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"Macroscopic" quantum superpositions: Atom-field entangled and steady states by two-photon processes

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The dynamics of an exact two-photon Hamiltonian is used to study the time evolution of an initially disentangled pure state of the atom-field system as it goes through cycles of entanglement separated by instances of disentanglement. For specific initial states of the electromagnetic field, the output state is a pure quantum superposition of a squeezed vacuum state and an orthogonal, odd-photon-number state. The odd-photon-number state, which is not a squeezed state, exhibits both nonclassical sub-Poissonian and classical super-Poissonian photon statistics. In the latter case the quantum superposition resembles a macroscopic superposition state. Conditions are obtained on the atom-cavity interaction time for such states to represent the steady states in the injection in a high-Q cavity of a monoenergetic, low-density beam of three-level atoms in a coherent state.

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I. INTRODUCTION

The Jaynes-Cummings model (JCM), applied to the micromaser, has demonstrated the existence of squeezed and sub-Poissonian photon fields, examples of nonclassical states of the electromagnetic field, as well as macroscopic quantum superpositions, the Schrödinger-cat states. In a micromaser, a monoenergetic low-density beam of two-level atoms in a coherent superposition of their upper and lowers states is injected inside a single-mode high Q cavity. Conditions on the interaction time τ, time atoms spend in the cavity, lead to trapping states whereby the field inside the cavity evolves to pure *tangent* and *cotangent* states [1]. The trapping conditions, which restrict the Fock space, give rise to normalizable steady states of the harmonic oscillator in the absence of dissipation. Macroscopic superposition persists in the micromaser even in the presence of dissipation [2].

Generation of pure states, the so-called *even* and *odd* states, have also been investigated for two-photon micromasers [3], in the limit of high detuning of the middle level of a three-level atom and trapping conditions, thus selecting special values of the interaction time. For finite detuning and no trapping condition, only an *even* photon number state was shown to exist for the two-photon micromaser. The *even* state leads to the squeezed vacuum state [4].

The question of entanglement is important for quantum information processing and its study in the JCM for mixed states has led to the conjecture that entanglement is present at all times for a system that was initially in a disentangled state [5]. This is to be contrasted to results for the time-evolution of an initially disentangled pure state in the JCM that yields an explicit form of the entangled atom-field state [6]. Approximate disentanglement occurs during the collapse region [6] and the atom-field system becomes asymptotically disentangled at precisely half the revival time [7]. Also, in a one-photon micromaser the field may evolve to pure states under appropriate conditions even for mixed-state initial conditions [2].

In the present paper, an exact two-photon Hamiltonian for an atom-field system [8] is used to study the time evolution of pure states and the question of entanglement and disentanglement. For particular initially disentangled pure states, the atom-field system goes through cycles of entanglement separated by instances of disentanglement at which time the system reverts back to its initial disentangled state. If the time of disentanglement is chosen to correspond to the interaction time τ of an atom traversing a high-Q cavity, then the initially disentangled pure states are the steady states reached after many atoms traverse the cavity.

II. TWO-PHOTON HAMILTONIAN

The two-photon Hamiltonian obtained by an exact unitary transformation [8] is

$$H=\hbar\omega N+E_0+\hbar\mu\sigma_{33}+\hbar\eta\sigma_{11}+\hbar\lambda(\sigma_{31}a^2+\sigma_{13}a^{\dagger 2}), \tag{2.1}$$

where a and $a^\dagger$ are photon operators, $\sigma_{ij}=|i\rangle\langle j|$ are the atomic transition operators, and the operator $N=a^\dagger a+\sigma_{33}-\sigma_{11}+1$ is a constant of the motion. The various parameters E_0, μ, η, and λ have been obtained in Ref. [8] and their explicit forms are not needed in the present paper.

The eigenvalues $E_n^\pm$ and eigenfunctions $|\Psi_n^\pm\rangle$ of H are best given in terms of the dressed-atom representation. One has [8] that

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$$|\Psi_n^+\rangle = \sin\theta_n|3, \quad n\rangle + \cos\theta_n|1, \quad n+2\rangle$$

$$|\Psi_n^-\rangle = \cos\theta_n|3, \quad n\rangle - \sin\theta_n|1, \quad n+2\rangle, \tag{2.2}$$

where $|1\rangle$ and $|3\rangle$ are the lower and upper atomic states, respectively, $|n\rangle$ is the photon number eigenstate and

$$\cos\theta_n = \frac{r(n+2)^{1/2}}{[n(r^2+1)+2r^2+1]^{1/2}},$$

$$\sin\theta_n = \frac{(n+1)^{1/2}}{[n(r^2+1)+2r^2+1]^{1/2}}, \tag{2.3}$$

with $r \equiv g_1/g_2$.

The respective eigenvalues are given by

$$E_n^+ = \hbar\omega(n+1) + \frac{E_1+E_3}{2} - \frac{\Delta}{2} + \frac{1}{2}\{\Delta^2 + 4\hbar^2[g_1^2(n+2) + g_2^2(n+1)]\}^{1/2},$$

$$E_n^- = \hbar\omega(n+1) + \frac{E_1+E_3}{2}, \tag{2.4}$$

where g_i's are the atom-photon coupling constants, E_1 is the energy of the lower state, E_3 is the energy of the upper state, and $\Delta = (E_1 - E_2) + \hbar\omega = (E_3 - E_2) - \hbar\omega$ is the detuning parameter of the midlevel of the three-level atom.

III. DISENTANGLED AND STEADY STATES

The time evolution of an arbitrary state of the atom-field system is determined in the interaction picture by the unitary operator $U(t) = \exp^{[-i(H-H_0)t/\hbar]}$, where $H_0 = \hbar\omega(a^\dagger a + \sigma_{33} - \sigma_{11}) + (E_1+E_3)/2$. The eigenstates $|\Psi_n^\pm\rangle$ are simultaneous eigenstates of H with eigenvalues $E_n^\pm$ and of H_0 with eigenvalue E_n^-. The set of eigenstates $|\Psi_n^\pm\rangle$, with $n = 0,1,2 \ldots$, together with the states $|1,0\rangle = -|\Psi_{-2}^-\rangle$ and $|1,1\rangle = |\Psi_{-1}^+\rangle$, where the first index refers to the lower atomic state and the second to the photon-number occupation, form a complete basis. Consider the initial disentangled state of the atom-field system at $t=0$,

$$|\Psi\rangle = \sum_{n=0}^{\infty} s_n|n\rangle(\alpha|1\rangle + \beta|3\rangle). \tag{3.1}$$

In terms of the atom-field dressed states, the initial state is given by

$$|\Psi\rangle = A|1, \quad 0\rangle + B|1, \quad 1\rangle + \sum_{n=0}^{\infty} A_n|\Psi_n^+\rangle + \sum_{n=0}^{\infty} B_n|\Psi_n^-\rangle, \tag{3.2}$$

where the coefficients in Eq. (3.2) are determined by α,β, and the set $\{s_n\}$ and so

$$A = \alpha s_0 \quad \text{and} \quad B = \alpha s_1,$$

$$A_n = \beta\sin\theta_n s_n + \alpha\cos\theta_n s_{n+2}, \tag{3.3}$$

$$B_n = \beta\cos\theta_n s_n - \alpha\sin\theta_n s_{n+2}.$$

The time development of $|\Psi\rangle$ is given by $|\Psi(t)\rangle = U(t)|\Psi\rangle$ and so

$$|\Psi(t)\rangle = A|1, \quad 0\rangle + Be^{-i\omega_{-1}t}|1, \quad 1\rangle + \sum_{n=0}^{\infty} A_n e^{-i\omega_n t}|\Psi_n^+\rangle + \sum_{n=0}^{\infty} B_n|\Psi_n^-\rangle, \tag{3.4}$$

where $\hbar\omega_n \equiv E_n^+ - E_n^-$. In general, $|\Psi(t)\rangle$ is an entangled state. However, we seek conditions under which $|\Psi(t)\rangle$ can become disentangled at $t=\tau$, and be of the form

$$|\Psi(\tau)\rangle = \sum_{n=0}^{\infty} s_n'|n\rangle(\alpha|1\rangle + \beta|3\rangle), \tag{3.5}$$

where $s_n' \equiv s_n(\tau)$. The atomic state is also required to be the same as that of the initial state $|\Psi\rangle$. Disentanglement would occur if the following conditions are satisfied:

$$A = \alpha s_0', \quad Be^{-i\omega_{-1}\tau} = \alpha s_1',$$

$$A_n e^{-i\omega_n\tau} = \beta(\sin\theta_n)s_n' + \alpha(\cos\theta_n)s_{n+2}', \tag{3.6}$$

$$B_n = \beta(\cos\theta_n)s_n' - \alpha(\sin\theta_n)s_{n+2}'.$$

Using Eq. (3.3), one has,

$$s_0' = s_0, \quad s_1' = e^{-i\omega_{-1}\tau}s_1, \tag{3.7}$$

and

$$s_n' = [e^{-i\omega_n\tau}\sin^2\theta_n + \cos^2\theta_n]s_n + \frac{\alpha}{\beta}\sin\theta_n\cos\theta_n[e^{-i\omega_n\tau} - 1]s_{n+2}. \tag{3.8}$$

The disentanglement conditions (3.3) and (3.6) give for $\tau > 0$ that

$$A_{n+2} = \frac{\beta}{\alpha}\frac{\cos\theta_n}{\sin\theta_{n+2}}\frac{(e^{-i\omega_n\tau} - 1)}{(e^{-i\omega_{n+2}\tau} - 1)}A_n, \tag{3.9}$$

and

$$B_{n+2} = -\frac{\beta}{\alpha}\frac{\sin\theta_n}{\cos\theta_{n+2}}B_n + \frac{\beta}{\alpha}\frac{\cos\theta_n}{\cos\theta_{n+2}}\frac{(e^{-i\omega_{n+2}\tau} - e^{-i\omega_n\tau})}{(e^{-i\omega_{n+2}\tau} - 1)}A_n, \tag{3.10}$$

for $n = 0,1,2 \ldots$. For given α,β, and τ, the set of initial coefficients A_0, A_1, B_0, and B_1 determines the rest of the initial coefficients A_n and B_n for $n = 2,3,4, \ldots$. Notice that the set $\{s_n\}$ for the initial state (3.1) are in turn determined uniquely by (3.3), (3.9), and (3.10) from the initial coefficients s_0, s_1, s_2, and s_3. Hence, disentanglement occurs at particular times τ only for specially prepared initial states.

Steady states can be achieved in a cavity by the injection of single atoms, in a coherent superposition of their upper and lower states, into a high-Q cavity. One assumes that the state of the outgoing atom is not measured by taking a trace over the atomic states [1,2]. The successive iteration give the following reduced field density matrix ρ_l after l such atoms have singly traversed the cavity:

$$\rho_l = \mathrm{tr}_a[U(\tau)\rho_{l-1}\rho_a U^{\dagger}(\tau)], \tag{3.11}$$

where the trace is over the atomic states $\rho_a = (\alpha|1\rangle + \beta|3\rangle)(\langle 1|\alpha^* + \langle 3|\beta^*)$, which is the state of the injected atoms. If the limit of the iterations exists, then the resulting state is called a steady state [1,2], which is a fixed point of the map (3.11). Note that our disentangled state (3.5) is a fixed point of this map provided $U(\tau)|\Psi\rangle = |\Psi\rangle$, that is,

$$s_n' = s_n. \tag{3.12}$$

Note that for such cases, representing the evolution of photonic steady states in a cavity as more and more identically prepared atoms traverse the cavity and the states of the outgoing atoms are not measured [1,2], we obtain the following recurrence relation from Eq. (3.8):

$$s_{n+2} = -\frac{\beta}{\alpha}(\tan\theta_n)s_n = -\frac{\beta}{\alpha r}\sqrt{\frac{n+1}{n+2}}s_n \quad (n=0,1,2,\ldots), \tag{3.13}$$

where the second equality follows with the aid of (2.3). The recursion relation (3.13) corresponds to the case $A_n = 0$ for $n = 0,1,2,\ldots$. For the case $s_1 \neq 0 (B \neq 0)$, result (3.12), together with Eq. (3.7), places the following condition on the interaction time τ that atoms spend in the cavity

$$\left[-\frac{\Delta}{2} + \frac{1}{2}\sqrt{\Delta^2 + 4\hbar^2 g_1^2}\right]\frac{\tau}{\hbar} = 2\pi l \quad (l = 1,2,3\ \ldots). \tag{3.14}$$

According to Eqs. (3.4) and (3.14), as the atom traverses the cavity for times $0 \leqslant t \leqslant \tau$, the atom-field system undergoes cycles of entanglement separated by instances of disentanglement at times $t = k\tau/l$, where $k = 0,1,2,\ldots l$. The atom finally emerges from the cavity at $t = \tau$ leaving the atom-field system in the same initially disentangled state.

It should be remarked that our condition on the cavity interaction time τ is not a trapping condition, as is the case for one-photon micromaser [1,2] and other two-photon micromaser models [3,4], but a requirement for the existence of disentangled states. For the one-photon micromaser, the trapping conditions are needed, in addition, since otherwise the steady states asymptotically reached by the cavity mode, viz., the *tangent* and *cotangent* states, would not be normalizable [1,2].

The coefficient s_0 generates the even series in photon numbers $|\Psi_{even}\rangle = \Sigma_{n=0}^{\infty} s_{2n}|2n\rangle$; whereas s_1 generates the odd photon number series $|\Psi_{odd}\rangle = \Sigma_{n=0}^{\infty} s_{2n+1}|2n+1\rangle$. Both states $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$ are normalizable. The recursion relation (3.13) was obtained earlier for a two-photon micromaser [3,4]. However, important differences with the present paper should be noted. In the high detuning limit considered in Ref. [3], trapping conditions limit the Fock space into isolated blocks. In contrast, disentanglement conditions considered in the present paper do not impose such restrictions and the sum over photon states is unrestricted. For finite detuning and no trapping [4], only $|\Psi_{even}\rangle$ occurs, while $|\Psi_{odd}\rangle$ is identically zero. It should be remarked that the even and the odd states considered here are not the so-called even and odd coherent states [9].

To summarize, if the initial state contains only even photon number states, then the unique disentangled steady state is given by $|\Psi_{even}\rangle$. However, if $e^{-i\omega_{-1}\tau} = 1$, then there exists two possible disentangled steady states, viz., $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$, and every disentangled steady state is a linear superposition of these two vectors. Note that the normalization conditions on the states $|\Psi_{even}\rangle$, $|\Psi_{odd}\rangle$, and $|\Psi_{even}\rangle + |\Psi_{odd}\rangle$ give rise to different values for the constants s_0 and s_1.

A particularly interesting case occurs in the high detuning limit when $\omega_n = \hbar(g_1^2 + g_2^2)n/\Delta + \hbar(2g_1^2 + g_2^2)/\Delta$ and where, contrary to the case of finite detuning, one can have $A_n \neq 0$ for $n = 0,1,2,\ldots$ for the steady states by requiring $e^{-i\omega_n\tau} = 1$ for $n = 0,1,2,\ldots$. The latter is accomplished for all n provided both $\hbar(g_1^2 + g_2^2)\tau/\Delta$ and $\hbar(2g_1^2 + g_2^2)\tau/\Delta$ are multiples of 2π, which restricts $r^2 = (g_1/g_2)^2$ to the set of rational numbers. If, in addition, one requires $e^{-i\omega_{-1}\tau} = 1$, then every initially disentangled state will go through periods of entanglement and instances of disentanglement.

The dynamical system also possesses nonseparable mixed states ρ of the form

$$\rho = \sum_{i,j} |\chi_i\rangle\rho_{ij}\langle\chi_j|, \tag{3.15}$$

where $U(\tau)\rho U^{\dagger}(\tau) = \rho$ with $\rho_{ij}^* = \rho_{ji}$ and $|\chi_i\rangle = (|\Psi_{even}\rangle + |\Psi_{odd}\rangle)(\alpha|1\rangle + \beta|3\rangle)$. The index $i = \{s_0, s_1, \alpha, \beta\}$ runs over all values of s_0, s_1, α, and β such that $|\alpha|^2 + |\beta|^2 = 1$ and s_0 and s_1 satisfy the normalization condition (6.2) (below). If $e^{-i\omega_{-1}\tau} \neq 1$, then only the even state appears in (3.15); however, both even and odd states appear when $e^{-i\omega_{-1}\tau} = 1$. The completely disentangled pure steady state corresponds to the case when the sum in Eq. (3.15) reduces to a single diagonal term, viz., $\rho_{ij} = \delta_{i,k} \quad \delta_{j,k}$. Separable mixed states occurs when ρ_{ij} is diagonal and so of the form $\rho = \Sigma_i \rho_{ii}\rho_i^{(f)}\rho_i^{(a)}$, where $\rho_i^{(f)} = (|\Psi_{even}\rangle + |\Psi_{odd}\rangle)(\langle\Psi_{even}| + \langle\Psi_{odd}|)$ and $\rho_i^{(a)} = (\alpha|1\rangle + \beta|3\rangle)(\langle 1|\alpha^* + \langle 3|\beta^*)$.

The disentangled states (3.5) are the states of the combined atom and electromagnetic field that occur when the atom entering the cavity exits the cavity in the same precise state and so the composite atom-field system disentangles as the atom exits the cavity. Note that this behavior cannot occur in the one-photon micromaser [1,2] since the outgoing atom can never exit the cavity in the same precise state it had before it entered the cavity. However, the field in the cavity does evolve to pure, disentangled states [1,2] . This result is to be contrasted with a recent conjecture [5] that for an originally disentangled mixed state in the JCM, entanglement is present at all times except at $t=0$. Note that in Ref. [5] the initial field is in a thermal state and what is needed for the system to evolve to a disentangled pure state is for both atom and field to be in an arbitrary initial state.

IV. EVEN AND ODD PHOTON STATES

As is evident from the recurrence relation (3.13), the disentangled pure steady state gives rise to two mutually orthogonal states. The general expression for the even and the odd expansion coefficients are

$$s_{2n}=\left(-\frac{\beta}{\alpha r}\right)^n\frac{\sqrt{(2n)!}}{2^n n!}\quad s_0$$

and

$$s_{2n+1}=\left(-\frac{\beta}{\alpha r}\right)^n\frac{2^n n!}{\sqrt{(2n+1)!}}\quad s_1\quad (n=1,2,3,\ldots). \tag{4.1}$$

The even photon number state $|\Psi_{even}\rangle$ is equal to the squeezed vacuum state $|\zeta\rangle$ since

$$|\Psi_{even}\rangle\equiv\sum_{n=0}^{\infty}\left(-\frac{\beta}{\alpha r}\right)^n\frac{\sqrt{(2n)!}}{2^n n!}s_0|2n\rangle=\hat{S}(\zeta)|0\rangle, \tag{4.2}$$

where $\hat{S}(\zeta)$ is the squeeze operator [10,11], $\zeta=\sigma e^{i\varphi}$, and

$$s_0=\frac{1}{\sqrt{\cosh\sigma}}\quad\text{and}\quad\frac{\beta}{\alpha r}=e^{i\varphi}\tanh\sigma. \tag{4.3}$$

Consider next the odd photon number state $|\Psi_{odd}\rangle$. Now

$$\begin{aligned}|\Psi_{odd}\rangle&\equiv\sum_{0}^{\infty}\left(-\frac{\beta}{\alpha r}\right)^n\frac{2^n n!}{\sqrt{(2n+1)!}}s_1|2n+1\rangle\\&=\sum_{0}^{\infty}\left(-\frac{\beta}{\alpha r}\right)^n\frac{2^n n!}{(2n+1)!}s_1(a^\dagger)^{2n+1}|0\rangle.\end{aligned} \tag{4.4}$$

Using the relation $\int_0^1 u^n du/\sqrt{1-u}=2^{2n+1}(n!)^2/(2n+1)!$, one obtains from (4.4) that

$$|\Psi_{odd}\rangle=\frac{s_1 a^\dagger}{2}\int_0^1\frac{du}{\sqrt{1-u}}e^{-\frac{\beta u a^{\dagger 2}}{2\alpha r}}|0\rangle. \tag{4.5}$$

The vacuum squeezed state $|\zeta\rangle$ with $\zeta=\sigma e^{i\varphi}$ is given by Ref. [11]:

$$|\sigma e^{i\varphi}\rangle=\frac{1}{\sqrt{\cosh\sigma}}\quad\exp\left[-\frac{a^{\dagger 2}e^{i\varphi}\tanh\sigma}{2}\right]|0\rangle, \tag{4.6}$$

hence,

$$|\Psi_{odd}\rangle=\frac{s_1 a^\dagger}{2}\int_0^1\frac{du}{\sqrt{1-u}}\frac{1}{\sqrt[4]{1-\left|\frac{\beta u}{\alpha r}\right|^2}}|\zeta\rangle, \tag{4.7}$$

where $\zeta=\sigma e^{i\varphi}$ with $\varphi=\arg(\beta u/\alpha r)=\arg(\beta/\alpha r)$ since u is real and $\sigma=\tanh^{-1}|\beta u/\alpha r|$. The state $|\Psi_{odd}\rangle$ has an extra photon added to a continuum of squeezed vacuum states that have constant argument and variable moduli.

V. PHOTON STATISTICS AND QUADRATURE OF THE ODD STATE

The photonic states $|\Psi_{odd}\rangle$ and $|\Psi_{even}\rangle$ form a complete basis for the space of pure disentangled and steady states governed by the dynamics of the two-photon Hamiltonian (2.1). The even photon number state $|\Psi_{even}\rangle$ is equal to the squeezed vacuum state, which exhibits the classical feature of super-Poissonian statistics and the quantum feature of quadrature squeezing [10,11]. The odd photon number state $|\Psi_{odd}\rangle$, on the other hand, exhibits nonclassical sub-Poissonian, as well as the classical super-Poissonian statistics. It is interesting that $|\Psi_{odd}\rangle$ cannot be squeezed and, in fact, its quadrature variance is always greater than one. A feature that makes the states $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$ analogous to the zero- and the one-photon states, respectively, in a quantum bit (qubit) and thus interesting for quantum computation and quantum information are the following results:

$$[a+a^\dagger e^{i\varphi}\tanh\sigma]|\Psi_{even}\rangle=0, \tag{5.1}$$

and

$$[a+a^\dagger e^{i\varphi}\tanh\sigma]|\Psi_{odd}\rangle=s_1|0\rangle, \tag{5.2}$$

where $\beta/\alpha r\equiv e^{i\varphi}\tanh\sigma$. One refers to $|\Psi_{even}\rangle$ as a squeezed vacuum owing to Eq. (5.1). However, $|\Psi_{odd}\rangle$ does not represent a squeezed one-photon state since the quadratures of $|\Psi_{odd}\rangle$ cannot be squeezed (see below).

A. Photon statistics

The state $|\Psi_{odd}\rangle$ of the electromagnetic field has an indefinite number of photons and its statistical description is based on the photon number probability amplitude given in Eq. (4.4). The photon number probability distribution $P(n)$ is

$$P(2n)=0$$

and

$$P(2n+1)=4^n\left|\frac{\beta}{\alpha r}\right|^{2n}\frac{(n!)^2}{(2n+1)!}|s_1|^2\quad(n=0,1,2\ldots), \tag{5.3}$$

where $\langle\Psi_{odd}|\Psi_{odd}\rangle=1$ gives that

$$|s_1|^{-2}=\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\left|\frac{\beta}{\alpha r}\right|^{2n}=\frac{\sin^{-1}\left|\frac{\beta}{\alpha r}\right|}{\left|\frac{\beta}{\alpha r}\right|\sqrt{1-\left|\frac{\beta}{\alpha r}\right|^2}}$$

$$\equiv f\left(\left|\frac{\beta}{\alpha r}\right|\right). \tag{5.4}$$

Now

$$\langle\hat{n}\rangle=\frac{xf'(x)+f(x)}{f(x)}, \tag{5.5}$$

and

$$\langle\hat{n}^2\rangle=\frac{x^2f''(x)+3xf'(x)+f(x)}{f(x)}, \tag{5.6}$$

where $f(x)$ is defined in Eq. (5.4), the number operator $\hat{n}=a^{\dagger}a$, $f^{(n)}(x)=d^nf(x)/dx^n$, and $x\equiv|\beta/\alpha r|$. The variance of $\hat{n}$ is given by

$$\Delta n^2=\frac{x^2f(x)f''(x)+xf(x)f'(x)-x^2(f'(x))^2}{[f(x)]^2}, \tag{5.7}$$

and the Mandel parameter Q by

$$Q=\frac{x^2f(x)f''(x)-x^2(f'(x))^2-[f(x)]^2}{f(x)(xf'(x)+f(x))}. \tag{5.8}$$

The Q parameter assumes negative values for nonclassical, sub-Poissonian statistics and positive for super-Poissonian fields having a classical description. Results (5.5)–(5.8) can all be expressed in terms of elementary functions. A numerical evaluation of Eq. (5.8) gives a monotonically increasing function for Q vs $|\beta/\alpha r|$ assuming negative values for $0\leqslant|\beta/\alpha r|\lesssim 0.547$ with $Q=-1$ at $|\beta/\alpha r|=0$. For $Q=-1$, $|\Psi\rangle=(s_0|0\rangle+s_1|1\rangle)|1\rangle$ with the atom in its ground state $|1\rangle$ and the radiation field in the qubit state $s_0|0\rangle+s_1|1\rangle$ with $|s_0|^2+|s_1|^2=1$. The latter state is the only other disentangled photon steady state besides $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$.

B. Quadrature squeezing

The quantization of the electromagnetic field gives sub-Poissonian photon statistics and nonclassical squeezed states that are used to improve accuracy measurements by limiting quantum noise. The state $|\Psi_{odd}\rangle$ exhibits quantum sub-Poissonian statistics but is not a squeezed state. Consider the phase-sensitive field operator

$$\hat{x}_\lambda=\frac{1}{\sqrt{2}}[ae^{-i\lambda}+a^{\dagger}e^{i\lambda}]. \tag{5.9}$$

Now $\langle\Psi_{odd}|\hat{x}_\lambda|\Psi_{odd}\rangle=0$ and so the variance of Eq. (5.9) for $|\Psi_{odd}\rangle$ is given by

$$\Delta x_\lambda^2=-\tanh\sigma\quad\cos(2\lambda-\varphi)\left[\frac{xf'(x)+2f(x)}{f(x)}\right]+\frac{2xf'(x)+3f(x)}{f(x)}, \tag{5.10}$$

with bounds

$$1\leqslant\langle\hat{n}\rangle=\frac{xf'(x)+f(x)}{f(x)}\leqslant\Delta x_\lambda^2\leqslant\frac{3xf'(x)+5f(x)}{f(x)}. \tag{5.11}$$

Accordingly, the electromagnetic field $|\Psi_{odd}\rangle$ can possess either sub-Poissonian or super-Poissonian statistics but does not represent a squeezed state since the uncertainty associated with each quadrature exceeds the value of $\frac{1}{2}$ associated with the vacuum and the coherent states. This is to be contrasted with the vacuum squeezed state $|\Psi_{even}\rangle$ whose statistics is always super-Poissonian.

C. Quadrature representation

The quadrature representation $\psi(x_\lambda)=\langle x_\lambda|\Psi_{odd}\rangle$ of the odd state, where $|x_\lambda\rangle$ is eigenstate of the quadrature operator (5.9), satisfies the differential equation given by Eq. (5.2)

$$(e^{i\lambda}-e^{-i\lambda}e^{i\varphi}\tanh\sigma)\frac{d\psi(x_\lambda)}{dx_\lambda}+(e^{i\lambda}+e^{-i\lambda}e^{i\varphi}\tanh\sigma)x_\lambda\psi(x_\lambda)=\frac{s_1\sqrt{2}}{\pi^{1/4}}e^{-x_\lambda^2/2}, \tag{5.12}$$

where $\psi(x_\lambda)$ is normalized to unity provided $|s_1|^{-2}=f(|\beta/\alpha r|)$. The solution of Eq. (5.12) is

$$\psi(x_\lambda)=\frac{\pi^{1/4}s_1\alpha(x_\lambda)}{\sqrt{C}}\quad\mathrm{erf}\left(x_\lambda\sqrt{\frac{1}{2}-\frac{D}{2}}\right)+\alpha(x_\lambda)\psi(0), \tag{5.13}$$

where erf(x) is the error function, $\alpha(x_\lambda)=e^{-(D/2)x_\lambda^2}$ is the solution of the vacuum quadrature equation (5.1),

$$D=\frac{1-i\sinh 2\sigma\sin(2\lambda-\varphi)}{\cosh 2\sigma-\sinh 2\sigma\cos(2\lambda-\varphi)}, \tag{5.14}$$

and

$$C=2\tanh\sigma\, e^{i\varphi}[e^{-2i\lambda}e^{i\varphi}\tanh\sigma-1]. \tag{5.15}$$

VI. PHOTON STATISTICS AND QUADRATURE OF THE STEADY STATE

The photonic part of the disentangled steady state is given by

$$|\psi\rangle = \sum_{n=0}^{\infty} s_n |n\rangle = |\Psi_{even}\rangle + |\Psi_{odd}\rangle, \tag{6.1}$$

with the normalization condition

$$1 = \frac{|s_0|^2}{\sqrt{1 - \left|\frac{\beta}{\alpha r}\right|^2}} + |s_1|^2 f\left(\left|\frac{\beta}{\alpha r}\right|\right), \tag{6.2}$$

by using $[1-y]^{-1/2} = \Sigma_{n=0}^{\infty}(2n)!y^n/2^{2n}(n!)^2$. The dynamics governed by the two-photon Hamiltonian (2.1) does not mix the space of even photon numbers with those with odd photon numbers. Accordingly, if the cavity field initially, prior to the entrance of the first atom in the cavity, had probabilities P_0 and P_1 to be in the even and the odd photon states respectively, then those probabilities will remain the same owing to the probabilities being separately constants of the motion. Therefore, $|s_0|^2 = \sqrt{1-|\beta/\alpha r|^2}P_0$ and $|s_1|^2 = f(|\beta/\alpha r|)P_1$ for the steady state.

The variance of $\hat{n}$ in the state $|\psi\rangle$ is given by

$$\Delta n^2 = \langle \Psi_{even}|\hat{n}^2|\Psi_{even}\rangle + \langle \Psi_{odd}|\hat{n}^2|\Psi_{odd}\rangle - (\langle \Psi_{even}|\hat{n}|\Psi_{even}\rangle + \langle \Psi_{odd}|\hat{n}|\Psi_{odd}\rangle)^2 \tag{6.3}$$

and the variance of $\hat{x}_\lambda$ by

$$\Delta x_\lambda^2 = \langle \Psi_{even}|\hat{x}_\lambda^2|\Psi_{even}\rangle + \langle \Psi_{odd}|\hat{x}_\lambda^2|\Psi_{odd}\rangle - (\langle \Psi_{even}|\hat{x}_\lambda|\Psi_{odd}\rangle + \langle \Psi_{odd}|\hat{x}_\lambda|\Psi_{even}\rangle)^2, \tag{6.4}$$

where $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$ are given by Eqs. (4.2) and (4.4), respectively. All the quantities that appear in Eqs. (6.3) and (6.4) are expressible in terms of elementary functions.

Consider the state $|\psi\rangle$ for which initially $P_0 = P_1 = \frac{1}{2}$, viz., $|\Psi_{even}\rangle$ and $|\Psi_{odd}\rangle$ are equally probable and so $|s_0|^2 = \sqrt{1-|\beta/\alpha r|^2}/2$ and $|s_1|^2 = f(|\beta/\alpha r|)/2$. A numerical evaluation of the Mandel Q parameter versus $|\beta/\alpha r|$ for $|\psi\rangle$ gives a monotonically increasing function of $|\beta/\alpha r|$, which assumes negative values for $0 \leqslant |\beta/\alpha r| \lesssim 0.36$ and positive values for $|\beta/\alpha r| \gtrsim 0.36$. In the former case $|\psi\rangle$ is sub-Poissonian and in the latter super-Poissonian. A numerical evaluation of the variance Δx_λ^2 versus $|\beta/\alpha r|$ also gives a monotonically increasing function that assumes the value of $\frac{3}{4}$ for $|\beta/\alpha r| = 0$. Accordingly, the state $|\psi\rangle$, with equal admixtures of the even and the odd states, is not a squeezed state. Therefore, for $|\beta/\alpha r| \gtrsim 0.547$ the photonic state $|\psi\rangle$ is not a squeezed state, its statistics is super-Poissonian, and is given by a linear superposition of two super-Poissonian states.

VII. SUMMARY AND CONCLUSIONS

The time evolution of initially pure disentangled atom-field states are studied under the action of two-photon processes. If the photonic part of the disentangled state is a superposition of a squeezed vacuum and the *odd* photonic state, then the composite system undergoes cycles of entanglement with instances of disentanglement. The period of such cycles is given by τ/l in Eq. (3.14). These disentangled composite states are the steady states that would evolve in a lossless cavity and the atom-field interaction time is given by τ. For appropriate coherent superpositions, the field radiation evolves into a pure state that is a macroscopic quantum superposition of two super-Poissonian photonic states, a Schrödinger-cat state.

The *odd* states have both classical and quantum features since they can possess both super-Poissonian and sub-Poissonian statistics, albeit the state is not a squeezed state. Accordingly, the steady state is a linear superposition of a squeezed, super-Poissonian state plus a state that can possess either super-Poissonian or sub-Poissonian statistics, but is not a squeezed state. The nature of the initial state of the electromagnetic field in the cavity determines the properties of the overall photonic steady state. One can produce steady states composed of a continuous admixture of a squeezed vacuum and an *odd* photonic states by the proper initial admixture in the cavity of even- and odd-photon-number states.

It is shown that the squeezed vacuum and the *odd* states are analogous to the zero-photon (the vacuum) and the one-photon states, respectively. This correspondence suggests the use of the squeezed vacuum and the *odd* states as basic constituents for qubits for quantum computation and quantum information.

[1] J.J. Slosser and P. Meystre, Phys. Rev. A **41**, 3867 (1990).
[2] J.J. Slosser, P. Meystre, and E.M. Wright, Opt. Lett. **15**, 233 (1990).
[3] M. Orszag, R. Ramirez, J.C. Retamal, and L. Roa, Phys. Rev. A **45**, 6717 (1992).
[4] M. Orszag, L. Roa, and R. Ramirez, Phys. Rev. A **48**, 4648 (1993).
[5] S. Bose, I. Fuentes-Guridi, P.L. Knight, and V. Vedral, Phys. Rev. Lett. **87**, 050401 (2001).
[6] S.J.D. Phoenix and P.L. Knight, Phys. Rev. A **44**, 6023 (1991).
[7] V. Bužek, H. Moya-Cessa, P.L. Knight, and S.J.D. Phoenix, Phys. Rev. A **45**, 8190 (1992).
[8] M. Alexanian, S.K. Bose, and L. Chow, J. Mod. Opt. **45**, 2519 (1998).
[9] J. Gea-Banacloche, Phys. Rev. Lett. **65**, 3385 (1990); M. Brune, S. Haroche, J.M. Raimond, L. Davidovich, and N. Zagury, Phys. Rev. A **45**, 5193 (1992); S.J.D. Phoenix and P.L. Knight, Phys. Rev. Lett. **66**, 2833 (1991).
[10] R. Loudon and P.L. Knight, J. Mod. Opt. **34**, 709 (1987).
[11] S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).