Two Ramsey-related Problems

Jingmei Zhang
University of Central Florida

TWO RAMSEY-RELATED PROBLEMS

by

JINGMEI ZHANG
MS University of Central Florida, 2016
MS Anhui University, 2011
BS Anhui University, 2008

A dissertation submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
in the College of Sciences
at the University of Central Florida
Orlando, Florida

Summer Term
2019

Major Professor: Zi-Xia Song

ABSTRACT

Extremal combinatorics is one of the central branches of discrete mathematics and has experienced an impressive growth during the last few decades. It deals with the problem of determining or estimating the maximum or minimum possible size of a combinatorial structure which satisfies certain requirements. In this dissertation, we focus on studying the minimum number of edges of certain co-critical graphs. Given an integer $r \geq 1$ and graphs G, $H_1, \ldots, H_r$, we write $G \to (H_1, \ldots, H_r)$ if every r-coloring of the edges of G contains a monochromatic copy of H_i in color i for some $i \in \{1, \ldots, r\}$. A non-complete graph G is $(H_1, \ldots, H_r)$-*co-critical* if $G \not\to (H_1, \ldots, H_r)$, but $G + uv \to (H_1, \ldots, H_r)$ for every pair of non-adjacent vertices u, v in G. Motivated in part by Hanson and Toft's conjecture from 1987, we study the minimum number of edges over all $(K_t, \mathcal{T}_k)$-co-critical graphs on n vertices, where $\mathcal{T}_k$ denotes the family of all trees on k vertices. We apply graph bootstrap percolation on a not necessarily K_t-saturated graph to prove that for all $t \geq 4$ and $k \geq \max\{6, t\}$, there exists a constant $c(t, k)$ such that, for all $n \geq (t-1)(k-1)+1$, if G is a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, then $e(G) \geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(t, k)$. We then show that this is asymptotically best possible for all sufficiently large n when $t \in \{4, 5\}$ and $k \geq 6$. The method we developed may shed some light on solving Hanson and Toft's conjecture, which is wide open.

We also study Ramsey numbers of even cycles and paths under Gallai colorings, where a *Gallai coloring* is a coloring of the edges of a complete graph without rainbow triangles, and a *Gallai k-coloring* is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the *Gallai-Ramsey number* $GR(H_1, \ldots, H_k)$ is the least integer n such that every Gallai k-coloring of the complete graph K_n contains a monochromatic copy of H_i in color i for some $i \in \{1, \ldots, k\}$. We completely determine the exact values of $GR(H_1, \ldots, H_k)$ for all $k \geq 2$ when each H_i is a path or an even cycle on at most 13 vertices.

ACKNOWLEDGMENTS

First, I would particularly like to express my sincere gratitude to my advisor, Professor Zi-Xia Song, for the guidance, advice, and encouragement she has given me during my entire time as a student at University of Central Florida.

Besides my advisor, I would like to thank the rest of my dissertation committee members, Professors Yue Zhao, Heath Martin, and Damla Turgut, for their service on my dissertation committee.

I also want to thank my friends Christian, Fangfang, Chathurani, Nathaniel, Van, Wei and many more who have made my stay at UCF wonderful.

Last but not the least, I would like to thank my husband, my parents, my parents-in-law and my sister. Without their constant encouragement and support, I would not have a chance to pursue my PhD in mathematics.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

Extremal combinatorics is one of the central branches of discrete mathematics and has experienced an impressive growth during the last few decades. It deals with the problem of determining or estimating the maximum or minimum possible size of a combinatorial structure which satisfies certain requirements. In this dissertation, we focus on studying two Ramsey-related problems.

1.1 Basic Definitions

Following the conventions set out in [81], a *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called it *endvertices* or *ends*, written as $G = (V(G), E(G))$ or $G = (V, E)$. See Figure 1.1 for an example of graph. The *order* of a graph G, written $|G|$, is the number of vertices in G. The *size* of a graph G, written $e(G)$, is the number of edges in G. Graphs are *finite*, *infinite*, *countable* and so on according to their order. A *loop* is an edge whose ends are both the same vertex. *Multiple edges* are distinct edges which share the same two ends. A graph is *simple* if it contains no loops or multiple edges. All graphs considered in this dissertation are finite and simple. We use the convention "$A :=$" to mean that A is defined to be the right-hand side of the relation.

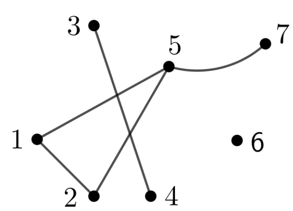


Figure 1.1: A graph on $V = \{1, \ldots, 7\}$ with edge set $E = \{12, 15, 25, 34, 57\}$

Let $G = (V, E)$. If an edge $e \in E$ has ends $x, y \in V$, we usually write $e = xy$ and say that x

and y are *incident* with e in G, and that x and y are *adjacent* or *neighbors* in G. If two vertices are not adjacent to each other in G, then we say that they are *non-adjacent.* The set of neighbours of a vertex v in G is denoted by $N_G(v)$, or briefly by $N(v)$. The *degree* $d_G(v) = d(v)$ of a vertex v is equal to the number of neighbours of v. The number $\delta(G) := \min\{d(v) : v \in V\}$ is the *minimum degree* of G, the number $\Delta(G) := \max\{d(v) : v \in V\}$ is its *maximum degree.* The *complement* $\overline{G}$ of G is the graph with vertex set V and edge set $\{uv : u, v \in V$ and $uv \notin E\}$. See Figure 1.2 for an example of this definition. Given disjoint sets $A, B \subseteq V$, we say that A is *complete to* B if for every $a \in A$ and every $b \in B$, $ab \in E$, and A is *anti-complete to* B if for every $a \in A$ and every $b \in B$, $ab \notin E$.

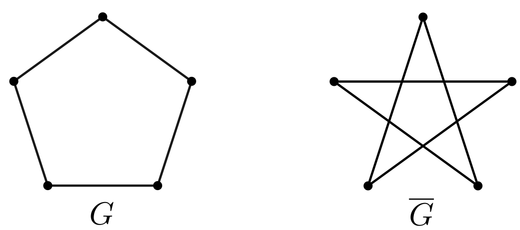


Figure 1.2: A graph G and its complement $\overline{G}$

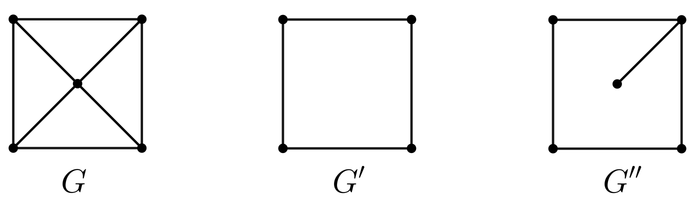


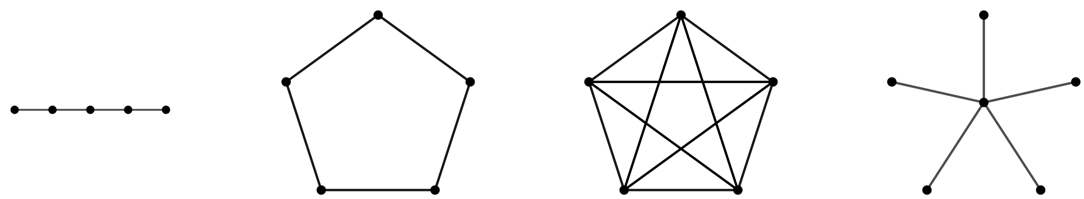
Figure 1.3: A graph G with subgraphs G' and G'': G' is an induced subgraph, but G'' is not

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Following the conventions of [25], if $V' \subseteq V$ and $E' \subseteq E$, then G' is a *subgraph* of G, written as $G' \subseteq G$. Less formally, we say that G *contains* G'. If $G' \subseteq G$ and $G' \neq G$, then G' is a *proper subgraph* of G. If $G' \subseteq G$ and G' contains all the edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G; we say that V' *induces* G' in G, and write $G' = G[V']$. See Figure 1.3 for examples of these definitions. If $U \subseteq V$ is any set of vertices, $G[U]$ is the subgraph of G obtained from G by deleting all vertices in $V \backslash U$. If W is any subset of V, we write $G \backslash W$ for $G[V \backslash W]$. In other words, $G \backslash W$ is obtained from G by deleting all the vertices in W and their incident edges. If $W = \{v\}$ is a singleton, we write $G \backslash v$ rather than $G \backslash \{v\}$. For a subset F of E, we write $G \backslash F = (V, E \backslash F)$ and $G + F = (V, E \cup F)$; as above, $G \backslash \{e\}$ and $G + \{e\}$ are abbreviated to $G \backslash e$ and $G + e$.

Let G and H be two vertex disjoint graphs. The *join* $G + H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. The *union* $G \cup H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. For an integer $t \geq 1$ and a graph H, we define tH to be the union of t disjoint copies of H.

A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices. The greatest integer r such that $K_r \subseteq G$ is the *clique number* $\omega(G)$ of G, and the greatest integer r such that $K_r \subseteq \overline{G}$ is the *independence number* $\alpha(G)$ of G. For any graph G, $\alpha(G) = \omega(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$.

Let us now describe some frequently used graphs. If all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is a K_n; a K_3 is called a *triangle*. A *path* P is an alternating sequence of all distinct vertices and edges, $v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k$, which begins and ends with vertices. We often refer to a path by the natural sequence of its vertices, writing, say $P := v_1 v_2 \ldots v_k$. The graph $C := P + v_1 v_k$ is called a *cycle*. As with paths, we often denote a cycle by its (cyclic) sequence of vertices; the above cycle C might be written as $v_1 \ldots v_{k-1} v_1$. A graph G is called *connected* if it is non-empty and there exists a path between any two of its vertices in G. A connected graph with no cycles is called a *tree*. A vertex of degree 1 in a tree is call a *leaf* of the tree. A tree on $k + 1$ vertices with k leaves is defined as a *star*, and is denoted by $K_{1,k}$ or S_k. We use P_n, C_n and T_n to denote the path, cycle and tree on n vertices, respectively. A graph G is *k-partite* if $V(G)$ is the union of k independent sets called *partite sets* of G. We say G is *bipartite* when $k = 2$. If the k independent sets of a k-partite graph G are complete to each other, then G is called a *complete k-partite graph.* Note that P_n, C_{2n} and T_n are bipartite graphs, and $K_{1,k}$ is a complete bipartite graph. Examples of some of these graphs are depicted in Figure 1.4.



(a) (b) (c) (d)

Figure 1.4: The path P_5, the cycle C_5, the complete graph K_5, and the star $K_{1,5}$

For any positive integer k, we write $[k]$ for the set $\{1, 2, \ldots, k\}$. A *k-coloring* of a graph $G = (V, E)$ is a map $c : V \to [k]$. If $c(v) \neq c(w)$ whenever v and w are adjacent, then c is said to be a *proper coloring.* The *chromatic number* of a graph G, written $\chi(G)$, is the minimum number of colors needed in a proper coloring of G. A *k-edge-coloring* of a graph $G = (V, E)$ or a *k-coloring* of edge set E of G is a map $\sigma : E \to [k]$. Similarly, if $\sigma(e) \neq \sigma(f)$ for any adjacent edges e and f, then σ is said to be a *proper edge coloring.* A subset of vertices assigned to the same color under c is called a *color class* of c, and similarly a subset of edges assigned to the same color under σ is called a *color class* of σ. Every color class of a proper coloring or a proper edge coloring forms an independent set. Note that a proper k-coloring is nothing but a vertex partition into k independent sets, and a proper k-edge-coloring is an edge partition into k independent sets. Let $H \subseteq G$ and σ be a k-edge-coloring of G. We say that G *contains a monochromatic copy of H* if all the edges of H have the same color under σ.

1.2 Pigeonhole Principle

In this section, we list an important, but elementary, combinatorial principle that can be used to solve a variety of interesting problems, often with surprising conclusions. This principle is commonly called the pigeonhole principle, the Dirichlet drawer principle, and the shoeboxprinciple. The first formalization of the idea is believed to have been made by Peter Gustav Lejeune Dirichlet in 1834 under the name Schubfachprinzip (German). Formulated as a principle about pigeonholes, it says roughly that if a lot of pigeons fly into not too many pigeonholes, then at least one pigeonhole will be occupied by more than one pigeons.

Theorem 1.2.1 (Pigeonhole principle (Simple form), Herstein [58]) *If $n+1$ (or more) objects are distributed into n boxes, then at least one box contains two or more of the objects.*

Theorem 1.2.2 (Pigeonhole principle (Strong form), Brualdi [11]) *Let $q_1, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ (or more) objects are distributed into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, $\ldots$, or the nth box contains at least q_n objects.*

The simple form of the pigeonhole principle can be obtained from the strong form by taking $q_1 = \cdots = q_n = 2$. Then $q_1 + q_2 + \cdots + q_n - n + 1 = n + 1$.

1.3 Ramsey Numbers of Graphs

Given an integer $k \geq 1$ and graphs $G, H_1, \ldots, H_k$, we write $G \to (H_1, \ldots, H_k)$ if every k-coloring of $E(G)$ contains a monochromatic copy of H_i in color i for some $i \in [k]$. If G has no monochromatic copy of H_i in color i for any $i \in [k]$ under some k-coloring σ of $E(G)$, then we write $G \nrightarrow (H_1, \ldots, H_k)$, and say that σ is a *critical k-coloring* of G with respect to $H_1, \ldots, H_k$;

when $k = 2$, we simply say *critical-coloring*. The classical *Ramsey number* $R(H_1, \ldots, H_k)$ is the minimum positive integer n such that $K_n \to (H_1, \ldots, H_k)$. If $H = H_1 = \cdots = H_k$, we simply write $R_k(H)$ to denote the k-*color Ramsey number* of H.

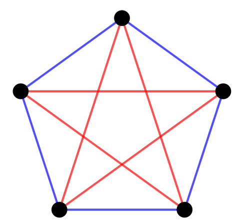


Figure 1.5: $K_5 \nrightarrow (K_3, K_3)$

From Figure 1.5, we see that $K_5 \nrightarrow (K_3, K_3)$. Note that the {red, blue}-coloring of K_5 depicted in Figure 1.5 is the unique critical-coloring of K_5, so $R_2(K_3) \geq 6$. Actually $R_2(K_3) = 6$ and is not difficult to prove. The task of proving $R_2(K_3) \leq 6$ was the second problem in Part I of the William Lowell Putnam Mathematical Competition held in March 1953 [13].

Ramsey theory stems from a deceptively simple problem, i.e., a problem that is very easy to state and that seems easy to solve, but turns out to be very difficult. In its general form, the problem is to determine the smallest integer $r = R(K_m, K_n)$, such that at any party of r people, we can find m mutual acquaintances (each one knows all $m - 1$ others) or n mutual strangers (each one does not know any of the $n - 1$ others). For small values of m and n the problem is easy. It is trivial that $R(K_1, K_n) = R(K_m, K_1) = 1$, and almost trivial that $R(K_2, K_n) = n$ and $R(K_m, K_2) = m$. The field is named for Frank P. Ramsey who proved its first result [72] in 1929. This paper [72] was published in 1930. Since then, the field has exploded.

Theorem 1.3.1 (Ramsey [72]) *For all $k \geq 1$ and any given graphs $H_1, \ldots, H_k$, there exists a $n \in \mathbb{N}$ such that $K_n \to (H_1, \ldots, H_k)$.*

Ramsey theory is a profound and important generalization of the Pigeonhole Principle. It studies conditions when a combinatorial object contains necessarily some smaller given objects. The role of Ramsey numbers is to quantify some of the general existential theorems in Ramsey Theory. The core idea of Ramsey theory is that complete disorder is impossible. Determining $R_k(K_t)$, or even $R_2(K_t)$, is one of the main open problems in Ramsey theory. However, computing Ramsey numbers is notoriously difficult.

Paul Erdős had a tremendous impact on the area of Ramsey theory. His contribution started with determining the Ramsey number $R(K_m, K_n)$. Ramsey [72] provided an upper bound of $R_2(K_t)$ which is $R_2(K_t) \leq 2^{2t-3}$. In 1947, Erdős [30] proved a lower bound of $R_2(K_t)$ which is $R_2(K_t) > 2^{t/2}$. The interesting feature of Erdős's proof is that he never presented a specific coloring. He simply proved that choosing a coloring at random almost always works. This was one of the first occurence of the probabilistic method in combinatorics. While there have been several improvements on these bounds (see for example [20] and [78]), the constant factors in the above exponents remain the same.

For 2-color Ramsey numbers of complete graphs, Greenwood and Gleason [49] established the initial values $R(K_3, K_4) = 9$, $R(K_3, K_5) = 14$ and $R_2(K_4) = 18$ in 1955. Kéry [61] obtained $R(K_3, K_6) = 18$ in 1964, and Graver and Yackel [48] proved that $R(K_3, K_7) = 23$ in 1968. The determination of other classical Ramsey numbers required the use of computers. Grinstead and Roberts [48] found that $R(K_3, K_9) = 36$ in 1982; Mckay and Zhang [68] computed $R(K_3, K_8) = 28$ in 1992; Mckay and Radziszowski [67] determined $R(K_4, K_5) = 25$ in 1995. Perhaps surprisingly, even the exact value of $R_2(K_5)$ remains unknown. The best known lower bound of $R_2(K_5)$ was provided by Exoo [34] in 1989, shown to be 43. More recently, Angeltveit and McKay [3] proved that $R_2(K_5) \leq 48$. For multiple colors, the only known nontrivial classical Ramsey number of complete graphs is $R_3(K_3)$, which is 17, as shown by Greenwood and Gleason [49]. 2-color Ramsey numbers of certain graphs are completely determined. We list some of the results below which will be used in this dissertation.

Theorem 1.3.2 (Chartrand, Schuster [16]) $R_2(C_4) = 6$ *and* $R_2(C_6) = 8$.

Theorem 1.3.3 (Faudree, Schelp [38]; Rosta [74] independtly)

$$R(C_m, C_n) = \begin{cases} 2n-1 & \textit{for } 3 \le m \le n,\ m \textit{ odd}, (m,n) \ne (3,3), \\ n-1+m/2 & \textit{for } 4 \le m \le n,\ m \textit{ and } n \textit{ even}, (m,n) \ne (4,4), \\ \max\{n-1+m/2, 2m-1\} & \textit{for } 4 \le m < n, m \textit{ even and } n \textit{ odd}. \end{cases}$$

Theorem 1.3.4 (Faudree, Lawrence, Parsons, Schelp [37])

$$R(C_m, P_n) = \begin{cases} 2n-1 & \textit{for } 3 \le m \le n,\ m \textit{ odd}, \\ n-1+m/2 & \textit{for } 4 \le m \le n,\ m \textit{ even}, \\ \max\{m-1+\lfloor n/2 \rfloor, 2n-1\} & \textit{for } 2 \le n \le m,\ m \textit{ odd}, \\ m-1+\lfloor n/2 \rfloor & \textit{for } 2 \le n \le m,\ m \textit{ even}. \end{cases}$$

Theorem 1.3.5 (Gerencsér, Gyárfás [46]) *For all integers* n, m *satisfying* $n \ge m \ge 2$, $R(P_m, P_n) = n + \lfloor m/2 \rfloor - 1$.

Theorem 1.3.6 (Chvátal [19]) *For all integers* $n, m \ge 2$, $R(K_m, T_n) = (m-1)(n-1) + 1$.

Determining the Ramsey number $R_k(C_n)$ is one of the earliest and well-known problems. For 3-color Ramsey numbers of even cycles, a lower bound $R_3(C_{2n}) \ge 4n$ for all $n \ge 2$ follows from a construction by Dzido, Nowik and Szuca [28]. In 2005, Dzido [27] posed the Triple Even Cycle Conjecture in his Ph.D. thesis.

Conjecture 1.3.7 (Triple Even Cycle, Dzido [27]) $R_3(C_{2n}) = 4n$ *for all integers* $n \ge 3$.

Benevides and Skokan [5] proved Conjecture 1.3.7 is true for sufficiently large n. For small value of n, only $R_3(C_4)$, $R_3(C_6)$ and $R_3(C_8)$ are known. For 3-color Ramsey numbers of odd cycles, we begin with the well know conjecture by Bondy and Erdős [7].

Conjecture 1.3.8 (Bondy, Erdős [7]) $R_k(C_{2n+1}) = n \cdot 2^k + 1$ *for all* $n \geq 2$ *and* $k \geq 3$.

When $k = 3$, Conjecture 1.3.8 is also known as Triple Odd Cycle Conjecture. Łuczak [66] employed the regularity method to prove that $R_3(C_{2n+1}) = 8n+o(n)$, as $n \to \infty$, and so the Triple Odd Cycle Conjecture holds asymptotically. Jenssen and Skokan [59] proved the Conjecture 1.3.8 holds for all fixed k and all n sufficiently large by using Łuczak's regularity method. However, Day and Johnson [24] recently proved Theorem 1.3.9 below which implies that Conjecture 1.3.8 is false when n is small with respect to k. For small value of n, only $R_3(C_3)$, $R_3(C_5)$ and $R_3(C_7)$ have been determined.

Theorem 1.3.9 (Day, Johnson [24]) *For all integers* n *there exists a constant* $\epsilon = \epsilon(n) > 0$ *such that, for all* k *sufficiently large,* $R_k(C_{2n+1}) > 2n \cdot (2+\epsilon)^{k-1}$.

For more detailed information on Ramsey numbers, and open problems on this topic, the readers are referred to the dynamic survey of Radziszowski [71]. For more information on Ramsey-related topics can be found in a very recent informative survey due to Conlon, Fox and Sudakov [21].

1.4 Co-critical Graphs

Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, a non-complete graph G is $(H_1, \ldots, H_k)$-*co-critical* if $G \nrightarrow (H_1, \ldots, H_k)$, but $G + e \to (H_1, \ldots, H_k)$ for every e in $\overline{G}$. This is a generalization to graphs whose edges are k-colored and saturated with respect to monochromatic subgraphs. Following Galluccio, Siminovits and Simonyi [45], the complete graphs in the definition of $(H_1, \ldots, H_k)$-co-critical graphs are excluded, else every complete graph on fewer than

$R(H_1, \ldots, H_k)$ vertices is $(H_1, \ldots, H_k)$-co-critical. It is worth noting that every $(H_1, \ldots, H_k)$-co-critical graph has at least $R(H_1, \ldots, H_k)$ many vertices.

For example, K_6^- is (K_3, K_3)-co-critical, where K_6^- denotes the graph obtained from K_6 by deleting exactly one edge. The {red, blue}-coloring of K_6^- depicted in Figure 1.6 below is a critical-coloring of K_6^- with respect to K_3, K_3, so $K_6^- \nrightarrow (K_3, K_3)$. However, we get a K_6 by adding the missing edge of K_6^- and $K_6 \rightarrow (K_3, K_3)$.

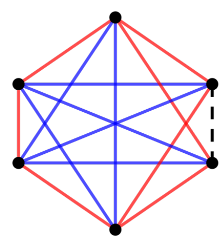


Figure 1.6: $K_6^- \nrightarrow (K_3, K_3)$

The notion of co-critical graphs was initiated by Nešetřil [70] in 1986 when he asked the following question regarding (K_3, K_3)-co-critical graphs:

> "Are there an infinite number of minimal co-critical graphs, i.e., co-critical graphs which lose this property when any vertex is deleted? Is K_6^- the only one? "

Galluccio, Siminovits and Simonyi [45] answered this question in the positive by constructing infinite many minimal (K_3, K_3)-co-critical graphs without containing K_5 as a subgraph and extended the notation (K_3, K_3)-co-critical to $(H_1, \ldots, H_k)$-co-critical. They [45] also mentioned that it's easy to construct (K_3, K_3)-co-critical graphs with a linear number of edges, so one should ask for constructing "almost regular" co-critical graphs with low maximum degree. In [45], they proved the existence of (K_3, K_3)-co-critical graphs with maximal degree $O(n^{3/4} \log n)$ by using a random graph construction. Szabó [79] then constructed infinite many (K_3, K_3)-co-critical graphs with maximal degree $O(n^{3/4})$. It remains unknown whether there are infinitely many strongly minimal co-critical graphs, where an $(H_1, \ldots, H_k)$-co-critical graph is strongly minimal co-critical if it contains no proper subgraph which is also $(H_1, \ldots, H_k)$-co-critical. This is one of the most intriguing open problems proposed by Galluccio, Siminovits and Simonyi in [45]. One interesting observation made in [45] is that if G is $(H_1, \ldots, H_k)$-co-critical, then $\chi(G) \geq R(H_1, \ldots, H_k)$. They also made some observations on the minimum degree of (K_3, K_3)-co-critical graphs and maximum number of possible edges of $(H_1, \ldots, H_k)$-co-critical graphs.

Hanson and Toft [56] in 1987 also studied the minimum and maximum number of edges over all $(H_1, \ldots, H_k)$-co-critical graphs on n vertices when $H_1, \ldots, H_k$ are complete graphs, under the name of *strongly* $(|H_1|, \ldots, |H_k|)$-*saturated* graphs. Recently, this topic has been studied under the name of $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-*saturated* graphs [17, 39, 73]. A graph G is $(H_1, \ldots, H_k)$-*Ramsey-minimal* if $G \to (H_1, \ldots, H_k)$, but for any proper subgraph G' of G, $G' \not\to (H_1, \ldots, H_k)$. We define $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ to be the family of all $(H_1, \ldots, H_k)$-Ramsey-minimal graphs. A graph G is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-*saturated* if no element of $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ is a subgraph of G, but for any edge e in $\overline{G}$, some element of $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ is a subgraph of $G+e$. It can be easily checked that a non-complete graph is $(H_1, \ldots, H_k)$-co-critical if and only if it is $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated. From now on, we shall use the notion of $(H_1, \ldots, H_k)$-co-critical other than $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated, as the former is much simpler and straightforward.

Let $r := R(K_{t_1}, \ldots, K_{t_k})$ be the classical Ramsey number for $K_{t_1}, \ldots, K_{t_k}$. Hanson and Toft [56] proved that every $(K_{t_1}, \ldots, K_{t_k})$-co-critical on n vertices has at most $e(T_{r-1,n})$ edges and this bound is best possible, where $T_{r-1,n}$ denotes the complete $(r-1)$-partite graphs on $n \geq r-1$ vertices whose partition sets differ in size by at most one. We will often refer to this graph as the *Turán graph* [80]. Note that $T_{r-1,n}$ contains no K_r.

In the same paper [56], Hanson and Toft also observed that for all $n \geq r$, the graph $H := K_{r-2} +$

$\overline{K}_{n-r+2}$ is $(K_{t_1}, \ldots, K_{t_k})$-co-critical. To obtain a critical k-coloring of H, fix a critical k-coloring σ of the edges of complete graph K_{r-1}, then duplicate any vertex of the complete graph K_{r-1} with $n-r+1$ times together with the edge colors (see Figure 1.7). One can see that H has no monochromatic copy of K_{k_i} in color i for any $i \in [t]$ under the edge coloring depicted in Figure 1.7. Note that for any edge e in $\overline{H}$, $H+e$ contains a monochromatic copy of K_r. By the definition of $R(K_{t_1}, \ldots, K_{t_k})$, $H+e \to (K_{t_1}, \ldots, K_{t_k})$. They further made the following conjecture that no $(K_{t_1}, \ldots, K_{t_k})$-co-critical graph on n vertices can have fewer than $e(K_{r-2}+\overline{K}_{n-r+2})$ edges.

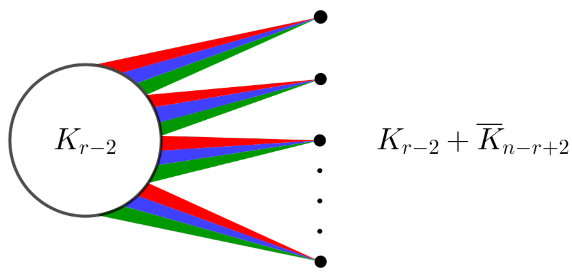


Figure 1.7: $K_{r-2}+\overline{K}_{n-r+2} \nrightarrow (K_{t_1}, \ldots, K_{t_k})$

Conjecture 1.4.1 (Hanson, Toft [56]) *Let G be a $(K_{t_1}, \ldots, K_{t_k})$-co-critical graph on n vertices and $r = R(K_{t_1}, \ldots, K_{t_k})$. Then*

$$e(G) \geq (r-2)(n-r+2) + \binom{r-2}{2}.$$

This bound is best possible for every n.

Conjecture 1.4.1 remains wide open, except that the first nontrivial case, (K_3, K_3)-co-critical graphs, has been settled in [17] for $n \geq 56$.

Theorem 1.4.2 (Chen, Ferrara, Gould, Magnant, Schmitt [17]) *If G is a (K_3, K_3)-co-critical graph on $n \geq 56$ vertices, then $e(G) \geq 4n-10$. This bound is sharp for every $n \geq 56$.*

Chen et al. also considered the minimum number of edges over all (K_3, P_3)-co-critical graphs on n vertices in [17]. Later, Ferrara, Kim and Yeager [39] determined the minimum number of edges over all $(m_1K_2, \dots, m_tK_2)$-co-critical graphs on n vertices.

Theorem 1.4.3 (Chen, Ferrara, Gould, Magnant, Schmitt [17]) *If G is a (K_3, P_3)-co-critical graph on $n \geq 11$ vertices, then $e(G) \geq \lfloor \frac{5n}{2} \rfloor - 5$. This bound is sharp for every $n \geq 11$.*

Theorem 1.4.4 (Ferrara, Kim, Yeager [39]) *For integers $m_1, \dots, m_t \geq 1$ and $n > 3(m_1 + \cdots + m_t - t)$, if G is a $(m_1K_2, \dots, m_tK_2)$-co-critical graph on n vertices, then $e(G) \geq 3(m_1 + \cdots + m_t - t)$. This bound is sharp for every $n > 3(m_1 + \cdots + m_t - t)$.*

Motivated by Conjecture 1.4.1, we study the following problem. Let $\mathcal{T}_k$ denote the family of all trees on k vertices. For all $t, k \geq 3$, we write $G \to (K_t, \mathcal{T}_k)$ if for every 2-coloring $\tau : E(G) \to \{\text{red, blue}\}$, G has either a red K_t or a blue tree $T_k \in \mathcal{T}_k$. A non-complete graph G is $(K_t, \mathcal{T}_k)$*-co-critical* if $G \not\to (K_t, \mathcal{T}_k)$, but $G + e \to (K_t, \mathcal{T}_k)$ for all e in $\overline{G}$. By a classic result of Chvátal [19], $R(K_t, \mathcal{T}_k) = (t-1)(k-1) + 1$. Hence, every $(K_t, \mathcal{T}_k)$-co-critical graph has at least $R(K_t, \mathcal{T}_k) = (t-1)(k-1) + 1$ many vertices. Following the observation made in both [45] and [56], every $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices has at most $e(T_{R(K_t, \mathcal{T}_k)-1, n})$ edges. Recently, Rolek and Song [73] studied the minimum number of edges over all $(K_3, \mathcal{T}_k)$-co-critical graphs on n vertices for all $k \geq 4$.

Theorem 1.4.5 (Rolek, Song [73]) *Let $n, k \in \mathbb{N}$.*

(i) *Every $(K_3, \mathcal{T}_4)$-co-critical graph on $n \geq 18$ vertices has at least $\lfloor 5n/2 \rfloor$ edges. This bound is sharp for every $n \geq 18$ (see Figure 1.8).*

(ii) *For all $k \geq 5$, if G is $(K_3, \mathcal{T}_k)$-co-critical on $n \geq 2k + (\lceil k/2 \rceil + 1)\lceil k/2 \rceil - 2$ vertices, then $e(G) \geq \left(\frac{3}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c(k)$, where $c(k) = \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil + \frac{3}{2}\right) k - 2$. This bound is asymptotically best possible (see Figure 1.9).*

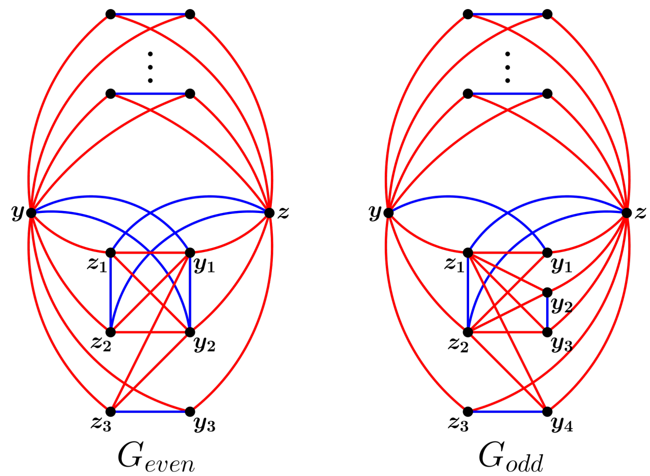


Figure 1.8: Two $(K_3, \mathcal{T}_4)$-co-critical graphs with a unique critical-coloring

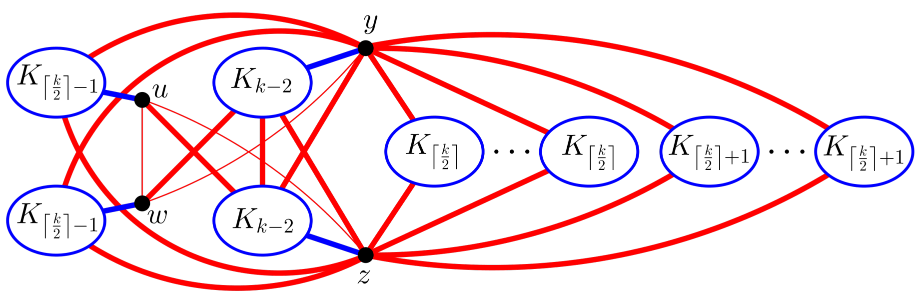


Figure 1.9: A $(K_3, \mathcal{T}_k)$-co-critical graph with a unique critical-coloring

Davenport and Song [22] considered the number of edges of $(K_3, K_{1,k})$-co-critical graphs on n vertices for all $k \geq 3$.

Theorem 1.4.6 (Davenport, Song [22]) *Let* $n, k \in \mathbb{N}$.

(i) *Every* $(K_3, K_{1,3})$*-co-critical graph on* $n \geq 13$ *vertices has at least* $3n - 4$ *edges. This bound is sharp for every* $n \geq 13$.

(ii) *For all* $k \geq 4$*, there exists a constant* $c(k)$*, if* G *is a* $(K_3, K_{1,k})$*-co-critical graph on* $n \geq 4k+2$ *vertices, then* $e(G) \geq (\frac{3}{2} + \frac{k}{2})n - c(k)$*. This bound is asymptotically best possible.*

We continue to study the size of $(K_t, \mathcal{T}_k)$-co-critical graphs for all $t \geq 4$ and $k \geq 3$. We first establish a number of important properties of such graphs in the hope that the method we develop here may shed some light on attacking Conjecture 1.4.1. The proof of Theorem 1.4.7 is given in Section 2.1.

Theorem 1.4.7 *For all $t, k \in \mathbb{N}$ with $t \geq 3$ and $k \geq 3$, let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices. Among all critical-colorings of G, let $\tau : E(G) \to \{$red, blue$\}$ be a critical-coloring of G with $|E_r|$ maximum. Let $D_1, \ldots, D_p$ be all components of G_b. Let $H := G \backslash (\bigcup_{i \in [p]} E(G[V(D_i)]))$. Then the following hold.*

(a) $\Delta(G_r) \leq n - 2$ *and* $\delta(G_r) \geq 2(t-2)$.

(b) *For all $i, j \in [p]$ with $i \neq j$, if there exist $u \in V(D_i)$ and $v \in V(D_j)$ such that $uv \notin E(H)$, then $H[N_H(u) \cap N_H(v)]$ contains a K_{t-2} subgraph.*

(c) *For every $uv \in E(H)$, if v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and $\{v\} = V(D_j)$ for some $j \in [p]$, then $|D_i| = k-1$ for all D_i with $u \notin D_i$ and $D_i \backslash N_H(u) \neq \emptyset$, where $i \in [p]$.*

(d) *If $\delta(H) \leq 2t-5$ and $k \geq t$, then for any vertex $u \in V(H)$ with $d_H(u) = \delta(H)$, no edge of $H[N_H(u)]$ is contained in all K_{t-2} subgraphs of $H[N_H(u)]$.*

(e) $k \geq 2t - 1 - \delta(H)$. *Moreover,* $\delta(H) \geq t-1$, *with equality when* $t = 4$.

(f) *If $k \geq t \geq 5$, then $\delta(H) \geq t + \min\{3, t-4\}$ or there exists an edge $uv \in E(H)$ such that $d_H(u) = \delta(H)$ and v is complete to $N_H(u) \backslash v$ in H.*

(g) $\displaystyle\sum_{i=1}^{p} e(G[V(D_i)]) > \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2 \rceil - 1)).$

(h) *H is connected.*

(i) *For every $q \in \mathbb{N}$ with $q \geq t-1$, there exists a constant $c(q,k)$ such that, if $\delta(H) \geq q$, then $e(H) \geq qn - c(q,k)$.*

Theorem 1.4.7(i) above is crucial in the proof of Theorem 1.4.8 and Theorem 1.4.9. Following Day [23], we apply the q-neighbour bootstrap percolation on a not necessarily K_t-saturated graph, to prove Theorem 1.4.7(i), but with more involved rules. Proof of Theorem 1.4.8 is given in Section 2.2 and proof of Theorem 1.4.9 is given in Section 2.3.

Theorem 1.4.8 *Let $t, k \in \mathbb{N}$ with $t \geq 4$ and $k \geq \max\{6, t\}$. There exists a constant $\ell(t, k)$ such that, for all $n \in \mathbb{N}$ with $n \geq (t-1)(k-1)+1$, if G is a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, then*

$$e(G) \geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - \ell(t, k).$$

When t is small, the linear bound given in Theorem 1.4.8 actually holds for more values of k. This is proved in Theorem 1.4.9 below.

Theorem 1.4.9 *Let $t, k \in \mathbb{N}$ with $t \in \{4, 5, 6, 7\}$ and $k \geq \max\{3, 4t-14\}$. There exists a constant $c(t, k)$ such that, for all $n \in \mathbb{N}$ with $n \geq (t-1)(k-1)+1$, if G is a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, then*

$$e(G) \geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c(t, k).$$

We then prove in Section 2.4 that the linear bound given in Theorem 1.4.9 is asymptotically best possible when $t \in \{4, 5\}$ and $k \geq 3t-9$.

Theorem 1.4.10 *For each $t \in \{4, 5\}$, all $k \geq 3$ and $n \geq (2t-3)(k-1) + \lceil k/2\rceil\lceil k/2\rceil - 1$, there exists a $(K_t, \mathcal{T}_k)$-co-critical graph G on n vertices such that*

$$e(G) \leq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n + C(t, k)$$

where $C(t, k) = \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left((2t-3)(k-1) - \left\lceil\frac{k}{2}\right\rceil\right)$.

With the support of Theorem 1.4.5 and Theorem 1.4.10, we believe that the linear bound given in Theorem 1.4.8 is asymptotically best possible for all $t \geq 4$ and $k \geq 3$.

1.5 Saturated Graphs

Given a family of graphs $\mathcal{F}$, a graph is *$\mathcal{F}$-free* if it does not contain any graph $F \subseteq \mathcal{F}$ as a subgraph. We simply say a graph is F-free when $F = \mathcal{F}$. A graph G is *$\mathcal{F}$-saturated* if G contains no member of $\mathcal{F}$ as a subgraph but for every edge $e \in E(\overline{G})$, there exists $F \in \mathcal{F}$ such that $G + e$ contains F as a subgraph. The *saturation number of $\mathcal{F}$*, written $sat(n, \mathcal{F})$, is the minimum number of edges over all $\mathcal{F}$-saturated graphs with n vertices. When $\mathcal{F} = F$, we simply use F-saturated and $sat(n, F)$, respectively. Note that the maximum number of edges over all $\mathcal{F}$-saturated graphs on n vertices, denoted by $ex(n, \mathcal{F})$, has been investigated extensively.

In 1964, Erdős, Hajnal and Moon [32] initiated the study of saturation number of K_p. Since then, saturation numbers have received considerable attention.

Theorem 1.5.1 (Erdős, Hajnal, Moon [32]) *If $2 \leq p \leq n$, then $sat(n, K_p) = (p-2)(n-p+2) + \binom{p-2}{2} = \binom{n}{2} - \binom{n-p+2}{2}$ and $K_{p-2} + \overline{K}_{n-p+2}$ is the only K_p-saturated graph with n vertices and $sat(n, K_p)$ edges.*

In 1986, Kászonyi and Tuza [60] found the best known general upper bound for sat$(n, \mathcal{F})$ by showing that there exists a constant $c = c(F)$ such that $sat(n, \mathcal{F}) < cn$. This means, in most cases, the order of magnitude of $sat(n, \mathcal{F})$ is n, while the order of magnitude of $ex(n, \mathcal{F})$ is n^2. In the same paper [60], they also pointed out that, among graphs on p vertices, $sat(n, F)$ is maximal if $F = K_p$, despite the fact that $sat(n, \mathcal{F})$ does not satisfy some simple monotonic properties. For every p, the extremal example for Theorem 1.5.1 contains a vertex of degree $n-1$ (such a vertex is called a *conical vertex*). Hajnal [54] asked the following question and proved Theorem 1.5.2.

"Let $2 \leq k \leq n$ be integers. What is the minimal number of edges of the K_p-saturated graph on n vertices which do not contain conical vertices? "

Theorem 1.5.2 (Hajnal [54]) *Let $t, n \in \mathbb{N}$. Let G be a K_p-saturated graph on n vertices. Then either $\Delta(G) = n - 1$ or $\delta(G) \geq 2(p-2)$.*

Some results of the above question for the case $p = 3$ can be found in [33] and [43]. The case $p \geq 4$ was considered by Alon, Erdős, Holzman, and Krivelevichin [1].

Theorem 1.5.3 (Alon, Erdős, Holzman, Krivelevich [1]) *Let G be a K_4-saturated graph on n vertices.*

(i) *If $\Delta(G) = n - 2$, then $e(G) \geq 4n - 13$ for $n \geq 7$.*

(ii) *If $\Delta(G) = n - 3$, then $e(G) \geq 4n - 14$ for $n \geq 7$.*

A similar problem was considered by Duffus and Hanson [26]. They asked that what is the minimum number of edges in a K_p-saturated graph on n vertices with minimum degree δ. They proved the following result.

Theorem 1.5.4 (Duffus, Hanson [26]) *Let G be a K_3-saturated graph on n vertices.*

(i) *If $\delta(G) = 2$, then $e(G) \geq 2n - 5$ for $n \geq 5$.*

(ii) *If $\delta(G) = 3$, then $e(G) \geq 3n - 15$ for $n \geq 10$. This bound is best possible, and if $e(G) = 3n - 15$, then G has a subgraph isomorphic to the Petersen graph.*

Rolek and Song [73] proved a stronger version of Theorem 1.5.4 by providing a structural characterizing of K_3-saturated graph with minimum degree at most 2.

Theorem 1.5.5 (Rolek, Song [73]) *Let G be a K_3-saturated graph with n vertices and $\delta(G) = \delta$.*

(i) *If $\delta = 1$, then $G = K_{1,n-1}$.*

(ii) *If $\delta = 2$, then $G = J$, where the graph J is depicted in Figure 1.10. Moreover, $J = K_{2,n-2}$ when $B = C = \emptyset$.*

(iii) *If $\delta \geq 3$, then $2e(G) \geq \max\{(\delta+1)n-\delta^2-1, (\delta+2)n-\delta(\delta+t)-2\}$, where $t := \min\{d(v) : v$ is adjacent to a vertex of degree δ in $G\}$.*

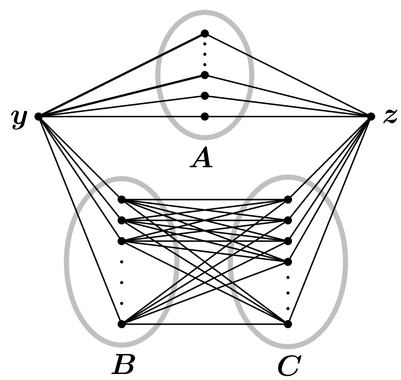


Figure 1.10: The graph J

Alon, Erdős, Holzman, and Krivelevich [1] showed that any K_4-saturated graph on $n \geq 11$ vertices with minimum degree 4 has at least $4n - 19$ edges. This has recently been generalized by Bosse, Song, and Zhang [10].

Theorem 1.5.6 (Bosse, Song, Zhang [10]) *Let G be a K_p-saturated graph on $n \geq p \geq 3$ vertices.*

(i) *If $\delta(G) = p - 2$, then $G = K_{p-2} + \overline{K}_{n-p+2}$, and $e(G) = (p-2)n - \binom{p-1}{2}$.*

(ii) *If $\delta(G) = p - 1$, then $G = K_{p-3} + J_{n-p+3}$, where J_{n-p+3} is isomorphic to J which is depicted in Figure 1.10. Therefore, $e(G) \geq (p-1)n - \binom{p}{2} - 2$, with equality only when $\min\{|B|, |C|\} = 1$.*

(iii) *If* $\delta(G) = p$ *and* $n \geq p + 7$*, then* $e(G) \geq pn - \binom{p+1}{2} - 9$*. The lower bound is sharp for all* p*.*

Theorem 1.5.7 below is a result of Day [23] on K_t-saturated graphs with prescribed minimum degree and it confirms a conjecture of Bollobás [6] when $t = 3$. It is worth noting that Day applied the r-neighbour bootstrap percolation on a K_t-saturated graph to prove Theorem 1.5.7, where graph bootstrap percolation was introduced in [15].

Theorem 1.5.7 (Day [23]) *Let* $q \in \mathbb{N}$*. There exists a constant* $c = c(q)$ *such that, for all* $3 \leq t \in \mathbb{N}$ *and all* $n \in \mathbb{N}$*, if* G *is a* K_t*-saturated graph on* n *vertices with* $\delta(G) \geq q$*, then* $e(G) \geq qn - c$*.*

For more detailed information on the intensive studies on saturated graphs, the readers are referred to the dynamic survey of J. R. Faudree, R. J. Faudree and Schmitt [35].

1.6 Gallai-Ramsey Numbers of Graphs

A *Gallai coloring* is an edge-coloring of a complete graph without rainbow triangles (that is, a triangle with all its edges colored differently). Gallai colorings naturally arise in several areas including: information theory [63]; the study of partially ordered sets, as in Gallai's original paper [44] (his result was restated in [53] in the terminology of graphs); and the study of perfect graphs [14]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [18, 41, 51, 52, 55, 12, 8, 9]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [40, 42].

A *Gallai* k*-coloring* is a Gallai coloring that uses at most k colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the *Gallai-Ramsey number* $GR(H_1, \ldots, H_k)$ is the least integer n such that every Gallai k-coloring of K_n contains a monochromatic copy of H_i in color i for some $i \in [k]$.

When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$. Clearly, $GR_k(H) \leq R_k(H)$ for all $k \geq 1$ and $GR(H_1, H_2) = R(H_1, H_2)$. In 2010, Gyárfás, Sárközy, Sebő and Selkow [52] proved the general behavior of $GR_k(H)$.

Theorem 1.6.1 (Gyárfás, Sárközy, Sebő, Selkow [52]) *Let H be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in k if H is not bipartite, linear in k if H is bipartite but not a star, and constant (does not depend on k) when H is a star.*

The lower bound for the case when H is not bipartite comes from the following inductive construction. Certainly there exists a small graph in one color containing no H. Suppose there exists G_k using k colors which contains no monochromatic copy of H. Then let G_{k+1} be two copies of G_k with all possible edges in between using the new color. The graph G_{k+1} also contains no monochromatic copy of H. For the lower bound when H is bipartite, the construction involves adding vertices to the graph with all edges in a single color. It turns out that for some graphs H (e.g., when $H = C_3$), $GR_k(H)$ behaves nicely, while the order of magnitude of $R_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $GR_k(H)$ is far from trivial, even when $|H|$ is small. Gallai [44] showed an important structural result on Gallai colorings of complete graphs.

Theorem 1.6.2 (Gallai [44]) *For any Gallai coloring c of a complete graph G with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, \ldots, V_p$ with $p > 1$ so that at most two colors are used on the edges in $E(G) \backslash (E(G[V_1]) \cup \cdots \cup E(G[V_p]))$ and only one color is used on the edges between any fixed pair (V_i, V_j) under c.*

The partition given in Theorem 1.6.2 is a *Gallai partition* of the complete graph G under c. Given a Gallai partition $V_1, \ldots, V_p$ of the complete graph G under c, let $v_i \in V_i$ for all $i \in [p]$ and let $\mathcal{R} := G[\{v_1, \ldots, v_p\}]$. Then $\mathcal{R}$ is the *reduced graph* of G corresponding to the given Gallai partition under c. Clearly, $\mathcal{R}$ is isomorphic to K_p. By Theorem 1.6.2, all the edges in $\mathcal{R}$ are colored by at most two colors under c. One can see that any monochromatic copy of H in $\mathcal{R}$ under c will result in a monochromatic copy of H in G under c. It is not surprising that Gallai-Ramsey numbers $GR_k(H)$ are closely related to the classical Ramsey numbers $R_2(H)$. Recently, Fox, Grinshpun and Pach [40] posed the following conjecture on $GR_k(H)$ when H is a complete graph.

Conjecture 1.6.3 (Fox, Grinshpun, Pach [40]) *For all $t \geq 3$ and $k \geq 1$,*

$$GR_k(K_t) = \begin{cases} (R_2(K_t) - 1)^{k/2} + 1 & \textit{if } k \textit{ is even} \\ (t-1)(R_2(K_t) - 1)^{(k-1)/2} + 1 & \textit{if } k \textit{ is odd.} \end{cases}$$

Recall that if $n < R_k(K_3)$, then there is a k-coloring c of the edges of K_n such that edges of every triangle in K_n are colored by at least two colors under c. A question of T. A. Brown (see [18]) asked:

> "What is the largest number $f(k)$ of vertices of a complete graph can have such that it is possible to k-color its edges so that edges of every triangle are colored by exactly two colors? "

Chung and Graham [18] answered this question in 1983.

Theorem 1.6.4 (Chung, Graham [18]) *For all $k \geq 1$,*

$$f(k) = \begin{cases} 5^{k/2} & \textit{if } k \textit{ is even} \\ 2 \cdot 5^{(k-1)/2} & \textit{if } k \textit{ is odd.} \end{cases}$$

Clearly, $GR_k(K_3) = f(k) + 1$. By Theorem 1.6.4, Conjecture 1.6.3 holds for $t = 3$. The proof of

Theorem 1.6.4 does not rely on Theorem 1.6.2. A simpler proof of this case using Theorem 1.6.2 can be found in [52]. The next open case, when $t = 4$, was recently settled in [65]. The case $t = 5$ was announced by Magnant and Schiermeyer in [69], and they observed that if $R_2(K_5) = 43$, then Conjecture 1.6.3 fails for K_5.

Theorem 1.6.5 (Liu, Magnant, Saito, Schiermeyer, Shi [65]) *For all $k \geq 1$,*

$$GR_k(K_4) = \begin{cases} 17^{k/2} + 1 & \textit{if } k \textit{ is even} \\ 17^{(k-1)/2}(t-1) + 1 & \textit{if } k \textit{ is odd.} \end{cases}$$

Motivated by Conjecture 1.6.3, Gallai-Ramsey numbers of cycles and paths have also been studied, as well as general upper bounds for $GR_k(P_n)$ and $GR_k(C_n)$ that were first studied in [36, 41] and later improved in [55]. Gregory [50] proved in his thesis that $GR_k(C_8) = 3k + 5$, but the proof was incomplete. We list some known results below.

Theorem 1.6.6 (Faudree, Gould, Jacobson, Magnant [36]) *For all $k \geq 1$,*

$$GR_k(C_4) = k + 4 \;\; \textit{and} \;\; GR_k(P_n) = \left\lfloor \frac{n-2}{2} \right\rfloor k + \left\lceil \frac{n}{2} \right\rceil + 1 \; \textit{for } n \in \{3, 4, 5, 6\}.$$

Theorem 1.6.7 (Fujita, Magnant [41]) *For all $k \geq 1$,*

$$GR_k(C_5) = 2^{k+1} + 1 \;\; \textit{and} \;\; GR_k(C_6) = 2k + 4.$$

Theorem 1.6.8 (Hall, Magnant, Ozeki, Tsugaki [55]) *For all $n \geq 3$ and $k \geq 1$,*

$$GR_k(C_{2n}) \leq (n-1)k + 3n \quad \textit{and} \quad GR_k(P_n) \leq \left\lfloor \frac{n-2}{2} \right\rfloor k + 3 \left\lfloor \frac{n}{2} \right\rfloor.$$

Theorem 1.6.9 (Bruce, Song [12]) *For all $k \geq 1$, $GR_k(C_7) = 3 \cdot 2^k + 1$.*

Theorem 1.6.10 (Bosse, Song [8]) *For all $k \geq 1$,*

$$GR_k(C_9) = 4 \cdot 2^k + 1 \text{ and } GR_k(C_{11}) = 5 \cdot 2^k + 1.$$

Theorem 1.6.11 (Bosse, Song, Zhang [9]) *For all $k \geq 1$,*

$$GR_k(C_{13}) = 6 \cdot 2^k + 1 \text{ and } GR_k(C_{15}) = 7 \cdot 2^k + 1.$$

Very recently, F. Zhang, Song and Chen [83] completely determined the Gallai-Ramsey numbers of all cycles.

Theorem 1.6.12 (F. Zhang, Song, Chen [83]) *For $n \geq 2$ and all $k \geq 1$,*

$$GR_k(C_{2n+1}) = n \cdot 2^k + 1 \text{ and } GR_k(C_{2n}) = (n-1)k + n + 1.$$

We study the Gallai-Ramsey numbers of even cycles and paths. For all $n \geq 3$ and $k \geq 2$, let $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$, $G_i := P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$. We want to determine the exact values of $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$. By reordering colors if necessary, we assume that $i_1 \geq i_2 \geq \cdots \geq i_k$. The construction for establishing a lower bound for $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$ for all $n \geq 3$ and $k \geq 2$ is similar to the construction given by Erdős, Faudree, Rousseau and Schelp in 1976 (see Section 2 in [31]) for classical Ramsey numbers of even cycles and paths. We recall their construction in the proof of Proposition 1.6.13 which is given in Section 3.1.

Proposition 1.6.13 *For all $n \geq 3$ and $k \geq 2$,*

$$GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j,$$

where $n-1 \geq i_1 \geq \cdots \geq i_k \geq 0$.

Song [76] recently conjectured that the lower bound established in Proposition 1.6.13 is also the desired upper bound for $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$ for all $n \geq 3$ and $k \geq 1$. We state it below.

Conjecture 1.6.14 (Song [76]) *For all* $n \geq 3$ *and* $k \geq 2$,

$$GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j,$$

where $n-1 \geq i_1 \geq \cdots \geq i_k \geq 0$.

Clearly, $GR_k(C_{2n}) \geq GR_k(P_{2n})$ and $GR_k(C_{2n}) \geq GR_k(M_n)$, where M_n denotes a set of n edges such that no two edges share the same vertex. It is worth noting that by letting $i_1 = \cdots = i_k = n-1$ and $G_{i_1} = C_{2n}$, the construction given in the proof of Proposition 1.6.13 yields that $(n-1)k+n+1 \leq GR_k(P_{2n})$ and $(n-1)k+n+1 \leq GR_k(M_n)$for all $n \geq 3$ and $k \geq 1$. The truth of Conjecture 1.6.14 implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k+n+1$ for all $n \geq 3$ and $k \geq 1$ and $GR_k(P_{2n+1}) = (n-1)k+n+2$ for all $n \geq 1$ and $k \geq 1$. As observed in [55], to completely solve Conjecture 1.6.14, one only needs to consider the case $G_{n-1} = C_{2n}$. We prove this in Proposition 1.6.15. The proof of Proposition 1.6.15 is similar to the proof of Theorem 7 given in [55]. We include a proof in Section 3.1 for completeness.

Proposition 1.6.15 *For all* $n \geq 3$ *and* $k \geq 2$*, if Conjecture 1.6.14 holds for* $G_{n-1} = C_{2n}$*, then it also holds for* $G_{n-1} = P_{2n+1}$.

We prove the Conjecture 1.6.14 is true for $n \in \{3, 4\}$ and all $k \geq 2$ in Section 3.2 and is true for $n \in \{5, 6\}$ and all $k \geq 2$ in Section 3.3.

Theorem 1.6.16 *For* $n \in \{3, 4\}$ *and all* $k \geq 2$*, let* $G_i = P_{2i+3}$ *for all* $i \in \{0, 1, \ldots, n-2\}$,

$G_{n-1} = C_{2n}$, *and* $i_j \in \{0, 1, \dots, n-1\}$ *for all* $j \in [k]$ *with* $i_1 \geq i_2 \geq \cdots \geq i_k$. *Then*

$$GR(G_{i_1}, G_{i_2}, \dots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

Theorem 1.6.17 *For* $n \in \{5, 6\}$ *and all* $k \geq 2$, *let* $G_i = P_{2i+3}$ *for all* $i \in \{0, 1, \dots, n-2\}$, $G_{n-1} = C_{2n}$, *and* $i_j \in \{0, 1, \dots, n-1\}$ *for all* $j \in [k]$ *with* $i_1 \geq \cdots \geq i_k$. *Then*

$$GR(G_{i_1}, \dots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.$$

Theorem 1.6.16 and Theorem 1.6.17 strengthen the results listed in Theorem 1.6.6 and Theorem 1.6.7. Our proof relies only on Theorem 1.6.2 and Ramsey numbers $R(H_1, H_2)$, where $H_1, H_2 \in \{C_{12}, C_{10}, C_8, C_6, P_{11}, P_9, P_7, P_5, P_3\}$. Theorem 1.6.16 and Theorem 1.6.17, together with Proposition 1.6.15, implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k + n + 1$ for $n \in \{3, 4, 5, 6\}$ and all $k \geq 1$, and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for $n \in [6]$ and all $k \geq 1$. Hence, Theorem 1.6.16 yields a new and simpler proof of the known results on Gallai-Ramsey numbers of C_8, C_6 and P_n with $n \leq 7$. As mentioned earlier, the proof of $GR_k(C_8) = 3k + 5$ given in [50] was incomplete. In our completely new strategy, we developed an extremely useful recoloring method (in the proof of Claim 6 which we believe will assist in solving other cases. Note that the method we developed here for even cycles and paths is very different from the method for odd cycles developed in [12, 8, 9].

CHAPTER 2: ON THE SIZE OF $(K_t, \mathcal{T}_k)$-CO-CRITICAL GRAPHS

2.1 Structural Properties of $(K_t, \mathcal{T}_k)$-co-critical Graphs

In this section, we establish a number of important properties of $(K_t, \mathcal{T}_k)$-co-critical graphs in the hope that the method we develop here may shed some light on attacking Conjecture 1.4.1.

We need to introduce more notation. For a graph G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a 2-edge-coloring of G and let E_r and E_b be the color classes of the coloring τ. We use G_r and G_b to denote the spanning subgraphs of G with edge sets E_r and E_b, respectively. We define τ to be a *critical-coloring* of G if G has neither a red K_t nor a blue $T_k \in \mathcal{T}_k$, that is, if G_r is K_t-free and G_b is $\mathcal{T}_k$-free. For every $v \in V(G)$, we use $d_r(v)$ and $N_r(v)$ to denote the degree and neighborhood of v in G_r, respectively. Similarly, we define $d_b(v)$ and $N_b(v)$ to be the degree and neighborhood of v in G_b, respectively. One can see that if G is $(K_t, \mathcal{T}_k)$-co-critical, then G admits at least one critical-coloring but $G + e$ admits no critical-coloring for every edge e in $\overline{G}$.

We first prove a lemma which will be used in the proofs of Theorem 1.4.7, Theorem 1.4.8, Theorem 1.4.9 and Theorem 1.4.10.

Lemma 2.1.1 *For all $t, k \in \mathbb{N}$ with $t \geq 3$ and $k \geq 3$, let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices. Let $\tau : E(G) \to \{$red, blue$\}$ be a critical-coloring of G. Then the following hold.*

(a) For every component D of G_b, $|D| \leq k-1$ and $G[V(D)] = K_{|D|}$.

(b) If $D_1, \ldots, D_q$ are the components of G_b with $|D_i| < k/2$ for all $i \in [q]$, then $V(D_1), \ldots, V(D_q)$ are complete to each other in G_r, and so $q \leq t-1$.

Proof. To prove (a), let D be a component of G_b. Since G_b is $\mathcal{T}_k$-free, we see that $|D| \leq k-1$.

Suppose next that $G[V(D)] \neq K_{|D|}$. Let $u, v \in V(D)$ be such that $uv \notin E(G)$. We obtain a critical-coloring of $G + uv$ from τ by coloring the edge uv blue, a contradiction.

To prove (b), let $D_1, \cdots, D_q$ be the components of G_b with $|D_i| < k/2$ for all $i \in [q]$. Since G is $(K_t, \mathcal{T}_k)$-co-critical, we see that $G + e$ admits no critical-coloring for every edge e in $\overline{G}$. Let $i, j \in [q]$ with $i \neq j$. We next show that $V(D_i)$ is complete to $V(D_j)$ in G_r. Suppose that there exist vertices $u \in V(D_i)$ and $v \in V(D_j)$ such that $uv \notin E_r$. Then $uv \notin E(G)$ and so we obtain a critical-coloring of $G + uv$ from τ by coloring the edge uv blue, a contradiction. Thus $V(D_i)$ is complete to $V(D_j)$ in G_r for all $i, j \in [q]$ with $i \neq j$. Since τ is a critical-coloring, it follows that G_r is K_t-free and so $q \leq t - 1$. ∎

We are now ready to prove Theorem 1.4.7.

Proof. Let G, τ, $D_1, \ldots, D_p$ and H be given as in the statement. Then $n \geq (t-1)(k-1)+1$. By Lemma 2.1.1(a), $|D_i| \leq k-1$ for all $i \in [p]$. Hence, G_b has at least t components because $|G_b| = n \geq (t-1)(k-1)+1$. We first prove Theorem 1.4.7(a). By the choice of τ, G_r is K_t-free but $G_r + e$ contains a copy of K_t for every $e \in E(\overline{G_r})$. Hence G_r is K_t-saturated. Suppose there exists a vertex $x \in V(G)$ such that $d_r(x) = n-1$. Note that $G_r \backslash x$ is K_{t-1}-free because G_r is K_t-free. Since $G \neq K_n$, there must exist $u, w \in N_r(x)$ such that $uw \notin E(G)$. By Lemma 2.1.1(a), u, w belong to different components of G_b. But then we obtain a critical-coloring of $G + uw$ from τ by first coloring the edge uw red, and then recoloring xu blue and all edges incident with u in G_b red, a contradiction. This proves that $\Delta(G_r) \leq n-2$. Since G_r is K_t-saturated, by Theorem 1.5.2, $\delta(G_r) \geq 2(t-2)$.

To prove Theorem 1.4.7(b), let $u \in V(D_i)$ and $v \in V(D_j)$ be such that $uv \notin E(H)$, where $i \neq j$. Suppose $H[N_H(u) \cap N_H(v)]$ is K_{t-2}-free. Since $|D_\ell| \leq k-1$ for all $\ell \in [p]$, we obtain a critical-coloring of $G + uv$ from τ by first coloring the edge uv red, and then recoloring all red edges in

$G[V(D_\ell)]$ blue for all $\ell \in [p]$, a contradiction. Therefore, $H[N_H(u) \cap N_H(v)]$ contains a K_{t-2} subgraph. This proves Theorem 1.4.7(b).

To prove Theorem 1.4.7(c), let $uv \in E(H)$ be such that v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and $\{v\} = V(D_j)$ for some $j \in [p]$. We may assume that $u \in V(D_p)$ and $\{v\} = V(D_{p-1})$. Note that $H[N_H(u)]\backslash v$ is K_{t-2}-free. Suppose there exists an $\ell \in [p-2]$ such that $D_\ell \backslash N_H(u) \neq \emptyset$ but $|D_\ell| \leq k-2$. Let $w \in V(D_\ell)\backslash N_H(u)$. Then $wv \in E_r$, else we obtain a critical-coloring of $G + wv$ from τ by coloring the edge wv blue. Since $H[N_H(u)]\backslash v$ is K_{t-2}-free, we then obtain a critical-coloring of $G + uw$ from τ by coloring the edge uw red, and then recoloring wv blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves Theorem 1.4.7(c).

To prove Theorem 1.4.7(d,e), let $u \in V(H)$ with $d_H(u) = \delta(H)$. We may assume that $u \in V(D_p)$. By Theorem 1.4.7(b), $d_H(u) \geq t-2$. Let $N_H(u) := \{u_1, \ldots, u_{\delta(H)}\}$. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p) \cup N_H(u))$, we see that $H[N_H(u)]$ must have a K_{t-2} subgraph. We may assume that $H[\{u_1, \ldots, u_{t-2}\}] = K_{t-2}$. Then we may further assume that $u_i \in V(D_{p-i})$ for all $i \in [t-2]$. Let $v \in V(H)\backslash(V(D_p) \cup N_H(u))$.

To proceed to prove Theorem 1.4.7(d), assume $d_H(u) \leq 2t-5$ and $k \geq t$. Suppose $H[N_H(u)]$ has an edge, say u_1u_2, that is contained in all K_{t-2} subgraphs of $H[N_H(u)]$. Then both $H[N_H(u)]\backslash u_1$ and $H[N_H(u)]\backslash u_2$ are K_{t-2}-free. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p) \cup N_H(u))$, $V(H)\backslash(V(D_p) \cup N_H(u))$ must be complete to $\{u_1, u_2\}$ in H. Then $V(D_{p-1}) \cup V(D_{p-2}) \subseteq N_H(u)\backslash\{u_3, \ldots, u_{t-2}\}$. Thus $|V(D_{p-1}) \cup V(D_{p-2})| = \delta(H) - (t-4) \leq t-1 \leq k-1$, because $\delta(H) \leq 2t-5$ and $t \leq k$. Then we obtain a critical-coloring of $G + uv$ from τ by first coloring the edge uv red, and then recoloring u_1u_2 blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves Theorem 1.4.7(d).

To proceed to prove Theorem 1.4.7(e), note that $|N_r(u) \cap V(D_p)| = |N_r(u)| - d_H(u)$. By Theorem 1.4.7(a), $|N_r(u)| \geq 2t-4$. Since D_p is a component of G_b, we see that $N_b(u) \cap V(D_p) \neq \emptyset$. It follows that $|V(D_p)| = |\{u\}| + |N_b(u) \cap V(D_p)| + |N_r(u) \cap V(D_p)| \geq 1+1+(2t-4) - d_H(u) = 2t-2-d_H(u)$. By Lemma 2.1.1(a), $2t-2-d_H(u) \leq |V(D_p)| \leq k-1$, which yields $k \geq 2t-1-d_H(u)$. Suppose next that $\delta(H) = t-2 < 2t-5$. Then $k \geq t+1$. But then $H[\{u_1, \ldots, u_{t-2}\}]$ is the only K_{t-2} subgraph of $H[N_H(u)]$, contrary to Theorem 1.4.7(d). Thus $\delta(H) \geq t-1$. Finally, suppose that $t \geq 5$ but $\delta(H) = t-1$. Since G_r is K_t-free, we see that $G_r[\{u, u_1, \ldots, u_{t-1}\}] \neq K_t$. We may assume that $u_{t-1}u_{t-2} \notin E(G_r)$. By Theorem 1.4.7(b) applied to u and any vertex in $V(H) \backslash (V(D_p) \cup N_H(u))$, $V(H) \backslash (V(D_p) \cup N_H(u))$ must be complete to $\{u_1, \ldots, u_{t-3}\}$ in H. This implies that $V(D_{p-1}) \cup V(D_{p-2}) \subseteq \{u_1, u_2, u_{t-1}\}$. Then we obtain a critical-coloring of $G+uv$ from τ by first coloring the edge uv red, and then recoloring u_1u_2 blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves that $\delta(H) \geq t$ if $t \geq 5$. This proves Theorem 1.4.7(e).

To proceed to prove Theorem 1.4.7(f), let $k \geq t \geq 5$. By Theorem 1.4.7(e), $\delta(H) \geq t-1$. Suppose $\delta(H) \leq t + \min\{2, t-5\} \leq 2t-5$ and for every $u \in V(H)$ with $d_H(u) = \delta(H)$, no vertex $v \in N_H(u)$ is complete to $N_H(u) \backslash v$ in H. Then for any $x \in A := \{u_1, \ldots, u_{t-2}\}$, $xy \notin E(H)$ for some $y \in B := \{u_{t-1}, \ldots, u_{\delta(H)}\}$. By Theorem 1.4.7(d), $H[N_H(u)]$ must contain at least three different K_{t-2} subgraphs. Then $|B| \geq 2$ and so $\delta(H) \geq t$. Let $K \neq H[A]$ be another K_{t-2} subgraph of $H[N_H(u)]$ and let $b := |V(K) \cap B|$. Then $1 \leq b \leq |B|-1$. Note that $2 \leq |B| = \delta(H) - |A| \leq 4$ because $t \leq \delta(H) \leq t + \min\{2, t-5\}$. We may assume that $K = H[\{u_1, \ldots, u_{t-2-b}, u_{t-1}, \ldots, u_{t-2+b}\}]$. Since H is K_t-free and no vertex $v \in N_H(u)$ is complete to $N_H(u) \backslash v$ in H, we see that

($*$) every vertex in $\{u_{t-1}, \ldots, u_{t-2+b}\}$ has a non-neighbor in $\{u_{t-1-b}, \ldots, u_{t-2}\}$ in H; and every vertex in $\{u_1, \ldots, u_{t-2-b}\}$ has a non-neighbor in $B \backslash \{u_{t-1}, \ldots, u_{t-2+b}\}$ in H.

Let $K' \neq H[A], K$ be another K_{t-2} subgraph of $H[N_H(u)]$. We next claim that $|B| = 4$. Suppose $|B| \leq 3$. Then $b \leq 2$ and $\delta(H) \leq t+1$. Suppose $b = 2$. Then $K = H[\{u_1, \ldots, u_{t-4}, u_{t-1}, u_t\}]$, $B = \{u_{t-1}, u_t, u_{t+1}\}$ and $t \geq 6$. Moreover, u_{t+1} is anti-complete to $\{u_1, \ldots, u_{t-4}\}$ in H. By $(*)$, we may assume that $u_{t-3}u_{t-1}, u_{t-2}u_t \notin E(H)$. But then every K_{t-2} subgraph of $H[N_H(u)]$ contains the edge u_1u_2, contrary to Theorem 1.4.7(d). This proves that $b = 1$. Then $K = H[\{u_1, \ldots, u_{t-3}, u_{t-1}\}]$. By the arbitrary choice of K, $|V(K') \cap B| = 1$, and so $u_{t-1} \notin V(K')$. We may assume that $u_1, \ldots, u_{t-4}, u_t \in V(K')$. Then every vertex in $\{u_1, \ldots, u_{t-4}\}$ has a non-neighbor in $B \backslash \{u_{t-1}, u_t\}$. Thus $B = \{u_{t-1}, u_t, u_{t+1}\}$ and $t \geq 6$. But then u_{t+1} is anti-complete to $\{u_1, \ldots, u_{t-4}\}$ in H, and thus $H[A], K, K'$ are the only K_{t-2} subgraphs of $H[N_H(u)]$, each containing the edge u_1u_2, contrary to Theorem 1.4.7(d). This proves that $|B| = 4$, as claimed. Then $B = \{u_{t-1}, u_t, u_{t+1}, u_{t+2}\}$, $t \geq 7$ and $\delta(H) = t+2$. If $b = 3$, then $K = H[\{u_1, \ldots, u_{t-5}, u_{t-1}, u_t, u_{t+1}\}]$. Moreover, u_{t+2} is anti-complete to $\{u_1, \ldots, u_{t-5}\}$ in H. Since H is K_t-free, no two vertices in $\{u_{t-1}, u_t, u_{t+1}\}$ have two common neighbors in $\{u_{t-4}, u_{t-3}, u_{t-2}\}$ in H. By $(*)$, we may then assume that $u_{t-4}u_{t-1}, u_{t-3}u_t, u_{t-2}u_{t+1} \notin E(H)$. But then every K_{t-2} subgraph of $H[N_H(u)]$ contains the edge u_1u_2, contrary to Theorem 1.4.7(d). This proves that $b \leq 2$. Suppose next that $b = 2$. Then $K = H[\{u_1, \ldots, u_{t-4}, u_{t-1}, u_t\}]$. By $(*)$, we may assume that $u_{t-3}u_{t-1}, u_{t-2}u_t \notin E(H)$. By the arbitrary choice of K, $|V(K') \cap B| \leq 2$, and so $\{u_{t-1}, u_t\} \nsubseteq V(K')$. By $(*)$, $N_H(u_{t+1}) \cap N_H(u_{t+2}) \cap A \subseteq \{u_{t-3}, u_{t-2}\}$. We may assume that $|\{u_1, \ldots, u_{t-4}\} \backslash N_H(u_{t+1})| \geq \lceil (t-4)/2 \rceil$. Then $|N_H(u_{t+1}) \cap A| \leq \lfloor (t-4)/2 \rfloor + 2 < t-3$, $|N_H(u_{t+1}) \cap N_H(u_j) \cap A| \leq \lfloor (t-4)/2 \rfloor + 1 < t-4$ for $j \in \{t-1, t\}$. Thus $u_{t+1} \notin V(K')$. Then $u_{t+2} \in V(K')$, else every K_{t-2} subgraph of $H[N_H(u)]$ contains $\{u_1, \ldots, u_{t-4}\}$, contrary to Theorem 1.4.7(d). We may assume that $u_1, \ldots, u_{t-5}, u_{t+2} \in V(K')$. Then either $u_{t+2}u_{t-1} \notin E(H)$ or $u_{t+2}u_t \notin E(H)$, else $K'' = H[\{u_1, \ldots, u_{t-5}, u_{t-1}, u_t, u_{t+2}\}]$ is a K_{t-2} subgraph of $H[N_H(u)]$ with $|V(K'') \cap B| = 3$. But then every K_{t-2} subgraph of $H[N_H(u)]$ contains the edge u_1u_2 because $t-5 \geq 2$, contrary to Theorem 1.4.7(d). This proves that $b = 1$. Then $K = H[\{u_1, \ldots, u_{t-3}, u_{t-1}\}]$. By the arbitrary choice of K, $|V(K') \cap B| = 1$

and $V(K') \cap B \neq V(K) \cap B$. We may assume that $u_1, \ldots, u_{t-4}, u_t \in V(K')$. Then u_1u_2 is contained in all of $H[A], K, K'$. By Theorem 1.4.7(d), there must exist a fourth K_{t-2} subgraph of $H[N_H(u)]$, say K''. Similarly, $|V(K'') \cap B| = 1$ by the arbitrary choice of K. We may assume that $u_1, \ldots, u_{t-5}, u_{t+1} \in V(K'')$. But then u_{t+2} is anti-complete to $\{u_1, \ldots, u_{t-5}\}$ in H, and thus $H[A], K, K', K''$ are the only K_{t-2} subgraphs of $H[N_H(u)]$, each containing the edge u_1u_2, contrary to Theorem 1.4.7(d). This proves Theorem 1.4.7(f).

We next prove Theorem 1.4.7(g). By Lemma 2.1.1(a,b), $|D_i| \leq k-1$, $G[V(D_i)] = K_{|D_i|}$ for all $i \in [p]$, and at most $t-1$ of the D_i's have less than $k/2$ vertices. Let r be the remainder of $n-(t-1)(\lceil k/2 \rceil - 1)$ when divided by $\lceil k/2 \rceil$, and let $s \geq 0$ be an integer such that

$$n-(t-1)(\lceil k/2 \rceil - 1) = s\lceil k/2 \rceil + r(\lceil k/2 \rceil + 1).$$

It is straightforward to see that $\sum_{i=1}^{p} e(G[V(D_i)])$ is minimized when: $t-1$ of the components, say $D_1, \ldots, D_{t-1}$ are such that $|D_1|, \ldots, |D_{t-1}| < k/2$; r of the components, say $D_t, \cdots, D_{r+t-1}$ are such that $|D_t| = \cdots = |D_{r+t-1}| = \lceil k/2 \rceil + 1$; and s of the components, say $D_{r+t}, \cdots, D_{r+s+t-1}$ are such that $|D_{r+t}| = \cdots = |D_{r+s+t-1}| = \lceil k/2 \rceil$. Using the facts that $s\lceil k/2 \rceil + r(\lceil k/2 \rceil + 1) = n-(t-1)(\lceil k/2 \rceil - 1)$ and $r \leq \lceil k/2 \rceil - 1$, it follows that

$$\begin{aligned}
\sum_{i=1}^{p} e(G[V(D_i)]) &> s\binom{\lceil k/2 \rceil}{2} + r\binom{\lceil k/2 \rceil + 1}{2} \\
&= \frac{s}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1\right) + \frac{r}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\
&= \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}\right)\left(s\left\lceil \frac{k}{2} \right\rceil + r\left(\left\lceil \frac{k}{2} \right\rceil + 1\right)\right) + \frac{r}{2}\left(\left\lceil \frac{k}{2} \right\rceil + 1\right) \\
&\geq \left(\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil - \frac{1}{2}\right)(n-(t-1)(\lceil k/2 \rceil - 1)).
\end{aligned}$$

This proves Theorem 1.4.7(g).

To prove Theorem 1.4.7(h), suppose that H is disconnected. Let $x, y \in V(H)$ be such that x and y are in different components of H. By Theorem 1.4.7(b), $\{x, y\} \subseteq D_i$ for some $i \in [p]$, and there must exist a vertex $w \in D_j$ such that $xw \notin E(H)$ and $yw \in E(H)$, where $j \in [p]$ with $j \neq i$. By Theorem 1.4.7(b), x and w have at least $t-2$ common neighbors in H. But then x and y must be in the same component of H, a contradiction. This proves Theorem 1.4.7(h).

It remains to prove Theorem 1.4.7(i). By Theorem 1.4.7(h), H is connected. Let $q \in \mathbb{N}$ with $q \geq t-1$. Assume $\delta(H) \geq q$. Following Day [23], we next apply the q-neighbour bootstrap percolation on H. Note that H is not necessarily K_t-saturated. Given a set $S \subseteq V(H)$ and any vertex $v \in V(H)$, let $N_S(v) := N_H(v) \cap S$ and $d_S(v) := |N_S(v)|$. Let $R \subseteq V(H)$ be any nonempty set. Let $R^0 := R$ and for $i \geq 1$, let

$$R^i := R^{i-1} \cup \{v \in V(H) : d_{R^{i-1}}(v) \geq q\}.$$

Let $\overline{R} := \bigcup_{i \geq 0} R^i$, the closure of R under the q-neighbor bootstrap percolation on H. Then

$$e(H[\overline{R}]) \geq q(|\overline{R}| - |R|),$$

because every vertex in $R^i \backslash R^{i-1}$ is adjacent to at least q vertices in R^{i-1}. Let $Y(R) := V(H) \backslash \overline{R}$. Finally, for any $v \in V(H)$, let

$$\omega_{_R}(v) := d_{\overline{R}}(v) + d_{Y(R)}(v)/2.$$

We call $\omega_{_R}(v)$ the weight of v (with respect to R). Then

$$e_H(\overline{R}, Y(R)) + e(H[Y(R)]) = \sum_{v \in Y(R)} \omega_{_R}(v).$$

Within $Y(R)$, we define $B(R)$ to be the set $\{v \in Y(R) : \omega_{_R}(v) < q\}$, which we call the set of bad vertices. We next show that there exists a constant $c_1(q, k)$ and a nonempty set $R \subseteq V(H)$ with $|R| \leq c_1(q, k)$ such that $B(R) = \emptyset$.

Assume $B(R) \neq \emptyset$ for our initial R. Our goal is to move a small number of vertices into R so that the remaining vertices in $B(R)$ have strictly larger weight. To achieve this, let

$$\mathcal{U}_R := \{U \subseteq R : U = N_R(v) \text{ for some } v \in B(R)\}.$$

Note that for every vertex $v \in B(R)$, $d_R(v) \leq q - 1$. Thus

$$|\mathcal{U}_R| \leq 1 + |R| + \binom{|R|}{2} + \cdots + \binom{|R|}{q-1}.$$

Let $\mathcal{U}_R := \{U_1, U_2, \ldots, U_{|\mathcal{U}_R|}\}$ and let $u_i \in B(R)$ with $N_R(u_i) = U_i$ for all $i \in \{1, 2, \ldots, |\mathcal{U}_R|\}$. Then $d_{\overline{R}}(u_i) < q$, and so $d_{Y(R)}(u_i) \geq 1$ because $d_H(u_i) \geq q$. Let $x_i \in Y(R)$ such that $u_i x_i \in E(H)$ for all $i \in \{1, \ldots, |\mathcal{U}_R|\}$, and let $X(R) := \{x_1, x_2, \ldots, x_{|\mathcal{U}_R|}\}$. By the choice of $\mathcal{U}_R$ and $u_1, u_2, \ldots, u_{|\mathcal{U}_R|}$, for every vertex $v \in B(R)$, we see that $N_R(v) = N_R(u_i)$ for some $i \in \{1, 2, \ldots, |\mathcal{U}_R|\}$. Finally, let

$$S(R) := \{v \in B(R) : N_R(v) = N_R(u_i) \text{ and } \{v, x_i\} \subseteq D_j \text{ for some } i \in \{1, \ldots, |\mathcal{U}_R|\} \text{ and } j \in [p]\}.$$

We next show that **Algorithm** 1 below yields a nonempty set $R \subseteq V(H)$ with $B(R) = \emptyset$.

Algorithm 1: Building a nonempty set $R \subseteq V(H)$ with $B(R) = \emptyset$

Data: $H := G \backslash (\bigcup_{i \in [p]} E(G[V(D_i)]))$ is a spanning subgraph of G_r with $\delta(H) \geq q$

Result: A nonempty set $R \subseteq V(H)$ with $B(R) = \emptyset$

1 Set R to be a set containing an arbitrary vertex in H;

2 **while** $B(R) \neq \emptyset$ **do**

3   Set R to be $R \cup X(R) \cup S(R) \cup \bigcup_{j=1}^{|\mathcal{U}_R|} N_{\overline{R}}(x_j)$;

4 **end**

Let R_i be the set R obtained in the i-th iteration of **Line 2** when running **Algorithm** 1. Then for all $i \geq 1$, $R_{i-1} \subseteq R_i$, $\overline{R}_{i-1} \subseteq \overline{R}_i$, $Y(R_i) \subseteq Y(R_{i-1})$ and $B(R_i) \subseteq B(R_{i-1})$. To see why $\omega_{_{R_i}}(v) > \omega_{_{R_{i-1}}}(v)$ for all $v \in B(R_i)$, we next introduce a control function on $V(H)$, because dealing with $\omega_{_R}(v)$ directly is difficult. Let $\phi_{_R}(v) := \sum_{x \in N_H(v)} f_{_R}(x)$ for all $v \in V(H)$, where for all $x \in V(H)$,

$$f_{_R}(x) = \begin{cases} 1, & \text{if } x \in R, \\ 1/2, & \text{if } x \in \overline{R} \backslash R, \\ d_R(x)/(2q), & \text{if } x \in Y(R). \end{cases}$$

It is worth noting that $\phi_{_R}(v) \leq \omega_{_R}(v)$ for every vertex $v \in V(H)$, because $d_{Y(R)}(x) \geq 1$ and $d_R(x) \leq q-1$ for all $x \in Y(R)$. Similarly, for all $i \geq 1$, $f_{_{R_{i-1}}}(x) \leq f_{_{R_i}}(x)$ for every $x \in V(H)$, because $Y(R_i) \subseteq Y(R_{i-1})$. We next claim that

($*$) for all $i \geq 1$ and every $v \in B(R_i)$, $\phi_{_{R_i}}(v) \geq \phi_{_{R_{i-1}}}(v) + 1/(2q)$.

Proof. Let $i \geq 1$ and $v \in B(R_i)$. Then $v \in B(R_{i-1})$, since $B(R_i) \subseteq B(R_{i-1})$. Let $\mathcal{U}_{R_{i-1}}$, $\{u_1, \ldots, u_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq B(R_{i-1})$, and $\{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq Y(R_{i-1})$ be defined accordingly for R_{i-1}. Then $N_{R_{i-1}}(v) = N_{R_{i-1}}(u_j)$ for some $j \in \{1, 2, \ldots, |\mathcal{U}_{R_{i-1}}|\}$. To prove $\phi_{_{R_i}}(v) \geq \phi_{_{R_{i-1}}}(v) + 1/(2q)$, it suffices to show that $f_{_{R_i}}(x) \geq f_{_{R_{i-1}}}(x) + 1/(2q)$ for some $x \in N_H(v)$. Since $\{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\} \subseteq Y(R_{i-1}) \cap R_i$, we see that $f_{_{R_{i-1}}}(x) = d_{_{R_{i-1}}}(x)/(2q) \leq (q-1)/(2q) = 1/2 - 1/(2q)$, and $f_{_{R_i}}(x) = 1 > f_{_{R_{i-1}}}(x) + 1/(2q)$ for all $x \in \{x_1, \ldots, x_{|\mathcal{U}_{R_{i-1}}|}\}$. We may assume that $vx_j \notin E(H)$ for all $j \in \{1, \ldots, |\mathcal{U}_{R_{i-1}}|\}$, otherwise we are done. Since $v \in B(R_i)$, by the choice of x_j and $S(R_{i-1})$, we see that $\{v, x_j\} \not\subseteq V(D_\ell)$ for all $\ell \in [p]$. By Theorem 1.4.7(b) applied to v and x_j, $H[N_H(v) \cap N_H(x_j)]$ has a K_{t-2} subgraph. Let W be the vertex set of such a K_{t-2} subgraph. It follows that $W \not\subseteq R_{i-1}$, else $G_r[W \cup \{u_j, x_j\}] = K_t$, since $N_{R_{i-1}}(v) = N_{R_{i-1}}(u_j)$ and $u_j x_j \in E(H)$. Let $x \in W \backslash R_{i-1}$.

If $x \in \overline{R}_{i-1} \backslash R_{i-1}$, then $f_{R_{i-1}}(x) = 1/2$ and $f_{R_i}(x) = 1$, and so $f_{R_i}(x) \geq f_{R_{i-1}}(x) + 1/(2q)$, as desired. If $x \in Y(R_{i-1})$, then either $x \in \overline{R}_i$ or $x \in Y(R_i)$. In both cases, we have $f_{R_{i-1}}(x) = d_{R_{i-1}}(x)/(2q) \leq 1/2 - 1/(2q)$. If $x \in \overline{R}_i$, then $f_{R_i}(x) \geq 1/2$ and so $f_{R_i}(x) \geq f_{R_{i-1}}(x) + 1/(2q)$. Finally, if $x \in Y(R_i)$, then $d_{R_i}(x) \geq d_{R_{i-1}}(x) + 1$ because $x_j \in R_i \backslash R_{i-1}$ and $R_{i-1} \subseteq R_i$. Hence, $f_{R_i}(x) = d_{R_i}(x)/(2q) \geq (d_{R_{i-1}}(x) + 1)/(2q) = f_{R_{i-1}}(x) + 1/(2q)$.

In all cases, we have shown that there exists some vertex $x \in N_H(v)$ such that $f_{R_i}(x) \geq f_{R_{i-1}}(x) + 1/(2q)$. Hence, $\phi_{R_i}(v) \geq \phi_{R_{i-1}}(v) + 1/(2q)$ for all $i \geq 1$ and $v \in B(R_i)$. ■

By (∗), **Algorithm** 1 stops after $m \leq 2q^2$ iterations of **Line 2**. Hence $R_m \subseteq V(H)$ with $R_m \neq \emptyset$ but $B(R_m) = \emptyset$. For all $i \geq 0$,

$$\begin{aligned}
|R_{i+1}| &= |R_i| + |X(R_i)| + |S(R_i)| + |\bigcup_{j=1}^{|\mathcal{U}_{R_i}|} N_{\overline{R}_i}(x_j)| \\
&\leq |R_i| + |\mathcal{U}_{R_i}| + (k-2)|\mathcal{U}_{R_i}| + (q-1)|\mathcal{U}_{R_i}| \\
&= |R_i| + (k+q-2)|\mathcal{U}_{R_i}| \\
&\leq |R_i| + (k+q-2)\left(1 + |R_i| + \binom{|R_i|}{2} + \cdots + \binom{|R_i|}{q-1}\right),
\end{aligned}$$

which only depends on q and k. It follows that by **Algorithm** 1, there exists a constant $c_1(q,k)$ and a non-empty set $R \subseteq V(H)$ with $|R| \leq c_1(q,k)$ such that $B(R) = \emptyset$. Then $\omega_R(v) \geq q$ for all $v \in Y(R)$ and so

$$e_H(\overline{R}, Y(R)) + e(H[Y(R)]) = \sum_{v \in Y(R)} \omega_R(v) \geq q|Y(R)|.$$

Therefore,

$$\begin{aligned}
e(H) &= e(H[\overline{R}]) + e_H(\overline{R}, Y(R)) + e(H[Y(R)]) \\
&\geq q(|\overline{R}| - |R|) + q|Y(R)|
\end{aligned}$$

$$
\begin{aligned}
&\geq q(|\overline{R}| - c_1(q,k)) + q|Y(R)| \\
&= q(n - c_1(q,k)) \\
&= qn - c(q,k)
\end{aligned}
$$

where $c(q,k) = qc_1(q,k)$. This proves Theorem 1.4.7(i).

This completes the proof of Theorem 1.4.7. ■

We end this section with a useful corollary which will be applied in the proof of Theorem 1.4.9.

Corollary 2.1.2 *Let $t, k, G, \tau, D_1, \ldots, D_p, H$ be given as in the statement of Theorem 1.4.7.*

(a) *There exists a constant $c_1(t,k)$ such that if $\delta(H) \geq 2t-4$, then*

$$e(G) \geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_1(t,k).$$

(b) *For every $t \geq 5$, $k \geq 4t-14$ and $n \geq (t-1)(k-1)+1$, there exists a constant $c_2(t,k)$ such that, if there exists an edge $uv \in E(H)$ with $d_H(u) = \delta(H)$ such that v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and $\{v\} = V(D_j)$ for some $j \in [p]$, then*

$$e(G) \geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_2(t,k).$$

(c) *There exists a constant $c_3(t,k)$ such that if $t \geq 6$ and $k \geq 4t-14$ and $n \geq (t-1)(k-1)+1$, then*

$$e(G) \geq \left(\frac{2t + \min\{5, 3(t-5)\}}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_3(t,k).$$

Proof. To prove Corollary 2.1.2(a), assume $\delta(H) \geq 2t-4$. By Theorem 1.4.7(i) applied to H and

$q = 2t-4$, there exists a constant $c(2t-4,k)$ such that $e(H) \geq (2t-4)n - c(2t-4,k)$. This, together with Theorem 1.4.7(g), yields that

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq (2t-4)n - c(2t-4,k) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n-(t-1)(\lceil k/2\rceil - 1)) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c(2t-4,k) - \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c_1(t,k),
\end{aligned}$$

as desired, where $c_1(t,k) = c(2t-4,k) + \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2$. This proves Corollary 2.1.2(a).

We next prove Corollary 2.1.2(b). Assume $t \geq 5$, $k \geq 4t-14$ and $n \geq (t-1)(k-1)+1$. Let $uv \in E(H)$ with $d_H(u) = \delta(H)$ such that $N_H(u)\backslash v$ is K_{t-2}-free and $\{v\} = V(D_j)$ for some $j \in [p]$. We may assume that $u \in V(D_p)$ and $\{v\} = V(D_{p-1})$. By Corollary 2.1.2(a), we may assume that $d_H(u) \leq 2t-5$. Since $t \geq 5$, by Theorem 1.4.7(e), $\delta(H) \geq t$. By Theorem 1.4.7(i) applied to H and $q = t$, there exists a constant $c(t,k)$ such that $e(H) \geq tn - c(t,k)$. Since $n \geq (t-1)(k-1)+1$ and $|V(D_i)| \leq k-1$ for all $i \in [p]$ with $i \neq p-1$, we see that $p \geq t$. If $p = t$, then $n = (t-1)(k-1)+1$ and $|V(D_i)| = k-1$ for $i \in [p]$ with $i \neq p-1$. In this case,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq (tn - c(t,k)) + (p-1)(k-1)(k-2)/2 \\
&= (tn - c(t,k)) + (n-1)(k-2)/2 \\
&= (t-1+k/2)n - c(t,k) - (k-2)/2 \\
&\geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - \ell_1(t,k)
\end{aligned}$$

for all $k \geq 4t-14$, as desired, where $\ell_1(t,k) = c(t,k) + (k-2)/2$.

Next assume $p \geq t+1$. Since $k \geq 2(2t-7)$, by Lemma 2.1.1(b), there are at most $t-1$ many D_i's satisfying $u \notin V(D_i)$ and $D_i \backslash N_H(u) = \emptyset$. We may assume that for all $i \in [p-t]$, $D_1, \ldots, D_{p-t}$ are such that $u \notin V(D_i)$ and $D_i \backslash N_H(u) \neq \emptyset$. By Theorem 1.4.7(c), $|D_i| = k-1$ for all $i \in [p-t]$. Thus

$$\sum_{i=1}^{p} e(G[V(D_i)]) \geq (p-t)(k-1)(k-2)/2.$$

Note that $n \leq (p-1)(k-1)+1$ because $\{v\} = V(D_{p-1})$ and $|D_i| \leq k-1$ for all $i \in [p]$ with $i \neq p-1$. Therefore,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq (tn - c(t,k)) + (p-t)(k-1)(k-2)/2 \\
&\geq (tn - c(t,k)) + \frac{1}{2}\left(\frac{n-1}{k-1} - t + 1\right)(k-1)(k-2) \\
&= (t-1+k/2)n - c(t,k) - (k-2)(tk-t-k+2)/2 \\
&\geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(t,k) - [(t-1)k^2 - (3t-4)k + 2t-4]/2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - \ell_2(t,k)
\end{aligned}$$

for all $k \geq 4t-14$, as desired, where $\ell_2(t,k) = c(t,k) + [(t-1)k^2 - (3t-4)k + 2t-4]/2$. Let $c_2(t,k) := \max\{\ell_1(t,k), \ell_2(t,k)\}$. This proves Corollary 2.1.2(b).

Finally, we prove Corollary 2.1.2(c). Assume $t \geq 6$, $k \geq 4t-14$ and $n \geq (t-1)(k-1)+1$. Then by Theorem 1.4.7(f), $\delta(H) \geq t + \min\{3, t-4\}$ or there exists an edge $uv \in E(H)$ such that $d_H(u) = \delta(H)$ and v is complete to $N_H(u)\backslash v$ in H. Assume first that there exists an edge $uv \in E(H)$ such that $d_H(u) = \delta(H)$ and v is complete to $N_H(u)\backslash v$ in H. Then v is contained in all K_{t-2} subgraphs of $H[N_H(u)]$, because H is K_t-free. We may assume that $u \in V(D_p)$. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p)\cup N_H(u))$, $V(H)\backslash(V(D_p)\cup N_H(u))$ must be complete to v in H. Thus $\{v\} = V(D_\ell)$ for some $\ell \in [p-1]$. By Corollary 2.1.2(b), there exists a constant $c_2(t,k)$ such that

$$\begin{aligned} e(G) &\geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_2(t,k) \\ &\geq \left(\frac{2t+\min\{5, 3(t-5)\}}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_2(t,k), \end{aligned}$$

for all $t \geq 6$, as desired.

Assume next that $\delta(H) \geq t + \min\{3, t-4\}$. By Theorem 1.4.7(i) applied to H and $q = t + \min\{3, t-4\}$, there exists a constant $c(q,k)$ such that $e(H) \geq (t + \min\{3, t-4\})n - c(q,k)$. By Theorem 1.4.7(g), we have

$$\begin{aligned} e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\ &\geq \begin{cases} 8n - c(q,k) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2\rceil - 1)) & \text{if } t = 6 \\ (t+3)n - c(q,k) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2\rceil - 1)) & \text{if } t \geq 7 \end{cases} \\ &= \begin{cases} \left(\frac{15}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(q,k) - \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2 & \text{if } t = 6 \\ \left(\frac{2t+5}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(q,k) - \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2 & \text{if } t \geq 7. \end{cases} \end{aligned}$$

This proves Corollary 2.1.2(c) and thus completes the proof of Corollary 2.1.2. ■

2.2 Proof of Theorem 1.4.8

We begin this section with a useful lemma, which may be of independent interest. It is worth noting that Lemma 2.2.2 is stronger than Theorem 2.2.1 when $\alpha(G) > |G|/2$. We include a proof here for completeness and the proof of Lemma 2.2.2 is due to Hehui Wu, which is completely different from the one of Hajnal [54].

Theorem 2.2.1 (Hajnal [54]) *Let G be a graph and let $\mathcal{F}$ be the family of all maximum stable sets of G. Then*

$$\left|\bigcap_{S\in\mathcal{F}} S\right| + \left|\bigcup_{S\in\mathcal{F}} S\right| \geq 2\alpha(G).$$

Lemma 2.2.2 *Let G be a graph with $\alpha(G) > |G|/2$ and let $\mathcal{F}$ be the family of all maximum stable sets of G. Then*

$$\left|\bigcap_{S\in\mathcal{F}} S\right| \geq \delta(G) + 2\alpha(G) - |G| \geq \delta(G) + 1.$$

Moreover, if $\bigcap_{S\in\mathcal{F}} S = \{u\}$, then $\alpha(G) = (|G|+1)/2$ and u is an isolated vertex in G.

Proof. Let $X \in \mathcal{F}$ and $Y := V(G)\backslash X$. Then $|X| = \alpha(G) > |G|/2$, and so $|X| > |Y|$. Let $H := G[X, Y]$ be the bipartite subgraph of G with $V(H) = X \cup Y$ and $E(H) = \{xy \in E(G) : x \in X, y \in Y\}$. Let T be a maximum stable set of H and let $X_1 := X\backslash T$, $Y_1 := Y \cap T$ and $Y_2 := Y\backslash T$. Then $|Y_1| + |X\backslash X_1| = |T| \geq |X| = |X_1| + |X\backslash X_1| > |Y| = |Y_1| + |Y_2|$, which implies that $|X_1| \leq |Y_1|$ and $|X\backslash X_1| > |Y_2|$. We next show that $H' := G[X\backslash X_1, Y_2]$ contains a matching that saturates Y_2. For any $S \subseteq Y_2$, we have $|N_{H'}(S)| \geq |S|$, else $T' := (T\backslash N_{H'}(S)) \cup S$ is a stable set of H with $|T'| > |T|$, a contradiction. By Hall's Theorem, there exists a matching, say M, of H' that saturates Y_2. Let $X_2 := V(M) \cap X$ and $X_3 := X\backslash(X_1 \cup X_2)$. Then

$$|X_3| = |X| - |X_1| - |X_2| \geq |X| - |Y| = 2\alpha(G) - |G| > 0,$$

because $|X_1| \leq |Y_1|$, $|X_2| = |Y_2|$ and $\alpha(G) > |G|/2$. Note that $X_1 \cup Y_1$ is anti-complete to $X \backslash X_1$ in H. By the choice of T, $\alpha(H[X_1 \cup Y_1]) \leq |X_1|$. Moreover, $\alpha(H[X_2 \cup Y_2]) \leq |X_2|$ because M is a perfect matching of $G[X_2, Y_2]$. Then for any $S \in \mathcal{F}$, $|S \cap (X_1 \cup Y_1)| \leq |X_1|$ and $|S \cap (X_2 \cup Y_2)| \leq |X_2|$. Therefore, $|X_3| \geq |S \cap X_3| = |S| - |S \cap (X_1 \cup Y_1)| - |S \cap (X_2 \cup Y_2)| \geq |X| - |X_1| - |X_2| = |X_3|$. It follows that $|S \cap X_3| = |X_3|$. Then $X_3 \subseteq S$. Hence, $X_3 \subseteq \bigcap_{S \in \mathcal{F}} S$ by the arbitrary choice of S.

Next, suppose there exists a vertex $u \in X_3$ with $d_G(u) = d > 0$. Let $N_G(u) := \{v_1, \ldots, v_d\}$. Then $\{v_1, \ldots, v_d\} \subseteq Y_2$. Let $u_1, \ldots, u_d \in X_2$ be such that $u_i v_i \in E(M)$ for all $i \in [d]$. For each $i \in [d]$, let $M^i := (M \backslash u_i v_i) \cup \{uv_i\}$, $X_2^i := V(M^i) \cap X$ and $X_3^i := X \backslash (X_1 \cup X_2^i)$. Then $u_i \in X_3^i$ and M^i is a perfect matching of $G[X_2^i, Y_2]$. By the arbitrary choice of M, $u_i \in \bigcap_{S \in \mathcal{F}} S$. Therefore, $|\bigcap_{S \in \mathcal{F}} S| \geq |\{u_1, \ldots, u_d\} \cup X_3| \geq d + (2\alpha(G) - |G|) \geq \delta(G) + 2\alpha(G) - |G| \geq \delta(G) + 1$, as desired.

Finally, if $\bigcap_{S \in \mathcal{F}} S = \{u\}$, then $1 = |\bigcap_{S \in \mathcal{F}} S| \geq d + 2\alpha(G) - |G|$. It follows that $d = 0$ and $\alpha(G) = (|G| + 1)/2$, because $2\alpha(G) - |G| > 0$. This completes the proof of Lemma 2.2.2. ■

We are now ready to prove Theorem 1.4.8.

Proof of Theorem 1.4.8: Let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices, where $t \geq 4$ and $k \geq \max\{6, t\}$. Then $n \geq (t-1)(k-1) + 1$ and G admits a critical-coloring. Among all critical-colorings of G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a critical-coloring of G with $|E_r|$ maximum. By the choice of τ, G_r is K_t-saturated and G_b is $\mathcal{T}_k$-free. By Theorem 1.4.7(a), $\delta(G_r) \geq 2t - 4$. Let $D_1, \cdots, D_p$ be all components of G_b. By Lemma 2.1.1(a), $|D_i| \leq k - 1$ for all $i \in [p]$. Then $(t-1)(k-1) + 1 \leq n \leq p(k-1)$. This implies that $p \geq t$. Let $H := G \backslash (\bigcup_{i \in [p]} E(G[V(D_i)]))$.

Then H is a spanning subgraph of G_r. Clearly, H is K_t-free.

Assume first that $\delta(H) \geq 2t-4$. By Theorem 1.4.7(i) applied to H and $q = 2t-4$, there exists a constant $c(2t-4, k)$ such that $e(H) \geq (2t-4)n - c(2t-4, k)$. This, together with Theorem 1.4.7(g), yields that

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq (2t-4)n - c(2t-4,k) + \left(\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil - \frac{1}{2}\right)(n - (t-1)(\lceil k/2\rceil - 1)) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c(2t-4,k) - \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right) n - c_1(t,k),
\end{aligned}$$

as desired, where $c_1(t,k) = c(2t-4,k) + \frac{1}{2}(t-1)(\lceil k/2\rceil - 1)^2$.

Assume next that $\delta(H) \leq 2t-5$. Note that $k \geq \max\{6, t\} \geq t$ for all $t \geq 4$. Let $u \in V(H)$ with $d_H(u) = \delta(H)$. We may assume that $u \in V(D_p)$. Let $N_H(u) = \{u_1, \ldots, u_{\delta(H)}\}$. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p) \cup N_H(u))$, we see that $H[N_H(u)]$ must have a K_{t-2} subgraph. We may assume that $H[\{u_1, \ldots, u_{t-2}\}] = K_{t-2}$. Then we may further assume that $u_i \in V(D_{p-i})$ for all $i \in [t-2]$. Note that $H[N_H(u)]$ is K_{t-1}-free and $\omega(H[N_H(u)]) = t-2 > |N_H(u)|/2$. Let $\mathcal{F}$ be the family of all K_{t-2} subgraphs of $H[N_H(u)]$. By Theorem 1.4.7(d), $|\bigcap_{A\in\mathcal{F}} A| \leq 1$. By Lemma 2.2.2 applied to the complement of $H[N_H(u)]$, we have $|\bigcap_{A\in\mathcal{F}} A| = 1$. We may assume that $\bigcap_{A\in\mathcal{F}} A = \{u_1\}$. By Lemma 2.2.2 again, $|N_H(u)| = 2t-5$, u_1 is complete to $N_H(u)\backslash u_1$ in H and u_1 is contained in all K_{t-2} subgraphs of $H[N_H(u)]$. Then $H[N_H(u)]\backslash u_1$ is K_{t-2}-free. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p) \cup N_H(u))$, $V(H)\backslash(V(D_p) \cup N_H(u))$ must be complete to u_1 in H. Thus $\{u_1\} = V(D_{p-1})$. By Theorem 1.4.7(i) applied to H and $q = 2t-5$, there exists a constant $c(2t-5, k)$ such that $e(H) \geq$

$(2t-5)n - c(2t-5,k)$. Since $n \geq (t-1)(k-1)+1$ and $|V(D_i)| \leq k-1$ for all $i \in [p]$ with $i \neq p-1$, we see that $p \geq t$. If $p = t$, then $n = (t-1)(k-1)+1$ and $|V(D_i)| = k-1$ for $i \in [p]$ with $i \neq p-1$. In this case,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq ((2t-5)n - c(2t-5,k)) + (p-1)(k-1)(k-2)/2 \\
&= ((2t-5)n - c(2t-5,k)) + (n-1)(k-2)/2 \\
&= (2t-6+k/2)n - c(2t-5,k) - (k-2)/2 \\
&\geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\right) n - c_2(t,k)
\end{aligned}$$

for all $k \geq 6$, as desired, where $c_2(t,k) = c(2t-5,k) + (k-2)/2$.

Next assume $p \geq t+1$. Since $k \geq t$, $|N_H(u)| \leq 2t-5$, and G_r is K_t-free, by Lemma 2.1.1(b), there are at most $t-1$ many D_i's satisfying $u \notin V(D_i)$ and $D_i \backslash N_H(u) = \emptyset$. We may assume that for all $i \in [p-t]$, $D_1, \ldots, D_{p-t}$ are such that $u \notin V(D_i)$ and $D_i \backslash N_H(u) \neq \emptyset$. By Theorem 1.4.7(c), $|D_i| = k-1$ for all $i \in [p-t]$. Thus

$$\sum_{i=1}^{p} e(G[V(D_i)]) \geq (p-t)(k-1)(k-2)/2.$$

Note that $n \leq (p-1)(k-1)+1$ because $\{u_1\} = V(D_{p-1})$ and $|D_i| \leq k-1$ for all $i \in [p]$ with $i \neq p-1$. Therefore,

$$\begin{aligned}
e(G) &= e(H) + \sum_{i=1}^{p} e(G[V(D_i)]) \\
&\geq ((2t-5)n - c(2t-5,k)) + (p-t)(k-1)(k-2)/2 \\
&\geq ((2t-5)n - c(2t-5,k)) + \frac{1}{2}\left(\frac{n-1}{k-1} - t + 1\right)(k-1)(k-2)
\end{aligned}$$

$$
\begin{aligned}
&= (2t-6+k/2)n - c(2t-5,k) - (k-2)(tk-t-k+2)/2 \\
&\geq \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c(2t-5,k) - [(t-1)k^2 - (3t-4)k + 2t - 4]/2 \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\right)n - c_3(t,k)
\end{aligned}
$$

for all $k \geq 6$, as desired, where $c_3(t,k) = c(2t-5,k) + [(t-1)k^2 - (3t-4)k + 2t - 4]/2$.

Let $\ell(t,k) := \max\{c_1(t,k), c_2(t,k), c_3(t,k)\}$. This completes the proof of Theorem 1.4.8. ■

2.3 Proof of Theorem 1.4.9

Let G be a $(K_t, \mathcal{T}_k)$-co-critical graph on $n \geq (t-1)(k-1)+1$ vertices, where $t \in \{4,5,6,7\}$ and $k \geq \max\{3, 4t-14\}$. Then G admits a critical-coloring. Among all critical-colorings of G, let $\tau : E(G) \to \{\text{red, blue}\}$ be a critical-coloring of G with $|E_r|$ maximum. By the choice of τ, G_r is K_t-saturated and G_b is $\mathcal{T}_k$-free. By Theorem 1.4.7(a), $\delta(G_r) \geq 2t-4$. Let $D_1, \cdots, D_p$ be all components of G_b. By Lemma 2.1.1(a), $|D_i| \leq k-1$ for all $i \in [p]$. Then $(t-1)(k-1)+1 \leq n \leq p(k-1)$. This implies that $p \geq t$. Let $H := G \backslash (\bigcup_{i \in [p]} E(G[V(D_i)]))$. Then H is a spanning subgraph of G_r. Since $k \geq \max\{3, 4t-14\}$, by Corollary 2.1.2(c), we may assume that $t \in \{4,5\}$. By Corollary 2.1.2(a), we may further assume that $\delta(H) \leq 2t-5$. Then by Theorem 1.4.7(e), $k \geq 4$. Thus $k \geq t$. By Theorem 1.4.7(e, f), $\delta(H) = 2t-5$ because $t \in \{4,5\}$ and $\delta(H) \leq 2t-5$.

Let $u \in V(H)$ with $d_H(u) = \delta(H)$. We may assume that $u \in V(D_p)$. Let $N_H(u) = \{u_1, \ldots, u_{2t-5}\}$. By Theorem 1.4.7(b) applied to u and any vertex in $V(H) \backslash (V(D_p) \cup N_H(u))$, we see that $H[N_H(u)]$ must have a K_{t-2} subgraph. We may assume that $H[\{u_1, \ldots, u_{t-2}\}] = K_{t-2}$. Then we may further assume that $u_i \in V(D_{p-i})$ for all $i \in [t-2]$. Since $\delta(H) = 2t-5$ and $k \geq t$, by Theorem 1.4.7(d), no edge of $H[N_H(u)]$ is contained in all K_{t-2} subgraphs of

$H[N_H(u)]$. Therefore, $H[N_H(u)]$ contains two different copies of K_{t-2} subgraphs other than $H[\{u_1, \ldots, u_{t-2}\}]$. Since H is K_t-free, $H[N_H(u)]$ has no K_{t-1} subgraph. It follows that there exists a vertex, say $u_2 \in \{u_1, \ldots, u_{t-2}\}$, such that u_2 is complete to $N_H(u)\backslash u_2$ in H. Then u_2 is contained in all K_{t-2} subgraphs of $H[N_H(u)]$ and so $H[N_H(u)]\backslash u_2$ is K_{t-2}-free. By Theorem 1.4.7(b) applied to u and any vertex in $V(H)\backslash(V(D_p) \cup N_H(u))$, $V(H)\backslash(V(D_p) \cup N_H(u))$ must be complete to u_2 in H. Thus $\{u_2\} = V(D_{p-2})$. Then $p \geq t + 1$ because $n \geq (t-1)(k-1) + 2$ and $|V(D_i)| \leq k - 1$ for all $i \in [p]$ with $i \neq p - 2$.

Assume first that $t = 5$ and $k \geq 6 = 4t - 14$. By Corollary 2.1.2(b), we obtain the desired lower bound for $e(G)$. We next consider the case $t = 4$ and $k \geq 4$. In this case, $H[\{u_1, u_2, u_3\}] = P_3$ with $\{u_2\} = V(D_{p-2})$ and $u_1u_3 \notin E(H)$. We next show that $u_3 \notin V(D_{p-1})$. Suppose $u_3 \in V(D_{p-1})$. Then $\{u_1, u_3\} = V(D_{p-1})$ because $|N_H(u) \cap N_H(w)| \geq 2$ for any $w \in V(H)\backslash(V(D_p) \cup \{u_1, u_2, u_3\})$. Then $u_1u_3 \in E_b$ and we obtain a critical-coloring of $G + uv$ from τ by first coloring the edge uv red, and then recoloring u_1u_2, u_2u_3 blue and all red edges incident with u in $G[V(D_p)]$ blue, a contradiction. This proves that $u_3 \notin V(D_{p-1})$. We may assume that $u_3 \in V(D_{p-3})$. For each $\ell \in \{1, 3\}$, by Theorem 1.4.7(b) applied to u and any vertex in $V(D_{p-(4-\ell)})\backslash u_{4-\ell}$, u_ℓ must be complete to $V(D_{p-(4-\ell)})\backslash u_{4-\ell}$ in H. Note that $p \geq t + 1 = 5$. For all $i \in [p-4]$, let

$$V_i^1 := \{w \in V(D_i) \mid N_H(w) \cap N_H(u) = \{u_1, u_2\}\},$$
$$V_i^2 := \{w \in V(D_i) \mid N_H(w) \cap N_H(u) = \{u_1, u_2, u_3\}\},$$
$$V_i^3 := \{w \in V(D_i) \mid N_H(w) \cap N_H(u) = \{u_2, u_3\}\}.$$

Let $A := \bigcup_{i\in[p-4]} V_i^1 \cup V_i^2$ and $B := \bigcup_{i\in[p-4]} V_i^3 \cup V_i^2$. Since G_r is K_4-free, we see that neither $G[A]$ nor $G[B]$ has red edges. This implies that for all $i \in [p-4]$, both $V_i^1 \cup V_i^2$ and $V_i^3 \cup V_i^2$ are blue cliques in D_i. We claim that

(†) for all $i \in [p-4]$, if $V_i^1 \neq \emptyset$ and $V_i^3 \neq \emptyset$, then $V_i^2 \neq \emptyset$.

Proof. Suppose there exists an $i \in [p-4]$, say $i=1$, such that $V_1^1 \neq \emptyset$ and $V_1^3 \neq \emptyset$ but $V_1^2 = \emptyset$. Then $V_1^1 \cup V_1^3 = V(D_1)$. Let $x \in V_1^1$ and $y \in V_1^3$. Since V_1^1 is anti-complete to $(A \cup \{u_3\}) \backslash V_1^1$ in G_r, and V_1^3 is anti-complete to $(B \cup \{u_1\}) \backslash V_1^3$ in G_r, we see that $N_H(x) \cap N_H(y) \subseteq \{u_2\} \cup (V(D_p) \backslash u)$. But then we obtain a critical-coloring of $G + ux$ from τ by first coloring the edge ux red, and then recoloring u_2x blue, and all red edges in $G[V(D_p)]$ blue, and all blue edges between V_1^1 and V_1^3 red, a contradiction. ■

Let $i \in [p-4]$. By Theorem 1.4.7(c) applied to the edge uu_2, $|V(D_i)| = k-1$. Since $V_i^1 \cup V_i^2$ and $V_i^3 \cup V_i^2$ are blue cliques in D_i, by (†), $e(G_b[V(D_i)])$ is minimized when $|V_i^2| = 1$, $||V_i^1| - |V_i^3|| \leq 1$. Note that $n \leq (p-1)(k-1)+1$ because $\{u_2\} = V(D_{p-2})$ and $|D_i| \leq k-1$ for all $i \neq p-2$. It follows that

$$\begin{aligned}
|E_b| &> \sum_{i=1}^{p-4} e(G_b[V(D_i)]) \\
&= \sum_{i=1}^{p-4} \left[e(G_b[V_i^1]) + e(G_b[V_i^3]) + e_{G_b}(V_i^2, V_i^1 \cup V_i^3]) \right] \\
&\geq \left\{ \frac{1}{2} \left\lceil \frac{k-2}{2} \right\rceil \left(\left\lceil \frac{k-2}{2} \right\rceil - 1 \right) + \frac{1}{2} \left\lfloor \frac{k-2}{2} \right\rfloor \left(\left\lfloor \frac{k-2}{2} \right\rfloor - 1 \right) + k - 2 \right\} (p-4) \\
&\geq \left\{ \frac{1}{2} \left\lceil \frac{k-2}{2} \right\rceil^2 + \frac{1}{2} \left\lfloor \frac{k-2}{2} \right\rfloor^2 + \frac{k-2}{2} \right\} \left(\frac{n-1}{k-1} - 3 \right) \\
&\geq \left(\frac{1}{2} \left\lceil \frac{k}{2} \right\rceil - \frac{1}{2} \right) n - (3k^2 - 5k + 2)/4.
\end{aligned}$$

Note that G_r is K_4-saturated and $\delta(G_r) \geq 4$. By Theorem 1.5.7, there exists a constant c such that $|E_r| \geq 4n - c$. Therefore,

$$e(G) = |E_r| + |E_b| \geq \left(\frac{7}{2} + \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \right) n - (3k^2 - 5k + 2)/4 - c, \text{ as desired.}$$

This completes the proof of Theorem 1.4.9. ■

2.4 Proof of Theorem 1.4.10

Let $t \in \{4, 5\}$, $k \geq 3$ and $n \geq (2t-3)(k-1) + \lceil k/2 \rceil \lceil k/2 \rceil - 1$. We will construct a $(K_t, \mathcal{T}_k)$-co-critical graph on n vertices which yields the desired upper bound in Theorem 1.4.10.

Let r, s be the remainder and quotient of $n - (2t-3)(k-1)$ when divided by $\lceil k/2 \rceil$, and let $A := K_{k-1}$. For each $i \in [t-2]$, let $B_i := K_{k-2}$ and $C_i := K_{k-2}$. Let H_1 be obtained from disjoint copies of $A, B_1, \ldots, B_{t-2}, C_1, \ldots, C_{t-2}$ by joining every vertex in B_i to all vertices in $A \cup C_i \cup B_j$ for each $i \in [t-2]$ and all $j \in [t-2]$ with $j \neq i$. Let $H_2 := (s-r)K_{\lceil k/2 \rceil} \cup rK_{\lceil k/2 \rceil + 1}$ when $k \geq 4$, and $H_2 := sK_2 \cup rK_1$ when $k = 3$. Finally, let G be the graph obtained from $H := H_1 \cup H_2$ by adding $2t-4$ new vertices $x_1, \ldots, x_{t-2}, y_1, \ldots, y_{t-2}$, and then, for each $i \in [t-2]$, joining: x_i to every vertex in $V(H)$ and all x_j; and y_i to every vertex in $V(H) \setminus V(A)$ and all x_j, where $j \in [t-2]$ with $j \neq i$. The construction of G when $t = 4$ and $k \geq 4$ is depicted in Figure 2.1, and the construction of G when $t = 5$ and $k \geq 4$ is depicted in Figure 2.2.

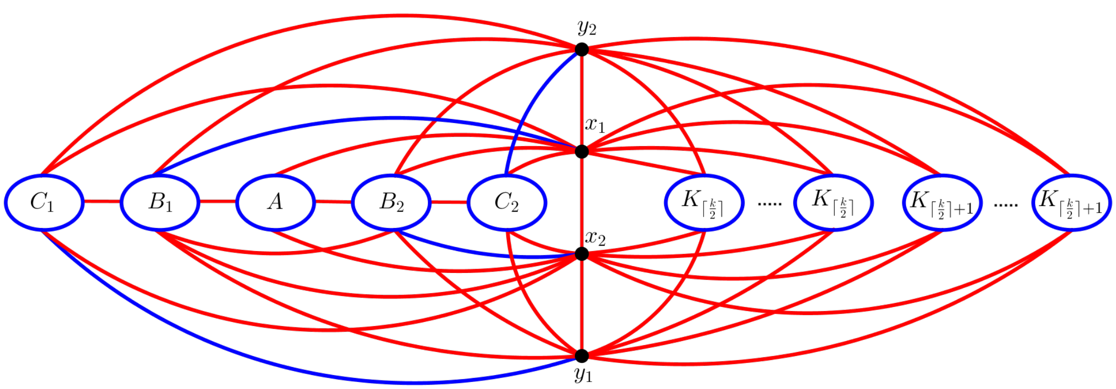


Figure 2.1: A $(K_4, \mathcal{T}_k)$-co-critical graph for all $k \geq 4$

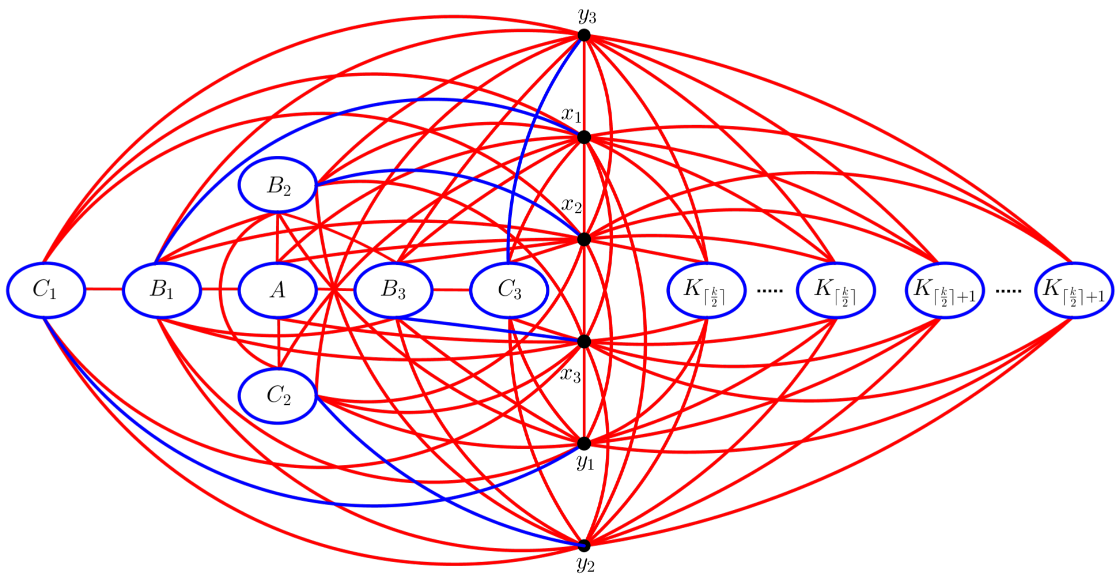


Figure 2.2: A $(K_5, \mathcal{T}_k)$-co-critical graph for all $k \geq 4$

Let $\sigma : E(G) \to \{\text{red, blue}\}$ be defined as follows: all edges in $A, B_1, \ldots, B_{t-2}, C_1, \ldots, C_{t-2}$ and H_2 are colored blue; for every $i \in [t-2]$, all edges between x_i and B_i are colored blue and all edges between y_i and C_i are colored blue; the remaining edges of G are all colored red. Note that the $\{$red, blue$\}$-coloring of G depicted in Figure 2.1 (resp. Figure 2.2) is σ when $t = 4$ (resp. $t = 5$) and $k \geq 4$. Clearly, σ is a critical-coloring of G. We next show that σ is the unique critical-coloring of G up to symmetry.

Let $X := \{x_1, \ldots, x_{t-2}\}$ and $Y := \{y_1, \ldots, y_{t-2}\}$. Let $\tau : E(G) \to \{\text{red, blue}\}$ be an arbitrary critical-coloring of G. It suffices to show that $\tau = \sigma$ upon to symmetry. Let G_r^τ and G_b^τ be G_r and G_b under the coloring τ, respectively. Note that $G[V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X] = K_{(t-1)(k-1)}$. By Lemma 2.1.1(a) and the fact that G_r^τ is K_t-free, $G_b^\tau[V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X]$ has exactly $t-1$ components, say $D_1, \ldots, D_{t-1}$, such that $V(D_i)$ is complete to $V(D_j)$ in G_r^τ for all $i, j \in [t-1]$ with $i \neq j$. Then each D_i is isomorphic to K_{k-1} in G_b^τ for all $i \in [t-1]$. Since every vertex in $V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X$ belongs to a blue K_{k-1} in G_b^τ, it follows that: for each $i \in [t-2]$, y_i is complete to $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup (X \backslash x_i)$ in G_r^τ; and $V(C_i)$ is complete to $V(B_i) \cup X$ in G_r^τ. We next prove three claims.

Claim 1. $A = D_i$ for some $i \in [t-1]$.

Proof. Suppose $A \neq D_i$ for all $i \in [t-1]$. Then for each $i \in [t-1]$, $(V(B_1) \cup \cdots \cup V(B_{t-2})) \cap V(D_i) \neq \emptyset$. Let $d_i \in (V(B_1) \cup \cdots \cup V(B_{t-2})) \cap V(D_i)$ for all $i \in [t-1]$. Then $d_1, \ldots, d_{t-1}$ are pairwise distinct, and $G_r^\tau[\{d_1, \ldots, d_{t-1}\}] = K_{t-1}$. But then $G_r^\tau[\{d_1, \ldots, d_{t-1}, y_1\}] = K_t$, because y_1 is complete to $V(B_1) \cup \cdots \cup V(B_{t-2})$ in G_r^τ, a contradiction. This proves that $A = D_i$ for some $i \in [t-1]$. ■

By Claim 1, we may assume that $A = D_{t-1}$. Then $V(A)$ is complete to $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X$ in G_r^τ. For each $i \in [t-2]$, since G_b^τ is $\mathcal{T}_k$-free, there must exist a vertex $c_i \in V(C_i)$ such that c_i is adjacent to at most one vertex of Y in G_b^τ. Then c_i is adjacent to at least $t-3$ vertices of Y in G_r^τ. We next show that

Claim 2. For each $i \in [t-2]$, $|X \cap V(D_i)| = 1$.

Proof. Suppose $|X \cap V(D_i)| \neq 1$ for some $i \in [t-2]$. Since $|X| = t-2$, we may assume that $|X \cap V(D_1)| \geq 2$ and $X \cap V(D_{t-2}) = \emptyset$. We may further assume that $x_1, x_2 \in V(D_1)$. Then $x_1x_2 \in E_b$. Since $X \cap V(D_{t-2}) = \emptyset$ and for all $i \in [t-2]$, $|V(B_i)| = k-2 < k-1 = |V(D_{t-2})|$, we may assume that $V(B_i) \cap V(D_{t-2}) \neq \emptyset$ for $i \in [2]$. Let $b_1 \in V(B_1) \cap V(D_{t-2})$. We may assume that $c_1 y_i \in E_r$ for some $i \in [2]$, because c_1 is adjacent to at least $t-3$ vertices of Y in G_r^τ. If $t = 4$, then $G_r^\tau[\{b_1, c_1, y_i, x_{3-i}\}] = K_4$, a contradiction. Thus $t = 5$. We claim that $V(B_1) \cap V(D_2) = \emptyset$ and $V(B_2) \cap V(D_2) = \emptyset$. Suppose, say $V(B_1) \cap V(D_2) \neq \emptyset$. Let $b_2 \in V(B_1) \cap V(D_2)$. Then $G_r^\tau[\{b_1, b_2, c_1, y_i, x_{3-i}\}] = K_5$, a contradiction. Thus $V(B_1) \cap V(D_2) = \emptyset$ and $V(B_2) \cap V(D_2) = \emptyset$. Then $V(D_2) = V(B_3) \cup \{x_3\}$. But then $G_r^\tau[\{b_1, c_1, y_i, x_{3-i}, x_3\}] = K_5$, a contradiction. ■

Claim 3. For each $i \in [t-2]$, $V(B_i) \subseteq V(D_j)$ for some $j \in [t-2]$.

Proof. Suppose there exists an $i \in [t-2]$ such that $V(B_i) \nsubseteq V(D_j)$ for every $j \in [t-2]$. We may assume $i = 1$. Since $V(B_1) \subseteq V(D_1) \cup \cdots \cup V(D_{t-2})$, we see that $k - 2 = |B_1| \geq 2$. Thus $k \geq 4$. We claim that $V(B_1) \cap V(D_j) = \emptyset$ for some $j \in [t-2]$. Suppose $V(B_1) \cap V(D_j) \neq \emptyset$ for all $j \in [t-2]$. Let $d_j \in V(B_1) \cap V(D_j)$ for all $j \in [t-2]$. But then $G_r^\tau[\{d_1, \ldots, d_{t-2}, c_1, y_\ell\}] = K_t$, where $c_1 y_\ell \in E_r$ for some $\ell \in [t-2]$, a contradiction. Thus $V(B_1) \cap V(D_j) = \emptyset$ for some $j \in [t-2]$, as claimed. We may assume that $V(B_1) \cap V(D_{t-2}) = \emptyset$. Since $V(B_1) \nsubseteq V(D_j)$ for every $j \in [t-2]$, it follows that $t = 5$, $V(B_1) \subseteq V(D_1) \cup V(D_2)$, and $V(B_1) \cap V(D_1) \neq \emptyset$ and $V(B_1) \cap V(D_2) \neq \emptyset$. Let $d_1 \in V(B_1) \cap V(D_1)$ and $d_2 \in V(B_1) \cap V(D_2)$. By Claim 2, let $x_i \in X \cap V(D_3)$. Then $G_r^\tau[\{d_1, d_2, x_i, c_1, y_j\}] = K_5$, where $c_1 y_j \in E_r$ for some $j \in [3]$ with $j \neq i$, a contradiction. ■

By Claim 2 and Claim 3, $V(B_i) \cup V(B_j) \nsubseteq D_\ell$ for any $i \neq j \in [t-2]$ and all $\ell \in [t-2]$. By symmetry, we may assume that $V(B_i) \subseteq V(D_i)$ for all $i \in [t-2]$. Then $V(B_i) \cup \{x_j\} = V(D_i)$ for some $j \in [t-2]$ since $|V(D_i)| = |V(B_i)| + 1$ and $V(B_1) \cup \cdots \cup V(B_{t-2}) \cup X = V(D_1) \cup \cdots \cup V(D_{t-2})$. By symmetry, we may assume that $V(B_i) \cup \{x_i\} = V(D_i)$ for all $i \in [t-2]$. It follows that for all $i, j \in [t-2]$ with $i \neq j$, B_i is complete to B_j in G_r^τ, x_i is complete to $X \backslash x_i$ and B_j in G_r^τ, y_i is complete to C_i in G_b^τ, y_i is complete to $C_j \cup (X \backslash x_i)$ in G_r^τ, x_i is complete to B_i in G_b^τ, $\{x_i, y_i\}$ is complete to H_2 in G_r^τ, all edges in $A, B_1, \ldots, B_{t-2}, C_1, \ldots, C_{t-2}$ and H_2 are colored blue under τ. This proves that $\tau = \sigma$ and thus σ is the unique critical-coloring of G upon to symmetry. It can be easily checked that adding any edge $e \in E(\overline{G})$ to G creates a red K_t if e is colored red, and a blue T_k if e is colored blue. Hence, G is $(K_t, \mathcal{T}_k)$-co-critical.

Note that $e_G(X \cup Y, V(G) \backslash (X \cup Y)) = (t-2)(n - (2t-4)) + (t-2)(n - (2t - 4 + k - 1)) = (t-2)(2n - 4t - k + 9)$; $e(G[X \cup Y]) = \binom{t-2}{2} + (t-2)(t-3)$; $e_G(V(B_1) \cup \cdots \cup V(B_{t-2}), V(C_1) \cup$

$\cdots \cup V(C_{t-2})) = (t-2)(k-2)^2$; $e(G[V(C_1) \cup \cdots \cup V(C_{t-2})]) = (t-2)\binom{k-2}{2}$; $e(G[V(A) \cup V(B_1) \cup \cdots \cup V(B_{t-2})]) = \binom{(t-2)(k-2)+k-1}{2}$. Using the facts that $s\lceil k/2 \rceil + r = n - (2t-3)(k-1)$ and $r \le \lceil k/2 \rceil - 1$, we see that

$$\begin{aligned}
e(G) &= (t-2)(2n-4t-k+9) + \binom{t-2}{2} + (t-2)(t-3) + (t-2)(k-2)^2 \\
&\quad + (t-2)\binom{k-2}{2} + \binom{(t-2)(k-2)+k-1}{2} + (s-r)\binom{\lceil k/2 \rceil}{2} + r\binom{\lceil k/2 \rceil + 1}{2} \\
&= (2t-4)n - (t-2)k - \frac{1}{2}(t-2)(5t-9) \\
&\quad + (k-2)\big((t-2)(k-2) + (t-2)(k-3)/2 + (t-1)(tk-k-2t+3)/2\big) \\
&\quad + \frac{s-r}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil - 1 \right) + \frac{r}{2}\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{k}{2} \right\rceil + 1 \right) \\
&= (2t-4)n - (t-2)k - \frac{1}{2}\big(t-2)(5t-9) + \frac{1}{2}(k-2)((t^2+t-5)k - 2t^2 - 2t + 11\big) \\
&\quad + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1 \right)\left(s\left\lceil \frac{k}{2} \right\rceil + r \right) + \frac{r}{2}\left(\left\lceil \frac{k}{2} \right\rceil + 1 \right) \\
&\le (2t-4)n + \frac{1}{2}\big((t^2+t-5)k^2 - (4t^2+6t-25)k - t^2 + 23t - 40\big) \\
&\quad + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1 \right)(n - (2t-3)(k-1)) + \frac{1}{2}\left(\left\lceil \frac{k}{2} \right\rceil - 1 \right)\left(\left\lceil \frac{k}{2} \right\rceil + 1 \right) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \right) n + \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k \\
&\quad - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left((2t-3)(k-1) - \left\lceil \frac{k}{2} \right\rceil \right) \\
&= \left(\frac{4t-9}{2} + \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \right) n + C(t,k),
\end{aligned}$$

where $C(t,k) = \frac{1}{2}(t^2+t-5)k^2 - (2t^2+2t-11)k - \frac{(t-2)(t-19)}{2} - \frac{1}{2}\left\lceil \frac{k}{2} \right\rceil \left((2t-3)(k-1) - \left\lceil \frac{k}{2} \right\rceil \right)$.

This completes the proof of Theorem 1.4.10. ■

CHAPTER 3: GALLAI-RAMSEY NUMBERS OF EVEN CYCLES AND PATHS

3.1 Proofs of Proposition 1.6.13 and Proposition 1.6.15

For all $n \geq 3$ and $k \geq 1$, let $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$, $G_i := P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$. We want to determine the exact values of $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$. By reordering colors if necessary, we assume that $i_1 \geq i_2 \geq \cdots \geq i_k$. Let $n^* := n$ when $G_{i_1} \neq P_{2n+1}$ and $n^* := n+1$ when $G_{i_1} = P_{2n+1}$. The construction for establishing a lower bound for $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$ for all $n \geq 3$ and $k \geq 1$ is similar to the construction given by Erdős, Faudree, Rousseau and Schelp in 1976 (see Section 2 in [31]) for classical Ramsey numbers of even cycles and paths. We recall their construction in the proof of Proposition 1.6.13 below (see Figure 3.1).

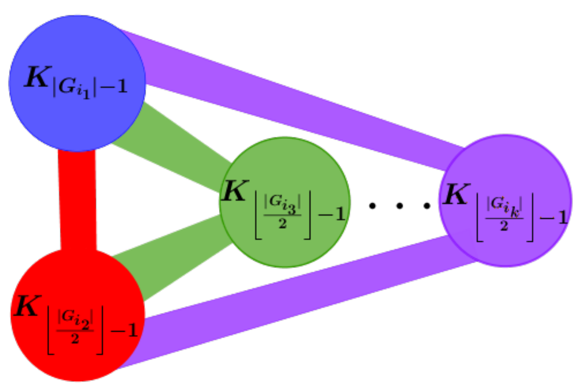


Figure 3.1: A lower bound construction for $GR(G_{i_1}, G_{i_2}, \ldots, G_{i_k})$

Proof of Proposition 1.6.13: By Theorems 1.3.3, 1.3.4 and 1.3.5, the statement is true when $k = 2$. So we may assume that $k \geq 3$. To show that $GR(G_{i_1}, \ldots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j$, we recall the construction given in [31]. Let G be a complete graph on $(|G_{i_1}| - 1) + \sum_{j=2}^{k} i_j$ vertices. Let

$V_1, \ldots, V_k$ be a partition of $V(G)$ such that $|V_1| = |G_{i_1}| - 1$ and $|V_j| = i_j$ for all $j \in \{2, 3, \ldots, k\}$. Let c be a k-edge-coloring of G by first coloring all the edges of $G[V_j]$ by color j for all $j \in [k]$, and then coloring all the edges between V_{j+1} and $\bigcup_{\ell=1}^{j} V_\ell$ by color $j+1$ for all $j \in [k-1]$. Then G contains neither a rainbow triangle nor a monochromatic copy of G_{i_j} in color j for all $j \in [k]$ under c. Hence, $GR(G_{i_1}, \ldots, G_{i_k}) \geq |G| + 1 = |G_{i_1}| + \sum_{j=2}^{k} i_j$, as desired. ∎

Proof of Proposition 1.6.15: By the assumed truth of Conjecture 1.6.14 for $G_{n-1} = C_{2n}$, we may assume that $G_{i_1} = P_{2n+1}$. Then $i_1 = n-1$. We may further assume that $n - 1 = i_1 = \cdots = i_t > i_{t+1} \geq \cdots \geq i_k$, where $t \in [k]$. By Proposition 1.6.13, $GR(G_{i_1}, \ldots, G_{i_k}) \geq (2n+1) + \sum_{j=2}^{k} i_j = 2 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$. We next show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq 2 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$.

Let G be a complete graph on $2 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$ vertices and let $c : E(G) \to [k]$ be any Gallai coloring of G. Suppose G does not contain a monochromatic copy of G_{i_j} in color j for all $j \in [k]$. By the assumed truth of Conjecture 1.6.14 for $G_{n-1} = C_{2n}$, $GR(C_{2n}, \ldots, C_{2n}, G_{i_{t+1}}, \ldots, G_{i_k}) = 2n + (t-1)(n-1) + \sum_{j=t+1}^{k} i_j = 1 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$. Thus G must contain a monochromatic copy of $H := C_{2n}$ in some color $\ell \in [t]$ under c. We may assume that $\ell = 1$. Then for every vertex $u \in V(G) \backslash V(H)$, all the edges between u and $V(H)$ must be colored by exactly one color j for some $j \in \{2, \ldots, k\}$, because G contains neither a rainbow triangle nor a monochromatic copy of P_{2n+1} in color 1 under c. Thus, $V(G) \backslash V(H)$ can be partitioned into $V_2, V_3, \ldots, V_k$ such that all the edges between V_j and $V(H)$ are colored by color j for all $j \in \{2, \ldots, k\}$. It follows that for all $j \in \{2, \ldots, k\}$, $|V_j| \leq i_j$, because G does not contain a monochromatic copy of G_{i_j} in color j. But then $|G| = |H| + \sum_{j=2}^{k} |V_j| \leq 2n + \sum_{j=2}^{k} i_j = 1 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$, contrary to $|G| = 2 + n + t(n-1) + \sum_{j=t+1}^{k} i_j$. ∎

3.2 Proof of Theorem 1.6.16

In this section, we prove Theorem 1.6.16 which shows that Conjecture 1.6.14 is true for $n \in \{3, 4\}$ and all $k \geq 2$.

Proof. Let $n \in \{3, 4\}$ and $k \geq 2$. By Proposition 1.6.13, it suffices to show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq |G_{i_1}| + \sum_{j=2}^{k} i_j$.

By Theorem 1.3.3, Theorem 1.3.4 and Theorem 1.3.5, $GR(G_{i_1}, G_{i_2}) = R(G_{i_1}, G_{i_2}) = |G_{i_1}| + i_2$. We may assume that $k \geq 3$. Let $N := |G_{i_1}| + \sum_{j=2}^{k} i_j$. Since $GR_k(P_3) = 3$, we may assume that $i_1 \geq 1$ and so $N \geq 2i_1 + 3 \geq 5$. Let G be a complete graph on N vertices and let $c : E(G) \to [k]$ be any Gallai coloring of G such that all the edges of G are colored by at least three colors under c. We next show that G contains a monochromatic copy of G_{i_j} in color j for some $j \in [k]$. Suppose G contains no monochromatic copy of G_{i_j} in color j for any $j \in [k]$ under c. Such a Gallai k-coloring c is called a *bad coloring*. Among all complete graphs on N vertices with a bad coloring, we choose G with N minimum.

Consider a Gallai partition of G with parts $A_1, \ldots, A_p$, where $p \geq 2$. We may assume that $|A_1| \geq \cdots \geq |A_p| \geq 1$. Let $\mathcal{R}$ be the reduced graph of G with vertices $a_1, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.6.2, we may assume that every edge of $\mathcal{R}$ is colored either red or blue. Since all the edges of G are colored by at least three colors under c, we see that $\mathcal{R} \neq G$ and so $|A_1| \geq 2$. By abusing the notation, we use i_b to denote i_j when the color j is blue. Similarly, we use i_r (resp. i_g) to denote i_j when the color j is red (resp. green). Let

$$A_r := \{a_j \in \{a_2, \ldots, a_p\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\} \text{ and}$$

$$A_b := \{a_i \in \{a_2, \ldots, a_p\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\}.$$

Let $R := \bigcup_{a_j \in A_r} A_j$ and $B := \bigcup_{a_i \in A_b} A_i$. Then $|A_1|+|R|+|B| = |G| = N$ and $\max\{|B|, |R|\} \neq 0$ because $p \geq 2$. Thus G contains a blue P_3 between B and A_1 or a red P_3 between R and A_1, and so $\max\{i_b, i_r\} \geq 1$. We next prove several claims.

Claim 1. Let $r \in [k]$ and let $s_1, \ldots, s_r$ be nonnegative integers with $s_1 + \cdots + s_r \geq 1$. If $i_{j_1} \geq s_1, \ldots, i_{j_r} \geq s_r$ for colors $j_1, j_2, \ldots, j_r \in [k]$, then for any $S \subseteq V(G)$ with $|S| \geq N - (s_1 + \cdots + s_r)$, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$, where $i^*_{j_q} = i_{j_q} - s_q$.

Proof. Let $i^*_{j_1} := i_{j_1} - s_1, \ldots, i^*_{j_r} := i_{j_r} - s_r$, and $i^*_j := i_j$ for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$. Let $i^*_\ell := \max\{i^*_j : j \in [k]\}$. Then $i^*_\ell \leq i_1$. Let $N^* := |G_{i^*_\ell}| + [(\sum_{j=1}^k i^*_j) - i^*_\ell]$. Then $N^* \geq 3$ and $N^* \leq N - (s_1 + \cdots + s_r) < N$ because $s_1 + \cdots + s_r \geq 1$. Since $|S| \geq N - (s_1 + \cdots + s_r) \geq N^*$ and $G[S]$ does not have a monochromatic copy of G_{i_j} in color j for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$ under c, by minimality of N, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$. ■

Claim 2. $|A_1| \leq n-1$ and so G does not contain a monochromatic copy of a graph on $|A_1|+1 \leq n$ vertices in any color $m \in [k]$ that is neither red nor blue.

Proof. Suppose $|A_1| \geq n$. We first claim that $i_b \geq |B|$ and $i_r \geq |R|$. Suppose $i_b \leq |B| - 1$ or $i_r \leq |R| - 1$. Then we obtain a blue G_{i_b} using the edges between B and A_1 or a red G_{i_r} using the edges between R and A_1, a contradiction. Thus $i_b \geq |B|$ and $i_r \geq |R|$, as claimed. Let $i^*_b := i_b - |B|$ and $i^*_r := i_r - |R|$. Since $|A_1| = N - |B| - |R|$, by Claim 1 applied to $i_b \geq |B|$, $i_r \geq |R|$ and A_1, $G[A_1]$ must have a blue $G_{i^*_b}$ or a red $G_{i^*_r}$, say the latter. Then $i_r > i^*_r$. Thus

$|R| > 0$ and $G_{i_r^*}$ is a red path on $2i_r^* + 3$ vertices. Note that

$$
\begin{aligned}
|A_1| &= |G_{i_1}| + \sum_{j=2}^{k} i_j - |B| - |R| \\
&\geq \begin{cases} |G_{i_r}| + i_b - |B| - |R| & \text{if } \ i_r \geq i_b \\ |G_{i_b}| + i_r - |B| - |R| & \text{if } \ i_r < i_b, \end{cases} \\
&\geq \begin{cases} |G_{i_r}| + i_b^* - |R| & \text{if } \ i_r \geq i_b \\ 2i_b + 2 + i_r - |B| - |R| \geq i_b^* + (2i_r + 3) - |R| & \text{if } \ i_r < i_b, \end{cases} \\
&\geq |G_{i_r}| - |R|.
\end{aligned}
$$

Then

$$
\begin{aligned}
|A_1| - |G_{i_r^*}| &\geq |G_{i_r}| - |G_{i_r^*}| - |R| \\
&= \begin{cases} (3 + 2i_r) - (3 + 2i_r^*) - |R| = |R| & \text{if } i_r \leq n - 2 \\ (2 + 2i_r) - (3 + 2i_r^*) - |R| = |R| - 1 & \text{if } i_r = n - 1. \end{cases}
\end{aligned}
$$

But then $G[A_1 \cup R]$ contains a red G_{i_r} using the edges of the $G_{i_r^*}$ and the edges between $A_1 \backslash V(G_{i_r^*})$ and R, a contradiction. This proves that $|A_1| \leq n - 1$. Next, let $m \in [k]$ be any color that is neither red nor blue. Suppose G contains a monochromatic copy of a graph, say J, on $|A_1| + 1$ vertices in color m. Then $V(J) \subseteq A_\ell$ for some $\ell \in [p]$. But then $|A_\ell| \geq |A_1| + 1$, contrary to $|A_1| \geq |A_\ell|$. ■

For two disjoint sets $U, W \subseteq V(G)$, we say U is *blue-complete* (resp. *red-complete*) to W if all the edges between U and W are colored blue (resp. red) under c. For convenience, we say u is *blue-complete* (resp. *red-complete*) to W when $U = \{u\}$.

Claim 3. $\min\{|B|, |R|\} \geq 1$, $p \geq 3$ and B is neither red- nor blue-complete to R under c.

Proof. Suppose $B = \emptyset$ or $R = \emptyset$. By symmetry, we may assume that $R = \emptyset$. Then $B \neq \emptyset$ and so $i_b \geq 1$. By Claim 2, $|A_1| \leq n - 1 \leq 3$ because $n \in \{3, 4\}$. Then $|A_1| \leq i_b + 2$. If $i_b \leq |A_1| - 1$, then $i_b \leq n - 2$ by Claim 2. Thus G_{i_b} is a blue path on $2i_b + 3$ and so

$$|B| = N - |A_1| \geq |G_{i_b}| - |A_1| = \begin{cases} i_b + 1 & \text{if } |A_1| = i_b + 2 \\ i_b + 2 & \text{if } |A_1| = i_b + 1. \end{cases}$$

But then we obtain a blue G_{i_b} using the edges between B and A_1. Thus $i_b \geq |A_1|$. Let $i_b^* := i_b - |A_1|$. By Claim 1 applied to $i_b \geq |A_1|$ and B, $G[B]$ must have a blue $G_{i_b^*}$. Since

$$|B| - |G_{i_b^*}| \geq |G_{i_b}| - |G_{i_b^*}| - |A_1| = \begin{cases} (3 + 2i_b) - (3 + 2i_b^*) - |A_1| = |A_1| & \text{if } i_b \leq n - 2 \\ (2 + 2i_b) - (3 + 2i_b^*) - |A_1| = |A_1| - 1 & \text{if } i_b = n - 1, \end{cases}$$

we see that G contains a blue G_{i_b} using the edges of the $G_{i_b^*}$ and the edges between $B \backslash V(G_{i_b^*})$ and A_1, a contradiction. Hence $R \neq \emptyset$ and so $p \geq 3$ for any Gallai partition of G. It follows that B is neither red- nor blue-complete to R, otherwise $\{B, R \cup A_1\}$ or $\{B \cup A_1, R\}$ yields a Gallai partition of G with only two parts. ∎

Claim 4. Let $m \in [k]$ be a color that is neither red nor blue. Then $i_m \leq 1$. In particular, if $i_m = 1$, then $n = 4$ and G contains a monochromatic copy of P_3 in color m under c.

Proof. By Claim 2, G contains no monochromatic copy of P_n in color m under c. Suppose $i_m \geq 1$. Let $i_m^* := i_m - 1$. By Claim 1 applied to $i_m \geq 1$ and $V(G)$, G must have a monochromatic copy of $G_{i_m^*}$ in color m under c. Since $n \in \{3, 4\}$ and G contains no monochromatic copy of P_n in color m, we see that $n = 4$ and $i_m^* = 0$. Thus $i_m = 1$ and G contains a monochromatic copy of P_3 in color m under c. ∎

By Claim 3, $B \neq \emptyset$ and $R \neq \emptyset$. Since $|A_1| \geq 2$, we see that G has a blue P_3 using edges between B and A_1, and a red P_3 using edges between R and A_1. Thus $i_b \geq 1$ and $i_r \geq 1$. Then $|G_{i_1}| \geq 5$ and so $N = |G_{i_1}| + \sum_{j=2}^{k} i_j \geq 6$. By Claim 2, $|A_1| \leq n-1$. If $|B| = |R| = 1$, then $N = |A_1| + |B| + |R| \leq n+1 \leq 5$, a contradiction. Thus $|B| \geq 2$ or $|R| \geq 2$. Since B is neither red- nor blue-complete to R, we see that G contains either a blue P_5 or a red P_5. Thus $i_1 \geq \max\{i_b, i_r\} \geq 2 \geq n-2$ because $n \in \{3,4\}$. By Claim 4, we may assume that $\{i_b, i_r\} = \{i_1, i_2\}$. Then

$$|G_{i_1}| = \begin{cases} 2i_1 + 2 = 1 + n + i_1 & \text{if } i_1 = n-1 \\ 2i_1 + 3 = 1 + n + i_1 & \text{if } i_1 = n-2. \end{cases}$$

Therefore $N = |G_{i_1}| + \sum_{j=2}^{k} i_j = 1 + n + \sum_{j=1}^{k} i_j \geq 1 + n + i_b + i_r$.

Claim 5. $|B| \leq n-1$ or $|R| \leq n-1$.

Proof. Suppose $|B| \geq n$ and $|R| \geq n$. Let $H = (B, R)$ be the complete bipartite graph obtained from $G[B \cup R]$ by deleting all the edges with both ends in B or both ends in R. Then H has no blue P_{2n-3} with both ends in B, else, we obtain a blue C_{2n} because $|A_1| \geq 2$. Similarly, H has no red P_{2n-3} with both ends in R. For every vertex $v \in B \cup R$, let $d_b(v) := |\{u : uv \text{ is colored blue in } H\}|$ and $d_r(v) := |\{u : uv \text{ is colored red in } H\}|$. Let $x_1, \ldots, x_n \in B$, $y_1, \ldots, y_n \in R$ and $a_1, a_1^* \in A_1$ be all distinct. We next claim that $d_r(v) \leq n-2$ for all $v \in B$. Suppose, say, $d_r(x_1) \geq n-1$. Then $n = 4$ because H has no red P_{2n-3} with both ends in R. We may assume that x_1 is red-complete to $\{y_1, y_2, y_3\}$. Since H has no red P_5 with both ends in R, we see that for all $i \in \{2,3,4\}$ and every $W \subseteq \{y_1, y_2, y_3\}$ with $|W| = 2$, no x_i is red-complete to W. We may further assume that x_2y_1, x_2y_2, x_3y_1 are colored blue. Then x_4y_2 must be colored red, else, H has a blue P_5 with vertices x_3, y_1, x_2, y_2, x_4 in order. Thus x_4y_1, x_4y_3 are colored blue. But then H has a blue P_5 with vertices x_2, y_2, x_3, y_1, x_4 in order (when x_3y_2 is colored blue) or vertices x_2, y_1, x_3, y_3, x_4 in order (when x_3y_3 is colored blue), a contradiction. Thus $d_r(v) \leq n-2$ for all $v \in B$. Similarly, $d_b(u) \leq n-2$ for all $u \in R$. Then $|B||R| = |E(H)| = \sum_{v\in B} d_r(v) + \sum_{u\in R} d_b(u) \leq (n-2)|B| + (n-2)|R|$. Using inequality of arithmetic and geometric means, we obtain that $n = 4$, $|B| = |R| = 4$ and $d_r(v) = d_b(v) = 2$ for each $v \in B \cup R$. Thus the set of all the blue edges in H induces a 2-regular spanning subgraph of H. Since H has no blue C_8, we see that H must contain two vertex-disjoint copies of blue C_4. We may assume that y_1 is blue-complete to $\{x_1, x_2\}$ and y_2 is blue-complete to $\{x_3, x_4\}$. But then G contains a blue C_8 with vertices $a_1, x_1, y_1, x_2, a_1^*, x_3, y_2, x_4$ in order, a contradiction. ■

Claim 6. $|A_1| = 3$ and $n = 4$.

Proof. By Claim 2, $|A_1| \leq n-1 \leq 3$ because $n \in \{3, 4\}$. Note that $|A_1| = 3$ only when $n = 4$. Suppose $|A_1| = 2$. By Claim 2, G has no monochromatic copy of P_3 in color j for any $j \in \{3, \ldots, k\}$ under c. By Claim 4, $i_3 = \cdots = i_k = 0$ and so $N = 1+n+\sum_{j=1}^{k} i_j = 1+n+i_b+i_r$. We may assume that $A_1, \ldots, A_t$ are all the parts of order two in the Gallai partition $A_1, \ldots, A_p$ of G, where $t \in [p]$. Let $A_i := \{a_i, b_i\}$ for all $i \in [t]$. By reordering if necessary, each of $A_1, \ldots, A_t$ can be chosen as the largest part in the Gallai partition $A_1, \ldots, A_p$ of G. For all $i \in [t]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored blue in } \mathcal{R}\} \text{ and}$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_ja_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Then $|B^i| + |R^i| = N - |A_1| = n+i_b+i_r-1 \geq n+2$, because $\max\{i_b, i_r\} \geq 2$ and $\min\{i_b, i_r\} \geq 1$. Since each of $A_1, \ldots, A_t$ †can be chosen as the largest part in the Gallai partition $A_1, \ldots, A_p$ of G, † by Claim 5, either $|B^i| \leq n-1$ or $|R^i| \leq n-1$ for all $i \in [t]$. We claim that $|B^i| \neq |R^i|$ for all $i \in [t]$. Suppose $|B^i| = |R^i|$ for some $i \in [t]$. By Claim 5, $n+2 \leq |B^i| + |R^i| \leq 2(n-1) \leq 6$. It follows that $|B^i| = |R^i| = 3$ and $n = 4$. Thus G

has a blue P_5 between B^i and A_i and a red P_5 between R^i and A_i. It follows that $\min\{i_b, i_r\} \geq 2$. But then $|B^i| + |R^i| = n + i_b + i_r - 1 \geq 7$, a contradiction. This proves that $|B^i| \neq |R^i|$ for all $i \in [t]$. Let

$$E_B := \{a_i b_i \mid i \in [t] \text{ and } |R^i| < |B^i|\} \text{ and } E_R := \{a_i b_i \mid i \in [t] \text{ and } |R^i| > |B^i|\}.$$

We next apply the recoloring method. Let c^* be an edge-coloring of G obtained from c by recoloring all the edges in E_B blue and all the edges in E_R red. Then every edge of G is colored either red or blue under c^*. Since $|G| = 1 + n + i_b + i_r \geq R(G_{i_b}, G_{i_r})$ by Theorem 1.3.3, Theorem 1.3.4 and Theorem 1.3.5, we see that G must contain a blue G_{i_b} or a red G_{i_r} under c^*. By symmetry, we may assume that G has a blue $H := G_{i_b}$ under c^*. Then H contains no edges of E_R but must contain at least one edge of E_B, else, we obtain a blue G_{i_b} in G under c. We choose H so that $|E(H) \cap E_B|$ is minimal. We may further assume that $a_1 b_1 \in E(H)$. By the choice of c^*, $|R^1| \leq n - 1$ and $|R^1| < |B^1|$. Then $|B^1| \geq 2$ and so G has a blue P_5 under c because B^1 is not red-complete to R^1. Thus $i_b \geq 2$. Let $W := V(G) \backslash V(H)$.

We next claim that $i_b = n - 1$. Suppose $2 \leq i_b \leq n - 2$. Then $n = 4$, $i_b = 2$, $H = P_7$ and $|G| = 1 + n + i_b + i_r = 7 + i_r$. Thus $|W| = i_r$. Let $x_1, \ldots, x_7$ be the vertices of H in order. By symmetry, we may assume that $x_\ell x_{\ell+1} = a_1 b_1$ for some $\ell \in [3]$. Then $W \cup \{x_7\}$ must be red-complete to $\{a_1, b_1\}$ under c, else, say a vertex $u \in W \cup \{x_7\}$, is blue-complete to $\{a_1, b_1\}$ under c, then we obtain a blue $H' := P_7$ under c^* with vertices $x_1, \ldots, x_\ell, u, x_{\ell+1}, \ldots, x_6$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, contrary to the choice of H. Thus $W \cup \{x_7\} \subseteq R^1$ and so $|R^1| \geq |W \cup \{x_7\}| = i_r + 1 \geq 2$. Note that G contains a red P_5 under c because $|R^1| \geq 2$ and R^1 is not blue-complete to B^1. Thus $i_r \geq 2$. Then $3 \leq i_r + 1 \leq |R^1| \leq 3$, which implies that $i_r = 2$ and $R^1 = W \cup \{x_7\}$. Thus $\{a_1, b_1\}$ is blue-complete to $V(H) \backslash \{x_\ell, x_{\ell+1}, x_7\}$. But then we obtain a blue $H' := P_7$ under c^* with vertices $x_1, \ldots, x_\ell, x_{\ell+2}, x_{\ell+1}, x_{\ell+3}, \ldots, x_7$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. This proves that $i_b = n-1$.

Since $i_b = n-1$, we see that $H = C_{2n}$. Then $|G| = 1+n+i_b+i_r = 2n+i_r$ and so $|W| = i_r$. Let $a_1, x_1, \ldots, x_{2n-2}, b_1$ be the vertices of H in order and let $W = V(G)\backslash V(H) := \{w_1, \ldots, w_{i_r}\}$. Then x_1b_1 and a_1x_{2n-2} are colored blue under c because $\{a_1, b_1\} = A_1$. Suppose $\{x_j, x_{j+1}\}$ is blue-complete to $\{a_1, b_1\}$ under c for some $j \in [2n-3]$. Then G has a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, \ldots, x_j, b_1, x_{2n-2}, \ldots, x_{j+1}$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, contrary to the choice of H. Thus, for all $j \in [2n-3]$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$. Since $\{x_1, x_{2n-2}\}$ is blue-complete to $\{a_1, b_1\}$ under c, we see that $x_2, x_{2n-3} \in R^1$ and then $|R^1 \cap \{x_2, \ldots, x_{2n-3}\}| = |R^1| = n-1$. Thus $R^1 = \{x_2, x_3\}$ when $n = 3$. By symmetry, we may assume that $R^1 = \{x_2, x_3, x_5\}$ when $n = 4$. Then $W \subseteq B^1$. Thus R^1 is red-complete to $\{a_1, b_1\}$ and W is blue-complete to $\{a_1, b_1\}$ under c. It follows that for any $w_j \in W$ and $x_m \in R^1$, $\{x_m, w_j\} \neq A_i$ for all $i \in [t]$. Then x_2 must be red-complete to W under c, else, say x_2w_1 is colored blue under c, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, x_2, w_1, b_1, x_4$ (when $n = 3$) and vertices $a_1, x_1, x_2, w_1, b_1, x_4, x_5, x_6$ (when $n = 4$) in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Similarly, x_3 is red-complete to W under c, else, say x_3w_1 is colored blue under c, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $b_1, x_4, x_3, w_1, a_1, x_1$ (when $n = 3$) and vertices $b_1, x_6, x_5, x_4, x_3, w_1, a_1, x_1$ (when $n = 4$) in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Thus $\{x_2, x_3\}$ is red-complete to W under c. Then for any $w_j \in W$, $\{x_1, w_j\} \neq A_i$ for all $i \in [t]$ since x_2x_1 is colored blue and x_2 is red-complete to W under c. If x_1w_j is colored blue under c for some $w_j \in W$, then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, w_j, x_1, \ldots, x_{2n-2}$ in order such that $|E(H') \cap E_B| < |E(H) \cap E_B|$, a contradiction. Thus $\{x_1, x_2, x_3\}$ is red-complete to W under c. Then $|W| = i_r \geq 2$ because G contains a red P_5 under c with vertices x_1, w_1, x_2, a_1, x_3 in order. But then we obtain a red C_{2n} under c with vertices $a_1, x_2, w_1, x_1, w_2, x_3$ in order (when $n = 3$) and $a_1, x_2, w_1, x_1, w_2, x_3, b_1, x_5$ in order (when $n = 4$), a contradiction. ■

By Claim 6, $|A_1| = 3$ and $n = 4$. Then $|B \cup R| = N - |A_1| \geq 2 + i_b + i_r \geq 5$ because $\max\{i_b, i_r\} \geq 2$ and $\min\{i_b, i_r\} \geq 1$. By symmetry, we may assume that $|B| \geq |R|$. Then $|B| \geq 3$ and so G has a blue P_7 because $|A_1| = 3$ and B is not red-complete to R. Thus $i_b = 3$. By Claim 5, $|R| \leq 3$. Then $i_r \geq |R|$, else, we obtain a red G_{i_r} because $|A_1| = 3$ and R is not blue-complete to B. Then $|B| \geq 2 + i_b + i_r - |R| \geq 5$. Thus $G[B \cup R]$ has no blue P_3 with both ends in B, else, we obtain a blue C_8 because $|A_1| = 3$ and $|B| \geq 5$. Let $i_b^* := 0$ and $i_r^* := i_r - |R| \leq 2$. By Claim 1 applied to $i_b = |A_1|$, $i_r \geq |R|$ and B, $G[B]$ must contain a red $P_{2i_r^*+3}$ with vertices, say $x_1, \ldots, x_{2i_r^*+3}$, in order. Let $R := \{y_1, \ldots, y_{|R|}\}$. Then no $y_j \in R$ is blue-complete to any $W \subseteq B$ with $|W| = 2$, in particular, when $W = \{x_1, x_{2i_r^*+3}\}$, because $G[B \cup R]$ has no blue P_3 with both ends in B. We may assume that $x_1 y_1$ is colored red. Note that $G[R \cup A_1]$ has a red $P_{2|R|}$ with y_1 as an end. Then $G[\{x_1, \ldots, x_{2i_r^*+3}\} \cup R \cup A_1]$ has a red P_{2i_r+3}. It follows that $i_r = 3$. Let $a_1^* \in A_1 \backslash \{a_1\}$.

Suppose first that $x_{2i_r^*+3}$ is blue-complete to $R = \{y_1, \ldots, y_{|R|}\}$. Since $G[B \cup R]$ has no blue P_3 with both ends in B, we see that $\{x_{2i_r^*+3}\} = A_\ell$ for some $\ell \in [p]$, $B \backslash \{x_{2i_r^*+3}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$, and $x_{2i_r^*+3}$ is adjacent to at most one vertex, say $w \in B$, such that $w x_{2i_r^*+3}$ is colored blue. Thus $x_{2i_r^*+3}$ is red-complete to $B \backslash \{w, x_{2i_r^*+3}\}$. Let $w^* \in B \backslash \{x_1, x_2, x_3, w\}$. Since $B \backslash \{x_{2i_r^*+3}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$, we see that $\{x_1, \ldots, x_{2i_r^*+2}\}$ is red-complete to $\{y_1, \ldots, y_{|R|}\}$. If $w \notin \{x_2, \ldots, x_{2i_r^*+1}\}$, then we obtain a red C_8 with vertices $y_1, x_1, x_2, x_7, x_3, \ldots, x_6$ (when $i_r^* = 2$), vertices $a_1, y_1, x_1, x_2, x_5, x_3, x_4, y_2$ (when $i_r^* = 1$), and vertices $a_1, y_1, x_2, x_3, w^*, y_2, a_1^*, y_3$ (when $i_r^* = 0$) in order, a contradiction. Thus $w \in \{x_2, \ldots, x_{2i_r^*+1}\}$. Then $i_r^* \geq 1$ and $x_1 x_{2i_r^*+1}$ is colored red. But then we obtain a red C_8 with vertices $y_1, x_2, x_3, x_4, x_5, x_6, x_7, x_1$ (when $i_r^* = 2$) and vertices $a_1, y_1, x_2, x_3, x_4, x_5, x_1, y_2$ (when $i_r^* = 1$) in order, a contradiction. This proves that $x_{2i_r^*+3}$ is not blue-complete to R. Then $|R| \geq 2$, else, $|R| = 1$, $i_r^* = 2$ and $x_7 y_1$ is colored red, which yields a red C_8 with vertices $y_1, x_1, \ldots, x_7$ in order, a contradiction. Thus $i_r^* \leq 1$. Next, suppose $x_{2i_r^*+3}$ is not blue-complete to $\{y_2, \ldots, y_{|R|}\}$, say $x_{2i_r^*+3}y_2$ is colored red. By assumption, x_1y_1 is red. We then obtain a red C_8 with vertices $a_1, y_1, x_1, \ldots, x_5, y_2$ (when $i_r^* = 1$) and vertices $a_1, y_1, x_1, x_2, x_3, y_2, a_1^*, y_3$ (when $i_r^* = 0$) in order, a contradiction. Thus $x_{2i_r^*+3}$ is blue-complete to $\{y_2, \ldots, y_{|R|}\}$ and so $x_{2i_r^*+3}y_1$ is colored red. By symmetry of x_1 and $x_{2i_r^*+3}$, x_1 must be blue-complete to $\{y_2, \ldots, y_{|R|}\}$. But then $G[B \cup R]$ has a blue P_3 with vertices $x_1, y_2, x_{2i_r^*+3}$ in order, a contradiction.

This completes the proof of Theorem 1.6.16. ∎

3.3 Proof of Theorem 1.6.17

In this section, we continue to establish more evidence for Conjecture 1.6.14. We prove Theorem 1.6.17 which shows that Conjecture 1.6.14 holds for $n \in \{5, 6\}$ and all $k \geq 2$.

Proof. Let $n \in \{5, 6\}$ and $k \geq 2$. By Proposition 1.6.13, it suffices to show that it suffices to show that $GR(G_{i_1}, \ldots, G_{i_k}) \leq |G_{i_1}| + \sum_{j=2}^{k} i_j$.

By Theorem 1.6.16 and Proposition 1.6.15, we may assume that $i_1 = n-1$. Then $|G_{i_1}| + \sum_{j=2}^{k} i_j = n + 1 + \sum_{j=1}^{k} i_j$. By Theorem 1.3.3 and Theorem 1.3.4, $GR(G_{i_1}, G_{i_2}) = R(G_{i_1}, G_{i_2}) = 1 + n + i_1 + i_2$. We may assume that $k \geq 3$. Let $N := |G_{i_1}| + \sum_{j=2}^{k} i_j$. Then $N \geq 2n$. Let G be a complete graph on N vertices and let $c : E(G) \to [k]$ be any Gallai coloring of G such that all the edges of G are colored by at least three colors under c. We next show that G contains a monochromatic copy of G_{i_j} in color j for some $j \in [k]$. Suppose G contains no monochromatic copy of G_{i_j} in color j for any $j \in [k]$ under c. Such a Gallai k-coloring c is called a *critical-coloring*. Among all complete graphs on N vertices with a critical-coloring, we choose G with N minimum.

Consider a Gallai-partition of G with parts $A_1, \ldots, A_p$, where $p \geq 2$. We may assume that $|A_1| \geq$

$\cdots \geq |A_p| \geq 1$. Let $\mathcal{R}$ be the reduced graph of G with vertices $a_1, \ldots, a_p$, where $a_i \in A_i$ for all $i \in [p]$. By Theorem 1.6.2, we may assume that the edges of $\mathcal{R}$ are colored either red or blue. Since all the edges of G are colored by at least three colors under c, we see that $\mathcal{R} \neq G$ and so $|A_1| \geq 2$. By abusing the notation, we use i_b to denote i_j when the color j is blue. Similarly, we use i_r (resp. i_g) to denote i_j when the color j is red (resp. green). Let

$$A_b := \{a_i \in \{a_2, \ldots, a_p\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\} \text{ and}$$
$$A_r := \{a_j \in \{a_2, \ldots, a_p\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\}.$$

Then $|A_b| + |A_r| = p - 1$. Let $B := \bigcup_{a_i \in A_b} A_i$ and $R := \bigcup_{a_j \in A_r} A_j$. Then $\max\{|B|, |R|\} \neq 0$ because $p \geq 2$. Thus G contains a blue P_3 between B and A_1 or a red P_3 between R and A_1, and so $\max\{i_b, i_r\} \geq 1$. We next prove several claims.

Claim 7. Let $r \in [k]$ and let $s_1, \ldots, s_r$ be nonnegative integers with $s_1 + \cdots + s_r \geq 1$. If $i_{j_1} \geq s_1, \ldots, i_{j_r} \geq s_r$ for colors $j_1, \ldots, j_r \in [k]$, then for any $S \subseteq V(G)$ with $|S| \geq |G| - (s_1 + \cdots + s_r)$, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$, where $i^*_{j_q} = i_{j_q} - s_q$.

Proof. Let $i^*_{j_1} := i_{j_1} - s_1, \ldots, i^*_{j_r} := i_{j_r} - s_r$, and $i^*_j := i_j$ for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$. Let $i^*_\ell := \max\{i^*_j : j \in [k]\}$. Then $i^*_\ell \leq i_1$. Let $N^* := |G_{i^*_\ell}| + [(\sum_{j=1}^k i^*_j) - i^*_\ell]$. Then $N^* \geq 3$ and $N^* \leq N - (s_1 + \cdots + s_r) < N$ because $s_1 + \cdots + s_r \geq 1$. Since $|S| \geq N - (s_1 + \cdots + s_r) \geq N^*$ and $G[S]$ does not have a monochromatic copy of G_{i_j} in color j for all $j \in [k] \backslash \{j_1, \ldots, j_r\}$ under c, by minimality of N, $G[S]$ must contain a monochromatic copy of $G_{i^*_{j_q}}$ in color j_q for some $j_q \in \{j_1, \ldots, j_r\}$. ■

Claim 8. $|A_1| \leq n - 1$ and so G does not contain a monochromatic copy of a graph on $|A_1| + 1 \leq n$ vertices in color m, where $m \in [k]$ is a color that is neither red nor blue.

Proof. Suppose $|A_1| \geq n$. We first claim that $i_b \geq |B|$ and $i_r \geq |R|$. Suppose $i_b \leq |B| - 1$ or $i_r \leq |R| - 1$. Then we obtain a blue G_{i_b} using the edges between B and A_1 or a red G_{i_r} using the edges between R and A_1, a contradiction. Thus $i_b \geq |B|$ and $i_r \geq |R|$, as claimed. Let $i_b^* := i_b - |B|$ and $i_r^* := i_r - |R|$. Since $|A_1| = N - |B| - |R|$, by Claim 7 applied to $i_b \geq |B|$, $i_r \geq |R|$ and A_1, $G[A_1]$ must have a blue $G_{i_b^*}$ or a red $G_{i_r^*}$, say the latter. Then $i_r > i_r^*$. Thus $|R| > 0$ and $G_{i_r^*}$ is a red path on $2i_r^* + 3$ vertices. Note that

$$
\begin{aligned}
|A_1| &= |G_{i_1}| + \sum_{j=2}^{k} i_j - |B| - |R| \\
&\geq \begin{cases} |G_{i_r}| + i_b - |B| - |R| & \text{if } i_r \geq i_b \\ |G_{i_b}| + i_r - |B| - |R| & \text{if } i_r < i_b, \end{cases} \\
&\geq \begin{cases} |G_{i_r}| + i_b^* - |R| & \text{if } i_r \geq i_b \\ 2i_b + 2 + i_r - |B| - |R| \geq i_b^* + (2i_r + 3) - |R| & \text{if } i_r < i_b, \end{cases} \\
&\geq |G_{i_r}| - |R|.
\end{aligned}
$$

Then

$$
\begin{aligned}
|A_1| - |G_{i_r^*}| \geq |G_{i_r}| - |G_{i_r^*}| - |R| \\
= \begin{cases} (3 + 2i_r) - (3 + 2i_r^*) - |R| = |R| & \text{if } i_r \leq n - 2 \\ (2 + 2i_r) - (3 + 2i_r^*) - |R| = |R| - 1 & \text{if } i_r = n - 1. \end{cases}
\end{aligned}
$$

But then $G[A_1 \cup R]$ contains a red G_{i_r} using the edges of the $G_{i_r^*}$ and the edges between $A_1 \backslash V(G_{i_r^*})$ and R, a contradiction. This proves that $|A_1| \leq n - 1$. Next, let $m \in [k]$ be any color that is neither red nor blue. Suppose G contains a monochromatic copy of a graph, say J, on $|A_1| + 1$ vertices in color m. Then $V(J) \subseteq A_\ell$ for some $\ell \in [p]$. But then $|A_\ell| \geq |A_1| + 1$, contrary to $|A_1| \geq |A_\ell|$. ■

For two disjoint sets $U, W \subseteq V(G)$, we say U is *blue-complete* (resp. *red-complete*) to W if all the edges between U and W are colored blue (resp. red) under c. For convenience, we say u is *blue-complete* (resp. *red-complete*) to W when $U = \{u\}$.

Claim 9. $\min\{|B|, |R|\} \geq 1$, $p \geq 3$, and B is neither red- nor blue-complete to R under c.

Proof. Suppose $B = \emptyset$ or $R = \emptyset$. By symmetry, we may assume that $R = \emptyset$. Then $B \neq \emptyset$ and so $i_b \geq 1$. By Claim 8, $|A_1| \leq n-1$. Thus $|B| = |G| - |A_1| = N - |A_1| \geq n+1+i_b - |A_1| \geq i_b + 2$. If $i_b \leq |A_1| - 1$, then $i_b \leq n-2$ by Claim 8. But then we obtain a blue G_{i_b} using the edges between B and A_1. Thus $i_b \geq |A_1|$. Let $i_b^* = i_b - |A_1|$. By Claim 7 applied to $i_b \geq |A_1|$ and B, $G[B]$ must have a blue $G_{i_b^*}$. Since

$$|B| - |G_{i_b^*}| \geq |G_{i_b}| - |G_{i_b^*}| - |A_1| = \begin{cases} (3+2i_b) - (3+2i_b^*) - |A_1| = |A_1| & \text{if } i_b \leq n-2 \\ (2+2i_b) - (3+2i_b^*) - |A_1| = |A_1| - 1 & \text{if } i_b = n-1, \end{cases}$$

we see that G contains a blue G_{i_b} using the edges of the $G_{i_b^*}$ and the edges between $B \backslash V(G_{i_b^*})$ and A_1, a contradiction. Hence $R \neq \emptyset$ and so $p \geq 3$ for any Gallai-partition of G. It follows that B is neither red- nor blue-complete to R, otherwise $\{B \cup A_1, R\}$ or $\{B, R \cup A_1\}$ yields a Gallai-partition of G with only two parts. ∎

Claim 10. Let $m \in [k]$ be the color that is neither red nor blue. Then $i_m \leq n-4$. In particular, if $i_m \geq 1$, then G contains a monochromatic copy of P_{2i_m+1} in color m under c.

Proof. By Claim 8, $|A_1| \leq n-1$ and G contains no monochromatic copy of $P_{|A_1|+1}$ in color m under c. Suppose $i_m \geq 1$. Let $i_m^* := i_m - 1$. By Claim 7 applied to $i_m \geq 1$ and $V(G)$, G must have a monochromatic copy of $G_{i_m^*}$ in color m under c. Since $n \in \{5, 6\}$, $|A_1| \leq n-1$ and G

contains no monochromatic copy of $P_{|A_1|+1}$ in color m, we see that $i_m^* \le n-5$. Thus $i_m \le n-4$ and G contains a monochromatic copy of P_{2i_m+1} in color m under c if $i_m \ge 1$. ∎

By Claim 9, $B \neq \emptyset$ and $R \neq \emptyset$. Since $|A_1| \ge 2$, we see that G has a blue P_3 using edges between B and A_1, and a red P_3 using edges between R and A_1. Thus $i_b \ge 1$ and $i_r \ge 1$. By Claim 10, $\max\{i_b, i_r\} = i_1 = n-1$. Then $N = 1 + n + \sum_{i=1}^{k} i_j \ge 1 + n + i_b + i_r \ge 2n+1$. For the remainder of the proof of Theorem 1.6.17, we choose $p \ge 3$ to be as large as possible.

Claim 11. If $|A_1| \ge n-3$, then $|B| \le n-1$ or $|R| \le n-1$.

Proof. Suppose $|A_1| \ge n-3$ but $|B| \ge n$ and $|R| \ge n$. By symmetry, we may assume that $|B| \ge |R| \ge n$. Let $B := \{x_1, x_2, \ldots, x_{|B|}\}$ and $R := \{y_1, y_2, \ldots, y_{|R|}\}$. Let $H := (B, R)$ be the complete bipartite graph obtained from $G[B \cup R]$ by deleting all the edges with both ends in B or in R. Then H has no blue P_7 with both ends in B and no red P_7 with both ends in R, else we obtain a blue C_{2n} or a red C_{2n} because $|A_1| \ge n-3$. We next show that H has no red $K_{3,3}$.

Suppose H has a red $K_{3,3}$. We may assume that $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is a red $K_{3,3}$ under c. Since H has no red P_7 with both ends in R, $\{y_4, \ldots, y_{|R|}\}$ must be blue-complete to $\{x_1, x_2, x_3\}$. Thus $H[\{x_1, x_2, x_3, y_4, y_5\}]$ has a blue P_5 with both ends in $\{x_1, x_2, x_3\}$ and $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ has a red P_5 with both ends in $\{y_1, y_2, y_3\}$. If $|A_1| \ge n-2$ or $\min\{i_b, i_r\} \le n-2$, then we obtain a blue G_{i_b} or a red G_{i_r}, a contradiction. It follows that $|A_1| = n-3$ and $i_b = i_r = n-1$. Thus $|B \cup R| \ge 1 + n + i_b + i_r - |A_1| = 2n+2$. If $|R| \ge 6$, then $\{y_4, y_5, y_6\}$ must be red-complete to $\{x_4, x_5, x_6\}$, else H has a blue P_7 with both ends in B. But then we obtain a red C_{2n} in G. Thus $|R| = 5$, $n = 5$, and so $|B| \ge 7$. Let $a_1, a_1^* \in A_1$. For each $j \in \{4, 5, 6, 7\}$ and every $W \subseteq \{x_1, x_2, x_3\}$ with $|W| = 2$, no x_j is red-complete to W under c, else, say, x_4 is red-complete to $\{x_1, x_2\}$, then we obtain a red C_{10} with vertices $a_1, y_1, x_1, x_4, x_2, y_2, x_3, y_3, a_1^*, y_4$ in order, a contradiction. We may assume that x_4x_1, x_5x_2 are colored blue. But then we obtain a blue C_{10} with vertices $a_1, x_4, x_1, y_4, x_3, y_5, x_2, x_5, a_1^*, x_6$ in order, a contradiction. This proves that H has no red $K_{3,3}$.

Let $X := \{x_1, x_2, \ldots, x_5\}$ and $Y := \{y_1, y_2, \ldots, y_5\}$. Let H_b and H_r be the spanning subgraphs of $H[X \cup Y]$ induced by all the blue edges and red edges of $H[X \cup Y]$ under c, respectively. By the Pigeonhole Principle, there exist at least three vertices, say x_1, x_2, x_3, in X such that either $d_{H_b}(x_i) \geq 3$ for all $i \in [3]$ or $d_{H_r}(x_i) \geq 3$ for all $i \in [3]$. Suppose $d_{H_r}(x_i) \geq 3$ for all $i \in [3]$. We may assume that x_1 is red-complete to $\{y_1, y_2, y_3\}$. Since $|Y| = 5$ and H has no red P_7 with both ends in R, we see that $N_{H_r}(x_1) = N_{H_r}(x_2) = N_{H_r}(x_3) = \{y_1, y_2, y_3\}$. But then $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is a red $K_{3,3}$, contrary to H has no red $K_{3,3}$. Thus $d_{H_b}(x_i) \geq 3$ for all $i \in [3]$. Since $|Y| = 5$, we see that any two of x_1, x_2, x_3 have a common neighbor in H_b. Furthermore, two of x_1, x_2, x_3, say x_1, x_2, have at least two common neighbors in H_b. It can be easily checked that H has a blue P_5 with ends in $\{x_1, x_2, x_3\}$, and there exist three vertices, say y_1, y_2, y_3, in Y such that y_ix_i is blue for all $i \in [3]$ and $\{x_4, \ldots, x_{|B|}\}$ is red-complete to $\{y_1, y_2, y_3\}$. Then H has a blue P_5 with both ends in $\{x_1, x_2, x_3\}$ and a red P_5 with both ends in $\{y_1, y_2, y_3\}$. If $|A_1| \geq n-2$ or $\min\{i_b, i_r\} \leq n-2$, then we obtain a blue G_{i_b} or a red G_{i_r}, a contradiction. It follows that $|A_1| = n-3$ and $i_b = i_r = n-1$. Thus $|B \cup R| \geq 1 + n + i_b + i_r - |A_1| = 2n+2$. Then $|B| \geq n+1$ and so $H[\{x_4, x_5, x_6, y_1, y_2, y_3\}]$ is a red $K_{3,3}$, contrary to the fact that H has no red $K_{3,3}$. ■

Claim 12. $|A_1| \geq 3$.

Proof. Suppose $|A_1| = 2$. Then G has no monochromatic copy of P_3 in color j for any $j \in \{3, \ldots, k\}$ under c. By Claim 10, $i_3 = \cdots = i_k = 0$. We may assume that $|A_1| = \cdots = |A_t| = 2$ and $|A_{t+1}| = \cdots = |A_p| = 1$ for some integer t satisfying $p \geq t \geq 1$. Let $A_i = \{a_i, b_i\}$ for all $i \in [t]$. By reordering if necessary, each of $A_1, \ldots, A_t$ can be chosen as the largest part in the Gallai-partition $A_1, A_2, \ldots, A_p$ of G. For all $i \in [t]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored blue in } \mathcal{R}\} \text{ and}$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Then $|B^i|+|R^i| = 1+n+i_b+i_r = 2n-2+\min\{i_b, i_r\}$. Let

$$E_B := \{a_i b_i \mid i \in [t] \text{ and } |R^i| < |B^i|\},$$
$$E_R := \{a_i b_i \mid i \in [t] \text{ and } |B^i| < |R^i|\},$$
$$E_Q := \{a_i b_i \mid i \in [t] \text{ and } |B^i| = |R^i|\}.$$

Let c^* be obtained from c by recoloring all the edges in E_B blue, all the edges in E_R red and all the edges in E_Q either red or blue. Then all the edges of G are colored red or blue under c^*. Since $|G| = n + 1 + i_b + i_r = R(G_{i_b}, G_{i_r})$ by Theorem 1.3.3 and Theorem 1.3.4, we see that G must contain a blue G_{i_b} or a red G_{i_r} under c^*. By symmetry, we may assume that G has a blue $H := G_{i_b}$. Then H contains no edges of E_R but must contain at least one edge of $E_B \cup E_Q$, else we obtain a blue G_{i_b} in G under c. We choose H so that $|E(H) \cap (E_B \cup E_Q)|$ is minimal. We may further assume that $a_1 b_1 \in E(H)$. Since $|B^1| + |R^1| = 2n - 2 + \min\{i_b, i_r\}$, by the choice of c^*, $|B^1| \geq n - 1 \geq 4$ and $|R^1| \leq n - 1 + \lfloor \frac{\min\{i_b, i_r\}}{2} \rfloor \leq 7$. So $i_b \geq 2$. By Claim 11, $|R^1| \leq 4$ when $n = 5$. Let $W := V(G) \backslash V(H)$.

We next claim that $i_b = n-1$. Suppose $i_b \leq n-2$. Then $H = P_{2i_b+3}$, $i_r = n-1$, $|G| = 2n+i_b$ and $|W| = 2n - 3 - i_b \geq n - 1$. Let $x_1, x_2, \ldots, x_{2i_b+3}$ be the vertices of H in order. We may assume that $x_\ell x_{\ell+1} = a_1 b_1$ for some $\ell \in [2i_b + 2]$. If a vertex $w \in W$ is blue-complete to $\{a_1, b_1\}$, then we obtain a blue $H' := G_{i_b}$ under c^* with vertices $x_1, \ldots, x_\ell, w, x_{\ell+1}, \ldots, x_{2i_b+2}$ in order (when

$\ell \neq 2i_b + 2$) or $x_1, x_2, \ldots, x_{2i_b+2}, w$ in order (when $\ell = 2i_b + 2$) such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus no vertex in W is blue-complete to $\{a_1, b_1\}$ under c and so W must be red-complete to $\{a_1, b_1\}$ under c. This proves that $W \subseteq R^1$. We next claim that $\ell = 1$ or $\ell = 2i_b + 2$. Suppose $\ell \in \{2, \ldots, 2i_b + 1\}$. Then $\{x_1, x_{2i_b+3}\}$ must be red-complete to $\{a_1, b_1\}$, else, we obtain a blue $H' := G_{i_b}$ with vertices $x_\ell, \ldots, x_1, x_{\ell+1}, \ldots, x_{2i_b+3}$ or $x_1, \ldots, x_\ell, x_{2i_b+3}, x_{\ell+1}, \ldots, x_{2i_b+2}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$. Thus $\{x_1, x_{2i_b+3}\} \subseteq R^1$ and so $W \cup \{x_1, x_{2i_b+3}\}$ is red-complete to $\{a_1, b_1\}$. If $n = 5$, then $4 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 6$, a contradiction. Thus $n = 6$ and $7 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 7$. It follows that $R^1 \cap V(H) = \{x_1, x_{2i_b+3}\}$ and thus either $\{x_{\ell-2}, x_{\ell-1}\}$ or $\{x_{\ell+2}, x_{\ell+3}\}$ is blue-complete to $\{a_1, b_1\}$. In either case, we obtain a blue $H' := G_{i_b}$ under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, a contradiction. This proves that $\ell = 1$ or $\ell = 2i_b + 2$. By symmetry, we may assume that $\ell = 1$. Then x_1x_3 is colored blue under c because $A_1 = \{a_1, b_1\}$. Similarly, for all $j \in \{3, \ldots, 2i_b + 2\}$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$, else we obtain a blue $H' := G_{i_b}$ with vertices $x_1, x_j, \ldots, x_2, x_{j+1}, \ldots, x_{2i_b+3}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$. It follows that $x_4 \in R^1$ and so $|R^1 \cap \{x_4, \ldots, x_{2i_b+3}\}| \geq i_b$. Then $|R^1| \geq |W| + |R^1 \cap \{x_4, \ldots, x_{2i_b+3}\}| \geq 2n - 3$, so $4 \geq |R^1| \geq 7$ (when $n = 5$) or $7 \geq |R^1| \geq 9$ (when $n = 6$), a contradiction. This proves that $i_b = n - 1$.

Since $i_b = n - 1$, we see that $H = C_{2n}$. Then $|G| = 2n + i_r$ and so $|W| = i_r$. Let $a_1, x_1, \ldots, x_{2n-2}, b_1$ be the vertices of H in order and let $W := \{w_1, \ldots, w_{i_r}\}$. Then x_1b_1 and a_1x_{2n-2} are colored blue under c because $A_1 = \{a_1, b_1\}$. Suppose $\{x_j, x_{j+1}\}$ is blue-complete to $\{a_1, b_1\}$ for some $j \in [2n - 3]$. We then obtain a blue $H' := C_{2n}$ with vertices $a_1, x_1, \ldots, x_j, b_1, x_{2n-2}, \ldots, x_{j+1}$ in order under c^* such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus, for all $j \in [2n - 3]$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$. Since $\{x_1, x_{2n-2}\}$ is blue-complete to $\{a_1, b_1\}$ under c, we see that $x_2, x_{2n-3} \in R^1$, and so $4 \geq |R^1 \cap V(H)| \geq 4$ (when $n = 5$) and $5 + \lfloor \frac{i_r}{2} \rfloor \geq |R^1 \cap V(H)| \geq 5$ (when $n = 6$). Thus, when

$n = 5$, we have $R^1 = \{x_2, x_4, x_5, x_7\}$ or $R^1 = \{x_2, x_4, x_6, x_7\}$, as depicted in Figure 3.2(a) and Figure 3.2(b); when $n = 6$, we have $R^1 \cap V(H) = \{x_2, x_9\} \cup \{x_j : j \in J\}$, where $J \in \{\{4, 6, 8\}, \{4, 6, 7\}, \{3, 4, 6, 7\}, \{3, 5, 6, 7\}, \{4, 5, 6, 7\}, \{4, 6, 7, 8\}, \{3, 5, 7, 8\}, \{3, 5, 6, 8\}, \{3, 4, 5, 6, 7\}, \{3, 4, 5, 6, 8\}, \{3, 4, 5, 7, 8\}\}$.

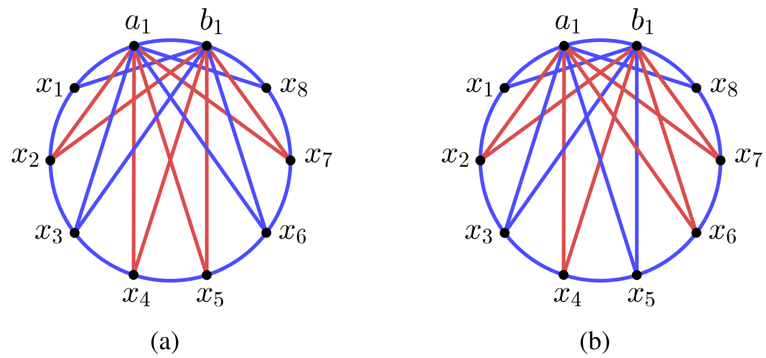


Figure 3.2: Two cases of R^1 when $i_b = 4$ and $n = 5$

Since $|R^1| \geq n - 1$ and R^1 is red-complete to $\{a_1, b_1\}$ under c, we see that $i_r \geq 2$. Let $W' := W \backslash R^1 \subset B^1$. It follows that $|W'| = i_r - |R^1 \backslash V(H)| \geq \lceil \frac{i_r}{2} \rceil \geq 1$. We may assume $W' = \{w_1, \ldots, w_{|W'|}\}$. We claim that $E(H) \cap (E_B \cup E_Q) = \{a_1 b_1\}$. Suppose, say $a_2 b_2 \in E(H) \cap (E_B \cup E_Q)$. Since $\{x_1, x_2\} \neq A_i$ and $\{x_{2n-3}, x_{2n-2}\} \neq A_i$ for all $i \in [t]$, we may assume that $a_2 = x_j$ and $b_2 = x_{j+1}$ for some $j \in \{2, \ldots, 2n - 4\}$. Then $x_{j-1} x_{j+1}$ and $x_j x_{j+2}$ are colored blue under c. But then we obtain a blue $H' := C_{2n}$ under c^* with vertices $a_1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n-2}, b_1, w_1$ in order such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of H. Thus $E(H) \cap (E_B \cup E_Q) = \{a_1 b_1\}$, as claimed.

($*$) Let $w \in W'$. For $j \in \{1, 2n - 2\}$, if $\{x_j, w\} \neq A_i$ for all $i \in [t]$, then $x_j w$ is colored red. For $j \in \{2, \ldots, 2n - 3\}$, if $\{x_j, w\} \neq A_i$ for all $i \in [t]$ and x_{j-2} or $x_{j+2} \in B^1$, then $x_j w$ is colored red.

Proof. Suppose there are some $j \in [2n - 2]$ such that $\{x_j, w\} \neq A_i$ for all $i \in [t]$, and x_{j-2} or

$x_{j+2} \in B^1$ if $j \in \{2, \ldots, 2n-3\}$, but $x_j w$ is colored blue. Then we obtain a blue C_{2n} under c with vertices $a_1, w, x_1, \ldots, x_{2n-2}$ (when $j = 1$) or $a_1, x_1, \ldots, x_{2n-2}, w$ (when $j = 2n-2$) in order if $j \in \{1, 2n-2\}$, and with vertices $b_1, x_{2n-2}, x_{2n-3}, \cdots, x_{j+2}, a_1, w, x_j, \cdots, x_1$ in order (when $x_{j+2} \in B^1$) or $a_1, x_1, \cdots, x_{j-2}, b_1, w, x_j, \cdots, x_{2n-2}$ in order (when $x_{j-2} \in B^1$) if $j \in \{2, \ldots, 2n-3\}$, a contradiction. ■

($**$) For $j \in [2n-4]$, $x_j x_{j+2}$ is colored red if $\{x_j, x_{j+2}\} \neq A_i$ for all $i \in [t]$.

Proof. Suppose $x_j x_{j+2}$ is colored blue for some $j \in [2n-4]$. Then we obtain a blue C_{2n} with vertices $a_1, x_1, \ldots, x_j, x_{j+2}, \ldots, x_{2n-2}, b_1, w$ in order, a contradiction, where $w \in W'$. ■

First if $n = 5$, then $W' = W$. Let $(\alpha, \beta) \in \{(5,7), (7,6)\}$. Suppose $R^1 = \{x_2, x_4, x_\alpha, x_\beta\}$. Since $\{x_{\alpha-1}, w_j\} \neq A_i$ and $\{x_\alpha, w_j\} \neq A_i$ for all $w_j \in W$ and $i \in [t]$, $x_{\alpha+1}, x_{\alpha-2} \in B^1$, by ($*$), $\{x_{\alpha-1}, x_\alpha\}$ must be red-complete to W under c. Then for any $w_j \in W$, $\{x_{\alpha-2}, w_j\} \neq A_i$ and $\{x_{\alpha+1}, w_j\} \neq A_i$ for all $i \in [t]$ since $x_{\alpha-1}x_{\alpha-2}$ and $x_\alpha x_{\alpha+1}$ are colored blue under c. Thus $\{x_{\alpha-2}, x_{\alpha+1}\}$ is red-complete to W by ($*$). So $\{x_{\alpha-2}, x_{\alpha-1}, x_\alpha, x_{\alpha+1}\}$ is red-complete to W under c. But then we obtain a red P_9 under c (when $i_r \leq 3$) with vertices $x_2, a_1, x_{\alpha-1}, b_1, x_\alpha, w_1, x_{\alpha-2}, w_2, x_{\alpha+1}$ in order or a red C_{10} under c (when $i_r = 4$) with vertices $a_1, x_2, b_1, x_{\alpha-1}, w_1, x_{\alpha-2}, w_2, x_{\alpha+1}, w_3, x_\alpha$ in order, a contradiction. This proves that $n = 6$. By ($*$), we may assume x_1 is red-complete to $W' \backslash w_1$ and x_{10} is red-complete to $W' \backslash w_{|W'|}$ because $|A_1| = 2$.

Case 1. $|R^1 \cap V(H)| = 5$. Let $(\alpha, \beta) \in \{(9,8), (7,9)\}$. Suppose $R^1 = \{x_2, x_4, x_6, x_\alpha, x_\beta\}$. Since $\{x_{\alpha-1}, w_j\} \neq A_i$ and $\{x_\alpha, w_j\} \neq A_i$ for all $w_j \in W'$ and $i \in [t]$, $x_{\alpha+1}, x_{\alpha-2} \in B^1$, $\{x_{\alpha-1}, x_\alpha\}$ must be red-complete to W' under c by ($*$). Then for any $w_j \in W'$, $\{x_{\alpha-2}, w_j\} \neq A_i$ and $\{x_{\alpha+1}, w_j\} \neq A_i$ for all $i \in [t]$ since $x_{\alpha-1}x_{\alpha-2}$ and $x_\alpha x_{\alpha+1}$ are colored blue under c. Thus

$\{x_{\alpha-2}, x_{\alpha+1}\}$ is red-complete to W' by $(*)$. So $\{x_{\alpha-2}, x_{\alpha-1}, x_\alpha, x_{\alpha+1}\}$ is red-complete to W' under c. We see that G has a red P_7 with vertices $x_{\alpha-1}, w_1, x_\alpha, a_1, x_2, b_1, x_4$ in order, and so $i_r \geq 3$ and $|W'| \geq 2$. Moreover, $x_{\alpha-1}x_{\alpha+1}$ and $x_{\alpha-2}x_\alpha$ are colored red by $(**)$. Then G has a red P_{11} with vertices $x_1, w_2, x_{\alpha-1}, x_{\alpha+1}, w_1, x_{\alpha-2}, x_\alpha, a_1, x_2, b_1, x_4$ in order under c. Thus $i_r = 5$ and so $|W'| \geq 3$. Since $|A_1| = 2$ and $x_{\alpha-6} \in B^1$, by $(*)$, we may assume $x_{\alpha-4}$ is red-complete to $W' \backslash w_2$. But then we obtain a red C_{12} with vertices $a_1, x_\alpha, x_{\alpha-2}, w_1, x_{\alpha-4}, w_3, x_1, w_2, x_{\alpha+1}, x_{\alpha-1}, b_1, x_2$ in order under c, a contradiction.

Case 2. $|R^1 \cap V(H)| = 6$, then $i_r \geq 3$ and $|W'| \geq 3$. Let $(\alpha, \beta, \gamma) \in \{(5,2,4), (4,7,5)\}$. Suppose $R^1 \cap V(H) = \{x_2, x_3, x_\alpha, x_6, x_7, x_9\}$. Since $\{x_\beta, w_j\} \neq A_i$, $\{x_3, w_j\} \neq A_i$ and $\{x_6, w_j\} \neq A_i$ for all $w_j \in W'$ and $i \in [t]$, by $(*)$, $\{x_\beta, x_3, x_6\}$ must be red-complete to W' under c. By $(**)$, x_γ is red-complete to $\{x_{\gamma-2}, x_{\gamma+2}\}$. But then we obtain a red C_{12} under c with vertices $a_1, x_2, x_4, x_6, w_1, x_{10}, w_2, x_1, w_3, x_3, b_1, x_5$ (when $\alpha = 5$) or $a_1, x_3, x_5, x_7, w_1, x_{10}, w_2, x_1, w_3, x_6, b_1, x_4$ (when $\alpha = 4$) in order, a contradiction. Let $(\alpha, \beta, \gamma, \delta) \in \{(3,8,5,6), (3,5,7,8), (4,6,8,2)\}$. Suppose $R^1 \cap V(H) = V(H) \backslash \{a_1, b_1, x_1, x_{10}, x_\alpha, x_\beta\}$. Since $\{x_\gamma, w\} \neq A_i$ and $\{x_\delta, w\} \neq A_i$ for all $w \in W'$ and $i \in [t]$, $\{x_\gamma, x_\delta\}$ must be red-complete to W' under c by $(*)$. Moreover, $x_\gamma x_{\gamma-2}$ and $x_\delta x_{\delta+2}$ are colored red by $(**)$. Since $|A_1| = 2$, there exists at least one of $x_1, x_{10}, x_\alpha, x_\beta$ is red-complete to $\{w_1, w_2, w_3\}$ by $(*)$. So we may assume x_α is red-complete to $W' \backslash w_2$ and x_β is red-complete to $\{w_1, w_2, w_3\}$. But then we obtain a red C_{12} with vertices $a_1, x_\gamma, x_{\gamma-2}, w_1, x_{10}, w_2, x_1, w_3, x_{\delta+2}, x_\delta, b_1, x_7$ in order if $(\alpha, \beta, \gamma, \delta) \in \{(3,8,5,6), (4,6,8,2)\}$ and $a_1, x_7, x_5, w_1, x_3, w_3, x_1, w_2, x_{10}, x_8, b_1, x_6$ in order if $(\alpha, \beta, \gamma, \delta) = (3,5,7,8)$, a contradiction. Finally if $R^1 \cap V(H) = \{x_2, x_3, x_5, x_6, x_8, x_9\}$. By $(*)$, $R^1 \cap V(H)$ is red-complete to W'. Then G has a red P_{11} with vertices $x_2, a_1, x_3, b_1, x_5, w_1, x_6, w_2, x_8, w_3, x_9$ in order. Thus $i_r = 5$ and so $|W'| \geq 4$. But then we obtain a red C_{12} with vertices $a_1, x_2, w_1, x_3, w_2, x_5, w_3, x_6, w_4, x_8, b_1, x_9$ in order, a contradiction.

Case 3. $|R^1 \cap V(H)| = 7$, then $i_r \geq 4$ and $|W'| = |W| = i_r$. Let $(\alpha, \beta) \in \{(6,5),(7,4)\}$. Suppose $R^1 \cap V(H)$=$\{x_2, x_3, x_4, x_5, x_\alpha, x_8, x_9\}$. Since $\{x_3, w_j\} \neq A_i$, $\{x_\beta, w_j\} \neq A_i$ and $\{x_8, w_j\} \neq A_i$ for all $i \in [t]$ and any $w_j \in W'$, $\{x_3, x_\beta, x_8\}$ must be red-complete to W' under c by $(*)$. But then we obtain a red C_{12} with vertices $a_1, x_3, w_1, x_{10}, w_2, x_1, w_3, x_\beta, w_4, x_8, b_1, x_2$ in order, a contradiction. Finally if $R^1 \cap V(H) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$. Since $\{x_3, w_j\} \neq A_i$ and $\{x_6, w_j\} \neq A_i$ for all $i \in [t]$ and any $w_j \in W'$, $\{x_3, x_6\}$ must be red-complete to W' under c by $(*)$. We may assume x_8 is red-complete to $W' \backslash w_2$ by $(*)$. But then we obtain a red C_{12} with vertices $a_1, x_3, w_1, x_{10}, w_2, x_1, w_3, x_8, w_4, x_6, b_1, x_2$ in order, a contradiction. This proves that $|A_1| \geq 3$. ■

Claim 13. For any A_i with $3 \leq |A_i| \leq 4$, $G[A_i]$ has a monochromatic copy of P_3 in some color $m \in [k]$ other than red and blue.

Proof. Suppose there exists a part A_i with $3 \leq |A_i| \leq 4$ but $G[A_i]$ has no monochromatic copy of P_3 in any color $m \in [k]$ other than red and blue. We may assume $i = 1$. Since $GR_k(P_3) = 3$, we see that $G[A_1]$ must contain a red or blue P_3, say blue. We may assume a_i, b_i, c_i are the vertices of the blue P_3 in order. Then $|A_1| = 4$, else $\{b_1\}, \{a_1, c_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p+1$ parts. Let $z_1 \in A_1 \backslash \{a_1, b_1, c_1, \}$. Then z_1 is not blue-complete to $\{a_1, c_1\}$, else $\{a_1, c_1\}, \{b_1, z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p+1$ parts. Moreover, $b_1 z_1$ is not colored blue, else $\{b_1\}, \{a_1, c_1, z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p+1$ parts. If $b_1 z_1$ is colored red, then $a_1 z_1$ and $c_1 z_1$ are colored either red or blue because G has no rainbow triangle. Similarly, z_1 is not red-complete to $\{a_1, c_1\}$, else $\{z_1\}, \{a_1, b_1, c_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p+1$ parts. Thus, by symmetry, we may assume $a_1 z_1$ is colored blue and $c_1 z_1$ is colored red, and so $a_1 c_1$ is colored blue or red because G has no rainbow triangle. But then $\{a_1\}, \{b_1\}, \{c_1\}, \{z_1\}, A_2, \ldots, A_p$ is a Gallai partition of G with $p+3$ parts, a contradiction. Thus $b_1 z_1$ is colored neither red nor blue. But then $a_1 z_1$ and $c_1 z_1$ must be colored blue because $G[A_1]$

has neither rainbow triangle nor monochromatic P_3 in any color $m \in [k]$ other than red and blue, a contradiction. ∎

For the remainder of the proof of Theorem 1.6.17, we assume that $|B| \geq |R|$. By Claim 11, $|R| \leq n-1$. Let $\{a_i, b_i, c_i\} \subseteq A_i$ if $|A_i| \geq 3$ for any $i \in [p]$. Let $B := \{x_1, \ldots, x_{|B|}\}$ and $R := \{y_1, \ldots, y_{|R|}\}$. We next show that

Claim 14. $i_r \geq |R|$.

Proof. Suppose $i_r \leq |R| - 1 \leq n-2$. Then $i_b = n-1$, $i_r \geq 3$, $|A_1| \leq 4$, else we obtain a red G_{i_r} because R is not blue-complete to B and $|A_1| \geq 3$. Moreover, there exist two edges, say x_1y_1, x_2y_2, that are colored red, else we obtain a blue C_{2n}. Then $G[A_1 \cup R \cup \{x_1, x_2\}]$ has a red P_9, it follows that $n = 6$, $i_r = 4$ and $|R| = 5$. By Claim 13, $G[A_1]$ has a monochromatic, say green, copy of P_3. By Claim 10, $i_g = 1$. Then $|A_1 \cup B| = |G| - |R| \geq 7 + i_b + i_r + i_g - |R| = 12$, and so $G[B]$ has no blue $G_{i_b - |A_1|}$, else we obtain a blue C_{12}. Let $i_b^* := i_b - |A_1| \leq 2$, $i_r^* := i_r - |R| + 2 = 1$, $i_j^* := i_j \leq 2$ for all color $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and let $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Observe that $|B| \geq N^*$. By minimality of N, $G[B]$ has a red P_5 with vertices, say $x_1, \ldots, x_5$, in order. Because there is a red P_7 with both ends in R by using edges between A_1 and R, we see that R is blue-complete to $\{x_1, x_2, x_4, x_5\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_5\}]$ has a red P_{11}. But then we obtain a blue C_{12} with vertices $a_1, x_1, y_1, x_2, y_2, x_4, y_3, x_5, b_1, x_3, c_1, x_6$ in order, a contradiction. ∎

Claim 15. $i_b > |A_1|$ and so $|A_1| \leq n-2$.

Proof. Suppose $i_b \leq |A_1|$. If $i_b \leq |A_1| - 1$, then $i_b \leq n-2$ by Claim 2 and so $i_r = n-1$. Thus $|B| \geq 2 + i_b$ because $|B| + |R| = |G| - |A_1| \geq n + 1 + i_b + (i_r - |A_1|) \geq 3 + 2i_b$. But then G has a blue G_{i_b} using edges between A_1 and B, a contradiction. Thus $i_b = |A_1|$. By Claim 11 and Claim 14, $|R| \le n-1$ and $i_r \ge |R|$. Observe that $|B| \ge 1+n+i_r-|R| \ge 1+n$. Then $G[B \cup R]$ has no blue P_3 with both ends in B, else we obtain a blue G_{i_b} in G. Let $i_b^* := i_b - |A_1| = 0$, $i_r^* := i_r - |R|$, and $i_j^* := i_j \le n-4$ for all color $j \in [k]$ other than blue and red. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*]$. Then $3 < N^* < N$. Suppose first that $|R| \ge 2$. Since B is not red-complete to R, we may assume that y_1x is colored blue for some $x \in B$. Note that $i_r^* \le n-3$ and $|B \backslash x| = N - |A_1| - |R| - 1 \ge N^*$. By minimality of N, $G[B\backslash x]$ must have a red $P_{2i_r^*+3}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i_r^* + 3$. Since $G[B \cup R]$ contains no blue P_3 with both ends in B and xy_1 is colored blue, we see that y_1 must be red-complete to $B\backslash x$ and y_2 is not blue-complete to $\{x_1, x_q\}$. We may assume that x_qy_2 is colored red in G. Then $n = 6$, $i_r = |R| = 5$ and $i_b = |A_1| = 3$, else we obtain a red G_{i_r} using vertices in $V(P_{2i_r^*+3}) \cup R \cup A_1$. Let $x' \in B\backslash\{x, x_1, x_2, x_3\}$. Then $\{x, x'\} \nsubseteq A_i$ and $\{x, x_1\} \nsubseteq A_i$ for all $i \in [p]$ because yx is colored blue and yx', yx_1 are colored red, and so xx' and xx_1 are colored red, else $G[A_1 \cup B \cup \{y_1\}]$ has a blue P_9. But then we obtain a red C_{12} with vertices $a_1, y_1, x', x, x_1, x_2, x_3, y_2, b_1, y_3, c_1, y_4$ in order, a contradiction. Thus $|R| = 1$. By Claim 7 applied to $i_b = |A_1|$, $i_r \ge |R|$ and B, $G[B]$ must have a red P_{2i_r+1} with vertices, say $x_1, x_2, \ldots, x_{2i_r+1}$, in order. Since $G[B \cup R]$ contains no blue P_3 with both ends in B, we may assume that y_1x_1 is colored red under c. Then $i_r = n-1$, else we obtain a red G_{i_r}, a contradiction. Moreover, y_1x_{2n-1} must be colored blue, else G has a red C_{2n} with vertices $y_1, x_1, \ldots, x_{2n-1}$ in order. Thus y_1 is red-complete to $\{x_1, \ldots, x_{2n-2}\}$, and so $\{x_j, x_{2n-1}\} \nsubseteq A_i$ for all $i \in [p]$ and $j \in [2n-2]$. So $x_{2n-1}x_i$ must be colored red for some $i \in [2n-3]$ because $G[B]$ has no blue P_3. But then we obtain a red C_{2n} with vertices $y_1, x_1, \ldots, x_i, x_{2n-1}, x_{2n-2}, \ldots, x_{i+1}$ in order, a contradiction. This proves that $i_b > |A_1|$, and so $|A_1| \le n-2$. ■

By Claim 12 and Claim 15, $3 \le |A_1| \le n-2$. Then by Claim 13, $G[A_1]$ has a monochromatic, say green, copy of P_3. By Claim 10, $i_g = 1$.

Claim 16. If $|A_1| = 3$, then $|A_2| = 3$, $|A_3| \leq 2$, and $i_j = 0$ for all color $j \in [k]\backslash[3]$.

Proof. Assume $|A_1| = 3$. To prove $|A_2| = 3$, we show that $G[B \cup R]$ has a green P_3. Suppose $G[B \cup R]$ has no green P_3. By Claim 15, $i_b \geq |A_1| + 1 = 4$. Let $i_g^* := 0$ and $i_j^* := i_j$ for all $j \in [k]$ other than green. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^{k} i_j^*) - i_\ell^*]$. Then $N^* = N - 1$ and $|G\backslash a_1| = N - 1 = N^*$. But then $G\backslash a_1$ has no monochromatic copy of $G_{i_j^*}$ in color j for all $j \in [k]$, contrary to the minimality of N. Thus $G[B \cup R]$ has a green P_3 and so $|A_2| = 3$.

Suppose $|A_3| = 3$. For all $i \in [3]$, let

$$A_b^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored blue in } \mathcal{R}\} \text{ and}$$
$$A_r^i := \{a_j \in V(\mathcal{R}) \mid a_j a_i \text{ is colored red in } \mathcal{R}\}.$$

Let $B^i := \bigcup_{a_j \in A_b^i} A_j$ and $R^i := \bigcup_{a_j \in A_r^i} A_j$. Since each of A_1, A_2, A_3 can be chosen as the largest part in the Gallai-partition $A_1, A_2, \ldots, A_p$ of G, by Claim 11, either $|B^i| \leq 5$ or $|R^i| \leq 5$ for all $i \in [3]$. Without loss of generality, we may assume that A_2 is blue-complete to $A_1 \cup A_3$. Let $X := V(G)\backslash(A_1 \cup A_2 \cup A_3) = \{v_1, \ldots, v_{|X|}\}$. Then $|X| \geq 1 + n + i_b + i_r + i_g - 9 = 2n - 8 + \min\{i_b, i_r\}$. Suppose $|X \cap B^1| \geq 2$. We may assume $v_1, v_2 \in X \cap B^1$. Then G has a blue C_{10} with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_2, b_3, c_2$ in order and a blue P_{11} with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_2, b_3, c_2, c_3$ in order, and so $n = 6$ and $i_b = 5$. Moreover, $X\backslash\{v_1, v_2\} \subseteq R^3$, else, say v_3 is blue-complete to A_3, then we obtain a blue C_{12} under c with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, v_3, b_3, b_2, c_3, c_2$ in order. Thus $|R^3| \geq |X\backslash\{v_1, v_2\}| \geq 2 + i_r$, and so $i_r \geq 3$, else G has a red G_{i_r} using the edges between A_3 and R^3. Then there exist at least two vertices in $X\backslash\{v_1, v_2\}$, say v_3, v_4, such that $\{v_3, v_4\}$ is blue-complete to A_1, else $G[A_1 \cup A_3 \cup (X\backslash\{v_1, v_2\})]$ contains a red G_{i_r}. Thus $|B^1| \geq |A_2 \cup \{v_1, \ldots, v_4\}| = 7$ and so $|R^1| \leq 5$. Moreover, $\{v_1, v_2\} \subset R^3$, else, say v_1 is blue-complete to A_3, we then obtain a blue C_{12} under c with vertices $a_1, v_3, b_1, v_4, c_1, a_2, a_3, v_1, b_3, b_2, c_3, c_2$ in order. Then $X \subseteq R^3$ and $|R^3| \geq |X| \geq 4 + i_r \geq 7$, and so $|B^3| \leq 5$ and A_1 is red-complete to A_3. Furthermore, $G[B^1 \backslash A_2]$ has no blue P_3, else, say v_1, v_2, v_3 is such a blue P_3 in order, we obtain a blue C_{12} with vertices $a_1, v_1, v_2, v_3, b_1, v_4, c_1, a_2, a_3, b_2, b_3, c_2$ in order. Therefore for any $U \subseteq B^1 \backslash A_2$ with $|U| \geq 4$, $G[U]$ contains a red P_3 because $|A_1| = 3$ and $GR_k(P_3) = 3$. Since $|R^1| \leq 5$ and $A_3 \subseteq R^1$, we may assume $v_1, \ldots, v_{|X|-2} \in B^1 \backslash A_2$. Then $G[\{v_1, \ldots, v_4\}]$ must contain a red P_3 with vertices, say v_1, v_2, v_3, in order. We claim that $X \subset B^1$. Suppose $v_{|X|} \in R^1$. Then $v_{|X|}$ is red-complete to A_1 and so G has a red P_{11} with vertices $c_1, v_{|X|}, a_1, a_3, b_1, b_3, v_1, v_2, v_3, c_3, v_4$ in order, it follows that $i_r = 5$. Thus $|X| \geq 9$, and $G[\{v_4, \ldots, v_7\}]$ has a red P_3 with vertices, say v_4, v_5, v_6, in order. But then we obtain a red C_{12} with vertices $a_1, v_{|X|}, b_1, a_3, v_1, v_2, v_3, b_3, v_4, v_5, v_6, c_3$ in order, a contradiction. Thus $X \subset B^1$ as claimed. Since $|X| \geq 7$, $G[\{v_4, \ldots, v_7\}]$ contains a red P_3 with vertices, say v_4, v_5, v_6, in order. Then G has a red P_{11} with vertices $a_1, a_3, b_1, b_3, v_1, v_2, v_3, c_3, v_4, v_5, v_6$ in order, and so $i_r = 5$, $|X| \geq 9$. Suppose $G[\{v_4, \ldots, v_9\}]$ has no red P_5. Then $G[\{v_4, \ldots, v_9\}]$ has at most one part with order three, say A_4, and we may assume $G[A_4]$ has a monochromatic P_3 in some color m other than red and blue if $|A_4| = 3$ by Claim 13. Let $i_r^* := 1$, $i_m^* := 1$, $i_j^* := 0$ for all color $j \in [k] \backslash \{m\}$ other than red. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^{k} i_j^*) - i_r^*] = 6 < N$. Then $G[\{v_4, \ldots, v_9\}]$ has no monochromatic copy of $G_{i_j^*}$ in any color $j \in [k]$, which contradicts to the minimality of N. Thus $G[\{v_4, \ldots, v_9\}]$ has a red P_5 with vertices, say $v_4, \ldots, v_8$, in order. But then we obtain a red C_{12} with vertices $a_3, v_1, v_2, v_3, b_3, v_4, \ldots, v_8, c_3, v_9$ in order, a contradiction. Therefore, $|X \cap B^1| \leq 1$. By symmetry, $|X \cap B^3| \leq 1$. Let $w \in X \cap B^1$ and $w' \in X \cap B^3$. Then $A_1 \cup A_3$ is red-complete to $X \backslash \{w, w'\}$. It follows that $n = 5$ and $|X \cap B^1| = |X \cap B^3| = 1$, else $G[A_1 \cup A_3 \cup (X \backslash \{w, w'\})]$ has a red G_{i_r} because $|X| \geq 2n - 8 + \min\{i_b, i_r\}$ and $i_b \geq 4$, a contradiction. But then we obtain a blue C_{10} with vertices $a_2, a_1, w, b_1, b_2, a_3, w', b_3, c_2, c_3$ in order, a contradiction. This proves that $|A_3| \leq 2$, and then both $G[A_1]$ and $G[A_2]$ have a green P_3, so $i_j = 0$ for all color $j \in [k]$ other than red, blue and green by Claim 10. ∎

Claim 17. If $i_b = |A_1| + 1$, then $|R| \le 2$.

Proof. Suppose $i_b = |A_1| + 1$ but $|R| \ge 3$. By Claim 14, $i_r \ge |R|$, it follows that $|B| \ge 1 + n + i_b + i_r + i_g - |A_1| - |R| \ge 3 + n$. Thus $G[B \cup R]$ has no blue P_5 with both ends in B, else we obtain a blue G_{i_b}. Let $i^*_b := i_b - |A_1| = 1$, $i^*_r := i_r - |R| + 1$ (when $n = 5$) or $i^*_r := \max\{i_r - |R| + 1, 2\}$ (when $n = 6$), $i^*_j := i_j$ for all $j \in [k]$ other than red and blue. Let $i^*_\ell := \max\{i^*_j : j \in [k]\}$ and $N^* := |G_{i^*_\ell}| + [(\sum_{j=1}^k i^*_j) - i^*_\ell]$. Then $3 < N < N^*$. Observe that $|B| \ge N^*$. By minimality of N, $G[B]$ has a red $G_{i^*_r}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i^*_r + 3$. If R is blue-complete to $\{x_1, x_q\}$, then R is red-complete to $B \backslash \{x_1, x_q\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then $G[A_1 \cup R \cup \{x_2, \ldots, x_{q-1}\}]$ has a red G_{i_r}, a contradiction. Thus R is not blue-complete to $\{x_1, x_q\}$, and so we may assume y_1x_1 is colored red. Then $i_r = n - 1$ and $R \backslash \{y_1\}$ is blue-complete to $\{x_{q-2}, x_q\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_q\}]$ has a red G_{i_r}. So $R \backslash \{y_1\}$ is red-complete to $B \backslash \{x_{q-2}, x_q\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then $G[A_1 \cup R \cup \{x_2, \ldots, x_{q-1}\}]$ has a red G_{i_r}, a contradiction. ∎

Claim 18. $i_b = n - 1$.

Proof. Suppose $i_b \le n - 2$. Then $i_r = 5$. By Claim 12 and Claim 15, $|A_1| \ge 3$ and $i_b > |A_1|$, it follows that $n = 6$, $i_b = 4$ and $|A_1| = 3$. By Claim 16, $|A_2| = 3$, $|A_3| \le 2$, $i_j = 0$ for all color $j \in [k] \backslash [3]$. By Claim 17, $|R| \le 2$ and so $A_2 \subset B$. It follows that $|B| = 7 + i_b + i_r + i_g - |A_1 \cup R| = 14 - |R| \ge 12$. Then $G[B \cup R]$ has no blue P_5 with both ends in B, else G has a blue P_{11} because $|A_1| = 3$. Thus there exists a set W such that $(B \cup R) \backslash (A_2 \cup W)$ is red-complete to A_2, where $W \subset (B \cup R) \backslash A_2$ with $|W| \le 1$. Let $i^*_b := i_b - |A_1| = 1$, $i^*_r := 2$, $i^*_j := 0$ for all $j \in [k]$ other than red and blue, and $N^* := |G_{i^*_r}| + [(\sum_{j=1}^k i^*_j) - i^*_r] = 8$. Then $N^* < N$. Observe that $|B \backslash (A_2 \cup W)| = 11 - |R| - |W| \ge N^*$. By minimality of N, $G[B \backslash (A_2 \cup W)]$ must contain a red $G_{i^*_r} = P_7$. But then $G[(B \cup R) \backslash W]$ has a red C_{12}, a contradiction. Thus $i_b = n - 1$. ∎

Claim 19. $|A_1| = n - 2$.

Proof. By Claim 15, $|A_1| \leq n-2$. Suppose $|A_1| \leq n-3$. By Claim 12, $n = 6$ and $|A_1| = 3$. By Claim 18, $i_b = 5$. By Claim 16, $|A_2| = 3$, $|A_3| \leq 2$ and $i_j = 0$ for all color $j \in [k] \backslash [3]$. By Claim 14, $i_r \geq |R|$. Then $|B| = 7 + i_b + i_r + i_g - |A_1| - |R| \geq 10$, and so $G[B \cup R]$ has neither blue P_7 nor blue $P_5 \cup P_3$ with all ends in B else we obtain a blue C_{12}.

Suppose $|R| \leq 2$. Then $A_2 \subset B$ and there exists a set $W \subset (B \cup R) \backslash A_2$ with $|W| \leq 3$ such that W is blue-complete to A_2 and $(B \cup R) \backslash (A_2 \cup W)$ is red-complete to A_2. Since $|B \backslash (A_2 \cup W)| \geq 4$, we see that there is a red P_7 using edges between A_2 and $B \backslash (A_2 \cup W)$, so $i_r \geq 3$ and $i_r - |R| \geq 1$. Let $i_b^* := 2$ (when $|B \cap W| \leq 1$) or $i_b^* := 0$ (when $|B \cap W| \geq 2$), $i_r^* := \min\{i_r - |R| - 1, 2\}$, $i_j^* := 0$ for all color $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*] = 3 + \max\{i_b^*, i_r^*\} + i_b^* + i_r^*$. Observe that $|B \backslash (A_2 \cup W)| = 7 + i_r - |R \cup W| \geq N^*$. By minimality of N, $G[B \backslash (A_2 \cup W)]$ has a red $G_{i_r^*}$ because $G[B]$ has neither blue P_7 nor blue $P_5 \cup P_3$ and $|A_3| \leq 2$. But then $G[(B \cup R) \backslash W]$ has a red G_{i_r} because $|(B \cup R) \backslash W| \geq 7 + i_r \geq |G_{i_r}|$ and A_2 is red-complete to $(B \cup R) \backslash (A_2 \cup W)$, a contradiction. Therefore, $3 \leq |R| \leq 5$ and so $i_r \geq 3$.

We claim that $i_r = 5$. Suppose $3 \leq i_r \leq 4$. Let $i_b^* := 2$, $i_r^* := 2$, $i_j^* := i_j$ for all color $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 10$. Observe that $|B| \geq 10 = N^*$. Since $G[B]$ has no blue P_7, by minimality of N, $G[B]$ has a red P_7 with vertices, say $x_1, \ldots, x_7$, in order. Then R is blue-complete to $\{x_1, \ldots, x_7\} \backslash x_4$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_7\}]$ has a red G_{i_r}. But then $G[B \cup R]$ has a blue P_7 with vertices $x_1, y_1, x_2, y_2, x_3, y_3, x_5$ in order, a contradiction. Thus $i_r = 5$ and so $|G| = 18$, $|B| = 15 - |R|$.

If $|R| = 3$. First suppose $A_2 \subseteq R$. Since R is not red-complete to B, we may assume that A_2 is blue-complete to x_1. Let $i_b^* := 2$, $i_r^* := 3$, $i_j^* := 0$ for all color $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 11$. Observe that $|B\backslash x_1| = 11 = N^*$. By minimality of N, $G[B\backslash x_1]$ has a red P_9 with vertices, say $x_2, \ldots, x_{10}$, in order. We claim that A_2 is blue-complete to $\{x_2, x_{10}\}$, else, say x_2 is red-complete to A_2. Then A_2 is blue-complete to $\{x_8, x_{10}\}$, else $G[A_1 \cup A_2 \cup \{x_2, \ldots, x_{10}\}]$ has a red C_{12}. Thus A_2 is red-complete to $B\backslash\{x_1, x_8, x_{10}\}$ because $G[B \cup R]$ has no blue P_7 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, a_2, x_3, \ldots, x_9, b_2, b_1, c_2$ in order, a contradiction. Thus, A_2 is blue-complete to $\{x_1, x_2, x_{10}\}$, and so A_2 is red-complete to $B\backslash\{x_1, x_2, x_{10}\}$ because $G[B \cup R]$ has no blue P_7 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, a_2, x_3, \ldots, x_9, b_2, b_1, c_2$ in order, a contradiction. This proves that $A_2 \subset B$. Then there exists a set $W \subset (B \cup R)\backslash A_2$ with $|W \cap B| \leq 3$ such that W is blue-complete to A_2 and $(B \cup R)\backslash(A_2 \cup W)$ is red-complete to A_2. Then $|W| \leq 3$ and $|W \cap B| \leq 3$ or $|W| = 4$ and $|W \cap B| = 1$ because $G[B \cup R]$ has no blue P_7 with both ends in B. Let

$$i_b^* := 2 - |W|,\ i_r^* := 2 \text{ when } |W| \in \{0, 1\},$$
$$i_b^* := 0,\ i_r^* := 2 \text{ when } |W| \geq 2 \text{ and } |W \cap B| \leq 2,$$
$$i_b^* := 0,\ i_r^* := 1 \text{ when } |W| = |W \cap B| = 3,$$

$i_j^* := 0$ for all color $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 3 + 2i_r^* + i_b^*$. Observe that $|B\backslash(A_2 \cup W)| \geq N^*$. By minimality of N, $G[B\backslash(A_2 \cup W)]$ has a red $G_{i_r^*}$ because $G[B \cup R]$ has neither blue P_7 nor blue $P_5 \cup P_3$ with all ends in B and $|A_3| \leq 2$. If $|W| \leq 3$ and $|W \cap B| \leq 2$, then $G[(B \cup R)\backslash W]$ has a red C_{12} because $|(B \cup R)\backslash W| \geq 12$ and A_2 is red-complete to $(B \cup R)\backslash(A_2 \cup W)$. Thus $|W| = |W \cap B| = 3$ or $|W| = 4$ and $|W \cap B| = 1$. For the former case, $G[B\backslash(A_2 \cup W)]$ has a red P_5 with vertices, say $x_1, \ldots, x_5$, in order. Let $W := \{w_1, w_2, w_3\} \subset B$. Then A_2 is blue-complete to W and red-complete to $\{x_1, \ldots, x_5\}$, and so W is red-complete to $\{x_1, \ldots, x_5\}$ because $G[B]$ has no blue P_7. But then we obtain a red C_{12} with vertices $a_2, x_1, w_1, x_2, w_2, x_3, w_3, x_4, b_2, x_5, c_2, x_6$ in order, where $x_6 \in B\backslash(A_2 \cup W \cup \{x_1, \ldots, x_5\})$, a contradiction. For the latter case, $G[B\backslash(A_2 \cup W)]$ has a red P_7 with vertices, say $x_1, \ldots, x_7$, in order. Let $W \cap B := \{w\}$. Then w is red-complete to $\{x_1, \ldots, x_7\}$ because $G[B]$ has no blue P_7. But then we obtain a red C_{12} with vertices $a_2, x_1, w, x_2, \ldots, x_6, b_2, x_7, c_2, x_8$ in order, where $x_8 \in B\backslash(A_2 \cup W \cup \{x_1, \ldots, x_7\})$, a contradiction. This proves that $|R| \in \{4, 5\}$. First we claim that $G[E(B, R)]$ has no blue P_5 with both ends in B. Suppose there is a blue $H := P_5$ with vertices, say x_1, y_1, x_2, y_2, x_3, in order. Then $G[(B \cup R)\backslash V(H)]$ has no blue P_3 with both ends in B. Let $i_b^* := 0$, $i_r^* := i_r - |R| + 1$, $i_j^* := i_j$ for all color $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*] = 6 + 2(i_r - |R|)$. Observe that $|B\backslash\{x_1, x_2, x_3\}| = 7 + i_r - |R| \geq N^*$ since $|R| \in \{4, 5\}$. By minimality of N, $G[B\backslash\{x_1, x_2, x_3\}]$ has a red $G_{i_r^*}$ with vertices, say $x_4, \ldots, x_q$, in order, where $q = 2i_r^* + 6$. Then y_3 is not blue-complete to $\{x_4, x_q\}$ because $G[(B \cup R)\backslash V(H)]$ has no blue P_3 with both ends in B. We may assume x_4y_3 is colored red. Then $R\backslash\{y_1, y_2, y_3\}$ is blue-complete to x_8, else we obtain a red C_{12} with vertices $a_1, y_3, x_4, \ldots, x_8, y_4, b_1, y_1, c_1, y_2$ in order, a contradiction. Since $G[(B\cup R)\backslash V(H)]$ has no blue P_3 with both ends in B, we see that $R\backslash\{y_1, y_2, y_3\}$ is red-complete to $\{x_4, \ldots, x_q\}\backslash\{x_8\}$. But then we obtain a red C_{12} with vertices $a_1, y_3, x_4, \ldots, x_{10}, y_4, b_1, y_1$ (when $|R| = 4$), or $a_1, y_3, x_4, x_5, x_6, y_4, x_7, y_5, b_1, y_1, c_1, y_2$ (when $|R| = 5$) in order, a contradiction. Thus, $G[E(B, R)]$ has no blue P_5 with both ends in B. Let $i_b^* := 2$, $i_r^* := 2$, $i_j^* := i_j$ for all color $j \in [k]$ other than red and blue, and $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 10$. Observe that $|B| \geq 10 = N^*$. By minimality of N, $G[B]$ has a red P_7 with vertices, say $x_1, \ldots, x_7$, in order. We claim that x_1 is blue-complete to R. Suppose x_1y_1 is colored red. Then $R\backslash y_1$ is blue-complete to $\{x_5, x_7\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_7\}]$ has a red C_{12}. Thus $R\backslash y_1$ is red-complete to $B\backslash\{x_5, x_7\}$ because $G[E(B, R)]$ has no blue P_5 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, y_2, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. Therefore, x_1 is blue-complete to R. By symmetry, x_7 is blue-complete to R. Then R is red-complete to $B\backslash\{x_1, x_7\}$ because $G[E(B, R)]$ has no blue P_5 with both ends in B. But then we obtain a red C_{12} with vertices $a_1, y_2, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. This proves that $|A_1| = n - 2$. ■

By Claim 18, Claim 19 and Claim 14, $i_b = n-1$, $|A_1| = n-2$, $i_r \geq |R|$. By Claim 17, $|R| \leq 2$. Then $|B| \geq 3+n+i_r-|R| \geq 3+n$, and so $G[B \cup R]$ has no blue P_5 with both ends in B.

Claim 20. $i_r = n-1$.

Proof. Suppose $i_r \leq n-2$. By Claim 9, B is not blue-complete to R. Let $x \in B$ and $y \in R$ such that xy is colored red. Let $i_b^* := i_b - |A_1| = 1$ and $i_r^* := i_r - |R| \leq n-3$, $i_j^* := i_j \leq n-4$ for all color $j \in [k]$ other than red and blue. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*]$. Then $3 < N^* < N$ and $|B \backslash x| = N - |A_1| - |R| - 1 \geq N^*$. By minimality of N, $G[B\backslash x]$ must have a red $P_{2i_r^*+3}$ with vertices, say $x_1, x_2, \ldots, x_{2i_r^*+3}$, in order. Then $\{x_1, x_{2i_r^*+3}\}$ must be blue-complete to $\{x, y\}$ and xx_2 must be colored blue under c, else we obtain a red P_{2i_r+3} using vertices in $V(P_{2i_r^*+3}) \cup \{x, y\}$ or in $V(P_{2i_r^*+3}\backslash x_1) \cup \{x, y\} \cup A_1$. But then $G[B \cup R]$ has a blue P_5 with vertices $x_2, x, x_1, y, x_{2i_r^*+3}$ in order, a contradiction. ∎

Let $A_1 := \{a_1, b_1, c_1\}$ (when $n = 5$) or $A_1 := \{a_1, b_1, c_1, z_1\}$ (when $n = 6$). By Claim 13, $G[A_1]$ has a monochromatic, say green, copy of P_3. By Claim 10, $i_g = 1$. We next show that $|A_2| \geq 3$. Suppose $|A_2| \leq 2$. Then by Claim 16, $|A_1| = 4$ and so $n = 6$. Let $i_b^* := i_b - |A_1|$, $i_r^* := i_r - |R| + 1$, $i_g^* := i_g - 1 = 0$ and $i_j^* := i_j$ for all $j \in [k]$ other than red, blue and green. Let $i_\ell^* := \max\{i_j^* : j \in [k]\}$ and $N^* := |G_{i_\ell^*}| + [(\sum_{j=1}^k i_j^*) - i_\ell^*]$. Then $3 < N^* < N$ and $|B| = |G| - |A_1| - |R| = N^*$. By minimality of N, $G[B]$ must contain a red $G_{i_r^*}$. It follows that $|R| = 2$ and $G_{i_r^*} = P_{11}$. Let $x_1, x_2, \ldots, x_{11}$ be the vertices of the red P_{11} in order. If R is blue-complete to $\{x_1, x_{11}\}$, then R is red-complete to $B\backslash\{x_1, x_{11}\}$ because $G[B \cup R]$ has no blue P_5 with both ends in B. But then G has a red C_{12} with vertices $a_1, y_1, x_2, \ldots, x_{10}, y_2$ in order, a contradiction. Thus, R is not blue-complete to $\{x_1, x_{11}\}$ and we may assume x_1y_1 is colored red. Then $x_{11}y_1$ and x_9y_2 are colored blue, else $G[\{x_1, \ldots, x_{11}\} \cup R \cup A_1]$ has a red C_{12}. If $x_{11}y_2$ is colored red, then x_1y_2 and x_3y_1 are colored blue by the same reasoning. But then we obtain a blue C_{12} with vertices $a_1, x_1, y_2, x_9, b_1, x_3, y_1, x_{11}, c_1, x_2, z_1, x_4$ in order, a contradiction. Thus $x_{11}y_2$ is colored blue. Then y_1 is red-complete to $B\backslash\{x_9, x_{11}\}$, else, say y_1w is colored blue with $w \in B\backslash\{x_9, x_{11}\}$, then $G[B \cup R]$ has a blue P_5 with vertices w, y_1, x_{11}, y_2, x_9 in order. It follows that $\{x_{11}, w\} \nsubseteq A_j$ for all $j \in [q]$, where $w \in B\backslash\{x_9, x_{11}\}$. Moreover, x_2y_2 is colored blue, else G has a red C_{12} with vertices $a_1, y_2, x_2, \dots, x_{10}, y_1$ in order, a contradiction. Thus, $G[B\backslash\{x_2, x_9\}]$ has no blue P_3, else $G[A_1 \cup B \cup \{y_2\}]$ has a blue C_{12}. Therefore, x_ix_{11} is colored red for some $i \in \{3, \dots, 7\}$. But then we obtain a red C_{12} with vertices $y_1, x_1, \dots, x_i, x_{11}, x_{10}, \dots, x_{i+1}$ in order, a contradiction. Thus $3 \le |A_2| \le n-2$ and $A_2 \subset B$ because $|R| \le 2$.

Since $G[B \cup R]$ has no blue P_5 with both ends in B, there exists at most one vertex, say $w \in B \cup R$, such that $(B \cup R)\backslash(A_2 \cup \{w\})$ is red-complete to A_2. Suppose $3 \le |A_3| \le n-2$. Then $n = 6$ by Claim 16, $A_3 \subseteq B$ and A_3 must be red-complete to A_2. Since $G[B \cup R]$ has no blue P_5 with both ends in B, there exists at most one vertex, say $w' \in B \cup R$, such that $(B \cup R)\backslash(A_3 \cup \{w'\})$ is red-complete to A_3. But then $G[(B \cup R)\backslash\{w, w'\}]$ has a red C_{12}, a contradiction. Thus $|A_3| \le 2$ and so $G[B\backslash A_2]$ has no monochromatic copy of P_3 in color j for all $j \in [k]$ other than red and blue. Let $i_b^* := 1$, $i_r^* := n - 1 - |A_2|$, and $i_j^* := 0$ for all colors $j \in [k]$ other than red and blue. Let $N^* := |G_{i_r^*}| + [(\sum_{j=1}^k i_j^*) - i_r^*] = 2i_r^* + 3 + i_b^* = 2n + 2 - 2|A_2|$. Then $3 < N^* < N$ and $|B\backslash(A_2 \cup \{w\})| \ge 2n + 1 - |R| - |A_2| \ge N^*$. By minimality of N, $G[B\backslash(A_2 \cup \{w\})]$ has a red $G_{i_r^*}$. But then $G[(B \cup R)\backslash\{w\}]$ has a red C_{2n}, a contradiction.

This completes the proof of Theorem 1.6.17. ■

CHAPTER 4: FUTURE WORK

In this chapter, we discuss possible extensions of our work in this dissertation, as well as other topics of interest.

4.1 More Open Problems on Co-critical Graphs

As we see that Conjecture 1.4.1 remains wide open, except that the first nontrivial case, so we are also interesting in the next open case (K_3, K_4)-co-critical. By considering the construction of K_4-saturated graphs and making use of the result of Theorem 1.5.7, hopefully we can obtain an asymptotic edge bound for (K_3, K_4)-co-critical graphs.

Galluccio, Siminovits and Simonyi proposed several interesting open problems in [45] which are listed below. G_n below denotes a graph on n vertices.

1. Are there infinitely many strongly minimal co-critical graphs?
2. Can one get a construction of a (K_3, K_3)-co-critical graph G_n without K_4?
3. Is it true that for every (K_3, K_3)-co-critical graph G_n adding any new edge we get a K_5? Or at least a K_4?
4. Assume that a (K_3, K_3)-co-critical graph G_n contains a K_5. Does this imply that G_n contains also a K_6^-?
5. Is it always true that duplicating a vertex of a co-critical graph we get a co-critical graph?

4.2 Rainbow Saturation Numbers of Graphs

We are also interested in saturation numbers of edge-colored graphs that are as far being monochromatic as possible. This problem was proposed by Barrus, Ferrara, Vandenbussche, and Wenger in [4]. Given a graph F, an edge coloring of F is called *rainbow* if every edge of F is colored differently. Note that it is not necessary to specify the set of colors that may be used in a rainbow-colored copy of F. Given a graph G and a t-edge-coloring τ of G, where $t \geq e(F)$. We say (G, τ) is *rainbow* (F, t)*-saturated* if G contains no rainbow copy of F under τ, but for any edge $e \in E(\overline{G})$ and any color $i \in [t]$, the addition of e to G in color i creates a rainbow copy of F. τ is called an F*-threshold coloring* if (G, τ) is rainbow (F, t)-saturated. The *t-rainbow saturation number* of F, denoted by $rsat_t(n, F)$, is the minimum number of edges in a rainbow (F, t)-saturated graph with n vertices. Barrus et al., in [4], proved the following results.

Theorem 4.2.1 (Barrus, Ferrara, Vandenbussche, Wenger [4]) *For every integer $k \geq 3$ and $t \geq \binom{k}{2}$, for all sufficiently large n, there exist two positive constants c_1, c_2 such that $c_1 \frac{n \log n}{\log \log n} \leq \mathrm{rsat}_t(n, K_k) \leq c_2 n \log n$.*

Theorem 4.2.2 (Barrus, Ferrara, Vandenbussche, Wenger [4])

(i) *If $t \geq k \geq 2$ and $n \geq (k+1)(k-1)/t$, then $\mathrm{rsat}_t(n, K_{1,k}) = \Theta(n^2)$.*

(ii) *For all $k \geq 4$, $\mathrm{rsat}_t(n, P_k) \geq n - 1$.*

(iii) *For $t \geq 8$, $\mathrm{rsat}_t(n, P_4) = n - 1$.*

(iv) *If T is a tree with at least four vertices that is not a star, then*

$$\mathrm{rsat}_t(n, T) \leq \left\lceil \frac{n}{k-1} \right\rceil \binom{k-1}{2}.$$

In the same paper, Barrus et al. conjectured that $\mathrm{rsat}_t(n, K_k) = \Theta(n \log n)$. This conjecture was verified by Girão et al. in [47] and Korándi in [62] independently. Recently, Shi et al. [75] improved the upper bound of $\mathrm{rsat}_t(n, P_k)$ to $\lceil \frac{n}{k} \rceil \binom{k-2}{2} + 4$ for $k \geq 5$ and $t \geq 2k - 5$. Motivated by these results, we are interested in the bound of $\mathrm{rsat}_t(n, \mathcal{T}_k)$, where $\mathcal{T}_k$ is the family of all trees on k vertices.

4.3 Antimagic Labeling of Graphs

Given a graph G with m edges. An *antimagic labeling* of a graph G is a bijection from $E(G)$ to $\{1, 2, ..., m\}$ such that for any distinct vertices u and v, the sum of labels on edges incident to u differs from that for edges incident to v. A graph G is *antimagic* if it has an antimagic labeling. Hartsfield and Ringel [57] introduced antigamic labelings in 1990 and made the following conjecture.

Conjecture 4.3.1 (Hartsfield, Ringel [57]) *Every connected graph, but K_2, is antimagic.*

The most significant progress on this problem is a result of Alon, Kaplan, Lev, Roditty, and Yuster [2] stated below.

Theorem 4.3.2 (Alon, Kaplan, Lev, Roditty, Yuster [2]) *There exists an absolute constant C such that every graph on n vertices with minimum degree at least $C \log n$ is antimagic.*

Theorem 4.3.3 (Alon, Kaplan, Lev, Roditty, Yuster [2]) *If G has $n \geq 4$ vertices and $\Delta(G) \geq n - 2$, then G is antimagic.*

Eccles [29] recently improved Theorem 4.3.2 by showing that there exists an absolute constant c_0 such that if G is a graph with average degree at least c_0, and G contains no isolated edge and at most one isolated vertex, then G is antimagic. In 2012, Yilma [82] proved the following result which improved Theorem 4.3.3.

Theorem 4.3.4 (Yilma [82]) *If G is connected, has $n \geq 9$ vertices, and $\Delta(G) \geq n-3$, then G is antimagic.*

Theorem 4.3.5 (Yilma [82]) *If G is a graph on n vertices, $\Delta(G) = d(x) = n-k$, where $k \leq n/3$, and there exists $y \in V(G)$ such that $N(x) \cup N(y) = V(G)$, then G is antimagic.*

As noted in [2], it is still an open problem to decide whether connected graphs with $\Delta(G) \geq n-k$ and $n > n_0(k)$ are antimagic, for any fixed $k \geq 4$. We are interested in showing Conjecture 4.3.1 is true for every graph G with $\Delta(G) = n-4$ and improving Theorem 4.3.5.

LIST OF REFERENCES

[1] N. Alon, P. Erdős, R. Holzman, M. Krivelevich, *On k-saturated graphs with restrictions on the degrees*, J. Graph Theory 23 (1996) 1–20.

[2] N. Alon, G. Kaplan, A. Lev, Y. Roditty, R. Yuster, *Dense graphs are antimagic*, J. Graph Theory 47 (2004) 297–309.

[3] V. Angeltveit, B. D. McKay, $R(5,5) \leq 48$, J. Graph Theory 89 (2018) 5–13.

[4] M. D. Barrus, M. Ferrara, J. Vandenbussche, P. S. Wenger, *Colored saturation parameters for rainbow subgraphs*, J. Graph Theory 86 (4) (2017) 375–386.

[5] F. S. Benevides, J. Skokan, *The 3-colored Ramsey number of even cycles*, J. Combin. Theory Ser. B 99 (2009) 690–708.

[6] B. Bollobás, *On generalized graphs*, Acta Math. Acad. Sci. Hungar. 16 (1965) 447–452.

[7] J. A. Bondy, P. Erdős, *Ramsey numbers for cycles in graphs*, J. Combin. Theory Ser. B 14 (1973) 46–54.

[8] C. Bosse, Z-X. Song, *Multicolor Gallai-Ramsey numbers of C_9 and C_{11}*. arXiv:1802.06503.

[9] C. Bosse, Z-X. Song, J. Zhang, *Improved upper bounds for Gallai-Ramsey numbers of odd cycles*. arXiv:1808.09963.

[10] C. Bosse, Z-X. Song, J. Zhang, *On the size of (K_3, K_4)-co-critical graphs*, in preparation.

[11] R. A. Brualdi, *Introductory Combinatorics* (5th ed.), Prentice Hall (Pearson), (2010) P74.

[12] D. Bruce, Z-X. Song, *Gallai-Ramsey numbers of C_7 with multiple colors*, Discrete Math. 342 (2019) 1191–1194.

[13] L. E. Bush, *The William Lowell Putnam Mathematical Competition* (question #2 in Part I asks for the proof of $R(3,3) \leq 6$), Amer. Math. Monthly 60 (1953) 539–542.

[14] K. Cameron, J. Edmonds, L. Lovász, *A note on perfect graphs*, Period. Math. Hungar. 17 (1986) 173–175.

[15] J. Chalupa, P. L. Leath, G. R. Reich, *Bootstrap percolation on a bethe latice*, J. Physics C. 12 (1979) L31–L37.

[16] G. Chartrand, S. Schuster, *On the existence of specified cycles in complementary graphs*, Bull. Amer. Math. Soc. 77 (1971) 995–998.

[17] G. Chen, M. Ferrara, R. J. Gould, C. Magnant, J. Schmitt, *Saturation numbers for families of Ramsey-minimal graphs*, J. Comb. 2 (2011) 435–455.

[18] F. R. K. Chung, R. L. Graham, *Edge-colored complete graphs with precisely colored subgraphs*, Combinatorica 3 (1983) 315–324.

[19] V. Chvátal, *Tree-complete graph Ramsey numbers*, J. Graph Theory 1 (1977) 93.

[20] D. Conlon, *A new upper bound for diagonal Ramsey numbers*, Ann. of Math. 170 (2009) 941–960.

[21] D. Conlon, J. Fox, B. Sudakov, *Recent developments in graph Ramsey theory*, Surveys in Combinatorics 424 (2015) 49–118.

[22] H. Davenport, Z-X. Song, *On the size of* $(K_3, K_{1,k})$*-co-critical graphs*, in preparation.

[23] A. N. Day, *Saturated graphs of prescribed minimum degree*, Combin. Probab. Comput. 26 (2017) 201–207.

[24] A. N. Day, J. R. Johnson, *Multicolour Ramsey numbers of odd cycles*, J. Combin. Theory Ser. B 124 (2017) 56–63.

[25] R. Diestel, *Graph Theory* (5th ed.), Graduate Texts in Mathematics, vol. 173, Springer Berlin Heidelberg, (2016).

[26] D. A. Duffus, D. Hanson, *Minimal* k*-saturated and color critical graphs of prescribed minimum degree*, J. Graph Theory 10 (1986) 55–67.

[27] T. Dzido, *Ramsey numbers for various graph classes* (in Polish), Ph.D. thesis, University of Gdańsk, Poland, (November 2005).

[28] T. Dzido, A. Nowik, P. Szuca, *New lower bound for multicolor Ramsey numbers for even cycles*, Electron. J. Combin. 12 (2005) # N13.

[29] T. Eccles, *Graphs of large linear size are antimagic*, J. Graph Theory 81 (2016) 236–261.

[30] P. Erdős, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. (N.S.) 53 (1947) 292–294.

[31] P. Erdős, R. J. Faudree, C. C. Rousseau, R. H. Schelp, *Generalized Ramsey theory for multiple colors*, J. Combin. Theory Ser. B 20 (1976) 250–264.

[32] P. Erdős, A. Hajnal, J. W. Moon, *A problem in graph theory*, Amer. Math. Monthly. 71 (1964) 1107–1110.

[33] P. Erdős, R. Holzman, *On maximal triangle-free graphs*, J. Graph Theory 18 (1994) 585–594.

[34] G. Exoo, *A lower bound for* $R(5,5)$, J. Graph Theory 13 (1989) 97–98.

[35] J. R. Faudree, R. J. Faudree, J. R. Schmitt, *A survey of minimum saturated graphs and hypergraphs*, Electron. J. Combin. 18 (2011) DS19.

[36] R. J. Faudree, R. J. Gould, M. S. Jacobson, C. Magnant, *Ramsey numbers in rainbow triangle free colorings*, Australas. J. Combin. 46 (2010) 269–284.

[37] R. J. Faudree, S. L. Lawrence, T. D. Parsons, R. H. Schelp, *Path-cycle Ramsey numbers*, Discrete Math. 10 (1974) 269–277.

[38] R. J. Faudree, R. H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. 8 (1974) 313–329.

[39] M. Ferrara, J. Kim, E. Yeager, *Ramsey-minimal saturation numbers for matchings*, Discrete Math. 322 (2014) 26–30.

[40] J. Fox, A. Grinshpun, J. Pach, *The Erdős-Hajnal conjecture for rainbow triangles*, J. Combin. Theory Ser. B 111 (2015) 75–125.

[41] S. Fujita, C. Magnant, *Gallai-Ramsey numbers for cycles*, Discrete Math. 311 (2011) 1247–1254.

[42] S. Fujita, C. Magnant, K. Ozeki, *Rainbow generalizations of Ramsey theory: a survey*, Graphs Combin. 26 (2010) 1–30.

[43] Z. Füredi, Á. Seress, *Maximal triangle-free graphs with restrictions on the degrees*, J. Graph Theory 18 (1994) 11–24.

[44] T. Gallai, *Transitiv orientierbare graphen*, Acta Math. Acad. Sci. Hung. 18 (1967) 25–66.

[45] A. Galluccio, M. Simonovits, G. Simonyi, *On the structure of co-critical graphs*, Graph theory, combinatorics, and algorithms, Vol. 1,2 (Kalamazoo, MI, 1992), 1053–1071, Wiley-Intersci. Publ., Wiley, New York, 1995.

[46] L. Gerencsér, A. Gyárfás, *On Ramsey-type problems*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 10 (1967) 167–170.

[47] A. Girão, D. Lewis, K. Popielarz, *Rainbow saturation of graphs*. arXiv:1710.08025.

[48] J. E. Graver, J. Yackel, *Some graph theoretic results associated with Ramsey's theorem*, J. Combin. Theory 4 (1968) 125–175.

[49] R. E. Greenwood, A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. 7 (1955) 1–7.

[50] J. Gregory, *Gallai-Ramsey number of an 8-Cycle*, Electronic Theses & Dissertations, Digital Commons@Georgia Southern (2016).

[51] A. Gyárfás, G. N. Sárközy, *Gallai colorings of non-complete graphs*, Discrete Math. 310 (2010) 977–980.

[52] A. Gyárfás, G. N. Sárközy, A. Sebő, S. Selkow, *Ramsey-type results for Gallai colorings*, J. Graph Theory 64 (2010) 233–243.

[53] A. Gyárfás, G. Simonyi, *Edge colorings of complete graphs without tricolored triangles*, J. Graph Theory 46 (2004) 211–216.

[54] A. Hajnal, *A theorem on k-saturated graphs*, Canad. J. Math. 17 (1965) 720-724.

[55] M. Hall, C. Magnant, K. Ozeki, M. Tsugaki, *Improved upper bounds for Gallai-Ramsey numbers of paths and cycles*, J. Graph Theory 75 (2014) 59–74.

[56] D. Hanson, B. Toft, *Edge-colored saturated graphs*, J. Graph Theory 11 (1987) 191–196.

[57] N. Hartsfield, G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston, (1990) 108-109 (revised version, 1994).

[58] I. N. Herstein, *Topics in Algebra*, Waltham: Blaisdell Publishing Company, (1964).

[59] M. Jenssen, J. Skokan, *Exact Ramsey numbers of odd cycles via nonlinear optimisation*. arXiv:1608.05705.

[60] L. Kászonyi, Z. Tuza, *Saturated graphs with minimal number of edges*, J. Graph Theory 10 (1986) 203–210.

[61] G. Kéry, *On a theorem of Ramsey* (in Hungarian), Matematikai Lapok 15 (1964) 204–224.

[62] D. Korándi, *Rainbow saturation and graph capacities*, Siam J. Discrete Math. 32 (2018) 1261–1264.

[63] J. Körner, G. Simonyi, *Graph pairs and their entropies: modularity problems*, Combinatorica 20 (2000) 227–240.

[64] H. Lei, Y. Shi, Z-X. Song, J. Zhang, *Gallai-Ramsey numbers of C_{10} and C_{12}*. arXiv:1808. 10282.

[65] H. Liu, C. Magnant, A. Saito, I. Schiermeyer, Y. Shi, *Gallai-Ramsey number for K_4*.

[66] T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, J. Combin. Theory Ser. B 75 (1999) 174–187.

[67] B. D. McKay, S. P. Radziszowski, $R(4, 5) = 25$, J. Graph Theory 19 (1995) 309–322.

[68] B. D. McKay, K. M. Zhang, *The value of the Ramsey number $R(3, 8)$*, J. Graph Theory 16 (1992) 99–105.

[69] C. Magnant, I. Schiermeyer, *Gallai-Ramsey number for K_5*. arXiv:1901.03622.

[70] J. Nešetřil, *Problem, in Irregularities of Partitions*, (eds G. Halász and V. T. Sós), Springer Verlag, Series Algorithms and Combinatorics, vol 8, (1989) P164. (Proc. Coll. held at Fertőd, Hungary 1986).

[71] S. P. Radziszowski, *Small Ramsey numbers*, Electron. J. Combin. (2017) DS1.15.

[72] F. P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society 30 (1930) 264–286.

[73] M. Rolek, Z-X. Song, *Saturation numbers for Ramsey-minimal graphs*, Discrete Math. 341 (2018) 3310–3320.

[74] V. Rosta, *On a Ramsey type problem of J. A. Bondy and P. Erdős, I & II*, J. Combin. Theory Ser. B 15 (1973) 94–120.

[75] Y. Shi, Z. Taoqiu, *A note on rainbow saturation number of paths*. arXiv:1902.05222.

[76] Z-X. Song, J. Zhang, *A conjecture on Gallai-Ramsey numbers of even cycles and paths*. arXiv:1803.07963.

[77] Z-X. Song, J. Zhang, *On the size of* $(K_t, \mathcal{T}_k)$*-co-critical graphs*. arXiv:1904.07825.

[78] J. Spencer, *Ramsey's theorem-a new lower bound*, J. Combin. Theory Ser. A 18 (1975) 108–115.

[79] T. Szabó, *On nearly regular co-critical graphs*, Discrete Math. 160 (1996) 279–281.

[80] P. Turán, *On an extremal problem in graph theory*, Matematikai és Fizikai Lapok (in Hungarian), 48 (1941) 436–452.

[81] D. B. West, *Introduction to Graph Theory* (2nd ed.), Prentice Hall, (September 2000).

[82] Z. B. Yilma, *Antimagic properties of graphs with large maximum degree*, J. Graph Theory 72 (2012) 367–373.

[83] F. Zhang, Z-X. Song, Y. Chen, *Multicolor Ramsey numbers of cycles in Gallai colorings*. arXiv:1906.05263.