{1/2} \big\| \nabla u - \boldsymbol{R}u^I \big\|_0 + \varepsilon^{1/2} \big\| \boldsymbol{R}(u^I - U) \big\|_0, \tag{5.42}$$

by a triangle inequality. For the second term in (5.42), the stability property (5.41b) of the recovery operator and the superconvergence result of Theorem 5.9 yield

$$\varepsilon^{1/2} \big\| \boldsymbol{R}(u^I - U) \big\|_0 \leq C \left(h + N^{-1} \max|\psi'|\right)^2 \ln^{1/2} N. \tag{5.43}$$

In the next result we estimate the first term in (5.42).

Lemma 5.16. *Let $\omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \geq 3$ that satisfies (1.11). Assume that the solution u of (5.2) can be decomposed as in (5.11) and that*

$$\min\left\{h_{qN}, k_{qN}\right\} \geq C\varepsilon N^{-1}. \tag{5.44}$$

Then

$$\varepsilon^{1/2} \big\| \nabla u - \boldsymbol{R}u^I \big\|_0 \leq C\big(h + N^{-1} \max|\psi'|\big)^2 \ln^{1/2} N. \tag{5.45}$$

Proof. For any $T \in \Omega^N$, the consistency property (5.41c) of the recovery operator yields

$$\big\| \nabla u - \boldsymbol{R}u^I \big\|_{0,T} \leq \big\| \nabla(u - Qu) \big\|_{0,T} + \big\| \boldsymbol{R}(u - Qu)^I \big\|_{0,T}, \tag{5.46}$$

since $\boldsymbol{R}(Qu)^I = \nabla Qu$. For the interpolation operator we can use the stability estimates

$$\big\| v^I \big\|_{\infty,T} \leq C \|v\|_{\infty,T} \quad \text{and} \quad \big\| v^I \big\|_{1,\infty,T} \leq C \big\| v \big\|_{1,\infty,T}.$$

To estimate the second term in (5.46), we bound the L_2 norm by the L_∞ norm and apply the stability property (5.41a) of the recovery operator:

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq (\operatorname{meas} T)^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{\infty,T} \leq C(\operatorname{meas} T)^{1/2}\left\|(u-Qu)^I\right\|_{1,\infty,\widetilde{T}}. \tag{5.47}$$

Thus, for $T \notin \Omega_{22}$ we have

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq C\varepsilon^{-1}(\operatorname{meas} T)^{1/2}\left(h+N^{-1}\max|\psi'|\right)^2, \tag{5.48}$$

by Lemma 5.15.

Next we consider $T \in \Omega_{22}$. We apply an inverse inequality and the L_∞ stability of Π to (5.47) to get

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq CN^{-1}\left(\min_{\widetilde{T}} h\right)^{-1}\left\|u-u^*\right\|_{\infty,\widetilde{T}}.$$

If $\min_{\widetilde{T}} h = \mathcal{O}\left(N^{-1}\right)$ then

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq C\left\|u-Qu\right\|_{\infty,\widetilde{T}} \leq CN^{-3}, \tag{5.49}$$

by Lemma 5.15. Otherwise — for the elements T along the transition from the fine to the coarse mesh — we have to estimate more carefully:

$$\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq \left\|(u-Qu)^I\right\|_{1,\widetilde{T}} \leq \sum_{T\in\widetilde{T}} \frac{(\operatorname{meas} T)^{1/2}}{\min_T h}\left\|u-Qu\right\|_{\infty,T}.$$

From (5.44), we have

$$\varepsilon^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq C\left\|u-Qu\right\|_{\infty,\widetilde{T}} \leq CN^{-3}, \tag{5.50}$$

by Lemma 5.15. Combining (5.48)–(5.50), we have

$$\varepsilon^{1/2}\left\|\boldsymbol{R}(u-Qu)^I\right\|_{0,T} \leq \begin{cases} C\varepsilon^{-1/2}(\operatorname{meas} T)^{1/2}\left(h+N^{-1}\max|\psi'|\right)^2 & \text{for} \quad T\subset\Omega\setminus\Omega_{22}, \\ CN^{-3} & \text{for} \quad T\subset\Omega_{22}. \end{cases}$$

We use the last estimate of Lemma 5.15 and (5.46) to obtain

$$\varepsilon^{1/2}\left\|\nabla u-\boldsymbol{R}u^I\right\|_{0,T} \leq \begin{cases} C\varepsilon^{-1/2}(\operatorname{meas} T)^{1/2}\left(h+N^{-1}\max|\psi'|\right)^2 & \text{for} \quad T\subset\Omega\setminus\Omega_{22}, \\ CN^{-3} & \text{for} \quad T\subset\Omega_{22}\,. \end{cases}$$

Recalling that

$$\left\|\nabla u-\boldsymbol{R}u^I\right\|_0^2 = \sum_{T\in\Omega^N}\left\|\nabla u-\boldsymbol{R}u^I\right\|_{0,T}^2$$

and $\operatorname{meas}\big(\Omega\setminus\Omega_{22}\big) = \mathcal{O}\left(\varepsilon\ln N\right)$, the proof of the lemma is complete. □

Remark 5.17. *The condition (5.44) is satisfied if for example $\tilde{\varphi}'$ in Section 1.3 is bounded from below by a positive constant independently of ε and N. Both the original Shishkin mesh and the Bakhvalov-Shishkin mesh satisfy this condition.*

As a consequence of (5.42), (5.43) and Lemma 5.16 we have

Theorem 5.18. *Let $\omega_x\times\omega_y$ be a tensor-product S-type mesh with $\sigma\geq 5/2$ that satisfies (1.11) and (5.44). Assume that the solution u of (5.2) can be decomposed as in (5.11). Then*

$$\varepsilon^{1/2}\left\|\nabla u-\boldsymbol{R}U\right\|_0 \leq C\left(h+N^{-1}\max|\psi'|\right)^2\ln^{1/2}N.$$

Similar to the one-dimensional case (Remark 3.6), using $\boldsymbol{R}U$ instead of ∇U, we get an asymptotically exact error estimator for the weighted H^1-seminorm of the finite element error $\varepsilon^{1/2}\left\|\nabla(u-U)\right\|_0$ on S-type meshes.

	$\|\|\|u-U\|\|\|_\varepsilon$		$\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$		$\varepsilon^{1/2}\left\|\nabla u-\nabla U\right\|_0$		$\varepsilon^{1/2}\left\|\nabla u-\boldsymbol{R}U\right\|_0$	
N	error	rate	error	rate	error	rate	error	rate
16	2.6900e-1	0.63	5.2110e-2	1.25	2.6898e-1	0.63	9.5425e-1	2.86
32	1.7359e-1	0.72	2.1896e-2	1.43	1.7359e-1	0.72	1.3141e-1	1.81
64	1.0556e-1	0.77	8.1467e-3	1.53	1.0556e-1	0.77	3.7507e-2	1.48
128	6.1881e-2	0.80	2.8137e-3	1.60	6.1881e-2	0.80	1.3479e-2	1.56
256	3.5421e-2	0.83	9.2543e-4	1.65	3.5421e-2	0.83	4.5685e-3	1.64
512	1.9936e-2	0.85	2.9398e-4	1.69	1.9936e-2	0.85	1.4687e-3	1.69
1024	1.1078e-2	—	9.0961e-5	—	1.1078e-2	—	4.5612e-4	—

Table 5.3: Shishkin mesh

	$\|\|\|u-U\|\|\|_\varepsilon$		$\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$		$\varepsilon^{1/2}\left\|\nabla u-\nabla U\right\|_0$		$\varepsilon^{1/2}\left\|\nabla u-\boldsymbol{R}U\right\|_0$	
N	error	rate	error	rate	error	rate	error	rate
16	1.2475e-1	1.00	7.9084e-3	2.00	1.2471e-1	1.00	5.0012e-1	3.43
32	6.2574e-2	1.00	1.9800e-3	2.00	6.2569e-2	1.00	4.6315e-2	3.09
64	3.1312e-2	1.00	4.9620e-4	2.00	3.1311e-2	1.00	5.4227e-3	2.43
128	1.5659e-2	1.00	1.2425e-4	2.00	1.5659e-2	1.00	1.0044e-3	2.08
256	7.8298e-3	1.00	3.1096e-5	2.00	7.8298e-3	1.00	2.3690e-4	2.01
512	3.9149e-3	1.00	7.7789e-6	2.00	3.9149e-3	1.00	5.8638e-5	2.00
1024	1.9575e-3	—	1.9460e-6	—	1.9575e-3	—	1.4624e-5	—

Table 5.4: Bakhvalov-Shishkin mesh

5.3.2.6 Numerical tests

Let us verify our theoretical results for the Galerkin FEM using bilinear trial and test functions on S-type meshes when applied to the test problem (5.6). In our computations we have chosen $\varepsilon = 10^{-8}$ and $\sigma = 3$ for the meshes. In the tables we compare both the error in the ε-weighted energy norm $\|\|\|u-U\|\|\|_\varepsilon$ with the error in the discrete energy norm $\left\|\left\|\left\|u^I-U\right\|\right\|\right\|_\varepsilon$ and the accuracy of the gradient approximation ∇U with that of the recovered gradient approximation $\boldsymbol{R}U$. The errors are estimated using a 4th-order Gauß-Legendre formula on each mesh rectangle. The rates of convergence are computed in the usual way. The tables are clear illustrations of Theorems 5.7, 5.9 and 5.18.

5.3.3 Upwind FEM

In Section 3.3 we have studied a FEM with artificial viscosity stabilization in one dimension. It can be generalized to two dimensions as follows. Set

$$\boldsymbol{\hbar} := \operatorname{diag}(\hbar, \bar{k}) \quad \text{with} \quad \hbar(x,y) := h_i \quad \text{for} \quad x \in (x_{i-1}, x_i) \quad \text{and} \quad \bar{k}(x,y) := k_j \quad \text{for} \quad y \in (y_{j-1}, y_j)$$

and let $\kappa \geq 0$ be arbitrary constants. Then we add artificial viscosity of order $\kappa\hbar$ in x-direction and of order $\kappa\bar{k}$ in y-direction, i. e., we consider the discretization: Find $U \in V^\omega$ such that

$$a_\kappa(U,v) := a(U,v) + \kappa\left(\boldsymbol{\hbar}\nabla U, \nabla v\right) = (f,v) \quad \text{for all} \quad v \in V^\omega.$$

The norm naturally associated with the bilinear form $a_\kappa(\cdot,\cdot)$ is

$$\|\|\|v\|\|\|_\kappa := \left[\|\|\|v\|\|\|_\varepsilon^2 + \kappa\left(\boldsymbol{\hbar}\nabla v, \nabla v\right)\right]^{1/2} \geq \|\|\|v\|\|\|_\varepsilon$$

and $a_\kappa(\cdot,\cdot)$ is coercive with respect to this norm, i. e.,

$$a_\kappa(v,v) \geq \|\|\|v\|\|\|_\kappa^2 \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{5.51}$$

In our analysis we follow Schneider et al. [92], but refine it by explicitly monitoring the dependence on κ. Let again $\eta = u^I - u$ denote the interpolation error and $\chi = u^I - U$ the difference between interpolated and exact solution. Because of the artificial viscosity the discretization does not satisfy the Galerkin orthogonality condition, but we have

$$a_\kappa(\chi,\chi) = a(\eta,\chi) + \kappa(\boldsymbol{\hbar}\nabla\eta,\nabla\chi) + \kappa(\boldsymbol{\hbar}\nabla u,\nabla\chi) \tag{5.52}$$

(i) For the first term we have two bounds from Sections 5.3.2.1 and 5.3.2.2:

$$|a(\eta,\chi)| \le C\,|||\chi|||_\varepsilon \begin{cases} h + N^{-1}\max|\psi'| & \text{for general linear and bilinear elements,} \\ h^2\ln^{1/2} N + N^{-2}\max|\psi'|^2 & \text{for bilinear elements.} \end{cases} \tag{5.53}$$

(ii) Next we bound $\kappa\big(\boldsymbol{\hbar}\nabla\eta,\nabla\chi\big)$. Let T_{ij} be arbitrary. Then

$$\big(\hbar\eta_x,\chi_x\big)_{T_{ij}} = h_i\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\eta_x\chi_x = \int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i}\eta_x\int_{x_{i-1}}^{x_i}\chi_x.$$

Thus

$$\big|\big(\hbar\eta_x,\chi_x\big)_{T_{ij}}\big| \le 2\big\|\eta\big\|_{L_\infty(T_{ij})}\big\|\chi_x\big\|_{L_1(T_{ij})}$$

Consequently we have

$$\begin{aligned}\big|\big(\hbar\eta_x,\chi_x\big)\big| &\le C\Big\{N^{-2}\big\|\chi_x\big\|_{\Omega_{22}} + \big(N^{-1}\max|\psi'|\big)^2\big(\varepsilon\ln N\big)^{1/2}\big\|\chi_x\big\|_{\Omega\setminus\Omega_{22}}\Big\} \\ &\le CN^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon\,,\end{aligned}$$

by an inverse inequality and (5.10). An analogous estimate holds for $\big|\big(\hbar\eta_y,w_y^N\big)\big|$. Thus

$$\kappa\big|\big(\boldsymbol{\hbar}\nabla\eta,\nabla\chi\big)\big| \le C\kappa N^{-1}\max|\psi'|\ln^{1/2}N\,|||\chi|||_\varepsilon\,. \tag{5.54}$$

(iii) Last $\big(\boldsymbol{\hbar}\nabla u,\nabla\chi\big)$ has to be considered. We restrict ourselves to bounding $\big(\hbar u_x,\chi_x\big)$ since the term $\big(\hbar u_y,\chi_y\big)$ can be treated analogously. Using the decomposition of Theorem 5.1, we get

$$\begin{aligned}\big(\hbar u_x,\chi_x\big) &= \big(\hbar(v+w_2)_x,\chi_x\big)_{\Omega_{11}\cup\Omega_{12}} + \big(\hbar(v+w_2)_x,\chi_x\big)_{\Omega_{21}\cup\Omega_{22}} \\ &\quad + \big(\hbar(w_1+w_{12})_x,\chi_x\big)_{\Omega_{11}\cup\Omega_{12}} + \big(\hbar(w_1+w_{12})_x,\chi_x\big)_{\Omega_{21}\cup\Omega_{22}}\end{aligned} \tag{5.55}$$

The Cauchy-Schwarz inequality and Theorem 5.1 yield

$$\big|\big(\hbar(v+w_2)_x,\chi_x\big)_{\Omega_{11}\cup\Omega_{12}}\big| \le Ch\big(\varepsilon\ln N\big)^{1/2}\big\|\chi_x\big\|_{\Omega_{11}\cup\Omega_{12}} \le Ch\ln^{1/2}N\,|||\chi|||_\varepsilon\,. \tag{5.56}$$

On $\Omega_{21}\cup\Omega_{22}$ we have

$$\begin{aligned}\big(\hbar(v+w_2)_x,\chi_x\big)_{\Omega_{21}\cup\Omega_{22}} &= H\int_0^1\int_{\lambda_x}^1(v+w_2)_x\chi_x\,dxdy \\ &= -H\int_0^1\Big\{\big((v+w_2)_x\chi\big)(\lambda_x,y) + \int_{\lambda_x}^1(v+w_2)_{xx}\chi\,dx\Big\}dy.\end{aligned}$$

Thus

$$\big|\big(\hbar(v+w_2)_x,\chi_x\big)_{\Omega_{21}\cup\Omega_{22}}\big| \le CN^{-1}\left\{\big\|\chi\big\| + \int_0^1|\chi(\lambda_x,y)|\,dy\right\}. \tag{5.57}$$

Note that

$$\int_0^1 |\chi(\lambda_x, y)|\, dy = \int_0^1 \left| \int_0^{\lambda_x} \chi_x dx \right| dy \le \|\chi_x\|_{L_1(\Omega_{11}\cup\Omega_{12})} \le C \ln^{1/2} N \, |||\chi|||_\varepsilon \, .$$

We apply this inequality to (5.57) to obtain

$$\left|(\hbar(v + w_2)_x, \chi_x)_{\Omega_{21}\cup\Omega_{22}}\right| \le CN^{-1} \ln^{1/2} N \, |||\chi|||_\varepsilon \, . \tag{5.58}$$

Now we bound the last two terms in (5.55). Using Theorem 5.1 we get, for any $T_{ij} \in \Omega^N$,

$$\left|\left(\hbar(w_1 + w_{12})_x, \chi_x\right)_{T_{ij}}\right| \le C \int_{y_{j-1}}^{y_j} \left\{ \int_{x_{i-1}}^{x_i} \varepsilon^{-1} e^{-\beta_1 x/\varepsilon} dx \int_{x_{i-1}}^{x_i} \left|\chi_x\right| dx \right\} dy \tag{5.59}$$

This implies that

$$\left|\left(\hbar(w_1 + w_{12})_x, \chi_x\right)_{T_{ij}}\right| \le \begin{cases} CN^{-1} \max|\psi'| \left\|\chi_x\right\|_{L_1(T_{ij})} & \text{for} \quad T_{ij} \subset \Omega_{11}^N \cup \Omega_{12}^N, \\ CN^{-2} \left\|\chi_x\right\|_{L_1(T_{ij})} & \text{for} \quad T_{ij} \subset \Omega_{21}^N \cup \Omega_{22}^N. \end{cases}$$

Thus

$$\begin{aligned} &\left|\left(\hbar(w_1 + w_{12})_x, \chi_x\right)_{\Omega_{11}\cup\Omega_{12}}\right| \\ &\qquad \le CN^{-1} \max|\psi'| \left\|\chi_x\right\|_{L_1(\Omega_{11}\cup\Omega_{12})} \le CN^{-1} \max|\psi'| \ln^{1/2} N \, |||\chi|||_\varepsilon \end{aligned} \tag{5.60}$$

and

$$\left|\left(\hbar(w_1 + w_{12})_x, \chi_x\right)_{\Omega_{21}\cup\Omega_{22}}\right| \le CN^{-1} \left\|\chi\right\|_0, \tag{5.61}$$

by an inverse inequality.

Combine the last two bounds with (5.55), (5.56) and (5.58) to get

$$\left|\left(\hbar u_x, \chi_x\right)\right| \le CN^{-1} \max|\psi'| \ln^{1/2} N \, |||\chi|||_\varepsilon \, .$$

With an analogous estimate for $\left(\hbar u_y, \chi_y\right)$ we have

$$\kappa \left|(\boldsymbol{\hbar}\nabla u, \nabla\chi)\right| \le C\kappa N^{-1} \max|\psi'| \ln^{1/2} N \, |||\chi|||_\varepsilon \, . \tag{5.62}$$

Finally combine (5.51)–(5.54) and (5.62) in order obtain the main result of this section.

Theorem 5.19. *Let $\omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \ge 2$ that satisfies (1.11) and (5.44). Then the upwind FEM solution U satisfies*

$$\left|\left|\left|u^I - U\right|\right|\right|_\kappa \le C \left(1 + \kappa \ln^{1/2} N\right) N^{-1} \max|\psi'|$$

and, for bilinear elements and $\sigma \ge 5/2$,

$$\left|\left|\left|u^I - U\right|\right|\right|_\kappa \le C \left\{ \kappa N^{-1} \max|\psi'| \ln^{1/2} N + h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2 \right\}.$$

A consequence of Theorem 5.19 and Corollary 5.6 is the following bound of the error in the ε-weighted energy norm:

$$|||u - U|||_\varepsilon \le C \left(h + N^{-1} \max|\psi'| \ln^{1/2} N\right) \quad \text{if} \ \ \kappa \le C.$$

Remark 5.20. (i) *The superconvergence property the Galerkin FEM with bilinear elements is not affected if we take $\kappa = \mathcal{O}\left(N^{-1}\right)$. For the efficient treatment of the discrete systems, however the choice $\kappa = \mathcal{O}\left(1\right)$ is more appropriate which then results in a loss of the superconvergence property.*

(ii) *The $|||\cdot|||_\kappa$ bounds imply that the method gives uniform convergent approximations of the gradient on the coarse mesh region Ω_{22}. For example for a Shishkin mesh, where* $\max|\psi'| \le C\ln N$ *and $h \le 2N^{-1}$, we have*

$$\kappa^{1/2}N^{-1/2}\left\|\nabla\left(u^I - U\right)\right\|_{0,\Omega_{22}} \le C\left\{\kappa N^{-1}\ln^{3/2} + N^{-2}\ln^2 N\right\}.$$

Thus

$$\left\|\nabla\left(u^I - U\right)\right\|_{0,\Omega_{22}} \le \begin{cases} CN^{-1/2}\ln^{3/2} & \textit{if } \kappa = \mathcal{O}\left(1\right), \\ CN^{-1}\ln^2 & \textit{if } \kappa = \mathcal{O}\left(N^{-1}\right). \end{cases}$$

Note that in contrast to the streamline-diffusion FEM we have full control of the gradient, while for SDFEM one has uniform bounds for the streamline derivative $\|\boldsymbol{b}\cdot\nabla(u^I - U)\|_{0,\Omega_{22}}$ only; see Section 5.3.4.

(iii) *Suboptimal maximum-norm error bounds on Ω_{22} can be obtained by application of the discrete Sobolev inequality*

$$\|\chi\|_{\infty,\Omega_{22}} \le C\ln^{1/2} N\,\|\nabla\chi\|_{0,\Omega_{22}}\,, \tag{5.63}$$

that holds true for piecewise-polynomial functions χ that vanish on a part of the boundary of finite length, see [101, Lemma 5.4] or [38]. We get

$$\|u - U\|_{\infty,\Omega_{22}} \le \begin{cases} CN^{-1/2}\ln^2 N & \textit{if } \kappa = \mathcal{O}\left(1\right), \\ CN^{-1}\ln^{5/2} N & \textit{if } \kappa = \mathcal{O}\left(N^{-1}\right). \end{cases}$$

5.3.4 Streamline-diffusion FEM

Introduced by Hughes and Brooks [34], this method is the most commonly used stabilized FEM for the discretisation of convection-diffusion and related problems. Starting from the weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \quad \text{for all} \quad v \in H_0^1(\Omega)$$

with $a(u,v) = \varepsilon(\nabla u, \nabla v) - (\boldsymbol{b}\cdot\nabla u - cu, v)$ and $f(v) = (f,v)$, we add weighted residuals in order to stabilize the method. Then the SDFEM reads: Find $U \in V^\omega$ such that

$$a_{SD}(U,v) = a(U,v) + a_{stab}(U,v) = f_{SD}(v) \quad \text{for all} \quad v \in V^\omega$$

with

$$a_{stab}(U,v) := \sum_{T\in\Omega^N} \delta_T(\mathcal{L}U, -\boldsymbol{b}\cdot\nabla v)_T \quad \text{and} \quad f_{SD}(v) := f(v) + \sum_{T\in\Omega^N}\delta_T(f, -\boldsymbol{b}\cdot\nabla v)_T$$

and user chosen stabilization parameters $\delta_T \ge 0$. We clearly have the Galerkin orthogonality property

$$a_{SD}(u - U, v) = 0 \quad \text{for all} \quad v \in V^\omega. \tag{5.64}$$

Let V^ω be our finite element space consisting of piecewise-linear and bilinear functions. It is shown in, e.g., [89, §III.3.2.1], that if

$$0 \le \delta_T \le \frac{\gamma}{\|c\|^2_{\infty,T}} \quad \text{for all} \quad T \in \Omega^N, \tag{5.65}$$

then

$$a_{SD}(v,v) \geq \frac{1}{2} |||v|||_{SD}^2 \quad \text{for all} \quad v \in V^\omega, \tag{5.66}$$

with the streamline-diffusion norm

$$|||v|||_{SD}^2 := |||v|||_\varepsilon^2 + \sum_{T\in\Omega^N} \delta_T(\boldsymbol{b}\cdot\nabla v, \boldsymbol{b}\cdot\nabla v)_T.$$

5.3.4.1 Convergence in the streamline-diffusion norm

Tobiska and Stynes [99, 100] analyse the SDFEM using piecewise-bilinear finite elements on standard Shishkin meshes for problems with regular layers. Here we shall extend the technique from [100] to our more general class of S-type meshes, but have to restrict ourselves to piecewise-bilinear test and trial functions.

Using standard recommendations [89, p.223] and recalling our partition $\bar{\Omega} = \Omega_{11} \cup \Omega_{21} \cup \Omega_{12} \cup \Omega_{22}$, see Figure 5.2, we set

$$\delta_T := \begin{cases} \delta & \text{if } T \subset \Omega_{22}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta := \begin{cases} \delta_0 N^{-1} & \text{if } \varepsilon \leq N^{-1}, \\ \delta_1 \varepsilon^{-1} N^{-2} & \text{otherwise.} \end{cases}$$

with positive constants δ_0 and δ_1. Clearly $\delta \leq \max\{\delta_0, \delta_1\} N^{-1}$ and therefore (5.65) is satisfied for N sufficiently large.

Note that in the layer regions $\Omega \setminus \Omega_{22}$ the stabilization is switched off because there the streamline-diffusion stabilization would be negligible compared to the natural stability induced by the discretization of the diffusion term in the differential equation.

Our error analysis again starts from the coercivity (5.66) and the Galerkin orthogonality (5.64). Let again $\eta = u^I - u$ and $\chi = u^I - U$. Then

$$\frac{1}{2} |||\chi|||_{SD}^2 \leq a(\eta,\chi) + a_{stab}(\eta,\chi).$$

For the first term we have

$$|a(\eta,\chi)| \leq C\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right) |||\chi|||_\varepsilon\,,$$

see Section 5.3.2.3, while the stabilization term

$$a_{stab}(\eta,\chi) = \delta \sum_{T\subset\Omega_{22}} (\varepsilon\Delta u + \boldsymbol{b}\cdot\nabla\eta - c\eta, \boldsymbol{b}\cdot\nabla\chi)_T$$

still has to be analysed. This was done in [100]. Using (5.14b) as a crucial ingredient, Stynes and Tobiska derive the bound

$$|a_{stab}(\eta,\chi)| \leq CN^{-2}\ln^{1/2} N\, |||\chi|||_{SD}\,.$$

Eventually we get the following convergence results.

Theorem 5.21. *Let $\omega_x \times \omega_y$ be a tensor-product S-type mesh with $\sigma \geq 5/2$ that satisfies (1.11). Then the SDFEM solution U satisfies*

$$\left|\left|\left|u^I - U\right|\right|\right|_{SD} \leq C\left(h^2 \ln^{1/2} N + N^{-2} \max|\psi'|^2\right)$$

From this bounds for the error $u - U$ can be easily estabilished. It is also possible to construct and analyse a gradient recovery operator in the flavour of Section 5.3.2.5.

N	$\|\|\|u-U\|\|\|_\varepsilon$ error	rate	$\|\|\|u^I-U\|\|\|_{SD}$ error	rate	$\|u-U\|_\infty$ error	rate
16	3.3542e-1	0.75	2.0654e-1	1.04	1.7673e-1	1.14
32	1.9932e-1	0.82	1.0021e-1	1.33	8.0261e-2	1.41
64	1.1259e-1	0.83	3.9957e-2	1.50	3.0251e-2	1.51
128	6.3418e-2	0.83	1.4151e-2	1.59	1.0635e-2	1.61
256	3.5718e-2	0.84	4.6849e-3	1.65	3.4956e-3	1.66
512	1.9989e-2	0.85	1.4886e-3	1.69	1.1063e-3	1.70
1024	1.1087e-2	—	4.5993e-4	—	3.4131e-4	—

Table 5.5: The SDFEM on a Shishkin mesh

5.3.4.2 Maximum-norm error bounds

Clearly the technique for the Galerkin FEM from Section 5.3.2.4 can be applied to give pointwise error bounds for the SDFEM with bilinear test and trial functions within the layer regions Ω_{12} and Ω_{21}, while on the coarse mesh region Ω_{22}, we can employ (5.63). We get

$$\|u-U\|_{\infty,\Omega\setminus\Omega_{11}} \le \begin{cases} CN^{-3/2}\ln^{5/2}N & \text{for standard Shishkin meshes} \\ CN^{-3/2}\ln N & \text{for Bakhvalov-Shishkin meshes with } \varepsilon \le CN^{-1}. \end{cases} \tag{5.67}$$

Adapting Niijima's technique [77], Linß & Stynes [65] study the SDFEM with piecewise-linear test and trial functions on Shishkin meshes. For technical reasons a modified version of the SDFEM with artifical crosswind diffusion added on $[\lambda_x,1]\times[\lambda_y,1]$ is studied. Furthermore it is assumed that the convective field $\boldsymbol{b}$ is constant. The method reads: Find $U\in V^\omega$ such that

$$a_{ACD}(U,v) = a_{SD}(U,v) + (\varepsilon^*\boldsymbol{b}^\perp\cdot\nabla U, \boldsymbol{b}^\perp\cdot\nabla v) = f_{SD}(v) \quad \text{for all} \quad v\in V^\omega$$

with

$$\boldsymbol{b}^\perp := \frac{1}{\|\boldsymbol{b}\|}\begin{pmatrix}-b_2\\ b_1\end{pmatrix} \quad \text{and} \quad \varepsilon^* := \begin{cases} \max\left\{0, N^{-3/2}-\varepsilon\right\} & \text{on } \Omega_{22}, \\ 0 & \text{otherwise.}\end{cases}$$

If $\varepsilon \le N^{-3/2}$, then for any point $(x,y)\in\Omega$ the analysis in [65] yields

$$|(u-U)(x,y)| \le \begin{cases} CN^{-1/2}\ln^{3/2}N & \text{if} \quad (x,y)\in\Omega_{22}, \\ CN^{-3/4}\ln^{3/2}N & \text{if} \quad (x,y)\in\Omega\setminus\Omega_{22}, \\ CN^{-11/8}\ln^{1/2}N & \text{if} \quad (x,y)\in(\lambda^*,1)^2, \end{cases}$$

where $\lambda^* = \mathcal{O}\left(N^{-3/4}\ln N\right)$. The analysis in [65] includes more detailed results and deals also with the case $\varepsilon \ge N^{-3/2}$. Numerical experiments in [64] show convergence of almost second order on the coarse part of the mesh, while inside the boundary layers, the rates are smaller than 1. For bilinear elements almost second-order convergence in the maximum norm is observed globally, but no rigorous analysis is yet available.

5.3.5 A numerical example

Let us verify our theoretical results when the SDFEM is applied to our test problem (5.6). In our computations we have chosen $\varepsilon = 10^{-8}$ and $\sigma = 3$. The tables display the error in the ε-weighted energy norm $|||u-U|||_\varepsilon$, in the discrete SD-norm $|||u^I-U|||_{SD}$ and in the maximum-norm. The tables are clear illustrations of Theorem 5.21, while for the maximum-norm errors our bounds (5.67) appear to be suboptimal: instead of convergence of order (almost) 3/2 we observe (almost) 2nd order.

N	$\lVert\lvert u-U\rvert\rVert_\varepsilon$ error	rate	$\lVert\lvert u^I-U\rvert\rVert_{SD}$ error	rate	$\lVert u-U\rVert_\infty$ error	rate
16	1.3415e-1	1.07	4.9909e-2	1.92	5.1204e-2	1.89
32	6.3934e-2	1.02	1.3161e-2	1.98	1.3793e-2	1.96
64	3.1488e-2	1.01	3.3354e-3	2.00	3.5346e-3	1.99
128	1.5681e-2	1.00	8.3621e-4	2.00	8.8983e-4	2.00
256	7.8326e-3	1.00	2.0910e-4	2.00	2.2291e-4	2.00
512	3.9153e-3	1.00	5.2263e-5	2.00	5.5756e-5	2.00
1024	1.9575e-3	—	1.3063e-5	—	1.3940e-5	—

Table 5.6: The SDFEM on a Bakhvalov-Shishkin mesh

5.4 Finite volume methods

In this section we consider an inverse-monotone finite volume discretization for (5.1). This scheme was introduced by Baba and Tabata [12] and later generalised by Angermann [8, 9]. For a detailed derivation of the method the reader is refered to [8] or [37]. Here we study convergence of the method in a discrete energy norm and in the maximum norm.

For the moment let $\Omega \subset I\!R^2$ be an arbitrary domain with polygonal boundary. Consider the problem

$$-\varepsilon\Delta u - \boldsymbol{b}\nabla u + cu = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma = \partial\Omega \tag{5.68}$$

with $0 < \varepsilon \ll 1$ and

$$c + \frac{1}{2}\operatorname{div}\boldsymbol{b} \geq \gamma > 0 \quad \text{on} \quad \Omega. \tag{5.69}$$

Let $\omega = \{\boldsymbol{x}_i\} \subset \bar{\Omega}$ be a set of mesh points. Let Λ and $\partial\Lambda$ be the sets of indices of interior and boundary mesh points, i.e., $\Lambda := \{i : \boldsymbol{x}_i \in \Omega\}$ and $\partial\Lambda := \{i : \boldsymbol{x}_i \in \partial\Omega\}$. Set $\bar{\Lambda} := \Lambda \cup \partial\Lambda$. We partition the domain Ω into subdomains

$$\Omega_i := \big\{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - \boldsymbol{x}_i\| < \|\boldsymbol{x} - \boldsymbol{x}_j\| \text{ for all } j \in \bar{\Lambda} \text{ with } i \neq j\big\} \quad \text{for } i \in \bar{\Lambda},$$

where $\|\cdot\|$ is the Euclidean norm in $I\!R^2$. We define $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and we say that two mesh nodes $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ are adjacent iff $m_{ij} := \operatorname{meas}_{1D}\Gamma_{ij} \neq 0$. By Λ_i we denote the set of indices of all mesh nodes that are adjacent to $\boldsymbol{x}_i$. Moreover we define $d_{ij} := \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$, $m_i = \operatorname{meas}_{2D}\Omega_i$ and we denote by n_{ij} the outward normal on the boundary part Γ_{ij} of Ω_i. Let h, the mesh size, be the maximal distance between two adjacent mesh nodes. For a reasonable discretisation of the boundary conditions we shall assume that $\Gamma \subset \bigcup_{i\in\partial\Lambda}\bar{\Omega}_i$. We set $N_{ij} := -n_{ij}\cdot\boldsymbol{b}\left((\boldsymbol{x}_i + \boldsymbol{x}_j)/2\right)$. Then our discretization of (5.68) is: Find U such that

$$[L_\varrho U]_i = f_i m_i \quad \text{for} \quad i \in \Lambda, \quad U_i = 0 \quad \text{for} \quad i \in \partial\Lambda, \tag{5.70a}$$

with

$$[L_\varrho U]_i := \sum_{j\in\Lambda_i} m_{ij}\left(\frac{\varepsilon}{d_{ij}} - N_{ij}\varrho_{ij}\right)(U_i - U_j) + c_i m_i U_i, \tag{5.70b}$$

$\varrho_{ij} = \varrho(N_{ij}d_{ij}/\varepsilon)$ and a function $\varrho : I\!R \to [0,1]$. Possible choices for ϱ are given in Section 3.4 which studies the one-dimensional version of the FVM. Again we shall assume that ϱ satisfies

$$\begin{aligned}
&(\varrho_0) && t \mapsto t\varrho(t) \text{ is Lipschitz continuous,} \\
&(\varrho_1) && \big[\varrho(t) + \varrho(-t) - 1\big]t = 0 \quad \text{for all } t \in I\!R, \\
&(\varrho_2) && \big[1/2 - \varrho(t)\big]t \geq 0 \quad \text{for all } t \in I\!R, \\
&(\varrho_3) && 1 - t\varrho(t) \geq 0 \quad \text{for all } t \in I\!R.
\end{aligned}$$

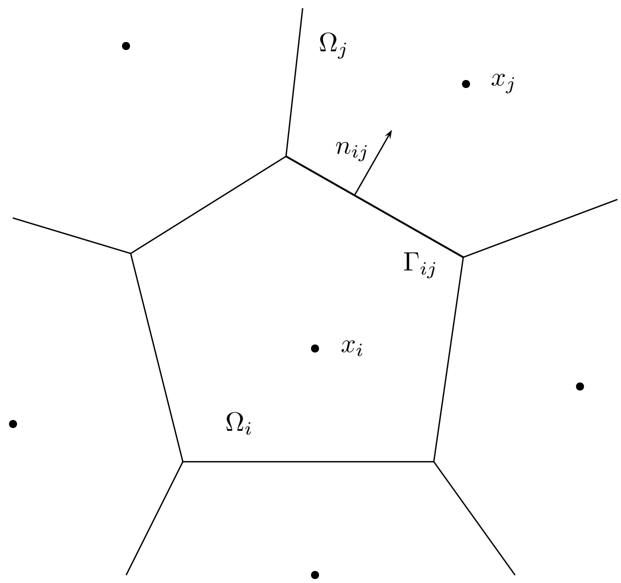


Figure 5.5: Mesh cell of the FVM

Note that the constant choice $\varrho \equiv \frac{1}{2}$, which generates a generalized central difference scheme, satisfies conditions (ϱ_1) and (ϱ_2), but not (ϱ_3). Conditions (ϱ_1) and (ϱ_2) guarantee the coercivity of the weak formulation associated with (5.70), while (ϱ_3) ensures the inverse monotonicity of the scheme when the coefficient c is strictly positive.

5.4.1 Coercivity of the method

The FVM can be written in variational form: Find

$$U \in V_0^\omega := \left\{ v \in I\!R^{\operatorname{card} \bar{\Lambda}} : v_k = 0 \text{ for } k \in \partial\Lambda \right\}$$

such that

$$a_\varrho(U, v) = f_\varrho(v) \quad \text{for all } v \in V_0^\omega,$$

with

$$a_\varrho(U, v) := \sum_{i \in \Lambda} [L_\varrho U]_i v_i \quad \text{and} \quad f_\varrho(v) := \sum_{i \in \bar{\Lambda}} f_i m_i v_i.$$

When studying the coercivity of the scheme we split the bilinear form into three parts representing the diffusion, convection and reaction terms:

$$a_\varrho(w, v) = \varepsilon d_\varrho(w, v) + c_\varrho(w, v) + r_\varrho(w, v)$$

with

$$d_\varrho(w, v) = \sum_{i \in \bar{\Lambda}} \sum_{j \in \Lambda_i} \frac{m_{ij}}{d_{ij}} (w_i - w_j) v_i,$$

$$c_\varrho(w, v) = -\sum_{i \in \bar{\Lambda}} \sum_{j \in \Lambda_i} m_{ij} N_{ij} \varrho_{ij} \left(w_i - w_j\right) v_i$$

and

$$r_\varrho(w,v) = \sum_{i\in\bar{\Lambda}} c_i m_i w_i v_i.$$

These three terms will be studied separately.

Changing the order of summation and renaming the indices yields

$$\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)v_i = -\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)v_j$$

Therefore

$$d_\varrho(v,v) = \frac{1}{2}\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} \frac{m_{ij}}{d_{ij}}(v_i - v_j)^2 =: |v|^2_{1,\omega} \tag{5.71}$$

which is a positive definite term.

Remark 5.22. *Given a mesh function $v \in V_0^\omega$ define a function $\tilde{v} \in H_0^1(\Omega)$ that coincides with v in the mesh points and that is piecewise linear on a Delaunay triangulation associated with the set of mesh points ω. Then $|v|^2_{1,\omega} = (\nabla\tilde{v}, \nabla\tilde{v}) = |\tilde{v}|_1^2$.*

Next consider the convection term. By definition we have $m_{ij} = m_{ji}$, $d_{ij} = d_{ji}$ and $N_{ij} = -N_{ji}$. Furthermore (ϱ_1) implies $N_{ji}\varrho_{ji} = N_{ij}(\varrho_{ij} - 1)$. Hence

$$\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} m_{ij}N_{ij}\varrho_{ij}(v_i - v_j)v_i = -\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(\varrho_{ij} - 1\right)(v_i - v_j)v_j$$

and

$$c_\varrho(v,v) = \frac{1}{2}\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(\frac{1}{2} - \varrho_{ij}\right)(v_i - v_j)^2 - \frac{1}{4}\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(v_i^2 - v_j^2\right)$$

Introducting

$$|v|^2_{\varrho,\omega} := \frac{1}{2}\sum_{i\in\bar{\Lambda}} \sum_{j\in\Lambda_i} m_{ij}N_{ij}\left(\frac{1}{2} - \varrho_{ij}\right)(v_i - v_j)^2,$$

which is a well-defined semi-norm when (ϱ_2) is satisfied, we have

$$c_\varrho(v,v) = |v|^2_{\varrho,\omega} - \frac{1}{2}\sum_{i\in\bar{\Lambda}} v_i^2 \sum_{j\in\Lambda_i} m_{ij}N_{ij}.$$

This and (5.71) yield

$$a_\varrho(v,v) = \varepsilon|v|^2_{1,\omega} + |v|^2_{\varrho,\omega} + \sum_{i\in\bar{\Lambda}} m_i v_i^2 \left(c_i - \frac{1}{2}\sum_{j\in\Lambda_i} m_{ij}N_{ij}\right).$$

Note that

$$m_i \operatorname{div} \boldsymbol{b}_i + \sum_{j\in\Lambda_i} m_{ij}N_{ij} = \mathcal{O}\left(h\right).$$

This implies

$$a_\varrho(v,v) \geq \varepsilon|v|^2_{1,\omega} + |v|^2_{\varrho,\omega} + \frac{\gamma}{2}\|v\|^2_{0,\omega} =: |||v|||^2_\varrho \quad \text{with} \quad \|v\|^2_{0,\omega} := \sum_{i\in\bar{\Lambda}} m_i v_i^2$$

provided h is sufficiently small, independent of the perturbation parameter ε.

We summarize the result of our stability analysis.

Theorem 5.23. *Assume the discretization (5.70) satisfies conditions (ϱ_1) and (ϱ_2). Suppose (5.69) holds true. Then the bilinear form $a_\varrho(\cdot,\cdot)$ is coercive with respect to the norm $|||\cdot|||_\varrho$, i.e.,*

$$a_\varrho(v,v) \geq |||v|||_\varrho^2 \quad \text{for all} \quad v \in V_0^\omega$$

uniformly with respect to h and the perturbation parameter ε.

Remark 5.24. *When $\varrho \equiv \frac{1}{2}$, i.e. when the stabilization is switched off, the bilinear form is coercive with respect to the discrete ε-weighted energy norm*

$$|||v|||_{\varepsilon,\omega}^2 := \varepsilon|v|_{1,\omega}^2 + \frac{\gamma}{2}\|v\|_{0,\omega}^2.$$

However when $\varrho \not\equiv \frac{1}{2}$ then we have coercivity of the scheme in a stronger norm which results in enhanced stability properties of the FVM.

5.4.2 Inverse monotonicity

Let the function ϱ which discribes the FVM method satisfy (ϱ_1) and (ϱ_3). Furthermore assume that $c > 0$ on $\bar{\Omega}$. Then recalling the definition (5.70), we have

$$\frac{\varepsilon}{d_{ij}} - N_{ij}\varrho_{ij} \geq 0.$$

Hence the diagonal entries of the matrix associated with L_ϱ are positive while the off-diagonal ones are non-positive. Therefore the system matrix is an L_0 matrix. Next application of the M-matrix criterion with the test function $v \equiv 1$ yields $[L_\varrho \mathbf{1}]_i = c_i m_i > 0$. Consequently L_ϱ is inverse monotone. That is if two mesh functions v and w satisfy

$$[L_\varrho v]_i \geq [L_\varrho w]_i \;\text{ for all }\; i \in \Lambda \quad \text{and} \quad v_i \geq w_i \;\text{ for all }\; i \in \partial\Lambda$$

then

$$v_i \geq w_i \;\text{ for all }\; i \in \bar{\Lambda}.$$

Note this result also holds true when no restriction on the convection field $\boldsymbol{b}$ is imposed.

Using this inverse monotonicity, we now study the Green's functions of the method and derive an anistropic stability inequality on a general tensor-product mesh $\omega := \omega_x \times \omega_y$. A stability result of this kind was first established by Andreev for a simple upwind difference scheme; see [7].

Setting

$$\varrho_{1;ij}^{+} := \varrho\left(-\frac{b_{1;i+1/2,j}h_{i+1}}{\varepsilon}\right) b_{1;i+1/2,j}, \quad \varrho_{1;ij}^{-} := \varrho\left(\frac{b_{1;i-1/2,j}h_{i}}{\varepsilon}\right) b_{1;i-1/2,j},$$

$$v_{\bar{x};ij} := \frac{v_{ij}-v_{i-1,j}}{h_i}, \quad v_{\hat{x};ij} = \frac{v_{i+1,j}-v_{ij}}{\hbar_i}, \quad v_{\check{x};ij} = \frac{v_{i,j}-v_{i-1,j}}{\hbar_i} \quad \text{and} \quad \hbar_i := \frac{h_{i+1}+h_i}{2},$$

with analogous definitions for ϱ_2^{+}, ϱ_2^{-}, $v_{\bar{y}}$, $v_{\hat{y}}$, $v_{\check{y}}$ and $\bar{k}$, we can rewrite (5.70) as: Find $U \in \left(I\!R_0^{N+1}\right)^2$ such that

$$\left[LU\right]_{ij} := -\varepsilon\left(U_{\bar{x}\hat{x};ij} + U_{\bar{y}\hat{y};ij}\right) - \varrho_{1,ij}^{+}U_{\hat{x},ij} - \varrho_{1,ij}^{-}U_{\check{x},ij} - \varrho_{2,ij}^{+}U_{\hat{y},ij} - \varrho_{2,ij}^{-}U_{\check{y},ij} + c_{ij}U_{ij} = f_{ij}$$

for $i,j = 1,\dots,N-1$. Any mesh function v that vanishes on the boundary can be represented using the Green's function:

$$v_{ij} = \left(v, G_{ij,\cdot\cdot}\right)_{\varrho} := \sum_{k,l=1}^{N-1} \hbar_k \bar{k}_l G_{ij,kl} \left[Lv\right]_{kl}, \tag{5.72}$$

where $G_{ij,kl} = G(x_i, y_j, \xi_k, \eta_l)$ solves for fixed k and l

$$\left[LG_{\cdot\cdot,kl}\right]_{ij} = \delta_{x;ik}\,\delta_{y;jl} \quad \text{for} \quad i,j = 1,\dots,N-1 \quad \text{and} \quad G_{\cdot\cdot,kl} = 0 \quad \text{on} \quad \omega \cap \partial\Omega$$

with

$$\delta_{x;ik} = \begin{cases} \hbar_i^{-1} & \text{if} \;\; i = k, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \delta_{y;jl} = \begin{cases} \bar{k}_j^{-1} & \text{if} \;\; j = l, \\ 0 & \text{otherwise.} \end{cases}$$

The adjoint operator to L is

$$\left[L^{*}v\right]_{kl} = -\varepsilon\left(v_{\bar{\xi}\hat{\xi};kl} + v_{\bar{\eta}\hat{\eta};kl}\right) + \left(\varrho_1^{+}v\right)_{\check{\xi};kl} + \left(\varrho_1^{-}v\right)_{\hat{\xi};kl} + \left(\varrho_2^{+}v\right)_{\check{\eta};kl} + \left(\varrho_2^{-}v\right)_{\hat{\eta};kl} + c_{kl}v_{kl}$$

and the Green's function solves for fixed i and j

$$\left[L^{*}G_{ij,\cdot\cdot}\right]_{kl} = \delta_{x;ik}\,\delta_{y;jl} \quad \text{for} \quad k,l = 1,\dots,N-1 \quad \text{and} \quad G_{ij,\cdot\cdot} = 0 \quad \text{on} \quad \omega \cap \partial\Omega. \tag{5.73}$$

In our subsequent analysis the following mean value theorem is used.

Lemma 5.25. *Let $\varphi, g \in I\!R^{N+1}$ be two mesh functions with $g_j \ge 0$ and $m \le \varphi_j \le M$ for $j = 1,\dots,N-1$. Then there exists a constant $\tilde{\varphi} \in [m, M]$ with*

$$\sum_{j=1}^{N-1} \bar{k}_j \varphi_j g_j = \tilde{\varphi} \sum_{j=1}^{N-1} \bar{k}_j g_j .$$

Let i and j be fixed. First, the inverse monotonicity of L yields $G_{ij,kl} \ge 0$. Next, multiplying (5.73) by $\bar{k}_l$ and summing for $l = 1,\dots,N-1$, we obtain the one-dimensional equation

$$\begin{aligned} &-\varepsilon\left(\sum_{l=1}^{N-1} \bar{k}_l G_{ij,\cdot l}\right)_{\bar{\xi}\hat{\xi},k} + \left(\sum_{l=1}^{N-1} \bar{k}_l \varrho_{1,\cdot l}^{+} G_{ij,\cdot l}\right)_{\check{\xi},k} + \left(\sum_{l=1}^{N-1} \bar{k}_l \varrho_{1,\cdot l}^{-} G_{ij,\cdot l}\right)_{\hat{\xi},k} + \sum_{l=1}^{N-1} \bar{k}_l c_{kl} G_{ij,kl} \\ &\quad = \delta_{x;ik} - F_k \end{aligned}$$

where

$$F_k = -\varepsilon \left[1 + \frac{\varrho^+_{2;k,N-1} h_N}{\varepsilon}\right] G_{\bar\eta;ij,kN} + \varepsilon \left[1 - \frac{\varrho^+_{2;k,0} h_1}{\varepsilon}\right] G_{\bar\eta;ij,k1} \geq 0,$$

by (ϱ_3) and since $G \geq 0$.

Defining

$$\tilde{G}_k := \sum_{l=1}^{N-1} \hbar_l G_{ij,kl} = \|G_{ij,\cdot l}\|_{\ell_1}, \quad \text{for } k = 0, \ldots, N,$$

we see that according to Lemma 5.25 there exist mesh functions $\tilde\varrho^+, \tilde\varrho^-, \tilde c \in \mathbb{R}^{N+1}$ with $\tilde\varrho^+ \geq \beta_1$, $\tilde\varrho^- \geq \beta_1$ and $\tilde c \geq \gamma$ such that

$$-\varepsilon \tilde G_{\bar\xi\check\xi,k} + \left(\tilde\varrho^+ \tilde G\right)_{\check\xi,k} + \left(\tilde\varrho^- \tilde G\right)_{\hat\xi,k} + \tilde c_k \tilde G_k = \delta_{x;ik} - F_k.$$

Let $\Gamma = \Gamma_{m,k}$ be the Green's function of the operator

$$[Lv]_k = -\varepsilon v_{\bar\xi\check\xi,k} - \tilde\varrho^+_k v_{\check\xi,k} - \tilde\varrho^-_k v_{\hat\xi,k} + \tilde c_k v_k.$$

Then $\tilde G$ can be written as

$$\tilde G_k = \Gamma_{i,k} - \sum_{m=1}^{N-1} \hbar_m \Gamma_{m,k} F_m$$

The nonnegativity of Γ and F gives

$$\tilde G_k \leq \Gamma_{i,k} \leq \frac{1}{\beta_1 \inf_{t<0} \varrho(t)},$$

by Theorem 3.11 and Remark 3.13. We get the first inequaltiy of the following theorem—the second one is proven analogously.

Theorem 5.26. *Suppose the control function ϱ enjoys properties (ϱ_0) and (ϱ_3). Then the Green's function associated with L satisfies*

$$\max_{i,j,k=1,\ldots,N-1} \sum_{l=1}^{N-1} \hbar_l G_{ij,kl} \leq \frac{\alpha}{\beta_1} \qquad \text{and} \qquad \max_{i,j,l=1,\ldots,N-1} \sum_{k=1}^{N-1} \hbar_k G_{ij,kl} \leq \frac{\alpha}{\beta_2}$$

with $\alpha = 1/\inf_{t<0} \varrho(t) \leq 2$.

Finally we shall use these bounds on the Green's function to derive stability estimates for the operator L. For any mesh function $v \in \left(\mathbb{R}^{N+1}\right)^2$ introduce the norm

$$\|v\|_A := \sum_{k=1}^{N-1} \hbar_k \max_{l=1,\ldots,N-1} |v_{kl}|.$$

Its dual norm with respect to the discrete scalar product $(\cdot,\cdot)_\varrho$ is

$$\|v\|_{A^*} = \max_{k=1,\ldots,N-1} \sum_{l=1}^{N-1} \hbar_k |v_{kl}|,$$

cf. [16, Theorem 2]. The representation (5.72) gives

$$|v_{ij}| \leq \|G_{ij,\cdot\cdot}\|_{A^*} \|v\|_A.$$

Application of Theorem 5.26 yields our final stability result which is an extension of the (l_∞, l_1) stability of Section 2.2.5:

Theorem 5.27. *Suppose the control function ϱ enjoys properties (ϱ_0) and (ϱ_3). Then*

$$\|v\|_{\infty,\omega} \leq \frac{\alpha}{\beta_1} \sum_{k=1}^{N-1} \hbar_k \max_{l=1,\dots,N-1} \left| \left[L_\varrho \right]_{kl} \right|$$

and

$$\|v\|_{\infty,\omega} \leq \frac{\alpha}{\beta_2} \sum_{l=1}^{N-1} \hbar_l \max_{k=1,\dots,N-1} \left| \left[L_\varrho \right]_{kl} \right|$$

with $\alpha = 1/\inf_{t<0} \varrho(t) \leq 2$.

5.4.3 Convergence

Energy norm. Starting from the coercivity of the bilinear form $a_\varrho(\cdot,\cdot)$, see Theorem 5.23, the analysis proceeds along the lines of Section 3.4.2 resembling many of the details also used for the Galerkin FEM in two dimensions, see Section 5.3.2. Eventually we get

$$|||u-U|||_\varrho + |||u-U|||_\varepsilon \leq CN^{-1} \max |\psi'| \ln^{1/2} N \quad \text{for S-type meshes with } \sigma \geq 2;$$

see also [112, 113].

Maximum norm. The pointwise errors can be bounded using the hybrid stability inequalities from Theorem 5.27. The truncation error is split according to the decomposition of Theorem 5.1. Then either of the two bounds from Theorem 5.27 is applied. Section 2.2.5 gives a flavour of the technical details. For a Shishkin mesh with $\sigma \geq 2$ we obtain

$$\|u-U\|_{\infty,\omega} \leq CN^{-1} \ln N.$$

If ϱ is Lipschitz continuous in $(-m,m)$ with $m>0$, then there exists an $N_m > 0$ independent of the perturbation parameter ε such that

$$\|u-U\|_{\infty,\omega} \leq CN^{-1} \quad \text{for} \quad N \geq N_m.$$

In the latter case the stabilization is reduced when the local mesh size is small enough, thus giving higher accuracy inside the layers. See also [57].

5.4.3.1 Numerical tests

We verify our theoretical results for the upwind FEM on Shishkin meshes when applied to the test problem (5.6). For our tests we take $\varepsilon = 10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problem.

We test the method for three different choices of the controlling function ϱ. The errors are measured in the discrete energy and maximum norm and in the FVM-norm. For $\varrho_{U,0}$ we observe convergence of almost first order, namely $N^{-1}\ln N$, in all three norms, while for both $\varrho_{U,2}$ and ϱ_I—which are Lipschitz continuous—the errors behave like $\mathcal{O}\left(N^{-1}\right)$. Though this is covered by our analysis only for the maximum norm.

N	$\|u-u^N\|_\varrho$ error	rate	$\|u-u^N\|_{\varepsilon,\omega}$ error	rate	$\|u-u^N\|_{\infty,\omega}$ error	rate
16	2.7575e-1	0.68	2.0623e-1	0.55	1.8112e-1	0.62
32	1.7198e-1	0.75	1.4052e-1	0.66	1.1770e-1	0.71
64	1.0230e-1	0.79	8.9046e-2	0.73	7.1880e-2	0.76
128	5.8999e-2	0.83	5.3575e-2	0.79	4.2537e-2	0.80
256	3.3292e-2	0.85	3.1081e-2	0.82	2.4483e-2	0.83
512	1.8493e-2	0.87	1.7579e-2	0.85	1.3786e-2	0.85
1024	1.0153e-2	0.88	9.7672e-3	0.87	7.6456e-3	0.87
2048	5.5247e-3	—	5.3576e-3	—	4.1908e-3	—

Table 5.7: FVM on Shishkin meshes, $\varrho = \varrho_{U,0}$

N	$\|u-u^N\|_\varrho$ error	rate	$\|u-u^N\|_{\varepsilon,\omega}$ error	rate	$\|u-u^N\|_{\infty,\omega}$ error	rate
16	1.5894e-1	0.83	8.9598e-2	0.80	7.5370e-2	0.70
32	8.9627e-2	0.92	5.1417e-2	0.90	4.6297e-2	0.84
64	4.7445e-2	0.96	2.7514e-2	0.95	2.5790e-2	0.92
128	2.4388e-2	0.98	1.4222e-2	0.98	1.3610e-2	0.96
256	1.2360e-2	0.99	7.2279e-3	0.99	6.9899e-3	0.98
512	6.2219e-3	1.00	3.6430e-3	0.99	3.5418e-3	0.99
1024	3.1214e-3	1.00	1.8288e-3	1.00	1.7827e-3	1.00
2048	1.5633e-3	—	9.1618e-4	—	8.9431e-4	—

Table 5.8: FVM on Shishkin meshes, $\varrho = \varrho_I$

N	$\|u-u^N\|_\varrho$ error	rate	$\|u-u^N\|_{\varepsilon,\omega}$ error	rate	$\|u-u^N\|_{\infty,\omega}$ error	rate
16	1.5359e-1	0.81	8.2430e-2	0.77	7.6384e-2	0.72
32	8.7574e-2	0.91	4.8263e-2	0.88	4.6337e-2	0.85
64	4.6686e-2	0.95	2.6272e-2	0.93	2.5790e-2	0.92
128	2.4120e-2	0.98	1.3773e-2	0.96	1.3610e-2	0.96
256	1.2270e-2	0.99	7.0752e-3	0.98	6.9899e-3	0.98
512	6.1928e-3	0.99	3.5935e-3	0.99	3.5418e-3	0.99
1024	3.1122e-3	1.00	1.8132e-3	0.99	1.7827e-3	1.00
2048	1.5605e-3	—	9.1144e-4	—	8.9431e-4	—

Table 5.9: FVM on Shishkin meshes, $\varrho = \varrho_{U,2}$

Notation

general

u	—	solution of boundary value problem
U	—	numerical approximation to u
ε	—	perturbation parameter
$\mathcal{L}$, $\mathcal{L}^*$, L, L^*	—	differential operator, its adjoint and their discretizations
$\mathcal{G}$, G	—	Green's functions associated with $\mathcal{L}$ and L
N	—	number of mesh intervals (in each coordinate direction)
C	—	generic constant, independent of ε and N

meshes

$$\omega : 0 = x_0 < x_1 < \cdots < x_N = 1, \quad h_i = x_i - x_{i-1}, \quad h = \max_i h_i$$

finite differences

$$\hbar_i = (h_i + h_{i+1})/2, \quad i = 1, \ldots, N-1, \quad \hbar_0 = \hbar_N = 0$$
$$\chi_i = \sigma_{i+1} h_{i+1} + \sigma_i h_i, \quad i = 1, \ldots, N-1, \quad \chi_0 = \chi_N = 0, \quad x_{\sigma,i-1/2} = x_{i-1} + \sigma_i h_i$$
$$v_{x,i} = \frac{v_{i+1} - v_i}{h_{i+1}}, \quad v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{\check{x},i} = \frac{v_i - v_{i-1}}{h_{i+1}},$$
$$v_{\hat{x},i} = \frac{v_{i+1} - v_i}{\hbar_i}, \quad v_{\tilde{x},i} = \frac{v_{i+1} - v_{i-1}}{2\hbar_i}, \quad v_{\breve{x},i} = \frac{v_{i+1} - v_i}{\chi_i}, \quad v_{-,i} = v_{i-1}$$

norms

$$\|v\|_\infty = \operatorname{ess\,sup} |v|, \quad \|v\|_1 = \|v\|_{L_1}, \quad \|v\|_* = \min_{V : V' = v} \|v\|_\infty, \quad |||v|||_{\varepsilon,\infty} = \frac{\beta}{2} \|v\|_\infty + \frac{\varepsilon}{2} \|v'\|_\infty$$
$$\|v\|_{\infty,\omega}, \quad \|v\|_{1,\omega}, \quad \|v\|_{*,\omega}, \quad |||v|||_{\varepsilon,\infty,\omega} \quad \text{– discrete versions}$$

Sobolev spaces

$$L_2(D): \ (u,v)_D = \int_D uv, \quad \|v\|_{0,D} = (v,v)_D^{1/2}$$
$$L_2(\Omega): \ (u,v) = \int uv, \quad \|v\|_0 = (v,v)^{1/2}$$
$$H^1(D),\ H_0^1(D): \ |v|_{1,D} = \|\nabla v\|_{0,D}, \quad |||v|||_{\varepsilon,D} := \left\{ \varepsilon\, |v|_{1,D}^2 + \|v\|_{0,D}^2 \right\}^{1/2}$$
$$H^1(\Omega),\ H_0^1(\Omega): \ |v|_1 = \|\nabla v\|_0, \quad |||v|||_\varepsilon := \left\{ \varepsilon\, |v|_1^2 + \|v\|_0^2 \right\}^{1/2}$$
$$V^\omega \subset H_0^1(\Omega): \ \text{finite element space on the mesh } \omega$$

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Hiermit erkläre ich, daß ich die mit dem heutigen Datum vorgelegte Habilitationsschrift “Layer-adapted meshes for convection-diffusion problems” selbst und ohne andere als die darin angegebenen Hilfsmittel angefertigt habe, sowie die wörtlich oder inhaltlich übernommenen Stellen als solche gekennzeichnet wurden.

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