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Aspects of Non-Perturbative Renormalization

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Abstract

At the heart of every quantum field theory, there is the need for renormalization. In the framework of the well-known perturbative renormalization procedure, the potentials – or interaction Lagrangians – are decomposed in a Taylor series in the fields; this series generates the vertices of the theory. If the Taylor expansion contains only a finite number of terms (this is the "normal" case), then each interaction vertex can be treated independently. However, there are theories which cannot be considered in this traditional way. In some theories, symmetries of the Lagrangian impose the requirement of taking infinitely many interaction vertices into account; any truncation of these infinite series would lead to an unacceptable violation of essential symmetries of the model. Still, these problematic theories have important physical realizations. For example, the Kosterlitz–Thouless phase transition can be described with theories of this type. In addition, the sine–Gordon model, whose interaction Lagrangian is periodic in the field variable, and the planar spin model, the X–Y model which is equivalent to the lattice regularized sine-Gordon model, and the two-dimensional Coulomb gas describing a plasma of static electric charges also belongs to this "problematic" class of theories.

Therefore, in some sense the requirement of perturbative renormalizability imposes too restrictive constraints on the "theory space", i.e. on the properties of theories which can be regarded as physically meaningful. The general question arises: which interaction Lagrangians are physically sensible? Are perturbatively renormalizable theories the only physically "important" models? Field theoric models with periodicity in the internal space (in the space of the field variable) are perturbatively non-renormalizable, but they are of relevance both for statistical physics and quantum field theory.

The goal of this Thesis is to give a presentation of some key issues regarding the non-perturbative renormalization of the periodic scalar field theories. As an example of the non-perturbative methods, we use the differential renormalization group approach, particularly the Wegner–Houghton and the Polchinski renormalization group equations, in order to investigate the renormalization of a one-component periodic scalar field theory. The Wegner–Houghton equation provides a resummation of the loop-expansion, and the Polchinski equation is based on the resummation of the perturbation series. Therefore, these equations are exact in the sense that they contain all quantum corrections. In the framework of these renormalization group equations, field theories with periodic self interaction can be considered without violating the essential symmetry of the model: the periodicity.

Both methods – the Wegner–Houghton and the Polchinski approaches – are inspired by Wilson's blocking construction in momentum space: the Wegner–Houghton method uses a sharp momentum cut-off and thus cannot be applied directly to non-constant fields (contradicts with the "derivative expansion"); the Polchinski method is based on a smooth cut-off and thus gives rise naturally to a "derivative expansion" for varying fields. However, the shape of the cut-off function (the "scheme") is not fixed a priori within Polchinski's ansatz. In this thesis, we compare the Wegner–Houghton and the Polchinski equation; we demonstrate the consistency of both methods for near-constant fields in the linearized level and obtain constraints on the regulator function that enters into Polchinski's equation. Analytic and numerical results are presented which illustrate the renormalization group flow for both

methods. We also briefly discuss the relation of the momentum-space methods to real-space renormalization group approaches. For the two-dimensional Coulomb gas (which is investigated by a real-space renormalization group method using the dilute-gas approximation), we provide a systematic method for obtaining higher-order corrections to the dilute gas result.

Contents

1 Introduction

1.1 Motivation and Thesis Overview

In this section we provide a brief summary of the motivations and the particular questions investigated in this Thesis. Our main goal is to give a presentation of the key issues in the non-perturbative renormalization of the periodic scalar field theory [1–4]. The renormalization of periodic potentials represents a challenge in quantum field theory. The usual perturbative renormalization is based on a Taylor expansion of the potential which generates the vertices of the model. Truncating the Taylor expansion, each monomial of the field is treated independently. This strategy is not necessarily sufficient when the symmetries of the theory require to take infinitely many interaction vertices into account because truncating of the infinite series violates these symmetries. The periodic field theory belongs to the type of the models which perturbatively cannot be considered in the traditional manner because, (i) any truncation of the Taylor expansion of the model violates the essential symmetry of the model, periodicity, and (ii) if the Taylor expansion of the potential is not truncated, one has to deal with infinitely many vertices which makes the perturbative treatment unacceptable.

The renormalization of the periodic real scalar field theory is considered by means of the differential renormalization group (RG) approach. The RG methods provide us with a tool to investigate the renormalization of a periodic potential without breaking the periodicity of the model. Particularly, we use the Wegner-Houghton and the Polchinski RG approach as applied in the momentum space in order to consider the renormalization of the periodic scalar field theory. The particular questions investigated by us can be summarized as follows,

1. We tried to find the connection between symmetries of the action and the shape of the effective potential under certain approximation. In general, the RG transformations should preserve the symmetries of the action, in our case the periodicity. In the local-potential approximation the solution of the RG equation reduces to the determination of the scale-dependent blocked potential $V_k(\phi_0)$ which tends to the effective potential $V_{\text{eff}}(\phi_0)$ moving the scale k to zero $k \to 0$. As it is well-known (see e. g. [5]), the effective potential should be convex (in the field variable). It is concluded, that for the periodic field theory, the convexity and periodicity are so strong constraints on the effective potential that it should be a constant, flat potential.

2. In order to check this general statement in a particular case, the Wegner-Houghton and the Polchinski RG equations are specified to the periodic scalar field theory. It is proven, that both the Wegner–Houghton and the Polchinski equations retain the periodicity of the potential.

3. Another natural question to ask what are the fixed points and the scaling operators and critical exponents of the periodic scalar field theory. Therefore, the linearized RG flow is determined around the UV trivial Gaussian fixed-point $[V^*(\phi_0) = 0]$. It is found that in the leading order of the gradient expansion, the Polchinski and the Wegner–Houghton equations give the same critical exponents and scaling operators around the trivial Gaussian fixed point. For dimension $d = 2$ the well-known phase structure and the fixed-point of the sine-Gordon model (which is the simplest realization of the periodic potentials) are recovered in a rather straightforward way. Using the Fourier-decomposition of the dimesnionless periodic potential,

$$\tilde{V}_k(\phi_0) = \sum_{n=1}^{\infty} \tilde{u}_n(k) \cos\left(n\tilde{\beta}\tilde{\phi}_0\right), \tag{1.1}$$

the linearized solutions for the dimensionless Fourier amplitudes are,

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda) \left(\frac{k}{\Lambda}\right)^{\alpha_d \tilde{\beta}^2 n^2 - d}, \tag{1.2}$$

where $2\pi/\tilde{\beta}$ is the length of period in the internal space, $\alpha_d = \Omega_d/(2(2\pi)^d)$ with the entire solid angle Ω_d in dimensions d. For the dimension $d = 2$, the fixed-point of the sine-Gordon model corresponds to the critical value of the length of period, $\tilde{\beta}_c^2 = 2/\alpha_2 = 8\pi$. The linearized solution for the Fourier amplitudes is found to be qualitatively the same for any dimensions $d \geq 2$. The dependence of the type (1.2) is essentially the same in any dimensions with the exception of its 'trivial' dependence on the entire solid angle in α_d.

Around the Gaussian fixed-point the scaling operators for the field-dependent wavefunction renormalization are found irrelevant for both phases of the model. The position of the fixed-point and the phase structure is independent of the choice of the regulator; this is independent of the renormalization scheme, while the actual flow depends on it.

4. Finally, the nonlinear RG flow is discussed by means of numerical integration of the Wegner-Houghton and the Polchinski RG equations. In the leading order of the gradient expansion, the solution for the periodic potential is compared to the known results reproduced by us for a non-periodic, polynomial potential. (This latter is chosen to approximate to periodic potential with a Taylor series at its minimum.) There is a remarkable difference in the behavior of the theory with a periodic potential and that with the corresponding polynomial potential: all the dimensionful coupling constants of the periodic potential tend to zero in contrary to those of the polynomial potential which remain finite as $k \to 0$. Therefore, the blocked periodic potential flattens out with decreasing scale k.

In the next-to-leading order of the gradient expansion, including the field-independent wavefunction renormalization, the results of two different RG methods are compared: the renormalization of the two-dimensional Coulomb gas by means of the real-space RG approach using the dilute gas approximation and the renormalization of the equivalent sine-Gordon model in the framework of the Wegner–Houghton RG approach. For small

values of the fugacity of the Coulomb gas, the two methods agree, and a systematic way of obtaining higher-order corrections to the dilute gas approximation is given. It is concluded, that at the linear level, the RG flow equations are identical irrespective of the blocking procedure by which they are obtained. Nevertheless, the RG equations obtained by Polchinski's method are rather different to those obtained by the real-space RG approach for the two-dimensional Coulomb gas. The differences between the various approaches occur when the non-linearities are kept that are responsible for the violation of the UV scaling laws.

The first part of the introduction in chapter 1, gives a general outline of the renormalization group approach to periodic scalar field theory. A few typical cases in which a field theorical model with periodicity in internal space is central issue, are mentioned. In the second part of chapter 1, the main properties of phase transitions and critical phenomena are discussed. In chapter 2, the pertinent features of the sine-Gordon model, X–Y model and the two-dimensional Coulomb gas, and the equivalences between these models are presented. The well-known phase structure of the two-dimensional Coulomb gas (or equivalently, the vortex gas) and the application of the vortex dynamics in experiments are given in chapter 2 as well. Chapter 3 contains a brief introduction into Wegner–Houghton's and Polchinski's renormalization group (RG) methods. The gradient expansion of the blocked action is used to reduce the Wegner–Houghton and the Polchinski RG equations into differential equations for the potential and the wavefunction renormalization. In chapter 4, these RG equations are applied for the periodic scalar field theory. The consequence of periodicity and convexity of the effective potential is also discussed. In the second part of chapter 4, the renormalization scheme-dependence of the Polchinski equation is considered. In chapter 5, the linearized RG flow is given around the Gaussian fixed-point and the periodic scaling operators of the potential are classified for dimensions $d = 2$ and $d > 2$ as well. The periodic scaling operators of the wavefunction renormalization are also determined in chapter 5. The nonlinear flow is discussed in chapter 6 by means of numerical integration of the Wegner–Houghton and the Polchinski RG equations. Finally, chapter 7 is for the summary of our findings.

In Apps. A–D, the conventions and symbols and the details of the derivation of the RG equations for periodic scalar field theory are described.

1.2 General Outline of the Renormalization of Periodic Potentials

In quantum field theory the usual strategies to investigate the dynamics of a field theorical model is based on the Taylor expansion of the potential. Truncating the Taylor expansion only polynomial interactions are taken into account. In this scheme, each monomial of the field variable (e.g. $(g_2/2)\phi^2$, $(g_4/4!)\phi^4$, $(g_6/6!)\phi^6$ etc.) corresponds to a vertex V in the Feynman graph representation of the theory. These vertices are treated independently. The theories with polynomial interactions can be classified into renormalizable, non-renormalizable or super-renormalizable theories according to the canonical dimensions of the vertices appearing in the model. The canonical dimension (see e. g. [6,7]) of the vertex $V_{\phi^n} \sim \int \mathrm{d}^d x\, \phi^n$

is defined as

$$\delta(V_{\phi^n}) = -d + n[\phi]_c \tag{1.3}$$

where $[\phi]_c$ is the canonical dimension of the field variable, [e.g. for scalar fields $[\phi]_c = (d-2)/2$ for dimension d]. If there is at least one vertex V whose canonical dimension is positive $\delta(V) > 0$ then the theory is non-renormalizable [6–8]. For super-renormalizable models all the vertices have negative canonical dimensions. If there is a vertex with zero canonical dimension $\delta(V) = 0$ then the theory is renormalizable. In the framework of the usual perturbation theory non-renormalizable theories cannot be considered consistently because a new type of vertex (interaction) should be introduced in each order of the perturbative expansion. This makes the perturbative treatment impossible, because the renormalization of the action would involve an infinite number of counterterms. Renormalizability imposes a strong constraint on the "theory space". Does it mean, that only perturbatively renormalizable interaction Lagrangians should be considered? Are they the only physically meaningful theories? This strategy of the classification of theories into renormalizable, super-renormalizable and non-renormalizable ones, where the vertices are treated independently, is not necessarily sufficient for any type of interaction. Let us take e.g. a field theory with an action exhibiting periodicity in internal space, like the sine-Gordon model and its generalizations. Such periodic field theories which are of relevance to physics, represent examples where the perturbative treatment for renromalizability does not work due to the following reasons:

1. Periodic potentials cannot be considered in this traditional perturbative manner because any truncation of the Taylor expansion of the model violates an essential symmetry of the problem: the periodicity.

2. If the Taylor expansion of the potential is not truncated, it generates infinitely many vertices. For dimension $d > 2$, if these vertices are treated independently, one has to deal with infinitely many irrelevant interaction terms (e.g. for dimension $d = 4$ the canonical dimension of the vertices V_{ϕ^n} is positive $\delta(V_{\phi^n}) = n - 4$ if $n > 4$). Therefore, the periodic field theory is found to be non-renormalizable in the perturbative sense. However, the symmetry (periodicity) imposes a strong constraint on this infinite series of non-renormalizable operators. They can be summed up and considered as one composite operator. In this case, the parameters of the theory can be chosen in such a way, that the periodic field theory becomes renormalizable.

Still field theorical models with periodic self-interaction are of relevance for physics. The dynamics of the periodic scalar field theory and the equivalent statistical models (e.g. sine-Gordon model [9], X–Y model [10] and the two-dimensional Coulomb gas [11] etc.) are investigated in several paper in great detail. These models play an important role in the phenomenology of superfluid He^4 films [12–14], and they can be of relevance in particle physics [17, 18]. Therefore, models with periodic self-interaction are physically meaningful but they cannot considered in the usual perturbative manner.

In this thesis, the renormalization of the periodic potential is investigated in the framework of the Euclidean one-component scalar field theory by means of the differential renormalization group (RG) approach [20–22]. The renormalization group (RG) method can be applied to

different scenarios [8]: to obtain the fixed-points and the scale dependence of the coupling constants of the theory; to perform at least a partial resummation of the perturbative expansion; or as a systematic non-perturbative method in order to handle highly non-linear, strongly correlated systems. An important property of the RG method that it provides us with a tool to consider periodic potentials without breaking the symmetry of the model, the periodicity. The renormalization group transformations can be realized by the Wilson's blocking procedure [20], which is a successive elimination of the degrees of freedom of the system.

Field theorical models with periodicity in internal space are of relevance both for statistical and quantum field theory. Below we mention a few typical cases in which a field theorical model with periodicity in internal space is a central issue.

1.2.1 Sine-Gordon Model in Dimension $d = 2$

In the framework of Wilson's renormalization group approach [20], the fixed-points, the scaling operators and generally the dynamics of the scalar field theory are discussed in the literature in great detail [8, 23–26]. These results are obtained for the scalar field theory which contains polynomial self-interactions. Nevertheless, the symmetries of the action like periodicity, can play an important role in the dynamics of the model, modifying the renormalization of the system, implying new phases, new fixed-points with new critical exponents. The simplest example for periodic potentials is the two-dimensional sine-Gordon model [9] which is a one-component scalar field theory with periodic self-interaction defined by the Euclidean action:

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial\phi)^2 + u_1 \cos(\beta\phi)\right], \tag{1.4}$$

with the coupling constants u_1 and β. The dynamics of the sine-Gordon model have received special attention due to its non-linear nature [6]. The classical sine-Gordon model possesses stationary, as well as time-dependent solutions called solitons, which represent stable configurations with a well defined energy [6,10]. The solutions have other important properties, e.g. since solitons are the solutions of non-linear wave equations, the superposition principle does not hold for them. Therefore, when two solitons meet the resultant wave form is a complicated one but then the solitons are separated out again, they somehow pass through one another [6].

1.2.2 X–Y Model and Coulomb gas in Dimension $d = 2$

The sine-Gordon model can be mapped into the statistical physical system, the X–Y model [10]. The X–Y model with an external magnetic field consists of classical two-component spins where the magnitude of the spin is $|\boldsymbol{S}_x| = 1$ at each site. The model is given by the action [10, 11, 13]:

$$S = \frac{1}{T} \sum_{<x,x'>} \boldsymbol{S}_x \cdot \boldsymbol{S}_{x'} + \frac{1}{T} \sum_x \boldsymbol{S}_x \cdot \boldsymbol{h}$$

$$= \frac{1}{T} \sum_{<x,x'>} \cos(\theta_x - \theta_{x'}) + \frac{h}{T} \sum_x \cos(\theta_x) \tag{1.5}$$

with the temperature T, the external field $h = |\boldsymbol{h}|$, and the angle θ of each spin with an arbitrarily chosen direction. In the model there exist topological excitations, called vortices, which interact via Coulomb interaction [11]. Therefore, the vortex gas can be thought as a gas of static electric charges (plasma) in dimension $d = 2$ so thus the X–Y model can be mapped by means of the Villain-transformation to a Coulomb gas [10, 11]. Although such a mapping is only valid up to irrelevant interaction terms, it is believed that these models (the sine-Gordon, the X–Y planar models and many other) belong to the same universality class, namely to that of the two-dimensional Coulomb gas. During the last three decades the renormalization of the two-dimensional Coulomb gas was investigated in a great detail by means of the differential renormalization group (RG) method, using the dilute-gas approximation [10, 27, 28]. Several intuitive approaches exist by which one has tried to go beyond the dilute gas approximation [29–31]. However, these attempts to improve the dilute gas approximation do not fit in a systematic scheme.

1.2.3 Periodic Haar-measure Potential

One reason why the periodic field theory may be of relevance in particle physics is quark confinement which is one of the basic features of strong interaction physics [17, 18]. Nevertheless, it has not yet a direct proof in the continuum QCD. The strong interaction which acts between the constituents of the hadrons (quarks and gluons) is a consequence of the local $SU_c(3)$ symmetry. According to non-perturbative lattice results [15, 16] the confinement - deconfinement phase transition is accompanied by the dynamical breaking of global center symmetry ($Z_3 \subset SU_c(3)$). An attempt to attack the confinement problem in the framework of QCD is based on the implementation of global center symmetry in the partition function [17–19]. This leads to an effective QCD action incorporating the cut-off dependent periodic Haar-measure potential. In $SU(2)$ the Haar-measure is a periodic function of the zero-th component of the vector potential [19] and after exponentialization it becomes a new additional term in the usual QCD action. Since the quark-confinement is a low energy effect it arises, therefore, the general question of the behavior of periodic potentials under RG transformations.

The goal of this thesis is to investigate the renormalization of the periodic one-component scalar field theory, (particularly the sine-Gordon model and the equivalent Coulomb gas) by means of the differential renormalization group method. The periodic fixed points of the RG transformations and the different phases of the periodic field theory are discussed and the periodic scaling operators and critical exponents are determined by solving the linearized renormalization group flow given around the Gaussian fixed-point. In order to make clear the interpretation of our findings obtained for the periodic scalar field theory, the main properties of phase transitions occurring in statistical physics and in field theory, are described briefly in the next section of the introduction.

1.3 Phase Transitions and the Renormalization Group Method

The renormalization group (RG) method has been originally introduced in hydrodynamics and later used to describe the critical phenomena of statistical systems near their phase transition point. In this Thesis, the RG method is applied for the periodic quantum field theory. Quantum field theories and the critical statistical systems have a common property. The quantum field as well as the critical statistical system has infinitely many relevant (important) degrees of freedom; the quantum fluctuations and the thermal fluctuations, respectively. The $d+1$ dimensional (d space-like and 1 time-like dimensional) quantum field theory is equivalent to a statistical system taken in $d+1$ space-like dimensions. Therefore, the understanding of the critical behavior of statistical systems close to their phase transitions provides us with a useful method to consider the phase structure of the equivalent quantum field theorical models. In order to give an introduction and summarize the main features of critical behavior and theory of phase transitions, here we follow the argumentation of Ref. [7, 32].

1.3.1 Order of the Phase Transitions

In the thermodynamical limit the statistical systems are determined by their thermodynamical potential Φ. The absolute minima of Φ corresponds to the state of equilibrium of the system. The thermodynamical potential is a continuous function of the parameters of the system (e.g. $\Phi = \Phi(T, p)$ with the temperature T, the pressure p). Therefore, if Φ_1 and Φ_2 are the thermodynamical potentials of the two different phases of the model, the following relation holds

$$\Phi_1(T, p) = \Phi_2(T, p). \tag{1.6}$$

This implies that a curve $p = p(T)$ curve on the (p – T) plane separating the two phases of the model. Crossing this separator $p(T)$ the system undergoes a phase transition. Due to the theorem of Ehrenfest, the phase transitions can be classified according to the partial derivatives of the thermodynamical potential Φ. The phase transition is of the first, second, third etc. order if the first, second, third etc, partial derivatives of Φ are discontinuous, respectively.

1. In case of first order phase transition, the first derivatives of Φ, e.g. the entropy $S = \partial G/\partial T$ or the volume of the system $V = \partial G/\partial p$ is discontinuous (where $\Phi = G$ is the free enthalpy of the system). Therefore, the energy of the system is changed under the phase transition by the latent heat:

$$Q = T_c(S_1 - S_2) = T_c \left(\left.\frac{\partial G}{\partial T}\right|_1 - \left.\frac{\partial G}{\partial T}\right|_2 \right), \tag{1.7}$$

with the critical temperature T_c.

2. Under second order phase transitions, the second derivatives of Φ (e.g. the magnetic susceptibility or the heat capacity), are discontinuous or singular at the critical temperature T_c. This has an important physical meaning. Since the susceptibility and the heat capacity diverge at T_c they have a critical asymptotic scaling behavior. Near T_c, they are power-law functions of the reduced temperature $t = (T - T_c)/T_c$ and the external field h. The exponents of these power-law functions (called critical exponents) fulfill certain relations found to be universal. Universality, can be understood as the independence of these relations of the particular choice of the system. The critical exponents can differ in various models but the relations between them are universal. The physical systems belong to the same universality class if their critical exponents are the same. Therefore, it is important to know what are the critical exponents of the model because they determine the behavior of the system near the phase transition.

3. One can define 'infinite' order phase transitions where each derivative of the thermodynamical potential Φ is continuous. An example is the Kosterlitz–Thouless phase transition [11, 27]. Although the derivatives of Φ are continuous, they diverge at T_c. In the case of the Kosterlitz–Thouless phase transition the type of the divergences are changed in the system undergoing phase transition.

In order to define the critical exponents and to discuss critical phenomena first we investigate the existence of long-range order in physical systems.

1.3.2 Long-range Order

In the systems undergoing second order phase transitions, usually one can define an ordered and a disordered phase. The ordered phase usually situated at low temperatures, below the critical temperature $T < T_c$, and the disordered phase corresponds to high temperatures $T > T_c$. What does it mean 'ordered' phase? The ordering can be thought as spatial ordering (e. g. long-range order) when some of the symmetries (e. g. rotational symmetry) of the model, is broken spontaneously in the ordered phase and remains unbroken in the disordered phase. The correlation function of the local physical quantity $X(\boldsymbol{r})$

$$G(\boldsymbol{r_1}, \boldsymbol{r_2}) \equiv \langle\, [X(\boldsymbol{r_1}) - \langle X \rangle\,]\, [X(\boldsymbol{r_2}) - \langle X \rangle\,]\, \rangle \tag{1.8}$$

can be used to determine long-range order. For isotropic and homogeneous systems the correlation function only depends on the magnitude of the relative distance $r = |\boldsymbol{r_1} - \boldsymbol{r_2}|$. There is long-range order in the system if the correlation function $G(r)$ tends to a finite but non-zero value if $r \to \infty$. This is, that local quantities situated infinitely far from each other are correlated. Therefore, a characteristic length ξ can be defined as a maximal distance within the correlation function differs from zero. This is the correlation length. In the long-range ordered phase, ξ diverges $\xi \to \infty$.

There is another important physical quantity which is used to distinguish between the ordered and the disordered phases of the model, called the order parameter $\Delta(T, h)$. The order parameter is zero in the disordered phase and different from zero in the ordered phase. It is increasing continuously moving the temperature far from the phase transition point in the ordered phase. For example, for ferromagnetic systems the order parameter

can be defined as the magnitude of the spontaneous magnetic momentum of the system $\Delta \equiv |M|$ at zero external magnetic field $h = 0$. In the disordered phase of the system there are small ordered and disordered domains so thus the average magnetic momentum of the system is zero. Moving close to the phase transition point one can find more and larger ordered domains. Finally after the phase transition in the ordered phase there is only one infinitely large ordered 'domain'. Therefore the spontaneous magnetic momentum of the system becomes non-zero.

The phase transitions strongly depend on the dimension of the system. The Mermin–Wagner theorem [33] states that models with short-range type interactions (where the number of the dynamical variable is $n \geq 2$) cannot possess spatial long-range order in dimension $d = 1$ and $d = 2$ at any finite temperature. This is because in low dimensions, the low frequency thermal fluctuations are strong enough to destroy the long-range order. For example, in two-component spin models (with short-range type interactions) the spontaneous magnetic momentum of the system is always found to be zero at finite temperature. Therefore, the order parameter of these systems introduced in a traditional way always vanishes. At a first sight, it could appear a contradiction that one can find phase transition in two dimensional systems with short-range type interactions (e.g. two-dimensional Coulomb gas, X–Y planar model etc.). This phase transition is of higher than second order. In the case of infinite order phase transitions (e.g. Kosterlitz–Thouless transition) there is a spatial ordering in the system but not of the long-range type one. This is a topological ordering.

In order to describe the properties of these phase transitions (second and higher order) in the next subsection we discuss the critical behavior of various physical quantities e.g. order parameter, correlation length etc. near the phase transition point.

1.3.3 Critical Phenomena

It is argued that in case of second order phase transitions various physical quantities (e.g. heat capacity, susceptibility etc.) have an asymptotic scaling behavior near the phase transition point T_c. They are power-law functions of the reduced temperature $t = (T - T_c)/T_c$ and the external field h. This is critical behavior. The correlation function $G(r)$, the correlation length ξ and the order parameter Δ have also critical scaling behavior near T_c. This asymptotic scaling behavior can be found in physically different systems. The exponents of the power-law functions (the critical exponents) can be different in various models but there are relations between these exponents (scaling laws) which are found to be universal. One can classify the models into different universality classes according to their critical exponents. Systems belonging to the same universality class, have the same critical exponents. The definition of the critical exponents are the following:

1. The critical exponent β is determined by the temperature-dependence of the order parameter $\Delta(t)$ at zero external fields in the ordered phase of the model:

$$\Delta(t, h) \equiv \left.\frac{\partial G}{\partial h}\right|_{p,T} \sim (-t)^{\beta}, \qquad \text{if} \quad h \to 0 \tag{1.9}$$

where G is the free enthalpy ($\Phi = G$) and $t = (T - T_c)/T_c$ is the reduced temperature.

2. The critical exponent δ is defined via the dependence of the order parameter Δ on the external field h at the critical temperature T_c:

$$\Delta(t=0,h) \sim h^{1/\delta}. \tag{1.10}$$

3. The temperature-dependence of susceptibility (which is the second partial derivative of the thermodynamical potential) determines the critical exponent γ:

$$\chi = \left.\frac{\partial \Delta}{\partial h}\right|_{p,T} = -\left.\frac{\partial^2 G}{\partial h^2}\right|_{p,T} \sim |t|^{-\gamma}, \qquad \text{if} \quad h \to 0. \tag{1.11}$$

4. The critical exponent α is defined via the temperature-dependence of the heat capacity:

$$C = T\left.\frac{\partial S}{\partial T}\right|_{p,h} = -T\left.\frac{\partial^2 G}{\partial T^2}\right|_{p,h} \sim |t|^{-\alpha}. \tag{1.12}$$

The critical exponents are not independent, one can find certain relations between them. These relations are universal, they are the same for various physical systems:

$$\gamma = \beta(\delta - 1), \tag{1.13}$$

$$2 = \alpha + \beta(\delta + 1). \tag{1.14}$$

Therefore, according to these scaling laws there are only two independent critical exponents. Further critical exponents can be defined via the correlation function of the order parameter at vanishing external field:

1. The correlation function $G(r)$ is a power-law function of the distance r at the critical temperature T_c. The definition for η is:

$$G(r, t=0) \sim r^{-(d-2+\eta)}, \qquad \text{if} \quad r \to \infty. \tag{1.15}$$

2. In the disordered phase ($t > 0$) the correlation function has an exponential asymptotic behavior near T_c:

$$G(r,t) \sim e^{-r/\xi}, \qquad \text{if} \quad r \to \infty. \tag{1.16}$$

This is the definition of the correlation length ξ. In the disordered phase the correlation length has also a power-law asymptotic behavior:

$$\xi \sim r^{-\nu}, \qquad \text{if} \quad t \to 0^+. \tag{1.17}$$

In the ordered phase the correlation length ξ tends to infinity, $\xi \to \infty$.

There are relations between the critical exponents of the correlation function and the correlation length but these scaling laws depend on the dimension d of the system:

$$\nu\, d = 2 - \alpha, \tag{1.18}$$

$$\gamma = (2 - \eta)\nu. \tag{1.19}$$

In case of higher than second order phase transitions the critical behavior, therefore, the critical exponents can be different. These scaling properties are the signatures of the phase transitions. For example, for infinite order phase transitions (e.g. Kosterlitz–Thouless phase transition) the correlation length ξ is infinite in the ordered phase, but decays exponentially in the disordered phase. Therefore one cannot introduce the critical exponent ν in the usual manner ($\xi \sim r^{-\nu}$).

1.3.4 Scale invariance

It is argued, that systems undergoing phase transitions can be classified according to their critical behavior near their phase transition point. The critical exponents determining the scaling properties of the model, can differ in various systems, but the relations between them are found to be universal. What is the physical reason for the critical behavior? The answer is scale invariance. The systems near their phase transition points are scale invariant. Changing the observational scale (e.g. changing the lattice site a), the thermodynamical potential of the model remains unchanged:

$$\Phi'(t', h') = \Phi(t, h), \tag{1.20}$$

where Φ is the thermodynamical potential of the original system depending on the reduced temperature t and the external field h, and the rescaled thermodynamical potential Φ' depends on the rescaled variables t' and h'. Therefore, the partition function of the model remains unchanged. One can find the analogy to quantum field theory where the generating functional is invariant under changing the observational scale (moving the momentum cut-off). Due to scale invariance (1.20), the thermodynamical potential is a homogeneous function of the reduced temperature t and the external field h,

$$\Phi(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda \Phi(t, h), \tag{1.21}$$

with arbitrary a_t, a_h and λ. This relation can be understood by the Wilson-Kadanoff blocking construction [20]. Let us consider a classical spin system on the lattice. In one blocking step, the lattice size of the original system is rescaled $a' = b\, a$. In the blocked system, one can define a block of spins with the new lattice size a', which contains b^d lattice points of the original system. In every block the spins of the original system are replaced by the 'average' of the spins. Every physical quantity is rescaled according to the new lattice size. The

thermodynamical potential Φ' of one block, is equal to the original Φ multiplied with the number of the lattice points in the block,

$$\Phi'(t', h') = b^d \, \Phi(t, h). \tag{1.22}$$

Due to scale invariance, the following relations hold for the blocked reduced temperature and the blocked external field

$$t' = b^{a_t d} t, \qquad h' = b^{a_h d} h. \tag{1.23}$$

Inserting Eq. (1.23) into the relation (1.22), the homogenity (1.21) of the thermodynamical potential is obtained.

The homogenity of the thermodynamical potential (1.21) implies critical scaling properties. The critical exponents can be derived from the relation (1.21) by derivation with respect to the external field or the reduced temperature. For example, the critical exponent δ is obtained in a following way. The equation (1.21) is differentiated with respect to the external field h

$$\frac{\partial \Phi(\lambda^{a_t} t, \lambda^{a_h} h)}{\partial h} = \lambda^{a_h} \Delta(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda \Delta(t, h). \tag{1.24}$$

Since λ is arbitrary, it can be defined via $1 = \lambda^{a_h} h$. Introducing $\lambda = h^{-1/a_h}$ into the equation (1.24) and setting $t = 0$, the equation reduces to

$$\Delta(0, 1) h^{(1-a_h)/a_h} = \Delta(0, h) \tag{1.25}$$

where $\Delta(0, 1)$ is constant. The critical exponent δ is obtained from (1.25) $\delta = a_h/(1-a_h)$. In this manner, every critical exponent can be derived from the relation (1.21). Therefore, we conclude the discussion of phase transitions with the important statement, that the scale invariance of the thermodynamical potential implies the critical behavior of the systems near their phase transition point.

2 The Coulomb gas and the Sine-Gordon Model

Several different models, like the sine-Gordon, Thirring, and the X–Y planar models etc. belong to the same universality class, namely to that of the two-dimensional Coulomb gas (plasma). The sine-Gordon model which is the one-component scalar field theory with periodic self-interaction in two dimensions, can be mapped exactly into the classical two-component spin-model called X–Y model with an external field. It can be shown with the help of the Villain transformation, that both models are equivalent to the two-dimensional Coulomb gas (two-dimensional plasma) which contains static positive and negative charges interacting with each other via the Coulomb interaction. During the last two decades the renormalization of the two-dimensional Coulomb gas was investigated in great detail using the dilute gas approximation. In this chapter the definition and the pertinent features of the sine-Gordon and the X–Y models, and the well-known phase structure of the two dimensional Coulomb gas are presented.

2.1 Model Equivalences

The dynamics generated by a periodic potential represents a challenge in quantum field theory because the Taylor expansion cannot be used to obtain the solution since it violates the essential symmetry of the problem, the periodicity. The simplest example for periodic potentials is the two-dimensional sine-Gordon model which is the one-component scalar field theory, described by the Euclidean action,

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)\right], \tag{2.1}$$

with $\mu = 0, 1$ and the scalar field $\phi(x_0, x_1)$, using the convention $\partial_\mu = \partial/(\partial x_\mu)$ and $x^\mu = x_\mu$, where only a single mode of the Fourier decomposition of the periodic self-interaction is kept:

$$V(\phi) = u_1 \cos(\beta\phi) \tag{2.2}$$

with the following "coupling constants": the Fourier amplitude u_1 and the length of period $\Delta = 2\pi/\beta$. The particular example reflects the invariance of the action under a shift of the field variable at each spacetime point with the same constant value $\phi(x) \to \phi(x) + \Delta$,

that is to say, its periodicity in internal space (which is the space where the field variable is represented). The dynamics and the renormalization of the model have been discussed by means of perturbative expansion [9] in the powers of u and by means of a semi-classical approach [34]. It was shown by Coleman [9], that the sine-Gordon model can be renormalized in every order of perturbative expansion through normal ordering in the weak coupling phase when $\beta^2 < 8\pi$. For $\beta^2 > 8\pi$, in the strong coupling phase, the energy of the system is unbounded from below. The particular choice $\beta_c^2 = 8\pi$ represents the well-known Coleman fixed-point (a single cosine mode with the critical length of period β_c, is the fixed-point of the linearized renormalization group equation).

The other important property of the sine-Gordon model, due to its non-linear nature, is that it possesses classical solutions, called solitons (kinks) which represent stable configurations with a well-defined energy [6]. In the kink configuration the field approaches different asymptotic values in opposite directions in space [10], and the difference is $N\Delta$, characterized by the integer N, called the kink number. The stability of the kinks is the consequence of the topology. There is a conservation law which corresponds to the stability of the kinks. The corresponding topological current $j_\mu(x)$ $(\mu = 0, 1)$:

$$j_\mu(x) = \frac{\beta}{2\pi}\epsilon_{\mu\nu}\partial^\nu\phi(x) \tag{2.3}$$

(where $\epsilon_{\mu\nu}$ is antisymmetric, with $\epsilon_{01} = 1$) is conserved in the semi-classical expansion: $\partial^\mu j_\mu = 0$. The conserved charge Q is equal to the kink number N:

$$Q = \int_{-\infty}^{\infty} \mathrm{d}x\; j_0(x) = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\; \frac{\partial\phi}{\partial x} = \frac{\beta}{2\pi}\left[\phi(\infty) - \phi(-\infty)\right] = N. \tag{2.4}$$

This conservation law can be violated in quantum theory, and the topological current can become anomalous.

There are mappings between the sine-Gordon, the X–Y planar models and the Coulomb gas [9, 31, 36]. Usually these mappings between the different models are not exact. In fact, the Coulomb gas representation of a periodic potential [35] is obtained perturbatively. There exists, however, an exact equivalence between the X–Y and the compact sine-Gordon model which is obtained by expressing (2.1) in terms of the compact variable [10]

$$z(x) = e^{i\beta\phi(x)}. \tag{2.5}$$

In terms of this variable the action of the compact sine-Gordon model can be written as,

$$S = \int \mathrm{d}x^2 \left[\frac{1}{2\beta^2}\,\partial_\mu z^*\,\partial^\mu z + \frac{u_1}{2}(z + z^*)\right]. \tag{2.6}$$

The models (2.1) and (2.6) are equivalent in any order of the perturbation expansion in continuous space-time. The lattice regularization of the compact model gives the action:

$$S = \frac{1}{\beta^2} \sum_{<x,x'>} \cos(\theta_x - \theta_{x'}) + \sum_x u_1 \cos(\theta_x), \tag{2.7}$$

where $\theta = \beta\phi$ and the first summation on the right hand side extends at nearest neighbor sites $< x, x' >$.

The lattice regularized theory (2.7) is in the same universality class as the two-component spin model, the X–Y model with external field. The isotropic planar X–Y model consists of classical two-component spins where the magnitude of the spin at each site is $|\boldsymbol{S}_x| = 1$ [11, 13]. The model can be described by the action:

$$S = \frac{1}{T} \sum_{<x,x'>} \boldsymbol{S}_x \cdot \boldsymbol{S}_{x'} \tag{2.8}$$

with the temperature T. Introducing the external field h and using the definition of θ which is the angle closed by each spin with an arbitrarily chosen direction, the action can be rewritten as:

$$S = \frac{1}{T} \sum_{<x,x'>} \cos(\theta_x - \theta_{x'}) + \sum_x \frac{h}{T} \cos(\theta_x), \tag{2.9}$$

with the external field h. Using the identities $T = \beta^2$, $h = u_1 T$, the expression (2.9) is equivalent to (2.7). The X–Y model also has singular classical solutions, called vortex solutions:

$$\theta(x) = N \arctan\left(\frac{x_1}{x_0}\right) \tag{2.10}$$

with the following property:

$$\int_c \mathrm{d}x^\mu \; \partial_\mu \theta(x) = 2\pi N, \tag{2.11}$$

with the integer N, called the vortex number. There is a vortex configuration plotted on Fig. 2.1. It is easy to see the above mentioned property of the vortex configuration, namely around a center of the vortex the spins can rotate along closed circles with the angle $2N\pi$, where N is the vortex number. Due to geometrical reasons, the vortex (anti-vortex) of the X–Y model is equivalent to the creation (annihilation) of the sine-Gordon soliton [10]. In this manner, the world lines of the sine-Gordon solitons end at the X–Y model vortices, therefore the integer N is called the vorticity and the soliton number in the same time. The relation between the sine-Gordon and the X–Y model is summarized in Table 2.1.

The renormalization of the X–Y model induces a non-trivial value for the vortex fugacity y which appears as an additional evolving coupling constant in the compact sine-Gordon

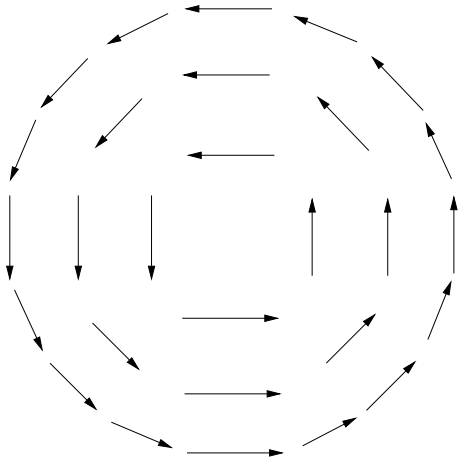

Figure 2.1: Spin configuration of an isolated $N = 1$ vortex in the planar X–Y model.

X–Y model with external field	compactified sine-Gordon model
external field h	Fourier amplitude $u_1 = -h/\beta^2$
temperature T	coupling constant β^2
molecular phase of the vortex gas $T < T_c$	weak coupling phase $\beta^2 < \beta_c^2$
ionized phase of the vortex gas $T > T_c$	strong coupling phase $\beta^2 > \beta_c^2$
vortex, anti-vortex	soliton creation and annihilation

Table 2.1: Comparison of the X–Y and the sine-Gordon model

model [10]. The vortex fugacity y controls the vortex activity, e.g. $y \to 0$ means the vortices are suppressed. Introducing the fugacity as a new coupling constant into the action (2.9) it is called the generalized X–Y model. Therefore the complete coupling constant space contains three coupling constants, the external field h, the temperature T, and the vortex fugacity y. Thus, the action for the generalized the X–Y model is [10]:

$$S = \frac{1}{T} \sum_{<x,x'>} \cos(\theta_x - \theta_{x'}) + \sum_x \frac{h}{T} \cos(\theta_x) + \log(y) \sum_x m_x^2 \tag{2.12}$$

where the vortex charge density m_x is defined as the vortex number of the plaquette at the lattice site x. A rigorous investigation of the lattice regularized X–Y model requires a redefinition of the vortex or kink number discussed in (2.4). In that sense the kink becomes an object defined on a link of the lattice. There are space-like and time-like kinks with the kink number k^μ. Then, introducing the lattice difference operator

$$\Delta^\mu \theta_x \equiv \theta_{x+\mu} - \theta_x \tag{2.13}$$

the vortex charge density can be defined as

$$m_x = \epsilon_{\mu\nu}\Delta^\mu k_x^\nu. \tag{2.14}$$

In order to investigate the lattice regularized generalized X–Y model (2.12) one can replace the cosine terms in (2.12) by quadratic one using the Villain transformation. Therefore, the vortex gas of the X–Y model can be mapped into the two-dimensional Coulomb gas using the Villain transformation [10]:

$$\exp\left(\cos(\theta)\right) = \sum_{n=-\infty}^{\infty} \exp\left(-\frac{1}{2}(\theta - 2n\pi)^2\right), \tag{2.15}$$

$$\exp\left(\frac{h}{T}\cos(\theta)\right) = \sum_{s=-\infty}^{\infty} \exp\left(s^2 \log\left(\frac{h}{2T}\right) + \mathrm{i}\, s\pi\theta\right). \tag{2.16}$$

After making these changes, the partition function for the generalized X–Y model can be rewritten as:

$$Z_{XY} = \sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\int \mathcal{D}[\theta]\exp\left[\frac{1}{2T}\sum_{x,\mu}(\Delta^\mu\theta_x - 2\pi n_x^\mu)^2 + \sum_x\left(s_x\log\left(\frac{h}{T}\right) + \mathrm{i}\, s_x\pi\theta_x\right) + \log(y)\sum_x m_x^2\right], \tag{2.17}$$

with lattice difference operator Δ^μ and the integer n^μ assigned to the link μ attached to the lattice site x, the integer s_x assigned to the site x as well, (where $\mu > 0$ denotes the positive directions). In the statistical sense the integer n^μ corresponds to the kink number k^μ because the most probable value of n^μ on the link is k^μ. Although n^μ can be any integer, form small fugacity only the values $0, 1, -1$ contribute. This is called the generalized Villain model. Such a mapping is valid only for small values of T and h. There is, however an important advantage of this modification. Namely, that the partition function (2.17) possesses certain duality properties. It satisfies the following duality relation for all T, y, h:

$$\left(\frac{T}{2\pi}\right)^n Z_{XY}\left(\frac{T}{2\pi}, y, h\right) = Z_{XY}\left(\frac{2\pi}{T}, h, y\right) \tag{2.18}$$

with the total number of the sites n. This duality transformation connects the sine-Gordon model and the X–Y model in the absence of the external field, $h = 0$ and provides us with the possibility to consider the Villain model without the external field. In a view of (2.17) the entire term $\sum_x\left(s_x\log\left(\frac{h}{T}\right) + \mathrm{i}\, s_x\pi\theta_x\right)$ is gone. Then, one can perform the Gaussian integration in (2.17) taking $h = 0$, and the partition function becomes:

$$Z_{XY}(h=0) = \sum_{m=-\infty}^{\infty}\exp\left[-\frac{(2\pi)^2}{2T}\sum_x n_x^\mu\left(\delta_{\mu\nu} - \frac{\Delta_\mu\Delta_\nu}{\Delta^2}\right)n_x^\nu + \log(y)\sum_x m_x^2\right], \tag{2.19}$$

where $\Delta^2 = \Delta^\mu \Delta_\mu$. Using the relation $m_x = \epsilon_{\mu\nu}\Delta^\mu n_x^\nu$ and $\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = \delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\mu\beta}\delta^{\nu\alpha}$, it is possible to show:

$$m_x \frac{1}{\Delta^2} m_{x'} = \epsilon_{\mu\nu}\epsilon_{\alpha\beta}\Delta^\mu n_x^\nu \frac{1}{\Delta^2}\Delta^\alpha n_{x'}^\beta = n_x^\mu \left(\delta_{\mu\nu} - \frac{\Delta_\mu \Delta_\nu}{\Delta^2}\right) n_x^\nu, \tag{2.20}$$

finally the partition function reads as follows:

$$Z_{CG} = \sum_{m=-\infty}^{\infty} \exp\left[\frac{\pi}{T}\sum_{x\neq x'} m_x m_{x'} \log\left(\frac{|\boldsymbol{x}-\boldsymbol{x'}|}{a}\right) + \log(y)\sum_x m_x^2\right], \tag{2.21}$$

with the lattice spacing a. Since in dimension $d = 2$, the Coulomb-like potential between the charges q_i can be written as:

$$U(|\boldsymbol{r_i} - \boldsymbol{r_j}|) = -2q_i q_j \ln\left(\frac{|\boldsymbol{r_i} - \boldsymbol{r_j}|}{a}\right), \tag{2.22}$$

one can conclude that in the partition function (2.21) the vortices are interacting via Coulomb interaction, therefore the partition function (2.21) is the partition function for the two-dimensional Coulomb gas (plasma) with the charges $q_x = \pi m_x/\sqrt{2}$ positioned at the site x. In the next section we discuss the renormalization of the Coulomb gas in dimension $d = 2$.

2.2 Real-space Renormalization Group Approach

The renormalization of the Coulomb gas in dimension $d = 2$ has been investigated by means of the real-space renormalization group (RG) approach in a great detail in the literature [10, 13, 27]. The RG method first introduced by Kadanoff and Wilson, provides us with a tool to consider the scaling of the coupling constants under the change of the length scale. The renormalization group transformation is a coarse-graining procedure and usually it is based on the blocking construction either in the coordinate space or in the momentum space. In every blocking step, the microscopic details of the model are integrated out successively, and this information is built in the change of the coupling constants. In this manner the transformations generate a trajectory in the parameter space of the model. The renormalization group transformations in momentum space are discussed in the chapter 3 in more detail.

During the last two decades the renormalization of the two-dimensional Coulomb gas was investigated in a great detail using the dilute gas approximation in coordinate space. The dilute vortex gas approximation assumes that all the vortex pairs (charges) are separated with at least to the minimum distance a in the coordinate space and this minimum distance is increased with infinitesimal steps during the blocking transformation. The contributions of the order higher than $\delta a/a$ are neglected. The real-space RG equations for the dilute

vortex-gas are well-known and their derivation is given in the literature [10, 13, 27]:

$$a\frac{d\tilde{h}}{da} = \left(2 - \frac{T}{4\pi}\right)\tilde{h}, \qquad a\frac{dT}{da} = -\pi T^2\tilde{h}^2 \tag{2.23}$$

with the dimensionless coupling constants $\tilde{h} = a^2 h$ and T. Using the duality transformations (2.18) it is possible to obtain the RG equations from (2.23) for the fugacity y and the temperature T. Several intuitive approaches exist by which one has tried to improve the dilute gas approximation [30, 31].

2.3 Phase Structure for the Coulomb gas

In this section we would like to describe the phase structure for the two-dimensional Coulomb gas or the equivalent two-dimensional classical X–Y model. Let us first note, as it is discussed in the introduction, the well-known theorem of Mermin and Wagner [33] states that two-dimensional systems with the number of spin vector components $2 \leq n$, and short range interactions between the spins cannot possess spatial long-range order at any finite temperature if the external field vanishes. This is, the spontaneous magnetic moment M of the system always vanishes at any finite temperature, but this does not mean that phase transitions cannot occur in the system. We consider here a two-component spin system ($n = 2$) with short-range type interaction in dimension $d = 2$. How can we discuss the phase structure of this system? Of course, this phase transition cannot be described in a traditional way, the order parameter introduced in a Landau sense $\Delta \equiv |M|$ at $h = 0$ field always vanishes, in agreement with the Mermin-Wagner theorem. However, there is spatial ordering in the system, but not of the long-range type for the pair correlation function of the dynamical variable in the limit $r \to \infty$. This is a topological ordering, and the phase transition is higher than second order.

In order to obtain the renormalization group flow for the two-dimensional Coulomb gas or the equivalent X–Y model, one can solve the RG equation (2.23) discussed in the previous section. As it is well-known, there are two phases in the $y - T$ plane of the X–Y model connected by the Kosterlitz–Thouless phase transition [27]. In the low temperature, molecular phase $T < T_{KT}$ the vortices and anti-vortices form closely bound pairs while above the transition temperature $T > T_{KT}$, in the ionized phase they dissociate into a plasma. There are two phases appearing in the $h - T$ plane of the X–Y model as well. The RG flow obtained solving (2.23) numerically is plotted in Fig. 2.2. One can connect the two phases of the X–Y model to those of the Coulomb gas in two different manner:

1. Due to the duality transformation (2.18) the corresponding two phases of the Coulomb gas appear in the $h - T$ plane of the X–Y model. Since the duality transformation connects the partition functions with inverse temperature, in the $h - T$ plane the ionized phase is realized at low temperature, i.e. at weak coupling $\beta^2 < \beta_c^2$, whereas the molecular phase is positioned at high temperature, i.e. at strong coupling $\beta^2 > \beta_c^2$. In this case, the inverse of the Kosterlitz–Thouless fixed-point T_{KT}^{-1} is proportional to the Coleman fixed-point $\beta_c^2 = T_c = 2\pi T_{KT}^{-1} = 8\pi$.

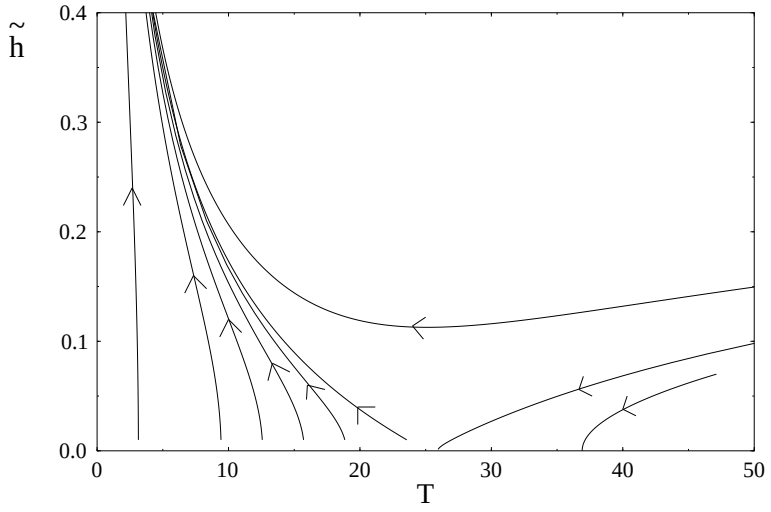

Figure 2.2: In this figure the renormalization group flow for the two-dimensional vortex gas (or Coulomb gas) is plotted, which is obtained by solving the differential equations (2.23), by a fourth-order Runge-Kutta method. The differential equations (2.23) are obtained by the real-space RG method using the dilute-gas approximation. The coupling constants are the dimensionless external field $\tilde{h}$, and the temperature T. We are using the unit system $c = \hbar = k_b = 1$. The solid lines represent the RG trajectories with different initial conditions and the arrows indicate the direction of the flow. The Coleman fixed-point corresponds to $T_c = 8\pi = 25.132$.

2. Instead of this duality transformation connecting the X–Y and the sine-Gordon models we can rely on the more direct equivalence of (2.6) and (2.9) as noted above [10], which identifies the weakly coupled (small β) and the strongly coupled (large β) phases of the sine-Gordon model with the molecular and the ionized phases of the X–Y model, respectively. The vortices of (2.6) and (2.9) correspond to each other in this scheme.

The transition between the phases can be characterized either perturbatively [9, 34, 37, 38] or by means of the vortex dynamics [10, 35]. Coleman showed, that the normal ordering is sufficient to remove the UV divergences for weak couplings as far as the perturbative expansion is concerned.

The inspection of the vortex dynamics allows us to follow the transition line for larger values of u_1. One way to identify the vortices is to use the equivalence of (2.6) and (2.9) in lattice regularization when the continuum limit is approached. This indicates that the weak and the strong coupling phases are the molecular and the ionized phases, respectively, from the point of view of the vortex gas and that the phases are separated by the Kosterlitz–Thouless transition. It is worthwhile to notice that this transition is higher than second order since the correlation length ξ is infinite in the molecular phase (see e.g. in [11, 13]) and diverges faster than any power of the reduced temperature (exponentially). Therefore, one cannot introduce the critical exponent ν in the usual manner because there is no relation like asymptotically $\xi \sim t^{-\nu}$ with the reduced temperature $t = (T - T_c)/T_c$.

The topological current, possessed by the sine-Gordon model $j_\mu(x) = \beta(2\pi)^{-1}\epsilon_{\mu\nu}\partial^\nu\phi(x)$ is conserved in the semi-classical expansion, but it can be anomalous in the quantum theory.

As we discussed in the first section, the flux defined by j_μ is the vortex number, called the vorticity and the soliton number in the same time. In this manner, the world lines of the sine-Gordon solitons end at the X–Y model vortices, making the solitons unstable and the topological current anomalous in the ionized phase where field configurations with singular space-time dependence survive the removal of the cutoff [10].

2.4 Application of the Vortex Dynamics in Experiments

In the previous sections we discussed the renormalization and some properties of the two-dimensional X–Y model or equivalently the two-dimensional Coulomb gas. In this section we conclude the investigation of the X–Y model giving an example when this model was used to explain experimental data. The vortex dynamics play an important role in the description of superfluid He^4 system, therefore here we discuss the model of thin films of superfluid He^4. The X–Y model was originally introduced to describe the superfluid-non superfluid phase transition of He^4 [11,13]. Based on experimental results taken from [14] for real helium films, the ratio of the superfluid density ρ_s and the critical temperature T_c found to be universal $\rho_s(T_c)/T_c$ =constant, that is independent of e.g. the substrate used or the areal density of the film.

The order parameter of this phase transition is a complex scalar field which distinguishes between the phases of the system. In the superfluid phase the order parameter differs from zero, and if one neglects fluctuations in its magnitude, the excitation of the system can be described in terms of its phase angle. The phase angle of the order parameter plays a role analogous to the field θ defined in the X–Y model (2.9). The field θ can be decomposed in a vortex and a spin wave part

$$\theta = \phi_{vortex} + \psi_{sw}. \tag{2.24}$$

There are two phases in the X–Y model. In the molecular phase ($T < T_c$) the vortices form closely bound pairs and they are "renormalized out" from the system, that is the fugacity of the vortex gas tends to zero towards the RG flow. Therefore, in the molecular phase of the X–Y model there are only spin waves. For $T < T_c$ the spin-spin correlation function,

$$G(r) = < S(r)\, S(0) > = < \cos(\theta(r) - \theta(0)) > \tag{2.25}$$

only depends on the spin wave part

$$G(r) = G_{sw}(r) = < \cos(\psi(r) - \psi(0)) > . \tag{2.26}$$

As it is defined in the introduction (see Eq. (1.15)), the correlation function has the following scaling property with the critical exponent η

$$G_{sw}(r, t = 0) = r^{-\eta}, \tag{2.27}$$

where the reduced temperature $t = (T - T_c)/T_c$ is set to be zero. In the framework of the X–Y model one can obtain the critical exponent η by calculating the expectation value in equation (2.26),

$$\eta = \frac{1}{4}. \tag{2.28}$$

Without going to details of the microscopic theory of superfluidity, we take the following relation from [13] holds for the superfluid density $\rho_s(T_c)$

$$\frac{m^2 k_B T_c}{\hbar^2 \rho_s(T_c)} = 2\pi\eta \tag{2.29}$$

where m is the mass of the He atom and k_B is the Boltzmann factor and T is the temperature and η is the critical exponent defined in (2.27). Introducing the result for the critical exponent η (2.28) into the equation (2.29) one obtains

$$\frac{m^2 k_B T_c}{\hbar^2 \rho_s(T_c)} = \frac{\pi}{2} \tag{2.30}$$

which implies that the ratio of $\rho_s(T_c)/T_c$ should be universal. Therefore, it was demonstrated, that this theoretical prediction for the superfluid density is in very good agreement with experimental results, e.g. [14] which illustrates the importance and the usefulness of the vortex dynamics discussed in the present chapter.

3 Differential Renormalization Group Approach in Momentum Space

In quantum field theory, the fundamental microscopic models and theories are usually defined at high energies (small distances) and the measurements, experiments are performed at low energies. Therefore, it is important to know how to obtain the low-energy effective theories from the model or theory given at a high-energy scale. One possible method is that of the renormalization group (RG) transformations, which provide us with a tool to make the connection between effective theories at various energy scales. The RG method can be applied to different scenarios: to obtain the fixed points and the scale dependence of the coupling constants of the theory; to perform at least a partial resummation of the perturbative expansion; or as a systematic non-perturbative method in order to handle highly non-linear, strongly correlated systems. In this thesis our goal is to consider the renormalization of the two-dimensional sine-Gordon model, which is a one-component scalar field theory with periodic self-interaction, i.e. a non-polynomial interaction which is periodic in the field variable.

In this chapter we discuss the differential renormalization group transformations for the one-component Euclidean scalar field theory. The realization of the renormalization group transformations is based on Wilson's blocking procedure, which is a successive elimination of the degrees of freedom of the system.

3.1 Renormalization Group Method

To describe an arbitrary physical system at a certain observational scale one has to find the *relevant* (important) interactions, and degrees of freedom of the system at this energy. If we change the scale, new interactions could become relevant. This is a consequence of the fact, that the description of a physical system strongly depends on the observational scale. For example, at the macroscopic length scale we can use the Newton equation to evaluate the motion of a macroscopic physical system. At the microscopic length scale, e.g. at the atomic one, the atoms inside the system are the relevant degrees of freedom, and new interactions become important therefore one has to use e.g. the Schrödinger equation in order to describe the components of that system. It is important, that usually the relevant interactions and degrees of freedom of a system at a certain energy scale are relatively independent of what is the behavior of the system at lower or at higher energy scales. Therefore, instead of using a kind of theory which contains all the interactions that could be relevant in any length scale, we use a chain of effective theories valid in different energy domains. Generally, during the renormalization we try to answer the question regarding the relation of the low-energy

effective theory to the model defined at the high-energy scale. In order to obtain the low-energy effective theory, a possible method is the usage of the renormalization group (RG) transformations, which relate the effective theories at different energy scales [7, 8].

According to the original definition of the renormalization group method, during the application of the RG transformations, the degrees of freedom which are responsible for the high-energy (microscopic) behavior of the system are integrated out and their impact are taken into account by influencing the scaling of the parameters (coupling constants) of the system. The solution of the renormalization group flow equations provides with the scaling of the coupling constants of the theory. The fixed-point of the RG transformation ($\mathcal{R}_k$) is defined as

$$\mathcal{R}_k(H^*) = H^* \tag{3.1}$$

where H^* is the fixed-point Hamiltonian of the system. Around H^*, one can classify the coupling constants of the model in the following way [7]. Using the assumption that the RG transformation $\mathcal{R}_k$ is an analytical function of the coupling constants of the theory, one can expand the transformation $\mathcal{R}_k$ around the fixed-point Hamiltonian H^*. Considering the action of $\mathcal{R}_k$ on the Hamiltonian $H = H^* + \epsilon O$ with ϵ infinitesimal,

$$\mathcal{R}_k(H^* + \epsilon O) = \mathcal{R}_k(H^*) + \epsilon \mathcal{L}_k(O) = H^* + \epsilon \mathcal{L}_k(O) \tag{3.2}$$

the linearized RG transformation $\mathcal{L}_k$ is defined around the fixed-point H^*. Then one has to find the eigenvectors (scaling operators) O_i of the linearized RG transformation $\mathcal{L}_k$,

$$\mathcal{L}_k(O_i) = \lambda_i(k)\, O_i \tag{3.3}$$

with the eigenvalues $\lambda_i(k)$ depending on the parameter k of the RG transformation (e.g. in case of differential RG transformation performed in momentum space, k is the momentum cut-off). The Hamiltonian of the model can be written

$$H = H^* + \sum_i g_{i,0} O_i \tag{3.4}$$

with the coupling constants g_i. The eigenvalues $\lambda_i(k)$ are the power-law function of the scale parameter of the RG transformation:

$$\lambda_i(k) = k^{y_i}. \tag{3.5}$$

Then performing the RG transformation on the Hamiltonian of the system,

$$\mathcal{R}_k(H(g_{i,0})) = \mathcal{R}_k(H^* + \sum_i g_{i,0} O_i) = H^* + \sum_i g_{i,0} k^{y_i} O_i, \tag{3.6}$$

the RG equation for the coupling constants g_i is obtained:

$$g_i(k) = g_{i,0}\, k^{y_i}. \tag{3.7}$$

Around the fixed point, the exponents y_i determine the scaling of the coupling constants. There are relevant, irrelevant or marginal scaling operators (coupling constants) corresponding to positive, negative or zero exponent of the eigenvalues (Fig. 3.1).

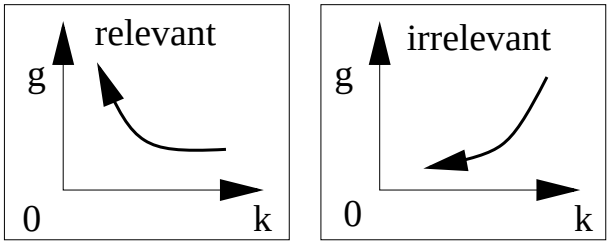

Figure 3.1: Changing the scale parameter of the RG transformation (e.g. decreasing the momentum scale k) around the fixed point, the relevant (irrelevant) coupling constant is increasing (decreasing).

It is possible to perform the renormalization group transformations in the coordinate or equivalently in the momentum space. In this chapter we discuss the RG transformations in the momentum space based on Wilson's RG approach [20], where the scale parameter of the RG transformation is the moving momentum cut-off k. During the differential RG transformation the momentum cut-off k is decreased with infinitesimal steps from the UV cut-off Λ towards $k = 0$ and the high-frequency quantum fluctuations of the field (ϕ_p, $p > k$) are integrated out in every infinitesimal steps. The infinitesimal changes of the cut-off k provide infinitesimal changes of the action, that is, infinitesimal changes of the coupling constants.

Changing the scale parameter, which implies moving the momentum cut-off k to the IR (low-energy) limit, the relevant coupling constants increase, the irrelevant coupling constants decrease. Since the coupling of the irrelevant scaling operators go to zero in the IR limit, they do not play any role in the low-energy behavior of the theory. Therefore, theories which are originally defined at an UV scale in a different way, but differ from each other only in irrelevant interactions, should have the same low-energy effective theory. This is called universality [7, 8]. The irrelevant interaction terms do not influence the critical behavior of the system. Therefore, theories belonging to the same universality class have the same critical exponents, they differs from each other only up to irrelevant terms.

In the framework of the usual, perturbative renormalization the theories which contain irrelevant interactions cannot be considered consistently, since the irrelevant coupling constants increase towards the high energies, and the corresponding vertices V have positive canonical dimensions $\delta(V) > 0$ (see Eq. (1.3)), so that the UV momentum cut-off cannot be removed to infinity. In this classification the irrelevant interactions corresponds to non-renormalizable theories (see chapter 1). On the other hand, the irrelevant interactions are unimportant, because due to universality they do not influence the low-energy effective theory of the model. Therefore, the view was held for a long time that renormalizable theories describe the real physical world containing only relevant and marginal interactions.

Universality may be lost, however if there is more than one fixed point in the theory. In this case, around every different fixed points one may find different classifications of the coupling constants into relevant, irrelevant or marginal coupling constants. It can happen e.g. that an irrelevant interaction at a high energy fixed point, becomes relevant at a low energy fixed-point (Fig. 3.2). If there is no universality, the irrelevant interactions at the UV fixed-point may become important at low energy, so one has to use a renormalization method which can follow the scaling of the irrelevant coupling constants, as well.

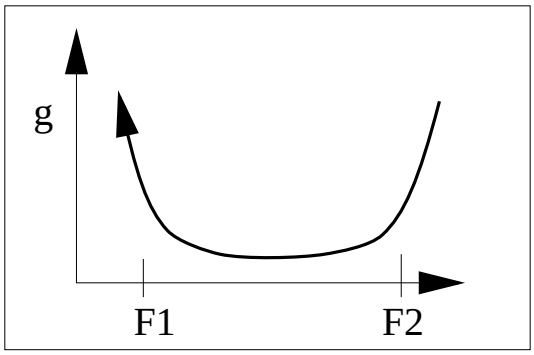

Figure 3.2: Decreasing the momentum scale k it may happen that the irrelevant coupling constants near the fixed-point $F2$ become relevant around another fixed-point $F1$.

The renormalization group method has several advantages:

1. It is one of the most important features of the RG method that it can handle models which contain irrelevant interactions.

2. The renormalization group method provides a scheme for summing up all the quantum contributions non-perturbatively.

3. It has a differential formulation.

In this work we consider the renormalization of the periodic scalar theory with the help of the differential RG method. In the next sections, we discuss two possible differential RG methods, the Wegner–Houghton and the Polchinski methods which are based on Wilson's renormalization group approach [20] applied in the momentum space.

3.2 Wegner–Houghton Renormalization Group Equation

In this section we apply the differential RG method in momentum space with sharp cut-off to the one-component scalar field theory, in order to obtain the scale dependence of the coupling constants. In the next chapter we shall specify the RG equations obtained here to a scalar field model which contains a periodic self-interaction (the simplest example is the sine-Gordon model, defined in the previous chapter). The challenge in developing an RG method for the sine-Gordon model, is that it should follow the mixing of all the operators which become relevant in the molecular and in the ionized phase of the model, as well, whereas the symmetry imposed by the periodicity should not be violated. Since there are

infinitely many relevant operators in two dimensions one needs the functional form of the evolution equation for the blocked Wilsonian action [21, 29, 44]. It is generally assumed that the blocked action $S_k[\phi]$ contains only local interactions, and that it can be expanded in the gradient of the field [8]:

$$S_k[\phi] = \int \mathrm{d}^d x \left[V_k(\phi) + \frac{1}{2} Z_k(\phi)\ (\partial_\mu \phi)(\partial^\mu \phi) + Y_k(\phi)\ ((\partial_\mu \phi)(\partial^\mu \phi))^2 + ... \right]. \tag{3.8}$$

First, we keep the leading order (LO) term of the gradient expansion in the Wegner–Houghton equation [21] ($Z_k(\phi) = 1, Y_k(\phi) = 0$), to study the renormalization group flow of a generalized model with an arbitrary periodic potential. Next, in order to clarify the supposed role of the wavefunction renormalization constant, we let evolve with the change of the scale the next-to-leading order term (NLO) in the gradient expansion but neglect its field dependence ($Z_k(\phi) = z(k)$, $Y_k(\phi) = 0$).

In the framework of Wilson's renormalization group approach the differential RG transformations are realized via a blocking construction, the successive elimination of the degrees of freedom which lie above the running ultraviolet momentum cut-off k. Consequently, the effective theory defined with the action $S_k[\phi]$ contains quantum fluctuations whose frequencies are smaller than the momentum cut-off k. The separation of the modes of the field according to their length scale is not a gauge invariant method, therefore the gauge symmetry is lost during the blocking. One possible solution for this problem could be the usage of the smooth cut-off [22], where the higher frequency modes of the field are suppressed partially, but not eliminated. Another solution could be the internal space renormalization group approach [44], where the quantum fluctuations are separated according to their amplitude and not to their frequencies or length scales. In this case the blocking procedure is performed in the space of the field variable and not in the space-time, which is understood as the external space. Here we consider the renormalization of the scalar field theory, where the gauge invariance has no importance, therefore we can use both methods, blocking in the external space, using the sharp or the smooth momentum cut-off, as well.

The generating functional Z, expressed in terms of the action $S_k[\phi]$ defined at the momentum cut-off k, reads as follows:

$$Z = \left(\prod_{|q|<k} \int \mathrm{d}\Phi_q \right) \exp\left[-\frac{1}{\hbar} S_k[\Phi_q] \right]. \tag{3.9}$$

The action $S_k[\Phi_q]$ depends on the field decomposed in Fourier series, contains modes ϕ_q with momenta less than the moving momentum cut-off $q < k$. Applying an infinitesimal RG transformation, the field is expanded in Fourier series and separated into a slow and a fast fluctuating part, which obey low-frequency and high-frequency Fourier modes, respectively:

$$\Phi(x) = \phi(x) + \hat{\phi}(x) = \sum_{|q|<k-\Delta k} \phi_q e^{iqx} + \sum_{k-\Delta k<|q|<k} \hat{\phi}_q e^{iqx}. \tag{3.10}$$

Then the momentum cut-off k is moved to $k - \Delta k$, and the high-frequency fluctuations of the field are integrated out in the momentum space. On the one hand, the expression (3.9)

can be rewritten:

$$Z = \left(\prod_{|q|<k-\Delta k} \int \mathrm{d}\phi_q \right) \left(\prod_{k-\Delta k<|q|<k} \int \mathrm{d}\hat{\phi}_q \right) \exp\left[-\frac{1}{\hbar} S_k[\Phi_q] \right]. \tag{3.11}$$

On the other hand, requiring the invariance of the generating functional Z:

$$Z = \left(\prod_{|q|<k-\Delta k} \int \mathrm{d}\phi_q \right) \exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi_q] \right], \tag{3.12}$$

one can read off the transformation for the blocked action when the cut-off is moved from the UV cut-off k to $k - \Delta k$,

$$\exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi] \right] = \int \mathcal{D}[\hat{\phi}] \;\; \exp\left[-\frac{1}{\hbar} S_k[\phi + \hat{\phi}] \right], \tag{3.13}$$

where the field variables ϕ and $\hat{\phi}$ contain Fourier components with momenta $|p| < k - \Delta k$, and $k - \Delta k < |p| < k$, respectively. In every infinitesimal step, the path integration in Eq. (3.13) is evaluated with the help of the saddle point approximation. There are two cases. The action has a saddle point either at $\hat{\phi} = 0$ (any constant saddle point can be transformed to zero by a constant shift of the field variable), or at $\hat{\phi} = \hat{\phi}_{cl} \neq 0$. In both cases, we keep only the linear and the quadratic terms, in the Taylor expansion at the saddle point therefore one can perform the Gaussian integration. The expansion around a general saddle point $\hat{\phi}_{cl}$ reads:

$$\begin{aligned} S_k[\phi + \hat{\phi}_{cl} + \hat{\phi}'] = \; & S_k[\phi + \hat{\phi}_{cl}] + \sum_{k-\Delta k<|p|<k} F_p \, \hat{\phi}'_p \\ & + \frac{1}{2} \sum_{k-\Delta k<|p|<k} \hat{\phi}'_p \, K_{p,-p} \, \hat{\phi}'_{-p} + \mathcal{O}(\hat{\phi}'^3) \end{aligned} \tag{3.14}$$

with the saddle point equation

$$F_p = \frac{\delta S_k[\phi + \hat{\phi}_{cl}]}{\delta \phi_p} = 0, \tag{3.15}$$

and

$$K_{p,-p} = \frac{\delta^2 S_k[\phi + \hat{\phi}_{cl}]}{\delta \phi_p \delta \phi_{-p}}. \tag{3.16}$$

Then one can evaluate the Gaussian integral in (3.13) and the result is:

$$\exp\left[-\frac{1}{\hbar} S_{k-\Delta k}[\phi + \hat{\phi}_{cl}] \right] = \left(\det K_{p,-p}[\phi + \hat{\phi}_{cl}] \right)^{-1/2} \exp\left[-\frac{1}{\hbar} S_k[\phi + \hat{\phi}_{cl}] \right]. \tag{3.17}$$

After taking the logarithm of both sides of (3.17) and using the identity: $\ln\det(K) = \mathrm{Tr}\,\ln(K)$, it becomes:

$$-\frac{1}{\hbar}S_{k-\Delta k}[\phi+\hat{\phi}_{cl}] = -\frac{1}{\hbar}S_k[\phi+\hat{\phi}_{cl}] - \frac{1}{2}\mathrm{Tr}\ln(K_{p,-p}[\phi+\hat{\phi}_{cl}]) + \mathcal{O}(\hbar^2). \tag{3.18}$$

If Δk is infinitesimal, the summation $\sum_p = V\,k^{d-1}\,\Delta k\,(2\pi)^{-d}\int \mathrm{d}\omega$ is $\mathcal{O}(\Delta k)$ and it can be replaced with the integral over the solid angle in dimension d. The second term on the r.h.s. of (3.18) is $\mathcal{O}(\Delta k)$ as well and one can see that the higher-loop contributions neglected in (3.18) can give only $\mathcal{O}(\Delta k^2)$ contributions for the blocked action [39]. Therefore taking the limit $\Delta k \to 0$ the equation (3.18) becomes an exact, integro-differential equation for the blocked action, which contains all the loop contributions. This is called the Wegner–Houghton renormalization group equation [21]:

$$k\partial_k S_k[\phi+\hat{\phi}_{cl}] = -\frac{k^d}{2}\int\frac{\mathrm{d}\omega}{(2\pi)^d}\,\hbar\,\ln(K[\phi+\hat{\phi}_{cl}])_{kn,-kn}, \tag{3.19}$$

where $\hat{\phi}_{cl} = \hat{\phi}_{cl}[\phi]$ as the functional of the background field ϕ is given by $\delta S_k[\phi+\hat{\phi}_{cl}]/\delta\phi_p = 0$. If the saddle point is trivial $\hat{\phi}_{cl} = 0$, then the Wegner–Houghton RG equation reduces to:

$$k\partial_k S_k[\phi] = \frac{k^d}{2}\hbar\int\frac{\mathrm{d}\omega}{(2\pi)^d}\ln(K[\phi])_{kn,-kn}. \tag{3.20}$$

When the saddle point is non-trivial $\hat{\phi}_{cl} \neq 0$, then the inverse propagator $K_{p,p'} = \delta^2 S_k[\phi+\hat{\phi}_{cl}]/\delta\phi_p\delta\phi_{-p}$ has at least one negative eigenvalue. The system becomes unstable against developing an inhomogeneous classical field configuration $\hat{\phi}_{cl}$, and the equation (3.20) loses its validity. In this case, one should expand the action $S_k[\phi+\hat{\phi}]$ around its true saddle point $\hat{\phi}_{cl}$, and one arrives at the system of equations (3.15) and (3.19). This is, however, of small practical use. Instead, the non-trivial saddle point can be determined by minimizing the action in Eq. (3.13) directly:

$$S_{k-\Delta k}[\phi] = \min_{\hat{\phi}_{cl}}\left[S_k[\phi+\hat{\phi}_{cl}]\right]. \tag{3.21}$$

For this one has to restrict the search of the minimum to a subspace of functions, e. g. to plane-wave like saddle points. In order to follow the evaluation of the action one can use the RG equation (3.20) until the eigenvalues of $K_{p,p'}$ are positive, $k_c[\phi] < k$. If $k < k_c[\phi]$ then one should use the tree-level blocking relation (3.21), [45]. When the saddle point becomes non-trivial, the argument of the logarithm in equation (3.20) becomes zero, which determines the critical scale k_c implicitly. The computation of the higher-order quantum corrections is difficult since the propagator is non-diagonal in momentum space when the saddle point differs from zero.

In the next two subsections we rewrite the Wegner–Houghton equation using the leading and the next-to-leading order expression for the Wilsonian action in the gradient expansion Eq. (3.8).

3.2.1 Gradient Expansion in LO

In the local potential approximation one uses the leading order expression for the action in the gradient expansion,

$$S_k = \int \mathrm{d}^d x \ \left[\frac{1}{2}\,(\partial_\mu\phi)(\partial^\mu\phi) + V_k(\phi)\right], \tag{3.22}$$

and the Wegner–Houghton equation (3.20) reduces to a differential equation for the scale dependent potential $V_k(\phi_0)$ and $(\phi(x) = \phi_0)$: [40]

$$k\partial_k V_k(\phi_0) = -k^d \hbar\, \alpha_d \,\ln\left(\frac{k^2 + \partial^2_{\phi_0} V_k(\phi_0)}{k^2}\right), \tag{3.23}$$

with $\alpha_d = \frac{1}{2}\Omega_d (2\pi)^{-d}$, and the solid angle Ω_d in dimension d. Let us note that the equation (3.23) is exact in that sense that it contains all the quantum corrections. One can read off the 1-loop contribution from (3.23) using the independent mode approximation where the k-dependence of the potential is ignored $(\partial^2_{\phi_0} V_k(\phi_0) \to \partial^2_{\phi_0} V_\Lambda(\phi_0))$ inside the argument of the logarithm:

$$k\partial_k V_k^{1-\mathrm{loop}}(\phi_0) = -k^d \hbar\, \alpha_d \,\ln\left(\frac{k^2 + \partial^2_{\phi_0} V_\Lambda(\phi_0)}{k^2}\right), \tag{3.24}$$

where the potential $V_\Lambda(\phi_0)$ is scale-independent. The solution of equation (3.23) is the scale dependent blocked potential $V_k(\phi_0)$, which tends to the effective potential $V_{\mathrm{eff}}(\phi_0)$ in the limit $k \to 0$. Furthermore, one can compare equation (3.24) with the usual form of the effective potential up to one-loop order, which reads as follows:

$$V_{\mathrm{eff}}(\phi_0) = V_{\mathrm{eff}}^{\mathrm{tree}}(\phi_0) + \hbar V_{\mathrm{eff}}^{1-\mathrm{loop}}(\phi_0), \tag{3.25}$$

with the tree-level potential $V_{\mathrm{eff}}^{\mathrm{tree}}(\phi_0) = V_\Lambda(\phi_0)$ and with the 1-loop correction:

$$V_{\mathrm{eff}}^{1-\mathrm{loop}}(\phi_0) = \alpha_d \int_0^\Lambda \mathrm{d}p\ p^{d-1} \ln\left(\frac{p^2 + \partial^2_{\phi_0} V_\Lambda(\phi_0)}{p^2}\right). \tag{3.26}$$

Integrating out both sides of equation (3.24) with respect to k and taking the IR limit $k \to 0$, equation (3.24) recover the equations (3.25) and (3.26).

There are dimensionful quantities in equation (3.23). In order to look for fixed-point solutions of (3.23) or to follow the scaling of an arbitrary potential one should remove the trivial scaling of the dimensionful coupling constants and rewrite equation (3.23). Therefore, one has to introduce dimensionless quantities via the following reparametrization:

$$\tilde{\phi} = k^{-\frac{d-2}{2}}\phi, \qquad \tilde{x}_\mu = k x_\mu, \tag{3.27}$$

with the changes in the derivate of the potential with respect to the field:

$$V_k(\phi) = k^d \tilde{V}_k(\tilde{\phi}), \qquad \partial_\phi^2 V_k(\phi) = k^d k^{-(d-2)} \partial_{\tilde{\phi}}^2 \tilde{V}_k(\tilde{\phi}). \tag{3.28}$$

One can rewrite equation (3.23) for dimensionless quantities

$$\left(d - \frac{d-2}{2} \tilde{\phi}_0 \partial_{\tilde{\phi}_0} + k\partial_k \right) \tilde{V}_k(\tilde{\phi}_0) = -\hbar\, \alpha_d \, \ln\left(1 + \partial_{\tilde{\phi}_0}^2 \tilde{V}_k(\tilde{\phi}_0)\right). \tag{3.29}$$

Notice that in dimension $d = 2$ the field has no trivial scale dependence, wherefore the second term on the left-hand side of (3.29) does not appear in the dimensionless equation. In the next chapter we specify equation (3.29) for periodic potentials. Since we consider the two-dimensional periodic scalar field theory it is useful to reduce the dimensionless equation (3.29) for dimension $d = 2$:

$$(2 + k\partial_k)\tilde{V}_k(\tilde{\phi}_0) = -\hbar\, \alpha_2 \, \ln\left(1 + \partial_{\tilde{\phi}_0}^2 \tilde{V}_k(\tilde{\phi}_0)\right) \tag{3.30}$$

with $\alpha_2 = (4\pi)^{-1}$.

Notice that the argument of the logarithm in (3.23) or in (3.29) must be non-negative for the expansion made around a stable saddle point. If the argument changes sign at a critical value $k_c > 0$, given by $k_c^2 = -\partial_\phi^2 V_{k_c}(\phi_0)$ then the Wegner–Houghton equation (3.23) loses its validity for $k < k_c$. The saddle point becomes non-zero and the tree-level blocking relation (3.21) has to be used. Restricting the search for the minimum to plane waves propagating in a given direction n_μ we find from Eq. (3.21):

$$V_{k-\Delta k}(\phi_0) = \min_\rho \left[k^2\rho^2 + \frac{1}{2}\int_{-1}^{1} \mathrm{d}u \;\; V_k(\phi_0 + 2\rho\cos(\pi u)) \right], \tag{3.31}$$

where ρ is the amplitude of the plane wave [45].

3.2.2 Gradient Expansion in NLO

In order to clarify the supposed role of the wavefunction renormalization ($Z_k(\phi)$) we let evolve the next-to-leading term in the gradient expansion, but neglect its field dependence (that is using the ansatz $Z_k(\phi) = z(k)$):

$$S_k = \int \mathrm{d}^2x \left[z(k)\, \frac{1}{2}\, (\partial_\mu\phi)(\partial^\mu\phi) + V_k(\phi) \right]. \tag{3.32}$$

Inserting $\phi(x) = \phi_0 + \epsilon(x)$ with $\phi_0 = \text{const.}$ and $\epsilon(x)$ infinitesimal inhomogeneous (therefore, $\epsilon(x)$ depends on the space-time) into both sides of Eq. (3.19), expanding them in powers of $\epsilon(x)$, and keeping the terms up to the second order, the Wegner–Houghton equation can be

reduced to [46]:

$$
\begin{aligned}
k\partial_k V_k(\phi_0) &= -k^d \hbar\, \alpha\, \ln\left(\frac{z(k)k^2 + \partial^2_{\phi_0} V_k(\phi_0)}{k^2}\right), \\
k\partial_k z(k) &= k^d \hbar\, \alpha\, [\partial^3_{\phi_0} V_k(\phi_0)]^2 \left[\frac{4\,[z(k)]^2\, k^2}{d\; A^4} - \frac{z(k)}{A^3}\right],
\end{aligned} \tag{3.33}
$$

with $A = (z(k)k^2 + \partial^2_{\phi_0} V_k(\phi_0))$. The last equation should hold only up to $\mathcal{O}(\phi_0^0)$ since the ansatz for the action contains only the field independent wavefunction renormalization. Equation (3.33) can be rewritten for dimensionless coupling constants:

$$
\begin{aligned}
&\left(d - \frac{d-2}{2}\tilde{\phi}_0 \partial_{\tilde{\phi}_0} + k\partial_k\right) \tilde{V}_k(\tilde{\phi}_0) = -\hbar\, \alpha\, \ln\left(z(k) + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0)\right), \\
&k\partial_k z(k) = \hbar\, \alpha\, [\partial^3_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0)]^2 \left[\frac{4\,[z(k)]^2}{d\; \tilde{A}^4} - \frac{z(k)}{\tilde{A}^3}\right],
\end{aligned} \tag{3.34}
$$

with $\tilde{A} = (z(k) + \partial^2_{\tilde{\phi}_0} \tilde{V}_k(\tilde{\phi}_0))$. Let us note here, that equation (3.34) does not determine the 'anomalous dimension' η due to the usage of the sharp momentum cut-off. The 'anomalous dimension' η is introduced as the non-trivial scale dependence of the wavefunction renormalization $\tilde{z}(k) = k^{-\eta} z(k)$. The critical exponent of the correlation function η defined in (1.15) is equivalent to the 'anomalous dimension'. Therefore, in the framework of the Wegner–Houghton method one cannot obtain the critical exponent η due to the usage of the sharp momentum cut-off.

In the next chapter we specify the RG equations obtained here for the periodic scalar field theory (sine-Gordon model). Using the equivalence between the sine-Gordon model and the Coulomb gas model ($T(k) = 1/z(k)$) described in chapter 2, one can compare the RG flow obtained by the Wegner–Houghton method with the well-known RG flow of the Coulomb gas.

Unfortunately the gradient expansion contradicts the usage of the sharp momentum cut-off. The higher order terms in the gradient expansion which correspond to the higher derivatives of the field, cannot be considered consistently due to the sharp momentum cut-off used in the Wegner–Houghton RG method [46]. Therefore it is not possible to obtain reliable RG equations for the field dependent wavefunction renormalization in the framework of the Wegner–Houghton method. One possibility to avoid the problems with the gradient expansion is the usage of a smooth cut-off. In order to consider the renormalization of $Z_k(\phi)$, in the next section, we discuss Polchinski's renormalization group method [22] which realizes the RG transformations in successive infinitesimal steps in momentum space using a smooth cut-off function.

3.3 Polchinski Renormalization Group Equation

In Polchinski's renormalization group method [22] the realization of the differential RG transformations is based on a non-linear generalization of the blocking procedure using

a smooth momentum cut-off. In the infinitesimal blocking step, the field variable $\Phi(x)$ is split into an IR (slow oscillating) and an UV (fast oscillating) part, but both fields contain low- and high-frequency modes due to the smoothness of the cut-off. The propagator for the IR component is suppressed by a properly chosen regulator function $K(p^2/k^2)$ at high frequency above the moving momentum scale k. The introduction of the regulator function generates infinitely many vertices with higher derivatives of the field. But these vertices are considered irrelevant and their flow is completely neglected (because the regulator function is not evolved under the RG transformations). One of the most important advantages of Polchinski's RG method is the usage of the smooth momentum cut-off which does not contradict with the gradient expansion. Therefore it is possible to consider the evolution of the field dependent wavefunction renormalization ($Z_k(\phi)$). In order to determine Polchinski's RG equation for the one-component scalar field theory, here we follow the method explained in Ref. [8].

The partition function for the scalar field Φ at the scale k is:

$$Z = \int \mathcal{D}[\Phi] \;\; \exp\left[-\frac{1}{\hbar} S_k[\Phi]\right] = \int \mathcal{D}[\Phi] \;\; \exp\left[-\frac{1}{2\hbar}\,\Phi\, G_k^{-1}\,\Phi - \frac{1}{\hbar}\, S_k^I[\Phi]\right], \tag{3.35}$$

where the blocked action is split into an interaction part: $S_k^I[\phi]$ and a quadratic part: $\frac{1}{2}\Phi\, G_k^{-1}\,\Phi \;=\; (2\pi)^{-d}\int \mathrm{d}^d p\, \frac{1}{2}\Phi_{-p}\, G_k^{-1}(p^2)\,\Phi_p$ containing the regularized inverse propagator $G_k^{-1}(p^2) = p^2 K^{-1}(p^2/k^2)$. The regulator function $K(z)$ suppresses the high-frequency modes ($|p| \gg k$) and keeps the low-frequency ones ($|p| \ll k$) unchanged due to the limiting behaviors $K(z) \to 0$ for $z \gg 1$ and $K(z) \to 1$ for $z \ll 1$, respectively. Then the field variable is split into an IR and an UV part:

$$\Phi = \phi + \hat{\phi} \tag{3.36}$$

where both ϕ and $\hat{\phi}$ contain high and low frequencies modes as well. The propagator is also split into two parts:

$$G_k = G_{k-\Delta k} + \hat{G}_k, \tag{3.37}$$

where $\hat{G}_k$ and $G_{k-\Delta k}$ correspond to $\hat{\phi}$ and ϕ, respectively. In the case of infinitesimal Δk the propagator can be written as $\hat{G}_k = \Delta k\; \partial_k G_k$. Due to the smooth cut-off, both ϕ and $\hat{\phi}$ are non-vanishing for all momenta it seems as if the degrees of freedom were doubled. In order to have the same number of degrees of freedom before the blocking one may introduce a new dummy field $\hat{\Phi}$ by inserting a trivial factor 1 in the partition function, written as a Gaussian path integral over the new field $\hat{\Phi}$, with the arbitrarily chosen propagator $G_D(p^2)$:

$$\begin{aligned} Z &= \int \mathcal{D}[\Phi] \;\; \exp\left[-\frac{1}{\hbar} S_k[\Phi]\right] \\ &= \mathcal{N} \int \mathcal{D}[\Phi]\mathcal{D}[\hat{\Phi}] \;\; \exp\left[-\frac{1}{\hbar} S_k[\Phi] - \frac{1}{2\hbar}\hat{\Phi}\, G_D^{-1}(p^2)\,\hat{\Phi}\right]. \end{aligned} \tag{3.38}$$

Then, the IR and UV fields ϕ and $\hat{\phi}$ are defined through a Bogoliubov-Valatin transformation

of the fields Φ and $\hat{\Phi}$:

$$\Phi = A_{1,1}\phi + A_{1,2}\hat{\phi}, \qquad \hat{\Phi} = A_{2,1}\phi + A_{2,2}\hat{\phi}, \tag{3.39}$$

with the requirement for the propagators:

$$\frac{1}{2}\phi\, G^{-1}_{k-\Delta k}\,\phi + \frac{1}{2}\hat{\phi}\,\hat{G}^{-1}_{k}\,\hat{\phi} = \frac{1}{2}\Phi\, G^{-1}_{k}\,\Phi + \frac{1}{2}\hat{\Phi}\,\hat{G}^{-1}_{D}\,\hat{\Phi}. \tag{3.40}$$

Using the definition of the interacting part of the action $S^I_k[\phi]$ the partition function can be written as:

$$Z = \int \mathcal{D}[\hat{\Phi}]\mathcal{D}[\Phi]\ \ \exp\left[-\frac{1}{2\hbar}\Phi\, G^{-1}_{k}\,\Phi - \frac{1}{2\hbar}\hat{\Phi}\,\hat{G}^{-1}_{D}\,\hat{\Phi} - \frac{1}{\hbar}S^I_k[\Phi]\right]; \tag{3.41}$$

after the inverse transformation of (3.39) the partition function becomes:

$$Z = \int \mathcal{D}[\hat{\phi}]\mathcal{D}[\phi]\ \ \exp\left[-\frac{1}{2\hbar}\phi\, G^{-1}_{k-\Delta k}\,\phi - \frac{1}{2\hbar}\hat{\phi}\,\hat{G}^{-1}_{k}\,\hat{\phi} - \frac{1}{\hbar}S^I_k[\phi+\hat{\phi}]\right]. \tag{3.42}$$

Rewriting this as:

$$Z = \int \mathcal{D}[\phi]\ \ \exp\left[-\frac{1}{2\hbar}\phi\, G^{-1}_{k-\Delta k}\,\phi - \frac{1}{\hbar}S^I_{k-\Delta k}[\phi]\right], \tag{3.43}$$

one can read off the result of the infinitesimal blocking step on the interaction part of the blocked action $S^I_{k-\Delta k}[\phi]$:

$$\exp\left[-\frac{1}{\hbar}S^I_{k-\Delta k}[\phi]\right] = \int \mathcal{D}[\hat{\phi}]\ \ \exp\left[-\frac{1}{2\hbar}\hat{\phi}\,\hat{G}^{-1}_{k}\,\hat{\phi} - \frac{1}{\hbar}S^I_k[\phi+\hat{\phi}]\right], \tag{3.44}$$

with the propagator $\hat{G}_k(p^2) = \Delta k\ \partial_k G_k(p^2)$.

One can perform the path integral (3.44) by the saddle point approximation. It is easy to read off the definition of the saddle point from (3.44):

$$\hat{\phi}_{cl} = -\hat{G}_k(p^2)\ \left.\frac{\delta S^I_k[\phi+\hat{\phi}]}{\delta\hat{\phi}}\right|_{\hat{\phi}=\hat{\phi}_{cl}} = -\hat{G}_k(p^2)\,\frac{\delta S^I_k[\phi]}{\delta\hat{\phi}} + \mathcal{O}(\Delta k^2). \tag{3.45}$$

As opposed to the Wegner–Houghton method, the saddle point (3.45) is always non-trivial $\hat{\phi}_{cl} \neq 0$, but it is of the order $\mathcal{O}(\Delta k)$. This saddle point is not the one seen by the Wegner–Houghton method as the extremum of the $S_k[\phi+\hat{\phi}]$. Thus, the possibility to see the spinodal instability seems to be lost in Polchinski's approach. In order to perform the path integral

(3.44), one has to expand the exponent of the integrand around the saddle point:

$$
\begin{aligned}
- & \frac{1}{\hbar} S_k^I[\phi+\hat{\phi}] - \frac{1}{2\hbar}\hat{\phi}\,\hat{G}_k^{-1}\,\hat{\phi} = \\
- & \frac{1}{2\hbar}\hat{\phi}_{cl}\,\hat{G}_k^{-1}\,\hat{\phi}_{cl} - \frac{1}{\hbar} S_k^I[\phi+\hat{\phi}_{cl}] - \frac{1}{\hbar}\phi'\left(\hat{G}_k^{-1} + \frac{\delta^2 S_k^I[\phi+\hat{\phi}_{cl}]}{\delta\phi\delta\phi}\right)\phi',
\end{aligned}
\tag{3.46}
$$

where $\hat{\phi} = \hat{\phi}_{cl} + \phi'$ and the first derivative of the exponent is vanishing due to the definition of the saddle point (3.45). Performing the Gaussian integral, one finds:

$$
\frac{1}{\hbar} S_k^I[\phi+\hat{\phi}_{cl}] - \frac{1}{\hbar} S_{k-\Delta k}^I[\phi] = -\frac{1}{2\hbar}\hat{\phi}_{cl}\,\hat{G}_k^{-1}\,\hat{\phi}_{cl} - \frac{1}{2}\mathrm{Tr}\,\ln\left[\hat{G}_k^{-1} + \frac{\delta^2 S_k^I[\phi+\hat{\phi}_{cl}]}{\delta\phi\delta\phi}\right]. \tag{3.47}
$$

Since the saddle point defined in (3.45) is $\mathcal{O}(\Delta k)$, therefore using the definition of the saddle point (3.45) one can expand the action in powers of $\hat{\phi}_{cl}$:

$$
\begin{aligned}
S_k^I[\phi+\hat{\phi}_{cl}] &= S_k^I[\phi] + \frac{\delta S_k^I[\phi]}{\delta\phi}\,\hat{\phi}_{cl} + \mathcal{O}(\Delta k^2) \\
&= S_k^I[\phi] - \frac{\delta S_k^I[\phi]}{\delta\phi}\,\hat{G}_k\,\frac{\delta S_k^I[\phi]}{\delta\phi} + \mathcal{O}(\Delta k^2).
\end{aligned}
\tag{3.48}
$$

Then the equation (3.47) can be rewritten as:

$$
\frac{1}{\hbar} S_k^I[\phi] - \frac{1}{\hbar} S_{k-\Delta k}^I[\phi] = \frac{1}{2\hbar}\frac{\delta S_k^I[\phi]}{\delta\phi}\hat{G}_k\frac{\delta S_k^I[\phi]}{\delta\phi} - \frac{1}{2}\mathrm{Tr}\,\ln\left[\hat{G}_k^{-1} + \frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right]. \tag{3.49}
$$

The logarithm in the last term of the right-hand side of Eq. (3.49) can be rewritten as:

$$
\ln\left[\hat{G}_k^{-1} + \frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right] = \ln[\hat{G}_k^{-1}] + \ln\left[1 + \hat{G}_k\frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right]. \tag{3.50}
$$

Expanding the logarithm in powers of the infinitesimal propagator $\hat{G}_k = \Delta k\partial_k G_k \sim \mathcal{O}(\Delta k)$, and keeping only the contribution $\mathcal{O}(\Delta k)$, the Polchinski equation is obtained:

$$
\partial_k S_k^I[\phi] = \frac{1}{2}\frac{\delta S_k^I[\phi]}{\delta\phi}\partial_k G_k\frac{\delta S_k^I[\phi]}{\delta\phi} - \frac{1}{2}\hbar\mathrm{Tr}\left[\partial_k G_k\frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right]. \tag{3.51}
$$

Using the definition:

$$
S_k[\phi] = \frac{1}{2}\phi G_k^{-1}\phi + S_k^I[\phi] \tag{3.52}
$$

one can rewrite Eq. (3.51) for the complete action:

$$\partial_k S_k[\phi] = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\partial_k G_k(p^2)\left[\frac{\delta S_k[\phi]}{\delta\phi_{-p}}\frac{\delta S_k[\phi]}{\delta\phi_p} - \right.$$
$$\left. -\hbar\,\frac{\delta^2 S_k[\phi]}{\delta\phi_{-p}\delta\phi_p} - 2\,\phi_p\,G_k^{-1}(p^2)\,\frac{\delta S_k[\phi]}{\delta\phi_p}\right]. \tag{3.53}$$

This is the Polchinski renormalization group equation, which is valid for arbitrary one-component scalar field theory. In this thesis our goal is, to obtain the renormalization of the two-dimensional sine-Gordon model, therefore in the next chapter we specify the equation (3.53) for the periodic scalar field theory.

The general feature of the Polchinski method is the presence of a non-trivial saddle point leading to non-vanishing tree-level renormalization in every infinitesimal blocking step. There is another possibility to obtain the tree-level renormalization group equations, namely to follow the same strategy what we used in the case of the Wegner–Houghton equation. The saddle point can be determined by minimizing the original action with respect to $\hat{\phi}_{cl}$:

$$S^I_{k-\Delta k}[\phi] = \min_{\hat{\phi}_{cl}}\left[-\frac{1}{2}\hat{\phi}_{cl}\,\hat{G}_k^{-1}\,\hat{\phi}_{cl} - S^I_k[\phi+\hat{\phi}_{cl}]\right]. \tag{3.54}$$

then one has to use the gradient expansion in order to reduce the tree-level blocking procedure Eq. (3.54) for the blocked potential and for the wavefunction renormalization of the field. If the saddle point $\hat{\phi}_{cl} \sim \mathcal{O}(\Delta k)$, then the two methods to obtain tree level renormalization group equations are equivalent since the part $\mathcal{O}(\hbar^0)$ of the differential equation is just the result of the minimization (3.54). The minimum at $\hat{\phi}_{cl}$ is given by Eq. (3.45) and by taking the functional on the right-hand side of Eq. (3.54) at its minimum the terms $\mathcal{O}(\hbar^0)$ of Eq. (3.49) are recovered. To find a saddle point of finite magnitude only the direct minimization would work. Thus, the first Eq. of (3.45) cannot be resolved for $\hat{\phi}_{cl}$, so that (3.45) and (3.47) cannot be rewritten in a closed form of (3.48). In the following we shall consider the $\mathcal{O}(\Delta k)$ saddle point, since we could not propose a mechanism how a finite magnitude saddle point would arise.

3.3.1 Gradient Expansion in LO

In order to solve the functional integro-differential equation (3.51) or (3.53), one has to project them to a functional subspace. In the local potential approximation one uses the leading order expression for the action in the gradient expansion, that means the wavefunction renormalization is set to one $Z_k(\phi) = 1$ and the field is set to a constant field: $\phi(x) = \phi_0$. Then the Polchinski equation reduces to a differential equation for the scale-dependent potential $V_k(\phi_0)$ [8]:

$$k\partial_k V_k(\phi_0) = \hbar V_k^{(2)}(\phi_0)\left(\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,K'(p^2/k^2)\right) - [V_k^{(1)}(\phi_0)]^2 K_0' \tag{3.55}$$

with a smooth regulator function $K(p^2/k^2)$, $K' = \partial_{p^2} K(p^2/k^2)$ and the derivatives of the potential $V_k^{(n)}(\phi_0) = \partial_{\phi_0}^n V_k(\phi_0)$. In the sharp cut-off limit $K_0' = K'(p^2/k^2)|_{p^2=0} = 0$ and $(2\pi)^{-d} \int \mathrm{d}^d p\, K' = -\alpha k^{d-2}$ therefore the Eq. (3.55) is equivalent to the linearized form of Eq. (3.23). Equation (3.55) can be rewritten in terms of dimensionless quantities [24, 25]:

$$\left(d - \frac{d-2}{2}\tilde{\phi}_0 \partial_{\tilde{\phi}_0} + k\partial_k\right) \tilde{V}_k(\tilde{\phi}_0) = \hbar \tilde{V}_k^{(2)}(\tilde{\phi}_0) \left(\int \frac{\mathrm{d}^d \tilde{p}}{(2\pi)^d} K'(\tilde{p}^2)\right) - [\tilde{V}_k^{(1)}(\tilde{\phi}_0)]^2 K_0', \quad (3.56)$$

with the dimensionless potential $\tilde{V}_k(\tilde{\phi}_0)$, dimensionless smooth regulator function $K'(\tilde{p}^2) = \partial_{\tilde{p}^2} K(\tilde{p}^2)$ and the dimensionless momentum $\tilde{p}^2 = p^2/k^2$.

The tree-level equation can be obtained by neglecting the loop-contribution term in equation (3.55) or by minimization of the action, that means using the expression (3.54).

3.3.2 Gradient Expansion in NLO

In order to consider the renormalization of the field-dependent wavefunction renormalization ($Z_k(\phi)$) we let evolve the next-to-leading term in the gradient expansion. Thus the Polchinski equation can be reduced using the following ansatz for the interaction part of the action S_k^I:

$$\begin{aligned}
S_k^I[\phi] &= S_k[\phi] - \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} G_k^{-1}(p^2)\phi_p \phi_{-p} \\
&= S_k[\phi] - \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} p^2 K^{-1}\left(\frac{p^2}{k^2}\right)\phi_p \phi_{-p} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d}(-p_1 p_2)\Big[Z_{-p_1-p_2}(\phi) \\
&\quad -\frac{1}{2}\left(K^{-1}\left(\frac{p_1^2}{k^2}\right) + K^{-1}\left(\frac{p_2^2}{k^2}\right)\right)\delta_{p_1+p_2}\Big]\phi_{p_1}\phi_{p_2} + \int \mathrm{d}^d x\ V_k(\phi)\,, \qquad (3.57)
\end{aligned}$$

with $Z_{-p_1-p_2}(\phi) = \int \mathrm{d}^d x\, Z_k(\phi_x)\, e^{-ix(p_1+p_2)}$. Following the method described previously in the case of the Wegner–Houghton equation, we insert $\phi(x) = \phi_0 + \epsilon(x)$ with ϕ_0 =const. and $\epsilon(x)$ infinitesimal and inhomogeneous into both sides of Eq. (3.51) and expand them in powers of $\epsilon(x)$ keeping the terms up to the second order [8]. Then, from Eq. (3.51) we obtain the following evolution equations for the potential $V_k(\phi_0)$ and the wavefunction renormalization $Z_k(\phi_0)$:

$$\begin{aligned}
k\partial_k V_k(\phi_0) &= -[V_k^{(1)}(\phi_0)]^2 K_0' + \hbar\,(Z_k(\phi_0) - K_0^{-1})\left(\int \frac{\mathrm{d}^d p}{(2\pi)^d} p^2 K'\right) \\
&\quad + \hbar\, V_k^{(2)}(\phi_0)\left(\int \frac{\mathrm{d}^d p}{(2\pi)^d} K'\right), \\
k\partial_k Z_k(\phi_0) &= -4\,(Z_k(\phi_0) - K_0^{-1})\, V_k^{(2)}(\phi_0)\, K_0' - 2\, V_k^{(1)}(\phi_0)\, Z_k^{(1)}(\phi_0)\, K_0'
\end{aligned}$$

$$-2\,[V_k^{(2)}(\phi_0)]^2\,K_0'' + \hbar\,Z_k^{(2)}(\phi_0)\left(\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,K'\right), \tag{3.58}$$

with $Z_k^{(n)}(\phi_0) = \partial_{\phi_0}^n Z_k(\phi_0)$, $V_k^{(n)}(\phi_0) = \partial_{\phi_0}^n V_k(\phi_0)$, and with the regulator function $K' \equiv \partial_{p^2} K(p^2/k^2)$, $K_0' = \partial_{p^2} K(p^2/k^2)|_{p^2=0}$.

One can rewrite equation (3.58) in terms of dimensionless quantities using the reparametrisation described in (3.27) [24, 25]:

$$\begin{aligned}
\left(d - \frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}_0) \;=\;& -\;[\tilde{V}_k^{(1)}(\tilde{\phi}_0)]^2\,K_0' \\
&+\;\hbar\,(\tilde{Z}_k(\tilde{\phi}_0) - K_0^{-1})\left(\int \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d}\,\tilde{p}^2\,K'(\tilde{p}^2)\right) \\
&+\;\hbar\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\left(\int \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d}\,K'(\tilde{p}^2)\right), \\
\left(\eta - \frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{Z}_k(\tilde{\phi}_0) \;=\;& -\;4\,(\tilde{Z}_k(\tilde{\phi}_0) - K_0^{-1})\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\,K_0' \\
&-\;2\,\tilde{V}_k^{(1)}(\tilde{\phi}_0)\,\tilde{Z}_k^{(1)}(\tilde{\phi}_0)\,K_0' - 2\,[\tilde{V}_k^{(2)}(\tilde{\phi}_0)]^2\,K_0'' \\
&+\;\hbar\,\tilde{Z}_k^{(2)}(\tilde{\phi}_0)\left(\int \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d}\,K'(\tilde{p}^2)\right),
\end{aligned} \tag{3.59}$$

with $\tilde{Z}_k^{(n)}(\tilde{\phi}_0) = \partial_{\tilde{\phi}_0}^n \tilde{Z}_k(\tilde{\phi}_0)$, $\tilde{V}_k^{(n)}(\tilde{\phi}_0) = \partial_{\tilde{\phi}_0}^n \tilde{V}_k(\tilde{\phi}_0)$, and with the regulator function $K' \equiv \partial_{\tilde{p}^2} K(\tilde{p}^2)$, $K_0' = \partial_{\tilde{p}^2} K(\tilde{p}^2)|_{\tilde{p}^2=0}$ and the 'dimensionless' momentum $\tilde{p}^2 = p^2/k^2$. The 'anomalous dimension' η is introduced as the non-trivial scaling of the wavefunction renormalization constant, $\tilde{Z}_k(\tilde{\phi}_0) = k^{-\eta} Z_k(\phi_0)$. The 'anomalous dimension' η is equivalent to the critical exponent of the correlation function defined in (1.15).

By setting $\hbar = 0$ one can read off the tree-level contributions from the renormalization group equation (3.58) or (3.59).

4 Periodicity of the Potential

In the previous chapter, we obtained exact differential renormalization group equations for the one-component scalar field theory in the Euclidean space-time. The Wilson-Kadanoff blocking construction was performed in momentum space in order to obtain the Wegner–Houghton and the Polchinski renormalization group equations, using the sharp and a smooth momentum cut-off, respectively. In this chapter we specify these RG equations for the periodic scalar field theory. These results are presented in [1–3]. In general, the RG transformations should preserve the symmetries of the action, namely the periodicity, therefore we can look for solutions of the RG equations among periodic functions. The solution of the renormalization group equations provides us the blocked potential $V_k(\phi_0)$ which tends to the effective potential $V_{\text{eff}}(\phi_0)$ in the IR limit, $k \to 0$. Therefore, if the action is periodic at the UV cut-off Λ, the effective potential should be periodic as well. On the other hand, as it is known, the effective potential is convex (in the field variable ϕ). In the first section the general properties of the effective potential are discussed, concluding that in case of a periodic field theory, the convexity and the periodicity are so strong constraints on the effective potential that it should become a flat, constant potential. The detailed study of the Wegner–Houghton and the Polchinski RG equations for the periodic scalar field theory is pursued in sections 2 and 3 in order to check this general conjecture in a particular case.

4.1 Periodicity and Convexity of the Effective Potential

The challenge in developing an RG method for the periodic scalar field theory is that the essential symmetry of the theory, namely the periodicity, should not be violated during the RG transformations. Indeed, one of the important general properties of the renormalization group methods is that they retain the symmetries of the action. Therefore, if the bare action $S_\Lambda(\phi)$ defined at the UV cut-off Λ has the symmetry under the transformation:

$$\phi(x) \to \phi(x) + \Delta \tag{4.1}$$

following

$$S_\Lambda \to S_\Lambda, \qquad \text{for} \quad \phi(x) \to \phi(x) + \Delta \tag{4.2}$$

then, the blocked action $S_k(\phi)$ which is the solution of the RG equations must be periodic with the same length of period Δ. In the local-potential approximation the blocked action reduces to the blocked potential $V_k(\phi_0)$ which is tending to the effective potential $V_{\text{eff}}(\phi_0)$

in the limit $k \to 0$. If the bare action is invariant under the symmetry transformation (4.1), then the effective potential should be periodic with the same length of period Δ.

On the other hand, as it is known, another important property of the effective potential is the convexity discussed in this section. The effective potential $V_{\text{eff}}(\phi)$ is convex even if the classical potential is non-convex, see [5]. In order to prove the convexity of $V_{\text{eff}}(\phi)$ we follow the argumentation described in [5] starting with the Euclidean generating functional

$$Z[j] = \exp\left[-\frac{1}{\hbar}W[j]\right] = \int \mathcal{D}[\phi]\ \exp\left[-\frac{1}{\hbar}S[\phi] - \frac{1}{\hbar}\int j\phi\right] \tag{4.3}$$

where $W[j]$ is the generating functional for the connected Green-functions. The effective action Γ_{eff} can be determined by the Legendre transformation

$$\Gamma_{\text{eff}}[\hat{\phi}] = W[j] - \int j\hat{\phi} \tag{4.4}$$

with

$$\hat{\phi}(x) = \frac{\delta W[j]}{\delta j(x)}, \qquad\qquad j(x) = \frac{\delta \Gamma_{\text{eff}}[\hat{\phi}]}{\delta \hat{\phi}(x)}. \tag{4.5}$$

In order to obtain the effective potential, the field and the source are set to be constant $\hat{\phi}(x) = \phi_0$ and $j(x) = j_0$ in the space-time volume Ω. In this case the effective action reduces to the effective potential

$$\Gamma_{\text{eff}}[\phi_0] = -\Omega\, V_{\text{eff}}(\phi_0). \tag{4.6}$$

On the one hand, since the source j_0 is constant, the space-time volume Ω can be detached from the generating functional of the connected Green-functions

$$W[j] = -\Omega\, w(j_0), \tag{4.7}$$

and the expression (4.4) reduces to

$$V_{\text{eff}}[\phi_0] = w[j_0] + j_0\,\phi_0 \tag{4.8}$$

with

$$\phi_0 = -\frac{\delta w[j_0]}{\delta j_0}, \qquad\qquad j_0 = \frac{\delta V_{\text{eff}}[\phi_0]}{\delta \phi_0}. \tag{4.9}$$

On the other hand, from expression (4.3) one can determine $w(j_0)$ in the presence of the constant source j_0

$$-\hbar\,\Omega\, w(j_0) = -\hbar^2 \log\left(\int \mathcal{D}[\phi]\ \ \exp\left[-\frac{1}{\hbar}\,S[\phi] - \frac{1}{\hbar}\,j_0\int\phi\right]\right) \tag{4.10}$$

and differentiating expression (4.10) twice with respect to the source j_0, we obtain

$$-\hbar\,\Omega\,\frac{\delta^2 w(j_0)}{\delta j_0 \delta j_0} = \left\langle \left(\int_\Omega \mathrm{d}x \;\; \phi(x) \right)^2 \right\rangle - \left\langle \left(\int_\Omega \mathrm{d}x \;\; \phi(x) \right) \right\rangle^2 , \tag{4.11}$$

where the definition for the expectation value is

$$\langle F[\phi] \rangle = \mathcal{N} \int \mathcal{D}[\phi] \;\; F[\phi] \;\; \exp\left[-\frac{1}{\hbar}\, S[\phi] - \frac{1}{\hbar}\, j_0 \int \phi \right] . \tag{4.12}$$

Although the integration measure in (4.12) is formally defined, requiring its 'positivity', the following relation holds $\left\langle (\int \phi)^2 \right\rangle \geq \langle (\int \phi) \rangle^2$. Therefore, from (4.11) one can read off the inequality

$$\frac{\delta^2 w(j_0)}{\delta j_0 \delta j_0} \leq 0. \tag{4.13}$$

Using the relation obtained from (4.9)

$$\left(\frac{\delta^2 V_{\text{eff}}}{\delta \phi_0 \delta \phi_0} \right) \left(\frac{\delta^2 w}{\delta j_0 \delta j_0} \right) = -1, \tag{4.14}$$

the convexity of the effective potential has been proven

$$\left(\frac{\delta^2 V_{\text{eff}}}{\delta \phi_0 \delta \phi_0} \right) \geq 0. \tag{4.15}$$

According to the proof presented above, the effective potential should be convex. As it is argued previously the effective potential should be periodic as well, if the bare theory is periodic. This conflict between convexity and periodicity can be resolved only in a trivial manner, namely if V_{eff} is a constant function, which is periodic and convex at the same time. In order to check this general statement we consider the renormalization of the periodic scalar field theory in the framework of the Wegner–Houghton and Polchinski RG method. For this purpose, the differential renormalization group equations obtained in the previous chapter are specified for the periodic scalar theory in the next two sections.

We conclude, that for the periodic field theory, the convexity and periodicity are so strong constraints on the effective potential that it should be constant.

4.2 Periodicity and the Wegner–Houghton Equation

In chapter 2, the Wegner–Houghton renormalization group equation (3.20) was derived for the one-component scalar field theory. Since such an integro-differential equation for functionals is unmanageable, we used the gradient expansion in order to reduce it to partial

differential equations for functions, the blocked potential $V_k(\phi_0)$ (the gradient expansion in leading order) and the blocked field independent wavefunction renormalization $z(k)$ (the gradient expansion in next-to-leading order). The symmetry of the action under the transformation

$$\phi(x) \to \phi(x) + \Delta \tag{4.16}$$

is to be preserved by the blocking and the potential $V_k(\phi_0)$ must be periodic with the period length Δ. It is actually obvious that the blocking, the transformation

$$kV_{k-\Delta k}(\phi_0) = kV_k(\phi_0) + \left[k^d \alpha \ln\left(z(k)k^2 + \partial_{\phi_0}^2 V_k(\phi_0)\right)\right] \Delta k \tag{4.17}$$

preserves the periodicity of the potential if the wavefunction renormalization $z(k)$ is independent of the field. Therefore, one can look for solutions of the Wegner–Houghton equation obtained in the local-potential approximation among periodic functions, if the initial condition for the action contains a periodic self-interaction at the UV cut-off Λ. The inclusion of the field independent wavefunction renormalization does not change the situation, since it is independent of the field.

It is proven, that the Wegner–Houghton equation obtained in next-to-leading order of the gradient expansion including the filed-independent wavefunction renormalization, retains the periodicity of the potential.

4.2.1 Tree-level Equation

In order to allow for a convenient truncation of the potential which is favorable in the numerical solution of the evolution equation and to preserve the periodicity we write $V_k(\phi_0)$ as a Fourier series,

$$V_k(\phi_0) = \sum_{n=0}^{\infty} u_n(k) \cos\left(n\beta\phi_0\right). \tag{4.18}$$

For the sake of simplicity we consider only potentials with a Z(2) symmetry, $V_k(\phi_0) = V_k(-\phi_0)$. The whole scale dependence occurs in the Fourier amplitudes $u_n(k)$, the 'coupling constants' of the scale dependent potential. The Fourier-amplitudes $u_n(k)$ are dimensionful quantities. One can introduce dimensionless coupling constants $\tilde{u}_n(k)$ using the reparametrisation described in (3.27):

$$\tilde{V}_k(\tilde{\phi}_0) = \sum_{n=0}^{\infty} \tilde{u}_n(k) \cos\left(n\tilde{\beta}\tilde{\phi}_0\right), \tag{4.19}$$

where $\tilde{u}_n(k) = k^{-d} u_n(k)$ and $\beta\phi_0 = \tilde{\beta}\tilde{\phi}_0$ are dimensionless. In the case of a non-trivial saddle point one can use Eq. (3.31), which reads as follows for the dimensionful periodic potential

(4.18)

$$V_{k-\Delta k}(\phi_0) = \min_{\rho}\left[k^2\rho^2 + \sum_{n=0}^{\infty} u_n(k)\cos(n\beta\phi_0)J_0(2n\beta\rho)\right], \tag{4.20}$$

where J_0 stands for the Bessel function and $\beta = 2\pi/\Delta$. In Appendix C we describe how to obtain Eq. (4.20) from Eq. (3.31). Let $\rho_k(\phi_0)$ denote the position of the minimum of the bracket on the r.h.s. of Eq. (4.20). Then $\rho_k(\phi_0)$ is periodic $\rho_k(\phi_0 + \Delta) = \rho_k(\phi_0)$, since one has to minimise the same expression of ρ for ϕ_0 and $\phi_0 + \Delta$.

We conclude, that the tree-level Wegner–Houghton equation (4.20) obtained in the local-potential approximation preserves the periodicity of the potential.

4.2.2 Local-potential Approximation

In the local-potential approximation we consider the renormalization of the action for a homogeneous field configuration where the wavefunction renormalization is set $Z_k(\phi) = 1$. It is easier to use the derivative of Eq. (3.29) with respect to $\tilde{\phi}_0$ rather than the original equation itself. In local-potential approximation the general form of the evolution equation for the Fourier amplitudes for dimension d reads as follows [1],

$$\alpha\tilde{\beta}^2 n^2\tilde{v}_n(k) = (d + k\partial_k)\tilde{v}_n(k) - \frac{1}{2}\tilde{\beta}^2\sum_{p=1}^{N} A_{np}(k)(d + k\partial_k)\tilde{v}_p(k), \tag{4.21}$$

where N is the truncation in the Fourier series in (4.19), with dimensionless $\tilde{\beta}$ and with dimensionless coupling constants $\tilde{v}_n(k) = n\tilde{u}_n(k)$ and

$$\begin{aligned} A_{np}(k) &= (n-p)\tilde{v}_{n-p}\Theta(n \geq p) \\ &\quad +(p-n)\tilde{v}_{p-n}\Theta(p \geq n) \\ &\quad -(n+p)\tilde{v}_{p+n}\Theta(N \geq n+p), \end{aligned} \tag{4.22}$$

with $\Theta(n \geq n') = \{1 \text{ if } n \geq n', 0 \text{ if } n < n'\}$. For details see Appendix B.

Reaching the critical value k_c one has to change automatically from the system of equations (4.21) to Eq. (4.20). Then the potential at the scale $k_c - \Delta k$ is found by minimizing the expression on the r.h.s. of Eq. (4.20). After the minimization, the potential $V_{k_c-\Delta k}(\phi_0)$ is expanded in Fourier-series to define the new Fourier amplitudes at the scale $k_c - \Delta k$. One repeats this algorithm step by step with the stepsize Δk until $k = 0$.

4.2.3 Including Wavefunction Renormalization

In the previous subsections we discussed the Wegner–Houghton equation for the periodic scalar field theory in the local-potential approximation. In order to retain the higher-order

contributions in the gradient expansion, the field independent wavefunction renormalization constant $z(k)$ is included into the kinetic energy of the action (3.32). Instead of considering an arbitrary periodic action, in this subsection, we derive RG equations in the framework of the Wegner–Houghton equation for the sine-Gordon model including the field-independent wavefunction renormalization constant $z(k)$ as well. The two-dimensional sine–Gordon model is the simplest example for a periodic scalar field theory, defined by the Euclidean action

$$S_k = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial\phi)^2 + u_1(k)\cos(\beta(k)\phi)\right] \tag{4.23}$$

with the two scale-dependent coupling constants $u_1(k)$ and $\beta(k)$. Rescaling the field $\phi' = \beta\phi$, the action (4.23) can be written as

$$S_k = \int \mathrm{d}^2x \left[\frac{1}{2}z(k)(\partial\phi')^2 + u_1(k)\cos(\phi')\right] \tag{4.24}$$

with the field-independent renormalization $z(k) = 1/\beta^2(k)$. The flow equations for the dimensionful coupling constants $u_1(k)$ and $z(k)$ are obtained by expanding both sides of Eq. (3.33) in Fourier series and neglecting the higher harmonics missed on the left-hand sides. It is straightforward to use the derivative of the first equation in (3.33) with respect to ϕ_0. Thus, we find

$$\begin{aligned} k\partial_k u_1(k) &= \frac{k^2}{2\pi}\frac{1}{x(k)}\left[1-\left(1-x^2(k)\right)^{1/2}\right], \\ k\partial_k z(k) &= -\frac{1}{8\pi}\frac{x^2(k)\left(1+x^2(k)\right)}{\left(1-x^2(k)\right)^{5/2}} \end{aligned} \tag{4.25}$$

with $x = u_1(k)/(k^2 z(k))$. In Appendix D we summarize the details of the calculation for obtaining Eq. (4.25).

As it is discussed in the previous subsections, the Wegner–Houghton equation loses its validity, when the saddle point becomes non-trivial in the course of the subsequent blocking steps. In this case, in the local potential approximation one should use equation (4.20) in order to determine the blocked potential. Unfortunately, the tree-level blocking relation (3.21) does not give an evolution for the field-independent wavefunction renormalization $z(k)$. Therefore, we do not consider the renormalization of the action (4.23) when the saddle point is non-trivial.

4.3 Periodicity and the Polchinski Equation

The Polchinski renormalization group equation [22] based on the Wilson-Kadanoff blocking procedure by using a smooth momentum cut-off was discussed and derived for the one-component scalar field theory in chapter 2. The functional integro-differential equations (3.53) and (3.51) are valid for an arbitrary scalar action $S_k[\phi]$ or for the interaction part of

the action $S_k^I[\phi]$, respectively. Under the RG transformation the symmetries of the action (periodicity) should be preserved. If the action $S_k[\phi]$ is periodic, then the interaction part of the action $S_k^I[\phi]$ should be periodic as well. The Polchinski equation (3.51) for the interaction part of the action retains the periodicity of $S_k^I[\phi]$

$$S^I_{k-\Delta k}[\phi] = S^I_k[\phi] + \Delta k \left(-\frac{1}{2}\frac{\delta S^I_k[\phi]}{\delta\phi}\,\partial_k G_k\,\frac{\delta S^I_k[\phi]}{\delta\phi} + \frac{1}{2}\,\mathrm{Tr}\left[\partial_k G_k \frac{\delta^2 S^I_k[\phi]}{\delta\phi\delta\phi}\right]\right). \tag{4.26}$$

Instead of equation (3.51), which is valid for the interaction part of the action $S_k^I[\phi]$, one can use the more usual form of the Polchinski equation (3.53) which is valid for the complete action. Although equation (3.53) could break the periodicity of the action due to the term $2\phi_p G_k^{-1}(p^2)\frac{\delta S_k[\phi]}{\delta\phi_p}$, this is not the case since

$$2\Big(\phi_p + \Delta\delta_{p,0}\Big)G_k^{-1}(p^2)\,\frac{\delta S_k[\phi+\Delta]}{\delta\phi_p} = 2\phi_p G_k^{-1}(p^2)\,\frac{\delta S_k[\phi]}{\delta\phi_p} \tag{4.27}$$

owing to the periodic nature of $S_k[\phi]$ and the property $G_k^{-1}(0) = 0$ (since $G_k^{-1}(p^2) = p^2K(p^2/k^2)$). Therefore equation (3.53) preserves the periodicity of the action as well.

In order to make the functional integro-differential equations (3.51) and (3.53) manageable, the action is expanded in the gradient of the field, up to next-to-leading order

$$S_k = \frac{1}{2}\int \mathrm{d}^d x\ Z_k(\phi(x))\ \partial_\mu\phi(x)\partial^\mu\phi(x) + \int \mathrm{d}^d x\ V_k(\phi(x)) \tag{4.28}$$

with the field-dependent renormalization $Z_k(\phi)$ and with the potential $V_k(\phi)$. In chapter 2, the Polchinski equation was reduced for $Z_k(\phi)$ and $V_k(\phi)$ by using the ansatz for the interaction part of the action (3.57). Let us note, that due to the quadratic term in (3.57), the interaction part of the action is also periodic if the action is periodic. Therefore, the renormalization group equations (3.58) retain the periodicity of the wavefunction renormalization $Z_k(\phi_0)$ and of the potential $V_k(\phi_0)$

$$Z_k(\phi_0) = Z_k(\phi_0+\Delta), \qquad V_k(\phi_0) = V_k(\phi_0+\Delta), \tag{4.29}$$

with the unchanged length of period Δ. The Polchinski equation (3.51) or (3.53) reduces to the RG equations (3.59) for the dimensionless functions $\tilde{Z}_k(\tilde{\phi}_0)$ and $\tilde{V}_k(\tilde{\phi}_0)$. Since in this Thesis we would like to consider the renormalization of the two-dimensional periodic scalar field theory, equation (3.59) should be written for the dimension dimension $d = 2$:

$$\begin{aligned}(2+k\partial_k)\tilde{V}_k(\tilde{\phi}_0) \;=\; & -[\tilde{V}_k^{(1)}(\tilde{\phi}_0)]^2\,K_0' + \hbar\,(\tilde{Z}_k(\tilde{\phi}_0) - K_0^{-1})\left(\int\frac{\mathrm{d}^2\tilde{p}}{(2\pi)^2}\tilde{p}^2K'\right) + \\ & +\hbar\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\left(\int\frac{\mathrm{d}^2\tilde{p}}{(2\pi)^2}\,K'\right),\end{aligned}$$

$$k\partial_k \tilde{Z}_k(\tilde{\phi}_0) = -4\,(\tilde{Z}_k(\tilde{\phi}_0) - K_0^{-1})\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\,K_0' - 2\,\tilde{V}_k^{(1)}(\tilde{\phi}_0)\,\tilde{Z}_k^{(1)}(\tilde{\phi}_0)\,K_0' - \\ -2\,[\tilde{V}_k^{(2)}(\tilde{\phi}_0)]^2\,K_0'' + \hbar\,\tilde{Z}_k^{(2)}(\tilde{\phi}_0)\left(\int \frac{\mathrm{d}^2\tilde{p}}{(2\pi)^2}\,K'\right) \tag{4.30}$$

where $\tilde{Z}_k^{(n)}(\tilde{\phi}_0) = \partial_{\tilde{\phi}_0}^n \tilde{Z}_k(\tilde{\phi}_0)$, $\tilde{V}_k^{(n)}(\tilde{\phi}_0) = \partial_{\tilde{\phi}_0}^n \tilde{V}_k(\tilde{\phi}_0)$, the regulator function $K' \equiv \partial_{\tilde{p}^2} K(\tilde{p}^2)$, $K_0' = \partial_{\tilde{p}^2} K(\tilde{p}^2)|_{\tilde{p}^2=0}$ and the dimensionless momentum $\tilde{p}^2 = p^2/k^2$. In the next subsection, the linearized form of equation (4.30) is discussed around the Gaussian fixed point where the 'anomalous dimension' η is zero, therefore η is not introduced in (4.30). One can find the same RG equations for the dimensionless potential and the wavefunction renormalization in an arbitrary dimension d in the literature by setting the anomalous dimension $\eta = 0$ in [24, 25].

It is proven that the Polchinski renormalization group equation retains the periodicity of the action.

4.3.1 Scheme-dependence of the Polchinski Equation

One of the most important advantages of Polchinski's RG method is the use of a smooth momentum cut-off, because this is compatible with the gradient expansion. Due to the smoothness of the cut-off, one has to introduce a properly chosen regulator function $K(p^2/k^2)$ which is a priori not defined. A regulator function can be introduced in the Wegner–Houghton RG method as well [8], but due to the usage of the sharp momentum cut-off, in this case, the regulator function is $K(p^2/k^2) = \Theta((p^2/k^2) - 1)$. Therefore, in order to obtain the solution of the Polchinski equation (3.59) or (4.30), one has to define the regulator $K(p^2/k^2)$, or in other words one has to fix the scheme. The scheme dependence of the Polchinski equation is discussed in the literature in great detail [23–25, 47, 48].

It is argued in [47] that the scheme-dependence of the Wilsonian blocked action can be regarded as a certain coordinate transformation in the theory space. To a certain extend, the Polchinski and the Wegner–Houghton equation are "equivalent" because it is possible to derive the Polchinski equation in the sharp cut-off limit which is identical to the Wegner–Houghton equation [47]. The fixed points and the critical exponents are invariant under these transformations in the theory space, but in the gradient expansion the scheme-dependence is enhanced and the physical quantities (fixed points, critical exponents) become scheme-dependent.

The leading order of the gradient expansion the Polchinski equation (and similarly the Wegner–Houghton equation) can be rescaled [23–25, 48] and around the trivial Gaussian fixed point $V^* = 0$ the linearized equations and the scaling operators are found to be independent of the scheme. However around other non-trivial fixed points the critical exponents obtained from the Polchinski and from the Wegner–Houghton method differ.

One can fix the scheme with optimization methods [24, 25] when the regulator function $K(p^2/k^2)$ is determined in such a way that the RG equations give the best estimate for the values of the fixed points and critical exponents obtained by other more traditional methods.

In the next subsection we determine the linearized Polchinski equation around the trivial Gaussian fixed point ($V^* = 0$) and we consider the renormalization scheme-dependence of the Polchinski equation by matching it with the Wegner–Houghton equation.

4.3.2 Linearized Polchinski Equation

Here we consider the linearized Polchinski equations. First, we show that they can be rescaled in such a way as to give the same classification of the scaling operators at the Gaussian fixed points as obtained in the Wegner–Houghton method. Second, we derive the linearized equations for the Fourier amplitudes of the periodic blocked potential and of the periodic wavefunction renormalization constant.

One can linearize equations (3.59) around the UV Gaussian fixed point: $V^* = 0$ and $Z^* = 1$ (the 'anomalous dimension' is zero, $\eta = 0$). Then the linearized equations are

$$\left(d - \frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}_0) = \hbar\, I_1\, \tilde{Z}_k(\tilde{\phi}_0) + \hbar\, I_2\, \tilde{V}_k^{(2)}(\tilde{\phi}_0),$$

$$\left(-\frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{Z}_k(\tilde{\phi}_0) = -4\,(1 - K_0^{-1})\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\,K_0' + \hbar\, I_2\, \tilde{Z}_k^{(2)}(\tilde{\phi}_0), \qquad (4.31)$$

with $I_1 = (2\pi)^{-d}\int \mathrm{d}^d\tilde{p}\,\tilde{p}^2\,K'(\tilde{p}^2)$ and $I_2 = (2\pi)^{-d}\int \mathrm{d}^d\tilde{p}\,K'(\tilde{p}^2)$. The RG equations (4.31) depend on the particular choice of the regulator function $K(\tilde{p}^2)$. Therefore, in order to calculate the integrals I_1 and I_2, first one has to specify the regulator function $K(\tilde{p}^2)$. The regulator $K(\tilde{p}^2)$ should be defined in such a manner, that around the Gaussian fixed point the classification of the scaling operators into relevant and irrelevant ones, be the same as those obtained with different RG methods. For this purpose, first we rewrite equation (4.31) for the field-independent wavefunction renormalization $\tilde{Z}_k(\tilde{\phi}_0) = \tilde{z}(k)$

$$\begin{aligned}\left(d - \frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}_0) &= \hbar\, I_1\, \tilde{z}(k) + \hbar\, I_2\, \tilde{V}_k^{(2)}(\tilde{\phi}_0),\\ k\partial_k\tilde{z}(k) &= -4\,(1 - K_0^{-1})\,\tilde{V}_k^{(2)}(\tilde{\phi}_0)\,K_0'. \end{aligned} \qquad (4.32)$$

Then we can compare equation (4.32) with the linearized form of the dimensionless Wegner–Houghton equation (3.34) around the Gaussian fixed point

$$\begin{aligned}\left(d - \frac{d-2}{2}\tilde{\phi}_0\partial_{\tilde{\phi}_0} + k\partial_k\right)\tilde{V}_k(\tilde{\phi}_0) &= -\hbar\,\alpha\,\log\left[1 + (\tilde{z}(k) + \tilde{V}_k^{(2)}(\tilde{\phi}_0))\right]\\ &= -\hbar\,\alpha\,(\tilde{z}(k) + \tilde{V}_k^{(2)}(\tilde{\phi}_0)),\\ k\partial_k\tilde{z}(k) &= 0, \end{aligned} \qquad (4.33)$$

with $\alpha = \Omega_d/2(2\pi)^d$ where Ω_d is the solid angle in dimension d. In order to get rid of the undesirable tree-level contribution in the linearized Polchinski equations, $K_0 = 1$ has to be chosen. Then, both methods give no wavefunction renormalization, $\tilde{z}(k) = Z^* = 1$. The field-independent term $-\hbar\,\alpha\,\tilde{z}(k)$ can be neglected on the right-hand side of the first equations (4.32) and (4.33). In order to obtain the same critical exponents and scaling operators from

the two different sets of RG equations (4.32) and (4.33), one has to rescale the field in the linearized Polchinski RG equations (4.32):

$$\tilde{\phi} \to \tilde{\phi} \left(\frac{-I_2}{\alpha}\right)^{\frac{1}{2}}. \tag{4.34}$$

This rescaling does not change the physical quantities (critical exponents); but only the shape of the potential. That means, the critical exponents around the Gaussian fixed points are independent of the choice of the regulator function. As argued in the previous subsection, they are renormalization scheme independent.

We conclude, that in leading order of the gradient expansion the Polchinski and the Wegner–Houghton equation give the same critical exponents around the trivial Gaussian fixed points.

If one requires that the two linearized sets of RG equations (4.32) and (4.33) give the same critical exponents without rescaling, one finds the constraint $I_2 = -\alpha$,

$$I_2 \equiv \int \frac{\mathrm{d}^d\tilde{p}}{(2\pi)^d} K'(\tilde{p}^2) = \alpha \int_0^\infty \mathrm{d}(\tilde{p}^2)\, \tilde{p}^{d-2}\, K'(\tilde{p}^2) \tag{4.35}$$

which is satisfied by only one regulator function $K(p^2/k^2) = \theta((p^2/k^2) - 1)$ because d is arbitrary.

In dimension $d = 2$ the linearized equations (4.32) do not evolve the field independent part of the wavefunction renormalization $\tilde{z}(k) = z = 1/\beta^2 = 1/\tilde{\beta}^2 =$ constant and $\tilde{\phi}_0 = \phi_0$. Therefore we can rescale the field $\phi_0 \to z^{-1/2}\phi_0 = \beta\phi_0$ and β becomes the length of period, which remains unchanged during the RG transformations. We can look for solutions of Eqs. (4.31) among the periodic functions. The potential $\tilde{V}_k(\phi_0)$ and the wavefunction renormalization $\tilde{Z}_k(\phi_0)$ are expanded in Fourier-series:

$$\begin{aligned} \tilde{V}_k(\phi_0) &= \sum_{n=0}^{\infty} \tilde{u}_n(k) \cos(n\beta\phi_0), \\ \tilde{Z}_k(\phi_0) &= \sum_{n=1}^{\infty} \tilde{z}_n(k) \cos(n\beta\phi_0). \end{aligned} \tag{4.36}$$

For the sake of simplicity we consider the potential $\tilde{V}_k(\phi_0)$ and the wavefunction renormalization $\tilde{Z}_k(\phi_0)$ possessing a Z(2) symmetry, ($\phi_0 \leftrightarrow -\phi_0$). The whole scale dependence occurs in the Fourier amplitudes $\tilde{u}_n(k)$ and $\tilde{z}_n(k)$, the 'coupling constants' of the blocked action. The linearized form of the evolution equations for the Fourier-amplitudes (when the field-dependent wavefunction renormalization is taken into account) reads as follows,

$$\begin{aligned} (2 + k\partial_k)\tilde{u}_n(k) &= \hbar\, I_1\, \tilde{z}_n(k) - \hbar\, I_2\, \tilde{u}_n(k)\, \beta^2\, n^2, \\ k\partial_k \tilde{z}_n(k) &= -\hbar\, I_2\, \tilde{z}_n(k)\, \beta^2\, n^2, \end{aligned} \tag{4.37}$$

for $n \geq 1$, where $I_2 = -\alpha$ and the actual value of the integral I_1 is not fixed and depends on

the particular choice of the regulator function $K(p^2/k^2)$. In order to obtain the numerical solution for (4.37) discussed in the next chapter, we set $I_1 = I_2 = -\alpha$.

Finally, let us note, that due to the usage of the smooth cut-off in the Polchinski renormalization group approach, there is always a non-trivial saddle point, $\phi_{cl} \neq 0$. The tree-level terms found in equation (4.30) depend on the derivatives of the regulator and appear due to the presence of this non-trivial saddle point. However, this tree-level renormalization has nothing to do with the instability found in the blocking procedure by using the sharp momentum cut-off. It seems to be rather a drawback of the method than to reflect some real physical content. This is a feature, significantly different from the Wegner–Houghton approach, where tree-level renormalization occurs only if the non-trivial saddle point does. That is why, here only the linearized Polchinski equations are considered, since they do not contain any tree-level terms.

5 Linearized Solution

In this chapter we consider the solution of the linearized renormalization group equations obtained in the previous chapters. In chapter 4 the Polchinski and the Wegner–Houghton RG equations were discussed for the periodic one-component scalar field theory. One of the most important advantages of these renormalization group equations is that in the framework of these RG equations it is possible to consider the renormalization of the periodic field theory without breaking the essential symmetry of the model, namely the periodicity. Results presented in [1–4] are summarized in this chapter. In the first section we discuss some general statements on universality. In the second and third section, the phase structure, and the fixed-points of the periodic scalar model are investigated by solving the linearized Wegner–Houghton and Polchinski renormalization group equations around the Gaussian fixed-point for dimension $d = 2$ and $d > 2$.

5.1 Universality

In the usual perturbative proof of the renormalizability each monomial vertex is treated independently. Therefore, this strategy is sufficient for polynomial interactions but it is not necessarily applicable for periodic potentials where the symmetry is destroyed by any truncation of the Taylor expansion. The treatment of an infinite series of operators instead of a single monomial may cause complications.

According to the power counting (see chapter 1), theories with polynomial interactions are super-renormalizable in dimension $d = 2$. It was shown in chapter 1, that in super-renormalizable theories there are vertices with negative canonical dimension (1.3). If there is at least one vertex V in the theory whose canonical dimension is positive $\delta(V) > 0$ then the theory is non-renormalizable. For renormalizable theories, there is at least one vertex in the theory with zero canonical dimension $\delta(V) = 0$. Furthermore, in chapter 3, the interactions are classified into relevant, marginal and irrelevant interactions. It was discussed, that theories containing irrelevant interactions are non-renormalizable. The renormalizable or super-renormalizable interactions correspond to relevant operators in the UV scaling regime.

However, this classification of monomial interactions via power counting which is in one-to-one correspondence with their RG classification into relevant, irrelevant operators, is not any more true for periodic potentials. As we show in the next sections, the coupling constants (Fourier amplitudes) of the periodic potentials scale differently. Indeed, for dimension $d = 2$ the coupling constants $\tilde{u}_n(k)$ are relevant, marginal or irrelevant for $\beta^2 < 8\pi/n^2$, $\beta^2 = 8\pi/n^2$, or $\beta^2 > 8\pi/n^2$, respectively

One can find the opposite effect. An infinite series of irrelevant operators can be defined in a renormalizable model, showing the possibility of the removal of the cutoff in the presence of non-renormalizable operators. The basic question in the removal of the cutoff is whether the series of the irrelevant operators is chosen in such a manner that the divergencies can be removed by the fine tuning of a finite number of parameters in the action. The infinite series of irrelevant operators is usually required by some symmetry of the theory, such as periodicity. As discussed in the next sections, for dimension $d > 2$ the scaling operator of the sine-Gordon model in the neighborhood of the cut-off is found to be relevant for $\beta^2 < 2/(\alpha_d)$, which is in contradiction with the usual perturbative treatment for monomial interactions. The symmetry imposes such constraints on the radiative corrections that the divergencies can in fact be removed within the given functional family of the action and the apparently non-renormalizable model becomes renormalizable.

In the next sections we show that these highly nontrivial effects can be reproduced in a very simple manner in the framework of a functional form of the renormalization group method. We shall use the Fourier amplitudes of the periodic potential $\tilde{u}_n(k)$ as coupling constants and we linearize the renormalization group flow (4.21) obtained in the previous chapter around the fixed-point $\tilde{u}_n = 0$ by assuming $|\partial_{\tilde{\phi}}^2 \tilde{V}_k(\tilde{\phi})| \ll 1$. In section 2, we look for periodic fixed-points and scaling operators for dimension $d = 2$ and $d > 2$. The phase transition obtained in dimension $d = 2$ is discussed in the third section. Finally, in the last section we consider the linearized Polchinski equation (4.37) obtained in the next-to-leading order of the gradient expansion.

5.2 Fixed-points and Scaling Operators in Local-potential Approximation

In the framework of Wilson's renormalization group approach, the fixed-points and the scaling operators of the scalar field theory are discussed in the literature in great detail [8], [23, 24, 26, 42]. Usually these results are obtained for the scalar field theory which contains polynomial interactions. Symmetries of the action like periodicity could imply new fixed-point solutions or new critical exponents. In this section, we look for periodic scaling operators and periodic fixed-point solutions of the Wegner–Houghton renormalization group equation (3.29), and the Polchinski renormalization group equation (3.56). The Wegner–Houghton (3.20) and the Polchinski equation (3.53) in the local-potential approximation (LPA) reduce to the differential equations (3.29) and (3.56) for the scale-dependent dimensionless potential $\tilde{V}_k(\tilde{\phi}_0)$, respectively.

We look for the solutions of Eq. (3.29) and (3.56) among the periodic potentials with Z(2) symmetry, $\tilde{V}_k(\phi_0) = \tilde{V}_k(-\phi_0)$. The potential can be expanded in Fourier modes:

$$\tilde{V}_k(\tilde{\phi}_0) = \sum_{n=1}^{\infty} \tilde{u}_n(k) \cos\left(n\tilde{\beta}\tilde{\phi}_0\right), \tag{5.1}$$

where $\tilde{\beta}$ is dimensionless, and the dimensionless coupling constants are the Fourier amplitudes $\tilde{u}_n(k)$. Let us note, that the dimensionful β has a trivial k-dependence $\beta = k^{-(d-2)/2}\tilde{\beta}$

due to $\tilde{\beta}\tilde{\phi} = \beta\phi$ and $\tilde{\beta}$ is constant. In dimension $d = 2$ the field has no scale-dependence, therefore $\tilde{\beta} = \beta$.

Among the polynomial interactions one can find the well-known fixed points like the trivial Gaussian fixed-point ($\tilde{V}^* = 0$), or the Wilson-Fisher fixed-points etc. Since in the local potential approximation the Polchinski equation (3.56) can be rescaled in such a way as to give the same scaling operators and fixed-points obtained in the Wegner–Houghton's method (see (4.34)), here we consider only the Wegner–Houghton equation (3.29). The trivial Gaussian fixed-point ($\tilde{V}^* = 0$) is the solution of the Wegner–Houghton fixed-point RG equation derived for periodic scalar potentials as well. In order to look for periodic scaling operators around the Gaussian fixed-point, we insert the Fourier expansion of the potential (5.1) into equation (3.29) and we find equation (4.21) for the dimensionless coupling constants $\tilde{u}_n(k)$ with the dimensionless $\tilde{\beta}$. Then linearizing equation (4.21) we obtain:

$$(d + k\partial_k)\,\tilde{u}_n(k) = \alpha_d\,\tilde{\beta}^2\,n^2\,\tilde{u}_n(k) \tag{5.2}$$

where $\tilde{\beta}$ is dimensionless therefore independent of the scale k. In the next two subsections we look for the scaling solutions of Eq. (5.2) around the Gaussian fixed-point $\tilde{V}^* = 0$:

$$\tilde{u}_n(k) = \left(\frac{k}{\Lambda}\right)^{\lambda_n} \tilde{u}_n(\Lambda). \tag{5.3}$$

Dimension $d = 2$

In order to consider the periodic scaling operators for dimension $d = 2$, we insert the ansatz (5.3) into the linearized equation (5.2) which takes the form:

$$(2 + \lambda_n)\,\tilde{u}_n(\Lambda) = \alpha_2\,\tilde{\beta}^2 n^2\,\tilde{u}_n(\Lambda). \tag{5.4}$$

From equation (5.4) the eigenvalue λ_n can be read off

$$\lambda_n = \alpha_2\,\tilde{\beta}^2\,n^2\,-\,2, \tag{5.5}$$

therefore the solution of the linearized equation (5.2) is

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda)\,\left(\frac{k}{\Lambda}\right)^{\alpha_2\,\tilde{\beta}^2\,n^2\,-\,2}, \tag{5.6}$$

where $\tilde{u}_n(\Lambda)$ is defined by the initial conditions and $\alpha_2 = 1/4\pi$ for dimension $d = 2$. Notice that according to the initial condition for $\tilde{\beta}$ the eigenvalue λ_n could be positive, negative or zero which determines the scaling properties around the Gaussian fixed-point. Although the consequences are discussed in the next section in more detail, let us note that the result (5.6) shows that the various coupling constants of the periodic potential scale differently around the Gaussian fixed-point. Indeed the coupling constants $\tilde{u}_n(k)$ are relevant, marginal

or irrelevant for $\tilde{\beta}^2 < 8\pi/n^2$, $\tilde{\beta}^2 = 8\pi/n^2$, or $\tilde{\beta}^2 > 8\pi/n^2$, respectively. Every $\tilde{u}_n(k)$ scales differently, with different λ_n.

If we consider the potential containing a single Fourier mode ($n = 1$), according to the solution (5.6) there is a phase transition at the critical length of period $\tilde{\beta}_c^2 = \beta_c^2 = 8\pi$. It is the phase transition found in the sine-Gordon model by Coleman [9].

For dimension $d = 2$ the well-known UV fixed-point of the sine-Gordon model is recovered in a rather straightforward way and the scaling operators are found, in the framework of the Wegner–Houghton and the Polchinski equation in the local-potential approximation.

Dimension $d > 2$

The periodic fixed-points of the RG equation (3.29) can be determined via the following equation:

$$d\,\tilde{V}^*(\theta) = -\hbar\,\alpha_d \ln\left(1 + \tilde{\beta}^2\,\partial_\theta^2 \tilde{V}^*(\theta)\right), \tag{5.7}$$

where $\theta = \tilde{\beta}\tilde{\phi}$ is dimensionless and $\tilde{\beta}$ is independent of k. The trivial Gaussian fixed point $\tilde{V}^*(\theta) = 0$ satisfies the fixed point equation (5.7). In this Thesis, we do not look for non-trivial periodic fixed points although the equation (5.7) is non-linear, therefore it might have non-trivial fixed points solutions.

In order to obtain the periodic scaling operators around the Gaussian fixed-point in dimension $d > 2$ one has to look for solutions of the linearized equation (5.2) using the ansatz (5.3). Then the solution of (5.2) is

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda)\left(\frac{k}{\Lambda}\right)^{\alpha_d\,\tilde{\beta}^2 n^2 - d}. \tag{5.8}$$

Therefore, near the initial value Λ of the cut-off, the scaling behavior of the coupling constants $\tilde{u}_n(k)$ are qualitatively the same as in dimension $d = 2$. There are relevant marginal and irrelevant coupling constants depending on the value $\tilde{\beta}^2 n^2$.

The similarity of the scaling in dimension $d = 2$ and $d > 2$ is reflected by the exponent $D = \alpha_d\,\tilde{\beta}^2$, determining the scaling via $\tilde{u}_n(k) = (k/\Lambda)^{n^2 D - d}$. The dependence of the type $(k/\Lambda)^{\alpha_d\,\tilde{\beta}^2 n^2 - d}$ is universal. This is essentially the same in any dimension with the exception of its 'trivial' dependence on the entire solid angle in $\alpha_d = \Omega_d/(2(2\pi)^d)$.

The scaling operators around the Gaussian fixed-point are determined in dimension $d \geq 2$. The dependence of the type $(k/\Lambda)^{\alpha_d\,\tilde{\beta}^2 n^2 - d}$ is essentially the same in any dimension excluding its 'trivial' dependence' on the entire solid angle in $\alpha_d = \Omega_d/(2(2\pi)^d)$.

5.3 Phase Transition for Dimension $d = 2$

In this section we consider the phase transition determined by the periodic solution with amplitudes (5.6) of the linearized Wegner–Houghton equation obtained in the local-potential

approximation for dimension $d = 2$. In order to investigate this phase transition let us recall the real-space renormalization group equation (2.23) for the two-dimensional Coulomb gas discussed in chapter 2. The two-dimensional Coulomb gas is equivalent to the two-dimensional sine-Gordon model which is the simplest realization of the periodic potentials. Linearizing the real-space equation (2.23) around $\tilde{h} = 0$ one can obtain:

$$a\frac{d\tilde{h}}{da} = \left(2 - \frac{T}{4\pi}\right)\tilde{h}, \qquad a\frac{dT}{da} = 0. \tag{5.9}$$

In the linearized equations (5.9) there is no evolution for the temperature T. Therefore $T = \beta^2 = \tilde{\beta}^2$ is constant, independent of the scale. Using the equivalences between the sine-Gordon and the Coulomb gas model ($\tilde{h} = \tilde{u}_1\beta^2$) explained in chapter 2, the equation (5.9) can be rewritten in momentum space:

$$k\partial_k\tilde{u}_1(k) = \left(\frac{\beta^2}{4\pi} - 2\right)\tilde{u}_1(k), \qquad k\partial_k\beta^2 = 0, \tag{5.10}$$

which is the same as equation (5.4) obtained in the framework of the Wegner–Houghton RG approach for periodic potentials. Therefore, the linearized real-space RG equations for the two-dimensional Coulomb gas are recovered in a rather straightforward way with the solution (5.6):

$$\tilde{u}_1(k) = \tilde{u}_1(\Lambda)\left(\frac{k}{\Lambda}\right)^{\alpha_2\beta^2 - 2}. \tag{5.11}$$

Thus, the coupling constant $\tilde{u}_1(k)$ is relevant, marginal, or irrelevant for $\beta^2 < 8\pi$, $\beta^2 = 8\pi$, or $\beta^2 > 8\pi$, respectively. There is a phase transition at the critical point $\beta_c^2 = 8\pi$. Furthermore, this is the phase transition found in the sine-Gordon model by Colemann in [9]. The details of the phase transition are discussed in chapter 2. Our result (5.6) is more general, because an arbitrary periodic potential is considered with infinitely many Fourier amplitude. From (5.6) one can read off the scaling of every $\tilde{u}_n(k)$.

In the strong coupling phase, for $\beta^2 > 8\pi$ all the dimensionless coupling constants $\tilde{u}_n(k)$ are irrelevant ($\tilde{u}_n(k) \to 0$ if $k \to 0$). Therefore the linearization around the trivial Gaussian fixed-point $\tilde{u}_n = 0$ remains valid in the $k \to 0$ limit as well. This keeps the trivial saddle point of the blocking stable $\phi_{cl} = 0$. This result indicates the inaccessibility of the Gaussian fixed-point in the UV limit $k \to \infty$, the non-renormalizability of the model. The infrared fixed-point is a trivial, non-interacting massless theory.

In the weak coupling phase, for $\beta^2 < 8\pi$, there are relevant and irrelevant coupling constants as well. Due to the relevant scaling operators, the solution loses its validity in the $k \to 0$ limit. Therefore, in the weak coupling phase the solution of the RG equation can be found only numerically. It will be discussed in the next chapter.

The critical point at $\beta_c^2 = 8\pi$ is a well-known result for the sine-Gordon model. What is interesting in this solution is that one sees a change in the scaling laws for the higher

harmonics at a finite distance away from this critical point. In fact, the n-th harmonic is found to be relevant for $\beta^2 \leq 8\pi/n^2$.

Finally, let us note, that these results are obtained in the local potential approximation where the wavefunction renormalization is not taken into account. In order to consider the scaling of the wavefunction renormalization $Z_k(\phi_0)$ which is a periodic function of the field ϕ_0, we let evolve the second term in the gradient expansion (3.8).

If $Z_k(\phi_0)$ is independent of the field $Z_k(\phi_0) = z(k)$, then the linearized Wegner–Houghton (4.33) and the linearized Polchinski equation (4.32) do not give an evolution for $z(k)$. This means, that the above described phase structure for the periodic scalar field theory remains unchanged. In Fig. 5.1 we plot the flow diagram obtained from the linearized form of the Wegner–Houghton equation (4.25) using the equivalences between the sine-Gordon model and the two-dimensional Coulomb gas discussed in chapter 2 ($T = 1/z = \beta^2$ and $\tilde{h} = \tilde{u}\,T$). One can compare this phase diagram to those obtained from the real-space RG approach

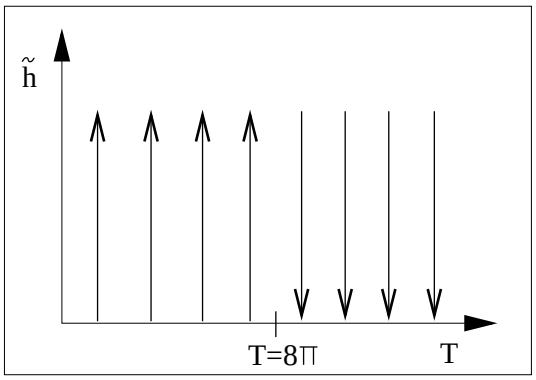

Figure 5.1: The flow diagram obtained from the linearized form of the Wegner–Houghton equation (4.25) is plotted in the two-dimensional coupling-constant space ($\tilde{h} = \tilde{u}T$ and $T = \beta^2 = 1/z$). There are two phases in the model, the weak ($T < 8\pi$) and the strong coupling phase ($T > 8\pi$) separated by the fixed-point $T = 8\pi$. When the non-linear terms are kept in the RG flow, the coupling constant T become scale-dependent plotted in Fig. 2.2.

for the two-dimensional Coulomb gas using the dilute gas approximation plotted in Fig. 2.2. Due to the non-linearity of the real-space RG equations (2.23) obtained in the dilute gas approximation the wavefunction renormalization $z = 1/T$ become scale-dependent. In the next chapter we go beyond the linearization and we discuss the numerical solution of the Wegner–Houghton equation (4.25) where the non-linear terms are taken into account.

It is discussed, that the linearized Wegner–Houghton, the linearized Polchinski and the linearized real-space RG equations give the same scaling operators for the periodic scalar theory around the Gaussian fixed-point when the field-independent wavefunction renormalization is taken into account.

If $Z_k(\phi_0)$ depends on the field ϕ_0, than one has to use the linearized Polchinski equation (4.37) in order determine the scaling operators around the Gaussian fixed-point. In the next section we discuss the solution of the linearized equation (4.37).

5.4 Wavefunction Renormalization for Dimension $d = 2$

Finally, in this section we investigate the periodic solution of the linearized Polchinski equation obtained by using the gradient expansion in next-to-leading order. Previously the fixed-points and the scaling operators of the periodic potential are considered in the local-potential approximation. One can ask, what are the changes in these scaling properties if the wavefunction renormalization is taken into account. Since in the framework of the Wegner–Houghton equation the field dependent wavefunction renormalization $Z_k(\phi)$ cannot be investigated consistently, therefore in order to determine the scaling of $Z_k(\phi)$ we use the linearized Polchinski RG equations (4.37) obtained in the previous chapter ($\hbar = 1, I_2 = -\alpha_d$):

$$\begin{aligned} (2 + k\partial_k)\tilde{u}_n(k) &= I_1\, \tilde{z}_n(k) + \alpha_2\, \beta^2\, n^2\, \tilde{u}_n(k), \\ k\partial_k \tilde{z}_n(k) &= \alpha_2\, \beta^2\, n^2\, \tilde{z}_n(k), \end{aligned} \tag{5.12}$$

for $n \geq 1$, where the actual value of the integral I_1 is not fixed. Notice, that equation (5.12) holds for the field-dependent wavefunction renormalization $\tilde{Z}_k(\phi_0)$.

The second equation in (5.12) can be solved analytically:

$$\tilde{z}_n(k) = \tilde{z}_n(\Lambda) \left(\frac{k}{\Lambda}\right)^{\alpha_2\, \beta^2 n^2}, \text{ for } n \geq 1. \tag{5.13}$$

Therefore, every $\tilde{z}_n(k)$ ($n \geq 1$) is irrelevant independently of the choice of β^2 and of the actual value of the integral I_1. This means that the field-dependent wavefunction renormalization is irrelevant for both phases of the model in the UV regime, where the linearized equations hold. Let us note, that the linearized Polchinski equation does not give an evolution for the field-independent wavefunction renormalization $\tilde{z}(k)$. Introducing the solution (5.13) into the first equation in (5.12) one can find the solution for the Fourier amplitude $\tilde{u}_n(k)$ analytically:

$$\tilde{u}_n(k) = \left(\tilde{u}_n(\Lambda) - \frac{I_1 \tilde{z}_n(\Lambda)}{2}\right) \left(\frac{k}{\Lambda}\right)^{\alpha_2\, \beta^2 n^2 - 2} + \frac{I_1 \tilde{z}_n(\Lambda)}{2} \left(\frac{k}{\Lambda}\right)^{\alpha_2\, \beta^2 n^2}, \text{ for } n \geq 1. \tag{5.14}$$

Depending on the choice of $\tilde{u}_n(\Lambda)$, $\tilde{z}_n(\Lambda)$ and I_1 the coupling constant $\tilde{u}_n(k)$ is relevant, irrelevant or marginal near the cut-off. Nevertheless, in the IR domain, far from the UV cut-off $k \ll \Lambda$, if the linearization still holds, the scaling of $\tilde{u}_n(k)$ only depends on the choice of β^2, since all the coupling constants $\tilde{z}_n(k)$ are irrelevant. For example, in the strong coupling phase $\beta^2 > \beta_c^2$ when all the Fourier amplitudes of the potential $\tilde{u}_n(k)$ become irrelevant in the IR domain, the linearized equation does not lose its validity, therefore the solution (5.13) and (5.14) remain valid in the $k \to 0$ limit. In the weak coupling phase $\beta^2 < \beta_c^2$ some of the $\tilde{u}_n(k)$ coupling constants become relevant far from the cut-off. Thus the linearization loses its validity and the solution of the RG equation (including the non-linear terms) can be found only numerically.

In Fig. 5.2, we plot the numerical solution of (5.12) for the scaling of the Fourier amplitude

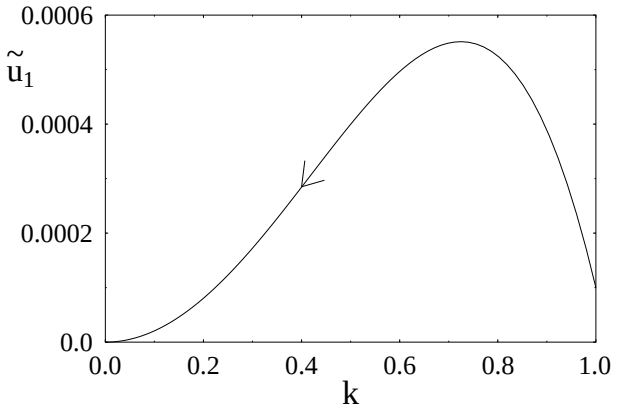

Figure 5.2: The scaling of the Fourier amplitude $\tilde{u}_1(k)$ is plotted versus the running momentum cut-off k from $k = 1$ to $k = 0$. The arrow indicates the direction of the flow. This result is obtained by solving the linearized Polchinski's RG equation (4.37) when $\beta^2 = 16\pi$ and the initial value for the Fourier amplitudes are $\tilde{u}_1 = 0.0001$, $\tilde{z}_1 = 0.001\beta^2$ and $\tilde{u}_n = \tilde{z}_n = 0$ if $n > 1$ and the integrals $I_1 = I_2 = -\alpha_2 = 1/(4\pi)$.

$\tilde{u}_1(k)$ when $\beta^2 = 2(8\pi) > \beta_c^2$ and the initial value for $\tilde{z}_1(k)$ is positive and the integral I_1 is set to be equal to: $I_1 = I_2 = -\alpha_2$.

The scaling operators for the field-dependent wavefunction renormalization around the Gaussian fixed-point are found irrelevant for both phases of the model. The position of the Coleman fixed-point and the phase structure is independent of the choice of the regulator function and of the renormalization scheme while the actual flow depends on it.

6 Numerical Solution

In the previous chapters we obtained renormalization group equations for the one-component periodic scalar field theory in the framework of Wegner–Houghton's and Polchinski's RG approaches. The solution of the linearized flow equations around the Gaussian fixed-point was discussed in chapter 5. For dimension $d = 2$ the well-known phase structure and scaling operators of the sine-Gordon model were recovered in a rather straightforward way. In order to compare the results obtained from the Wegner–Houghton and from the Polchinski methods to those obtained for the two-dimensional Coulomb gas in the framework of the real-space renormalization group method one needs to go beyond the linearization, since the real-space RG equations are non-linear. Therefore, in this chapter we investigate the numerical solution (presented in [1–3]) of the Wegner–Houghton's and Polchinski's RG equations obtained for periodic scalar field theory. In the first section we consider the numerical solution of the Wegner–Houghton equation obtained in the local-potential approximation. The wavefunction renormalization is taken into account for the numerical solution of the Wegner–Houghton and the Polchinski equation in sections 2 and 3.

6.1 Wegner–Houghton Equation in Local-potential Approximation

In the previous chapter the solution of the linearized Wegner–Houghton equation (5.2) obtained for the periodic scalar field theory was investigated in the local-potential approximation. The scaling operators were found around the trivial Gaussian fixed-point. The various coupling constants, the Fourier amplitudes $\tilde{u}_n(k)$ of the periodic potential scale differently. Indeed, the amplitudes $\tilde{u}_n(k)$ are relevant, irrelevant or marginal for $\tilde{\beta}^2 < 8\pi/n^2$, $\tilde{\beta}^2 = 8\pi/n^2$, or $\tilde{\beta}^2 > 8\pi/n^2$, respectively. For $\tilde{\beta}^2 < 8\pi$ there are relevant Fourier amplitudes, that is they increase if $k \to 0$. For example, in dimension $d = 2$ the Fourier amplitude $\tilde{u}_1(k)$ is relevant for $\tilde{\beta}^2 = \beta^2 < 8\pi$,

$$\tilde{u}_1(k) = \tilde{u}_1(\Lambda) \left(\frac{k}{\Lambda}\right)^{\frac{\beta^2}{4\pi} - 2}, \tag{6.1}$$

because the exponent in (6.1) is negative $(\beta^2/(4\pi)) - 2 < 0$. Therefore, the linearization ceases to be reliable for $\beta^2 < 8\pi$. The solution can be found only numerically in this case. In this section we investigate the numerical solution of the Wegner–Houghton equation (4.21) obtained for the periodic potential in the local-potential approximation for dimension $d = 2$.

For the numerical integration of (4.21), the Fourier expansion of the periodic potential has to be truncated with the truncation N. Monitoring numerically the coupling constants $u_n(k)$ of the periodic field theory ($1 \leq n \leq N$), we compare the solution of Eq. (4.21) with the result obtained from the analogous relation for the polynomial potential [45, 51] defined as

$$V_k(\phi_0) = \sum_m \frac{1}{(2m)!} g_{2m}(k)\ \phi_0^{2m}, \tag{6.2}$$

with the coupling constants $g_{2m}(k)$. The initial conditions for the polynomial potential were chosen as $g_2(k = \Lambda) = -0.001$, $g_4(k = \Lambda) = 0.01$ and $g_m(k = \Lambda) = 0$, if $m > 4$. In case of these initial conditions for $g_m(k)$, the saddle point $\hat{\phi}_{cl}$ of (3.13), remains trivial for any k and the singularity of Eq. (3.23) at $k_c^2(\phi_0) = -\partial_{\phi_0}^2 V_{k_c}(\phi_0)$ is avoided. In order to compare the periodic and the polynomial cases we choose the initial conditions for the Fourier amplitudes $u_n(k = \Lambda)$ for $\beta^2 = 0.1\beta_c^2$ such that after Taylor expansion of the periodic potential, the initial conditions for the polynomial case are recovered:

$$\begin{aligned} V_\Lambda(\phi_0) &= \sum_n^N u_n(\Lambda)\ \cos(n\beta\phi_0) = \sum_n^N u_n(\Lambda) \sum_m \frac{(-1)^m}{(2m)!}\ (n\beta\phi_0)^{2m} \\ &= \sum_m \frac{1}{(2m)!} \left[(-1)^m \beta^{2m} \sum_n^N u_n(\Lambda) n^{2m} \right] \phi_0^{2m} \equiv \sum_m \frac{1}{(2m)!} g_{2m}(\Lambda)\ \phi_0^{2m} \end{aligned} \tag{6.3}$$

Therefore, the initial conditions for various truncations N of the Fourier series are different. After fixing the initial conditions, the Wegner–Houghton RG equation obtained for the periodic potential (4.21) and for the polynomial potential [45, 51] is solved numerically. Notice, that for periodic potentials, equation (4.21) retains the periodicity of the potential. In order to compare the solutions obtained for the periodic and for the polynomial cases, the periodic potential is decomposed again in Taylor series. In this way, one can investigate, whether the violation of the periodicity can modify the renormalization of the periodic potential.

Eq. (4.21) is solved numerically, starting from the UV cutoff $\Lambda = 1$ by using a Runge-Kutta method. The Runge-Kutta method is commonly used for solving initial value problems for ordinary differential equations. This method provides a solution over an interval by combining the information from several Euler-style step and then using the information obtained to match a Taylor series expansion up to some higher order. Here we use a fourth-order Runge-Kutta method with the step size $\Delta k = 10^{-p} k$ with $p = 4$. There were no appreciable changes in the numerical results by increasing p further. The following results are obtained:

1. For increasing values of N the coupling constants $g_m(k)$ determined from the periodic potential by the Taylor expansion approach the running coupling constants of the polynomial potential in the UV regime. This is understandable since the quantum field ϕ_0 does not feel the global properties of the potential, i.e. the periodicity due to its small fluctuations in the UV regime.

2. Just the opposite holds in the IR regime, where the field fluctuations become larger and they make the global features of the potential manifest.

We conclude, that the solutions for periodic and polynomial potentials become different in the IR regime.

3. In Fig. 6.1 and Fig. 6.2 we show the scaling of the dimensionful coupling constants $g_2(k)$ and $g_4(k)$ for different truncations N. Increasing the value of N the differences in the results obtained for the periodic potential decrease.

   The numerical solution converges rapidly increasing the truncation N of the Fourier series of the periodic potential .

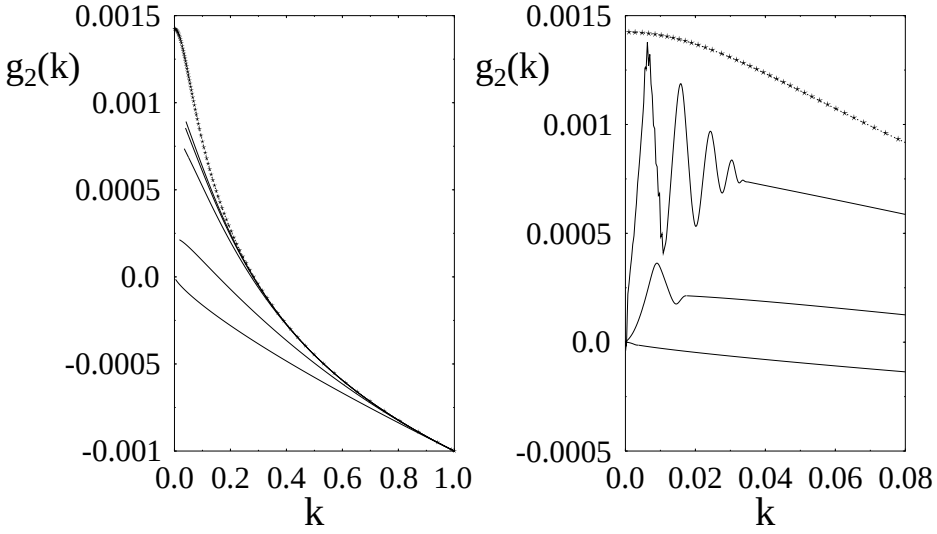

Figure 6.1: Comparison of the scaling of the dimensionful coupling constant g_2 obtained for the polynomial (dashed-line with stars) and for the periodic potential for various values of the truncation $N = 2, 3, 10, 20, 30$ in the Fourier series with $\beta^2 = 0.1\beta_c^2$. The figures on the left (right) show the scaling of the coupling above (below) k_c. Below k_c we show the scaling of g_2 for the cases $N = 2, 3, 10$. The increasing order of the full-line curves corresponds to increasing N.

4. Furthermore, in case of periodic potentials, there are relevant coupling constants for $\beta^2 < 8\pi$ which become large enough to destabilize the trivial saddle point of the blocking and we reach a non-vanishing critical value k_c where the saddle point becomes non-trivial and the tree-level blocking Eq. (4.20) must be used. By following the solution of this equation all dimensionful Fourier amplitudes are found to approach zero as $k \to 0$. The typical behavior is depicted in Fig. 6.1 and Fig. 6.2.

   There is a remarkable difference in the behavior of the theory with a periodic potential and that with the corresponding polynomial potential [51]. Namely, that all the dimensionful coupling constants $g_n(k)$ obtained for the periodic potential tend to zero contrary to those of the polynomial potential which remain finite as $k \to 0$. Therefore, the blocked potential flattens out with decreasing scale k.

5. In Fig. 6.3 we show this flattening starting from the value k_c. The periodic potential becomes a constant potential during the evolution in k. This is in agreement with the argumentation given in chapter 4. Namely, periodicity and convexity are so strong constraints on the effective potential, that it becomes a flat, constant potential.

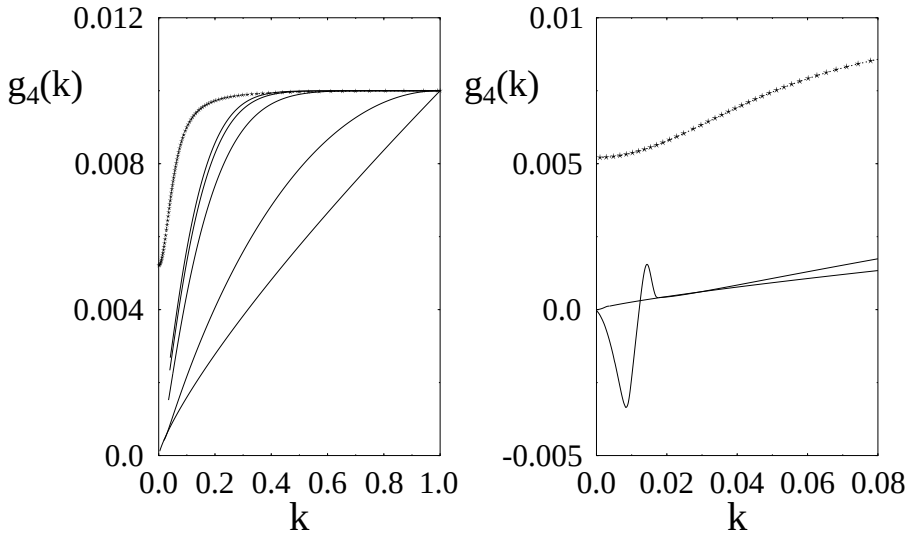

Figure 6.2: The scaling of the dimensionful coupling constant g_4 for the same cases as for g_2 in Fig. 6.1. The figures on the left (right) show the scaling of the coupling above (below) k_c. Below k_c, the scaling of g_4 are plotted for the cases $N = 2, 3$.

Integrating numerically the tree-level equation (4.20) we have shown that under the successive infinitesimal RG transformations the periodic potential becomes a constant potential up to an accuracy of 10^{-5}.

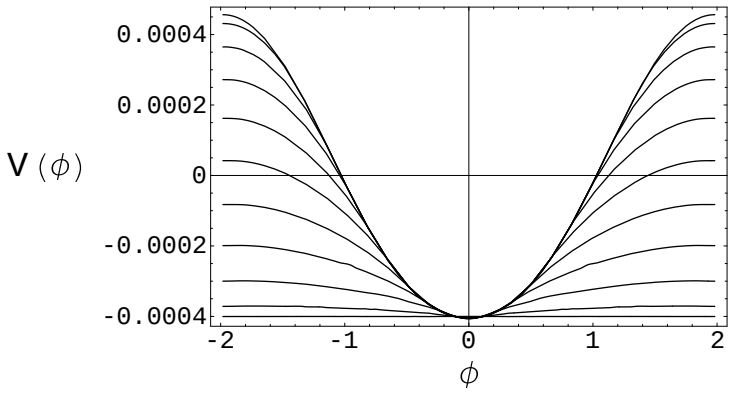

Figure 6.3: Flattening of the periodic potential below k_c. The decreasing order of the curves corresponds to decreasing values of the scale k with the step size $\Delta k = k_c/10$ starting with $k = k_c$.

6. In order to test the numerical solution we compared our results with the analytic formula for the amplitude $\rho_k(\phi_0)$ of the plane-wave like saddle point obtained for the polynomial potential [45]:

$$\rho_k(\phi_0) = \frac{1}{2}\left(\phi_{vac}(k) - |\phi_0|\right). \tag{6.4}$$

It is argued in [45], that this relation for $\rho_k(\phi_0)$ is universal with respect to the choice of the potential. Therefore, we can use it as a theoretical prediction to compare it to our numerical results obtained for periodic potentials. However, in the relation

(6.4) the first term on the right-hand side $\phi_{vac}(k)$ and the sign of the second term on the right-hand side depends on the particular case considered. For periodic potentials $V(\phi_0+\Delta)=V(\phi_0)$, the amplitude $\rho_k(\phi_0)$ should be periodic in the field variable with the length of period Δ for any scale k (see Eq. (4.20)). Thus, for periodic potentials, an expression similar to Eq. (6.4) is valid in the period $\phi_0 \in [-\Delta/2, \Delta/2]$, and then the same pattern of the function $\rho_k(\phi_0)$ is repeated in all other periods.

In the particular case investigated here, the minus sign in expression (6.4) should be changed to a plus sign and the field independent term is $\phi^{per}_{vac}(k_c) = -\Delta/2$ and $\phi^{per}_{vac}(0) = 0$. Therefore we can compare the theoretical result $\rho^{per}_{k=0}(\phi_0) = \frac{1}{2}|\phi|$ with that obtained by numerical integration of Eq. (4.20), see Fig. 6.4. We have established that with the increasing number of Fourier modes taken into account the computed curves get closer to the dashed line defined by $\rho^{per}_{k=0}(\phi_0) = \frac{1}{2}|\phi_0|$.

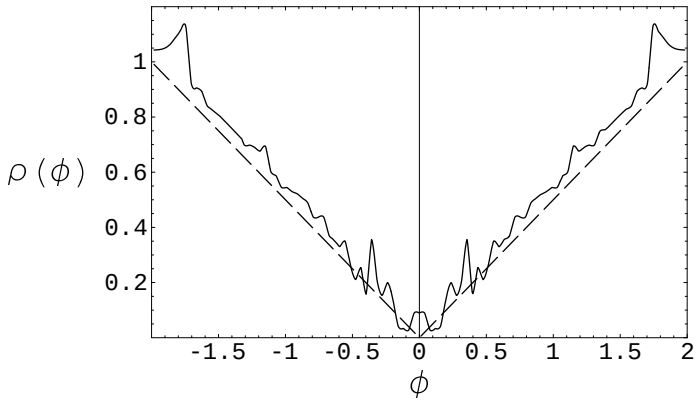

Figure 6.4: Comparison of the function $\rho_{k=0}(\phi_0)$ obtained by numerical integration of Eq. (4.20) with the analytic expression $\rho^{per}_{k=0}(\phi_0) = \frac{1}{2}|\phi|$ (dashed-line).

The amplitude of the non-trivial, plane-wave like saddle point is computed numerically and is found to be in a good agreement with the analytic result.

In this section we investigated the renormalization of the periodic potential by the numerical solution of the Wegner–Houghton renormalization group equations (4.21) and (4.20) obtained in the local-potential approximation where the wavefunction renormalization is not taken into account. In the next section we consider the numerical solution of the Wegner–Houghton equation including the field-independent wavefunction renormalization.

6.2 Wavefunction Renormalization in Wegner–Houghton's Method

In the previous chapter we have shown that the linearized Wegner–Houghton, the linearized Polchinski and the linearized real space RG equations for the periodic scalar field theory yield the same scaling laws around the Gaussian fixed-point including that for the field independent wavefunction renormalization. In this section we go beyond the linearization, we integrate numerically the Wegner–Houghton equation (4.25) reduced for the Fourier amplitude

of the sine-Gordon model $u_1(k)$ and for the field-independent wavefunction renormalization $z(k)$.

In order to compare Eq. (4.25) with the real-space equation (2.23), one should rewrite the Wegner–Houghton equation (4.25) using the formal equivalences between the momentum scale k (momentum cut-off) and the scale parameter a of the space-time (cut-off in the coordinate space), $k \sim 1/a$. Therefore, one can introduce the dimensionless couplings via $u_1(a) = -a^{-2}\tilde{h}/T$ and $\tilde{z} = 1/T$. Choosing the moving distance scale according to $1/a = (8\pi^2)^{1/4}k$, we get

$$\begin{aligned} a\frac{d\tilde{h}}{da} &= -\frac{1}{16\pi^3}\frac{T}{\tilde{h}}\left[1-\left(1-8\pi^2\tilde{h}^2\right)^{1/2}\right] + 2\tilde{h} - \pi T\tilde{h}^3\frac{\left(1+8\pi^2\tilde{h}^2\right)}{\left(1-8\pi^2\tilde{h}^2\right)^{5/2}}, \\ a\frac{dT}{da} &= -\pi T^2\tilde{h}^2\frac{\left(1+8\pi^2\tilde{h}^2\right)}{\left(1-8\pi^2\tilde{h}^2\right)^{5/2}}. \end{aligned} \tag{6.5}$$

Expanding (6.5) in series of $\tilde{h}(a)$, one finds:

$$\begin{aligned} a\frac{d\tilde{h}}{da} &= \tilde{h}\left[2-\frac{T}{4\pi}\left(1+2\pi^2\tilde{h}^2+...\right)-\pi T\tilde{h}^2\left(1+...\right)\right], \\ a\frac{dT}{da} &= -\pi T^2\tilde{h}^2\left(1+28\pi^2\tilde{h}^2+...\right). \end{aligned} \tag{6.6}$$

One can see that the leading order terms on the right-hand sides are those of Eqs. (2.23). The terms of higher order in $\tilde{h}$ yield the infinite series of corrections to the dilute-gas result and are summed up in a closed form on the right-hand sides of Eqs. (6.5).

The RG flow of $z(k)$ and $u_1(k)$ (i.e. $\tilde{h}(k)$, $T(k)$), obtained by the two different blocking transformations (real-space and Wegner–Houghton RG) are qualitatively the same. The real-space RG method (see Eqs. (2.23)) yielded the well-known phase structure for the Coulomb gas, (see Fig. 6.5) which has already been obtained in the literature [10,27] using the dilute gas approximation. There are two phases connected by the Kosterlitz–Thouless transition. In the molecular phase the vortices and anti-vortices form closely bound pairs while in the ionized phase they dissociate into a plasma. Similar results were obtained in Ref. [29], using momentum space RG with smooth cut-off, without the dilute gas approximation.

The flow diagram for the sine-Gordon model obtained by the Wegner–Houghton RG approach (see Eqs. (6.5)) using a sharp momentum cut-off, is plotted in Fig. 6.6. In order to compare the results obtained by the two different RG methods, four RG trajectories calculated by the real-space RG method are plotted in Fig. 6.6, as well. For small values of $\tilde{h}$ and T, the trajectories for both methods are the same for the same initial conditions. For larger values of $\tilde{h}$ the dilute gas approximation, for large values of T the Villain transformation loose their validity, so that the different RG trajectories belonging to the same initial values start to diverge. The flow diagram for the sine-Gordon model is valid for $\tilde{h}^2 < 8\pi^2$. At $\tilde{h}^2 = 8\pi^2$ a non-trivial saddle point occurs in the path integral (3.44) and the Wegner–Houghton equation loses its validity.

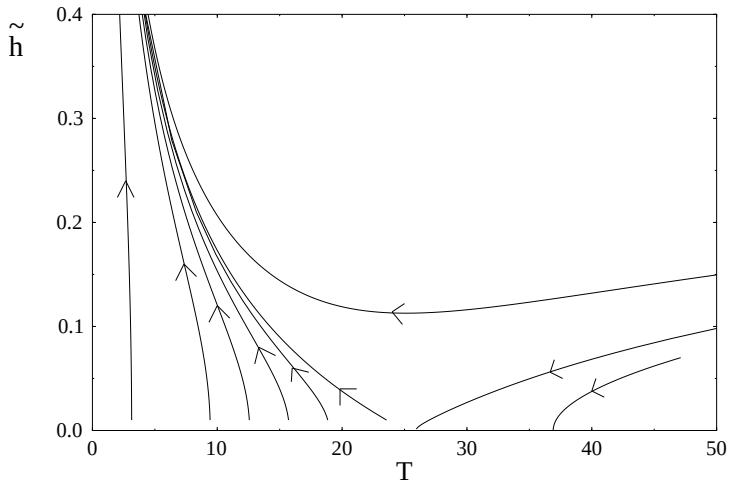

Figure 6.5: The phase structure for the two-dimensional Coulomb gas (or vortex-gas) is plotted, which was obtained by solving the differential equation (2.23), by the fourth-order Runge-Kutta method. The differential equations (2.23) were obtained by the real-space RG method using the dilute-gas approximation. The coupling constants are the dimensionless external field $\tilde{h}$ and the temperature T. We are using the unit system $c = \hbar = k_b = 1$. The solid lines represent the RG trajectories for different initial conditions (from left to the right: $\tilde{h}_{\rm init} = 0.01$, and $T_{\rm init} = 0.1\pi, 0.3\pi, 0.4\pi, 0.5\pi, 0.7\pi, 8\pi$ and $\tilde{h}_{\rm init} = 0.15$, $T_{\rm init} = 16\pi$; $\tilde{h}_{\rm init} = 0.1$, $T_{\rm init} = 16\pi$; $\tilde{h}_{\rm init} = 0.07$, $T_{\rm init} = 15\pi$) and arrows indicate the direction of the flow.

The results of two different RG methods were compared: The renormalization of the two-dimensional Coulomb gas by means of the real-space RG approach using the dilute gas approximation and the renormalization of the equivalent sine-Gordon model in the framework of the Wegner–Houghton RG approach including the field-independent wavefunction renormalization. For small values of the fugacity the two methods agree and a systematic way of obtaining higher-order corrections to the dilute gas approximation was provided.

6.3 Wavefunction Renormalization in Polchinski's Method

As it is discussed in chapter 5 the linearized Polchinski equation reproduces the linearized real-space RG equation for the two-dimensional Coulomb gas. In this section we compare the RG flow obtained by Polchinski's RG method to those obtained by the real-space RG approach using the dilute gas approximation for the two-dimensional Coulomb gas when the non-linear terms are kept.

Due to the equivalence between the sine-Gordon model and the two-dimensional vortex or Coulomb gas, it is possible to compare Polchinski's RG equations for the sine-Gordon model with equations (2.23). Since the real-space RG equations for the Coulomb gas are non-linear, one has to go beyond the linearized equations (4.37) in order to compare the equations obtained by the two different methods.

In the real-space RG equations (2.23) for the two-dimensional Coulomb gas only the field-

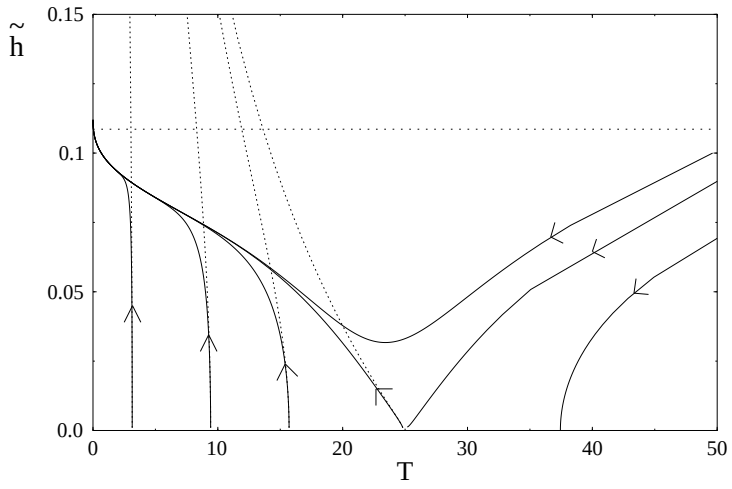

Figure 6.6: Phase structure for the sine-Gordon model. The full (dashed) lines correspond to the RG trajectories obtained by the Wegner–Houghton (real-space) RG methods (from left to the right the initial conditions are $\tilde{h}_{\rm init} = 0.001$, and $T_{\rm init} = 0.1\pi, 0.3\pi, 0.5\pi, 8\pi$ and $\tilde{h}_{\rm init} = 0.1$, $T_{\rm init} = 16\pi$; $\tilde{h}_{\rm init} = 0.09$, $T_{\rm init} = 16\pi$; $\tilde{h}_{\rm init} = 0.07$, $T_{\rm init} = 16\pi$). The flow diagram for the sine-Gordon model is valid for $\tilde{h}_c^2 < 8\pi^2$ (see horizontal dotted line), above this the Wegner–Houghton equation looses its validity.

independent wavefunction renormalization $\tilde{Z}_k(\phi_0) = \tilde{z}(k) = 1/\tilde{\beta}^2 = 1/\beta^2 = 1/T$ is taken into account, therefore we consider Polchinski's equations for the following two-dimensional model:

$$S_k[\phi] = \int \mathrm{d}^2x \left[\frac{1}{2} z(k)(\partial\phi)^2 + u_1(k)\, \cos(\phi)\right] \tag{6.7}$$

where the two dimensionful coupling constants are the Fourier amplitude $u_1(k)$ and the field-independent wavefunction renormalization $z(k)$. After changing to dimensionless coupling constants and inserting the ansatz (6.7) in Polchinski's equations (4.30) and neglecting the terms on the right-hand sides containing higher harmonics, we find

$$\begin{aligned} (2 + k\,\partial_k)\tilde{u}_1(k) &= \hbar\,\alpha_2 \tilde{u}_1(k), \\ k\,\partial_k \tilde{z}(k) &= -K_0''\,\tilde{u}^2(k). \end{aligned} \tag{6.8}$$

The terms containing the derivatives of the wavefunction renormalization with respect to the field ϕ_0 do not appear in (6.8), since $\tilde{Z}_k(\phi_0) = \tilde{z}(k)$ is independent of the field. In the first equation of (6.8), the field-independent term $\tilde{z}(k) I_1$ and the term $[\tilde{V}_k^{(1)}(\phi_0)]^2 K_0'$ do not give contributions for the Fourier mode $\cos(\phi_0)$ since $[\tilde{V}_k^{(1)}(\phi_0)]^2 = \tilde{u}_1^2(k)\sin^2(\phi_0) = \tilde{u}_1^2(k)(1/2 - 1/2\cos(2\phi_0))$. In the second equation of (6.8), only $2[\tilde{V}_k^{(2)}(\phi_0)]^2 K_0''$ gives field-independent contribution, since $[\tilde{V}_k^{(2)}(\phi_0)]^2 = \tilde{u}_1^2(k)\cos^2(\phi_0) = \tilde{u}_1^2(k)(1/2 + 1/2\cos(2\phi_0))$.

These equations should be compared to the flow equations of the sine-Gordon model ob-

tained in the Wegner–Houghton approach [2],

$$(2 + k\,\partial_k)\tilde{u}_1(k) = \hbar\left[\alpha_2 \frac{\tilde{u}_1(k)}{\tilde{z}(k)} + \mathcal{O}(\tilde{u}_1^3)\right],$$

$$k\,\partial_k \tilde{z}(k) = -\hbar\left[\frac{\alpha_2}{2}\frac{\tilde{u}_1(k)^2(k)}{\tilde{z}^2(k)} + \mathcal{O}(\tilde{u}_1^4)\right]. \tag{6.9}$$

These are the equations (4.25) rewritten for the dimensionless parameters $\tilde{u}_1(k)$ and $\tilde{z}(k)$. The significant difference is that the field-independent wavefunction renormalization in (6.8) occurs due to tree-level renormalization. Therefore, the field-independent piece of the wavefunction renormalization depends on the 'scheme', Which is equivalent to saying that it dpends on the details of the blocking procedure (of Polchinski's type with various regulators, or of Wegner–Houghton's type). For the choice $K_0'' = 0$, and on the linear level the scheme gets closer to the WH approach as also to the evolution of the local-potential, but the field-independent wavefunction renormalization does not alter during the blocking as opposed to the results obtained with the WH method, Eqs. (6.9). Then the choice $\tilde{z}(k) = 1/T(k) \equiv 1$ is consistent.

Using the equivalence between the lattice regularized compact sine-Gordon model and the X-Y model defined in (2.1),

$$\tilde{z} = 1/T = 1/\beta^2, \qquad \tilde{u}_1 = \tilde{h}/T\,, \tag{6.10}$$

one finds that Polchinski's equations (6.8) can be rewritten in the form of the two-dimensional Coulomb gas as follows:

$$a\,\frac{d\tilde{h}(a)}{da} = [2 - \hbar\alpha_d T]\ h, \qquad a\,\frac{dT(a)}{da} = 0 \ \rightarrow\ \ T(a) = T \equiv 1. \tag{6.11}$$

These equations (6.11) are rather different from those obtained by the real-space RG approach for the two-dimensional Coulomb gas (2.23).

It was concluded, that at the linear level, the RG flow equations are identical irrespectively of the blocking procedure by which they are obtained. Nevertheless, the RG equations obtained by Polchinski's method are rather different to those obtained by the real-space RG approach for the two-dimensional Coulomb gas. The differences between the various approaches occurs when the non-linearities are kept that are responsible for the violation of the UV scaling laws.

Summary of the results

In this Thesis, the non-perturbative renormalization of the Euclidean one-component scalar field theory model with periodic local self-interaction has been investigated. Field theorical models with periodicity in the internal space are of relevance both for statistical physics and quantum field theory. The simplest example for such models is the two-dimensional sine-Gordon model which is the one-component scalar field theory with the periodic, harmonic self-interaction defined in (2.2). The sine-Gordon model can be mapped to the statistical physical model, the X–Y model (2.8) describing the system of classical two-component spins. The sine-Gordon-model and the X–Y spin model belong, among others, to the same universality class, namely to that of the two-dimensional Coulomb gas. A short overview of the pertinent features of the sine-Gordon model, X–Y model and the two-dimensional Coulomb gas, and that of the mappings relating these models was given in chapter 2.

The main issue of the present Thesis is the non-perturbative renormalization of scalar field theorical models with periodicity in the internal space without breaking this symmetry, using various renormalization group (RG) approaches. Keeping periodicity explicit and unbroken requires the handling of infinitely many vertex-operators that cannot be achieved by the usual perturbative manner. Therefore, various versions of the differential renormalization group approach have been applied to the investigation of the renormalization of the periodic field theory. A comparison of the results obtained in the framework of the various renormalization schemes, has also been performed. Chapter 3 contains a brief introduction into Wegner-Houghton's and Polchinski's RG methods, and these are then specified for periodic potentials in chapter 4. The linearized RG flow around the Gaussian fixed-point is given and the periodic scaling operators of the potential are classified for $d = 2$ and $d > 2$ dimensions in chapter 5. The non-linear flow is discussed in chapter 6 by means of the numerical integration of the Wegner–Houghton and the Polchinski RG equations. This is the framework enabling one to compare the results obtained with the help of the Wegner–Houghton and the Polchinski methods to those well-known in the literature for the two-dimensional Coulomb gas. This is because most of the results for the Coulomb gas were obtained in the literature by means of the real-space RG method leading to non-linear RG equations.

Our findings can be grouped into two types of statements: those generically valid for any periodic scalar field theory models in any dimensions and those valid only in the 2-dimensional case.

1. *General results for periodic, one-component scalar field models.*

   (a) *The effective potential for a periodic scalar field theory model is constant.* This statement is related to the very general aspects of the renormalization of field theorical models with periodicity in the internal space. On the one hand, the RG

transformations preserve the periodicity of the action, like its other symmetries. It is also well-known that the solution of the RG equation providing the scale-dependent blocked potential $V_k(\phi_0)$ tends to the effective potential $V_{eff}(\phi_0)$ in the limit $k \to 0$. Therefore, the effective potential in a periodic scalar field theory should be periodic. On the other hand, the effective potential should also be convex (in the field variable). Periodicity and convexity are so strong constraints imposed on the effective potential that it should be a constant, i.e. a flat potential. This behaviour was verified by us in detail in particular cases.

(b) *The Wegner-Houghton and the Polchinski RG equations do not violate periodicity.* The Wegner–Houghton and the Polchinski RG equations were used to investigate the periodic scalar field theory. It was shown in chapter 4, that both types of flow equations retain the periodicity of the local potential. In the Polchinski method, periodicity of the blocked action is also kept in the next-to-leading order of the gradient expansion, i.e. when field-dependent wavefunction renormalization is included.

(c) *The UV scaling laws and the phase structure have been determined for any $d \geq 2$ dimensions by means of the linearized RG equations.* The phase structure, and the fixed-points of the periodic scalar model were considered by solving the linearized Wegner–Houghton and Polchinski renormalization group equations around the Gaussian fixed-point for $d = 2$ and $d > 2$ dimensions, as well.

i. *For $d = 2$ dimensions the well-known phase structure and scaling operators of the sine-Gordon model were recovered in a rather straightforward way.* There are two phases in the model: the strong coupling phase (where the coupling constant $\tilde{\beta} = \beta$ is larger than a critical value $\beta > \beta_c$) and the weak coupling phase ($\beta < \beta_c$). Here β is the coupling defining the length of period $2\pi/\beta$ of the blocked action in the internal space. In the strong coupling phase, the model was found to be non-renormalizable and trivial because the coupling constants approach zero in the IR limit ($k \to 0$) and this leads to the flattening of the effective potential. In the weak coupling phase there are coupling constants which increase with decreasing value of the scale parameter k, therefore the linearized flow equations and the UV scaling laws loose their validity. The correct scaling laws should be obtained by solving the exact non-linear RG equations.

ii. *For $d > 2$ dimensions in the local-potential approximation the UV scaling laws were found similar to those for $d = 2$ dimensions.* The similarity of the scaling for $d = 2$ and $d > 2$ dimensions is reflected by the exponent $D = \alpha_d \tilde{\beta}^2$, determining the scaling of the Fourier amplitudes of the blocked potential via $\tilde{u}_n(k) = (k/\Lambda)^{n^2 D - d}$. This is essentially the same in any dimensions disregarding its 'trivial' dependence on the entire solid angle Ω_d in $\alpha_d = \Omega_d/(2(2\pi)^d)$.

2. *Results for 2-dimensional periodic, one-component scalar field models.* Our results are related on the one hand, to the UV scaling laws and, on the other hand to the global features of the flow including its features in the IR regime.

(a) *UV scaling.*

i. *The UV scaling laws for the local potential and the field-independent part of the wavefunction renormalization are independent of the RG scheme used.* The UV scaling laws are the solutions of the linearized RG equations. It was shown, that the linearized forms of the Wegner–Houghton and the Polchinski equations for the 2-dimensional sine-Gordon model, and those of the real-space RG equations for the 2-dimensional Coulomb gas in the dilute gas approximation give the same scaling operators around the Gaussian fixed-point. The field-independent part of the wavefunction renormalization does not scale, i.e. appears to be marginal in the linear approximation.

ii. *An approximation scheme is proposed that enables one to find corrections to the dilute gas results for the 2-dimensional Coulomb gas in a systematic manner.* In the next-to-leading order of the gradient expansion and retaining only the field-independent part of the wavefunction renormalization, the flow diagram obtained by the Wegner–Houghton RG approach for the sine-Gordon model agrees well with that obtained by the real-space RG approach in the dilute gas approximation for the Coulomb gas for small values of the fugacity. Making use of the equivalence of these models, an approximation scheme for obtaining corrections in a systematic manner to the dilute gas results was proposed, which is based on expanding the non-linear terms of the Wegner-Houghton equations in power series in the fugacity. This approximation scheme has the advantage that it is free of the ambiguities generally present in the rather intuitive approaches by which similar corrections have been obtained in the literature.

iii. *The field-dependent part of the wavefunction renormalization is irrelevant in both phases in the UV scaling regime.* Including the wavefunction renormalization, the periodic scaling operators for the field-dependent part of the wavefunction renormalization were determined around the Gaussian fixed-point by solving the linearized RG equations. The coupling constants of the periodic field-dependent part of the wavefunction renormalization monotonically decrease for decreasing scale k in this regime.

(b) *Global features of the RG flow.*

i. *The IR scaling laws were found for the weak coupling phase in the local potential approximation solving the Wegner-Houghton RG equations for the periodic scalar field model numerically, and the flattening of the blocked potential was shown.* For the weak coupling phase of the model ($\beta^2 < 8\pi$), where the linearization ceases to be reliable, the solution was found numerically in the local-potential approximation, monitoring the coupling constants $u_n(k)$ $(1 \leq n \leq N)$ of the periodic blocked potential. A thoroughful comparison of the evolution has been performed in terms of the polynomial couplings $g_n(k)$ to the evolution of the polynomial potential defined in (6.2). The latter was chosen to approximate the periodic potential at the UV scale in the neighbourhood of one of its minima. A remarkable difference was found in the evolution of the periodic potential as compared to that of the corresponding polynomial potential. Namely, that all the dimensionful coupling constants $g_n(k)$ obtained for the periodic potential tend to zero in contrast to those of the polynomial potential which remain finite in the limit $k \to 0$. Thus, it was

found that the blocked periodic potential flattens out with decreasing scale k.

ii. *The flattening of the blocked potential is due to the large-amplitude tree-level fluctuations of the field in the weak coupling phase, while it occurs due to the small-amplitude fluctuations filling the various minima of the potential in the strong coupling phase.* For periodic models in the weak coupling phase, there are coupling constants which become sufficiently large to destabilize the trivial saddle point of the blocking when the scale k falls below the non-vanishing critical value k_c. Then, a non-trivial saddle point occurs and the tree-level blocking Eq. (4.20) must be used. Integrating the tree-level equation (4.20) numerically we have shown that the periodic potential becomes a constant one up to an absolute accuracy of 10^{-5} as a result of the successive infinitesimal RG transformations.

iii. *Due to the non-linear nature of the RG equations, the RG flow depends heavily on the renormalization scheme used.* The renormalization of the two-dimensional Coulomb gas by means of the real-space RG approach in the dilute gas approximation was compared to the renormalization of the equivalent sine-Gordon model in the framework of the Wegner–Houghton RG approach in next-to-leading order of the gradient expansion, only including the field-independent part of the wavefunction renormalization. As stated previously, the two methods agree well for small values of the fugacity of the Coulomb gas. It was shown, however, that retaining the non-linear terms leads to significant discrepancies. Determining corrections to the dilute gas results for the Coulomb gas as described under 2(a)ii., we took the point of view that the Wegner–Houghton approach is the more reliable one concerning the non-linear terms.

An attempt to perform the renormalization by means of the Polchinski method including the field-dependent part of the wavefunction renormalization has been made, as well. For this purpose the renormalization scheme-dependence of the Polchinski RG equation, i.e. its dependence on the regulator function was discussed. The position of the fixed-point and the phase structure of the periodic models are independent of the choice of the regulator function, while the actual flow depends on it. The conclusion has been drawn that the differences occur between the various RG schemes when the non-linearities are kept that are responsible for the violation of the UV scaling laws.

It was noticed that the tree-level renormalization present at all scales in the Polchinski approach depends on the regulator and appears as if there was an instability for the zero mode of the field. Since the Polchinski method treats all the modes above and below the moving cut-off k on equal footing, it is unclear wheather this method can be applied at all for finding the spinodal instability for some particular mode with finite momentum k_c, that is the instability found by the Wegner–Houghton approach in the weak coupling phase. Therefore, an independent verification of the results under 2(b)i., ii. for the flattening of the blocked potential in the weak coupling phase could not be obtained by the Polchinski method.

iv. *The field-dependent part of the wavefunction renormalization remains irrelevant over all scales down to the IR limit for the strong coupling phase.* Including the wavefunction renormalization, the periodic scaling operators for the field-dependent part of the wavefunction renormalization were determined and found to decrease monotonically in the strong coupling phase by means of the linearized Polchinski equations. The closer the scale k is to the IR limit $k \to 0$, the more the linear approximation gets then justified. This means that the monotonic decrease of the couplings is exhibited over all scales. For the weak coupling phase the linearized flow equations loose their validity, and our last remark under point 2(b)iii. applies.

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Appendix A Conventions and Symbols

In this appendix we collect the symbols and conventions used in this Thesis. The renormalization of the periodic field theory is considered in the framework of the differential renormalization group approach. We follow the conventions of the rewiews [8, 23] on renormalization group methods. The symbols and conventions are:

- x, x_1, x_0 x^μ – **spacetime coordinates,**
- d – **dimension,**
- $\phi(x)$, $\theta(x)$ – **spacetime-dependent real field variable,**
- ϕ_o – **constant real field, independent of the spacetime coordinate** x**,**
- $\hat{\phi} = \hat{\phi}_{cl}$ – **saddle point of the path integral,**
- $\int \mathrm{d}^d x$ – **integration in spacetime for dimension** d**,**
- p, q – **momentum space coordinates,**
- $\int \mathrm{d}^d p$ – **integration in momentum space for dimension** d**,**
- k – **moving momentum cut-off; scale-parameter of the RG transformations,**
- Λ – **UV cut-off; initial value for** k**,**
- a – **lattice space,**
- ξ – **correlation length**
- Ω_d – **solid angle for dimension** d**.**

The parameters of the models are defined in the following way:

- T – **temperature,**
- h – **magnitude of the external field,**
- y – **fugacity of the Coulomb gas or vortex gas,**
- β – **length of period of the periodic potential,**
- $Z_k(\phi)$ – **field-dependent wavefunction renormalization,**

- $V_k(\phi)$ – **field-dependent potential,**
- $u_n(k)$ – **scale-dependent Fourier amplitudes of the periodic potential,**
- $z_n(k)$ – **Fourier amplitudes of the periodic wavefunction renormalization,**
- $z(k)$ – **field-independent wavefunction renormalization.**

These are dimensionful parameters. One can define dimensionless quantities removing the trivial scale-dependence of the dimensionful quantities. We distinguish between the dimensionful and the dimensionless parameters in a following way:

- $u_n(k)$, $z_n(k)$, $V_k(\phi)$ **etc. – are dimensionful quantities,**
- $\tilde{u}_n(k)$, $\tilde{z}_n(k)$, $\tilde{V}_k(\tilde{\phi})$ **etc. – are dimensionless quantities.**

Appendix B Wegner–Houghton Equation for Periodic Potentials in LPA

In chapter 4, it is proven, that the Wegner–Houghton equation obtained in the leading order of the gradient expansion (local-potential approximation) for a periodic scalar field theory, retains the periodicity of the potential. In the local-potential approximation, the renormalization of the action is considered for a homogeneous field configuration ($\phi(x) \to \phi_0$) where the wavefunction renormalization is set $Z_k(\phi) = 1$. In this case the Wegner–Houghton equation reduces to the following differential equation for the scale-dependent potential $V_k(\phi_0)$ [see Eq. (3.23)],

$$k\partial_k V_k(\phi_0) = -k^d\, \alpha_d\, \ln\left(\frac{k^2 + \partial_{\phi_0}^2 V_k(\phi_0)}{k^2}\right), \tag{B.1}$$

with $\alpha_d = \frac{1}{2}\Omega_d(2\pi)^{-d}$, and the solid angle Ω_d in dimension d (using the unit system $\hbar = 1$). Since equation (B.1) retains the periodicity of the potential, one can look for a solution of equation (B.1) among the periodic functions. For the sake of simplicity we consider periodic potentials with $Z(2)$ symmetry. One can reduce the RG equation (B.1) to the differential equation for the Fourier amplitudes of the potential which are the coupling constants of the theory (4.21). In this appendix we give the details of the derivation of (4.21).

There are dimensionful quantities in equation (B.1). In order to look for fixed-point solutions of (B.1) or to follow the scaling of an arbitrary potential one should remove the trivial scaling of the dimensionful coupling constants and rewrite equation (B.1). Since equation (B.1) retains the periodicity of the potential with the dimensionless length of period $\tilde{\beta}$, therefore, we use the following dimensionless form of the potential

$$\tilde{V}_k(\tilde{\phi}_0) = \sum_{n=1}^{\infty} \tilde{u}_n(k) \cos\left(n\tilde{\beta}\tilde{\phi}_0\right). \tag{B.2}$$

where the dimensionless coupling constants are the Fourier amplitudes $\tilde{u}_n(k) = k^{-d}u_n(k)$ and the dimensionless length of period $\tilde{\beta}$. Then the equation (B.1) can be rewritten:

$$(d + k\partial_k)\,\tilde{V}_k(\theta) = -\alpha_d\, \ln\left(1 + \tilde{\beta}^2\, \partial_\theta^2 \tilde{V}_k(\theta)\right), \tag{B.3}$$

with $\theta = \tilde{\beta}\tilde{\phi}_0 = \beta\phi_0$ dimensionless. It is easier to use the derivative of Eq. (B.3) with respect to θ rather than the original equation itself,

$$(d + k\partial_k)\tilde{V}_k^{(1)}(\theta) + \alpha_d\,\tilde{\beta}^2\tilde{V}_k^{(3)}(\theta) + \tilde{\beta}^2\tilde{V}_k^{(2)}(\theta)(d + k\partial_k)\tilde{V}_k^{(1)}(\theta) = 0. \tag{B.4}$$

with $\tilde{V}_k^{(n)}(\theta) = \partial_\theta^n \tilde{V}_k(\theta)$. In order to consider the numerical solution of the equation (B.4) the Fourier series of the potential has to be truncated:

$$\begin{aligned}
\tilde{V}_k^{(1)}(\theta) &= -\sum_{n=1}^{N} \tilde{u}_n(k) n \,\sin(n\,\theta)\,, \\
\tilde{V}_k^{(2)}(\theta) &= -\sum_{n=1}^{N} \tilde{u}_n(k) n^2 \,\cos(n\,\theta)\,, \\
\tilde{V}_k^{(3)}(\theta) &= \sum_{n=1}^{N} \tilde{u}_n(k) n^3 \,\sin(n\,\theta)\,.
\end{aligned} \tag{B.5}$$

In order to reduce the renormalization group equation (B.4) to a differential equation for the Fourier amplitudes $\tilde{u}_n(k)$, one has to multiply two Fourier series,

$$\begin{aligned}
&\tilde{V}_k^{(2)}(\theta)\,(d + k\partial_k)\,\tilde{V}_k^{(1)}(\theta) = \\
&\qquad -\sum_{n}^{N}\sum_{m}^{N} \tilde{u}_n(k) n^2 (d + k\partial_k)\tilde{u}_n(k) n \,\cos(n\,\theta)\,\sin(n\,\theta)\,.
\end{aligned} \tag{B.6}$$

Using the identity,

$$\cos(n\theta)\,\sin(n\theta) = \frac{1}{2}\sin[(n+m)\theta] - \frac{1}{2}\sin[(n-m)\theta] \tag{B.7}$$

the following result can be obtained,

$$\begin{aligned}
\tilde{V}_k^{(2)}(\theta)\,(d + k\partial_k)\,\tilde{V}_k^{(1)}(\theta) &= \frac{1}{2}\sum_{l=1}^{N}\sum_{j=1}^{n} \tilde{u}_{l-j}(k)(l-j)^2\,(d + k\partial_k)\,\tilde{u}_j(k) j \,\sin(l\,\theta) \\
&\quad + \frac{1}{2}\sum_{l=1}^{N}\sum_{j=1}^{N-l} \Big[\tilde{u}_j(k) j^2\,(d + k\partial_k)\,\tilde{u}_{l+j}(k)(l+j) \\
&\quad - \tilde{u}_{l+j}(k)(l+j)^2\,(d + k\partial_k)\,\tilde{u}_j(k) j\Big] \sin(l\,\theta)\,.
\end{aligned} \tag{B.8}$$

Then the Wegner–Houghton equation for the Fourier amplitudes of the periodic potentials reads as follows,

$$0 = (d + \alpha\tilde{\beta}^2 l^2 + k\partial_k) l\tilde{u}_l(k)$$

$$-\frac{1}{2}\,\tilde{\beta}^2\Big[\sum_{j=1}^{i}(l-j)^2 j\tilde{u}_{l-j}(k)(d+k\partial_k)\tilde{u}_j(k)$$

$$-\sum_{j=1}^{N-i} j(l+j)[j\tilde{u}_j(k)\,(d+k\partial_k)\,\tilde{u}_{l+j}(k)-(l+j)\tilde{u}_{l+j}(k)\,(d+k\partial_k)\tilde{u}_j(k)]\Big]. \tag{B.9}$$

Introducing a new variable

$$\tilde{v}_l(k)=\,l\,\tilde{u}_l(k), \tag{B.10}$$

and using the following relation

$$\sum_{j=1}^{N-l} l\tilde{v}_l(k)(d+k\partial_k)\tilde{v}_{l+j}(k)=$$

$$\sum_{f=l}^{N}(f-l)\tilde{v}_{f-l}(k)(d+k\partial_k)\tilde{v}_f(k)=\sum_{j=l}^{N}(j-l)\tilde{v}_{j-l}(k)(d+k\partial_k)\tilde{v}_j(k), \tag{B.11}$$

the general form of the evolution equation for the Fourier amplitudes can be obtained in arbitrary dimension d, using the local-potential approximation [1],

$$\alpha\tilde{\beta}^2 l^2\tilde{v}_l(k)=(d+k\partial_k)\tilde{v}_l(k)-\frac{1}{2}\tilde{\beta}^2\sum_{j=1}^{N}A_{lj}(k)(d+k\partial_k)\tilde{v}_j(k), \tag{B.12}$$

where N is the truncation in the Fourier series in (B.2), with dimensionless $\tilde{\beta}$ and with dimensionless coupling constants $\tilde{v}_j(k)=j\tilde{u}_j(k)$ and

$$\begin{aligned} A_{ij}(k) \;=\; & (l-j)\,\tilde{v}_{l-j}\,\Theta(l\geq j) \\ & +(j-l)\,\tilde{v}_{j-l}\,\Theta(j\geq l) \\ & -(l+j)\,\tilde{v}_{l+j}\,\Theta(N\geq l+j), \end{aligned} \tag{B.13}$$

with $\Theta(l\geq l')=1$ if $l\geq l', 0$ if $l<l'$. This is the equation (4.21) of chapter 4.

Appendix C Tree-level Blocking Relation for Periodic Potentials in LPA

The Wegner–Houghton renormalization group equation (3.20) is derived and discussed in chapter 3. It is obtained by using the Kadanoff-Wilson blocking construction which is a successive elimination of infinitely many degrees of freedom. In order to derive the Wegner–Houghton equation one has to evaluate a path-integral (3.13) using the saddle point approximation. If the saddle point of (3.13) is trivial ($\hat{\phi}_{cl} = 0$), the resulting integro-differential equation is the Wegner–Houghton equation. If the saddle point is non-trivial ($\hat{\phi}_{cl} \neq 0$), the tree-level blocking relation (3.21) has to be used,

$$S_{k-\Delta k}[\phi] = \min_{\hat{\phi}_{cl}} \left[S_k[\phi + \hat{\phi}_{cl}] \right], \tag{C.1}$$

which is determined by minimizing the action in the path-integral (3.13). In this appendix we specify the equation (C.1) for the periodic scalar field theory taken in the local potential approximation.

In local potential approximation ($\phi(x) \to \phi_0$) and when the system is considered in a box with the linear size L with periodic boundary conditions for the saddle point, equation (C.1) can be rewritten,

$$L^d V_{k-\Delta k}[\phi_0] = \min_{\hat{\phi}_{cl}} \left[\int_{-L/2}^{L/2} \mathrm{d}^d x \left(\frac{1}{2} (\partial \hat{\phi}_{cl}(x))^2 + V_{k-\Delta k}[\phi_0 + \hat{\phi}_{cl}(x)] \right) \right]. \tag{C.2}$$

One can restrict the search of the minimum to a subspace of functions, e. g. to plane-wave like saddle points [45] propagating in a given direction $n_{\mu=1}$

$$\hat{\phi}_{cl}(x) = 2\rho \cos(k x_1). \tag{C.3}$$

As a consequence of the boundary conditions for the saddle point the following relation holds

$$\hat{\phi}_{cl}(x) = \hat{\phi}_{cl}(x + L) \qquad \to \qquad L = \frac{2\pi}{k}. \tag{C.4}$$

Then the right-hand side of equation (C.2) can be rewritten for the kinetic part and for the potential,

$$\begin{aligned}\int_{-L/2}^{L/2} \mathrm{d}^d x\ \frac{1}{2}\,(\partial\,\hat{\phi}_{cl}(x))^2 &= 2\rho^2 k^2 L^{d-1}\int_{-L/2}^{L/2}\mathrm{d}x_1\ \sin^2(kx_1) = \rho^2 k^2 L^d,\\ \int_{-L/2}^{L/2}\mathrm{d}^d x\, V_k(\phi_0+\hat{\phi}_{cl}(x)) &= L^d\frac{1}{2}\int_{-1}^{1}\mathrm{d}u\, V_k(\phi_0+2\rho\cos(\pi u)).\end{aligned} \tag{C.5}$$

Therefore, in local-potential approximation, by restricting the search for the minimum to plane-waves, one can obtain the following relation

$$V_{k-\Delta k}(\phi_0) = \min_{\rho}\left[k^2\rho^2 + \frac{1}{2}\int_{-1}^{1}\mathrm{d}u\ \ V_k(\phi_0+2\rho\cos(\pi u))\right], \tag{C.6}$$

where ρ is the amplitude of the plane wave. This is equation (7) of Ref. [45].

It is argued in chapter 4, that for periodic potentials equation (C.6) retains the periodicity of the potential. In this appendix the relation (C.6) is specified for periodic potentials. In order to allow for a convenient truncation of the potential which is favorable in the numerical solution of the evolution equation and to preserve the periodicity we write $V_k(\phi_0)$ as a Fourier series,

$$V_k(\phi_0) = \sum_{n=0}^{\infty} u_n(k)\cos\left(n\beta\phi_0\right). \tag{C.7}$$

For the sake of simplicity we consider only potentials with Z(2) symmetry, $V_k(\phi_0) = V_k(-\phi_0)$. The whole scale dependence occurs in the Fourier amplitudes $u_n(k)$, the 'coupling constants' of the scale dependent potential. The Fourier amplitudes $u_n(k)$ are dimensionful quantities. Inserting (C.7) into the relation (C.6), one obtain

$$V_{k-\Delta k}(\phi_0) = \min_{\rho}\left[k^2\rho^2 + \frac{1}{2}\sum_{n=0}^{\infty}u_n(k)\int_{-1}^{1}\mathrm{d}u\ \ \cos[n\beta(\phi_0+2\rho\cos(\pi u))]\right]. \tag{C.8}$$

Using the identity,

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{C.9}$$

equation (C.8) is written as,

$$\begin{aligned}V_{k-\Delta k}(\phi_0) = \min_{\rho}\Big[k^2\rho^2 + \frac{1}{2}\sum_{n=0}^{\infty}u_n(k)\\ \int_{-1}^{1}\mathrm{d}u\,[\cos(2n\beta\rho\cos(\pi u))\cos(n\beta\phi_0) - \sin(2n\beta\rho\cos(\pi u))\sin(n\beta\phi_0)]\Big]\end{aligned} \tag{C.10}$$

Then one can perform the integrals,

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}u \ \sin(2n\beta\rho\cos(\pi u)) &= 0,\\ \int_{-1}^{1} \mathrm{d}u \ \cos(2n\beta\rho\cos(\pi u)) &= 2\,J_0(2n\beta\rho),\end{aligned} \tag{C.11}$$

where J_0 is the zeroth order Bessel function.

Then the tree-level equation (C.6), in the local-potential approximation for dimensionful periodic potentials reads as follows,

$$V_{k-\Delta k}(\phi_0) = \min_{\rho}\left[k^2\rho^2 + \sum_{n=0}^{\infty} u_n(k)\cos(n\beta\phi_0)J_0(2n\beta\rho)\right], \tag{C.12}$$

where J_0 stands for the Bessel function. This is equation (4.20) of chapter 4.

Appendix D Wegner–Houghton Equation for Periodic Potentials Including $z(k)$

It is proven in chapter 4, that the Wegner–Houghton renormalization group equation obtained for the periodic scalar field theory in local potential approximation, preserves the periodicity of the theory. In order to retain the higher-order contributions in the gradient expansion, the field-independent wavefunction renormalization constant $z(k)$ is included into the kinetic energy of the action (3.32). Since $z(k)$ is field-independent, it does not change the situation and the Wegner–Houghton equation retains the periodicity of the action in this case as well. Therefore, keeping the next-to-leading order term in the gradient expansion, the Wegner–Houghton equation reduces to a differential equation for the potential $V_k(\phi_0)$ and for the field-independent wavefunction renormalization $z(k)$,

$$\begin{aligned} k\partial_k V_k(\phi_0) &= -k^d \hbar\, \alpha\, \ln\left(\frac{z(k)k^2 + \partial_{\phi_0}^2 V_k(\phi_0)}{k^2}\right), \\ k\partial_k z(k) &= k^d \hbar\, \alpha\, [\partial_{\phi_0}^3 V_k(\phi_0)]^2 \left[\frac{4\,[z(k)]^2\, k^2}{d\ A^4} - \frac{z(k)}{A^3}\right], \end{aligned} \tag{D.1}$$

with $A = (z(k)k^2 + \partial_{\phi_0}^2 V_k(\phi_0))$. This is equation (3.33) of chapter 3. The last equation should hold only up to $\mathcal{O}(\phi_0^0)$ since the ansatz for the action contains only the field-independent wavefunction renormalization.

Instead of considering an arbitrary periodic action, in this appendix, we derive RG equations (4.25) in the framework of the Wegner–Houghton equation for the sine-Gordon model including the field-independent wavefunction renormalization constant $z(k)$ as well. The two-dimensional sine–Gordon model is the simplest example for a periodic scalar field theory, defined with the Euclidean action

$$S_k = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial\phi)^2 + u_1(k)\cos(\beta(k)\phi)\right] \tag{D.2}$$

with the two scale-dependent coupling constants are $u_1(k)$ and $\beta(k)$. Rescaling the field $\phi' = \beta\phi$, the action (4.23) can be written as

$$S_k = \int \mathrm{d}^2x \left[\frac{1}{2}z(k)(\partial\phi')^2 + u_1(k)\cos(\phi')\right] \tag{D.3}$$

with the field independent renormalization $z(k) = 1/\beta^2(k)$. The flow equations for the dimensionful coupling constants $u_1(k)$ and $z(k)$ are obtained by expanding both sides of Eq. (D.1) in Fourier series and neglecting the higher harmonics missed on the left-hand sides. Therefore, in order to derive the flow equation for $z(k)$ from the second equation of (D.1), one has to calculate the following integrals,

$$I_n = \frac{1}{\pi}\int_0^\pi \mathrm{d}\phi_o \frac{\left[V_k^{(3)}(\phi_0)\right]^2}{\left[z(k)k^2 + V_k^{(2)}(\phi_0)\right]^n}, \qquad \mathrm{n} = 3,4 \tag{D.4}$$

where $V_k^{(m)}(\phi_0) = \partial_{\phi_0}^m V_k(\phi_0)$. Inserting the ansatz for the action (D.3) into (D.4), the integral I_n can be written,

$$I_n = \frac{1}{\pi}\int_0^\pi \mathrm{d}\phi_0 \frac{u_1^2(k)\sin^2(\phi_0)}{[z(k)k^2 - u_1(k)\cos(\phi_0)]^n}, \qquad \mathrm{n} = 3,4. \tag{D.5}$$

Let us discuss the integral $I_{n=3}$ first. It can be separated into two parts ($I_3 = I_a + I_b$),

$$\begin{aligned} I_a &= \frac{1}{\pi}\int_0^\pi \mathrm{d}\phi_0 \frac{u_1^2(k)}{[z(k)k^2 - u_1(k)\cos(\phi_0)]^3}, \\ I_b &= -\frac{1}{\pi}\int_0^\pi \mathrm{d}\phi_0 \frac{u_1^2(k)\cos^2(\phi_0)}{[z(k)k^2 - u_1(k)\cos(\phi_0)]^3}. \end{aligned} \tag{D.6}$$

which can be calculated separately using the tables of integrals [52]. The final result for the integral I_3 is,

$$I_3 = -\frac{1}{2}\frac{x^2(k)}{(z(k)\,k^2)\,[1-x^2(k)]^{3/2}}, \tag{D.7}$$

with $x(k) = u_1(k)/(z(k)\,k^2)$. Then the integral I_4 can be obtained by derivating the result for I_3, with respect to $(z(k)\,k^2)$,

$$I_4 = -\frac{1}{3}\left(\frac{\mathrm{d}}{\mathrm{d}(z(k)\,k^2)}\ I_3\right) = -\frac{1}{2}\frac{x^2(k)}{(z(k)\,k^2)^2\,[1-x^2(k)]^{5/2}}. \tag{D.8}$$

Then the renormalization group flow equation for $z(k)$ reads as follows,

$$k\partial_k z(k) = \frac{1}{4\pi}\,2\,(z(k)\,k^2)^2\,I_4 - \frac{1}{4\pi}\,(z(k)\,k^2)\,I_3 = -\frac{1}{8\pi}\frac{x^2(k)\,(1+x^2(k))}{[1-x^2(k)]^{5/2}}, \tag{D.9}$$

with $\alpha = 1/(4\pi)$ and using the unit system $\hbar = 1$.

Then we derive the RG flow equation for the coupling constant $u_1(k)$. It is straightforward to use the derivative of the first equation in (D.1) with respect to ϕ_0,

$$k\partial_k V_k^{(1)}(\phi_0) = -k^d \alpha\, \hbar \frac{V_k^{(3)}(\phi_0)}{z(k)k^2 + V_k^{(2)}(\phi_0)}. \tag{D.10}$$

Inserting the ansatz for the action (D.3) into equation (D.10) we find,

$$-k\partial_k u_1(k)\sin(\phi_0) = -k^d \alpha\, \hbar \frac{u_1(k)\sin(\phi_0)}{z(k)k^2 - u_1(k)\cos(\phi_0)}. \tag{D.11}$$

In order to obtain the flow equation for $u_1(k)$ the following integral has to be performed,

$$I = \frac{2}{\pi}\int_0^\pi \mathrm{d}\phi_0 \; \frac{u_1(k)\sin(\phi_0)}{z(k)k^2 - u_1(k)\cos(\phi_0)} \; \sin(\phi_0). \tag{D.12}$$

For this, one can use the relation taken from [52],

$$\int_0^\pi \mathrm{d}\phi_0 \frac{\sin^2(\phi_0)}{p + q\cos(\phi_o)} = \frac{p\pi}{q^2}\left[1 - \left(1 - \frac{q^2}{p^2}\right)^{1/2}\right]. \tag{D.13}$$

Then, the differential equation for the Fourier amplitude $u_1(k)$ can be read off,

$$k\partial_k u_1(k) = k^2 \frac{1}{2\pi}\frac{1}{x(k)}\left[1 - (1 - x(k))^{1/2}\right]. \tag{D.14}$$

Therefore, the flow equation for the dimensionful Fourier amplitude $u_1(k)$ and for the dimensionful wavefunction renormalization $z(k)$ reads as follows,

$$\begin{aligned} k\partial_k u_1(k) &= \frac{k^2}{2\pi}\frac{1}{x(k)}\left[1 - \left(1 - x^2(k)\right)^{1/2}\right], \\ k\partial_k z(k) &= -\frac{1}{8\pi}\frac{x^2(k)\left(1 + x^2(k)\right)}{\left(1 - x^2(k)\right)^{5/2}} \end{aligned} \tag{D.15}$$

with $x = u_1(k)/(k^2 z(k))$ which is equation (4.25) in chapter 4.

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