



University of Central Florida  
STARS

---

Faculty Bibliography 2000s

Faculty Bibliography

---

1-1-2005

## The ascending double cone: a closer look at a familiar demonstration

Sohang C. Gandhi

*University of Central Florida*

Costas J. Efthimiou

*University of Central Florida*

Find similar works at: <https://stars.library.ucf.edu/facultybib2000>

University of Central Florida Libraries <http://library.ucf.edu>

This Article is brought to you for free and open access by the Faculty Bibliography at STARS. It has been accepted for inclusion in Faculty Bibliography 2000s by an authorized administrator of STARS. For more information, please contact [STARS@ucf.edu](mailto:STARS@ucf.edu).

---

### Recommended Citation

Gandhi, Sohang C. and Efthimiou, Costas J., "The ascending double cone: a closer look at a familiar demonstration" (2005). *Faculty Bibliography 2000s*. 5202.

<https://stars.library.ucf.edu/facultybib2000/5202>





## Related content

### The ascending double cone: a closer look at a familiar demonstration

To cite this article: Sohang C Gandhi and Costas J Efthimiou 2005 *Eur. J. Phys.* **26** 681

- [Mechanical paradox](#)  
Emilio Cortés and D Cortés-Poza
- [The double cone: a mechanical paradox or a geometrical constraint?](#)  
Aurelio Agliolo Gallitto and Emilio Fiordilino
- [Small surprises in ‘rolling-physics’ experiments](#)  
Jirí Bartoš and Jana Musilová

View the [article online](#) for updates and enhancements.

## Recent citations

- [Solution to the problem of contact between a double-cone and two eccentric circular rails used in the construction of a wave-powered electrical generator](#)  
Barenten SUCIU
- [The motion of two identical masses connected by an ideal string symmetrically placed over a corner](#)  
Constantin Rasinariu and Asim Gangopadhyaya
- [Estudo do movimento de um duplo cone sobre uma pista em V](#)  
F. Pascoal et al



**IOP | ebooks™**

Bringing you innovative digital publishing with leading voices to create your essential collection of books in STEM research.

Start exploring the collection - download the first chapter of every title for free.

# The ascending double cone: a closer look at a familiar demonstration

Sohang C Gandhi and Costas J Efthimiou

Department of Physics, University of Central Florida, Orlando, FL 32816, USA

Received 8 March 2005

Published 10 June 2005

Online at [stacks.iop.org/EJP/26/681](http://stacks.iop.org/EJP/26/681)

## Abstract

The double cone ascending an inclined V-rail is a common exhibit used for demonstrating concepts related to centre of mass in introductory physics courses (see <http://www.physics.ncsu.edu/pira/>). While the conceptual explanation is well known—the widening of the ramp allows the centre of mass of the cone to drop, overbalancing the increase in altitude due to the inclination of the ramp—there remains rich physical content waiting to be extracted through deeper exploration. Such an investigation seems to be absent from the literature. This paper seeks to remedy the omission.

(Some figures in this article are in colour only in the electronic version)

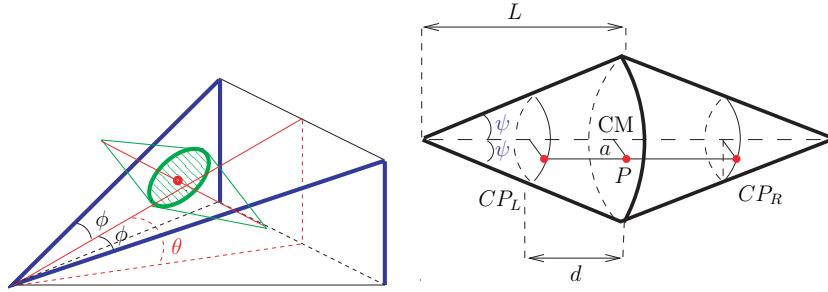
## 1. Introduction

The, familiar, double-cone demonstration is illustrated in figure 1. The set-up consists of a ramp, inclined at an angle  $\theta$ , composed of two rails forming a ‘V’ and each making an angle  $\phi$  with their bisector. A double cone of angle  $2\psi$  is placed upon the ramp. For certain values of the angles  $\theta$ ,  $\phi$  and  $\psi$ , the double cone will spontaneously roll upward.

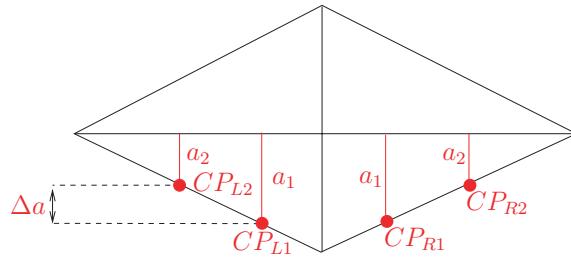
A qualitative description of the mechanism responsible for the phenomenon is easily given. As the cone moves up the ramp and gains altitude  $\Delta h$ , the distance between the rails increases and the contact points move nearer the symmetry axis, thus reducing the elevation of the centre of mass (CM) relative to the ramp by an amount  $\Delta H$ , which is determined by (but not necessarily equal to) the change of radius  $\Delta a$  resulting from the outward shifting of the contact points (figure 2). The net change of the CM height is thus  $\Delta h - \Delta H$ , which may be positive, negative or zero. When it is negative the cone rolls uphill since the effect of the widening of the ramp *overbalances* the increase in altitude.

Despite the above well-known and well-understood explanation for the motion of the double cone, it seems—to the best of our knowledge—that no detailed, quantitative account has been written. Our work seeks to remedy this omission.

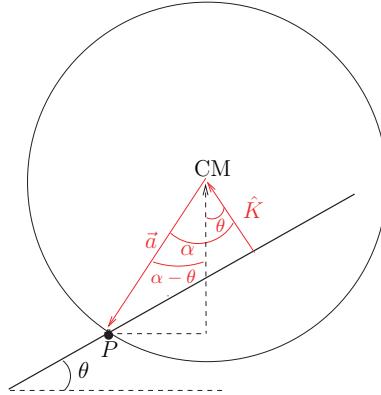
Towards this goal we present some notation which will facilitate the discussions in the following sections. As illustrated in figure 1 (right-hand side) and figure 3, we shall denote



**Figure 1.** A double cone that rolls up a V-shaped ramp. The figure shows the principal parameters,  $\theta, \phi, \psi$ , determining the geometry of the demo. Also in the figure are several quantities used in our study of the motion. In particular,  $a$  is the radius of a cross-section taken at one of the contact points and perpendicularly to the axis of the cone.

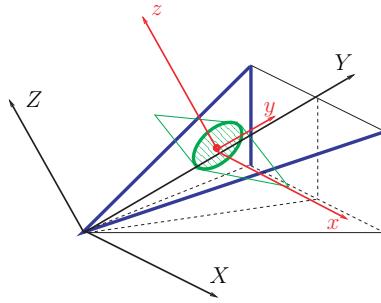


**Figure 2.** The shifting of the contact points as the double cone rolls uphill. The reader is cautioned. As shall be explained, the contact points do not lie on the vertical plane that contains the symmetry axis of the cone; instead, with the symmetry axis of the cone, they define a plane that intersects the vertical plane at an angle. See also figure 1.



**Figure 3.** The circle represents part of the middle cross-section of the cone. (See right-hand side of figure 1.) The solid thick straight line is the intersection of the plane defined by the two rails and a vertical plane perpendicular to the axis of the cone.

the left and right points of contact as  $CP_L$  and  $CP_R$  respectively, the half length of the cone as  $L$ , the distance between the middle cross-section of the cone and the contact points as  $d$  and the distance from the axis of symmetry to the points of contact as  $a$ . Also, let  $P$  be the point where the line defined by the two contact points intersects the middle cross-section of the cone



**Figure 4.** The ramp and CM systems.

and  $\vec{a}$  be the vector that points from the CM of the cone to  $P$ . The magnitude  $a$  of  $\vec{a}$  is equal to the radius of the cross-section of the cone taken at the contact points perpendicularly to the axis of the cone. When the cone is at the vertex of the ramp, the two contact points coincide and  $a$  has a value of  $L \tan \psi$ . Also, we shall denote by  $\pi - \alpha$  the angle the vector  $\vec{a}$  makes with the unit normal vector  $\hat{K}$  to the ramp. All this is better illustrated in figure 3.

## 2. Net change of height

To carry out our calculation we shall introduce two coordinate systems: first, a fixed system, which we shall call *the ramp coordinate system XYZ*, with origin at the vertex of the ramp, the  $Y$ -axis along the bisector of the rails, the  $Z$ -axis perpendicular to the plane defined by the rails and the  $X$ -axis parallel to the symmetry axis of the double cone. We also adopt a CM system with axes parallel to those of the ramp system but whose origin is attached to the CM of the double cone. The systems are shown in figure 4.

Our experience with uniform spheres, discs or cylinders rolling on an incline suggests that a line drawn at a contact point perpendicularly to the incline on which the objects roll passes through the CM (or the symmetry axis). Indeed, for these particular shapes, this is implied by the fact that the incline must be tangent to the surface of the object at the contact point and that the radii of a circle are normal to the circle. However, the conical shape on the V-shaped ramp does not possess the same geometric property (although a double-cone rolling on two parallel rails would). Consequently, the contact points,  $CP_R$  and  $CP_L$ , have a different spatial relation to the CM of the cone (figure 3).

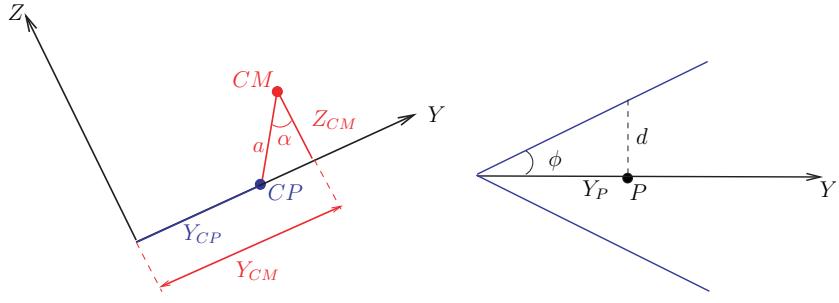
Instead of working with the two points  $CP_R$ ,  $CP_L$ , it is more convenient to work with the point  $P$  defined previously. Note that all three points  $CP_R$ ,  $CP_L$ ,  $P$  have the same  $z$  and  $y$  coordinates and only differ in their  $x$  coordinates. In fact, due to symmetry—we assume the cone to be placed symmetrically between the rails so that there is no motion in the  $X$ -direction—we are really dealing with a two-dimensional problem.

To determine the angle  $\alpha$  of figure 3, we must find the  $y$  coordinate  $y_P$  in the CM frame. This will give us the location of  $P$  relative to the CM of the cone. To this end, we write the equation

$$z^2 + y^2 - \tan^2 \psi (L - x)^2 = 0$$

that describes the cone and, from this, we construct the surface

$$z = -\sqrt{(L - x)^2 \tan^2 \psi - y^2} \equiv f(x, y)$$



**Figure 5.** The relation between some of the parameters used in the study of the motion of the cone.

which represents the lower right quarter of the cone in the CM frame. We then calculate the gradient of the function  $f(x, y) - z$ ,

$$\nabla z = \frac{1}{\sqrt{(L-x)^2 \tan^2 \psi - y^2}} ((L-x) \tan^2 \psi, y, -1),$$

that is normal to the cone. From the definition of the two frames  $xyz$  and  $XYZ$  we immediately see that  $x = X$  and therefore  $x_{CP} = X_{CP} = d$ . Thus, at the right-hand contact point

$$\nabla z|_{CP} = \frac{1}{\sqrt{(L-d)^2 \tan^2 \psi - y_P^2}} ((L-d) \tan^2 \psi, y_P, -1).$$

We also construct the unit vector  $\hat{u}$  whose direction coincides with the right-hand rail and points away from the vertex of the ramp. In other words, the vector  $\hat{u}$  lies in the  $xy$ -plane and makes an angle  $\phi$  with the  $y$ -axis; it can therefore be written as

$$\hat{u} = (\sin \phi, \cos \phi, 0).$$

As the cone rolls on the rails, the rails are always tangent to the cone's surface and, therefore, perpendicular to the normal  $\nabla z$ :

$$\nabla z|_{CP} \cdot \hat{u} = 0,$$

from which we locate the point  $P$ ,

$$y_P = -(L-d) \tan^2 \psi \tan \phi. \quad (1)$$

From figure 3, it can be seen that  $y_P$  may be written in terms of  $\alpha$  as

$$y_P = -a \sin \alpha. \quad (2)$$

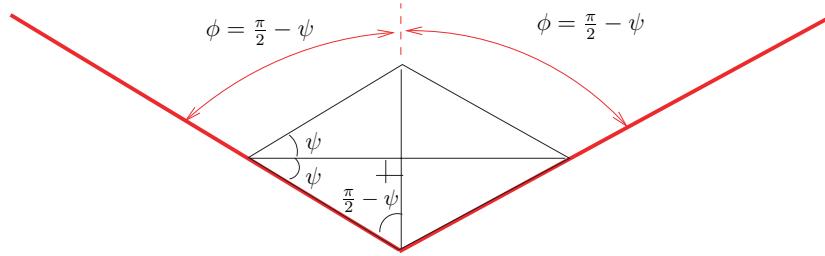
Also, from figure 5 it can be seen that  $a$  can be written as

$$a = (L-d) \tan \psi. \quad (3)$$

From equations (1)–(3), we thus find

$$\sin \alpha = \tan \phi \tan \psi, \quad (4)$$

which expresses the angle  $\alpha$  in terms of the known angles  $\phi$  and  $\psi$ . Since  $-1 \leq \sin \alpha \leq 1$ , there will be a solution for  $\alpha$  if the angles  $\phi$  and  $\psi$  are such that  $-1 \leq \tan \phi \tan \psi \leq 1$ . Since  $\phi, \psi$  must all lie in  $[0, \frac{\pi}{2})$ , we have that  $\tan \phi, \tan \psi$  are non-negative and therefore  $0 \leq \tan \phi \tan \psi$ . We use figure 6 to examine the other side of the inequality. In the figure, we draw a projection of the cone onto the ramp plane when  $\phi = \pi/2 - \psi$ . In this case, the cone



**Figure 6.** When a cone of angle  $\psi = \frac{\pi}{2} - \phi$  is projected on the ramp plane, then the cone exactly ‘fills’ the wedge formed by the rails. Increasing the angle  $\phi$ , that is if  $\phi \geq \pi/2 - \psi$ , then the cone is allowed to fall through the rails. Therefore, the demo is well made if

$$\phi < \frac{\pi}{2} - \psi.$$

For angles in the first quadrant, the tangent function is strictly increasing. Therefore, the previous condition implies

$$\tan \phi < \tan\left(\frac{\pi}{2} - \psi\right) = \cot \psi = \frac{1}{\tan \psi} \Rightarrow \tan \phi \tan \psi < 1.$$

Therefore, for a well-made demo, the inequality

$$0 \leq \tan \phi \tan \psi < 1 \quad (5)$$

is always satisfied and the angle  $\alpha$  has a well-defined value.

We are now able to see the reasoning behind the statement  $\Delta a \neq \Delta H$  made in the introduction. The angle between  $\vec{a}$  and the vertical is  $\alpha - \theta$  (figure 3). Thus, the change in the height  $\Delta H$  of the CM relative to the points of contact is in fact given by  $\Delta a \cos(\alpha - \theta)$ . It follows that the net change in height of the CM is

$$\text{Net change of height} = \Delta h - \Delta a \cos(\alpha - \theta).$$

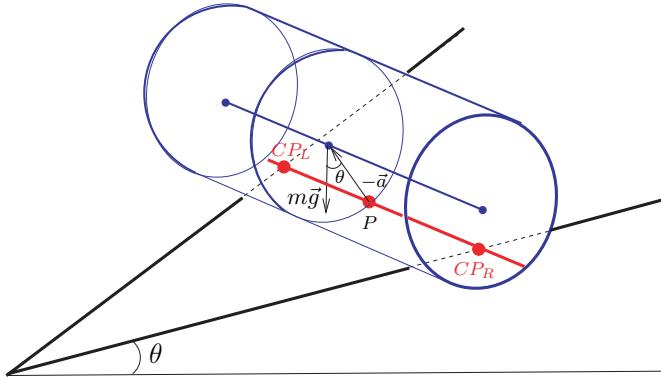
### 3. A dynamical explanation

With our newfound knowledge regarding the way in which the double cone sits on the ramp, we are in a position to provide an account from a dynamical perspective.

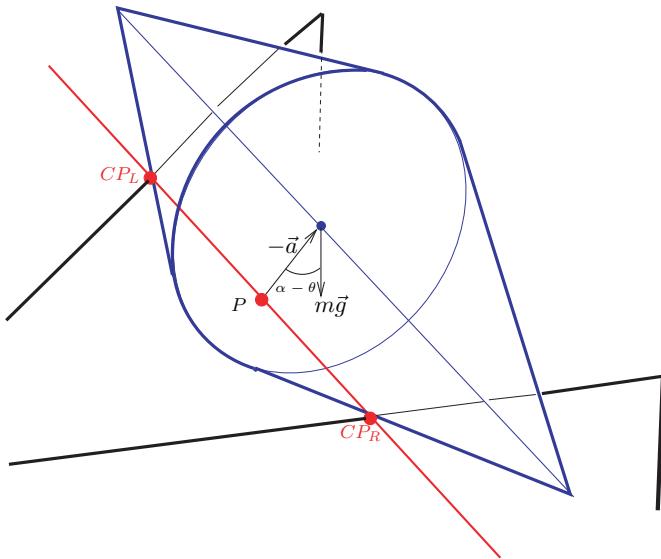
Any solid placed on the ramp will contact it at two points. Since we assume that the solid rolls without slipping, the contact points must be instantaneously at rest. The rigidity of the solid then implies that all points lying on the line connecting the two contact points are also at rest instantaneously, and, hence, this line defines the instantaneous axis of rotation. The torque about this axis will solely be due to gravity since the frictional forces and the reaction forces of the ramp are exerted at the axis.

The situation for a cylinder placed on the ramp is shown in figure 7. The angle between the weight and the vector  $\vec{a}$ , which enters the torque, is determined by the inclination  $\theta$  of the ramp only. Thus the resultant torque of the weight,  $-mga \sin \theta$ , always tends the cylinder to roll down the ramp.

The situation for the double cone is shown in figure 8. The angle between the weight and the vector  $\vec{a}$  which enters the torque is now  $\pi - (\alpha - \theta)$ . The resultant torque of the



**Figure 7.** The resultant torque always pulls a cylinder down the ramp.



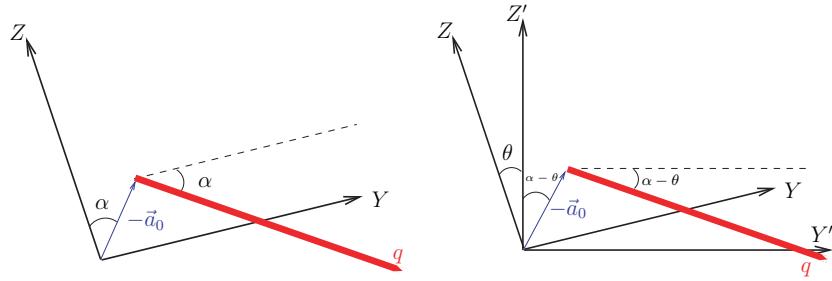
**Figure 8.** The resultant torque in the case of a double cone can be positive, negative or zero depending on the value of  $\alpha$ .

weight,  $mga \sin(\alpha - \theta)$ , depends strongly on the value of  $\alpha$ , which can change by changing the parameters  $\phi$  and  $\psi$ . If  $\alpha > \theta$ , the resultant torque will tend the solid up the ramp. We can thus see that it is the unusual way in which the double cone sits on the ramp that distinguishes it from other solids and provides the mechanism with which to roll up the ramp.

#### 4. The motion of the CM

Since  $Z$  is oriented perpendicular to the plane defined by the rails and  $P$  lies on the ramp plane,  $Z_P$  is always zero. Thus  $Z_{\text{CM}}$  is given by

$$Z_{\text{CM}} = a \cos \alpha$$



**Figure 9.** The motion of the CM in the  $XYZ$  frame and in the  $X'Y'Z'$  frame—a clockwise rotation of  $XYZ$  about the  $X$ -axis by  $\theta$ . The thick straight line represents the trajectory of the CM.

(see figure 5). If the point  $P$  has coordinate  $Y_P$  in the ramp frame, then the half-distance between the rails,  $d$ , must be given by

$$d = Y_P \tan \phi, \quad (6)$$

as is evident from figure 5. Using equations (6), (3) and (4), we write for  $a$ ,

$$a = L \tan \psi - Y_P \sin \alpha. \quad (7)$$

Hence,

$$Z_{\text{CM}} = (L \tan \psi - Y_P \sin \alpha) \cos \alpha.$$

Referring again to figure 5, we obtain

$$Y_{\text{CM}} = Y_P + a \sin \alpha = Y_{\text{CP}} + (L \tan \psi - Y_{\text{CP}} \sin \alpha) \sin \alpha,$$

or

$$Y_P = \frac{Y_{\text{CM}} - L \tan \psi \sin \alpha}{\cos^2 \alpha}. \quad (8)$$

Plugging this into our previous result we obtain

$$Z_{\text{CM}} = \lambda Y_{\text{CM}} + b, \quad (9)$$

where  $\lambda$  and  $b$  are constants determined by the geometry of the demo:

$$\lambda = -\tan \alpha, \quad b = L \tan \psi (\cos \alpha - \tan \alpha \sin \alpha).$$

Equation (9) implies that the CM is constrained to move along a line on the  $zy$ -plane with slope  $\lambda$  and intercept  $b$ .

This result is also made apparent through physical reasoning. The cone's motion is that of rigid rotation about the symmetry axis plus a translational motion of the CM with velocity  $\vec{v}_{\text{CM}}$ . Thus, the total velocity of the point  $P$  is  $\vec{v}_{\text{rot},P} + \vec{v}_{\text{CM}}$ , where  $\vec{v}_{\text{rot},P}$  is its rotational velocity about the symmetry axis. The rolling without slipping condition implies that the velocity of the point  $P$  must be zero. Therefore

$$\vec{v}_{\text{CM}} = -\vec{v}_{\text{rot},P}.$$

Since  $\vec{v}_{\text{rot},P}$  must be perpendicular to  $\vec{a}$ , we conclude that

$$\vec{v}_{\text{CM}} \perp \vec{a}.$$

Since  $\vec{a}$  is fixed in direction and the velocity of the CM must always lie on the  $YZ$ -plane (by symmetry), the CM's path is a line declined (inclined if  $\alpha < \theta$ ) below the horizontal by an angle  $\alpha - \theta$  (see figure 9).

## 5. Energy analysis

As we have seen, the path of the CM is a straight line that makes an angle  $\alpha$  with the  $Y$ -axis as shown in figure 9. We now introduce a new generalized coordinate: the distance along this line,  $q$ . We shall take its zero to correspond to the position of the CM when the cone is at the bottom of the ramp. In figure 9,  $\vec{a}_0$  represents the value of  $\vec{a}$  when the cone is at the bottom of the ramp and  $d = 0$ . It can be seen that, in terms of  $q$ ,  $Y_{CM}$  is given by

$$Y_{CM} = Y_{CM}^0 + q \cos \alpha,$$

where  $Y_{CM}^0$  is the coordinate of the CM when the cone is at the bottom of the ramp—that is when  $q = 0$ . It is also clear from the figure that

$$Y_{CM}^0 = a_0 \sin \alpha = L \tan \psi \sin \alpha.$$

Combining the results with equations (7) and (8), we obtain

$$a = L \tan \psi - q \tan \alpha. \quad (10)$$

The rolling without slipping constraint implies that the angular velocity  $\omega$  of the cone about the symmetry axis is given by

$$\omega = -\frac{\dot{q}}{a} = -\frac{\dot{q}}{L \tan \psi - q \tan \alpha}.$$

Thus, we obtain for the kinetic energy

$$T = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} \left[ m + \frac{I}{(L \tan \psi - q \tan \alpha)^2} \right] \dot{q}^2,$$

where  $I$  is the moment of inertia of the cone with respect to the symmetry axis.

For the potential energy, we are concerned with the vertical height of the CM; that is, we must find the coordinate  $Z'_{CM}$  in terms of  $q$ , where the primed frame  $X'Y'Z'$  is one that is obtained from the unprimed frame  $XYZ$  by rotating the latter by an angle  $\theta$  about the  $X$ -axis (see figure 9). From the figure we see that

$$Z'_{CM} = Z_{CM}^0 - q \sin(\alpha - \theta).$$

Thus, we have for the potential

$$U = -mgq \sin(\alpha - \theta),$$

where we have set the potential to zero at the initial position of the cone.

For the cone to roll up the ramp, the potential must be a decreasing function of  $q$ , that is

$$\sin(\alpha - \theta) > 0.$$

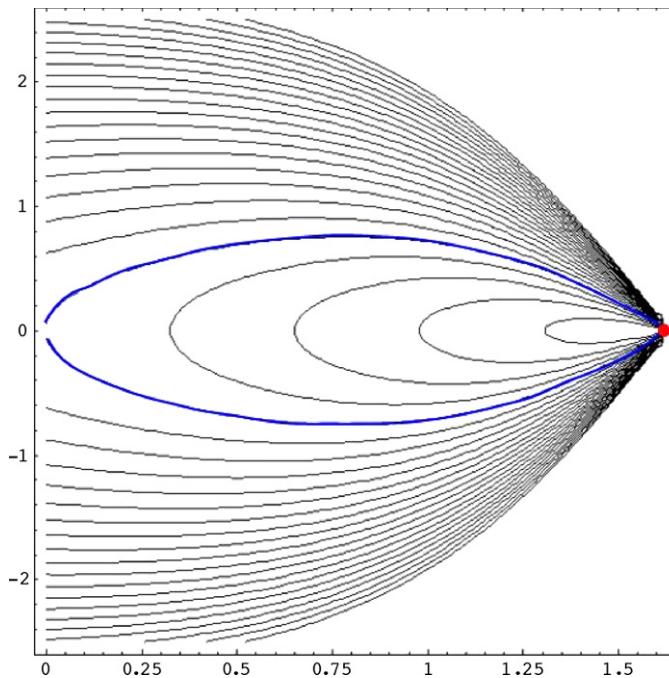
With the use of equation (4) and some trigonometric identities, this condition may equivalently be written as

$$\tan \theta < \frac{\tan \phi \tan \psi}{\sqrt{1 - \tan^2 \phi \tan^2 \psi}}. \quad (11)$$

Recalling the inequality (5), we see the denominator is well defined and is not singular or imaginary.

We may now write down the energy:

$$E = T + U = \frac{1}{2} \left[ m + \frac{I}{(L \tan \psi - q \tan \alpha)^2} \right] \dot{q}^2 - mgq \sin(\alpha - \theta). \quad (12)$$



**Figure 10.**  $\dot{q}$  versus  $q$  phase portrait for  $\phi = \psi = \frac{\pi}{6}$ ,  $\theta = \frac{\pi}{12}$ ,  $L = m = 1$ .

It should be noted that  $q$  may not take on arbitrary values. Indeed, as we have defined it,  $q$  has a minimum value of 0. There will also be a maximum value at which point the cone falls through the rails. This point corresponds to the double cone contacting the ramp at its very tips. Hence  $a = 0$  and, from equation (10),

$$0 \leq q \leq L \frac{\tan \psi}{\tan \alpha}.$$

In the next section, we shall see an unexpected property of the motion about the fall-through point,  $q = L \tan \psi / \tan \alpha$ .

## 6. A surprising feature

The reader should note the position-dependent coefficient in the inertial term of our expression for energy (12). It is this term that gives the system unique behaviour beyond what has been discussed so far.

Figure 10 shows the  $\dot{q}$  versus  $q$  phase portrait for the system with  $\phi = \psi = \pi/6$ ,  $\theta = \pi/12$ , and  $L = m = 1$ . The first astonishing fact is that *all* of the curves go through the same point. This is the fall-through point and it will be the subject of a more exhaustive discussion below.

Shown in figure 10 as a thick curve is the zero energy trajectory. Those curves contained within it correspond to progressively more negative energies. Those exterior, progressively more positive. One may be tempted to identify, as cycles, the negative energy trajectories. However, the fall-through point is a singularity. While all the trajectories approach it arbitrarily as  $t \rightarrow +\infty$ , none of them actually reach it. Thus, the negative energy curves are all open

by one point. Their ‘pen loop’ shape merely illustrates the simple fact that, when the cone is placed at the top of the ramp and given an initial shove, the cone will roll down for a bit and then roll back up.

To illustrate just what it is about the motion that is so startling, let us focus on a single trajectory. Consider the  $E = 0$  energy curve corresponding to releasing the cone from rest at the bottom of the ramp (as the demonstration is typically performed). This predicts that, at first, the cone’s translational speed will be increasing reaching a maximum value about midway along its path after which the translational speed will begin to ebb, decreasing to zero at the fall-through point. The cone will indefinitely approach but never reach the fall-through point. Though, at first, difficult to accept, with a little thought, one may see that this behaviour does in fact make physical sense.

To make things more concrete, let us divide the path of the CM into equal tiny increments of  $\Delta q = \epsilon$ . Since the potential is linear, the cone will gain a fixed amount of energy,  $\delta$ , over each increment. Also, since  $a$  is a linear function of  $q$ , it will decrease by a fixed amount,  $\Delta a$ , over each increment.

Now, let us for the moment, consider a simplified situation; namely, we shall adjust the inclination of the ramp so that the system is potentially neutral and have a physicist take over gravity’s job. We shall stipulate:

- (i) the cone will be given an initial speed  $v$ ;
- (ii) the physicist will provide enough and only enough energy over each increment so as to keep the cone translating at speed  $v$ , unless this conflicts with (iii);
- (iii) the physicist shall not give the cone more than  $\delta$  energy per increment.

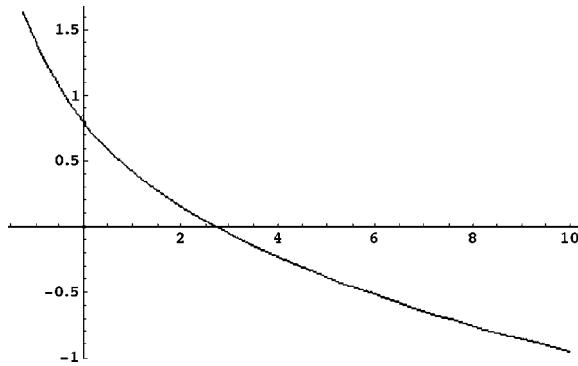
From the rolling without slipping constraint we may write the cone’s rotational energy as  $T_{\text{rot}} = \frac{1}{2} I \frac{v^2}{a^2}$ . Differentiating we obtain

$$\Delta T_{\text{rot}} = I \frac{v^2}{a^3} \Delta a, \quad (13)$$

for the amount of energy needed over each increment to keep the cone translating at speed  $v$ . Since  $a$  goes from a maximum to zero, we can always find a position along the ramp for which this amount exceeds  $\delta$ . At this point, the physicist will fail to maintain the cone at its current translational speed. However, the physicist may transfer some of the translational energy into rotation, causing the cone to slow down and repeat the process for each of the following intervals.

Returning to the actual situation, with the cone rolling freely on the ramp, it is the same principle that is responsible for the cone’s peculiar motion. However, the accountant that keeps track of energy transactions is no longer the physicist but gravity. Also, in this case, energy is initially being proportioned between translational and rotational energy. We may, for the sake of argument, use the speed of the cone, say, one microsecond after release, as the  $v$  in equation (13). No matter how small it is, we may still find a position for which the right-hand side of equation (13) exceeds  $\delta$ . In reality, the initial increase of the speed of the cone as it moves along only serves to bring about the cross-over point sooner. As seen in figure 10, the higher the energy, the more the curve ‘flattens’ at the beginning, eventually losing the increasing part altogether.

By straightforward extension of the above argument, we thus see more and more energy has to be transferred from translation to rotation as  $a$  shrinks to zero. This results in asymptotic behaviour of pure rotation about the symmetry axis. The energy transfer mechanism in all of this is ultimately friction—which we assume to be ideal—working to keep the cone from slipping.



**Figure 11.** Cross-over point versus energy.

## 7. Integration of the equation of motion

Equation (12) can be solved for the velocity,

$$\dot{q} = \pm \sqrt{\frac{2(E + mgq \sin(\alpha - \theta))}{m + \frac{I}{(L \tan \psi - q \tan \alpha)^2}}}. \quad (14)$$

Note that, except in the degenerate cases in which  $\phi$  or  $\psi = 0$  (and thus also  $\alpha = 0$ ),

$$\lim_{q \rightarrow L \frac{\tan \psi}{\tan \alpha}} \dot{q} = 0.$$

Thus the ebb in the cone speed is a general feature.

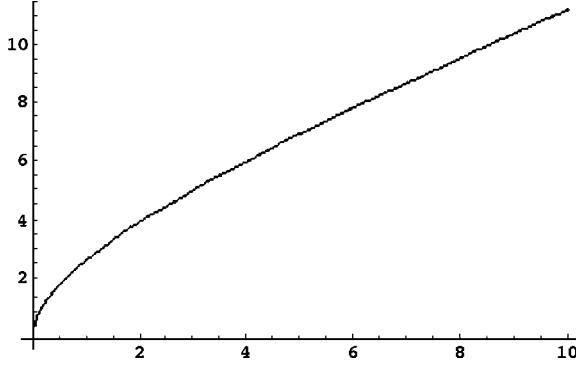
Equation (14) gives the velocity of the CM as a function of position,  $\dot{q} = \dot{q}(q)$ . We can find where the cross-over point occurs by taking the derivative of this equation and setting it equal to zero. The resulting expression is quite turbid. More meaningful is a plot of the cross-over point versus energy. This is shown in figure 11 where we used the same values of the parameters as for the phase plot of figure 10.

Note that the cross-over point is a decreasing function of energy. As we have already pointed out, increasing energy means increasing the starting velocity which brings the cross-over point closer. Also interesting is that, above a certain energy, the cross-over point occurs at negative values of  $q$ . For large enough energy, the initial velocity is so high that gravity can never provide enough energy over an increment to keep the cone translating at its current speed. Therefore, at these energies the speed will be a strictly decreasing function of time. This may be quickly confirmed using the extreme case  $E \rightarrow +\infty$  in (14).

We may now integrate equation (14) to get

$$\beta \pm t = \int \sqrt{\frac{m + \frac{I}{(L \tan \psi - q \tan \alpha)^2}}{2(E + mgq \sin(\alpha - \theta))}} dq, \quad (15)$$

where  $\beta$  is a constant of integration. There is no simple analytical solution to this integral. We may, however, make approximations in two regimes: small values of  $q$  (cone close to the vertex) and values of  $q$  close to  $L \tan \psi / \tan \alpha$  (cone close to the fall-through point).



**Figure 12.** The graph of  $t$  versus  $q$  close to the vertex of the ramp.

### 7.1. Near vertex regime

For small values of  $q$ , we may apply the binomial expansion to the  $\frac{I}{(L \tan \psi - q \tan \alpha)^2}$  term in the numerator of (14) and approximate the integral as

$$\beta \pm t \approx \int \sqrt{\frac{A + Bq}{E + Cq}} \frac{dq}{2},$$

where

$$A \equiv m + \frac{I}{L^2 \tan^2 \psi}, \quad B \equiv \frac{2I \tan \alpha}{L^3 \tan^3 \psi}, \quad C \equiv mg \sin(\alpha - \theta).$$

The last integral is elementary and gives

$$\begin{aligned} \beta \pm t \approx & \frac{\sqrt{\frac{A+Bq}{E+Cq}}(E+Cq)}{C} \\ & + \frac{(AC - BE)\sqrt{\frac{A+Bq}{E+Cq}}\sqrt{E+Cq}\ln\left(2\sqrt{(A+Bq)(E+Cq)} + \frac{BE+AC+2BCq}{\sqrt{BC}}\right)}{2C^{3/2}\sqrt{B(A+Bq)}}. \end{aligned}$$

Figure 12 plots the solution for the example that we have been using throughout, with initial conditions  $q = 0$  and  $\dot{q} = 0$  at  $t = 0$ . The inverse curve  $q = q(t)$  is found by reflection with respect to the diagonal  $q = t$  of the unit ‘square’. Note that the figure indicates that the cone is speeding up.

If one makes an additional binomial expansion on equation (15), a simpler expression is obtained

$$q(t) \approx \frac{e^{\sqrt{E}(\beta \pm t)}(B^2 - 4AE - 2B\sqrt{E} e^{\sqrt{E}(\beta \pm t)} + E e^{2\sqrt{E}(\beta \pm t)})}{4E^{3/2}}.$$

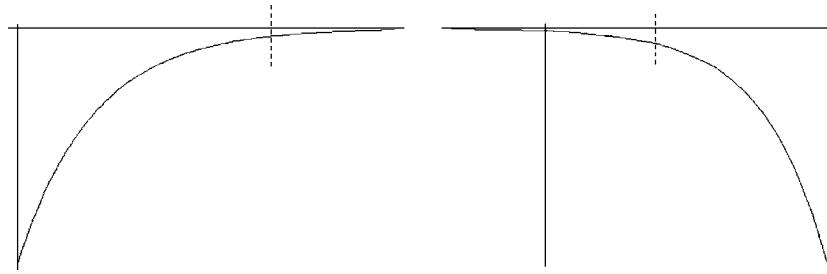
### 7.2. Near fall-through regime

The integral of equation (15) may be written in terms of  $a$  as

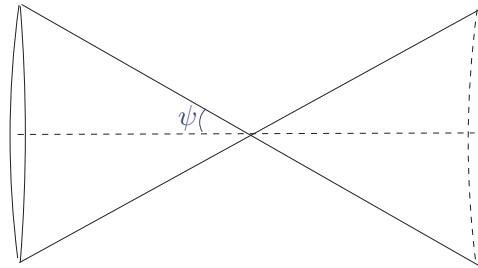
$$-\beta \pm t = \int \cot \alpha \sqrt{\frac{m + \frac{I}{a^2}}{D - Fa}} da,$$

where

$$D \equiv 2E + 2mgL \frac{\sin(\alpha - \theta) \tan \psi}{\tan \alpha}, \quad F \equiv 2mg \sin(\alpha - \theta) \cot \alpha.$$



**Figure 13.**  $q$  versus  $t$  near the fall-through point.



**Figure 14.** The simplest variation of the double-cone demo would be an hourglass sitting on a V-shaped ramp that has its vertex elevated.

For  $q \approx L \frac{\tan \psi}{\tan \alpha} \Rightarrow a \approx 0$ ,

$$-\beta \pm t \approx \sqrt{\frac{I}{D}} \cot \alpha \int \frac{da}{a}.$$

Integrating and inverting, we obtain

$$q(t) \approx L \frac{\tan \psi}{\tan \alpha} - \cot \alpha \exp \left[ \tan \alpha \sqrt{\frac{D}{I}} (-\beta \pm t) \right].$$

The general form of the solution is that of the left side of figure 13, when the minus sign is chosen in the above expression, representing the cone moving up the ramp towards the fall-through point. It is that of the right side of figure 13, when the positive sign is chosen, representing the cone moving down the ramp, starting at some point very close to the fall-through point at  $t = 0$ . Note that the graph on the left depicts the cone speeding up as it gains potential and that on the right slowing down as it drops in potential. In the figures, the dotted vertical lines roughly mark off the regions in which the solutions are valid. The vertical axes are at  $q = L \frac{\tan \psi}{\tan \alpha}$ , the point at which the cone falls through the rails. Due to this asymptotic behaviour, it would take infinite time for the cone to reach the fall-through point or reach any other point starting from the fall-through point.

## 8. The hourglass

It is not difficult to invent variations on this system (see, for example, [2]). In particular, one may obtain an essentially identical motion by ‘inverting’ the double cone as well as our ramp—that is, flipping the ramp up side down so that its vertex is at the top, and placing upon it an hourglass-shaped object such as that depicted in figure 14. If we define our parameters in

a similar manner as before (adjusting, as in the figure, the definition of  $\psi$  appropriately), and redefine  $q$  so that its zero now corresponds to the location of the CM when the hourglass is at the top of the ramp and so that it increases in the direction in which the cone travels down the ramp, we obtain for the energy,

$$E = \frac{1}{2} \left[ m + \frac{I}{q^2 \sec^4 \alpha \tan^2 \alpha} \right] \dot{q}^2 + mgq \sin(\alpha - \theta).$$

The result can be seen to possess essentially the same form as the energy of the double cone, and thus the system's motion will be similar, with the cone, now, tending to roll in the direction of decreasing  $q$ —again, up the ramp—and the singularity, now, occurring at  $q = 0$ .

## 9. Concluding remarks

We see that the implications of the unique geometry of this system extend far beyond the simple conceptual explanation relying on a rough trend of the CM. Despite the simplicity of the demo, the dynamical explanation, in terms of the consequences of the cone's peculiar way of sitting on the ramp, has revealed surprising properties. We hope that instructors of introductory physics will be as charmed as we are by this demo and its rich physical content.

## Acknowledgments

This research was sponsored by generous grants from the University of Central Florida Honors College and Office of Undergraduate Studies. SG would like to thank Rick Schell and Alvin Wang for this financial support. SG would like to thank CE for the opportunity to work on this project, his generous time and guidance throughout this work and SG's studies.

## References

- [1] PIRA demonstration 1J11.50. See <http://www.physics.ncsu.edu/pira/>
- [2] Balta N 2002 New versions of the rolling double cone *Phys. Teach.* **40** 156