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## A STRUCTURE THEOREM FOR THE GROMOV-WITTEN INVARIANTS OF KÄHLER SURFACES

JUNHO LEE & THOMAS H. PARKER

### Abstract

We prove a structure theorem for the Gromov-Witten invariants of compact Kähler surfaces with geometric genus  $p_g > 0$ . Under the technical assumption that there is a canonical divisor that is a disjoint union of smooth components, the theorem shows that the GW invariants are universal functions determined by the genus of this canonical divisor components and the holomorphic Euler characteristic of the surface. We compute special cases of these universal functions.

Much of the work on the Gromov-Witten invariants of Kähler surfaces has focused on rational and ruled surfaces, which have geometric genus  $p_g = 0$ . This paper focuses on surfaces with  $p_g > 0$ , a class that includes most elliptic surfaces and most surfaces of general type. In this context we prove a general “structure theorem” that shows (with one technical assumption) how the GW invariants are completely determined by the local geometry of a generic canonical divisor.

The structure theorem is a consequence of a simple fact: the “Image Localization Lemma” of Section 3. Given a Kähler surface  $X$  and a canonical divisor  $D \in |K_X|$ , this lemma shows that the complex structure  $J$  on  $X$  can be perturbed to a *non-integrable* almost complex structure  $J_D$  with the property that the image of all  $J_D$ -holomorphic maps lies in the support of  $D$ . This immediately gives some striking vanishing theorems for the GW invariants of Kähler surfaces (see Section 3). More importantly, it implies that the Gromov-Witten invariant of  $X$  for genus  $g$  and  $n$  marked points is a sum

$$GW_{g,n}(X, A) = \sum GW_{g,n}^{\text{loc}}(D_k, A_k)$$

over the connected components  $D_k$  of  $D$  of “local invariants” that count the contribution of maps whose image lies in or (after perturbing to a generic moduli space) near  $D_k$ . These local invariants have not been previously defined. The proof of their existence relies on using non-integrable structures and geometric analysis techniques.

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Our structure theorem characterizes the local invariants by expressing them in terms of usual GW invariants of certain standard surfaces. For this we make the mild assumption that one can deform the Kähler structure on  $X$  and choose a canonical divisor  $D$  so that all components of  $D$  are smooth. With this assumption, the restriction of  $K_X$  to each component  $D_k$  of multiplicity  $m_k$  is the normal bundle  $N_k$  to  $D_k$ , and

$$N_k^{m_k+1} = K_{D_k},$$

that is,  $N_k$  is a holomorphic  $(m_k + 1)$ -th root of the canonical bundle of the curve  $D_k$ . The local invariants are given by universal functions

$$(0.1) \quad L^i(t) \in \prod_{g,n} H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n)[[t, \lambda]],$$

depending only on topological invariants of the pair  $(D_k, N_k)$ . Four types of such universal functions are relevant. Each is defined in terms of the GW invariant of a rational surface or a local GW invariant of a line bundle over a curve (they can also be defined using obstruction bundles, but we do not pursue that approach here). There is one universal function  $L^0(t)$  for exceptional curves. This enters into the blow-up formula for GW invariants proved in Section 5. The blow-up formula reduces the computation of the GW invariants to the case of minimal surfaces. In light of the vanishing results given in Section 3, there are only two types of minimal surfaces to consider: properly elliptic surfaces and surfaces of general type.

A minimal properly elliptic surface can be deformed to guarantee the existence of a canonical divisor whose support is the union of smooth elliptic fibers. The structure theorem separates these into two types: regular fibers and multiple fibers with multiplicity  $m \geq 2$ ; the corresponding universal functions are  $L^1(t)$  and  $L_m^2(t)$  respectively. For a minimal surface of general type all canonical divisors are connected; we assume that one such divisor  $D$  is smooth and reduced. By the adjunction formula,  $D$  has genus  $h = K_X^2 + 1$ . The GW invariant is then given by one of two universal functions  $L_{h,\pm}^3(t)$  for this  $h$ .

The GW invariants of a Kähler surface  $X$  can be regarded as a power series in formal variables  $t_A$  with  $A \in H_2(X; \mathbb{Z})$  as described in Section 1. Each smooth component  $D_k$  of the canonical class, we can replace  $t$  by  $t_{D_k}$  in the appropriate universal function (0.1), taking  $h$  to be the genus  $D_k$ . Pushing forward under the map  $(\iota_{D_k})_* : H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n) \rightarrow H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$  induced by the inclusion  $D_k \hookrightarrow X$  then gives a series (suppressing  $(\iota_{D_k})_*$  in the notation)

$$L^i(t_{D_k}) \in \prod_{g,n} H_*(\overline{\mathcal{M}}_{g,n} \times X^n)[[t_{D_k}, \lambda]],$$

which gives the local contribution of  $D_k$  to the GW series. For each surface, only a few of these series are needed. The structure theorem

lists the possibilities. The contribution  $GW_X^0$  of the class  $A = 0$  (that is, the evaluation of (1.4) at  $t = 0$ ) must be separated out. Note that  $GW_X^0$  has been explicitly computed (see [16]). Also note that surfaces with  $p_g > 0$  have a unique minimal model ([4], p. 243).

**Theorem 0.1** (Structure Theorem). *Let  $X$  be a closed Kähler surface with  $p_g > 0$  and smooth canonical divisor  $D$ . Write  $D = \sum_i E_i + D'$  where  $\{E_i\}$  are the exceptional curves in  $D$ . Then the GW invariant of  $X$  is a sum*

$$GW_X = GW_X^0 + \sum_{E_i} L^0(t_{E_i}) + GW'_X,$$

where  $GW'_X$  is given as follows according to the type of the minimal model  $X'$  of  $X$ :

- 1) If  $X'$  is K3 or abelian, then  $GW'_X = 0$ .
- 2) If  $\pi : X' \rightarrow C$  is properly elliptic, we can assume that the canonical divisor  $D'$  has the form  $\sum n_j F_j + \sum (m_k - 1) F_k$  for regular fibers  $F_j$  and smooth multiple fibers  $F_k$  of multiplicity  $m_k$ . We then have

$$GW'_X = k_\pi L^1(t_F) + \sum_k L^2_{m_k}(t_{F_k}),$$

where  $F$  is a regular fiber,  $t_{F_k}^{m_k} = t_F$ , and  $k_\pi = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$ .

- 3) If  $X'$  is general type and we can choose  $D'$  to be smooth with multiplicity 1, then  $D'$  has genus  $h = K_{X'}^2 + 1 \geq 2$  and

$$GW'_X = \begin{cases} L^3_{h,+}(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is even} \\ L^3_{h,-}(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is odd.} \end{cases}$$

A more detailed version of the structure theorem is given, with proofs, in Sections 4-7. Section 8 contains some analytic results about the linearization of the  $J_\alpha$ -holomorphic map equation, which has some remarkable properties. Those are used in Sections 9 and 10 to explicitly compute the contribution to the GW invariants of special types of covers. We do this for double covers, then for all etale covers of elliptic fibers.

The structure theorem shows that, under the stated hypotheses, the GW invariants are determined by the map  $H^*(X, \mathbb{Q}) \rightarrow H^*(D, \mathbb{Q})$  induced by the inclusion  $D \subset X$  and by the parity of the holomorphic Euler characteristic, which is given in terms of the Betti numbers by  $\chi(\mathcal{O}_X) = \frac{1}{2}(1 - b^1 + b^+)$ . In the case when  $X$  is a simply-connected surface of general type with a smooth reduced canonical divisor, this information is determined by the homology and Seiberg-Witten invariants of  $X$ , and hence depends only on the differentiable structure of  $X$ . Furthermore, the SW invariants are equivalent to Taubes'  $Gr$  invariants, which correspond to a subset of GW invariants [23], [12]. But we learn

from the structure theorem that the full set of GW invariants contain exactly the same information as the  $Gr$  and SW invariants.

Our structure theorem applies for surfaces with  $p_g > 0$ . This is exactly the case when the dimension of the spaces of stable maps differs from the dimension of the generalized Severi variety, and thus the GW invariants are not enumerative invariants. The  $J_\alpha$ -holomorphic map equation can also be used to define a set of “Family GW invariants” that are directly related to enumerative invariants. That context is explained in [17], [18] and [19].

One might hope that the structure theorem extended to non-Kähler symplectic manifolds with  $b^+ > 1$ . Unfortunately, as M. Usher observed ([24], page 4), McMullen and Taubes have constructed a symplectic four-manifold whose GW invariant is not the sum of local invariants supported on the components of the canonical class.

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## 1. Gromov-Witten invariants

We will use the definitions and notation of [14] for stable maps and the Gromov-Witten invariants; these are based on the approach developed by Ruan-Tian [22] and Li-Tian [20]. In summary, the key definitions go as follows. A *bubble domain*  $B$  is a finite connected union of smooth oriented 2-manifolds  $B_i$  joined at nodes together with  $n$  marked points, none of which are nodes. Collapsing the unstable components to points gives a connected domain  $st(B)$  with some arithmetic genus  $g$ . Let  $\bar{U}_{g,n} \rightarrow \bar{\mathcal{M}}_{g,n}$  be the universal curve over the Deligne-Mumford space of genus  $g$  curves with  $n$  marked points. We can put a complex structure  $j$  on  $B$  by specifying an orientation-preserving map  $\varphi_0 : st(B) \rightarrow \bar{U}_{g,n}$ , which is a diffeomorphism onto a fiber of  $\bar{U}_{g,n}$ . We will often write  $C$  for the curve  $(B, j)$ . A  $(J, \nu)$ -holomorphic map from  $B$  is then a map  $(f, \varphi) : B \rightarrow X \times \bar{U}_{g,n}$  where  $\varphi = \varphi_0 \circ st$  and which satisfies

$$\bar{\partial}_J f = \varphi^* \nu$$

(here the perturbation  $\nu$  is a tensor on  $X \times \bar{U}_{g,n}$ ; see [14]). Such a map is a *stable map* if the restriction of  $(f, \varphi)$  to each component of  $B$  is non-trivial in homology. For generic  $(J, \nu)$  the moduli space  $\mathcal{M}_{g,n}(X, A)$  of stable  $(J, \nu)$ -holomorphic maps representing a class  $A \in H_2(X)$  is a smooth orbifold of (real) dimension

$$(1.1) \quad -2K_X \cdot A + (\dim X - 6)(1 - g) + 2n.$$

Its compactification carries a (virtual) fundamental class whose push-forward under the map

$$(1.2) \quad \overline{\mathcal{M}}_{g,n}(X, A) \xrightarrow{\text{st} \times \text{ev}} \overline{\mathcal{M}}_{g,n} \times X^n$$

defined by stabilization and evaluation at the marked points is the Gromov-Witten invariant

$$GW_{g,n}(X, A) \in H_*(\overline{\mathcal{M}}_{g,n} \times X^n).$$

This is equivalent to the collection of “GW numbers”

$$(1.3) \quad GW_{g,n}(X, A)(\mu; \gamma^1, \dots, \gamma^n)$$

obtained by evaluating on classes  $\mu \in H^*(\overline{\mathcal{M}}_{g,n})$  and  $\gamma^j \in H^*(X)$  whose total degree is the dimension (1.1) of the space of stable maps. The number (1.3) is obtained by choosing (generic) geometric representatives  $M \subset \overline{\mathcal{M}}_{g,n}$  and  $\Gamma_i$  of the classes Poincaré dual to  $\mu \in H^*(\overline{\mathcal{M}}_{g,n})$  and  $\gamma^i \in H^*(X)$  and counting, with sign, the finite set of maps  $f : C \rightarrow X$  in  $st(C) \in M$  and  $f(x_i) \in \Gamma_i$  for each marked point  $x_i$ .

It is convenient to assemble these into a single invariant by introducing variables  $\lambda$  to keep track of the Euler class and  $t_A$  satisfying  $t_A t_B = t_{A+B}$  to keep track of  $A$ . The GW series of  $(X, \omega)$  is then the formal series

$$(1.4) \quad GW_X = \sum_{A,g,n} \frac{1}{n!} GW_{g,n}(X, A) t_A \lambda^{2g-2}.$$

### 2. $J_\alpha$ -holomorphic maps into Kähler surfaces

Fix a Kähler surface  $(X, J, g)$ . On  $X$ , holomorphic sections of the canonical bundle are holomorphic  $(2, 0)$  forms, and the dimension of the space  $H^{2,0}(X)$  of such forms is the geometric genus  $p_g$  of  $X$ . We will always assume that  $p_g > 0$ . Each  $\alpha \in H^{2,0}(X)$  can be identified with an element of the  $2p_g$ -dimensional real vector space

$$\mathcal{H} = \text{Re}(H^{2,0} \oplus H^{0,2}).$$

Using the metric, each  $\alpha \in \mathcal{H}$  defines an endomorphism  $K_\alpha$  of  $TX$  by the equation

$$(2.1) \quad \langle u, K_\alpha v \rangle = \alpha(u, v).$$

These endomorphisms  $K_\alpha$  are central to our discussion, and we will frequently use the following properties. Denote by  $\nabla$  the Levi-Civita connection of the given metric.

**Lemma 2.1.** *The  $K_\alpha$  are skew-adjoint and anti-commute with  $J$  ( $K_\alpha J = -JK_\alpha$ ). Furthermore,*

$$(a) \quad \nabla K_\alpha = K_{\nabla \alpha} \quad \text{and} \quad (b) \quad K_\alpha^2 = -|\alpha|^2 Id.$$

Consequently, they satisfy the pointwise Clifford relations

$$K_\alpha K_\beta + K_\beta K_\alpha = -2\langle \alpha, \beta \rangle Id.$$

*Proof.* The first two statements and (a) are immediate from (2.1). The Clifford relations follow by polarization from (b), which is easily proved (cf. [17]). q.e.d.

Now consider holomorphic maps  $f : C \rightarrow X$  from a connected complex curve with complex structure  $j$  into  $X$ . It is standard in geometric analysis to consider solutions of the perturbed  $J$ -holomorphic map equation

$$\bar{\partial}_J f = \nu,$$

where  $\bar{\partial}_J f = \frac{1}{2}(df + Jdfj)$  and where  $\nu$  is an appropriate perturbation term. In [17] the first author observed that, on a Kähler surface with  $p_g > 0$ , there is a natural family of such perturbations parameterized by  $\mathcal{H}$ . Specifically, we can consider the pairs  $(f, \alpha)$  satisfying

$$(2.2) \quad \bar{\partial}_J f = K_\alpha \partial_J f j.$$

This can equally well be viewed as a set of unperturbed holomorphic map equations for a family of almost complex structures  $\{J_\alpha\}$  parameterized by  $\mathcal{H}$ . For each  $\alpha \in \mathcal{H}$  the endomorphism  $JK_\alpha$  is skew-adjoint, so  $Id + JK_\alpha$  is injective, and hence invertible. Thus there is a family of almost complex structures

$$(2.3) \quad J_\alpha = (Id + JK_\alpha)^{-1} J (Id + JK_\alpha)$$

on  $X$  parameterized by  $\alpha \in \mathcal{H}$ . A simple computation shows that (2.2) is equivalent to the  $J_\alpha$ -holomorphic map equation

$$(2.4) \quad \bar{\partial}_{J_\alpha} f = 0$$

for maps  $f : C \rightarrow X$ . Our structure theorem for GW invariants will emerge from studying the solutions of this equation for a fixed  $\alpha \in \mathcal{H}$ . Note that while  $\alpha$  itself is holomorphic, the corresponding almost complex structure  $J_\alpha$  need not be integrable. On the other hand,  $J_\alpha$  is generally not a generic almost complex structure on  $X$ , so the moduli space of  $J_\alpha$  holomorphic maps does not directly define the GW invariants.

### 3. The Localization Lemma and vanishing results

The discussion in this section builds on the following simple principle about Gromov-Witten invariants.

**Vanishing Principle 3.1.** If for *some*  $\omega$ -tamed almost complex structure  $J$ , a class  $A \in H_2(X)$  cannot be represented by a  $J$ -holomorphic curve of genus  $g$ , then  $GW_{g,n}(X, A)$  vanishes.

The proof is straightforward: if some  $GW_{g,n}(X, A)$  were not zero, we could choose sequences  $\{J_n\}$  of generic almost complex structures converging to  $J$  for which there were  $J_n$ -holomorphic maps representing  $A$ . But then, by the compactness theorem for pseudo-holomorphic maps, a subsequence of those maps would limit to a  $J$ -holomorphic map representing  $A$ , contradicting the assumption. As a simple application, note that for a Kähler surface  $(X, J)$ , any  $J$ -holomorphic curve represents a  $(1,1)$  class, so  $GW_{g,n}(X, A) = 0$  unless  $A$  is a  $(1,1)$  class. This observation allows us to restrict attention to  $(1,1)$  classes for all our results.

**Lemma 3.2** (Image Localization Lemma). *Fix a Kähler surface  $(X, J)$  with  $p_g > 0$  and  $\alpha \in \mathcal{H}$ . If  $f : C \rightarrow X$  is a  $J_\alpha$ -holomorphic map with connected domain that represents a  $(1,1)$  class  $A \neq 0$ , then  $f$  is in fact a  $J$ -holomorphic map whose image  $f(C)$  lies in the support of the zero divisor  $D_\alpha$  of  $\alpha$ .*

*Proof.* For any  $C^1$  map  $f : C \rightarrow X$  we have the pointwise equality

$$(3.1) \quad \langle \bar{\partial}f, K_\alpha \partial f j \rangle \, d\text{vol} = f^* \alpha$$

(see Proposition 1.3 of [17]). Integrating over the domain and using (2.2) gives

$$\int_C |\bar{\partial}f|^2 = \int_C \langle \bar{\partial}f, K_\alpha \partial f j \rangle = \int_C f^* \alpha.$$

Because  $\alpha$  is closed, the last integral is the homology pairing  $\alpha[A]$ . This vanishes on the  $(1,1)$  class  $A$  because  $\alpha$  is a linear combination of  $(2,0)$  and  $(0,2)$  forms. Thus  $\bar{\partial}f \equiv 0$  on  $C$ . Then using (2.2), Lemma 2.1 and the equality  $|df|^2 = |\bar{\partial}f|^2 + |\partial f|^2$ , we obtain

$$(3.2) \quad 0 = \int_C |\bar{\partial}f|^2 = \int_C |K_\alpha \partial f j|^2 = \int_C |\alpha|^2 |df|^2.$$

Since  $A \neq 0$ , there is at least one irreducible component of  $C$  with  $df \not\equiv 0$ . On each such component  $C_i$ ,  $df$  has finitely many zeros, so (3.2) implies that  $f(C_i)$  lies in the support of  $D_\alpha$ . Each of the remaining components is taken to a single point by  $f$ ; since  $C$  is connected those points also lie in the support of  $D_\alpha$ . q.e.d.

Lemma 3.2 leads directly to some striking vanishing results for GW invariants. For example, K3 and abelian surfaces have trivial canonical bundle, so they admit  $(2,0)$  forms that vanish nowhere. Lemma 3.2 and Principle 3.1 then give:

**Corollary 3.3.** *For K3 and abelian surfaces, all GW invariants  $GW_{g,n}(X, A)$  vanish for  $A \neq 0$ .*

We also obtain a vanishing result for the GW numbers (1.3). This follows from the Vanishing Principle and the geometric interpretation of the GW numbers.



**Corollary 3.4.** *On a Kähler surface  $X$  with  $p_g > 0$ , any GW invariant constrained to pass through (generic) points or circles vanishes. Equivalently,  $GW_{g,n}(X, A)(\mu; \gamma^1, \dots, \gamma^k) = 0$  whenever one of the  $\gamma^j$  lies in  $H^3(X)$  or  $H^4(X)$ .*

*Proof.* When  $PD(\gamma^j)$  is a point or 1-dimensional class, we can fix a geometric representative  $\Gamma_j$  disjoint from  $D_\alpha$ . Then, if the invariant  $GW_{g,n}(X, A)(\gamma^1, \dots, \gamma^k)$  were not zero, we could find a sequence  $\{J_n\}$  of generic almost complex structures converging to  $J_\alpha$  and  $J_n$ -holomorphic maps  $\{f_n\}$  representing  $A$  with  $f_n(x_i) \in \Gamma_i$  for all  $i$  and  $n$ . The compactness theorem would then yield a limit  $J_\alpha$ -holomorphic map  $f$  satisfying  $f(x_j) \in \Gamma_j$ , contradicting Lemma 3.2. q.e.d.

The Image Localization Lemma allows us to localize the GW invariants for Kähler surfaces with  $p_g > 0$ . When  $X$  is such a surface and  $\alpha \in \mathcal{H}$ , the support of the zero divisor  $D_\alpha$  of  $\alpha$  is a union of disjoint topological components  $D_\alpha^k$ . Lemma 3.2 implies that, for generic  $(J, \nu)$  near  $(J_\alpha, 0)$ , the image of any  $(J, \nu)$ -holomorphic map with connected domain lies in an open neighborhood  $U_k$  of one and only one of the  $D_\alpha^k$ . Thus the compactified moduli space of  $(J, \nu)$ -holomorphic maps representing a non-zero class  $A$  is a disjoint union

$$(3.3) \quad \overline{\mathcal{M}}_{g,n}(X, A) = \coprod \overline{\mathcal{M}}_{g,n}(U_k, A_k)$$

over all  $A_k$  with  $(\iota_k)_*A_k = A$  under the inclusion  $\iota_k : U_k \rightarrow X$ . Note that each  $U_k$  is an open symplectic four-manifold with  $H_*(U_k) = H_*(D_\alpha^k)$ . As in Section 1, the image of each  $\overline{\mathcal{M}}_{g,n}(U_k, A_k)$  under the map (1.2) defines a homology class

$$(3.4) \quad GW_{g,n}^{\text{loc}}(D_\alpha^k, A_k) \in H_*(\overline{\mathcal{M}}_{g,n} \times D_k^n)$$

that we call the *local GW invariant of  $D_\alpha^k$  for the (non-zero) class  $A_k$* . These local invariants depend on the choice of the canonical divisor  $D_\alpha$ , rather than on the choice of  $\alpha$  itself. Indeed, if  $\beta \in \mathcal{H}$  also has zero divisor  $D_\alpha$ , then  $\beta = c\alpha$  for some constant. Thus,  $J_\alpha$  and  $J_\beta$  are connected by a path  $J_t = J_{\alpha_t}$  with  $\alpha_0 = \alpha$  and  $\alpha_1 = \beta$ , for which every  $J_t$ -holomorphic map lies in the support of  $D_\alpha$ . The standard cobordism argument then shows that the local invariants  $GW_{g,n}^{\text{loc}}(D_\alpha^k, A_k)$  associated with  $J_\alpha$  and  $J_\beta$  are the same.

We remark in passing that the local invariants (3.4) can also be regarded as elements of the homology of the space  $\overline{\mathcal{M}}_{g,n}(D_\alpha^k, d_k(A))$  of stable maps into the curve  $D_\alpha^k$  with degree determined by the equation  $(\iota_k)_*A_k = d_k(A)[D_\alpha^k]$ . From that perspective, (3.4) is the image of the local invariant under the homology map induced by the evaluation map (1.2) with  $X = D_\alpha^k$ .

Pushing (3.3) forward under the evaluation map (1.2) and passing to homology shows that, for  $A \neq 0$ ,

$$(3.5) \quad GW_{g,n}(X, A) = \sum_{(\iota_k)_* A_k = A} GW_{g,n}^{\text{loc}}(D_\alpha^k, A_k)$$

for any choice of the canonical divisor  $D_\alpha$ . This formula is the first step toward our structure theorem. It shows that the GW invariants can be expressed as a sum of local contributions associated with the components of a canonical divisor.

### 4. Local GW invariants

The local invariants in the sum (3.5) depend, at least *a priori*, on the local geometry of  $J_\alpha$  around the components of the canonical divisor  $D_\alpha$ . In the rest of this paper we will write  $D_\alpha = \sum m_k D_k$  and assume that the  $D_k$  are smooth and disjoint. We will show that the local invariants depend only on discrete data  $g, n, d$  and the multiplicities  $m_k$ . When  $D_k$  is smooth every map with image in  $D_k$  represents a multiple  $d$  of  $[D_k]$ , so we will write the local invariant (3.4) as

$$GW_{g,n}^{\text{loc}}(D_k, m_k, d)$$

or simply  $GW_{g,n}^{\text{loc}}(D_k, d)$  when  $m_k = 1$ . Then, for  $A \neq 0$ , equation (3.5) reads

$$(4.1) \quad GW_{g,n}(X, A) = \sum_{d_k [D_k] = A} GW_{g,n}^{\text{loc}}(D_k, m_k, d_k).$$

Using arguments like those in the previous section, one can also define local GW invariants of some open complex surfaces. Fix a smooth curve  $D$  with canonical bundle  $K_D$  and a line bundle  $\pi : N \rightarrow D$  satisfying  $N^{m+1} = K_D$ . The total space of  $N$  is a complex manifold; from the exact sequence  $0 \rightarrow \pi^* N \rightarrow TN \rightarrow \pi^* TD \rightarrow 0$  we see that its canonical bundle is

$$(4.2) \quad K_N = \wedge^2 T^* N = \pi^* K_D \otimes \pi^* N^* = \pi^* N^{m+1} \otimes \pi^* N^* = \pi^* N^m.$$

The bundle  $\pi^* N$  has a tautological section  $\sigma$  whose zero divisor is exactly  $D$ . Then  $\alpha = \sigma^m$  is a section of the canonical bundle  $K_N$ , and so is a holomorphic (2,0)-form on  $N$ . The argument used to prove Lemma 3.2 then shows that the image of any  $J_\alpha$ -holomorphic map into  $N$  lies in  $D$ . On the other hand, an open neighborhood  $U$  of  $D \subset N$  is isomorphic to some open neighborhood  $V$  of the zero section  $D_0$  in the projectivization  $\mathbb{P} = \mathbb{P}(N \oplus \mathcal{O}_D)$  by an isomorphism taking  $D$  to  $D_0$ . The pull-back of the Kähler form on  $V$  by that isomorphism gives a Kähler form on  $U$ . Thus, for any generic  $(J, \nu)$  sufficiently close to  $(J_\alpha, 0)$ , the moduli space  $\mathcal{M}_{g,n}(U, d[D])$  can be compactified by standard geometric analysis

techniques. Taking the image as in (1.2) yields homology classes

$$(4.3) \quad L_{g,n}(N, m, d) \in H_*(\overline{\mathcal{M}}_{g,n} \times D^n)$$

that we call the *local GW invariants of  $N$  associated with  $mD$*  for maps representing  $d[D]$ ,  $d > 0$ . When  $m = 1$  we will often write (4.3) as simply  $L_{g,n}(N, d)$ . These local invariants depend on the zero divisor of  $\alpha$  but not on  $\alpha$  itself by the following reasoning. Let  $\beta$  be a section of the canonical bundle  $K_N$ , defined on a neighborhood  $U$  of  $D \subset N$ , such that the zero divisor of  $\beta$  is  $mD$ . Then  $\beta = h\alpha$  for some holomorphic function  $h$  whose restriction of  $h$  to  $D$  is a non-zero constant. Hence, after shrinking  $U$  if necessary,  $J_\alpha$  and  $J_\beta$  can be connected by a path  $J_{\alpha_t}$  where the zero divisor of each  $\alpha_t$  on  $U$  is  $mD$ . As in the previous section, the usual cobordism argument then shows that the local invariants associated with  $J_\alpha$  and  $J_\beta$  are the same.

A similar cobordism argument gives the following fact.

**Lemma 4.1.** *If  $\{(N_t, D_t)\}_{0 \leq t \leq 1}$  is a smooth path of line bundles satisfying  $N_t^{m+1} = K_{D_t}$  then*

$$L_{g,n}(N_0, m, d) = L_{g,n}(N_1, m, d).$$

Thus the local invariants (4.3) depend only on the discrete data  $g, n$ , and  $d$  and the deformation class of the pair  $(N, D)$ .

**Example 4.2.** Consider the line bundle  $\mathcal{O}(-1)$  on  $\mathbb{P}^1$ . The complex structure  $J_0$  on the total space of  $\mathcal{O}(-1)$  is not of the form  $J_\alpha$ , but nevertheless has the property that any  $J_0$ -holomorphic map representing the class  $d[\mathbb{P}^1]$  has an image in the zero section in the total space of the bundle  $\mathcal{O}(-1)$ . The argument used above thus applies for  $J_0$  as well as for  $J_\alpha$ , showing that  $J_0$  itself defines the local GW invariants  $L_{g,n}(\mathcal{O}(-1), d)$ .

We can relate the local invariants of  $D_k$  defined in (3.4) with the local invariants of its normal bundle defined in (4.3), as follows.

**Lemma 4.3.** *Let  $X$  be a Kähler surface with  $p_g > 0$  and  $D_\alpha = \sum m_k D_k$  be the zero divisor of  $\alpha \in \mathcal{H}$ . If  $D_k$  is smooth with normal bundle  $N_k$  and  $D_k \cap D_\ell = \emptyset$  for all  $\ell \neq k$ , then*

$$GW_{g,n}^{\text{loc}}(D_k, m_k, d) = L_{g,n}(N_k, m_k, d).$$

*Proof.* Fix  $D = D_k$ . By the adjunction formula, the normal bundle  $N$  of  $D$  satisfies  $N^{m+1} = K_D$  with  $m = m_k$ . Let  $Z$  be the blow-up of  $X \times \mathbb{C}$  along  $D \times \{0\}$ . The projection  $X \times \mathbb{C} \rightarrow \mathbb{C}$  lifts to a map  $p: Z \rightarrow \mathbb{C}$  whose fibers  $Z_\lambda = p^{-1}(\lambda)$  are isomorphic to  $X$  for  $\lambda \neq 0$  and whose central fiber  $Z_0$  is a singular surface  $X \cup_D \mathbb{P}$  where  $\mathbb{P}$  is the ruled surface  $\mathbb{P}(N \oplus \mathcal{O}_D) \rightarrow D$  defined by fiber projectivization. The proper transform of  $D \times \mathbb{C}$  is a smooth divisor  $\tilde{D} \subset Z$ , disjoint from the proper transforms  $\tilde{D}_\ell$  of the other  $D_\ell \times \mathbb{C}$ , and  $\alpha$  gives rise to a section  $\tilde{\alpha}$  of the

canonical bundle  $K_Z$  of  $Z$  whose zero divisor is  $m\tilde{D} + \sum_{\ell \neq k} m_\ell \tilde{D}_\ell + \mathbb{P}$ . Now fix a tubular neighborhood  $U$  of  $\tilde{D}$  that is disjoint from the  $\tilde{D}_\ell$ . Let  $\kappa$  be the line bundle of the divisor  $m\tilde{D}$ , and let  $\beta \in \Gamma(\kappa)$  be a section with zero divisor  $m\tilde{D}$ . For each  $\lambda$ , the intersection  $U_\lambda = U \cap Z_\lambda$  is a tubular neighborhood of  $D_\lambda = \tilde{D} \cap Z_\lambda$ . The restriction

$$\kappa_\lambda = \kappa|_{U_\lambda}$$

is the line bundle on  $U_\lambda$  with divisor  $mD_\lambda$ . Observe that:

- For  $\lambda \neq 0$ , the normal bundle  $N_\lambda$  to  $Z_\lambda$  in  $Z$  is trivial. Restricting the exact sequence  $0 \rightarrow TZ_\lambda \rightarrow TZ \rightarrow N_\lambda \rightarrow 0$  to  $U_\lambda$  then shows that the canonical bundle of  $U_\lambda$  is the restriction of the canonical bundle of  $Z$ , which is the bundle of the divisor  $m\tilde{D} \cap U_\lambda = mD_\lambda$ .
- For  $\lambda = 0$  we use a different argument. By the definition of blow-up,  $U_0$  is biholomorphic to a neighborhood of the zero section in the total space of the bundle  $N \rightarrow D$ ; in fact, this identifies the zero section with  $D_0$ . But by (4.2) the canonical bundle of  $N$  has a tautological section whose divisor is  $m$  times that zero section.

Thus,  $\kappa_\lambda$  is the canonical bundle of  $U_\lambda$  for each  $\lambda$ .

Restricting  $\beta$  to  $U_\lambda$  gives a section  $\beta_\lambda$  of  $\kappa_\lambda$  whose zero divisor is  $mD_\lambda$ , and a corresponding almost complex structure  $J_\lambda = J_{\beta_\lambda}$  on  $U_\lambda$ . Then the image of any  $J_\lambda$ -holomorphic map lies in  $D_\lambda$ , so  $J_\lambda$  determines local invariants  $GW_{g,n}^{loc}(D_\lambda, m, d)$  of  $U_\lambda$  for the class  $d[D_\lambda]$  (with  $d > 0$ ). Because  $\beta_\lambda$  and  $J_\lambda$  vary smoothly in  $\lambda$ , we then have

$$GW_{g,n}^{loc}(D_\lambda, m, d) = GW_{g,n}^{loc}(D_0, m, d)$$

for each  $\lambda$ . The righthand side of the above equals  $L_{g,n}(N, m, d)$  by definition, while for  $\lambda \neq 0$  the lefthand side is  $GW_{g,n}^{loc}(D, m, d)$  because  $Z_\lambda$  is biholomorphic to  $X$  by a map that takes  $D_\lambda$  to  $D$ . This completes the proof of the lemma. q.e.d.

**Example 4.4.** Let  $\pi : E(m+2) \rightarrow \mathbb{P}^1$  be an elliptic surface with  $12(m+2)$  singular fibers which are all nodal. This surface is K3 if  $m = 0$  and properly elliptic if  $m > 0$ . By the canonical divisor formula (see (6.1) below) the canonical bundle of  $E(m+2)$  is  $\pi^*\mathcal{O}(m)$ . Thus the generic canonical divisor is the sum of  $m$  disjoint regular fibers  $F_i$ , and for any regular fiber  $F$  the divisor  $mF$  is also a canonical divisor. Using Lemma 4.3 and equation (4.1), we then have

$$L_{g,n}(\mathcal{O}, m, d[F]) = GW_{g,n}(E(m+2), d[F]) = m L_{g,n}(\mathcal{O}, d[F]).$$

### 5. Exceptional curves and blowups

This section establishes a “blowup formula” that reduces the problem of computing GW invariants to the case of minimal surfaces. This extends some previous partial blowup formulas, cited at the end of this

section. In our approach the blowup formula is a consequence of the localization Lemma 3.2.

First consider a closed symplectic 4-manifold  $X$  with an almost complex structure  $J$  and an exceptional  $J$ -holomorphic curve  $E$ . We can then consider the (global) invariants

$$(5.1) \quad GW_{g,n}(X, d[E]),$$

which give the contributions to  $GW_X$  of all maps whose image represents a multiple of  $[E]$ . Fix a diffeomorphism  $\iota : \mathbb{P}^1 \rightarrow E$  and let  $\iota_*$  denote the map  $H_*(\overline{\mathcal{M}}_{g,n} \times (\mathbb{P}^1)^n) \rightarrow H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$  induced by  $\iota$ .

**Lemma 5.1.** *For  $d > 0$ , (5.1) is given by the local invariant of Example 4.2:*

$$GW_{g,n}(X, d[E]) = \iota_* L_{g,n}(\mathcal{O}(-1), d).$$

*Proof.* Since  $E^2 = -1$ , any  $J$ -holomorphic curve representing a class  $d[E]$  has an image in  $E$ . Thus,

$$GW_{g,n}(X, d[E]) = GW_{g,n}^{\text{loc}}(E, d).$$

After rescaling the symplectic form on  $\mathbb{P}^1$  we may assume that  $\iota : \mathbb{P}^1 \rightarrow E$  is a symplectomorphism. By the Symplectic Neighborhood Theorem this extends to a symplectomorphism  $\varphi : U \rightarrow V$  from a neighborhood  $U$  of the zero section in  $\mathcal{O}(-1) \rightarrow \mathbb{P}^1$  to a neighborhood  $V$  of  $E$  in  $X$ . Pushing the standard complex structure  $J_0$  on  $\mathcal{O}(-1)$  forward by  $\varphi$  gives an almost complex structure  $J'_0$  on  $V$  that makes  $\varphi$  an isomorphism of almost complex neighborhoods. Furthermore,  $E$  is a  $J'_0$  holomorphic curve, so the local invariant above can be calculated using  $J'_0$ . Thus, when  $d > 0$ ,

$$GW_{g,n}^{\text{loc}}(E, d) = \iota_* L_{g,n}(\mathcal{O}(-1), d).$$

q.e.d.

Let  $X$  be a compact Kähler surface with  $p_g > 0$  and let  $\pi : \tilde{X} \rightarrow X$  be the blowup of  $X$  at a point  $p$ . Different choices of the point  $p$  yield surfaces  $\tilde{X}$  that are symplectic deformation equivalent, so the GW invariants of  $\tilde{X}$  are independent of the choice of  $p$ . Note that every  $A \in H_2(\tilde{X})$  can be uniquely written as  $A = B + dE$  where  $E$  is the class of the exceptional curve and  $B \cdot E = 0$  and the invariant  $GW_{g,n}(X, \pi_* B)$  can be regarded as a homology class in  $H_*(\overline{\mathcal{M}}_{g,n} \times (X \setminus \{p\})^n)$ .

**Proposition 5.2.** *Let  $X$  be a compact Kähler surface with  $p_g > 0$  and let  $\pi : \tilde{X} \rightarrow X$  be its blowup at a point  $p$ . Then the GW invariant of each class  $A = B + dE$  as above is given by*

$$(5.2) \quad GW_{g,n}(\tilde{X}, A) = \begin{cases} L_{g,n}(\mathcal{O}(-1), d) & \text{if } A = dE \text{ with } d > 0 \\ \pi'_* GW_{g,n}(X, \pi_* A) & \text{if } A \cdot E = 0 \\ 0 & \text{otherwise} \end{cases}$$

where  $\pi'_*$  is the induced homology map by the composition of the isomorphism  $X \setminus \{p\} \rightarrow \tilde{X} \setminus E$  and the inclusion  $\tilde{X} \setminus E \rightarrow \tilde{X}$ .

*Proof.* Fix a holomorphic  $(2, 0)$  form  $\alpha$  on  $\tilde{X}$  with zero divisor  $D \in |K_X|$  and a blowup point  $p \notin D$ . Then  $\tilde{\alpha} = \pi^*\alpha$  is a holomorphic  $(2, 0)$  form on  $\tilde{X}$  whose zero divisor  $\tilde{D} \in |K_{\tilde{X}}|$  is the disjoint union of  $\pi^*(D)$  and the exceptional curve  $E$ . Each class  $A \in H_2(\tilde{X})$  with non-zero GW invariant can be represented by a  $J_{\tilde{\alpha}}$ -holomorphic map  $f : C \rightarrow \tilde{X}$  from a connected curve  $C$ . By Lemma 3.2, the image of  $f$  lies in  $\tilde{D}$ . Hence either  $A \cdot E = 0$  or  $A = dE$  with  $d > 0$ . The case  $A = dE$  was done in Lemma 5.1.

If  $A \cdot E = 0$ , choose a sequence of almost complex structures  $J_\ell$  converging to  $J_{\tilde{\alpha}}$ . As  $\ell \rightarrow \infty$ , the  $J_\ell$ -holomorphic maps converge pointwise to  $J_{\tilde{\alpha}}$ -holomorphic maps. These limit maps lie in  $\tilde{D}$  but not in  $E$  because of the condition  $A \cdot E = 0$ . Thus for large  $\ell$  the images are bounded away from  $E$ ; in fact, they are uniformly bounded away from  $E$  for  $f$  in the compact space  $\overline{\mathcal{M}}_{g,n}^{J_\ell}(\tilde{X}, A)$  of stable maps. Consequently, the condition that  $J_\ell$  is generic for this space of stable curves is the same as the condition that an almost complex structure that agrees with  $\pi_*J_\ell$  outside a sufficiently small neighborhood of the blowup point is generic for the corresponding space of stable maps into  $X$ . When both are generic, composition with  $\pi$  gives a diffeomorphism

$$\mathcal{M}_{g,n}^{J_\ell}(\tilde{X}, A) \xrightarrow{\cong} \mathcal{M}_{g,n}^{J_\ell}(X, \pi_*A)$$

that respects orientations and the stabilization and evaluation maps. Hence the corresponding GW invariants are equal. q.e.d.

**Remark 5.3.** The hypothesis  $p_g > 0$  is needed in Proposition 5.2. For example, when  $X$  is  $\mathbb{P}^2$  and  $L$  is the class of the line, the invariants  $GW_{g,n}(\tilde{X}, aL + bE)$  with  $b > 1$  are non-zero: they are enumerative counts of the curves in  $\mathbb{P}^2$  satisfying certain contact and tangency conditions at the blowup point (see Gathmann [7]). Jianxun Hu showed that the part of Proposition 5.2 pertaining to classes  $A$  with  $A \cdot E = 0$  and  $A \cdot E = 1$  hold on any symplectic manifold ([11]). For other classes, however, the contrast between Proposition 5.2 and Gathmann’s results for  $\mathbb{P}^2$  shows that any universal blowup formula for GW invariants must distinguish rational surfaces from those with  $p_g > 0$ .

The first part of the Structure Theorem 0.1 is a version of the blowup formula (5.2). Given a compact Kähler surface  $X$  with  $p_g > 0$ , let  $\pi : X \rightarrow X'$  be the projection to the minimal model. By perturbing the blowup points, we can insure that there is a canonical divisor on  $X$  whose support is a disjoint union of exceptional curves  $\{E_k\}$  and other curves  $D_\ell$ . Define a formal power series with coefficients in  $H_*(\overline{\mathcal{M}}_{g,n} \times$

$(\mathbb{P}^1)^n$ ) by setting

$$(5.3) \quad L^0(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} L_{g,n}(\mathcal{O}(-1), d) t^d \lambda^{2g-2}$$

and another with coefficients in  $H_*(\overline{\mathcal{M}}_{g,n} \times X^n)$  by

$$(5.4) \quad GW'_X = \sum_{A \neq 0} \sum_{g,n} \frac{1}{n!} \pi'_* GW_{g,n}(X, \pi_* A) t_A \lambda^{2g-2}.$$

The blowup formula then gives the following succinct equation (cf. Theorem 0.1).

**Proposition 5.4.** *The GW invariant of  $X$  is a sum*

$$GW_X = GW_X^0 + \sum_{E_i} L^0(t_{E_i}) + GW'_X.$$

### 6. The Structure Theorem for properly elliptic surfaces

In light of the blowup formula of the previous section, we can henceforth assume that all surfaces  $X$  are minimal. Furthermore, the GW invariants of a K3 or abelian surface are trivial by Corollary 3.3. The Enriques-Kodaira classification then shows that, among minimal surfaces with  $p_g > 0$ , there are two cases left to consider: minimal properly elliptic surfaces and minimal surfaces of general type. We will consider these separately.

Let  $\pi : X \rightarrow C$  be a minimal properly elliptic surface. Then the sheaf  $L = (R^1 \pi_* \mathcal{O}_X)^{-1}$  is a line bundle on  $C$  with  $\deg L = \chi(\mathcal{O}_X) \geq 0$ , and the canonical bundle is

$$K_X = \pi^*(L \otimes K_C) \otimes \mathcal{O}(\sum_k (m_k - 1)F'_k),$$

where  $F'_k$  are multiple fibers of multiplicity  $m_k$  ([6] pages 47-49). Correspondingly, each canonical divisor of  $X$  has the form

$$(6.1) \quad \sum_j n_j F_j + \sum_k (m_k - 1)F'_k$$

where  $\sum_j n_j F_j$  is the pullback of a divisor in  $|L + K_C|$  of degree  $k_\pi = \chi(\mathcal{O}_X) - 2\chi(\mathcal{O}_C)$ . In general, the fibers  $F_j$  need not to be smooth or disjoint from the  $F'_k$ .

**Proposition 6.1.** *Every minimal properly elliptic surface  $\pi : X \rightarrow C$  can be deformed to a minimal properly elliptic surface whose generic canonical divisor has the form (6.1) where the  $F_j$  and  $F'_k$  are disjoint smooth fibers.*

*Proof.* By a theorem of Moishezon  $X$  can be deformed to a minimal properly elliptic surface whose only singular fibers are reduced nodal curves and multiple fibers with smooth reduction (see [6] p. 113 and [4] p. 266). This deformed  $X$  is obtained by log transforms on an elliptic surface  $\pi : S \rightarrow C$  without multiple fibers whose canonical bundle is  $K_S = \pi^*(L + K_C)$  for the same line bundle  $L$  ([6] pp. 102–103). By deforming the fibers on which the logarithmic transformations are done, we can assume that none of the fibers  $F'_k$  lie over the base points of the linear system  $|L + K_C|$ , and hence the generic canonical divisor of  $X$  has the form (6.1) with  $F_j \cap F'_k = \emptyset$  for all  $j$  and  $k$ . It therefore suffices to prove Proposition 6.1 for the surface  $S$ .

Next note that  $|L + K_C|$  is empty when  $\deg(L + K_C) = \deg L + 2g(C) - 2 \leq 0$  and has a base point at  $p \in C$  if and only if  $h^0(L + K_C - p) = h^0(L + K_C)$  (see [10] p. 308). By Riemann-Roch and Serre duality, this last condition is equivalent to  $h^0(p - L) = h^0(-L) + 1$ . Hence,  $|L + K_C|$  has no base points when  $\deg L \geq 2$ , and also when  $\deg L = 1$  and  $L \neq \mathcal{O}(p)$  for any  $p \in C$ . In these cases Bertini's Theorem implies the generic canonical divisor is the disjoint union of distinct smooth fibers. This leaves only two specific cases:

- a)  $\deg L = 0$  and  $g = g(C) \geq 2$ , and
- b)  $L = \mathcal{O}(p)$  for some  $p \in C$  and  $g \geq 1$ .

In fact, case a) occurs only when  $S$  has no singular fibers ([6], p. 48). Thus the proposition is true in case a).

In case b), choose points  $p, q \in C$  that are not linearly equivalent, and let  $L$  be any one of the  $2^{2g}$  line bundles on  $C$  with  $L^2 = \mathcal{O}(p + q)$ . Following [6] p. 60, one can construct an elliptic surface  $\pi_L : S_L \rightarrow C$  with section with  $(R^1\pi_{L*}\mathcal{O}_{S_L})^{-1} = L$  whose only singular fibers are the fibers over  $p$  and  $q$ . It follows from Seiler's Theorem (Corollary I.5.14 of [6]) that each  $S_L$  is deformation equivalent to  $S$ . Since  $L$  is not isomorphic to  $\mathcal{O}(p)$  or  $\mathcal{O}(q)$ , the generic element of  $|L + K_C|$  has support disjoint from  $p$  and  $q$ . The corresponding canonical divisor of  $S_L$  is then a union of smooth fibers. q.e.d.

**Remark 6.2.** R. Friedman (private communication) has proved a stronger version of Proposition 6.1: one can assume, after further deformations, that  $n_j = 1$  for all  $j$ . This is a more natural statement, but is not needed for our purposes in light of the calculation of Example 4.4.

Proposition 6.1 is useful because *Kähler surfaces that are deformation equivalent as complex surfaces have the same GW invariants*. This is true because deformation equivalent surfaces are smoothly isotopic ([6] page 18) and, because the space of Kähler forms with a fixed orientation is convex, that isotopy lifts to give a symplectic deformation equivalence. Consequently, the GW invariants are the same.



Thus we may assume that the generic canonical divisor  $D$  has the form (6.1) where

- each  $F_j$  is a regular fiber with holomorphically trivial normal bundle, and
- smooth multiple fiber  $F_{m_k}$  whose normal bundle  $N_k$  is torsion of order  $m_k$  in the group  $\text{Pic}^0(F_{m_k})$  of line bundles of degree zero

(cf. Section III.8 of [4]). Then for a regular fiber  $F_j$  with  $n_j = 1$ , we have the local GW invariants (4.3) with  $m = 1$  and  $N = \mathcal{O}$ . These define a function  $L^1$  as follows.

**Definition 6.3.** Let  $\mathcal{O}$  is the trivial line bundle over  $T^2$  and set

$$L^1(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} L_{g,n}(\mathcal{O}, d) t^d \lambda^{2g-2}.$$

For a regular fiber  $F_j$  with  $n_j > 1$ , one can form the corresponding power series with  $L_{g,n}(\mathcal{O}, d)$  replaced by  $L_{g,n}(\mathcal{O}, n_j, d)$ . The result is simply  $n_j L^1(t)$  by the calculation of Example 4.4.

For multiple fibers, we will define similar functions  $L_m^2(t)$  in terms of the GW invariants of a “model space” constructed by a logarithmic transformation. To that end, fix an elliptic K3 surface  $X \rightarrow \mathbb{P}^1$ , a regular  $F$  of  $X$  and a torsion line bundle  $\xi \in \text{Pic}^0(F)$  of order  $m > 1$ . Applying the logarithmic transformation defined by this data yields an elliptic surface  $X(F, \xi)$ . This surface

- is simply connected and therefore Kähler (see [8] and Theorem 3.1 of [4]), and
- has  $\chi(\mathcal{O}_X) = 2$ , so by (6.1) its canonical divisor  $D = (m - 1)F'_m$  is supported on a single multiple fiber  $F'_m$  of multiplicity  $m$ .

Changing the choices of  $X$ ,  $F$  and  $\xi$  yields a surface that is deformation equivalent to  $X(F, \xi)$  (Theorem I.7.6 of [6]) and hence has the same GW invariants. We will write  $K3(m)$  for the generic surface in this deformation class.

**Definition 6.4.** With  $K3(m)$  and  $F'_m$  as above, set

$$L_m^2(t) = \sum_{d>0} \sum_{g,n} \frac{1}{n!} GW_{g,n}(K3(m), d[F'_m]) t^d \lambda^{2g-2}.$$

The following proposition shows that the local invariants at any smooth multiple fiber  $F_m$  of multiplicity  $m$  can be expressed in terms of GW invariants of  $K3(m)$  that are encoded in the function  $L_m^2(t)$ .

**Proposition 6.5.** *Let  $X$  be a properly elliptic surface with a smooth multiple fiber  $F_m$  of multiplicity  $m \geq 2$ . Then*

$$GW_{g,n}^{\text{loc}}(X, m - 1, d[F_m]) = GW_{g,n}(K3(m), d[F'_m]).$$

*Proof.* Recall that there is a local model for a neighborhood  $U$  of  $F_m$  (cf. Prop. 6.2 of [6]). Specifically, there is a (smooth) elliptic fibration  $\pi_0 : U_0 \rightarrow \Delta$  over a unit disk  $\Delta \subset \mathbb{C}$  and a torsion line bundle  $\xi$  of order  $m$  on  $\pi_0^{-1}(0)$  such that  $U$  is isomorphic, as an elliptic fibration, to the elliptic fibration obtained by performing the  $m$ -logarithmic transformation defined by  $\xi$  on the central fiber  $\pi_0^{-1}(0)$ . In particular,  $\pi_0$  and  $\xi$  completely determine the  $m$ -spin curve  $(F_m, N_m)$ , that is, determine the curve  $F_m$  and a normal bundle  $N_m$  satisfying  $N_m^m = K_{F_m}$ .

Furthermore, there is a holomorphic function  $h_0$  on  $\Delta$  satisfying  $\text{Im } h_0(s) > 0$  such that  $\pi_0 : U_0 \rightarrow \Delta$  is the quotient  $(\mathbb{C} \times \Delta) / (\mathbb{Z} \times \mathbb{Z}) \rightarrow \Delta$  with the action of  $\mathbb{Z} \times \mathbb{Z}$  given by

$$(m, n)(z, s) = (z + m + nh_0(s), s)$$

(p. 202 of [4]). Now fix a normal neighborhood of a smooth fiber of  $K3 \rightarrow \mathbb{P}^1$ . One can then choose an isomorphic (smooth) elliptic fibration  $\pi_1 : U_1 \rightarrow \Delta$  over the unit disk  $\Delta$  under which the fixed smooth fiber of  $K3$  corresponds to the central fiber  $\pi_1^{-1}(0)$ . As above, this fibration is determined by a holomorphic function  $h_1$  on  $\Delta$  with  $\text{Im } h_1(s) > 0$ .

Since for each  $t \in [0, 1]$  the function  $h_t = (1 - t)h_0 + th_1$  is holomorphic on  $\Delta$  and satisfies  $\text{Im } h_t(s) > 0$ , using  $h_t$  one can obtain a family of elliptic fibrations  $\pi_t : U_t \rightarrow \Delta$ . Then, performing  $m$ -logarithmic transformation on each fiber  $\pi_t^{-1}(0)$  using a family of line bundles  $\xi_t$  of order  $m$  on  $\pi_t^{-1}(0)$  with  $\xi_0 = \xi$  shows that the  $m$ -spin curves defined by  $F_m \subset X$  and a multiple fiber  $F'_m \subset K3(m)$  are deformation equivalent. Therefore, we have

$$\begin{aligned} GW_{g,n}^{\text{loc}}(X, m - 1, d[F_m]) &= GW_{g,n}^{\text{loc}}(K3(m), m - 1, d[F'_m]) \\ &= GW_{g,n}(K3(m), d[F'_m]), \end{aligned}$$

where the first equality follows from Lemmas 4.1 and 4.3, and the second follows from (3.5) and the fact that the canonical divisor of  $K3(m)$  is  $(m - 1)F'_m$ . q.e.d.

The structure theorem for minimal properly elliptic surfaces follows immediately from (4.1), Lemmas 4.1 and 4.3, and Proposition 6.5. The result is the following case of Theorem 0.1.

**Theorem 6.6.** *If  $X$  is a minimal properly elliptic surface whose canonical divisor  $D$  is given as in (6.1), then*

$$GW_X = GW_X^0 + k_\pi L^1(t_F) + \sum_k L_{m_k}^2(t_{F_k}),$$

where  $F$  is a regular fiber and  $t_{F_k}^{m_k} = t_F$ .

## 7. The Structure Theorem for surfaces of general type

When  $X$  is a minimal surface of general type, every canonical divisor is connected and has arithmetic genus  $h \geq 2$  ([4]). Unlike the case of elliptic surfaces, it is not always possible to deform a surface of general type to insure the existence of a smooth canonical divisor. For example, Bauer and Catanese have shown that there is a surface  $S$  with  $p_g = 4$ ,  $K^2 = 45$  that has no complex deformations and such that each canonical divisor is singular and reducible ([3]). It is not presently understood how common such examples are. To avoid this complication we make the following assumption.

**Assumption.** For some Kähler structure in the deformation class of  $X$ , there is a smooth canonical divisor  $D$  with multiplicity 1.

(Of course, if this is true for some Kähler structure then it is true for the generic one.) When  $D$  is smooth with multiplicity 1, the adjunction formula shows that the normal bundle  $N$  of  $D$  is a holomorphic square root of  $K_D$ :

$$(7.1) \quad N^2 = K_D.$$

Recall that a *theta characteristic* on a smooth curve  $D$  is a line bundle  $N$  with  $N^2 = K_D$ . In the special case when  $K_D = \mathcal{O}$  is trivial, the set  $S(D)$  of all theta characteristics is the same as the group  $J_2(D)$  of points of order 2 in the Jacobian. In general,  $S(D)$  is a principal homogeneous space for  $J_2(D)$  with the obvious action: if  $N$  is a theta characteristic and  $L^2 = \mathcal{O}$ , then  $N \otimes L$  is another theta characteristic. Since  $J_2(D)$  is naturally isomorphic to  $H^1(D; \mathbb{Z}_2)$ , there are  $2^{2h}$  theta characteristics on a curve of genus  $h$ . A theta characteristic  $N$  is *even* or *odd* according to the parity of  $h^0(D, N)$ .

A *spin curve* is a pair  $(D, N)$  consisting of a curve with a theta characteristic. The spaces  $\mathcal{S}_{h,+}$  (resp.  $\mathcal{S}_{h,-}$ ) of all genus  $h$  even (resp. odd) spin curves have compactifications  $\overline{\mathcal{S}}_{h,\pm}$ . The following three facts are classical.

**Proposition 7.1** (see [1], [2], and [5]). *Let  $D$  be a smooth curve of genus  $h$ .*

- (a) *There are  $2^{h-1}(2^h + 1)$  even and  $2^{h-1}(2^h - 1)$  odd theta characteristics.*
- (b)  *$h^0(D_t, N_t) \bmod 2$  is constant along any smooth family  $(D_t, N_t)$  of spin curves.*
- (c)  *$\overline{\mathcal{S}}_{h,\pm}$  is an irreducible projective variety and  $\partial\mathcal{S}_{h,\pm} = \overline{\mathcal{S}}_{h,\pm} \setminus \mathcal{S}_{h,\pm}$  is a proper analytic subvariety.*

**Corollary 7.2.** *The invariants  $L_{h,n}(N, d)$ , defined by (4.3) when  $m = 1$ , depend only on the genus  $h$  and the parity of  $h^0(D, N)$ .*

*Proof.* Since  $\overline{\mathcal{S}}_{h,\pm}$  is irreducible, the smooth part  $\overline{\mathcal{S}}_{h,\pm}^*$  is connected ([9] page 21), and hence  $\overline{\mathcal{S}}_{h,\pm}^* \setminus \partial\mathcal{S}_{h,\pm}$  is connected. Thus any two smooth spin curves of the same parity can be joined by a path of spin curves. The corollary then follows from Lemma 4.1. q.e.d.

For our case — a surface of general type with a smooth canonical divisor  $D$  with multiplicity 1 — the parity of  $h^0(D, N)$  is actually a *global* invariant, as the following lemma shows.

**Lemma 7.3.** *If  $X$  is a minimal surface of general type and  $D \subset X$  is a smooth canonical divisor with normal bundle  $N$ , then*

$$h^0(D, N) \equiv \chi(\mathcal{O}_X) \pmod{2}.$$

*Proof.* Since  $N$  is the restriction of  $K$  to  $D$ , there is an exact sequence  $0 \rightarrow \mathcal{O}_X \xrightarrow{m} \mathcal{O}_X(K) \xrightarrow{r} \mathcal{O}_D(N) \rightarrow 0$ , where  $m(f) = f\alpha$  and  $r(\beta) = \beta|_D$ . This induces a long exact sequence of cohomology which, using the isomorphisms  $H^{0,1}(X) \cong H^1(\mathcal{O}_X)$  and  $H^1(K) \cong H^{2,1}(X)$ , begins

$$0 \rightarrow H^0(\mathcal{O}_X) \rightarrow H^0(K) \rightarrow H^0(N) \rightarrow H^{0,1}(X) \xrightarrow{m_*} H^{2,1}(X) \rightarrow \dots$$

where  $m_*$  is given by  $m_*(\lambda) = \lambda \wedge \alpha$ . The hermitian inner product on  $H^{0,1}(X)$  gives an orthogonal splitting  $H^{0,1}(X) = \ker m_* \oplus V$  and, by the above sequence,  $h^0(N) = p_g + q - 1 - \dim V$ . Since  $\chi(\mathcal{O}_X) = 1 - q + p_g$ , it suffices to show that  $V$  is even dimensional. After composing with the star operator,  $L = *m_* : H^{0,1}(X) \rightarrow H^{0,1}(X)$  satisfies

$$(7.2) \quad \langle \lambda, L(\delta) \rangle = -\langle \delta, L(\lambda) \rangle.$$

Thus,  $L$  induces a nondegenerate symplectic pairing on  $H^{0,1}(X)/\ker m_* \cong V$ , so  $\dim V$  is even. q.e.d.

We can proceed as we did for elliptic surfaces. Again, we first define invariants associated with a spin curve.

**Definition 7.4.** For each smooth genus  $h \geq 2$  spin curve  $(D, N)$ , let  $L_{g,n}(N, d)$  be the local GW invariant (4.3) and set

$$(7.3) \quad L_{h,\pm}^3(t) = \sum_{g,n} \sum_{d \geq 1} \frac{1}{n!} L_{g,n}(N, d) t^d \lambda^{2g-2}.$$

This notation incorporates the fact that, by Corollary 7.2, this series depends only on  $h$  and the parity of  $(D, N)$ .

For minimal surfaces of general type, the statement of the structure theorem is especially simple because the canonical divisor of  $X$  has a single component. The GW series is obtained from one of the series (7.3).

**Theorem 7.5.** *Suppose that  $X$  is a minimal surface of general type with a smooth, multiplicity 1 canonical divisor  $D$ . Let  $h = K_X^2 + 1$  be the genus of  $D$ . Then (again suppressing inclusion maps)*

$$GW_X = GW_X^0 + \begin{cases} L_{h,+}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is even} \\ L_{h,-}^3(t_{D'}) & \text{if } \chi(\mathcal{O}_X) \text{ is odd.} \end{cases}$$

Consequently, the GW series of  $X$  depends only on  $h$  and  $\chi(\mathcal{O}_X)$ .

*Proof.* This follows directly from (4.1), Lemma 4.3, Corollary 7.2 and Lemma 7.3. q.e.d.

## 8. Moduli spaces and linearizations

For each fixed  $\alpha \in \mathcal{H}$ , we can consider the linearization  $D_f$  of the  $J_\alpha$ -holomorphic map equation at each  $J_\alpha$ -holomorphic map  $f : C \rightarrow X$ . This operator is important for local descriptions of the moduli space. After a brief discussion of moduli spaces, we will write down the formula for  $D_f$  and show that it has some remarkable analytic properties.

Consider a smooth component  $D$  of a canonical divisor of  $X$ . When  $D$  has multiplicity 1, we have  $N^2 = K_D$  as in (7.1). When  $D = F_m$  is a multiple elliptic fiber with multiplicity  $m$ , the normal bundle satisfies  $N^m = \mathcal{O}_D$ . Taking Chern classes, both cases give the formula

$$(8.1) \quad c_1(N)[D] = h - 1.$$

**Lemma 8.1.** *Fix a smooth genus  $h$  component  $D \subset X$  of a canonical divisor. Then the (formal real) dimensions of the moduli spaces  $\mathcal{M}_g(D, d)$  (of degree  $d$  genus  $g$  covers of the curve  $D$ ) and  $\mathcal{M}_g(X, d[D])$  (of maps from a genus  $g$  curve representing  $d[D] \in H_2(X)$ ) are*

$$(8.2) \quad \dim \mathcal{M}_g(D, d) = 4\beta \quad \text{and} \quad \dim \mathcal{M}_g(X, d[D]) = 2\beta$$

where  $\beta = d(1 - h) + g - 1$ .

*Proof.* The restriction of  $TX$  to  $D$  decomposes as  $TD \oplus N$ . Using (8.1) we then have  $K_X \cdot D = K_D \cdot D - c_1(N)[D] = h - 1$ . Both parts of (8.2) then follow from the dimension formula (1.1). q.e.d.

To interpret the number  $\beta$  geometrically, consider a  $J_\alpha$ -holomorphic map  $f : C \rightarrow D$  from a smooth genus  $g$  curve onto  $D$ . The canonical classes of  $C$  and  $D$  are then related by the Riemann-Hurwitz formula  $K_C = f^*K_D + B$  where  $B$  is the ramification divisor. Consequently, the number of branch points, counted with multiplicity, is

$$(8.3) \quad |B| = 2\beta \quad \text{where } \beta = d(1 - h) + g - 1.$$

To proceed, we need explicit formulas. By a standard calculation (cf. [15], [21]), the linearization of the  $J_\alpha$ -holomorphic map equation,

evaluated at a map  $f$  and applied to a variation  $\xi$  of the map and a variation  $k$  of the complex structure on the domain, is

$$(8.4) \quad D_f(\xi, k) = L_f(\xi) + J_\alpha dfk$$

where the operator  $L_f : \Omega^0(f^*TX) \rightarrow \Omega^{0,1}(f^*TX)$  is given by

$$L_f(\xi)(w) = \bar{\partial}_f \xi(w) + \left( \frac{1}{2} J \nabla_\xi J + \nabla_\xi K_\alpha \right) (dfjw) + K_\alpha(\nabla \xi)jw$$

for each  $w \in \Omega^0(TC)$  (here  $\bar{\partial}_f \xi(w)$  is  $\frac{1}{2}(\nabla_w \xi + J \nabla_{jw} \xi)$ ). In our case  $\nabla J = 0$  and  $\alpha$  vanishes along the image of  $f$ , so that

$$(8.5) \quad L_f = \bar{\partial}_f + R_\alpha$$

with

$$(8.6) \quad R_\alpha(\xi) = -(\nabla_\xi K_\alpha) \circ df \circ j.$$

**Lemma 8.2.** *Let  $D$  be a smooth component of a canonical divisor  $D_\alpha$  and  $N$  be the normal bundle of  $D$ . Then, for each  $p \in D$ ,  $u \in T_p D$  and  $\xi \in N_p$  we have*

$$(a) \nabla_u K_\alpha = 0, \quad (c) \nabla_{J\xi} K_\alpha(u) = -J \nabla_\xi K_\alpha(u),$$

$$(b) \nabla_\xi K_\alpha(u) \text{ is orthogonal to } T_p D, \quad (d) |\nabla_\xi K_\alpha(u)|^2 = |\nabla \alpha|^2 |\xi|^2 |u|^2.$$

*Proof.* Since  $\alpha \equiv 0$  along  $D$ , (a) follows from Lemma 2.1a. Next, the fact  $\alpha$  is a closed 2-form and  $\nabla_u \alpha = 0$  gives the formula

$$0 = d\alpha(u, \xi, \eta) = (\nabla_\xi \alpha)(\eta, u) - (\nabla_\eta \alpha)(\xi, u)$$

for any  $\eta \in T_p X$ . Applying the definition of  $K_\alpha$ , this becomes

$$(8.7) \quad \langle \eta, \nabla_\xi K_\alpha(u) \rangle = \langle \xi, \nabla_\eta K_\alpha(u) \rangle.$$

When  $\eta \in T_p D$ , we have  $\nabla_\eta K_\alpha = 0$  and thus (8.7) shows (b). Then, because  $J$  is skew-adjoint with  $\nabla J = 0$ , and  $K_\alpha J = -J K_\alpha$ , (8.7) implies that

$$\begin{aligned} \langle \eta, \nabla_{J\xi} K_\alpha(u) \rangle &= \langle J\xi, \nabla_\eta K_\alpha u \rangle = \langle \xi, \nabla_\eta K_\alpha(Ju) \rangle \\ &= \langle \eta, \nabla_\xi K_\alpha(Ju) \rangle = -\langle \eta, J \nabla_\xi K_\alpha(u) \rangle. \end{aligned}$$

This gives (c). Finally, noting  $\nabla K_\alpha(Ju) = -J \nabla K_\alpha(u)$  and using (c), the fact  $TD$  and  $N$  are  $J$ -invariant and Lemma 2.1, we have

$$|\nabla_\xi K_\alpha(u)|^2 = |\nabla K_\alpha|^2 |\xi|^2 |u|^2 = |\nabla \alpha|^2 |\xi|^2 |u|^2.$$

q.e.d.

As an immediate corollary, we have:

**Corollary 8.3.** *If  $f : C \rightarrow D$  is a  $J_\alpha$ -holomorphic map onto a smooth component  $D$  of a canonical divisor  $D_\alpha$ , then  $R_\alpha$  vanishes on  $f^*TD$  and defines a (real) bundle map*

$$R_\alpha : f^*N \rightarrow T^{0,1}C \otimes f^*N,$$

where  $N$  is the normal bundle to  $D$ . This  $R_\alpha$  satisfies  $R_\alpha J = -JR_\alpha$  and

$$(8.8) \quad |R_\alpha(\xi)|^2 = |\nabla\alpha|^2 |\xi|^2 |df|^2.$$

On a Kähler surface  $X$ , each  $\alpha \in \mathcal{H}$  has an associated almost complex structure  $J_\alpha$  and canonical divisor  $D_\alpha$ . Let  $V$  be a smooth component of the support of  $D_\alpha$ . Following [13], one can use the space of  $(J_\alpha, \nu)$ -holomorphic maps to define the relative GW invariant for the pair  $(X, V)$  provided  $(J_\alpha, \nu)$  is a generic “ $V$ -compatible” pair as defined in Section 3 of [13].

**Corollary 8.4.** *Let  $V$  be a smooth component of the support of  $D_\alpha$ . If  $\nabla\alpha \equiv 0$  on  $V$ , then  $(J_\alpha, 0)$  is a  $V$ -compatible pair, while if  $\nabla\alpha \not\equiv 0$  then  $(J_\alpha, \nu)$  is  $V$ -compatible for no choice of  $\nu$ .*

*Proof.* Let  $\pi_N$  denote the orthogonal projection onto the normal bundle  $N$  of  $V$ . A pair  $(J_\alpha, \nu)$  is  $V$ -compatible if it satisfies three conditions:  $J_\alpha$  preserves  $TV$ ,  $\nabla J_\alpha$  satisfies

$$(8.9) \quad \pi_N [(\nabla_\xi J_\alpha + J_\alpha \nabla_{J_\alpha \xi} J_\alpha)(u)] = \pi_N [(\nabla_u J_\alpha + J_\alpha \nabla_{J_\alpha u} J_\alpha)(\xi)]$$

for all  $u \in TD$  and  $\xi \in N$ , and  $\nu$  and  $\nabla\nu$  satisfy conditions that are automatically true when  $\nu = 0$ . Since  $\alpha = 0$  along  $V$ , the definition (2.3) of  $J_\alpha$  shows that  $J_\alpha = J$  and  $\nabla J_\alpha = -2\nabla K_\alpha$  at each point in  $V$ . Thus  $V$  is  $J_\alpha$ -holomorphic. One can then use Lemma 8.2 to see that Condition (8.9) is equivalent to

$$\nabla_\xi K_\alpha(u) = 0 \quad \forall u \in TD, \quad \forall \xi \in N.$$

Lemma 8.2d then implies that  $V$ -compatibility conditions hold only if  $\nabla\alpha = 0$  along  $V$ , and that if  $\nabla\alpha = 0$  along  $V$  then  $(J_\alpha, 0)$  satisfies the  $V$ -compatibility conditions. q.e.d.

The two terms of the operator (8.5) satisfy a remarkable property under the  $L^2$  pairing:

**Lemma 8.5.** *Let  $D$  be a smooth component of a canonical divisor  $D_\alpha$  with normal bundle  $N$ . Then for each  $J_\alpha$ -holomorphic map  $f : C \rightarrow D$  we have*

$$\int_C \langle \bar{\partial}\xi, R_\alpha\eta \rangle + \int_C \langle \bar{\partial}\eta, R_\alpha\xi \rangle = 0 \quad \forall \xi, \eta \in \Omega^0(f^*N).$$

*Proof.* Let  $f_{s,t}$  be a 2-parameter family of deformations of the map  $f = f_{0,0}$  with  $\frac{d}{ds}f|_{s=t=0} = \xi$  and  $\frac{d}{dt}f|_{s=t=0} = \eta$ . Then  $\bar{\partial}f_{s,t} = \bar{\partial}(s\xi + t\eta) + Q(s,t)$  where  $Q$  is at least quadratic in  $(s,t)$ . Since the image of  $f$  represents a multiple of the (1,1) class  $[D]$ , equation (3.1) gives

$$0 = \int f^*\alpha = \int \langle \bar{\partial}f, K_\alpha(dfj) \rangle$$

for each  $f = f_{s,t}$ . Now differentiate this equation with respect to both  $s$  and  $t$  and evaluate at  $s = t = 0$ , noting that  $\bar{\partial}f$  and  $K_\alpha(dfj)$  both vanish at  $s = t = 0$ . The result is

$$0 = \int \langle \bar{\partial}\xi, \nabla_\eta K_\alpha(dfj) \rangle + \int \langle \bar{\partial}\eta, \nabla_\xi K_\alpha(dfj) \rangle.$$

The lemma follows by the definition of  $R_\alpha$ . q.e.d.

We finish this section by discussing the operator given by the normal component of the linearization (8.5). For each map  $f : C \rightarrow D$  as in Corollary 8.3 the pullback  $f^*TX$  of the tangent bundle decomposes orthogonally as  $f^*TX = f^*TD \oplus f^*N$ . Let  $\pi^N$  be the projection onto  $f^*N$ . The normal component  $\pi^N \circ \nabla$  of the connection on  $f^*TX$  is a hermitian connection on  $f^*N$ ; its (0,1) part defines an operator  $\bar{\partial}_f^N$  and hence a holomorphic structure on  $f^*N$ . The restriction of  $\bar{\partial}_f$  to  $f^*N$  then has the form

$$\bar{\partial}_f|_{f^*N} = \bar{\partial}_f^N + A$$

where  $A$  is a bundle map  $f^*N \rightarrow T^{0,1}C \otimes f^*TD$  (which vanishes if  $f^*N$  is a holomorphic subbundle; see [9] pg. 78). On the other hand, since  $f^*TD$  is a holomorphic subbundle, the restriction of  $\bar{\partial}_f$  to  $f^*TD$  is an operator  $\bar{\partial}_f^T$  on  $f^*TD$  which is the usual  $\bar{\partial}$ -operator. Corollary 8.3 then implies that the linearization (8.4), as an operator

$$D_f : \Omega^0(f^*TD \oplus f^*N) \oplus H^{0,1}(TC) \rightarrow \Omega^{0,1}(f^*TD \oplus f^*N),$$

is given by

$$(8.10) \quad D_f = \begin{pmatrix} \bar{\partial}_f^T & A \\ 0 & L_f^N \end{pmatrix} \oplus Jdf$$

where  $L_f^N = \bar{\partial}_f^N + R_\alpha$ . The next result shows that  $L_f^N$  is injective.

**Proposition 8.6.** *Suppose that  $f : C \rightarrow D$  is a  $J_\alpha$ -holomorphic map from a smooth curve onto a smooth component  $D$  of a canonical divisor  $D_\alpha$ , and that either (i)  $\nabla\alpha \neq 0$  somewhere on  $D$ , or (ii)  $\ker \bar{\partial}_f^N = 0$ . Then*

$$(8.11) \quad \ker L_f^N = 0.$$



*Proof.* Suppose there is a non-zero  $\xi \in \ker L_f^N$ . Then the integral

$$\|L_f^N \xi\|^2 = \|(\bar{\partial}_f^N + R_\alpha)\xi\|^2 = \int_C |\bar{\partial}_f^N \xi|^2 + |\nabla\alpha|^2 |\xi|^2 |df|^2$$

vanishes (here we have used (8.8) and noted that, because  $R_\alpha \xi$  is normal and  $A\xi$  is tangent, Lemma 8.5 holds with  $\bar{\partial}_f \xi = \bar{\partial}_f^N \xi + A\xi$  replaced by  $\bar{\partial}_f^N \xi$ ). But both  $\xi$  and  $f$  satisfy elliptic equations, so by the Unique Continuation Theorem for elliptic equations  $|\xi|^2 |df|^2$  is not zero on any open set. We conclude that  $\bar{\partial}_f^N \xi = 0$  and  $\nabla\alpha \equiv 0$  along  $D$ . q.e.d.

### 9. Zero-dimensional spaces of stable maps

The simplest GW invariants are those associated with a space of stable maps whose formal dimension is zero. Such stable maps are especially simple: Lemma 9.1 below shows that they are unramified maps from smooth domains, and that the linearization  $D_f$  is invertible. Thus all zero-dimensional GW invariants are signed counts of the number of connected etale covers. This section establishes some basic facts needed to make these counts. Specific computations are done in Section 10.

The formal dimension of a space  $\mathcal{M}_{g,n}(X, A)$  of stable maps is the index of linearization  $D_f$  at each  $f \in \mathcal{M}_{g,n}(X, A)$ . Calculating as in the proof of Lemma 8.1, one finds that  $\text{index } D_f = 2\beta + 2n$  and similarly  $\text{index } L_f^N = -2\beta + 2n$ . Consequently, when the space of stable maps is formally 0-dimensional and the domain curve is smooth, we have

$$(9.1) \quad \text{index } D_f = \text{index } L_f^N = 0.$$

Now fix a smooth canonical divisor satisfying the conditions of Proposition 8.6. The Image Localization Lemma 3.2 implies that all invariants  $GW_{g,n}(X, A)$  vanish unless  $A$  is a multiple  $d[D]$  of the class of a component  $D$  of that canonical divisor. These invariants also vanish whenever the formal dimension of  $\overline{\mathcal{M}}_{g,0}(X, A)$  is negative because the space  $\overline{\mathcal{M}}_{g,0}(X, A)$ , and therefore  $\overline{\mathcal{M}}_{g,n}(X, A)$ , is then empty for generic  $(J, \nu)$ . Thus, using the dimension formula of Lemma 8.1, we may assume that  $A = d[D]$  and

$$(9.2) \quad \beta = n = 0 \quad \text{with} \quad \beta = d(1 - h) + g - 1,$$

where  $h$  is the genus of  $D$ .

**Lemma 9.1.** *Suppose that  $D \subset X$  is a smooth component of a canonical divisor  $D_\alpha$ . Then any non-constant stable map  $f : C \rightarrow D$  satisfying (9.2) is an etale cover from a smooth curve  $C$  and the linearization  $D_f$  is invertible.*

*Proof.* By (9.2) we have  $g = dh - d + 1$ . Suppose that  $C$  has  $\ell$  irreducible components  $\{C_i\}$ . Restricting  $f$  to each component and

lifting to the normalization gives maps  $\tilde{f}_i : \tilde{C}_i \rightarrow D$ . Suppose that exactly  $k$  of these have degree  $[\tilde{f}_i] = d_i > 0$ . Then  $\sum d_i = d$  and (8.3), applied to each  $C_i$ , gives

$$g = dh - d + 1 \leq \sum (d_i h - d_i + 1 + \beta_i) = \sum g'_i \leq \sum g_i \leq g$$

where  $\beta_i$  is the ramification index of  $\tilde{f}_i$ ,  $g'_i$  is the geometric genus of  $C_i$ , and  $g_i$  is the arithmetic genus of  $C_i$ . This shows that  $k = 1$  and  $C_1$  has the same geometric and arithmetic genus. Consequently,  $C_1$  is smooth of genus  $g$  and the remaining  $\ell - k$  components have genus 0. Stability then implies that  $\ell - k = 0$ . Thus  $C$  is smooth and  $f : C \rightarrow D$  has no critical points.

Recall that the linearization  $D_f$  is given by (8.10). The normal operator  $L_f^N$  is injective by (8.11) and hence is surjective by (9.1). Furthermore,  $Jdf$  induces an isomorphism from  $H^{0,1}(TC)$  to  $H^{0,1}(f^*TD) = \text{coker } \bar{\partial}_f^T$ , and therefore

$$\bar{\partial}_f^T \oplus Jdf : \Omega^0(f^*TD) \oplus H^{0,1}(TC) \rightarrow \Omega^{0,1}(f^*TD)$$

is also onto. Thus  $D_f$  is surjective with index zero, so is an isomorphism between the appropriate Sobolev spaces. q.e.d.

When  $D_f$  is invertible, there is an associated invariant: its mod 2 spectral flow. That spectral flow is computed in the next proposition. This calculation is crucial to the discussion in the next section.

The mod 2 spectral flow of  $D_f$  is determined by choosing a path  $D_t$  of first order elliptic operators from an invertible complex linear operator  $D_0$  to  $D_1 = D_f$  so that  $D_t$  is invertible except at finitely many  $t_i$  along the way, and taking

$$(9.3) \quad SF(D_f) = \sum_i \dim \ker D_{t_i} \pmod{2}.$$

This is a homotopy invariant of the path, and is independent of  $D_0$  because any two choices of  $D_0$  can be connected by a path  $D_t$  of complex linear first order elliptic operators, and at each point along such a path  $\ker D_t$  is even-dimensional.

**Proposition 9.2.** *Under the conditions of Lemma 9.1,*

$$(9.4) \quad SF(D_f) \equiv h^0(f^*N) \pmod{2}.$$

*Proof.* First deform  $D_f$  to a diagonal operator along the path

$$D_t = \begin{pmatrix} \bar{\partial}_f^T & tA \\ 0 & L_f^N \end{pmatrix} \oplus Jdf.$$

Because both  $\bar{\partial}_f^T \oplus Jdf$  and  $L_f^N$  are surjective, each  $D_t$  is surjective with index zero, so  $\ker D_t = 0$  for all  $t$ . Noting that  $\bar{\partial}_f^T \oplus Jdf$  is complex-linear, we then have

$$SF(D_f) = SF(D_0) = SF(L_f^N).$$

Next, since  $L_f^N$  is invertible by Lemma 8.6,  $SF(L_f^N) = SF(L_f^N + B)$  for any sufficiently small compact perturbation  $B$ . Now write  $L_f^N$  as  $\bar{\partial} + R_\alpha$  with  $\bar{\partial} = \bar{\partial}_f^N$ . Because  $\text{index } \bar{\partial} = 0$  for etale covers, we can choose a complex-linear isomorphism  $\bar{B} : \ker \bar{\partial} \rightarrow \text{coker } \bar{\partial}$  and set  $B = \bar{B}P$  where  $P$  is the  $L^2$  orthogonal projection onto  $\ker \bar{\partial}$ . Then

$$D_t = \bar{\partial} + \delta B + tR_\alpha$$

is a path from  $D_0 = \bar{\partial} + \delta B$  to  $D_1 = L_f^N + \delta B$ . Using Lemma 8.5, we have

$$\int |D_t \xi|^2 = \int |\bar{\partial} \xi|^2 + |(\delta B + tR_\alpha) \xi|^2.$$

This shows that  $D_0$  is invertible and that  $\ker D_t$  lies in  $\ker \bar{\partial}$  and in  $\ker(\delta \bar{B} + tR_\alpha)$  for each  $t$ . Taking  $\delta$  sufficiently small, we then have

$$SF(L_f^N) = SF(D_1) = SF(\delta \bar{B} + \bar{R}_\alpha)$$

where  $\bar{R}_\alpha$  is the restriction of  $R_\alpha$  to  $\ker \bar{\partial}$ . But  $\bar{R}_\alpha$  is injective and anti-commutes with  $J$  by Lemma 8.3. Furthermore, its image is  $L^2$  perpendicular to the image of  $\bar{\partial}$  by Lemma 8.5 and  $\text{index } \bar{\partial} = 0$ , so  $\bar{R}_\alpha : \ker \bar{\partial} \rightarrow \text{coker } \bar{\partial}$  is an isomorphism. This means that  $SF(\delta \bar{B} + \bar{R}_\alpha)$  is the same as  $SF(\bar{R}_\alpha)$  and, from the definition (9.3), the same as  $SF(\bar{B}^{-1} \bar{R}_\alpha)$ . Here  $\bar{B}^{-1} \bar{R}_\alpha$  is an isomorphism of  $H^0(C, f^*N)$  that anti-commutes with  $J$ . The lemma is completed using two simple facts about the spectral flow of finite-dimensional matrices:

- (a)  $(-1)^{SF(A)} = \text{sign det } A$  for all  $A \in GL(n, \mathbb{R})$ .
- (b) If  $A \in GL(2n, \mathbb{R})$  satisfies  $JA = -AJ$  then  $SF(A) = n \pmod 2$ .

To see (a), choose a path  $A_t$  in the space of  $n \times n$  matrices from  $A$  to  $Id$ ; for a generic such path each kernel in (9.3) is 1-dimensional, so the spectral flow is the number of sign changes in  $\det A_t$ . For (b), choose a basis  $\{v_1, Jv_1, \dots, v_n, Jv_n\}$  and set  $w_i = Av_i$ . Then  $v_1 \wedge Jv_1 \wedge \dots \wedge Jv_n$  and  $w_1 \wedge Jw_1 \wedge \dots \wedge Jw_n$  both represent the complex orientation, so the calculation

$$\begin{aligned} \det A \cdot v_1 \wedge Jv_1 \wedge \dots \wedge Jv_n &= Av_1 \wedge AJv_1 \wedge \dots \wedge AJv_n \\ &= (-1)^n w_1 \wedge Jw_1 \wedge \dots \wedge Jw_n \end{aligned}$$

shows that  $\text{sign det } A = (-1)^n$ . q.e.d.

In Gromov-Witten theory, the GW invariant associated with a zero-dimensional space of stable maps is the signed count of the maps in

that space with the sign of each map  $f$  specified by the mod 2 spectral flow of the linearization  $D_f$  (provided each  $D_f$  is an isomorphism). By Proposition 9.2 this sign is

$$(9.5) \quad (-1)^{SF(D_f)} = (-1)^{h^0(f^*N)}.$$

This sign is well-defined even though  $h^0(f^*N)$  may change under deformations of the holomorphic structure on  $f^*N$ . This is because, for etale covers  $f : C \rightarrow D$ , we have  $f^*K_D = K_C$  and hence the equation  $N^2 = K_{D_0}$  pulls back to  $(f^*N)^2 = K_C$ . Thus  $(C, f^*N)$  is a spin curve, so by Lemma 7.1 the parity of  $h^0(f^*N)$  does not change as  $(D, N)$  is deformed.

Formula (9.5) is a key difference between GW invariants in two and four dimensions. The finite set of etale covers of  $D$  contribute to both the Gromov-Witten invariants of the curve  $D$ , and to the GW invariants of  $X$  through the inclusion  $D \subset X$ . But in the first case, each etale cover contributes  $+1/|\text{Aut}(f)|$  to the invariant, while in the second case the signs vary according to (9.5).

### 10. Zero-dimensional GW invariants: computations

The facts established in the previous section are enough to compute the contributions of etale covers to the GW series in some cases. We do this for the canonical class itself, for double covers, and for general etale covers for elliptic fibers.

#### The canonical class.

When  $X$  and  $D$  are as in Lemma 7.3,  $D$  is an embedded genus  $g = K^2 + 1$  curve representing the canonical class  $K$ . For that genus, the GW invariant has dimension 0 by (1.1) and is immediately computable using (4.1), Proposition 9.2, and Lemma 7.3:

$$GW_g(X, K) = GW_g^{\text{loc}}(D, 1) = (-1)^{h^0(N)} = (-1)^{\chi(\mathcal{O}_X)}.$$

This fact is well-known from other perspectives. In the context of Taubes'  $Gr$  invariant (see [23]),  $g = K^2 + 1$  is the "embedded genus" case. In that case the  $Gr$  invariant is the same as the Seiberg-Witten invariant and is given by  $Gr(K) = SW(K) = (-1)^{\chi(\mathcal{O}_X)}$ . On the other hand, because  $D$  is embedded and connected, we also have  $Gr(K) = GW_g(X, K)$ .

#### Double covers.

The etale double covers of a curve  $D$  are classified by either  $H^1(D; \mathbb{Z}_2)$  or, equivalently, by  $J_2(D)$ . In fact, if the square of a line bundle  $L$  is trivial, then  $L$  has a bisection  $s$  satisfying  $s^2 = 1$  and the image of  $s$  is a smooth unramified double covering  $f : C_L \rightarrow D$  that is connected

whenever  $L \neq \mathcal{O}_D$ . Such double coverings satisfy  $f_*\mathcal{O}_{C_L} = \mathcal{O}_D \oplus L^{-1}$ , and thus for any line bundle  $N$  on  $D$

$$(10.1) \quad \begin{aligned} h^0(C_L, f^*N) &= h^0(D, f_*f^*N) = h^0(D, N \otimes f_*\mathcal{O}_{\tilde{C}_L}) \\ &= h^0(D, N) + h^0(D, NL^{-1}). \end{aligned}$$

Now suppose that  $D$  is a smooth component of a canonical divisor of genus  $h$  with normal bundle satisfying  $N^2 = K_D$ . Since each map  $f$  in the moduli space  $\overline{\mathcal{M}}(D, 2)$  of etale double covers with connected domains has automorphism group  $\mathbb{Z}_2$ , each contributes  $\pm \frac{1}{2}$  to the GW invariant, with the sign given by Proposition 9.2. Thus Proposition 7.1a, Lemma 7.3 and equation (10.1) yield

$$(10.2) \quad \begin{aligned} GW_g^{loc}(D, 2) &= \sum_{f \in \overline{\mathcal{M}}(D, 2)} \frac{1}{2} (-1)^{h^0(f^*N)} \\ &= \frac{1}{2} \left[ (-1)^{h^0(N)} 2^h - 1 \right], \end{aligned}$$

where  $g = 2h - 1$ . For surfaces of general type the sign  $(-1)^{h^0(N)}$  can be calculated from the global invariant  $\chi(\mathcal{O}_X) = 1 - q + p_g$  by Lemma 7.3.

**Example 10.1.** Exceptional curves have no etale double covers, while elliptic fibers have three connected double covers, all etale with genus 1. Thus (10.2) gives

- 1) A regular fiber  $F$  has trivial normal bundle, so  $GW_1^{loc}(F, 2) = -\frac{3}{2}$ .
- 2) A multiple fiber  $F_2$  of order 2 has  $h^0(N) = 0$ , so  $GW_1^{loc}(F_2, 2) = \frac{1}{2}$ .
- 3) Formula (10.2) does not apply to multiple fiber  $F_m$  with multiplicity  $m > 2$  because the normal bundle to  $F_m$  is not a theta characteristic, but instead satisfies  $N^m = \mathcal{O}$ . Nevertheless, we have  $h^0(f^*N) = 0$  for each of the three nontrivial double covers  $f$  of  $F_m$ , so

$$GW_1^{loc}(F_m, m - 1, 2) = \frac{3}{2}.$$

- 4) When  $D$  is a smooth multiplicity 1 canonical divisor in a surface of general type,  $D$  has genus  $h = K^2 + 1$  and a connected double cover  $C \rightarrow D$  is etale if and only if  $C$  has genus  $g = 2K^2 + 1$ . By (10.2) the genus  $g = 2h - 1$  invariant is

$$GW_g(X, 2K) = GW_g^{loc}(D, 2) = \frac{1}{2} \left[ (-1)^{\chi(\mathcal{O}_X)} 2^h - 1 \right].$$

### Etale Covers of Elliptic Fibers.

When  $(X, J)$  is a generic complex structure on a minimal properly elliptic surface, the generic canonical divisor has components of two types: smooth elliptic fibers and multiple fibers with smooth reduction. The simplest cases are regular fibers and multiple fibers of multiplicity

two. For those, we can give explicit formulas for the contributions to the GW invariants of smooth etale covers.

**Regular Fibers.** Every holomorphic map  $f : C \rightarrow F$  onto a regular elliptic fiber has  $f^*N = \mathcal{O}$ , so  $h^0(f^*N) = 1$ . Such a map  $f$  is an etale cover if and only if  $C$  has genus  $g = 1$ . The stable moduli space  $\overline{\mathcal{M}}_{1,0}(F, d)$  consists of  $\sigma(d)$  points, where  $\sigma(d) = \sum_{k|d} k$  is the sum of the divisors of  $d$ . Each of these is a generic as  $J_\alpha$ -holomorphic map (Lemma 9.1 implies that  $\ker D_f = 0$ ) with automorphism group of order  $d$ , and each is counted with a minus sign by Lemma 9.2 because  $f^*N = \mathcal{O}_C$ . Thus the contribution of the etale covers to the local GW invariant of  $F$  is

$$\sum_{d>0} GW_1^{\text{loc}}(F, d) t_F^d = - \sum_{d>0} \frac{\sigma(d)}{d} t_F^d = - \int \frac{G(t_F)}{t_F} dt_F,$$

where

$$G(t) = \sum \sigma(d) t^d = \prod_{k>0} \frac{kt^k}{1 - t^k}.$$

**$F_2$  Fibers.** As in (6.1), every elliptic fiber  $F_2$  of multiplicity 2 is a component of each canonical divisor  $D_\alpha$  with multiplicity 1. In particular,  $\nabla\alpha$  does not vanish identically along  $F_2$ . Thus, by Lemma 9.1,  $\ker D_f = 0$  and  $\text{coker } D_f = 0$  for every  $J_\alpha$ -holomorphic etale cover  $f : C \rightarrow F_2$ . Consequently, the (local) GW invariants of etale covers are determined by their Taubes' type. Since the degree 1 map has positive sign, and two of (nontrivial) double covers have positive sign and one has negative sign, we have

$$\begin{aligned} \sum_{d>0} GW_1^{\text{loc}}(F_2, d) t_{F_2}^d &= \sum_{d>0} \frac{1}{d} \left[ \sigma(d) - 2\sigma\left(\frac{d}{2}\right) \right] t_{F_2}^d \\ &= \int \frac{G(t_{F_2}) - 2G(t_{F_2}^2)}{t_{F_2}} dt_{F_2} \end{aligned}$$

(see Proposition 4.4 of [12]).

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