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ABSTRACT. We introduce the concept of the extension spectrum of a Hilbert space operator. This is a natural subset of the spectrum which plays an essential role in dealing with certain extension properties of operators. We prove that it has spectral-like properties and satisfies a holomorphic version of the Spectral Mapping Theorem. We establish structural theorems for algebraic extensions of triangular operators which use the extension spectrum in a natural way. The extension spectrum has some properties in common with the Kato spectrum, and in the final section we show how they are different and we examine their inclusion relationships.

0. INTRODUCTION

Let $B(H)$ be the algebra of all bounded operators acting on a separable complex Hilbert space H. An *extension* of an operator $A \in B(H)$ by an operator $C \in B(K)$ is an operator of the form

$$(*) \qquad T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

acting on $H \oplus K$ for some $B \in B(K, H)$. The extension is called *null* if C is the zero operator on K. The extension is called *finite* if the extension space K is finite dimensional. An operator T in $B(H)$ is called *triangular* if H has an orthonormal basis $\{e_1, e_2, \dots\}$ with the property that $Te_n \in \operatorname{span}\{e_1, \dots, e_n\}$ for each $n \in \mathbb{N}$. Then T is said to be triangular *with respect to* $\{e_n\}$. This article has its roots in the earlier papers [14, 15, 21]. In [21] the fourth author proved several counterexamples which answered some old open questions in operator theory. A few of the counterexamples had the form of finite extensions of triangular operators. Further investigation of these and other examples led to the papers [14], [15] and [12], and subsequently [16], [17] and [7]. The term *semitriangular* was first used in [14] to denote a finite extension of a triangular operator.

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An operator is called *algebraic* if it satisfies a nontrivial polynomial identity. Algebraic operators are easily shown to be triangular, and in fact have a wide family of triangular bases. A finite extension of an algebraic operator is algebraic, hence is triangular. However, it is a curious fact that finite extensions of triangular operators need not be triangular. Indeed, this "fact" is at the bottom of some of the interesting pathology mentioned above (including several of the counterexamples to well-known open questions) that has been discovered concerning single operators on Hilbert space and their invariant subspace and reflexivity properties (cf. [3, 4, 9, 12, 14, 15, 21]). We refer to [2], [8] [10], [18] and [19], etc. for more related work on reflexivity and triangularization of operators and subspaces of operators.

This paper is a new, much improved version of an earlier unpublished article "The triangular extension spectrum and algebraic extensions of operators" which dealt only with extensions of operators which were triangular. This paper supercedes that article and is far less restrictive. We found to our surprise that many of the concepts and results make sense and are valid for arbitrary operators, sometimes with only a slight degree of increase in technical difficulty of proofs, and other times with the need for new innovative techniques. So this present version is more general, and also more natural. Likewise, although much of our interest lies in finite extensions of operators, we discovered that many proofs go through sometimes with no more difficulty, for the wider class of extension by algebraic operators; i.e., the case where C in $(*)$ satisfies a polynomial identity. So when appropriate we state and prove our results in the wider context.

This paper is organized as follows: In Section 2 we shall prove a spectral mapping theorem and two stability results for the extension spectrum. The stability results are needed in obtaining the structural theorems for algebraic extensions of triangular operators in Section 3. Section 4 is devoted to examining the relationship between the extension spectrum and the Kato spectrum of operators.

1. PRELIMINARIES

Semi-triangular operators frequently fail to be triangular. For instance, if $A = \mathrm{diag}(1, \frac{1}{2}, \frac{1}{3}, \dots)$, B is the column vector with entries $(1, \frac{1}{2}, \frac{1}{3}, \dots)$, and C is the one-dimensional zero matrix, then it is not hard to show that there is no orthonormal basis for the direct sum space for which the operator T in $(*)$ is triangular (The operator T is triangular in the generalized sense of having a multiplicity free nest of invariant subspaces, but not in the standard, more restrictive, sense of triangularity defined in the first paragraph). In fact, one of the main results in [12] states that if a triangular operator is not algebraic, then it has a 1-dimensional extension which is not triangular. (And hence some scalar translate of A has a null extension which is not triangular.)

For the special case where A is *triangular* we define the *extension spectrum* of A, denoted by $\sigma_\Delta(A)$, to be the set of all complex numbers λ such that $A - \lambda I$ has a 1-dimensional *null* extension which is not triangular. So an algebraic operator has empty extension spectrum, and the above mentioned result from [12] implies that a non-algebraic triangular operator has non-empty extension spectrum. For a general (not-necessarily-triangular) operator T the appropriate definition of extension spectrum is *necessarily* a bit more abstract (see Definition 2.1). With this more general definition, it remains true (see Remark 2.4) that the extension spectrum is non-empty for an *arbitrary* non-algebraic operator. This is a consequence of a result in [12, Corollary 2.6], which was one of the motivating factors for the present paper.

A vector x in H is called *an algebraic vector* for an operator $A \in B(H)$ if there is a non-zero polynomial p in one variable satisfying $p(A)x = 0$. We use E_A to denote the set of all algebraic vectors for A. Clearly E_A is an invariant linear manifold of H. Let $[X]$ denote the closed linear span of X for any subset $X \subseteq H$. It is known that A is triangular if and only if $[E_A] = H$, and A is semi-triangular if and only if $[E_A]$ has finite co-dimension in H.

The following observation, and its "converse", will be useful: Let $T \in B(H)$ have the form $(*)$ with respect to a decomposition $H = M \oplus K$ with $A \in (\Delta)$ and C algebraic. Thus $M \oplus 0 = [E_A] \oplus 0 \subseteq [E_T]$. Let p be a non-zero polynomial such that $p(C) = 0$. Then we have that $p(T)H \subseteq M \oplus 0 \subset [E_T]$. This has a converse: if $T \in B(H)$ is an operator such that $p(T)H \subset [E_T]$ for some non-zero polynomial p, then T has the form $(*)$ with $M = [E_T]$ and $A \in (\Delta)$ and C algebraic. Indeed, since $[E_T]$ is an invariant subspace of T, it follows that T has the form $(*)$. Thus every element in E_T is an algebraic vector for A. So A is triangular. We claim that $p(C) = 0$. In fact, for any $x \in [E_T]^\perp$, we have $p(T)x = y \oplus p(C)x \in [E_T]$ for some $y \in [E_T]$. However, $p(C)x \in [E_T]^\perp$. Thus $p(C)x = 0$. So $p(C) = 0$.

If T is an algebraic extension of a triangular operator, the fact that there exists a nonzero polynomial p such that $p(T)H \subseteq [E_T]$ implies that there is a unique monic polynomial which is minimal with respect to this property. We denote this by p_T. It is clear that if p is any polynomial with $p(T)H \subseteq [E_T]$, then p_T divides p.

We will define the *triangular part* of an operator $T \in B(H)$ to be $T|_{[E_T]}$, and we will call $[E_T]$ the *domain of triangularity* of T. We write $i_\Delta(T) = \operatorname{codim}[E_T]$. This generalizes the index of semitriangularity, which was written $i_{S\Delta}(T)$ in [12], to operators that are not necessarily semitriangular. An operator is semitriangular precisely when $i_\Delta(T) < \infty$.

An operator T is said to be *bi-triangular* if both T and T^* are triangular. We use (Δ) to denote the set of all triangular operators. For an operator $T \in B(H)$, we also use $\sigma(T)$ and $\sigma_e(T)$ to denote the spectrum and essential spectrum of T, respectively.

Let $A \in (\Delta)$ and let

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

be an algebraic extension of A on $H \oplus K$. We call T a *minimal algebraic extension* of A on $H \oplus K$ if $[E_T] = H \oplus 0$. In this case p_T is precisely the minimal polynomial of the "pure extension part" C. It is useful to note that an algebraic extension T of a triangular operator A is a minimal algebraic extension if and only if the triangular part of T is A.

2. The Extension Spectrum

The extension spectrum of a *triangular* operator was defined in Section 1. The following is the corresponding definition of the extension spectrum for an *arbitrary* operator.

Definition 2.1. Let $A \in B(H)$ be an operator. The extension spectrum of A, denoted by $\sigma_\Delta(A)$, is the set of all $\lambda \in \mathbb{C}$ for which there exists $b \in B(\mathbb{C}, H)$ with the property that $E_T = E_A \oplus 0$, where

$$(**) \qquad T = \begin{pmatrix} A & b \\ 0 & \lambda \end{pmatrix}.$$

Loosely put, λ is an element of the extension spectrum of A if and only if T and A have the "same" domain of triangularity when one regards H as a subspace of $H \oplus \mathbb{C}$. In the case when A is triangular, $[E_A] = H$ and so $\lambda \in \sigma_\Delta(A)$ if and only if $[E_{T-\lambda I}] = H \oplus 0$, which is in turn equivalent to the condition that $A - \lambda I$ has a 1-dimensional *null* extension that is not triangular. So this definition is consistent with the extension spectrum of triangular operators given in Section 1. The following lemma will be frequently used in the rest of the paper.

Lemma 2.2. *Let* $A \in B(H)$. *Then*

(i) $\lambda \in \sigma_\Delta(A)$ *if and only if* $E_A + \operatorname{ran}(\lambda I - A) \neq H$.

(ii) $\lambda \notin \sigma_\Delta(A)$ *if and only if there exists an* n_0 *such that* $\ker(\lambda I - A)^{n_0} + \operatorname{ran}(\lambda I - A) = H$.

(iii) *Let* $A \in B(H)$. *If* $E_A + \operatorname{ran}(\lambda I - A) = H$, *then* $E_A + \operatorname{ran}(\lambda I - A)^n = H$ *for all positive integers* n.

Proof. (i) Let $T = \left(\begin{smallmatrix} A & b \\ 0 & \lambda \end{smallmatrix}\right)$. Then it follows from Corollary 2.2 of [12] that $E_T = E_A \oplus 0$ if and only if $b \notin E_T + \operatorname{ran}(A - \lambda I)$.

(ii) " $\Leftarrow$ " Clearly, for any n we have that $\ker(\lambda I - A)^n + \operatorname{ran}(\lambda I - A) \subseteq E_A + \operatorname{ran}(\lambda I - A) \subseteq H$. If there exists an n_0 such that $\ker(\lambda I - A)^{n_0} + \operatorname{ran}(\lambda I - A) = H$, then $E_A + \operatorname{ran}(\lambda I - A) = H$. By (i), $\lambda \notin \sigma_\Delta(A)$.

(ii) " $\Rightarrow$ " Suppose that $\lambda \notin \sigma_\Delta(A)$. By (i), $E_A + \operatorname{ran}(\lambda I - A) = H$. Note that $E_A = E_{\lambda I - A}$. Therefore, $E_{\lambda I - A} + \operatorname{ran}(\lambda I - A) = H$. By Theorem 2.4(i) in [12], $H = \bigcup_1^\infty (\ker(\lambda I - A)^n + \operatorname{ran}(\lambda I - A))$. Let P_n be the orthogonal projection on to $\ker(\lambda I - A)^n$. Then $H = \bigcup_1^\infty (\operatorname{ran} P_n + \operatorname{ran}(\lambda I - A))$. By Theorem 2.2 in [6], $\operatorname{ran} P_n + \operatorname{ran}(\lambda I - A) = \operatorname{ran}\sqrt{(\lambda I - A)(\lambda I - A)^* + P_n P_n^*}$. Thus, each subspace $\ker(\lambda I - A)^n + \operatorname{ran}(\lambda I - A))$ is an operator range and hence an F_σ set. An application of the Baire Category Theorem shows that there exists an n_0 such that $\ker(\lambda I - A)^{n_0} + \operatorname{ran}(\lambda I - A) = H$.

(iii) Suppose that $E_A + \operatorname{ran}(\lambda I - A) = H$. Let $x \in H$ be arbitrary. Write $x = e_1 + (\lambda I - A)y$ and $y = e_2 + (\lambda I - A)z$ for some e_1, $e_2 \in E_A$ and some y, $z \in H$. Then $x = e_1 + (\lambda I - A)e_2 + (\lambda I - A)^2 z$. Note that $e_1 + (\lambda I - A)e_2 \in E_A$. So $x \in E_A + \operatorname{ran}(\lambda I - A)^2$. Repeating the above process, we have that $x \in E_A + \operatorname{ran}(\lambda I - A)^n$ for all n. □

Note that from Lemma 2.2(i) we immediately have $\sigma_\Delta(A) \subseteq \sigma(A)$.

Theorem 2.3. *For any $T \in B(H)$, $\sigma_\Delta(T)$ is a closed subset of $\sigma(T)$. If T is triangular, then $\sigma_\Delta(T)$ is contained in $\sigma_e(T)$.*

Proof. As noted above, $\sigma_\Delta(T) \subseteq \sigma(T)$. To show that $\sigma_\Delta(T)$ is closed, let $\lambda \notin \sigma_\Delta(T)$. We will show that there is a neighborhood of λ which has empty intersection with $\sigma_\Delta(T)$. Without losing generality, we can assume that $\lambda = 0$. So $E_T + \operatorname{ran} T = H$. By Lemma 2.2(ii), there exists k such that $\ker T^k + \operatorname{ran} T = H$. Let P be the orthogonal projection of H onto $(\ker T^k)^\perp$. Then $P(\operatorname{ran} T) = \operatorname{ran}(PT) = PH$, so $\operatorname{ran}(PT)$ is closed.

It follows that there exists $\varepsilon > 0$ such that

$$\operatorname{ran}(PT - \mu P) = PH = \operatorname{ran}(PT)$$

for all μ with $|\mu| < \varepsilon$. To see this, let $Q = \operatorname{support}(PT) = \operatorname{proj}((\ker(PT))^\perp)$. By the open mapping theorem, $PT|_{QH}$ is invertible as a mapping from QH onto PH, so for sufficiently small μ, $(PT - \mu P)|_{QH}$ is also invertible as a mapping onto PH. Hence $PT - \mu P$ has range PH.

Now for $|\mu| < \varepsilon$, since $E_{T-\mu I} = E_T \supseteq \ker T^k$, we have that

$$\begin{aligned} E_{T-\mu I} + \operatorname{ran}(T - \mu I) &\supseteq \ker T^k + \operatorname{ran}(T - \mu I) \\ &= (I - P)H + (I - P)\operatorname{ran}(T - \mu I) + P\operatorname{ran}(T - \mu I) \\ &\supseteq P^\perp H + \operatorname{ran}(P(T - \mu I)) = H. \end{aligned}$$

So, by Lemma 2.2(i), $\mu \notin \sigma_\Delta(T)$ for all $|\mu| < \varepsilon$.

Thus $\sigma_\Delta(T)$ is closed.

In the case T is triangular, E_T is dense in H. If $\lambda \in \rho_e(T)$, then $\lambda I - T$ is a Fredholm operator so has closed range with finite codimension. It follows that $E_T + \operatorname{ran}(\lambda I - T) = H$. Therefore $\lambda \notin \sigma_\Delta(T)$ by Lemma 2.2(i). Hence $\sigma_\Delta(T) \subseteq \sigma_e(T)$. □

Remark 2.4. We note that the extension spectra of all non-algebraic operators are non-empty. This follows from Lemma 2.2(i), and a result from [12, Corollary 2.6], which states that if $E_T + \operatorname{ran}(T - \lambda I) = H$ for all values of λ in the boundary of the essential spectrum of T, then T must be algebraic.

Also, if T is not triangular, then $\sigma_\Delta(T)$ need not be contained in $\sigma_e(T)$. For instance, let T be the forward unilateral shift. Then $\sigma_e(T)$ is the unit circle, $\sigma(T)$ is the unit disk, and $E_T = \{0\}$. Moreover, $\operatorname{ran}(T - \lambda I) \neq H$ for all $|\lambda| \leq 1$. Since

$\lambda \in \sigma_\Delta(T)$ if and only if $E_T + \operatorname{ran}(T - \lambda I) \neq H$, it follows that $\sigma_\Delta(T)$ is the unit disk.

Proposition 2.5. *If $T \in B(H)$ is a compact operator which is not algebraic, then $\sigma_\Delta(T) = \{0\}$.*

Proof. If T is triangular, this follows from Theorem 2.3. Suppose $T \notin (\Delta)$ and $\lambda \in \sigma_\Delta(T)$ is nonzero. Then λ is an isolated point of $\sigma(T)$. Let P_λ be the spectral idempotent corresponding to $\{\lambda\}$. Then P_λ is finite rank and $P_\lambda H$, $P_\lambda^\perp H$ are invariant for T. Also $(T - \lambda I)|_{P_\lambda^\perp H}$ is invertible. We have $P_\lambda H \subset E_T$, and also

$$P_\lambda^\perp H = (T - \lambda)P_\lambda^\perp H \subseteq (T - \lambda I)H.$$

Hence $E_T + \operatorname{ran}(T - \lambda I) = H$, which implies $\lambda \notin \sigma_\Delta(T)$, a contradiction. ∎

Proposition 2.6. *If T is a diagonal operator, then $\sigma_\Delta(T)$ is the set of limit points of $\sigma(T)$. For any compact set K in $\mathbb{C}$, there is a triangular operator T such that $\sigma_\Delta(T) = K$.*

Proof. Write $T = \operatorname{diag}(\lambda_1, \lambda_2, \ldots)$ with respect to an orthonormal basis of H. Let $\Omega = \{\lambda_1, \lambda_2, \ldots\}$ and let Ω' be the set of limit points of Ω. We need to show $\sigma_\Delta(T) = \Omega'$. Note that $\ker(T - \lambda I) + \operatorname{ran}(T - \lambda I) = H$ if and only if $\lambda \in \mathbb{C} \setminus \Omega'$. If $\lambda \in \sigma_\Delta(T)$, then

$$\ker(T - \lambda I) + \operatorname{ran}(T - \lambda I) \subseteq E_{T-\lambda I} + \operatorname{ran}(T - \lambda I) \neq H.$$

Thus $\lambda \in \Omega'$.

Conversely, if $\lambda \notin \sigma_\Delta(T)$, then $E_{T-\lambda I} + \operatorname{ran}(T - \lambda I) = H$. By Lemma 2.2, there is a positive integer k such that $\ker(T - \lambda)^k + \operatorname{ran}(T - \lambda I) = H$. Since $\ker(T - \lambda I) = \ker(T - \lambda I)^k$ for our case, we get $\ker(T - \lambda I) + \operatorname{ran}(T - \lambda I) = H$, which implies that $\lambda \in \mathbb{C} \setminus \Omega'$. For the second statement, use the fact that every compact set is the set of limit points of some bounded countable set, and let T be the diagonal operator with the elements of an enumeration of the countable set as diagonal terms. ∎

The classical Spectral Mapping Theorem states that if f is holomorphic in a neighborhood G of $\sigma(T)$, then $\sigma(f(T)) = f(\sigma(T))$, where $f(T)$ is defined by the Riesz functional calculus. If f is a constant function, then $\sigma_\Delta(f(T))$ and $f(\sigma_\Delta(T))$ can be different simply because the extension spectrum of a scalar operator is the empty set. However, we can prove an "Extension Spectral Mapping Theorem" valid for arbitrary holomorphic functions f such that f is not constant on each component of G which meets $\sigma(T)$.

Lemma 2.7. *Let f be holomorphic on G such that f is not constant on each component of G which meets $\sigma(T)$. Then $E_T = E_{f(T)}$. Thus T is triangular if and only if $f(T)$ is triangular.*

Proof. We first show that E_T is the linear span of the generalized eigenvectors of T. That is,

$$E_T = \operatorname{span}\{\ker(T-\lambda)^n \mid n \in \mathbb{N},\ \lambda \in \sigma(T)\}.$$

In fact, if $x \in E_T$ and if $M = \{p(T)x \mid p \text{ is a polynomial}\}$, then M is finite dimensional. The Jordan decomposition of $T|_M$ tells us that M is the linear span of the generalized eigenvectors of $T|_M$. Thus x is contained in the linear span of all the generalized eigenvectors of T. The inverse inclusion is obvious.

Now we show that $E_T = E_{f(T)}$ using the previous paragraph. Suppose that $(T-\lambda)^n x = 0$ for some $\lambda \in \mathbb{C}$ and some $n \in \mathbb{N}$. Write $f(z)-f(\lambda) = (z-\lambda)g(z)$ for some holomorphic function g on G. Then $(f(T)-f(\lambda)I)^n x = (g(T))^n(T-\lambda I)^n x = 0$. Thus $x \in E_{f(T)}$.

Conversely, suppose that $(f(T)-\mu I)^n x = 0$ for some $\mu \in \mathbb{C}$, $n \in \mathbb{N}$, and $x \neq 0$. Thus $\mu \in \sigma(f(T))$. Note that by the assumption on f, $f(z)-\mu$ has finitely many roots in $\sigma(T)$. Thus there is a finite subset $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of σ_T such that

$$f(z) - \mu = \prod_{j=1}^{k}(z-\lambda_j)g(z),$$

where g is holomorphic on G and $g(z) \neq 0$ on $\sigma(T)$. Hence $g(T)$ is invertible. But $(g(T))^n \prod_{j=1}^{k}(T-\lambda_j)^n x = 0$ implies that $\prod_{j=1}^{k}(T-\lambda_j)^n x = 0$. So $x \in E_T$. ◻

Theorem 2.8 (Extension Spectral Mapping Theorem). *Let T, f, G be as in Lemma 2.7. Then $\sigma_\Delta(f(T)) = f(\sigma_\Delta(T))$. In particular, if T is invertible, then $E_{T^{-1}} = E_T$ and $\sigma_\Delta(T^{-1}) = \{1/\lambda \mid \lambda \in \sigma_\Delta(T)\}$.*

Proof. Suppose that $\lambda \in \sigma_\Delta(T)$. Then $E_T + \operatorname{ran}(T-\lambda I) \neq H$ by Lemma 2.2. As in the proof of Lemma 2.7, we can write $f(z)-f(\lambda) = (z-\lambda)g(z)$.

Note that $E_T = E_{f(T)}$ by Lemma 2.7. We have

$$\begin{aligned} E_{f(T)} + \operatorname{ran}(f(T)-f(\lambda)I) &= E_T + \operatorname{ran}((T-\lambda I)g(T)) \\ &\subseteq E_T + \operatorname{ran}(T-\lambda I). \end{aligned}$$

Thus $E_{f(T)} + \operatorname{ran}(f(T)-f(\lambda)I) \neq H$, which implies that $f(\lambda) \in \sigma_\Delta(f(T))$ by Lemma 2.2.

Conversely, assume that $\mu \in \sigma_\Delta(f(T))$. Then, again by Lemma 2.2, $E_{f(T)} + \operatorname{ran}(f(T)-\mu) \neq H$. As in the proof of Lemma 2.7 we write

$$f(z) - \mu = \prod_{j=1}^{k}(z-\lambda_j)g(z),$$

with g holomorphic and $g(z) \neq 0$ on $\sigma(T)$. We show that at least one of the numbers λ_j must be in $\sigma_\Delta(T)$. Suppose, to the contrary, that none of λ_j is in $\sigma_\Delta(T)$. Then $E_T + \operatorname{ran}(T - \lambda_j I) = H$ for each j by Lemma 2.2. Thus

$$\begin{aligned} H &= E_T + (T - \lambda_1 I)H \\ &= E_T + (T - \lambda_1 I)(E_T + (T - \lambda_2)H) \\ &\subseteq E_T + E_T + (T - \lambda_1)(T - \lambda_2)H \\ &= E_T + (T - \lambda_1)(T - \lambda_2)H. \end{aligned}$$

Iterating this step k times gives us $H = E_T + p(T)H$, where $p(z) = \prod_{j=1}^k (z - \lambda_j)$. But $(f(T) - \mu I)H = p(T)g(T)H = p(T)H$ since $g(T)$ is invertible. So we have $E_T + (f(T) - \mu)H = H$, contradicting our assumption on μ. Thus $\lambda_j \in \sigma_\Delta(T)$ for some j, which implies that $\mu = f(\lambda_j) \in f(\sigma_\Delta(T))$, as required. ◻

Next, we prove two stability results for the extension spectrum. The first one, Proposition 2.9, is needed in the proof of Theorem 3.5; and the second one, Theorem 2.12, shows that certain finite dimensional extensions of an operator can be made without changing the extension spectrum.

Proposition 2.9. *Let $A \in B(H)$ and let*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

with $B \in B(K, H)$ and $C \in B(K)$. If C is algebraic, then $\sigma_\Delta(T^*) = \sigma_\Delta(A^*)$.

Proof. If A is algebraic, then $\sigma_\Delta(A^*) = \sigma_\Delta(T^*) = \varnothing$. So we assume that A is not algebraic. Let $\lambda \in \sigma_\Delta(A^*)$. Then there exists $b \in B(\mathbb{C}, H)$ such that

$$\bar{A} = \begin{pmatrix} A^* & b \\ 0 & \lambda \end{pmatrix}$$

satisfies $E_{\bar{A}} = E_{A^*} \oplus 0$. Let us consider the operator

$$\bar{T} = \begin{pmatrix} A^* & 0 & b \\ B^* & C^* & 0 \\ 0 & 0 & \lambda \end{pmatrix}.$$

For any $x \oplus y \oplus z \in E_{\bar{T}}$, there exists a non-zero polynomial p such that $p(\bar{T})(x \oplus y \oplus z) = 0$. Since C is algebraic, we can also require that $p(C^*) = 0$. If we write

$$p(\bar{A}) = \begin{pmatrix} p(A^*) & d \\ 0 & p(\lambda) \end{pmatrix},$$

then

$$p(\bar{T}) = \begin{pmatrix} p(A^*) & 0 & d \\ * & 0 & 0 \\ 0 & 0 & p(\lambda) \end{pmatrix}.$$

Thus $x \oplus z \in E_{\bar{A}} = E_{A^*} \oplus 0$, so $z = 0$. This implies that $E_{\bar{T}} = E_{T^*} \oplus 0$. Therefore $\lambda \in \sigma_\Delta(T^*)$.

Conversely, let $\lambda \in \sigma_\Delta(T^*)$. Then there exist $b_1 \in B(\mathbb{C}, H)$ and $b_2 \in B(\mathbb{C}, K)$ such that

$$L = \begin{pmatrix} A^* & 0 & b_1 \\ B^* & C^* & b_2 \\ 0 & 0 & \lambda \end{pmatrix}$$

satisfying $E_L = E_{T^*} \oplus 0$. We show that

$$\hat{A} = \begin{pmatrix} A^* & b_1 \\ 0 & \lambda \end{pmatrix}$$

satisfies $E_{\hat{A}} = E_{A^*} \oplus 0$. Indeed, for any $x \oplus z \in E_{\hat{A}}$, there exists a non-zero polynomial p such that $p(\hat{A})(x \oplus z) = 0$ and $p(C^*) = 0$. This implies that $p(L)^2(x \oplus 0 \oplus z) = 0$. Thus $x \oplus 0 \oplus z \in E_L = E_{T^*} \oplus 0$, so $z = 0$. Hence $\sigma_\Delta(T^*) \subseteq \sigma_\Delta(A^*)$. ◻

Lemma 2.10 ([12, Lemma 2.4])**.** *If the linear manifold $E_A + \operatorname{ran}(A - \lambda I)$ has finite codimension in H, then it is closed. If it has infinite codimension in H, then it is contained in an operator range of infinite algebraic codimension in H.*

Lemma 2.11. *Let $A \in B(H)$ and $\lambda \notin \sigma_\Delta(A)$. Then $\sigma_\Delta(A) = \sigma_\Delta(T)$, where*

$$T = \begin{pmatrix} A & b \\ 0 & \lambda \end{pmatrix}$$

for any $b \in B(\mathbb{C}, H)$.

Proof. Suppose $\lambda_1 \in \sigma_\Delta(A)$. Then there is an operator $b_1 \in B(\mathbb{C}, H)$ so that

$$T_1 = \begin{pmatrix} A & b_1 \\ 0 & \lambda_1 \end{pmatrix}$$

satisfies $E_{T_1} = E_A \oplus 0$. Let

$$S = \begin{pmatrix} A & b & b_1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} \in B(H \oplus \mathbb{C} \oplus \mathbb{C}).$$

We show that $E_S = E_T \oplus 0$, and thus $\lambda_1 \in \sigma_\Delta(T)$. In fact, let $x \oplus y \oplus z \in E_S$. Then there is a non-zero polynomial p so that $p(S)(x \oplus y \oplus z) = 0$. Thus $p(S)(S - \lambda I)(x \oplus y \oplus z) = 0$. Write

$$p(T_1) = \begin{pmatrix} p(A) & d \\ 0 & p(\lambda_1) \end{pmatrix}.$$

Then

$$p(S) = \begin{pmatrix} p(A) & * & d \\ 0 & p(\lambda) & 0 \\ 0 & 0 & p(\lambda_1) \end{pmatrix}.$$

Note that $(S - \lambda I)(x \oplus y \oplus z) = u \oplus 0 \oplus (\lambda_1 - \lambda)z$ for some element $u \in H$. Then, from $p(S)(S - \lambda I)(x \oplus y \oplus z) = 0$, we obtain that

$$\begin{pmatrix} p(A) & D \\ 0 & p(\lambda_1) \end{pmatrix} \begin{pmatrix} u \\ (\lambda_1 - \lambda)z \end{pmatrix} = 0.$$

This means that $u \oplus (\lambda_1 - \lambda)z \in E_{T_1}$. By the assumption on T_1, we have $(\lambda_1 - \lambda)z = 0$, which implies that $z = 0$ since $\lambda_1 \neq \lambda$. Thus $E_S = E_T \oplus 0$, as claimed.

Conversely, let $\lambda_1 \in \sigma_\Delta(T)$. Then $E_T + \operatorname{ran}(T - \lambda_1 I)$ has infinite algebraic codimension in $H \oplus \mathbb{C}$, by Lemmas 2.2 and 2.10. It follows that $E_A + \operatorname{ran}(A - \lambda_1 I)$ also has infinite algebraic codimension in H. By Lemma 2.2, $\lambda_1 \in \sigma_\Delta(A)$. ◻

Theorem 2.12. *Let $A \in B(H)$. Suppose that*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

is a finite dimensional extension of A on $H \oplus K$ such that $\sigma(C) \cap \sigma_\Delta(A) = \emptyset$. Then $\sigma_\Delta(T) = \sigma_\Delta(A)$.

Proof. The conclusion follows by applying induction on the dimension n of K, since, by Lemma 2.11, it is true for $n = 1$. ◻

3. ALGEBRAIC EXTENSIONS

Given a triangular operator $A \in B(H)$ and an algebraic operator $C \in B(K)$, we show that C is the pure extension part for a minimal algebraic extension T of A on $H \oplus K$ if and only if the spectrum of C is contained in the extension spectrum of A.

Theorem 3.1. *Let $A \in B(H)$ be a triangular operator and let C be an algebraic operator on K. Then there is an operator $B \in B(K, H)$ such that*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

is a minimal algebraic extension of A on $H \oplus K$ if and only if $\sigma(C) \subseteq \sigma_\Delta(A)$.

We will complete the proof of this theorem by proving several lemmas.

Lemma 3.2. *Let A, H and K be as in Theorem 3.1 and let $C \in B(K)$. Suppose that $\sigma(C) = \{\lambda\}$ for some $\lambda \in \sigma_\Delta(A)$. Then there is an operator $B \in B(K, H)$ such that $E_T = E_A \oplus 0$, where*

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Proof. Since $\lambda \in \sigma_\Delta(A)$, we have that $E_A + \operatorname{ran}(A - \lambda I) \neq H$ by Lemma 2.2. So from $[E_A] = H$ and by Lemma 2.10, we get that $E_A + \operatorname{ran}(A - \lambda I)$ has infinite algebraic codimension and moreover, there is a dense operator range M of infinite algebraic codimension such that $E_A + \operatorname{ran}(A - \lambda I) \subseteq M$. By [6], there is a unitary operator U satisfying $M \cap UM = \{0\}$. Choose $B \in B(K, H)$ such that B is injective and $\operatorname{ran}(B) \subseteq UM$. We claim that B will satisfy our requirements.

Let $x \oplus y \in E_T$. We need to prove that $y = 0$. Let p be a nonzero polynomial such that $p(T)(x \oplus y) = 0$. First assume that $p(t) = (t - \lambda)^m$ for some positive integer m. Note that

$$p(T) = \begin{pmatrix} (A - \lambda I)^m & q(T) \\ 0 & (C - \lambda I)^m \end{pmatrix},$$

where $q(T)$ is

$$\begin{aligned} &(A - \lambda I)^{m-1} B + (A - \lambda I)^{m-2} B (C - \lambda I) \\ &\qquad + \cdots + (A - \lambda I) B (C - \lambda I)^{m-2} + B (C - \lambda I)^{m-1}. \end{aligned}$$

Thus $p(T)(x \oplus y) = 0$ implies that $(A - \lambda I)^m x + q(T) y = 0$. Hence we have that

$$\begin{aligned} &(A - \lambda I)^m x + (A - \lambda I)^{m-1} B y \\ &\qquad + \cdots + (A - \lambda I) B (C - \lambda I)^{m-2} y = -B (C - \lambda I)^{m-1} y, \end{aligned}$$

which belongs to $M \cap UM = \{0\}$.

So we obtain

$$(A - \lambda)[(A - \lambda I)^{m-1} x + (A - \lambda I)^{m-2} B y + \cdots + B (C - \lambda I)^{m-2} y] = 0.$$

This implies

$$(A - \lambda I)^{m-1} x + (A - \lambda I)^{m-2} B y + \cdots + B (C - \lambda I)^{m-2} y \in E_A.$$

Thus $B(C - \lambda I)^{m-2} y \in \operatorname{ran}(A - \lambda I) + E_A \subseteq M$ and also $B(C - \lambda I)^{m-2} y \in UM$. Therefore $B(C - \lambda I)^{m-2} y = 0$. Repeating the above process, we obtain that $By = 0$. Thus $y = 0$, since B is injective.

Now consider an arbitrary non-zero polynomial p such that $p(T)(x \oplus y) = 0$. If $p(\lambda) \neq 0$, then $p(C)$ is invertible by the assumption. In this case, from $p(T)(x \oplus y) = 0$, we obtain $p(C)y = 0$. Hence $y = 0$, as expected.

Suppose that $p(t) = (t - \lambda)^m r(t)$ with $m > 0$ and $r(\lambda) \neq 0$. Then

$$r(T) = \begin{pmatrix} r(A) & * \\ 0 & r(C) \end{pmatrix}.$$

So $r(T)(x \oplus y) = w \oplus r(C)y$ for some $w \in H$ and $(T - \lambda I)^m (w \oplus r(C)y) = 0$. Therefore, by the first part of the proof, we have that $r(C)y = 0$. Hence $y = 0$, since $r(C)$ is invertible. ◻

Lemma 3.3. *Let $A \in B(H)$ be a triangular operator and let $C_k \in B(H_k)$ be algebraic operators with minimal polynomials $(t - \lambda_k)^{m_k}$ for $k = 1, 2, \dots, n$. Suppose that λ_k are distinct and $\lambda_k \in \sigma_\Delta(A)$ for all k. Then there exist $B_k \in B(H_k, H)$ such that $E_T = E_A \oplus 0$, where*

$$T = \begin{pmatrix} A & B_1 & B_2 & \cdots & B_n & \\ & C_1 & 0 & \cdots & 0 & \\ & & C_2 & \cdots & 0 & \\ & & & \cdots & \cdot & \\ & & & & & C_n \end{pmatrix} \in B(H \oplus H_1 \oplus \cdots \oplus H_n).$$

Proof. Let

$$T_k = \begin{pmatrix} A & B_k \\ 0 & C_k \end{pmatrix}$$

be constructed as in Lemma 3.2 such that $E_{T_k} = E_A \oplus 0 \subset H \oplus H_k$.

Suppose that $y = x \oplus x_1 \oplus \cdots \oplus x_n \in E_T$. We need to show that $x_k = 0$ for all k. Let p be a non-zero polynomial such that $p(T)y = 0$. We note that if

$$p(T_k) = \begin{pmatrix} p(A) & D_k \\ 0 & p(C_k) \end{pmatrix},$$

then

$$p(T) = \begin{pmatrix} p(A) & D_1 & D_2 & \cdots & D_n & \\ & p(C_1) & 0 & \cdots & 0 & \\ & & & \cdots & \cdot & \\ & & & & & p(C_n) \end{pmatrix}.$$

Let $p_k = \prod_{i \neq k}(t - \lambda_i)^{m_i}$. Then, by $p_k(C_i) = 0$ for $i \neq k$, we obtain that $p_k(T)y = w \oplus 0 \oplus \cdots \oplus 0 \oplus p_k(C_k)x_k \oplus 0 \oplus \cdots \oplus 0$ for some element $w \in H$. Thus, from $p(T)p_k(T)y = p_k(T)p(T)y = 0$, we obtain

$$\begin{pmatrix} p(A) & D_k \\ 0 & p(C_k) \end{pmatrix} \begin{pmatrix} w \\ p_k(C_k)x_k \end{pmatrix} = 0.$$

This implies that $p_k(C_k)x_k = 0$ by Lemma 3.2. Hence $x_k = 0$ since $p_k(C_k)$ is invertible. Since k is arbitrary, we have that $y \in E_A \oplus 0$. ◻

Lemma 3.4. *Let M be a finite dimensional Hilbert space and let $S \in B(M)$ be a strictly upper-triangular matrix with respect to an orthonormal basis of M. Let A be a triangular operator on H. Then $\sigma_\Delta(A)$ is equal to the set of all λ with the property that there exists $B \in B(M, H)$ such that*

$$T = \begin{pmatrix} A & B \\ 0 & \lambda I + S \end{pmatrix}$$

is a minimal algebraic extension of A on $H \oplus M$.

Proof. Let $\{f_1, \ldots, f_n\}$ be an orthonormal basis for M so that S has the form $S = (s_{ij})$, with $s_{ij} = 0$ when $i \geq j$, with respect to this basis.

Suppose that λ is a number such that T is a minimal algebraic extension of A on $H \oplus M$ and write $B = (b_1, \ldots, b_n)$. If $x \oplus \alpha f_1$ $(\alpha \in \mathbb{C})$ is an algebraic element for

$$\begin{pmatrix} A & b_1 \\ 0 & \lambda \end{pmatrix},$$

then $x \oplus \alpha f_1 \oplus 0 \oplus \cdots \oplus 0$ is an algebraic vector for T. Thus $\alpha = 0$, which implies that $\lambda \in \sigma_\Delta(A)$.

Conversely, let $\lambda \in \sigma_\Delta(A)$. We use induction on n to complete the proof. We write $M_k = \operatorname{span}\{f_1, \ldots, f_k\}$.

By the definition of $\sigma_\Delta(A)$, there is an element $b_1 \in H$ such that

$$T_1 = \begin{pmatrix} A & b_1 \\ 0 & \lambda \end{pmatrix}$$

has index 1. Thus $E_{T_1} = E_A \oplus 0$ and moreover, $\operatorname{ran}(A - \lambda I) + E_A \neq H$ by Lemma 2.2. But $[E_A] = H$. Thus Lemma 2.10 implies that $\operatorname{ran}(A - \lambda I) + E_A$ has infinite algebraic codimension in H. Hence $(\operatorname{ran}(A - \lambda I) \oplus 0) + E_{T_1} + \mathbb{C}(b_1 \oplus 0)$ also has infinite algebraic codimension in $H \oplus M_1$.

Assume that

$$T_k = \begin{pmatrix} A & b_1 & b_2 & \cdots & b_k & \\ & \lambda & s_{1i} & \cdots & s_{1k} & \\ & & & \cdots & \cdot & \\ & & & & & \lambda \end{pmatrix}$$

has been constructed so that $E_{T_k} = E_A \oplus 0$ and $(\operatorname{ran}(A - \lambda I) \oplus 0) + E_{T_k} + \operatorname{span}\{c_1, \ldots, c_k\}$ has infinite algebraic codimension in $H \oplus M_k$, where $c_1 = b_1 \oplus 0 \oplus \cdots \oplus 0$, $c_i = b_i \oplus s_{11} \oplus \cdots \oplus s_{i-1,i} \oplus 0 \oplus \cdots \oplus 0 \in H \oplus M_k$, $i = 2, \ldots, k$. Hence $Q = (\operatorname{ran}(A - \lambda I) \oplus 0) + E_{T_k} + \operatorname{span}\{c_1, \ldots, c_k\} + (0 \oplus M_k)$ has infinite algebraic codimension in $H \oplus M_k$. Take $b_{k+1} \oplus 0 \notin Q$. Then, by Lemma 2.2 in [12], we have that $E_{T_{k+1}} = E_{T_k} \oplus 0 = E_A \oplus 0$ and also $(\operatorname{ran}(A-\lambda I) \oplus 0) + E_{T_{k+1}} + \operatorname{span}\{c_1, \dots, c_{k+1}\}$ has infinite algebraic codimension. Therefore, we complete the induction argument. ◻

Proof of Theorem 3.1. Let $\sigma(C) = \{\lambda_1, \dots, \lambda_n\}$.

" $\Rightarrow$ " We assume that C has the Jordan form $C = \lambda_1 I_0 \oplus \sum_k (\lambda_1 I_k + J_k) \oplus C_1$ with $\sigma(C_1) = \{\lambda_2, \dots, \lambda_n\}$. Write

$$T = \begin{pmatrix} A & B_0 & B_1 & \cdots & B_\infty & \\ & \lambda_1 I_0 & 0 & \cdots & 0 & \\ & & \lambda_1 I_1 + J_1 & \cdots & 0 & \\ & & & \cdots & \cdot & \\ & & & & & C_1 \end{pmatrix}.$$

If I_0 acts on a non-zero space, then

$$\begin{pmatrix} A & B_0 \\ 0 & \lambda_1 I_0 \end{pmatrix}$$

is not triangular since $E_T = E_A \oplus 0$. Hence $\lambda_1 \in \sigma_\Delta(A)$.

Suppose that I_0 acts on the zero space. Then there is a k so that I_k acts on a non-zero space. We consider

$$D_k = \begin{pmatrix} A & B_k \\ 0 & \lambda_1 I_k + J_k \end{pmatrix}.$$

Then D_k is a minimal extension of A. Thus, by Lemma 3.4, $\lambda_1 \in \sigma_\Delta(A)$. Similarly, $\lambda_k \in \sigma_\Delta(A)$ for $k = 2, \dots, n$.

" $\Leftarrow$ " Let C have the Jordan form $C_1 \oplus C_2 \oplus \cdots \oplus C_n$ with $\sigma(C_k) = \{\lambda_k\}$ for $k = 1, \dots, n$. Then the conclusion follows from Lemma 3.3.

A general algebraic operator C is bi-triangular. By [5], C is quasisimilar to its Jordan form. Then the conclusion follows easily. ◻

Theorem 3.5. *Let $A \in B(H)$ be a bi-triangular operator and let $C_1 \in B(K_1)$ and $C_2 \in B(K_2)$ be algebraic operators. Then there exist X, Y, D such that*

$$T = \begin{pmatrix} C_1 & X & Y \\ 0 & A & D \\ 0 & 0 & C_2 \end{pmatrix} \in B(K_1 \oplus H \oplus K_2)$$

has the property that $[E_T]^\perp = K_2$ and $[E_{T^*}]^\perp = K_1$ if and only if $\sigma(C_1^*) \subseteq \sigma_\Delta(A^*)$ and $\sigma(C_2) \subseteq \sigma_\Delta(A)$.

Proof. " $\Rightarrow$ " Let

$$T_1 = \begin{pmatrix} C_1 & X \\ 0 & A \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} A & D \\ 0 & C_2 \end{pmatrix}.$$

By Proposition 2.9, $\sigma_\Delta(T_2^*) = \sigma_\Delta(A^*)$. We also have that $\sigma_\Delta(T_1) = \sigma_\Delta(A)$ since C_1 is algebraic.

By Theorem 3.1, we have $\sigma(C_2) \subseteq \sigma_\Delta(T_1)$ and $\sigma(C_1^*) \subseteq \sigma_\Delta(T_2^*)$. Thus the necessity follows.

" $\Leftarrow$ " By Theorem 3.1, there exists $D \in B(K_2, H)$ so that

$$T_2 = \begin{pmatrix} A & D \\ 0 & C_2 \end{pmatrix}$$

has the property that $E_{T_2} = E_A \oplus 0$. Since T_2^* is triangular, again by Theorem 3.1, there exist X, Y such that

$$T^* = \begin{pmatrix} C_1^* & 0 & 0 \\ X^* & A^* & 0 \\ Y^* & D^* & C_2 \end{pmatrix}$$

has the property that $E_{T^*} = 0 \oplus E_{T_2^*}$. Thus we get $[E_T]^\perp = K_2$ and $[E_{T^*}]^\perp = K_1$. $\square$

Using Theorem 3.1, we can prove the following:

Corollary 3.6. *Suppose that T is an algebraic extension of a triangular operator and A is the triangular part of T. Then $\sigma_\Delta(T) = \sigma_\Delta(A)$.*

Proof. Let

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with respect to the Hilbert space decomposition $H \oplus K$.

Suppose that $\lambda \in \sigma_\Delta(T)$. Then there exist two elements, $b_1 \in H$ and $b_2 \in K$, such that

$$\begin{pmatrix} A & B & b_1 \\ 0 & C & b_2 \\ 0 & 0 & \lambda \end{pmatrix}$$

is the minimal algebraic extension of T, and so it is the minimal algebraic extension of A. Thus, by Theorem 3.1, $\lambda \in \sigma_\Delta(A)$.

Conversely, let $\lambda \in \sigma_\Delta(A)$. By similarity, we can assume that C has the the form $\operatorname{diag}(C_1, \ldots, C_n)$ such that $\sigma(C_i) = \{t_i\}$ and $t_1, t_2, \ldots, t_n$ are distinct. Now if λ is different from t_i for all i, then, as in Lemma 3.3, we can find $b \in H \oplus K$ such that

$$\begin{pmatrix} T & b \\ 0 & \lambda \end{pmatrix}$$

is a minimal algebraic extension of A, and so it is a minimal algebraic extension of T since $[E_T] = [E_A] \oplus 0$. Thus $\lambda \in \sigma_\Delta(T)$.

If $\lambda \in \sigma(C)$, we can assume that C has the form

$$\begin{pmatrix} D & 0 \\ 0 & F \end{pmatrix}$$

such that $\sigma(F) = \{\lambda\}$ and $(F - \lambda I)^k = 0$ for some positive integer k. Write

$$T = \begin{pmatrix} G & B_1 \\ 0 & F \end{pmatrix}.$$

Then $E_T + \operatorname{ran}(T - \lambda I)^k \neq H \oplus K$. So, by Lemma 2.2(iii), $E_T + \operatorname{ran}(T - \lambda I) \neq H \oplus K$. Thus $\lambda \in \sigma_\Delta(T)$ by Lemma 2.2(i). □

4. THE KATO SPECTRUM

We recall (cf [1]) that an operator $T \in B(H)$ is said to be of *Kato type of degree d*, where d is a positive integer, if there exist two closed subspaces M, N, invariant under T, such that the following properties hold:

(a) $H = M \oplus N$ (here $\oplus$ means Banach space direct sum).
(b) If T_0 denotes the restriction $T|_M$ of T on M, then $\operatorname{ran}(T_0)$ is closed and the inclusion $\ker(T_0^n) \subseteq \operatorname{ran}(T_0)$ holds for all positive integer n.
(c) The restriction $T|_N$ is nilpotent of degree d.

The pair (M, N) is called a *Kato decomposition* associated with T. For any operator $T \in B(H)$, the *Kato spectrum*, denoted by $\sigma_K(T)$, of T is defined by

$$\{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not of Kato type}\}.$$

It was known that the Kato spectrum of an operator T is empty if and only if T is algebraic (cf [1]). We also know that T is algebraic if and only if $\sigma_\Delta(T)$ is empty. These properties suggest that there might be some connection between the Kato spectrum and the extension spectrum, although the definitions are completely different. For triangular operators we have the following result.

Theorem 4.1. *Suppose that T is a triangular operator. Then the extension spectrum is contained in the Kato spectrum.*

Proof. Suppose that $\lambda \notin \sigma_K(T)$. We need to show that $\lambda \notin \sigma_\Delta(T)$. Without loss of generality we can assume that $\lambda = 0$. Let T be of Kato type of degree d, and let (M, N) be a Kato type decomposition associated with T and d. We claim that $T^d H = M$.

In fact, since $M \in \operatorname{Lat} T$, we have $T^d M \subset M$. Note that $T^d N = 0$. Thus $T^d H = T^d M \subset M$. To get the other inclusion, it suffices to show that $TM = M$.

Note that TM is closed. Let $M_1 = M \ominus TM$. For arbitrary $x \in E_T$, write $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in N \subseteq \ker T^d \subseteq E_T$. Then $x_1 = x - x_2 \in E_T$. Let $p(t) = \sum_{i=0}^{n} a_i t^i$ be a non-zero polynomial such that $p(T)x_1 = 0$. If $a_0 \neq 0$, then $x_1 \in (p(T) - p(0))M \subseteq TM$. If $a_0 = 0$, we write $p(t) = t^k p_1(t)$ with $p_1(0) \neq 0$. Then $p_1(T)x_1 \in \ker(T|_M)^k \subseteq TM$. Thus

$$x_1 \in \ker(T|_M)^k - (p_1(T) - p_1(0))M \subseteq TM.$$

Since E_T is dense in H, we have that $M_1 = \{0\}$. Thus $TM = M$ and so $T^d H = M$, which implies that $\operatorname{ran}(T) + \ker T^d = H$. By Lemma 2.2, $0 \notin \sigma_\Delta(T)$. ∎

The following result is an easy consequence of the proof of Theorem 4.1.

Corollary 4.2. *If T is triangular and of Kato type, then $T^d H$ is closed for some $d \geq 1$ and $T^{k+d}H = T^d H$ for all $k \in \mathbb{N}$.*

The following examples show that in general there are no definite inclusion relations between the extension spectrum and the Kato spectrum. Example A also shows that the Kato spectrum and the extension spectrum can be different for triangular operators.

Example A. Let H be an infinite dimensional Hilbert space and $W \in B(H)$ such that the range of W is not closed. Let

$$T = \begin{pmatrix} 0 & W & 0 & 0 & \cdots & \\ & 0 & I & 0 & \cdots & \\ & & 0 & I & \cdots & \\ & & & \cdots & \cdot & \\ & & & & & \cdot \end{pmatrix}$$

be in $B(K)$, where K is the direct sum of infinitely many copies of H. Then T is triangular. Clearly the range of T^n is not closed for all $n \geq 1$. Since $\operatorname{ran}(T) + \ker T = K$, we have $0 \notin \sigma_\Delta(T)$. We claim that T is not of Kato type. It follows from Corollary 4.2 that if a triangular operator T is of Kato type of degree d, then $\operatorname{ran}(T^d)$ is closed. Hence $0 \in \sigma_K(T)$. Therefore $\sigma_K(T) \nsubseteq \sigma_\Delta(T)$.

Example B. Let $S \in B(H)$ be the forward shift operator defined by $Se_n = e_{n+1}$. Then $\ker S^n = \{0\}$ for all $n \geq 1$ and $\operatorname{ran}(S) = H \ominus \mathbb{C}e_1$. Thus $\ker S^n + \operatorname{ran}(S) \neq H$. Thus $0 \in \sigma_\Delta(S)$. Since $\operatorname{ran}(S)$ is closed and $\ker(S^n) = \{0\}$ for all positive integers n, we have that $(H, 0)$ is a Kato decomposition associated with S. Thus S is of Kato type and so $0 \notin \sigma_K(S)$. Therefore $\sigma_\Delta(S) \nsubseteq \sigma_K(S)$.

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