Singularly perturbed problems with characteristic layers
Supercloseness and postprocessing

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Notation

Ω	computational domain
$D \subset \Omega$	arbitrary domain
L	differential operator
∇	gradient
Δ	Laplacian
∂	partial derivative
C	generic constant, independent of ε and N
$C^k(D)$	space of functions over D with continuous k-th order derivatives
$C^{k,\alpha}(D)$	subspace of $C^k(D)$, k-th order derivatives are Hölder-continuous with exponent α
$L_p(D)$	$p < \infty$: Lebesgue space of p-power integrable functions over D $p = \infty$: Lebesgue space of piecewise bounded functions over D
$W^{k,p}(D)$	standard Sobolev space, derivatives up to order k lie in $L_p(D)$
$H^k(D)$	Sobolev space $W^{k,2}(D)$
$H_0^1(D)$	subspace of $H^1(D)$, vanishing boundary traces
$\mathcal{Q}_p(D)$	space of polynomials of degree p in each variable over D
$\mathcal{P}_p(D)$	space of polynomials of absolute degree p over D
ε	perturbation parameter
β	lower bound for convection
σ	mesh parameter for S-type meshes
N	number of cells in each coordinate direction
λ_x, λ_y	mesh-transition points
$\mathcal{O}(\cdot)$, $o(\cdot)$	Landau symbols

ϕ	mesh generating function
ψ	mesh characterising function, related to ϕ
$T^N(\Omega)$	tensor-product mesh on Ω
Ω_{11}, Ω_{12}, Ω_{21}, Ω_{22}	subdomains of Ω, see page 7
τ, $\tau_{i,j}$	general and special rectangle of $T^N(\Omega)$
h_i, k_j	sizes of rectangle $\tau_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$
h, k	maximal mesh sizes inside layer regions
$\hbar$, $\bar{k}$	mesh sizes in coarse mesh region Ω_{11}
$\operatorname{meas} D$	measure of the area of D
$v + w_1 + w_2 + w_{12}$	decomposition of solution u, see page 5
$lt(y)$	abbreviation for $\begin{cases} e^{y/(\sigma\varepsilon^{1/2})}, & y \le \lambda_y \\ e^{(1-y)/(\sigma\varepsilon^{1/2})}, & y \ge 1 - \lambda_y \end{cases}$
u^I	piecewise nodal bilinear interpolation of u
πu	piecewise bilinear local L_2-projection of u
$(\cdot,\cdot)_D$	L_2-scalar product on D
$a_*(\cdot,\cdot)$	several bilinear forms
$\lVert\cdot\rVert_{0,D}$, $\lVert\cdot\rVert_{L_p(D)}$	L_2- and L_p-norm on D
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_\varepsilon$	energy norm
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{SD}$	streamline-diffusion norm
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{GLS,1}$, $\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{GLS,2}$	Galerkin least-squares norms
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{CIP}$, $\lvert v\rvert_J$	continuous interior penalty norm and seminorm
$[\cdot]_e$	jump across edge e
$\tilde{a}$, $\tilde{b}$	reference marks on reference macro elements, see 38

Chapter 1

Introduction

In the area of numerical simulation, computational fluid dynamics represents one of the most challenging tasks. This field ranges from aerodynamics and simulation of gas flows in engines to simulation of liquids in complex channels. The physical model describing the behaviour of fluids is mainly the Navier-Stokes equations, either for compressible or for incompressible flows. Although they have been known since the early 19th century, the existence of global solutions in general domains has not been proven yet. Therefore, numerical simulations are used to approximate possible solutions.
Generally the solution exhibits layers at the boundaries and they can be seen in experiments with flows. Especially for high Reynolds numbers, the treatment of such phenomena is important, but complicated. Moreover, experiments can hardly be conducted for high Reynolds numbers. Consequently, understanding the numerical handling of boundary layers is important.

A model problem to the Navier-Stokes equations is the convection-diffusion equation in the unit square $\Omega = (0,1)^2$

$$Lu := -\varepsilon\Delta u - bu_x + cu = f \tag{1.1a}$$

with Dirichlet boundary conditions on $\Gamma = \partial\Omega$

$$u|_\Gamma = 0. \tag{1.1b}$$

This model equation applies to other fields of simulation too, for example to time-dependent chemical reaction equations. These are time-dependent partial differential equations whose time discretisation leads to (1.1).

We suppose the data in (1.1) to satisfy $b \in W^{1,\infty}(\Omega)$, $c \in L_\infty(\Omega)$ and $b, c = \mathcal{O}(1)$. Additionally, let $b \geq \beta$ on $\bar{\Omega}$ with some positive constant β and $0 < \varepsilon \ll 1$, a small perturbation parameter.
To ensure coercivity of the bilinear form associated with the differential operator L we shall assume that

$$c + \tfrac{1}{2}b_x \geq \gamma > 0. \tag{1.2}$$

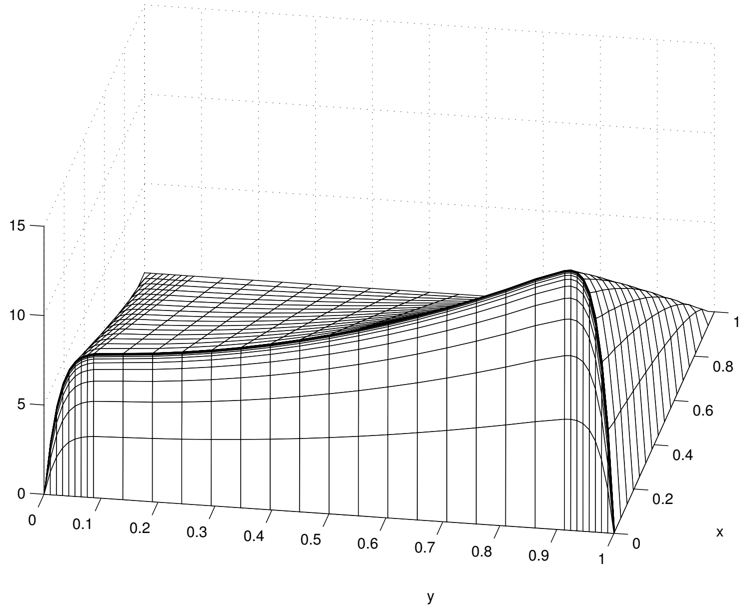


Figure 1.1: Typical solution to (1.1) with parabolic layers (left and right) and an exponential layer (front)

Then (1.1) possesses a unique solution in $H_0^1(\Omega)$. Note that (1.2) can always be ensured by a simple transformation $\tilde{u}(x,y) = u(x,y)e^{\varkappa x}$ with $\varkappa$ chosen suitably.
The unique solution $u \in H_0^1(\Omega)$ depends on ε. Moreover, in the limit $\varepsilon \to 0$ the type of the differential equation changes and the reduced problem for $\varepsilon = 0$ can only fulfill the boundary conditions at $x = 1$. Thus we have a singularly perturbed problem according to the following definition.

Definition 1.1. *Consider the reduced problem to* (1.1)

$$-br_x + cr = f \qquad \text{in } \Omega$$

with Dirichlet boundary conditions on the inflow boundary

$$r|_{x=1} = 0$$

where $\varepsilon = 0$ *and not all boundary conditions of* (1.1) *are invoked. The partial differential equation* (1.1) *is* singularly perturbed, *if the limit of the solution* u *for* $\varepsilon = 0$ *does not tend to the solution* r *of the reduced problem.*

The presence of ε and the orientation of convection give rise to an exponential layer in the solution of width $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ near the outflow boundary at $x = 0$ and to two parabolic layers of width $\mathcal{O}(\sqrt{\varepsilon}|\ln\varepsilon|)$ near the characteristic boundaries at $y = 0$ and $y = 1$; see Fig. 1.1.
Discretisation of (1.1) on standard meshes and with standard methods leads to numerical solutions with non-physical oscillations unless the mesh size is of order of the perturbation

parameter ε which is impracticable. Instead we shall aim at *robust* or *uniformly convergent* methods in the sense of the following definition.

Definition 1.2. *Let u be the solution of* (1.1) *and u^N the solution of its discretisation with N^k degrees of freedom, $k > 0$. The numerical method is said to be* uniformly convergent *or* robust *with respect to ε in a given norm $\|\cdot\|$ if*

$$\|u - u^N\| \leq \vartheta(N) \quad \textit{for} \quad N > N_0$$

with a function ϑ and a constant $N_0 > 0$, both independent of ε and

$$\lim_{N\to\infty} \vartheta(N) = 0.$$

We will focus on layer-adapted meshes combined with standard methods. The meshes considered here are generalisations of the standard Shishkin mesh [23, 29], see Chapter 2. In [9], we showed that for (1.1) the unstabilised Galerkin finite element method on a Shishkin mesh is uniformly convergent in the energy norm

$$|||v|||_\varepsilon := (\varepsilon\|\nabla v\|_0 + \gamma\|v\|_0)^{1/2}$$

with $\|v\|_{0,D}$ denoting the usual L_2-norm on D. If $D = \Omega$ we drop the index from the notation. The Galerkin method is convergent of order one up to a logarithmic factor, i.e.,

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \leq CN^{-1} \ln N,$$

where here and throughout the thesis C denotes a generic constant that is independent of both the perturbation parameter ε and N. Moreover, the numerical solution u^N satisfies

$$\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \leq C(N^{-1} \ln N)^2$$

with the nodal bilinear interpolant u^I. This property is known as *supercloseness* and can be used to prove *superconvergence*

$$\left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon \leq C(N^{-1} \ln N)^2$$

for a suitable postprocessing operator P, see [9, Section V].
Unfortunately, the Galerkin method lacks stability, resulting in linear systems that are hard to solve. Therefore, we are looking for stabilisation methods, improving stability of the underlying Galerkin method without destroying its good approximation properties. Basically these methods add an stabilisation term to the Galerkin bilinear form.
For problems of type (1.1) with only *exponential* layers in its solution the numerical analysis with respect to uniform convergence and supercloseness is well understood, see for Galerkin FEM [18, 30, 33], for streamline diffusion FEM [31] and for the continuous interior penalty FEM [12, 28].
In the present thesis, *parabolic* (or *characteristic*) layers will be considered. Unlike the exponential-layer problems little is known about supercloseness and stabilised methods in literature. Nevertheless these layer structures play an important role in fluid dynamics

and can be considered as a flow past a surface with a no-slip condition. Moreover, they are similar in structure to *interior* boundary layers that stem from discontinuous boundary conditions or point sources.
In [16, 20] an analysis of streamline diffusion FEM on a Shishkin mesh is given, but without a rigorous analysis of possible supercloseness effects. In [10, 11] we proved that both streamline diffusion and central interior penalty FEM on Shishkin meshes possess a supercloseness property. We will extend these results to more general S-type meshes in Chapter 3.

Discretisation of (1.1) will be done using piecewise bilinear elements. For higher-order elements supercloseness results are only known for streamline diffusion FEM, see [32]. In the case of Shishkin meshes, exponential-layer problems and $\mathcal{Q}_p$-elements with $p > 1$, supercloseness of order $N^{-(p+1/2)}$ was proved using a special interpolant.

The organisation of this thesis is as follows. In Chapter 2 layer-adapted meshes and a decomposition of the solution to (1.1) using a priori information will be described. The main part will be in Chapter 3, where we analyse the supercloseness property of several methods and address for residual based stabilisation methods the optimal choice of parameters. In Chapter 4 we compare different postprocessing methods and prove superconvergence on the meshes introduced before. Finally, in Chapter 5 numerical simulations illustrate the theoretical results.

Chapter 2

Solution decomposition and layer-adapted meshes

As mentioned in the introduction, the solution u of (1.1) exhibits boundary layers. In order to construct layer-adapted meshes and to establish uniform convergence, it is convenient to have a decomposition of u into different parts corresponding to the layers and a smooth part.

2.1 Solution decomposition

For problems like (1.1) we propose the following decomposition.

Assumption 2.1. *The solution u of* (1.1) *can be decomposed as*

$$u = v + w_1 + w_2 + w_{12},$$

where for all $x, y \in [0,1]$ and $0 \le i+j \le 2$ we have the pointwise estimates

$$\left.\begin{aligned} \left|\partial_x^i \partial_y^j v(x,y)\right| &\le C, \quad \left|\partial_x^i \partial_y^j w_1(x,y)\right| \le C\varepsilon^{-i} e^{-\beta x/\varepsilon}, \\ \left|\partial_x^i \partial_y^j w_2(x,y)\right| &\le C\varepsilon^{-j/2} \left(e^{-y/\varepsilon^{1/2}} + e^{-(1-y)/\varepsilon^{1/2}}\right), \\ \left|\partial_x^i \partial_y^j w_{12}(x,y)\right| &\le C\varepsilon^{-(i+j/2)} e^{-\beta x/\varepsilon} \left(e^{-y/\varepsilon^{1/2}} + e^{-(1-y)/\varepsilon^{1/2}}\right) \end{aligned}\right\} \tag{2.1}$$

and for $0 \le i+j \le 3$ the L_2 bounds

$$\left.\begin{aligned} \left\|\partial_x^i \partial_y^j v\right\|_0 &\le C, & \left\|\partial_x^i \partial_y^j w_1\right\|_0 &\le C\varepsilon^{-i+1/2}, \\ \left\|\partial_x^i \partial_y^j w_2\right\|_0 &\le C\varepsilon^{-j/2+1/4}, & \left\|\partial_x^i \partial_y^j w_{12}\right\|_0 &\le C\varepsilon^{-i-j/2+3/4}. \end{aligned}\right\} \tag{2.2}$$

Remark 2.2. *For $i+j \le 2$ the L_2 bounds* (2.2) *follow clearly from the pointwise bounds* (2.1)*.*

As we know the structure of the solution u a priori, the idea of decomposing u in this way seems convincing. However, it should be clarified under which circumstances such a decomposition exists. For solutions of (1.1) with exponential layers only the existence of such a decomposition with bounds up to second order derivatives was proved in [22] using the idea of matched asymptotic expansion.
For the case of characteristic layers, Kellogg and Stynes [15] proved the following Lemma.

Lemma 2.3. *Assume b and c in* (1.1) *are constant. Let $f \in C^{8,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ satisfy the compatibility conditions*

$$f(0,0) = f(1,0) = f(1,1) = f(0,1) = 0.$$

Then Assumption 2.1 holds true with the only exception of the bound on $\partial_x^2 \partial_y w_2$. For this the weaker bound

$$\left\| \partial_x^2 \partial_y w_2 \right\|_0 \le C \varepsilon^{-1/2} \tag{2.2'}$$

holds.

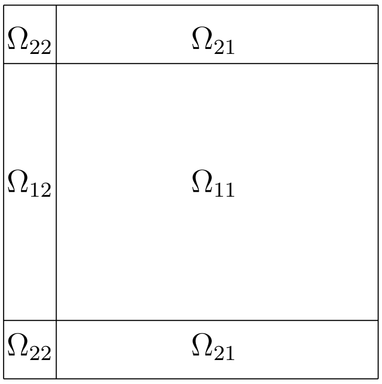
As already mentioned in [15], the bound (2.2') suffices. Thus, the estimates of Assumption 2.1 are appropriate even for non-constant b and c.

2.2 Layer-adapted meshes

The history of layer-adapted meshes began 1969 with a paper by Bakhvalov [2] followed by several publications of other authors. In '88 Shishkin [29] proposed the use of piecewise uniform meshes, later called *Shishkin meshes*. For a survey of layer-adapted meshes for convection-diffusion see [19, 21].
The performance of Shishkin meshes is inferior compared to Bakhvalov meshes. Therefore, much effort has been made to improve the results while retaining aspects of the simple construction. In this variety of meshes a simple criterion to deduce the order of convergence for standard methods is useful. In [25] such a general criterion on generalised Shishkin-type meshes, so called *S-type meshes*, is derived.
The analysis of FEM on Bakhvalov-type meshes is much more complicated than on S-type meshes. So far only one optimal result in 1d is known, see [24], where a quasi interpolant was used—rather than the more common nodal interpolant—to theoretically establish the optimal order of convergence. The application of this idea to 2d is still under research.
Here we shall consider S-type meshes only. These generalise the original Shishkin mesh. The transition point is unchanged, but inside the fine mesh region the mesh needs not to be uniform.

Let us start with the mesh-transition points

$$\lambda_x := \min\left\{\frac{1}{2}, \frac{\sigma\varepsilon}{\beta}\ln N\right\} \quad \text{and} \quad \lambda_y := \min\left\{\frac{1}{4}, \sigma\sqrt{\varepsilon}\ln N\right\}$$



$$\begin{aligned}
\Omega_{11} &:= [\lambda_x, 1] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{12} &:= [0, \lambda_x] \times [\lambda_y, 1-\lambda_y] \\
\Omega_{21} &:= [\lambda_x, 1] \times ([0, \lambda_y] \cup [1-\lambda_y, 1]) \\
\Omega_{22} &:= [0, \lambda_x] \times ([0, \lambda_y] \cup [1-\lambda_y, 1])
\end{aligned}$$

Figure 2.1: Subregions of Ω

with some user-chosen positive parameter σ that will be fixed later. Typically σ is chosen equal to the formal order of the numerical method or to accommodate the analysis. For the mere sake of simplicity in our subsequent analysis we shall assume that

$$\lambda_x = \frac{\sigma\varepsilon}{\beta} \ln N \leq \frac{1}{2} \quad \text{and} \quad \lambda_y = \sigma\sqrt{\varepsilon} \ln N \leq \frac{1}{4} \tag{2.3}$$

as is typically the case for (1.1).
The domain Ω is divided into four (resp. six) subregions —see Fig. 2.1— with Ω_{12} covering the exponential layer, Ω_{21} the parabolic layers, Ω_{22} the corner layers and Ω_{11} the remaining region.
These subdomains will be dissected by a tensor product mesh, according to

$$x_i := \begin{cases} \frac{\sigma\varepsilon}{\beta}\phi\left(\frac{i}{N}\right), & i = 0, \dots, N/2, \\ 1 - 2(1-\lambda_x)(1-\frac{i}{N}), & i = N/2, \dots, N, \end{cases}$$

$$y_j := \begin{cases} \sigma\sqrt{\varepsilon}\phi\left(\frac{2j}{N}\right), & j = 0, \dots, N/4, \\ (1-2\lambda_y)(\frac{2j}{N} - 1) + \frac{1}{2}, & j = N/4, \dots, 3N/4, \\ 1 - \sigma\sqrt{\varepsilon}\phi\left(2 - \frac{2j}{N}\right), & j = 3N/4, \dots, N \end{cases}.$$

where ϕ is a monotone increasing mesh-generating function with $\phi(0)$=0 and $\phi(1/2)$=$\ln N$. The final mesh is constructed by drawing lines parallel to the coordinate axes through these mesh points. Thus, Ω_{11} is uniformly dissected and the dissection in the other subdomains depends on ϕ.
Related to ϕ we define a new function ψ by

$$\phi = -\ln \psi.$$

Then ψ is decreasing with $\psi(0) = 1$ and $\psi(1/2) = N^{-1}$.

Assumption 2.4. *Let the mesh-generating function ϕ be piecewise differentiable with*

$$\max \phi' \leq CN \ \ \textit{(or equivalently } \max \frac{|\psi'|}{\psi} \leq CN). \tag{2.4}$$

All S-type meshes we shall consider here satisfy Assumption 2.4. We can use (2.4) to estimate $h_i := x_i - x_{i-1}$ inside the fine mesh region. Let $t_i = i/N$. Then for $i = 1, \dots, N/2$ holds (with $\max \phi'$ taken over $[t_{i-1}, t_i]$)

$$\psi(t_i) = e^{-\phi(t_i)} = e^{-(\phi(t_i)-\phi(t))}e^{-\phi(t)} \geq e^{-(\phi(t_i)-\phi(t_{i-1}))}\psi(t) \geq e^{-N^{-1}\max\phi'}\psi(t) \geq C\psi(t)$$

for all $t \in [t_{i-1}, t_i]$. Furthermore

$$x = \frac{\sigma\varepsilon}{\beta}\phi(t) = -\frac{\sigma\varepsilon}{\beta}\ln\psi(t)$$

implies

$$\psi(t) = e^{-\beta x/(\sigma\varepsilon)}$$

and

$$\begin{aligned} h_i = \frac{\sigma\varepsilon}{\beta}(\phi(t_i) - \phi(t_{i-1})) &\leq \frac{\sigma}{\beta}\varepsilon N^{-1}\max\phi' \leq \frac{\sigma}{\beta}\varepsilon N^{-1}\max|\psi'|/\psi(t_i) \\ &\leq C\varepsilon N^{-1}\max|\psi'|/\psi(t) = C\varepsilon N^{-1}\max|\psi'|e^{\beta x/(\sigma\varepsilon)} \end{aligned} \tag{2.5}$$

with $x \in [x_{i-1}, x_i]$. Similarly for $j = 1, \dots, N/4$ and $j = 3N/4 + 1, \dots, N$ we can bound

$$k_j := y_j - y_{j-1} \leq C\varepsilon^{1/2}N^{-1}\max|\psi'|lt(y) \tag{2.6}$$

using

$$lt(y) := \begin{cases} e^{y/(\sigma\varepsilon^{1/2})}, & y \leq \lambda_y \\ e^{(1-y)/(\sigma\varepsilon^{1/2})}, & y \geq 1 - \lambda_y \end{cases}$$

and $y \in [y_{j-1}, y_j]$. Of course the simpler bounds

$$\begin{aligned} h_i &\leq C\varepsilon N^{-1}\max\phi' \quad \leq C\varepsilon \\ k_j &\leq C\varepsilon^{1/2}N^{-1}\max\phi' \leq C\varepsilon^{1/2} \end{aligned}$$

for $i = 1, \dots, N/2$ and $j = 1, \dots, N/4, 3N/4 + 1, \dots, N$ follow from (2.4) too.

Notation: Let

$$h := \max_{i=1,\dots,N/2} h_i \quad \text{and} \quad k := \max_{j=1,\dots,N/4} k_j$$

be the maximal mesh sizes inside the layer regions.

Table 2.1 gives some examples of S-type meshes. We use the naming convention introduced in [25]. The polynomial S-mesh has an additional parameter $m > 0$ to adjust the grading inside the layer.

Name	$\phi(t)$	$\max \phi'$	$\psi(t)$	$\max \lvert\psi'\rvert$
Shishkin mesh	$2t \ln N$	$2 \ln N$	N^{-2t}	$2 \ln N$
B-S mesh	$-\ln(1-2t(1-N^{-1}))$	$2N$	$1-2t(1-N^{-1})$	2
polynomial S-mesh	$(2t)^m \ln N$	$2m \ln N$	$N^{-(2t)^m}$	$C(\ln N)^{1/m}$
modified B-S-mesh	$\frac{t}{q-t}$, $q=\frac{1}{2}(1+\frac{1}{\ln N})$	$3\ln^2 N$	$e^{-\frac{t}{q-t}}$	$3/(2q) \leq 3$

Table 2.1: Some examples of S-type meshes

2.3 Interpolation errors on layer-adapted meshes

Let u^I be the nodal bilinear interpolant. In [1, Theorem 2.7] the following anisotropic interpolation error bounds for $\tau = \tau_{i,j} \in T^N$ are given

$$\|w - w^I\|_{L_p(\tau)} \leq C \left\{ h_i^2 \|w_{xx}\|_{L_p(\tau)} + k_j^2 \|w_{yy}\|_{L_p(\tau)} \right\} \tag{2.7}$$

and

$$\|(w - w^I)_x\|_{L_p(\tau)} \leq C \left\{ h_i \|w_{xx}\|_{L_p(\tau)} + k_j \|w_{xy}\|_{L_p(\tau)} \right\} \tag{2.8}$$

which hold true for $p \in [1, \infty]$ and arbitrary $w \in W^{2,p}(\Omega)$.
Furthermore, for any $w \in L_\infty(\Omega)$ we have

$$\left\|w - w^I\right\|_{L_\infty(\tau)} \leq \|w\|_{L_\infty(\tau)} + \left\|w^I\right\|_{L_\infty(\tau)} \leq 2\,\|w\|_{L_\infty(\tau)}. \tag{2.9}$$

For w_x and $w_y \in L_\infty(\Omega)$ follows

$$\|w_x^I\|_{L_\infty(\tau)} \leq \|w_x\|_{L_\infty(\tau)} \quad \text{and} \quad \|w_y^I\|_{L_\infty(\tau)} \leq \|w_y\|_{L_\infty(\tau)}. \tag{2.10}$$

For $\tau \subset \Omega_{12} \cup \Omega_{22}$ the cell width h_i depends on the position inside the region. Using (2.5) we bound the terms on the right-hand-side of (2.7) and (2.8) by

$$h_i^\alpha \|w\|_{L_p(\tau)} = \|h_i^\alpha w(x,y)\|_{L_p(\tau)} \leq C(\varepsilon N^{-1} \max|\psi'|)^\alpha \|e^{\alpha\beta x/(\sigma\varepsilon)} w(x,y)\|_{L_p(\tau)}. \tag{2.11a}$$

For $\tau \subset \Omega_{21} \cup \Omega_{22}$ the cell height k_j varies and thus (2.6) gives

$$k_j^\alpha \|w\|_{L_p(\tau)} \leq C(\varepsilon^{1/2} N^{-1} \max|\psi'|)^\alpha \|lt(y)^\alpha w(x,y)\|_{L_p(\tau)}. \tag{2.11b}$$

Moreover, for any $w \in L_\infty(\Omega)$ we have by an inverse inequality

$$\|(w-w^I)_x\|_{0,\tau} \leq C(\|w_x\|_{0,\tau} + h_i^{-1}\|w^I\|_{0,\tau}) \leq C(\|w_x\|_{0,\tau} + h_i^{-1}(\operatorname{meas}\tau)^{\frac{1}{2}}\|w\|_{L_\infty(\tau)}). \tag{2.12}$$

In order to increase the readability of this thesis, the proofs of the following lemma and theorem are deferred to Section 2.4.
For the supercloseness analysis we need sufficiently sharp L_2-estimates of the interpolation error of certain layer parts of $u = v + w_1 + w_2 + w_{12}$, see Assumption 2.1.

Lemma 2.5. *Let $\sigma \geq 5/2$ and $\tilde{w} = w_1 + w_{12}$. Then the interpolation errors in the L_2-norm can be bounded by*

$$\|(w_2 + \tilde{w}) - (w_2 + \tilde{w})^I\|_{0,\Omega_{11}} \leq CN^{-\sigma}(\varepsilon^{1/4} + N^{-1/2}), \tag{2.13a}$$

$$\|(w_2 + \tilde{w}) - (w_2 + \tilde{w})^I\|_{0,\Omega_{12}} \leq C\varepsilon^{1/2}(N^{-1} \max |\psi'|)^2, \tag{2.13b}$$

$$\|\tilde{w} - \tilde{w}^I\|_{0,\Omega_{21}} \leq C\varepsilon^{1/4}N^{-\sigma} \ln^{1/2} N \tag{2.13c}$$

and

$$\|\tilde{w} - \tilde{w}^I\|_{0,\Omega_{22}} \leq C\varepsilon^{1/2}(k + N^{-1} \max |\psi'|)^2. \tag{2.13d}$$

Theorem 2.6 (Interpolation error)**.** *The error of bilinear nodal interpolation on an S-type mesh with $\sigma \geq 5/2$ satisfies in the maximum norm*

$$\|u - u^I\|_{L_\infty(\Omega_{11})} \leq CN^{-2}, \tag{2.14a}$$

$$\|u - u^I\|_{L_\infty(\Omega_{12})} \leq C(h + N^{-1} \max |\psi'|)^2, \tag{2.14b}$$

$$\|u - u^I\|_{L_\infty(\Omega_{21})} \leq C(k + N^{-1} \max |\psi'|)^2 \tag{2.14c}$$

and

$$\|u - u^I\|_{L_\infty(\Omega_{22})} \leq C(h + k + N^{-1} \max |\psi'|)^2, \tag{2.14d}$$

in the L_2-norm

$$\|u - u^I\|_{0,\Omega_{11}} \leq CN^{-2}, \tag{2.15a}$$

$$\|u - u^I\|_{0,\Omega_{12}} \leq C\varepsilon^{1/2}((h + N^{-1}) \ln^{1/4} N + N^{-1} \max |\psi'|)^2, \tag{2.15b}$$

$$\|u - u^I\|_{0,\Omega_{21}} \leq C\varepsilon^{1/4} \ln^{1/2} N(k + N^{-1} \max |\psi'|)^2 \tag{2.15c}$$

and

$$\|u - u^I\|_{0,\Omega_{22}} \leq C\varepsilon^{1/2} \ln^{1/2} N(h + k + N^{-1} \max |\psi'|)^2, \tag{2.15d}$$

and in the energy-norm

$$|||u - u^I|||_\varepsilon \leq C(\varepsilon^{1/4}h + k + N^{-1} \max |\psi'|). \tag{2.16}$$

Moreover, we have the following estimates for the x-derivative

$$\|(u - u^I)_x\|_{0,\Omega_{12}} \leq C(\varepsilon^{1/2} \ln^{1/2} Nh + \varepsilon^{-1/2}N^{-1} \max |\psi'|) \tag{2.17a}$$

and

$$\|(u - u^I)_x\|_{0,\Omega_{22}} \leq C\varepsilon^{1/4} \ln^{1/2} N(\varepsilon^{1/2} \ln^{1/2} Nh + \varepsilon^{-1/2}(k + N^{-1} \max |\psi'|)). \tag{2.17b}$$

Remark 2.7. *In the proof of Theorem 2.6 we use the decomposition of u and bound the different parts of the interpolation errors on the subregions separately. The energy-norm error can also be estimated as in [21, Theorem 5.4] giving*

$$|||u - u^I|||_\varepsilon \leq C(\|u - u^I\|_{L_\infty(\Omega)})^{1/2} \leq C(h + k + N^{-1} \max |\psi'|).$$

2.4 Proofs

Proof of Lemma 2.5.
(1) Let us start with Ω_{11}. We follow an idea by Zhang [33], that enables us to assume $\sigma \geq 5/2$ rather than $\sigma \geq 3$.
Let $w = w_1 + w_2 + w_{12}$. Clearly

$$\|w - w^I\|_{0,\Omega_{11}} \leq \|w\|_{0,\Omega_{11}} + \|w^I\|_{0,\Omega_{11}}. \tag{2.18}$$

A direct calculation gives

$$\|w\|_{0,\Omega_{11}} \leq C\varepsilon^{1/4}N^{-\sigma}.$$

Let $\hbar$ and $\bar{k}$ be the mesh width and mesh height in Ω_{11}. The domain Ω_{11} is divided into $S = [\lambda_x + \hbar, 1] \times [\lambda_y + \bar{k}, 1 - \lambda_y - \bar{k}]$ and $\Omega_{11} \setminus S$.
Note that $\Omega_{11} \setminus S$ consists of only a single ply of $\mathcal{O}(N)$ mesh elements adjacent to the boundary layer regions and $\hbar$, $\bar{k} \leq CN^{-1}$. Thus,

$$\|w^I\|^2_{0,\Omega_{11}\setminus S} \leq \sum_{\tau \subset \Omega_{11}\setminus S} \hbar\bar{k}\|w^I\|^2_{L_\infty(\tau)} \leq CN^{-1}\|w\|^2_{L_\infty(\Omega_{11})} \leq CN^{-2\sigma-1}. \tag{2.19}$$

For $\tau_{i,j} \subset S$ we have

$$\begin{aligned}
\|w^I\|^2_{0,\tau_{i,j}} &\leq h_i k_j \|w^I\|^2_{L_\infty(\tau_{i,j})} \\
&\leq Ch_i k_j (e^{-2\beta x_{i-1}/\varepsilon} + (e^{-2y_{j-1}/\sqrt{\varepsilon}} + e^{-2(1-y_j)/\sqrt{\varepsilon}})(1 + e^{-2\beta x_{i-1}/\varepsilon})) \\
&\leq C\left\{\int_{\tau_{i-1,j-1}} (e^{-2\beta x/\varepsilon} + e^{-2y/\sqrt{\varepsilon}}(1 + e^{-2\beta x/\varepsilon})) + \int_{\tau_{i-1,j}} e^{-2(1-y)/\sqrt{\varepsilon}}(1 + e^{-2\beta x/\varepsilon})\right\}.
\end{aligned}$$

Summing over S yields

$$\|w^I\|^2_{0,S} \leq C\int_{\Omega_{11}} (e^{-2\beta x/\varepsilon} + (e^{-2y/\sqrt{\varepsilon}} + e^{-2(1-y)/\sqrt{\varepsilon}})(1 + e^{-2\beta x/\varepsilon})) \leq C\varepsilon^{1/2}N^{-2\sigma}.$$

This together with (2.18) and (2.19) completes the proof of (2.13a).
(2) On Ω_{12} use (2.7) and (2.11) to establish

$$\begin{aligned}
\|w_1 - w_1^I\|^2_{0,\Omega_{12}} &\leq C(\|h_i^2 w_{1xx}\|^2_{0,\Omega_{12}} + \|k_j^2 w_{1yy}\|^2_{0,\Omega_{12}}) \\
&\leq C(\varepsilon(N^{-1}\max|\psi'|)^4 + \varepsilon N^{-4}) \leq C\varepsilon(N^{-1}\max|\psi'|)^4
\end{aligned} \tag{2.20}$$

while for w_2 and w_{12} we use (2.9) to obtain

$$\|(w_2 + w_{12}) - (w_2 + w_{12})^I\|_{0,\Omega_{12}} \leq C(\operatorname{meas}\Omega_{12})^{1/2}\|w_2 + w_{12}\|_{L_\infty(\Omega_{12})} \leq C\varepsilon^{1/2}N^{-\sigma}\ln^{1/2}N.$$

Together with $\sigma > 2$ and (2.20) we obtain (2.13b).
(3) Similarly, on Ω_{21} we have

$$\|(w_1 + w_{12}) - (w_1 + w_{12})^I\|_{0,\Omega_{21}} \leq C\varepsilon^{1/4}N^{-\sigma}\ln^{1/2}N.$$

(4) On Ω_{22} we use (2.7) and (2.11) for w_1 and w_{12}.

$$\|w_1 - w_1^I\|_{0,\Omega_{22}} \le C\varepsilon^{3/4} \ln^{1/2} N(k + N^{-1} \max |\psi'|)^2,$$
$$\|w_{12} - w_{12}^I\|_{0,\Omega_{22}} \le C\varepsilon^{3/4}(N^{-1} \max |\psi'|)^2$$

Applying (2.3), we get (2.13d). □

Proof of Theorem 2.6.
Let us start with the pointwise estimates. We separately bound the errors for the various terms of the decomposition $u = v + w_1 + w_2 + w_{12}$ and the different subregions of Ω. The anisotropic estimate (2.7) gives

$$\|v - v^I\|_{L_\infty(\Omega_{11})} \le CN^{-2}, \qquad \|v - v^I\|_{L_\infty(\Omega_{12})} \le C(h + N^{-1})^2,$$
$$\|v - v^I\|_{L_\infty(\Omega_{21})} \le C(k + N^{-1})^2, \qquad \|v - v^I\|_{L_\infty(\Omega_{22})} \le C(h + k)^2.$$

On $\Omega_{11} \cup \Omega_{21}$ we use (2.9) and $\|w_1\|_{L_\infty(\Omega_{11}\cup\Omega_{21})} \le CN^{-\sigma}$ to get

$$\|w_1 - w_1^I\|_{L_\infty(\Omega_{11}\cup\Omega_{21})} \le CN^{-\sigma}.$$

For Ω_{12} and Ω_{22} application of (2.7) and (2.11) with $\sigma \ge 2$ gives

$$\|w_1 - w_1^I\|_{L_\infty(\Omega_{12})} \le C \max_{\tau\subset\Omega_{12}} (h_i^2 \|w_{1,xx}\|_{L_\infty(\tau)} + k_j^2 \|w_{1,yy}\|_{L_\infty(\tau)}) \le C(N^{-1} \max |\psi'|)^2$$

and

$$\|w_1 - w_1^I\|_{L_\infty(\Omega_{22})} \le C(k + N^{-1} \max |\psi'|)^2.$$

Similarly we have

$$\|w_2 - w_2^I\|_{L_\infty(\Omega_{11}\cup\Omega_{12})} \le CN^{-\sigma},$$
$$\|w_2 - w_2^I\|_{L_\infty(\Omega_{21})} \le C(N^{-1} \max |\psi'|)^2$$

and

$$\|w_2 - w_2^I\|_{L_\infty(\Omega_{22})} \le C(h + N^{-1} \max |\psi'|)^2.$$

For w_{12} apply (2.9) on $\Omega \setminus \Omega_{22}$ and (2.11) on Ω_{22} in both directions. This gives

$$\|w_{12} - w_{12}^I\|_{L_\infty(\Omega\setminus\Omega_{22})} \le CN^{-\sigma}$$

and

$$\|w_{12} - w_{12}^I\|_{L_\infty(\Omega_{22})} \le C(N^{-1} \max |\psi'|)^2.$$

Combining these estimates we get (2.14) since $\sigma \ge 2$.

The L_2-norm errors are obtained using (2.13) and the pointwise estimates (2.14) for those terms of the decomposition not contained in (2.13).
The estimates for the H^1-seminorm error can be derived in different ways. One possibility is shown in the proof of Theorem 5.4 in [21]. The key idea is to bound the square of the H^1-seminorm by the L_∞-norm. We will use a different approach, bounding the parts individually on the subregions, because this provides sharper bounds.
With the anisotropic error estimates (2.8) we have

$$\|\nabla(v-v^I)\|_{0,\Omega_{11}} \le CN^{-1}, \tag{2.21a}$$
$$\|\nabla(v-v^I)\|_{0,\Omega_{12}} \le C\varepsilon^{1/2}\ln^{1/2} N(h+N^{-1}), \tag{2.21b}$$
$$\|\nabla(v-v^I)\|_{0,\Omega_{21}} \le C\varepsilon^{1/4}\ln^{1/2} N(k+N^{-1}) \tag{2.21c}$$

and

$$\|\nabla(v-v^I)\|_{0,\Omega_{22}} \le C\varepsilon^{3/4}\ln N(h+k). \tag{2.21d}$$

On Ω_{11} and Ω_{21} inequality (2.12) and $\|w_1\|_{L_\infty(\Omega_{11})} \le CN^{-\sigma}$ gives

$$\begin{aligned}\varepsilon^{1/2}\|(w_1-w_1^I)_x\|_{0,\Omega_{11}} &\le C\varepsilon^{1/2}(\|w_{1,x}\|_{0,\Omega_{11}} + N\|w_1\|_{L_\infty(\Omega_{11})})\\ &\le C(N^{-\sigma}+\varepsilon^{1/2}N^{-\sigma+1}), \qquad (2.22a)\\ \varepsilon^{1/2}\|(w_1-w_1^I)_x\|_{0,\Omega_{21}} &\le C(\varepsilon^{1/4}\ln^{1/2} NN^{-\sigma}+\varepsilon^{1/2}N^{-\sigma+1}), \qquad (2.22b)\end{aligned}$$

while on Ω_{12} and Ω_{22} we use (2.8) and (2.11) to get

$$\varepsilon\|(w_1-w_1^I)_x\|^2_{0,\Omega_{12}} \le C\varepsilon \sum_{\tau_{i,j}\subset\Omega_{12}} (\|h_i w_{1,xx}\|^2_{\tau_{i,j}} + \|k_j w_{1,xy}\|^2_{\tau_{i,j}}) \le C(N^{-1}\max|\psi'|)^2 \tag{2.22c}$$

and

$$\varepsilon\|(w_1-w_1^I)_x\|^2_{0,\Omega_{22}} \le C\varepsilon^{1/2}\ln N(N^{-1}\max|\psi'|+k)^2. \tag{2.22d}$$

When estimating the norm of y-derivative we can proceed on all regions except Ω_{21} as above and obtain

$$\varepsilon^{1/2}\|(w_1-w_1^I)_y\|_{0,\Omega_{11}} \le C\varepsilon^{1/2}N^{-\sigma+1} \tag{2.22e}$$

and

$$\varepsilon^{1/2}\|(w_1-w_1^I)_y\|_{0,\Omega_{12}\cup\Omega_{22}} \le C\varepsilon(k+N^{-1}\max|\psi'|). \tag{2.22f}$$

On Ω_{21} follows

$$\begin{aligned}\varepsilon^{1/2}\|(w_1-w_1^I)_y\|_{0,\Omega_{21}} &\le C\varepsilon^{1/2}(N^{-1}\|w_{1,xy}\|_{0,\Omega_{21}} + k\|w_{1,yy}\|_{0,\Omega_{21}})\\ &\le C\varepsilon^{1/4}\ln^{1/2} NN^{-\sigma}(N^{-1}+\varepsilon k). \qquad (2.22g)\end{aligned}$$

For w_2 we get in a similar manner for the x-derivatives

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_x\|_{0,\Omega_{11}} \le C\varepsilon^{1/2}(\varepsilon^{1/4}N^{-\sigma} + N^{-\sigma+1}), \tag{2.23a}$$

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_x\|_{0,\Omega_{12}} \le C\varepsilon^{3/4}\ln^{1/2} N N^{-\sigma}(N^{-1} + \varepsilon^{1/2}h), \tag{2.23b}$$

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_x\|_{0,\Omega_{21}} \le C\varepsilon^{3/4}N^{-1}\max|\psi'| \tag{2.23c}$$

and

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_x\|_{0,\Omega_{22}} \le C\varepsilon^{5/4}\ln^{1/2} N(h + N^{-1}\max|\psi'|) \tag{2.23d}$$

and for the y-derivatives

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_y\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\varepsilon^{1/4}(N^{-\sigma} + \varepsilon^{1/4}N^{-\sigma+1}) \tag{2.23e}$$

and

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_y\|_{0,\Omega_{21}\cup\Omega_{22}} \le C\varepsilon^{1/4}(h + N^{-1}\max|\psi'|). \tag{2.23f}$$

Finally, for w_{12} on all regions except Ω_{12} bound as above to get for the x-derivatives

$$\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_x\|_{0,\Omega_{11}} \le C\varepsilon^{1/4}(N^{-2\sigma} + \varepsilon^{1/4}N^{-2\sigma+1}), \tag{2.24a}$$

$$\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_x\|_{0,\Omega_{21}} \le C\varepsilon^{1/4}(N^{-\sigma} + \varepsilon^{1/2}\ln^{1/2} N N^{-\sigma+1}) \tag{2.24b}$$

and

$$\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_x\|_{0,\Omega_{22}} \le C\varepsilon^{1/4}N^{-1}\max|\psi'|. \tag{2.24c}$$

On Ω_{12} the stability estimate (2.10) yields

$$\begin{aligned}\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_x\|_{0,\Omega_{12}} &\le C\varepsilon^{1/2}(\|w_{12,x}\|_{0,\Omega_{12}} + (\operatorname{meas}\Omega_{12})^{\frac{1}{2}}\|w_{12,x}\|_{L_\infty(\Omega_{12})}) \\ &\le CN^{-\sigma}\ln^{1/2} N.\end{aligned} \tag{2.24d}$$

Similarly for the y-derivatives we get

$$\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_y\|_{0,\Omega_{11}\cup\Omega_{12}} \le C\varepsilon^{1/2}(\varepsilon^{1/4}N^{-\sigma} + N^{-\sigma+1}), \tag{2.24e}$$

$$\varepsilon^{1/2}\|(w_2 - w_2^I)_y\|_{0,\Omega_{22}} \le C\varepsilon^{3/4}N^{-1}\max|\psi'| \tag{2.24f}$$

and again by the stability estimate (2.10)

$$\begin{aligned}\varepsilon^{1/2}\|(w_{12} - w_{12}^I)_y\|_{0,\Omega_{21}} &\le C\varepsilon^{1/2}(\|w_{12,y}\|_{0,\Omega_{21}} + (\operatorname{meas}\Omega_{21})^{\frac{1}{2}}\|w_{12,y}\|_{L_\infty(\Omega_{21})}) \\ &\le C\varepsilon^{1/4}\ln^{1/2} N N^{-\sigma}.\end{aligned} \tag{2.24g}$$

Combining (2.21)–(2.24) and the L_2-norm estimates (2.15) we are done. □

Chapter 3

Supercloseness

This chapter is devoted to one of the main themes of this thesis. For several methods we analyse a property named *supercloseness*.

Definition 3.1. *The numerical solution u^N to a given solution u of a problem like* (1.1) *is called* ***superclose*** *with respect to a given interpolation u^I and norm $\|\cdot\|$, if*

$$\|u^N - u^I\| = o(\|u^N - u\|),$$

i.e. the numerical solution u^N is closer to the interpolant u^I than to the exact solution in the given norm.

In Chapter 4 this property is used to enhance the accuracy of the numerical solution by means of simple postprocessing techniques.
We will start by analysing the underlying Galerkin finite element method and continue with stabilised methods.

3.1 Galerkin Finite Element Method

The foundation of all stabilised methods presented in this thesis is the Galerkin finite element method (GFEM). Therefore, in this section we analyse the supercloseness property of GFEM.
In [9] we present a supercloseness analysis of GFEM on Shishkin meshes that is generalised to S-type meshes here.
Start with the variational formulation of (1.1): Find $u \in H_0^1(\Omega)$ such that

$$a_{Gal}(u, v) := \varepsilon(\nabla u, \nabla v) - (bu_x, v) + (cu, v) = f(v) =: (f, v), \text{ for all } v \in H_0^1(\Omega), \quad (3.1)$$

where $(\cdot,\cdot)_D$ denotes the standard scalar product in $L_2(D)$. If $D = \Omega$ we drop the Ω from the notation again.
Due to (1.2) the bilinear form $a_{Gal}(\cdot,\cdot)$ is coercive with respect to the ε-weighted energy norm, i.e.,

$$a_{Gal}(v, v) = \varepsilon(\nabla v, \nabla v) + ((c + \frac{1}{2}b_x)v, v) \geq \varepsilon\|\nabla v\|_0^2 + \gamma\|v\|_0^2 = |||v|||_\varepsilon^2 \quad \text{for all } v \in H_0^1(\Omega).$$

Because of the coercivity the Lax-Milgram Lemma ensures the existence of a unique solution $u \in H_0^1(\Omega)$ of the variational formulation.
Let $V^N \subset H_0^1(\Omega)$ be a finite-element space consisting of piecewise bilinear elements over the S-type mesh. Then the discretisation is: Find $u^N \in V^N$ such that

$$a_{Gal}(u^N, v^N) = f(v^N) \quad \text{for all } v^N \in V^N. \tag{3.2}$$

The uniqueness of this solution is guaranteed by the coercivity of $a_{Gal}(\cdot,\cdot)$.
The following integral identities from [17] are a crucial ingredient to the analysis.

Lemma 3.2. *Let $\tau_{i,j} \in T^N$ be an arbitrary mesh rectangle with midpoint $(\tilde{x}_i, \tilde{y}_j)$ and edges ℓ_1, ℓ_2 that run parallel to the y-axis (with ℓ_1 the "western" boundary).
For any function $w \in C^3(\bar{\tau}_{i,j})$ and any bilinear function χ there holds*

$$\int_{\tau_{i,j}} (w - w^I)_x \chi_x = \int_{\tau_{i,j}} \left[F_j \chi_x - \frac{1}{3}(F_j^2)' \chi_{xy} \right] w_{xyy} \tag{3.3}$$

and

$$\int_{\tau_{i,j}} (w - w^I)_x \chi = H_{i,j}(w, \chi) + \frac{h_i^2}{12} \left(\int_{\ell_1} - \int_{\ell_2} \right) \chi w_{xx} \, \mathrm{d}y \tag{3.4}$$

with

$$H_{i,j}(w,\chi) := \int_{\tau_{i,j}} \left[F_j(\chi - E_i'\chi_x) - \frac{(F^2)'}{3}(\chi_y - E_i'\chi_{xy}) \right] w_{xyy} + \int_{\tau_{i,j}} \left[\frac{(E_i^2)'}{6}\chi_x - \frac{h_i^2}{12}\chi \right] w_{xxx}$$

and

$$E_i(x) := \frac{(x - \tilde{x}_i)^2}{2} - \frac{h_i^2}{8} \quad \text{and} \quad F_j(y) := \frac{(y - \tilde{y}_j)^2}{2} - \frac{k_j^2}{8}.$$

Remark 3.3. *The Cauchy-Schwarz inequality and the inverse inequalities*

$$\|\chi_x\|_{L_p(\tau_{i,j})} \le C h_i^{-1} \|\chi\|_{L_p(\tau_{i,j})} \quad \text{and} \quad \|\chi_y\|_{L_p(\tau_{i,j})} \le C k_j^{-1} \|\chi\|_{L_p(\tau_{i,j})} \tag{3.5}$$

applied to (3.3) *give*

$$\left| \left((w - w^I)_x, \chi_x \right)_{\tau_{i,j}} \right| \le C k_j^2 \|w_{xyy}\|_{0,\tau_{i,j}} \|\chi_x\|_{0,\tau_{i,j}} \quad \text{for all } w \in C^3(\bar{\tau}_{i,j}); \tag{3.6}$$

similarly

$$|H_{i,j}(w,\chi)| \le C \Big\{ k_j^2 \|w_{xyy}\|_{0,\tau_{i,j}} + h_i^2 \|w_{xxx}\|_{0,\tau_{i,j}} \Big\} \|\chi\|_{0,\tau_{i,j}}. \tag{3.7}$$

name	$\|\|\|u^I - u^N\|\|\|_\varepsilon$	$\|\|\|u - u^N\|\|\|_\varepsilon$
Shishkin mesh	$CN^{-2}\ln^2 N$	$CN^{-1}\ln N$
polynomial S-mesh	$CN^{-2}(\ln^{1/2} N + \ln^{2/m} N)$	$CN^{-1}\ln^{1/m} N$
Bakhvalov-Shishkin mesh	$C(\varepsilon + N^{-2}\ln^{1/2} N)$	$C(\varepsilon^{1/2} + N^{-1})$
modified B-S-mesh	$C(\varepsilon + N^{-2}\ln^{1/2} N)$	$C(\varepsilon^{1/2} + N^{-1})$

Table 3.1: Expected rates for GFEM on different S-type meshes

3.1.1 Supercloseness and convergence

As in Chapter 2 the proof to the following Lemma is deferred to a separate subsection.

Lemma 3.4. *Let $\sigma \geq 5/2$, $\chi = u^I - u^N$ and u be the solution to* (1.1). *Then we have*

$$\left|\varepsilon(\nabla(u-u^I),\nabla\chi)\right| \leq C(\varepsilon^{1/4}h + k + N^{-1} + \varepsilon^{1/8}N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,, \tag{3.8a}$$

$$\left|(b(u-u^I)_x,\chi)\right| \leq C((h+N^{-1})\ln^{1/4}N + k + N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon \tag{3.8b}$$

and

$$\left|(c(u-u^I),\chi)\right| \leq C(h + k + N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,. \tag{3.8c}$$

With this auxiliary result we are in a position to state the main theorem of this section.

Theorem 3.5 (Supercloseness GFEM)**.** *Let $\sigma \geq 5/2$. Then the GFEM-solution u^N on an S-type mesh satisfies*

$$\left|\left|\left|u^N - u^I\right|\right|\right|_\varepsilon \leq C((h+N^{-1})\ln^{1/4}N + k + N^{-1}\max|\psi'|)^2. \tag{3.9}$$

Proof. By coercivity and Galerkin orthogonality we have

$$\begin{aligned}\left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon^2 &\leq a_{Gal}(u-u^I, u^I-u^N)\\ &\leq \varepsilon|(\nabla(u-u^I),\nabla\chi)| + |(b(u-u^I)_x,\chi)| + |(c(u-u^I),\chi)|\end{aligned}$$

and (3.9) follows by Lemma 3.4. □

Remark 3.6 (Convergence GFEM)**.** *We can conclude convergence by supercloseness of Theorem 3.5 and the interpolation error of Theorem 2.6.*

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \leq \left|\left|\left|u - u^I\right|\right|\right|_\varepsilon + \left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon \leq C(\varepsilon^{1/4}h + k + N^{-1}\max|\psi'|). \tag{3.10}$$

Table 3.1 shows the expected rates of the errors in the energy norm for different S-type meshes in accordance with Theorem 3.5 and Remark 3.6. Numerical simulations indicate that the logarithmic factor $\ln^{1/2} N$ in these estimates is caused by the analysis, which seems not to be sharp enough, see Chapter 5.

3.1.2 Proofs

Proof of Lemma 3.4.
(1) We start by bounding the diffusive term separately for each part of the decomposition of u.
(i) Inequality (3.6) and a discrete Cauchy-Schwarz inequality give

$$|((v-v^I)_x,\chi_x)| \le C(k+N^{-1})^2\|\chi_x\|_0 \quad \text{and} \quad |((v-v^I)_y,\chi_y)| \le C(h+N^{-1})^2\|\chi_y\|_0.$$

This leads to

$$\varepsilon|(\nabla(v-v^I),\nabla\chi)| \le C\varepsilon^{1/2}(h+k+N^{-1})^2\,|||\chi|||_\varepsilon\,. \tag{3.11}$$

(ii) Similarly we get for w_1 and the x-derivative

$$|((w_1-w_1^I)_x,\chi_x)| \le C(k+N^{-1})^2\|w_{1,xyy}\|_0\|\chi_x\|_0 \le C(k+N^{-1})^2\varepsilon^{-1}\,|||\chi|||_\varepsilon\,.$$

With inequality (2.10) applied in y-direction follows

$$|((w_1-w_1^I)_y,\chi_y)_{\Omega_{11}\cup\Omega_{21}}| \le C\|w_{1,y}\|_{L_\infty(\Omega_{11}\cup\Omega_{21})}\|\chi_y\|_{0,\Omega_{11}\cup\Omega_{21}} \le CN^{-\sigma}\varepsilon^{-1/2}\,|||\chi|||_\varepsilon$$

and on the remaining region the analogue of (3.6), (2.11) and $\sigma > 2$ give

$$\begin{aligned}|((w_1-w_1)_y,\chi_y)_{\Omega_{12}\cup\Omega_{22}}| &\le C(N^{-1}\max|\psi'|)^2\|e^{\beta x(2/\sigma-1)/\varepsilon}\|_{0,\Omega_{12}\cup\Omega_{22}}\|\chi_y\|_{0,\Omega_{12}\cup\Omega_{22}}\\ &\le C(N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,.\end{aligned}$$

The combination of these three estimates yields

$$\varepsilon|(\nabla(w_1-w_1^I),\nabla\chi)| \le C(k+N^{-1}+\varepsilon^{1/2}N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,. \tag{3.12}$$

(iii) Similar to the previous estimates we get

$$\begin{aligned}|((w_2-w_2^I)_x,\chi_x)_{\Omega_{11}\cup\Omega_{12}}| &\le C\varepsilon^{-1/2}N^{-\sigma}\,|||\chi|||_\varepsilon\,,\\ |((w_2-w_2^I)_x,\chi_x)_{\Omega_{21}\cup\Omega_{22}}| &\le C\varepsilon^{-1/4}(N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,,\\ |((w_2-w_2^I)_y,\chi_y)_{\Omega_{11}\cup\Omega_{21}}| &\le C\varepsilon^{-3/4}N^{-2}\,|||\chi|||_\varepsilon\end{aligned}$$

and

$$|((w_2-w_2^I)_y,\chi_y)_{\Omega_{12}\cup\Omega_{22}}| \le C\varepsilon^{-1/2}h^2\,|||\chi|||_\varepsilon$$

that summarises to

$$\varepsilon|(\nabla(w_2-w_2^I),\nabla\chi)| \le C(\varepsilon^{1/4}h+\varepsilon^{1/8}N^{-1}+\varepsilon^{3/8}N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,. \tag{3.13}$$

(iv) For w_{12} apply the inverse inequalities (2.12) and (3.5) on Ω_{11} to get

$$\begin{aligned}|((w_{12}-w_{12}^I)_x,\chi_x)_{\Omega_{11}}| &\le C\|w_{12,x}\|_{L_\infty(\Omega_{11})}\|\chi_x\|_{0,\Omega_{11}} \le C\varepsilon^{-1}N^{-2\sigma+1}\|\chi\|_{0,\Omega_{11}}\\ &\le C\varepsilon^{-1}N^{-2\sigma+1}\,|||\chi|||_\varepsilon\,,\end{aligned}$$

while on Ω_{12} follows directly

$$\begin{aligned}|((w_{12}-w_{12}^I)_x,\chi_x)_{\Omega_{12}}| &\le C\|w_{12,x}\|_{L_\infty(\Omega_{12})}(\operatorname{meas}\Omega_{12})^{1/2}\|\chi_x\|_{0,\Omega_{12}}\\ &\le C\varepsilon^{-1}N^{-\sigma}\ln^{1/2}N\,|||\chi|||_\varepsilon .\end{aligned}$$

In the remaining region (2.8) and (2.11) with $\sigma > 2$ yield

$$|((w_{12}-w_{12}^I)_x,\chi_x)_{\Omega_{21}\cup\Omega_{22}}| \le C\varepsilon^{-3/4}(N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon .$$

The y-derivative of w_{12} is treated similar to the one of w_1.

$$\begin{aligned}|((w_{12}-w_{12}^I)_y,\chi_y)_{\Omega_{11}\cup\Omega_{21}}| &\le C\varepsilon^{-1}N^{-\sigma}\,|||\chi|||_\varepsilon\\ |((w_{12}-w_{12}^I)_y,\chi_y)_{\Omega_{12}\cup\Omega_{22}}| &\le C\varepsilon^{-1/4}(N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\end{aligned}$$

Combining these estimates with $\sigma \ge 5/2$ we get

$$\varepsilon|(\nabla(w_{12}-w_{12}^I),\nabla\chi)| \le C(N^{-1}+\varepsilon^{1/8}N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon . \tag{3.14}$$

Estimates (3.11)-(3.14) give (3.8a).

(2) Before bounding the convective part we prove (3.8c) using the Cauchy-Schwarz inequality and the L_2-interpolation error (2.15)

$$|(c(u-u^I),\chi)| \le C\|u-u^I\|_0\|\chi\|_0 \le C(\varepsilon^{1/8}h+k+N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon .$$

(3) Now let us handle the convective part. Let $\tilde w = w_1 + w_{12}$. Start with a decomposition of the convective part and apply integration by parts.

$$\begin{aligned}(b(u-u^I)_x,\chi) = {} & (b(v-v^I)_x,\chi) + (b(w_2-w_2^I)_x,\chi)_{\Omega_{21}\cup\Omega_{22}}\\ & - (b_x(\tilde w-\tilde w^I),\chi) - (b(\tilde w-\tilde w^I),\chi_x)\\ & - (b_x(w_2-w_2^I),\chi)_{\Omega_{11}\cup\Omega_{12}} - (b(w_2-w_2)^I,\chi_x)_{\Omega_{11}\cup\Omega_{12}}\end{aligned}$$

(i) A Cauchy-Schwarz inequality and Lemma 2.5 give

$$\begin{aligned}|(b_x(\tilde w-\tilde w^I),\chi)| + |(b_x(w_2-w_2^I),\chi)_{\Omega_{11}\cup\Omega_{12}}| &\le C(\|\tilde w-\tilde w^I\|_0 + \|w_2-w_2^I\|_{0,\Omega_{11}\cup\Omega_{12}})\|\chi\|_0\\ &\le C(\varepsilon^{1/2}(k+N^{-1}\max|\psi'|)^2 + N^{-\sigma})\,|||\chi|||_\varepsilon .\end{aligned} \tag{3.15}$$

(ii) Cauchy-Schwarz, Lemma 2.5 and the following inverse inequality

$$\varepsilon^{1/4}N^{-1/2}\|\chi_x\|_{0,\Omega_{11}\cup\Omega_{21}} = (\varepsilon^{1/2}\|\chi_x\|_{0,\Omega_{11}\cup\Omega_{21}}N^{-1}\|\chi_x\|_{0,\Omega_{11}\cup\Omega_{21}})^{1/2} \le C\,|||\chi|||_\varepsilon$$

yield

$$\begin{aligned}&|(b(\tilde w-\tilde w^I),\chi_x)| + |(b(w_2-w_2)^I,\chi_x)_{\Omega_{11}\cup\Omega_{12}}|\\ &\qquad\le C(\|\tilde w-\tilde w^I\|_{0,\Omega_{11}}\|\chi_x\|_{0,\Omega_{11}} + \|\tilde w-\tilde w^I\|_{0,\Omega_{12}}\|\chi_x\|_{0,\Omega_{12}} +\\ &\qquad\qquad \|\tilde w-\tilde w^I\|_{0,\Omega_{21}}\|\chi_x\|_{0,\Omega_{21}} + \|\tilde w-\tilde w^I\|_{0,\Omega_{22}}\|\chi_x\|_{0,\Omega_{22}} +\\ &\qquad\qquad \|w_2-w_2^I\|_{0,\Omega_{11}}\|\chi_x\|_{0,\Omega_{11}} + \|w_2-w_2^I\|_{0,\Omega_{12}}\|\chi_x\|_{0,\Omega_{12}})\\ &\qquad\le C(N^{-2}(\varepsilon^{1/4}N^{-1/2}+N^{-1})\|\chi_x\|_{0,\Omega_{11}} + (N^{-1}\max|\psi'|)^2\varepsilon^{1/2}\|\chi_x\|_{0,\Omega_{12}} +\\ &\qquad\qquad N^{-\sigma}\ln^{1/2}N\varepsilon^{1/4}\|\chi_x\|_{0,\Omega_{21}} + (k+N^{-1}\max|\psi'|)^2\varepsilon^{1/2}\|\chi_x\|_{0,\Omega_{22}})\\ &\qquad\le C((k+N^{-1}\max|\psi'|)^2 + N^{-\sigma+1/2}\ln^{1/2}N)\,|||\chi|||_\varepsilon .\end{aligned} \tag{3.16}$$

The term including the logarithmic factor can be neglected if $\sigma > 5/2$. But as we will see later, this is not the only occurrence of this logarithmic term.
(iii) Let $\bar{b}|_{\tau_{i,j}} := b_{i,j} := b(x_i, y_j)$ be a piecewise constant approximation to b. With the Lin-identity (3.4) follows

$$\begin{aligned}(b(v-v^I)_x,\chi) &= \sum_{\tau_{i,j}\subset\Omega}\left(b_{i,j}((v-v^I)_x,\chi)_{\tau_{i,j}} + ((b-b_{i,j})(v-v^I)_x,\chi)_{\tau_{i,j}}\right)\\ &= \sum_{\tau_{i,j}\subset\Omega} b_{i,j}H_{i,j}(v,\chi) + ((b-\bar{b})(v-v^I)_x,\chi)\\ &\qquad + \frac{1}{12}\sum_{i=1}^{N-1}\sum_{j=1}^{N}(b_{i+1,j}h_{i+1}^2 - b_{i,j}h_i^2)\int_{y_{j-1}}^{y_j}(\chi v_{xx})(x_i,y)\,\mathrm{d}y\\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Use (3.7) to obtain

$$|I_1| \le C\sum_{\tau_{i,j}\subset\Omega}(k_j^2\|v_{xyy}\|_{0,\tau_{i,j}} + h_i^2\|v_{xxx}\|_{0,\tau_{i,j}})\|\chi\|_{0,\tau_{i,j}} \le C(h+k+N^{-1})^2\,|||\chi|||_\varepsilon$$

while a Taylor expansion gives

$$\|b-b_{i,j}\|_{L_\infty(\tau_{i,j})} \le C(h_i+k_j).$$

Thus, with (2.21) we have

$$|I_2| \le \|b-\bar{b}\|_{L_\infty(\Omega)}\|(v-v^I)_x\|_0\|\chi\|_0 \le C(h+k+N^{-1})^2\,|||\chi|||_\varepsilon\,.$$

When studying I_3 for $i \le N/2$, use

$$\int_{y_{j-1}}^{y_j}(\chi v_{xx})(x_i,y) = \sum_{k=1}^{i}\int_{\tau_{k,j}}(\chi_x v_{xx} + \chi v_{xxx})$$

that implies

$$\begin{aligned}&\sum_{i=1}^{N/2}(b_{i+1,j}h_{i+1}^2 - b_{i,j}h_i^2)\int_{y_{j-1}}^{y_j}(\chi v_{xx})(x_i,y)\,\mathrm{d}y =\\ &\qquad\qquad = \sum_{i=1}^{N/2}(b_{N/2+1,j}h_{N/2+1}^2 - b_{i,j}h_i^2)\int_{\tau_{i,j}}(\chi_x v_{xx} + \chi v_{xxx}).\end{aligned}$$

For $i > N/2$ apply an inverse inequality to obtain

$$\left|\int_{y_{j-1}}^{y_j}(\chi v_{xx})(x_i,y)\right| \le \|v_{xx}\|_{L_\infty(\tau_{i,j})}\int_{y_{j-1}}^{y_j}|\chi(x_i,y)| \le Ch_i^{-1}\|v_{xx}\|_{L_\infty(\tau_{i,j})}\|\chi\|_{L_1(\tau_{i,j})}.$$

Recall $|b_{i+1,j} - b_{i,j}| \le Ch_{i+1}$ and $h_i = h_{i+1} = \hbar \le CN^{-1}$ for $i > N/2$. We get

$$\begin{aligned}
|I_3| &\le C \sum_{j=1}^{N} \Bigg(\sum_{i=1}^{N/2} |b_{N/2+1,j} h_{N/2+1}^2 - b_{i,j} h_i^2| \int_{\tau_{i,j}} |(\chi_x v_{xx} + \chi v_{xxx})| \\
&\qquad + \sum_{i=N/2+1}^{N-1} |b_{i+1,j} h_{i+1}^2 - b_{i,j} h_i^2| h_i^{-1} \|v_{xx}\|_{L_\infty(\tau_{i,j})} \|\chi\|_{L_1(\tau_{i,j})} \Bigg) \\
&\le C((\hbar + h)^2 \|\chi_x v_{xx} + \chi v_{xxx}\|_{L_1(\Omega_{12}\cup\Omega_{22})} + \hbar^2 \|\chi\|_{L_1(\Omega_{11}\cup\Omega_{21})}) \\
&\le C(h + N^{-1})^2 \ln^{1/2} N \, |||\chi|||_\varepsilon \qquad (3.17)
\end{aligned}$$

due to $\operatorname{meas} \Omega_{12} \cup \Omega_{22} \le C\varepsilon \ln N$. Combining the estimates for the I's yields

$$|(b(v - v^I)_x, \chi)| \le C(k + (h + N^{-1}) \ln^{1/4} N)^2 \, |||\chi|||_\varepsilon \, . \qquad (3.18)$$

(iv) The last term can be bounded the same way as done in *(iii)*

$$\begin{aligned}
&(b(w_2 - w_2^I)_x, \chi)_{\Omega_{21}\cup\Omega_{22}} \\
&\quad = \sum_{\tau_{i,j}\subset\Omega_{21}\cup\Omega_{22}} b_{i,j} H_{i,j}(w_2, \chi) + ((b - \bar b)(w_2 - w_2^I)_x, \chi)_{\Omega_{21}\cup\Omega_{22}} \\
&\qquad + \frac{1}{12} \sum_{i=1}^{N-1} \left(\sum_{j=1}^{N/4} + \sum_{j=3N/4+1}^{N} \right) (b_{i+1,j} h_{i+1}^2 - b_{i,j} h_i^2) \int_{y_{j-1}}^{y_j} (\chi w_{2,xx})(x_i, y) \, \mathrm{d}y \\
&\quad =: J_1 + J_2 + J_3.
\end{aligned}$$

We have

$$\begin{aligned}
|J_1| &\le C \sum_{\tau_{i,j}\subset\Omega_{21}\cup\Omega_{22}} (k_j^2 \|w_{2,xyy}\|_{0,\tau_{i,j}} + h_i^2 \|w_{2,xxx}\|_{0,\tau_{i,j}}) \|\chi\|_{0,\tau_{i,j}} \\
&\le C\varepsilon^{1/4} (h + N^{-1} \max |\psi'|)^2 \, |||\chi|||_\varepsilon \, , \\
|J_2| &\le \|b - \bar b\|_{L_\infty(\Omega_{21}\cup\Omega_{22})} \|(w_2 - w_2^I)_x\|_{0,\Omega_{21}\cup\Omega_{22}} \|\chi\|_{0,\Omega_{21}\cup\Omega_{22}} \\
&\le C\varepsilon^{1/4} (h + k + N^{-1})(h + N^{-1} \max |\psi'|) \, |||\chi|||_\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
|J_3| &\le C((h + N^{-1})^2 \|\chi_x w_{2,xx} + \chi w_{2,xxx}\|_{L_1(\Omega_{22})} + N^{-2} \|\chi\|_{L_1(\Omega_{21})}) \\
&\le C(h + N^{-1})^2 \, |||\chi|||_\varepsilon \, .
\end{aligned}$$

Combining (3.15)-(3.18) with the estimates for the J's gives (3.8b). □

Remark 3.7. *Although in the decomposition of $u = v + w_1 + w_2 + w_{12}$ the term v is modest in view of growth, its analysis is complicated and not sharp yet. The logarithmic factor in the estimates due to* (3.17) *cannot be seen in the numerical results, see Chapter 5.*

3.2 Streamline Diffusion Finite Element Method

The streamline-diffusion FEM (SDFEM) was first proposed by Hughes and Brooks [14]. For problems of type (1.1) with *exponential* layers only, SDFEM on Shishkin meshes is well understood. Stynes and Tobiska [31] derived uniform supercloseness in the streamline-diffusion norm of second order up to a logarithmic factor.
Since we focus on problems with parabolic layers, particular attention will be paid to the choice of the streamline-diffusion parameter inside the parabolic layers where the mesh is aligned to the flow and anisotropically refined. It was observed [16, 20] that when the stabilization parameter is chosen according to standard recommendations [8, p. 132] proportional to the streamline diameter of the mesh cell, the accuracy is adversely affected. An alternative—and significantly smaller—choice based on residual free bubbles is advertised in [20]. However the argument is not rigorous and uses heuristics.
The SDFEM can be introduced in two different ways. In the sense of a Petrov-Galerkin method we multiply (1.1) for the weak formulation with test functions $v + \sum_\tau \delta_\tau b v_x$ and $v \in V$. Therefore, the method is also known as Streamline Upwind Petrov-Galerkin method—SUPG.
On the other hand it can be regarded as adding weighted residuals to the standard GFEM in order to stabilise the discretisation.

$$a_{Gal}(u, v) + \sum_{\tau \in T^N} \delta_\tau \left(f - Lu, bv_x\right)_\tau = f(v), \tag{3.19}$$

with user chosen parameter $\delta_\tau \geq 0$ for all $\tau \subset \Omega$.
This modification is consistent with (1.1), i.e., its solution solves (3.19) too.
Our discretization reads: Find $u^N \in V^N$ such that

$$a_{SD}(u^N, v^N) := a_{Gal}(u^N, v^N) + a_{SDstab}(u^N, v^N) = f_{SD}(v^N) \quad \text{for all } v^N \in V^N,$$

with

$$a_{SDstab}(u, v) := \sum_{\tau \in T^N} \delta_\tau \big(\varepsilon(\Delta u, bv_x)_\tau + (bu_x - cu, bv_x)_\tau\big)$$

and

$$f_{SD}(v) := f(v) - \sum_{\tau \in T^N} \delta_\tau (f, bv_x)_\tau.$$

The SDFEM satisfies the orthogonality condition

$$a_{SD}(u - u^N, v^N) = 0 \quad \text{for all } v \in V^N$$

and is coercive with respect to the streamline diffusion norm

$$|||v|||_{SD}^2 := |||v|||_\varepsilon^2 + \sum_{\tau \in T^N} \delta_\tau (bv_x, bv_x)_\tau.$$

The proof of coercivity is shown, e.g., in [27, §III 3.2.1]:

$$\begin{aligned} a_{SD}(v^N, v^N) &= \varepsilon\|\nabla v^N\|_0^2 + \left(\left(c + \frac{1}{2}b_x\right)v^N, v^N\right) + \sum_{\tau\subset\Omega}\delta_\tau(bv_x^N - cv^N, bv_x^N)_\tau \\ &\geq \left|\left|\left|v^N\right|\right|\right|_{SD}^2 - \sum_{\tau\subset\Omega}\delta_\tau(cv^N, bv_x^N)_\tau, \end{aligned}$$

$$\begin{aligned} \left|\sum_{\tau\subset\Omega}\delta_\tau(cv^N, bv_x^N)_\tau\right| &\leq \sum_{\tau\subset\Omega}\delta_\tau\|cv^N\|_{0,\tau}\|bv_x^N\|_{0,\tau} \\ &\leq \frac{1}{2}\sum_{\tau\subset\Omega}\delta_\tau(\|c\|_{L_\infty(\tau)}^2\|v^N\|_{0,\tau}^2 + \|bv_x^N\|_{0,\tau}^2). \end{aligned}$$

If

$$0 \leq \delta_\tau \quad \text{and} \quad \delta_\tau\|c\|_{L_\infty(\tau)}^2 \leq \gamma \quad \text{for all } \tau \in T^N \tag{3.20}$$

then

$$a_{SD}(v^N, v^N) \geq \frac{1}{2}\left|\left|\left|v^N\right|\right|\right|_{SD}^2 \quad \forall v \in V^N.$$

Remark that $|||v|||_\varepsilon \leq |||v|||_{SD}$ for all $v \in H_0^1(\Omega)$. Thus $a_{SD}(\cdot,\cdot)$ enjoys a stronger stability than $a_{Gal}(\cdot,\cdot)$. Roughly speaking, the larger δ the more stability is introduced into the method.

3.2.1 Supercloseness and convergence

A supercloseness analysis for SDFEM and singularly perturbed problems with characteristic layers was first done in [10]. Therein we investigated the SDFEM on Shishkin meshes. Here we generalise the results to S-type meshes.
Our error analysis starts from coercivity and Galerkin orthogonality:

$$\frac{1}{2}\left|\left|\left|u^I - u^N\right|\right|\right|_{SD}^2 \leq a_{Gal}\left(u^I - u, u^I - u^N\right) + a_{SDstab}\left(u^I - u, u^I - u^N\right). \tag{3.21}$$

For the first term on the right-hand side applies

$$\left|a_{Gal}\left(u^I - u, \chi\right)\right| \leq C((h + N^{-1})\ln^{1/4} N + k + N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon \quad \text{for all } \chi \in V^N \tag{3.22}$$

by Lemma 3.4 if $\sigma \geq 5/2$.
The streamline-diffusion parameter δ is chosen to be constant on each subdomain of the decomposition of Ω, i.e.,

$$\delta\Big|_\tau = \delta_\tau = \delta_{k\ell} \quad \text{if } \tau \subset \Omega_{k\ell}, \quad k, \ell = 1, 2.$$

The stabilisation will now be analysed in each region separately. The proof of the following lemma can be found in the next subsection.

Lemma 3.8. *Let $\sigma \geq 5/2$, $\chi = u^I - u^N$ and u be the solution to* (1.1)*. Then we have*

$$|a_{SD}(u-u^I,\chi)_{\Omega_{11}}| \leq C(\delta_{11}^{1/2}N^{-\sigma+1} + \varepsilon\delta_{11}\ln^{1/2}N)\,|||\chi|||_{SD}\,, \tag{3.23a}$$

$$|a_{SD}(u-u^I,\chi)_{\Omega_{12}}| \leq C(\delta_{12}\varepsilon^{-1} + \delta_{12}^{1/2}(N^{-\sigma+1} + \varepsilon^{1/2}\ln^{1/2}N(h+N^{-1}\max|\psi'|)^2))\,|||\chi|||_{SD}\,, \tag{3.23b}$$

$$|a_{SD}(u-u^I,\chi)_{\Omega_{21}}| \leq C\varepsilon^{1/4}\ln^{1/2}N(\delta_{21} + \delta_{21}^{1/2}(N^{-\sigma+1} + (k+N^{-1}\max|\psi'|)^2))\,|||\chi|||_{SD} \tag{3.23c}$$

and

$$|a_{SD}(u-u^I,\chi)_{\Omega_{22}}| \leq C\ln^{1/2}N(\delta_{22}\varepsilon^{-3/4} + \delta_{22}^{1/2}\varepsilon^{-1/4}(h+k+N^{-1}\max|\psi'|)^2)\,|||\chi|||_{SD}\,. \tag{3.23d}$$

The main theorem of this section follows immediately from Lemma 3.8, (3.21) and (3.22).

Theorem 3.9 (Supercloseness SDFEM)**.** *Let u^N be the streamline-diffusion approximation to u on an S-type mesh with $\sigma \geq 5/2$. Suppose the stabilisation parameter δ satisfies* (3.20)*,*

$$\delta_{12} \leq C^*\varepsilon(h+N^{-1}\max|\psi'|)^2, \quad \delta_{21} \leq C^*\varepsilon^{-1/4}N^{-2}, \quad \delta_{22} \leq C^*\varepsilon^{3/4}N^{-2}$$

and

$$\delta_{11} \leq \begin{cases} C^*N^{-1} & \text{if } \varepsilon \leq N^{-1}, \\ C^*\varepsilon^{-1}N^{-2} & \text{if } \varepsilon \geq N^{-1} \end{cases}$$

with some positive constant C^* independent of ε and the mesh. Then

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{SD} \leq C((h+N^{-1})\ln^{1/4}N + k + N^{-1}\max|\psi'|)^2.$$

Moreover we have the same convergence result as for GFEM.

Remark 3.10 (Convergence SDFEM)**.** *Theorems 2.6 and 3.9, and the triangle inequality provide bounds for the error in the ε-weighted energy norm:*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon} \leq C(\varepsilon^{1/4}h + k + N^{-1}\max|\psi'|).$$

3.2.2 Proofs

Note that because of $0 < \beta \leq b \leq \|b\|_{L_\infty(\Omega)}$ the two semi norms $\|\chi_x\|_{L_p(D)}$ and $\|b\chi_x\|_{L_p(D)}$ are equivalent with constants independent of ε and N. This fact will be used repeatedly in our analysis without special reference.

Proof of Lemma 3.8.
The terms to be estimated are

$$a_{SDstab}(u-u^I,\chi)_{\Omega^*} = \sum_{\tau\subset\Omega^*} \delta_\tau(\varepsilon(\Delta u, b\chi_x)_\tau + (b(u-u^I)_x, b\chi_x)_\tau + (c(u-u^I), b\chi_x)_\tau) \tag{3.24}$$

for Ω^* being one of the four subregions of Ω. Due to δ_τ being constant inside each subregion, all terms in (3.24) can be written as scalar products over Ω^*.
Start with the third term and apply the L_2-interpolation error estimates (2.15) to obtain

$$\delta_{11}|(c(u-u^I), b\chi_x)_{\Omega_{11}}| \le C\delta_{11}\|u-u^I\|_{0,\Omega_{11}}\|b\chi_x\|_{0,\Omega_{11}} \le C\delta_{11}^{1/2}N^{-2}\,|||\chi|||_{SD}\,, \tag{3.25a}$$
$$\delta_{12}|(c(u-u^I), b\chi_x)_{\Omega_{12}}| \le C\delta_{12}^{1/2}\varepsilon^{1/2}((h+N^{-1})\ln^{1/4}N + N^{-1}\max|\psi'|)^2\,|||\chi|||_{SD}\,, \tag{3.25b}$$
$$\delta_{21}|(c(u-u^I), b\chi_x)_{\Omega_{21}}| \le C\delta_{21}^{1/2}\varepsilon^{1/4}\ln^{1/2}N(k + N^{-1}\max|\psi'|)^2\,|||\chi|||_{SD} \tag{3.25c}$$

and

$$\delta_{22}|(c(u-u^I), b\chi_x)_{\Omega_{22}}| \le C\delta_{22}^{1/2}\varepsilon^{1/2}\ln^{1/2}N(h+k+N^{-1}\max|\psi'|)^2\,|||\chi|||_{SD}\,. \tag{3.25d}$$

The other two terms of (3.24) will be estimated on each subregion separately.
(1) On $\Omega^*=\Omega_{11}$ we set $w = w_1 + w_2 + w_{12}$ using the decomposition of Assumption 2.1. A Hölder and an inverse inequality yield for the first term of (3.24)

$$\begin{aligned}\delta_{11}\varepsilon|(\Delta w, b\chi_x)_{\Omega_{11}}| &\le C\varepsilon\delta_{11}\|\Delta w\|_{L_1(\Omega_{11})}\|b\chi_x\|_{L_\infty(\Omega_{11})}\\ &\le C\delta_{11}N^{-\sigma}\|\chi_x\|_{L_\infty(\Omega_{11})} \le C\delta_{11}N^{-\sigma+1}\|\chi_x\|_{0,\Omega_{11}}\\ &\le C\delta_{11}N^{-\sigma+1}\|b\chi_x\|_{0,\Omega_{11}} \le C\delta_{11}^{1/2}N^{-\sigma+1}\,|||\chi|||_{SD}\,.\end{aligned} \tag{3.26}$$

The second term is estimated using inverse estimates

$$\begin{aligned}\delta_{11}|(b(w-w^I)_x, b\chi_x)_{\Omega_{11}}| &\le C\delta_{11}(\|w_x\|_{L_1(\Omega_{11})}\|\chi_x\|_{L_\infty(\Omega_{11})} + \|w_x^I\|_{0,\Omega_{11}}\|b\chi_x\|_{0,\Omega_{11}})\\ &\le C\delta_{11}(\|w_x\|_{L_1(\Omega_{11})}N\|b\chi_x\|_{0,\Omega_{11}} + N\|w^I\|_{L_\infty(\Omega_{11})}\|b\chi_x\|_{0,\Omega_{11}})\\ &\le C\delta_{11}^{1/2}N^{-\sigma+1}\,|||\chi|||_{SD}\,.\end{aligned} \tag{3.27}$$

Now only the non-layer part v is left. Although its derivatives can be bounded independent of ε, the terms containing v must be treated with care. Therefore, use

$$(\Delta v, b\chi_x)_{\Omega_{11}} + (\Delta v, b\chi_x)_{\Omega_{12}} = -((b\Delta v)_x, \chi)_{\Omega_{11}\cup\Omega_{12}}$$

to get for the first term

$$\begin{aligned}\delta_{11}\varepsilon|(\Delta v, b\chi_x)_{\Omega_{11}}| &\le C\varepsilon\delta_{11}(\|\chi\|_{0,\Omega_{11}\cup\Omega_{12}} + \|\chi_x\|_{L_1(\Omega_{12})})\\ &\le C\varepsilon\delta_{11}(|||\chi|||_\varepsilon + \operatorname{meas}^{1/2}\Omega_{12}\|\chi_x\|_{0,\Omega_{12}}) \le C\varepsilon\delta_{11}\ln^{1/2}N\,|||\chi|||_\varepsilon\,.\end{aligned} \tag{3.28}$$

The Lin-identities (3.6) applied to

$$(b(v-v^I), b\chi_x)_{\tau_{i,j}} = ((b^2-b_\tau^2)(v-v^I)_x, \chi_x)_{\tau_{i,j}} + b_\tau^2((v-v^I)_x, \chi_x)_{\tau_{i,j}} \tag{3.29}$$

with b_τ being a constant approximation to b on τ and a discrete Cauchy-Schwarz inequality for the second term of (3.24) give

$$\begin{aligned}\delta_{11}|(b(v-v^I)_x, b\chi_x)_{\Omega_{11}}| &\le C\delta_{11}\sum_{\tau\subset\Omega_{11}} |(((b^2-b_\tau^2)(v-v^I)_x,\chi_x)_\tau| + b_\tau^2|((v-v^I)_x,\chi_x)_\tau)| \\ &\le C\delta_{11}\sum_{\tau\subset\Omega_{11}} ((h_\tau+k_\tau)\|(v-v^I)_x\|_{0,\tau} + k_\tau^2\|v_{xyy}\|_{0,\tau})\|\chi_x\|_{0,\tau} \\ &\le C\delta_{11}N^{-2}\|b\chi_x\|_{0,\Omega_{11}} \le C\delta_{11}^{1/2}N^{-2}\,|||\chi|||_{SD}\,. \end{aligned} \tag{3.30}$$

Combining (3.26)– (3.28) and (3.30) with (3.25a) proves (3.23a).
(2) Next consider the exponential layer region $\Omega^* = \Omega_{12}$. Let $w = w_2+w_{12}$ and $\tilde{w} = v+w_1$. The first term of (3.24) is bounded similarly to ***(1)*** by

$$\begin{aligned}\delta_{12}\varepsilon|(\Delta w, b\chi_x)_{\Omega_{12}}| &\le C\varepsilon\delta_{12}\|\Delta w\|_{L_1(\Omega_{12})}\|b\chi_x\|_{L_\infty(\Omega_{12})} \\ &\le C\delta_{12}^{1/2}N^{-\sigma+1}\ln^{-1/2}N\,|||\chi|||_{SD} \end{aligned} \tag{3.31a}$$

and

$$\begin{aligned}\delta_{12}\varepsilon|(\Delta\tilde{w}, b\chi_x)_{\Omega_{12}}| &\le C\varepsilon\delta_{12}\|\Delta\tilde{w}\|_{0,\Omega_{12}}\|b\chi_x\|_{0,\Omega_{12}} \\ &\le C\delta_{12}\varepsilon^{-1/2}\|b\chi_x\|_{0,\Omega_{12}} \le C\delta_{12}\varepsilon^{-1}\,|||\chi|||_{\varepsilon}\,. \end{aligned} \tag{3.31b}$$

With (2.17a) we get

$$\begin{aligned}\delta_{12}|(b(u-u^I)_x, b\chi_x)_{\Omega_{12}}| &\le C\delta_{12}\|(u-u^I)_x\|_{0,\Omega_{12}}\|\chi_x\|_{0,\Omega_{12}} \\ &\le C\delta_{12}\varepsilon^{-1}(h+N^{-1}\max|\psi'|)\,|||\chi|||_{\varepsilon}\,. \end{aligned} \tag{3.32}$$

(3.31), (3.32) together with (3.25b) yield (3.23b).
(3) On $\Omega^* = \Omega_{21}$ set $w = w_1+w_{12}$ and $\tilde{w} = v+w_2$. The first term of (3.24) is treated similar to (3.26) and (3.28).

$$\begin{aligned}\delta_{21}\varepsilon|(\Delta w, b\chi_x)_{\Omega_{21}}| &\le C\delta_{21}\varepsilon\|\Delta w\|_{L_1(\Omega_{21})}\|b\chi_x\|_{L_\infty(\Omega_{21})} \\ &\le C\delta_{21}^{1/2}\varepsilon^{1/4}\ln^{1/2}N N^{-\sigma+1}\,|||\chi|||_{SD} \end{aligned} \tag{3.33a}$$

$$\begin{aligned}\delta_{21}\varepsilon|(\Delta\tilde{w}, b\chi_x)_{\Omega_{21}}| &\le C\delta_{21}\varepsilon(\|(b\Delta\tilde{w})_x\|_{0,\Omega_{21}\cup\Omega_{22}}\|\chi\|_{0,\Omega_{21}\cup\Omega_{22}} + \|\Delta\tilde{w}\|_{0,\Omega_{22}}\|\chi_x\|_{0,\Omega_{22}}) \\ &\le C\delta_{21}\varepsilon^{1/4}\ln^{1/2}N\,|||\chi|||_{\varepsilon} \end{aligned} \tag{3.33b}$$

With

$$\begin{aligned}\|w_x^I\|_{0,\Omega_{21}} \le CN\|w^I\|_{0,\Omega_{21}} &\le C\varepsilon^{1/4}\ln^{1/2}N N\|w^I\|_{L_\infty(\Omega_{21})} \\ &\le C\varepsilon^{1/4}\ln^{1/2}N N\|w\|_{L_\infty(\Omega_{21})} \\ &\le C\varepsilon^{1/4}\ln^{1/2}N N^{-\sigma+1} \end{aligned}$$

and

$$\|\chi_x\|_{L_\infty(\Omega_{21})} \le CN\varepsilon^{-1/4}\ln^{-1/2}N\|\chi_x\|_{0,\Omega_{21}}$$

we have

$$\begin{aligned}\delta_{21}|(b(w-w^I)_x, b\chi_x)_{\Omega_{21}}| &\leq C\delta_{21}(\|w_x\|_{L_1(\Omega_{21})}\|\chi_x\|_{L_\infty(\Omega_{21})} + \|w_x^I\|_{0,\Omega_{21}}\|b\chi_x\|_{0,\Omega_{21}}) \\ &\leq C\delta_{21}(\varepsilon^{1/2}\ln N N^{-\sigma}\|\chi_x\|_{L_\infty(\Omega_{21})} + \varepsilon^{1/4}\ln^{1/2} N N^{-\sigma+1}\|b\chi_x\|_{0,\Omega_{21}}) \\ &\leq C\delta_{21}^{1/2}\varepsilon^{1/4}\ln^{1/2} N N^{-\sigma+1}\,|||\chi|||_{SD}\,. \end{aligned} \tag{3.34a}$$

For v and w_2 use (3.29) to obtain

$$\begin{aligned}\delta_{21}|(b(v-v^I)_x, b\chi_x)_{\Omega_{21}}| &\leq C\delta_{21}\Big((N^{-1}+k)(N^{-1}\|v_{xx}\|_{0,\Omega_{21}} + k\|v_{xy}\|_{0,\Omega_{21}}) + \\ &\qquad k^2\|v_{xyy}\|_{0,\Omega_{21}}\Big)\|b\chi_x\|_{0,\Omega_{21}} \\ &\leq C\delta_{21}^{1/2}\varepsilon^{1/4}\ln^{1/2} N(N^{-1}+k)^2\,|||\chi|||_{SD} \end{aligned} \tag{3.34b}$$

and

$$\begin{aligned}\delta_{21}|(b(w_2-w_2^I)_x, b\chi_x)_{\Omega_{21}}| &\leq C\delta_{21}\sum_{\tau\subset\Omega_{21}}\Big((N^{-1}+k)(N^{-1}\|w_{2,xx}\|_{0,\tau} + k_\tau\|w_{2,xy}\|_{0,\tau}) + \\ &\qquad k_\tau^2\|w_{2,xyy}\|_{0,\tau}\Big)\|b\chi_x\|_{0,\tau} \\ &\leq C\delta_{21}\Big((N^{-1}+k)(N^{-1}\|w_{2,xx}\|_{0,\Omega_{21}} + \\ &\qquad \varepsilon^{1/2}N^{-1}\max|\psi'|\|lt(y)\,w_{2,xy}\|_{0,\Omega_{21}}) + \\ &\qquad \varepsilon(N^{-1}\max|\psi'|)^2\|lt(y)^2\,w_{2,xyy}\|_{0,\Omega_{21}}\Big)\|b\chi_x\|_{0,\Omega_{21}} \\ &\leq C\delta_{21}^{1/2}\varepsilon^{1/4}(k+N^{-1}\max|\psi'|)N^{-1}\max|\psi'|\,|||\chi|||_{SD} \end{aligned} \tag{3.34c}$$

Combining (3.33), (3.34) and (3.25c) gives (3.23c).
(4) Finally, set $\Omega^* = \Omega_{22}$. For the first term in (3.24) holds immediately

$$\delta_{22}\varepsilon|(\Delta u, b\chi_x)_{\Omega_{22}}| \leq C\delta_{12}\varepsilon\|\Delta u\|_{0,\Omega_{22}}\|\chi_x\|_{0,\Omega_{22}} \leq C\delta_{22}\varepsilon^{-3/4}\ln^{1/2} N\,|||\chi|||_{\varepsilon}\,. \tag{3.35}$$

The second term is bounded by applying the Lin identities and (3.29) to all parts of the decomposition of u.

$$\delta_{22}|(bv_x, b\chi_x)_{\Omega_{22}}| \leq C\delta_{22}\,\mathrm{meas}^{1/2}\,\Omega_{22}(h+k)^2\|b\chi_x\|_{0,\Omega_{22}} \leq C\delta_{22}^{1/2}\varepsilon^{3/4}\ln N(h+k)^2\,|||\chi|||_{SD} \tag{3.36a}$$

For $w = w_1 + w_2 + w_{12}$ holds

$$\begin{aligned}h_i\|w_{xx}\|_{0,\tau_{i,j}} &\leq C(\varepsilon N^{-1}\max|\psi'|\|e^{\beta x/(\sigma\varepsilon)}(w_1+w_{12})_{xx}\|_{0,\tau_{i,j}} + h\|w_{2,xx}\|_{0,\tau_{i,j}}), \\ k_j\|w_{xy}\|_{0,\tau_{i,j}} &\leq C(\varepsilon^{1/2}N^{-1}\max|\psi'|\|lt(y)\,(w_2+w_{12})_{xy}\|_{0,\tau_{i,j}} + k\|w_{1,xy}\|_{0,\tau_{i,j}})\end{aligned}$$

and

$$k_j^2\|w_{xyy}\|_{0,\tau_{i,j}} \leq C(\varepsilon(N^{-1}\max|\psi'|)^2\|lt(y)^2\,(w_2+w_{12})_{xxy}\|_{0,\tau_{i,j}} + k^2\|w_{1,xyy}\|_{0,\tau_{i,j}}).$$

Thus we get

$$\begin{aligned}
\delta_{22}|(bv_x, b\chi_x)_{\Omega_{22}}| &\leq C\delta_{22} \sum_{\tau\subset\Omega_{22}} \Big((h+k)(h_\tau\|w_{xx}\|_{0,\tau} + k_\tau\|w_{xy}\|_{0,\tau}) + k_\tau^2\|w_{xyy}\|_{0,\tau}\Big)\|b\chi_x\|_{0,\tau} \\
&\leq C\delta_{22}\varepsilon^{-1/4}\Big(\ln^{1/2} N(h+k)(k+\varepsilon h + N^{-1}\max|\psi'|)+ \\
&\qquad (N^{-1}\max|\psi'|)^2 + k^2\ln^{1/2} N\Big)\|b\chi_x\|_{0,\Omega_{22}} \\
&\leq C\delta_{22}^{1/2}\varepsilon^{-1/4}\ln^{1/2} N(h+k+N^{-1}\max|\psi'|)^2\, |||\chi|||_{SD}\,. \qquad (3.36b)
\end{aligned}$$

(3.35), (3.36) and (3.25d) prove (3.23d). □

3.3 Galerkin Least-Squares Finite Element Method

Another possibility to use residual-based stabilisation is the so called Galerkin least-squares FEM (GLSFEM). Like the SDFEM, this method can be seen either as adding a stabilisation term to the Galerkin bilinear form or using a Petrov-Galerkin method with test functions $v + \sum_\tau \delta_\tau Lv$ and $v \in V$. We get

$$a_{Gal}(u,v) + \sum_{\tau\in T^N} \delta_\tau\,(Lu - f, Lv)_\tau = f(v), \qquad (3.37)$$

where $\delta_\tau \geq 0$ is a user chosen parameter for each $\tau \in T^N$.
This modification is consistent with (1.1), i.e., its solution solves (3.37) too.
Our discretization reads: Find $u^N \in V^N$ such that

$$a_{GLS}(u^N, v^N) := a_{Gal}(u^N, v^N) + a_{GLSstab}(u^N, v^N) = f_{GLS}(v^N) \quad \text{for all } v^N \in V^N,$$

with

$$a_{GLSstab}(u,v) := \sum_{\tau\in T^N} \delta_\tau(-\varepsilon\Delta u - bu_x + cu, -\varepsilon\Delta v - bv_x + cv)_\tau$$

and

$$f_{GLS}(u,v) := f(v) + \sum_{\tau\in T^N} \delta_\tau(f, -\varepsilon\Delta v - bv_x + cv)_\tau.$$

The GLSFEM satisfies the orthogonality condition

$$a_{GLS}(u - u^N, v^N) = 0 \quad \text{for all } v \in V^N.$$

By

$$|||v|||^2_{GLS,1} := |||v|||^2_\varepsilon + \sum_{\tau\in T^N} \delta_\tau(Lv, Lv)_\tau$$

we define a norm and have coercivity without restrictions on the parameter δ_τ:

$$a_{GLS}(v^N, v^N) \geq \left|\left|\left|v^N\right|\right|\right|_{GLS,1}^2 .$$

Unfortunately, the streamline and L_2-control are mixed in the above norm. Moreover, since the SDFEM stabilisation is part of the GLS stabilisation terms it would be beneficial to reuse the results of the last section. Therefore, define a second norm by

$$|||v|||_{GLS,2}^2 := |||v|||_\varepsilon^2 + \sum_{\tau \in T^N} \delta_\tau \Big((-\varepsilon \Delta v + cv, -\varepsilon \Delta v + cv)_\tau + (bv_x, bv_x)_\tau \Big).$$

Obviously, we have $|||v|||_{GLS,1} \leq 2\,|||v|||_{GLS,2}$ and $|||v|||_{SD} \leq |||v|||_{GLS,2}$.
With a restriction on δ_τ similar to the one in the previous section, we have coercivity

$$\begin{aligned} a_{GLS}(v^N, v^N) &\geq \left|\left|\left|v^N\right|\right|\right|_\varepsilon^2 + \sum_{\tau \in T^N} \delta_\tau \Big(\|bv_x^N\|_{0,\tau}^2 + \|cv^N\|_{0,\tau}^2 - 2(cv^N, bv_x^N)_\tau \Big) \\ &= \left|\left|\left|v^N\right|\right|\right|_{GLS,2} - 2 \sum_{\tau \in T^N} \delta_\tau (cv^N, bv_x^N)_\tau \end{aligned}$$

$$\begin{aligned} 2\left| \sum_{\tau \subset \Omega} \delta_\tau (cv^N, bv_x^N)_\tau \right| &\leq 2 \sum_{\tau \subset \Omega} \delta_\tau \|cv^N\|_{0,\tau} \|bv_x^N\|_{0,\tau} \\ &\leq \frac{1}{2} \sum_{\tau \subset \Omega} \delta_\tau (4\|c\|_{L_\infty(\tau)}^2 \|v^N\|_{0,\tau}^2 + \|bv_x^N\|_{0,\tau}^2). \end{aligned}$$

If

$$0 \leq \delta_\tau \quad \text{and} \quad 4\delta_\tau \|c\|_{L_\infty(\tau)}^2 \leq \gamma \quad \text{for all } \tau \in T^N \tag{3.38}$$

then

$$a_{GLS}(v^N, v^N) \geq \frac{1}{2} \left|\left|\left|v^N\right|\right|\right|_{GLS,2}^2 \quad \forall v^N \in V^N.$$

Another consequence of restriction (3.38) is the equivalence of the streamline and GLS-norm in its second version for $v^N \in V^N$:

$$\left|\left|\left|v^N\right|\right|\right|_{SD}^2 \leq \left|\left|\left|v^N\right|\right|\right|_{GLS,2}^2 \leq 5/4 \left|\left|\left|v^N\right|\right|\right|_{SD}^2 .$$

3.3.1 Supercloseness and convergence

Again our error analysis starts from coercivity, Galerkin orthogonality and the splitting of the stabilisation terms into streamline stabilisation and the remaining terms:

$$\begin{aligned} \frac{1}{2} \left|\left|\left|u^I - u^N\right|\right|\right|_{GLS,2}^2 &\leq a_{GLS}\left(u^I - u, u^I - u^N\right) \\ &= a_{SD}\left(u^I - u, u^I - u^N\right) + \\ &\qquad \sum_{\tau \in T^N} \delta_\tau \Big(-\varepsilon \Delta(u^I - u) - b(u^I - u)_x + c(u^I - u), c(u^I - u^N) \Big)_\tau . \end{aligned} \tag{3.39}$$

The first term can be estimated by means of Lemma 3.8 while the remaining terms are bounded by

Lemma 3.11. *Let $\sigma \geq 5/2$, $\chi = u^I - u^N$ and u be the solution to* (1.1). *Then we have*

$$|(a_{GLS} - a_{SD})(u - u^I, \chi)_{\Omega_{11}}| \leq C\delta_{11}(\varepsilon + N^{-1}) \, |||\chi|||_{\varepsilon} \,, \tag{3.40a}$$

$$|(a_{GLS} - a_{SD})(u - u^I, \chi)_{\Omega_{12}}| \leq C\delta_{12}(\varepsilon^{-1/2} + N^{-\sigma+1}) \, |||\chi|||_{\varepsilon} \,, \tag{3.40b}$$

$$|(a_{GLS} - a_{SD})(u - u^I, \chi)_{\Omega_{21}}| \leq C\delta_{21}\varepsilon^{1/4}(\varepsilon^{1/2} + N^{-1} \max |\psi'| + \ln^{1/2} N(k + N^{-1})) \, |||\chi|||_{\varepsilon} \tag{3.40c}$$

and

$$|(a_{GLS} - a_{SD})(u - u^I, \chi)_{\Omega_{22}}| \leq C\delta_{22}(\varepsilon^{-1/4} \ln^{1/2} N + \varepsilon^{1/4}(h + k)) \, |||\chi|||_{\varepsilon} \,. \tag{3.40d}$$

The proof of Lemma 3.11 can be found in the following subsection.

Theorem 3.12 (Supercloseness GLSFEM)**.** *Let u^N be the GLSFEM approximation to u on an S-type mesh with $\sigma \geq 5/2$. Suppose the stabilisation parameter δ satisfies* (3.38), *and the assumptions of Theorem 3.9, namely*

$$\delta_{12} \leq C^*\varepsilon(h + N^{-1} \max |\psi'|)^2, \quad \delta_{21} \leq C^*\varepsilon^{-1/4}N^{-2}, \quad \delta_{22} \leq C^*\varepsilon^{3/4}N^{-2}$$

and

$$\delta_{11} \leq \begin{cases} C^*N^{-1} & \text{if } \varepsilon \leq N^{-1}, \\ C^*\varepsilon^{-1}N^{-2} & \text{if } \varepsilon \geq N^{-1} \end{cases}$$

with some positive constant C^* independent of ε and N. Then

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{GLS,1} \leq C \left|\left|\left|u^I - u^N\right|\right|\right|_{GLS,2} \leq C((h + N^{-1}) \ln^{1/4} N + k + N^{-1} \max |\psi'|)^2.$$

Proof. The first inequality is obvious by definition and the second follows from Lemmas 3.8 and 3.11. □

Moreover, we have the same convergence result as for GFEM and SDFEM.

Remark 3.13 (Convergence GLSFEM)**.** *Theorems 2.6 and 3.12, and the triangle inequality provide bounds for the error in the ε-weighted energy norm:*

$$\left|\left|\left|u - u^N\right|\right|\right|_{\varepsilon} \leq C(\varepsilon^{1/4}h + k + N^{-1} \max |\psi'|).$$

3.3.2 Proofs

Proof of Lemma 3.11.
We have to bound

$$\sum_{\tau\subset\Omega}\delta_\tau\varepsilon(\Delta u, c\chi)_\tau + \sum_{\tau\subset\Omega}\delta_\tau(b(u-u^I)_x, c\chi)_\tau - \sum_{\tau\subset\Omega}\delta_\tau(c(u-u^I), c\chi)_\tau. \tag{3.41}$$

Recall δ to be constant on each region and start treating the first term in two different ways on each subregion. For those terms of the decomposition of u that are decayed, we use a Hölder inequality to obtain

$$\begin{aligned}\delta_{11}\varepsilon|(\Delta(w_1+w_2+w_{12}), c\chi)_{\Omega_{11}}| &\le C\delta_{11}\varepsilon\|\Delta(w_1+w_2+w_{12})\|_{L_1(\Omega_{11})}\|c\chi\|_{L_\infty(\Omega_{11})}\\ &\le C\delta_{11}N^{-\sigma}\|\chi\|_{L_\infty(\Omega_{11})} \le C\delta_{11}N^{-\sigma+1}|||\chi|||_\varepsilon, \qquad (3.42a)\\ \delta_{12}\varepsilon|(\Delta(w_2+w_{12}), c\chi)_{\Omega_{12}}| &\le C\delta_{12}N^{-\sigma+1}\ln^{-1/2}N\,|||\chi|||_\varepsilon \qquad (3.42b)\end{aligned}$$

and

$$\delta_{21}\varepsilon|(\Delta(w_1+w_{12}), c\chi)_{\Omega_{21}}| \le C\delta_{21}N^{-\sigma+1}\varepsilon^{1/4}\ln^{1/2}N\,|||\chi|||_\varepsilon. \tag{3.42c}$$

A Cauchy-Schwarz inequality holds for the remaining parts

$$\begin{aligned}\delta_{11}\varepsilon\,|(\Delta v, c\chi)_{\Omega_{11}}| &\le C\varepsilon\delta_{11}\|\Delta v\|_{0,\Omega_{11}}\|\chi\|_{0,\Omega_{11}} \le C\varepsilon\delta_{11}\,|||\chi|||_\varepsilon, \qquad (3.42d)\\ \delta_{12}\varepsilon|(\Delta(v+w_1), c\chi)_{\Omega_{12}}| &\le C\delta_{12}\varepsilon^{-1/2}\,|||\chi|||_\varepsilon, \qquad (3.42e)\\ \delta_{21}\varepsilon|(\Delta(v+w_2), c\chi)_{\Omega_{21}}| &\le C\delta_{21}\varepsilon^{3/4}\,|||\chi|||_\varepsilon \qquad (3.42f)\end{aligned}$$

and

$$\delta_{22}\varepsilon|(\Delta u, c\chi)_{\Omega_{22}}| \le C\delta_{22}\varepsilon^{-1/4}\ln^{1/2}N\,|||\chi|||_\varepsilon. \tag{3.42g}$$

We proceed by bounding the second term of (3.41) separately on the four different subregions of Ω with different methods and start with Ω_{11}. For $w = w_1 + w_2 + w_{12}$ proceed as in (3.27) to get

$$\begin{aligned}\delta_{11}\left|(b(w-w^I)_x, c\chi)_{\Omega_{11}}\right| &\le C\delta_{11}\|b(w-w^I)_x\|_{L_1(\Omega_{11})}\|c\chi\|_{L_\infty(\Omega_{11})}\\ &\le C\delta_{11}N^{-\sigma+1}\,|||\chi|||_\varepsilon \qquad (3.43a)\end{aligned}$$

while for v the interpolation error result (2.21a) yields

$$\delta_{11}\left|(b(v-v^I)_x, c\chi)_{\Omega_{11}}\right| \le C\delta_{11}\|b(v-v^I)_x\|_{0,\Omega_{11}}\|c\chi\|_{0,\Omega_{11}} \le C\delta_{11}N^{-1}\,|||\chi|||_\varepsilon. \tag{3.43b}$$

On Ω_{21} and $w = w_1 + w_{12}$ estimate as done for (3.34) obtaining

$$\begin{aligned}\delta_{21}|(b(w-w^I)_x, c\chi)_{\Omega_{21}}| &\le C\delta_{21}(\|w_x\|_{L_1(\Omega_{21})}\|\chi\|_{L_\infty(\Omega_{21})} + \|w_x^I\|_{0,\Omega_{21}}\|\chi\|_{0,\Omega_{21}})\\ &\le C\delta_{21}\varepsilon^{1/4}\ln^{1/2}N N^{-\sigma+1}\,|||\chi|||_\varepsilon, \qquad (3.44a)\end{aligned}$$

while for v and w_2 we use the error estimates (2.21c) and (2.23c), respectively to get

$$\begin{aligned}\delta_{21}|(b(v-v^I)_x, c\chi)_{\Omega_{21}}| &\le C\delta_{21}\|(v-v^I)_x\|_{0,\Omega_{21}}\|\chi\|_{0,\Omega_{21}}\\ &\le C\delta_{21}\varepsilon^{1/4}\ln^{1/2}N(k+N^{-1})\,|||\chi|||_\varepsilon \end{aligned} \tag{3.44b}$$

and

$$\begin{aligned}\delta_{21}|(b(w_2-w_2^I)_x, c\chi)_{\Omega_{21}}| &\le C\delta_{21}\|(w_2-w_2^I)_x\|_{0,\Omega_{21}}\|\chi\|_{0,\Omega_{21}}\\ &\le C\delta_{21}\varepsilon^{1/4}N^{-1}\max|\psi'|\,|||\chi|||_\varepsilon\,. \end{aligned} \tag{3.44c}$$

On Ω_{12} we have with the interpolation-error estimate (2.17a)

$$\begin{aligned}\delta_{12}|(b(u-u^I)_x, c\chi)_{\Omega_{12}}| &\le C\delta_{12}\|(u-u^I)_x\|_{0,\Omega_{12}}\|\chi\|_{0,\Omega_{12}}\\ &\le C\delta_{12}\varepsilon^{-1/2}(h+N^{-1}\max|\psi'|)\,|||\chi|||_\varepsilon \end{aligned} \tag{3.45}$$

and on Ω_{22} with the estimate (2.17b)

$$\delta_{22}|(b(u-u^I)_x, c\chi)_{\Omega_{22}}| \le C\delta_{22}\ln^{1/2}N(\varepsilon^{1/2}h+\varepsilon^{-1/4}(k+N^{-1}\max|\psi'|))\,|||\chi|||_\varepsilon\,. \tag{3.46}$$

The third term is estimated by the L_2-interpolation error estimates (2.15) and holds

$$\delta_{11}|(c(u-u^I), c\chi)_{\Omega_{11}}| \le C\delta_{11}\|u-u^I\|_{0,\Omega_{11}}\|\chi\|_{0,\Omega_{11}} \le C\delta_{11}N^{-2}\,|||\chi|||_\varepsilon \tag{3.47a}$$

$$\delta_{12}|(c(u-u^I), c\chi)_{\Omega_{12}}| \le C\delta_{12}\varepsilon^{1/2}((h+N^{-1})\ln^{1/4}N+N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon \tag{3.47b}$$

$$\delta_{21}|(c(u-u^I), c\chi)_{\Omega_{21}}| \le C\delta_{21}\varepsilon^{1/4}\ln^{1/2}N(k+N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon \tag{3.47c}$$

and

$$\delta_{22}|(c(u-u^I), c\chi)_{\Omega_{22}}| \le C\delta_{22}\varepsilon^{1/2}\ln^{1/2}N(h+k+N^{-1}\max|\psi'|)^2\,|||\chi|||_\varepsilon\,. \tag{3.47d}$$

Collecting (3.42)–(3.47) proves the estimates of Lemma 3.11. □

Remark 3.14. *The estimates in the proof above are not sharp. I.e., one could apply the Lin-formulas* (3.4) *to* (3.44b) *and* (3.44c) *to obtain* $(k+N^{-1})^2$ *and* $(N^{-1}\max|\psi'|)^2$ *on the right-hand side instead of first order terms. But due to the choice of* δ *from the SDFEM the estimates are sufficiently sharp.*

3.4 Continuous Interior Penalty FEM

The methods considered in the previous sections were residual based stabilisation methods. For more complicated problems than (1.1) this is a drawback. In such cases a number of additional terms have to be analysed and implemented. Therefore, it would be better to add a simpler stabilising term.
The Continuous Interior Penalty finite element method (CIPFEM) was introduced by Burman and Hansbo in [6] for convection-diffusion problems. They used a method by Douglas and Dupont [7] to stabilise singularly perturbed problems. At this time the method was called “Edge stabilisation” describing in its name where the stabilisation effect is located.
The technique is based on least-squares stabilisation of the gradient jumps across interior edges. By definition it has the advantage of symmetry in the stabilising term, see (3.48)—a property shared by GLSFEM. This can be beneficial, for instance in optimal-control problems with constraints like (1.1). There the primal and dual problem can be discretised using the same stabilisation method and parameter; see [3] for a local-projection method, that is symmetric too.
Burman and Hansbo prove convergence for the CIPFEM for singularly perturbed problems in the induced norm of order $(\varepsilon^{1/2}h + h^{3/2})\|u\|_{H^2(\Omega)}$ for quasi-uniform meshes. Of course, for singularly perturbed problems the H^2-norm depends badly on the mesh parameter ε. They based their analysis on the possibility to shift the error from the convection term into the jump term by using a special quasi-interpolant of Clement- or Oswald-type. Further work on problems with exponential layers was done in [12, 28] using Shishkin meshes. There it was showed that in the analysis similar ideas as above can be applied if the stabilisation is used only in the coarse region of a Shishkin mesh. Then uniform convergence can be achieved.
In [11] we proved a supercloseness property using the supercloseness of GFEM on problems with characteristic layers. This analysis will now be extended to S-type meshes.

On a Shishkin mesh it was mentioned in [11] that stabilisation on all interior edges leads to a linear system, that is hard to solve. Moreover, it was shown that the best way to stabilise with CIPFEM is to add stabilising terms only on edges perpendicular to the streamline direction (i.e. x-axis) inside the coarse mesh and the characteristic layer region. Therefore, on S-type meshes we will also stabilise in these regions only.
Let $\Omega^* = \Omega_{11} \cup \Omega_{21}$ be the region, where we want to stabilise. With $\mathcal{E}^*$ we denote the interior edges of T^N inside Ω^* that are perpendicular to the streamline direction.
Define the stabilisation term for $u, v \in V^N \cup H^2(\Omega)$

$$J(u,v) := \hbar^2 \sum_{\tau\subset\Omega^*} \sum_{e\in\partial\tau\setminus\partial\Omega^*} \int_e \gamma_e [u_x]_e [v_x]_e = \hbar^2 \sum_{e\in\mathcal{E}^*} \int_e \gamma_e [u_x]_e [v_x]_e \tag{3.48}$$

with $[q]_e$ denoting the jump of a function q across the edge e and $\hbar \le CN^{-1}$ the mesh size in x-direction inside Ω^*. The equivalence in (3.48) follows by $[u_x]_{e'} = 0$ for $u \in V^N \cup H^2(\Omega)$ and edges e' parallel to the x-axis.

Note that for all $u, v \in H^2(\Omega)$ the traces of ∇u and ∇v are well defined and therefore

$$J(u, v) = 0. \tag{3.49}$$

The final discretisation reads: Find $u^N \in V^N$ such that

$$a_{CIP}(u^N, v^N) := a_{Gal}(u^N, v^N) + J(u^N, v^N) = f(v^N), \text{ for all } v^N \in V^N. \tag{3.50}$$

We define the induced mesh dependent CIPFEM-norm by

$$|||v|||_{CIP}^2 := |||v|||_\varepsilon^2 + |v|_J^2, \text{ for all } v \in H_0^1(\Omega)$$

with the seminorm $|v|_J := J(v, v)^{1/2}$.
We immediately have coercivity in the CIPFEM-norm

$$a_{CIP}(v^N, v^N) \geq \left|\left|\left|v^N\right|\right|\right|_\varepsilon^2 + |v^N|_J^2 = \left|\left|\left|v^N\right|\right|\right|_{CIP}^2, \text{ for all } v^N \in V^N \tag{3.51}$$

and Galerkin orthogonality by (3.49)

$$a_{CIP}(u - u^N, v^N) = a_{Gal}(u - u^N, v^N) + J(u - u^N, v^N) = 0, \text{ for all } v^N \in V^N. \tag{3.52}$$

3.4.1 Analysis of the method

Let us start the analysis by bounding the interpolation error in the CIP semi-norm. The proof of the following Lemma is given in Section 3.4.2.

Lemma 3.15. *Let u^I be the bilinear interpolant to the solution u of* (1.1) *on an S-type mesh with parameter $\sigma \geq 2$. Then*

$$|u - u^I|_J^2 \leq CN^{-3}$$

and

$$\left|\left|\left|u - u^I\right|\right|\right|_{CIP} \leq \left|\left|\left|u - u^I\right|\right|\right|_\varepsilon + |u - u^I|_J \leq C(\varepsilon^{1/4}h + k + N^{-1}\max|\psi'|).$$

Theorem 3.16 (Supercloseness CIPFEM). *Let $\sigma \geq 5/2$. Then the CIP solution u^N satisfies*

$$\left|\left|\left|u^I - u^N\right|\right|\right|_{CIP} \leq C(N^{-3/2} + (h\ln^{1/4} N + k + N^{-1}\max|\psi'|)^2).$$

Proof. Coercivity (3.51) and Galerkin orthogonality (3.52) yield

$$\begin{aligned}
\left|\left|\left|u^I - u^N\right|\right|\right|_{CIP}^2 &\leq a_{Gal}(u^I - u^N, u^I - u^N) + J(u^I - u^N, u^I - u^N) \\
&= a_{Gal}(u^I - u, u^I - u^N) + J(u^I - u, u^I - u^N) \\
&\leq a_{Gal}(u^I - u, u^I - u^N) + |u^I - u|_J |u^I - u^N|_J.
\end{aligned}$$

Applying Lemma 3.15 and Theorem 3.5, we are done. □

name	$\lvert\lvert\lvert u^I - u^N \rvert\rvert\rvert_{CIP}$	$\lvert\lvert\lvert u - u^N \rvert\rvert\rvert_{CIP}$
Shishkin mesh	$CN^{-3/2}$	$CN^{-1}\ln N$
polynomial S-mesh	$CN^{-3/2}$	$CN^{-1}\ln^{1/m} N$
Bakhvalov-Shishkin mesh	$C(\varepsilon + N^{-3/2})$	$C(\varepsilon^{1/2} + N^{-1})$
modified B-S-mesh	$C(\varepsilon + N^{-3/2})$	$C(\varepsilon^{1/2} + N^{-1})$

Table 3.2: Expected rates for CIPFEM on different S-type meshes

Remark 3.17 (Convergence CIPFEM)**.** *Convergence in the CIPFEM-norm follows from Lemma 3.15, interpolation-error Theorem 2.6 and Theorem 3.16*

$$\begin{aligned}\lvert\lvert\lvert u - u^N \rvert\rvert\rvert_{CIP} &\leq \lvert\lvert\lvert u - u^I \rvert\rvert\rvert_{CIP} + \lvert\lvert\lvert u^I - u^N \rvert\rvert\rvert_{CIP} \\ &\leq \lvert\lvert\lvert u - u^I \rvert\rvert\rvert_{\varepsilon} + \left|u - u^I\right|_J + \lvert\lvert\lvert u^I - u^N \rvert\rvert\rvert_{CIP} \\ &\leq C(\varepsilon^{1/4} h + k + N^{-1}\max|\psi'|).\end{aligned}$$

Additionally the jump term yields

$$\begin{aligned}|u - u^N|_J^2 &= J(u - u^I, u - u^N) + J(u^I - u^N, u - u^N) \\ &\leq (|u - u^I|_J + |u^I - u^N|_J)|u - u^N|_J,\end{aligned}$$

which immediately gives

$$|u - u^N|_J \leq C(N^{-3/2} + (h\ln^{1/4} N + k + N^{-1}\max|\psi'|)^2).$$

Table 3.2 shows the expected rates for different S-type meshes corresponding to Theorem 3.16 and Remark 3.17.

3.4.2 Proofs

Proof of Lemma 3.15.
For any $w \in H^2(\Omega)$ apply (3.49) and (3.48) to obtain

$$|w - w^I|_J^2 \leq C\hbar^2 \sum_{e\in\mathcal{E}^*} \|[w_x^I]\|_{0,e}^2. \tag{3.53}$$

It remains to bound $\|[w_x^I]\|_{0,e}^2$. Let $e = \tau \cap \tau'$ with $\tau = \tau_{i,j}$, $\tau' = \tau_{i+1,j}$ and recall $h_i = h_{i+1} = \hbar$. Then

$$\begin{aligned}\hbar^2\|[w_x^I]\|_{0,e}^2 &= \hbar^2 \int_e (w_x^I|_{\tau,e} - w_x^I|_{\tau',e})^2 \,\mathrm{d}y \\ &= \hbar^2 \int_e \Bigg(\left(\frac{w_{i+1,j-1} - 2w_{i,j-1} + w_{i-1,j-1}}{\hbar}\right) \frac{y_j - y}{k_j} + \\ &\qquad \left(\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\hbar}\right) \frac{y - y_{j-1}}{k_j} \Bigg)^2 \mathrm{d}y \\ &\leq \int_e \left(|w_{i+1,j-1} - 2w_{i,j-1} + w_{i-1,j-1}| + |w_{i+1,j} - 2w_{i,j} + w_{i-1,j}|\right)^2 \mathrm{d}y. \quad (3.54)\end{aligned}$$

Using integration by parts we have for any function $g \in C^2(x_{i-1}, x_{i+1})$

$$\begin{aligned} g(x_i) &= g(x_{i+1}) - g'(x_{i+1})\hbar + \int_{x_i}^{x_{i+1}} g''(x)(x - x_i)\,\mathrm{d}x, \\ g(x_{i-1}) &= g(x_{i+1}) - 2g'(x_{i+1})\hbar + \int_{x_{i-1}}^{x_{i+1}} g''(x)(x - x_{i-1})\,\mathrm{d}x, \\ g(x_{i+1}) - 2g(x_i) + g(x_{i-1}) &= -2\int_{x_i}^{x_{i+1}} g''(x)(x - x_i)\,\mathrm{d}x + \int_{x_{i-1}}^{x_{i+1}} g''(x)(x - x_{i-1})\,\mathrm{d}x. \end{aligned} \tag{3.55}$$

If g'' is positive and monotonically non-increasing on (x_{i-1}, x_{i+1}) then follows

$$\int_{x_{i-1}}^{x_{i+1}} g''(x)(x - x_{i-1})^{(k-1)}\,\mathrm{d}x \le \frac{1}{k}\left(\int_{x_{i-1}}^{x_{i+1}} g''(x)^{1/k}\,\mathrm{d}x\right)^k \tag{3.56}$$

for any $k \in \mathbb{N}^+$, see [4].
Let $w = u$. Then (3.54)—(3.56) yield with $|u_{xx}(x,y)| \le C(1 + \varepsilon^{-2}e^{-\beta x/\varepsilon})$

$$\hbar^2\|[u_x^I]\|_{0,e}^2 \le C\int_e \left(\int_{x_{i-1}}^{x_{i+1}} \left(1 + \varepsilon^{-1}e^{-\beta x/(2\varepsilon)}\right)\,\mathrm{d}x\right)^4\,\mathrm{d}y. \tag{3.57}$$

With $\sigma \ge 2$ follows

$$|u - u^I|_{J,\Omega_{11}}^2 \le \sum_{i=N/2-1}^{N-1}\int_{\lambda_y}^{1-\lambda_y}\left(\int_{x_{i-1}}^{x_{i+1}}\left(1 + \varepsilon^{-1}e^{-\beta x/(2\varepsilon)}\right)\right)^4 \le CNN^{-4} \le CN^{-3}$$

and similarly

$$|u - u^I|_{J,\Omega_{21}}^2 \le CN^{-3}$$

by (3.53) and (3.57).
The second estimate of Lemma 3.15 follows from the first one and Theorem 2.6. □

Chapter 4

Postprocessing and enhancement of accuracy

In Chapter 3 it was shown, that the difference between numerical solution and the bilinear interpolant of the exact solution is much smaller than the real error. However, we are interested in an approximation of the exact solution and not of its interpolation.
This chapter explores possibilities of using the supercloseness property in order to obtain better approximations of the exact solution or some related information. This will be achieved by means of postprocessing.
We study different approaches starting with a method first used by Stynes and Tobiska [31] for convection-diffusion problems. There the numerical solution is projected into a higher-order space on a macro mesh.
The second approach was considered by Roos and Linß [26] for two-dimensional convection-diffusion problems. Usually the L_2-error is $\mathcal{O}(N^{-1})$ better than the ε-weighted H^1-seminorm error. Therefore, the finite element user is interested especially in recovery of the gradient. The method of [26] interpolates the gradient of the numerical solution on a patch of elements.
Finally, a postprocessing operator that originates from the area of discontinuous Galerkin described in [13] will be applied. Although developed for recovery of discontinuous functions, the recovered discontinuous solution has a very good error-behaviour. A connection between this approach and a modification of the previous method is shown, too.

4.1 Postprocessing of u^N

Suppose N is divisible by 8. We construct a coarser mesh $\tilde{T}^{N/2}$ composed of macrorectangles M, each consisting of four rectangles of T^N. Furthermore each macroelement M is supposed to belong to one of the four subdomains of Ω only; see Figure 4.1. Remark that in general $\tilde{T}^{N/2} \neq T^{N/2}$ due to the transition points λ_x and λ_y and the mesh-generating function ϕ.
Let P_M denote the projection/interpolation operator on a macro M that maps any continuous function v onto that biquadratic function $P_M v$ that coincides with v at the nine

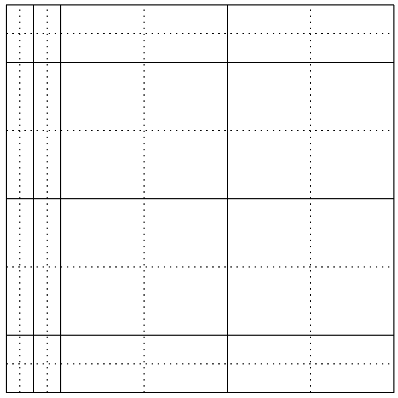


Figure 4.1: Macroelements M of $\tilde{T}^{N/2}$ constructed from T^N

mesh nodes of T^N in M. This piecewise projection is extended to give a global continuous function by setting

$$\big(Pv\big)(x,y) := \big(P_M v\big)(x,y) \quad \text{for } (x,y) \in M.$$

Recall that depending on the mesh-generating function ϕ the macroelement M may consist of rectangles with different sizes. Therefore, the results of [31], where this recovery method was used on piecewise uniform meshes, have to be checked carefully.

We start by defining the linear mapping from the macro $M = [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$ to the reference macro $\widetilde{M} = [-1,1]^2$.

$$\xi(x) = \frac{2x - x_{i-1} - x_{i+1}}{h_i + h_{i+1}} \quad \text{and} \quad \eta(y) = \frac{2y - y_{j-1} - y_{j+1}}{k_j + k_{j+1}} \tag{4.1}$$

The reference marks $\tilde{a}$ and $\tilde{b}$ are defined by

$$\tilde{a} = \xi(x_i) = \frac{h_i - h_{i+1}}{h_i + h_{i+1}} \quad \text{and} \quad \tilde{b} = \eta(y_j) = \frac{k_j - k_{j+1}}{k_j + k_{j+1}}. \tag{4.2}$$

If M consists of uniform rectangles, then the reference marks are $\tilde{a} = \tilde{b} = 0$.
Let us number the nodes on $\widetilde{M}$ corresponding to Figure 4.2 and prove a stability estimate similar to (2.9) and (2.10).

Lemma 4.1. *Let $|\tilde{a}| \le q$ and $|\tilde{b}| \le q$ for a fixed $q \in (0,1)$. Then the interpolation operator P is L_∞-stable, that is*

$$\|Pw\|_{L_\infty(M)} \le C\|w\|_{L_\infty(M)} \ \text{ for } w \in L_\infty(M). \tag{4.3}$$

Moreover, if $w_x \in L_\infty(M)$, we have additionally

$$\|(Pw)_x\|_{L_\infty(M)} \le C\|w_x\|_{L_\infty(M)} \tag{4.4}$$

and similarly for w_y.

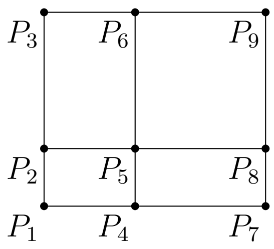


Figure 4.2: Numbering of nodes on $\widetilde{M}$

Proof. We proof the estimates on the reference macroelement. Let $\{\hat{\varphi}_i\}$ be the set of Lagrange-basis functions of $\mathcal{Q}_2(\widetilde{M})$ with $\hat{\varphi}_i(P_i) = 1$ and $\hat{\varphi}_i(P_j) = 0$ for $i \neq j$. With $\hat{w}_i = \hat{w}(P_i)$ we have

$$\|\hat{P}\hat{w}\|_{L_\infty(\widetilde{M})} \leq \sum_{i=1}^{9} |\hat{w}_i| \|\hat{\varphi}_i\|_{L_\infty(\widetilde{M})} \leq 9\|\hat{w}\|_{L_\infty(\widetilde{M})} \max_{i=1,\dots,9} \|\hat{\varphi}_i\|_{L_\infty(\widetilde{M})}.$$

A direct calculation shows the maximum to be bounded by

$$\max_{i=1,\dots,9} \|\hat{\varphi}_i\|_{L_\infty(\widetilde{M})} \leq \max\left\{1, \frac{1}{2(1-|\tilde{a}|)}\right\} \max\left\{1, \frac{1}{2(1-|\tilde{b}|)}\right\}.$$

Due to $|\tilde{a}|, |\tilde{b}| \leq q < 1$ and the transformation (4.1) we obtain the stability (4.3). Remark, that

$$(\hat{\varphi}_i + \hat{\varphi}_{i+3} + \hat{\varphi}_{i+6})_\xi = 0, \; i = 1, 2, 3$$

holds. Thus,

$$\begin{aligned}
(\hat{P}\hat{w})_\xi &= \sum_{i=1}^{3} (\hat{w}_i\hat{\varphi}_i + \hat{w}_{i+3}\hat{\varphi}_{i+3} + \hat{w}_{i+6}\hat{\varphi}_{i+6})_\xi \\
&= \sum_{i=1}^{3} ((\hat{w}_i - \hat{w}_{i+3})(\hat{\varphi}_i)_\xi + (\hat{w}_{i+6} - \hat{w}_{i+3})(\hat{\varphi}_{i+6})_\xi).
\end{aligned}$$

We get

$$\begin{aligned}
\|(\hat{P}\hat{w})_\xi\|_{L_\infty(\widetilde{M})} &\leq \sum_{i=1}^{3} \left(\underbrace{|\hat{w}_i - \hat{w}_{i+3}|}_{\leq \int_{P_i}^{P_{i+3}} |\hat{w}_\xi|} \|(\hat{\varphi}_i)_\xi\|_{L_\infty(\widetilde{M})} + \underbrace{|\hat{w}_{i+3} - \hat{w}_{i+6}|}_{\leq \int_{P_{i+3}}^{P_{i+6}} |\hat{w}_\xi|} \|(\hat{\varphi}_{i+6})_\xi\|_{L_\infty(\widetilde{M})} \right) \\
&\leq \sum_{i=1}^{3} \left(\max\left\{ \|(\hat{\varphi}_i)_\xi\|_{L_\infty(\widetilde{M})}, \|(\hat{\varphi}_{i+6})_\xi\|_{L_\infty(\widetilde{M})} \right\} \int_{P_i}^{P_{i+6}} |\hat{w}_\xi| \right) \\
&\leq 6\|\hat{w}_\xi\|_{L_\infty(\widetilde{M})} \max_{i=1,2,3,7,8,9} \|(\hat{\varphi}_i)_\xi\|_{L_\infty(\widetilde{M})} \\
&\leq \frac{24}{(1-|\tilde{a}|)(1-|\tilde{b}|)} \|\hat{w}_\xi\|_{L_\infty(\widetilde{M})}
\end{aligned}$$

and with $|\tilde{a}|, |\tilde{b}| \leq q < 1$ and the transformation (4.1) follows the stability (4.4). □

The proof of Lemma 4.1 shows that we have to take care of the constants occurring in estimates that are trouble-free for locally equidistant meshes.
Let us turn to interpolation error estimates. We like to use the anisotropic interpolation error estimate [1, Theorem 2.7] for biquadratic interpolation. However, because of the possible non-uniformity in our macroelements, the constants in these estimates have to be analysed.
The proof of [1, Theorem 2.7] refers the reader to the proof of the crucial Lemma [1, Lemma 2.14]. We cite it for the convenience of the reader.

Lemma 4.2 (Lemma 2.14 of [1])**.** *Assume that $\hat{e}$ is a square or cube. Let $I : C(\bar{\hat{e}}) \to \mathcal{Q}_k(\hat{e})$ be a linear operator. Fix $m, l \in \mathbb{N}$, $p \in [1, \infty)$ and $q \in [1, \infty]$ such that $0 \le m \le l \le k+1$ and $W^{l-m,p}(\hat{e}) \hookrightarrow L_q(\hat{e})$ hold. Consider a multi-index γ with $|\gamma| = m$ and define $j := \dim \hat{D}^\gamma \mathcal{Q}_{k,\hat{e}}$. Assume that there are linear functionals F_i, $i = 1, \ldots, j$, such that*

$$F_i \in \left(W^{l-m,p}(\hat{e})\right)', \ \forall i = 1, \ldots, j, \tag{4.5a}$$

$$F_i(\hat{D}^\gamma(\hat{u} - I\hat{u})) = 0, \ \forall i = 1, \ldots, j \qquad \forall \hat{u} \in C(\hat{e}) : \hat{D}^\gamma \hat{u} \in W^{l-m,p}(\hat{e}) \tag{4.5b}$$

$$\hat{w} \in \mathcal{Q}_k(\hat{e}) \text{ and } F_i(\hat{D}^\gamma \hat{w}) = 0, \ \forall i = 1, \ldots, j \ \Rightarrow \ \hat{D}^\gamma \hat{w} = 0. \tag{4.5c}$$

Then the error can be estimated for all $\hat{u} \in C(\bar{\hat{e}})$ with $\hat{D}^\gamma \hat{u} \in W^{l-m,p}(\hat{e})$ by

$$\|\hat{D}^\gamma(\hat{u} - I\hat{u})\|_{L_q(\hat{e})} \le C[\hat{D}^\gamma \hat{u}]_{W^{l-m,p}(\hat{e})},$$

where $[\hat{u}]_*$ is a special seminorm including pure derivatives only.

We are interested in the case $k = 2$, $I = \hat{P}$, $\hat{e} = \widetilde{M}$ and $p = q = 2$. In the proof of Lemma 4.2 the constants in the following steps for $\hat{v} \in \mathcal{Q}_2(\widetilde{M})$ need to be checked only:

$$\|\hat{D}^\gamma(\hat{v} - \hat{P}\hat{u})\|_{0,\widetilde{M}} \le C_1 \sum_{i=1}^{j} |F_i(\hat{D}^\gamma(\hat{v} - \hat{P}\hat{u}))| \tag{4.6a}$$

$$= C_1 \sum_{i=1}^{j} |F_i(\hat{D}^\gamma(\hat{v} - \hat{u}))| \tag{4.6b}$$

$$\le C_2 C_1 \|\hat{D}^\gamma(\hat{v} - \hat{u})\|_{l-m,\widetilde{M}}. \tag{4.6c}$$

Before investigating C_1 and C_2 we mention that step (4.6b) is a direct conclusion of (4.5b). The proof of the following lemma can be found in Section 4.4.

Lemma 4.3. *Let $|\tilde{a}|, |\tilde{b}| \le q$ for a fixed $q \in (0, 1)$. Then the constant C_1 of (4.6a) is bounded. The constant C_2 of (4.6c) is bounded independent of $\tilde{a}$ and $\tilde{b}$.*

Remark 4.4. *It is not clear whether $|\tilde{a}|$ and $|\tilde{b}|$ need to be bounded away from 1. Consider $\hat{v} \in C^3(\widetilde{M})$ and its Taylor series. As our projector P is the nodal biquadratic interpolation we have*

$$\hat{v}(\xi, \eta) - \hat{P}\hat{v}(\xi, \eta) = \hat{v}_{\xi\xi\xi}(\xi_1, \eta_1)\frac{\xi^3 - \hat{P}\xi^3}{6} + \hat{v}_{\eta\eta\eta}(\xi_2, \eta_2)\frac{\eta^3 - \hat{P}\eta^3}{6}$$

with $-1 \le \xi_1, \xi_2, \eta_1, \eta_2 \le 1$. *An explicit calculation gives* $\hat{P}\xi^3 = \xi^2\tilde{a} + \xi - \tilde{a}$ *and*

$$\|\xi^3 - \hat{P}\xi^3\|_{L_\infty(\widetilde{M})} \le \frac{32}{27}$$

independent of $\tilde{a}$. *A similar result follows for the term in* η. *Eventually we get a pointwise bound*

$$\|\hat{v} - \hat{P}\hat{v}\|_{L_\infty(\widetilde{M})} \le \left(\frac{2}{3}\right)^4 \left(\|\hat{v}_{\xi\xi\xi}\|_{L_\infty(\widetilde{M})} + \|\hat{v}_{\eta\eta\eta}\|_{L_\infty(\widetilde{M})}\right) = C[\hat{v}]_{W^{3,\infty}(\widetilde{M})}$$

for any $|\tilde{a}| \le 1$ *and* $|\tilde{b}| \le 1$.

Theorem 4.5 (Interpolation error). *Let* T^N *be an S-type mesh with* $\sigma \ge 3$. *Suppose* $1 < p \in \mathbb{R}$ *and*

$$\frac{\max\{h_i, h_{i+1}\}}{\min\{h_i, h_{i+1}\}} \le p, \quad \text{for all } i = 1, \dots, N/2 - 1,$$
$$\frac{\max\{k_j, k_{j+1}\}}{\min\{k_j, k_{j+1}\}} \le p, \quad \text{for all } j = 1, \dots, N/4 - 1 \text{ and } j = 3N/4 + 1, \dots, N - 1.$$

Then

$$|||u - Pu|||_\varepsilon \le C(\varepsilon^{1/8}h + k + N^{-1}\max|\psi'|)^2. \tag{4.7}$$

Proof. If the stepsizes are bounded then $|\tilde{a}|$, $|\tilde{b}| \le q < 1$ and due to Lemma 4.3 the constant C_1 is bounded. Indeed, for

$$1 \le \frac{\max(h_i, h_{i+1})}{\min(h_i, h_{i+1})} \le p$$

follows with (4.2)

$$|\tilde{a}| = \frac{\frac{\max(h_i,h_{i+1})}{\min(h_i,h_{i+1})} - 1}{\frac{\max(h_i,h_{i+1})}{\min(h_i,h_{i+1})} + 1} = 1 - \frac{2}{1 + \frac{\max(h_i,h_{i+1})}{\min(h_i,h_{i+1})}} =: q$$

and finally (cf. (4.20) and (4.21))

$$1 \le \frac{1}{1 - \tilde{a}^2} = \frac{1}{(1 - |\tilde{a}|)(1 + |\tilde{a}|)} \le \frac{1}{1 - |\tilde{a}|} \le \frac{p+1}{2}.$$

Apply Lemma 4.3 to Lemma 4.2 in order to get estimates on the reference macroelement with constants independent of N and ε.
With the transformation (4.1) we get for $M = [x_{i-1}, x_{i+1}] \times [k_{j-1}, k_{j+1}]$

$$\|u - Pu\|_{0,M} \le C((h_i + h_{i+1})^3\|u_{xxx}\|_{0,M} + (k_j + k_{j+1})^3\|u_{yyy}\|_{0,M}) \tag{4.8}$$

and

$$\|(u - Pu)_x\|_{0,M} \le C((h_i + h_{i+1})^2\|u_{xxx}\|_{0,M} + (k_j + k_{j+1})^2\|u_{xyy}\|_{0,M}).$$

The estimate (4.7) now follows in the same way the interpolation estimates in Theorem 2.6 were proved using (4.3) and (4.4) instead of (2.9) and (2.10) □

Remark 4.6. *In [9] above interpolation error estimate was proved on Shishkin meshes using $\sigma \geq 5/2$ giving*

$$|||u - Pu|||_\varepsilon \leq C(\varepsilon N^{-\sigma+1} + N^{-2}\ln^2 N).$$

For general S-type meshes $\sigma \geq 3$ is needed in the following step. Recall the decomposition $u = v + w_1 + w_2 + w_{12}$. Applying (4.8) to w_1 on $M \subset \Omega_{12}$ gives

$$\|w_1 - Pw_1\|_{0,M} \leq C\left((\varepsilon N^{-1}\max|\psi'|)^3 \|e^{3\beta x/(\sigma\varepsilon)} w_{1,xxx}\|_{0,M} + N^{-3}\|w_{1,xyy}\|_{0,M}\right).$$

Summing over all macroelements in Ω_{12} results in

$$\|w_1 - Pw_1\|_{0,\Omega_{12}} \leq C\left((\varepsilon N^{-1}\max|\psi'|)^3 \|e^{3\beta x/(\sigma\varepsilon)} w_{1,xxx}\|_{0,\Omega_{12}} + N^{-3}\|w_{1,xyy}\|_{0,\Omega_{12}}\right)$$

with

$$\|e^{3\beta x/(\sigma\varepsilon)} w_{1,xxx}\|^2_{0,\Omega_{12}} \leq C\varepsilon^{-6}\int_{\Omega_{12}} e^{\beta x/\varepsilon(6/\sigma-2)}$$

bounded for $\sigma \geq 3$.

Remark 4.7. *All the S-type meshes listed in Table 2.1 have a bounded stepsize.*

- ***Shishkin mesh**: Because $h_i = h_{i+1}$ for all $i = 1, \dots, N/2-1$ the macroelements consist of uniform rectangles. Here the extra factor is 1.*
- ***polynomial S-mesh**: For $m > 1$ holds $h_i < h_{i+1}$ and*
$$\frac{h_{i+1}}{h_i} \leq \frac{h_2}{h_1} \leq p = 2^m - 1 \Rightarrow \frac{1}{1-|a|} \leq 2^{m-1}.$$
- ***Bakhvalov-S-mesh**: We have $h_i < h_{i+1}$ and*
$$\frac{h_{i+1}}{h_i} \leq \frac{h_{N/2}}{h_{N/2-1}} \leq p = \frac{\ln 3}{\ln(5/3)} \Rightarrow \frac{1}{1-|a|} < 1.58.$$
- ***mod. B-S-mesh**: Again $h_i < h_{i+1}$ holds and $N \geq 8$ yields*
$$\frac{h_{i+1}}{h_i} \leq \frac{h_{N/2}}{h_{N/2-1}} \leq 1 + 4\frac{\ln N}{N} < p = 2.04 \Rightarrow \frac{1}{1-|a|} < 1.52.$$

Now that we have interpolation-error estimates, let us look at further properties of P. The proof of the following lemma is given in Section 4.4.

Lemma 4.8. *The projection operator P is consistent in the following sense*

$$P(v^I) = Pv \qquad \text{for all } v \in C(\bar{\Omega}) \tag{4.9}$$

with v^I denoting the usual nodal bilinear interpolant.
Furthermore, let $|\tilde{a}|, |\tilde{b}| \leq q$ with a fixed $q \in (0,1)$. Then it has the stability property

$$|||Pv^N|||_\varepsilon \leq C\,|||v^N|||_\varepsilon \qquad \text{for all } v^N \in V^N. \tag{4.10}$$

Theorem 4.9. *Let T^N be an S-type mesh with $\sigma \geq 3$ and*

$$\frac{\max\{h_i, h_{i+1}\}}{\min\{h_i, h_{i+1}\}} \leq p, \quad \text{for all } i = 1, \dots, N/2 - 1,$$
$$\frac{\max\{k_j, k_{j+1}\}}{\min\{k_j, k_{j+1}\}} \leq p, \quad \text{for all } j = 1, \dots, N/4 - 1 \text{ and } j = 3N/4 + 1, \dots, N - 1.$$

for $1 < p \in \mathbb{R}$. Then depending on the numerical method that generates the numerical solution u^N we have

$$\left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon \leq C((h + N^{-1}) \ln^{1/4} N + k + N^{-1} \max |\psi'|)^2 \tag{4.11a}$$

for GFEM, SDFEM and GLSFEM and

$$\left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon \leq C(N^{-3/2} + (h \ln^{1/4} N + k + N^{-1} \max |\psi'|)^2) \tag{4.11b}$$

for CIPFEM.

Proof. The triangle inequality and Lemma 4.8 implies

$$\begin{aligned}\left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon &\leq |||u - Pu|||_\varepsilon + \left|\left|\left|Pu - Pu^N\right|\right|\right|_\varepsilon \leq |||u - Pu|||_\varepsilon + \left|\left|\left|Pu^I - Pu^N\right|\right|\right|_\varepsilon \\ &\leq |||u - Pu|||_\varepsilon + C \left|\left|\left|u^I - u^N\right|\right|\right|_\varepsilon .\end{aligned}$$

The interpolation error is estimated by Theorem 4.5 and the second term—depending on the method—by Theorems 3.5, 3.9, 3.12 and 3.16, respectively. □

Remark 4.10. *We have*

$$\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon \leq \left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon + \left|\left|\left|Pu^N - u^N\right|\right|\right|_\varepsilon .$$

As the order of convergence for $\left|\left|\left|u - Pu^N\right|\right|\right|_\varepsilon$ is higher than for $\left|\left|\left|u - u^N\right|\right|\right|_\varepsilon$ we can use $\left|\left|\left|Pu^N - u^N\right|\right|\right|_\varepsilon$ as an asymptotically exact a posteriori error estimator.

4.2 Postprocessing of ∇u^N

In this section we will adopt the postprocessing operator and results from [26]. Recalling the decomposition $u = v + w_1 + w_2 + w_{12}$ of Assumption 2.1 we now need pointwise estimates like (2.1) up to order 3. Furthermore, let us assume for the sake of simplicity in the presentation of the results, that $k, h \leq CN^{-1}$.
Let τ be a rectangle of T^N and $\tilde{\tau}$ be the patch that consists of all rectangles having a common corner with τ, see Figure 4.3. In the midpoints of these mesh rectangles the gradient is computed ($\gamma_{i,j} := \nabla w^N(x_{i-1/2}, y_{j-1/2})$, see "∘" in Figure 4.3). Bilinear interpolation (or an area-weighted mean) of these values gives the values of the recovered gradient Rw^N for $w^N \in V^N$ at the mesh points (see "•" in Figure 4.3)

$$(Rw^N)_{i,j} = \alpha_{i,j} := \frac{(\gamma_{i,j} h_{i+1} + \gamma_{i+1,j} h_i) k_{j+1} + (\gamma_{i,j+1} h_{i+1} + \gamma_{i+1,j+1} h_i) k_j}{(h_i + h_{i+1})(k_j + k_{j+1})}.$$

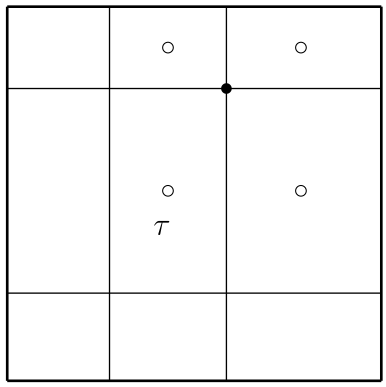


Figure 4.3: A rectangle τ and its associated patch $\tilde{\tau}$

Finally, by bilinear interpolation the recovered gradient is extended to Ω

$$(Rw^N)(x,y) := \left(\alpha_{i-1,j-1}\frac{x_i - x}{h_i} + \alpha_{i,j-1}\frac{x - x_{i-1}}{h_i}\right)\frac{y_j - y}{k_j} + \left(\alpha_{i-1,j}\frac{x_i - x}{h_i} + \alpha_{i,j}\frac{x - x_{i-1}}{h_i}\right)\frac{y - y_{j-1}}{k_j}, \text{ for } (x,y) \in \tau_{i,j},\ 2 \le i,j \le N-1.$$

For boundary rectangles the well-defined bilinear function of the adjacent rectangles is extrapolated. We quote the following lemma from [26].

Lemma 4.11. *The recovery operator $R : V^N \to V^N \times V^N$ is a linear operator with the following properties:*

$$\begin{array}{rl} \textit{(locality)} & Rw^N \text{ on } \tau \text{ depends only on values of } \nabla w^N \text{ on the patch } \tilde{\tau}, \\ \textit{(stability)} & \|Rw^N\|_{L_\infty(\tau)} \le C\|\nabla w^N\|_{L_\infty(\tilde{\tau})}, \\ & \|Rw^N\|_{0,\tau} \le C\|\nabla w^N\|_{0,\tilde{\tau}}, \\ \textit{(consistency)} & R(w^I) = \nabla w \text{ on } \tau \text{ for all } w \text{ that are quadratic on } \tilde{\tau}. \end{array}$$

The stability properties hold for the components of $Rw^N = (R_1, R_2)$ independently, i.e. $\|R_1\|_{0,\tau} \le C\|(w^N)_x\|_{0,\tilde{\tau}}$ and $\|R_2\|_{0,\tau} \le C\|(w^N)_y\|_{0,\tilde{\tau}}$.
As the consistency property holds for quadratic functions, we define a quadratic interpolation v^* to a given continuous function v on $\tilde{\tau}$ by

$$v^*(P_k) = v(P_k),\ k = 1,\dots 6, \tag{4.12}$$

where P_k are the corners of $\tilde{\tau}$ and midpoints of two adjacent sides of $\tilde{\tau}$, see Figure 4.4.

Lemma 4.12. *Let v^* be the quadratic function interpolating $v \in C(\tilde{\tau})$ as defined in* (4.12)*. Then*

$$\|v^*\|_{L_\infty(\tilde{\tau})} \le C\|v\|_{L_\infty(\tilde{\tau})} \quad \textit{and} \quad \|\nabla v^*\|_{L_\infty(\tilde{\tau})} \le C\|\nabla v\|_{L_\infty(\tilde{\tau})}.$$

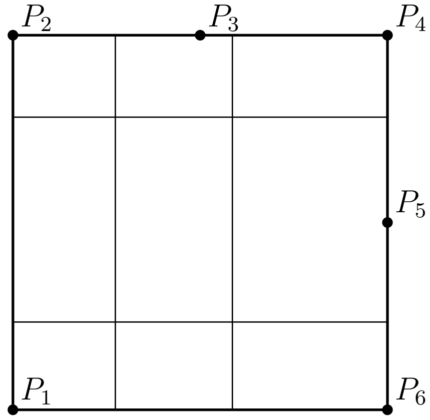


Figure 4.4: Interpolation points for v^* on $\tilde{\tau}$

Proof. We prove the stability results on the reference macro $\widetilde{T} = [0,1]^2$ to the patch $\tilde{\tau}$. Denoting $v_i = v(P_i)$ and $\hat{v}^*$ as the mapped biquadratic function v^* we obtain

$$\begin{aligned}\hat{v}^*(\xi,\eta) = & v_1 \frac{(1-\xi)(1-\eta)}{4} + v_2 \frac{(1-\xi)(\eta-2\xi-1)}{4} + v_3(1-\xi^2) + \\ & v_4 \frac{2\xi^2+2\eta^2+\xi\eta+\xi+\eta-3}{4} + v_5(1-\eta^2) + v_6(1-\eta)(\xi-2\eta-1).\end{aligned}$$

The first stability result follows with $C = 25/7$ on $\widetilde{T}$ and thus on $\tilde{\tau}$ too. Moreover, the explicit formula for $\hat{v}^*$ yields

$$\begin{aligned}\hat{v}^*_\xi(\xi,\eta) &= (v_6 - v_1)\frac{\eta-1}{4} + (v_4 - v_2)\frac{1+\eta}{4} + (v_4 - v_3)\xi + (v_2 - v_3)\xi \\ &= \int_{-1}^{1} \hat{v}_\xi(\xi,-1)\,\mathrm{d}\xi \frac{\eta-1}{4} + \int_{-1}^{1} \hat{v}_\xi(\xi,1)\,\mathrm{d}\xi \frac{\eta+1}{4} + \\ &\quad \int_0^1 \hat{v}_\xi(\xi,1)\,\mathrm{d}\xi \cdot \xi - \int_{-1}^{0} \hat{v}_\xi(\xi,1)\,\mathrm{d}\xi \cdot \xi.\end{aligned}$$

Thus,

$$\|\hat{v}^*_\xi\|_{L_\infty(\widetilde{T})} \le 3\|\hat{v}_\xi\|_{L_\infty(\widetilde{T})}.$$

Similarly the estimate for the η-derivative can be established. By transformation from $\widetilde{T}$ to $\tilde{\tau}$ we are done. □

Lemma 4.13. *Let u^* be the quadratic interpolant to the solution u of* (1.1)*. Recalling the decomposition $u = v + w_1 + w_2 + w_{12}$ of Assumption 2.1 we have*

$$\begin{aligned}
\|\nabla(v - v^*)\|_{L_\infty(\tilde\tau)} &\le CN^{-2}, && \tau \subset \Omega,\\
\|w_1 - w_1^*\|_{L_\infty(\tilde\tau)} &\le CN^{-3}, && \tau \subset \Omega_{11} \cup \Omega_{21},\\
\|(w_1 - w_1^*)_x\|_{L_\infty(\tilde\tau)} &\le C\varepsilon^{-1}(N^{-1}\max|\psi'|)^2, && \tau \subset \Omega_{12} \cup \Omega_{22},\\
\|(w_1 - w_1^*)_y\|_{L_\infty(\tilde\tau)} &\le C(N^{-1}\max|\psi'|)^2, && \tau \subset \Omega,\\
\|\nabla(w_2 - w_2^*)\|_{L_\infty(\tilde\tau)} &\le C\varepsilon^{-1/2}(N^{-1}\max|\psi'|)^2, && \tau \subset \Omega,\\
\|w_{12} - w_{12}^*\|_{L_\infty(\tilde\tau)} &\le CN^{-3}, && \tau \subset \Omega_{11} \cup \Omega_{21},\\
\|(w_{12} - w_{12}^*)_x\|_{L_\infty(\tilde\tau)} &\le C\varepsilon^{-1}(N^{-1}\max|\psi'|)^2, && \tau \subset \Omega_{12} \cup \Omega_{22},\\
\|(w_{12} - w_{12}^*)_y\|_{L_\infty(\tilde\tau)} &\le C\varepsilon^{-1/2}(N^{-1}\max|\psi'|)^2, && \tau \subset \Omega
\end{aligned}$$

and

$$\|\nabla(u - u^*)\|_0 \le C\varepsilon^{-1/2}(N^{-1}\max|\psi'|)^2.$$

Proof. The proof is similar to the one of the Interpolation Theorem 2.6. □

The proof of the following lemma is deferred to Section 4.4.

Theorem 4.14. *Let T^N be an S-type mesh with $\sigma \ge 3$ and*

$$h_{N/2} \ge C\varepsilon N^{-1}. \tag{4.13}$$

Then depending on the numerical method used we have

$$\varepsilon^{1/2}\|\nabla u - Ru^N\|_0 \le C(N^{-2}\ln^{1/2} N + (N^{-1}\max|\psi'|)^2) \tag{4.14a}$$

for GFEM, SDFEM and GLSFEM and

$$\varepsilon^{1/2}\|\nabla u - Ru^N\|_0 \le C(N^{-3/2} + (N^{-1}\max|\psi'|)^2) \tag{4.14b}$$

for CIPFEM.

Remark 4.15. *The additional assumption* (4.13) *is satisfied for all meshes considered in this thesis. It is sufficient if the derivative of the mesh generating function ϕ is bounded in* $[1/4, 1/2]$ *from below by a constant independent of ε and N.*

Remark 4.16. *The analysis of Theorem 4.14 is involved because some patches consist of rectangles from the fine and the coarse mesh. The analysis can be simplified, if the operator is changed and a macro mesh is used.*
Let us assume N to be divisible by 8. Then we construct a macro mesh $\widetilde{T}^{N/2}$ as in Figure 4.1. Suppose $M \in \widetilde{T}^{N/2}$. As before, in the midpoints of the mesh rectangles the gradient $\gamma_{i,j} := \nabla w^N(x_{i-1/2}, y_{j-1/2})$ is computed for $w^N \in V^N$. Our recovery operator

$\widetilde{R}w^N$ *is locally the bilinear interpolation of these values on a macro cell* $M = [x_{i-1}, x_{i+1}] \times [k_{j-1}, k_{j+1}]$

$$
\begin{aligned}
(\widetilde{R}_M w^N)(x,y) := & \frac{4}{(h_i + h_{i+1})(k_j + k_{j+1})} \cdot \\
& \big((\gamma_{i,j}(x_{i+1/2} - x) + \gamma_{i+1,j}(x - x_{i-1/2}))(y_{j+1/2} - y) + \\
& (\gamma_{i,j+1}(x_{i+1/2} - x) + \gamma_{i+1,j+1}(x - x_{i-1/2}))(y - y_{j-1/2}) \big).
\end{aligned}
$$

The piecewise definition is extended to a global discontinuous function by setting

$$\big(\widetilde{R}w^N\big)(x,y) := \big(\widetilde{R}_M w^N\big)(x,y) \quad \textit{for } (x,y) \in M.$$

This new recovery operator has similar properties as the operator R *in Lemma 4.11, especially the same consistency for quadratic functions on* M*. Therefore, the quadratic interpolant* u^* *of* (4.12) *can be used with* $\tilde{\tau} = M$*.*
We estimate

$$\|\nabla u - \widetilde{R}u^N\|_0 \le \|\nabla(u - u^*)\|_0 + \|\widetilde{R}(u - u^*)^I\|_0 + \|\widetilde{R}(u^I - u^N)\|_0$$

and have the same results as in Theorem 4.14 without taking care of condition (4.13) *and macroelements crossing the transition line between fine and coarse mesh.*

4.3 Discontinuous recovery

In [13] we developed a general framework for recovery of a piecewise $\mathcal{Q}_p$-function that is globally discontinuous. We review the recovery method for $p = 1$.
Let us assume N to be divisible by 8 and construct a macro mesh $\widetilde{T}^{N/2}$ as in Figure 4.1. Define a family of operators $L^i_{k,t} : L_1(t_{i-1}, t_i) \to \mathbb{R}$ by

$$L^i_{k,t} v = \int_{t_{i-1}}^{t_i} \eta^i_k(t) v \, \mathrm{d}t, \; k = 0, 1$$

with the k-th Legendre polynomial $\eta^i_k(t)$ on $[t_{i-1}, t_i]$. Using these operators we construct 9 local degrees of freedom named $N^{i,j}_k : L_1(M_{i,j}) \to \mathbb{R}$ on $M_{i,j} = [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$.

$$
\begin{aligned}
N^{i,j}_1(v) &= (L^{i+1}_{0,x} + L^i_{0,x}) \circ (L^{j+1}_{0,y} + L^j_{0,y}) v & N^{i,j}_5(v) &= (L^{i+1}_{1,x} - L^i_{1,x}) \circ (L^{j+1}_{0,y} + L^j_{0,y}) v \\
& & N^{i,j}_6(v) &= (L^{i+1}_{1,x} - L^i_{1,x}) \circ (L^{j+1}_{1,y} + L^j_{1,y}) v \\
N^{i,j}_2(v) &= (L^{i+1}_{1,x} + L^i_{1,x}) \circ (L^{j+1}_{0,y} + L^j_{0,y}) v & N^{i,j}_7(v) &= (L^{i+1}_{1,x} - L^i_{1,x}) \circ (L^{j+1}_{1,y} - L^j_{1,y}) v \\
N^{i,j}_3(v) &= (L^{i+1}_{1,x} + L^i_{1,x}) \circ (L^{j+1}_{1,y} + L^j_{1,y}) v & N^{i,j}_8(v) &= (L^{i+1}_{1,x} + L^i_{1,x}) \circ (L^{j+1}_{1,y} - L^j_{1,y}) v \\
N^{i,j}_4(v) &= (L^{i+1}_{0,x} + L^i_{0,x}) \circ (L^{j+1}_{1,y} + L^j_{1,y}) v & N^{i,j}_9(v) &= (L^{i+1}_{0,x} + L^i_{0,x}) \circ (L^{j+1}_{1,y} - L^j_{1,y}) v
\end{aligned}
$$

In [13] it is shown, that for $M_{i,j}$ consisting of 4 congruent rectangles the sets $\{N^{i,j}_k\}_k$ are unisolvent and therefore a local basis $\psi_k \in \mathcal{Q}_2(M_{i,j})$, $k = 1, \ldots, 9$ exists with

$$N^{i,j}_k(\psi_l) = \begin{cases} 1, & k = l \\ 0, & k \ne l. \end{cases}$$

The same arguments hold for the macroelements M_{ij} considered here.
The recovery operator $P^d_{i,j} : L_1(M_{i,j}) \to \mathcal{Q}_2(M_{i,j})$ is characterised by

$$N^{i,j}_k(P^d_{i,j}v) = N^{i,j}_k(v),\ k = 1,\dots,9, \quad \Leftrightarrow \quad P^d_{i,j}v = \sum_{k=1}^{9} N^{i,j}_k(v)\psi_k$$

with a basis $\{\psi_k\}$ of $\mathcal{Q}_2(M_{i,j})$ that fulfills $N^{i,j}_k(\psi_{k'}) = 0$ for $k \neq k'$ and $N^{i,j}_k(\psi_k) = 1$.
This locally defined operator can be extended to a global operator P^d on Ω as usual.
Remark that the sets $\{N^{i,j}_k\}_k$ consist of 9 out of 16 possible combinations of the local integral means. Therefore this choice is not unique.
Let $\pi u \in V^{disc,N} := \{v \in L_2(\Omega) : v|_\tau \in \mathcal{Q}_1(\tau)\, \forall\, \tau \subset \Omega\}$ be the globally discontinuous local L_2-projection of u. Then the recovery operator P^d enjoys consistency

$$P^d u = P^d \pi u, \tag{4.15}$$

stability

$$\left|\left|\left|P^d v^N\right|\right|\right|_\varepsilon \le C \left|\left|\left|v^N\right|\right|\right|_\varepsilon \text{ for all } v^N \in V^{disc,N} \tag{4.16}$$

and the anisotropic error estimates

$$\|P^d u - u\|_{0,M} \le C(h_M^3 \|u_{xxx}\|_{0,M} + k_M^3 \|u_{yyy}\|_{0,M}), \tag{4.17a}$$

$$\|(P^d u - u)_x\|_{0,M} \le C(h_M^2 \|u_{xxx}\|_{0,M} + k_M^2 \|u_{xyy}\|_{0,M}) \tag{4.17b}$$

on all macro elements $M \subset \widetilde{T}^{N/2}$ with width h_M and height k_M, see [13]. Remark that the constants C in (4.16) and (4.17) depend on the mesh sizes inside M in the same way the constants in the analysis of the nodal biquadratic interpolation in Section 4.1 depend on them.

Remark 4.17. *Numerical simulations indicate a connection between the recovery operators P^d and $\widetilde{R}$ of Remark 4.16, namely*

$$\nabla P^d u^N = \widetilde{R} u^N,\ \forall u^n \in V^N. \tag{4.18}$$

Indeed we proof this in 1d for $M = [-h_0, h_1]$ and $v^N \in V_1^{disc,N}(M) = \{v \in L_2(M) : v|_{[-h_0,0]} \in \mathcal{P}_1([-h_0,0]), v|_{[0,h_1]} \in \mathcal{P}_1([0,h_1])\}$. The equivalence (4.18) follows immediately by $V^N \subset V^{disc,N}$ and the tensor-product character of the recovery operator and the mesh. Let $v^N \in V_1^{disc,N}(M)$ be given as

$$v^N(x) = \begin{cases} v_1 + \frac{x}{h_0}(v_1 - v_0), & x \in (-h_0, 0) \\ v_2 + \frac{x}{h_1}(v_3 - v_2), & x \in (0, h_1) \end{cases}$$

with $v_0, \dots, v_3 \in \mathbb{R}$. Then follows

$$N_1(v^N) = \int_{-h_0}^{h_1} \frac{1}{h_0 + h_1} v^N(x)\,\mathrm{d}x = \frac{h_0(v_0 + v_1) + h_1(v_2 + v_3)}{2},$$

$$N_2(v^N) = \int_{-h_0}^{0} \frac{h_0 + 2x}{h_0 + h_1} v^N(x)\,\mathrm{d}x + \int_0^{h_1} \frac{h_1 - 2x}{h_0 + h_1} v^N(x) = \frac{h_0^2(v_1 - v_0) + h_1^2(v_3 - v_2)}{6(h_0 + h_1)},$$

$$N_3(v^N) = -\int_{-h_0}^{0} \frac{h_0 + 2x}{h_0 + h_1} v^N(x)\,\mathrm{d}x + \int_0^{h_1} \frac{h_1 - 2x}{h_0 + h_1} v^N(x) = \frac{h_0^2(v_0 - v_1) + h_1^2(v_3 - v_2)}{6(h_0 + h_1)}$$

and for the derivatives of the basis functions holds

$$\begin{aligned}
\psi_1'(x) &= 0,\\
\psi_2'(x) &= \frac{3(h_0^4 + h_1^4 + 2x(h_0^3 - h_1^3))}{h_0^3 h_1^3},\\
\psi_3'(x) &= \frac{3(h_0^4 - h_1^4 + 2x(h_0^3 + h_1^3))}{h_0^3 h_1^3}.
\end{aligned}$$

Thus

$$(P^d v^N)' = N_2(v^N)\psi_2'(x) + N_3(v^N)\psi_3'(x) = \frac{(h_1 - 2x)(v_1 - v_0)}{h_0(h_0 + h_1)} + \frac{(h_0 + 2x)(v_3 - v_2)}{h_1(h_0 + h_1)}.$$

The recovery method using $\widetilde{R}$ *interpolates the derivatives in* $x = -h_0/2$ *and* $x = h_1/2$ *linearly. We obtain*

$$\widetilde{R}v^N = \frac{v_1 - v_0}{h_0}\frac{h_1 - 2x}{h_0 + h_1} + \frac{v_3 - v_2}{h_1}\frac{h0 + 2x}{h0 + h1} = (P^d v^N)'.$$

Theorem 4.18. *Let* T^N *be an S-type mesh with* $\sigma \geq 3$ *and* $h, k \leq CN^{-1}$. *Suppose*

$$\begin{aligned}
&\frac{\max\{h_i, h_{i+1}\}}{\min\{h_i, h_{i+1}\}} \leq p, \quad \textit{for all } i = 1, \dots, N/2 - 1,\\
&\frac{\max\{k_j, k_{j+1}\}}{\min\{k_j, k_{j+1}\}} \leq p, \quad \textit{for all } j = 1, \dots, N/4 - 1 \textit{ and } j = 3N/4 + 1, \dots, N - 1
\end{aligned}$$

for $1 < p \in \mathbb{R}$. *Then depending on the numerical method that generates the numerical solution* $u^N \in V^N$ *we have*

$$|||u - P^d u^N|||_\varepsilon \leq C(N^{-2}\ln^{1/2} N + (N^{-1}\max|\psi'|)^2) \tag{4.19a}$$

for GFEM, SDFEM and GLSFEM and

$$|||u - P^d u^N|||_\varepsilon \leq C(N^{-3/2} + (N^{-1}\max|\psi'|)^2) \tag{4.19b}$$

for CIPFEM.

Proof. We estimate the L_2-norm and the H^1-seminorm differently. Applying the triangle inequality, consistency (4.15) and stability (4.16) yields

$$\begin{aligned}
\|u - P^d u^N\|_0 &\leq \|u - P^d u\|_0 + \|P^d \pi u - P^d u^I\|_0 + \|P^d u^I - P^d u^N\|_0\\
&\leq \|u - P^d u\|_0 + C\|\pi u - u^I\|_0 + C\|u^I - u^N\|_0.
\end{aligned}$$

The first term is estimated locally by (4.17) and the last term–depending on the method–by Theorems 3.5, 3.9, 3.12 and 3.16, respectively.
With the triangle inequality the second term is bounded by

$$\|\pi u - u^I\|_0 \leq \|\pi u - u\|_0 + \|u - u^I\|_0.$$

On the reference element $\tilde{\tau}$ of $\tau \in T^N$ the interpolation error $\|u - u^I\|_{0,\tilde{\tau}}$ and the projection error $\|u - \pi u\|_{0,\tilde{\tau}}$ have similar estimates, see [1, Lemma 2.15] and [5, Theorem 4.6.11] respectively. Thus the results of Theorem 4.5 hold for both terms.
The H^1-seminorm is estimated using Remark 4.17

$$\|\nabla(u - P^d u^N)\|_0 = \|\nabla u - \widetilde{R} u^N\|_0$$

and Remark 4.16. Altogether proves Theorem 4.18. □

4.4 Proofs

Proof of Lemma 4.3.
i) The existence of the constant C_1 in (4.6a) is due to the equivalence of norms in finite dimensional spaces as $\hat{v} - \hat{P}\hat{u} \in \mathcal{Q}_2(\widetilde{M})$.
We estimate C_1 by defining linear functionals F_i fulfilling (4.5) for $m = 0$ and linear functionals G_i for $m = 1$. Start with the case $m = 0$—the L_2-norm—where $j = 9$ functionals are needed. Let

$$F_i(\hat{v}) = \hat{v}(P_i),\ i = 1, \dots, 9.$$

These functionals fulfil (4.5) obviously. Suppose $\hat{w} \in \mathcal{Q}_2(\widetilde{M})$. Using the Lagrange-basis $\{\hat{\varphi}_i\}$ with $\hat{\varphi}_i(P_i) = 1$ and $\hat{\varphi}_i(P_j) = 0$, $i \neq j$ yields

$$\|\hat{w}\|_{0,\widetilde{M}} \leq \sum_{i=1}^{9} |F_i(\hat{w})| \|\hat{\varphi}_i\|_{0,\widetilde{M}} \leq \max_{i=1,\dots,9} \|\hat{\varphi}_i\|_{0,\widetilde{M}} \sum_{i=1}^{9} |F_i(\hat{w})|.$$

Thus, in the L_2-case we have

$$C_1^0 = \max_{i=1,\dots,9} \|\hat{\varphi}_i\|_{0,\widetilde{M}} = \frac{16}{15} \frac{1}{(1-\tilde{a}^2)(1-\tilde{b}^2)}. \tag{4.20}$$

We see, that for $|\tilde{a}| \to 1$ or $|\tilde{b}| \to 1$ the constant C_1 becomes infinity, while for bounded $|\tilde{a}|, |\tilde{b}| \leq q < 1$ it is finite.
In the case $m = 1$ we show the proof for $\gamma = (1, 0)$—that means $\hat{D}^\gamma \hat{v} = \hat{v}_\xi$—only. For $\gamma = (0, 1)$ the proof is similar. Here $j = 6$ functionals are needed. Because in (4.5b) and (4.5c) the functionals are applied to derivatives, we define them as follows

$$G_i(\hat{v}) = \int_{P_i}^{P_{i+3}} \hat{v}(\xi, \eta)\, \mathrm{d}\xi.$$

Property (4.5a) follows right by definition while (4.5b) holds with an auxiliary function $g \in C(\mathbb{R})$ by

$$G_i((\hat{u} - \hat{P}\hat{u})_\xi) = (\hat{u} - \hat{P}\hat{u})|_{P_i}^{P_{i+3}} + g(\eta)|_{P_i}^{P_{i+3}} = 0$$

due to the interpolation property and P_i and P_{i+3} sharing the same value of η.

In order to show (4.5c) assume $G_i(\hat{w}_\xi) = 0$ for all $i = 1, \dots, 6$. Then

$$\begin{aligned} G_1(\hat{w}_\xi) = 0 &\Rightarrow \hat{w}(P_1) = \hat{w}(P_4) = \hat{w}(P_7) \Leftarrow G_4 = 0, \\ G_2(\hat{w}_\xi) = 0 &\Rightarrow \hat{w}(P_2) = \hat{w}(P_5) = \hat{w}(P_8) \Leftarrow G_5 = 0, \\ G_3(\hat{w}_\xi) = 0 &\Rightarrow \hat{w}(P_3) = \hat{w}(P_6) = \hat{w}(P_9) \Leftarrow G_6 = 0 \end{aligned}$$

and $\hat{w}(\xi,\eta) = \hat{w}(\eta)$. Therefore $\hat{w}_\xi = 0$ and the functionals fulfil (4.5).
Recall

$$(\hat{\varphi}_i + \hat{\varphi}_{i+3} + \hat{\varphi}_{i+6})_\xi = 0,\ i = 1,2,3.$$

With $\hat{w} = \sum_{i=1}^{9} \hat{w}(P_i)\hat{\varphi}_i$ follows

$$\begin{aligned} \hat{w}_\xi &= \sum_{i=1}^{3} (\hat{w}(P_i)\hat{\varphi}_i + \hat{w}(P_{i+3})\hat{\varphi}_{i+3} + \hat{w}(P_{i+6})\hat{\varphi}_{i+6})_\xi \\ &= \sum_{i=1}^{3} ((\hat{w}(P_i) - \hat{w}(P_{i+3}))(\hat{\varphi}_i)_\xi + (\hat{w}(P_{i+6}) - \hat{w}(P_{i+3}))(\hat{\varphi}_{i+6})_\xi) \end{aligned}$$

and therefore

$$\begin{aligned} \|\hat{w}_\xi\|_{0,\widetilde{M}} &\le \sum_{i=1}^{3} (|\hat{w}(P_i) - \hat{w}(P_{i+3})| \|(\hat{\varphi}_i)_\xi\|_{0,\widetilde{M}} + |\hat{w}(P_{i+6}) - \hat{w}(P_{i+3})| \|(\hat{\varphi}_{i+6})_\xi\|_{0,\widetilde{M}}) \\ &\le \max_{i=1,2,3,7,8,9} \|(\hat{\varphi}_i)_\xi\|_{0,\widetilde{M}} \underbrace{\sum_{i=1}^{3} (|\hat{w}(P_i) - \hat{w}(P_{i+3})| + |\hat{w}(P_{i+3}) - \hat{w}(P_{i+6})|)}_{= \sum_{i=1}^{6} |G_i(\hat{w}_\xi)|}. \end{aligned}$$

Thus, computing the maximum gives in case $m = 1$

$$C_1^1 = \frac{1}{6}\sqrt{3 + \frac{4}{(1-|\tilde{a}|)^2}}\sqrt{3 + \frac{4}{(1-|\tilde{b}|)^2}} \le \frac{1}{2} + \frac{2}{3(1-\max(|\tilde{a}|,|\tilde{b}|))^2}. \tag{4.21}$$

The combination of the results for $m = 0$ and $m = 1$ yields the first part of Lemma 4.3.

ii) The constant C_2 occurs in (4.6c) as a consequence of (4.5a). In case $m = 0$ we use the embedding $H^3(\widetilde{M}) \hookrightarrow C^0(\widetilde{M})$ to conclude

$$\|F_i\|_{H^3(\widetilde{M})'} \le C_e^0$$

and for $m = 1$ the embedding $H^2(\widetilde{M}) \hookrightarrow C^0(\widetilde{M})$ to get

$$\|G_i\|_{H^2(\widetilde{M})'} \le 2C_e^1.$$

Therein the embedding constants C_e^0 and C_e^1 do not depend on $|\tilde{a}|$ and $|\tilde{b}|$. We conclude

$$\sum_{i=1}^{9} |F_i(\hat{v}-\hat{u})| \leq \sum_{i=1}^{9} \|F_i\|_{H^3(\widetilde{M})'} \|\hat{v}-\hat{u}\|_{l,\widetilde{M}} \qquad \leq 9\, C_e^0 \|\hat{v}-\hat{u}\|_{l,\widetilde{M}}$$

and

$$\sum_{i=1}^{6} |G_i((\hat{v}-\hat{u})_\xi)| \leq \sum_{i=1}^{6} \|G_i\|_{H^2(\widetilde{M})'} \|(\hat{v}-\hat{u})_\xi\|_{l-1,\widetilde{M}} \leq 12\, C_e^1 \|(\hat{v}-\hat{u})_\xi\|_{l-1,\widetilde{M}}.$$

Thus C_2 is independent of $\tilde{a}$ and $\tilde{b}$. □

Proof of Lemma 4.8.
Equality (4.9) is a direct consequence of the definitions of Pv and v^I. They both interpolate at the mesh nodes.
The energy-norm stability estimate rests on the L_∞-stabilities (4.3) and (4.4). We have

$$\begin{aligned} |||Pv^N|||_{\varepsilon,M} &\leq C\,\mathrm{meas}^{1/2}\, M \left(\varepsilon \|\nabla(Pv^N)\|_{L_\infty(M)} + \|Pv^N\|_{L_\infty(M)}\right) \\ &\leq C\,\mathrm{meas}^{1/2}\, M \left(\varepsilon \|\nabla(v^N)\|_{L_\infty(M)} + \|v^N\|_{L_\infty(M)}\right). \end{aligned}$$

Now the L_∞-norms have to be estimated by L_2-norms.

1. Let us start with $\|v^N\|_{L_\infty(M)}$ on the reference macroelement.

   With the Lagrange-basis $\{\hat{\varphi}_i\}$ of piecewise bilinears on $\widetilde{M}$ we have $\hat{v}^N = \sum_{i=1}^{9} v_i \hat{\varphi}_i$.

   (a) Let $\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_1|$. On $[-1,\tilde{a}] \times [-1,\tilde{b}]$ follows

   $$\|\hat{v}^N\|^2_{0,[-1,\tilde{a}]\times[-1,\tilde{b}]} = \frac{(1+\tilde{a})(1+\tilde{b})}{18} f(v_1,v_2,v_4,v_5)$$

   with

   $$f(v_1,v_2,v_4,v_5) = (v_1+v_2)^2 + (v_1+v_4)^2 + (v_2+v_5)^2 + (v_4+v_5)^2 + v_1 v_5 + v_2 v_4.$$

   We compute the minimum of f with respect to v_1 and get

   $$f(v_1,v_2,v_4,v_5) \geq f\left(v_1, -\frac{1}{2}v_1, -\frac{1}{2}v_1, \frac{1}{4}v_1\right) = \frac{9}{8}v_1^2.$$

   Thus,

   $$\|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \|\hat{v}^N\|^2_{0,[-1,\tilde{a}]\times[-1,\tilde{b}]} \geq \frac{(1+\tilde{a})(1+\tilde{b})}{16} \|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}.$$

Similarly follows

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_3| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{(1+\tilde{a})(1-\tilde{b})}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})},$$

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_7| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{(1-\tilde{a})(1+\tilde{b})}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}$$

and

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_9| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{(1-\tilde{a})(1-\tilde{b})}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}.$$

(b) Now let $\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_2|$. Similar to (a) we bound

$$\begin{aligned}\|\hat{v}^N\|^2_{0,\widetilde{M}} &\geq \|\hat{v}^N\|^2_{0,[-1,\tilde{a}]\times[-1,\tilde{b}]} + \|\hat{v}^N\|^2_{0,[-1,\tilde{a}]\times[\tilde{b},1]}\\ &\geq \frac{(1+\tilde{a})(1+\tilde{b})}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})} + \frac{(1+\tilde{a})(1-\tilde{b})}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}\\ &= \frac{1+\tilde{a}}{8}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}.\end{aligned}$$

Analogously follows

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_4| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{1+\tilde{b}}{8}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})},$$

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_6| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{1-\tilde{b}}{8}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}$$

and

$$\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_8| \;\Rightarrow\; \|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{1-\tilde{a}}{8}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}.$$

(c) Finally, let $\|\hat{v}^N\|_{L_\infty(\widetilde{M})} = |v_5|$. We get in the same way as in (a)

$$\|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{1}{4}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})}.$$

Combining above estimates with the transformation (4.1) yields

$$\|\hat{v}^N\|^2_{0,\widetilde{M}} \geq \frac{(1-|\tilde{a}|)(1-|\tilde{b}|)}{16}\|\hat{v}^N\|^2_{L_\infty(\widetilde{M})} \Longrightarrow \|v^N\|^2_{L_\infty(M)} \leq \frac{64}{\operatorname{meas} M(1-|\tilde{a}|)(1-|\tilde{b}|)}\|v^N\|^2_{0,M}.$$

2. Consider the estimate for $\|(v^N)_x\|_{L_\infty(M)}$ on the reference macroelement. The estimate for the y-derivative can be derived analogously.

(a) Let $\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_4-v_1}{1+\tilde{a}}\right|$ and calculate

$$\|(\hat{v}^N)_\xi\|^2_{0,[-1,\tilde{a}]\times[-1,\tilde{b}]} = \frac{(1+\tilde{a})(1+\tilde{b})}{3} g(v_1, v_2, v_4, v_5)$$

with

$$g(v_1, v_2, v_4, v_5) = \left(\left(\frac{v_4 - v_1}{1+\tilde{a}}\right)^2 + \left(\frac{v_4 - v_1}{1+\tilde{a}}\right)\left(\frac{v_5 - v_2}{1+\tilde{a}}\right) + \left(\frac{v_5 - v_2}{1+\tilde{a}}\right)^2\right).$$

We compute the minimum of g with respect to $\|(v^N)_\xi\|_{L_\infty(\widetilde{M})}$ and obtain

$$\|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \|(\hat{v}^N)_\xi\|^2_{0,[-1,\tilde{a}]\times[-1,\tilde{b}]} \geq \frac{(1+\tilde{a})(1+\tilde{b})}{4}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})}.$$

Similarly follows

$$\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_6 - v_3}{1+\tilde{a}}\right| \Rightarrow \|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{(1+\tilde{a})(1-\tilde{b})}{4}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})},$$

$$\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_7 - v_4}{1-\tilde{a}}\right| \Rightarrow \|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{(1-\tilde{a})(1+\tilde{b})}{4}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})}$$

and

$$\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_9 - v_6}{1+\tilde{a}}\right| \Rightarrow \|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{(1-\tilde{a})(1-\tilde{b})}{4}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})}.$$

(b) Now let $\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_5-v_2}{1+\tilde{a}}\right|$. It follows

$$\|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{1+\tilde{a}}{2}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})}$$

and similarly

$$\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})} = \left|\frac{v_8 - v_5}{1+\tilde{a}}\right| \Rightarrow \|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{1-\tilde{a}}{2}\|(\hat{v}^N)_\xi\|_{L_\infty(\widetilde{M})}.$$

Combining these results with the transformation (4.1) gives

$$\|(\hat{v}^N)_\xi\|^2_{0,\widetilde{M}} \geq \frac{(1-|\tilde{a}|)(1-|\tilde{b}|)}{4}\|(\hat{v}^N)_\xi\|^2_{L_\infty(\widetilde{M})}$$
$$\Rightarrow \|(v^N)_x\|^2_{L_\infty(M)} \leq \frac{16}{\text{meas}\, M(1-|\tilde{a}|)(1-|\tilde{b}|)}\|(v^N)_x\|^2_{0,M}.$$

The proof is finished by

$$\begin{aligned}\left|\left|\left|Pv^N\right|\right|\right|_{\varepsilon,M} &\le C\,\mathrm{meas}^{1/2} M \left(\varepsilon\|\nabla(v^N)\|_{L_\infty(M)} + \|v^N\|_{L_\infty(M)}\right)\\ &\le C\left(\varepsilon\|\nabla(v^N)\|_{0,M} + \|v^N\|_{0,M}\right) \le C\left|\left|\left|v^N\right|\right|\right|_{\varepsilon,M}\end{aligned}$$

with a finite constant C for $|\tilde{a}|, |\tilde{b}| \le q < 1$. □

Proof of Theorem 4.14.
We start with the triangle inequality

$$\|\nabla u - Ru^N\|_0 \le \|\nabla(u-u^*)\|_0 + \|R(u-u^*)^I\|_0 + \|R(u^I - u^N)\|_0 \tag{4.22}$$

where the consistency of R was used. The first term can be estimated by Lemma 4.12 and the last term—depending on the method used—by Theorems 3.5, 3.9, 3.12 and 3.16, respectively. So only the second term needs to be estimated now. We follow the proof of [26, Lemma 4] but estimate the parts of $u = v + w_1 + w_2 + w_{12}$ separately.
Start with $\tilde{w} = v + w_2$. With L_∞-stability (2.10) of the nodal bilinear interpolation and the L_∞-stability of R we have

$$\begin{aligned}\|R(\tilde{w}-\tilde{w}^*)^I\|_{0,\tau} &\le \mathrm{meas}^{1/2}\,\tau\|R(\tilde{w}-\tilde{w}^*)^I\|_{L_\infty(\tau)} \le C\,\mathrm{meas}^{1/2}\,\tau\|\nabla(\tilde{w}-\tilde{w}^*)^I\|_{L_\infty(\tilde{\tau})}\\ &\le C\,\mathrm{meas}^{1/2}\,\tau\|\nabla(\tilde{w}-\tilde{w}^*)\|_{L_\infty(\tilde{\tau})} \le C\,\mathrm{meas}^{1/2}\,\tau\varepsilon^{-1/2}(N^{-1}\max|\psi'|)^2\end{aligned}$$

for any $\tau \in T^N$.
Similarly we get for $w = w_1 + w_{12}$ and $R(w-w^*)^I = (R_1, R_2)$

$$\begin{aligned}\|R_2\|_{0,\tau} &\le \mathrm{meas}^{1/2}\,\tau\|((w-w^*)^I)_y\|_{L_\infty(\tilde{\tau})} \le C\,\mathrm{meas}^{1/2}\,\tau\|(w-w^*)_y\|_{L_\infty(\tilde{\tau})}\\ &\le C\,\mathrm{meas}^{1/2}\,\tau\varepsilon^{-1/2}(N^{-1}\max|\psi'|)^2,\ \forall\tau\in T^N.\end{aligned}$$

For $\tau \subset \Omega_{12} \cup \Omega_{22}$ holds

$$\|R_1\|_{0,\tau} \le C\,\mathrm{meas}^{1/2}\,\tau\|(w-w^*)_x\|_{L_\infty(\tilde{\tau})} \le C\,\mathrm{meas}^{1/2}\,\tau\varepsilon^{-1}(N^{-1}\max|\psi'|)^2.$$

Let $S := [x_{N/2}, x_{N/2+1}] \times [0,1]$. For $\tau \subset \Omega_{11} \cup \Omega_{21} \setminus S$ an inverse inequality gives

$$\begin{aligned}\|R_1\|_{0,\tau} &\le \mathrm{meas}^{1/2}\,\tau\|((w-w^*)^I)_x\|_{L_\infty(\tilde{\tau})} \le C\,\mathrm{meas}^{1/2}\,\tau N\|(w-w^*)^I\|_{L_\infty(\tilde{\tau})}\\ &\le C\,\mathrm{meas}^{1/2}\,\tau N^{-\sigma+1} \le C\,\mathrm{meas}^{1/2}\,\tau N^{-2}.\end{aligned}$$

The region S consists of a ply of at most CN elements. Here we apply an inverse inequality again, but have to estimate more carefully due to elements of $\tilde{\tau}$ reaching into $\Omega_{12} \cup \Omega_{22}$. We obtain for $\tau \subset S$

$$\begin{aligned}\|R_1\|_{0,\tau} &\le C\|((w-w^*)^I)_x\|_{0,\tilde{\tau}} \le C\sum_{\tau\subset\tilde{\tau}}\frac{\mathrm{meas}^{1/2}\,\tau}{h(\tau)}\|w-w^*\|_{L_\infty(\tau)}\\ &\le C\varepsilon^{-1/2}\|w-w^*\|_{L_\infty(\tilde{\tau})} \le C\varepsilon^{-1/2}N^{-3}\end{aligned}$$

due to the assumption $h_{N/2} \ge C\varepsilon N^{-1}$.
Recalling $\|R(u-u^*)^I\|_0^2 = \sum_{\tau\in T^N}\|R(u-u^*)^I\|_{0,\tau}^2$ and $\mathrm{meas}(\Omega_{12}\cup\Omega_{22}) \le C\varepsilon\ln N$, the estimates above give

$$\|R(u-u^*)^I\|_0 \le C\varepsilon^{-1/2}\ln^{1/2}N(N^{-1}\max|\psi'|)^2.$$

Together with the estimates for the first and the last term of (4.22) we are done. □

Chapter 5

Numerical results

The theoretical results of the previous chapters will be illustrated by numerical simulations of the following test problem with a given exact solution.

Problem:

$$-\varepsilon\Delta u - (1+x)(1+y)u_x + (1+xy)u = f \quad \text{in } \Omega = (0,1)^2 \tag{5.1a}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{5.1b}$$

with f such that

$$u(x,y) = \left(\cos(x\pi/2) - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}}\right)(1+y^2)\frac{(1-e^{-y/\varepsilon^{1/2}})(1-e^{-(1-y)/\varepsilon^{1/2}})}{(1-e^{-1/(2\varepsilon^{1/2})})^2}.$$

The solution u exhibits two characteristic layers and one outflow layer of exponential type. Assumption 2.1 applies immediately.

We start our numerical survey in Section 5.1 addressing the question, in which regions it is useful to stabilise. In the second part convergence and supercloseness for the different numerical methods are illustrated. Finally, in Section 5.3 different recovery techniques are compared.

In all numerical computations the MATLAB environment and its built-in solver biCGstab were used to solve the resulting sparse linear system. We apply an incomplete LU-decomposition as preconditioner for the iterative solver.
The stiffness matrix is assembled locally on each cell of the mesh T^N. There the scalar products, e.g.

$$(cu^N, v^N)_\tau = \int_\tau (cu^N v^N) \quad \text{and} \quad (f, v^N)_\tau = \int_\tau (fv^N), \qquad \tau \in T^N,\ u^N, v^N \in V^N$$

are approximated. Numerical simulations indicate that a midpoint rule is advantageous in comparison to a Gaussian rule, i.e.,

$$(cu^N, v^N)_\tau \approx \bar{c}\int_\tau u^N v^N \qquad \text{vs.} \quad (cu^N, v^N)_\tau \approx \frac{\operatorname{meas}\tau}{4}\sum_{i,j=0}^{1} c(x_i^g, y_j^g)u^N(x_i^g, y_j^g)v^N(x_i^g, y_j^g)$$

and

$$(f, v^N)_\tau \approx \bar{f} \int_\tau v^N \qquad \text{vs.} \qquad (f, v^N)_\tau \approx \frac{\operatorname{meas} \tau}{4} \sum_{i,j=0}^{1} f(x_i^g, y_j^g) v^N(x_i^g, y_j^g)$$

where $\{(x_i^g, y_j^g)\}_{i,j=0,1}$ denotes the Gaussian points in τ and $\bar{g}$ the value of a function g at the center of τ. The remaining integrals are computed explicitly because the integrands are polynomial. The background of the different behaviour of the quadrature methods is topic of ongoing research.

As parameter of the S-type meshes we chose $\sigma = 3$. In Chapter 3 it was shown, that $\sigma = 5/2$ is sufficient for supercloseness of the numerical methods. Nevertheless, the analysis of the postprocessing methods in Chapter 4 needs $\sigma \geq 3$.

The constant C^* in the stabilisation parameters of SDFEM and GLSFEM is set either to 1 if in the corresponding region shall be stabilised or to 0 if not.

Let the perturbation parameter ε = 1e-8. This choice is sufficiently small to bring out the singularly perturbed nature of (5.1). It is much smaller than N^{-1} in our simulations too, and additionally, the maximum mesh sizes satisfy $h, k \leq CN^{-1}$.

We do not look at ε-uniformity of our results, as this was already addressed in different publications, see for example [9, 10, 21].

5.1 Stabilisation regions

In [11] we showed for CIPFEM, that stabilisation inside all subregions can influence the iterative solvers performance negatively. Therein it was advised to stabilise in the coarse-mesh and the parabolic-layer region only.

For SDFEM we will analyse the different strategies where to stabilise in this section. GLSFEM can be analysed similarly yielding the same conclusions.

Fix $N = 512$ and let the threshold for the preconditioner be 5e-3, that means in the incomplete LU-decomposition all values below 5e-3 are set to zero. In order to analyse the numerical costs, the number of non-zeros in the preconditioner—nnz(L+U)—will be considered as measure for the memory consumptions. The time needed to generate the preconditioner and to solve the system—t_{pre} and t_{sol}, respectively— are taken as second measure of costs.

The results on a Shishkin mesh are given in Figure 5.1. Clearly the amount of memory used is almost the same for GFEM and SDFEM with stabilisation inside the coarse-mesh region Ω_{11} only. If we stabilise in the parabolic layer region too, less memory is needed and the system is solved faster.

The stabilisation parameter in the exponential layer region is $\delta_{12} = \varepsilon(h + N^{-1} \max |\psi'|)^2$ and in the corner layer region $\delta_{22} = \varepsilon^{3/4} N^{-2}$. They are much smaller than the natural diffusion ε. From this point of view they could be set to zero.

Indeed, Figure 5.1 shows that additional stabilisation inside $\Omega_{12} \cup \Omega_{22}$ has no further advantage in terms of numerical costs.

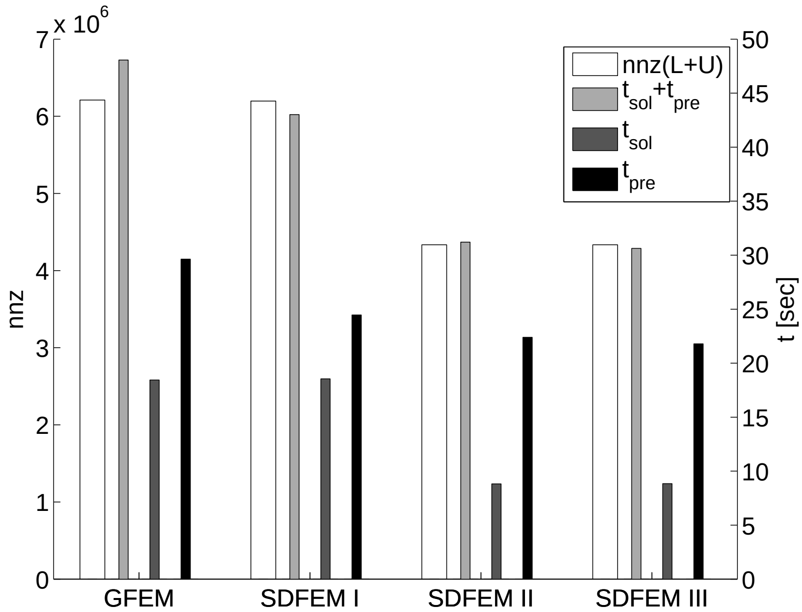


Figure 5.1: Numerical behaviour on a Shishkin mesh for GFEM and SDFEM with stabilisation in I: Ω_{11}, II: $\Omega_{11} \cup \Omega_{21}$ and III: Ω

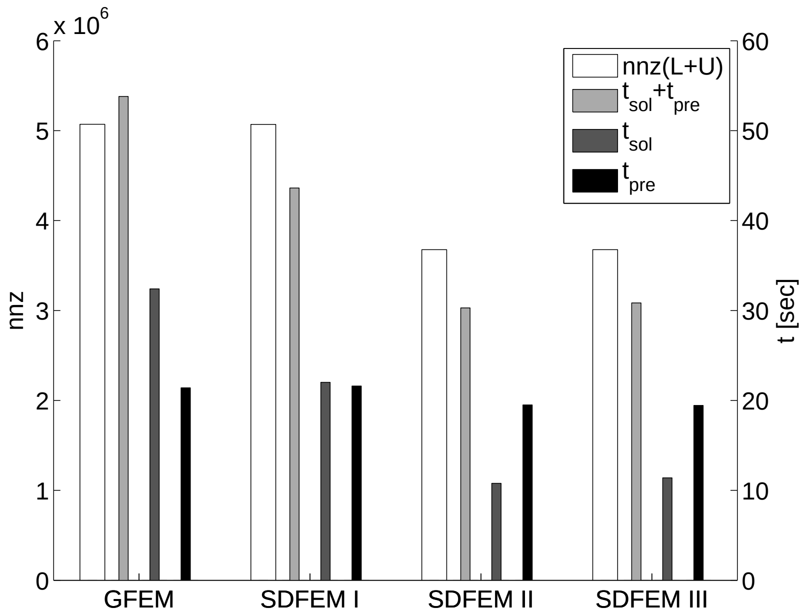


Figure 5.2: Numerical behaviour on a modified BS-mesh for GFEM and SDFEM with stabilisation in I: Ω_{11}, II: $\Omega_{11} \cup \Omega_{21}$ and III: Ω

For the other S-type meshes a similar analysis can be done, see Figure 5.2 for the modified BS-mesh. Again we have minimal costs for stabilisation in the regions $\Omega_{11} \cup \Omega_{21}$.
Thus, in the following tests stabilisation of SDFEM and GLSFEM will only be activated in $\Omega_{11} \cup \Omega_{21}$.

5.2 Supercloseness and convergence

Let us investigate numerically the convergence of the four methods considered in this thesis. We will look at the (estimated) orders of convergence on various S-type meshes separately, starting with the standard Shishkin mesh.
Remarks 3.6, 3.10, 3.13 and 3.17 apply and convergence of order $N^{-1}\ln N$ in the ε-weighted energy norm can be expected for each corresponding numerical solution u^N.
Supercloseness is provided for GFEM by Theorem 3.5 giving $|||u^I - u^N|||_\varepsilon \le C(N^{-1}\ln N)^2$, for SDFEM by Theorem 3.9 with $|||u^I - u^N|||_{SD} \le C(N^{-1}\ln N)^2$ and for GLSFEM by Theorem 3.12 with $|||u^I - u^N|||_{GLS,2} \le C(N^{-1}\ln N)^2$. For CIPFEM Theorem 3.16 states $|||u^I - u^N|||_{CIP} \le CN^{-3/2}$.
Table 5.1 shows the results achieved by these methods on a family of Shishkin meshes for $N = 128, \dots, 2048$. Therein, for each method the convergence and supercloseness errors are given with the corresponding estimated order of convergence (EOC).
The EOC is calculated for a given sequence of errors E_N by

$$EOC = \frac{\ln(E_N) - \ln(E_{2N})}{\ln 2}.$$

In Table 5.2 for the model sequences $E_N = N^{-1}\ln N$ and $E_N^2 = (N^{-1}\ln N)^2$ the computed EOCs are given for comparison.

	GFEM				SDFEM			
N	$\lvert\lvert\lvert u - u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I - u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u - u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I - u^N\rvert\rvert\rvert_{SD}$	
128	4.633e-2	0.81	2.418e-3	1.61	4.633e-2	0.81	2.419e-3	1.61
256	2.651e-2	0.83	7.902e-4	1.66	2.651e-2	0.83	7.905e-4	1.66
512	1.492e-2	0.85	2.500e-4	1.70	1.492e-2	0.85	2.500e-4	1.70
1024	8.290e-3	0.86	7.712e-5	1.73	8.290e-3	0.86	7.713e-5	1.73
2048	4.560e-3		2.332e-5		4.560e-3		2.332e-5	
	GLSFEM				CIPFEM			
N	$\lvert\lvert\lvert u - u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I - u^N\rvert\rvert\rvert_{GLS,2}$		$\lvert\lvert\lvert u - u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I - u^N\rvert\rvert\rvert_{CIP}$	
128	4.633e-2	0.81	2.420e-3	1.61	4.633e-2	0.81	2.418e-3	1.61
256	2.651e-2	0.83	7.906e-4	1.66	2.651e-2	0.83	7.902e-4	1.66
512	1.492e-2	0.85	2.501e-4	1.70	1.492e-2	0.85	2.500e-4	1.70
1024	8.290e-3	0.86	7.714e-5	1.73	8.290e-3	0.86	7.712e-5	1.73
2048	4.560e-3		2.332e-5		4.560e-3		2.332e-5	

Table 5.1: Supercloseness and convergence on Shishkin meshes

N	$EOC(N^{-1}\ln N)$	$EOC((N^{-1}\ln N)^2)$
128	0.81	1.61
256	0.83	1.66
512	0.85	1.70
1024	0.86	1.73
2048		

Table 5.2: Estimated orders of convergence for model errors of Shishkin mesh

Comparing the figures in Table 5.2 with the respective ones in Table 5.1, we see that Table 5.1 is a clear illustration of the expected orders for GFEM, SDFEM and GLSFEM. For CIPFEM we observe almost first order convergence and second order supercloseness, too. This indicates, that the supercloseness result of Theorem 3.16 may not be sharp.

The definition of the polynomial S-mesh contains an additional parameter m. For our calculations let $m = 3$ to illustrate the better performance for $m > 1$ compared to the original Shishkin mesh.
Our theoretical results state convergence $|||u - u^N|||_\varepsilon \le CN^{-1}\ln^{1/3} N$ for all numerical solutions u^N and supercloseness of order $N^{-2}\ln^{2/3} N$ except for CIPFEM, where again $|||u^I - u^N|||_{CIP} \le CN^{-3/2}$ was proved only.
Table 5.3 lists the results of the simulations on a family of polynomial S-type meshes for $N = 128, \dots, 2048$ and in Table 5.4 the EOCs for model sequences $E_N = N^{-1}\ln^{1/3} N$ and $E_N^2 = N^{-2}\ln^{2/3} N$ are provided. As in the case of Shishkin meshes, these rates are comparable to those of Table 5.3 even for CIPFEM.

	GFEM				SDFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{SD}$	
128	1.605e-2	0.94	3.917e-4	1.88	1.605e-2	0.94	3.979e-4	1.88
256	8.395e-3	0.94	1.067e-4	1.89	8.395e-3	0.94	1.081e-4	1.89
512	4.366e-3	0.95	2.876e-5	1.90	4.366e-3	0.95	2.909e-5	1.90
1024	2.261e-3	0.95	7.713e-6	1.89	2.261e-3	0.95	7.774e-6	1.86
2048	1.167e-3		2.083e-6		1.167e-3		2.144e-6	
	GLSFEM				CIPFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{GLS,2}$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{CIP}$	
128	1.605e-2	0.94	4.001e-4	1.88	1.605e-2	0.94	3.919e-4	1.88
256	8.395e-3	0.94	1.086e-4	1.89	8.395e-3	0.94	1.067e-4	1.89
512	4.366e-3	0.95	2.922e-5	1.90	4.366e-3	0.95	2.877e-5	1.88
1024	2.261e-3	0.95	7.806e-6	1.86	2.261e-3	0.95	7.835e-6	1.94
2048	1.167e-3		2.153e-6		1.167e-3		2.047e-6	

Table 5.3: Supercloseness and convergence on polynomial S-meshes

N	$EOC(N^{-1}\ln^{1/3} N)$	$EOC(N^{-2}\ln^{2/3} N)$
128	0.94	1.87
256	0.94	1.89
512	0.95	1.90
1024	0.95	1.91
2048		

Table 5.4: Estimated orders of convergence for model errors of polynomial S-mesh

Finally, consider the two meshes combining ideas of Bakhvalov and Shishkin. Table 5.5 shows the errors on a family of modified B-S-meshes and Table 5.6 for the B-S-mesh. Clearly convergence of order N^{-1} as stated by Remarks 3.6, 3.10, 3.13 and 3.17 is observed. Moreover, supercloseness of order N^{-2} can be seen where Theorems 3.5, 3.9 and 3.12 state $\mathcal{O}\left(N^{-2}\ln^{1/2}N\right)$ and Theorem 3.16 states $\mathcal{O}\left(N^{-3/2}\right)$ only. These results indicate, that the $\ln^{1/2}N$ factor is not necessary and that CIPFEM has similar supercloseness properties as the other methods. Further work is needed to improve these theoretical results.

	GFEM				SDFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{SD}$	
128	1.158e-2	0.97	1.898e-4	1.94	1.158e-2	0.97	2.034e-4	1.95
256	5.915e-3	0.98	4.932e-5	1.95	5.915e-3	0.98	5.274e-5	1.96
512	3.009e-3	0.98	1.273e-5	1.96	3.009e-3	0.98	1.356e-5	1.97
1024	1.525e-3	0.98	3.277e-6	1.95	1.525e-3	0.98	3.467e-6	1.97
2048	7.716e-4		8.508e-7		7.716e-4		8.831e-7	
	GLSFEM				CIPFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{GLS,2}$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{CIP}$	
128	1.158e-2	0.97	2.057e-4	1.95	1.158e-2	0.97	1.901e-4	1.95
256	5.915e-3	0.98	5.331e-5	1.96	5.915e-3	0.98	4.936e-5	1.96
512	3.009e-3	0.98	1.371e-5	1.97	3.009e-3	0.98	1.272e-5	1.96
1024	1.525e-3	0.98	3.502e-6	1.97	1.525e-3	0.98	3.261e-6	1.97
2048	7.716e-4		8.916e-7		7.716e-4		8.333e-7	

Table 5.5: Supercloseness and convergence on modified B-S-meshes

	GFEM				SDFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{SD}$	
128	1.163e-2	0.99	1.834e-4	1.99	1.163e-2	0.99	1.977e-4	1.98
256	5.837e-3	1.00	4.627e-5	1.99	5.837e-3	1.00	4.996e-5	1.99
512	2.924e-3	1.00	1.162e-5	1.99	2.924e-3	1.00	1.256e-5	1.99
1024	1.463e-3	1.00	2.928e-6	1.95	1.463e-3	1.00	3.157e-6	2.00
2048	7.321e-4		7.576e-7		7.321e-4		7.889e-7	
	GLSFEM				CIPFEM			
N	$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{GLS,2}$		$\lvert\lvert\lvert u-u^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u^I-u^N\rvert\rvert\rvert_{CIP}$	
128	1.163e-2	0.99	1.998e-4	1.98	1.163e-2	0.99	1.840e-4	1.99
256	5.837e-3	1.00	5.049e-5	1.99	5.837e-3	1.00	4.634e-5	1.99
512	2.924e-3	1.00	1.269e-5	1.99	2.924e-3	1.00	1.163e-5	2.00
1024	1.463e-3	1.00	3.193e-6	2.00	1.463e-3	1.00	2.912e-6	1.97
2048	7.321e-4		7.973e-7		7.321e-4		7.415e-7	

Table 5.6: Supercloseness and convergence on B-S-meshes

5.3 Postprocessing

In this section different postprocessing methods are applied to the numerical solutions of the test problem (5.1). We restrict our numerical experiments to the GFEM solution. Simulations using the numerical solutions of the stabilised methods show similar results. In Chapter 4 two postprocessing procedures to recover the exact solution and two to recover the gradient are described. In Table 5.7 the results of these methods applied to the GFEM solution on a family of Shishkin meshes are given.
In the left block Pu^N denotes the standard biquadratic interpolation on a macro mesh considered in Section 4.1 and $P^d u^N$ the biquadratic projection from Section 4.3. Both recovery techniques give comparable results with almost second order superconvergence—$\mathcal{O}\left((N^{-1}\ln N)^2\right)$—in the ε-weighted energy norm.
The right block focuses on recovery of the gradient. In its first column the gradient of the biquadratic interpolation operator Pu^N is used. Due to Remark 4.17 the gradient $\nabla P^d u^N$ is equal to $\widetilde{R}u^N$ whose results are given in the last column. In between the original gradient-recovery method from [26] is applied. All approaches give the same estimated order of superconvergence $\mathcal{O}\left((N^{-1}\ln N)^2\right)$. Moreover, ∇Pu^N and $\widetilde{R}u^N$ give better results than Ru^N, by a factor of approximately 4 in the ε-weighted H^1-seminorm.
Due to our theoretical results, we expect on B-S-meshes even better errors and convergence rates. In Table 5.8 the results for GFEM on a family of B-S-meshes are presented in the same manner as before. Clearly superconvergence of order N^{-2} can be observed for all postprocessing methods without the logarithmic factor of the supercloseness result. Again Ru^N is slightly inferior to the other gradient-recovery techniques, by a factor 4 in the ε-weighted H^1-seminorm.

	GFEM									
N	$\lvert\lvert\lvert u - Pu^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u - P^d u^N\rvert\rvert\rvert_\varepsilon$		$\sqrt{\varepsilon}\lVert\nabla(u - Pu^N)\rVert_0$		$\sqrt{\varepsilon}\lVert\nabla u - Ru^N\rVert_0$		$\sqrt{\varepsilon}\lVert\nabla u - \widetilde{R}u^N\rVert_0$	
128	2.507e-3	1.65	2.509e-3	1.65	2.506e-3	1.65	9.901e-3	1.55	2.507e-3	1.65
256	8.002e-4	1.67	8.008e-4	1.67	8.000e-4	1.67	3.385e-3	1.63	8.002e-4	1.67
512	2.510e-4	1.70	2.512e-4	1.70	2.509e-4	1.70	1.096e-3	1.68	2.510e-4	1.70
1024	7.721e-5	1.73	7.729e-5	1.73	7.720e-5	1.73	3.421e-4	1.72	7.723e-5	1.73
2048	2.333e-5		2.335e-5		2.332e-5		1.040e-4		2.333e-5	

Table 5.7: Postprocessing on Shishkin meshes

	GFEM									
N	$\lvert\lvert\lvert u - Pu^N\rvert\rvert\rvert_\varepsilon$		$\lvert\lvert\lvert u - P^d u^N\rvert\rvert\rvert_\varepsilon$		$\sqrt{\varepsilon}\lVert\nabla(u - Pu^N)\rVert_0$		$\sqrt{\varepsilon}\lVert\nabla u - Ru^N\rVert_0$		$\sqrt{\varepsilon}\lVert\nabla u - \widetilde{R}u^N\rVert_0$	
128	1.845e-4	1.99	1.845e-4	1.99	1.837e-4	1.99	1.911e-3	2.95	1.834e-4	1.99
256	4.635e-5	1.99	4.634e-5	2.00	4.613e-5	1.99	2.466e-4	2.30	4.606e-5	1.99
512	1.163e-5	1.99	1.162e-5	1.99	1.157e-5	1.99	5.006e-5	2.04	1.155e-5	1.99
1024	2.930e-6	1.95	2.919e-6	1.96	2.911e-6	1.96	1.214e-5	2.00	2.901e-6	1.96
2048	7.577e-7		7.521e-7		7.475e-7		3.031e-6		7.430e-7	

Table 5.8: Postprocessing on B-S-meshes

	GFEM			
N	$\|u-u^N\|_0$	$\|u^N-Pu^N\|_0$	$\sqrt{\varepsilon}\|\nabla(u-u^N)\|_0$	$\sqrt{\varepsilon}\|\nabla(u^N-Pu^N)\|_0$
128	9.267e-5	1.359e-4	4.632e-2	4.623e-2
256	3.004e-5	4.438e-5	2.651e-2	2.649e-2
512	9.465e-6	1.399e-5	1.492e-2	1.492e-2
1024	2.915e-6	4.304e-6	8.290e-3	8.290e-3
2048	8.870e-7	1.299e-6	4.560e-3	4.560e-3

Table 5.9: A-posteriori error estimates on a Shishkin mesh

Comparing Tables 5.1 and 5.7 as well as Tables 5.6 and 5.8 it is observable that the supercloseness error dominates the superconvergence error.

Remark 4.10 states, that $\left|\left|\left|u^N - Pu^N\right|\right|\right|_\varepsilon$ can be used as an asymptotically exact *a-posteriori* error estimator. Its L_2- and ε-weighted H^1-norm of $u^N - Pu^N$ are estimators too. Table 5.9 displays the actual error $u - u^N$ in the L_2-norm and the ε-weighted H^1-seminorm compared to the estimators for GFEM on a family of Shishkin meshes. Clearly, the estimators capture the error well with

$$\|u-u^N\|_0 \approx 1.47\|u^N - Pu^N\|_0 \quad \text{and} \quad \varepsilon^{1/2}\|\nabla(u-u^N)\|_0 \approx \varepsilon^{1/2}\|\nabla(u^N-Pu^N)\|_0.$$

The other postprocessing approaches can be used as a-posteriori error-estimators in above sense as well.

Outlook

We have seen, that standard Galerkin FEM and the presented stabilisation techniques provide for bilinear elements superclose numerical solutions. For higher order elements only in the case of SDFEM similar results are known. Further research is needed to investigate other stabilisation methods in the bilinear and higher order case with respect to the supercloseness property.
Moreover, some presented theoretical results seem not to be sharp, namely

- the logarithmic factor $\ln^{1/2} N$ in the supercloseness estimates for Galerkin FEM and its stabilisations, see Theorem 3.5,
- the reduced supercloseness order 3/2 for CIPFEM, see Theorem 3.16,
- the logarithmic factor $\ln^{1/2} N$ in the recovery of the gradient, see Theorem 4.14

and how to improve them is an open task.
For SDFEM and GLSFEM the stabilisation parameter inside the parabolic layers can contain the factor $\varepsilon^{-1/2}$ without disturbing the supercloseness properties. This seems to indicate a weakness of the energy norm in presence of parabolic boundary layers. Therefore, analysis for supercloseness properties should also be done in the L_∞-norm.

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Affirmation

1. Hereby I affirm, that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this nor in any other country.
2. The present thesis has been developed at the Institut für Numerische Mathematik, Departement of Mathematics, Faculty of Science, TU Dresden under supervision of Jun.-Prof. T. Linß and Prof. H.-G. Roos.
3. There have been no prior attempts to obtain a PhD degree at any university.
4. I accept the requirements for obtaining a PhD (Promotionsordnung) of the Faculty of Science of the TU-Dresden, issued March 20, 2000 with changes in effect since April 16, 2003.

Versicherung

1. Hiermit versichere ich, dass ich die vorliegene Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilsmittel angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.
2. Die vorliegende Arbeit wurde am Institut für Numerische Mathematik, Fachbereich Mathematik, Fakultät Mathematik und Naturwissenschaften, TU Dresden unter Betreuung von Jun.-Prof. T. Linß und Prof. H.-G. Roos angefertigt.
3. Es wurden zuvor keine Promotionsvorhaben unternommen.
4. Ich erkenne die Promotionsordnung der Fakultät Mathematik und Naturwissenschaften der TU-Dresden vom 20. März 2000, in der geänderten Fassung mit Gültigkeit vom 16. April 2003 an.

Dresden, den 08. April 2008.