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Abstract. It is well known that laser beams spread as they propagate through free space due to natural diffraction, and that there is additional spreading when optical waves propagate through atmospheric turbulence. Previous studies on Gaussian beams have mainly involved the lowest order mode (zero order). The study of higher order mode Gaussian beams has involved Hermite-Gaussian and Laguerre-Gaussian beams for rectangular and cylindrical geometry, respectively. These studies have developed expressions for the field and intensity in free space, in addition to developing new definitions of beam size in the receiver plane for the higher order modes. We calculate the mean intensity of higher order mode Gaussian beams propagating through atmospheric turbulence, and, based on previously developed definitions for beam radius, we calculate the additional beam spreading due to random media. It is shown that higher order mode Gaussian beams experience less percentage of additional broadening due to atmospheric fluctuations than the zero-order mode beams. [DOI: 10.1117/1.1465427]

Subject terms: beam spreading; optical waves; Gaussian beams; atmospheric turbulence.

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1 Introduction

In optical systems, it is well known that laser beams spread as they propagate through free space due to natural diffraction, and that there is additional spreading when the beam propagates through atmospheric turbulence. This spreading corresponds to a loss of power over a receiver. Previous studies of Gaussian beams have mainly involved the most common type of optical wave emanating from a laser known as the TEM_{00} mode. Beam spreading of zero-order Gaussian beams in all regimes of atmospheric fluctuations is well documented and summarized in Andrews and Phillips.[1] In practice, there are certain scenarios to minimize excitation of nonlinearities within the crystal of a laser, or when the received spot needs to have a multiple spot pattern. In addition, the conversion efficiency, power out/power in, is greater for higher order modes. In those cases, we mathematically model the optical field as a multimode beam with Hermite and Laguerre polynomials in rectangular (CO_2 laser) and cylindrical (HeNe laser) coordinates, respectively. Some investigators have studied higher order Hermite and Laguerre Gaussian beams in free space. Carter[2] developed expressions for the intensity of a Hermite-Gaussian beam of any order and developed a definition for beam spot size, which is consistent with known results for the zero-order mode beam. Phillips and Andrews[3] followed that with a similar study for the Laguerre-Gaussian beam. These studies are for propagation through free space. The purpose of this study is to develop expressions for the mean intensity of higher order mode Hermite- and Laguerre-Gaussian beams based on the extended Huygens-Fresnel principle, which is valid in all regimes of atmospheric turbulence. Based on the definitions of beam spot size suggested by Carter[2] and by Phillips and Andrews,[3] we develop expressions governing beam spreading induced by atmospheric turbulence.

2 Higher Order Mode Gaussian Beams: Free Space

The lowest order Gaussian beam mode is a solution to the paraxial wave equation. Higher order solutions of the paraxial wave equation can take the form of either Hermite-Gaussian functions in rectangular coordinates or Laguerre-Gaussian functions in cylindrical coordinates. In real lasers, the Brewster windows and other tilted surfaces or distorted elements usually provide a small but inherent rectangular symmetry to the laser cavity.[4] Real lasers, therefore, overwhelmingly elect to oscillate in near Hermite-Gaussian rather than near Laguerre-Gaussian modes. Experiments with very carefully aligned gas lasers having internal mirrors and no Brewster windows have, however, clearly demonstrated oscillations in Laguerre-Gaussian modes with higher order radial and azimuthal symmetry. In this study we develop expressions for both Hermite- and Laguerre-Gaussian higher order mode beams.

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2.1 *Hermite-Gaussian Beams of Any Order*

Higher order modes of a collimated beam at the exit aperture $(z=0)$ of a laser are described in rectangular coordinates by

$$U_{mn}(x,y,0)=H_m\left[\frac{x}{\sigma_x(0)}\right]H_n\left[\frac{y}{\sigma_y(0)}\right]\exp\left[-\frac{x^2}{2\sigma_x^2(0)}-\frac{y^2}{2\sigma_y(0)^2}\right], \tag{1}$$

where the spot size along the x and y axes at the transmitter is given by $\sigma_x(0)$ and $\sigma_y(0)$, respectively, and H_m and H_n are Hermite polynomials determining the field distributions in the x and y directions, respectively. It can be shown that the intensity of the optical wave after some propagation distance, L, by means of the Huygens-Fresnel integral

$$\langle I(r,L)\rangle =\left(\frac{k}{2\pi L}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2s_1d^2s_2U_0(s_1,0)U_0^*(s_2,0)\times\exp\left[\frac{ik|s_1-r|^2}{2L}\right]\exp\left[\frac{-ik|s_2-r|^2}{2L}\right] \tag{2}$$

is given by

$$I(x,y,L)=\frac{\sigma_x(0)\sigma_y(0)}{\sigma_x(L)\sigma_y(L)}H_m^2\left[\frac{x}{\sigma_x(L)}\right]H_x^2\left[\frac{y}{\sigma_y(L)}\right]\times\exp\left[-\frac{x^2}{\sigma_x^2(L)}-\frac{y^2}{\sigma_y^2(L)}\right], \tag{3}$$

where, using Carter's definition, the spot size along the x and y axes in any plane of constant L is defined by

$$\sigma_s(L)=\sigma_s(0)\left\{1+\left[\frac{\lambda L}{2\pi\sigma_s^2(0)}\right]^2\right\}^{1/2}, \tag{4}$$

in which s represents either x or y and λ is the optical wavelength. If we take the special case of a TEM_{00} beam and assume the parameters in the x and y directions are the same, then Eq. (3) reduces to the known result[1]

$$I(x,y,L)=\frac{W_0^2}{W^2}\exp\left[-\frac{2(x^2+y^2)}{W^2}\right], \tag{5}$$

W_0 and W are the beam radius at the transmitter and receiver, respectively, and are defined by the $1/e^2$ halfwidth of the intensity. The diffractive spot size is related to the transmitted spot size by $W=W_0(1+\Lambda_0^2)^{1/2}$ and $\Lambda_0=2L/kW_0^2$. In Eq. (5) we have used the identity that $W=2^{1/2}\sigma_s$.

For the lowest order Gaussian beam, for which $m=0$, $n=0$, W is actually the $1/e^2$ halfwidth for the beam intensity along the x or y axes in any plane of constant L. However, for higher order beams, Carter[2] pointed out that W has no such simple meaning. We see in Fig. 1 a two-dimensional cross section of the intensity of a Hermite-Gaussian beam. In Figs. 2, 3, and 4 we see the mode patterns of a laser oscillator. Higher order beams do not form simple spots of light but form more complicated patterns.

These beams diverge more rapidly from focus, so that for fixed $\sigma_s(0)$ the illuminated area in any plane of constant $L\neq 0$ increases with n or m. To account for this effect, it is necessary to modify the usual definition of spot size.

Carter[2] characterized the illuminated area of a higher order beam by use of twice the variance of x or y within a plane of constant L, i.e.,

$$\sigma_s^2(L)_l=\frac{2\int\int_{-\infty}^{\infty}s^2I(x,y,L)dxdy}{\int\int_{-\infty}^{\infty}I(x,y,L)dxdy}, \tag{6}$$

where l represents m or n, and s represents x or y. On substituting Eq. (3) into Eq. (6), we find that

$$\sigma_s^2(L)_l=\frac{2\int_{-\infty}^{\infty}s^2H_l^2\left[\frac{s}{\sigma_s(L)}\right]\exp\left[-\frac{s^2}{\sigma_s^2(L)}\right]ds}{\int_{-\infty}^{\infty}H_l^2\left[\frac{s}{\sigma_s(L)}\right]\exp\left[-\frac{s^2}{\sigma_s^2(L)}\right]ds}, \tag{7}$$

which yields

$$\sigma_s(L)_l=\sigma_s(L)(2l+1)^{1/2}. \tag{8}$$

If we first set $m=0$, $n=0$ in Eq. (3) and then substitute that into Eq. (7), we find that for the TEM_{00} wave we get $\sigma_s{}^2(L)=W^2/2$, so when we later compute the additional broadening due to atmospheric turbulence, it is important to remember the relation $W=2^{1/2}\sigma_s$ in our analysis as Carter[2] noted.

2.2 *Laguerre-Gaussian Beams of Any Order*

Higher order modes of a collimated beam at the exit aperture $(z=0)$ of a laser are described in cylindrical coordinates by

$$U_{mn}(r,\theta,0)=\left(\frac{\sqrt{2}r}{W_0}\right)^m(-i)^m\exp im\theta\exp\left(-\frac{r^2}{W_0^2}\right)\times L_n{}^m\left(\frac{2r^2}{W_0^2}\right), \tag{9}$$

where U represents the optical field, r is the distance off axis in the transverse plane at angle θ, W_0 is the radius of the TEM_{00} mode beam, $L_n{}^m$ is the associated Laguerre polynomial, and n and m are the radial and angular mode numbers. It can be shown that the intensity of the optical wave after some propagation distance, L, by means of the Huygens-Fresnel integral, Eq. (2), is given by

$$I(r,L)=\frac{W_0^2}{W^2}\left(\frac{2r^2}{W^2}\right)^m\exp\left(-\frac{2r^2}{W^2}\right)\left[L_n{}^m\left(\frac{2r^2}{W^2}\right)\right]^2. \tag{10}$$

When we take the limit of $m=n=0$, we find that Eq. (10) reduces to the TEM_{00} mode of Eq. (5). We see in Figs. 5–7 the intensity of Laguerre-Gaussian beams and the ring pattern of the laser for $m=n$.

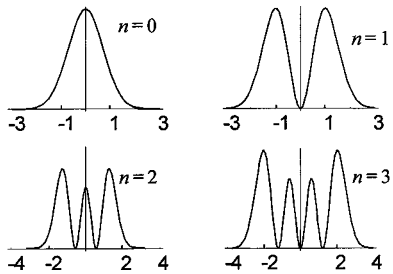

Fig. 1 2-D view of higher mode Hermite-Gaussian beams.

Again, for the TEM_{00} mode we see that W is the $1/e^2$ point of the intensity, but for higher order modes there are multiple rings and hence a new definition of spot size is needed. Phillips and Andrews[3] developed a parallel definition of spot size for higher order mode beams in cylindrical coordinates similar to that of Carter[2] in rectangular coordinates. The spot size of a higher order mode beam is given by

$$\sigma_r^2(L)_{mn}=\frac{2\int_0^{2\pi}\int_{-\infty}^{\infty} r^2 I(r,\theta,L)\, r\, dr\, d\theta}{\int_0^{2\pi}\int_{-\infty}^{\infty} I(r,\theta,L)\, r\, dr\, d\theta}. \tag{11}$$

On substituting Eq. (10) into Eq. (11), Phillips and Andrews[3] found the spot size of a higher order mode beam is given by

$$\sigma_r(L)_{mn}=W(2n+m+1)^{1/2}. \tag{12}$$

We see that in the limiting case of the TEM_{00} wave, Eq. (12) reduces to W, which is the known diffractive beam size for lowest order Gaussian beams.

3 Higher Order Mode Gaussian Beams in Atmospheric Turbulence

In the presence of atmospheric fluctuations, expressions for the mean intensity of a laser beam can be obtained by use of the extended Huygens-Fresnel principle given by[1]

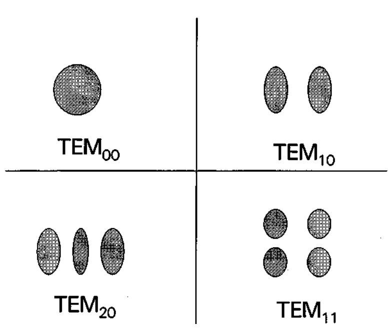

Fig. 2 Multiple spots associated with Hermite-Gaussian multiple mode beams.

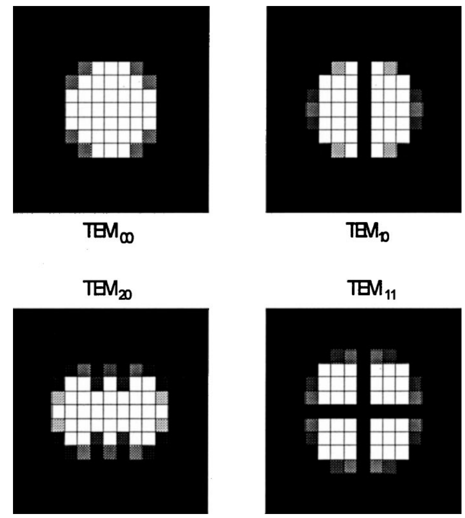

Fig. 3 Density plots of 3-D intensity for multiple spot Hermite-Gaussian beams.

$$\langle I(r,L)\rangle=\left(\frac{k}{2\pi L}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2s_1 d^2s_2 U_0(s_1,0) \times U_0^*(s_2,0)\exp\left[\frac{ik|s_1-r|^2}{2L}\right]\exp\left[\frac{-ik|s_2-r|^2}{2L}\right] \times\exp\left[-\frac{1}{2}D_{sp}(Q)\right], \tag{13}$$

where U_0 denotes the optical field at the transmitter, k is the optical wavenumber, $Q=s_1-s_2$, and D_{sp} is the spherical wave structure function

$$D_{sp}(Q)=1.093C_n^2k^2LQ^{5/3}=1.093C_n^2k^{7/6}L^{11/6}\left(\frac{kQ^2}{L}\right)^{5/6}. \tag{14}$$

By making the approximation $5/6\approx 1$, Eq. (14) can be written in terms of the Rytov variance, $\sigma_1{}^2$, as

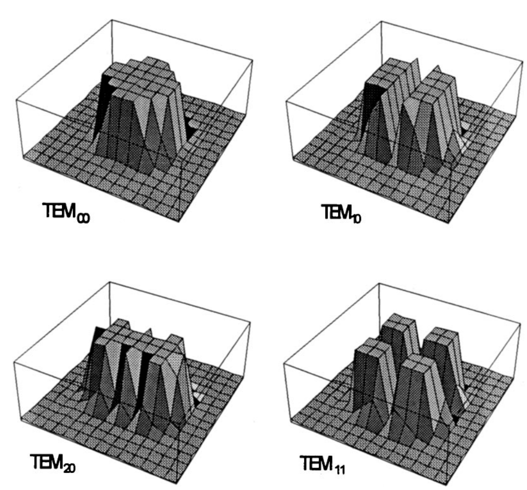

Fig. 4 3-D intensity plots for higher mode Hermite-Gaussian beams.

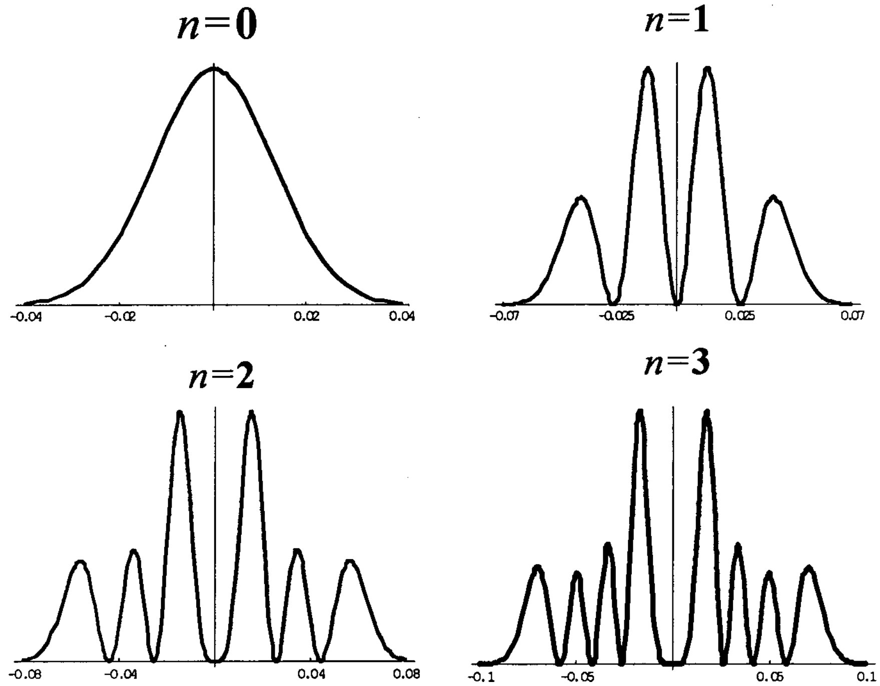

Fig. 5 2-D multiple mode Laguerre-Gaussian beam.

$$D_{\mathrm{sp}}(Q) = .89\sigma_1^2\left(\frac{kQ^2}{L}\right), \quad \sigma_1^2 = 1.23C_n^2 k^{7/6} L^{11/6}. \tag{15}$$

It has been shown[1] that for either the quadratic or 5/3 structure functions, a good approximation of the mean intensity of a zero-order Gaussian beam in either weak or strong fluctuations is given by

$$\langle I(r,L)\rangle = \frac{W_0^2}{W_e^2}\exp\left(-\frac{2r^2}{W_e^2}\right), \tag{16}$$

where $W_e = W(1+4q\Lambda/3)^{1/2}$, W is the diffractive spot size, $q = 1.22\ \sigma_1{}^{12/5}$, and $\Lambda = 2L/kW^2$.

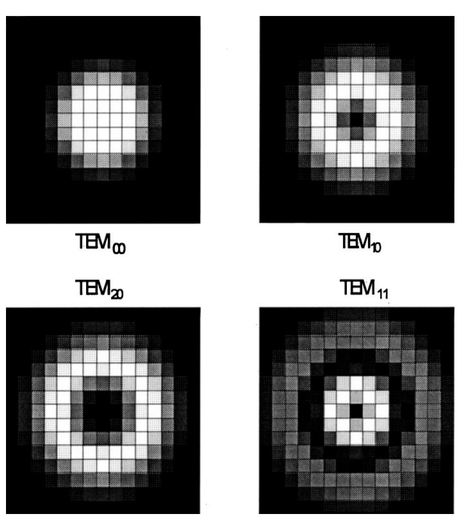

Fig. 6 Density plot Laguerre-Gaussian multiple mode laser.

3.1 *Hermite-Gaussian* ($m=1, n=0$)

When subsituting the spherical wave structure function, Eq. (15), and the field of a Hermite-Gaussian beam, Eq. (1), into the mean intensity via the extended Huygen's Fresnel principle, Eq. (13), we find that the mean intensity of a TEM_{10} Hermite-Gaussian beam is given by

$$\langle I(x,y,L)\rangle_{10} = \left(\frac{8x^2W_0^2}{W^4[1+1.78\sigma_1^2\Lambda]^3} + \frac{3.56\sigma_1^2\Lambda(1+\Lambda\sigma_1^2)}{[1+1.78\sigma_1^2\Lambda]^3}\right) \times \exp\left[-\frac{2(x^2+y^2)}{W^2[1+1.78\sigma_1^2\Lambda]}\right], \tag{17}$$

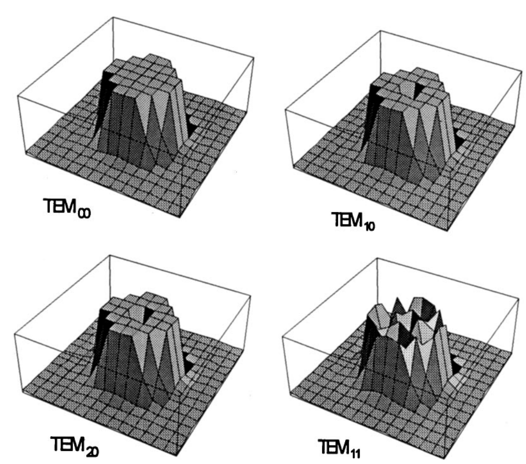

Fig. 7 Intensity of multiple mode Laguerre-Gaussian beam.

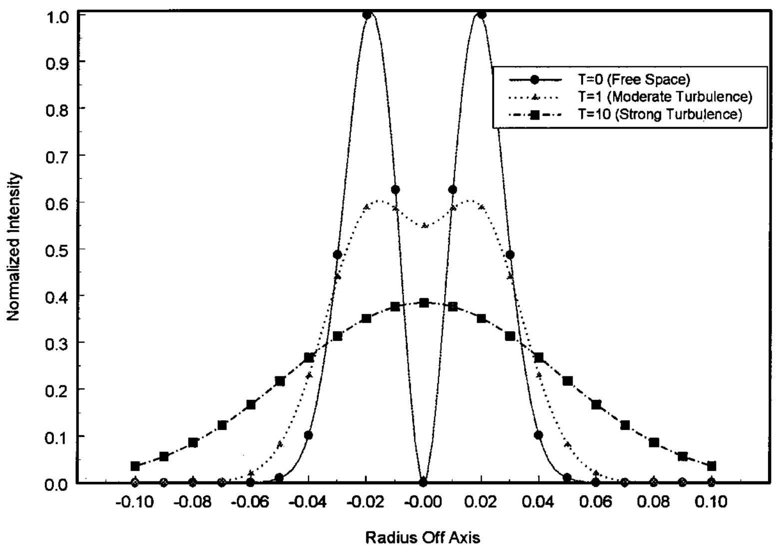

Fig. 8 Normalized intensity of a TEM_{10} Hermite-Gaussian beam for various levels of atmospheric turbulence.

where x and y represent the rectangular coordinates and $\Lambda = 2L/kW^2$. If we take the limit of free space $(\sigma_1{}^2 = 0)$, then we find that Eq. (17) reduces to

$$\langle I(x,y,L)\rangle_{10} = \frac{8x^2W_0^2}{W^4}\exp\left[-\frac{2r^2}{W^2}\right], \quad (18)$$

where W is the beam radius of the TEM_{00} beam in the absence of turbulence. Equation (18) agrees with Eq. (3) when $m=1$ and $n=0$.

We see in Fig. 8 the mean intensity of the TEM_{10} Hermite-Gaussian beam in the presence of turbulence, Eq. (17), compared with the mean intensity in free space, Eq. (18). We see that as the turbulence strength increases, the beam becomes more Gaussian.

If we now calculate the beam radius based on Carter's[2] definition, Eq. (6), using Eq. (17) for the mean intensity we find that the beam radius of a TEM_{10} Hermite beam is given by

$$\sigma_{sT} = W[1+1.78\sigma_1^2\Lambda]^{1/2} \times \left(\frac{6W_0^2[1+1.78\sigma_1^2\Lambda]+3.56\sigma_1^2W^2\Lambda(1+\sigma_1^2\Lambda)}{4W_0^2[1+1.78\sigma_1^2\Lambda]+7.12\sigma_1^2W^2\Lambda(1+\sigma_1^2\Lambda)}\right)^{1/2}. \quad (19)$$

In the limit of free space, Eq. (19) reduces to $\sigma_s = 3^{1/2}W/2^{1/2}$, which agrees with Eq. (8) when $m=1$.

For comparison, we look at the ratios of Eq. (8) to Eq. (19) for the TEM_{10} Hermite-Gaussian beam, and Eq. (16) yields W/W_e for the TEM_{00} Gaussian beam. We see in Fig. 9 that the higher order mode beam experiences less additional broadening beyond diffraction due to atmospheric turbulence.

3.2 *Laguerre-Gaussian Beam* $(m=0, n=1)$

Let us now proceed for cylindrical coordinates using the TEM_{01} Laguerre Gaussian beam. When substituting Eq. (9) into Eq. (13), we find the mean intensity can be written as

$$\langle I(r,L)\rangle = \left[A + B\frac{r^2}{W^2} + C\frac{r^4}{W^4}\right]\exp\left[-\frac{2r^2}{W^2(1+2\Lambda T)}\right], \quad (20)$$

where A, B, and C are given by

$$A = \frac{\Theta^2(1+T^2)-\Lambda^2}{(1+2\Lambda T+\Theta T^2)(1+2\Lambda T)} + \frac{2\Lambda^2-2\Lambda\Theta T}{(1+2\Lambda T)^2} - \frac{2\Lambda^2\Theta T^2+2\Lambda\Theta^2T^3}{(1+2\Lambda T+\Theta T^2)(1+2\Lambda T)^2} + \frac{8\Lambda^2\Theta T^2}{(1+2\Lambda T)^3}, \quad (21)$$

$$B = -\frac{4\Theta(1-2\Lambda T)}{(1+2\Lambda T)^4}, \quad (22)$$

$$C = \frac{4\Theta}{[1+2\Lambda T]^5}, \quad (23)$$

where

$$T = .89\sigma_1^2 \quad \Theta = \frac{1}{1+\Lambda_0^2} \quad \Lambda = \frac{\Lambda_0}{1+\Lambda_0^2} \quad \Lambda_0 = \frac{2L}{kW_0^2}. \quad (24)$$

We see in Fig. 10 the mean intensity of the TEM_{01} Laguerre-Gaussian beam, Eq. (20), compared with the free space intensity, Eq. (10).

When substituting Eq. (20) into Eq. (11), we find the received beam size in the presence of turbulence is given by

$$\sigma_{rt}(L)_{01} = \sqrt{3}W\left(1+\frac{2\Lambda T}{3}\right)^{1/2}. \quad (25)$$

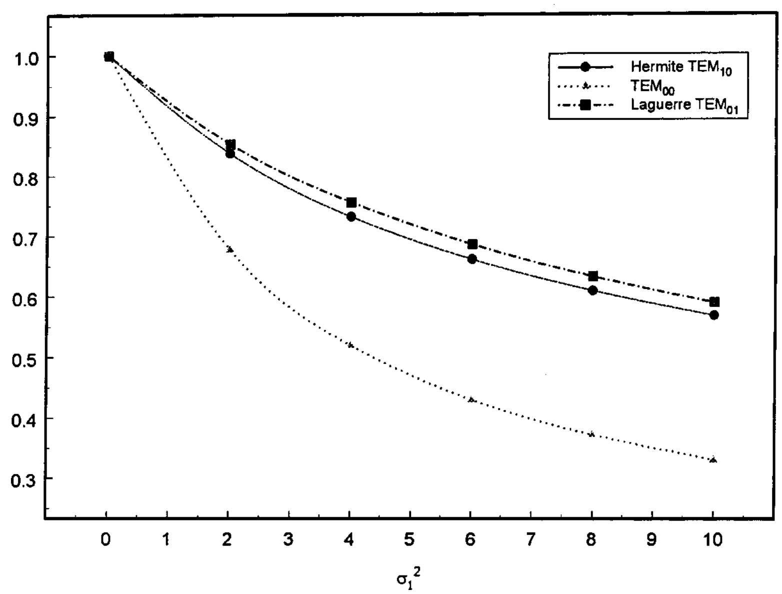

Fig. 9 Ratio of spot size in free space to spot size in turbulence, i.e., W_e/W.

Note that in the limiting case of free space, Eq. (25) reduces to $\sqrt{3}W$, which agrees with Eq. (12) when $m=0$ and $n=1$. Let us investigate the quantity σ_r/σ_{rt}, which is the ratio of the spot size in free space to the spot size in turbulence. We see in Fig. 9 that the higher order Laguerre beam experiences less predicted additional broadening due to atmospheric turbulence than the TEM_{00} mode Gaussian beam.

It is expected that higher order mode beams experience less additional broadening due to turbulence, because the mode structure already places more energy away from the beam axis; hence the incremental spreading due to turbulence is smaller.

4 Discussion

Previous studies involving higher order mode Gaussian beams primarily involved free space conditions. Carter[2] developed a modified definition for the spot size of a higher order mode Hermite-Gaussian beam, and Phillips and Andrews[3] developed a similar definition for the higher order mode Laguerre-Gaussian beams. We developed expressions governing the mean intensity for both Hermite and Laguerre higher mode Gaussian beams based on the extended Huygens Fresnel principle. We then used these expressions for mean intensity in conjunction with the modified definitions for spot size. We found that the Laguerre

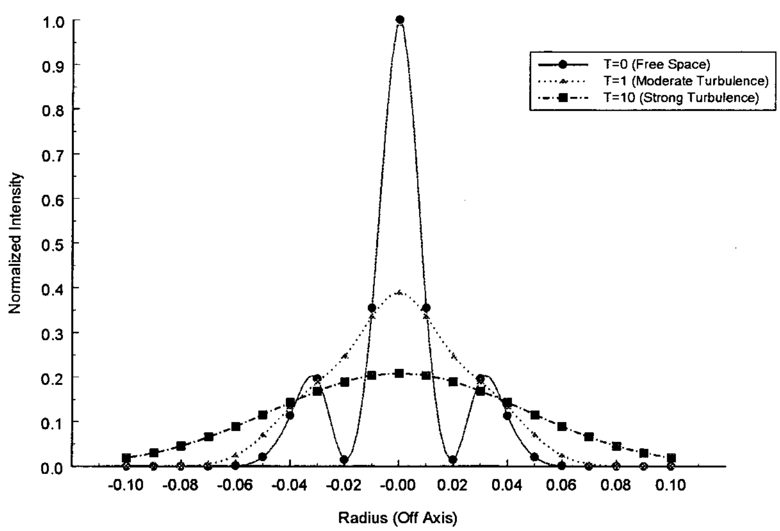

Fig. 10 Normalized mean intensity for TEM_{01} Laguerre-Gaussian beam for various levels of atmospheric turbulence.

TEM_{01} and the Hermite TEM_{10} beam waves experience less percentage of predicted additional broadening due to atmospheric turbulence than the TEM_{00}, which is expected because they have larger initial beam sizes. Less additional broadening due to atmospheric fluctuations corresponds to less additional power loss.

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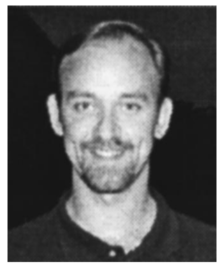

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